METHODS OF INVESTIGATING PHENOMENA ARISING FROM NON-LINEARITIES IN POWER SYSTEMS.

by

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of

KENNETH STEPHEN JULIEN

B.Sc., University of Nottingham, 1957

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ABSTRACT

Mathematical methods for investigating power-system phenomena arising from non-linearities are developed in this thesis.

Most information available about power-system phenomena arising from non-linear effects is obtained from two main sources of research: field tests and miniature representation experiments.

The use of equivalent circuits describing the physical system and the application of circuit analysis techniques is another approach to this problem. This thesis is concerned with the establishment of procedures for methods based on this approach.

The incremental method is simple in theory but its application was difficult in the past because of the necessity of numerous calculations. The facilities of the digital computer overcome this difficulty and this method is fully explored. Certain aspects of the phenomena are investigated and some programming details of the method discussed.

In contrast, the other methods require less calculations as the solutions are in the form of simple algebraic expressions. An insight into the system behaviour rather than accurate numerical results are obtained. Under the broad heading of analytical methods, the Method of Isoclines, the Principle of Harmonic Balance and the Method of Integral curves are investigated and used.

The establishment of the equivalent circuits representing the physical system is studied and the adequacy of these representations is discussed. An interesting method of approaching the transient solution of the long-line equations is also developed. Comparison between the representations of the power transmission line by a finite number of T-sections and the use of distributed parameters is made.

Underlying the whole study is the growing importance of non-linear effects and transient phenomena in power-system planning, design and operation.

GRADUATE STUDIES

Field of Study: Power Systems Analysis

Electric Power Systems	F. Noakes
Network Theory	A.D. Moore
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Performance of Electric Machines	J.F. Szablya
Design of Electric Machines	J.F. Szablya
Analogue Computers	E.V. Bohn
Thermal Power	H.M. McIlroy
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Atomic and Nuclear Physics	J.C. Giles
Theory and Applications of Differential Equations	C.A. Swanson
Functions of a Complex Variable	W.H. Simons

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CHAPTER I.

1-1 Introduction

The title of this thesis is purposely general although the subject matter itself is quite specific. The justification for this is based on the fact that whereas a particular non-linear effect under definite system conditions is treated in detail, the emphasis is on establishing a method of treatment rather than discussing the solution of this particular problem. The ideas developed are themselves quite general and certainly will have application in the rather broad field of non-linear effects in power-system analysis.

The two key-words in this title are non-linearities and power systems and it is important that the sense in which these are used be established. This in turn requires a discussion of the circuit concept which will be used exclusively in this study.

1-2 The Circuit Concept.

Electric circuit theory finds its true foundation in Maxwell's theory of the electromagnetic field. This theory expressed by the well known Maxwell's equations has been very successful in explaining all electric and magnetic phenomena in terms of fields resulting from charges and currents. Despite the success of this field theory, the circuit concept has found widespread use and acceptance especially in the formulation of many phenomena of engineering interest. Its simplicity of application and use of easily measurable quantities--voltages and current--have been responsible for its development independent of field theory and this fact has hidden its foundation, its limitations and approximations.

High frequency phenomena, transient behavior and non-linear effects have been responsible for a more critical examination of the circuit concept. Its practicability in analysis of these effects is still a big attraction but its value as an approximation must be examined and the results of any study must be coloured by its limitations. There are five main parameters which usually enter a circuit analysis--current, voltage, resistance, inductance and capacitance. The other quantities of interest--charge, fields, power, etc.--can be computed from these.

In defining these quantities, three different approaches are possible, each finally giving the same mathematical result, and it is a matter of choice whether one starts with the easily visualizable concept of charge, the more physically significant concept of current or the more mathematical basis of Maxwell's equation.[1] Two of these concepts will be used and the choice is based on convenience.

1-2-1 The Electric Current.

Using the concept of charge as understood in present day atomic theory, a single electron is credited with having the basic unit of charge. The phenomenon of charge transfer or velocity is described by the term electric current. If q denotes the charge and i the electric current, in equation form, the relation

$$\text{current } i = \frac{dq}{dt} \text{ is established} \tag{1-1}$$

1-2-2 The Electric Potential or Voltage.

The movement of charge requires energy of one form or another and the term voltage or potential is defined as the energy per unit charge--i.e. it is the amount of energy required to transport a unit charge under certain definite conditions. In equation form, if dW_e is the energy required to move dq charge, then

$$\text{voltage } v = \frac{dW_e}{dq} \tag{1-2}$$

also

$$vi = \frac{dW_e}{dq}\frac{dq}{dt} = \frac{dW_e}{dt} = \text{rate of doing work or expending energy} = \text{power}$$

$$\therefore \quad \text{power} \quad p = vi \tag{1-3}$$

$$\text{energy} \quad W_e = \int p\,dt = \int vi\,dt \tag{1-4}$$

1-2-3 The Magnetic Field. The Inductance Parameter.

For a fuller appreciation of the description non-linear as applied to an inductive element, the concept of energy and Maxwell's equations are used as a basis.

Maxwell's equation[1] state

$$\text{Curl } \bar{H} = \bar{J} + \frac{d\bar{D}}{dt} \tag{1-5}$$

$$\text{Curl } \bar{E} = - \frac{d\bar{B}}{dt} \tag{1-6}$$

from which

$$\text{div } \bar{B} = 0 \tag{1-7}$$

$$\text{div } \bar{D} = \rho \tag{1-8}$$

As expressed in these equations, the basic quantities are:

$\bar{H}$ = magnetic field intensity (1-9)

$\bar{E}$ = electric field intensity (1-10)

$\bar{B}$ = flux density (1-11)

$\bar{D}$ = displacement density (1-12)

$\bar{J}$ = current density (1-13)

ρ = charge density (1-14)

In the circuit concept, all relations must be expressed by quantities of voltage, current and circuit parameters. A necessary step is to establish the correspondence between the field quantities and the quantities used in the circuit concept.

Considering equation (1-6) and referring to Fig (1-1),

$$\text{Curl } \bar{E} = - \frac{d\bar{B}}{dt} \tag{1-15}$$

then

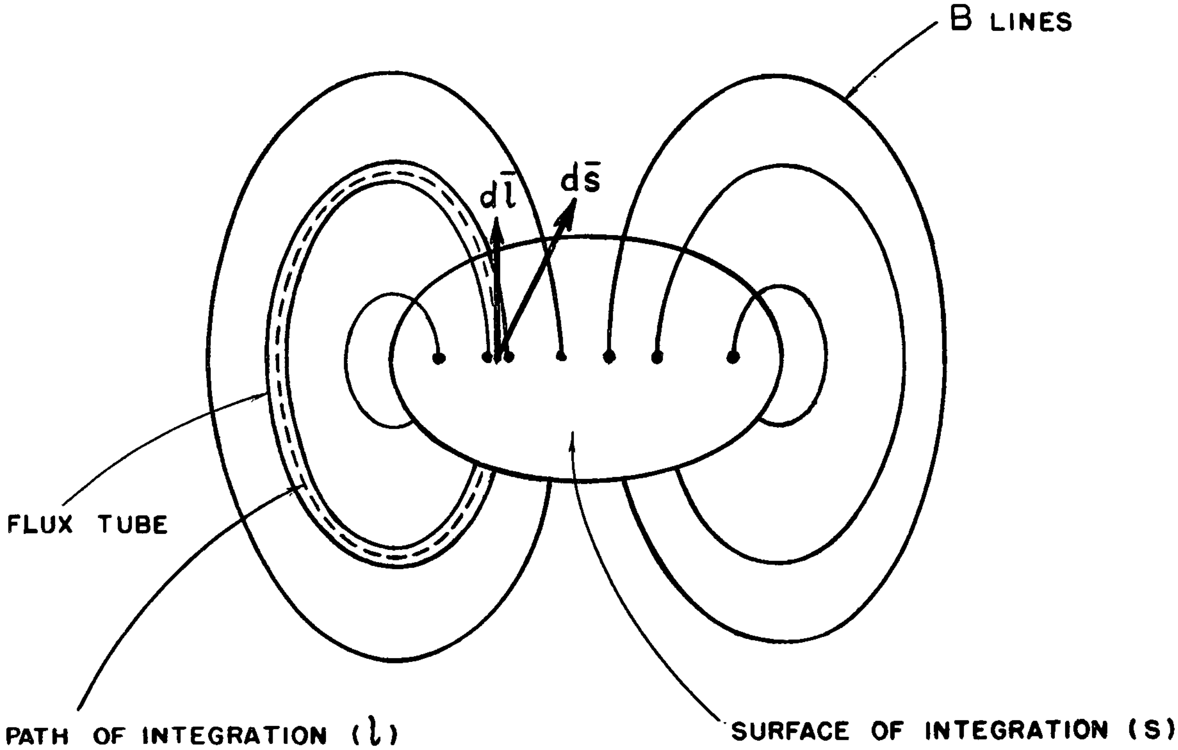

B LINES
$d\bar{l}$
$d\bar{s}$
FLUX TUBE
PATH OF INTEGRATION (l)
SURFACE OF INTEGRATION (S)

FIG. 1-1

$$\bar{B}.\overline{ds} = \text{constant} \tag{1-9}$$

$\therefore$ equation (1-8) can be written

$$W + W' = \int_s \bar{B}.\overline{ds} \oint_l \bar{H}.\overline{dl} \tag{1-10}$$

Now

$$\oint \bar{H}.\overline{dl} = \int_a \bar{J}.\overline{da}$$

where $\bar{a}$ = surface area bounded by a $\bar{B}$ line, therefore a quantity M can be defined

$$M \triangleq \oint \bar{H}.\overline{dl} = \int_a \bar{J}.\overline{da} \tag{1-11}$$

$$\therefore \quad W + W' = \int_s M\ \bar{B}.\overline{ds} \tag{1-12}$$

from which

$$d(W + W') = \int_s (M\ \overline{dB}).\overline{ds} + \int_s (dM\ \bar{B}).\ \overline{ds} \tag{1-13}$$

Comparing this equation with (1-6) suggests that

$$dW = \int_s M\ \overline{dB}.\overline{ds}. \tag{1-14}$$

and

$$dW' = \int_s dM\ \bar{B}.\overline{ds}. \tag{1-15}$$

In circuit theory, the variables are the terminal voltages and currents, and a relationship must be established betweeen these and the field quantities.

$$\text{Now terminal current } i \doteq \int_{a^*} \bar{J}.\overline{da}^* \tag{1-16}$$

where a* = area of the cross-section of the conductor.

From the form of equations (1-11) and (1-16), it follows

Considering equation (1-5) and neglecting the term $\frac{d\bar{D}}{dt}$, a procedure justifiable in this context, a relation

$$\oint_l \bar{H}.\bar{dl} = \int_s \bar{J}.d\bar{s} = \mathcal{N}.i \tag{1-23}$$

can be established. $\mathcal{N}$ is the constant of proportionality between $\int_s \bar{J}.\bar{ds}$ and i, the current.

Eq. (1-22) gives a relation between voltage and flux-linkage; eq. (1-23) gives a relation between current and magnetic field intensity. In any medium, there is a correspondence between $\bar{B}$ and $\bar{H}$. This correspondence can then be used to relate v_{ind} and i.

In some media, the correspondence between $\bar{B}$ and $\bar{H}$ is linear. In such a case, the relation between λ and i is linear. Then

$$v_{ind} = \frac{d\lambda}{dt} = \frac{d\lambda}{di} \cdot \frac{di}{dt} = L \cdot \frac{di}{dt} \tag{1-24}$$

The constant L is called the <u>inductance parameter</u>.

In other media such as iron, the correspondence between $\bar{B}$ and $\bar{H}$ is rather complex, especially when hysteresis effects are present. The simple relationship given by eq. (1-24) is no longer valid. Instead some complex function referred to as

$$\lambda = \lambda(i)$$
$$\text{or} \qquad i = i(\lambda) \tag{1-25}$$

has to be used. It may not even be possible to express such a function analytically and the methods of handling such functions form an important part of this study.

1-2-4 The Electric Field. The Capacitance Parameter.

Using the same approach as is sec. (1-2-3) and referring to Fig (1-2), the relation

$$\oint_s \bar{D}.d\bar{s} = \int_V \text{div } \bar{D}.dV \qquad (1-26)$$

is used.

Now

$$\text{div } \bar{D} = \rho \qquad (1-27)$$

therefore

$$\oint_s \bar{D}.d\bar{s} = \int_V \rho \, dV \qquad (1-28)$$

Defining

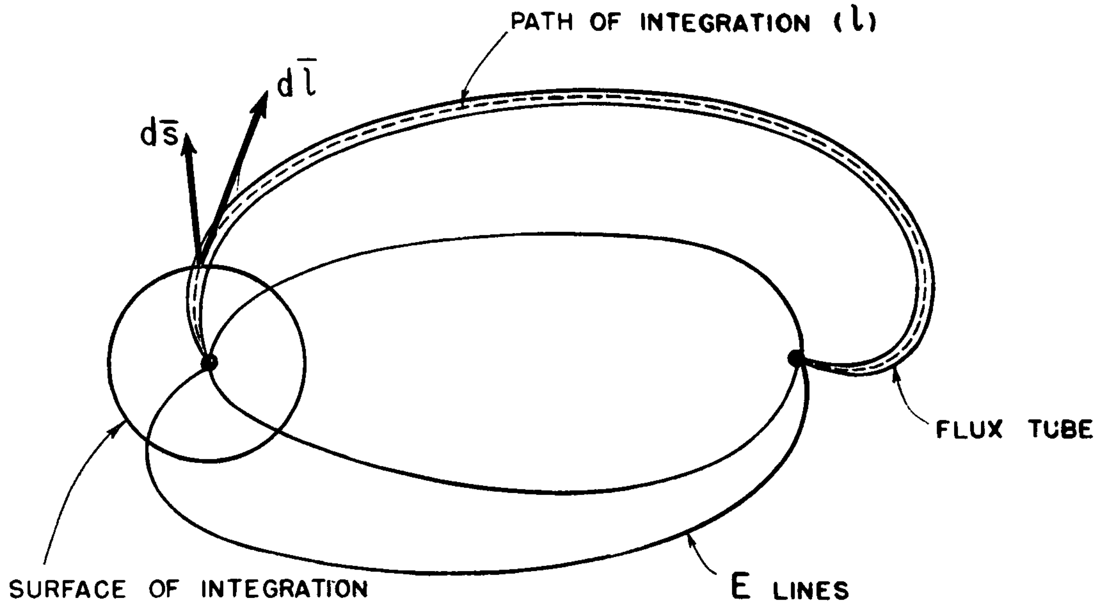
$$\text{charge} = q \doteq \int_V \rho \, dV \qquad (1-29)$$

the relation

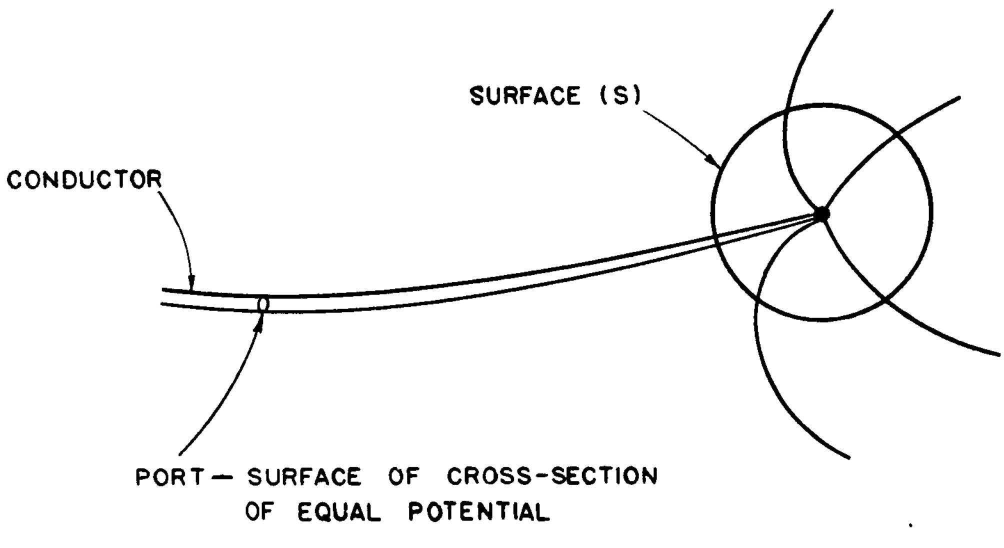
$$q = \oint_s \bar{D}.d\bar{s} \qquad (1-30)$$

is established.

Considering a potential difference v_c between two terminals,


PATH OF INTEGRATION (l)
$d\bar{l}$
$d\bar{s}$
FLUX TUBE
SURFACE OF INTEGRATION
E LINES

(a)


SURFACE (S)
CONDUCTOR
PORT — SURFACE OF CROSS-SECTION
OF EQUAL POTENTIAL

(b)

FIG. 1-2

$$\int_{l} \bar{E}.d\bar{l} = v_c \tag{1-31}$$

Eq. (1-30) establishes the relation between v_c and $\bar{E}$.

If the relation between $\bar{D}$ and $\bar{E}$ in a medium is simply given by

$$\bar{D} = \varepsilon.\bar{E} \tag{1-32}$$

where ε = some constant (permittivity), (1-33)

then the relation between q and v_c is expressed by the linear equation

$$q = C\ v_c \tag{1-34}$$

C is called the capacitance parameter.

For other cases, some complex relationship

$$q = q(v_c)$$

or $$v_c = v_c(q) \tag{1-35}$$

must be used. This type of non-linearity does not arise in this investigation and will not be considered any further.

1-2-5 Simulation of Losses. The Resistance Parameter.

In power-systems, losses are always present. These losses may arise from current flow in conductors, the corona phenomenon, secondary effects of the magnetising field, etc.

The currents and voltages corresponding to these phenomena satisfy the principle of conservation of energy,

$$\frac{dW_R}{dt} = P_R = i_R \, v_R \tag{1-36}$$

If the relation between i_R and v_R is linear, this relation may be expressed as

$$v_R = R \, i_R \tag{1-37}$$

However, with the increasing importance of the more complex phenomena of corona, arc-behaviour, hysteresis losses etc., for the simulation of these losses, eq. 1-37 is not valid. In this study, the question of the hysteresis losses is investigated with this in mind and some interesting results are obtained.

Except for this case, it will be assumed in this study that the <u>resistance parameter</u> R is constant.

<u>1-3 Non-Linearity</u>.

In the circuit concept, the circuit parameters are used to represent a physical system which displays certain electrical phenomena. It is realized that such representation can only be an approximation, as a complete and true description of the physical behaviour will introduce endless combinations of circuit parameters and thus defeat the simplicity of the method.

The same physical system may be represented by different parameters depending on the phase of the behaviour being studied. The choice of the representation depends on experimental fact and engineering judgement but a wide range of phenomena can be studied by the use of linear analysis. This implies that the physical system is represented in a circuit by parameters which are constant for variations of the magnitude of current, charge and voltage.

The mathematics and procedure of linear analysis are well developed and follow a set pattern but, as will be seen, the introduction of non-linear elements requires individual mathematical treatment and the establishment of a procedure to suit the specific problem.

Before being able to study a system, there are other questions to be asked about the elements or parameters. Are they bilateral, time-invariant, lumped?[2] In this study, the assumption will be made that all parameters are bilateral and time-invariant but the question of a lumped-network representation of a transmission line will be fully discussed in Chapter 2.

1-4 Mathematical Implication of Non-Linearity.

If linear behaviour is assumed, then equations (1-22), (1-35), (1-37) can be written

$$\begin{aligned} v_{ind} &\triangleq e_\lambda = L\,\frac{di}{dt} \\ v_c &= \frac{1}{C}\int i\,dt \\ v_r &= R\,i \end{aligned} \qquad (1\text{-}38)$$

where C, L, R are constants and depend only on the geometry of the physical system.

If however, any of these elements are non-linear, no values of C, L, R, can be defined, therefore the functions $\lambda(i)$, $q(v_c)$, and $v_R(i_R)$ have to be used directly and a suitable mathematical approach has to be developed.

If any form of mathematical analysis is to be attempted,

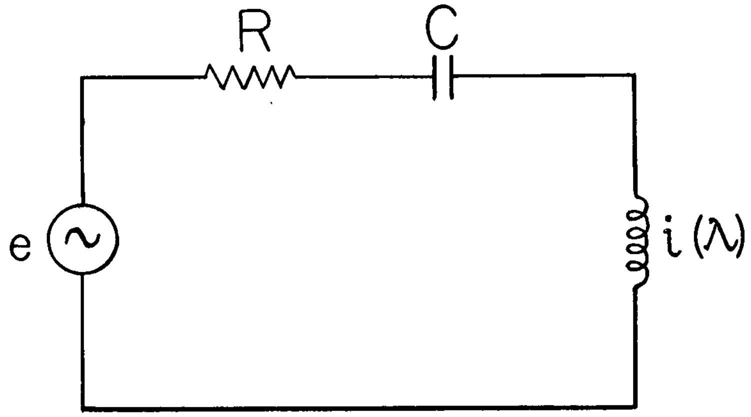


FIG. 1-3

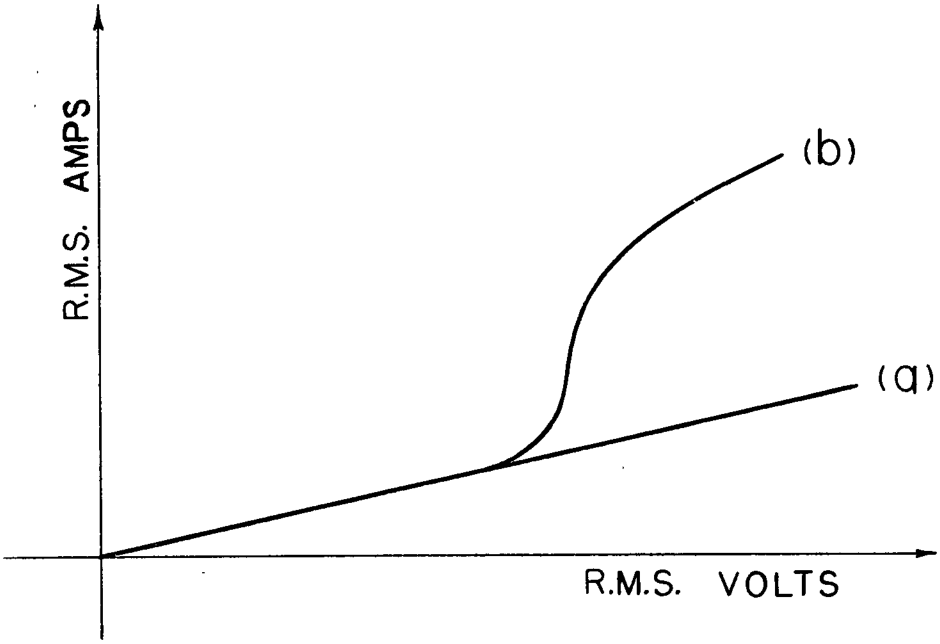


FIG. 1-4

the non-linear characteristic must be established for the non-linear element. This requires further approximation and the use of engineering judgment as the mathematical complexities must be balanced against the usefulness of the result.

To clarify some of the points discussed, an example will be briefly discussed.

Consider the circuit shown in Fig. (1-3). All the elements are linear. This is recognised as the simple series resonant circuit, with a single frequency excitation.

The mathematical equation governing this circuit is given by

$$e = E \sin(\omega t + \Theta) = iR + \frac{1}{C}\int i dt + \frac{d\lambda}{dt} \tag{1-39}$$

Differentiating,

$$\omega E \operatorname{Cos}(\omega t + \Theta) = R\frac{di}{dt} + \frac{1}{C} i + \frac{d^2\lambda}{dt^2} \tag{1-40}$$

Assuming linearity throughout,

$$\frac{d\lambda}{dt} = \frac{d\lambda}{di} \frac{di}{dt} = L\frac{di}{dt} \tag{1-41}$$

where

$$L \triangleq \frac{d\lambda}{di} \quad \text{(since linear)} \tag{1-42}$$

$\therefore$ Equation (1-40) reduces to

$$L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{1}{C} i = \omega E \cos(\omega t + \Theta) \tag{1-43}$$

This is a simple linear second-order differential equation and the various methods available for the solution of this equation yield an explicit solution for i as a function of t.

The complete solution gives both transient and steady-state behaviour ; the terms in the solution containing exponential decrement factors become insignificant when the circuit reaches its steady-state ($t \longrightarrow \infty$).

If instead of the previous conditions, the inductor exhibits non-linearity, before the solution of the equation can be attempted, a relationship between i and λ must be established. The function $i = i(\lambda)$ is usually given in the form of a graph but for the purpose of this example, it will be assumed that an exact relationship

$$i = a_1\lambda + a_3\lambda^3 + a_5\lambda^5 \tag{1-44}$$

exists, where a_1, a_3, and a_5 are known constants.

This relationship can be substituted in equation (1-40) yielding

$$\omega E \cos(\omega t + \theta) = R(a_1 \frac{d\lambda}{dt} + 3a_3 \lambda^2 \frac{d\lambda}{dt} + 5a_5 \lambda^4 \frac{d\lambda}{dt}) + \frac{1}{C} (a_1\lambda + a_3\lambda^3 + a_5 \lambda^5) + \frac{d^2\lambda}{dt^2} \tag{1-45}$$

Further manipulation of this equation yields

$$\frac{d^2\lambda}{dt^2} + R(a_1 + 3a_3\lambda^2 + 5a_5\lambda^4) \frac{d\lambda}{dt} + \frac{1}{C} (a_1\lambda + a_3\lambda^3 + a_5\lambda^5) = \omega E \cos(\omega t + \theta). \tag{1-46}$$

The solution of this non-linear differential equation provides an interesting mathematical exercise[3] and the results are certainly not as straight forward or simple as in the previous example.

Although the introduction of a non-linear element complicates the mathematics, the basic problem of what effect this has on the physical phenomena still has to be answered. In all cases, verification of the validity of any analysis depends ultimately on experimental results but, in non-linear problems, there is the added difficulty that general methods for direct interpretation of non-linear equations are not available.

Fig (1-4) shows the volt-ampere relationships obtained experimentally[3] for the circuit of Fig.(1-3). Curve (a), a straight line characteristic, is obtained if an air-cored reactor is used--i.e. one that can be truly represented by a linear inductance. Curve (b) is obtained with the same parameters but an iron-cored reactor is used.[3] Obviously, obtaining a solution for such a physical system will involve more complex representation and techniques. Eq. (1-46) as an approximate description of such a system is one example of the difficult mathematical problems involved in studying such phenomena.

In the simple series linear circuit shown in Fig. (2) a familiar term is the resonance frequency. If either ω, L or C is varied, there occurs a point when $\omega^2 = \frac{1}{LC}$ and the current i reaches a maximum value, $i_{max} = \frac{E}{R}$; this value of ω is called the resonant frequency.

There is no such simple occurrence in a non-linear circuit. Instead, the terms higher harmonics, sub-harmonics, ferro-resonance and jump phenomenon are common throughout the literature of non-linear studies. The significance of these terms will be considered as they occur in this study.

1-5 Power-Systems.

The transfer of energy from one point to another requires a system diagrammatically shown in Fig. (1-5).[4] Three main

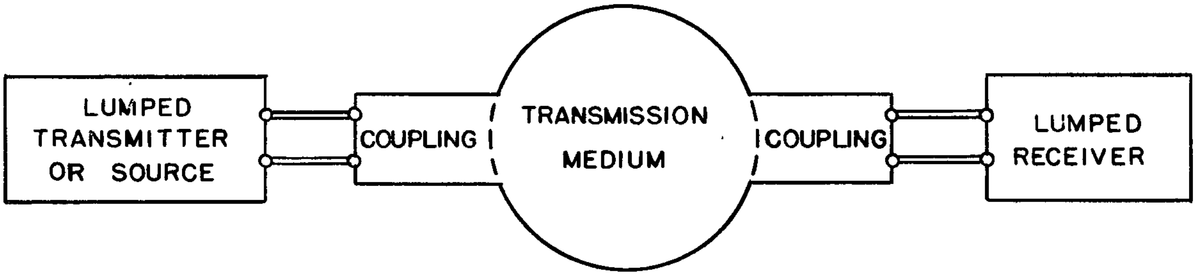

LUMPED
TRANSMITTER
OR SOURCE
COUPLING
TRANSMISSION
MEDIUM
COUPLING
LUMPED
RECEIVER

FIG. 1-5

features characterise an electrical power system as used in this context:

(1) The transmission medium and coupling are solid conductors.

(2) Relatively low frequencies (100 c.p.s.) are used for transmission.

(3) The quantities of energy transferred are usually very large.

Because of these factors, the circuit concept is admirably suited for studying power systems. If a proper representation of the elements of the system is determined, the application of the simple laws of Kirchhoff in combination with the voltage-current relations for the lumped parameters reduces the problem to the solution of a set of differential equations.

The lumped transmitter or source is usually a 3-phase a.c. generator or a combination of such generators. The couplings at either end depend on the system but in most cases power transformers are required.

In d.c. transmission, a combination of transformers, rectifying and conversion equipment is necessary. The transmission system is solid conductors with the earth sometimes being used as part of the system.

Each component in a power system as described is capable of complex behaviour. In the representations of these devices using the circuit concept, many simplifications have to be introduced but even then, non-linear techniques must be employed if the entire range of performance is to be studied.

A review of the literature pertinent to this field reveals the necessity of such studies. It is remarkable that power system design and analysis has reached its present advanced stage relying principally on linear techniques.

Rotating machines with their complex field distribution and geometrical asymmetries, transformers characterised by the non-linear flux-current relationship, non-linear resistors used as protective devices are all essential elements of any modern power system. Under steady-state conditions, the mode of operation of these devices and phenomena experienced are such that linear differential equations can usually be used to describe them. The solution of these equations in combination with some empirical information provide useful and sufficiently accurate results for normal, steady-state analysis of power systems. The size and complexity of the system may warrant the use of network analysers or digital computers, using the techniques of linear analysis.

1-6 Non-Linear Effects in Power Systems.

Fault occurrences, switching operations can set up disturbances on a system resulting in phenomena which cannot easily be represented by linear parameters. Experimental tests have proved that knowledge of the effects caused by these disturbances is necessary for the design and operation of power systems.

In the economic design and operation of any power system, two factors are of prime importance: firstly, the coordination of insulation level throughout the system; secondly, the setting up of a reliable, selective and quick-acting protective system. The first factor requires a knowledge of the voltages that will be experienced by the equipment in all parts of the system under expected conditions, and the second requires a knowledge of both voltage, current and associated relationships mainly at key points of the system.

Field tests verified in part by theoretical considerations have shown that abnormal voltages may either be externally induced--principally from lightning--or result from switching operations. Lightning surges have been the most frequent cause of outages, following insulation breakdown, and a great deal of research work has been devoted to this field. Development of ingenious transformer designs, improvement of insulating materials and the introduction of lightning arresters have all contributed to making power systems more immune from such failures.

With the growing use of extra high voltages, long transmission distances, and such devices as series capacitors and d. c. conversion equipment, prediction of abnormal voltages caused by switching operations has become increasingly important in system planning. The term switching operation as used here implies a change of state of the system from one set of conditions to another. Associated with this change of state is a new distribution of voltages and currents which is accompanied by a transient behaviour. The causes of such changes of state are varied and the consequent behaviour covers a very wide range but there is one essential difference between this phenomenon and lightning surges. In this case, exponential or near-exponential voltages and currents are experienced whereas in lightning phenomena, wave-shapes of varied patterns occur.

It must be noted at this point that in fact the phenomena of abnormal voltages and currents need not necessarily be associated with non-linear behaviour and in fact, some severe overvoltages can be studied completely by means of linear techniques. Specifically, fundamental frequency voltages that occur due to the occurrence of a system fault can be determined by the use of symmetrical components.[5] This technique may rely on certain assumptions which exclude non-linearity, e.g., the non-existence of arcing or assigning a constant impedance to rotating machines. However, in this study, attention will be confined to the particular case of the energising of a system in which the non-linear flux-current relationship of a transformer plays an important part. This particular problem will be studied in some detail with the emphasis on methods of approach to certain problems rather than on the significance of the results. Some of the ideas explored will be pertinent to most non-linear behaviour in power systems and similar types of switching conditions, e.g., the dropping of load and the paralleling of transformers.

To appreciate how some of these effects arise, two interesting phenomena will be examined.

Consider a transmission line, fed by a single-phase a.c. generator. The line is terminated by a reactor. The flux-linkage vs current relationship of the reactor is given by curve (a) Fig (1-6). The circuit representation of such a system is shown in Fig (1-7) where R and L (both linear) are the resistance and inductance respectively of the line.

The equation describing such a circuit is

$$e = E_{max} \operatorname{Sin}(\omega t + \Theta) = i\ R + L\,\frac{di}{dt} + e_{\lambda}$$

$$= i\ R + L\,\frac{di}{dt} + \frac{d\lambda}{dt} \qquad (1\text{-}47)$$

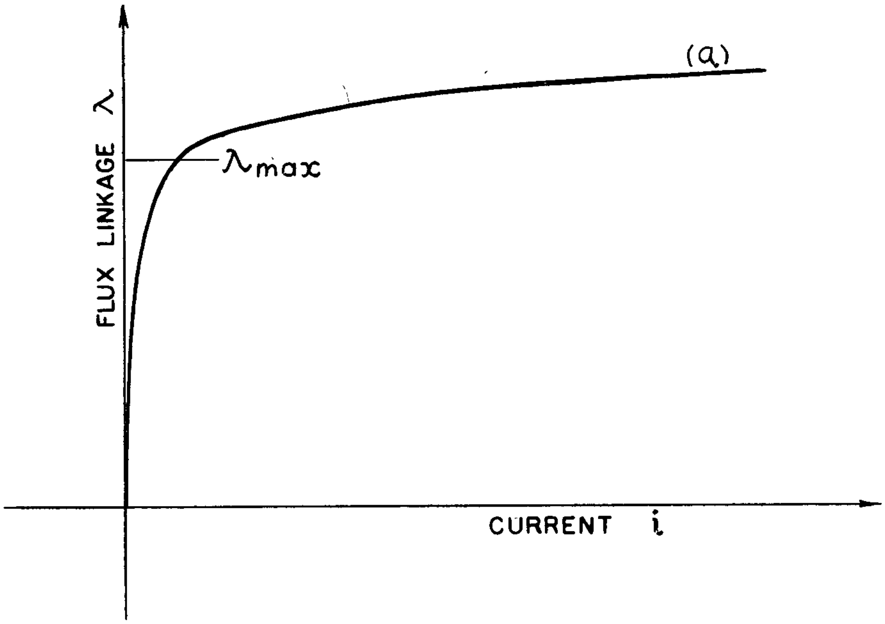


FIG. 1-6

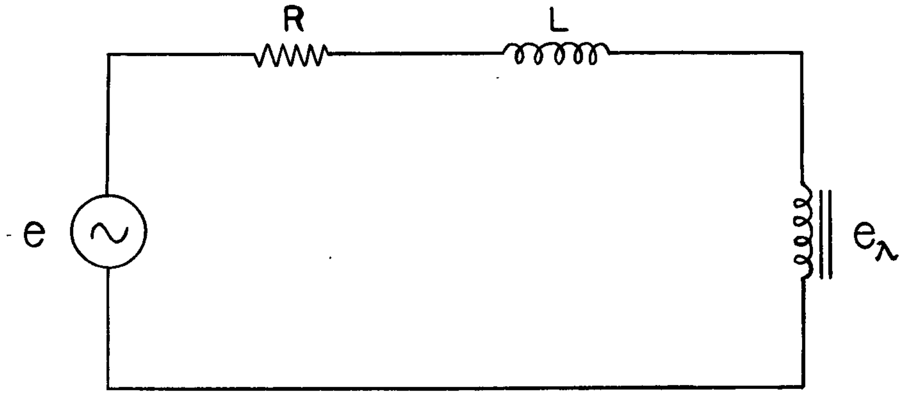


FIG. 1-7

As a first approximation, it can be assumed that

$$e_\lambda >> i\ R \tag{1-48}$$

$$e_\lambda >> L \frac{di}{dt} \tag{1-49}$$

Then eq. (1-47) simplifies to

$$E_{max} \operatorname{Sin} (\omega t + \theta) = \frac{d\lambda}{dt} \tag{1-50}$$

i.e.

$$\lambda = \frac{-E_{max}}{\omega} \operatorname{Cos} (\omega t + \theta) + K_1 \tag{1-51}$$

If the initial conditions are

$$\lambda = 0 \tag{1-52}$$

$$\theta = 0 \tag{1-53}$$

at

$$t = 0 \tag{1-54}$$

then

$$K_1 = \frac{E_{max}}{\omega} \tag{1-55}$$

i.e.

$$\lambda = \frac{E_{max}}{\omega} (1 - \operatorname{Cos} \omega t) \tag{1-56}$$

Eq. (1-56) shows that for

$$\omega t = \pi \tag{1-57}$$

$$\lambda = \frac{2\ E_{max}}{\omega} \tag{1-58}$$

$= 2$ x normal steady-state value of λ

$$= 2\ \lambda_{max} \tag{1-59}$$

Curve (a) Fig. (1-6) shows that as λ approacnes this value $2\ \lambda_{max}$ large values of current result. The inclusion of line resistance in these considerations results in a damping effect, the peak value of λ approaching its steady-state value λ_{max} with increase in time. However, in any case, large values

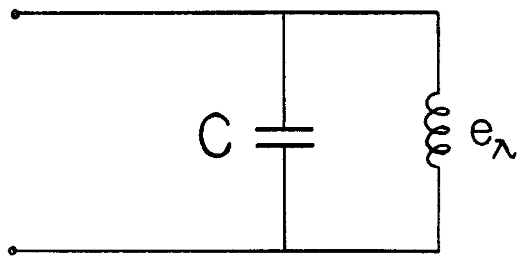

C
e_λ

FIG. 1-8

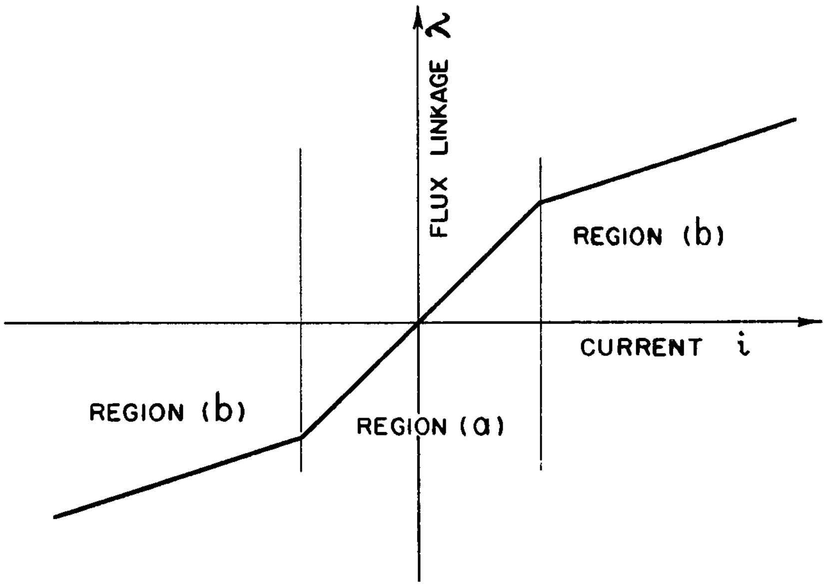

FLUX LINKAGE λ
REGION (b)
CURRENT i
REGION (b)
REGION (a)

FIG. 1-9

of current occur initially.

A transformer terminating a transmission line behaves like a reactor with a non-linear flux-current relationship. This phenomenon of large currents occurring initially on energising is the well known <u>transformer current inrush</u>.

Because of the fact that the peak value of this inrush current may exceed the current rating of the transformer, knowledge of the wave shape and magnitude of this current is necessary in the setting up of protective devices for the transformer.

Some attention has been given to this problem--the more significant contributors in this field being Turner[6], Finzi[7] and Blume[8]. The approach has been primarily experimental as a complete rigorous mathematical treatment will be complex and difficult. This phenomenon described over 60 years ago-[9] has a continued interest for system designers and planners. Because of the use of higher flux densities and the need for more exacting specifications of protective equipment, more precise results must be obtained.

A simple demonstration of the occurrence of overvoltages requires the use of the circuit shown in Fig (1-8). A long transmission line terminated by a non-linear reactor is represented by Fig (1-8). Resistance is neglected and the $\lambda(i)$ function is given by Fig. (1-9). The system is assumed to be suddenly de-energised.

The equation describing such a circuit is

$$\frac{1}{C}\int i\ dt + e_\lambda = 0. \qquad (1\text{-}60)$$

Differentiating eq. (1-60)

$$\frac{i}{C} + \frac{de_\lambda}{dt} = 0 \qquad (1\text{-}61)$$

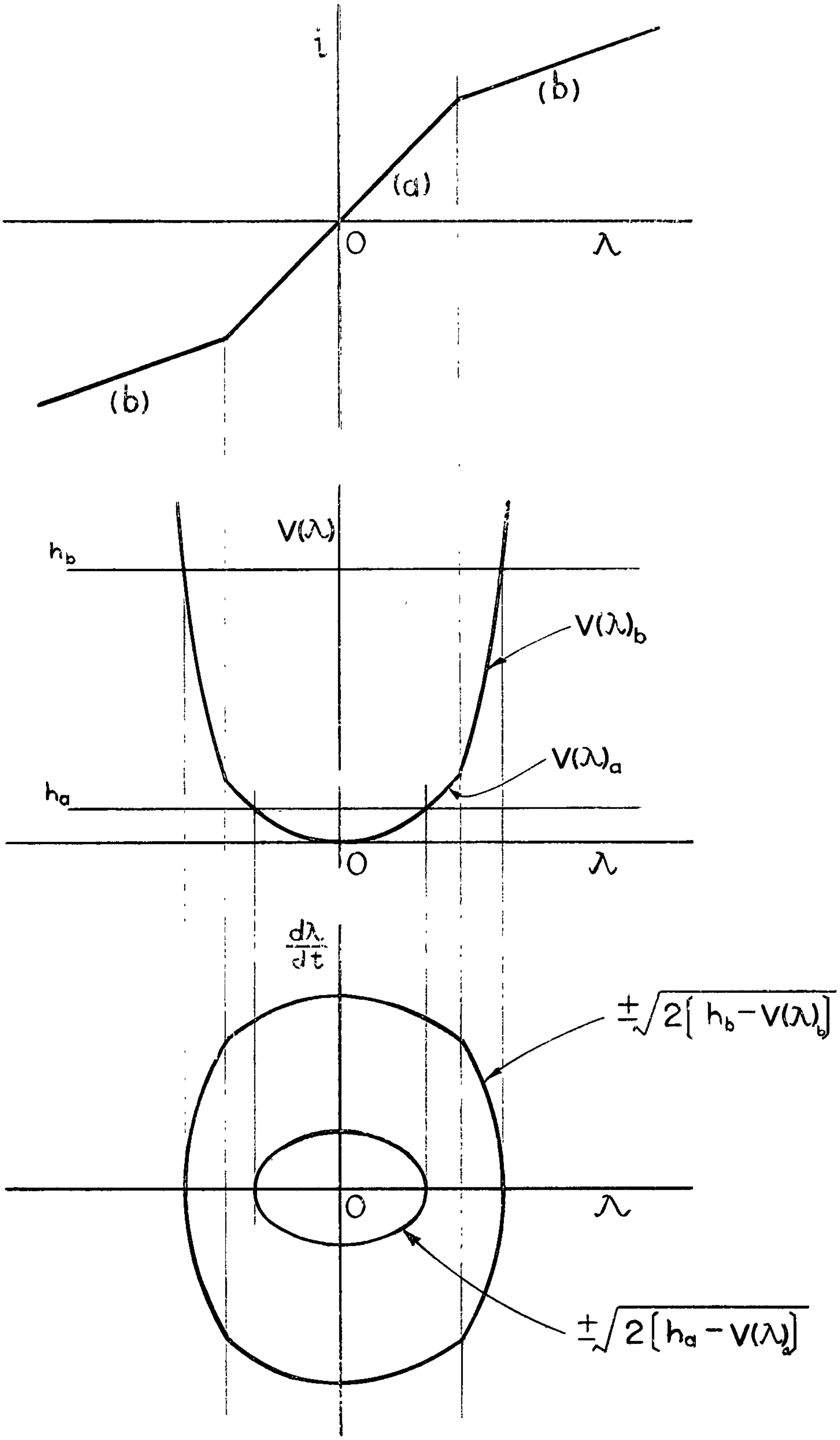

i
(b)
(a)
O
λ
(b)
V(λ)
h_b
$V(\lambda)_b$
$V(\lambda)_a$
h_a
O
λ
$\frac{d\lambda}{dt}$
$\pm\sqrt{2[h_b - V(\lambda)_b]}$
O
λ
$\pm\sqrt{2[h_a - V(\lambda)_a]}$

FIG. 1-10

Now, referring to Fig (1-9)

$$i = i(\lambda)$$

$$i = k\lambda \text{ in region (a)} \tag{1-62}$$

and

$$i = m\lambda + b \text{ in region (b)} \tag{1-63}$$

$$e_\lambda = \frac{d\lambda}{dt} \tag{1-64}$$

$\therefore$

$$\frac{de_\lambda}{dt} = \frac{d^2\lambda}{dt^2} \tag{1-65}$$

$\therefore$ Eq. (1-61) becomes

$$\frac{d^2\lambda}{dt^2} + \frac{1}{C}\, i(\lambda) = 0 \tag{1-66}$$

Multiplying this equation by $\frac{2d\lambda}{dt}$,

$$2\,\frac{d\lambda}{dt}\,\frac{d^2\lambda}{dt^2} + \frac{2}{C}\, i(\lambda).\,\frac{d\lambda}{dt} = 0 \tag{1-67}$$

Integrating eq. (1-67),

$$\left(\frac{d\lambda}{dt}\right)^2 + \frac{2}{C}\int i(\lambda)\, d\lambda = 2h \tag{1-68}$$

where h is some constant determined by initial conditions.

Eq. (1-68) can be put into a more convenient form

$$\frac{d\lambda}{dt} = \pm\sqrt{2\left[h - V(\lambda)\right]} \tag{1-69}$$

where

$$V(\lambda) = \frac{1}{C}\int i(\lambda)\, d\lambda. \tag{1-70}$$

In region (a)

$$V(\lambda)_a = \frac{1}{C}\int k\lambda\, d\lambda = \frac{k\lambda^2}{2C}. \tag{1-71}$$

In region (b)

$$V(\lambda)_b = \frac{1}{C}\int (m\lambda + b)\, d\lambda = \frac{m\lambda^2}{2C} + \frac{b\lambda}{C} \tag{1-72}$$

The significance of these results is more easily shown graphically.

A plot of $V(\lambda)$ versus λ is shown in Fig (1-10); both $V(\lambda)_a$ and $V(\lambda)_b$ are parabolas but $V(\lambda)_b$ increases more rapidly. The phase-plane trajectory is then plotted.

For some initial condition given by $V(\lambda) = h_a$, the phase-plane trajectory $\frac{d\lambda}{dt}$ versus λ can be plotted using the relation given by eq. (1-69).

This gives the closed trajectory (a) and no abnormal conditions occur.

However, for some initial condition h_b the phase-plane trajectory takes the shape given by curve (b) and very high values of $\frac{d\lambda}{dt}$ and hence e_λ are possible just outside the normal operating region (a). These over-voltages are often referred to in the literature as ferro-resonance voltages but in fact since a resonance frequency is rather difficult to define under such circuit conditions, the term ferro-resonance is misleading.

1-7 Methods of Investigation.

In the investigation of problems of this nature, the methods used can be classified under the broad headings of

(1) Field Tests.

(2) Mathematical Analysis.

(3) Model Methods.

Field tests, in fact, provide the most accurate information about a particular system and have to be relied upon eventually to verify any results obtained otherwise. However, the expense and inconvenience of performing these field tests, exclude them from most investigations.

Another method which has proved very successful is the simulation of the system to be studied in miniature and carrying out measurements of voltages and currents where needed. The success of this method lies in the fact that with the system set up, many different conditions can be tried and the effects seen almost immediately. Great ingenuity has gone into the design of these very specialized analogue computers and the transient analyser(10) and the "Anacom"(11) are some of the better known developments in this field. Efforts have continued to improve the miniature representation of transformers and generators and the simulation of effects like corona(12), arcing etc.

Peterson, (referring to this type of problem) in the preface to his book(13) published in 1951 wrote "any method of calculation is either extremely and prohibitively time consuming or is so riddled with simplifying assumptions that the finally calculated result is itself of questionable value". In view of this statement, the question arises whether today, ten years later, a mathematical analysis is any more attractive.

A significant part of this study-Chapter 3-will be devoted to the use of the digital computer in investigating this problem and the answer to the above question is a very definite affirmative. The digital computer has made possible the exploring of a few mathematical methods which were previously considered unattractive because of their tediousness. The continuing improvement in digital computer design as regards speed, capacity and simplicity in programming will establish it as a very useful tool in these investigations.

The comparison between these two methods will not be discussed here as the accuracy desired, economics and equipment availability will have to be considered but, although the analogue method is well established and has proved successful, it is felt that there will be a place for analyses which rely on the digital computer.

Quite often, an insight to the influence of changes of design or operating conditions is needed without having to resort to either analogue or digital computers. This naturally involves some simplifying assumptions and minimum numerical computation. Chapter (4) will be devoted to discussing such an approach. Available non-linear analytical techniques are examined with the possibility of applying them to this problem. Some numerical examples demonstrate some of these techniques and several interesting general conclusions arise.

CHAPTER 2

2-1 Introduction

The circuit conceptual scheme is used exclusively in this study, and before attempting a solution of the behaviour of the system, a circuit representation or equivalent circuit must be established. Such a circuit, describing many complex phenomena, will be the result of some simplifications and approximations; therefore, another objective will be to obtain an appreciation of the adequacy of the representation by a particular equivalent circuit.

As described by Fig (1-5), a power system requires principally four different types of physical apparatus: a source, a coupling, a transmission medium and a load.

In this study, emphasis will be on the coupling--the power-transformer--and the transmission medium--the high-voltage transmission line.

The 3-phase synchronous generator which normally is the generating source, will be represented by its simplest equivalent circuit--a sinusoidal voltage of constant amplitude in series with a linear inductance. Proper representation of synchronous machines under different conditions has been extensively studied and forms an important and interesting part of machine theory.[(14), (15), (16)]

The transformer's flux-current relationship provides the non-linearity of concern in this study and thus, the circuit representation of this device will be considered fully with particular emphasis on the simulation of the hysteresis losses.

In contrast with the 3-phase generator and transformer, little attention has been given in the literature to the proper representation of the power-transmission line. Its conventional representation as a simple lumped resistance in series with a lumped inductance or as a four-terminal T or π circuit has always been adopted and the adequacy of such representations has not often been studied. This can be somewhat justified by the fact that the behaviour of the transmission line itself is usually overshadowed by that of more complex components of the system.

However, the use of longer lines and extra-high voltages, the growing interest in transient behaviour of power systems and the need for more knowledge of system losses have created greater interest in the transmission line as a circuit element.

In this Chapter, transmission line behaviour and the various circuit representations of such lines will be studied, relying principally on numerical results for any conclusions drawn.

2-2 The Power-Transformer.

A transformer is a device which transfers electrical energy at one terminal voltage to another. Its operation depends on the existence of a magnetic field. Thus, there are two major interdependent phenomena associated with such a device--the transfer of electrical energy and the existence of a magnetic flux. All the effects of interest in this study occur when the transformer is unloaded or at very light load; therefore, the transformer as an energy transfer device will not be considered in any detail.

A very useful and widely accepted concept is the treatment of the transformer as the combination of an 'ideal' energy transfer-device with a circuit simulating the other effects. This section is devoted to the development of the circuit that simulates the more important effects.

The most significant component of the field of magnetic flux is established in specially shaped structures of ferromagnetic material with appropriately located current-carrying conductors. The analysis of such a device with the necessary computation of flux density $\bar{B}$ and the field intensity $\bar{H}$ is essentially a field problem. The quantities $\bar{B}$ and $\bar{H}$ are functions of both space and time; they depend on the geometry, magnetic properties and history of the structure and the currents in the conductors.

In effect, stated as a field problem, three equations have to be satisfied:-[17]

$$\int_S \bar{B} \cdot \overline{ds} = 0 \qquad (2\text{-}1)$$

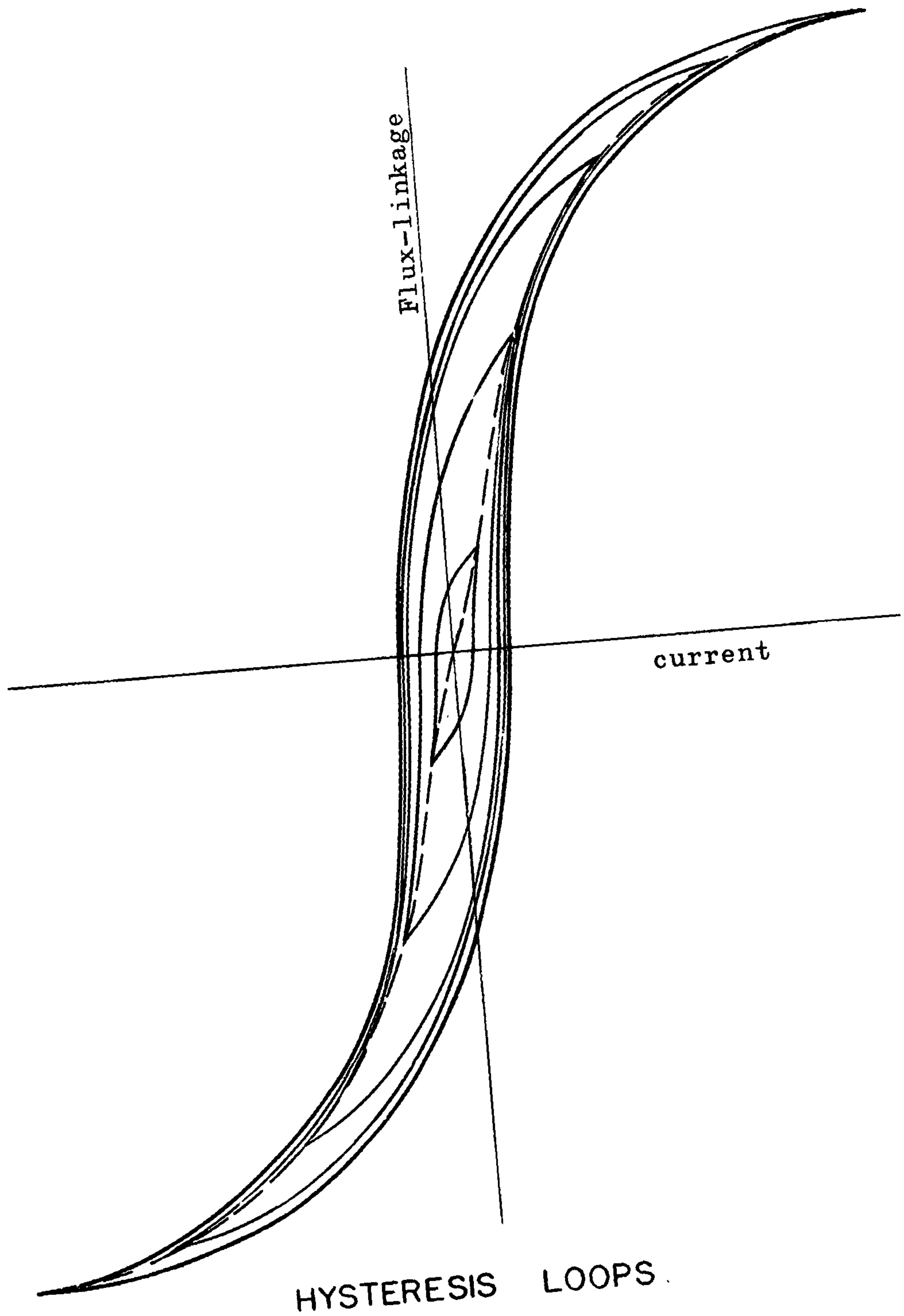
over any closed surface
in the region,

$$\oint \bar{H} \cdot \overline{dl} = M \qquad (2\text{-}2)$$

and thirdly, the inherent relationship between the flux density and field intensity. This depends on the material of the transformer core. (2-3)

Examination of the normal construction of a power transformer reveals that for the frequencies and currents involved, one major simplification can be immediately introduced. The flux lines are confined almost entirely to the high permeability paths of the ferromagnetic material and in fact, this three-dimensional field problem can be simplified to a one-dimensional magnetic circuit problem.

Equations (2-1) and (2-2) are now open to easier interpretation and with the use of a given B(H) relationship of the material,



HYSTERESIS LOOPS

FIG. 2-1

analysis of the field problem is theoretically possible. However, at this point, the existence of certain other effects has to be kept in mind because these are of some influence if the electric circuit representation is to be used.

These effects are:-

(1) The nature of the exciting current.

(2) The hysteresis and eddy-current losses in the core.

(3) The conductors' losses.

(4) Magnetic leakage.

(5) The Capacitive effect of the windings at high frequencies.

Effects (1) and (2) are closely related and tied to the condition given by eq. (2-3).

Obtaining the B(H) relationship for a transformer core is an interesting measurement problem[18] [19]. A set of curves as shown in Fig (2-1) is obtained when a sample of core material is excited through complete cycles in continuous succession. These curves reveal that

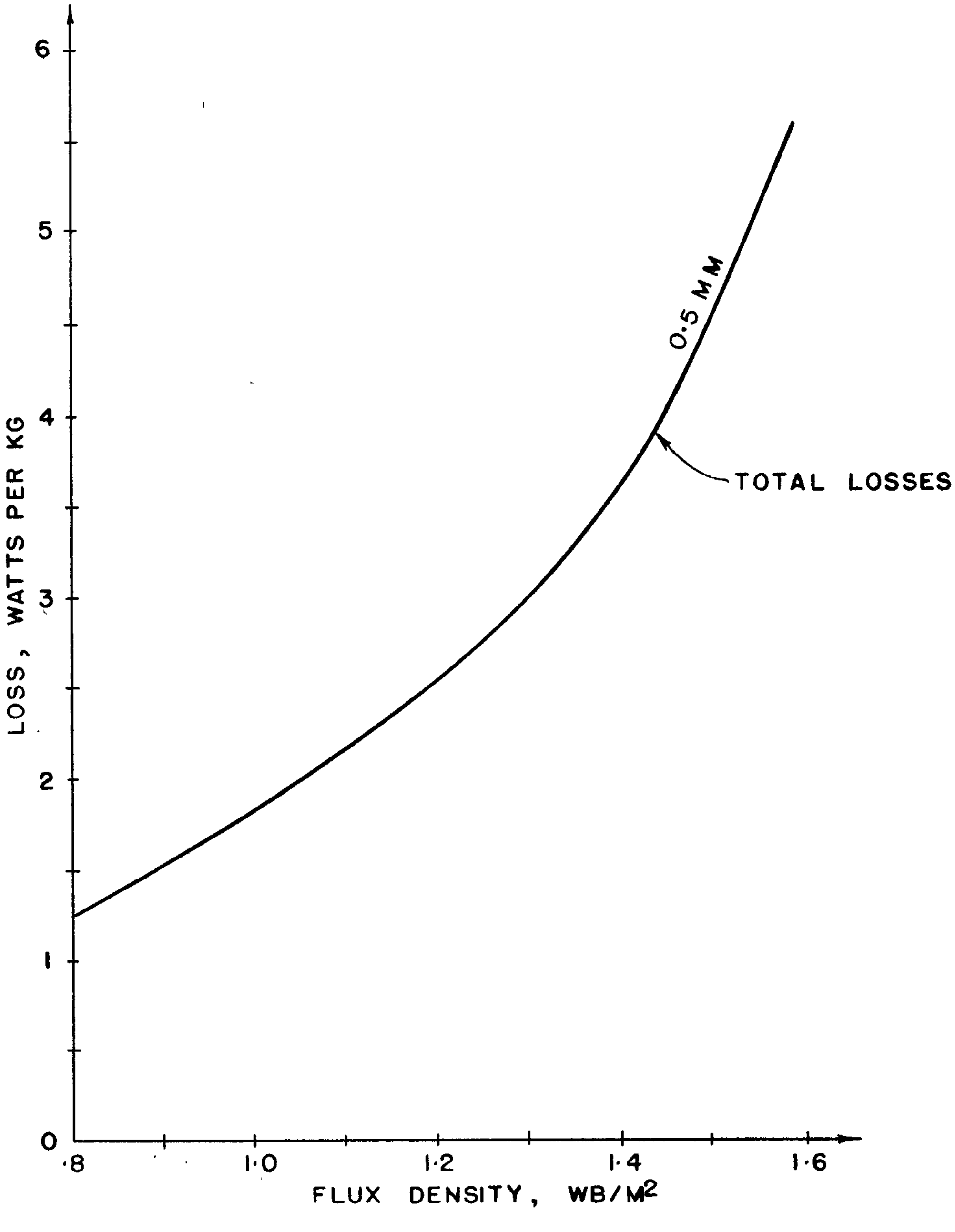
(1) $\oint \bar{H} \cdot \bar{dB} \neq 0.$

(2) The B(H) relationship is not single-valued.

(3) After the first few cycles, a steady-state condition is reached when the B(H) contour becomes a closed loop--the hysteresis loop.

(4) The size and shape of the steady-state B(H) locus varies with the peak value of H.

The integral $\int \bar{H} \cdot \bar{dB} = \frac{\partial W}{\partial V}$ is interpreted



TRANSFORMER CORE LOSSES

FIG. 2-2

as an energy loss per unit volume of ferromagnetic material per cycle. This loss, called the hysteresis loss is the result of the material's property of retaining magnetism or opposing a change in magnetic state.

The eddy-current losses are produced by currents in the magnetic material, and these currents result from the varying flux density $\bar{B}$ under a.c. excitation.

Fig. (2-2) shows these total losses for a particular transformer at a definite frequency and impressed voltage of excitation. Formulae[20] have been derived, giving expressions for these losses in terms of the circuit parameters but these have been mostly empirical and in any case are difficult to simulate using circuit elements.

These physical phenomena present a major problem in circuit representation[21] and mathematical analysis and three different sets of assumptions are made if any computation or analysis is to be carried out.

(1)

(a) The B(H) relationship is at most, a two-valued function.

(b) The eddy-current losses at no load for a particular transformer are simulated by a resistive element.

(2)

(a) The B(H) relationship is single-valued.
This single-valued curve is obtained by drawing a curve through the tips of a series of increasingly larger symmetrical hysteresis loops. This curve is called the magnetisation characteristic and is further assumed to be symmetrical about the origin.

(b) The total core losses are represented by a lumped resistor.

(3)

(a) The magnetisation characteristic is used.

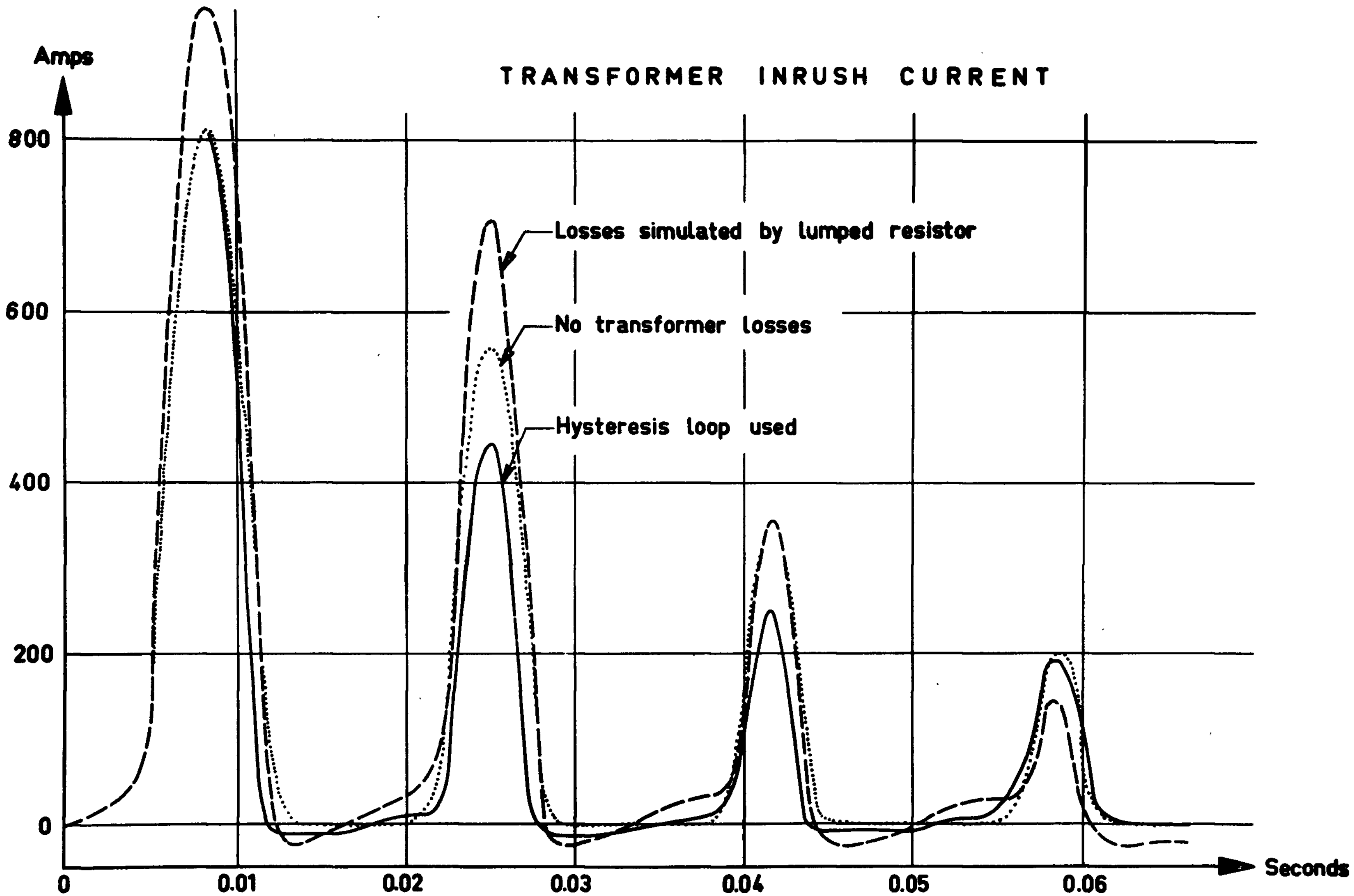
(b) All core losses are neglected.

Assumption (1) can rarely be used as it is difficult to formulate an analytical expression for the two-valued relationship and in any case such an expression would be awkward to use. In Chapter 3, a method utilising the digital computer is described through which an actual hysteresis loop can be followed in the computation. Such an approach is based on assumption (1).

In most steady-state considerations, the circuit representation arising from assumption (2) is used. Such a circuit consists of a single inductive element in parallel (or in series) with a resistive element. The inductive element simulates the magnetic phenomena and the resistor accounts for all core losses. The transient response of the series circuit does not approximate the transient behaviour of the transformer and is of no value in transient studies. The parallel combination indicates two separate currents where only one exists but is a more reasonable approximation.

When non-linear analytical techniques are employed, assumption (3) provides the simplest possible representation--a single non-linear element.

Using the incremental method developed in Chapter 3, three separate computations of the inrush-current to a transformer are done. The graphs of Fig. (2-3) show the results. Curves (a), (b),



TRANSFORMER INRUSH CURRENT
Amps
800
600
400
200
0
Losses simulated by lumped resistor
No transformer losses
Hysteresis loop used
0
0.01
0.02
0.03
0.04
0.05
0.06
Seconds

FIG. 2-3

(c) are obtained using assumption (1), (2) and (3) respectively. Current waveforms are plotted as these give a better indication of the differences and are also the quantities of interest.

The losses of the system chosen are principally the transformer core losses as this allows an appreciation of the differences between the three assumptions.

With all transformer losses ignored, curve (c) Fig. (2-3) is obtained. Although having a slightly lower maximum peak value than curve (b), its decay is much less rapid since curve (b) results from a circuit in which the losses are simulated by a constant resistive element.

The actual tracing of the hysteresis loop which can be done using the incremental method (Chapter 3)--results in curve (a). The fact that decrease in current magnitude results in a reduction of the size of the hysteresis loop and hence the hysteresis losses, is demonstrated by the initial decay of curve (a) being more rapid than that of curve (c) and the tendency for these two curves to coincide at smaller peak current values.

The use of a linear resistor to simulate the core losses results in the continuing rapid decay of curve (b). The fact that the amplitude of the voltage across the transformer does not change appreciably keeps the R. M. S. losses simulated by the resistor almost constant. These losses, in fact, vary with the peak value of the current and the accuracy of such a representation for transient studies will depend on the value of the resistive element chosen.

The losses of the conductors of the transformer can be simulated by a lumped resistance in which the total current flows and the magnetic leakage can be represented by a linear inductance.[22]

The capacitive effects of the windings are of no significance in the frequency range experienced in the phenomena being investigated and will be neglected in these considerations.

Finally, then, the equivalent representation of the transformer as a circuit element consists of an ideal energy transfer device, a non-linear element and a set of linear resistive and inductive elements. The choice of combination of these is based on the type of behaviour being studied and the extent of simplification that can be tolerated.

2-3 The Transmission Line.

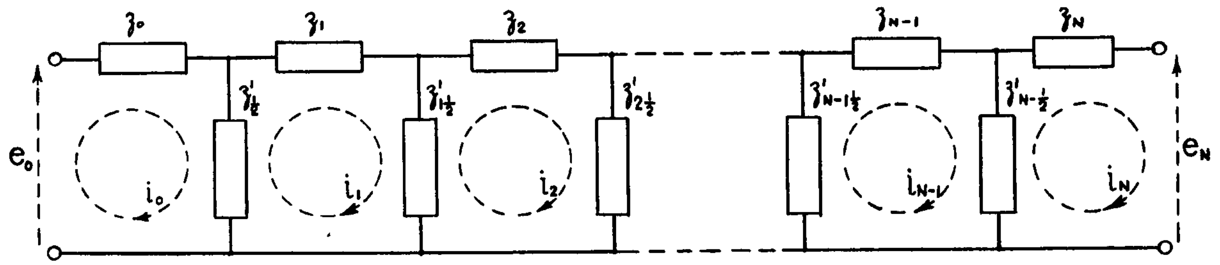
In any problem that involves the parameters of a transmission line, two approaches are possible. The familiar T or π-section representation using lumped elements can be used as an approximate circuit representation, or the solution of the actual partial differential equations describing the transmission line behaviour, with prescribed boundary and initial conditions, can be sought.

Both these methods will be described and the results of numerical examples are presented in graphical form to obtain appreciation of the magnitude of the differences involved.

In particular, a solution of the partial differential equations is fully developed. The method used here can be quite useful in other power system applications, especially under transient conditions. As will be realized, its success depends on many numerical calculations and the use of a digital computer is almost essential.

2-3-1 The Transmission Line--Lumped Parameters.

Consider the network shown in Fig. (2-4), in which the elements z_k and $z'_{k+\frac{1}{2}}$ denote arbitrary impedances.



z_0
z_1
z_2
z_{N-1}
z_N
$z'_{\frac{1}{2}}$
$z'_{1\frac{1}{2}}$
$z'_{2\frac{1}{2}}$
$z'_{N-1\frac{1}{2}}$
$z'_{N-\frac{1}{2}}$
e_0
e_N
i_0
i_1
i_2
i_{N-1}
i_N

FIG. 2-4

Applying Kirchhoff's laws and with the use of the La-place Transformation, mesh equations are established.

Let

$$I_k(s) \triangleq \mathcal{L}\left[i_k\right] \qquad (2\text{-}4)$$
$$E_k(s) \triangleq \mathcal{L}\left[e_k\right]$$

Then considering any interior mesh,

$$Z_k\ I_k + Z'_{k+\frac{1}{2}}\ (I_k - I_{k+1}) + Z'_{k-\frac{1}{2}}\ (I_k - I_{k-1}) = 0$$
$$(k = 1,\ 2,\ \ldots\ldots.N-1). \qquad (2\text{-}5)$$

For the zeroth mesh i.e. $k = 0$.

$$(Z_0 + Z'_{\frac{1}{2}})\ I_0 - Z'_{\frac{1}{2}}\ I_1 = E_0 \qquad (2\text{-}6)$$

and for the Nth mesh.

$$-Z'_{N-\frac{1}{2}}\ I_{N-1} + (Z'_{N-\frac{1}{2}} + Z_N)\ I_N = -\ E_N \qquad (2\text{-}7)$$

These N + 1 equations can be written more systematically as

$$\begin{aligned}
&(Z_0 + Z'_{\frac{1}{2}})\ I_0 - Z'_{\frac{1}{2}}\ I_1 && = E_0\\
&-Z'_{\frac{1}{2}}\ I_0\ (Z'_{\frac{1}{2}} + Z'_{1\frac{1}{2}} + Z_1)\ I_1 - Z'_{1\frac{1}{2}}\ I_2 && = 0\\
&-Z'_{1\frac{1}{2}}\ I_1 \quad (Z'_{1\frac{1}{2}} + Z'_{2\frac{1}{2}} + Z_2)\ I_2 - Z'_{2\frac{1}{2}}\ I_3 && = 0\\
&\qquad\vdots\\
&-Z'_{N-1\frac{1}{2}}\ I_{N-2}\ (Z'_{N-1\frac{1}{2}} + Z'_{N-\frac{1}{2}} + Z_{N-1})\ I_{N-1} - Z'_{N-\frac{1}{2}}\ I_N && = 0\\
&-Z'_{N-\frac{1}{2}}\ I_{N-1} + (Z'_{N-\frac{1}{2}} + Z_N)\ I_N && = -\ E_N
\end{aligned}$$

These are a set of difference equations and if

$$Z_1 = Z_1 = Z_3 = \ldots\ldots.\ Z_{N-1} = Z \qquad (2\text{-}9)$$

and $Z'_{\frac{1}{2}} = Z'_{1\frac{1}{2}} = Z'_{2\frac{1}{2}} \ldots = Z'_{N-\frac{1}{2}} = Z'$

with $\quad Z_0 = Z_N = \frac{1}{2}\ Z \qquad (2\text{-}10)$

the solution for I_k is given by [23]

$$I_k(s) = \frac{E_0 \cosh\left[2(N-k)\ \alpha\right] - E_N \cosh(2k\alpha)}{(ZZ' + \tfrac{1}{4}Z^2)^{\frac{1}{2}}\ \sinh(2N\alpha)} \tag{2-11}$$

where $\sinh(\alpha) = \left(\dfrac{Z}{4Z'}\right)^{\frac{1}{2}}$ (2-12)

In the case of a transmission line, E_0 is usually some specified voltage and

$$E_N = Z_R\ I_N \tag{2-13}$$

where

(2-14)

Z_R = an impedance function.

Substituting this relation in eq. (2-11) and putting k = N

$$I_N(s) = \frac{E_0 \cosh(0) - Z_R\ I_N \cosh(2N\alpha)}{(ZZ' + \tfrac{1}{4}Z^2)^{\frac{1}{2}}\ \sinh(2N\alpha)} \tag{2-15}$$

Further

$$I_N(s) = \frac{E_0}{(ZZ' + \tfrac{1}{4}Z^2)^{\frac{1}{2}}\ \sinh(2N\alpha) + Z_R \cosh(2N\alpha)} \tag{2-16}$$

Hence, if a transmission line is to be represented by T or π-sections, N in number, eq. (2-16) can be used to determine the current response at the end of the line when some voltage $e_0(t)$ is applied.

Defining

$$\Delta_N \triangleq (ZZ' + \tfrac{1}{4}Z^2)^{\frac{1}{2}}\ \sinh(2N\alpha) + Z_R \cosh(2N\alpha) \tag{2-17}$$

for N number of T-sections,

$Z = R + sL$

$Z' = \dfrac{1}{sC}$

where

$$R = \frac{\text{total lumped resistance of line in ohms}}{N}$$

$$L = \frac{\text{total lumped inductance of line in henries}}{N} \qquad (2\text{-}18)$$

$$C = \frac{\text{total lumped shunt-capacitance of line in farads}}{N}$$

Then

$$\Delta_N = \left(\left(\frac{R + sL}{sC}\right) + \left(\frac{R + sL}{4}\right)^2\right)^{\frac{1}{2}} \sinh (2N\alpha) + Z_R \cosh (2N\alpha)$$

and

$$\sinh \alpha = \left(\frac{sC\ (R + sL)}{4}\right)^{\frac{1}{2}}$$

$$\cosh \alpha = \left(1 + \frac{sC\ (R + sL)}{4}\right)^{\frac{1}{2}} \qquad (2\text{-}19)$$

To find the current response for any number of sections when a step-function voltage is applied at one end of the line, the inverse Laplace transform of $\frac{E_0}{\Delta_N}$ has to be found.

i.e.
$$i_N(t) = \mathcal{L}^{-1}\left[\frac{v_0}{s\ \Delta_N}\right]$$

where v_0 = amplitude of the step-voltage.

In this case, the problem amounts to finding $\mathcal{L}^{-1}\left[\frac{1}{s\ \Delta_N}\right]$.

This requires putting the expression $\frac{1}{s\ \Delta_N}$ into partial fractions and then finding the inverse transforms of the simpler fractions. For a terminating inductance, the polynomial $s\ \Delta_N$ has 2 real zeros one of which is $s = 0$, and N conjugate pairs of complex zeros. Expressing $\frac{1}{s\ \Delta_N}$ as a sum of partial fractions allows the evaluation of $i_N(t)$ as a function of time.

$$\frac{1}{s\ \Delta_N} = \frac{A_0}{s} + \frac{A_1}{s + \alpha_1} + \sum_{2}^{N+1} \frac{B_k s + A_k}{(s + \alpha_k + j\beta_k)(s + \alpha_k - j\beta_k)} \qquad (2\text{-}20)$$

$$\therefore \quad i_N(t) = v_0 \left(A_0 + A_1 e^{-\alpha_1 t} + \sum_{k=1}^{N} D_k e^{-\alpha_k} \cos(\beta_k + \phi_k) \right) \tag{2-21}$$

where

$$D_k \triangleq \left(B_k^2 + \frac{(B_k \alpha_k - A_k)^2}{\beta_k^2} \right)^{\frac{1}{2}}$$

and (2-22)

$$\phi_k \triangleq \tan^{-1} \frac{(B_k \alpha_k - A_k)}{B_k \beta_k} \; .$$

The solution of the polynomial s Δ_n of order $2(n + 1)$, the conversion to partial fractions and the final evaluation of $i_N(t)$ become an almost impossible task without the aid of machine computation. Programmes to perform these numerical tasks were written and a wide range of behaviour could be studied.

2-4 T-sections terminated by a linear inductance.

For 1, 2, and 4 T-sections, the current and voltage responses at the end of various lengths of lines are computed. Each length of line was terminated by various values of linear inductances, varying from 0.1 henry to 9000 henries and the more interesting of these results are presented graphically in Figs. (2-6) to (2-22). In each case, a 1000 volt step-input was applied at the beginning of the line.

The algebraic expressions for Δ_N in terms of R, L, C and L_R are given by equations (2-24), (2-25), (2-26) and (2-27) for 1, 2, 3 and 4 T-sections respectively. R, L, C are as defined previously by equations (2-18) and L_R is the value of the terminating inductance in henries.

The results of the current responses as given by equation (2-21) and the voltage responses as given by

$$v_R = L_R \frac{di_N}{dt} \tag{2-23}$$

are discussed in greater detail in section 2-7 .

One T-section

$$\Delta_1 = s^3 \, CL \, (0.25L + 0.5L_R) + s^2 \, CR \, (0.5L + 0.5L_R) + s \, (L + L_R + 0.25CR^2) + R \tag{2-24}$$

Two T-sections

$$\begin{aligned} \Delta_2 = {} & s^5 \, L^2C^2 \, (0.25L + 0.5L_R) + s^4 \, LC^2R \, (0.75L + L_R) \\ & + s^3\left(C^2R^2 \, (0.75L + 0.5L_R) + LC(1.5L + 2L_R)\right) \\ & + s^2\left(CR(3L + 2L_R) + 0.25C^2R^3\right) \\ & + s \left((2L + L_R) + 1.5 \, CR^2\right) + 2R. \end{aligned} \tag{2-25}$$

Three T-sections

$$\begin{aligned} \Delta_3 = {} & s^7 \, L^3 \, C^3 \, (0.25L + 0.5L_R) + s^6 \, L^2 \, C^3 \, R \, (L + 1.5L_R) \\ & + s^5\left(L^2 \, C^2 \, (2L + 3L_R) + LR^2 \, C^3 \, (1.5L_R + 1.5L)\right) + \\ & + s^4\left(C^3 \, R^3 \, (L + 0.5L_R) + C^2 \, RL \, (6L + 6L_R)\right) \\ & + s^3\left(C^2 \, R^2 \, (6L + 3L_R) + LC \, (4.75L + 4.5_R) + 0.25C^3 \, R^4\right) \\ & + s^2\left(CR \, (9.5L + 4.5L_R) + 2R^3 \, C^2 \right) \\ & + s \left(4.75 \, C \, R^2 + 3L + L_R\right) + 3R \end{aligned} \tag{2-26}$$

Four T-sections

$$\begin{aligned} \Delta_4 = {} & s^9 \, L^4 \, C^4 \, (0.25L + 0.5L_R) + s^8 \, R \, L^3 \, C^4 \, (1.25L + 2L_R) \\ & + s^7\left(R^2 \, L^2 \, C^4 \, (2.5L + 3L_R) + L^3 \, C^3 \, (2.5L + 4L_R)\right) \\ & + s^6\left(C^4 \, R^3 \, L \, (2.5L + 2L_R) + L^2 \, C^3 \, R \, (10L + 12L_R)\right) \\ & + s^5\left(C^4 \, R^4 \, (1.25L + 0.5L_R) + C^3 \, R^2 \, L \, (15L + 12L_R) \right. \\ & \left. \qquad + L^2 \, C^2 \, (8.5L + 10L_R)\right) \end{aligned} \tag{2-27}$$

$$+ s^4 \left(C^3 R^3 (10L + 4L_R) + L C^2 R (25.5L + 20L_R) + \frac{C^4 R^5}{4} \right)$$
$$+ s^3 \left(C^2 R^2 (25.5L + 10L_R) + LC (8L_R + 11L) + 2.5C^3 R^4 \right)$$
$$+ s^2 \left(CR (22L + 8L_R) + 8.5C^2 R^3 \right) \tag{2-27}$$
$$+ s \left(L_R + 4L + 11C R^2 \right) + 4R$$

2-5 The transmission Line--Distributed Parameters.

Nomenclature.

r = line resistance, ohms per unit distance

l = line inductance, henries per unit distance

g = line conductance, mhos per unit distance

c = line capacitance, farads per unit distance

d = length of line

x = distance along the line measured from the sending-end

t = time in sec.

s = complex frequency variable

$Z_0 = Z_0(s) = \sqrt{\dfrac{r + sl}{g + sc}}$

$\gamma = \sqrt{(r + sl)(g + sc)}$

$Z_R = Z_R(s)$ = terminating impedance

$N(s) = \dfrac{Z_0 - Z_R}{Z_0 + Z_R}$

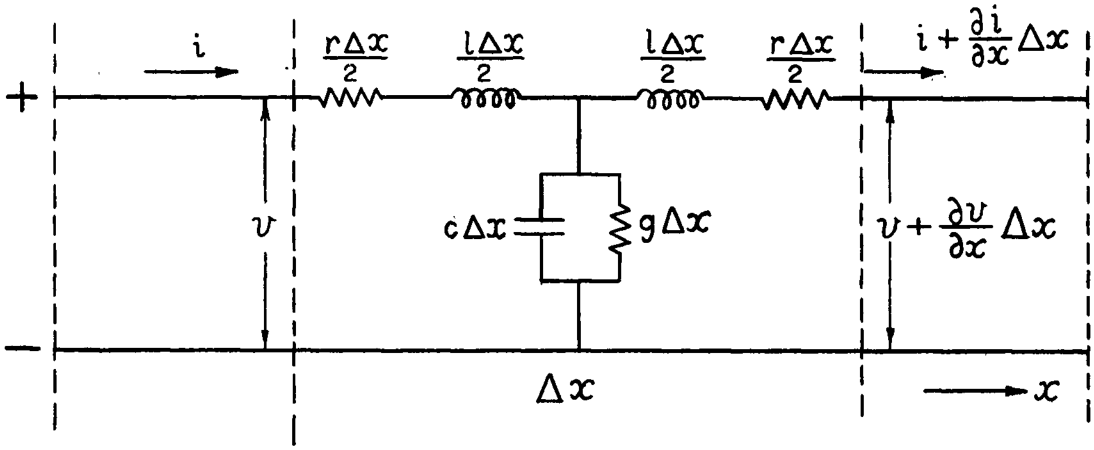
$i_0(x)$ = value of $i(x,t)$ at $t = 0$

$v_0(x)$ = " " $v(x,t)$ at $t = 0$

Consider an element of a transmission line as shown in Fig. (2-5).

Then using notation shown in Fig. (2-5)

$$-\frac{\partial v}{\partial x} \cdot \Delta x = (ri + l \frac{\partial i}{\partial t}) \cdot \Delta x \tag{2-28}$$



i
$\frac{r\Delta x}{2}$
$\frac{l\Delta x}{2}$
$\frac{l\Delta x}{2}$
$\frac{r\Delta x}{2}$
$i + \frac{\partial i}{\partial x}\Delta x$
+
v
$c\Delta x$
$g\Delta x$
$v + \frac{\partial v}{\partial x}\Delta x$
−
Δx
x

FIG. 2-5

$$- \frac{\partial i}{\partial x} \cdot \Delta x = (gv + c \frac{\partial v}{\partial t}) . \Delta x \tag{2-29}$$

Transforming these equations,

$$\frac{\partial}{\partial x} V(x,s) = -(r + sl) I (x,s) + li_0(x) \tag{2-30}$$

$$\frac{\partial}{\partial x} I(x,s) = -(g + sc) V (x,s) + cv_0(x) \tag{2-31}$$

The distributed-parameter solution to the transmission line requires the solution of the equations (2-28) and (2-29) with the appropriate initial and boundary conditions.

Three steps are required for a complete solution.

(1) Determination of the functions $V(x,s)$ and $I(x,s)$ satisfying equations (2-30) and (2-31).

(2) Satisfying initial and boundary conditions.

(3) Determination of the inverse transformations $v(x,t)$ and $i(x,t)$.

The final step usually provides the greatest difficulty and an important feature of this section is the development of a method for accomplishing this final step.

The general solution of equations (2-30), (2-31) is well known[24] and will not be repeated here. Let $V_0(x,s)$ be the transforms representing the effect of the initial voltage and initial current distribution along the line respectively; the complete solution of the equations (2-30), (2-31) yields,

$$V(x,s) = V_0(x,s) + A \cosh(\gamma x) + B \sinh(\gamma x) \tag{2-32}$$

$$I(x,s) = I_0(x,s) - \frac{A}{Z_0} \sinh (\gamma x) - \frac{B}{Z_0} \cosh (\gamma x) \tag{2-33}$$

where A and B are to be determined from the boundary conditions.

At $x = 0$ $\quad V(x,s) = V(0,s) \triangleq V_S \triangleq \mathcal{L}$ [sending-end voltage]

$\qquad\qquad I(x,s) = I(0,s) \triangleq I_S \triangleq \mathcal{L}$ [sending-end current]

$$\therefore \quad V_S = V_0\ (0,s) + A \tag{2-34}$$

$$I_S = I_0\ (0,s) - \frac{B}{Z_0} \tag{2-35}$$

$$\therefore \quad A = V_s - V_0\ (0,s) \tag{2-36}$$

$$B = -Z_0 \left| I_s - I_0\ (0,s) \right| \tag{2-37}$$

$$\therefore \quad V(x,s) = V_0(x,s) - V_0(0,s) \cosh\ (\gamma x) + Z_0 I_0(0,s) \sinh(\gamma x) \tag{2-38}$$

$$+ V_S \cosh\ (\gamma x) - Z_0 I_S \sinh\ (\gamma x).$$

and

$$I(x,s) = I_0(x,s) - I_0(0,s) \cosh(\gamma x) + \frac{V_0}{Z_0}\ (0,s) \sinh(\gamma x) \tag{2-39}$$

$$- \frac{V_S}{Z_0} \sinh(\gamma x) + I_S \cosh(\gamma x).$$

At $x = d$, $V(x,s) = V(d,s) \triangleq V_R \triangleq \mathcal{L}$ [receiving-end voltage]

$I(x,s) = I(d,s) \triangleq I_R \triangleq \mathcal{L}$ [receiving-end current]

Putting this condition in eqs. (2-38) and (2-39), an exact relationship between V_S, I_S, and V_R and I_R is obtained.

This is given by

$$V_R = V_0(d,s) - V_0(0,s) \cosh\ (\gamma d) + Z_0 I_0\ (0,s) \sinh\ (\gamma d) \tag{2-40}$$

$$+ V_S \cosh\ (\gamma d) - Z_0 I_S \sinh\ (\gamma d)$$

$$I_R = I_0(d,s) - I_0(0,s) \cosh\ (\gamma d) + \frac{V_0}{Z_0}\ (0,s) \sinh\ (\gamma d)$$

$$- \frac{V_S}{Z_0} \sinh\ (\gamma d) + I_S \cosh\ (\gamma d)$$

At this point, depending on the range and type of behaviour of interest, these equations can be put into various forms.

For example, if changes in conditions after some disturbance are of interest, the most convenient form is

$$\Delta V_R\ (d,s) = \Delta V_S\ (0,s) \cosh\ (\gamma d) - Z_0 \Delta I_S\ (0,s) \sinh\ (\gamma d) \tag{2-42}$$

$$\Delta I_R\ (d,s) = -\frac{\Delta V_s}{Z_0} \sinh\ (\gamma d) + \Delta I_S \cosh\ (\gamma d) \tag{2-43}$$

where

$$\Delta V_K \triangleq V_K - V_0$$

$$\Delta I_K \triangleq I_K - I_0$$

for K=S and R

On the other hand, if steady-state sinusoidal behaviour is of interest, the correct mathematical relationships are established if the following substitutions are made

$$\begin{aligned} &(1)\quad V_0\ (0,s) = 0; \qquad I_0\ (0,s) = 0 \\ &(2)\quad s = j\omega \end{aligned} \tag{2-44}$$

where ω = frequency of the excitation

Such substitutions result in the equations involving the familiar generalised line circuit constants[25].

In this study, of particular interest is transient behaviour, i.e. behaviour for small values of t after t = 0. There are two reasons for this interest. Firstly, the main purpose of this chapter is to evaluate the T or π-section as a circuit representation of the power transmission line in comparison with the distributed parameter solution. It is recognised that in the steady-state or for large t, the T and π-section representations are accurate certainly as regards terminal conditions but that inaccuracies do occur for small t. Secondly, in the larger field of transients in power-systems, it is felt that the method developed here may have some definite applications.

In this comparison, the initial conditions are not significant and both V_0 and I_0 will be put = 0.

Therefore Equations (2-40) and (2-41) can now be written

$$V\ (x,s) = V_S \cosh (\gamma x) - Z_0\ I_0 \sinh (\gamma x) \tag{2-45}$$

$$I\ (x,s) = \frac{V_S}{Z_0} \sinh (\gamma x) + I_S \cosh (\gamma x) \tag{2-46}$$

If the line of length 'd' is terminated by some finite impedance Z_R,

then at x = d

$$V_R = Z_R\ I_R \tag{2-47}$$

$$\therefore\quad V_R = Z_R\ I_R = V_S \cosh (\gamma d) - Z_0\ I_S \sinh (\gamma d) \tag{2-48}$$

and from (2-46),

$$Z_0\ I_R = - V_S \sinh (\gamma d) + Z_0\ I_S \cosh (\gamma d) \tag{2-49}$$

$$\therefore\quad \frac{Z_R}{Z_0} = \frac{V_S \cosh (\gamma d) - Z_0\ I_S \sinh (\gamma d)}{-V_S \sinh (\gamma d) + Z_0\ I_S \cosh (\gamma d)} \tag{2-50}$$

I_S is determined from this equation in terms of V_S and substituted in eqs. (2-45) and (2-46). Sinh (γx) and cosh (γx) are replaced by their exponential equivalents and the following equations result,

$$V(s,x) = V_S(s) \Big(e^{-\gamma x} - N(s)\ e^{-\gamma(2d-x)} + N(s)\ e^{-\gamma(2d+x)} - N(s)^2\ e^{-\gamma(4d-x)} + N(s)^2\ e^{-\gamma(4d+x)} + \ldots\ldots\ldots \Big) \tag{2-51}$$

and

$$I(s,x) = \frac{V_s(s)}{Z_0 + Z_R} \Big(e^{-\gamma x} + N(s)\ e^{-\gamma(2d-x)} + N(s)\ e^{-\gamma(2d+x)} + N(s)^2\ e^{-\gamma(4d-x)} + N(s)^2\ e^{-\gamma(4d+x)} + \ldots\ldots\ldots\ldots \Big)$$

where $$N(s) \triangleq \frac{Z_0 - Z_R}{Z_0 + Z_R} \tag{2-52}$$

$\therefore$ for $x = d$

$$V(s,d)=V_R=V_S(s)\Big(e^{-\gamma d}-N(s)\ e^{-\gamma d}+ N(s)\ e^{-\gamma 3d} -N^2(s)\ e^{-\gamma 3d} +N^2(s)e^{-\gamma 5d} -N^3(s)\ e^{-\gamma 5d}+ \dots \Big) \tag{2-53}$$

$$I(s,d)=I_R = \frac{V_s(s)}{Z_0+Z_R}\Big(e^{-\gamma d} +N(s)\ e^{-\gamma d} +N(s)\ e^{-\gamma 3d}+N^2(s)\ e^{-\gamma 3d} +N^2(s)\ e^{-\gamma 5d} +N^3(s)\ e^{-\gamma 5d} + \dots \Big) \tag{2-54}$$

This form of representing the expressions for $V(s,x)$ and $I(s,x)$ is considered more convenient than the hyperbolic functions as the functions $N(s)$ and $\gamma(s)$ can then be thought of as reflection and loss coefficients respectively. This idea correlates the above approach with the travelling wave concept normally used in this type of analysis.

Since all the distributed circuit parameters are considered in arriving at equations (2-51) and (2-52), exact values for $v(x,t)$ and $i(x,t)$ can be obtained for all t if the inverse transforms of the expressions for $V(s,x)$ and $I(s,x)$ can be determined. Since these expressions are infinite series in s and x, exact determination of the inverse transforms will be difficult for a general case,but for small t an approximate evaluation of $i(x,t)$ and $v(x,t)$ is possible.

Now,

$$\gamma = \Big((r+sl)(g+sc) \Big)^{\frac{1}{2}} = (lc)^{\frac{1}{2}}\Big(s^2 + x(\frac{r}{l} + \frac{g}{c}) + \frac{rg}{cl} \Big)^{\frac{1}{2}}$$
$$=\beta\Big((s + \mu)^2 - \alpha^2 \Big)^{\frac{1}{2}} \tag{2-55}$$

and

$$Z_0 = \left(\frac{r + sl}{g + sc}\right)^{\frac{1}{2}} = \left(\frac{l}{c}\right) \frac{(s+\mu)^2 - \alpha^2)^{\frac{1}{2}}}{(s+\mu) - \alpha} = R_0 \frac{(s+\mu)^2 - \alpha^2)^{\frac{1}{2}}}{(s+\mu - \alpha)} \tag{2-56}$$

where

$$\begin{aligned} \beta &\triangleq (lc)^{\frac{1}{2}} \\ R_0 &\triangleq \left(\frac{l}{c}\right)^{\frac{1}{2}} \\ \mu &\triangleq \frac{1}{2}\left(\frac{r}{l} + \frac{g}{c}\right) \\ \alpha &\triangleq \frac{1}{2}\left(\frac{r}{l} - \frac{g}{c}\right) \end{aligned} \tag{2-57}$$

also

$$N(s) = \frac{Z_0 - Z_R}{Z_0 + Z_R} \tag{2-58}$$

$$= \frac{\left((s+\mu)^2 - \alpha^2\right)^{\frac{1}{2}} - \frac{Z_R}{R_0}(s + \mu - \alpha)}{\left((s+\mu)^2 - \alpha^2\right)^{\frac{1}{2}} + \frac{Z_R}{R_0}(s + \mu - \alpha)} \tag{2-59}$$

Since in all these expressions s occurs coupled with μ, use of the s-translation theorem(26) simplifies the computation.

Then

$$i(d,t) = e^{-\mu t} \left(\mathcal{L}^{-1} \left[\frac{(s-\alpha)\ V_s(s-\mu)}{R_0\ (s^2-\alpha^2)^{\frac{1}{2}}} \times e^{-\beta d(s^2-\alpha^2)^{\frac{1}{2}}} + N(s-\mu)\ e^{-\beta d(s^2-\alpha^2)^{\frac{1}{2}}} + N(s-\mu)\ e^{-2\beta d(s^2-\alpha^2)^{\frac{1}{2}}} + \ldots\ldots \right] \right) \tag{2-60}$$

$\therefore$ in fact, evaluation of $i(d,t)$ amounts to the evaluation of terms of the form

$$e^{-\mu t} \times \left(\mathcal{L}^{-1} \left[\left(\frac{(s-\alpha)\ V_S\ (s-\mu)}{R_0\ (s^2 - \alpha^2)^{\frac{1}{2}}} \right) \times \left(N(s-\mu)\right)^n \left| e^{-t_j} (s^2-\alpha^2)^{\frac{1}{2}} \right] \right) \tag{2-61}$$

where

$$N(s-\mu) = \frac{\left|s^2-\alpha^2\right|^{\frac{1}{2}} \quad \frac{Z_R}{R_0}\,(s-\mu)}{\left|s^2-\alpha^2\right|^{\frac{1}{2}} + \frac{Z_R}{R_0}\,(s-\mu)}$$

and t_j = some constant for a particular term.

In this expression for $N(s-\mu)$, the substitution

$$z = \frac{s - (s^2-\alpha^2)^{\frac{1}{2}}}{\alpha} \quad \text{or } s = \frac{-\alpha}{2}\left(z + \frac{1}{z}\right) \tag{2-62}$$

is made.

Then $N(s-\mu) = \frac{M(z)}{D(z)}$

$$= 1 + m_1 z + m_2 z^2 + m_3 z^3 + \ldots\ldots . m_n z^n + \ldots .$$

Therefore

$$N(s-\mu)^n = (1+m_1 z + m_2 z^2 + m_3 z^3 + \ldots\ldots\ldots\ldots . m_n z^n + \ldots .)^n \tag{2-63}$$

$$= \sum_{k=0}^{\infty} c_{jk}\, z^k = \sum_{k=0}^{\infty} c_{jk} \left(\frac{s-(s^2-\alpha^2)^{\frac{1}{2}}}{\alpha}\right)^k \tag{2-64}$$

where c_{jk} is determined from the coefficients m_k.

and

$$i(d,t) = e^{-\mu t} \times \sum_{j=0}^{\infty} \mathcal{L}^{-1}\left[\left(\frac{(s-\alpha)\ V_S\ (s-\mu)}{R_0}\right) \times \right.$$

$$\left. \left(\frac{1}{s^2-\alpha^2}\right)\left(\sum_{k=0}^{\infty} c_{jk}\ \frac{s-(s^2-\alpha^2)^{\frac{1}{2}k}}{\alpha}\ e^{-t_j}\ (s^2-\alpha^2)^{\frac{1}{2}}\right)\right] \tag{2-65}$$

The purpose behind this manipulation is to obtain functions of s for which inverse transforms are available.

The transform pair

$$\mathcal{L}^{-1}\left[\left(\frac{\alpha^k}{\left|s^2-\alpha^2\right|^{\frac{1}{2}}}\right)\left(s + (s^2-\alpha^2)^{\frac{1}{2}}\right)^{-k}\left(e^{-t_j}\ (s^2-\alpha^2)^{\frac{1}{2}}\right)\right]$$

$$= \left(\frac{t-t_j}{t-t_j}\right)^{k/2} \quad I_k\left(\alpha(t^2-t_j^2)^{\frac{1}{2}}\right) \tag{2-66}$$

$$t > t_j$$

is given in Bateman's[27] tables.

Multiplying the term $\left|s + (s^2-\alpha^2)\right|^{\frac{1}{2}}$ by $\dfrac{s-(s^2-\alpha^2)^{\frac{1}{2}}}{s-(s^2-\alpha^2)^{\frac{1}{2}}}$

gives the following pair

$$\mathcal{L}^{-1}\left[\left(\frac{s-(s^2-\alpha^2)^{\frac{1}{2}}}{\alpha}\right)^k \left(\frac{e^{-t_j(s^2-\alpha^2)^{\frac{1}{2}}}}{(s^2-\alpha^2)^{\frac{1}{2}}}\right)\right] = \left(\frac{t-t_j}{t+t_j}\right)^{k/2} I_k\left|\alpha(t^2-t_j^2)^{\frac{1}{2}}\right| \tag{2-67}$$

where I_k is the modified Bessel function of the first kind of order k, given by

$$I_k(x) = \sum_{p=0}^{\infty} \frac{(\frac{x}{2})^{(2p+k)}}{p\,!\,(k+p)!} \tag{2-68}$$

$\therefore$ all the terms of the form $\dfrac{N(s-\mu)^n \, e^{-t_j(s^2-\alpha^2)^{\frac{1}{2}}}}{(s^2-\alpha^2)^{\frac{1}{2}}}$

can be evaluated in time. To evaluate the complete expression for i(d,t) exactly, an infinite number of terms will have to be evaluated. In practice, for small $(t-t_j)$, the series can be terminated at a finite value of k

as

$$\left(\frac{t-t_j}{t+t_j}\right)^{k/2} I_k\left|\alpha\ (t^2-t_j^2)^{\frac{1}{2}}\right| \longrightarrow 0 \tag{2-69}$$

as k becomes large.

The value of k at which the series is terminated depends on the accuracy required and the maximum value of $(t-t_j)$ for which results are to be computed. The value of t_j which has the dimension of time, is some multiple of βd.

The evaluation of the coefficients c_{jk}, the Bessel functions I_k and the final computed value of i(d,t) require the use of a digital computer for speed and accuracy.

2-6 A Finite Line Terminated By a Linear Inductor.

Using the method developed in sec. 2-5, the current responses as given by eq. (2-65), are found for various lengths of lines, terminated by a linear inductance with a 1000 volt step-input applied.

Algebraically, the above conditions imply

$$V_S(s) = \frac{1000}{s}$$

$$\therefore \quad V_S(s-\mu) = \frac{1000}{s-\mu} \tag{2-70}$$

$$Z_R(s-\mu) = (s-\mu)\cdot L_R$$

$\therefore$ from eq. (2-58)

$$N(s-\mu) = \frac{R_0 \left(s^2-\alpha^2\right)^{\frac{1}{2}} - (s-\mu)(s-\alpha)\ L_R}{R_0 \left(s^2-\alpha^2\right)^{\frac{1}{2}} + (s-\mu)(s-\alpha)\ L_R} \tag{2-71}$$

The substitution

$$s = \frac{-\alpha}{2}\left(z + \frac{1}{z}\right)$$

gives

$$N(s-\mu) = \frac{z^4 + 2z^3\left(1+\frac{\mu}{\alpha} - \frac{R_0}{\alpha L_R}\right) + z^2\left(2+\frac{4\mu}{\alpha}\right) + 2z\left(\frac{R_0}{\alpha L_R} + 1 + \frac{\mu}{\alpha}\right) + 1}{-z^4 - 2z^3\left(1+\frac{\mu}{\alpha} + \frac{R_0}{\alpha L_R}\right) - z^2\left(2+\frac{4\mu}{\alpha}\right) - 2z\left(\frac{-R_0}{\alpha L_R} + 1 + \frac{\mu}{\alpha}\right) - 1} \tag{2-72}$$

$$= \sum_0^\infty c_{jk}\ z^k.$$

It was found by numerical computation that for the time range of interest, only terms up to the order z^6 need be considered and for high values of L_R, terms up to z^4 proved sufficient.

Equation (2-65) now becomes

$$i(d,t) = e^{-\mu t} \sum_{j=0} \mathcal{L}^{-1}\left[\left(\frac{(s-\alpha)\ x\ 1000}{R_0\ (s-\mu)}\right) x \left(\frac{1}{s^2-\alpha^2}\right)^{\frac{1}{2}} \sum_{j=0}^{n} c_{jk}\ z^k\ e^{-t_j (s^2-\alpha^2)^{\frac{1}{2}}}\right] \tag{2-73}$$

where n now has some finite value and $z \triangleq \frac{s-(s^2-\alpha^2)^{\frac{1}{2}}}{\alpha}$

If the line conductance g is neglected--a reasonable approximation--then eq. (2-57) becomes

$$\begin{aligned} \beta &= (lc)^{\frac{1}{2}} \\ R_0 &= \left(\frac{l}{c}\right)^{\frac{1}{2}} \\ \alpha &= 0.5\,\frac{r}{l} \\ \mu &= 0.5\,\frac{r}{l} \end{aligned} \tag{2-74}$$

$$\therefore \qquad \alpha = \mu.$$

Hence

$$i(d,t) = e^{-\mu t}\left(\mathcal{L}^{-1}\left[\sum_{j=0}^{\infty} \frac{1000}{R_0(s^2-\alpha^2)^{\frac{1}{2}}} \sum_{k=0}^{n} c_{jk}(z)^k\, e^{-t_j(s^2-\alpha^2)^{\frac{1}{2}}}\right]\right)$$

$$= \frac{e^{-\mu t}1000}{R_0} \sum_{j=0}^{\infty} \sum_{k=0}^{n} c_{jk} \left(\frac{t-t_j}{t+t_j}\right)^{k/2} I_k\left(\alpha(t^2-t_j^2)^{\frac{1}{2}}\right)$$

The coefficients c_{jk} are tabulated in Table 2-1 up to k=6 for j=0 to j=4.

With the tabulated c_{jk} values computed and the use of the series for the modified Bessel function I_k -- eq. (2-68), i(d,t) can now be computed.

The voltage across the inductance, v(d,t) can be computed from the relationship

$$v(d,t) = L_R\,\frac{di(d,t)}{dt}. \tag{2-78}$$

2-7 Numerical Results.

Equations (2-21), (2-23), (2-76), (2-78) can provide four different results for the current and voltage response at the end of a long line terminated by a linear inductance.

k =	0	1	2	3	4	5	6
j=0	1	0	0	0	0	0	0
j=1	-1	c_1	c_2	c_3	c_4	c_5	c_6
j=2	1	$2c_1$	$2c_2 + c_1^2$	$2c_3 + c_1c_2$	$2c_4 + c_1c_3 + c_2^2$	$c_1c_4 + c_2c_3 + c_5$	$2c_1c_5 + c_2c_4 + c_6 + c_3^2$
j=3	-1	$3c_1$	$3c_2 + c_1^2$	$3c_3 + 6c_1c_2 + c_1^3$	$c_4 + 2c_1c_3$ $c_2^2 + c_1^2c_2$	$3c_5 + 6c_1c_4$ $6c_2c_3 + c_1^2c_3$ $+ c_1c_2^2$	$6c_1c_5 + 2c_2c_4 + c_6$ $c_3^2 + c_1^2c_4$ $+ c_2^3 + 2c_1c_2c_3$
j=4	1	$4c_1$	$4c_2 + 6c_1^2$	$4c_3 + 12c_1c_2 + 4c_1^3$	$4c_4 + 12c_1c_3$ $+12c_1^2c_2 + c_1^4$ $+ 6c_2^2$	$4c_5 + 12c_1c_4$ $12c_2c_3 + 12c_1^2c_3$ $12c_1c_2^2 + 4c_2c_1^3$	$12c_1c_5 + 12c_2c_4 + 4c_6$ $12c_1^2c_4 + 24c_1c_2c_3$ $4c_2^3 + 4c_1^3c_3 + 6c_3^2$ $6c_1^2c_2$

Table 2-1. ($c_1, c_2, c_3, c_4, c_5, c_6$, defined in Appendix 1.)

Inspection of the algebraic expressions given by these equations does not reveal the differences between the results and to obtain some appreciation of these differences, computation of numerical examples is necessary. The computation is involved but the speed and accuracy of the digital computer permits the calculation of various examples and thus a very wide range of line lengths and inductance values can be covered.

The design data of a typical high-voltage power transmission line was chosen and the various parameters calculated.
The line constants are

$$\begin{aligned} r &= 0.0376 \text{ ohms per mile} \\ l &= 1.52 \times 10^{-3} \text{ henries per mile} \\ c &= 1.43 \times 10^{-9} \text{ farads per mile} \\ g &= 0 \end{aligned} \tag{2-79}$$

For lengths of line, from 40 miles to 600 miles, the current and voltage responses are calculated when a 1000 volt step-input is applied at one end of the line and the other end is terminated by an inductance.

The more interesting of these results are presented graphically in Figs. (2-6) to (2-22). These are the results for line lengths of 40, 80, 160 and 320 miles terminated in inductances of values 0.1, 10, and 100 henries. It is felt that this range of line lengths and inductance values covers the behaviour experienced in this particular study.

<u>2-8 The Current Response.</u> Figs. (2-6) to (2-17).

These results show that both in magnitude and wave-shape, there are definite differences between the current values obtained using the long-line equations and those computed from the 1, 2, or

4 T-section representations. The differences are more pronounced at the lower values of inductances and for longer line lengths.

2-9 The Voltage Response.

These results obtained for the voltage response reveal the significant differences between a lumped circuit and the distributed parameter approach. Using the expressions developed from the long line equations, the familiar travelling wave phenomenon is demonstrated. The time delay and then doubling of the voltage wave is seen in these graphs.

On the other hand, the 1, 2 and 4 T-section representations give an oscillating voltage wave shape and in no case approach the behaviour pattern obtained from the distributed parameter method over the period of time considered.

Only four sets of results are presented in this case and these represent the extreme range of behaviour. Fig. (2-18) gives the voltage response for a 40 mile length of line terminated by an inductance of 0.1 henry, while Fig. (2-21) is from results for a 320 mile line and 100 henry terminating inductance.

Since normally, the main quantity of interest in voltage behaviour is the peak value, Fig. (2-22) provides a better indication of the errors involved if a lumped circuit representation is used. These graphs of maximum value of the voltages show that large, intolerable errors occur at the lower values of inductances, especially for the longer lengths of line. The reason for this is simple. In the final section of a lumped circuit representation, if the current flowing is i_N then the voltage across the load inductance

$$v_R = L_R \frac{di_N}{dt} \tag{2-80}$$

and across the inductance of the section itself is

$$v_N = L_N \frac{di_N}{dt} \tag{2-81}$$

Therefore, if L_R is of comparable value with L_N, although the current response i_N may be correct, the voltage response will be incorrect. Accurate representation requires an infinite number of sections to ensure that $L_R \gg L_N$, and hence

$$V_R \gg V_N.$$

Therefore, for small L_R and a small number of T-sections, (i.e. relatively large L_N), large errors in voltage values will occur.

2-10 General Conclusions.

Examination of these numerical results provides some general conclusions.

Firstly, the differences in the current response are not intolerable and certainly for the larger values of terminating inductances, either the 1, 2, or 4 T-sections are good approximations to the transmission line so far as current behaviour is concerned. Secondly, the simple one T-section seems to give just as good results as the 2 or 4 T-sections and because of the easier analysis involved, will be a suitable choice for many analytical methods.

The errors in the wave-shapes of the voltage responses are pronounced, when the T-section representation is used. However, correct peak values are obtained, provided L_R is large and these limitations must be borne in mind in the interpretation of results obtained using a finite number of T-sections.

2-11 Equivalent Circuit.

In light of these considerations, the components of the power system are represented as follows

(1) The source is represented as a sinusoidal voltage of constant amplitude and frequency in series with a linear inductive element.

(2) Whenever possible, a double-valued hysteresis curve is used to account for the non-linear flux characteristic and the hysteresis losses of the transformer. The single-valued magnetisation characteristic is used in the analytical methods and hysteresis losses are neglected.

(3) A single T-section is used as an equivalent for the transmission line although the same method of solution--the incremental method--is applicable to a 2 or 4 T-section representation. The analytical methods as used in Chapter 4 become too involved if any but a single T-section is used. The effect of neglecting the lumped shunt-capacitive element in dealing with shorter lengths of lines is discussed in Chapter 3 (sec 3-8-6).

(4) Corona losses which are becoming increasingly significant in high-voltage systems are not taken into account in this study. Suggestions are made how this non-linear effect can be simulated when a technique such as the incremental method is used.

(5) A single-phase rather than the more complex three-phase equivalent circuit is used. Extension of these investigations to three-phase systems is beyond the scope of this thesis. However, a single phase analysis is of quite general nature as any general three-phase linear electro-

magnetic system can always be transformed into four eigenvectors[28], each of which, by definition of the eigenvector, has a set of single-phase equivalent circuits.

In conclusion, it must be borne in mind that the final evaluation of the adequacy of any equivalent circuit whether used in mathematical analysis or for miniature representation must depend on results from comparative field tests.

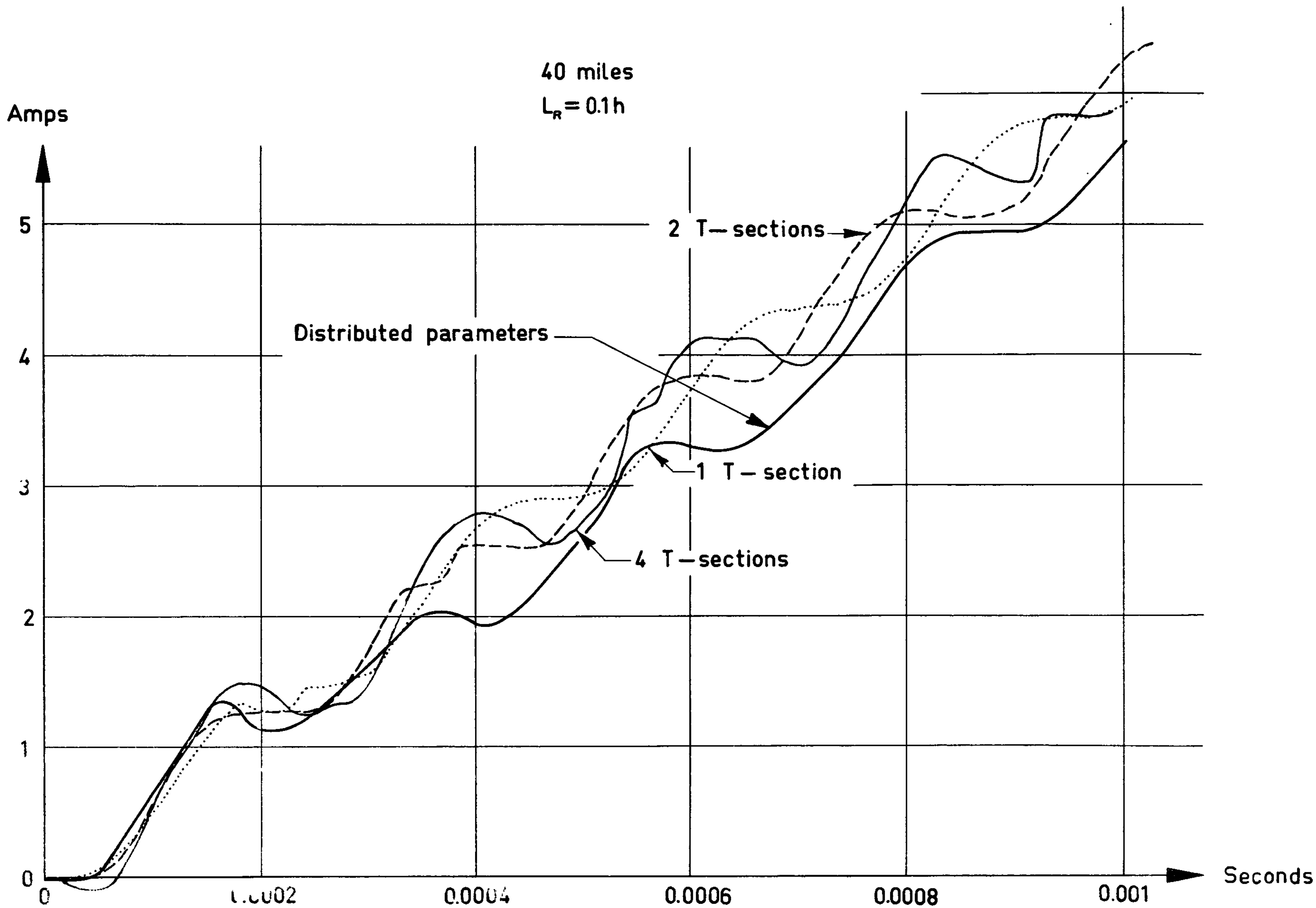

40 miles
$L_R = 0.1h$
Amps
5
4
3
2
1
0
2 T— sections
Distributed parameters
1 T — section
4 T —sections
0
0.0002
0.0004
0.0006
0.0008
0.001
Seconds

FIG. 2-6

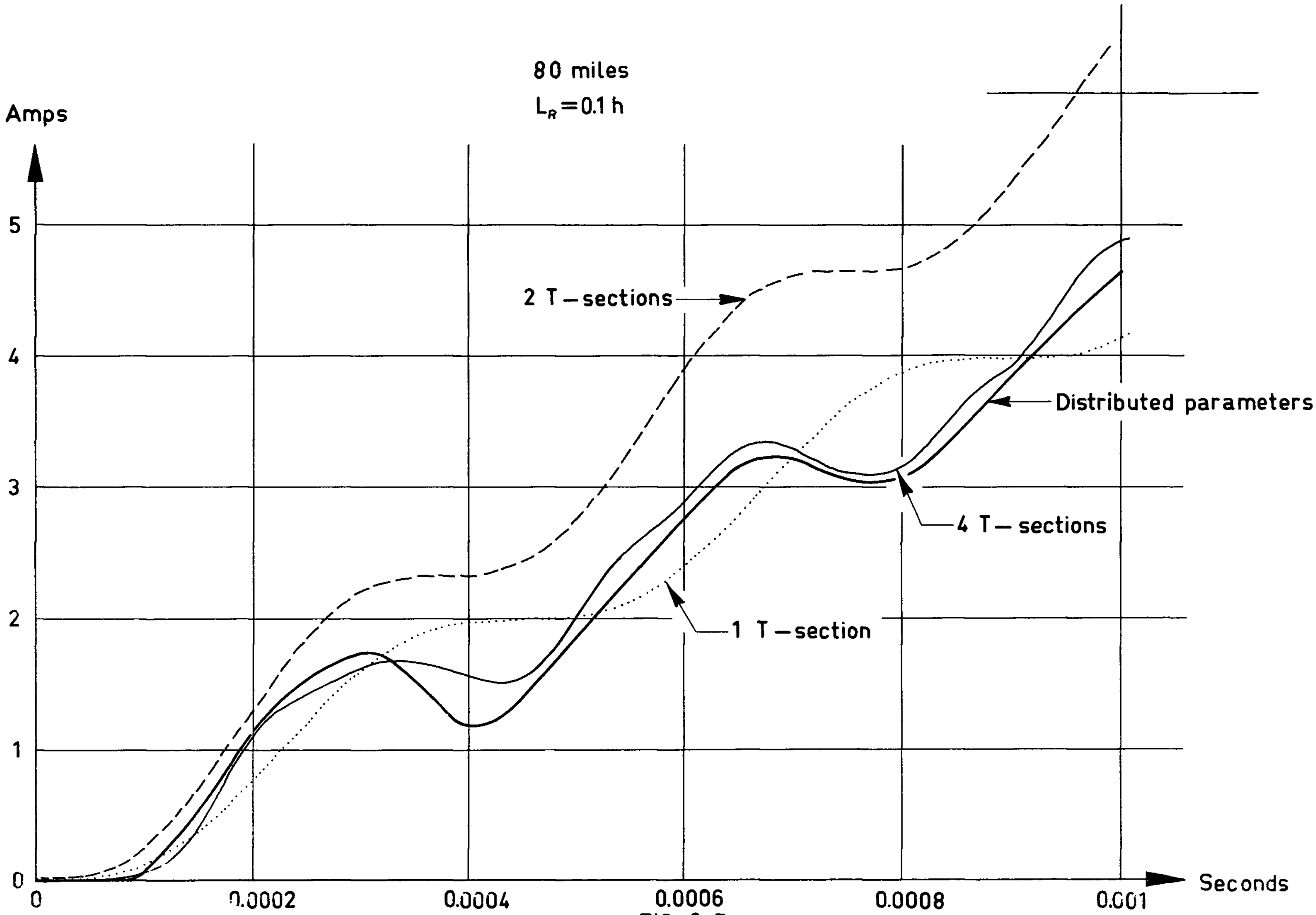

80 miles
$L_R = 0.1$ h
Amps
5
4
3
2
1
0
2 T – sections
Distributed parameters
4 T – sections
1 T – section
0
0.0002
0.0004
0.0006
0.0008
0.001
Seconds

FIG. 2-7

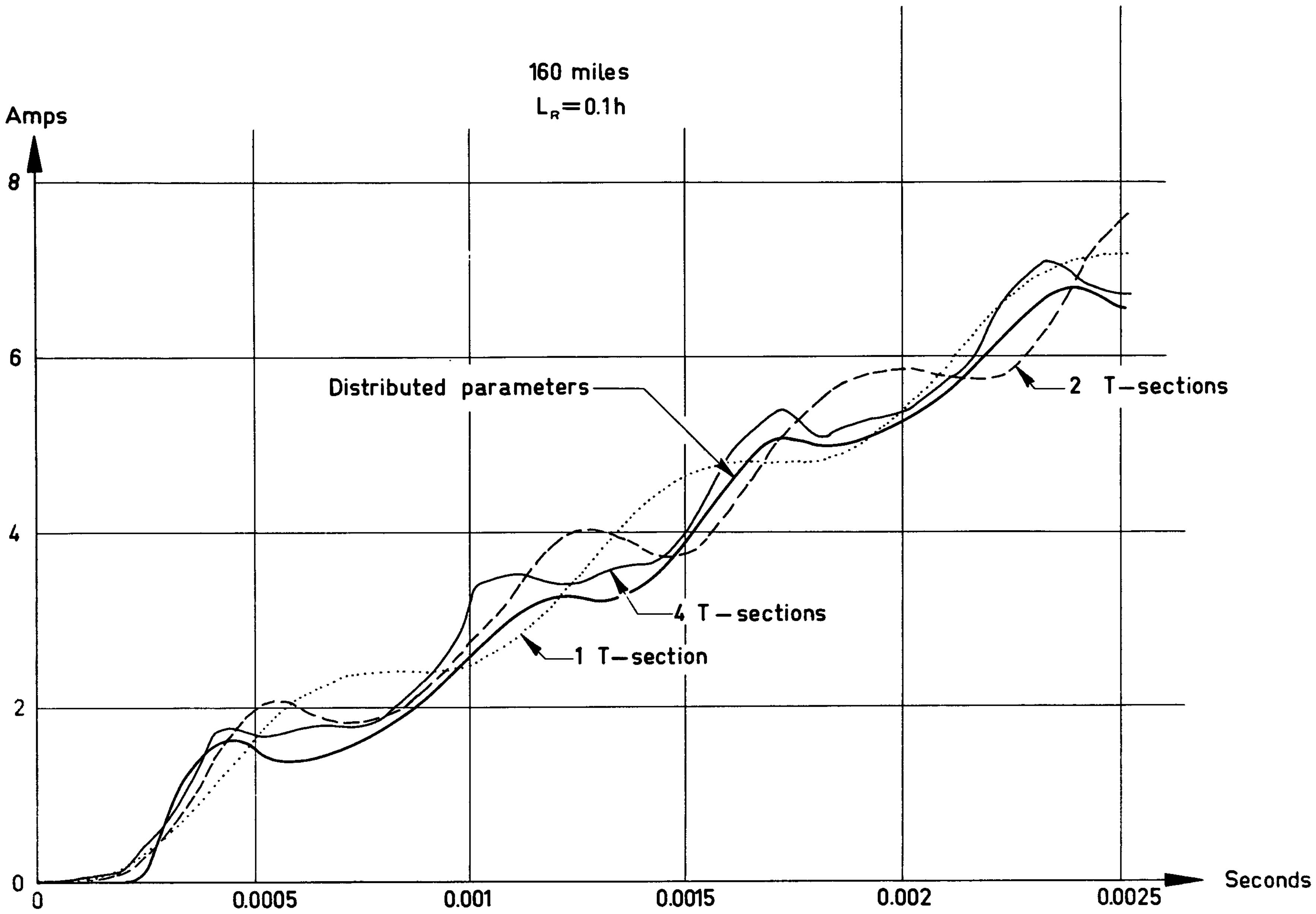

160 miles
$L_R = 0.1h$
Amps
8
6
4
2
0
0
0.0005
0.001
0.0015
0.002
0.0025
Seconds
Distributed parameters
2 T-sections
4 T-sections
1 T-section

FIG. 2-8

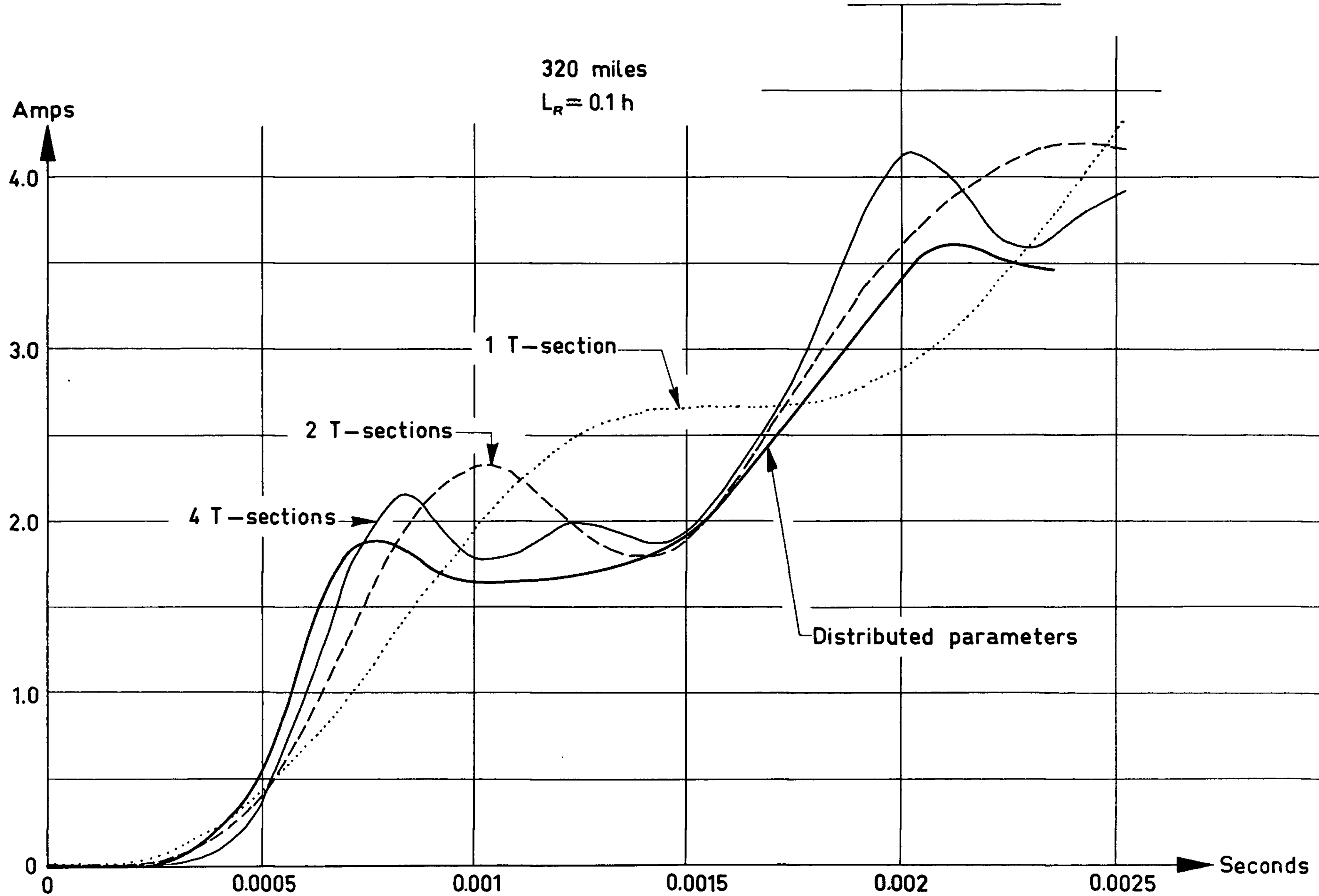

320 miles
$L_R = 0.1$ h
Amps
4.0
3.0
2.0
1.0
0
1 T—section
2 T—sections
4 T—sections
Distributed parameters
0
0.0005
0.001
0.0015
0.002
0.0025
Seconds

FIG. 2-9

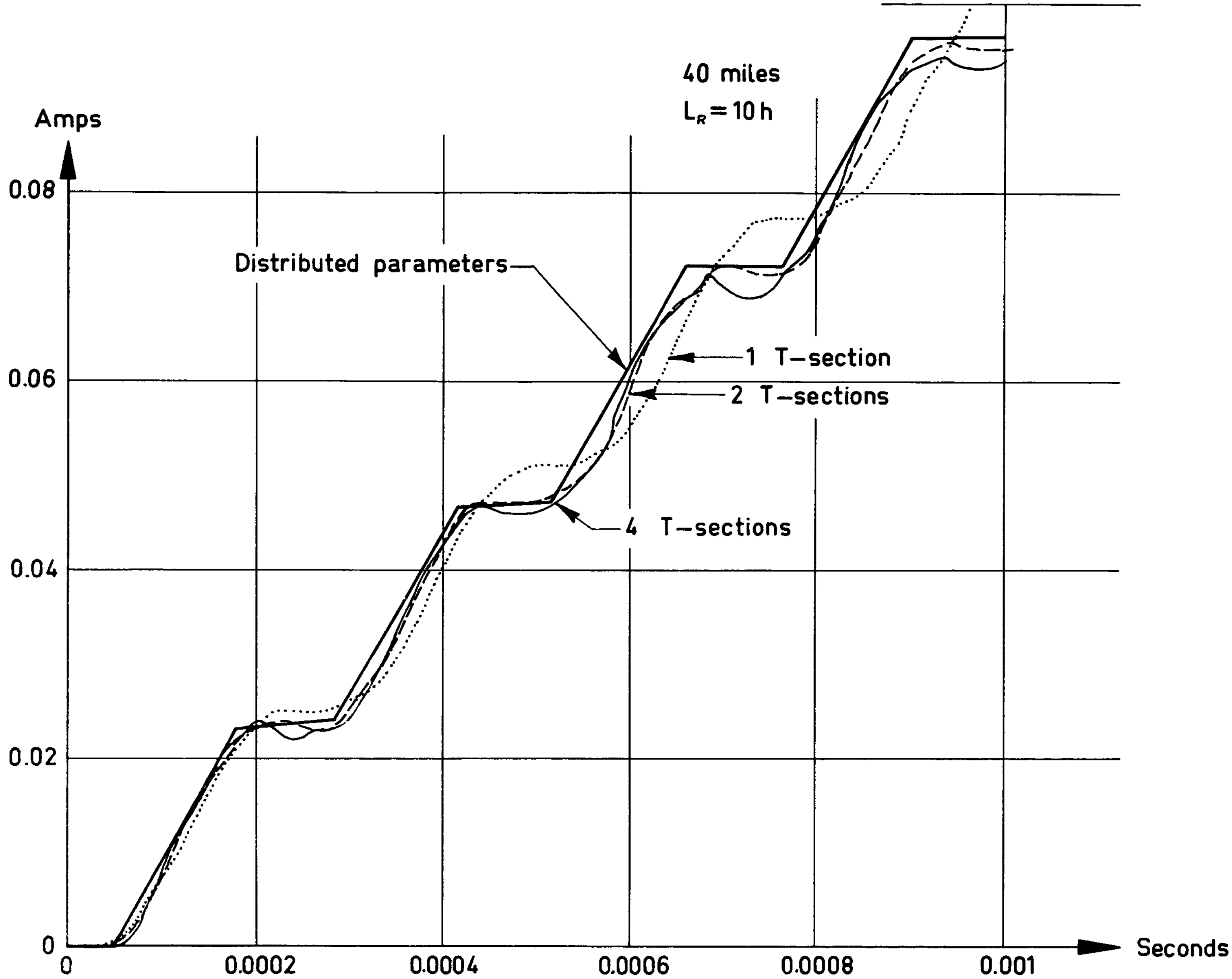

40 miles
$L_R = 10$ h
Amps
0.08
0.06
0.04
0.02
0
Distributed parameters
1 T-section
2 T-sections
4 T-sections
0
0.0002
0.0004
0.0006
0.0008
0.001
Seconds

FIG. 2-10

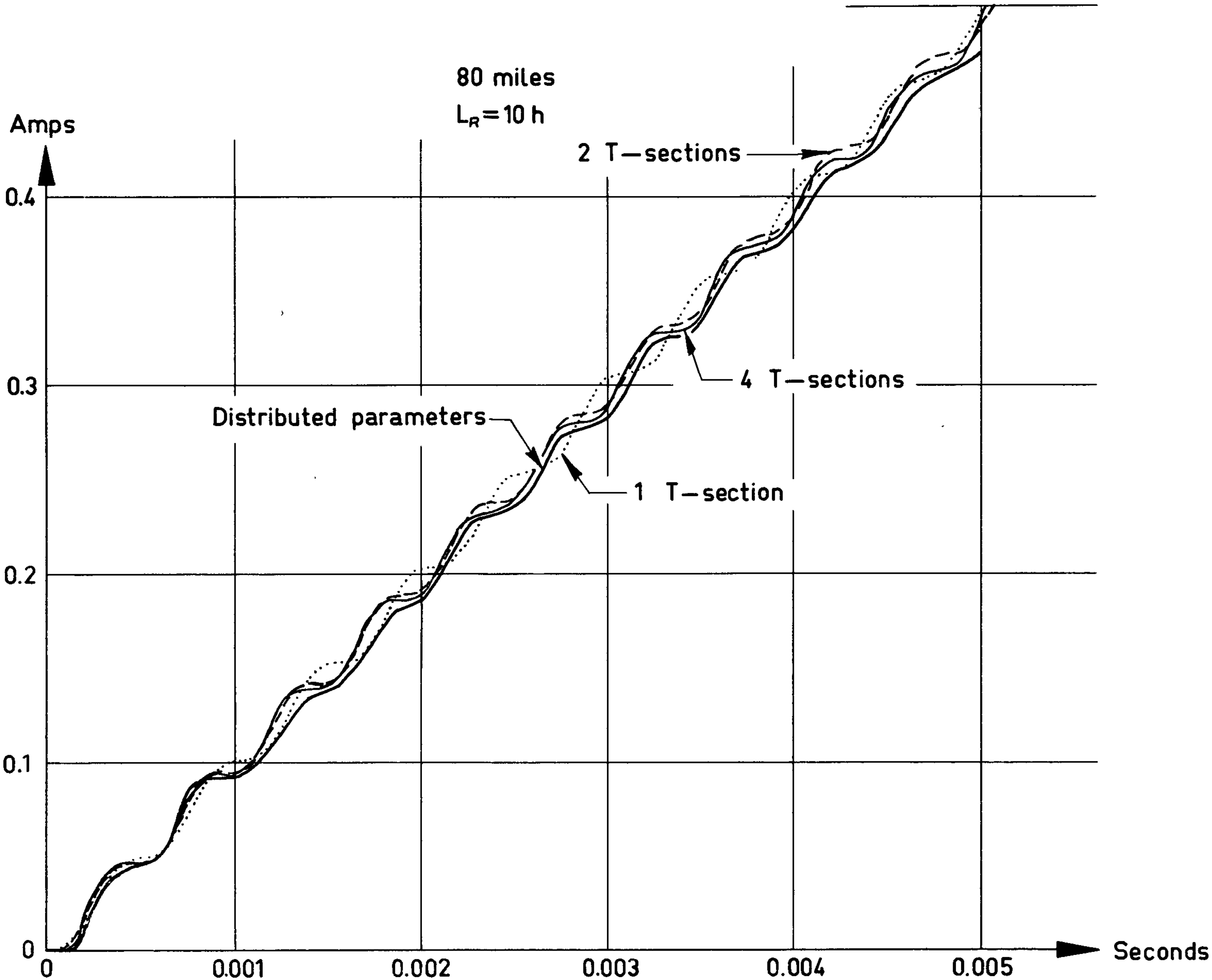

80 miles
$L_R = 10$ h
Amps
0.4
0.3
0.2
0.1
0
2 T—sections
4 T—sections
Distributed parameters
1 T—section
0.001
0.002
0.003
0.004
0.005
Seconds

FIG. 2-11

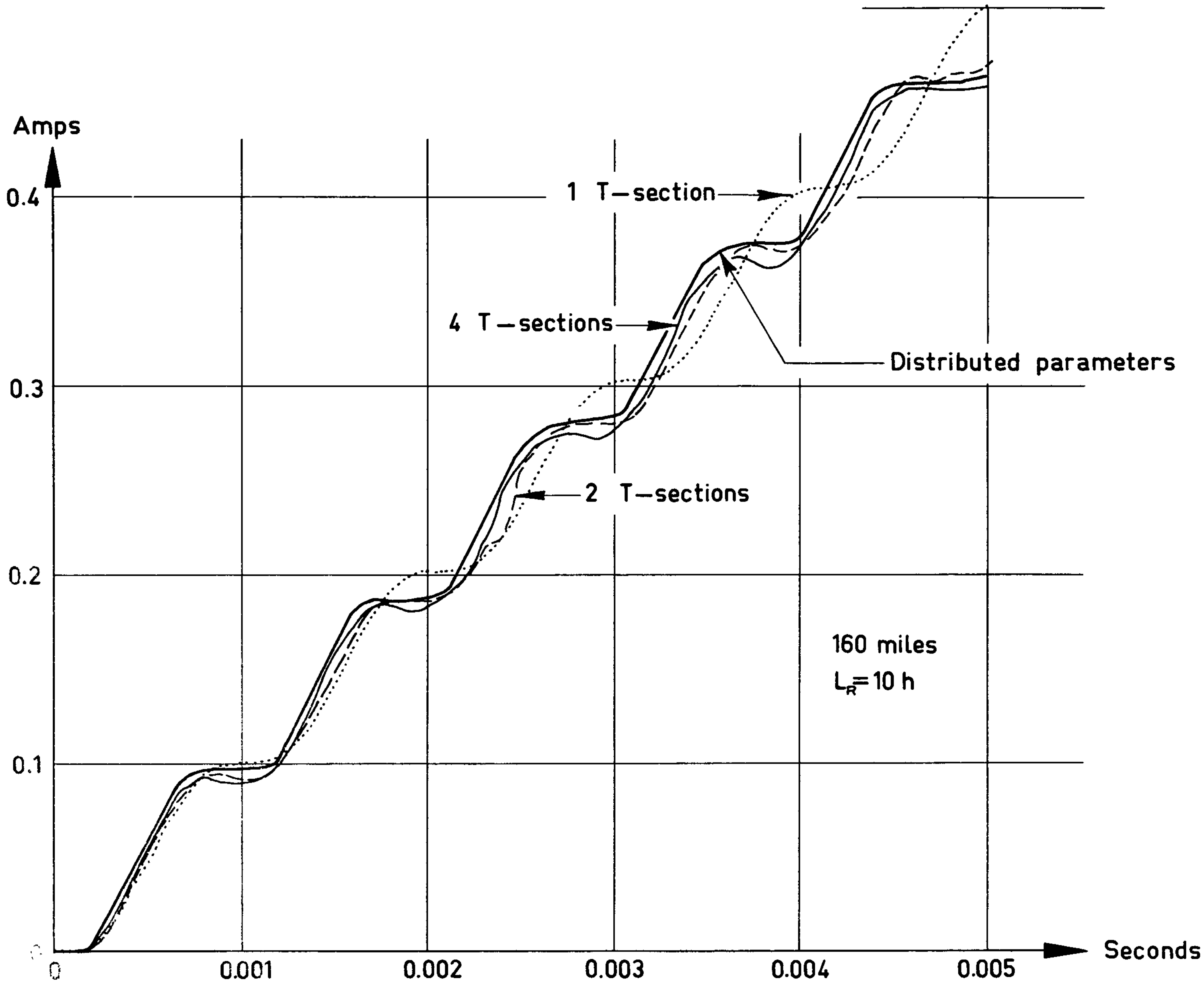
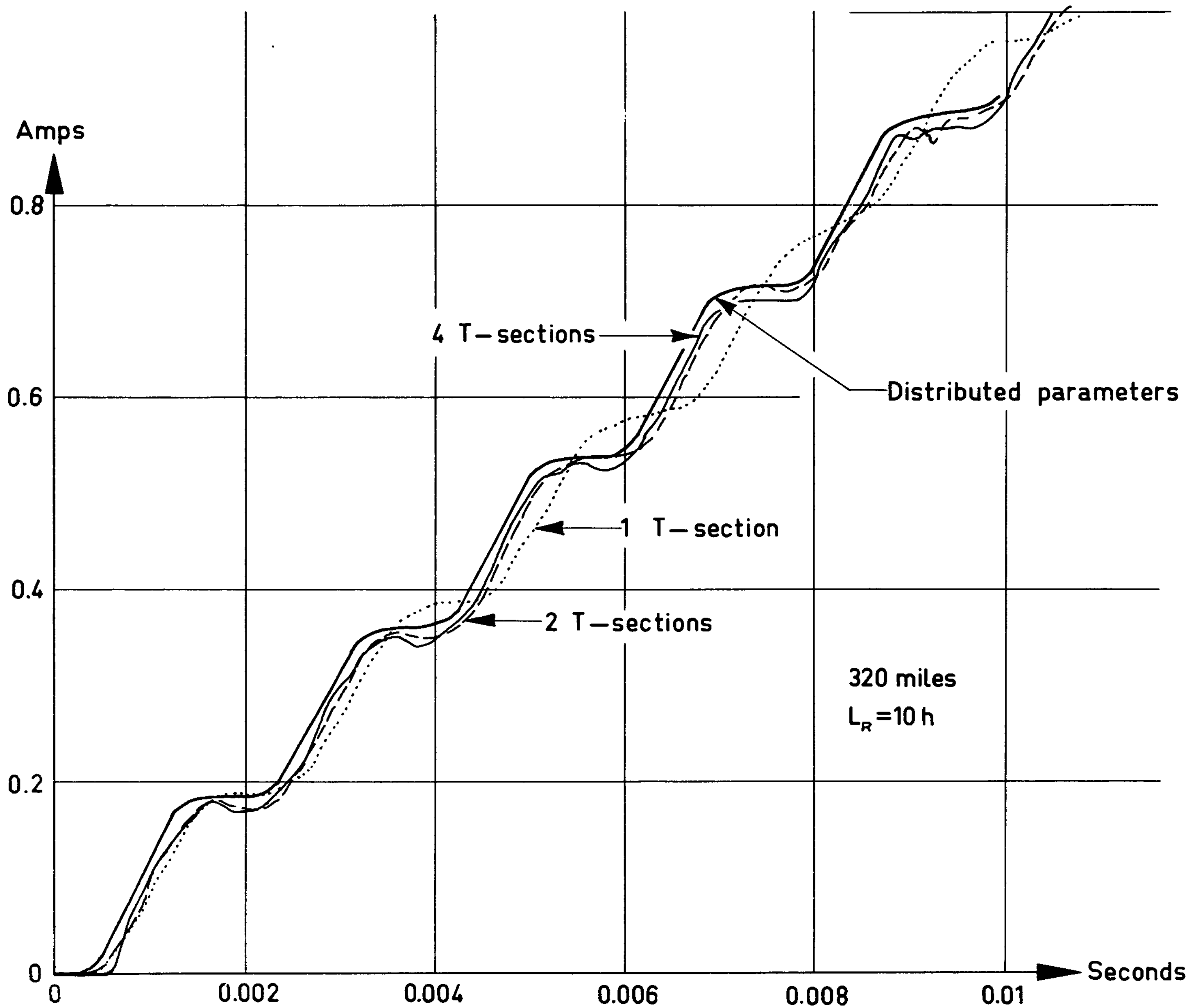

Amps
0.4
0.3
0.2
0.1
0
0.001
0.002
0.003
0.004
0.005
Seconds
1 T—section
4 T—sections
2 T—sections
Distributed parameters
160 miles
$L_R = 10$ h

FIG. 2-12


Amps
0.8
0.6
0.4
0.2
0
0
0.002
0.004
0.006
0.008
0.01
Seconds
4 T—sections
Distributed parameters
1 T—section
2 T—sections
320 miles
L_R=10 h
FIG. 2-13

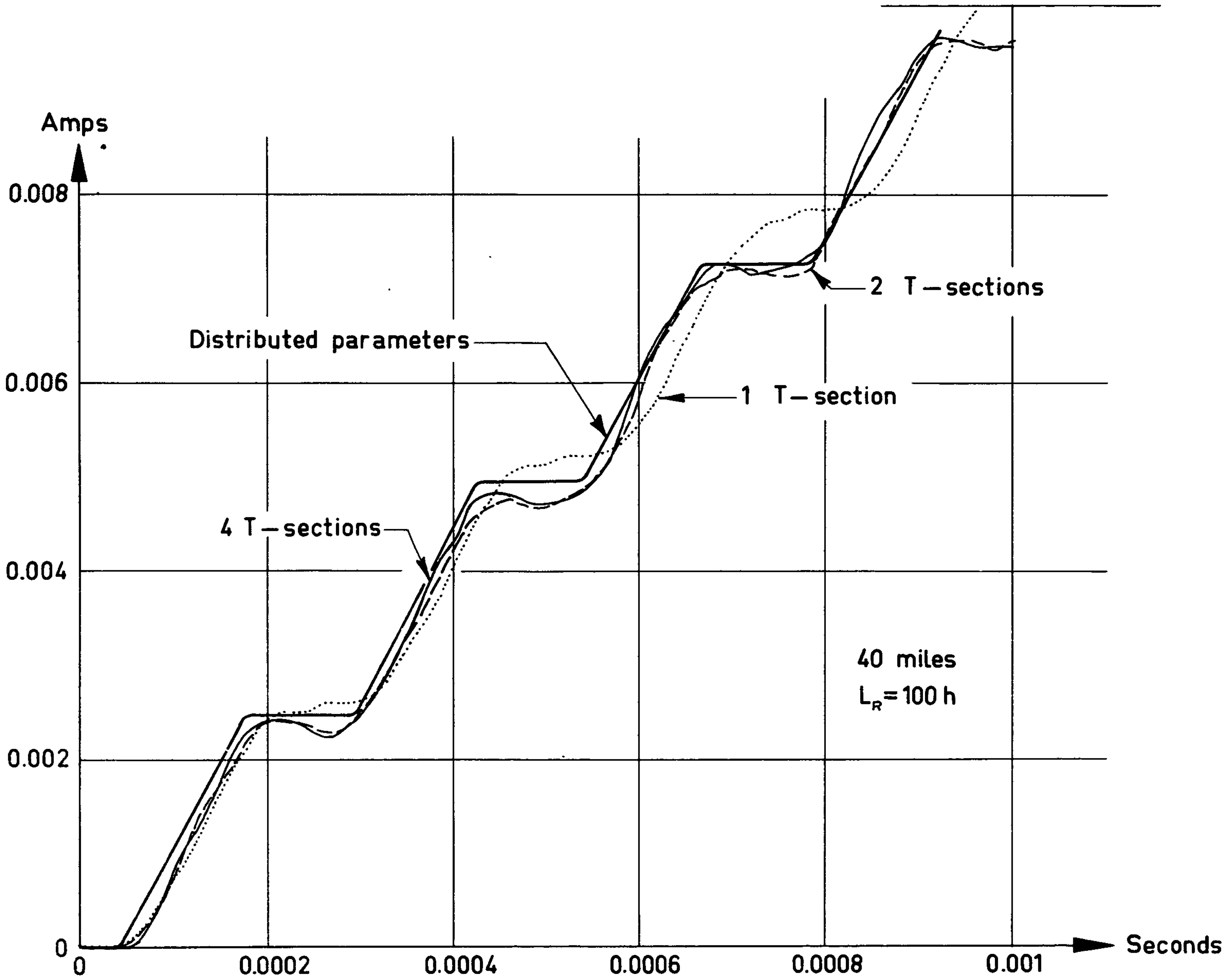

Amps
0.008
0.006
0.004
0.002
0
0
0.0002
0.0004
0.0006
0.0008
0.001
Seconds
2 T—sections
Distributed parameters
1 T—section
4 T—sections
40 miles
$L_R = 100$ h

FIG. 2-14

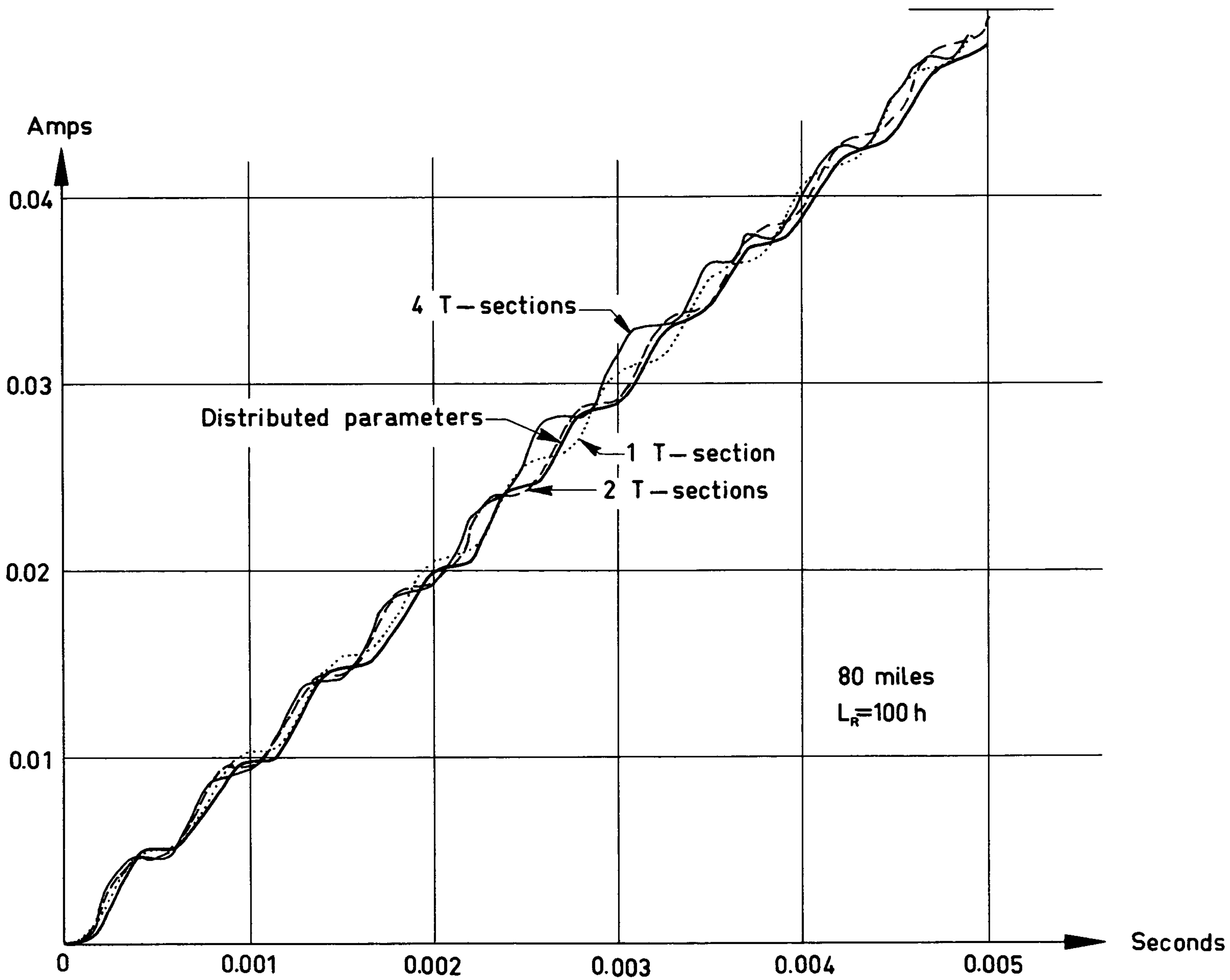
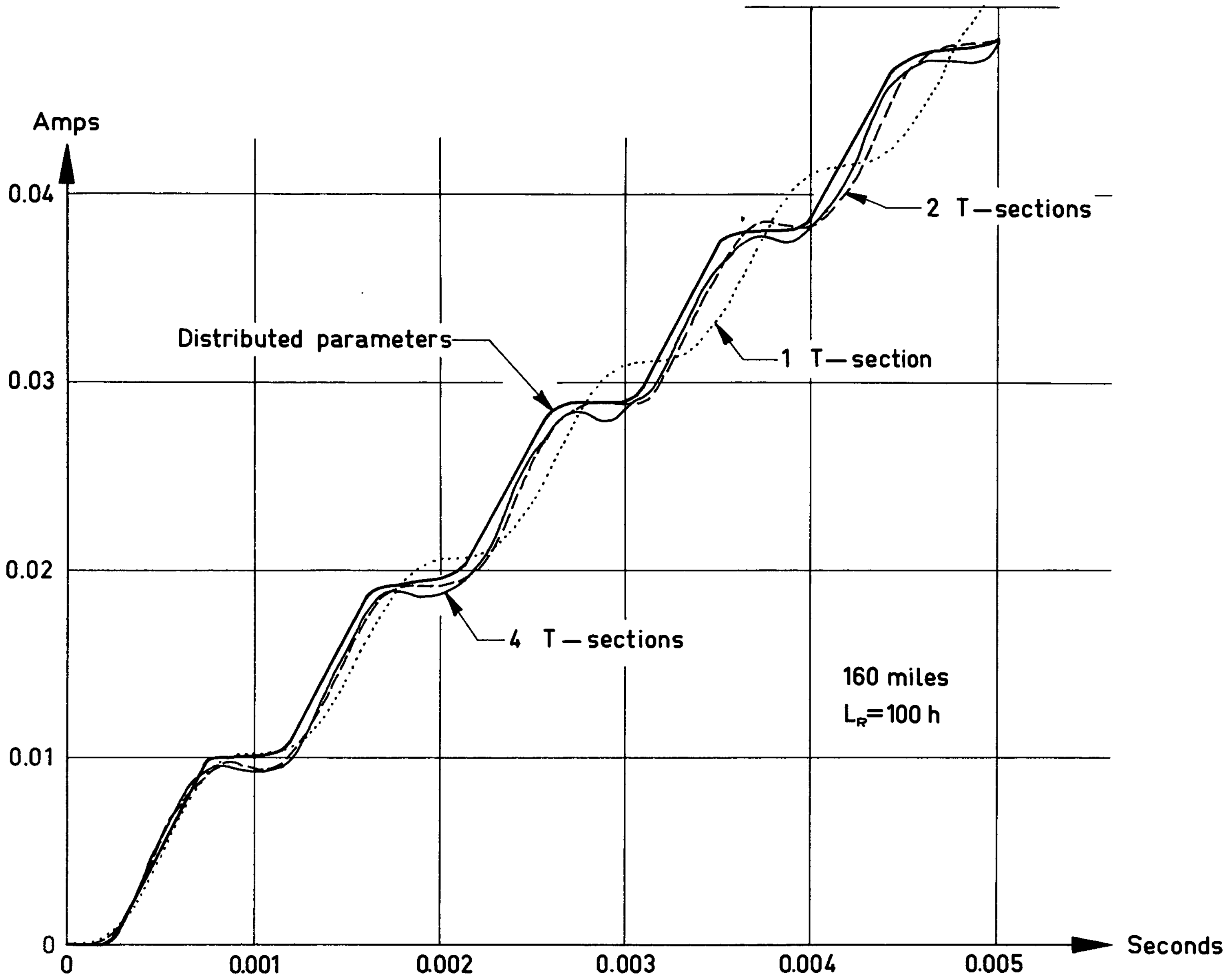
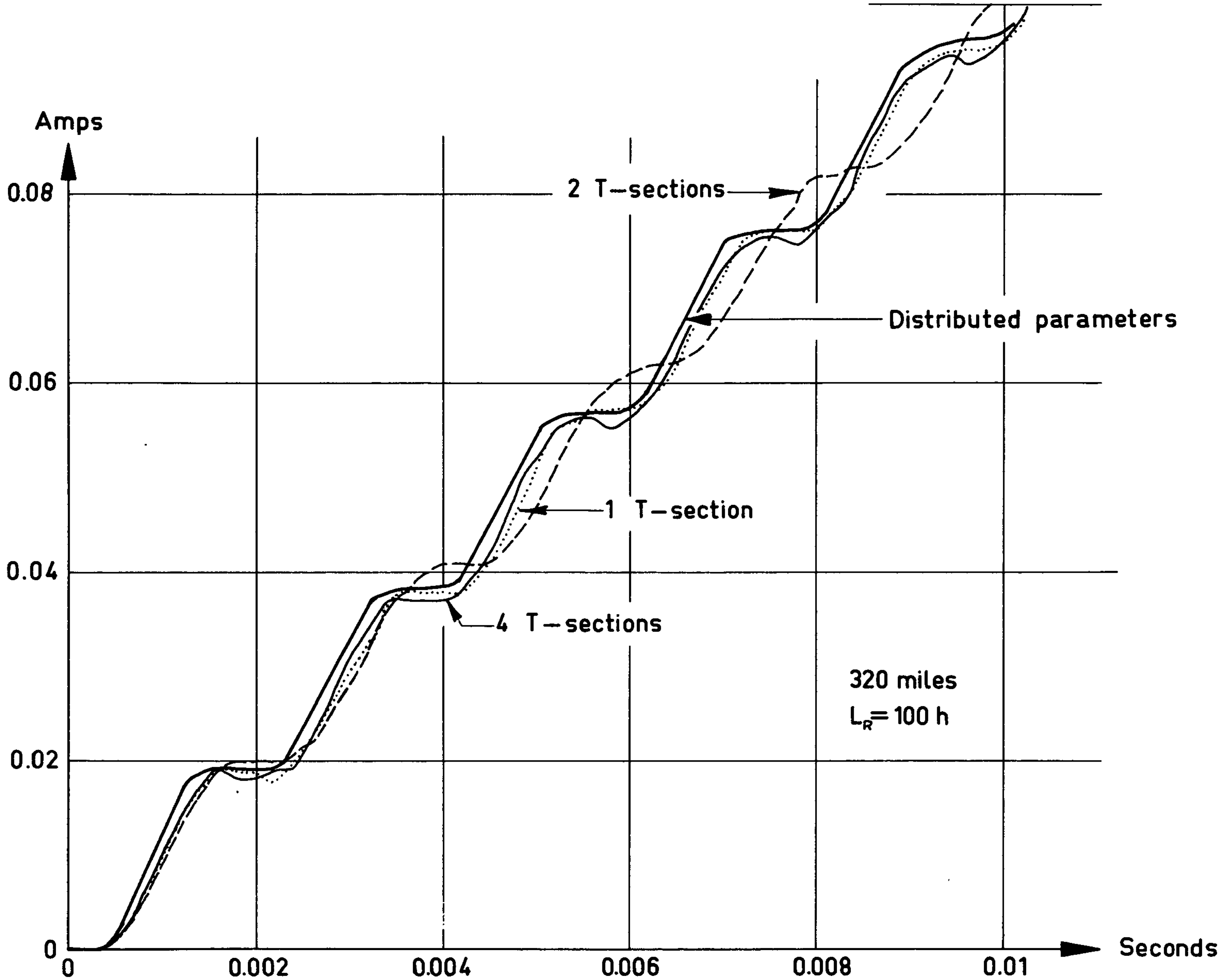

Amps
0.04
0.03
0.02
0.01
0
0.001
0.002
0.003
0.004
0.005
Seconds
4 T—sections
Distributed parameters
1 T—section
2 T—sections
80 miles
L_R=100 h

FIG. 2-15



Amps
0.04
0.03
0.02
0.01
0
0
0.001
0.002
0.003
0.004
0.005
Seconds
Distributed parameters
1 T—section
2 T—sections
4 T—sections
160 miles
L_R=100 h

FIG. 2-16



Amps
0.08
0.06
0.04
0.02
0
0
0.002
0.004
0.006
0.008
0.01
Seconds
2 T—sections
Distributed parameters
1 T—section
4 T—sections
320 miles
L_R= 100 h

FIG. 2 -17

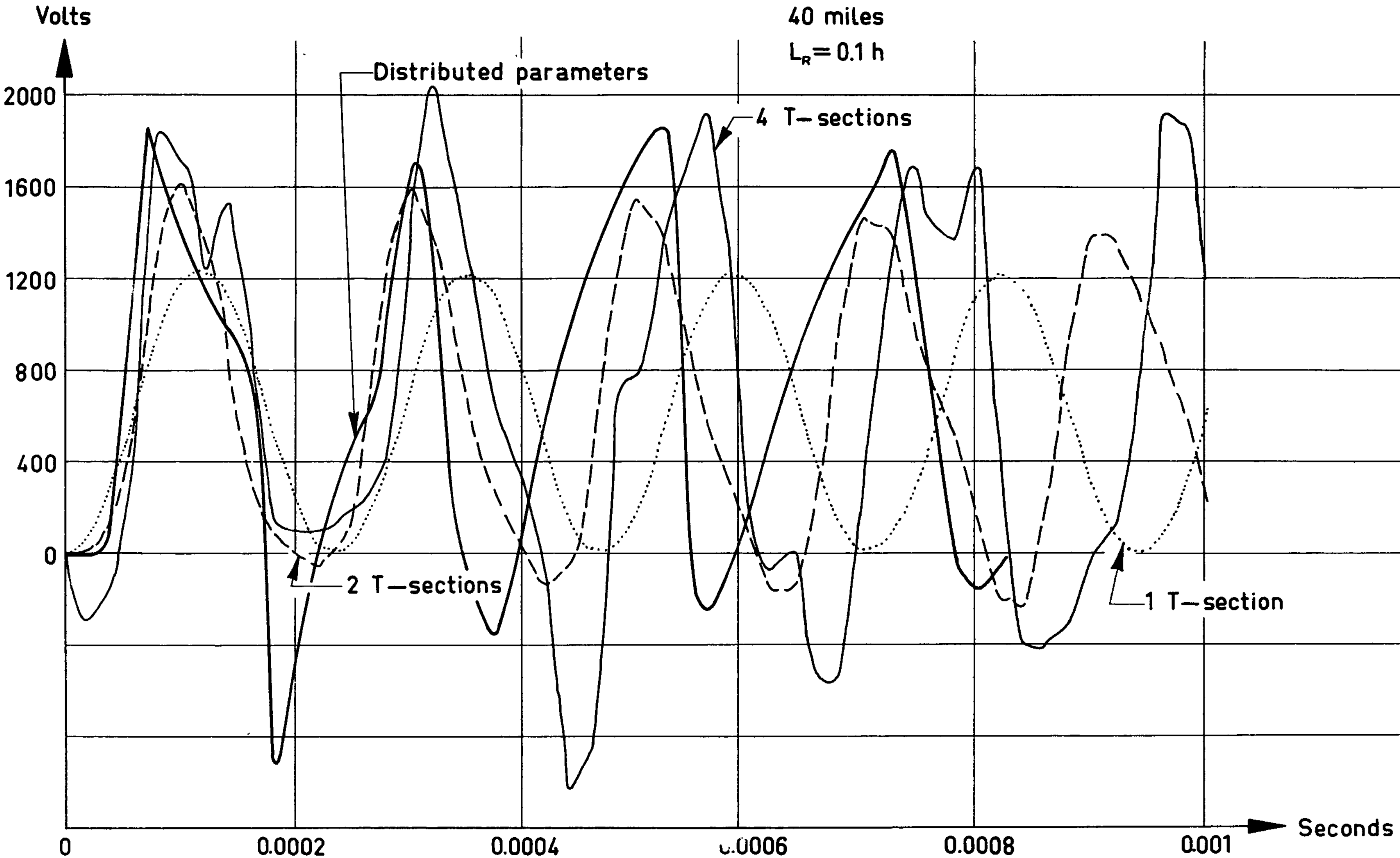

Volts
40 miles
$L_R = 0.1$ h
Distributed parameters
4 T-sections
2 T-sections
1 T-section
2000
1600
1200
800
400
0
0
0.0002
0.0004
0.0006
0.0008
0.001
Seconds

FIG. 2-18

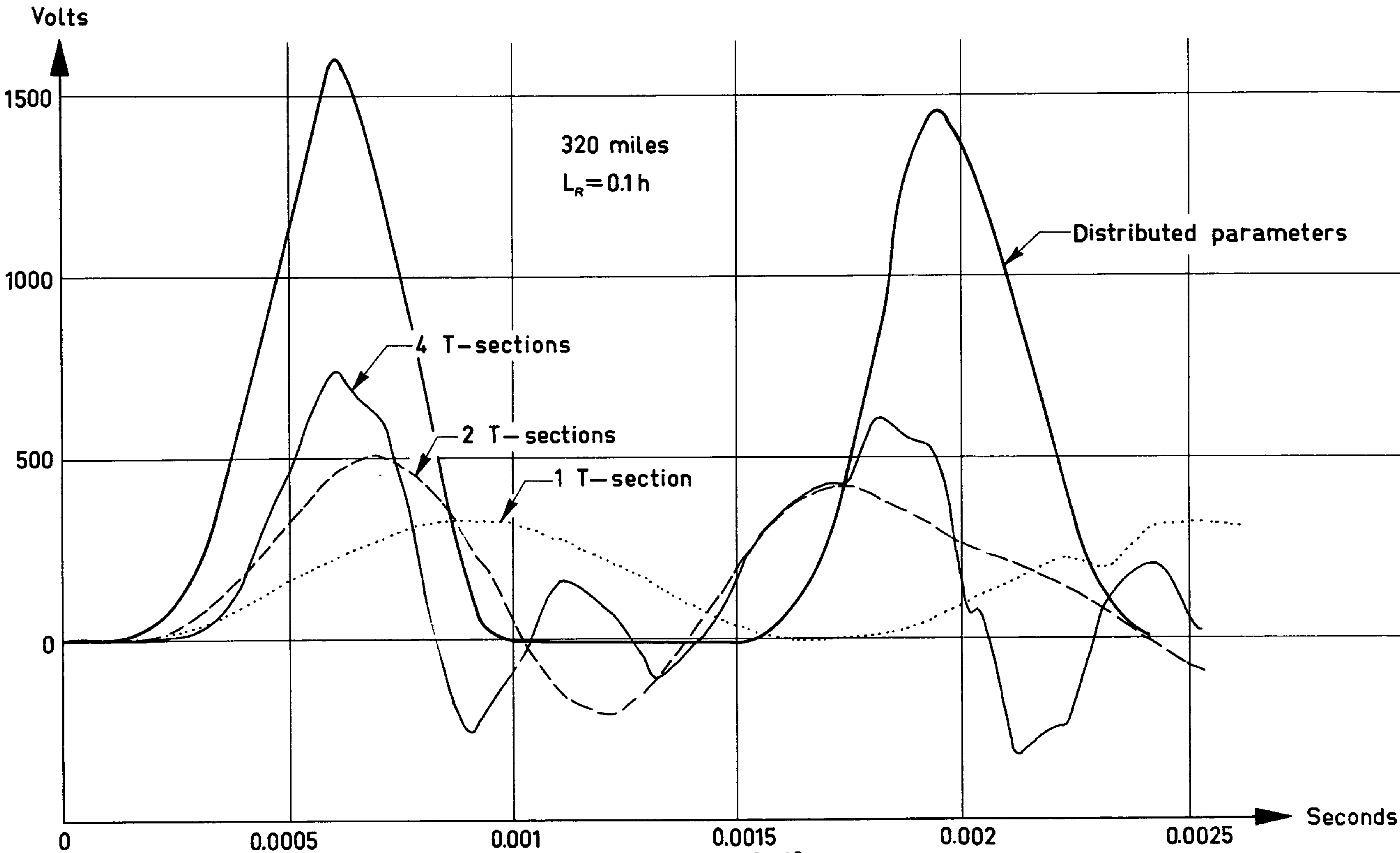
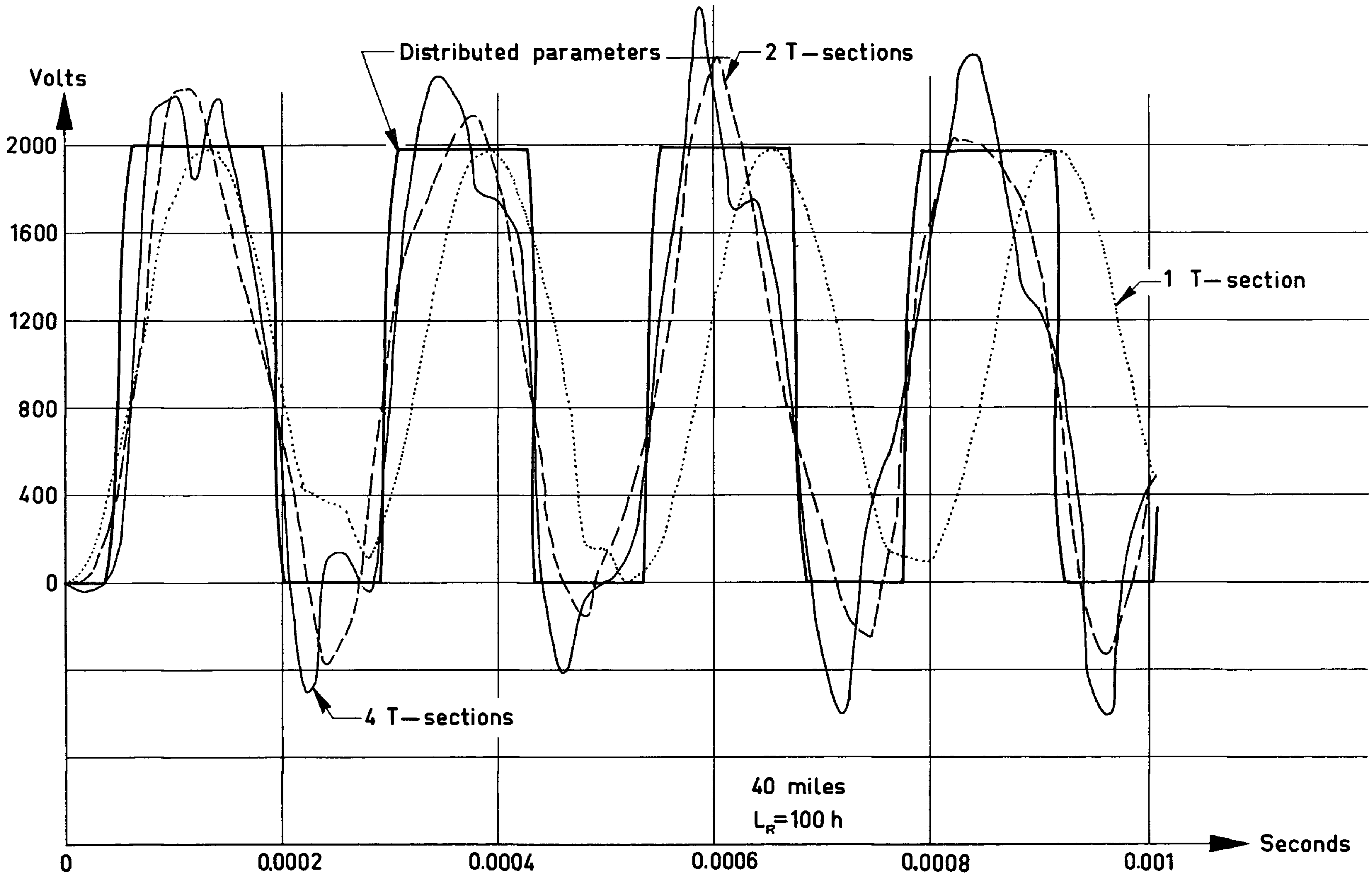

Volts
1500
1000
500
0
320 miles
$L_R = 0.1\,h$
Distributed parameters
4 T-sections
2 T-sections
1 T-section
0
0.0005
0.001
0.0015
0.002
0.0025
Seconds

FIG. 2-19



Volts
2000
1600
1200
800
400
0
Distributed parameters
2 T—sections
1 T—section
4 T—sections
40 miles
L_R=100 h
0
0.0002
0.0004
0.0006
0.0008
0.001
Seconds

FIG. 2-20

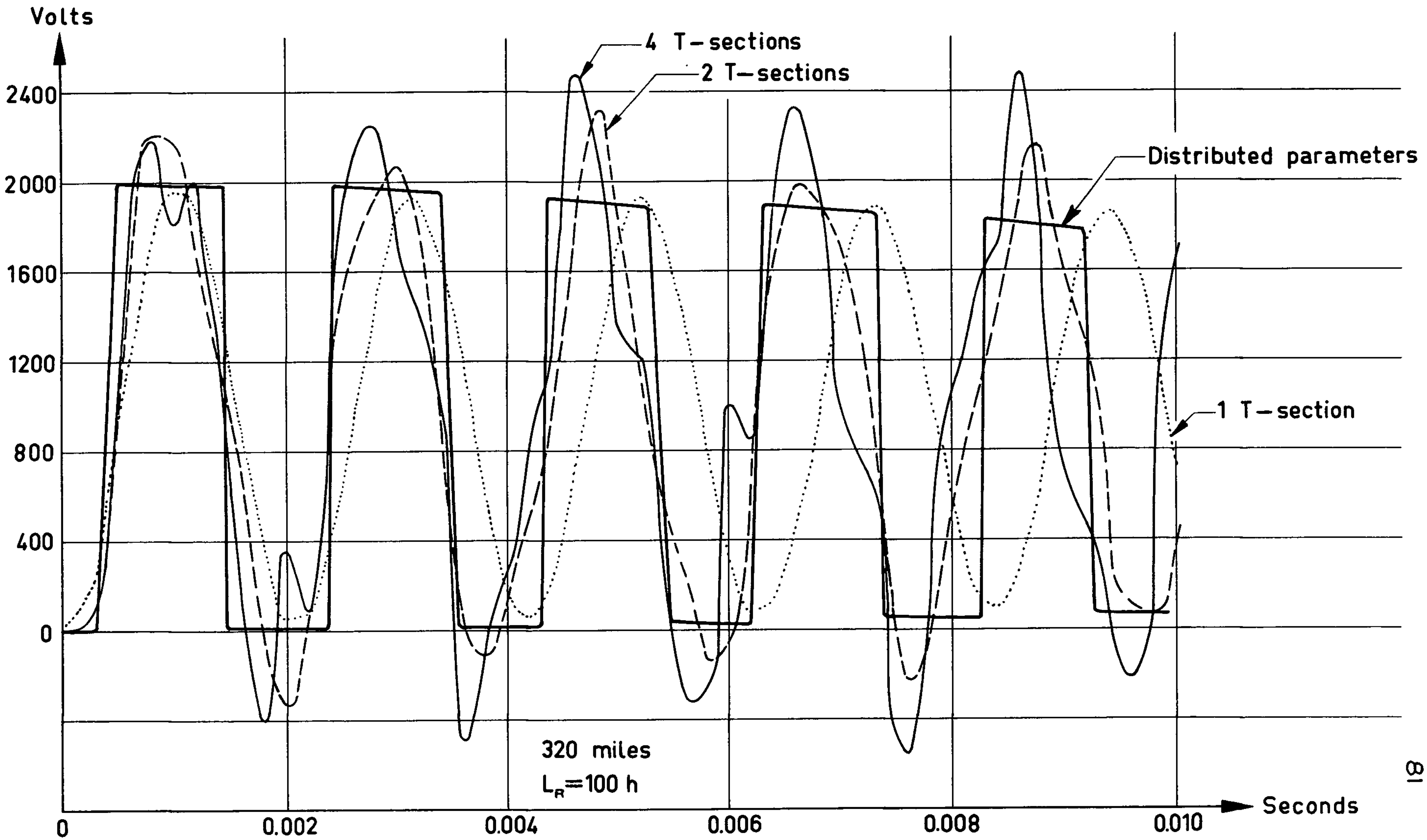
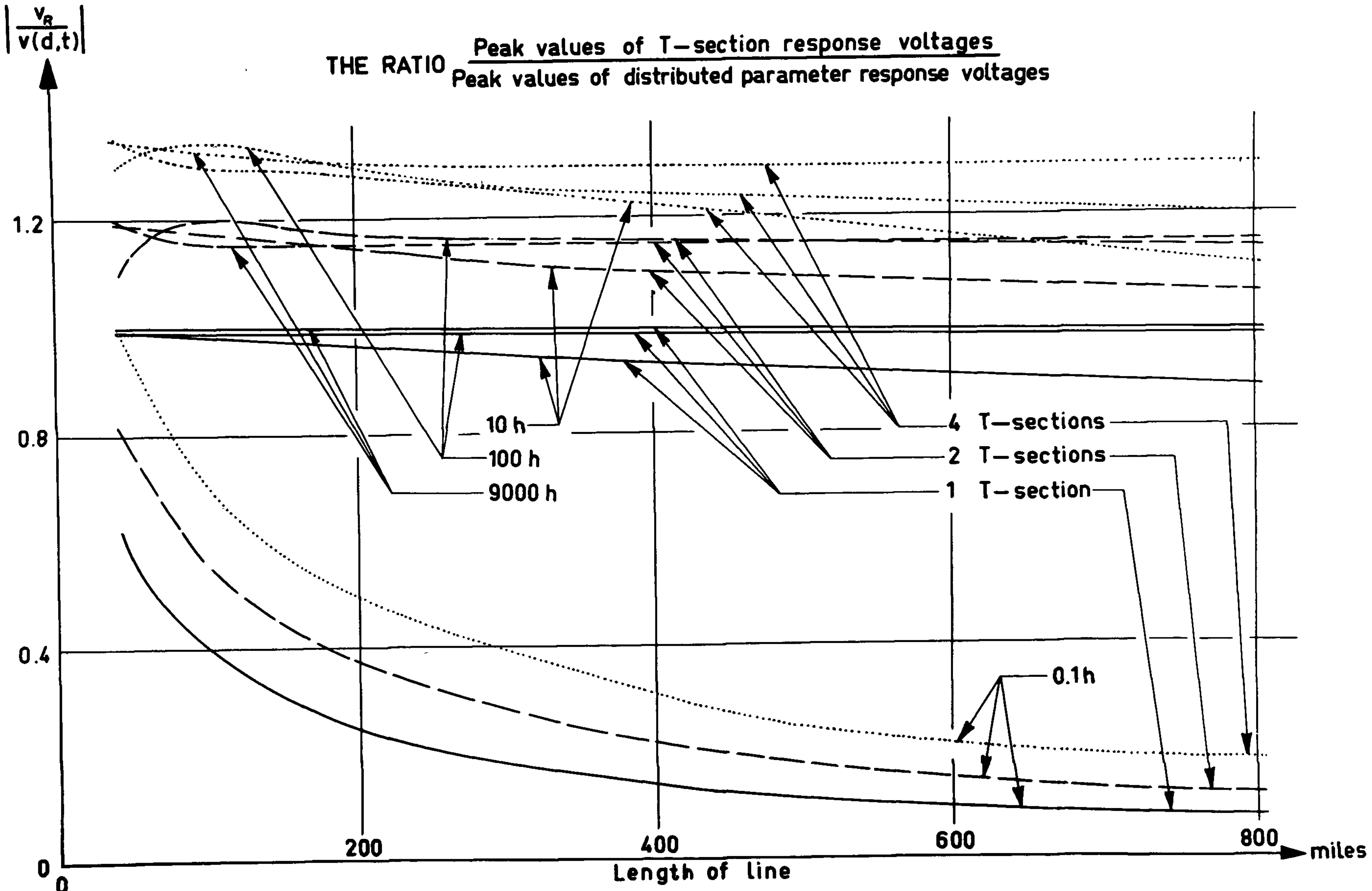

Volts
2400
2000
1600
1200
800
400
0
4 T-sections
2 T-sections
Distributed parameters
1 T-section
320 miles
L_R=100 h
0
0.002
0.004
0.006
0.008
0.010
Seconds

FIG. 2-21



$\left|\frac{v_R}{v(d,t)}\right|$
THE RATIO $\frac{\text{Peak values of T-section response voltages}}{\text{Peak values of distributed parameter response voltages}}$
1.2
0.8
0.4
0
0
200
400
600
800
miles
Length of line
10 h
100 h
9000 h
4 T-sections
2 T-sections
1 T-section
0.1h

FIG. 2-22

CHAPTER 3.

3-1 Introduction

In investigating for a solution to the type of non-linear problem under study, two approaches are possible. An analytical solution with its simple numerical calculations and its appreciation of the system behaviour without too many calculations has the disadvantage of depending on too many approximations and simplifications. On the other hand, there may be available a mathematical method, simple in theory but with many difficulties in practical application. These difficulties may stem principally from the tedious and numerous calculations necessary to obtain a complete solution but if such objections are overcome, a minimum of approximations is necessary and a reasonable degree of accuracy can be obtained.

In this chapter, the latter approach is fully explored and the facilities of the digital computer are utilised.

Certain familiar linear techniques are based on more general methods which can be applied to non-linear problems. One approach to the solution is the application of some such general method. The actual procedure is usually similar to some linear method and this is an advantage. The equivalent linearization process,[29] the incremental method and certain iterative techniques are examples of this approach.

In this study, the incremental method and an iterative technique are decided upon and although both are described and used, the incremental method is far superior in dealing with this type of problem. The iterative technique has some attractive features and may prove more successful in some other applications.

In deciding upon a particular method, the following requirements are desirable:

(1) Need for few approximations and simplifications of the circuits.

(2) Possibility of studying the entire range of system behaviour in time i.e. both transient and steady-state.

(3) Suitability for programming on a digital computer.

3-2 Incremental Method.

Consider any circuit in which there is a non-linear element--e.g. an iron-cored reactor about which is known the relationship between λ, the flux linkage and i, the current flowing in the reactor.

Consider any value of current i, which occurs at time t. The Taylor's expansion around i=i(t) yields

$$\lambda\ (i) = \lambda\ \left\{i(t)\right\} + \left|i - i(t)\right| \cdot \frac{d\lambda\ \left\{i(t)\right\}}{di} + \dots \qquad (3\text{-}1)$$

and differentiating

$$\frac{d\lambda(i)}{di} = \frac{d\lambda\left\{i(t)\right\}}{di} + \left|i - i(t)\right| \cdot \frac{d^2\lambda\ \left\{i(t)\right\}}{di^2} + \dots \qquad (3\text{-}2)$$

But e_λ = voltage induced

$$= \frac{d\lambda}{dt}$$

$$= \frac{d\lambda}{di} \cdot \frac{di}{dt}$$

$$= \frac{d\lambda\ \left\{i(t)\right\}}{di} \cdot \frac{di}{dt} + \left|i - i(t)\right| \cdot \frac{d^2\lambda\ \left\{i(t)\right\}}{dt^2} \cdot \frac{di}{dt} + \dots (3\text{-}4)$$

therefore, if i - i(t) is chosen small enough, terms with higher powers of i - i(t) approach zero and

$$e_\lambda = \frac{d\lambda\ \left\{i(t)\right\}}{di} \cdot \frac{di}{dt} \qquad (3\text{-}5)$$

Any non-linear element can be dealt with in a similar manner and this forms the basis of the incremental method.

In the particular case of a reactor, the relationship (3-5) is valid over a small excursion from i(t) and therefore, over that corresponding period of time the circuit can be treated by a linear technique.

At t=0, initial conditions of voltages and currents are known; the differential equations describing the circuit are established and a solution over some increment of time or current is obtained. After the valid range of i is exceeded the new initial conditions are calculated; $\frac{d\lambda}{di}\left\langle i(t)\right\rangle$ for a new t=0 is found and the process is repeated. This approach proves highly successful and is admirably suited for digital computing techniques. Various examples, with their complete solutions, will show the flexibility and usefulness of this method.

The Laplace operational calculus is used exclusively as it offers two distinct advantages;

(1) A systematic solution of simultaneous differential equations.

(2) Simplicity in taking into account various initial conditions.

To demonstrate the validity of this approach, a simple <u>linear</u> example of a resistance R in series with an inductance L energised by a voltage $e = E_{max} \sin \omega t$ is considered in Appendix- 2.

3-3 The Equivalent Circuits.

Of concern in this study is the system which includes a generator, a transmission line and an unloaded transformer. Figs. (3-1) to Figs. (3-7) give the equivalent circuits of some typical arrangements.

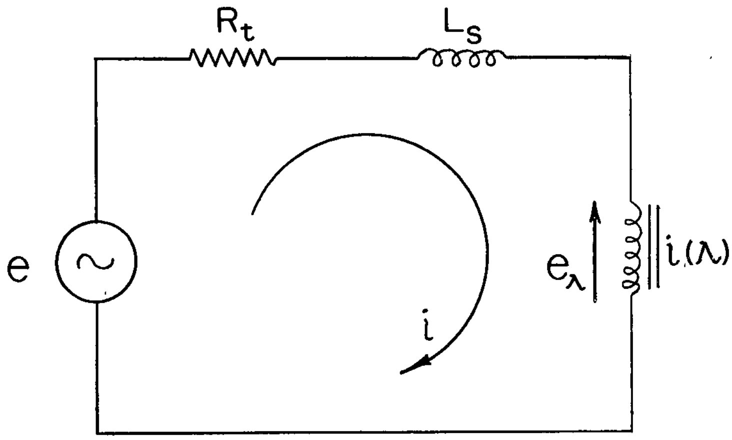


FIG. 3-1

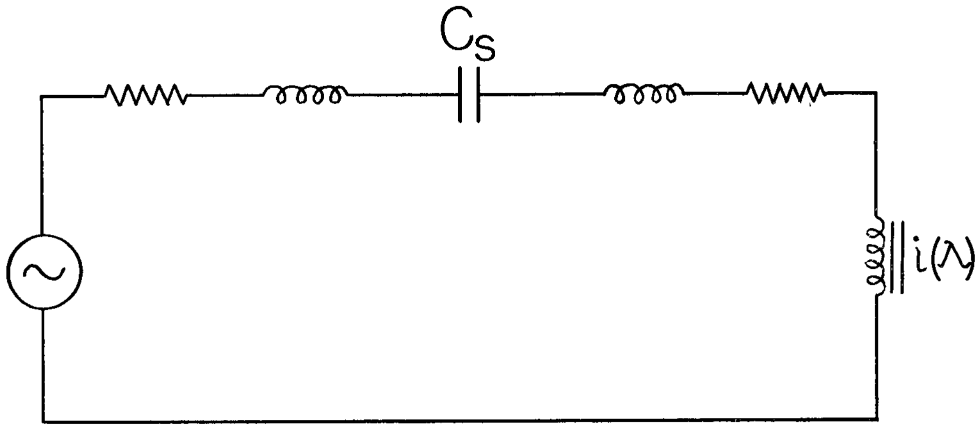


FIG. 3-2

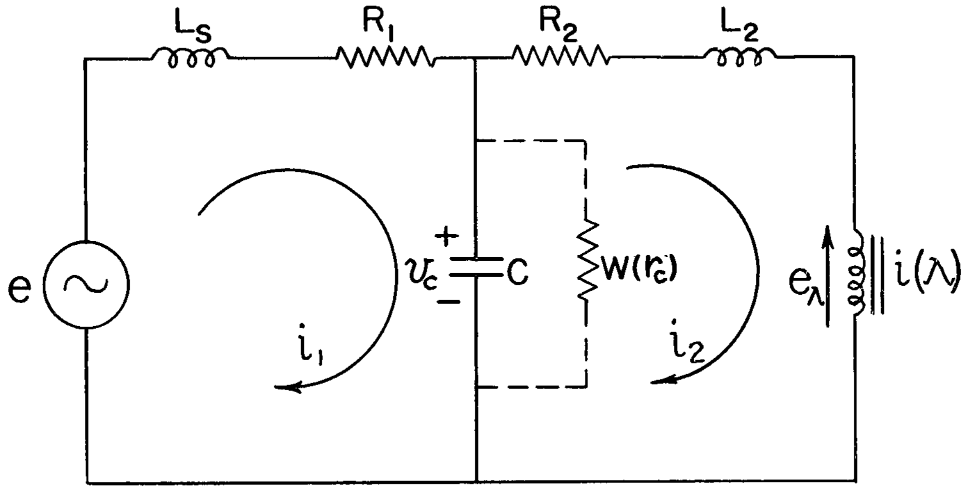


FIG. 3-3

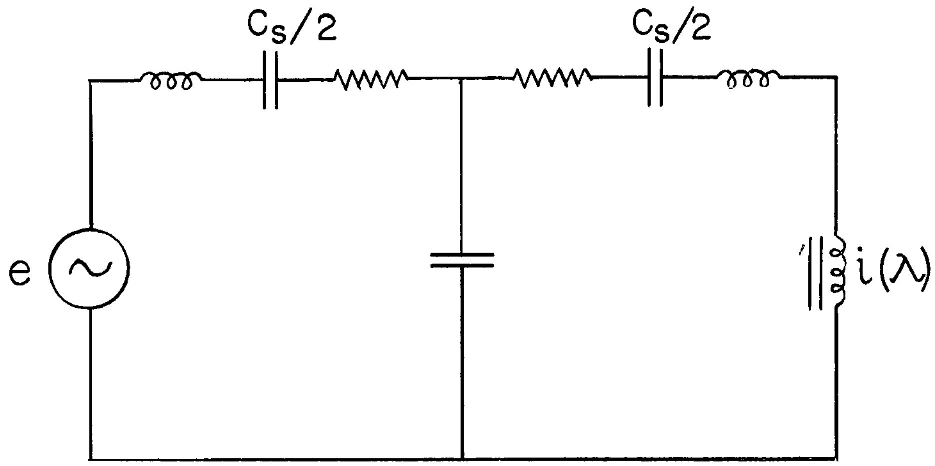


FIG. 3-4

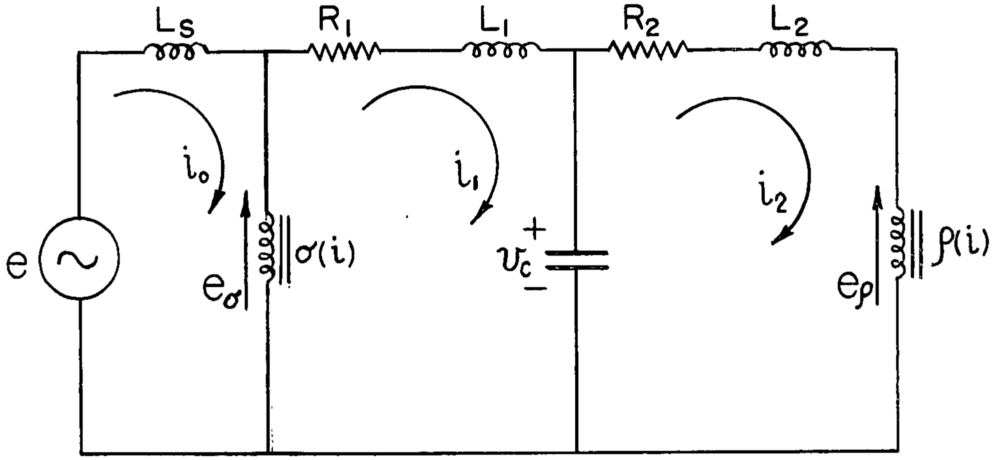


FIG. 3-5

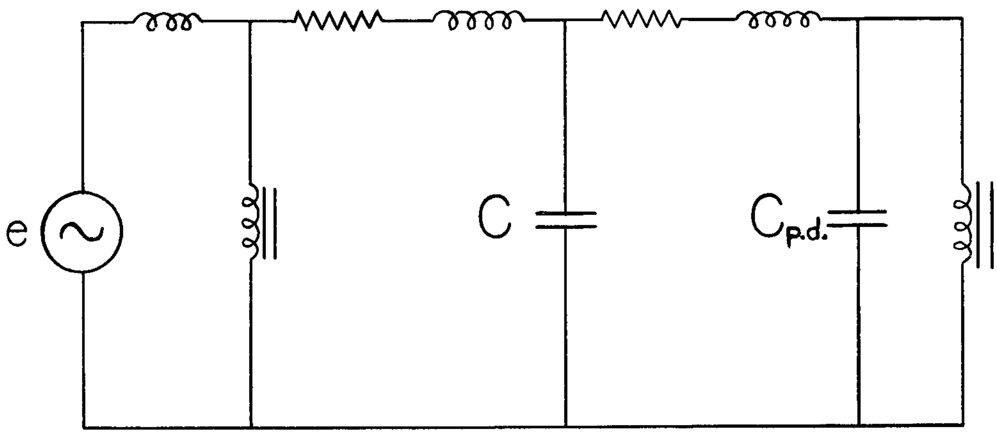


FIG. 3-6

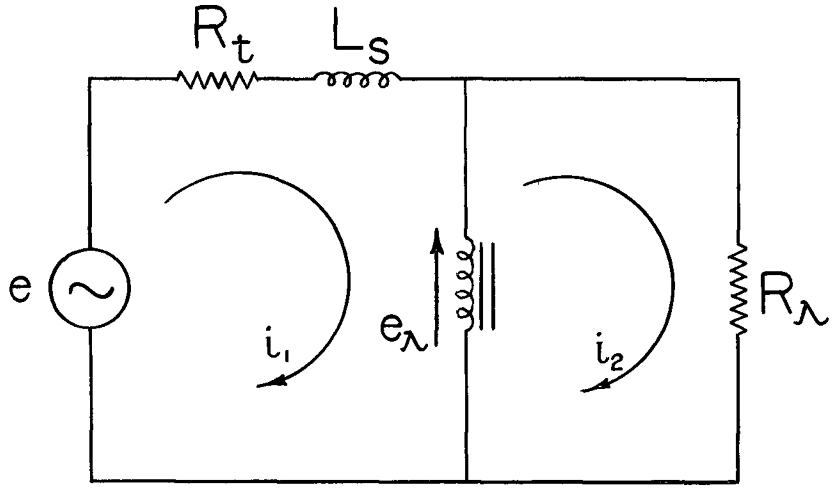


FIG. 3-7

Fig. (3-1) is the familiar representation of a transformer at the end of a 'short-line'; the line is considered short enough so that the shunt capacitance may be neglected. This circuit is often used in the investigation of inrush current phenomena.

Fig. (3-2) represents a series-capacitor compensated version of Fig. (3-1). This circuit although of great interest in certain fields is of not too great importance in this study as usually in energisation of a power transmission line, the series-capacitors are out of circuit.

Fig. (3-3) represents a 'long-line' terminated by a transformer; the line is considered long enough so that the effects of the shunt-capacitance must be taken into account. The lumped element C simulates these effects. Simulation of corona losses is not investigated in this study but introduction of some non-linear element, $W(V_c)$ - is a possible method of so doing.

Fig. (3-4) represents the compensated 'long-line'.

Fig. (3-5) represents a system which includes two transformers. This system is rather typical of the normal arrangement in a transmission system, although both transformers need not be in circuit simultaneously on energisation.

Fig. (3-6) is equivalent to a network when a capacitor potential divider is used for relaying or measuring purposes. The non-linear element in this case is a high-voltage potential transformer.

In each of these representations, the transformer losses can be simulated by the inclusion of a resistor R_T in parallel with the non-linear element.

As observed in sec. 2-2 the use of this incremental method allows the actual hysteresis loop to be closely approximated but eddy-current losses still have to be accounted for. In Fig. (3-7) a resistor R_λ is included to simulate such transformer losses.

The systems represented by Figs. (3-1), (3-3), (3-5), (3-7) are completely solved using the incremental method and numerical examples are worked out using actual network parameters and characteristics.

3-4 The Laplace Transforms and Their Inverses.

In the solutions of the differential equations arising from these circuits, certain Laplace transforms and their inverses are used frequently. For convenience in referring to them, they are tabulated and classified in Appendix-3. This idea also proved useful in programming as the various inverses are programmed as sub-routines, and used when necessary.

These inverses are obtained from two sources[(31), (32)] but as will be noticed in some cases, 'the multiplication-by-s-theorem'[(26)] is used in obtaining some inverses.

3-5 The Equations.

3-5-1 The Uncompensated 'Short-Line' (Fig. 3-1).

Using the following nomenclature

R_t ohms = (sum of source and line resistances) per phase

L_s henries = (sum of source, line and transformer leakage inductances) per phase

e = $E_{max} \sin(\omega t + \theta)$

θ = voltage switching angle,

i = current

λ = total flux linkage in transformer

T = total time

the basic equation is

$$e = iR_t + L_s \frac{di}{dt} + \frac{d\lambda}{dt} \qquad (3\text{-}6)$$

$$\therefore \quad E_{max} \sin(\omega t + \theta) = iR_t + L_s \frac{di}{dt} + \frac{d\lambda}{di} \cdot \frac{di}{dt}$$

$$= iR_t + \frac{di}{dt}\left(L_s + \frac{d\lambda}{di}\right). \qquad (3\text{-}7)$$

Defining

$$\frac{d\lambda}{di} \triangleq L_\lambda(i)$$

Equation (3-16) can be transformed

$$E_{max}\left[\frac{\omega\cos\theta}{s^2+\omega^2} + \frac{s\sin\theta}{s^2+\omega^2}\right] = \left(R_t + s\left(L_s + L_\lambda(i)\right)\right). I(s)$$

$$- \left(L_s + L_\lambda(i)\right) . i(0). \qquad (3\text{-}8)$$

where $i(0)$ = value of i at $t=0$.
and $I(s) = \mathcal{L}\left[i(t)\right]$.
From eq. (3-8),

$$I(s) = \frac{E_{max}}{L_t} \; \frac{\omega\cos\theta + s\sin\theta}{(s^2+\omega^2)(s+\frac{R_t}{L_t})} + \frac{u}{L_t(s+\frac{R_t}{L_t})}. \qquad (3\text{-}9)$$

where

$$L_t \triangleq L_s + L_\lambda(i).$$

and

$$u \triangleq L_t . i(0). \qquad (3\text{-}10)$$

$i(t)$ can be computed from the transform pairs 1 and 3 given in Appendix-3.

The value of $i(t)$ after a small increment of time Δt is computed; this value $i(t)_{t=\Delta t}$ is used to determine

(1) a new value of $L_\lambda(i)$ and hence L_t.

(2) a new value of $i(0)$ and hence u.

θ is increased to $\theta_0 + \omega.T$ where θ_0 = initial voltage switching angle and the process repeated. The corresponding value of

$$e_\lambda = L_\lambda(i) \cdot \frac{di}{dt} \tag{3-11}$$

can be easily computed as it involves simple differentiation of trigonometric functions.

3-5-2 The Uncompensated 'Short-Line' with Simulated Transformer Core Losses.

Using the nomenclature of sec. 3-5-1 with the additional term

$R_\lambda \triangleq$ resistive element to simulate core losses,

circuit equations are

$$E_{max} \operatorname{Sin} (\omega t + \theta) = R_t \cdot i_1 + L_s \cdot \frac{di_1}{dt} + e_\lambda$$

$$= R_t \cdot i_1 + L_s \cdot \frac{di_1}{dt} + L_\lambda (i_1 - i_2) \cdot \left(\frac{di_1}{dt} - \frac{di_2}{dt}\right) \tag{3-12}$$

and

$$0 = -e_\lambda + R_\lambda \cdot i_2$$

$$= L_\lambda (i_1 - i_2) \cdot \left|\frac{di_2}{dt} - \frac{di_1}{dt}\right| + R_\lambda \cdot i_2 \tag{3-13}$$

Transforming eqs. (3-12) and (3-13), with the further definitions

$$I_1 \triangleq \mathcal{L}\left[i_1(t)\right] \tag{3-14}$$

$$I_2 \triangleq \mathcal{L}\left[i_2(t)\right] \tag{3-14a}$$

$$u_1 \triangleq \left|L_s + L_\lambda (i_1 - i_2)\right| \cdot i_1(0) - \left|L_\lambda(i_1 - i_2)\right| \cdot i_2(0) \tag{3-15}$$

$$u_2 \triangleq \left|L_\lambda (i_1 - i_2)\right| \cdot \left|i_2(0) - i_1(0)\right| \tag{3-16}$$

and

$$L_t \triangleq L_s + L_\lambda (i_1 - i_2) \tag{3-17}$$

then

$$\frac{E_{max}}{s^2 + \omega^2} (s \operatorname{Sin} \theta + \omega \operatorname{Cos} \theta) + u_1 = (R_t + sL_t) \cdot I_1 - sL_\lambda \cdot I_2 \tag{3-18}$$

$$u_2 = -sL_\lambda . I_1 + (R_\lambda + sL_\lambda) . I_2 \tag{3-19}$$

Solving these two equations yields

$$I_1(s) = \frac{E_{max}}{L_s} \frac{(s^2 \sin \Theta + \omega s \cos \Theta)}{(s^2 + \omega^2)(s + \alpha)(s + \mu)} + \frac{E_{max}}{L_s L_\lambda} \frac{R_\lambda (s \sin \Theta + \omega \cos \Theta)}{(s^2 + \omega^2)(s + \alpha)(s + \mu)} + \frac{s(u_1 + u_2) + \frac{R_L}{L_\lambda} u_1}{L_s (s + \alpha)(s + \mu)} \tag{3-20}$$

and

$$I_2 = \frac{E_{max}}{L_s} \frac{(s^2 \sin \Theta + \omega s \cos \Theta)}{(s^2 + \omega^2)(s + \alpha)(s + \mu)} + \frac{s(L_t u_2 + L_\lambda u_1) + R_t u_2}{L_s L_\lambda (s + \alpha)(s + \mu)} \tag{3-21}$$

where $-\alpha$ and $-\mu$ are the roots (always real and negative) of the quadratic equation

$$s^2 + s \frac{(R_\lambda L_t + R_t L_\lambda)}{L_\lambda L_s} + \frac{R_t R_\lambda}{L_\lambda L_s} = 0 \tag{3-22}$$

In computing the currents $i_1(t)$ and $i_2(t)$, the transform pairs 5 and 2 in Appendix-3 are used.

Computation of $i_1(t)$ and $i_2(t)$ over a period Δt is carried out and the values $i_1(t)_{t=\Delta t}$ and $i_2(t)_{t=\Delta t}$ are used to determine new values of u_1 and u_2. The process is then repeated.

$$e_\lambda = \left| L_\lambda (i_1 - i_2) \right| . \left| \frac{di_1}{dt} - \frac{di_2}{dt} \right| \text{ can also be computed.} \tag{3-23}$$

3-5-3 Transformer at the Receiving-End of a 'Long-Line' Fig. (3-3)

Nomenclature

L_s henries = (sum of source inductance + ½ total line inductance) per phase.

L_2 henries = (½ total line inductance + transformer leakage inductance) per phase

$R_1 = R_2$ ohms $= \frac{1}{2}$ total line resistance per phase

C farads = lumped shunt capacitance per phase

v_c volts = voltage across the capacitance at $t=0$

λ refers to the transformer

$L_\lambda = \frac{d\lambda}{di_2}$ for some value of i_2.

Differential equations of the system:

$$E_{max} \operatorname{Sin} (\omega t+\theta) = L_s . \frac{di_1}{dt} + R_1 . i_1 + \frac{1}{C}\int(i_1-i_2)\,dt \tag{3-24}$$

$$0 = - \frac{1}{C}\int(i_1-i_2)dt + R_2 . i_2 + L_2 . \frac{di_2}{dt} + e_\lambda$$

$$= - \frac{1}{C}\int(i_1-i_2)dt + R_2 . i_2 + (L_2 + L_\lambda) . \frac{di_2}{dt} \tag{3-25}$$

Transforming these equations and using

$$u_1 \triangleq L_s . i_1(0) \tag{3-26}$$

$$L_q \triangleq L_2 + L_\lambda \tag{3-27}$$

$$u_2 \triangleq L_q . i_2(0), \tag{3-28}$$

equations (3-24) and (3-25) become

$$\frac{s\, E_{max} \sin \theta}{s^2 + \omega^2} + \frac{\omega E_{max} \cos \theta}{s^2 + \omega^2} + u_1 - \frac{v_c}{s} = (r_1 + sL_s + \frac{1}{sC}) . I_1 \; \frac{-I_2}{sC} \tag{3-29}$$

$$u_2 + \frac{v_c}{s} = \frac{-I_1}{sC} + (R_2 + s\, L_q + \frac{1}{sC}) . I_2 \tag{3-30}$$

The determinant Δ_1 of the impedance matrix is given by

$$\Delta_1 = \frac{L_s\, L_q}{s} \left(s^3 + s^2 \left(\frac{L_s\, R_2 + L_q\, R_1}{L_s\, L_q}\right) + s\, \frac{R_1\, R_2 + \frac{L_s + L_q}{C}}{L_s\, L_q} \right.$$

$$\left. + \frac{R_1 + R_2}{CL_s\, L_q} \right) \tag{3-31}$$

which can be put in the form

$$\Delta_1 = \frac{K_1}{s}(s+\mu)\left[(s+\alpha)^2+\beta^2\right] \tag{3-32}$$

μ, α, and β are obtained from the factorization of the cubic

$$s^3 + s^2\left(\frac{L_s R_2 + L_q R_1}{L_s L_q}\right) + s\,\frac{R_1 R_2 + \frac{L_s + L_q}{C}}{L_s L_q} + \frac{R_1 + R_2}{CL_s L_q} \tag{3-33}$$

and will always be real and positive.

The solution of these equations yields,

$$I_1 = \frac{E_{max}\, s\left(\omega R_2 \operatorname{Cos}\theta + \frac{\operatorname{Sin}\theta}{C} - \omega^2 L_q \operatorname{Sin}\theta\right) + \omega\left(\frac{\operatorname{Cos}\theta}{C} - \omega^2 L_q \operatorname{Cos}\theta - \omega R_2 \operatorname{Sin}\theta\right)}{K_1 (s+\mu)\left[(s+\alpha)^2+\beta^2\right](s^2+\omega^2)}$$
$$+ \frac{s^2 L_q u_1 + s(E_{max} L_q \operatorname{Sin}\theta + u_1 R_2 - v_c L_q) + E_{max}(R_2 \operatorname{Sin}\theta + \omega L_q \operatorname{Cos}\theta) +}{K_1 (s+\mu)\left[(s+\alpha)^2+\beta^2\right]}$$
$$+ \frac{\frac{u_1+u_2}{C} - v_c R_2}{K_1 (s+\mu)\left[(s+\alpha)^2+\beta^2\right]} \tag{3-34}$$

$$I_2 = \frac{E_{max}(s \operatorname{Sin}\theta + \omega \operatorname{Cos}\theta)}{K_1 C (s+\mu)\left[(s+\alpha)^2+\beta^2\right](s^2+\omega^2)}$$
$$+ \frac{s^2 L_s u_2 + s(R_1 u_2 + L_s v_c) + \frac{u_1+u_2}{C} + v_c R_1}{K_1 (s+\mu)\left[(s+\alpha)^2+\beta^2\right]} \tag{3-35}$$

$i_1(t)$ and $i_2(t)$ can be computed from the expressions for the inverse transformations as given by 4 and 6 -Appendix 3.

$i_1(t)$ and $i_2(t)$ are evaluated for some small time Δt; the voltages

$$e_\lambda = L_\lambda \frac{di_2}{dt} \tag{3-36}$$

$$v_c = E_{max} \sin(\omega t + \theta) - R_1 i_1 - L_s \frac{di_1}{dt} \tag{3-37}$$

are calculated.

With these values, new values of u_1 and u_2 are then calculated and the process repeated for a new t=0 and $\theta=\theta_0 + \omega T$.

where θ_0 = initial voltage switching angle. (3-38)

3-5-4 Transformers at Both Ends of a 'Long-Line'. Fig. (3-5).

Nomenclature

L_s henries = source inductance per phase.

$R_1=R_2$ ohms = $\frac{1}{2}$ total line resistance per phase.

L_1 henries = $\frac{1}{2}$ total line inductance per phase.

L_2 henries = ($\frac{1}{2}$ total line inductance + transformer leakage inductance)per phase.

C farads = lumped shunt capacitance per phase

v_C volts = voltage across the capacitance at t=0.

i_0, i_1, i_2 = currents in amps. as marked.

σ refers to sending-end transformer--$L_\sigma = \frac{d\lambda_\sigma}{d(i_0-i_1)}$ for some value of i_0-i_1.

ρ refers to receiving-end transformer--$L_\rho = \frac{d\lambda_\rho}{di_2}$ for some value of i_2.

The basic equations are

$$e = (L_s + L_\sigma).\frac{di_0}{dt} - L_\sigma \frac{di_1}{dt} \tag{3-39}$$

$$0 = L_\sigma.\frac{di_0}{dt} + R_1.i_1 + (L_1 + L_\sigma).\frac{di_1}{dt} + \frac{1}{C}\int(i_1-i_2)\,dt. \tag{3-40}$$

$$0 = R_2.i_2 + (L_2 + L_\rho)\frac{di_2}{dt} - \frac{1}{C}\int(i_1-i_2)\,dt. \tag{3-41}$$

Introducing,

$$L_p \triangleq L_s + L_\sigma \tag{3-42}$$

$$L_q \triangleq L_1 + L_\sigma \tag{3-43}$$

$$L_r \triangleq L_2 + L_\rho \tag{3-44}$$

$$u_1 \triangleq (L_s + L_\sigma)\ i_0(0) - L_\sigma\ i_1(0) \tag{3-45}$$

$$u_2 \triangleq (L_1 + L_\sigma)\ i_2(0) - L_\sigma\ i_0(0) \tag{3-46}$$

$$u_3 \triangleq (L_2 + L_\rho)\ i_2(0) \tag{3-47}$$

where $i_0(0)$, $i_1(0)$, $i_2(0)$ are the values of i_0, i_1 and i_2 respectively for any t=0, the transformed equations are

$$\frac{E_{max}\ (s \sin \Theta + \omega \cos \Theta)}{s^2 + \omega^2} + u_1 = s\ L_p\ I_0 - s\ L_\sigma I_1 \tag{3-48}$$

$$-\frac{v_C}{s} + u_2 = -s\ L_\sigma I_0 + (R_1 + s\ L_q + \frac{1}{sC}).I_1 - \frac{I_2}{sC} \tag{3-49}$$

$$+\frac{v_C}{s} + u_3 = 0 - \frac{I_1}{sC} + (R_2 + s\ L_r + \frac{1}{sC}).I_2 \tag{3-50}$$

The determinant Δ_2 of the impedance matrix is given by

$$\Delta_2 = s^3\ (L_p\ L_q\ L_r - L_r\ L_\sigma^{\ 2}) + s^2 \left| L_p\ (R_1\ L_r + R_2\ L_q) - R_2 L_\sigma^{\ 2} \right|$$
$$+ s \left| L_p\ (R_1\ R_2 + \frac{L_q + L_r}{C}) - \frac{L_\sigma^2}{C} \right| + \frac{L_p}{C}\ (R_1 + R_2) \tag{3-51}$$

which can be put in the form

$$\Delta_2 = K_2\ (s+\mu) \left| (s+\alpha)^2 + \beta^2 \right| . \tag{3-52}$$

The values of μ, α and β will always be real and positive. The solution of these equations yields

$$I_0 = \frac{E_{max}\ s^3\ L_q\ L_r\ \text{Sin}\ \Theta}{K_2(s+\mu)\left| (s+\alpha)^2 + \beta^2 \right|\ (s^2+\omega^2)} +$$

$$+\frac{E_{max}\ s\left[s\left(\sin\Theta(R_1L_r+R_2L_q)+\omega\cos\Theta L_qL_r\right)+\sin\Theta\left(R_1R_2+\frac{L_q+L_r}{C}\right)\right]}{K_2(s+\mu)\left[(s+\alpha)^2+\beta^2\right](s^2+\omega^2)}$$

$$+\frac{E_{max}\ s\omega\cos\Theta\ (R_1L_r+R_2L_q)}{K_2(s+\mu)\left[(s+\alpha)^2+\beta^2\right](s^2+\omega^2)}$$

$$+\frac{E_{max}\ \sin\Theta\ \frac{R_1+R_2}{C}+\omega\cos\Theta\left(R_1R_2+\frac{L_q+L_r}{C}\right)}{K_2(s+\mu)\left[(s+\alpha)^2+\beta^2\right](s^2+\omega^2)}$$

$$+\frac{E_{max}\ \omega\cos\Theta\ \frac{R_1+R_2}{C}}{K_2\ s\ (s+\mu)\left[(s+\alpha)^2+\beta^2\right](s^2+\omega^2)} \tag{3-53}$$

$$+\frac{u_1\ \ s^2L_qL_r+s(R_1L_r+R_2L_q)+R_1R_2+\frac{L_r+L_q}{C}}{K_2(s+\mu)\left[(s+\alpha)^2+\beta^2\right]}$$

$$+\frac{u_1\ \frac{(R_1+R_2)}{C}}{K_2\ s\ (s+\mu)\left[(s+\alpha)^2+\beta^2\right]}$$

$$I_1=\frac{E_{max}\ s^3L_rL_\sigma\sin\Theta}{K_2(s+\mu)\left[(s+\alpha)^2+\beta^2\right](s^2+\omega^2)}$$

$$+\frac{E_{max}\ sL_\sigma\left[s(R_2\sin\Theta+\omega L_r\cos\Theta)+\omega R_2\cos\Theta+\frac{\sin\Theta}{C}\right]}{K_2(s+\mu)\left[(s+\alpha)^2+\beta^2\right](s^2+\omega^2)} \tag{3-54}$$

$$+\frac{E_{max}\ \frac{\omega L_\sigma\cos\Theta}{C}}{K_2(s+\mu)\left[(s+\alpha)^2+\beta^2\right](s^2+\omega^2)}$$

$$+ \frac{s^2(L_\sigma L_r u_1 + L_p L_r u_2) + s(R_2 L_\sigma u_1 + R_2 L_q u_2 - v_c L_p L_r) + \frac{L_p(u_2+u_3) + L_\sigma u_1}{C}}{K_2(s+\mu)\left[(s+\alpha)^2 + \beta^2\right]}$$

$$\frac{- v_c L_p R_2}{K_2 (s+\mu) \quad (s+\alpha)^2 + \beta^2} \tag{3-54}$$

$$I_2 = \frac{E_{max} L_\sigma \frac{s \text{ Sin } \Theta + \omega \text{ Cos } \Theta}{C}}{K_2 (s+\mu)\left[(s+\alpha)^2 + \beta^2\right](s^2+\omega^2)}$$

$$+ \frac{s^2 L_q L_p u_3 + s(u_3 R_1 L_p + L_p L_q v_c - L_\sigma^2 v_c) + \frac{u_2 L_p + u_1 L_\sigma + u_3 L_p}{C} + v_c R_1 L_p}{K_2 (s+\mu)\left[(s+\alpha)^2 + \beta^2\right]} \tag{3-55}$$

Using the appropriate transform-pairs of Appendix-3 , i_0, i_1 and i_2 can be evaluated as functions of time and computed over some small increment of time dt.

The voltages e_σ, e_ρ and v_c are given by

$$e_\sigma = e - L_s \frac{di_0}{dt} \tag{3-56}$$

$$e_\rho = L_\rho \cdot \frac{di_2}{dt} \tag{3-57}$$

$$v_c = e_\rho + R_2 i_2 + L_2 \cdot \frac{di_2}{dt} \tag{3-58}$$

The process is repeated as in previous sections.

3-6 Programming Notes.

For results of any accuracy when the incremental method is used, a digital computer is necessary. The solution of the cubic equation; the wide range of numbers utilised (from the very small microfarad values to the large megavolt values); the accurate evaluation of the trigonometric functions, all point to the need for machine computation.

In the actual solution of these problems, two points are to be noted in the programming.

(1) The solution of the cubic equation.

The knowledge that the values of α, β and μ are real and positive in all cases, eliminates any testing of the coefficients of the cubic. The evaluation of α, β and μ follows directly as shown below [33].

The cubic expression

$$s^3 + ps^2 + qs + r \tag{3-59}$$

is considered. α, β and μ is obtained as follows:

$$d \triangleq \frac{3q - p^2}{9} \tag{3-60}$$

$$e \triangleq \frac{2p^3 - 9pq + 27r}{54} \tag{3-61}$$

$$f \triangleq e^2 + d^3 \tag{3-62}$$

$$g = \sqrt[3]{(f - e)} \tag{3-63}$$

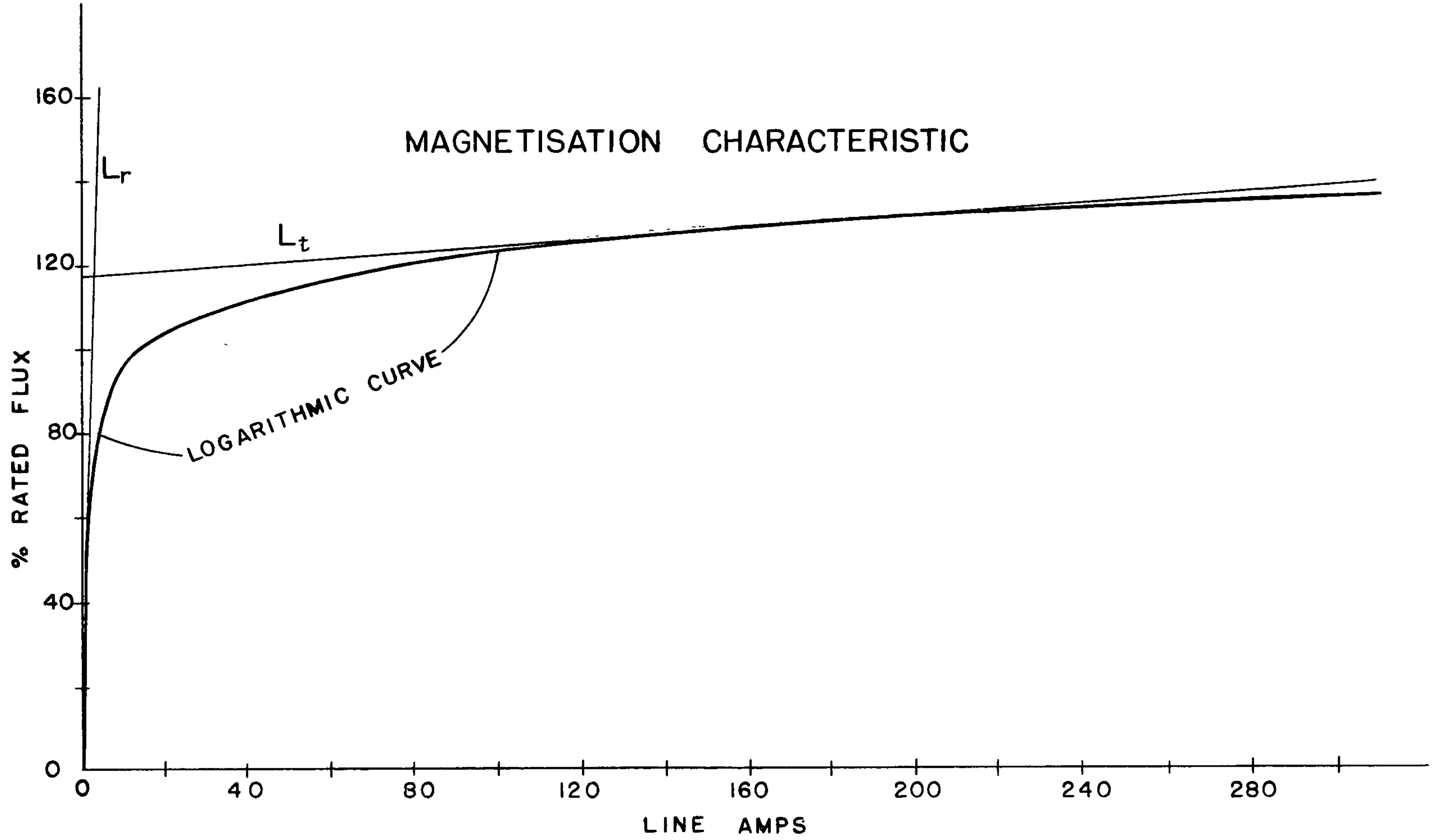
$$h = \sqrt[3]{(-f - e)} \tag{3-64}$$

$$\mu = -g - h + \frac{p}{3} \; ; \; \alpha = \frac{g+h}{2} + \frac{p}{3} \; ; \; \beta = \frac{\sqrt{3}}{2} \cdot (g - h) \tag{3-65}$$

(2) Determination of L_λ from the corresponding value of current.

For each increment of current or time, a value of L_λ has to be determined. The obvious method of doing this is the use of some form of 'table look-up' routine. The complexity of such a routine will depend on the flux-linkage vs. current characteristic and the accuracy required. Considerable time saving can be achieved if the correct routine is used since this routine will have to be used many times in a complete solution.

Using the magnetisation characteristic as shown in Fig. (3-8),



MAGNETISATION CHARACTERISTIC
L_r
L_t
LOGARITHMIC CURVE
% RATED FLUX
160
120
80
40
0
0
40
80
120
160
200
240
280
LINE AMPS

FIG. 3-8

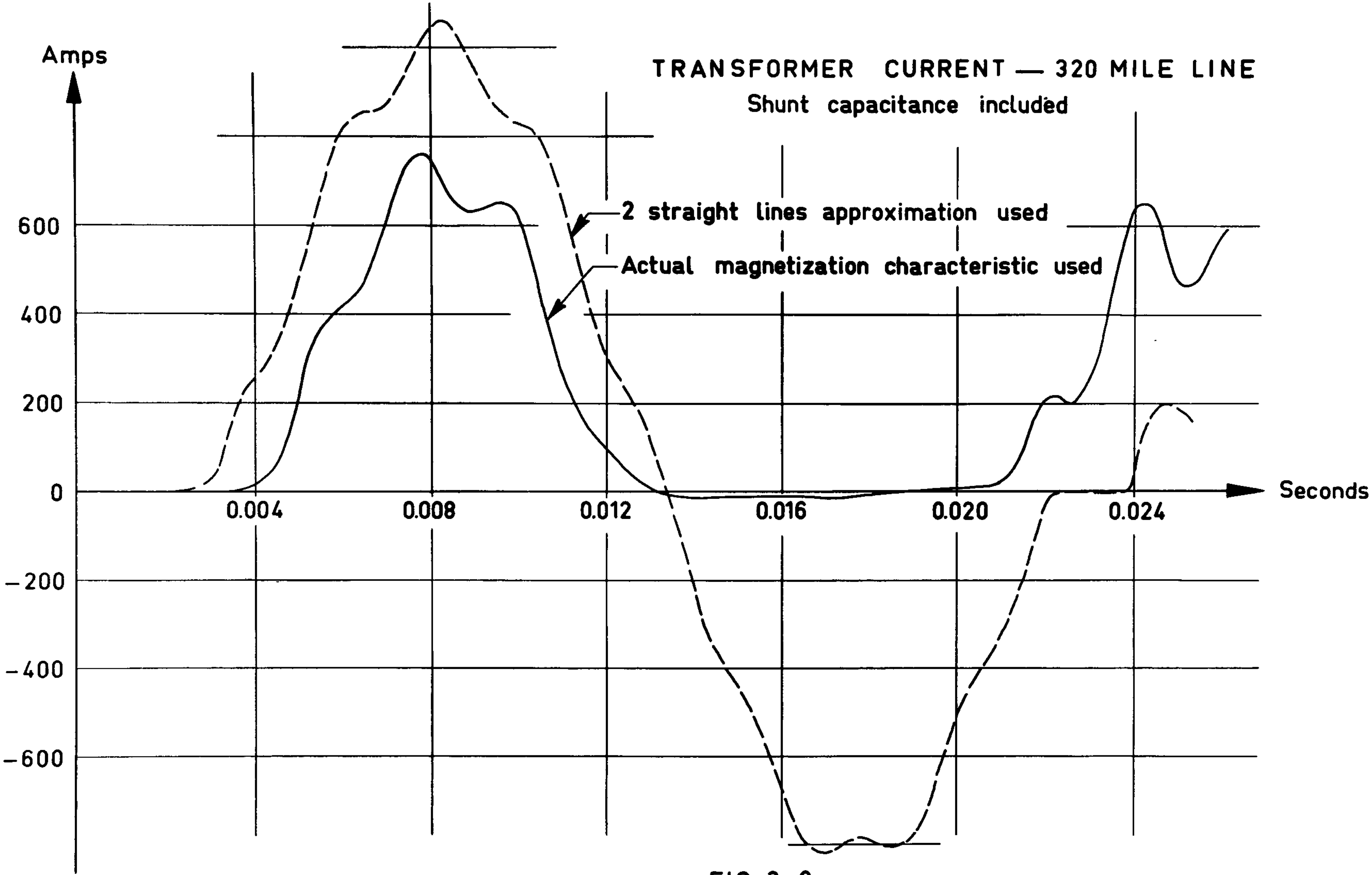

TRANSFORMER CURRENT — 320 MILE LINE
Shunt capacitance included
2 straight lines approximation used
Actual magnetization characteristic used
Amps
600
400
200
0
−200
−400
−600
0.004
0.008
0.012
0.016
0.020
0.024
Seconds

FIG. 3-9

instead of a 'table look-up' routine, a simpler and time saving method is possible. The magnetisation characteristic curve (a) Fig. (3-8) is divided into 3 sections--two straight line sections, joined by a logarithmic curve. For each straight line section, L_λ is a constant and for the curved section, L_λ is determined from the relationship

$$L_\lambda = \frac{\text{constant}}{i} \tag{3-66}$$

This method which is suitable for this particular curve, eliminated a great deal of calculation. In the constant L_λ sections of the curve, all calculations using L_λ are required to be done only once. These values are stored and used as required.

A further simplification is the use of two straight lines to approximate the magnetisation characteristic. Two such lines are marked L_r and L_t in Fig. (3-8). This approximation to the curve (a), Fig. (3-8) is used and the resulting current wave-form over the first cycle are shown in Fig. (3-9)--curve (a). These results are for a 320 mile length of line and corresponding wave-forms obtained using the complete characteristic are plotted on the same graph (curve (b)). The positive peak values are similar but there are wide differences on the negative side of the wave-forms. Since in the incremental method, any computed value of current or voltage depends on the previous computed values, an error in one result may set up a pattern of divergence. For this reason, approximation of the magnetisation characteristic by two straight lines is likely to give incorrect results.

3-7 The Iterative Method.

Iterative methods for the solution of differential equations are well known and well developed but for problems that arise from some physical phenomena, the following requirements are desirable in the choice of a method:

(1) Few approximations in establishing the differential equations in a form for solution.

(2) Close tie between the method and the actual phenomena.

(3) Accurate answers in a reasonable time.

One method that satisfies these conditions in this type of investigation is described in this section but is not used extensively as the incremental method proved far superior.

In this iterative method, the terms involving the non-linear element are separated from the others. The linear elements are then ignored and a solution obtained. A correction computed from the linear elements is then applied to the solution and the process repeated. There are a few variations [34] [35] of this approach but the following example shows the method as used in this investigation.

3-V-1 Transformer Terminating a T-section.- The Iterative Method.

The equivalent circuit is shown in Fig. (3-3).

The transformed equations for this circuit are

$$\mathcal{L}[e] + u_1 - \frac{v_C}{s} = (R_1 + sL_s + \frac{1}{sC}).I_1 - \frac{I_2}{sC} \tag{3-67}$$

$$u_2 + \frac{v_C}{s} = -\frac{I_1}{sC} + (R_2 + sl_2 + \frac{1}{sC}).I_2 + E_\lambda \tag{3-68}$$

where

L_s henries $\triangleq$ (sum of lumped source inductance + $\frac{1}{2}$ total lumped line inductance)per phase

L_2 henries $\triangleq$ ($\frac{1}{2}$ total lumped line inductance + transformer leakage inductance)per phase

$R_1 = R_2$ ohms $\triangleq \frac{1}{2}$ total lumped line resistance per phase

C farads $\triangleq$ lumped shunt capacitance of the line per phase

v_C volts $\triangleq$ voltage across the capacitance at $t=0$

i_1, i_2 amps $\triangleq$ currents as marked

$i_1(0)$ $\triangleq$ value of i_1 at any $t=0$

$i_2(0)$ $\triangleq$ value of i_2 at any $t=0$

$$I_1 = \mathcal{L}\left[i_1(t)\right]$$

$$I_2 = \mathcal{L}\left[i_2(t)\right]$$

$$E_\lambda = \mathcal{L}\left[e_\lambda(t)\right]$$

Eliminating I_1 from eqs. (3-67) and (3-68)

$$E_\lambda + \frac{\Delta_3 \; I_2}{R_1 + sl_s + \frac{1}{sC}} = \frac{\mathcal{L}\left[e(t)\right] + u_1 - \frac{v_c}{s}}{sC(R_1 + sL_s + \frac{1}{sC})} + u_2 + \frac{v_c}{s} \quad (3\text{-}69)$$

where

$$\Delta_3 \triangleq L_s \, L_2 \, s^2 + (R_1 \, L_2 + R_2 \, L_s) \, s + R_1 \, R_2 + \frac{L_s + L_2}{C} \quad (3\text{-}70)$$
$$+ \frac{R_1 + R_2}{sC}$$

The function $e'(t) \triangleq$

$$\mathcal{L}^{-1}\left[\frac{\mathcal{L}\left[e(t)\right] + u_1 - \frac{v_c}{s}}{sC\;(R_1 + sL_s + \frac{1}{sC})} + u_2 + \frac{v_c}{s}\right] \quad (3\text{-}71)$$

is expressed as a function of time.

$$e'(t) = \frac{E_{max}}{L_s C \;\left((\beta_0^2 - \omega^2)^2 + 4\alpha^2 \, \omega^2\right)^{\frac{1}{2}}}\Bigg(\sin\Theta.\ \sin\left(\omega t + \frac{\pi}{2} - \beta_1\right) \quad (3\text{-}72)$$
$$+ \frac{e^{-\alpha t}}{\beta} \cdot \beta_0 . \sin(\beta t + \beta_2 - \beta_3) + \omega \cos\Theta\left(\frac{1}{\omega}.\sin(\omega t - \beta_1) + \frac{e^{-\alpha t}}{\beta}.\sin(\beta t + \beta_3)\right)\Bigg)$$
$$+ \frac{u_1 e^{-\alpha t} \sin\beta t}{L_s \; C \; \beta} - \frac{v_c}{L_s C \beta_0}\left(\frac{1}{\beta_0} + \frac{e^{-\alpha t}}{\beta}.\sin(\beta t + \beta_2)\right) + u_2 d(t) + v_c u(t).$$

where

$$\beta_0^2 \triangleq \alpha^2 + \beta^2. \tag{3-73}$$

$$\beta_1 \triangleq \tan^{-1} \frac{2\alpha\omega}{\beta_0^2 - \omega^2} \tag{3-74}$$

$$\beta_2 \triangleq \tan^{-1} \frac{\beta}{-\alpha} \tag{3-75}$$

$$\beta_3 \triangleq \tan^{-1} \frac{-2\,\alpha\,\beta}{\alpha^2 - \beta^2 + \omega^2} \tag{3-76}$$

$$d(t) = 1 \quad t=0 \qquad\qquad u(t) = 1 \qquad t \geqslant 0 \tag{3-77}$$

$$d(t) = 0 \quad t > 0 \tag{3-78}$$

$$\alpha \triangleq \frac{R_1}{2L_s} \; ; \; \beta \triangleq \left(\frac{1}{L_s C} - \alpha^2\right)^{\frac{1}{2}} \tag{3-79}$$

Ignoring the linear elements, eq. (3-69) is solved.

$$e_\lambda = e'(t) \tag{3-80}$$

$$\frac{d\lambda}{dt} = e'(t) \tag{3-81}$$

therefore,

$$\lambda = \int_0^T e'(t)\, dt \tag{3-82}$$

For some chosen value of T, λ can be determined by a numerical integration process.

With values of λ tabulated over the period of time T, corresponding values of i_2 can then be determined.

From eq. (3-72), the correction term is

$$\Delta\, e'(t) = \mathcal{L}^{-1}\left[\frac{\Delta_3 \cdot I_2}{R_1 + sL_s + \frac{1}{sC}}\right] \tag{3-83}$$

Defining

$$f(t) \triangleq \mathcal{L}^{-1}\left[\frac{\Delta_3}{R_1 + sL_s + \frac{1}{sC}}\right] \tag{3-84}$$

from Appendix-3 and eq. (3-70)

$$f(t) = L_2 \left[d(t) + \frac{R_2 e^{-\alpha t}}{\beta L_2} (a_0 - \alpha)^2 + \beta^2 \sin(\beta t + \beta_4) + \frac{R_2 + R_1}{CL_s L_2 \beta_0} (\frac{1}{\beta_0} + \frac{1}{\beta} e^{-\alpha t} \sin(\beta t - \beta_2) \right] \quad (3\text{-}85)$$

where

$$a_0 = \frac{R_1}{L_s} + \frac{1}{R_2 C} \text{ and } \beta_4 = \text{arc tan} \frac{\beta}{a_0 - \alpha} \quad (3\text{-}86)$$

Using the convolution theorem, $\Delta e'(t) = \int_0^T i_2(t).f(T-t)\,dt$ (3-87)

values of $\Delta e'(t)$ can be tabulated as a function of time.

These correction values are subtracted from $e'(t)$ and a new set of values for $e_1'(t) = e'(t) - \Delta e'(t)$ found.

The process is then repeated, starting from eq. (3-82) and a successful iterative procedure depends on decreasing values of $\Delta e'(t)$.

This method, admirably suited for machine computation has in many cases the advantage of speed over the incremental method. However, the value of T is critical. Too large a value of T leads to divergence[34] and incorrect results. If this condition occurs, then the process can be stopped and a smaller value of T chosen. To obtain a continuous solution, after a successful iteration for some value of T, with new initial conditions, the process is repeated.

In the numerical example attempted, it was found that the maximum suitable values of T were so small that any advantage of speed over the incremental method was lost.

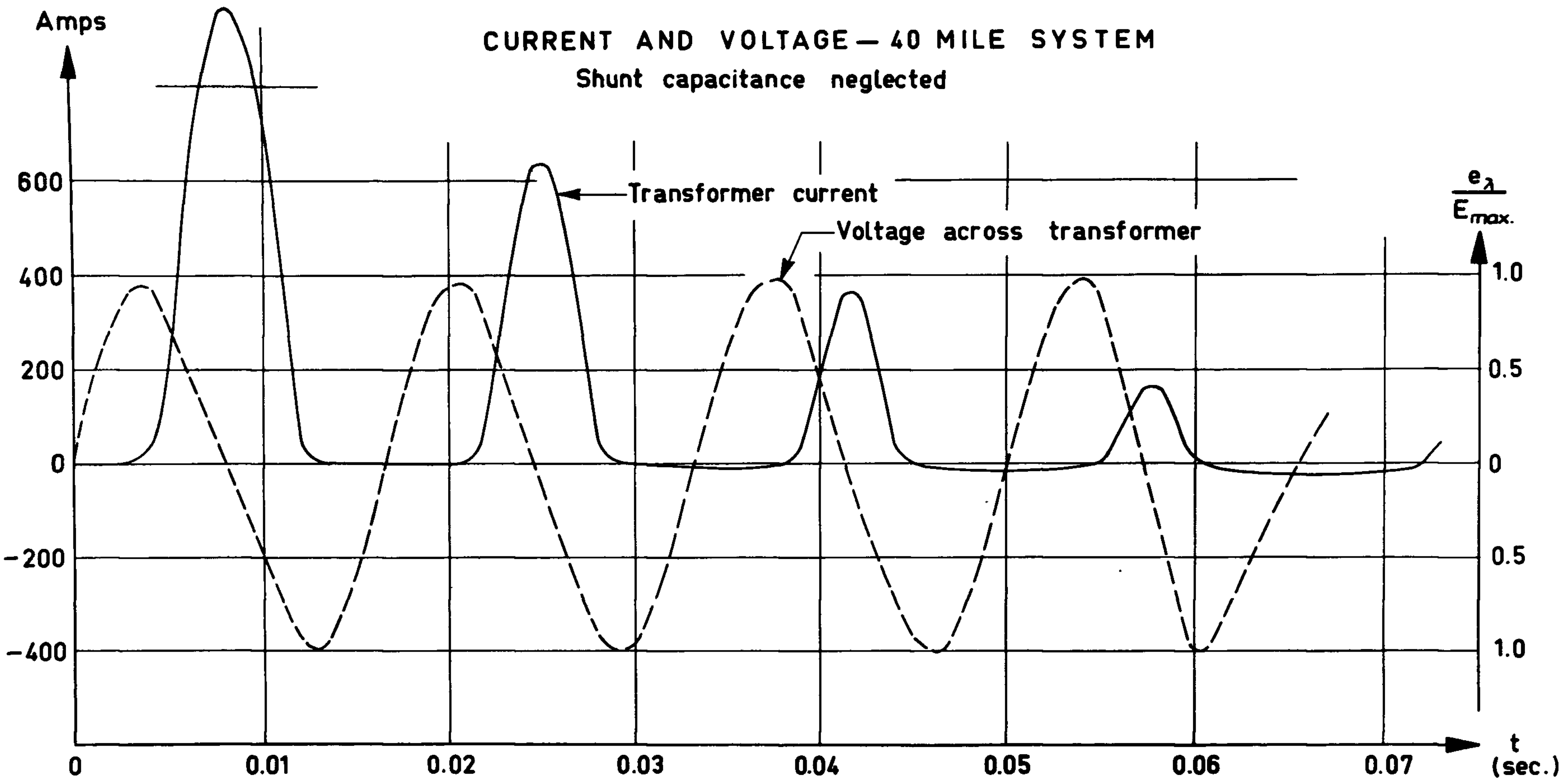

CURRENT AND VOLTAGE — 40 MILE SYSTEM
Shunt capacitance neglected
Amps
600
400
200
0
−200
−400
Transformer current
Voltage across transformer
$\frac{e_\lambda}{E_{max.}}$
1.0
0.5
0
0.5
1.0
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
t
(sec.)

FIG. 3–10

3-8 Discussion of Results.

Solutions of voltages and currents for some specific examples are obtained for a two-fold purpose. Some general ideas about the phenomenon itself can be obtained and secondly, some detailed procedures about the use of the different equations, equivalent circuits, programming techniques etc. . . can be established.

In this study, greater emphasis is on the second objective. The use of the techniques established in the investigation of various aspects of system behaviour will not be considered in any detail.

All results for this discussion are obtained using the incremental method.

3-8-1 The 'Short-Line' Current and Voltage Wave-Forms.

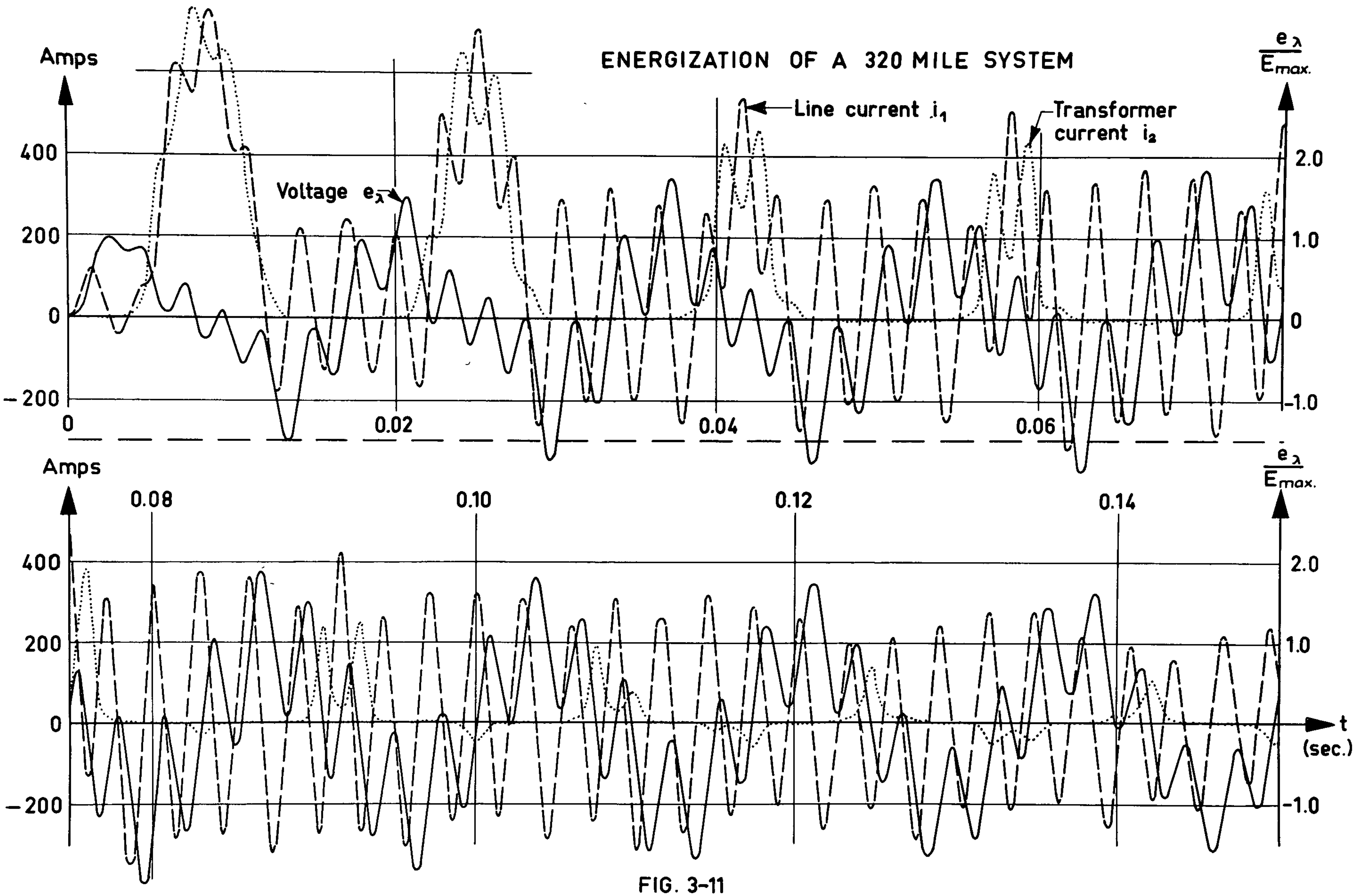
Fig. (3-10) shows the current and voltage wave-form for a 40 mile system. The 'short-line' equivalent circuit is used - Fig. (3-1), and the system is energised when the instantaneous value of the sinusoidal driving voltage is zero with positive slope. i.e. $\theta=0$ at $t=0$.

The current wave-form displays high initial positive peak values and low initial negative peak values. The decay of the current peaks as t increases is also evident.

The voltage wave-form displays no peak values greater than E_{max}. This is to be expected from the equivalent circuit used.

3-8-2 The 'Long-Line' Current and Voltage Wave-Forms.

Using the 'Long-Line' equivalent T-section (Fig. 3-3), solution wave-forms are obtained for the energisation of a 320 mile system.



ENERGIZATION OF A 320 MILE SYSTEM
Amps
Line current i_1
Transformer current i_2
Voltage e_λ
$\frac{e_\lambda}{E_{max.}}$
400
200
0
−200
2.0
1.0
0
−1.0
0
0.02
0.04
0.06
0.08
0.10
0.12
0.14
t (sec.)

FIG. 3-11

Voltage across Receiving End Transformer / $E_{max.}$

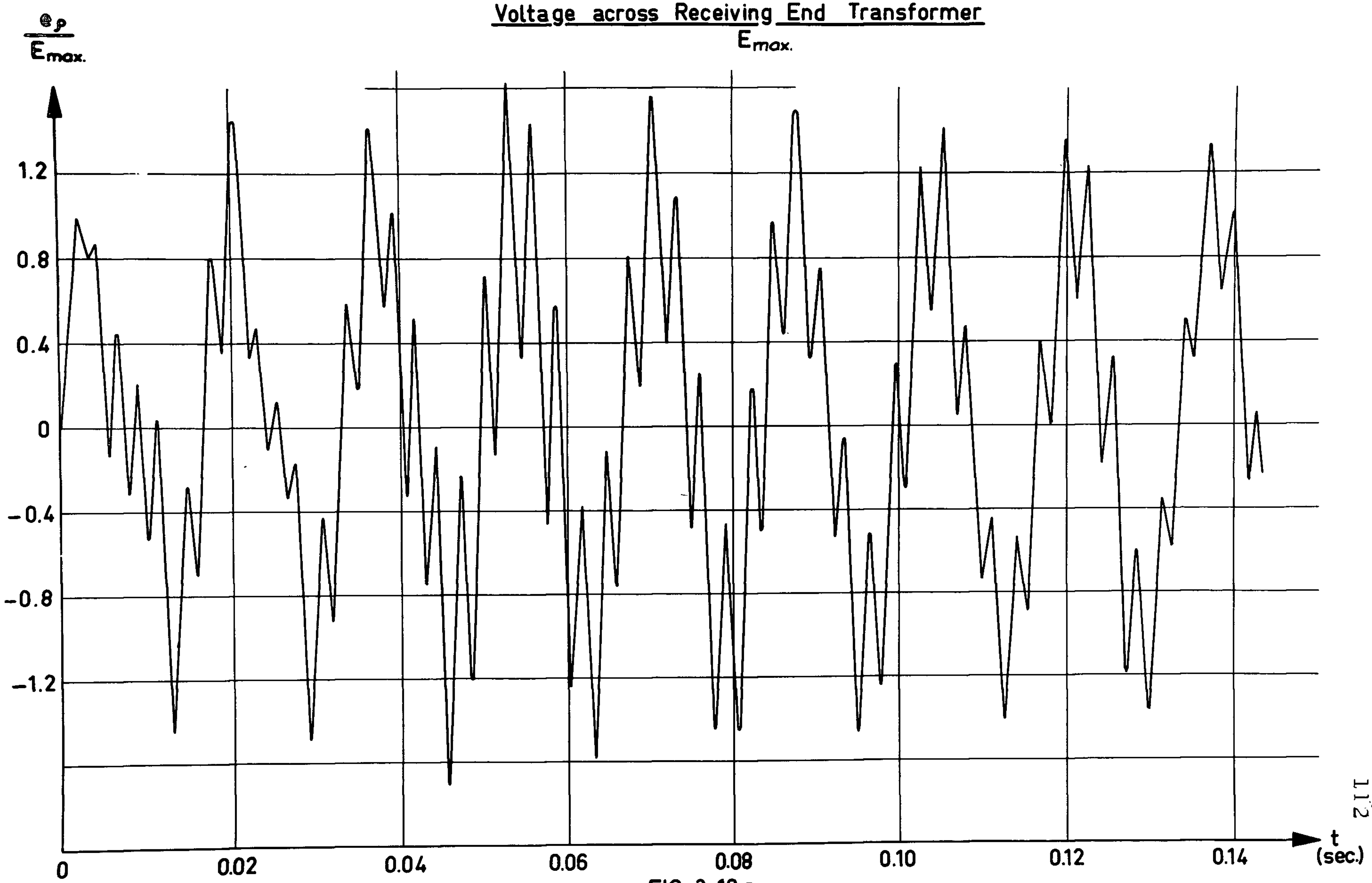


FIG. 3-12 a

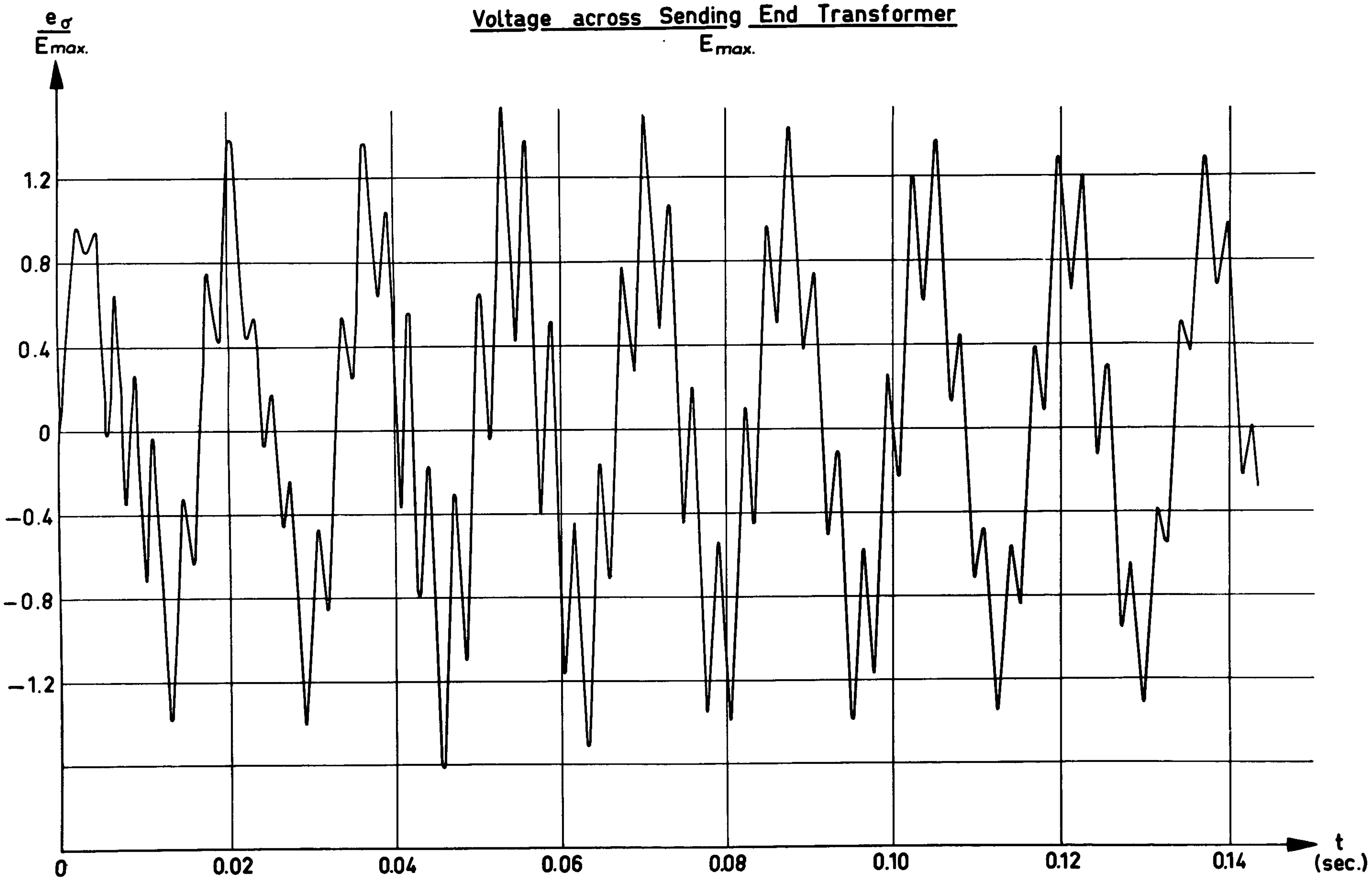

Voltage across Sending End Transformer
$\frac{e_\sigma}{E_{max.}}$
$E_{max.}$
1.2
0.8
0.4
0
−0.4
−0.8
−1.2
0
0.02
0.04
0.06
0.08
0.10
0.12
0.14
t
(sec.)

FIG. 3-12 b

Fig. (3-11) shows the wave-forms of the line current 'i_1', the transformer current 'i_2' and the voltage across the transformer e_λ. The system is considered to be energised again at $\theta=0$ for $t=0$.

Both current wave-forms display the familiar inrush current phenomenon, reaching almost the same peak values but there are distinct differences in the wave-shapes.

Peak values of the transformer terminal voltage are of the order of 1.6 E_{max} and values greater than E_{max} persist throughout the oscillations. This phenomenon of overvoltages is experienced in the energisation of long-lines and their persistence can be dangerous.

3-8-3 Energisation of a System With a Transformer at Both Ends of the Line.

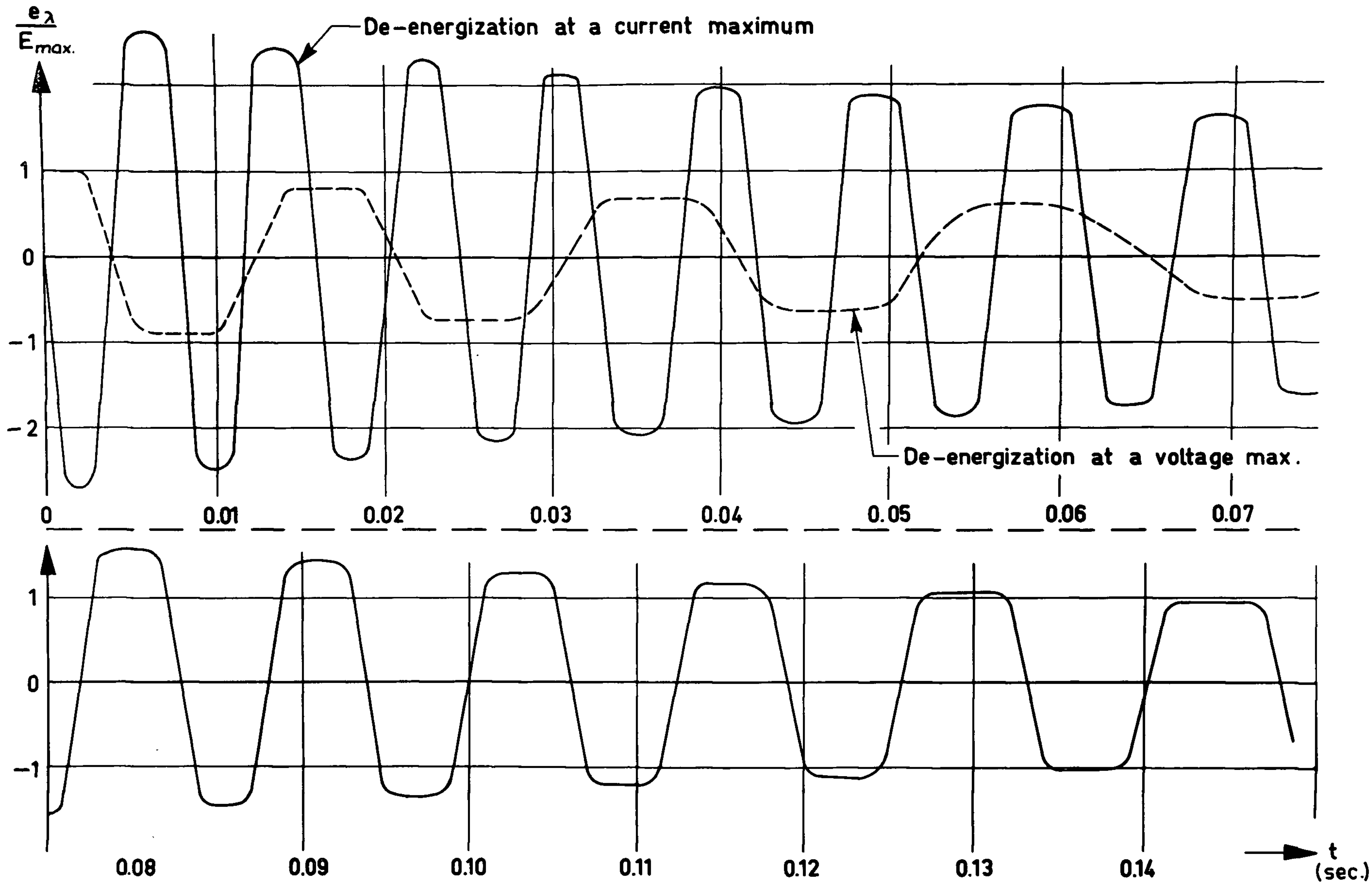
Fig. 3-6, shows the equivalent circuit for a 'long-line' with transformers at either end of a 320 mile line.

Energisation of such a system with a $\theta=0$, $t=0$ initial condition, results in the voltage wave-forms displayed in Figs. (3-12a) and (3-12b). e_ρ refers to the receiving-end transformer and e_σ to the sending-end transformer.

In both cases, peak values of over 1.6 E_{max} are experienced and these overvoltages persist for some time. Associated with these overvoltages are high - frequency oscillations in the transient state.

3-8-4 De-energisation of a 'Long-Line' System.

When a system such as that represented by Fig. (3-3) is de-energised, transient overvoltages may be experienced depending on the initial conditions.



$\frac{e_\lambda}{E_{max.}}$
De-energization at a current maximum
De-energization at a voltage max.
1
0
-1
-2
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
1
0
-1
0.08
0.09
0.10
0.11
0.12
0.13
0.14
t
(sec.)

FIG. 3-13

A 320 mile system is considered to be de-energised with two different sets of initial conditions.

(1) $t = 0$

Transformer voltage $e_\lambda = 0$

Transformer current $i_2 = 750$ amps. (maximum transient value).

(2) $t = 0$

Transformer voltage $e_\lambda = 1.1\ E_{max}$

Transformer current $i_2 = 0$.

Fig. (3-13) shows the transformer voltage wave-forms. Curve (a) corresponds to conditions (1). Overvoltages exceeding $2\ E_{max}$ are experienced and these decrease as t increases. Curve (b), corresponding to conditions (2), shows a decay of the voltage oscillations and no overvoltages are experienced.

3-8-5 The Transient Current Inrush.

The term 'inrush current' is often used in describing the transient current of a system which consists of a transmission line terminated by an unloaded transformer.

If the 'short-line' equivalent circuit is used - Fig. (3-1) - this term refers to current 'i'. If however, the 'long-line' equivalent circuit is used, two currents 'i' and 'i_2' then exist and the term 'inrush current' can be misleading as the behaviour of these two currents differs in some respects.

If the behaviour of the transient current for such a system is an important system design criterion, then it becomes important to distinguish between these currents.

In this study, when a T-section equivalent circuit is used,

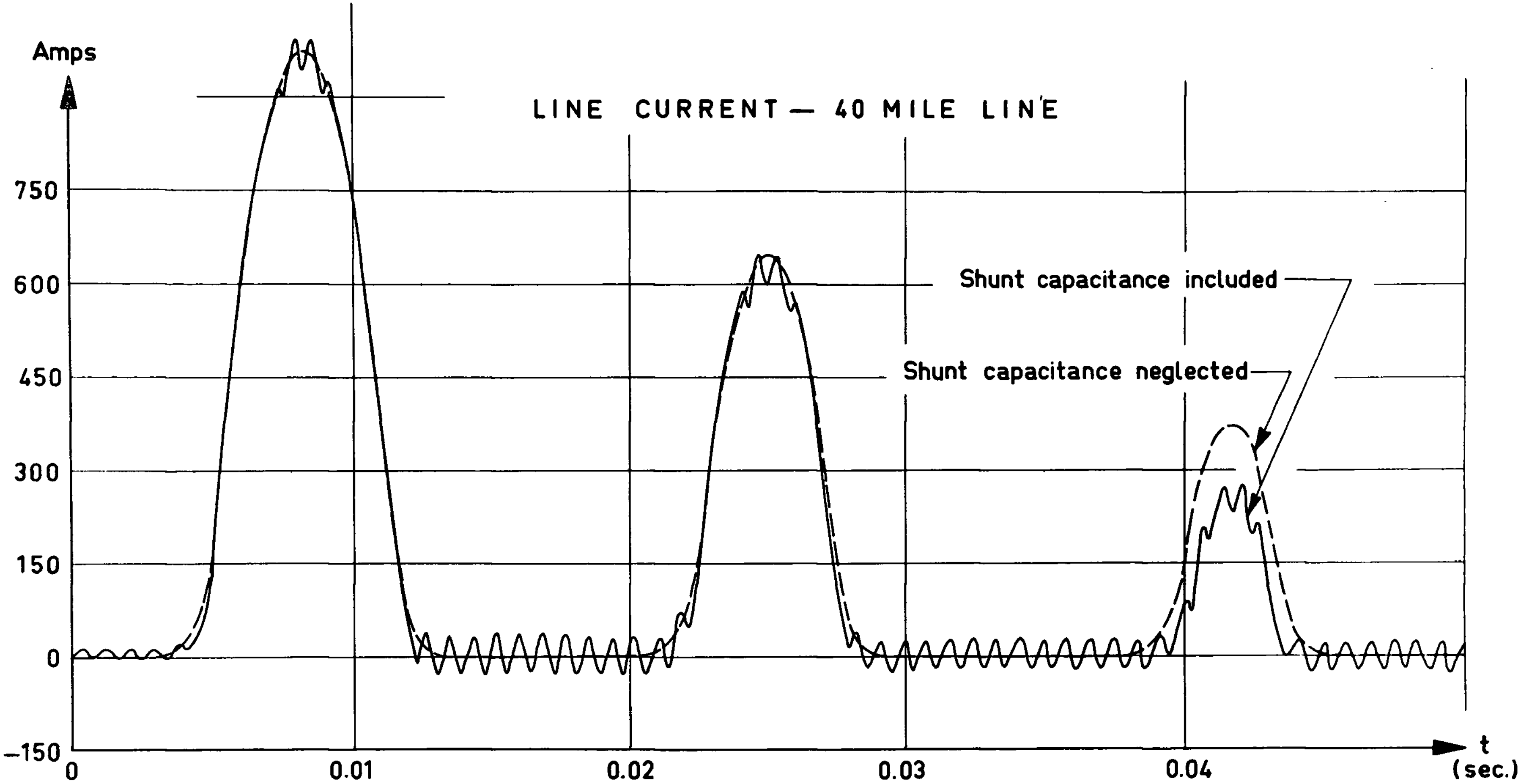

LINE CURRENT — 40 MILE LINE
Amps
750
600
450
300
150
0
−150
0
0.01
0.02
0.03
0.04
t
(sec.)
Shunt capacitance included
Shunt capacitance neglected

FIG 3–14

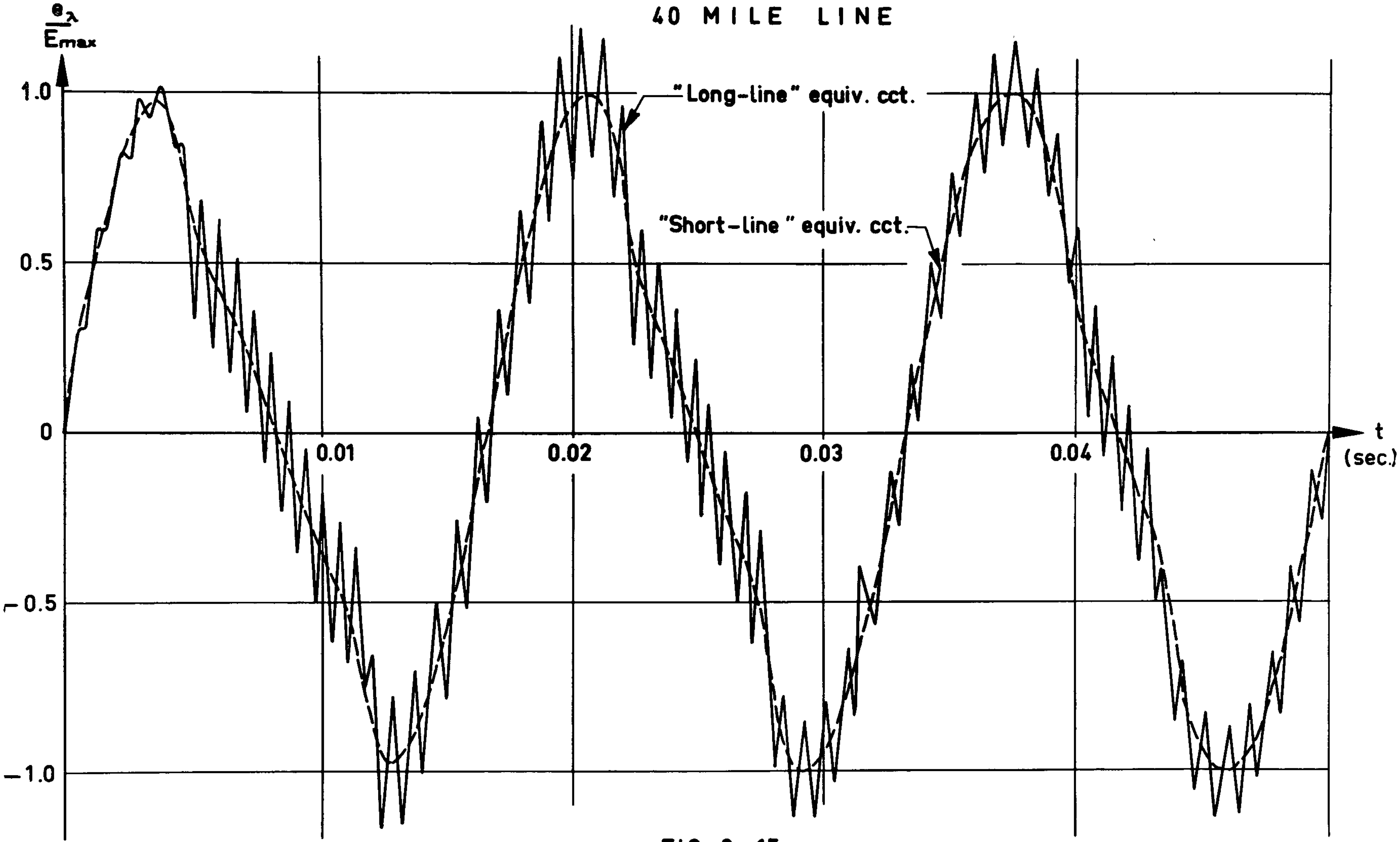

40 MILE LINE
$\frac{e_\lambda}{E_{max}}$
1.0
0.5
0
-0.5
-1.0
0.01
0.02
0.03
0.04
t (sec.)
"Long-line" equiv. cct.
"Short-line" equiv. cct.

FIG. 3-15

LINE CURRENT — 320 MILE LINE

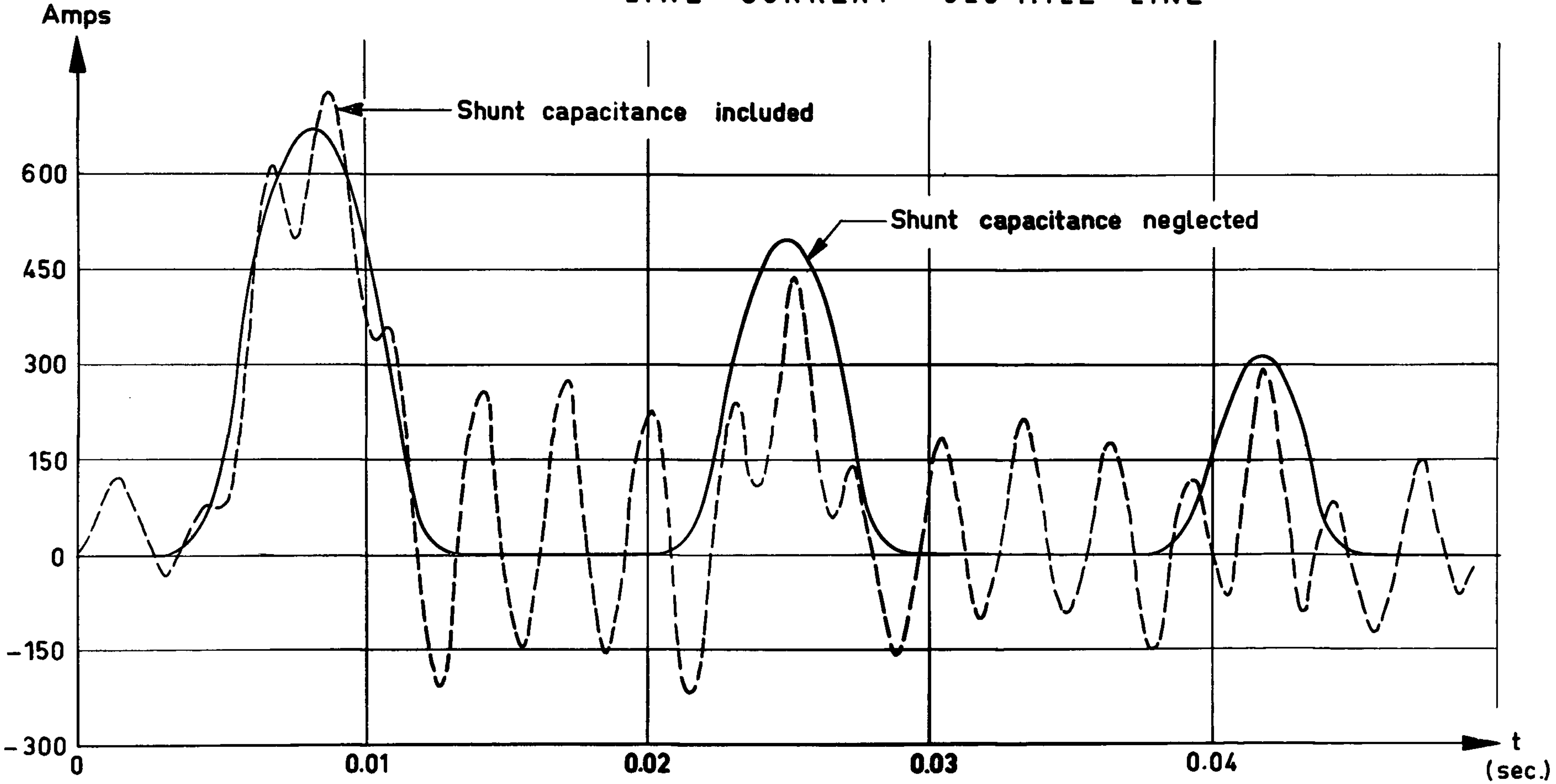
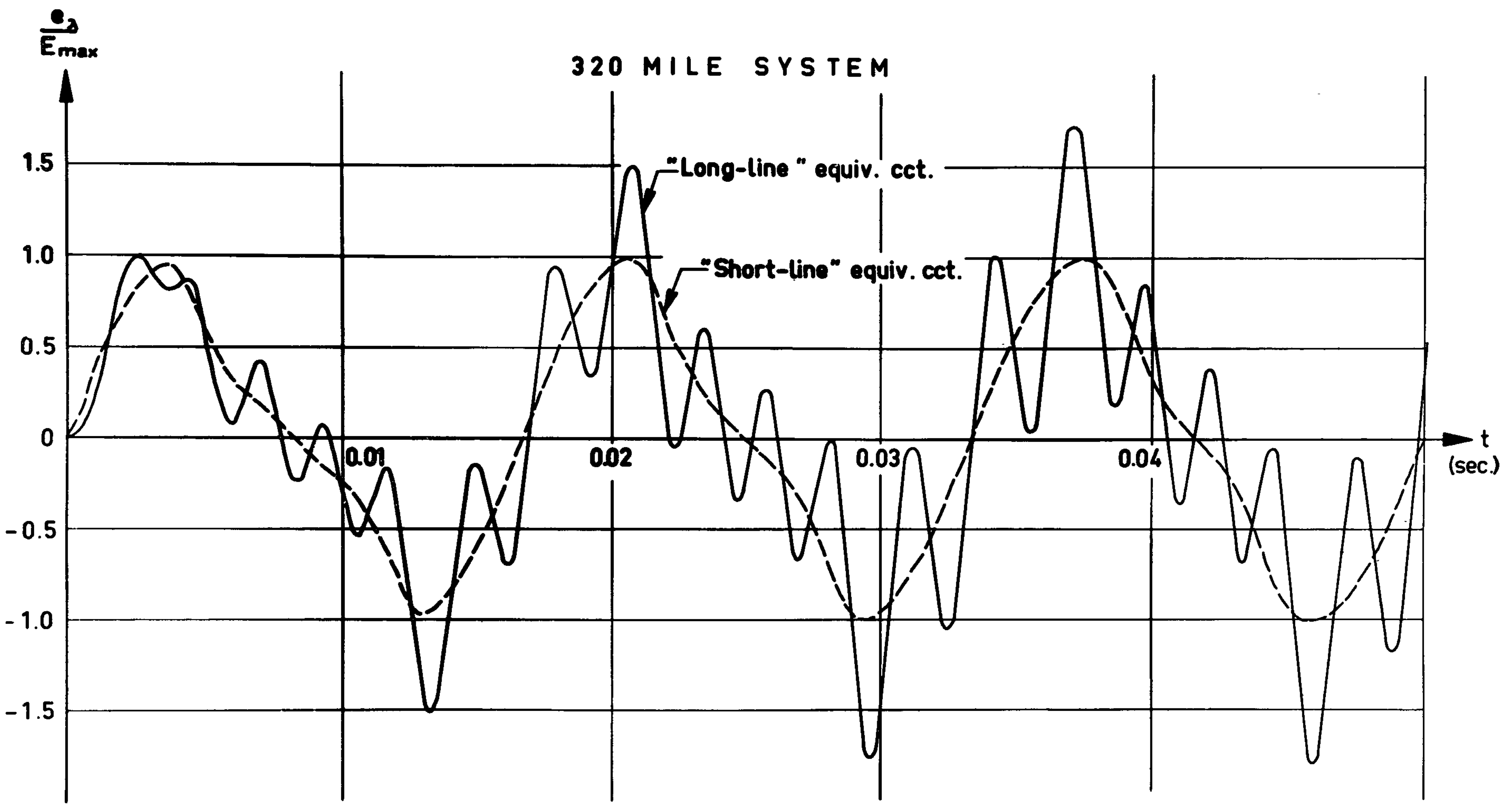


FIG. 3–16



$\frac{e_\lambda}{E_{max}}$
320 MILE SYSTEM
"Long-line" equiv. cct.
"Short-line" equiv. cct.
1.5
1.0
0.5
0
-0.5
-1.0
-1.5
0.01
0.02
0.03
0.04
t
(sec.)

FIG. 3—17

'i_1' will be referred to as the line inrush current and the term transformer inrush current will refer to i_2.

3-8-6 The 'Short-Line' and the 'Long-Line'.

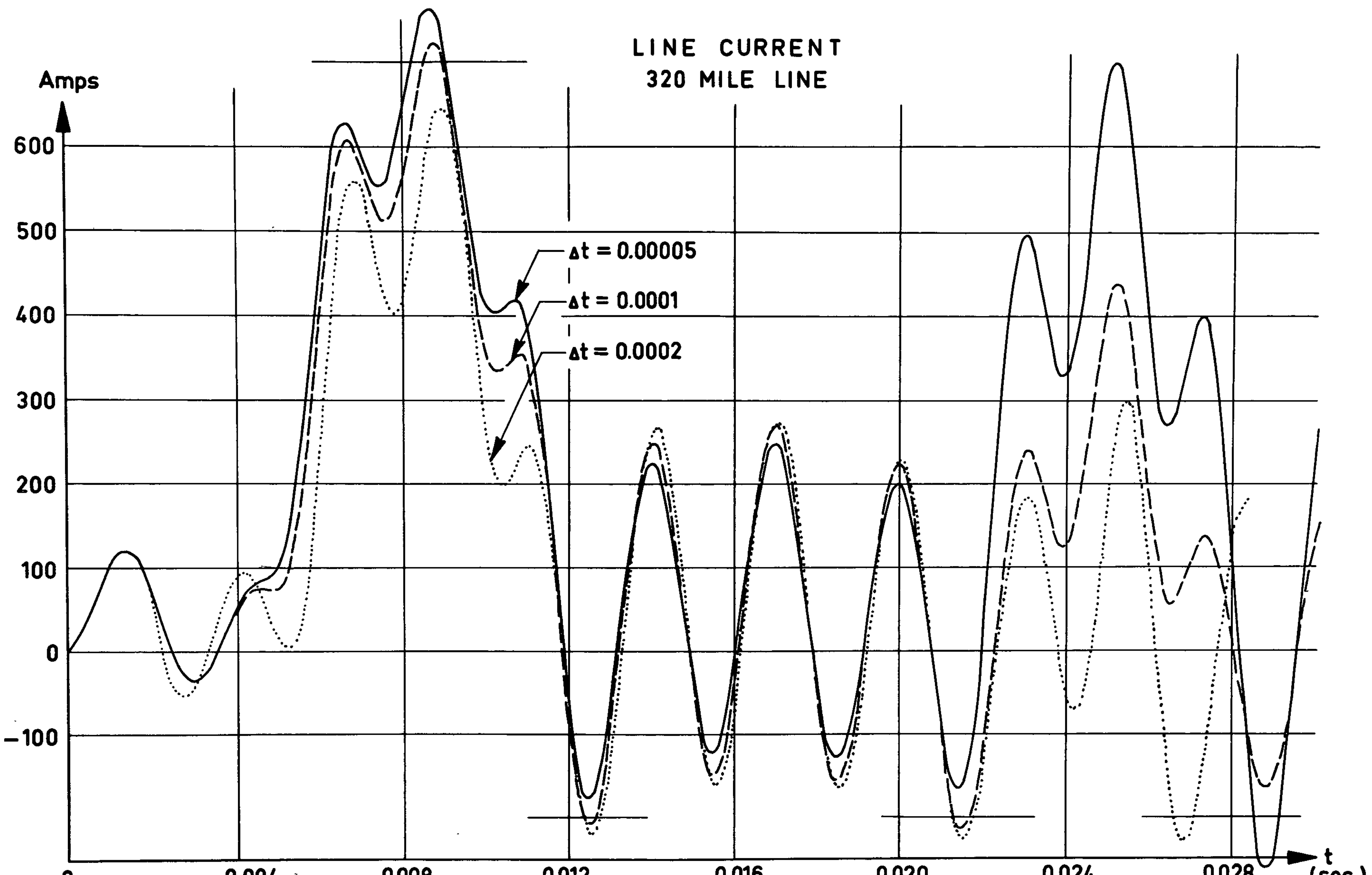
Figs. (3-1) and (3-2) show the equivalent circuits for the 'short-line' and 'long-line' systems respectively. In the first case, the line is considered short enough so that shunt capacitance may be neglected; in the second case, a lumped capacitance at the middle of the line is used to simulate the effects of the line shunt capacitance.

To offer some appreciation of the differences between these two representations, a 40 mile system and a 320 mile system are solved using one representation, then the next.

Fig. (3-14) shows the line current wave-forms for a 40 mile system. There are no appreciable practical differences between these wave-forms and certainly the important peak values are nearly identical. The corresponding voltage wave-forms are shown in Fig. (3-15). The differences in this case are more pronounced but not large enough to be of concern.

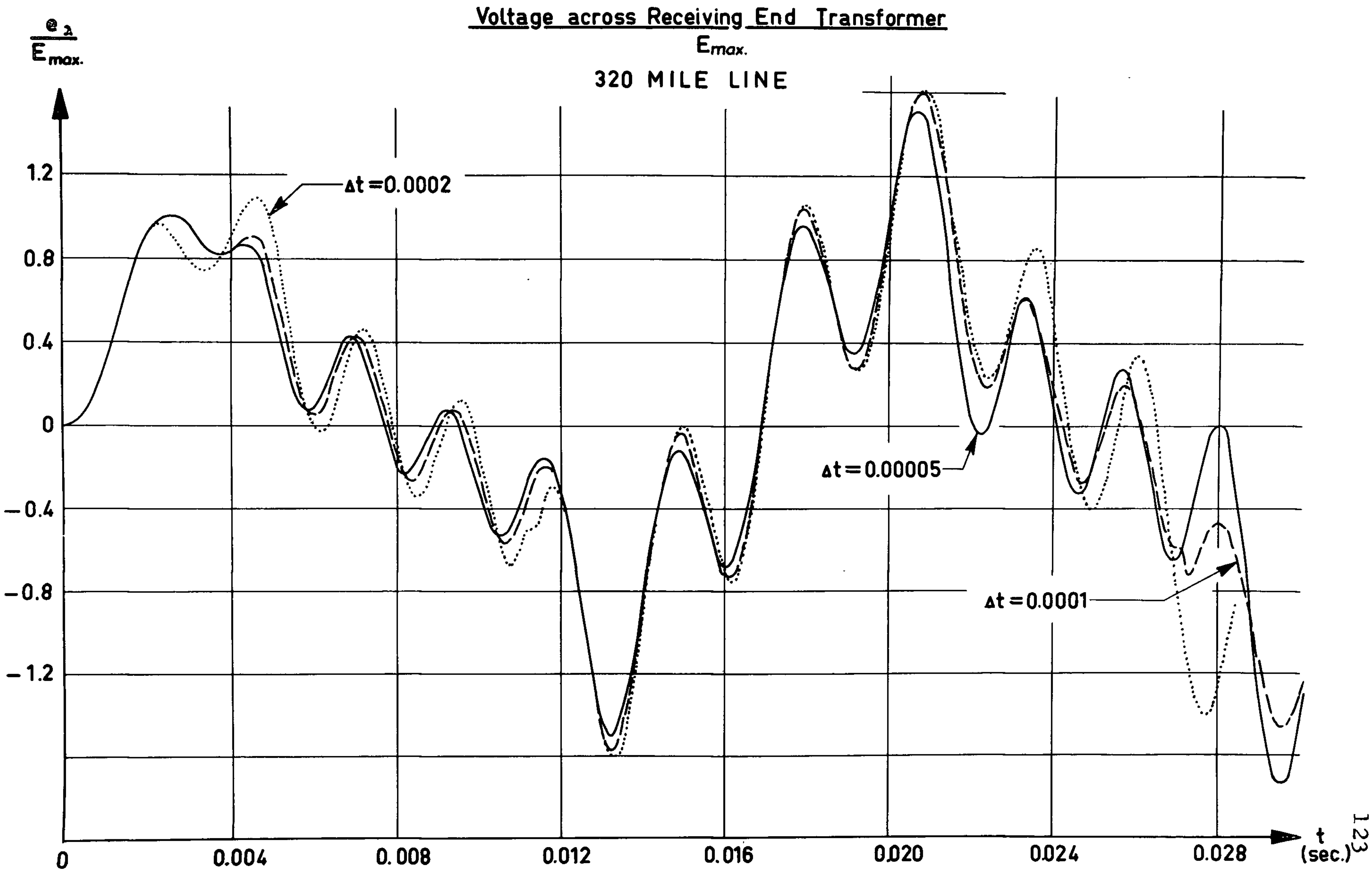
Similar wave-forms are shown in Figs. (3-16) and (3-17) for a 320 mile system. The differences in the current wave-forms are in shape rather than in the more important quantity of peak values. The voltage wave-forms are, however, quite different both in shape and amplitude. Inspection of the 'short-line' equivalent circuit reveals that no transformer voltages greater than E_{max} are possible whereas, in fact, transformer voltages of the order of 1.6 E_{max} are experienced.

These considerations indicate that if transformer voltages



LINE CURRENT
320 MILE LINE
Amps
600
500
400
300
200
100
0
−100
Δt = 0.00005
Δt = 0.0001
Δt = 0.0002
0
0.004
0.008
0.012
0.016
0.020
0.024
0.028
t
(sec.)

FIG. 3-18



Voltage across Receiving End Transformer
Emax.
320 MILE LINE
e λ
Emax.
1.2
0.8
0.4
0
-0.4
-0.8
-1.2
Δt=0.0002
Δt=0.00005
Δt=0.0001
0
0.004
0.008
0.012
0.016
0.020
0.024
0.028
t
(sec.)

FIG. 3-19

are important in a particular investigation, the 'long-line' equivalent circuit is necessary for reasonable results. Another pertinent factor is that in this particular study, use of the 'long-line' equivalent circuit rather than the 'short-line' requires approximately three times as much computing time on a digital computer.

3-8-7 The Accuracy of the Incremental Method.

The accuracy of the incremental method depends almost entirely on small enough current changes over each interval. If the programming is arranged so that these current changes are extremely small to ensure maximum accuracy, the computation time involved may become excessive.

In all the programmes that use the incremental method, a time increment rather than a current increment is used. This choice is based solely on programming convenience. The value of the time increment then becomes the criterion for accuracy and speed of computation.

A 320 mile system is solved using three different time increments - $\Delta t = 0.0002$ sec.; $\Delta t = 0.0001$ sec.; $\Delta t = 0.00005$ sec. Figs. (3-18) and (3-19) show the resulting current and voltage wave-forms for the system.

A time increment of 0.00005 sec. gives the most accurate results but on the other hand, computation time for a solution is twice the time needed for a 0.0001 sec. increment. The time increment of 0.0002 sec. is too large and gives inaccurate results.

Since computation time is an important factor in digital computer techniques, it is necessary to carry out some preliminary investigations to decide upon the optimum time increment. Its value may change not only from system to system but may also be influenced by the stage of behaviour being studied.

3-9 Computation Procedures.

The numerical results of chapters 2 and 3 are computed by digital computer programmes.

With the exception of the programme used for the finding of polynomial zeros, all programmes were written especially for this study.

The programme for determining the zeros of the polynomials is based on Bairstow's method and a slightly modified version is used to suit the wide range of coefficients that occurred in this study.

An I.B.M. 1620 machine was used for most of the computation and the programmes were written in the language of Fortran 1A.

Most of the programmes are written in a general form so that they may be available to other researchers in this field. One set of input data cards is necessary for all the programmes based on the incremental method; the programme make use of the necessary data, ignoring the rest.

For efficient, accurate and quick plotting of the results, an X-Y plotter and tape converter were utilised. These devices removed a great deal of the tediousness of graph-plotting.

For certain utility routines, preliminary runs and checks, an Alwac III-E machine was used and this was programmed in machine language.

CHAPTER 4

4-1 Introduction

There are many advantages in an analytical solution of a non-linear differential equation. A good solution in algebraic form gives a better insight into the range of behaviour of the system without the need for tedious numerical calculations and the effects of change of parameters and new initial conditions can be better visualized. These reasons justify the efforts spent in searching for analytical methods which suit a particular problem.

There has been widespread interest in non-linear analysis and some very useful methods have been developed for the solution of various types of differential equations. The work done by Kryloff and Bogoliuboff, Minorsky etc. is finding application in all fields.

The concepts and techniques used in engineering analysis can be broadly classified into two groups: those of a general nature applicable at all times and those that are only valid in linear analysis. In non-linear analysis, the general concepts and techniques are immediately applicable but the familiar linear techniques have to be examined for validity and in most cases, a more general form must be determined before application.

Methods of particular usefulness in non-linear analysis have been developed but unlike linear analysis where differential equations can be easily classified into groups, each particular equation has to be treated individually, in either of two ways: a completely new method may be developed to suit that equation or various well developed methods may be tried in search for a reasonable solution.

The latter approach is adopted in this study and the procedures for three different methods applicable to this type of problem are established. Each method is particularly useful over a definite range of behaviour and numerical examples demonstrate this usefulness.

4-2 The Flux-Linkage vs Current Relationship.

An important requirement before an analytical solution is sought is the expressing of all relations describing the physical system in analytical terms. The use of the linear circuit concept in itself does this for all linear behaviour but for non-linear phenomena, some additional mathematical relations must be established. In the problem of interest, the non-linearity is introduced by the flux-linkage vs current relationship.

The first approximation is the use of the single-valued magnetisation characteristic--$i(\lambda)$--instead of the multi-valued flux-linkage vs current relationship. Various expressions have been used to describe the magnetisation characteristic of a transformer. These include polynomial expressions, exponential and hyperbolic functions. The use of one function or another obviously depends on the shape of the characteristic but in some cases, a particular function is used with the possible sacrifice of some accuracy to take advantage of some special analytical technique. For example,

$$i = a\, e^{k\lambda} \tag{4-1}$$

although it is not a good approximation for the magnetisation characteristic near the origin and for negative λ values can[36] and will be used with reasonable success.

One of the more successful approximations to the functions $i(\lambda)$ is the polynomial fit

$$i = a_1\lambda + a_n\lambda^n \tag{4-2}$$

where n is some positive integer.

There are three main reasons for this success:

(1) Such a function gives a reasonably good fit to most magnetisation characteristics.

(2) a_1 and a_n are easily determined.

(3) In many cases, the use of this expression results in a differential equation to which well developed solution techniques are applicable.

The inclusion of additional terms of the form $a_k \lambda^k$ is sometimes considered necessary but the techniques employed in the solutions are the same although the algebra and trigonometry become somewhat more involved.

In this study, except for one instance, it will be assumed that the flux-linkage current relationship can always be expressed by the relationship

$$i = a_1\lambda + a_n \lambda^n \quad .$$

Sec.(4-6)deals with the relationship given by eq. (4-1) in the discussion of transient current inrush phenomena.

4-2-1 Determination of a_1 and a_n.

For a given set of values of λ and i, many methods exist for the determination of the constants a_1 and a_n for a particular value of n.

The art of interpolation is quite a developed one and many of these elegant and accurate methods make very good use of the facilities of a digital computer.(37) Since the intention behind this chapter is the keeping of numerical calculations to the minimum, the method suggested here is one of the simplest and most familiar--the method of least squares(38).

FLUX LINKAGE VERSUS CURRENT RELATIONSHIP

Magnetization characteristic and approximations of the form $i = a_1\lambda + a_n\lambda^n$

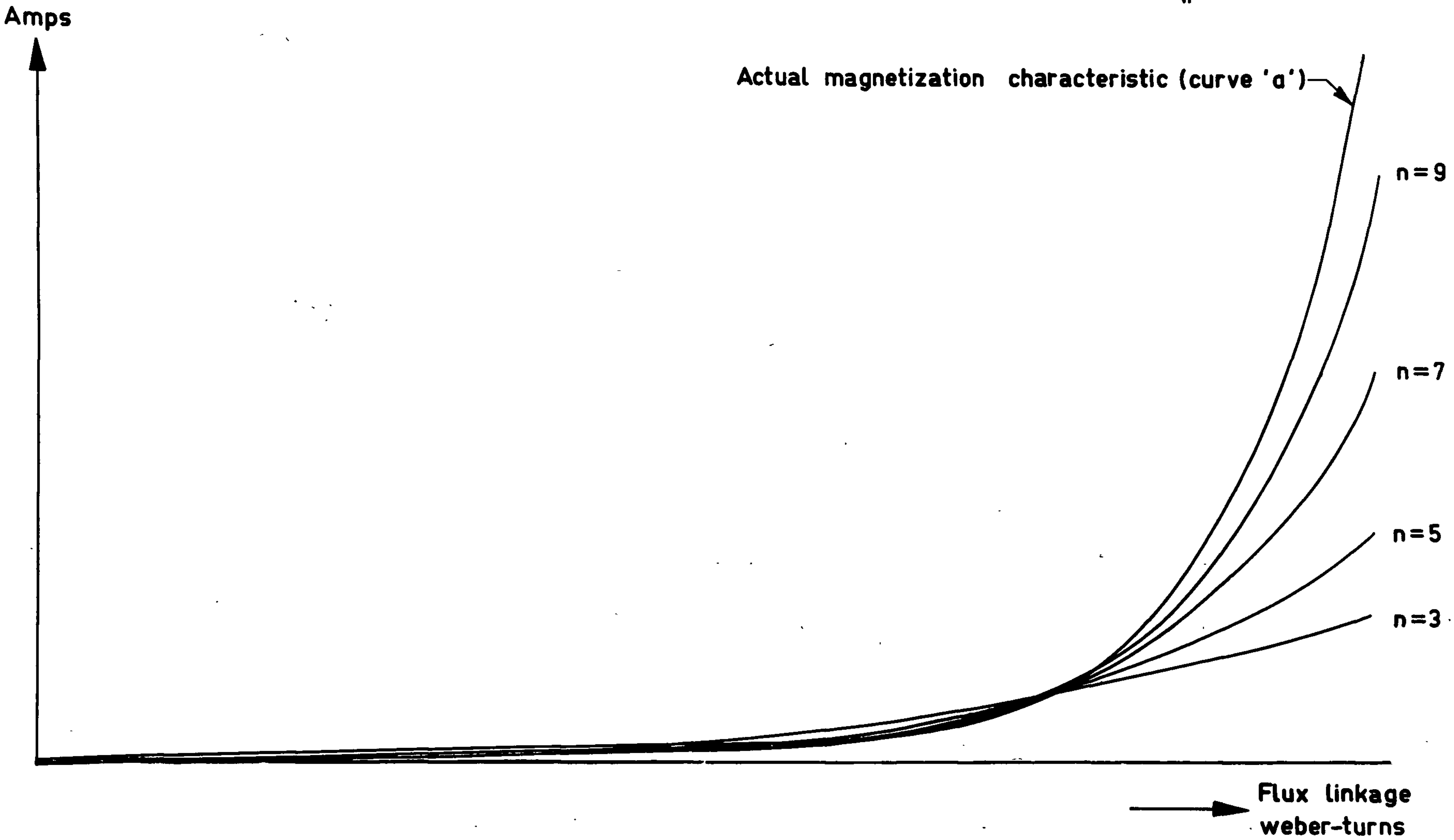


FIG. 4-1

If the magnetisation characteristic is some function $i(\lambda)$, then a_1 and a_n here have to be determined so that $a_1\lambda + a_n\lambda^n$ is the best approximation to $i(\lambda)$. In the method of least squares, this requirement is fulfilled if

$$\frac{d}{da_1}\int_0^{\lambda_m}\left(i(\lambda) - a_1\lambda - a_n\lambda^n\right)^2 d\lambda = 0 \qquad (4\text{-}3)$$

and

$$\frac{d}{da_n}\int_0^{\lambda_m}\left(i(\lambda) - a_1\lambda - a_n\lambda^n\right)^2 d\lambda = 0 \qquad (4\text{-}4)$$

where 0 to λ_m gives the range of values of interest. A simple differentiation and integration yields two simultaneous equations the solution of which gives values for a_1 and a_n[38].

If $i(\lambda)$ is expressed as a set of tabulated data or in the form of a graph, the integration process may have to be done numerically.

It is realized that the difficulty of numerical calculations depends to a great extent on the relation assumed between i and λ. The higher the value of n, the higher the order of the resulting equations in λ with the added difficulties in solutions.

A typical magnetisation characteristic for a power transformer is shown in Fig. (4-1) curve (a). Polynomial expressions of the form $i = a_1\lambda + a_n\lambda^n$ for various values of n were fitted and the resulting curves are shown for n = 3, 5, 7, and 9.

Use of the expression $i = a_1\lambda + a_9\lambda^9$ gives a better approximation than any of the others and certainly, if large current values are expected, intolerable errors will result if the expression $i = a_1\lambda + a_3\lambda^3$ is used in calculations. A further effect of the choice of polynomial expression will be discussed in sec. 4-9.

4-3 The Ritz Method and the Principle of Harmonic Balance.

One of the techniques used in this chapter is based on the well-established Ritz Method[39]. In this particular case, the procedure followed is equivalent to that of the Principle of Harmonic Balance which arises naturally from the Ritz method when oscillatory systems are considered.[40]

Consider a non-linear equation of the form

$$f\ (D,\ x,\ t) = 0 \tag{4-5}$$

An approximate solution $X(t)$, a linear combination of suitably chosen linearly independent functions, is assumed:

$$x(t) \triangleq X(t) = C_0\ \phi_0(t) + C_1\ \phi_1(t) \ldots\ldots + C_m\ \phi_m(t) \tag{4-6}$$

The choice of functions ϕ_0, ϕ_1 etc. is not arbitrary but presupposes some knowledge of the properties of the solution and may also be influenced by initial conditions. The constants C_0, C_1, $C_2 \ldots C_m$ are to be adjusted to optimize the solution.

The assumed solution $X(t)$ is substiuted in eq. (4-6) and the residual

$$\varepsilon(t) = f\ (D,\ X(t),\ t) \tag{4-7}$$

gives a measure of unbalance in the differential equation.

The interval over which the assumed solution is a reasonable approximation may be limited and in the case of oscillatory solutions in a steady-state, this interval will normally be the length of a period.

There are various approaches possible in the determination of the constants C_0, C_1, C_2 ... C_m, and the solutions obtained to a particular problem need not be identical.

In Galerkin's method, the minimization of the integral

$$J = \int_{t_0}^{t_1} \varepsilon^2(t) \tag{4-8}$$

is the criterion for an optimum solution.

In the Ritz method, the minimization of the integral

$$I = \int_{t_0}^{t_1} F(D,\ X(t),\ t)\ dt \tag{4-9}$$

is the criterion and this leads to the condition(41)

$$\int_{t_0}^{t_1} \varepsilon(t)\ \phi_i(t)\ dt = 0 \tag{4-10}$$

$$i = 0,1,2,3,\ .\ .\ .\ m.$$

provided the functions ϕ_i' s are periodic

i.e.

$$\phi_i(t_0) = \phi_i(t_1) \tag{4-11}$$

where $t_1 - t_0$ = length of a period.

If a solution $X = C_1 \cos \omega t + C_m \cos (n\omega t + \theta_m)$ (4-12)

is assumed, then the Ritz Method requires

that

$$\int_0^{2\pi} \varepsilon(t) \cos \omega t d(\omega t) = 0 \tag{4-13}$$

and $$\int_0^{2\pi} \varepsilon(t) \cos n\omega t \, d(\omega t) = 0 \qquad (4\text{-}14)$$

Because of orthogonality all terms of the form

$$\int_0^{2\pi} \cos \omega t \cos n\omega t \, d(\omega t) = 0 \text{ for } m \neq n \qquad (4\text{-}15)$$

Therefore, in fact, the requirement is equivalent to the simple condition that the coefficients of terms in cos ωt and cos $n\omega t$ appearing in the residual must individually sum to zero. This interpretation is referred to as the Principle of Harmonic Balance.

4-4 The Fundamental, The Higher-Harmonics and The Sub-Harmonics.

The expressions Fundamental, Higher-Harmonics and Sub-Harmonics are commonly used in electrical engineering literature; in this chapter, these terms appear frequently. It is, therefore, necessary to emphasize their origin and significance as used in this study.

In determining the Fourier components of some wave-form, integration has to be carried out over a full-cycle (i.e. the curve repeats itself exactly after its period--the length of one cycle). In such a Fourier analysis, one can obtain an infinite number of components where frequencies are multiples of the full-cycle frequency.

One of the components thus obtained may have some special significance in the physical system being studied. It may have the largest amplitude or be associated with one of the natural frequencies or may be the frequency of one of the sources. In any case, in engineering practice, such a chosen component is called the Fundamental and the components with higher frequencies are called the Higher-Harmonics while those with lower frequencies are called the Sub-Harmonics.

It must also be observed that if ν is defined as the ratio of the frequency of the ν^{th} harmonic to the fundamental frequency, it is possible that fractional ν's will appear both for higher and lower harmonics.

4-5 The Method of Isoclines.

The method of Isoclines[42] is a graphical method that can be used for the solution of some types of differential equations.

Consider the first-order equation

$$\frac{dx}{dt} = f(x_1 t) \tag{4-16}$$

For any given point on the x-t plane, the numerical value of $\frac{dx}{dt}$ can be calculated. This value can be interpreted as 'm' - the slope of the solution curve at that point. With a sufficient value of points, the significant isoclines (curves of constant m) can be plotted and using these, the solution curve x(t) can be drawn.

4-6 The Method of Integral Curves.

A combination of graphical and algebraic methods often gives a better insight into some problems. The method of Integral Curves[43] combines the Principle of Harmonic Balance with a graphical technique and proves very useful in the study of higher harmonics, sub-harmonics and the effect that initial conditions have on these. It can also establish graphically the relation between the transient and steady-state behaviours.

In dealing with oscillatory motion, a solution made up of terms of the form $x_n(t)$ Sin$n\omega t$ and $y_n(t)$ Cos$n\omega t$ is assumed. $x_n(t)$ and $y_n(t)$ are constants in the final steady-state but may be some function of t in the transient state.

Using the Principle of Harmonic Balance, expressions are obtained for

$$\frac{dx}{dt} \triangleq X(x,y)$$

and $$\frac{dy}{dt} \triangleq Y(x,y) \qquad (4\text{-}17)$$

The curves given by the equation

$$\frac{dy}{dx} = \frac{Y(x,y)}{X(x,y)} \qquad (4\text{-}18)$$

are called the integral curves and since these functions are not explicit functions of t, these curves can be drawn in the x-y plane.

The singular points given by

$$Y(x,y) = 0 \qquad (4\text{-}19)$$

and $$X(x,y) = 0 \qquad (4\text{-}20)$$

are then determined and the behaviour of the solution can be discussed.

An important step in this method, having determined the singular points is the determination of the nature of the singularities as the stability of a particular solution depends on the nature of the singularities.

The question of singular point and their behaviour is well covered in the literature[44] and only the briefest outline is given here.

A numerical application of this method amplifies this approach.

4-6-1 Singular Points.

The Singular Points are values of x and y which satisfy the simultaneous equations (4-19), (4-20). These values may be denoted by x_0, y_0.

i.e. $X(x_0, y_0) \equiv 0$ (4-20)

and $Y(x_0, y_0) \equiv 0$ (4-21)

Small variations near a singularity, i.e. at a point $(x_0 + \eta_x, y_0 + \eta_y)$ are considered, these conditions determine whether these deviations approach zero or not with increase in time.

Now, using eq. (4-17),

$$\frac{d\eta_x}{dt} = \eta_x \cdot \frac{\partial X}{\partial x} + \eta_y \cdot \frac{\partial x}{\partial y} \tag{4-22}$$

$$\frac{d\eta_y}{dt} = \eta_x \cdot \frac{\partial X}{\partial y} + \eta_y \cdot \frac{\partial Y}{\partial y} \tag{4-23}$$

The solutions[44] for η_x and η_y are of the form $e^{\nu t}$ where ν is given by

$$\begin{vmatrix} \frac{\partial X}{\partial x} - \nu & \frac{\partial X}{\partial y} \\ \frac{\partial Y}{\partial x} & \frac{\partial Y}{\partial y} - \nu \end{vmatrix} = 0 \tag{4-24}$$

Defining

$$\left(\frac{\partial X}{\partial x}\right)_{\substack{x=x_0 \\ y=y_0}} \triangleq A_1$$

$$\left(\frac{\partial X}{\partial y}\right)_{\substack{x=x_0 \\ y=y_0}} \triangleq A_2$$

$$\left(\frac{\partial Y}{\partial x}\right)_{\substack{x=x_0 \\ y=y_0}} \triangleq B_1 \tag{4-25}$$

$$\left(\frac{\partial Y}{\partial y}\right)_{\substack{x=x_0 \\ y=y_0}} \triangleq B_2$$

then

$$\lambda_1, \lambda_2 = A_1 + B_2 \pm \frac{\left[(A_1+B_2)^2 + 4(A_2B_1-A_1B_2)\right]^{\frac{1}{2}}}{2} \qquad (4-26)$$

The nature of λ_1 and λ_2 whether real, imaginary etc. decides the type of singularity.[44]

(a) λ_1, λ_2 real and of same sign - Node

(b) λ_1, λ_2 real and of opposite sign - Saddle

(c) λ_1, λ_2 imaginary - Vortex

(d) λ_1, λ_2 complex conjugates - Focus

4-7 The Non-Linear Equations.

Conditions leading to three different non-linear differential equations are considered. Suitable and reasonable approximations can be made and using the three techniques discussed, an insight into the system behaviour is possible.

The choice of these three equations is based on two reasons. Firstly, the conditions described are of practical importance and secondly, the different methods previously outlined can be effectively applied. The procedures for their use are established, an important objective of this study.

4-7-1 The 'Short-Line' - Shunt-Capacitance Neglected.

Nomenclature

R_t ohms = (sum of source and line resistances) per phase

L_s henries = (sum of source, line, and transformer leakage inductances) per phase

$e = E_{max} \mathrm{Sin}\ (\omega t + \theta)$

θ = Voltage switching angle

i = current

Referring to Fig. (4-2), the equation describing this circuit is

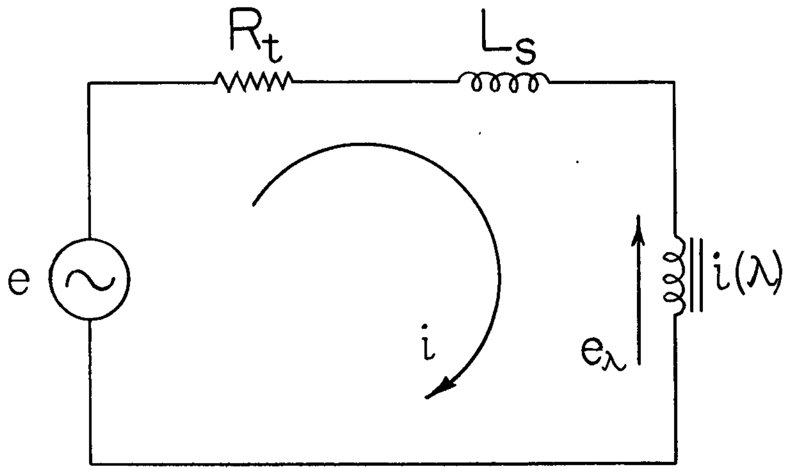


FIG. 4-2

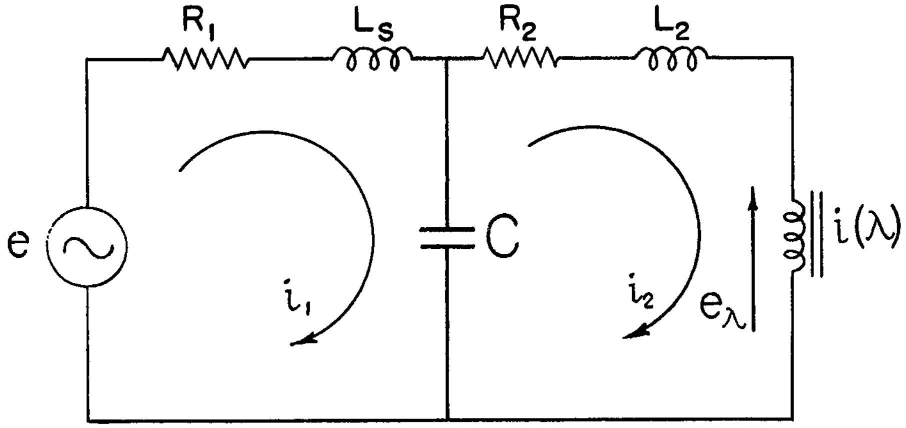


FIG. 4-3

$$e = L_s \frac{di}{dt} + i R_t + e_\lambda . \tag{4-27}$$

Putting

$$e_\lambda = \frac{d\lambda}{dt} \quad , \tag{4-28}$$

equation (4-27) becomes

$$\frac{d\lambda}{dt} + L_s \frac{di}{d\lambda} \cdot \frac{d\lambda}{dt} + R\, i(\lambda) - E_{max} \operatorname{Sin} (\omega t + \theta) = 0 \tag{4-29}$$

(1) The substitutions

$$i(\lambda) = a_1 \lambda + a_n \lambda^n \tag{4-30}$$

and $\theta = 0$ (its value at $t=0$ is used as an initial condition). (4-31)

and the approximation

$$R_t = 0 \quad \text{(losses neglected)} \tag{4-32}$$

result in the equation

$$\frac{d\lambda}{dt} = \frac{E_{max} \operatorname{Sin} \omega t}{1 + L_s (a_1 + n a_n \lambda^{n-1})} . \tag{4-33}$$

(2) The substitutions

$$i(\lambda) = a\, e^{\lambda/k} \tag{4-34}$$

and $\theta = 0$ (4-35)

result in the equation for current

$$i = \frac{E_{max} \operatorname{Sin} \omega t - L_s . \frac{di}{dt} \pm \left(E_{max} \operatorname{Sin} \omega t - L_s . \frac{di}{dt}^2 - 4 R_t . k . \frac{di}{dt}\right)^{\frac{1}{2}}}{2 R_t} \tag{4-36}$$

or the equation for flux

$$\lambda = k \operatorname{Log}_e \left(E_{max} \operatorname{Sin} \omega t - \frac{d\lambda}{dt}\right) - \operatorname{Log}_e a . \left(\frac{L_s}{k} . \frac{d\lambda}{dt} + R_t\right) \tag{4-37}$$

Either of these equations (4-36) or (4-37) can be used in an isocline plot and it may appear that eq. (4-36) is more suitable as it gives a direct solution for current, the quantity that is normally of interest.

However, the high rates of change in current that are experienced make the use of eq. (4-36) impractical from the point of view of a graphical method. On the the other hand, putting eq. (4-37) in the form

$$\frac{\lambda}{E_{max}} = k\left[\mathrm{Log}_e \; (\mathrm{Sin}\, \omega t - p) - \mathrm{Log}_e \; a\left(\frac{L_s}{k} + \frac{R_t}{E_{max}}\right)\right] \qquad (4\text{-}38)$$

where

$$p = \frac{d\lambda}{dt} \Big/ E_{max} \qquad (4\text{-}39)$$

allows an easy graphical construction as p then lies in the range $-1 < p < 1$.

The use of eq. (4-36) or eq. (4-38) depends on which approximation can be more easily tolerated. In equation (4-36), system losses are neglected but a good approximation is possible for the magnetisation characteristic. In equation (4-38), losses can be considered but the expression for

$$i(\lambda) = a e^{\lambda/k}$$

is not as good a fit.

Solution curves are obtained for both these equations using the method of isoclines and the differences between these solutions are discussed in section 4-8. .

<u>4-7-2. The 'Long-Line' - Energisation of the System.</u>

Nomenclature

L_s henries = (sum of source inductance + $\frac{1}{2}$ total line inductance)per phase.

L_2 henries = ($\frac{1}{2}$ total line inductance + transformer leakage inductance) per phase

R_1=R_2 ohms = $\frac{1}{2}$ total line resistance per phase

C farads = Lumped shunt capacitance per phase

i_1, i_2 = currents in amps. as marked

$e = E_{max} \sin \omega t$

Referring to Fig. (4.3), the equations describing this circuit are

$$e = i_1R_1 + L_s \frac{di_1}{dt} + \frac{1}{C} \int (i_1 - i_2)dt \tag{4-40}$$

$$0 = i_2R_2 + L_2 \frac{di_2}{dt} - \frac{1}{C} \int (i_1 - i_2)dt + e_\lambda. \tag{4-41}$$

Eliminating i_1 yields,

$$\frac{L_s d^2 e_\lambda}{dt^2} + R_1 . \frac{de_\lambda}{dt} + \frac{e_\lambda}{C} + L_sL_2 . \frac{d^3 i_2}{dt^2} + (L_sR_2 + L_2R_1) . \frac{d^2 i_2}{dt^2} + + \left(\frac{L_s + L_2}{C} + R_1R_2\right) . \frac{di_2}{dt} + i_2 \frac{R_1 + R_2}{C} - \frac{e}{C} = 0. \tag{4-42}$$

The substitutions

$$e_\lambda = \frac{d\lambda}{dt} \tag{4-43}$$

and

$$i = i(\lambda) \tag{4-44}$$

yield

$$L_s . \frac{d^3\lambda}{dt^3} + R_1 . \frac{d^2\lambda}{dt^2} + \frac{1}{C} . \frac{d\lambda}{dt} + (L_s R_2 + L_2 R_1) . \left(i'(\lambda) . \frac{d^2\lambda}{dt^2} + i''(\lambda) \left(\frac{d\lambda}{dt}\right)^2 \right)$$

$$+ L_sL_2 \left(i'''(\lambda) . \frac{d\lambda}{dt}^3 + 3i''(\lambda) . \frac{d\lambda}{dt} . \frac{d^2\lambda}{dt^2} + i'(\lambda) . \frac{d^3\lambda}{dt^3} \right) + \left(\frac{R_2 + R_1}{C} \right) . i(\lambda)$$

$$+\left(\left(\frac{L_2+L_s}{C}\right)+R_1\ R_2\right)\circ i'(\lambda)\circ\frac{d\lambda}{dt}\ -\ \frac{E_{max}\ \mathrm{Sin}\ (\omega t+\theta)}{C}\ = 0 \qquad (4\text{-}45)$$

where the notation

$$i'(\lambda) = \frac{di}{d\lambda}$$
$$i''(\lambda) = \frac{d^2 i}{d\lambda^2} \qquad (4\text{-}46)$$
$$i'''(\lambda) = \frac{d^3 i}{d\lambda^3}$$

is used.

Equation (4-45) as it stands, even with the simple expression for $i(\lambda) = a_1\lambda + a_3\lambda^3$ is rather complex and certain simplific-ations will have to be introduced before any attempt at an analytical solution is made.

The approximations

$$L_2 = 0 \quad \text{(neglecting line inductance)} \qquad (4\text{-}47)$$

$$R_1 = R_2 = 0 \quad \text{(neglecting losses)}$$

result in the modification of equation (4-45) to

$$L_s\ \frac{d^3\lambda}{dt^3} + \frac{L_s}{C}\left(i'(\lambda) + 1\right)\frac{d\lambda}{dt} - e = 0. \qquad (4\text{-}48)$$

Integrating this equation,

$$\frac{d^2\lambda}{dt^2} + \frac{1}{L_s C}\left(\lambda + L_s\ i(\lambda)\right) - \frac{1}{L_s C}\int e\ dt = J. \qquad (4\text{-}49)$$

J = a constant

The substitution for $i(\lambda)$,

$$i(\lambda) = a_1\ \lambda + a_n\lambda^n \qquad (4\text{-}50)$$

further yields

$$\frac{d^2\lambda}{dt^2} + \omega_0^2\lambda + a_1 \lambda L_s + \omega_0^2 L_s a_n\lambda^n - \frac{\omega_0^2 e}{\omega} = J \tag{4-51}$$

where

$$\omega_0^2 \triangleq \frac{1}{L_s C} \quad . \tag{4-52}$$

This simplified form of the equation is investigated using both the Principle of Harmonic Balance and the Method of Integral Curves.

4-7-3 The 'Long-Line'. De-energisation of the System.

Using the nomenclature of section 4-7-2 and the same approximations (4-47), the equation describing such a system condition is

$$\frac{de_\lambda}{dt} + \frac{1}{C} \; i(\lambda) = 0 \tag{4-53}$$

and integrating

$$\left(\frac{d\lambda}{dt}\right)^2 + \frac{2}{C}\int i(\lambda) \; d\lambda = 2h \tag{4-54}$$

where 2h = some constant fixed by initial conditions.

Hence

$$\frac{d\lambda}{dt} = \pm \left(2\left(h - V(\lambda)\right)\right)^{\frac{1}{2}} \tag{4-55}$$

where

$$V(\lambda) \triangleq \frac{1}{C}\int i(\lambda) \; d\lambda . \tag{4-56}$$

The substitution $i(\lambda) = a_1\lambda + a_n\lambda^n$ (4-57)

yields

$$\frac{1}{C}\int i(\lambda) \; d\lambda = \frac{1}{C}\left(\frac{a_1\lambda^2}{2} + \frac{a_n\lambda^{n+1}}{n+1}\right) \tag{4-58}$$

and hence

$$\frac{d\lambda}{dt} = \pm \left(2\left(h - \frac{1}{C} \quad \frac{a_1\lambda^2}{2} + \frac{a_n\lambda^{n+1}}{n+1} \right)\right)^{\frac{1}{2}} \tag{4-59}$$

In the form given by eq. (4-59), this equation can be solved graphically by the Method of Isoclines. However, using the original form of the equation (4-53) and the substitution (4-57), the following equation results

$$\frac{d^2\lambda}{dt^2} + \frac{1}{C}\,(a_1\lambda + a_n\lambda^n) = 0 \tag{4-60}$$

This final eq. (4-60) will be used in this study.

4-8 The Transient Behaviour.-Energisation of a 'Short-Line'.

Equations (4-33) and (4-38) are used and solved by the Method of Isoclines.

Equation (4-33) states

$$\frac{d\lambda}{dt} = \frac{E_{max} \quad \mathrm{Sin}\ \omega t}{1 + L_s(a_1 + n\ a_n\lambda^{n-1})} \tag{4-61}$$

Defining

$$\frac{d\lambda}{dt} \triangleq m \tag{4-62}$$

and putting n=9

$$\text{i.e. } i(\lambda) = a_1\lambda + a_9\lambda^9. \tag{4-63}$$

Then

$$m + m\,L_s\ (a_1 + 9a_9\lambda^8) = E_{max}\ \mathrm{Sin}\ \omega t. \tag{4-64}$$

$$\therefore \quad \lambda = \left(\frac{q\ \mathrm{Sin}\ \omega t - 1 - a_1\ L_s}{9\ a_9\ L_s} \right)^{\frac{1}{8}} \tag{4-65}$$

where

$$q \triangleq \frac{E_{max}}{m}. \tag{4-66}$$

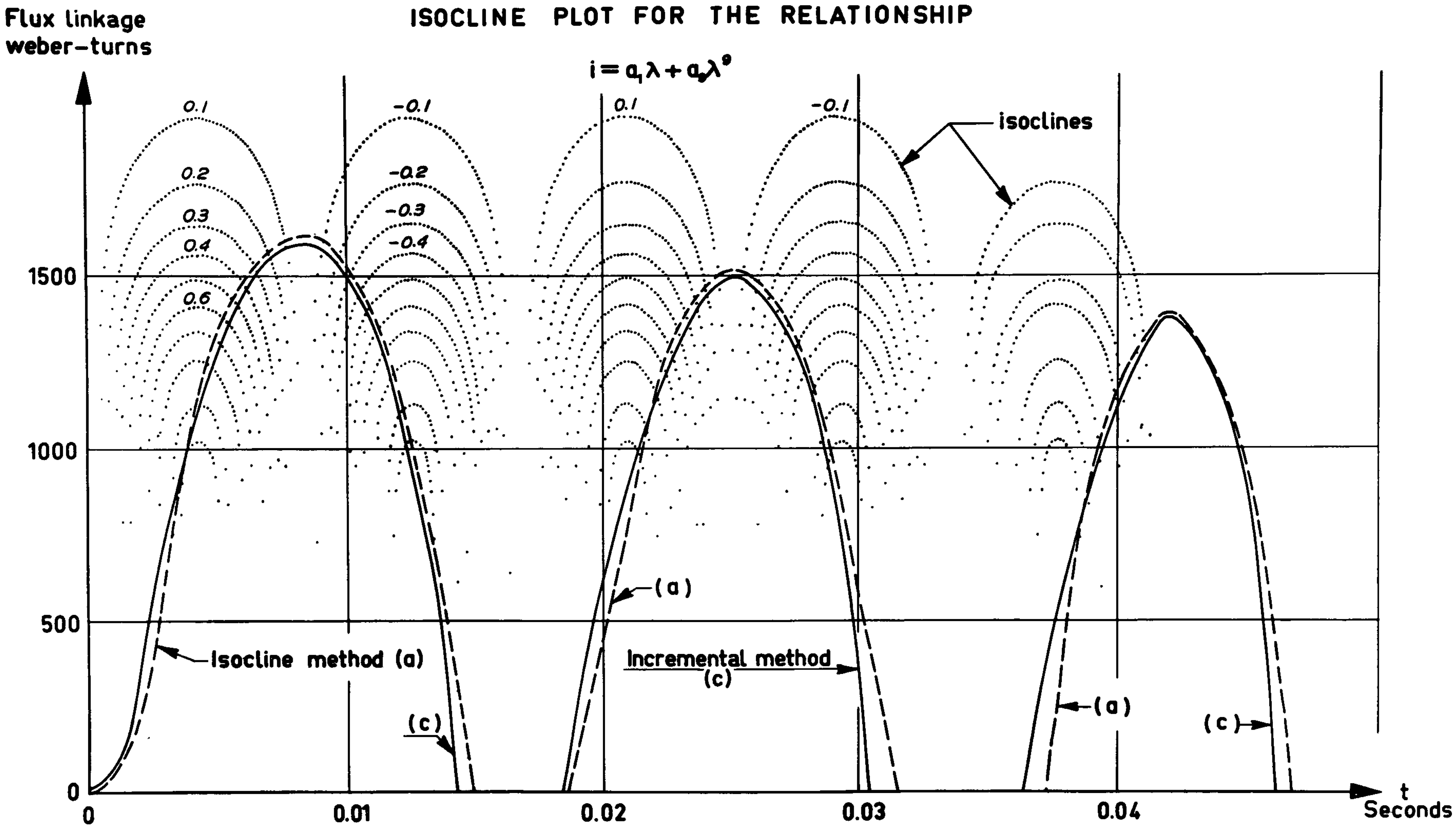

Flux linkage
weber-turns
ISOCLINE PLOT FOR THE RELATIONSHIP
$i = a_1\lambda + a_9\lambda^9$
isoclines
0.1
0.2
0.3
0.4
0.6
-0.1
-0.2
-0.3
-0.4
0.1
-0.1
1500
1000
500
0
0
0.01
0.02
0.03
0.04
t
Seconds
Isocline method (a)
Incremental method
(c)
(a)
(c)

FIG. 4-4

ISOCLINE PLOT FOR THE RELATIONSHIP

$$i = ae^{\lambda/k}$$

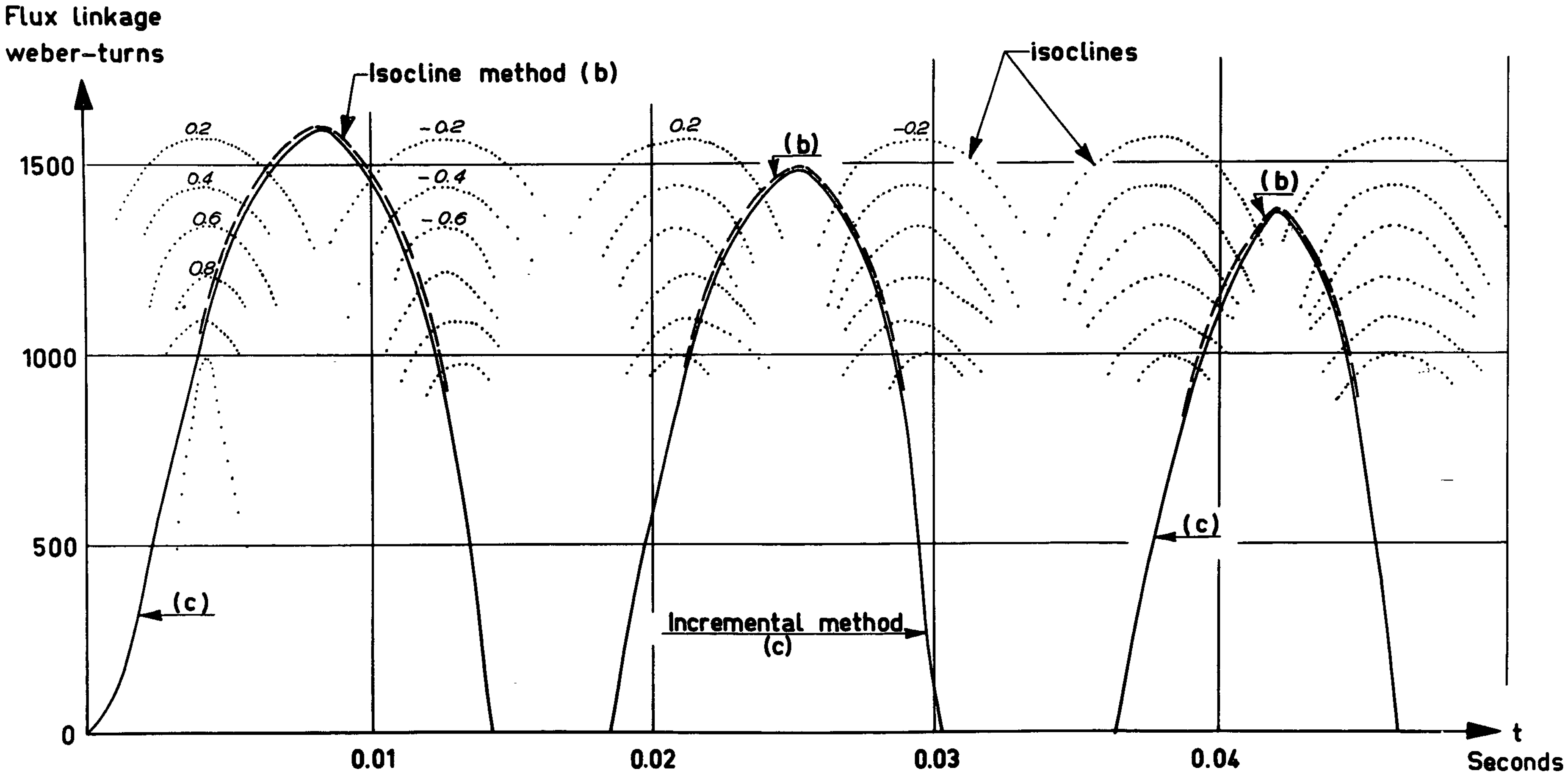


FIG. 4-5

Using eq. (4-38) based on the substitution

$$i = ae^{\lambda/k} \tag{4-67}$$

the corresponding equation is

$$\frac{\lambda}{E_{max}} = k\ \mathrm{Log}_e(\mathrm{Sin}\ \omega t - p) - \mathrm{Log}_e\ a\left(\frac{L_s}{k}\ p + \frac{R_t}{E_{max}}\right) \tag{4-68}$$

where

$$p \triangleq \frac{d\lambda}{dt}\Big/E_{max} \tag{4-69}$$

The isoclines and solution curves using these relations for a 60 mile system are shown in Figs. (4-4) and (4-5). Fig. (4-4) curve (a) is obtained using equation (4-61) and Fig. (4-5) curve (b) is obtained using equation (4-68). Curve (c) on both figures is the result obtained using the incremental method.

In both cases, the initial condition corresponds to a switching condition - e=0 for t=0.

4-9 The Steady-State - Energisation of a 'Long-Line'. The Principle of Harmonic Balance.

In this section, it is assumed that a steady-periodic state is reached and the approach will be to determine the amplitude and nature of the various components of flux and from these, voltages and currents. The basis for this treatment is the Principle of Harmonic Balance.

In the steady-state, the differential equation of interest when the system is energised, is eq. (4-49) with J=0, thus,

$$\frac{d^2\lambda}{dt^2} + \omega_0^2\ (\lambda + a_{1n}\lambda\ L_s) + L_s\ a_n\lambda^n - \frac{\omega_0^2\ e}{\omega} = 0. \tag{4-70}$$

where

$$\omega_0^2 \triangleq \frac{1}{L_s C} \tag{4-71}$$

4-9-1 The Fundamental Component.

As a first approximation, it will be assumed that

$$\lambda = \lambda_{1n} \text{ Cos } \omega t \tag{4-72}$$

Substituting this in equation (4-70)

$$-\omega^2 \lambda_{1n} \text{ Cos } \omega t + \omega_0^2 (\lambda_{1n} \text{ Cos } \omega t + a_1 \lambda_{1n} L_s) + L_s a_n \lambda_{1n}^n \text{ Cos}^n \omega t - $$

$$- G \text{ Cos } \omega t = 0 \tag{4-73}$$

where

$$G \triangleq \frac{\omega_0^2 E_{max}}{\omega}. \tag{4-74}$$

$\text{Cos}^n \omega t$ can be expressed as $\sum_1^m a_m \text{ Cos } m \omega t$(45) (4-75)

and using the Principle of Harmonic Balance, the individual components of Cos ωt are equated.

For practical values of a_1 and L_s, $1 \gg a_1 L_s$. This approximation is made without any loss of generality in the method. Then, $\omega_0^2 (1 + a_1 L_s) \triangleq \omega_0^2 = \frac{1}{L_s C}$ (4-76)

With this substitution, and putting n=3

i.e. $$i = a_{13}\lambda + a_3 \lambda^3 \tag{4-77}$$

$$\lambda_{13}(\omega_0^2 - \omega^2) + \frac{3}{4} a_3 L_s \omega_0^2 \lambda_{13}^3 - G = 0 \tag{4-78}$$

and with n=9

i.e. $$i = a_{19}\lambda + a_9 \lambda^9 \tag{4-79}$$

$$\lambda_{19}(\omega_0^2 - \omega^2) + \frac{126}{256} a_9 L_s \omega_0^2 \lambda_{19}^9 - G = 0 \tag{4-80}$$

Note that in the notation λ_{mn}, m refers to the harmonic order of the component and n refers to the order of the polynomial being used.

For any assumed polynomial relation between i and λ, an equation of the above type will be obtained.

The solution of the n^{th} order equation, besides giving an approximate value of the fundamental amplitude of the flux, can also provide an insight into the behaviour of the system. If for example, there is more than one real value of λ for particular values of ω, the familiar 'backbone' response curves[46] are obtained indicating the possibility of jump-phenomenon. Under these conditions, for these values of ω, the amplitude of λ can change quite rapidly with resulting over-voltages.

The question now arises whether this jump phenomenon can occur in the system described by eq. (4-78) and (4-80).

For n=3, a direct answer to this question is possible.

Consider the cubic equation

$$x^3 + p\,x^2 + q\,x + r = 0. \tag{4-81}$$

This equation has 3 real, unequal roots if[33]

$$p_1^2 + q_1^3 < 0 \tag{4-82}$$

where

$$p_1 = \frac{2\,p^3 - 9\,pq + 27r}{54}\ ;\ q_1 = \frac{3q - p^2}{9}. \tag{4-83}$$

Equation (4-78) is

$$\lambda_{13}^3 + \frac{4(\omega_0^2-\omega^2)\lambda_{13}}{3a_3\ L_s\omega_0^2} - \frac{4\ E_{max}}{3a_3\ L_s\omega} = 0 \tag{4-84}$$

Therefore, in eq. (4-81)

$$p = 0 \tag{4-85}$$

$$q = \frac{4(\omega_0^2 - \omega^2)}{3a_3 L_s \omega_0^2}. \tag{4-86}$$

$$r \doteq - \frac{4E_{max}}{3a_3 L_s \omega} \tag{4-87}$$

The relationship given by eq. (4-82) will occur only if

$$\omega^2 > \omega_0^2. \tag{4-88}$$

In power systems, since $\omega_0^2 \triangleq \frac{1}{L_s C}$ and ω is the power frequency, this condition is unlikely to occur for practical values of L_s and C and thus occurrence of the jump phenomenon is improbable in the energisation of such a power-system.

For the case n=9, a direct answer is not possible but examination of the coefficients of the equation showed that except for a real root of the order of $\lambda_{19} = \frac{E_{max}}{\omega}$, all the other roots of the equation are imaginary. This fact was verified by actual numerical solution of various examples.

4-9-2 The Higher Harmonics.

A solution is now assumed

$$\lambda = \lambda_{1n} \text{ Cos } \omega t + \lambda_{mn} \text{ Cos } m\,\omega t \tag{4-89}$$

where n and m are integers.

This expression for λ is substituted in equation (4-70) and using the Principle of Harmonic Balance, the individual components of Cos ωt and Cos m ωt are equated. Two simultaneous equations result, the solution of which determines values of λ_{1n} and λ_{mn}.

As will be seen, the solution of the equations can present certain practical difficulties which may eliminate some of the niceties of an analytical solution.

4-9-2-1 The Third Harmonic.

Applying the above procedure in investigation of the 3^{rd} harmonic, the resulting equations are:

(1) with $i = a_{13}\lambda + a_3\lambda^3$ (4-90)

$$(\omega_0^2-\omega_1^2)\lambda_{13}+ \tfrac{3}{4}h_3\lambda_{13}^3+3h_3\lambda_{13}\lambda_{33}(\lambda_{13}+2\lambda_{33}) - G = 0 \tag{4-91}$$

and

$$(\omega_0^2-9\omega_1^2)\lambda_{33}+ \frac{3h_3}{4}\lambda_{33}^3 + \frac{h_3\lambda^2{}_{13}}{4}(\lambda_{13}+6\lambda_{33}) = 0 \tag{4-92}$$

where

$$h_3 \triangleq a_3 L_s \omega_0^2 \tag{4-93}$$

(2) with $i = a_{19}\lambda + a_9\lambda^9$ (4-94)

$$(\omega_0^2-\omega_1^2)\lambda_{19}+ \frac{126}{256} h_9\lambda_{19}^9 + \frac{378}{128} h_9\lambda_{19}^8\lambda_{39} - G = 0 \tag{4-95}$$

and

$$(\omega_0^2-9\omega_1^2)\lambda_{39}+ \frac{84}{256}h_9\lambda_{19}^9 + \frac{351}{128} h_9\lambda_{19}^8\lambda_{39} = 0 \tag{4-96}$$

where

$$h_9 \triangleq a_9 L_s \omega_0^2 \tag{4-97}$$

4-9-3 Amplitudes of the Higher Harmonics.

Numerical analysis solutions of equations (4-91), (4-92), (4-95), (4-96) give values of the fundamental and 3^{rd} harmonic component of λ. The computation necessary for such exact answers is tedious and will not be carried out.

Instead, the assumption that $\lambda_{1n} >> \lambda_{mn}$ is made. i.e. the fundamental predominates over the higher harmonics and certainly in the steady state, this is almost always a correct assumption. Then equations (4-91), (4-92), (4-95), and (4-96) simplify to

$$(\omega_0^2 - \omega_1^2)\lambda_{13} + \frac{3}{4}h_3\lambda_{13}^3 - G = 0 \tag{4-98}$$

$$\frac{\lambda_{33}}{\lambda_{13}} = -\frac{h_3\lambda_{13}^2}{4(\omega_0^2 - 9\omega^2)} \tag{4-99}$$

$$(\omega_0^2 - \omega_1^2)\lambda_{19} + \frac{126}{256}h_9\lambda_{19}^9 - G = 0 \tag{4-100}$$

$$\frac{\lambda_{39}}{\lambda_{19}} = -\frac{84h_9\lambda_{19}^8}{256(\omega_0^2 - 9\omega^2)} \tag{4-101}$$

These simplified equations determine the procedure for obtaining approximate values of the fundamental and any other higher harmonic component. The fundamental component can be determined from an equation of the form given either by equations (4-98) or (4-100). With the fundamental component determined, the amplitude of any harmonic component results from an equation of the type (4-100) or (4-101).

Equations (4-99) and (4-101) give rise to a useful criterion. Consider equation (4-99),

$$\frac{\lambda_{33}}{\lambda_{13}} = -\frac{h_3\lambda_{13}^2}{4(\omega_0^2-9\omega^2)}\,. \tag{4-102}$$

λ_{33}, i.e. the amplitude of the 3^{rd} harmonic will be large if

$$\omega_0^2 = 9\omega^2 \tag{4-103}$$

i.e. if

$$(3\omega)^2 = \frac{1}{L_s C} \tag{4-104}$$

An interpretation of this condition is that dangerous 3^{rd} harmonic over-voltages occur if the 3^{rd} harmonic output impedance from the transformer terminals is infinite. This conclusion can also be reasoned from physical considerations.[47] The value of λ_{33} for this condition is not infinite as eq. (4-102) suggests but is some finite value. This is realized if the more exact form of the third harmonic equation is used - eq. (4-91)

Then, with,

$$\omega_0^2 - 9\omega^2 = 0 \qquad (4\text{-}105)$$

eq. (4-92) becomes

$$\lambda_{33}^3 + 2\lambda_{33}\lambda_{13}^2 + \frac{1}{3}\lambda_{13}^3 = 0 \qquad (4\text{-}106)$$

Eq. (4-106) gives one real root

$$\lambda_{33} = 0.16\ \lambda_{13} \qquad (4\text{-}107)$$

Hence under the condition,

$$\frac{1}{L_s C} = (3\omega)^2 \qquad (4\text{-}108)$$

the amplitude of the third harmonic component of the flux-linkage increases to 16% of the peak value of the fundamental. Such a high third harmonic content can lead to dangerous persistent over-voltages.

A similar procedure can be adopted in the investigation of any higher harmonics but an important observation must be made.

Assuming a polynomial fit of order 3,

i.e. $i(\lambda) = a_1\lambda + a_3\lambda^3 \qquad (4\text{-}109)$

if the procedure outlined in this section is used in the investigation of the fifth harmonic, equations corresponding to eqs. (4-91) and (4-92) are

$$(\omega_0^2-\omega_1^2)\lambda_{13} + \frac{3}{4} h_3 \lambda_{13}^3 - G = 0 \tag{4-110}$$

and

$$(\omega_0^2-25\omega_1^2)\lambda_{53} + \frac{3h_3\lambda_{13}^2\lambda_{53}}{2} + \frac{3h_3\lambda_{53}^3}{4} = 0. \tag{4-111}$$

Eq. (4-110) gives a result for λ_{13} identical with eq. (4-91) but inspection of eq. (4-111) reveals that one real root is given by

$$\lambda_{53} = 0 \tag{4-112}$$

and that other real roots exists only if

$$25\omega_1^2 > \omega_0^2 \tag{4-113}$$

This condition may lead to some interesting conclusions if the polynomial fit

$$i(\lambda) = a_{13}\lambda + a_3\lambda^3 \tag{4-114}$$

is an exact one. However, assuming a polynomial fit of order 9

$$\text{i.e. } i(\lambda) = a_{19}\lambda + a_9\lambda^9 \tag{4-115}$$

the equation corresponding to eq. (4-111) is

$$\left|\frac{\lambda_{59}}{\lambda_{19}}\right| = \frac{h_9\lambda_{19}^8}{256(\omega_0^2-25\omega^2)} \tag{4-116}$$

This equation gives real roots for λ_{59} with no such condition as given by eq. (4-113). Since in this particular case, eq. (4-115) is a better approximation to the magnetisation characteristic than the one given by eq. (4-109), the condition suggested by eq. (4-113) is incorrect and may lead to misleading conclusions.

Examination of the algebra of the method suggests that if a particular higher harmonic is of interest, then a polynomial approximation of order greater than or equal to the order of the harmonics being investigated must be used to avoid incorrect results. This statement assumes that the higher order polynomial approximation used is a better fit to the magnetisation characteristic than any of lower order.

4-9-4 A Numerical Example.

The circuit to be studied is shown in Fig. (4-6) and represents a transmission line of length 320 miles terminated by a transformer whose i(λ) characteristic is given by Fig.(4-1), curve (a). The following values are also used.

L_s = 0.75 henry

C = 4.54 x 10^{-7} Farads.

e = 375000 Sin 377t.

Two values for n will be considered

n=3

$$i = 0.00024\lambda + 9.84\left(\frac{\lambda}{1000}\right)^3 \qquad (4\text{-}117)$$

n=9

$$i = 0.003\lambda + 13\left(\frac{\lambda}{1000}\right)^9. \qquad (4\text{-}118)$$

$$a_{13} = 0.24 \times 10^{-3} \qquad a_3 = 0.984 \times 10^{-8} \qquad (4\text{-}119)$$

$$a_{19} = 0.3 \times 10^{-2} \qquad a_9 = 0.13 \times 10^{-25} \qquad (4\text{-}120)$$

$$\omega_0^2 = \frac{1}{L_s C} = \frac{10^7}{0.75 \times 4.54} = 2.94 \times 10^6 \qquad (4\text{-}121)$$

$$\omega^2 = (377)^2 = 0.142 \times 10^6 \quad \text{(radians per sec)}^2 \qquad (4\text{-}122)$$

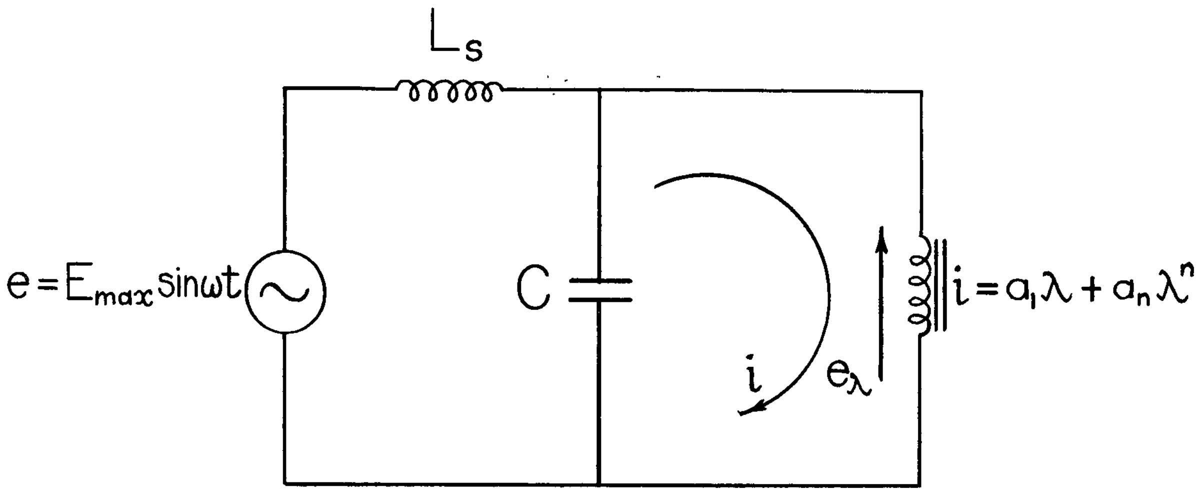

L_s
$e = E_{max} \sin\omega t$
C
i
e_λ
$i = a_1\lambda + a_n\lambda^n$

FIG. 4-6

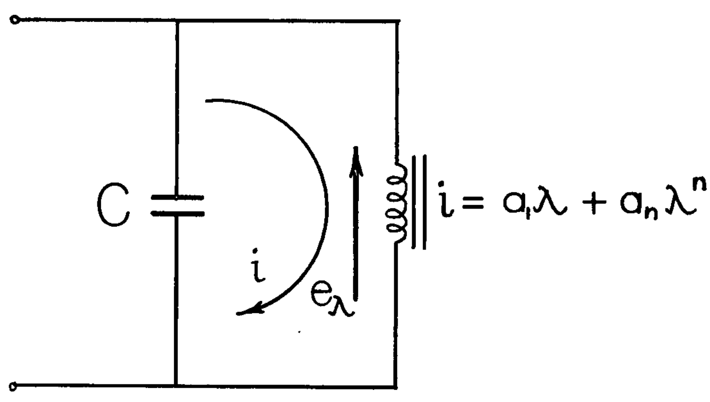

C
i
e_λ
$i = a_1\lambda + a_n\lambda^n$

FIG. 4-7

$$G = \frac{E_{max}\omega_0^2}{\omega} = \frac{375000 \times 2.94 \times 10^6}{377} = 2.92 \times 10^9 \tag{4-123}$$

$$h_3 = a_9 L_s \omega_0^2 = 2.16 \times 10^{-2} \tag{4-124}$$

$$h_9 = a_9 L_s \omega_0^2 = 2.86 \times 10^{-20}. \tag{4-125}$$

$$\frac{4}{3h_3} = 61.6 \tag{4-126}$$

$$\frac{256}{126h_9} = 0.71 \times 10^{-20} \tag{4-127}$$

For n=3, the fundamental component λ_{13} results from

$$\lambda_{13}^3 + \frac{4(\omega_0^2 - \omega_1^2)\lambda_{13}}{3h_3} - \frac{G \times 4}{3h_3} = 0. \tag{4-128}$$

i.e.

$$\lambda_{13}^3 + 1.72 \times 10^8 \lambda_{13} - 1.80 \times 10^{11} = 0. \tag{4-129}$$

The only real root of this equation is given by λ_{13} = 1050 wb-turns. (4-130)

For n=9, the corresponding equation is

$$\lambda_{19}^9 + 1.99 \times 10^{26}\lambda_{13} - 2.08 \times 10^{29} = 0. \tag{4-131}$$

The only real root of this equation is given by

$$\lambda_{19} = 1010 \text{ wb-turns} \tag{4-132}$$

For the higher harmonics

n=3

$$\frac{\lambda_{33}}{\lambda_{13}} = \frac{h_3 \lambda_{13}^2}{4(\omega_0^2 - 9\omega^2)} = 0.324 \times 10^{-2} \tag{4-133}$$

$$\therefore \quad \lambda_{33} = 0.324\% \text{ of the fundamental.} \tag{4-134}$$

n=9

$$\frac{\lambda_{39}}{\lambda_{19}} = \frac{84h_9\lambda_{19}^8}{256(\omega_0^2 - 9\omega^2)} = 0.55 \times 10^{-2} \tag{4-135}$$

$\therefore \quad \lambda_{39} = 0.55\%$ of the fundamental. (4-136)

Using the polynomial of order 3 instead of the 9^{th} order results in a small difference in the amplitude of the fundamental flux but an appreciable difference in that of the third harmonic.

The added accuracy in using the more complex relationship may or may not be justified but in any case, an appreciation of the order of magnitude of a particular harmonic can be obtained.

4-10 Transient to Steady-State. The Method of Integral Curves.

Examination of some of the current and voltage wave-forms obtained using the incremental method (Chapter 3) shows that in general, the transient behaviour of the systems under study can be extremely complex. This complexity does not encourage an analytical solution of the transient state and there have been few efforts in this regard, recorded in the literature.[48] [49] [50]

The Method of Integral Curves can give an insight into the behaviour of a particular harmonic component and its main usefulness is the prediction of the existence of some higher harmonic or sub-harmonic in the steady-state.

To establish the procedure and show its usefulness, the method is applied in the investigation of the third harmonic and the one third sub-harmonic.

4-10-1 Energisation of a 'Long-Line'. The Third Harmonic.

The equation describing the energisation of a 'long-line' is (eq. (4-49)- sec. 4-7-2)

$$\frac{d^2\lambda}{dt^2} + \omega_0^2\lambda + h_3\lambda^3 - G \cos \omega t = 0 \qquad (4\text{-}137)$$

A solution

$$\lambda = x(t) \text{ Sin } 3\omega t + y(t) \text{ Cos } 3\omega t + \lambda_1 \text{ Cos } \omega t \qquad (4\text{-}138)$$

is sought, where x(t), y(t) functions of time are associated with the 3^{rd} harmonic.

The substitutions

$$\begin{aligned} x(t) &= a(t).\lambda_1 \\ y(t) &= b(t).\lambda_1 \end{aligned} \qquad (4\text{-}139)$$

are made, then eq. (4-138) becomes

$$\lambda = \lambda_1 \left| \text{ Cos } 3\omega t + a(t). \text{ Sin } 3\omega t + b(t). \text{ Cos } \omega t \right| \qquad (4\text{-}140)$$

Substituting this value of λ in eq. (4-137), the Principle of Harmonic Balance is applied and the following equations result from equating the 3^{rd} harmonic terms,

$$\lambda_1 . \frac{d^2a}{dt^2} + \lambda_1 \left| 6\omega . \left| \frac{da}{dt} - 9\omega^2 b \right| + \lambda_1 . \omega_0^2 b + \frac{3h\lambda_1^3}{4} \left| \frac{1}{3} + b^3 + a^2 b \right| = 0 \qquad (4\text{-}141)$$

$$\lambda_1 . \frac{d^2b}{dt^2} - \lambda_1 \left| 6\omega . \frac{db}{dt} - 9\omega^2 a \right| + \lambda_1 . \omega_0^2 a + \frac{3h\lambda_1^3}{4} \left| 2a + a^3 + ab^2 \right| = 0 \qquad (4\text{-}142)$$

Assuming that a(t) and b(t) are slowly varying

$\frac{d^2a}{dt^2}$ and $\frac{d^2b}{dt^2}$ can be neglected.

Equations (4-141) and (4-142) are then put in the form

$$\frac{da}{dt} = - \frac{1}{6\omega} \left| b(\omega_0^2 - 9\omega^2) + \frac{3h\lambda_1^2}{4} \left(\frac{1}{3} + b^3 + 2b + a^2 b \right) \right| \qquad (4\text{-}143)$$

$$\frac{db}{dt} = \frac{1}{6\omega} \left| a(\omega_0^2 - 9\omega^2) + \frac{3h\lambda_1^2}{4} \quad (a^3 + 2a + ab^2) \right| \qquad (4\text{-}144)$$

Therefore, following the procedure outlined in sec (4-6) the functions Y(a,b), X(a,b) are now defined

$$Y(a,b) \triangleq \frac{db}{dt} = \frac{1}{6\omega}\left| a(\omega_0^2-9\omega^2) + \frac{3h\lambda_1^2}{4}\ (a^3+2a+ab^2)\right| \tag{4-145}$$

$$X(a,b) \triangleq \frac{da}{dt} = \frac{-1}{6\omega}\left| b(\omega_0^2-9\omega^2) + \frac{3h\lambda_1^2}{4}\ (\tfrac{1}{3} + b^3+2b+a^2b)\right| \tag{4-146}$$

4-10-1-1 The Integral Curves of the 3rd Harmonic.

From eqs. (4-145) and (4-146), the equations to the Integral Curves are

$$\frac{Y(a,b)}{X(a,b)} = \frac{db}{dt} = \frac{a(\omega_0^2-9\omega^2) + \frac{3h_3\lambda_1^2}{4}\ (a^3+2a+ab^2)}{-\left|b(\omega_0^2-9\omega^2) + \frac{3h_3\lambda_1^2}{4}\ (\frac{1}{3} + b^3+2b+a^2b)\right|} \tag{4-147}$$

also

$$\frac{\partial Y(a,b)}{\partial b} = \frac{1}{6\omega}\left| \frac{3h_3\lambda_1^2}{2} \cdot ab \right| \tag{4-148}$$

$$\frac{\partial X(a,b)}{\partial b} = -\frac{1}{6\omega}\ (\frac{3h_3\lambda_1^2}{2} \cdot ab) \tag{4-149}$$

$$\therefore \quad \frac{\partial Y}{\partial b} + \frac{\partial X}{\partial a} = 0 \tag{4-150}$$

$\therefore$ the eq.

$$Y\ da - X\ db = 0 \tag{4-151}$$

is an exact integral and the integral curves are given by

$$\frac{a^2}{2}\ (\omega_0^2-9\omega^2) + \frac{3h_3\lambda_1^2}{4}\ (\frac{a^4}{4} + a^2 + \frac{a^2b^2}{2}) +$$
$$+ \frac{b^2}{2}\ (\omega_0^2-9\omega^2) + \frac{3h_3\lambda_1^2}{4}\ (\frac{b}{3} + \frac{b^4}{4} + b^2 + \frac{a^2b^2}{2}) = C_2 \tag{4-152}$$

where C_2 is a constant.

The substitution

$$r_3^2 \triangleq a^2 + b^2 \tag{4-153}$$

in eq. (4-152), gives

$$r_3^4 + r_3^2\left(4 + 2b^2 + \frac{8}{3h_3\lambda_{13}^2}(\omega_0^2-9\omega^2)\right) + \frac{4b}{3} - 2b^4 - 4\ C_2 = 0 \tag{4-154}$$

Eq. (4-154) is a quadratic in r_3^2 and for various values of C_2, solution of this quadratic as b is varied gives values of r_3^2. Using these values, an a-b plot can be made.

4-10-1-2 The Singularities of the Third Harmonic.

The singularities (a_0, b_0) are given by the solution of the simultaneous equations,

$$b_0(\omega_0^2-9\omega^2) + \frac{3h_3\lambda_{13}^2}{4}\left(\frac{1}{3} + b_0^3+2b_0+a_0^2b_0\right) \equiv 0 \tag{4-155}$$

and

$$a_0(\omega_0^2-9\omega^2) + \frac{3h_3\lambda_{13}^2}{4}\left(a_0^3 + 2a_0+a_0b_0^2\right) \equiv 0 \tag{4-156}$$

From (4-156), $a_0 = 0$ is one root.

Putting $a_0=0$ in eq. (4-155)

$$b_0(\omega_0^2-9\omega^2) + \frac{3h_3\lambda_{13}^2}{4}\left(\frac{1}{3} + b_0^3 + 2b_0\right) = 0 \tag{4-157}$$

$$\therefore \quad b_0^3 + b_0\left(2 + \frac{4}{3h_3\lambda_{13}^2}(\omega_0^2-9\omega^2)\right) + \frac{1}{3} = 0 \tag{4-158}$$

Equation (4-158) gives 3 values for b.

Using the condition stated in eq. (4-82), for normal power-system values of ω_0 and ω, eq. (1-58) has only one real root. Therefore, only one singularity exists and this is given by

$$a_0 = 0$$
$$b_0 = \frac{h_3\lambda_{13}^2}{4(\omega_0^2-9\omega^2)} \tag{4-159}$$

4-10-1-3 The Nature of the 3rd Harmonic Singularity.

From eq. (4-145) and (4-146),

$$\frac{\partial X}{\partial a} = - \frac{1}{6\omega} \left(\frac{3h_3\lambda_{13}^2}{4} . \; 2\,ab \right) \triangleq A_1 \tag{4-160}$$

$$\frac{\partial X}{\partial b} = - \frac{1}{6\omega} \left| (\omega_0^2 - 9\omega^2) + \frac{3h_3\lambda_{13}^2}{4} (3b^2 + 2 + a^2) \right| \triangleq A_2 \tag{4-161}$$

$$\frac{\partial Y}{\partial a} = - \frac{1}{6\omega} \left| (\omega_0^2 - 9\omega^2) + \frac{3h_3\lambda_{13}^2}{4} (3a^2 + 2 + b^2) \right| \triangleq B_1 \tag{4-162}$$

$$\frac{\partial Y}{\partial b} = \frac{1}{6\omega} \left(\frac{3h_3\lambda_{13}^2}{4} . \; 2\,ab \right) \triangleq B_2 \tag{4-163}$$

From eq. (4-26),

$$\nu_1, \nu_2 = A_1 + B_2 \pm \frac{((A_1 - B_2)^2 + 4A_2B_1)^{\frac{1}{2}}}{2} \tag{4-164}$$

For the singularity,

$$a_0 = 0 \qquad b_0 = \text{some finite value}$$

$$\therefore \quad A_1 = B_2 = 0 \tag{4-165}$$

$$A_2 = - \frac{1}{6\omega} \left| (\omega_0^2 - 9\omega^2) + \frac{3h_3\lambda_{13}^2}{4} (3b_0^2 + 2) \right| \tag{4-166}$$

$$B_1 = \frac{1}{6\omega} \left| (\omega_0^2 - 9\omega^2) + \frac{3h_3\lambda_{13}^2}{4} (3b_0^2 + 2) \right| \tag{4-167}$$

Hence, provided $\omega_0^2 > 9\omega^2$ - a usual condition in power systems,

$$4A_2B_1 < 0 \tag{4-168}$$

and thus, ν_1, ν_2 are imaginary and the singularity is a vortex.

<u>4-10-1-4 The Isoclines. The 3^rd^ Harmonic .</u>

The isoclines are sometimes helpful in the a-b plot of a particular harmonic.

<u>The isoclines</u> for the 3rd Harmonic plot are given by

$$m = \frac{db}{da} = \frac{a(\omega_0^2-9\omega^2) + \frac{3h_3\lambda_{13}^2}{4}(a^3 + 2a + ab^2)}{b(\omega_0^2-9\omega^2) + \frac{3h_3\lambda_{13}^2}{4}(\frac{1}{3} + b^3 + 2b + a^2b)} \qquad (4\text{-}169)$$

This equation can give corresponding values of a and b for various values of m.

4-10-1-5 The Transient to Steady-State behaviour of the 3rd Harmonic.

Sections (4-10-1-1) to (4-10-1-4) provide information for a plot of the Integral Curves of the 3rd Harmonic component of the flux-linkage. However, some interesting results can be derived from inspection of some of the equations.

The equation for the integral curves - eq. (4-154) states

$$r_3^4 + r_3^2 \left(4 + 2b^2 + \frac{8}{3h_3\lambda_{13}^2}(\omega_0^2-9\omega^2) \right) + \frac{4b}{3} - 2b^4 - 4\,C_2 = 0 \qquad (4\text{-}170)$$

Normally,

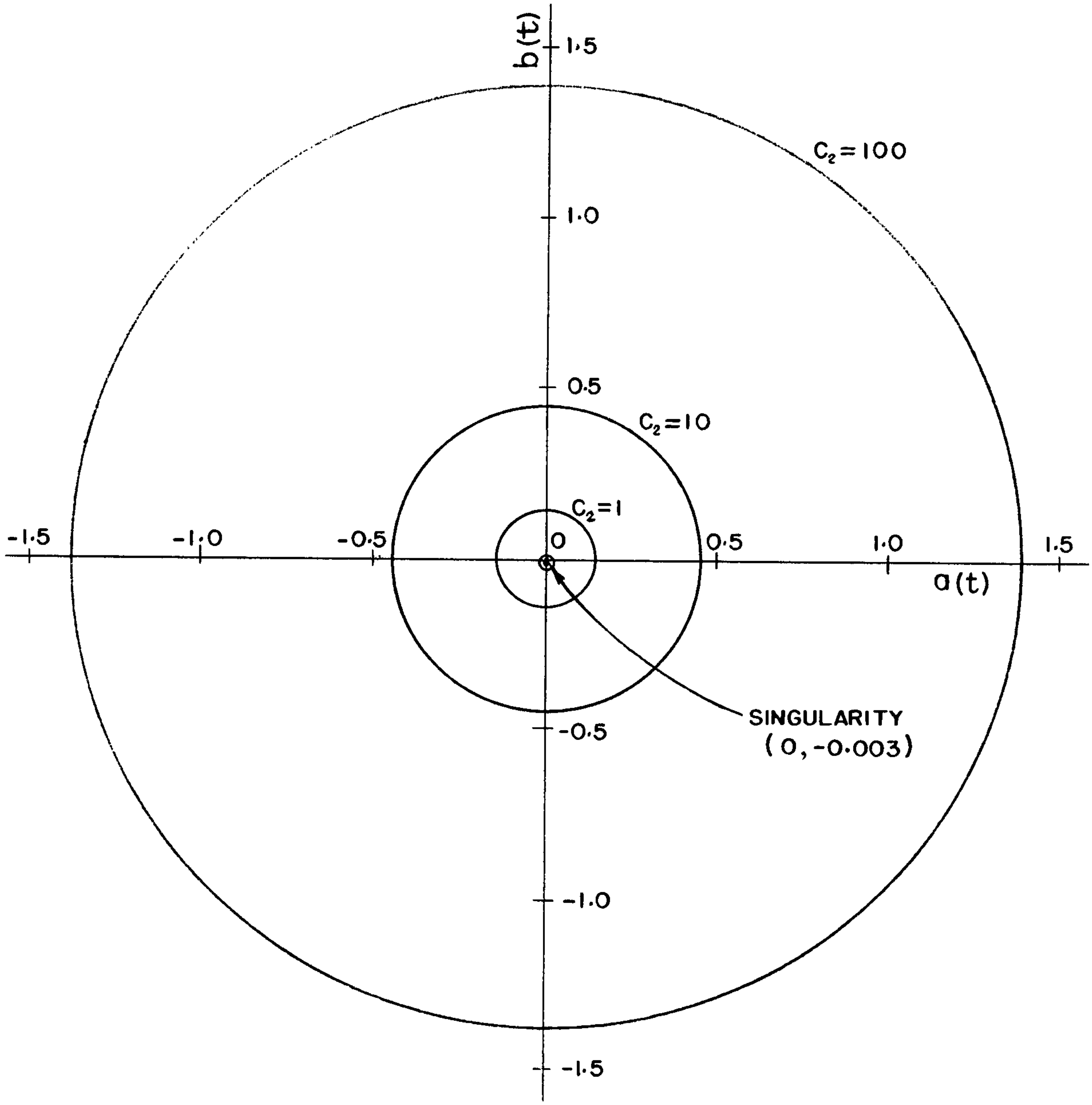
$$\frac{8(\omega_0^2-9\omega^2)}{3h_3\lambda_{13}^2} \gg 4 + 2b^2 \;; \qquad (4\text{-}171)$$

eq. (4-170) then becomes

$$r_3^4 + r_3^2 \; (\omega_0^2-9\omega^2) . \frac{8}{3h_3\lambda_{13}^2} + \frac{4b}{3} - 2b^4 - 4\,C_2 = 0 \qquad (4\text{-}172)$$

If C_2 is very small, and $|b| < 1$, this equation has a large negative root and a small ($<|b|^2$) root for r_3^2. In neither case, are real values for a and b obtained.

If C_2 is large, and $|b| \ll 1$, the roots are then independent of the values of b and r_3^2 remains reasonably constant as b is varied.



INTEGRAL CURVES

FOR

THIRD HARMONIC $\omega_0^2 - 9\omega^2 = 1.66 \times 10^6$

FIG. 4-8

The integral curves are then very nearly circles. The approximate value of the singularity obtained in sec (4-10-1-2) is

$$a_0 = 0 \tag{4-173}$$

$$b_0 = \frac{h_3\lambda_{13}^2}{4(\omega_0^2-9\omega^2)} \tag{4-174}$$

This corresponds exactly to the peak value of the steady-state third harmonic component given by eq. (4-99) obtained in the steady-state analysis.

These facts suggest that the integral curves are a set of near-circular loci, concentric around the steady-state value of the 3rd harmonic. If losses are considered, these loci will become a continous curve spiralling towards the singularity.

Using the system parameters of the numerical example - sec. (4-9-4)- integral curves for three different values of C_2 are shown in Fig. (4-8). This plot verifies the conclusions arrived at in this section.

4-11-1 Energisation of a 'Long-Line'. The Third Sub-Harmonic.

Eq. (4-49) is repeated

$$\frac{d^2\lambda}{dt^2} + \omega_0^2\lambda + h_3\lambda^3 - G \text{ Cos } 3\omega t = 0 \tag{4-175}$$

As previously, a solution

$$\lambda = x(t) \text{ Sin } \omega t + y(t) \text{ Cos } \omega t + \lambda_{13} \text{ Cos } \omega t \tag{4-176}$$

is sought. Note that to avoid fractions, the driving frequency is now 3ω.

The substitutions

$$x(t) = a(t).\ \lambda_{13} \tag{4-177}$$

$$y(t) = b(t)\ldots\lambda_{13} \tag{4-177}$$

are made and with these substitutions, the expression for λ - eq. (4-176) - is substituted in eq. (4-175).

Assuming that a(t), and b(t) are slowly varying

$\frac{d^2a}{dt^2}$ and $\frac{d^2b}{dt^2}$ can be neglected.

The equations corresponding to eqs. (4-143) and (4-144) are then

$$\frac{da}{dt} = \frac{-1}{2\omega}\left| b(\omega_0^2-\omega^2) + \frac{3h_3\lambda_{13}^2}{4}(b^3+a^2b+2b+b^2-a^2)\right| \tag{4-178}$$

$$\frac{db}{dt} = \frac{1}{2\omega}\left| a(\omega_0^2-\omega^2) + \frac{3h_3\lambda_{13}^2}{4}(a^3+ab^2+2a-2ab)\right| \tag{4-179}$$

The functions Y(a,b), X(a,b), are then

$$Y(a,b) \triangleq \frac{db}{dt} = \frac{1}{2\omega}\left| a(\omega_0^2-\omega^2) + \frac{3h_3\lambda_{13}^2}{4}(a^3+ab^2+2a-2ab)\right| \tag{4-180}$$

$$X(a,b) \triangleq \frac{da}{dt} = -\frac{1}{2\omega}\left| b(\omega_0^2-\omega^2) + \frac{3h_3\lambda_{13}^2}{4}(b^3+a^2b+2b+b^2-a^2)\right| \tag{4-181}$$

4-11-1-1 The Integral Curves of the Third Sub-Harmonic.

The equation $Y.da-X.db = 0$ (4-182)

is again a complete integral and the integral curves are given by

$$\frac{a^4}{4} + \frac{a^2b^2}{2} + \frac{b^4}{4} + (a^2+b^2)\left|1+\frac{2}{3h_3\lambda_{13}^2}(\omega_0^2-\omega^2)\right| + \frac{a^2b^2}{2}$$

$$-2a^2b + \frac{b^3}{3} - C_3 = 0 \tag{4-183}$$

where C_3 is some constant.

The substitution

$r^2_{1/3} = a^2 + b^2$ is made and the resulting equation

is

$$r^4_{1/3} + r^2_{1/3}\left(4 - 8b + 2b^2 + \frac{8}{3h_3\lambda_1^2}(\omega_0^2-\omega^2)\right) + \frac{28}{3}b^3 - 2b^4 - 4C_3 = 0 \tag{4-184}$$

This quadratic in $r^2_{1/3}$ can be solved for various values of b and C_3 and an a-b plot is made from these results.

4-11-1-2 The Singularities of the Third Sub-Harmonic.

The singularities (a_0, b_0) are given by the simultaneous equations

$$a_0(\omega_0^2-\omega^2) + \frac{3h_3\lambda_{13}^2}{4}(a_0^3+a_0b_0^2 + 2a_0-2a_0b_0) \equiv 0 \tag{4-185}$$

and

$$b_0(\omega_0^2-\omega^2) + \frac{3h_3\lambda_{13}^3}{4}(b_0^3+a_0^2b_0+ 2b_0+b_0^2-a_0^2) \equiv 0 \tag{4-186}$$

$a_0=0$ $b_0=0$ is one singularity.

The other roots are given by

$$a_0^2 = 3b_0^2 \tag{4-187}$$

and

$$4b_0^2-2b_0+2 + \frac{3h_3\lambda_{13}^2}{4}(\omega_0^2-\omega^2) = 0 \tag{4-188}$$

Solving for b_0 from eq. (4-188)

$$b_0 = 0.25 \pm \left(-0.4375 - \frac{3}{4}h_3\lambda_{13}^2(\omega_0^2-\omega^2)\right)^{\frac{1}{2}} \tag{4-189}$$

Hence, singularities other than (0,0) can exist only if $\omega^2 > \omega_0^2$, an unlikely condition in power-systems.

4-11-1-3 The Nature of the Singularity. Third Sub-Harmonic.

For the singularity (0,0) from eqs. (4-178), (4-179)

$$A_1 = B_2 = 0 \tag{4-190}$$

$$A_2 = \frac{-3h_3\lambda_{13}^2}{4\omega} \tag{4-191}$$

$$B_1 = \frac{3h_3\lambda_{13}^2}{4\omega} + \frac{1}{2\omega}(\omega_0^2-\omega^2) \tag{4-192}$$

$$\nu_1,\ \nu_2 = \pm\ (4A_2B_1)^{\frac{1}{2}} \tag{4-193}$$

From eqs. (4-190), (4-191), (4-192), (4-193), ν_1, ν_2 are equal and imaginary for normal values of ω_0^2. The singularity (0,0) is therefore a vortex.

<u>4-11-1-4 The Isoclines. The Third Sub-Harmonic.</u>

The isoclines in this case are given by

$$M = \frac{db}{da} = \frac{a\left(\frac{4}{3h_3\lambda_{13}^2}(\omega_0^2-\omega^2) + a^2+b^2+2-2b\right)}{-b\left(\frac{4}{3h_3\lambda_{13}^2}(\omega_0^2-\omega^2) + a^2+b^2+2 \quad +a^2-b^2\right)} \tag{4-194}$$

or in another form

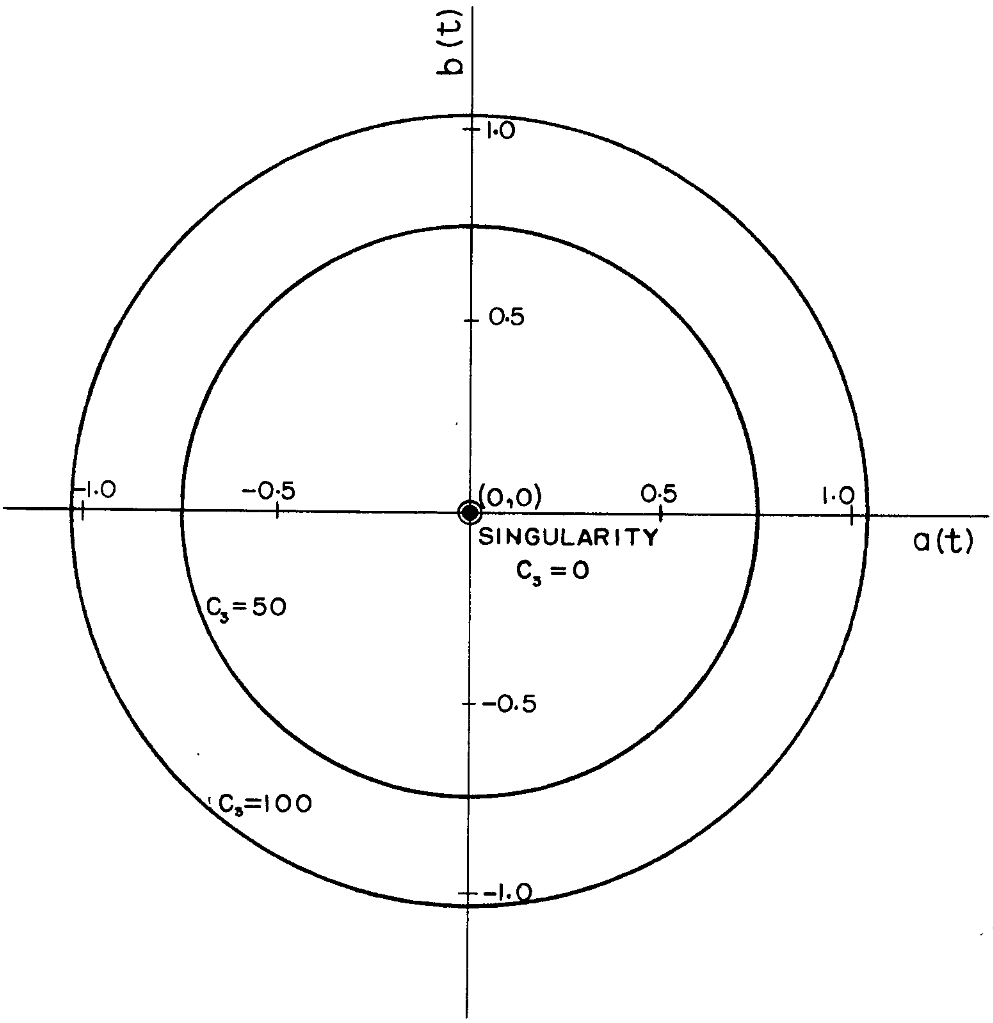
$$a^3+a^2(mb-m) + a\left(\frac{4}{3h_3\lambda_{13}^2}(\omega_0^2-\omega^2)+2(1-b)+b^2\right)$$

$$+ m\left(b^3+b^2+\frac{4b}{3h_3\lambda_{13}^2}(\omega_0^2-\omega^2) + 2b\right) = 0 \tag{4-195}$$

The solution of this cubic equation in 'a' for fixed values of m as 'b' is varied gives points for plotting the isoclines which are helpful in the integral curve plot.

<u>4-11-1-5 Transient to Steady-State Behaviour of the Third Sub-Harmonic.</u>

Similar considerations as used in discussing the Third Harmonic show that the integral curves are very nearly circular



INTEGRAL CURVES
FOR
THIRD SUB-HARMONIC $\omega_0^2 - \omega^2 = 2.924 \times 10^6$

FIG. 4-9

about the singularity - in this case the point (0,0). This suggests that no stable sub-harmonic oscillation of flux-linkage exists in the steady-state.

The integral curves corresponding to the $\frac{1}{3}$ Sub-Harmonic is shown in Fig. (4-9).

4-12 The Condition for the Maximum Third Harmonic Component.

In sec. (4-9), the condition

$$\omega_0^2 = 9\omega^2 \tag{4-196}$$

is discussed. This condition leads to a maximum value of the third harmonic component.

Substituting this condition in eq. (4-155) and (4-156), the singularities are given by

$$\frac{1}{3} + b_0^3 + 2b_0 + a_0^2 b_0 \equiv 0 \tag{4-197}$$

and

$$a_0^3 + 2a_0 + a_0 b_0^2 \equiv 0 \tag{4-198}$$

These equations give one singularity

$$a_0 = 0 \tag{4-199}$$

and

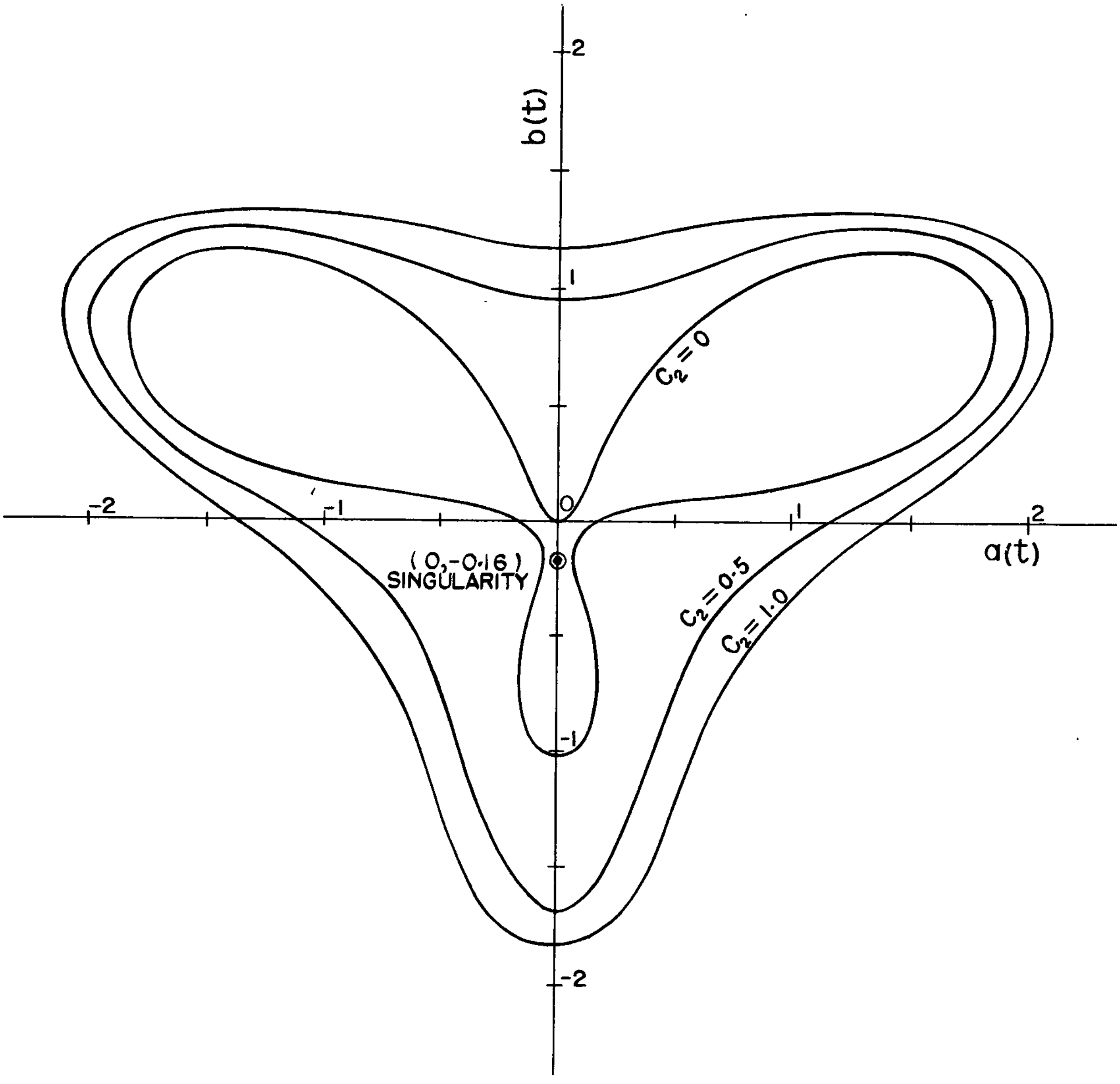
b_0 - the only real root of the equation

$$b_0^3 + 2b_0 + \frac{1}{3} = 0 \tag{4-200}$$

Since $b_0 = x_0(t).\lambda_{13}$, eq. (4-200) corresponds exactly with eq. (4-106) arrived at in sec. 4-9.

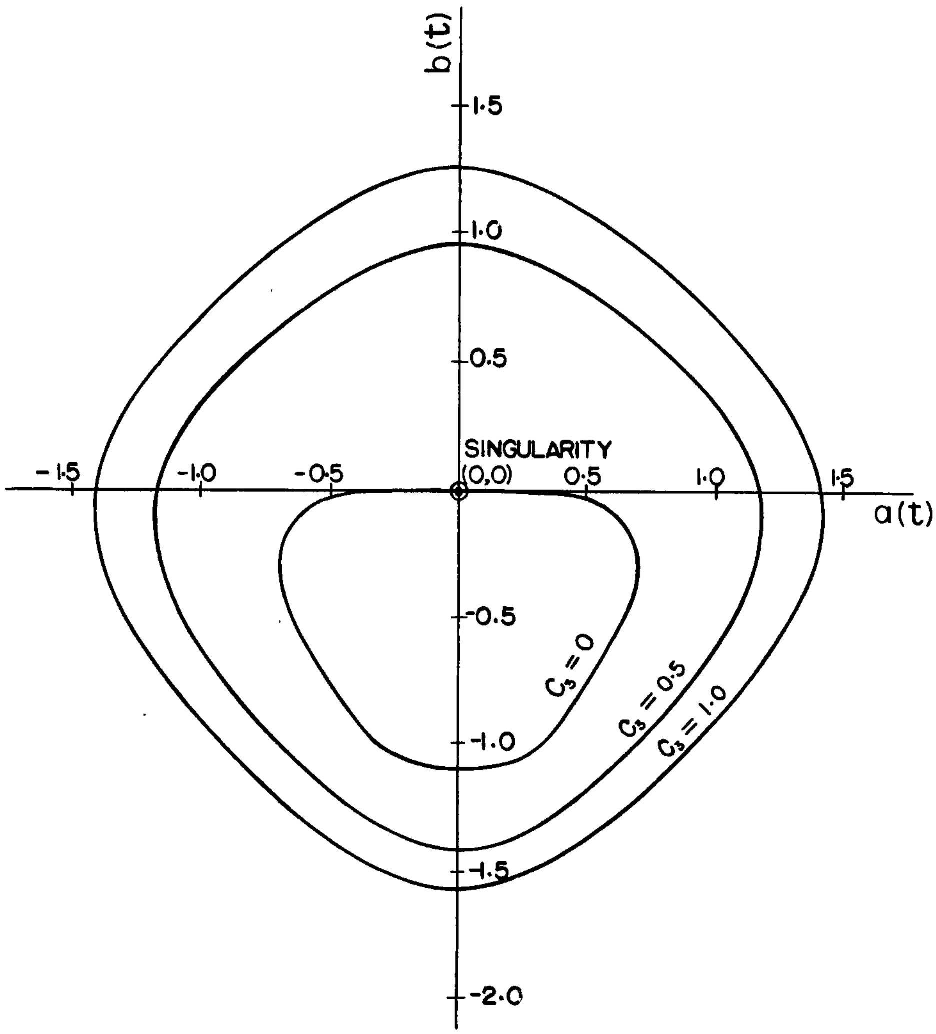
This behaviour is shown graphically in the integral curve plot - Fig. (4-10). The singularity given by

$$a_0 = 0 \qquad b_0 = 0.16\lambda_{13}$$



INTEGRAL CURVES
FOR
THIRD HARMONIC $\omega_0^2 = 9\omega^2$

FIG. 4-10



INTEGRAL CURVES
FOR
THIRD SUB-HARMONIC $\omega_0^2 - \omega^2 = 0$

FIG. 4-11

represents the steady-state and the loci represent the a(t), b(t) behaviour for various values of C_2.

For comparison, the integral-curve plot corresponding to the Third Sub-Harmonic for condition $\omega_0^2-\omega^2$ is shown in Fig. 4-11.

4-13. De-energisation of a 'Long-Line'. The Method of Integral Curves.

A similar procedure can be adopted in considering the flux-linkage behaviour when a 'long-line' terminated by a transformer is de-energised.

Such a system is considered in Appendix 3. The system is lossless and this factor, in this particular case, detracts from the usefulness of the result. However, the establishing of the procedure when more than one singularity exists (in this case, five singularities are found) is considered important enough to include such an example.

The numerical details and integral curves are in Appendix 4.

4-14 Discussion of the Analytical Methods.

The three methods used in this chapter are of definite usefulness in the investigation of this type of problem but their application may be limited by the desired accuracy.

4-14-1 Use of the Method of Isoclines.

The Method of Isoclines has the disadvantages of any graphical method. The accuracy is limited and the graphical construction can be time consuming.

These two factors suggest that this method as used in this investigation may only be suitable for obtaining good approximations for peak values rather than solutions for the entire waveform. This is not entirely correct as with the computing devices available, reasonably accurate results are possible.

In this study, the numerical results for the isocline plot were computed by a digital computer and plotted by an x-y plotter. The use of these devices provided in a reasonable time, quite accurate results.

4-14-2 Use of The Principle of Harmonic Balance.

The main usefulness of this method is in giving an indication of possible abnormal conditions arising in the steady-state.

The criterion established for abnormally high flux and voltage values of a particular higher harmonic can be useful in preliminary investigations. The accuracy of the actual value computed by this method will be limited by the approximations that are made in arriving at a solution.

4-14-3 Use of The Method of Integral Curves.

This method can give an insight into the system behaviour over a wider range of behaviour than the other two. The numerical computation needed is greater than in the previous methods but the fact that it gives an insight into the complex transient behaviour of the system justifies this.

Further comments on this method are included in Chapter 5.

CHAPTER 5

5-1 Introduction

This chapter will be devoted to some general conclusions arising from this study. It will necessarily be brief as the final sections of each previous chapter are used in discussions and conclusions arising from the topics studied in it.

In this chapter, attention will be restricted to some general ideas about the phenomena being investigated; the approach used in this study; and suggestions for future work.

5-2 The Phenomena.

The behaviour of the transformer current inrush has been for a long time a matter of great engineering interest* and the question of overvoltages as a result of 'saturation effects' has become increasingly important over the past ten years.+ The results presented in chapter 3 indicate to some extent the reasons for this.

Figs. (3-10) to (3-17) show the large transient current peaks and the associated overvoltages that are expected under certain normal system conditions. Information about this particular aspect of system behaviour will be an important necessity in most power system designs and the accuracy of this information will become more critical as systems grow larger, transmission distances become longer and transmission voltages are increased to higher levels.

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* See references (36), (51), (52), (8), (53), (54), (55), (6).

+ See references (56), (57), (58), (59), (60), (61), (62).

Present day economic considerations do not permit large design tolerances, and techniques for dealing with the most complex phenomena must be developed.

5-3 The Methods.

The digital computer has proved itself an extremely useful tool in certain aspects of power system studies.[63] [64] Since the success of the incremental method is completely dependent on the availability of a fast and accurate digital computer, this type of investigation can be another example of its usefulness.

The incremental method has some inherent advantages which makes it an attractive approach to these problems. Firstly, more than one non-linear effect can be taken into account and this is of particular importance in power-systems.(sec. 3-5-4, is an example where this is utilised). With the increasing importance of certain non-linear effects such as corona losses, this advantage may become quite an important one. Secondly, closely related as it is to a familiar linear technique, the mathematical details necessary for computation are relatively simple and the availability of comprehensive tables of Laplace transform pairs etc. simplifies some of the mathematical computation necessary. Thirdly, strict control of the accuracy and of the solution time, continuity of the solution - all important factors in programming - are available.

In contrast, the iterative method lacks these three advantages to some extent and its main attraction lies in its speed and the independence of a final solution on any small errors made during the computation.

Some comments about the analytical methods have already been made in sec. 4-14 but further remarks about the method of integral curves are justified.

Because of the complexity of the transient phenomena associated with this problem, any analytical method that gives an insight into the transient behaviour is worthy of consideration. The method of integral curves, providing a graphical picture of the transient behaviour of the harmonics with definite information about the steady-state condition offers a quick and simple approach to this problem. As with most analytical methods for non-linear problems, it does not give precise answers but can provide helpful information for further studies or in preliminary investigations.

5-4 Suggestions for Future Work.

In the course of this study, many ideas and questions arose and these, although promising to be of great interest could not be dealt with in these investigations.

The more interesting of these are included here as suggestions for future work.

5-4-1 The Distributed-Parameter Solution for a Transmission Line.

Interest in transient behaviour of power systems has been growing and because of this, the need for adequate representations of the system components under transient conditions has become more necessary.

Representation of the power transmission line by its distributed parameters and the solution of the equations arising from this has received some recent attention.[65]

In the past, there have been some worthy contributions and in some cases, termination of a line by a non-linear element has received consideration.[50]

5-4-2 The Source-Reactance.

The simplest representation of the a.c. synchronous generator is used in this study.

The adequacy of such a representation sec.2-11 under these system conditions can be a matter of investigation. The simplicity of the representation is an attraction but the question of the values and characteristics of the source reactance will need further study.

5-4-3 System Losses.

Because of the close system design tolerances demanded by economic considerations, system losses and the effect these have on system behaviour are becoming increasingly important.

In this study, attention is given to transformer and line losses. Corona losses are not considered but a suggestion how these can be accounted for using the incremental method is made in sec. 3 - 3 . With sufficient information of the magnitude and variation of corona losses, some such suggestion can be used to account for these additional losses.

5-4-4 The Analytical Methods.

The application of methods of non-linear analysis will be helpful in many aspects of this type of investigation.

Exact solutions for equations similar to the types studied are receiving some attention[66][67] and in this regard, the method of integral curves offers some possibilities. The inclusion of a term in the equation to account for system losses, the establishing of a relationship between the constant of integration C and actual switching initial conditions are subjects which may provide fruitful research.

Appendix 1

The Coefficients for Table 2-1

$$\alpha_1 \triangleq -2\left(\frac{\mu}{\alpha} + 1 - \frac{R_0}{\alpha L_R}\right) \qquad \text{(A 1-1)}$$

$$\alpha_2 = 2\left(1 + \frac{2\mu}{\alpha}\right) \qquad \text{(A 1-2)}$$

$$\alpha_3 = -2\left(\frac{\mu}{\alpha} + 1 + \frac{R_0}{\alpha L_R}\right) \qquad \text{(A 1-3)}$$

$$c_1 = \alpha_3 - \alpha_1 \qquad \text{(A 1-4)}$$

$$c_2 = \alpha_3 (\alpha_1 - \alpha_3) \qquad \text{(A 1-5)}$$

$$c_3 = (\alpha_3 - \alpha_1)(\alpha_3^2 - \alpha_2 - 1) \qquad \text{(A 1-6)}$$

$$c_4 = (\alpha_3 - \alpha_1)(2\alpha_3\alpha_2 - \alpha_1 - \alpha_3^3 + \alpha_3) \qquad \text{(A 1-7)}$$

$$c_5 = (\alpha_3 - \alpha_1)(\alpha_2^2 + \alpha_2 - 3\alpha_2\alpha_3^2 - \alpha_3^2 + \alpha_3^4 + 2\alpha_1\alpha_3 - 1) \qquad \text{(A 1-8)}$$

$$c_6 = \alpha_3^3(1 - \alpha_3^2 + 4\alpha_2) - 2\alpha_2(\alpha_3 - \alpha_1) - \alpha_3 + \alpha_1 + 3\alpha_3(1 - \alpha_2^2) \qquad \text{(A 1-9)}$$

Appendix 2

A Linear Example Solved Using the Incremental Method.

To demonstrate the validity of the Incremental method used in Chapter 3, a simple linear example of a resistance R in series with an inductance L energised by a voltage.

$$e = E_{max}(\text{Sin } \omega t + \theta) \qquad \text{(A 2-1)}$$

is considered.

Assuming that the initial conditions are

$$\begin{aligned} &t=0 \\ &\theta=\theta_0 \\ &i(0)=i_0 \end{aligned} \qquad \text{(A 2-2)}$$

then after a time t_2, using any suitable linear technique, the current $i(t_2)$ is given by

$$i(t_2)= i_0 e^{-\frac{R}{L}t_2} + \frac{E_{max}}{(R^2 + \omega^2 L^2)^{\frac{1}{2}}} e^{-\frac{R}{L}t_2} \text{ Sin}(\phi-\theta_0)$$

$$+ \frac{E_{max}}{(R^2 + \omega^2 L^2)^{\frac{1}{2}}} \text{ Sin}(\omega t_2 + \theta_0 - \phi) \qquad \text{(A 2-3)}$$

where

$$\tan \phi = \frac{\omega L}{R}$$

Similarly

$$i(t_1) = i_0 e^{-\frac{R}{L}t_1} + \frac{E_{max}}{(R^2 + \omega^2 L^2)^{\frac{1}{2}}} e^{-\frac{R}{L}t_1} \text{ Sin}(\phi-\theta_0)$$

$$+ \frac{E_{max}}{(R^2 + \omega^2 L^2)^{\frac{1}{2}}} \text{ Sin}(\omega t_1 + \theta_0 - \phi) \qquad \text{(A 2-4)}$$

In using the incremental method to obtain $i(t_2)$, initial conditions at $t=t_1$ $(t_2 > t_1)$ are considered. These are

$$t = t_1$$

$$i = i(t_1) \tag{A 2-5}$$

$$\theta = \theta_0 + \omega t_1$$

$\therefore$ after a time $t_2 - t_1$,

$$i = i(t_1) e^{-\frac{R}{L}(t_2 - t_1)} + \frac{E_{max}}{(R^2 + \omega^2 L^2)^{\frac{1}{2}}} e^{-\frac{R}{L}(t_2 - t_1)} \operatorname{Sin}(\phi - \omega t_1 - \theta_0)$$

$$+ \frac{E_{max}}{(R^2 + \omega^2 L^2)^{\frac{1}{2}}} \operatorname{Sin}(\omega(t_2 - t_1) + \omega t_1 + \theta_0 - \phi) \tag{A 2-6}$$

Substituting the value of $i(t_1)$ from eq. (4) and simplifying this expression for i yields

$$i = i_0 e^{-\frac{Rt_2}{L}} + \frac{E_{max}}{(R^2 + \omega^2 L^2)^{\frac{1}{2}}} e^{-\frac{Rt_2}{L}} \operatorname{Sin}(\phi - \theta_0)$$

$$+ \frac{E_{max}}{(R^2 + \omega^2 L^2)^{\frac{1}{2}}} e^{-\frac{Rt_2}{L}} \operatorname{Sin}(\omega t_2 + \theta_0 - \phi) \tag{A 2-7}$$

$$= i(t_2) \text{ as given by eq. (3)} \tag{A 2-8}$$

Appendix 3

Laplace Transform Pairs for Chapter 3

The following definitions are used:

$$\varepsilon_1 = e^{-\mu t}$$

$$\varepsilon_2 = e^{-\alpha t}$$

$$\beta_0^2 = \alpha^2 + \beta^2$$

$$\mu_0^2 = \mu^2 + \omega^2$$

$$\mu_1^2 = (\alpha-\mu)^2 + \beta^2$$

$$\theta_1 = \mathrm{Tan}^{-1} \frac{\omega}{\mu}$$

$$\theta_2 = \mathrm{Tan}^{-1} \frac{2\alpha\omega}{\beta_0^2-\omega^2}$$

$$\theta_3 = \mathrm{Tan}^{-1} \frac{\beta}{\mu-\alpha}$$

$$\theta_4 = \mathrm{Tan}^{-1} \frac{-2\alpha\beta}{\alpha^2-\beta^2+\omega^2}$$

$$\theta_5 = \mathrm{Tan}^{-1} \frac{\beta}{-\alpha}$$

c,f - constants

The following nine pairs and their combinations are used in the computations for Chapter 3.

	F(s)	f(t)
1	$\frac{1}{s+\alpha}$	$e^{-\alpha t}$
2	$\frac{s+c}{(s+\alpha)(s+\mu)}$	$\frac{(c-\mu)e^{-\mu t}-(c-\alpha)e^{-\alpha t}}{\alpha-\mu}$
3	$\frac{s+c}{(s+\mu)\ (s+\alpha)^2+\beta^2}$	$\frac{\varepsilon_1(c-\mu)}{\mu_1^2} + \frac{\varepsilon_2\sqrt{(c-\alpha)^2+\beta^2}\ \mathrm{Sin}(\beta t+\phi_1-\theta_3)}{\beta\mu_1}$
		$\phi_1 = \mathrm{Tan}^{-1} \frac{\beta}{c-\alpha}$

4 $\dfrac{s^2 + fc + c}{(s+\mu)\left[(s+\alpha)^2+\beta^2\right]}$ $\qquad$ $\dfrac{(\mu^2 - f\mu + c)\varepsilon_1}{\mu_1^2}$

$$+ \frac{\varepsilon_2}{\beta\mu_1}\sqrt{(\alpha^2-\beta^2-f\alpha+c)^2+\beta^2(f-2\alpha)^2}\ \mathrm{Sin}(\beta t+\emptyset_2-\theta_3)$$

$$\emptyset_2 = \mathrm{Tan}^{-1}\ \frac{\beta(f-2\alpha)}{\alpha^2-\beta^2-f\alpha+c}$$

5 $\dfrac{s^2 + fs + c}{(s+\mu)(s+\delta)\quad(s+\alpha)^2+\beta^2}$ $\qquad$ $\dfrac{(\mu^2 - f\mu + c)\varepsilon_1}{(\delta-\mu)\left((\alpha-\mu)^2+\beta^2\right)}$

$$+ \frac{(\delta^2 - f\mu + c)\varepsilon_2}{(\mu-\delta)\left((\alpha-\delta)^2+\beta^2\right)}$$

$$+ \frac{1}{\beta}\left[\frac{(\alpha^2-\beta^2-f\alpha+c)^2+\beta^2(f-2\alpha)^2}{\left[(\delta-\alpha)^2+\beta^2\right]\left[(\mu-\alpha)^2+\beta^2\right]}\right]^{\frac{1}{2}} \varepsilon_2 \mathrm{Sin}(\beta t+\emptyset_3-\theta_3$$

$$\emptyset_3 = \mathrm{Tan}^{-1}\ \frac{\beta(f-2\alpha)}{\alpha^2-\beta^2-f\alpha+c} - \mathrm{Tan}^{-1}\ \frac{\beta}{-\alpha}$$

6 $\dfrac{s + c}{(s+\mu)(s^2+\omega^2)\left[(s+\alpha)^2+\beta^2\right]}$ $\qquad$ $\dfrac{(c-\mu)\varepsilon_1}{\mu_0^2\mu_1^2} + \dfrac{\sqrt{c^2+\omega^2}}{\omega\mu_0\mu_2^2}\ \mathrm{Sin}(\omega t+\emptyset_3-\theta_1-\theta_2)$

$$+ \frac{\varepsilon_2\sqrt{(c-\alpha)^2+\beta^2}}{\beta\ \mu_1\mu_2^2}\ \mathrm{Sin}(\beta t+\emptyset_4-\theta_3-\theta_4)$$

$$\emptyset_4 = \mathrm{Tan}^{-1}\ \frac{\omega}{c}$$

$$\emptyset_5 = \mathrm{Tan}^{-1}\ \frac{\beta}{c-\alpha}$$

7 $\dfrac{s(s+c)}{(s+\mu)(s^2+\omega^2)\ (s+\alpha)^2+\beta^2}$ $\qquad$ This inverse function is obtained by differentiating the inverse function (6).

8	$\frac{s(s^2 + cs)}{(s+\mu)(s^2+\omega^2)\left[(s+\alpha)^2+\beta^2\right]}$	Differentiation of the inverse function (7) yields this function.
9	$\frac{1}{(s+\mu)(s^2+\omega^2)\left[(s+\alpha)^2+\beta^2\right]}$	$\frac{\varepsilon_1}{\mu_0^2\mu_1^2} + \frac{\mathrm{Sin}(\omega t-\theta_1-\theta_2)}{\omega\ \mu_0\mu_2^2} + \frac{\varepsilon_2}{\beta\ \mu_1\mu_2^2}\ \mathrm{Sin}(\beta t-\theta_3-\theta_4)$

Appendix 4

De-energisation of a 'Long-Line' System. The Method of Integral Curves.

To establish the procedure when more than one stable singularity exists, an example of a loss-less system is considered. The 30 c.p.s. frequency is investigated when a 320 mile system is de-energised.

The fact that losses are excluded in this example detracts from the usefulness of the result. The inclusion of losses naturally exclude the existence of any singularities other than (0,0).

The equation describing such a system (eq. 4-60) is

$$\frac{d^2\lambda}{dt^2} + \frac{1}{C}\left(a_{13}\lambda + a_3\lambda^3\right) = 0 \qquad \text{(A 4-1)}$$

Using the parameters of sec. 4-9 and proceeding in the manner of sec. 4-11 the following equations result:

$$Y(a,b) \triangleq \frac{db}{dt} = \frac{1}{2\omega}\left[\, a(\omega_0^2-\omega^2) + \frac{3h_3\lambda_{13}^2}{4}(a^3+ab^2+2a-2ab)\right] \qquad \text{(A 4-2)}$$

$$X(a,b) \triangleq \frac{da}{dt} = -\frac{1}{2\omega}\left[\, b(\omega_0^2-\omega^2) + \frac{3h_3\lambda_{13}^2}{4} \qquad + \frac{a^2b^2}{2} - 2a^2b + \frac{b^3}{3} - C\right] \qquad \text{(A 4-3)}$$

The integral curves are given by

$$\frac{a^4}{4} + \frac{a^2b^2}{2} + \frac{b^4}{4} + (a^2+b^2)\left(1 + \frac{2}{3h_3\lambda_{13}^2}(\omega_0^2-\omega^2)\right) + \frac{a^2b^2}{2} - 2a^2b + \frac{b^3}{3} - C = 0 \qquad \text{(A 4-4)}$$

where C is a constant.

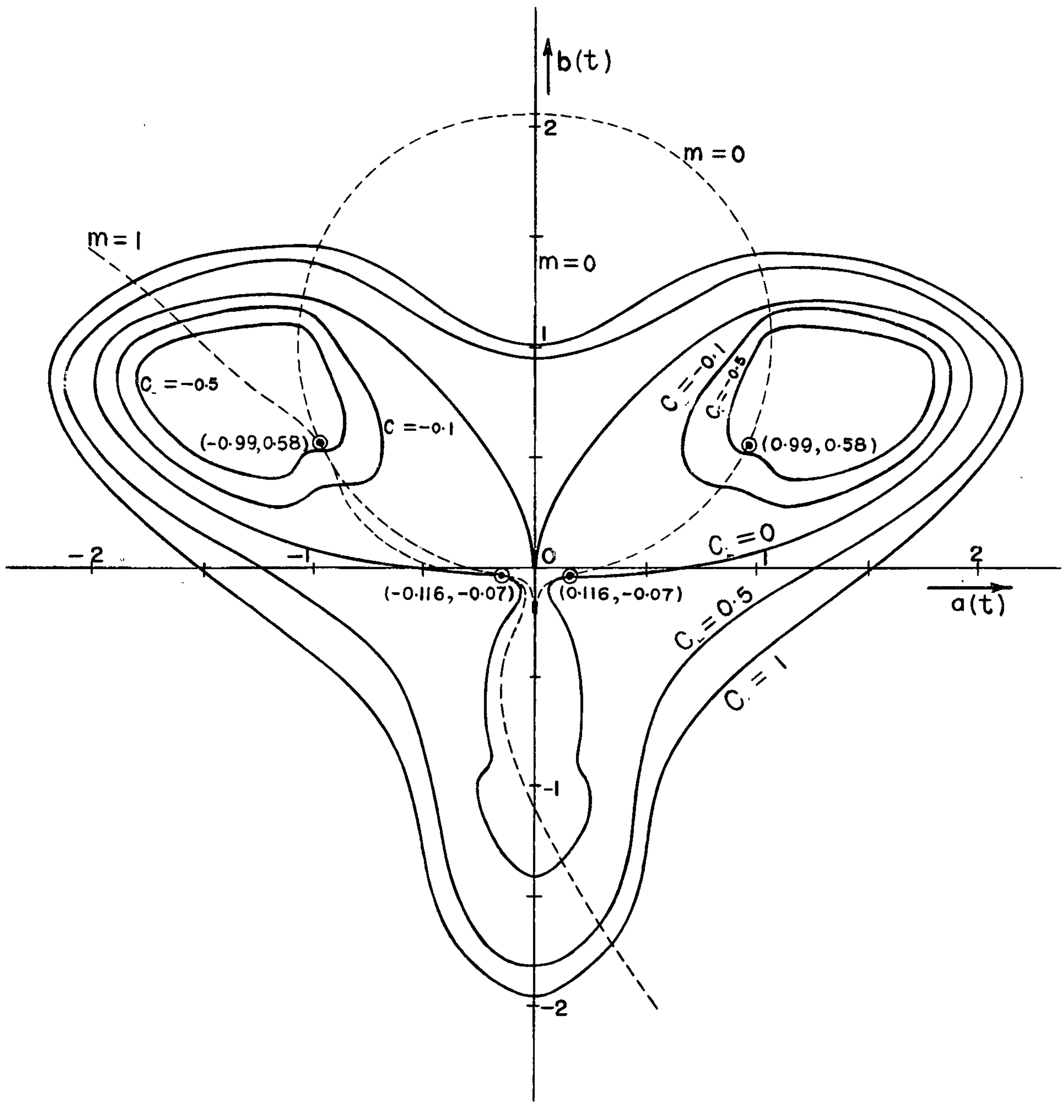
The Isoclines are given by eq. (4-195).

Singularities exist at the five points

(0,0); (0.99,0.57); (-0.99,0.58); (0.116,-0.07); (-0.116,-0.07)

Application of the criterion to determine the nature of the singularities establishes the three points (0,0); (0.99,0.57); (-0.99,0.58); as vortices and the remaining two (0.116,-0.07); (-0.116, -0.07) as unstable saddle points.

The integral curve plot - Fig.(1) shows these results in graphical form.



INTEGRAL CURVES, 30 C/S

FIG. 1

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