SOME PROPERTIES OF A COSMOLOGICAL MODEL CONTAINING
ANTI-MATTER

by

DETLEF MATZ

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ABSTRACT

The chief aim of this work is to investigate cosmological consequences of a hypothesis put forward by Morrison and Gold in 1956. These authors postulate the existence of equal amounts of matter and antimatter in our universe. Abandoning the principle of equivalence, they attribute negative gravitational mass to anti-nucleons. The result is a drastic alteration in the field equation for the gravitational potential.

In the first three chapters Newtonian Cosmology is developed from basic principles. The equations describing a universe consisting of matter are set up and solved. In chapter IV the hypothesis of Morrison and Gold is introduced, and the resulting model for the universe is compared with models obtained in chapter III.

It is concluded that within the framework of the model considered, the hypothesis of Morrison and Gold is incompatible with the observational evidence, because it leads to an age of the universe of between 1.3 and 1.95 billion years, which is less than the age derived from other geological and astrophysical data.
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Department of Physics

The University of British Columbia,
Vancouver 8, Canada.

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CHAPTER 1: TERMINOLOGY AND POSTULATES.

I. 1. The Universe. The universe consists of a huge collection of galaxies which are separated by distances much larger than their maximum diameters. We can therefore choose an element of volume $dV$ such that:

volume occupied by one galaxy $\ll dV \ll$ volume of universe.

It is then reasonable to describe the universe in terms of the density $\rho(x, y, z; t)$, the velocity $\mathbf{v}(x, y, z; t)$, and if desirable, in terms of the pressure $p(x, y, z; t)$.

We shall also restrict ourselves to Euclidean Geometry.

Assuming further the validity of the laws of Newtonian Mechanics, including Newton's law of gravitation, one obtains a description of the universe known as "Newtonian Cosmology", which was first studied in this form by Milne and McCrea in 1934 (1).

I. 2. The Cosmological Principle. Milne and McCrea make the following two basic assumptions:

(1) **Choice of the coordinate system:** As a suitable choice of "observers" (or coordinate systems) we describe the universe exclusively from the point of view of observers swimming with the universe.

(2) **Basic Postulate:** At any given time all observers have the same view of the universe. This important assumption is called the Cosmological Principle (or the Homogeneity Postulate).
As a direct consequence of (2) the velocity fields are restricted to functions linear in the coordinates.

For with the aid of the diagram we see that what observer 1 sees at \( x \) should be identical with what observer 2 sees at \( x + \delta \). We must thus have:

\[
\begin{align*}
I.1 & : & f(x + \delta, t) &= f(x, t) \\
 & & n(x + \delta, t) &= n(x, t).
\end{align*}
\]

It also follows that:

\[
\begin{align*}
I.2 & : & \mathbf{u}(x + \delta, t) &= \mathbf{u}(x, t) + \mathbf{u}(\delta, t).
\end{align*}
\]

Since \( x \) and \( \delta \) are arbitrary vectors, we find that the most general functions satisfying these postulates are:

\[
\begin{align*}
I.3 & : & f &= f(t) \\
& & n &= n(t) \\
& & \mathbf{u}_1 &= a_{11} x_1 + a_{12} x_2 + a_{13} x_3 \\
& & \mathbf{u}_2 &= a_{21} x_1 + a_{22} x_2 + a_{23} x_3 \\
& & \mathbf{u}_3 &= a_{31} x_1 + a_{32} x_2 + a_{33} x_3
\end{align*}
\]

where \( a_{ik}(t) \) are arbitrary functions. From now on we shall use the usual summation conventions and write:

\[
\begin{align*}
\mathbf{u}_i &= a_{ik}(t) x_k \\
& \quad (i = 1, 2, 3; \quad k = 1, 2, 3).
\end{align*}
\]

Thus for a complete description of the universe we must determine ten unknowns, the \( a_{ik}(t) \)'s and \( f(t) \).

I.3. Matrix representation of velocity fields. To visualize the velocity fields represented by the matrix \( (a_{ik}) \), it is convenient to write the matrix elements in the form:
\[(a_{i\kappa}) = \frac{1}{2} (a_{i\kappa} + a_{\kappa i}) + \frac{1}{2} (a_{i\kappa} - a_{\kappa i}),\]

where \(\frac{1}{2}(a_{i\kappa} + a_{\kappa i}) = a_{i\kappa}\) is the symmetrical part of \((a_{i\kappa})\), that is: \(a_{i\kappa} = a_{\kappa i}\), and \(\frac{1}{2}(a_{i\kappa} - a_{\kappa i}) = a_{i\kappa}\) is the skew-symmetrical part of \((a_{i\kappa})\), that is: \(a_{i\kappa} = -a_{\kappa i}\).

Let us take the special case of \(a_{i\kappa} = 0\). Then \((a_{i\kappa})\) describes an irrotational velocity field. For, taking the vector curl of \(v\), we have:

\[
(curl \ v)_i = \frac{\partial v_j}{\partial x_i} - \frac{\partial v_i}{\partial x_j} = a_{i3} - a_{3i} = 2a_{33} = 0,
\]

and similarly:

\[
(curl \ v)_2 = 0 ; \quad (curl \ v)_3 = 0.
\]

Thus in an irrotational field we have only seven unknowns left, since \((a_{i\kappa})\) is symmetric.

I.4. Special subcase and example. A particularly simple and important subcase is that of spherically symmetric velocity fields, allowing only contraction or expansion in radial direction, represented by:

\[a_{i\kappa}(t) = f(t) \delta_{i\kappa},\]

or:

\[\mu_1 = \frac{f(t)}{t} x_1 ; \quad \mu_2 = \frac{f(t)}{t} x_2 ; \quad \mu_3 = \frac{f(t)}{t} x_3.\]

Example: To illustrate the Cosmological Principle, graph I and II have been drawn, that is, one depicting an irrotational, and one a rotational velocity field for two different observers swimming with the universe. In each graph the velocity field as observer 0 sees it was drawn first (the type of field is indicated on each graph).
Graph II: Rotational velocity field \((v_x = -2x_2, v_y = -2x_1)\).
Then observer $\Omega$'s field was constructed as follows.

From the Cosmological Principle it follows (see diagram):

$$\mathbf{v}(y) = \mathbf{v}(x) - \mathbf{v}(x).$$

Since $x = \mathbf{x}(-3, 3)$, the velocity components for observer $\Omega$ are:

$$v_i(x) = a x_i + 2a; \quad v_i(\Omega) = b x_i - 3b,$$

where $a, b$ are constants (for example: $a = 0.8$, $b = -0.1$ in graph I). Thus if observer $\Omega$'s field is drawn in this manner, he will have the same view of the universe as does observer $\Omega$. 
CHAPTER II: DYNAMICS OF THE UNIVERSE.

II. 1. Field equations. We now state the fundamental laws which are assumed to hold universally.

(1) Conservation of mass. This law is most conveniently expressed as an equation of continuity, which in absence of either creation or annihilation of matter reads:

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \]

(2) Conservation of momentum. Let \( \rho k \) be the density of the \( k \) component of momentum. Then the equation of continuity for each component independently is:

\[ \frac{\partial (\rho k v_k)}{\partial t} + \nabla \cdot (\rho k \mathbf{v} v_k) = \frac{\text{external force}}{\text{volume}} = -\rho \frac{\partial \phi}{\partial x_k} \]

where \( \phi \) is the gravitational potential per unit mass.

(3) Newton's law of gravitation. The law in differential form is:

\[ \nabla^2 \phi = 4\pi G \rho \]

* If there is any amount of creation of matter per unit time and per unit volume, \( Q \), the equation should be:

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = \pm Q \]

where the positive sign is for creation and the negative sign for annihilation of matter. The quantity \( Q \) may be either a constant or a function of time \( t \). It is particularly important in the Steady-State Theory proposed by Bondi and Gold in 1948 (2). For reasons of simplicity, these authors assume that the creation \( Q \) is constant in space and time. \( Q \) is of such order that the mean density of matter of our expanding universe remains constant. It follows that the amount of matter created in a volume element of space-time is proportional to this volume element. The factor of proportionality is:

\[ 3 \times (\text{Hubble's constant}) \times (\text{mean density of the universe}) \times 10^{-43} \text{g m}^{-3} \text{cm}^2 \text{sec}^{-1} \]

Any further discussion of this case, however, would take us beyond the range of this work.
Graph I: Irrotational velocity field \((v_1 = -2x_1, v_2 = -1x_2)\).
where $g = 6.664 \times 10^{-8} \text{ dyn cm}^{-2} \text{ gm}^{-2}$, is the gravitational constant.

Writing out equation II.2), we have:

$$v_k \frac{df}{dt} + \frac{f}{\partial t} + \nu_k \frac{\partial (f v_i)}{\partial x_i} + \nu_i \frac{\partial v_k}{\partial x_i} = - \frac{\partial \phi}{\partial x_k}.$$ 

By use of II.1) it simplifies to:

$$\text{II.4)} \quad \frac{dv_k}{dt} + \nu_i \frac{\partial v_k}{\partial x_i} = - \frac{\partial \phi}{\partial x_k}.$$ 

Equations II.1), II.3), and II.4) are the fundamental equations determining the dynamics of the universe. These equations are non-linear and in general cannot be solved. With the help of the Cosmological Principle, however, they can be simplified to the extent of permitting solution by elementary means. For, as a direct result of the Cosmological Principle, we have velocity fields linear in the coordinates. This fact, and some further simplifying assumptions, will allow us to integrate the equations.

It is instructive to consider first some simple special cases.

II.2. Irrotational velocity field. Here we have no rotation, that is $a_{ik} = a_{ki}$.

Substituting $v_i = a_{ik}(t)x_k$; $f = f(t)$, we get from II.1):

$$\text{II.5)} \quad \frac{df}{dt} = \phi (a_{ii} + a_{22} + a_{33}) = 0$$

and from II.4):

$$\text{II.6)} \quad \frac{d(a_{ki})}{dt} x_i + a_{ki} a_{il} x_l = - \frac{\partial \phi}{\partial x_k}.$$
Differentiating II.6, we get:

\[-\frac{d^2 \phi}{dx^2} = \frac{d^2 a_{uu}}{dt^2} + a_{ii} a_{ii} = \dot{a}_{uu} + \sum a_{ii}^2 \quad \text{for } k = 1.\]

There are similar expressions for \( k = 2, 3 \). Adding these, and using II.3, we have:

II.7) \[ \dot{a}_{uu} + \dot{a}_{zz} + \dot{a}_{ss} + \sum a_{iik} a_{ik} = -4\pi \delta \rho. \]

This last equation enables us to prove a theorem of Neumann and Von Seeliger which states that the universe cannot be static.

**Proof:** Equation II.5 can be written:

\[ \dot{\rho}/\rho = - \left( a_{uu} + a_{zz} + a_{ss} \right), \]

and after differentiation:

\[ \dot{a}_{uu} + \dot{a}_{zz} + \dot{a}_{ss} = - \frac{d}{dt}(\dot{\rho}/\rho), \]

or with the help of II.7:

II.8) \[ \frac{d}{dt}(\dot{\rho}/\rho) = \sum a_{iik} a_{ik} + 4\pi \delta \rho > 0, \]

with equality only if there is no velocity, and \( \rho = 0 \). From II.6 it can readily be seen that if \( \delta \rho \neq 0 \), and for as long as there is any non-vanishing velocity field at all, \( \rho \) cannot be constant, which completes the proof.

It must be pointed out, however, that this theorem is not generally true in case of rotational motion, because if \( a_{ii} \neq 0 \), then the equations would read as follows.

Equations II.6 still holds, but II.7 should now read:

\[ \dot{a}_{uu} + \dot{a}_{zz} + \dot{a}_{ss} + \sum a_{iik} a_{ik} = -4\pi \delta \rho. \]

Similarly II.8 should read:

\[ \frac{d}{dt}(\dot{\rho}/\rho) = \sum a_{iik} a_{ik} + 4\pi \delta \rho = \sum \left( a_{iik} + a_{ijk} \lambda a_{ik} + a_{ikj} \right) + 4\pi \delta \rho \]

\[ = \sum (a_{iik}^2 - a_{iik}^2) + 4\pi \delta \rho \geq 0. \]
The equation II.7) can be integrated most easily in a special subcase.

II.3. Isotropic velocity field. Here: \( a_{i\kappa} = \delta_{i\kappa} \frac{f(t)}{R} \), or: \( a_{i} = a_{22} = a_{33} = \frac{f}{R} \). Equation II.5) then becomes:

II.9) \[ \frac{d^2 f}{dt^2} + 3 \frac{df}{dt} = 0 \]

and II.6) is:

II.10) \[ x_k \left( \frac{d^2 f}{dt^2} + \frac{df}{dt} \right) = -\frac{\partial \Phi}{\partial x_k} \]

By differentiating, and with the help of II.7) we obtain:

II.11) \[ \frac{d^2 f}{dt^2} + \frac{df}{dt} = -\frac{\omega f}{3} \frac{df}{dt} \]

Thus in using II.9) and II.11) we can find \( f \) and then by integrating II.10) find \( \rho \).

Let us now introduce a new dimensionless variable \( R(t) \) such that: \( f = \frac{\dot{R}}{R} \). We determine the meaning of \( R(t) \) as follows. Upon integrating:

\[ \nu_i = \frac{dx_i}{dt} = x_i \frac{d}{dt} = x_i \frac{\dot{R}}{R} \]

we get:

\[ x_i(t) = x_i(t_0) R(t) \]

where \( x_i(t_0) \) is a constant of integration obtained by the arbitrary condition that \( R(t_0) = 1 \).

From these considerations it follows that \( R(t) \) is a universal dimensionless (time dependent only) scale factor, that is, it shows how much the universe has expanded or contracted.

\[ \star \]

Another way of seeing the meaning of \( R(t) \) is the following way: suppose that for the motion of a particle we write in polar coordinates:

II.12) \[ \nu = \frac{d\xi}{dt} = \frac{\dot{\xi}}{\frac{d\xi}{dt}} = \frac{\dot{\xi}}{} \]

Upon replacing \( f(t) \) by \( \frac{\dot{\xi}}{} \) and integrating II.12): \( \zeta = \xi R(t) \), where \( \xi \) is a constant of integration. If we now choose the condition that at some time \( t = t_0 \), \( R(t_0) = 1 \), and \( \xi = \zeta_0 \), then \( \zeta = \frac{\dot{\xi}}{} \). Therefore \( \zeta = R(t) \xi \) where \( \xi \) is the position vector of the particle at \( t = t_0 \).
Substituting $R$ in our equations, we then have

for II.9):

$$\dot{\rho} + 3 \frac{\dot{R}}{R} = 0.$$ Integrating:

$$\frac{d}{dt}(\ln \rho) = -3 \frac{d}{dt}(\ln R),$$ or:

$$\ln (\rho R^3) = \ln \text{(constant)}.$$

Therefore:

$$\frac{4\pi}{3} \rho R^3 = M = \text{constant} > 0.$$

This expresses the conservation of mass. In fact if we apply the condition $R(t_0) = 1$, $\frac{3M}{4\pi}$ represents the mass contained in a unit volume at the time $t = t_0$.

Similarly for the equation II.11):

$$\frac{\dot{R}}{R} = -\frac{4\pi \rho s}{3}.$$

Combination with the equation above gives:

II.13) $$\dot{R} = -\frac{\rho M}{R^2}.$$

The solution of this equation will give us $R$ as a function of $t$. 
CHAPTER III: SOLUTION OF THE COSMOLOGICAL EQUATION.

III.1. First integration of the equation. To solve the cosmological equation, let $\gamma M = \frac{b}{k}$, multiply through by $k$: 

$$\dot{a}^2 = -\frac{\frac{b}{k}}{a^2},$$

and integrate:

III.1) $$(\dot{a})^2 = \frac{b}{k} + a,$$

where $a$ is a constant of integration, or:

$$\frac{d}{dt} = \pm \sqrt{\frac{b}{k} + a} = \pm \sqrt{a k^2 + b k}$$

which gives

III.2) $$t = \int_{a_k}^{t} \frac{d\rho}{a k^2 + b k} + \text{constant}.$$

Before giving the different solutions to this integral we will attempt to find at least a partial interpretation of $b$ and $a$. The constant $b$ was defined as $b = 2\gamma M > 0$, where $\gamma$ is the constant of gravity. From the previous chapter $M = \frac{4\pi}{3} \rho (t_0)$, where $\rho (t_0) > 0$ is the density at time $t_0$. The dimension of $b$ is $(\text{time})^{-2}$ as can easily be verified. The meaning of $a$ is obtained from equation II.1), and by applying to it the condition $K(t_0) = 1$. We had: $\mathcal{T} = \dot{K} = \gamma v$, which gives: 

$$\left(\frac{d\mathcal{T}}{dt}\right)^2 = \frac{1}{\gamma^2}$$

where $v$ is the velocity. Hence at $t = t_0$ III.1) becomes:

$$v^2 = \frac{\gamma^2}{\gamma^2} (b + a)$$

which can be interpreted as the kinetic energy per unit mass of matter at $t = t_0$. If we choose $t_0 = 0$, then $a$ gives a measure of the energy per unit mass of matter at "Creation." It also follows that $a$ must have dimension $(\text{time})^{-2}$. The full meaning of $a$ and $b$ will become apparent in Chapter V.
Graph III: Hyperbolic Universe ($a > 0$)

Graph IV: Elliptic Universe ($a < 0$)

Graph V: Parabolic Universe ($a = 0$)
We return now to the integral III.2 and discuss its three different solutions according to whether $a > 0$, $a < 0$, $a = 0$.

III.2. Hyperbolic universe. The integral has

$$t = \frac{\sqrt{\alpha R^2 + \beta R}}{\alpha} - \frac{\beta}{2\alpha^2} \ln \left[ \sqrt{\alpha R^2 + \beta R} + \frac{2\alpha R + \beta}{\alpha} \right] + C.$$

For this curve to start at the origin of a plot of $R(t)$ versus $t$, we choose $C_1$ such that at $t = 0, R = 0$, that is:

$$C_1 = \frac{\beta}{2\alpha^2} \ln \frac{\beta}{2\alpha}.$$

To get a qualitative picture of the curve we note that for small $R$ we have: $\frac{dR}{dt} \sim \sqrt{R}$ and the curve behaves like: $R \sim \left( \frac{9b^2}{4t} \right)^{1/3}$. On the other hand, for large $R$ we have $\frac{dR}{dt} \sim \sqrt{R}$. Therefore: $R = \sqrt{a} t + O(1)$. Also, since $a > 0$, and $b > 0$, we have $\frac{dR}{dt} > 0$, always. Hence $R(t)$ is always concave downwards with no points of inflections. At $R = 0$, $\frac{dR}{dt} = \infty$. (see graph III)

III.3. Elliptic universe. Here we have $a < 0$.

The solution is:

$$t = \frac{\sqrt{\alpha R^2 - \beta R}}{\alpha} - \frac{\beta}{2\alpha^2} \sin^{-1} \left[ \frac{-2\alpha R - \beta}{\beta} \right] + C_2.$$

Again we choose $C_2$ such that at $t = 0, R = 0$:

$$C_2 = -\frac{\pi \beta}{4a \sqrt{-\alpha}}.$$
Graph VI: Newtonian World Models.
At $R=0$, $\frac{dR}{dt} = \infty$. From $\frac{dR}{dt} = \sqrt{\frac{L}{R} + a}$ we see that there is a maximum at $R = -\frac{L}{a}$. Thus at $t = t_{\text{max}}$, $\frac{dR}{dt} = 0$.

Because of the condition of reality we must take $\frac{dR}{dt} = \sqrt{\frac{L}{R} + a}$ between $t = 0$ and $t = t_{\text{max}}$, and between $t_{\text{max}}$ and the time at which the second zero of $R$ occurs we must take $\frac{dR}{dt} = -\sqrt{\frac{L}{R} + a}$. The two branches are thus symmetrical about $t = t_{\text{max}}$. Substituting $R = -\frac{L}{a}$ into the equation we obtain:

III.7) $t_{\text{max}} = -\frac{\pi \sqrt{L}}{2a \sqrt{-a}}$

Since we have symmetry about $t_{\text{max}}$, the second zero of $R$ occurs at: $t = 2t_{\text{max}} = -\frac{\pi \sqrt{L}}{a\sqrt{-a}}$. (see graph IV)

III.4. Parabolic universe. This is the simplest of the three cases, since $a = 0$. We have: $t = \frac{2}{3\sqrt{L}} R^{\frac{3}{2}} + C$.

Therefore:

III.8) $R = \left(\frac{8}{9} \frac{L}{a} t^{3}\right)^{\frac{1}{3}}$

Since at $t = 0$, $L = 0$, we have:

III.9) $C_{3} = 0$.

At $t = 0$, $\frac{dR}{dt} = \infty$. Also $\frac{dR}{dt}$ is always concave downward and $\frac{dR}{dt} = 0$ at $R = \infty$. (see graph V)

For the purpose of comparing all three curves, graph VI has been plotted with arbitrary values for $a$ and $b$, that is, $|a| = 1 \left(10^{9} \text{ yrs}\right)^{-2}$, $b = 3 \left(10^{9} \text{ yrs}\right)^{-2}$. 
CHAPTER IV: ANTI-MATTER.

IV.1. Alteration of the gravitational field equation.

We are now prepared to tackle the main purpose of this work, namely to consider consequences of the possible presence of large amounts of anti-matter in the universe. In particular we wish to investigate a universe consisting of equal amounts of matter and anti-matter as proposed by Morrison and Gold in 1956\(^{(3)}\).

A universe in which anti-matter has positive gravitational mass could be described by the models treated previously provided the smallest units of matter or anti-matter are at least of the size of galaxies, so that no annihilation processes will have to be taken into account in the dynamical equations.

However, Morrison and Gold, abandoning the principle of equivalence, proposed that anti-nucleons have negative gravitational rest mass. In addition, to preserve charge symmetry, they argued that all other forms of energy (that is, electrons, positrons binding energy) ought to be attracted gravitationally by both nucleons and anti-nucleons. At present it is impossible to test this hypothesis in the laboratory because of the short life of anti-matter in the presence of matter; one cannot make an anti-proton live long enough to see it rise or fall under gravity.

Questioning the Principle of Equivalence requires
examining the extent to which this principle is established to date. The term "Principle of Equivalence" is, unfortunately, used not always with the same meaning attached to it. The General Theory of Relativity rests very heavily on two versions of the Equivalence Principle, the strong and the weak one, as pointed out by Dicke. In the strong version it is assumed that the laws of physics are independent of time and position in a freely falling laboratory. In the weak form on the other hand it is assumed that in a uniform gravitational field, all bodies fall with equal acceleration independently of their structure. This was the statement that Eotvos tried to justify in his famous experiment.

The strong and the weak form of the Equivalence Principle appear quite different. Actually they are related as can be seen from the following considerations. The definition of the strong version of the Equivalence Principle implies the position and time constancy of both the strong and the weak interactions, including gravity. This statement is partially substantiated by experiment. From the accurate Eotvos experiments it can be deduced that nuclear binding energy is position independent to within one part in one hundred thousand. Further the coupling constant of electromagnetic interaction, $\frac{e^2}{\hbar c}$, is known to be constant with great accuracy.
by observation of light from distant galaxies. Thus, at least for the strong interactions, we expect position independency. If this were not so, the following situation would arise. First one must keep in mind that all matter consists of atoms composed of protons, neutrons, and electrons, held together almost exclusively by the strong forces. Now if an atom is displaced, a change in its binding energy and hence in its total rest energy, (that is, binding energy plus energy due to mass) should result. As a consequence the inertial mass of the atom will alter. Then, since binding energy and total rest energy are equivalent to inertial mass, one should expect a variation in the ratio of binding energy to total rest energy of the atom. But this is contrary to the experimental data given above.

On the other hand when we consider the weak interactions and gravity, the situation is quite different. Their contribution to the binding energy of atoms is so negligible that it cannot be measured. Also one

*Unless one wishes to assume the main part of the total rest energy also to be caused by coupling via self-energy effects. The mass difference between isotopic spin multiplets at least can be explained by the difference in coupling to the electromagnetic field.
cannot make any deductions from the Eotvos experiment as far as the constancy of these interactions is concerned. To test whether the gravitational constant and the weak coupling constants vary with time, Dicke investigated the consequences of such a time dependency by examining geological, astronomical, and biological data. He could find no evidence in favour of their constancy.

Thus, except in the case of the strong interactions, there is no definite experimental support for the strong version of the Equivalence Principle. However, if we neglect the contributions of the weak interactions and gravitation to the binding energy of atoms, we arrive at a weakened form of the strong version of the Equivalence Principle that is backed by the accurate Eotvos experiment, and others. It then appears that the weak form of the Equivalence Principle is an approximation of the strong form of the Equivalence Principle.

Since the only substantial evidence available is for the weak form of the Equivalence Principle, let us describe this evidence.

The diagram represents a cross-section of a simple Eotvos torsion balance. The various substances are suspended from ends a and b.
The difference in height of weights at a and b allows for the spatial variation of the earth's gravitational field. If there is a structure dependency for different bodies in the earth's gravitational field, then this would show in a deflection of the beam, since the gravitational constant $\gamma$ can then be written $\gamma = \gamma_0(1 + \chi)$, $\chi$ being a constant, and $\gamma_0$ the gravitational constant for a standard substance such as water. Further, a deflection in the horizontal plane should also be noticed due to the sun's or moon's field when the beam is placed parallel to a meridian. If for two substances such as platinum and magnalium (90% Au, 10% Mg) at ends a and b the gravitational constants are $\gamma_a$, $\gamma$, then $\gamma_a = \gamma_a(1 + \chi_a)$ and $\gamma_b = \gamma_b(1 + \chi_b)$. Then obviously $\frac{\gamma_a - \chi_b}{\gamma_b} = \chi_a - \chi_b$. Eotvos found that within experimental limits $\chi_a - \chi_b < 5 \times 10^{-8}$. Hence it follows that the weak form of the Equivalence Principle is satisfied to within one part in twenty million.

*It should be mentioned that Eotvos performs his experiments on what Bondi calls the passive gravitational mass of a body. According to Bondi a body's passive gravitational mass is the mass acted upon by the gravitational fields. On the other hand, he calls the mass from which gravitational fields originate the active gravitational mass. The point to note is that Eotvos sets up the apparatus in such a way as to detect a rotation of the torsion balance beam caused by the action of the gravitational field of the earth, the sun and/or the moon.*
In connection with the above experiments Dicke reports that he is presently working on two Eotvos type experiments that represent an order of magnitude improvement over Eotvos findings (11).

It must be pointed out, however, that the above mentioned tests for the Equivalence Principle apply to one kind of matter only, namely, matter prevailing in our part of the universe. It would be difficult to test the Equivalence Principle for anti-matter by using present day experimental techniques.

There is nevertheless one possible approach to test the postulated anti-gravity of anti-matter based on field theoretical arguments. The general belief is that through interaction with the meson field the nucleon is dissociated part of the time into nucleon-anti-nucleon pairs. Let us denote the dissociation constant by \( \theta \). Then taking the hypothesis of Morrison and Gold into account, the inertial mass \( I \) and the weight \( W \) of a complete atom are:

\[
I = A M + Z m
\]

\[
W = A M (1 - \theta) + Z m,
\]

where \( A \) is the mass number, \( M \) the nucleon mass, \( Z \) the atomic number, and \( m \) the electron mass. Taking the ratio \( E \) of the inertial to gravita-

*The much weaker effect of electron-positron vacuum polarization has been neglected since it will not affect the argument.*
tional mass and considering only first terms in \( \Theta \) we obtain: 
\[
E = 1 + \Theta \left\{ 1 - \frac{Z_m}{\hat{M}} \right\}
\]
If \( \Theta \) were known accurately we could perform an Eotvos type experiment on two substances differing in \( \frac{A}{Z} \) values and thus obtain a difference in their \( E \) values.
On the other hand since for nucleons \( \Theta \) is uncertain, the Eotvos experiments would set an upper limit to \( \Theta \).
As an example consider Carbon \((Z=6, \, A=12)\) and Uranium \((Z=92, \, A=238)\). This gives:
\[
\Theta < \frac{E_C - E_U}{\hat{M} \left\{ \frac{Z}{Z} - \frac{\hat{E}}{\hat{F}} \right\}} \approx 0.00077.
\]
Thus the virtual nucleon-anti-nucleon formation would be no greater than \( 0.77\% \) if anti-nucleons had negative gravitational mass.
Schiff(12) argues that on grounds of the vacuum-polarization in quantum-electrodynamics and the Eotvos experiments, positrons cannot have negative gravitational mass. This, however, does not invalidate Morrison and Gold's hypothesis, for it postulated that positrons are attracted both to nucleons and anti-nucleons for reasons of charge symmetry. On the other hand, as Schiff also points out the vacuum polarization by the meson field is little understood and calculations based on conventional meson field-theory cannot be considered as reliable.
Thus, on grounds of the preceding general arguments one cannot exclude the hypothesis of Morrison
and Gold. In the present work an attempt is made therefore to investigate cosmic consequences of a partial violation of the Equivalence Principle, which was first considered by Morrison and Gold. This requires a drastic change in the field equation II.3). Considering only volume elements containing large numbers of galaxies and supposing that the smallest units of matter or anti-matter are of the size of galaxies, the average gravitational rest mass density, appearing as $\mathcal{f}$ in II.3), should be zero. Because of their motion, however, both galaxies and anti-galaxies should contribute a term to the right hand side of II.3), representing the gravitational effect of the kinetic energy density of the universe. Since it is known from experiment that anti-matter has positive inertial mass, we can represent the kinetic energy density by $\mathcal{T} = \frac{1}{2} \mathcal{f} \mathcal{v}^2$ (we consider only non-relativistic approximations), where $\mathcal{f}$ is now the inertial mass density of both matter and anti-matter. According to the hypothesis of Morrison and Gold this kinetic energy density constitutes positive gravitational mass density $\mu = \frac{\mathcal{T}}{c^4} = \frac{1}{2} \mathcal{f} \frac{\mathcal{v}^2}{c^4}$. Hence the field equation II.3) changes into:

IV.1) \[ \nabla^2 \phi = 4\pi \mathcal{V}^{\frac{1}{2}} \mathcal{f} \frac{\mathcal{v}^2}{c^4} \]
equations II.1) and II.4) remaining unchanged.

IV.2. General solution of the cosmological 
Equation.
The equations for an isotropic universe are now:

IV.2) \[ \frac{dv}{dt} + \frac{3v}{R} \frac{dR}{dt} = 0 \]

IV.3) \[ \frac{d}{dt} \left( \frac{\dot{R}}{R} \right) + \left( \frac{\dot{R}}{R} \right)^2 = - \frac{4\pi G \rho}{3} \int \frac{\nu^2}{c^2} \]

Since \( \nu^2 = A \ddot{R}(t) \), where \( A = \sum_i x_i(t) \) is constant (see paragraph II.3), IV.3) becomes upon simplification:

IV.4) \[ \frac{\ddot{R}}{R} = - \frac{2\pi G A}{3c^2} \int \dot{R}^2 \]

Further, a simple integration of IV.2) gives: \( \dot{r}(t) = \frac{3M}{4\pi R(t)^2} \), where the constant \( M \) has the same meaning as in paragraph II.3. Substituting this value of \( \dot{r}(t) \) into IV.4), one obtains:

\[ \ddot{R} = - \frac{YMA}{2c^2} \left( \frac{\dot{R}}{R} \right)^2 \]

or simply:

IV.5) \[ \ddot{R} = - \beta \left( \frac{\dot{R}}{R} \right)^2 \]

where \( \beta = \frac{YMA}{2c^2} \)

Obviously, \( \beta \) is a positive dimensionless constant.

Equation IV.5) can now be easily integrated in the form \( \frac{\ddot{R}}{R} = - \beta \frac{\dot{R}}{R^2} \) which is equivalent to:

IV.6) \[ \frac{dR}{dt} = \alpha e^{\beta R} \]

where \( \alpha \) is a constant of integration with dimension of \((\text{time})^{-1}\), the meaning of which will become clear later (see paragraph V.3).

The simplest way to solve IV.6) is to integrate term by term:

\[ t = \frac{1}{\alpha} \int e^{-\beta R} dR + \frac{C}{\alpha} \]

where \( C \) is a constant of integration,

\[ t = \frac{1}{\alpha} \left\{ t_0 - \frac{\beta}{\alpha} + \frac{\beta^2}{2!} R^2 - \frac{\beta^3}{3!} R^3 + \cdots \right\} + \frac{C}{\alpha} \]

that is:

IV.7) \[ t = \frac{1}{\alpha} \left\{ R - \beta \ln R - \sum_{n=0}^{\infty} \frac{(-1)^n \beta^n}{n! (n+1)! R^n} \right\} + \frac{C}{\alpha} \]
Graph VII: Case 1, $\alpha > 0$.

$R(t) = \alpha t + c.$

Graph VIII: Case 2, $\alpha < 0$.

$R(t) = -\alpha t + c.$

Graph IX: Case 3, $\alpha = 0$.
The constant \( C \) is so chosen that at \( t=0, K=0 \). However it cannot be evaluated directly. The method for its calculation will be shown in the next chapter, appendix A. Let it suffice here to give its value as:

\[
C = \beta (e^{\alpha \beta} - 4^{3/5}), \quad \text{which is dimensionless.}
\]

The complete solution of equation IV.5) can now be written:

\[
IV.9) \quad t = \frac{1}{\alpha} \left\{ R - \beta \ln R - \sum_{n=1}^{\infty} \frac{(-1)^n \beta^n}{n \cdot (n+1)! R^n} + \beta (e^{\alpha \beta} - 4^{3/5}) \right\}
\]

The shape of the curve of equation IV.9) can be determined if one notes that for \( R = \infty, \dot{R} = 0 \). Therefore for large \( R, R \) has the form \( R = \alpha t + c, c \) being a constant. Further for \( R = 0, \dot{R} = 0 \). Graph VII, VIII, and IX illustrate these qualitative features.

As in chapter III, here one also has three possible models, according to whether \( \alpha > 0, \alpha = 0, \alpha < 0 \).

IV.3. Types of universes.

Case 1: \( \alpha > 0 \). This case represents expansion and is shown on graph VII. The curve approaches the straight line \( R = \alpha t + c \) asymptotically.

Case 2: \( \alpha < 0 \). This case is illustrated by graph VIII which shows contraction of the universe. If this case would correspond to observation, we could fit it with the observed rate of contraction of the universe, and then choose some zero of time, which is taken arbitrarily here.
Case 3: $a=0$. If $a=0$, we have $\frac{dR}{dt} = 0$, therefore $R = \text{constant} = k$. This obviously is a static universe as seen from graph IX. It is however in unstable equilibrium. To show this, we expand $R$ in terms of $\alpha$ about $\alpha = 0$ in a Taylor series: $R(\alpha) = R(0) + \frac{R'(0)}{1!}\alpha + \frac{R''(0)}{2!}\alpha^2 + \ldots$

From above we have for $\alpha = 0$, $R = k$. Therefore: $R(0) = k$.

Also $\frac{dR}{d\alpha} = t e^{\frac{R}{k}}$. Using the previous result of $R(0) = k$:

$R'(0) = t e^{\frac{k}{k}}$.

For the higher derivatives:

$\frac{d^2 R}{d\alpha^2} = \frac{dR}{d\alpha} = \ldots = \frac{d^n R}{d\alpha^n} = 0$.

Hence:

$R(\alpha) = k + (te^{\frac{R}{k}})\alpha$.

Equation IV.10) shows that for $\alpha = 0$ we get the static case. But for a non-vanishing $\alpha$, $R$ goes to infinity. There is therefore unstable equilibrium.

Present day experimental observation indicates that the universe can only be described by an expanding model. Hence, of the three models discussed here, the only one to be considered is the one with positive $\alpha$. 
CHAPTER V: COMPARISON WITH EXPERIMENTAL EVIDENCE.

V.1. Introduction. In this concluding chapter it will be shown that out of the three cosmological models discussed in chapter three, the elliptic model is the only one compatible with present day astronomical data. This model will then be fitted to the actual universe and compared with the model containing antimatter.

In order to carry out the above plan it is necessary to determine the two parameters occurring in the equation for $R(t)$. This is readily done by employing the experimentally known parameters

$$V.1 \quad h_1 = \frac{\dot{R}(t)}{R(t)}$$

and

$$h_2 = \frac{\ddot{R}(t)}{R(t)}$$

$t_0$ being the present time.

If one adopts McVittie's notation, (13) $h_1$ is the Hubble constant, that is, the constant of proportionality in the velocity-distance relation for nebulae. Its reciprocal has the dimension of time, and it is in case of negative $h_2$ an upper limit to the age of the universe. $h_1$ has the range of values:

\[ 0 < h_1 \leq 1 \]

This appears immediately if one performs the following integration: $\int 4\pi R^2 \, d\Omega = h_1 \int 4\pi V \, dt$, which leads to $t = (h_1)^{-1}$, where the definition $R(t_0) = 1$ at the present time $t_0$ is adopted. Further, if the same rate of expansion since "Creation" is assumed, and since $h_2$ is known to be negative, $t = (h_1)^{-1}$ sets an upper limit to the age of the universe.
\[0.146 \left(10^9 \text{yrs}\right)^{-1} < h_2 < 0.232 \left(10^7 \text{yrs}\right)^{-1}\]

\(h_2\), on the other hand, is the acceleration parameter, that is, it measures the acceleration of the elements in the universe, and hence has dimensions of \((\text{time})^{-2}\). In the following discussion \(h_2\) will appear only in the dimensionless number \(\frac{k_2}{k_1}\), the value of which is \(-\left(3 \pm 0.8\right)\) or \(-\left(1.6 \pm 0.8\right)\), as calculated by McVittie (13) and Sandage (11) respectively. \(\frac{k_2}{k_1}\) will therefore be considered to have the range: \(-3.8 < \frac{k_2}{k_1} < -1.8\).

It is now proposed to obtain a minimum and maximum curve for each model, corresponding to the experimentally known upper and lower limits of \(h_1\) and \(h_2\).

V.2. Exclusion of the hyperbolic and parabolic models. In chapter three the solution to equation II.13) gave three cosmological models, for \(a = 0\), \(a > 0\), \(a < 0\). It is now simple to show that the models with \(a > 0\), \(a = 0\), are incompatible with astronomical observation. For this purpose, equations II.13) and III.1) will be substituted into the equations V.1) which define \(h_1\) and \(h_2\). Two equations result:

\[
\frac{k_2}{k_1} + a = h_2^2 \sqrt{R_p}
\]

\[
-\frac{k_2^2}{2k_1} = h_2 R_p
\]

\(R_p\) denoting \(R(t_p)\), that is, the value of \(R\) at the present time \(t_p\). The second equation immediately gives:

V.2) \[\ell = -2h_2 R_p^3\]
which upon substitution into the first equation gives:
\[ a = \frac{a_i^2}{2} \left( \frac{1}{2} + \frac{a_i^2}{a_i^2} \right) \] or simply:
V.3) \[ a = \frac{a_i^2}{2} \left( 1 + \frac{a_i^2}{a_i^2} \right). \]

Equation V.3) inevitably leads to the conclusion that \( a < 0 \), since \( a_i^2 > 0 \), \( a > 0 \), and \( a_i^2 < -3.6 \). \( a < 0 \) represents of course the elliptic model, equation III.5).

The universe containing anti-matter can therefore be compared only with the elliptic universe.

V.3. Comparison of the elliptic with the anti-matter model. Calculations of \( \alpha \) and \( \beta \) of equation IV.9) lead to:
V.4) \[ \alpha = \frac{a_i}{2} e^{a_i^2}, \]
\[ \beta = -\frac{a_i}{2} e^{a_i^2}, \]
where the definition \( R(t_o) = 1 \) is adopted. (see appendix B for detailed computations.)

Similarly one finds:
V.5) \[ a = \frac{a_i}{2} \left( 1 + \frac{a_i}{a_i^2} \right), \]
\[ \beta = -\frac{3}{2} a_i. \]

The adoption of this definition has the following advantage. The scale factor \( R(t) \) is related to the radial distance \( r \) of a given element of the universe by the equation (see paragraph II.3): \( \tau = \frac{a_i}{2} R(t) \). \( r \) is the position vector of the element at \( t = t_o \), and thus if \( R(t_o) = 1 \), (here \( t_o = t_p \)) it gives the radial distance of any particle at the present time. It is then clear that the above definition enables one to describe the path of any particle throughout its entire history, if only its present distance \( \tau_0 = \tau_p \) is known.
Graph X: Elliptic and anti-matter models.

Legend:
- : anti-matter model
- : elliptic model.
Equations V.4) and V.5) immediately suggest a meaning for the constants \(\alpha, \beta, a, b\). Thus \(\alpha\) is directly related to the Hubble constant \(h_1\), and would be equal to it if the deceleration \(h_2\) were zero; \(\beta\) measures the deceleration; \(a\) would equal \(h_1^2\) for vanishing deceleration; \(b\) is twice the deceleration.

The minimum and maximum values of \(\alpha, \beta, a, b\), corresponding to minimum and maximum \(h_1\) and \(h_2\) are now computed. They are:

<table>
<thead>
<tr>
<th></th>
<th>(\frac{h_1}{(10^6\text{yrs})^{-1}})</th>
<th>(\frac{h_2}{(10^6\text{yrs})^{-1}})</th>
<th>(\alpha)</th>
<th>(\beta)</th>
<th>(\frac{\alpha}{(10^6\text{yrs})^{-2}})</th>
<th>(\frac{\beta}{(10^6\text{yrs})^{-2}})</th>
</tr>
</thead>
<tbody>
<tr>
<td>minimum values</td>
<td>(0.146)</td>
<td>(-3.8)</td>
<td>(3.25 \times 10^{-3})</td>
<td>(3.8)</td>
<td>(-0.143)</td>
<td>(0.164)</td>
</tr>
<tr>
<td>maximum values</td>
<td>(0.232)</td>
<td>(-1.8)</td>
<td>(3.62 \times 10^{-2})</td>
<td>(1.8)</td>
<td>(-0.140)</td>
<td>(0.196)</td>
</tr>
</tbody>
</table>

These values are then substituted into their corresponding equations resulting in a minimum and maximum curve for each model. The curves are plotted on graph X. The ages of the universe for the models are given by the abcissa's for the ordinate \(\mathcal{R}=1\). Thus for the elliptic model the present age of the universe lies between 2.1 and 2.7 billion years, and for the model containing anti-matter, between 1.3 and 1.95 billion
years. Hence, for a given \( h_1 \) and \( h_2 \) the anti-matter model leads to a much younger universe than the corresponding matter universe.

This result considerably weakens the hypothesis of Morrison and Gold who considered a universe which (a) consists of equal amounts of matter and anti-matter, and in which (b) the anti-matter has negative gravitational rest mass. For, there is independent astronomical, and even geological evidence that the age of the universe must be in excess of at least \( 2 \times 10^9 \) years. Thus, for instance, the age of the earth is estimated to be between \( 3.4 \) and \( 5 \times 10^9 \) years, \( (15) \) the ages of meteorites between \( 4.5 \) and \( 5 \times 10^9 \) years, \( (16) \) and the ages of the globular clusters NGC 5272 and NGC 6205 are given as \( 5 \times 10^9 \) years \( (17) \) and \( 2 \times 10^9 \) years respectively.

Analytically the present time \( t \) for the elliptic model is given by:

\[
t_n = \frac{1}{f_0(1 + \frac{h_2}{h_1})} + \frac{h_2}{h_1} \left( \frac{1}{f_0(1 + h_2)} \right) \sin^{-1} \left( \frac{h_2}{h_1} + 1 \right) + \frac{\pi}{2} \left( \frac{h_2}{h_1} \right) \left( \frac{1}{f_0(1 + h_2)} \right) \left( \frac{1}{f_0(1 + h_2)} \right).
\]

Then for maximum \( h_1 \) and \( h_2 \), \( t_n = 2.1 \times 10^9 \) years, and for minimum \( h_1 \) and \( h_2 \), \( t_n = 2.7 \times 10^9 \) years. Similarly for the anti-matter model:

\[
t_n = \frac{e^{-\frac{1}{4} h_1}}{\frac{1}{h_1}} \left\{ 1 - \sum_{m=1}^{\infty} \left( \frac{h_2}{h_1} \right)^{m-1} \right\} - \frac{h_2}{h_1} \left[ \ln \left( \frac{h_1}{h_2} \right) - .4225 \right],
\]

which for corresponding minimum and maximum \( h_1 \) and \( h_2 \) fixes the age of the universe between \( 1.95 \times 10^9 \) years and \( 1.3 \times 10^9 \) years.
Obviously, the result of the present work does not exclude the possibility that only one of the hypothesis is true, namely (a) or (b). But a universe in which both (a) and (b) are true, seems to be incompatible with observational data available at present.
APPENDIX.

This appendix is added with the intention of presenting some of the details of calculations of chapters four and five, which otherwise would have distracted too much from the main line of the argument.

A. Determination of C. C is the constant of integration occurring in equation IV.7). Under the transformation

\[ \frac{r}{\beta} = f \quad \frac{x}{\alpha} = \phi \]

equation IV.7) becomes:

\[ x = f - \ln f - \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} f^n + A \]

where \( x \) and \( y \) are now dimensionless variables, and the constant \( A \) is:

\[ A = \frac{c}{\beta} - \ln \beta \]

The transformation A.1) has the advantage of permitting quick calculations of points of equation IV.7) for any \( \alpha \) and \( \beta \), once equation A.2) is fixed. It also allows the evaluation of \( C \), which by a direct substitution of \( r=0, t=0 \) is rather difficult to determine.

Suppose now that equation A.2) is subjected to the condition: \( f=1 \) at \( x=0 \). Then \( A \) assumes the value \(-.5713\). Further, from a graph of A.2) with this value
of $A$, it is found that $y=0$ at $x=-1.1488$. If now .1488 is added to all values of $A$, one obtains the initial condition $y=0$ at $x=0$, which transforms into the desired initial condition of equation IV.7), namely: $y=0$ at $t=0$. The constant $A$ hence changes to:

$$A + .1488 = -.5713 + .1488 = -.4225.$$  

With the help of equation (A.2), $C$ can then be determined:

$$IV.8) \quad C = \beta (\ln \beta - .4225).$$

**B. Calculation of $\alpha$ and $\beta$.** For the purpose of computing $\alpha$ and $\beta$ equations V.1), which define $h_1$ and $h_2$, are employed. Further, as stated in paragraph V.3, the definition $R(t_0)=1$ holds. One then arrives at the equations:

$$\alpha e^\beta = h_1, \quad \text{and} \quad \beta 2^2 = -h_2,$$

since $R(t_0) = e^{h_1}$ and $R(t_0) = -\frac{\beta^2 e^{h_1}}{R(t_0)}$. Upon taking logs, these two equations take on the form:

$$\ln \alpha + \beta = \ln h_1,$$

$$2 \ln \alpha + 2 \beta + \ln \beta = \ln (-h_2),$$

which are easily solved for $\alpha$ and $\beta$. Thus:

$$B.1) \quad \alpha = \frac{h_1 e^{h_2 / h_1}}{h_2 h_1},$$

$$\beta = -\frac{h_2}{h_1}.$$  

Direct evaluation of $x$ for $y=1$ in A.2) gave $x=-1488$. At $y=1$, however, the slope is $\frac{dy}{dx} = e^y = e^{1/2} = 2.118$. Thus an upper limit to the intersection of the curve with the $x$-axis is given by $x=-1488 - \Delta x$, where $\Delta x =\frac{A x}{87100}$. But $\Delta y = .1$, and therefore $\Delta x=0.00005$. Hence at $y=0$, $x=-1488$, since the effect of $\Delta x$ is less than 5 in the sixth figure.
Graph II: General anti-matter model, equation II.7)
C. Points for the maximum and minimum curves of both models. In the case of the antimatter universe, because of the great reduction in computation, the points of the minimum and maximum curve were calculated with the help of equation A.2) and the transformation, equation A.1). These calculations are summarized in table I. Graph XI represents equation A.2).

In the case of the elliptic model the computations were short enough not to warrant the use of a transformation into dimensionless variables. Direct substitution of the values of a and b into equation III.5) lead to the coordinates tabulated in table II.
<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\gamma$</th>
<th>$t(\text{yr})$</th>
<th>$R$</th>
<th>$t(\text{yr})$</th>
<th>$R$</th>
</tr>
</thead>
<tbody>
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### TABLE II

COORDINATES FOR ELLIPTIC MODEL

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<thead>
<tr>
<th>MINIMUM CURVE</th>
<th>MAXIMUM CURVE</th>
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<tr>
<td>( a = -1.143 \times 10^{-5} \text{ y}^2 )</td>
<td>( a = -1.140 \times 10^{-5} \text{ y}^2 )</td>
</tr>
<tr>
<td>( \beta = 0.164 \times 10^{-4} \text{ y}^2 )</td>
<td>( \beta = 0.176 \times 10^{-4} \text{ y}^2 )</td>
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</table>

<table>
<thead>
<tr>
<th>( t (10^3 \text{ yrs}) )</th>
<th>( R )</th>
<th>( t (10^3 \text{ yrs}) )</th>
<th>( R )</th>
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</thead>
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<td>0.19</td>
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<tr>
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<td>0.37</td>
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<tr>
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<td>0.6</td>
<td>0.82</td>
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<td>2.09</td>
<td>1.0</td>
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<tr>
<td>4.91</td>
<td>1.15</td>
<td>3.27</td>
<td>1.2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5.88</td>
<td>1.4</td>
</tr>
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</table>

Maximum \( R \),

\[ R_{\text{max}} = -\frac{\beta}{a} = 1.15 \]

at time

\[ t_{\text{max}} = -\frac{\beta}{2a \sqrt{a}} = 4.91 \times 10^3 \text{ yrs} \]

Maximum \( R \),

\[ R_{\text{max}} = -\frac{\beta}{a} = 1.4 \]

at time

\[ t_{\text{max}} = -\frac{\beta}{2a \sqrt{a}} = 5.88 \times 10^3 \text{ yrs} \]
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