INTENSITY FLUCTUATIONS AND PHOTOELECTRIC MIXING OF LIGHT BEAMS

BY

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We accept this thesis as conforming to the required standard

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September, 1964
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Date September 1964
ABSTRACT

Photoelectric mixing in a photodiode is used in this work as a statistical spectroscopic tool. A number of experiments were performed to determine the fluctuation spectrum generated by this process and the statistical properties of the light which might be deduced from the data.

Due to practical limitations in attainable temperatures, blackbody sources were not able to produce an observable mixing above shot noise. Experiments were also carried out using line spectra from gas discharge lamps. The best source available was a 300 W Xenon lamp, emitting lines in the red, which under optimum conditions produced excess current fluctuations equal to 60% of shot noise.

The observation of photoelectric mixing using a gas laser source has already been reported in the literature, but the Gaussian distributed electric field model usually applied does not fit the experimental results. A new model was proposed in this thesis which considers the laser light as a narrow band of coherent light embedded in a relatively broad band of spontaneous light. Mixing between the signal and the spontaneous emission was considered to be the only observable effect due to experimental limitations. This model appeared to fit the data and gave some information about the statistical properties of the laser beam.
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CHAPTER 1

INTRODUCTION

1-1 Photoelectric Mixing

Photoelectric mixing (or optical heterodyning) involves the interaction of two light beams of different frequencies in any non-linear device sensitive to them. If the frequency spectrum of the light beam is continuous the self-mixing signals produced appear as a continuous spectrum with the low frequency portion extending from zero frequency.

The same effect has been used in radio communications to change the frequency of signals. No difficulties are encountered since very monochromatic signal sources are available up to microwave frequencies. With the possible exception of the laser, no such light sources exist so mixing at optical frequencies was hardly considered to be of practical value. In recent years interest has arisen about the possible use of optical mixing as a spectroscopic tool.

A photodiode is one device that can be used to produce mixing. A semiconductor photodiode, biased in the reverse direction, can be fabricated with a response that is extremely linear with light intensity. Since the light intensity is proportional to the square of the electric field strength $E$ of the light waves, this type of photodiode will act as a square law device.
The dc light current \((I_L)\) generated in the photodiode is given by

\[ I_L = aEE^* \]

Although the validity of this square law assumption was doubted, an experiment performed by Forrester (1955) proved that it is correct since mixing was observed.

The derivation of equations for photoelectric mixing has been carried out (Alkemade, 1959 & Forrester, 1961) for a single spectral line from a light source producing a Gaussian distribution of the electric field strengths. The results are essentially the same as in a previous paper (Rice, 1944) concerned with the effect of a square law device on a noise spectrum input. A brief discussion and review of the derivation is given in Appendices A & B.

The results show that for a given light spectrum of width \(\Delta \nu\) (cps) the ratio \(M(f)\) of the excess current fluctuations to shot noise \((2eI_L)\) at very low frequencies \((f \ll \Delta \nu)\) is given by

\[ M(0) = s(\gamma^2/A \Omega)(I_L/e \Delta \nu) \]

where

- \(\gamma\) = the mean wavelength of the light
- \(\Omega\) = the solid angle subtended by the source at the detector
- \(A\) = the illuminated area of the detector
- \(s\) = a numerical factor near unity depending on the shape of the spectrum

The same sort of result was deduced by Mandel (1958) using Bose-Einstein statistics of the light photons. If
the arrival of a photon could be exactly associated with
the ejection of an electron the photoelectric current would
also have the same statistics. If $\bar{n}$ is the mean number of
photons arriving in a time $T_m$ the variance of the number
of electrons observed in the same time interval will be
\[ \text{var } n = \bar{n}(1 + b) \]
where $b$ is the mean number of photons per unit cell in
phase space. However, as Mandel points out, due to the
uncertainty principle and a quantum efficiency (electrons
out per photon in) less than unity, the photons and
photoelectrons cannot be directly associated with each
other. He derives two results depending on the length of
the measuring time. For the case of interest $\Delta \gamma T_m \gg 1$
the variance of the number of electrons observed becomes
\[ \bar{n}(1 + sb) \]
where again $s$ is a shape factor. The first term is the
same as given by Poisson statistics and is just the usual
shot noise. The second term is some excess fluctuation
that can be associated with the mixing derived using the
wave interaction model. In Mandel's model the excess
current fluctuations are directly related to fluctuations
in the intensity (or number of photons) in the light beam
which are greater than those given by Poisson statistics.

The equivalence of these two models can easily
be seen if a particular case is worked out. Take for
example blackbody radiation in which the average number
of photons of energy $h \nu$ in a unit frequency interval and leaving a unit source area per second into a unit solid angle per independent polarization component is given by Planck's radiation law

$$n(\nu) = \left(\frac{1}{\gamma^2}\right) \left[\exp\left(\frac{h \nu}{kT}\right) - 1\right]^{-1} = \frac{b}{\gamma^2}$$

where $T$ is the radiation temperature. If a small portion of this spectrum with a width $\Delta \nu$ is allowed to fall on the detector so that $n(\nu)$ is nearly constant over the range then the dc current generated will be

$$I_L = A \mathcal{R} \Delta \nu n(\nu)$$

where

- $A =$ the source area
- $\mathcal{R} =$ the solid angle subtended by the detector
- $\nu =$ the mean frequency of the light

Here it is assumed that the quantum efficiency is unity and there are no losses in the optics. Note that the value of $A \mathcal{R}$ defined here is equal to the one defined in the mixing equation since $A \mathcal{R}$ is determined by the product of the source area and the detector area. Therefore, the source and detector can be interchanged without changing the value of $A \mathcal{R}$. Putting this value of the dc current into the mixing equation gives a mixing ratio

$$M(0) = s/\left[\exp(h \nu/kT) - 1\right] = sb$$

This is exactly the same answer as given by Mandel's treatment so the two methods are equivalent.

This formula is the same as the one used by Alkemade (1959) to calculate for sunlight ($T = 6000^\circ$K)
that \( M(0) = 0.3 \) for a wavelength of 18,000 Å which is beyond the response of the silicon and germanium diodes used in the present experiments. For this \( \Delta \nu \) must be small enough that the mixing is uniform over the interval yet large enough that background noise in the detector is negligible. At 7000 Å this ratio becomes only 0.03 and other losses are expected to reduce this still more. Obviously there is little hope of observing photoelectric mixing with a blackbody source using photodiodes whose threshold lies in the visible spectrum.

Using a spectral line the mixing ratio should increase by at least an order of magnitude. The width of a gas discharge spectral line is expected to be of the order of \( 10^{10} \) cps with the lamps used since no precautions were taken to reduce broadening. A mercury green line at 5460 Å with a Gaussian line shape and a lamp giving \( I_L/AJ_2 = 1 \) A/cm\(^2\)-steradian the ratio will be 0.5. Again losses have been neglected but this should be observable if a powerful enough lamp can be found. This implies an equivalent radiation temperature of 23,000 °K which is very high even for a gas discharge lamp. The factor \( I_L/AJ_2 \) is a true figure of merit for a lamp since it is invariant under simple optical transformations if losses are neglected.

If the radiation contains two or more spectral lines the mixing ratio for the same current will be less.
This is because inter-mixing is not observed in low frequency measurements if the lines are separated by a sufficiently large frequency interval. In fact, for \( N \) identical spectral lines (same intensity and width), the observed value of \( M(0) \) will be a factor \( 1/N \) less than the value if only one spectral line was producing the same dc current. The only advantage of using a number of spectral lines is if the extra current is necessary to make the other background noise negligible or if the lines cannot be separated without large optical losses.

Forrester (1961) discusses at some length the effect of the shape of the light spectrum on the low frequency mixing. There is some ambiguity as to the definition of the width of the spectral line but he points out that in terms of the observed quantities

(1) the dc light current 
(2) the upper cut-off frequency of the mixing spectrum

the value of the shape factor \( s \) can be determined

<table>
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<tr>
<td>Rectangular</td>
<td>(.5)</td>
</tr>
<tr>
<td>Gaussian</td>
<td>( \sqrt{\ln 2/\pi} \approx .48 )</td>
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<tr>
<td>Lorentzian</td>
<td>( 1/\pi \approx .32 )</td>
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It will be very difficult to distinguish between the first two cases. Forrester also computes graphs showing the shape of the mixing spectrum for these line shapes. In this thesis a Gaussian line shape is assumed since it is probably the closest to the actual shape of a spectral line because of Doppler broadening.
Coherence of Light Sources

"Ordinary" light can be considered as being made up of a large number of waves. Each wave is produced in a different part of the source at a different time and is therefore unrelated to any other. The resultant light has a continuous spectrum restricted to a frequency band determined by the characteristics of the source. If the electric field strength of the light can be represented by (Eqn 1, Appendix A)

\[ E = \sum_{n=1}^{N} c_n \cos(2\pi \nu_n t - \psi_n) \]

where the \( \psi_n \) are phase angles uniformly distributed over the range 0 to \( 2\pi \) the source is said to be Gaussian random. This is because as \( N \to \infty \) the "central limit theorom" (Sec 2.10, Rice, 1944) can be applied. This theorom states that the sum of a large number of independent random vectors approaches a normal distribution as the number tends to infinity.

On the other hand in laser light, produced by stimulated emission, there is a definite phase and wave vector relation between the production of different waves. The light is then said to be coherent because the number of independent radiators has been theoretically reduced to one. However, even a laser beam is not strictly monochromatic (as inferred by a single radiator) so that in that sense it is only highly coherent.
The degree of coherence is discussed in length in Chapter X of a book by Born and Wolf (1959) in connection with the visibility of interference fringes. Consider two points $P_1$ and $P_2$ in the radiation field at some distance from the source. If $E(P_1, t)$ represents the electric field strength at the point $P_1$ and a time $t$, then a mutual coherence function

$$\langle \gamma \rangle = \frac{E(P_1, t+\gamma)E(P_2, t)}{\langle E^2 \rangle}$$

can be defined where $\gamma$ is the time delay introduced by the separation of the two points. In the present experiments the detector area is assumed to be plane and perpendicular to the incident radiation so that $\gamma = 0$ and only spatial coherence is of interest.

In this case $|\langle \gamma \rangle| = 0$ unless the two points are very close together and the radiation at the two points is said to be non-coherent. The area to which $P_1$ and $P_2$ must be restricted is called the coherence area and is of the same order of magnitude as the area of the diffraction pattern ($\lambda^2 / \Omega$) formed by a single source point.

Another, possibly better, way of considering the area of coherence is from the point of view of the number of independent oscillators required to duplicate the radiation field (Forrester, 1956). The number of degrees of freedom $N$ associated with the radiation emitted in a time $t$ from an area $A$ into a solid angle $\Omega$
\[ N = 2ARt \Delta \lambda / \pi^2 \]

In a coherence time \( t_c = 1/\Delta \lambda \) this is just the number of independent oscillators \( n \) setting up each polarization component of the field:

\[ n = AR/\pi^2 \]

Therefore, the area corresponding to each oscillator is

\[ A_c = \pi^2 / \Delta \lambda \]

In Appendix A an attempt is made to introduce the coherence area into the mixing equations in a more natural manner. To do this two separate assumptions about electric field vectors not in the same coherence area must be made:

(i) there is no relation between their phases
(ii) they cannot interact non-linearly with each other

These assumptions are equivalent to considering the photodiode as being broken up into a number of independent detectors each of area \( \pi^2 / \Delta \lambda \). The mean square fluctuations from each coherence area then simply add.

If the detector area \( A \gg A_c \) the photomixing spectrum (as compared with the coherent case) will be reduced by a factor

\[ \pi^2 / A \Delta \lambda \]

In this case the source is said to be resolved by the detector.
Review of Previous Experiments

The only experiment reported to date with a single illuminated photodiode and a non-coherent light source was performed by Forrester et al. (1955). They used the mercury green line as a source but had to use Zeeman splitting to increase the apparent value of $M(f)$ from $10^{-4}$ to 2. This was accomplished by modulation of the mixing signals but leaving the shot noise constant by making use of the polarization properties of the Zeeman components. Since their observations were at frequencies near $10^{10}$ cps they were not observing self-mixing but instead mixing between the Zeeman components. The only important result of their observations is that optical mixing in a photodiode does occur, a fact that had previously been doubted.

A recent experiment (Bolwijn et al., 1963) was performed using a laser source. They tried to explain their results by the theory for Gaussian distributed electric field strengths but assuming $\Phi/A \lambda = 1$ and a very narrow line width. The awkward plot of the results makes it impossible to tell how good an agreement was achieved. In any case, the bandwidth and the low frequency mixing ratio observed are not consistent with the equations used. The authors comment on the rather large optical bandwidth (19 kc) obtained but their explanation,
attributing it to plasma resonances, appears to be unsatisfactory.

In another paper (Cummins, 1963) this rather large optical bandwidth is said to be caused by "an admixture of a large number of off-axis modes" present as well as the purely axial ones.

In this thesis it is proposed that the model is fundamentally incorrect when applied to a laser source. After all, a laser can hardly be considered to be a source of Gaussian random light.

The same conclusion was reached by Bellisio et al (1964) from the results of their experiments. They observed three types of laser photomixing spectrums

(i) high noise similar to Bolwijn's observations
(ii) spectrum with spikes and little excess noise between the spikes
(iii) low noise spectrum

The last case is assumed to be the only fundamental one and they observed no excess photocurrent fluctuations above shot noise. However, their observations are restricted by enhanced shot noise from the photomultiplier so that they could only set a rough upper limit to the amount of mixing. They attributed the high noise operation to instabilities in the plasma because this type of spectrum was not observed in rf excited lasers.

For these reasons it was considered necessary to
repeat these experiments and to try to develop different models for the results obtained. At the same time since no one has reported successful observation of self-mixing with an ordinary light source, these experiments were also performed.
Fig 2-1  Block Diagram of the Experimental Apparatus
APPARATUS AND EXPERIMENTAL TECHNIQUE

A block diagram of the experimental apparatus is shown in Fig 2-1. The amplifier is a Model 103 Keithley Low Noise amplifier with a gain of 1000 and a pass band from 100 cps to a maximum of 100 kc. The noise spectrum was measured using a Quan-Tech Wave Analyzer with a frequency range of 30 cps to 100 kc. Fixed bandwidths of 10, 30, 100 or 1000 cps are available. To give a full scale deflection a minimum signal of 0.1 mV is required. A meter time constant of 1 sec provides a smooth response.

A Singer Model SB-12b Panalyzor was used as a continuous monitor of the frequency spectrum from zero to 100 kc. The voltage vs frequency spectrum of the signals is displayed on a cathode ray tube. The panalyzor operates in a standard pass band of 450 kc to 550 kc and in order to operate outside this range a local oscillator has to be used to translate the signal frequency.

The panalyzor was designed for radio frequency work where the signal frequency either lies in or above the pass band. Therefore, the local oscillator frequency usually does not fall in this range so no attempt was made to suppress the carrier amplitude. However, to display the spectrum desired here a local oscillator frequency of 550 kc is required and the extreme magnitude
Fig 2-2 Transistor Balanced Modulator
of the carrier (.3 V) overshadows the frequency spectrum below 10 kc. All signals below 20 kc were observed to lose amplitude on the panalyzor display as compared with the same input signal strength at higher frequencies. Since a great deal of information about instabilities and other harmful effects is contained in this portion of the frequency spectrum, some means of removing this had to be devised.

A balanced modulator, external to the panalyzor, was built which allows the signals to be in the required pass band. It was found that to remove the undesirable condition only a rough balance to reduce the carrier amplitude to near the signal level is required which is an advantage since the carrier can be used as a marker for zero frequency. This type of modulator can simply be built using transistor circuits eliminating the need for transformers and high voltage as well as filament supplies.

The circuit built is shown in Fig 2-2. The transistors T1 and T2 which have equal load resistances in their collector and emitter circuits give the phase inversion instead of the usual transformers. The carrier input is then applied to two gating transistors T3 and T4 in the emitter leads of the mixing transistors T5 and T6. The gating transistors give some gain and allow the carrier to be introduced to the mixing transistors in
much the same way as with the second grid of a vacuum
tetrode. The mixed output of the two transistors is
taken off a common load resistor. The carrier is
suppressed since under ideal balance conditions two
equal and out of phase components appear in the output.
The mixed signals will however be in phase.

Carrier balance is obtained by setting the
variable resistances R2 and R3 to their matching values.
The carrier could be balanced to full scale on the log
scale of the panalyzor and is quite stable over periods
of hours. Further balance by using precision resistors
and matched transistors was not considered necessary
for the present purposes. Sufficient improvement was
obtained to allow resolution from the carrier of a
500 cps signal on the 100 kc bandwidth. There was no
decrease in the amplitude of these low frequency signals.

The panalyzor was used as a monitor only and
no actual measurements were made from its display. For
this reason no attempt was made to account for losses
in the modulator or to calibrate the panalyzor.

2-2 The Standard Noise Source

A direct comparison method was used to compare
the noise output of an illuminated photodiode with that
from a standard noise diode. In order to make this
method more accurate as many components as possible were
Fig 2-3  Circuit Diagram of the Noise Source

Fig 2-4  Ac Equivalent Circuit of the Noise Source
made common to both circuits. The circuit used is shown in Fig 2-3. Dry cells are used as power supplies and the heater current for the noise diode is supplied by a regulated 6.3 V power supply. The noise diode standard is a Sylvania 5722 vacuum diode.

The ac equivalent circuit for the noise diode is shown in Fig 2-4.

\[ \begin{align*}
R_d &= \text{the diode plate resistance} \\
C_d &= \text{the diode capacitance} \\
R_{in} &= \text{the amplifier input resistance} \\
C_{in} &= \text{the amplifier input capacitance}
\end{align*} \]

Using the usual approximations as well as

\[ R_d \gg R, \quad R_1 \ll R_{in} \quad \text{and} \quad C_d \ll C_{in} \]

the low frequency cut-off of the circuit is given by

\[ f_L = \frac{1}{2\pi C_1(R_1 + R)} \]

and the high frequency cut-off by

\[ f_H = \frac{(R_1 + R)}{2\pi C_{in}R_1R} \]

The mid-frequency noise voltage spectrum from the circuit is

\[ S_V = S_I \left[ \frac{R_1R}{(R_1 + R)} \right]^2 \]

Inserting the values of the circuit elements used it is found that the cut-off frequencies are about 1 kc and 400 kc for \( R_{in} = 10 \, \text{M}\Omega \) and \( C_{in} = 20 \, \text{pF} \). The low frequency response is then determined by this circuit while the amplifier restricts the high frequencies.

If the vacuum diode is operating in the temperature limited region it will generate full shot
noise which is produced since the current is made up of
discrete electrons. Theoretically pure shot noise has
a uniform spectrum which means that $S_f$ has the same
magnitude at all frequencies. This will only be true
up to frequencies of the order of $1/\tau$ where $\tau$ is the
electron transit time. Since this frequency is much
larger than 100 kc in a vacuum diode this need not be
considered here.

The measured spectrum will not be uniform due
to circuit limitations. Shot noise from the diode will
be given by

$$S_f(f) = 2eI_{dc}$$

but if the circuit response is $A(f)$ the measured current
spectrum will be

$$S(f) = 2eI_{dc}A^2(f)$$

For a load resistance $R$, a wave analyzer bandwidth $\Delta f$
(an interval small enough that $S(f)$ can be assumed to
be constant over it) and an amplifier gain $G$ the mean
square voltage will be

$$\overline{(V^2)_{ND}} = 2eI_{dc}R^2A^2(f)\Delta fG^2$$

If the photodiode current contains pure shot
noise and other signals with a spectral density $W(f)$,
the measured mean square voltage in the wave analyzer
will be

$$\overline{(V^2)_{PD}} = \left[2eI_{dc} + W(f)\right] R^2A^2(f)\Delta fG^2$$
The ratio of the two readings will be
\[ 1 + \frac{W(f)}{2eI_{dc}} = 1 + M(f) \]
Here the value of the current in the noise diode is assumed to be made equal to the light current generated in the photodiode. This function \( M(f) \) will be completely determined from the experimental data and independent of the circuit response. Although \( M(f) \) has the same frequency dependence as \( W(f) \) it is a dimensionless spectrum referred to the value of shot noise.

2-3 Semiconductor Photodiodes

2-3-1 Theory of Operation

The photodiode consists of a p-n junction which has one side exposed to the incident radiation. The remaining sides are usually surrounded by a non-transparent material. A larger area photodiode may typically be made by covering one surface of a sheet of n-type material with a thin layer of p-type material. A very thin layer of non-reflective material may also be applied to the surface to reduce radiation losses and to protect the surface from humidity effects.

The p-n junction is operated in a reverse bias condition when used as a light detector in order to approach a linear characteristic response with the light intensity. If the reverse bias exceeds a few tenths of
Reverse biased p-n junction

Fig 2-5 Photodiode V-I Characteristic
a volt the resulting current is very nearly independent of
the bias voltage. With no illumination there is still a
current \(-I_o\), the dark current) which is the normal diode
reverse saturation current produced by the thermally
excited carriers.

If light falls on the junction additional hole-
electron pairs are formed by the absorption of a photon
and corresponding excitation of an electron from the
valence band to the conduction band. This can happen as
long as the photon energy \(h\nu\) exceeds the width of the
energy gap (1.1 eV for silicon and .72 eV for germanium).
For a well saturated photodiode the light current \(I_L\) is
proportional to the rate of photon absorption or
equivalently proportional to the light intensity. The
total photodiode current is then

\[
I = -(I_o + I_L)
\]

A typical V-I characteristic is shown in Fig 2-5.

The current in the photodiode depends on the
diffusion of the minority carriers across the junction.
These minority carriers may recombine before reaching the
junction so that only a small area near the junction is
effective in producing a current. This current will also
be extremely temperature dependent due to variations in
the density, mobility and lifetime of the carriers with
temperature.
Fig 2-6 Equivalent Noise Circuit of a Photodiode
2-3-2 Noise Considerations

Noise in semiconductor devices and in particular the photodiode is considered in detail by van der Ziel (1959). The noise equivalent circuit is shown in Fig 2-6 where the current generators represent the mean square fluctuations in the photocurrent.

The light current $I_L$ shows full shot noise plus some other noise represented as a fraction of shot noise by $M(f)$ which may be a function of frequency. The detector noise equivalent dark current $I_d$ is used to represent all circuit noise sources when there is no illumination.

$$I_d = I_0 + \frac{2kT}{eR} + \frac{2kT_d}{eR_d}$$

Again in a photodiode with good saturation it is expected that $R_d \gg R$. The following amplifier will also introduce a noise into the circuit. If the amplifier noise figure is $F$ the equivalent noise current (at its input) will be

$$I_A = (F-1) \frac{2kT}{Re}$$

following the usual assumptions about the relative magnitudes of the circuit resistances.

The total noise spectrum out of the circuit with an illuminated photodiode is then

$$2e \left[ I_d + I_A + (M + 1)I_L \right]$$

From all this the excess fluctuations $M(f)I_L$ must be deduced. It is the ratio of these excess fluctuations to the total noise that determines the resolution limit.
Fig 2-7 1/f Noise in a TP50 Photodiode

Conditions: \( I_L = 500 \, \mu A \)
\( \Delta f = 100 \, \text{cps} \)
Fig 2-8  Spectral Response of Various Photodiode Types
of the experiments.

At low frequencies so-called 1/f noise was observed in many of the photodiodes. At 1 kc this noise might be as high as 20 times pure shot noise. The graph in Fig 2-7, taken with one of the worst photodiodes illuminated with light from a tungsten filament lamp, the frequency dependence of the excess noise is shown to be almost exactly 1/f.

1/f noise has been generally attributed to surface effects either at the diode surface, grain boundaries or dislocations in the crystal (van der Ziel, 1959). Even with the same material, processed in the same way, the amount of excess noise differs from diode to diode by a large factor. 1/f noise also depends on the bias voltage and operation at voltages near breakdown may increase the ratio by as much as a factor of ten.

2-3-3 Comparison of Photodiodes

In the selection of a photodiode for the experiments a number of different properties were considered. In each case at least two diodes of each type were tested.

Each material has its own characteristic spectral response. Approximate curves for each of the three general types are shown in Fig 2-8. For the vacuum type this is limited by the material on the
cathode and the glass used in the envelope. The semiconductor types are limited by the width of the energy gap and the depth of the depletion layer.

Vacuum photodiodes were not used in any of the experiments because of their low quantum efficiency which was measured in the RCA 929 type to be about .08 and their low allowable currents for reliable operation. Currents above the maximum permissible (about 20 \( \mu \)A) can be used but this leads to very short lives of the diodes.

The germanium type (TP 50) used is quite sufficient for most of the experiments. Its small area limits its usefulness in some cases but the main disadvantages are a high dark current and 1/f noise. By choosing the best diode and operating at as low a bias as possible the latter can be reduced to almost negligible amounts for frequencies above 1 kc.

In the range from 7000 A\(^0\) to 10,000 A\(^0\) the silicon diodes are probably the best. They have low dark currents due to the wider energy gap of silicon. However, the LS 223 silicon photodiodes could not be used due to their poor frequency response. There are two main reasons main reasons for frequency limitations in any diode

1. a finite transit time
2. the diode capacitance

The transit time probably limits the frequency response
<table>
<thead>
<tr>
<th>Manufacturer and Type #</th>
<th>TP 50 F10D</th>
<th>TI LS223</th>
<th>SSR 10L267</th>
<th>RCA SJ2411</th>
</tr>
</thead>
<tbody>
<tr>
<td>Material</td>
<td>Ge</td>
<td>Si</td>
<td>Si</td>
<td>Si</td>
</tr>
<tr>
<td>Sensitive Area</td>
<td>.1 mm²</td>
<td>1 mm²</td>
<td>25 mm²</td>
<td>5 mm²</td>
</tr>
<tr>
<td>Dark Current</td>
<td>2 μA</td>
<td>1 μA</td>
<td>.3 μA</td>
<td>.075 μA</td>
</tr>
<tr>
<td>Operating Voltage</td>
<td>22.5 V</td>
<td>9 V</td>
<td>22.5 V</td>
<td>9 V</td>
</tr>
<tr>
<td>Peak Current</td>
<td>1 mA</td>
<td>.5 mA</td>
<td>1 mA</td>
<td>200 μA</td>
</tr>
<tr>
<td>Frequency Response</td>
<td>very high</td>
<td>0-10 kc</td>
<td>0-100 kc</td>
<td>beyond 100 kc</td>
</tr>
<tr>
<td>1/f noise</td>
<td>low in some</td>
<td>low</td>
<td>very low</td>
<td>negligible</td>
</tr>
</tbody>
</table>

* TP  Transistor Products  
   TI  Texas Instruments  
   SSR  Solid State Radiations

Fig 2-9  Table of Photodiode Characteristics
of the LS 223 but with the large area SSR 10L267 nuclear detectors the capacitance appears to be the limiting factor. Using the usual parallel plate capacitance formula

\[ C_d = \varepsilon A/d \]

where

- \( \varepsilon \) = the dielectric constant for silicon
- \( A \) = the detector area
- \( d \) = depth of the depletion layer

and the data given for the detector the capacitance of the 10L267 is calculated to be 50 pF. This reduces the circuit cut-off to slightly less than 100 kc.

The RCA SJ2411 nuclear particle detectors were the best photodiodes tested. No 1/f noise could be observed down to a frequency of 500 ops. Their dark current was measured to be 0.075 \( \mu A \) which makes this negligible in all measurements and the diode capacitance is sufficiently low that there is no reduction in the high frequency circuit response. The quantum efficiency of these detectors was measured to be around 0.8. The reduction from unity is probably mostly due to reflection from the surface.

A table of the properties of all the photodiodes considered is shown in Fig 2-9.
Fig 3-1  Photodiode Noise Spectra

Source: Tungsten Filament lamp and #700 optical filter

Conditions:

\[ I_L = 75 \, \mu A \]

\[ \Delta f = 100 \, \text{cps} \]
CHAPTER 3

EXPERIMENTAL RESULTS (NON-COHERENT LIGHT)

3-1 Blackbody Light Source

A blackbody spectrum is defined as a continuous spectrum of light with a frequency distribution obeying Planck's radiation law such as exists in a vacuum cavity in thermal equilibrium. Blackbody-type sources were chosen as a starting point for the experiments even though calculations show (Sec 1-1) that no observable mixing can occur. This type of source served as a test for the photodiodes and an indication that the apparatus was working properly.

The lamp used was a prefocused, 150 W, Sylvania DFA type with a tungsten filament. This will actually be a greybody source since tungsten has an emissivity of .45 at 2000 °K. The radiation has the same frequency distribution but only .45 the intensity as that from a blackbody source at the same temperature.

The optics used consisted of a simple condensing lens with optical filters protected by a heat resistant glass. Power for the lamp was supplied by the dc mains in the lab.

An Optics Technology Monopass filter #700 was used to restrict the spectrum to a 150 A° band around 7000 A°. The shape of the filter pass band is somewhat Gaussian. The transmission of the filter was about 35% at its peak. The results are shown in Fig 3-1 for the three main types of photodiodes used and under the conditions noted. The shot noise from the noise diode at same operating current is
identical with the curve for the SJ 2411 shown. The excess noise at low frequencies for the other diodes is due to 1/f noise and the high frequency falloff of the 10L267 is due to its high diode capacitance.

This light source was also used to check the quantum efficiency of the photodiodes. In each case a rough calculation showed that for the semiconductor photodiodes the quantum efficiency is between 0.5 and 0.8.

The best blackbody source available is probably still the sun. Its temperature is about 6000 °K which is a factor of three larger than the tungsten filament lamp. The light could be focused to about 2 mm² area but due to the rotation of the earth the spot moved considerably during the measuring time. For this reason the largest area diode (10L267) had to be used. With the same optical filter as before and a solid angle of .02 steradian a light current of 75 μA was generated in the detector. The expected ratio of the mixing fluctuations to shot noise is given by Eqn 2, Appendix B to be

\[ M(0) = \left( \frac{\lambda}{A} \right) R (I_L/2e\sqrt{B}) = 2 \times 10^{-4} \]

This is much less than the ratio calculated by Alkemade's formula (Sec 1-1) but may be attributed to losses and the shape of the optical filter pass band. This is much too small to be measured and only the usual discrepancies for the diode from pure shot noise were observed. To observe any mixing an improvement of at least a factor of 1000 will be necessary.
3-2 Mercury Vapour Lamp

An Osram mercury vapour lamp operating at 50 V and 1.2 A was used to generate the mercury spectrum. Again the dc mains in the lab were used as a power supply. An Electro Products power supply was also tried but the excessive 120 cps ripple prevented measurements.

Operation on dc current proved to be extremely difficult. A great deal of instability was observed at frequencies below 10 kc and definite oscillations were observed at 3.9 kc and 750 cps. The 750 cps oscillations were traced to fluctuations at this frequency in the power supply. This was done by inserting the photodiode output on the horizontal plates of an oscilloscope and a sample of the power supply voltage on the vertical plates and checking for correlations at different frequencies. At 3.9 kc no such correlation was found so these must be plasma oscillations inside the lamp.

Insertion of a 70 $\mu$F capacitor directly across the lamp tended to remove much of the instability. The 3.9 kc oscillations disappeared and would not return when the capacitor was removed unless the lamp was moved suddenly or the operating point changed slightly. The amount of instability was also noted to depend strongly on the operating current. The lamp would extinguish if the current varied by more than 10% from the operating value. This was especially critical during the warmup period which lasted
nearly 10 minutes.

Using the large area 10L267 detector considerable current could be generated when a single spectral line was isolated. The best line was the 5790 Å one isolated by the #566 filter. This monopass filter has a bandwidth of 100 Å and a transmission of 25% at 5790 Å. Measurements at 75 μA showed only full shot noise in the part of the spectrum undisturbed by instabilities. This is to be expected since a large solid angle (about a steradian) and a large detector area (25 mm²) were necessary to obtain this current. A coherence factor \( \frac{\lambda^2}{A \lambda} = 10^{-8} \) thus removes any possibility of observing mixing.

The other extreme was also tried where the complete spectrum was allowed to fall on the photodiode. In this case the solid angle was reduced to such an extent that the same amount of current was generated in a small area (.1 mm²) TP 50 photodiode. Again only pure shot noise and plasma instabilities were observed.

External modulation experiments were also carried out by applying a few tenths of a volt signal across the ballast resistor in the lamp circuit. This modulation was picked up by the detector and displayed on the panalyzeor. Harmonic free signals up to 100 mV could be generated in the detector output. This is about a 4% modulation of the dc current. The width of the modulation signal was less than 10 cps as shown by the variable bandwidth of the analyzer.
When modulation was carried out in the neighbourhood of 3.9 kc, the internal oscillations were observed to start up again. They sometimes remained for a minute or so after the modulation was removed. After this time they died out again if the capacitor across the lamp was in the circuit. The discharge seemed to be somewhat more stable when modulation was applied. This may be due to power being dissipated in this way and not being allowed to build up in other modes.

Use of higher power mercury lamps with better optics might lead to observation of some mixing. However, due to the uncertainty in the widths of the spectral lines this is somewhat doubtful. An electrodeless rf discharge might provide a better source because the discharge is expected to be more stable and there will be no darkening of the vapour envelope due to electrode sputtering.

Further experiments using other Osram discharge lamps (Cd, Cs and K) were performed but the lower power from these lamps again made observation of mixing impossible. Similar instabilities were also observed in these lamps.

3-3 The Xenon Lamp

Due to the appearance of a factor \( \frac{\gamma}{B} \approx \frac{\lambda}{c} \Delta \lambda \) in the mixing equation (Eqn 2, Appendix B) it was decided that it would be better to use a lamp giving strong radiation in the red and infrared portion of the spectrum and an xenon lamp was chosen. A Hanovia discharge lamp operating at 20 V
Fig 3-2 Photomixing Spectrum

Source: Xenon Lamp

Photodiode: TP 50

Conditions:
\[ I_L = 40 \, \mu A \]
\[ \Delta f = 100 \, \text{cps} \]
and 15 A was used with a Trygon M36-30 A power supply with 0.05% regulation and constant current facilities.

The optical arrangement used a solid angle at the detector of 0.01 steradian. A Corning #2-64 optical filter was used to restrict the spectrum to above 7000 Å. The current generated in a TP 50 photodiode was 40 μA. Cooling of the lamp was provided by a small fan. This produced a noise spectrum in the photodiode current about 60% higher than pure shot noise and the results are shown in Fig 3-2. At frequencies below 5 kc plasma instabilities were observed to dominate the spectrum but the value of M(f) remained quite constant over the range from 10 kc to 100 kc.

The mixing ratio also remained relatively constant over long periods of time but variations from as low as 20% to as high as 100% were observed to occur at irregular intervals. This might be explained by changes in the line width (due to fluctuations in the temperature and pressure in the lamp) or possibly due to changes in the intensity of the spectral lines.

Tests were also made to determine if the excess noise is due to photoelectric mixing. Using a SJ 2411 detector with 50 times the area only pure shot noise was observed. Again using the TP 50 diode but with the solid angle increased by a factor of 100 no excess noise could be measured. Note that in actual fact it is not these changes that decrease the mixing ratio since the light
current will increase proportionally but the decrease is due to the light attenuators inserted to hold the current constant. No other phenomena could be found which might produce excess noise that would be affected in these ways. For this reason it is concluded that photoelectric mixing was observed with this light source.

There are about nine main spectral lines in the light allowed to fall on the diode. Their wavelengths and relative intensities quoted from "MIT Wavelength Tables, 1939" are

<table>
<thead>
<tr>
<th>( \lambda (\text{A}^\circ) )</th>
<th>Relative Intensity</th>
</tr>
</thead>
<tbody>
<tr>
<td>10,838</td>
<td>1000 arbitrary units</td>
</tr>
<tr>
<td>9,923</td>
<td>2000</td>
</tr>
<tr>
<td>9,799</td>
<td>2000</td>
</tr>
<tr>
<td>8,952</td>
<td>1000</td>
</tr>
<tr>
<td>8,819</td>
<td>5000</td>
</tr>
<tr>
<td>8,409</td>
<td>2000</td>
</tr>
<tr>
<td>8,346</td>
<td>2000</td>
</tr>
<tr>
<td>8,280</td>
<td>5000</td>
</tr>
<tr>
<td>8,231</td>
<td>5000</td>
</tr>
</tbody>
</table>

Which of these lines were actually present and contributing to the mixing was not determined.

Assuming a single spectral line at say 8,500 \( \text{A}^\circ \) the line width required to give a mixing ratio of .6 would be (using Eqn 2, Appendix B which assumes a Gaussian line shape)

\[
M(0) = \left( \frac{\lambda^2}{\lambda^2} \right) \left( \frac{I_L}{2eB} \right) = 0.6
\]

\[
B = 8.5 \times 10^{10} \text{ cps}
\]

\[
\Delta \lambda = 2\left( \frac{\lambda^2}{c} \right) \sqrt{21n2/B} = 4.8 \text{ A}^\circ
\]
If all nine spectral lines had the same intensity and width then neglecting the different wavelengths the width of each would be about .55 Å (Sec 1-1). These values are reasonable for the conditions assumed.

Due to the variations in the amount of mixing over which there seemed to be no control, accurate measurements of the dependence of $M(0)$ on $I_L$, $A$, or $\Omega$ could not be made.
Fig 4-1 He-Ne Energy Level Diagram (Simplified)
CHAPTER 4

EXPERIMENTAL RESULTS (COHERENT LIGHT)

4-1 Operation of the Laser

A Spectra-Physics Model 130 gas phase laser was used as a coherent source of light. It is a continuous emission He-Ne laser using dc plasma excitation. The theory of gaseous optical masers is discussed by Bennett (1962). The output of this laser at 6328 Å is plane polarized due to the Brewster window termination of the plasma tube. The cavity resonator is formed by a pair of highly reflective, multilayer dielectric mirrors (one plane and one spherical) mounted outside the tube.

This laser is essentially a four level system with one He and three Ne levels participating. A simplified energy level diagram for the production of the 6328 Å line is shown in Fig 4-1. The He atoms are excited by the dc induced plasma discharge to level #2 which is 20.66 eV above the ground level. Since the Ne atoms have an energy level very close to the same energy (20.66 eV) they can be excited by collisions and absorption of some thermal energy. The Ne atom in level #3 then de-excites by one of the possible transitions. However, if a photon with a wavelength of 6328 Å interacts with it first, induced emission of another identical photon can occur. The Ne atom in level #4 then de-excites to the ground level ready to start the cycle again.
Since the gas pressure in the plasma tube is low, Doppler broadening is expected to determine the width of the spontaneous emission line that would occur if there was no resonant cavity present.

\[ \Delta \nu_D = \frac{(2/\gamma)}{\sqrt{2kT(1n2)/M_{Ne}}} \]

where \( M_{Ne} \) is the mass of the neon atoms. Based on a radiation temperature of 325 °K the full width at half maximum \( \Delta \nu_D \) of the Doppler broadened emission line is 1400 Mc. The distance between the mirrors is \( L = 30 \text{ cm} \) giving an interval of

\[ c/2L = 500 \text{ Mc} \]

between successive longitudinal modes. Using a standard Q representation the width of the cavity resonance at half maximum will be (Eqn 2, Appendix E)

\[ \Delta \nu_c = cq/4\pi L = 800 \text{ kc} \]

where \( q = 0.01 \) is the fractional energy lost per transit of the cavity. The width of the cavity resonance is then small enough to separate the different modes and simultaneous oscillation in three longitudinal modes within the Doppler width is then possible.

The output beam is taken off one end of the cavity by making that mirror only 99% reflective. The power output of this laser is rated at .2 mW. The beam diameter is 2.2 mm at the exit aperture and has a divergence of 1.3 min of arc. This is equivalent to a solid angle of \( 1.1 \times 10^{-7} \text{ steradian} \) and a coherence factor \( \gamma^2/\Delta \lambda \approx 1 \).
Fig 4-2 Photomixing Spectrum

Source: Laser
Photodiode: SJ 2411
Conditions: $I_L = 7.8 \ \mu A$

$\Delta f = 10 \ \text{cps}$
4-2 Experimental Results

Measurements were made on the frequency spectrum, from 500 cps to 100 kc, of the photodiode current generated by the laser radiation. Three different photodiodes were used but all the results are similar. The experimental results for the SJ 2411 detector are shown in Fig 4-2 under the conditions indicated. This graph shows the variation of M(f) with frequency plotted on a log-log scale. Both the background noise and the dark current have been subtracted from the data.

Plasma resonances at about 40 kc and 80 kc were observed with the latter probably just the second harmonic of the former. These oscillations have a width of about 10 kc and made observation of an important part of the spectrum impossible. Also 120 cps ripple with harmonics up to the 8'th from the laser power supply made low frequency measurements very difficult. To reduce this effect a narrow bandwidth of 10 cps was used in the wave analyzer for the measurements.

No lenses were used to concentrate the beam since the SJ 2411 detector collected all the light and in fact light attenuators (passing about 10%) had to be used to prevent overloading of the amplifier by the 120 cps hum. Because of the low light current (about 10 μA) a current in the noise diode that gave about the same amount of noise was used instead of equal currents as previously. The
Fig 4-3  Current Characteristic of Photomixing Spectrum

Source: Laser
Photodiode: SJ 2411
Conditions: $f = 5$ kc

$\Delta f = 10$ cps
photodiode current was measured with a Keithley Model 409 Picoammeter.

When the laser intensity control was varied the plasma resonances were observed to change frequency, but they always remained in the range from 40 kc to 50 kc with corresponding second harmonics. Also at certain positions of the intensity dial other spurious signals were observed on the oscilloscope used to monitor the photodiode current. These signals appeared to be co-ordinated with the 120 cps hum so may have been a defect in the power supply. However, this is quite similar to the high noise operation observed by Bellisio et al (1964). At other settings the operation of the laser remained stable over periods of hours once the laser had warmed up.

The graph in Fig 4-3 shows the variation of $M(f)$ at 5 kc with changes in the photodiode current. The relation is quite linear as expected but at currents above $50 \mu A$ some levelling off occurs. This was attributed to overloading of the amplifier by extreme signals at frequencies below the measurements. The different currents were obtained by leaving the laser intensity constant and placing light attenuators in the path of the beam.

Another experiment was performed to determine if the factor $\frac{\gamma^2}{AN}$ should be included in the equations for the mixing of laser light. The value of this factor for the conditions is almost exactly unity and will remain so as
long as all the laser light is collected by the diode (see Sec. 1.1). The diameter of the beam was further restricted by an aperture so that the area of the photodiode that was illuminated was reduced by a factor of 10. In this case the value of $I_L/A_\lambda$ is expected to remain constant and thus also the value of $M(f)$ if the coherence factor is included. However, the value of $M(f)$ was observed to decrease exactly in proportion to the decrease in $I_L$. It must then be concluded that a factor of $\gamma^2/A_\lambda > 1$ does not appear in the equations for the mixing of laser light. No method could be found to obtain a factor $\gamma^2/A_\lambda < 1$ with this gas laser.

4-3 Interpretation of the Results

The discrepancy of these results from the model suggested by Bolwijn et al. (1963) is even more striking than their results. Using a value of $B = 40$ kHz, which is reasonable from the data, the low frequency value of $M(f)$ for a Gaussian random light source would be

$$M(0) = I_L / 2eB_{\gamma} = 3.4 \times 10^8$$

which is greater than the observed value by a factor of $4 \times 10^6$. Obviously this model cannot be applied and a laser is anything but a Gaussian random source.

An alternative model was suggested (Burgess, 1964) in which the output of the laser is assumed to consist of a substantially monochromatic stimulated signal embedded in a background of spontaneous emission. A discussion of
Fig 4-4 Laser Photomixing Spectrum
(data as in Fig 4-2)
the application of this model to photoelectric mixing is given in Appendices C, D and E.

With this model if \( r \) is the ratio of the signal power to the total power then the total area under the low frequency mixing spectrum is (Eqn 1, Appendix C)

\[
(1 - r^2)I_L^2
\]

which from Fig 4-2 is approximately

\[
M \times 2eI_L \times f_{\text{max}} = 7.0 \times 10^{-18} \quad \text{A}^2
\]

Therefore \((1 - r) \simeq (1 - r^2)/2 = 5.7 \times 10^{-8}\)

giving a signal to noise ratio for this laser (Eqn 3, Appendix C)

\[
S/N \simeq 1/(1 - r) = 1.8 \times 10^7
\]

This can alternatively be considered as the ratio of the light current generated by the stimulated signal to that generated by the spontaneous emission background.

The low frequency mixing spectrum for a symmetrical stimulated signal is given by this model to be (Eqn 2, Appendix C)

\[
W_L(f) = \left[4(1 - r)I_L^2/\sqrt{2\pi B}\right] \exp(-f^2/2B^2)
\]
or the dependence of \( M(f) \) on frequency has the form

\[
M(f) = \left[2(1-r)I_L/\sqrt{2\pi} eB\right] \exp(-f^2/2B^2)
\]

The graph in Fig 4-4 shows a plot of \( \log M(f) \) vs \( f^2 \) for the same data as in Fig 4-2. For a true Gaussian spectrum of spontaneous emission this graph would be a straight line but the curve becomes flat at low frequencies and tends to fall off too rapidly at higher frequencies. Therefore,
either the spontaneous emission line does not have a Gaussian spectrum or this symmetrical signal model does not apply. However, in Appendix D considerations are given to the possibility that the stimulated signal may not be centred on the spontaneous emission line. The graphs (Figs D-1 and D-2) computed appear to fit this type of experimental curve. The solid line in Fig 4-4 shows the theoretical conditions which best fit the data. The parameters for this curve are

\[ B = 27.5 \text{ kc} \]
\[ g = |\nu_p - \nu_0| / B = 1.0 \]

This means that the spontaneous emission line has a full width at half maximum

\[ \Delta \nu_s = 2 \sqrt{2 \ln 2} \cdot B = 65 \text{ kc} \]

and the stimulated signal is displaced 27.5 kc from the central position.

The agreement of this curve with the experimental data, except for the plasma resonances, is quite remarkable. Similar curves and agreement were obtained using the other types of photodiodes and different light currents. The value of B remained constant within 10% and the same value of g had to be used in each case.

The width of the spontaneous emission line is expected to be determined by the cavity Q since the Doppler broadened line width and the mode spacing are much larger than the cavity resonance width. However, the experimental results give a spontaneous line width of 65 kc which is a
Fig 4-5  Simplified Spectral Diagram of the
Laser Radiation

(the relative magnitudes on the vertical scale are not intended to have any meaning)
a factor of 12 lower than the expected cavity resonance width of 800 kc. This may be due to the fact that the cavity acts as an active device instead of a passive one as assumed in the derivation of $\Delta \nu$.

In Appendix E considerations are given to the output power of a laser in terms of the observed data (Eqn 1, Appendix E)

$$P = (S/N)h\nu qc/2L$$

Inserting the values of these quantities gives $P = 0.03$ mW. This is almost a factor of ten down from the rated value of 0.2 mW, confirmed from the observed maximum photocurrent. The main assumptions in the derivation of this formula are linearity rate equation and a single mode in oscillation. Both of these are probably incorrect, leading to the above discrepancy. However, this does indicate that the observed S/N is at least in the correct range of values.

A possible explanation for the non-central stimulated signal may be the phenomena of "mode pulling" which is discussed by Bennett (1962). The basis for this is that under certain conditions the cavity is forcing an oscillation at a large frequency interval from the natural resonance frequency of the material. The resonant modes then tend to be pulled towards this central frequency.

A simplified spectral diagram of the laser light is shown in Fig 4-5. The dashed line is the Doppler emission line that would occur if the cavity was not present. Here it is assumed for simplicity that three modes, separated by
500 Mc because of the cavity length, are in oscillation and that the central mode is at the actual material resonance frequency $\nu_b$. The cavity mode width $\Delta \nu_c$ is the width of the spontaneous spectrum measured in the experiment to be 65 kc. The spikes on these modes are the stimulated signals.

To a first order approximation the displacement of the stimulated signal is proportional to the ratio of the material Q to the cavity Q and the displacement of the cavity resonance from the material resonance. This leads to an equation for the amount of pulling (Eqn 40, Bennett, 1962)

$$|\nu_p - \nu_o| = |\nu_D - \nu_o| \Delta \nu_c / \Delta \nu_o$$

if $\Delta \nu_c \ll \Delta \nu_o$

An appreciable displacement will only occur in the outer modes for which

$$|\nu_D - \nu_p| \sim |\nu_D - \nu_o| = 500 \text{ Mc}$$

giving

$$|\nu_p - \nu_o| \approx 23 \text{ kc}$$

This value is close to the value of 27.5 kc deduced from the observations. However, no attempt was made to separate the mixing spectrums of the 3 separate modes assumed in this explanation. This may change the observed value of $|\nu_p - \nu_o|$ slightly. Also in fitting the theoretical curve to the data there is some insensitivity (within 10%) to the value of the parameter $g$.

In this explanation, it is assumed that the spontaneous emission in the mode is not affected by mode
pulling. This seems reasonable since the stimulated signal is in oscillation and is thus strongly coupled to both the material in the cavity and the cavity itself, but once the spontaneous light is emitted it is coupled only to the cavity.

The model has now yielded three separate properties of the laser radiation which are at least within an order of magnitude of the values deduced from purely theoretical arguments. The discrepancies are probably due to factors not included in the theoretical derivations.
The results of the experiments prove that under certain conditions photoelectric mixing can be observed with a non-coherent light source emitting visible radiation. However, this is only possible with high power lamps operating in the long wavelength portion of the visible spectrum.

The information obtained is of little use since a single spectral line could not be isolated. In order to isolate a single spectral line a higher power lamp, than the one used here, will be necessary so that the generated noise will be sufficiently above the background noise to give an observable signal to noise ratio in the analyzer. In the experiment, even though the current fluctuation spectrum is 60% higher than pure shot noise, this corresponds to an analyzer signal to noise ratio (see Sec 2.3-2) of only .35. Another means of increasing this ratio for a given light current is to go to longer wavelengths in the infrared because of the appearance of a factor $\lambda^2/\beta$ in the mixing equation for a Gaussian-distributed source field (Eqn 2, Appendix B).

With a spectral line there are two unknowns, the line shape and width. To determine these unambiguously two independent observations, the low frequency magnitude and the upper cut-off frequency of the mixing spectrum, are required. The upper cut-off frequency is approximately of
the same order of magnitude as the width of the spectral line which is around $10^{10}$ cps and beyond easy measurement. If the spectral line width was known from spectroscopic methods, very accurate low frequency measurements might be used to determine the line shape. However, it is expected that the interpretation of the data would be difficult if not impossible. If both the line shape and width were known a check of the photoelectric mixing theory could be made. In particular, this method could be used to determine if the output electric field of the light does in fact have a Gaussian amplitude distribution.

With laser light it is possible to easily cover the entire low frequency mixing spectrum. Both the line shape and width should then readily be determined but the interpretation of the data is still in doubt. The model applied in this thesis is highly idealistic but seems to fit the experimental results extremely well. In fact the model is probably much more general than it appears at first sight.

The electric field of the stimulated emission was assumed to have a constant amplitude. If the stimulated line was assumed to have a finite width (this is expected to be at most a few cps) and the electric field some statistical distribution, the resultant mixing spectrum would not be expected to change very much. A further mixing spectrum (much stronger than the observed one) would appear at very low frequencies with a cut-off at the
assumed width of the stimulated line.

Any intensity distribution is expected to have a time constant $\tau \approx 1/\Delta v_{stim}$. Therefore, any variations in intensity should be observable from fluctuations of the photocurrent since $\Delta v_{stim} \approx 1$ cps. Meter observations suggest that $\Delta I/I < 0.05$ putting an upper limit on the intensity fluctuations.

The type of experiment performed by Cummins (1963) is of limited value in determining the laser line shape or width. Here two separate modes are mixed together to give a beat spectrum at a frequency equal to the mode spacing. These observations can be used to determine the number of modes that are oscillating (Bennett, 1962) but there is no reason for assuming that these modes have the same width and intensity distribution. In this case unless detailed knowledge about one of the modes is known, little can be deduced about the other. Also additional factors like the relative frequency stability of the modes come into the observations. Also note that according to the model applied in this thesis it is expected that only the mixing between the stimulated signals will be observed since this will overshadow any effect from the spontaneous emission.

In another experiment by Bellisio et al (1964) the complete frequency spectrum from 14 cps to 12 Mc was observed with a number of different lasers. They represent their data in terms of modulation of the electric field
strength. They used a photomultiplier which has a poor quantum efficiency and gives enhanced shot noise thus reducing their ability to detect modulation of less than 1%. For this reason they were not able to detect any mixing when the laser was in so-called quiet operation. In the present experiment the sensitivity has been increased so that mixing could be observed quite readily.

No mention has been made in the literature about using large area nuclear detectors as photodiodes. For the low frequency mixing experiment they have a number of advantages over the ordinary type of diode. Because of the large area they allow some experiments to be performed which previously required vacuum photodiodes. There is no window on these detectors and the surface is extremely flat. However, their high capacitance reduces the upper cut-off frequency to well below a megacycle. Detectors with a much higher resistivity than 1,000 ohm-cm (as used in the present experiments) would allow the use of a higher reverse bias and because this increases the depletion layer depth the capacitance is reduced.

There are still unanswered questions about the properties of laser radiation and the concept of coherence which cannot be resolved without further experiments and theoretical work beyond the scope of this thesis. It has only been shown that photoelectric mixing can be an important tool for these investigations.
APPENDIX A

PHOTOELECTRIC MIXING EQUATION FOR A SPECTRAL LINE

Critique of Alkemade (1959) and Forrester (1961)

Consider a general spectrum of light with a power per unit bandwidth \( G(\nu) \) at a frequency \( \nu \). Split this band into a large number of equal intervals each of width \( \Delta \nu \).

Let the intensity of the \( m \)'th interval be

\[
(\Delta I)_m = 2 G(\nu_m) \Delta \nu = C_m^2
\]

corresponding to an electric field representation

\[
E = \sum_{m=1}^{M} C_m \cos (2\pi \nu_m t - \phi_m)
\]

where \( \phi \rightarrow \phi_m \) are phase angles distributed uniformly from 0 to \( 2\pi \).

This assumption along with letting \( M \rightarrow \infty \) as \( \Delta \nu \rightarrow 0 \) are the conditions necessary for the "central limit theorem" which states that a large number of independent random vectors tends to a normal distribution as the number tends to infinity. This is identical with assuming a Gaussian random source (see Sec 1-2).

The output current of a photodiode is given by

\[
I = a E^2
\]

(See Sec 1-1)

These assumptions make the theory identical to the "Noise Through a Square Law Device" considered by Rice (Sec 4.5, 1944). Up to this point the treatment has been the same as done by Forrester (1961) but at this time it seems to be advantageous to introduce the concept of coherence areas.

If the light covers a detector area \( A \) which is larger than the coherence area \( A_c = \lambda^2 / 4 \), there will be no
relation between a $\gamma_m$ on one coherence area and a $\gamma_m$ on another. Divide the detector area into $A/A_c = A/R/\gamma^2$ areas of coherence and represent the electric field by

$$E = \sum_{m=1}^{M} \sum_{q=1}^{A/A_c} C_m \cos \left(2\pi \chi_n t - \gamma_m a \right)$$

In terms of the light spectrum

$$G(\gamma_n) \Delta \nu = (\Delta I)_m = \frac{E^2(\gamma_n)}{2}$$

$$= \sum_{a}^{(1/2)} \sum_{b}^{\Delta} \sum_{c}^{A/A_c} \sum_{m}^{C_m} \cos \left[2\pi \chi_n t \right]$$

There are two separate averaging processes here. The time average removes the first term for all values of $\gamma_n \neq 0$. The ensemble average removes the second term unless $\gamma_m a = \gamma_m b$, which only occurs if $a = b$.

$$2G(\gamma_n) \Delta \nu = \sum_{a}^{(1/2)} \sum_{b}^{\Delta} \sum_{c}^{C_m} \sum_{m}^{\cos \left[2\pi (\chi_n + \chi_n) t \right] - (\gamma_m a + \gamma_m b)}$$

The photodiode current is

$$I = (a/2) \sum_{m}^{(1/2)} \sum_{n}^{\Delta} \sum_{b}^{A/A_c} C_n \cos \left[2\pi \chi_n t \right] - (\gamma_m a + \gamma_m b)$$

Taking the low frequency component

$$\gamma_m - \gamma_n = f_k$$

or $m - n = k$

$$I(f_k) = (a/2) \sum_{n}^{(1/2)} \sum_{a}^{A/A_c} C_n \cos \left[2\pi f_k t \right] - (\gamma_m a - \gamma_m b)$$

The dc light current is the ensemble average of $I(f_k)$ when $k = 0$.

$$I_\alpha = \sum_{a}^{(1/2)} \sum_{n}^{A/A_c} C_n \cos \left[2\pi (\gamma_m a - \gamma_m b) \right]$$

$$= (a/2) \sum_{n}^{A/A_c} C_n \sum_{a}^{\cos \left[2\pi \gamma_n \Delta \nu \right]}$$

$$\Rightarrow I_\alpha \rightarrow a \int G(\gamma) d\gamma \quad \text{as} \quad \Delta \nu \rightarrow 0 \quad \text{and} \quad M \rightarrow \infty$$
In going over to the integral in the last line it is
assumed that \( G(\nu) \) is negligible for values of \( \nu \) near 0.

If \( k \neq 0 \) there are two separate terms in \( I(f_k) \)

\[
\begin{align*}
(1) & \quad m > n \text{ or } k > 0 \\
(11) & \quad m < n \text{ or } k < 0
\end{align*}
\]

These are identical since cosine is an even function.

\[
I(f_k) = a \sum_{n} \sum_{a} \sum_{b} \sum_{c} \sum_{d} c_{n+k} c_{n} \cos \left( 2 \pi f_k t - (\nu_{n,a} - \nu_{n,b}) \right)
\]

where \( k > 0 \) always. The mean square value of the current
fluctuations is

\[
\overline{I^2(f_k)} = a \sum_{n} \sum_{a} \sum_{b} \sum_{c} \sum_{d} c_{n+k} c_{n} \frac{c_{q} c_{q+k}}{\cos \left( 2 \pi (\nu_{n,a} - \nu_{n,b} - \nu_{p,c} + \nu_{p,d}) \right)}
\]

When all the averages are taken the coherence factor drops out. However, if it is further assumed that mixing cannot occur between two different areas of coherence this is not so. This is the same as saying that the detector is made up of \( \frac{\lambda}{\lambda_0} \) separate detectors and the mean square current fluctuations generated on each detector are independent of that generated on any other. Therefore

\[
\overline{I^2(f_k)} = (2a \Delta f) \sum_{n} \sum_{a} \sum_{c} \sum_{d} \frac{c_{n+k}^2 c_n^2}{c_{q+k} c_q}
\]

\[
= 2a \sum_{n} \sum_{a} \sum_{c} \sum_{d} \left( \frac{\lambda}{\lambda_0} \right)^2 G(\nu) G(\nu + \Delta f + k) \Delta \nu^2
\]

\[
\int_{0}^{\infty} I^2(f) \rightarrow 2a \Delta f \int_{0}^{\infty} \left( \frac{\lambda}{\lambda_0} \right)^2 G(\nu) G(\nu + \Delta f) d\nu
\]

where \( \Delta f \) is the bandwidth of the circuit used to measure
the fluctuations. If \( G(\nu) \) is only non-zero over an
interval \( \Delta \nu \ll \nu_0 \) (as will be true for most optical spectra)
the coherence factor may be taken outside the integral.

The ac spectral density of the photodiode current
fluctuations becomes,

\[ W_L(f) = 2 \alpha (\chi/A) \int G(\nu) G(\nu + f) \, d\nu \quad (2) \]

which is the same answer as derived by Forrester (1961) and Alkemade (1959).

In the derivation of this result a number of assumptions were necessary

(i) A Gaussian distribution of electric field vectors
(ii) The photodiode has a perfect square law response to the electric field strength
(iii) No relation between the phases of the electric field vectors in different coherence areas
(iv) \( G(\nu) \) is negligible near \( \nu = 0 \).
(v) Each coherence area acts as an independent diode
(vi) If \( \chi^2/A \lambda \) is greater than unity the whole diode area lies within an area of coherence so the factor should be replaced by unity in the above equation
(vii) \( \Delta \nu \ll \nu \) where \( \Delta \nu \) is the full width at half maximum of \( G(\nu) \)

The validity of the above equation necessarily depends on how good these assumptions correspond to actual fact.
APPENDIX B

PROPERTIES OF THE PHOTOMIXING SPECTRUM OF A SPECTRAL LINE

If the entire low frequency spectrum of the current fluctuations is observed the variance of the current is just the area under the spectrum

\[ \text{var } I = \int W_L(f) \, df \]

If the mixing spectrum is given by Eqn 2, Appendix A then

\[ \text{var } I = 2a^2(\gamma^2/4\pi\lambda) \int g(\nu) \, d\nu \int g(\nu + f) \, df \]

\[ = a^2(\gamma^2/4\pi\lambda) \left[ \int g(\nu) \, d\nu \right]^2 = (\gamma^2/4\pi\lambda)I_L^2 \quad \text{---(1)} \]

Therefore, for any source with a Gaussian-distributed electric field output, the relative variance of I is equal to the coherence factor \((\gamma^2/4\pi\lambda)\).

Assuming a Gaussian line shape (corresponding to a Doppler broadened emission line) centred at a frequency \(\nu_0\) and with a standard deviation \(\Delta\nu\)

\[ g(\nu) = (\nu/\sqrt{2\pi}B) \exp \left[-(\nu - \nu_0)^2/2B^2\right] \]

then

\[ I_L = a \int_0^n g(\nu) \, d\nu = aD \]

and the low frequency mixing spectrum is

\[ W_L(f) = (\gamma^2/4\pi\lambda)(I_L^2/\sqrt{\pi}B) \exp -(f^2/2B^2) \]

If \(\Delta\nu\) is the full width at half maximum then

\[ \Delta\nu = 2\sqrt{2\ln2} \, B \]

The ratio of the excess current fluctuations to shot noise at very low frequencies \((f \ll B)\) is then

\[ M(0) = (\gamma^2/4\pi\lambda)(I_L/2a/\sqrt{\pi}B) \quad \text{---(2)} \]
APPENDIX C

PHOTOELECTRIC MIXING EQUATION FOR A LASER SOURCE

For a source with some degree of coherence new equations will have to be derived in which the $\gamma$'s are not assumed to have a uniform distribution. Only the extreme case in which the stimulated light is represented as a pure sinusoidal signal (perfect coherence) will be considered. A model of the laser radiation might be a stimulated signal embedded in a spontaneous emission line.

Assuming that the electric field strength of the light can be represented by

$$ E = C \cos 2\pi \nu_p t + \sum_m C_m \cos (2\pi \nu_m t - \gamma_m) $$

this again reduces to one of the input spectra considered by Rice (1944) in Sec 4.5. For a square law device he derives the low frequency mixing spectrum to be

$$ W_L^T(f) = a^2 C^2 \left[ g(f - \nu_p) + g(f + \nu_p) \right] $$

$$ + 2a^2 \int g(\nu) g(f+\nu) \, d\nu $$

(4.5-13, Rice)

The average light current is given by

$$ I_L = aC^2/2 + a \int g(\nu) \, d\nu $$

(4.5-11, Rice)

$$ = aC^2/2 + aD $$

Integrating the low frequency spectrum gives

$$ \int W_L^T(f) \, df = I_L^2 \left[ 1 - \left( \frac{aC^2}{2I_L} \right)^2 \right] $$

$$ = (1 - r^2) I_L^2 $$

(1)

where

$$ r = \frac{aC^2}{2I_L} $$

is the ratio of the signal power to the total power in the laser radiation.
A "signal to noise ratio" of the laser radiation may be defined as

\[ \frac{S}{N} = \frac{C^2}{2D} = \frac{r}{(1-r)} \]

This is the ratio of the current generated by the stimulated emission to that generated by the spontaneous emission.

Assuming a spontaneous emission line with a Gaussian shape and a centrally located stimulated signal the low frequency mixing spectrum becomes

\[ W_L(f) = \left[ (1-r) I_L^2 / \sqrt{\pi B} \right] \left[ \left( \frac{4r}{\sqrt{\pi}} \right) \exp \left( -\frac{f^2}{2B^2} \right) + (1-r) \exp \left( -\frac{f^2}{4B^2} \right) \right] \]

If the signal power is much greater than the spontaneous power (i.e. \( r \approx 1 \)) only the first term is appreciable

\[ W_L(f) = \left[ 4(1-r) I_L^2 / \sqrt{\pi B} \right] \exp \left( -\frac{f^2}{2B^2} \right) \]

and \( \frac{S}{N} = \frac{1}{(1-r)} \)

If the signal is very weak (i.e. \( r \ll 0 \)) the spectrum reduces to exactly the same as derived for a single spectral line.

Throughout this derivation it has been assumed that the light falls within one coherence area as this is the usual case for a gas laser. If this is not so a similar argument to that in Appendix A may be used.
Fig D-1  Mixing of Gaussian Line with Non-central Signal
Fig D-2  Mixing of Gaussian Line with Non-central Signal
The equations derived at the end of Appendix C assume that the stimulated signal is centrally located on the spontaneous emission line. For a solid state laser the displacement of the signal from a central position has been observed and there is no reason to assume that this could not occur in a gas laser as well.

If the signal is not centred and if the value of $r$ is sufficiently close to unity that the first term in $W_L(f)$ dominates, the low frequency mixing spectrum will be given by

$$W_L(f) = a^2c^2 \left[ G(f + \nu_p) + G(\nu_p - f) \right]$$

Again assuming a Gaussian line shape centred at a frequency $\nu_0$ for the spontaneous emission as in Appendix B

$$D = \int G(\nu) \, d\nu = I_L(1-r)/a$$

Define a new parameter $g$ such that

$$g = \frac{|\nu_p - \nu_0|}{B}$$

representing the amount that the signal is displaced.

The graphs in Fig D-1 and D-2 have been computed for various values of $g$ to show the shape of $W_L(f)$. All the curves are normalized to have unit area under them. For purposes of comparison the dashed line shows the corresponding Lorentzian shape from Forrester (1961).
APPENDIX E

OUTPUT POWER OF A LASER

Consider two levels separated by an energy difference $E = h\nu$ and enclosed in a cavity. Let $n$ be the total number of photons of energy $h\nu$ in this cavity. If $N_2$ is the population of the upper level and $N_1$ the lower, the linear rate equation for this system

$$\frac{dn}{dt} = AN_1 + BnN_2 - CN_1 - pn$$

where

- $AN_1$ = spontaneous emission rate
- $BnN_2$ = stimulated emission rate
- $CN_1$ = rate of absorption
- $pn$ = rate at which other losses occur

In steady state $\frac{dn}{dt} = 0$ and the average number of photons $\bar{n}$ in the cavity becomes

$$\bar{n} = \frac{AN_2 + BnN_2}{CN_1/AN_2 - B/A}$$

Neglecting all other losses in the cavity (i.e. $p = 0$) then

$$\bar{n} = 1/(CN_1/AN_2 - B/A)$$

Assuming a Boltzmann distribution of populations

$$\frac{N_1}{N_2} = \exp\left(-\frac{h\nu}{kT}\right)$$

and that for a single mode

$$\bar{n} = 1/\left[\exp\left(\frac{h\nu}{kT}\right) - 1\right] = \frac{1}{b}$$

this gives $A = B = C$. The ratio of the rate of stimulated emission to spontaneous emission is then equal to the average number of photons in the cavity.

The output power of the laser is just the losses
incurred by the fact that one mirror is only 99% reflective. 

$p_0$ is the rate of loss of photons from the cavity and if $q$ is the loss per pass and $c/2L$ is the number of passes per second then $p = qc/2L$

\[ c = \text{velocity of light in the medium} \]

\[ L = \text{length of the cavity between the mirrors} \]

The power output of the laser is then

\[ P = \pi h \gamma qc/2L = \pi h \gamma p \]

The signal to noise ratio of the laser light is just equal to the ratio of the rate of stimulated emission to the spontaneous emission and is then equal to $\pi$.

\[ P = (S/N) h \gamma qc/2L \quad --(1) \]

Now for a cavity

\[ Q = 2 \pi \left[ \frac{\text{energy stored}}{\text{energy lost per cycle}} \right] = \gamma \Delta \nu_c \]

Therefore

\[ \Delta \nu_c = qc/4\pi L = p/2\pi \quad --(2) \]

and

\[ P = (S/N) 2\pi h \gamma \Delta \nu_c \]

The main assumptions in this derivation are a linear rate equation for the system and a single mode in oscillation neither of which may be very good.
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