SYNTHESIS METHODS FOR ANTENNA ARRAYS WITH NON-UNIFORMLY-SPACED ELEMENTS

by

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ABSTRACT

A study is made of some of the existing techniques for determining the positions and excitations of the elements in an array, in order to produce a desired radiation pattern. A new method based on numerical quadrature is proposed. Equal-sidelobe patterns are synthesized and a comparison of the methods is made on this basis. Results indicate that the proposed quadrature method is an improvement over the other tested methods in two ways, (i) it produces a more accurately synthesized pattern and (ii) it is simple to compute. It is therefore useful for synthesizing arrays containing large numbers of elements.

However, for the particular radiation pattern selected, none of the methods gave an improvement over the corresponding uniformly-spaced array.

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LIST OF PRINCIPAL SYMBOLS

a, a_i	constants
A	a normalizing constant
A_n	normalized current
b, b_i	constants
c_i	complex Fourier coefficients
d	a constant spacing
E	the normalized electric field strength pattern of a non-uniformly spaced array
F	the normalized desired pattern
$g(x)$	a normalized current distribution
$G(x)$	$= g(x)/w(x)$
H	a weighting factor
I_n	a normalized current
$J_m(x)$	Bessel's function of order m and argument x
k	an integer variable
L	a line source aperture length
m	an integer variable
M	$(N-1)/2$, for odd N or $N/2$ for even N
n	an integer variable
N	the number of elements in an array or as defined
p	a normalized current density
$P_n(x)$	an n^{th} order polynomial in x
u	$= (\cos \theta - \cos \theta_o)$, unless otherwise defined
v	a dummy variable of integration
$w(x)$	a weighting function
x	the aperture position variable

x_n	the position of the n^{th} element
y	the source number function
z	$= \exp(j\beta d \sin \phi)$
α	an aperture phase constant
β	$= 2\pi/\lambda$
δ	the unit impulse function
Δ	a residual error term
ε_n	the fractional change from uniform spacing
θ	the angle between the line of the array and a given direction
λ	wavelength
μ_i	the i^{th} moment of the excitation function
ξ	a dummy variable of integration
ϕ	the complement of θ
ψ	$= \beta d \cos\theta$

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1. INTRODUCTION

Antenna systems composed of a family of identical individual radiators are termed arrays. The field produced by an antenna system as a function of some spatial variable, generally in one of several principal planes, is called the radiation pattern, though this name is also given to the curve relating the power radiated and the spatial variable. The main factors influencing the radiation patterns of such a system are the type of radiators, the distribution and orientation of radiators and their current excitation in amplitude and phase. The theory of two and three-dimensional arrays follows readily from one-dimensional array theory by the principle of pattern multiplication and so only linear array theory will be discussed here.

Antenna synthesis is the problem of determining the parameters of an antenna system that will produce a radiation pattern which accurately approximates some desired pattern. For the linear-array synthesis of a given pattern, it is required to find, in general, the positions of the radiators along a line and their current excitations in magnitude and phase. This presents great difficulty, however. Present practice is to simplify the problem by making two assumptions, (i) that the element spacing is uniform and (ii) that the phase of the current excitation is not a design variable. This reduces the parameters to the magnitudes of the current excitations and the constant spacing. The purpose of this work is (i) to study some of the recent synthesis methods that have the non-uniform spacing of the elements included in the design parameters, (ii) to evaluate the relative usefulness of these methods and (iii) to propose a new synthesis method based on numerical quadrature.

The radiation pattern of an antenna system depends on the individual patterns of the elements composing the system. In the case of a linear array the principle of pattern multiplication applies and so the system pattern is the product of the element pattern and the array factor. This latter is the pattern of a similar array containing isotropic elements, and is nearly always the more directive pattern. As there is no conceptual difficulty in taking the element pattern into account, array theory is generally based on isotropic sources.

Throughout this work the effects of mutual coupling between the elements of the array are not considered. Some work on this topic has been done by Allen[2].

1.1 Beam Scanning

Antenna radiation patterns consist in general, of a main beam and a number of smaller beams, the sidelobes. In many applications spatial scanning of the main beam is required. This is achieved by two main techniques, mechanical and electronic scanning. The former requires no elaboration. Electronic beam scanning in arrays is realized by arranging that contributions from the elements are all in phase in a direction other than broadside or endfire. A uniform progressive phase shift along the aperture will produce this effect.

When all elements of a uniformly-spaced array are in phase, the array factor is[3]

$$E(\theta) = \sum_{n=0}^{N} I_n \, e^{j\beta nd\cos\theta} \qquad 1.1$$

where I_n is the magnitude of the current excitation of the n^{th} element,

d is the constant spacing of the elements,

θ is the angle from the line of the array,

$N+1$ is the number of elements,

$\beta = \frac{2\pi}{\lambda}$, λ being the wavelength .

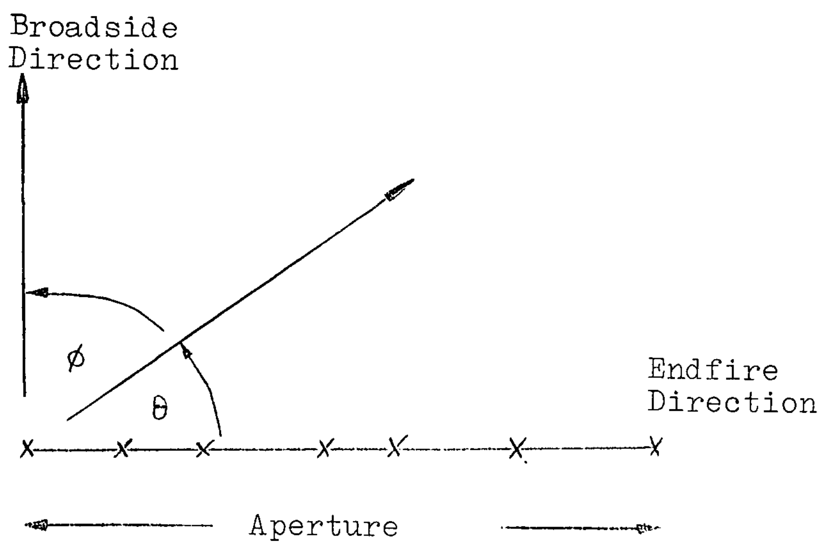


Fig. 1.1 Array Geometry

If the phase of the n^{th} element is now $-n\alpha$ where α is a constant, then the pattern becomes

$$E(\theta) = \sum_{n=0}^{N} I_n e^{jn(\beta d \cos\theta - \alpha)}$$

If θ_o is the angle between the array and the direction of the main beam, then

$$\beta d \cos \theta_o = \alpha$$

hence

$$E(\theta) = \sum_{n=0}^{N} I_n e^{jn\beta d u} \qquad 1.2$$

where

$$u \triangleq \cos \theta - \cos \theta_o$$

The "visible" region of the pattern lies between $\emptyset = \pm \frac{\pi}{2}$. For the broadside direction this corresponds to values of u in the range $1 \geqslant u \geqslant -1$. The region having values of u in the range $1 < u \leqslant 2$ is termed the "scanning" region, since, when the main beam is scanned from broadside to endfire, sidelobes from this region move into visible space. Hence if an array is to have beam scanning out to endfire, then the sidelobe level (defined as the height of the highest sidelobe) must be controlled for all u values in the range $-2 \leqslant u \leqslant 2$. If no scanning is required then only the sidelobe level for u values between -1 and 1 need be controlled. Thus scanning requirements need to be taken into consideration when pattern synthesis is being attempted.

1.2 Previous Work

With the advent of radio communications, the importance of directive antennas was realized. It was not until 1937 however, when Wolff[4] showed how to determine the radiating system that would produce a specified directive characteristic, that the first significant step was taken. In 1943 Schelkunoff[5] published his classic paper on the mathematical theory of linear arrays in which he put forward a synthesis technique of a completely different nature from that of Wolff. He also demonstrated that the effect of using different current excitations on elements in an array was to alter both the width of the main beam and the level of the sidelobes. The next logical step was the optimization of these two parameters, a problem overcome by Dolph[6] who utilized the equal-ripple property of Chebychev polynomials. Dolph (1946) showed that the minimum beam width for a given sidelobe level is obtained when all the sidelobes are of equal height, and also gave a method for determining the element excitations required to attain this optimum.

In the decade that followed Woodward[7] and Woodward and Lawson[8] produced a synthesis method based on the addition of terms of the $\frac{\sin \theta}{\theta}$ type, each term having its main beam in a different direction. This proved to be a useful technique. Work has also been done on apertures containing continuous current sources. Taylor[9] (1955) showed how to determine the continuous-current excitation function which would have the optimal property as defined by Dolph. Realizability limitations prevented an ideal solution but Taylor showed how the ideal equal-sidelobe pattern could be obtained arbitrarily closely.

Previous to 1956 all discrete arrays were assumed to consist of uniformly-spaced elements, but in that year Unz[10] proposed that the use of arbitrarily positioned elements would be advantageous since it gives the array designer an extra degree of freedom, namely, the element positions.

In view of this extra degree of freedom Unz suggested that the non-uniformly-spaced array needs, in general, fewer elements in order to achieve the same performance as an array with uniformly-spaced elements. Also in this paper Unz proposed a synthesis method for non-uniformly-spaced arrays based on a Fourier-Bessel expansion. In 1960, King et al.[11] did some numerical studies on empirically designed non-uniformly-spaced arrays and demonstrated that some advantages could be obtained over uniformly-spaced arrays with respect to the number of elements required and the scanning properties. Andreasen[12] (1962) has done similar studies using both analog and digital computer techniques and has produced arrays with beam scanning over wide angles, a modest sidelobe level and a significant reduction in the number of elements. An optimal design procedure put forward in Andreasen's paper was independently applied by Lo[13], who pointed out that, at best, the solution is only locally optimum. This technique was, however, used in the design of the University of Illinois radio telescope[14] which appears to be the only practical application of non-uniformly-spaced arrays to date.

Further work on the synthesis problem was done by Harrington[15] (1961) who used a perturbation approach by assuming that the element positions of the non-uniformly-spaced array were not far removed from those of some uniformly-spaced array. Sandler[16] (1960) and Willey[17] (1962) also conducted their synthesis procedures from the basis of a reference pattern produced by a uniformly-spaced array. Ishimaru[18], Maffett[19] and others considered a continuous-current distribution to be the source of their reference pattern and directed their analytical synthesis procedures accordingly.

A great deal of work has been done that is not referred to above. Notably, Lo[20] (a probabilistic approach), Pokrovskii et al[21]. (an "optimal" theory), Unz[22], Ma[23] and Yanpolskii[24] (synthesis methods) have contributed to the theory of the subject.

Before discussing the theory of some of the non-uniformly-spaced array synthesis methods, the simpler problem of synthesizing uniformly-spaced arrays will be discussed.

1.3 Synthesis of Uniformly-Spaced Arrays

There are two basic methods for synthesizing uniformly-spaced arrays. The first, due to Schelkunoff, is based on the fact that the space factor of a linear array is characterized completely by an associated polynomial in a complex variable, whose range is the unit circle. The second utilizes the Fourier series representation of the space factor.

1.3.1 Schelkunoff's Method

The space factor of a uniformly-spaced array of N elements is given by

$$E(\phi) = \sum_{i=0}^{N-1} I_i \, e^{jd\beta i \sin\phi}$$

where the coefficients I_i may be complex to allow for the phasing of the elements.

Let $$z \triangleq e^{j\beta d \sin\phi}$$

then $$E(z) = \sum_{i=0}^{N-1} I_i z^i$$

i.e. we have expressed the space factor as a polynomial in a complex variable z. This polynomial has N-1 roots and can therefore be expressed in normalized form, as

$$|E| = |z - z_1| \; |z - z_2| \; \cdots\cdots \; |z - z_{N-1}|$$

The locus of z corresponding to real space is that part of the unit circle given by z values ranging from $z = 1$ to $z = e^{\pm j2\beta d}$.

The synthesis problem then becomes the question of the location of the roots of the polynomial in the complex plane. The zeros for a uniform broadside array as shown in Figure 1.2.

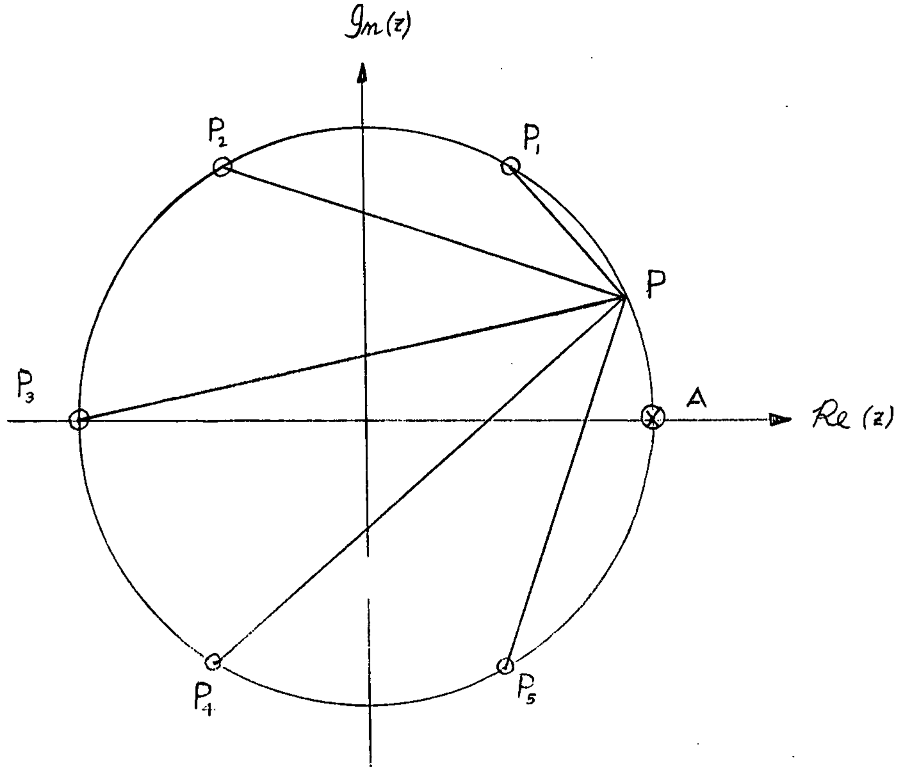


Fig. 1.2 Complex Plane for 6 Element Uniformly-Spaced Array

As P moves round the unit circle, the space factor has nulls when P passes through each root and a maximum when P is at A. A corresponds to the centre of the main beam and the regions between the roots correspond to the sidelobes. The separation of P_1 and P_5 is a measure of the width of the main beam, so for a narrow beam these should be close together. However for low sidelobes the remaining roots should be situated close together, so from the above diagram it can be seen that by clustering the roots towards the L.H. plane the side-lobe level will be reduced but the main beam width will be increased.

In a classic paper, Dolph[6] derived the excitation function which yields the optimum relationship between beam-width and side-lobe level, and this has been the basis for many designs in the past twenty years.

1.3.2 Fourier Series Method

Consider an array of N = 2M + 1 elements; its space factor is

$$E = \sum_{i=0}^{2M} I_i z^i \qquad 1.3$$

Dividing through by z^M, which leaves $|E|$ unchanged,

$$|E| = \left| I_o z^{-M} + I_1 z^{-M+1} \ldots I_{2M} z^M \right| \qquad 1.4$$

Taking the phase reference at the centre of the array and assuming a progressive phase shift from one end of the array, it can be seen that symmetrically corresponding elements have complex conjugate excitations, enabling pairs of terms of equation (1.4) to be added:

$$I_{M-k} z^{-k} + I_{M+k} z^k = a_k (z^k + z^{-k}) + jb_k (z^k - z^{-k})$$

$$= 2a_k \cos k\psi - 2b_k \sin k\psi \qquad (\text{where } z^k = e^{jk\psi})$$

$$\therefore \quad |E| = 2\left\{\frac{a_o}{z} + \sum_{k=1}^{M} \left[a_k \cos k\psi + (-b_k)\sin k\psi\right]\right\} \qquad 1.5$$

Any radiation pattern, $f(\psi)$, may be expanded as a Fourier series. Thus by equating terms in $f(\psi)$ with those in equation (1.5) the excitation distribution required to approximate the radiation pattern can be found.

The Fourier series representation of the desired pattern, as used here, is also the basis of the first of the methods of synthesizing non-uniformly spaced arrays, as described in the following chapter.

2. THEORY OF THE SYNTHESIS METHODS

In its general form, the synthesis problem is the determination of both the complex excitations and the element positions yielding an array factor, $E(\phi)$, which accurately approximates the desired pattern, $F(\phi)$. The approach to the problem is determined mainly by the mathematical form of the specified pattern, $F(\phi)$, and it is this form which will be used to characterize the three basic types of methods.

For an antenna having a continuous current distribution over its aperture, the far field radiation pattern is readily obtained from the Fourier transform relationship[25] and as a result, many continuous current distributions yielding useful patterns are known. The far field pattern of such a continuous distribution may be the pattern required to be synthesized.

In this case we have

$$F(\phi) = \int_{x_1}^{x_2} g(x) e^{j\beta x \sin \phi} \, dx \qquad 2.1$$

where x represents a position in the aperture, the limits of the aperture being x_1 and x_2.

Alternatively, the desired pattern may be given as that of a uniformly spaced array, in the form

$$F(\phi) = \sum_{n=0}^{N} I_n e^{j\beta n d \sin\phi} \qquad 2.2$$

A third form for $F(\emptyset)$ is the complex Fourier series whose coefficients can be obtained for any periodic function:

$$F(\emptyset) = \sum_{m=-M}^{M} C_m \, e^{jm\emptyset} \qquad 2.3$$

Synthesis requires the matching of one of these three expressions to that of a non-uniformly spaced array,

$$E(\emptyset) = \sum_{n=0}^{N} I_n e^{j\beta x_n \sin\emptyset} \qquad 2.4$$

where x_n is the position of the n^{th} element.

2.1 Methods Based on the Fourier Series Representation of the Desired Pattern

2.1.1 Unz's Method

The first non-uniformly-spaced array synthesis technique was put forward by Unz in 1956 and utilizes a Fourier-Bessel expansion to match equations (2.3) and (2.4).

The expression for generating Bessel functions[26] is

$$e^{\frac{1}{2}z(t-\frac{1}{t})} = \sum_{m=-\infty}^{\infty} t^m J_m(z)$$

Putting $t=e^{j\emptyset}$, then $\frac{1}{2}(t-\frac{1}{t}) = j\sin\emptyset$ and we obtain

$$e^{jz\sin\emptyset} = \sum_{m=-\infty}^{\infty} e^{jm\emptyset} J_m(z)$$

Using this relationship equation (2.4) can be transformed:

$$\sum_{n=0}^{N} I_n \, e^{j\beta x_n \sin\emptyset} = \sum_{n=0}^{N} I_n \sum_{m=-\infty}^{\infty} e^{jm\emptyset} J_m(\beta x_n)$$

giving

$$E(\emptyset) = \sum_{m=-\infty}^{\infty} \sum_{n=0}^{N} I_n J_m(\beta x_n) \; e^{jm\emptyset} \qquad 2.5$$

It can be seen that this is the form of equation (2.3) if the series is truncated to 2M+1 terms. For equivalence we then require

$$\sum I_n J_m(\beta x_n) = c_m \; , \qquad 2.6$$

where the c_m are the complex Fourier coefficients of the desired pattern. Equation (2.6) represents a set of non-linear transcendental equations which are to be solved for N+1 values of the x_n and/or I_n.

2.2 Methods Based on the Line-Source Representation of the Desired Pattern

2.2.1 Ishimaru's Method

A procedure which relates equations (2.1) and (2.4) is described by Ishimaru[18] and is briefly as follows. The non-uniformly-spaced array pattern is given in equation (2.4) as

$$E(\phi) = \sum_{n=1}^{N} I_n e^{j\beta x_n \sin\phi} \qquad 2.7$$

and can be rewritten as

$$E(\phi) = \sum_{n=1}^{N} f(n) \qquad 2.8$$

The Poisson Sum formula is[26]

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} f(v) e^{j2m\pi v} \, dv \qquad 2.9$$

Applying this to equation 2.8 we obtain

$$E(\emptyset) = \sum_{m=-\infty}^{\infty} \int_{0}^{N} f(v) e^{j2m\pi v} \, dv \qquad 2.10$$

(where the integration is from 0 to N since f(v) vanishes for $v < 0$ and $v > N$).

The source position function, $s = s(v)$, gives the position of the n^{th} element when $v = n$ and the source number function, $v = u(s)$, gives the numbering of each element when s is at the correct position of the element.

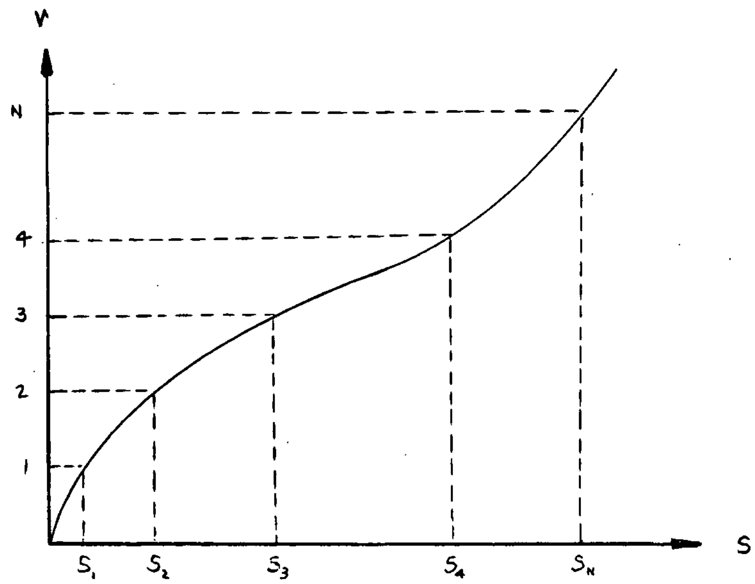


Figure 2.1 Plot of a Typical Source Number Function

Transforming equation (2.10) we obtain

$$E(\emptyset) = \sum_{m=-\infty}^{\infty} \int_{s_0}^{s_N} f(s) \frac{dv}{ds} e^{j2m\pi v(s)} ds \qquad 2.11$$

If equation (2.7) is rewritten as

$$E(\phi) = \sum_{m=-\infty}^{\infty} E_m(\phi), \qquad 2.12$$

where

$$E_m(\phi) = \int_{s_o}^{s_N} A(s) \frac{dv}{ds} e^{-j(\psi(s)-2m\pi v(s))} e^{j\beta s \sin\phi} \, ds, \qquad 2.13$$

then the physical significance of the formulation can be seen. ($A(s)$ is a function which yields A_n, the amplitude of the current in the n^{th} element, at $s=s_n$, and $\psi(s)$ is a function which yields ψ_n, the phase of the current, at $s=s_n$). Equation (2.13) is similar in form to equation (2.1) and represents the pattern due to a continuous current distribution of amplitude $A(s) \frac{dv}{ds}$ and phase $\psi(s)-2m\pi v(s)$.

Thus the non-uniformly-spaced array pattern has been reduced to an infinite series whose terms correspond to continuous current distributions. Normalizing the variables as follows:

u	$= \beta a \sin\phi$	
$2a$	$= s_N - s_o$	(the array aperture)
x	$= x(y)$	(normalized source position function)
y	$= y(x)$	(normalized source number function)

equations (2.12) and (2.13) become

$$E(u) = \sum_{m=-\infty}^{\infty} (-1)^{m(N-1)} E_m(u) \qquad 2.14$$

and

$$E_m(u) = \frac{1}{2}\int_{-1}^{1} A(x)\,\frac{dy}{dx}\,e^{-j\psi(x)+jm\pi N(y-x)}\,e^{j(n+m\pi N)x}\,dx \tag{2.15}$$

The actual position of the n^{th} element is

$$s_n = ax(y_n)$$

For odd N, i.e. $N = 2M+1$,

$$y_n = \frac{n}{M+\frac{1}{2}} \quad , \quad n = 0, \pm 1, \pm 2, \ldots \pm M$$

For even N, i.e. $N = 2M$,

$$y_n = \frac{n-\frac{1}{2}}{M} \quad n > 0$$

$$n = \pm 1, \pm 2, \ldots \pm M$$

$$y_n = \frac{n+\frac{1}{2}}{M} \quad n < 0$$

The total length of the array is then

$$L_o = a\left[x(y_M) - x(y_{-M})\right]$$

Ishimaru then argued that the series of equation (2.14) is so rapidly convergent that, to a good approximation, it may be truncated after the first term, so that

$$E(u) \simeq E_0(u)$$

Considering the uniformly-excited broad-side array, where $A(x) = 1$ and $\psi(x) = 0$, we have

$$E(u) = E_o(u) = \frac{1}{2}\int_{-1}^{1} \frac{dy}{dx}\, e^{jux}\, dx \qquad , \qquad 2.16$$

which is the radiation pattern of a continuous-source distribution of amplitude $\frac{dy}{dx}$ and zero phase shift. Hence if the continuous source distribution giving the desired pattern is known, the element positions in the equivalent non-uniformly-spaced array can be calculated.

2.2.2 Maffett's Method[19]

If the pattern to be synthesized is generated by a symmetrical line source $g(x)$ of aperture length L then instead of equation (2.1) we have

$$E(u) = 2\int_{0}^{\frac{L}{2}} g(x)\, \cos\beta ux\, dx \qquad 2.17$$

(using the normalized variable $u = \sin\emptyset$).

Define a normalized current density,

$$p(x) = \frac{1}{A}\, g(x), \quad \text{where } A = 2\int_{0}^{\frac{L}{2}} g(x)dx \; ; \qquad 2.18$$

then the cumulative current distribution is

$$y(x) = \int_{-\frac{L}{2}}^{x} p(\xi)d\xi \quad , \quad -\frac{L}{2} \leqslant x \leqslant \frac{L}{2} \qquad 2.19$$

In equation (2.17) putting $x=x(y)$, i.e.

$$dx = \frac{dx}{dy}\,dy,$$ and differentiating equation (2.19)

we obtain $\frac{dy}{dx} = p(x)$.

Changing the variable of integration in (2.17) we get

$$E(u) = 2\int_{\frac{1}{2}}^{1} A\, p(x)\cos\beta ux \,\frac{dy}{p(x)}$$

$$= 2A\int_{\frac{1}{2}}^{1} \cos\beta ux\, dy \text{ where } x=x(y) \qquad 2.20$$

This is approximated by the trapezoidal rule to yield the approximate pattern F(u). The increment in y is chosen to be $\frac{1}{2M}$, i.e. M+1 points are chosen at $y_o = \frac{1}{2}, y_1, y_2, \ldots\ldots y_M = 1$, then

$$F(u) = \frac{A}{2M}\sum_{n=1}^{M} \cos\beta ux_n \quad ,$$

where the x_n are given by

$$y_n = \tfrac{1}{2} + \int_0^{x_n} p(\xi)\,d\xi \qquad n=0,1,2\ldots M \qquad 2.21$$

For some distributions, equation (2.19) can be inverted to give x_n explicitly[27]; but in general iteration techniques have to be used.

2.3 Methods Based on the Uniform-Array Representation of the Desired Pattern

2.3.1 Harrington's Method

A perturbation approach was used by Harrington[15] who determined the fractional change from uniform spacing required to synthesize the desired pattern.

For a uniformly spaced and excited array of N elements, we have

$$E_u = \frac{1}{N} \sum_{n} \alpha_n \cos\left(\frac{nu}{2}\right) \quad , \qquad 2.22$$

where $\sum\limits_{n}$ means $\begin{cases} \sum\limits_{n=1,3,5,\ldots}^{N-1} \\ \sum\limits_{n=0,2,4}^{N-1} \end{cases}$ for $\begin{cases} N \text{ even} \\ N \text{ odd} \end{cases}$, $u = \beta d \sin\phi$

and $\alpha_n = \begin{cases} 1 & n = 0 \\ 2 & n \quad 0 \end{cases}$

For non-uniform spacing a convenient "base" separation, d, can be chosen and the element spacing expressed as

$x_n = \left(\frac{n}{2} + \varepsilon_n\right)d$, i.e. ε_n is the fractional change from uniform spacing.

Then

$$F(u) = \frac{1}{N} \sum_{n} \alpha_n \cos\left\{\left(\frac{n}{2} + \varepsilon_n\right)u\right\}$$

Expanding the cosine and assuming $\varepsilon_n u$ is small (so that $\cos \varepsilon_n u \simeq 1$ and $\sin \varepsilon_n u \simeq \varepsilon_n u$)

$$F(u) = E_u - \frac{u}{N} \sum_{n} \alpha_n \varepsilon_n \sin\left(\frac{nu}{2}\right)$$

i.e. $$\sum_{n} \alpha_n \varepsilon_n \sin\left(\frac{nu}{2}\right) = \frac{N}{u}\left(E_u - F(u)\right)$$

Using a Fourier expansion of the right hand side we obtain

$$\varepsilon_n = \frac{N}{\pi} \int_0^{\pi} \frac{1}{u} \left(E_u - F(u)\right) \sin nu \, du \quad , \quad n = \begin{cases} 1,3,5,\ldots N-1, & N \text{ even} \\ 2,4,6,\ldots N-1, & N \text{ odd} \end{cases} \qquad 2.23$$

Hence the positional perturbations can be calculated.

2.3.2 Willey's Method[17]

The pattern of a symmetrical uniformly-spaced array can be expressed as

$$E(\emptyset) = \sum_{-M}^{M} A_m e^{j \frac{m}{M} u} \quad , \text{ where } u = \beta d M \sin\emptyset$$

Expanding the exponential,

$$e^{j \frac{m}{M} u} = \sum_{k=0}^{\infty} \frac{(j \frac{m}{M} u)^k}{k!}$$

$$\therefore \quad E(u) = \sum_{k=0}^{\infty} \frac{(ju)^k}{k!} \mu_k \quad , \qquad 2.24$$

where $\mu_k = \sum_{-M}^{M} A_m \left(\frac{m}{M}\right)^k$ is the k^{th} moment of the excitation function.

Consider a uniformly-excited non-uniformly-spaced array, whose pattern is given by

$$F(\emptyset) = \sum_{i=1}^{N} e^{j\beta x_i \sin\emptyset} \qquad (N = \text{no. of elements in array})$$

$$= \sum_{i=1}^{N} e^{j \frac{x_i}{M} u}$$

Expanding as before

$$F(\emptyset) = \sum_{k=0}^{\infty} \frac{(ju)^k}{k!} \mu_k' \quad , \qquad 2.25$$

where $u_k' = \sum_{i=1}^{N} (\frac{x_i}{M})^k$ is the k^{th} moment of the non-uniformly-spaced array excitation function. For equivalence of the patterns of equations (2.24) and (2.25), we require $\mu_k = \mu_k'$, i.e.

$$\sum_{m=-M}^{M} A_m (\frac{m}{M})^k = \sum_{i=1}^{N} (\frac{k_i}{M})^k \qquad 2.26$$

This set of non-linear algebraic equations can be solved for N, the number of elements required in the non-uniformly-spaced array, and the positions, x_i, of the N elements.

3. ARRAY SYNTHESIS USING NUMERICAL QUADRATURE

3.1 Introduction

In chapter 2 the theory behind some existing techniques of array synthesis was investigated. In this chapter a novel procedure is proposed. The basis of the method is numerical quadrature, a technique for evaluating definite integrals.

3.2 Quadrature Theory

The problem of finding the numerical value of the integral of a function of one variable, because of its geometrical meaning, is often called quadrature.

Consider an integral of the form

$$\int_a^b F(x)\ dx$$

This might be approximated by dividing the interval into n equal segments and evaluating F(x) at n equally spaced values of x, one within each segment. This gives the approximation

$$\int_a^b F(x)\ dx \simeq \frac{b-a}{n} \sum_{j=1}^{n} F(x_j) \qquad 3.1$$

More generally the x_j could be irregularly spaced. Then each of the $F(x_j)$ would be multiplied by a weighting factor, H_j, and the integral could be written in the form

$$\int_a^b F(x)dx = \sum_{j=1}^{n} H_j\ F(x_j) + \Delta \quad , \qquad 3.2$$

where Δ is a residual error term. Since there are 2n unknowns in the quadrature sum it might be suspected that the H_j and x_j could be chosen such that Δ be zero for all polynomials of order 2n-1 or less. Krylov[28] has shown that this is in fact the case. The quadrature methods that have this degree of accuracy for polynomials are termed Gaussian. For Gaussian quadrature, a set of 2n equations in the 2n unknown constants can be obtained by substituting $F(x) = x^k$, $k = 0,1,\ldots\ 2n-1$ into equation (3.2) and setting $\Delta=0$. Instead of considering the set of the x_j, it is convenient to consider the polynomial, $P_n(x)$, which has these as roots. It is also convenient to consider the integrand of expression (3.1) to be broken up into two factors, $w(x)$ and $f(x)$. The quadrature expression then becomes

$$\int_a^b w(x)\ f(x)\ dx \simeq \sum_{k=1}^{n} A_k\ f(x_k) \qquad 3.3$$

and the coefficients A_k are given by

$$A_k = \int_a^b w(x)\ \frac{P_n(x)}{(x-x_k)P_n'(x_k)}\ dx$$

Utilizing the Christoffel-Darboux identity[24] yields

$$A_k = -\frac{a_{n+1}}{a_n}\frac{1}{P_n'(x_k)P_{n+1}(x_k)}$$ where a_n is the

coefficient of x^n in $P_n(x)$. Use of the recursion relationships[30] for orthonormal polynomials alters this expression to

$$A_k = \frac{a_n}{a_{n-1}} \frac{1}{P_n'(x_k)P_{n-1}(x_k)} \qquad 3.4$$

3.3 <u>Gauss-Chebychev Quadrature</u>

By a linear transformation, the limits (a,b) of the region of integration can be transformed into an chosen segment of the x-axis. In order to make use of the symmetry of the nodes x_k and of the coefficients A_k, the standard segment will be taken to be (-1, 1). By judicious selection of the polynomial type, the coefficients A_k can be greatly simplified.

The weight function that is orthogonal to the Chebychev polynomials is

$$w(x) = (1 - x^2)^{-\frac{1}{2}}$$

Using the Chebychev polynomials in equation (3.4) it can be readily shown that the A_k are constant, i.e.

$$A_k = \frac{\pi}{n}, \quad k = 1, \ldots . n$$

The x_k, being the roots of the n^{th} order Chebychev polynomial, are given by

$$x_k = \cos\left(\frac{2k-1}{2n}\right)\pi$$

Thus the complete Gauss-Chebychev quadrature formula is

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\,dx \simeq \frac{\pi}{n} \sum_{k=1}^{n} f\left(\cos\left(\frac{2k-1}{2n}\right)\pi\right) \qquad 3.5$$

This will be illustrated by means of an example, the evaluation of

$$I = \int_{-1}^{1} \frac{x^8}{\sqrt{1 - x^2}} \, dx$$

Using n=4 in equation (3.5)

$$I \simeq \frac{\pi}{4} \quad \cos^8 \frac{\pi}{8} + \cos^8 \frac{3\pi}{8} + \cos^8 \frac{5\pi}{8} + \cos^8 \frac{7\pi}{8}$$

$$\simeq 0.834$$

The actual value of the integral is

$$I = \int_{-\pi}^{\pi} \cos^8 \theta \, d\theta = 0.859$$

The use of a 5-point quadrature will yield the result:

$$I \simeq \frac{\pi}{5} \quad \cos^8 \frac{\pi}{10} + \cos^8 \frac{3\pi}{10} + \cos^8 \frac{5\pi}{10} + \cos^8 \frac{7\pi}{10} +$$

$$+ \cos^8 \frac{9\pi}{10} = 0.859$$

3.4 Application of Gauss-Chebychev Quadrature to Array Synthesis

The pattern of a symmetrically excited line source in a normalized aperture is given by

$$F(u) = \int_{-1}^{1} g(x) \cos ux \, dx \,,$$ where $g(x)$ is the

normalized current-distribution function. Putting $G(x) \triangleq g(x) \sqrt{1-x^2}$, we obtain

$$F(u) = \int_{-1}^{1} (1-x^2)^{-\frac{1}{2}} G(x) \cos ux \, dx.$$

This is of the form

$$\int_{-1}^{1} w(x)\ f(x)dx \quad , \qquad \text{with } w(x) = (1-x^2)^{-\frac{1}{2}} \text{ being the}$$

weight function of Chebychev polynomials. Hence the quadrature formula is

$$F(u) \simeq \sum_{k=1}^{n} A_k\ f(x_k) \quad , \quad \text{where } f(x) =$$

$G(x)\cos ux$ and the x_k are the roots of the n^{th} order Chebychev polynomial. As stated previously (and shown by Krylov[31]) the coefficients are all equal to $\frac{\pi}{n}$ and so

$$F(u) \simeq \frac{\pi}{n} \sum_{k=1}^{n} G(x_k)\ \cos ux_k \qquad 3.6$$

This is of the same form as the radiation pattern of a non-uniformly-spaced symmetric array with the elements positioned at $\pm\ x_k$ and having normalized excitations given by $\frac{\pi}{n}\ G(x_k)$. Thus the excitation of the k^{th} element, I_k, is given by

$$I_k = g(x_k)\ \sqrt{1 - x_k^{\ 2}} \qquad 3.7$$

By this procedure the pattern of a continuous line source has been transformed into that of a non-uniformly-spaced array, enabling the element positions and current excitations to be readily calculated.

It is worth noting that though the resulting array is non-uniformly-spaced, this spacing does not depend on the desired pattern but only on the number of elements in the array. In this respect this method is similar to the uniformly-spaced array in which the spacings are preassigned.

4. RESULTS

4.1 Introduction

This chapter begins by specifying a certain antenna pattern to be synthesized. Each of the various methods of Chapters 2 and 3 is then applied to the synthesis of this pattern and the success of the methods in approximating the pattern is examined.

4.2 Choice of Pattern to be Synthesized

In order to be able to make a critical comparison of the various methods, it is desirable

(i) that the pattern to be synthesized can be mathematically represented as the pattern of a line source, as the pattern of a uniformly-spaced array and as a Fourier series sum

(ii) that parametric variations of the pattern function will give rise to a wide range of pattern forms.

The pattern chosen was an equal-sidelobe pattern. Dolph has determined the element excitations of a uniformly-spaced array which will give this pattern, and his work has been generalized by others. The continuous current source counterpart of this array has been determined by Taylor. However, the line source producing this ideal pattern with constant level sidelobes for all values of n, is unrealizable in practice since the remote sidelobes do not decay. Taylor resolved this problem by making the pattern have equal sidelobes up to a point and then causing them to decay as $\frac{\sin u}{u}$.

The transition point at which the sidelobes start to decay can be placed at will so that the ideal pattern can be approached arbitrarily closely. This transition point can be placed outside the visible and scanning regions of the pattern, thus effectively producing an equal-sidelobe pattern. The theory behind Taylor's line source is complicated and a summary is given in Appendix I. Thus the equal-sidelobe pattern satisfies the first criterion above. By varying the parameter controlling the sidelobe level, patterns can be obtained ranging from the binomial array with no sidelobes, all the way to the interferometer case with sidelobes equal in height to the main beam, thus satisfying the second criterion.

4.3 Unz's Method

Unz's method synthesizes both the positions and excitations of the elements and requires the solution of the following set of equations:

$$\begin{array}{l} I_o J_{-M}(\beta x_o) + I_1 J_{-M}(\beta x_1) \ldots\ldots + I_N J_{-M}(\beta x_N) = C_{-M} \\ \vdots \\ I_o J_o(\beta x_o) + I_1 J_o(\beta x_1) \ldots\ldots \quad I_N J_o(\beta x_N) = C_o \\ \vdots \\ I_o J_M(\beta x_o) + I_1 J_M(\beta x_1) \ldots\ldots \quad I_N J_M(\beta x_N) = C_M \end{array}$$

Where the X_i are the element positions, the I_i are their excitations and the C_k are the Fourier coefficients of the desired radiation pattern. In general, the solution of this set is extremely difficult. To gain insight into the solution, a simple pattern, $F(\phi) = 32 \cos^4\phi$, was synthesized using 5 elements in a 2λ aperture. Since the chosen pattern has symmetrical Fourier coefficients the resulting array will be symmetric about the centre element. This reduces the unknowns to one spacing (that of the inner pair of elements) and three excitations, i.e.

$$I_1 J_4\ (\beta x_1) + I_2 J_4(2\pi) - 1 = 0$$

$$I_1 J_2\ (\beta x_1) + I_2 J_2(2\pi) - 4 = 0$$

$$I_1 J_o\ (\beta x_1) + I_2 J_o(2\pi) - 6 + \tfrac{1}{2}\, I_o = 0$$

Solutions can be obtained graphically as shown in fig. 4.1a.

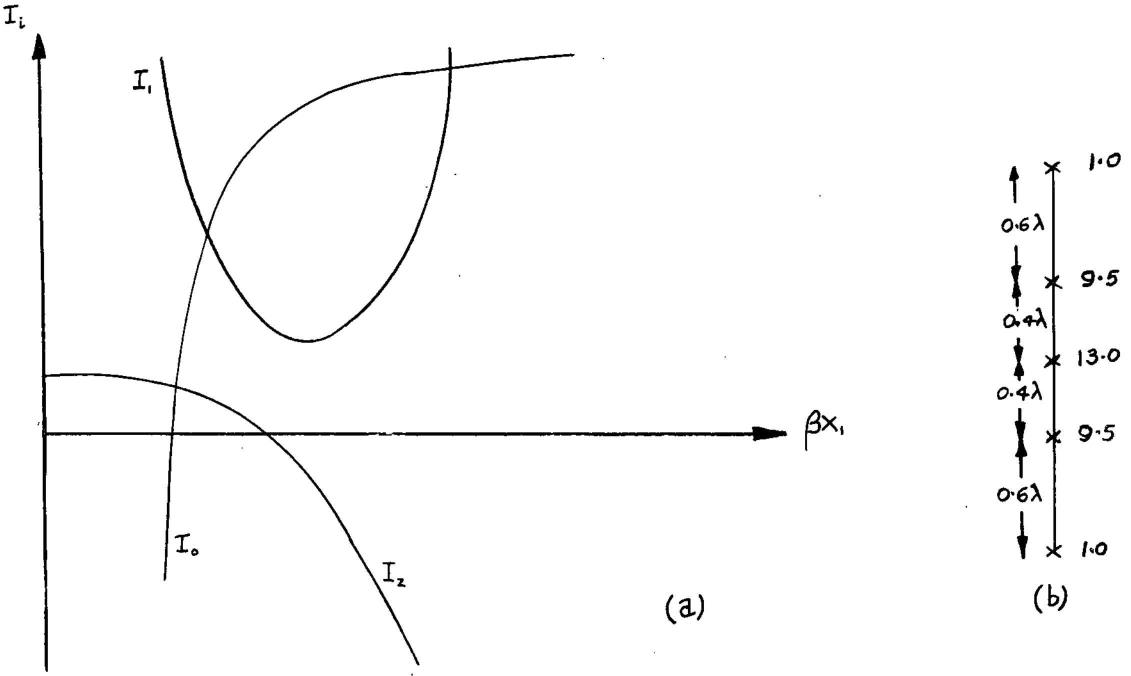


Fig. 4.1a Example of Unz's Method
4.1b Array Synthesized by Unz's Method

It is of interest to note that one value of x_1 corresponds to $I_2 = 0$. In that case the outer elements are redundant. This happens when $\beta x_1 = 2.9$; i.e. $x_1 = 0.47\lambda$. The central element then has twice the current excitation of the outer elements.

Several array configurations were taken from Fig. 4.1; the array yielding the best pattern is shown in Fig. 4.1b. Its

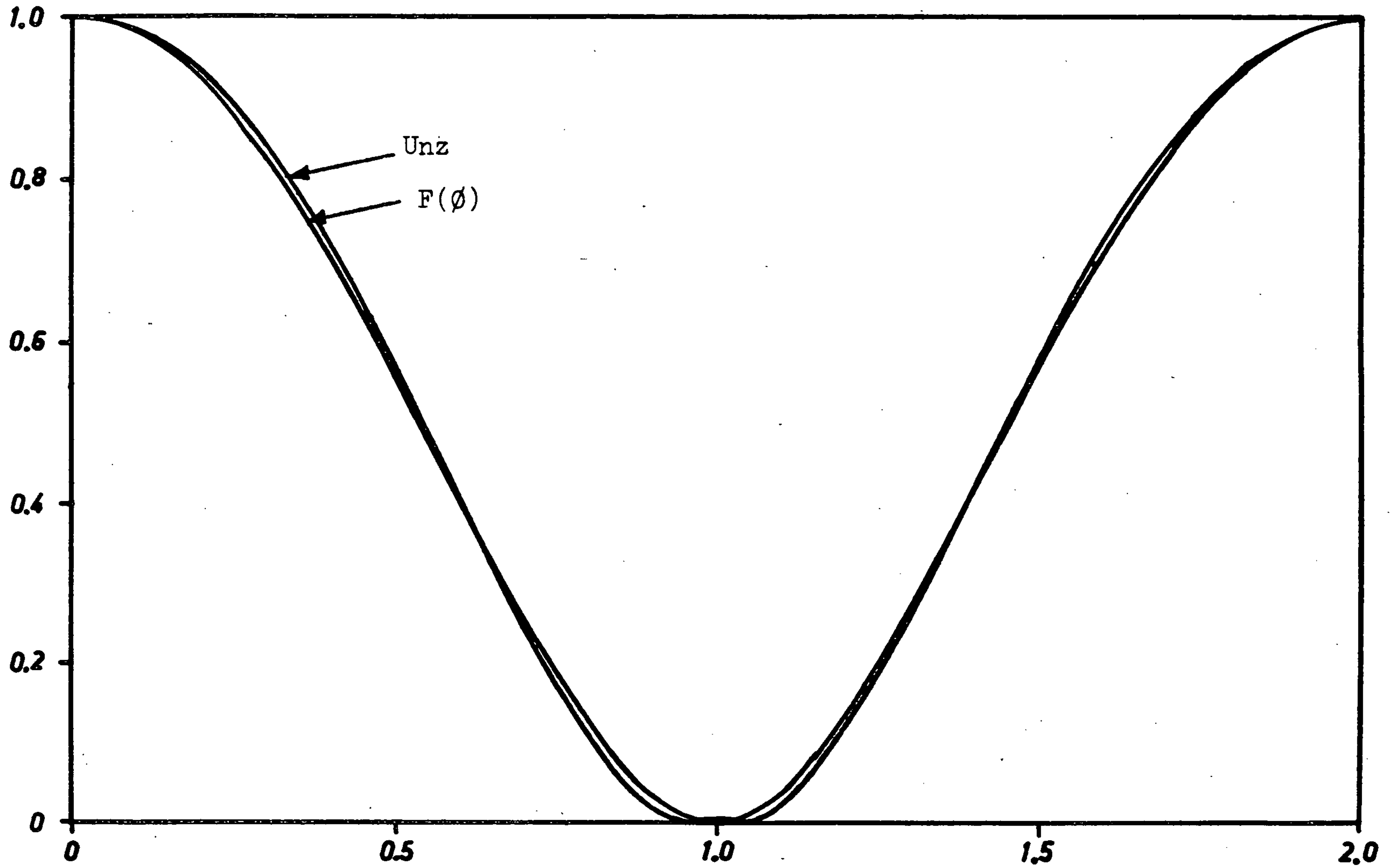


Fig. 4.2 Unz Synthesis of $F(\emptyset) = 32 \cos^4\emptyset$

pattern and the desired pattern are shown in Fig. 4.2. Agreement between the two patterns is good over the range plotted, i.e. the visible and scanning regions.

A far more difficult problem is the synthesis of the pattern $F(\emptyset) = 2^{10} \cos\left(\frac{\pi}{2} \cos \emptyset\right)$, which is the pattern of an 11 element binomial array in a 5λ aperture. The equations reduce to

$$I_3 J_{2n}(\beta x_3) + I_2 J_{2n}(\beta x_2) + I_1 J_{2n}(\beta x_1) = \tfrac{1}{2} C_{2n}, \quad n \geqslant 1$$

$$\frac{I_o}{2} + I_3 J_o(\beta x_3) + I_2 J_o(\beta x_2) + I_1 J_o(\beta x_1) = \tfrac{1}{2} C_o$$

Approximate solutions were found by graphical means and final values were reached by correcting these. The synthesized array is shown in Fig. 4.3 and the corresponding patterns are shown in Fig. 4.4.

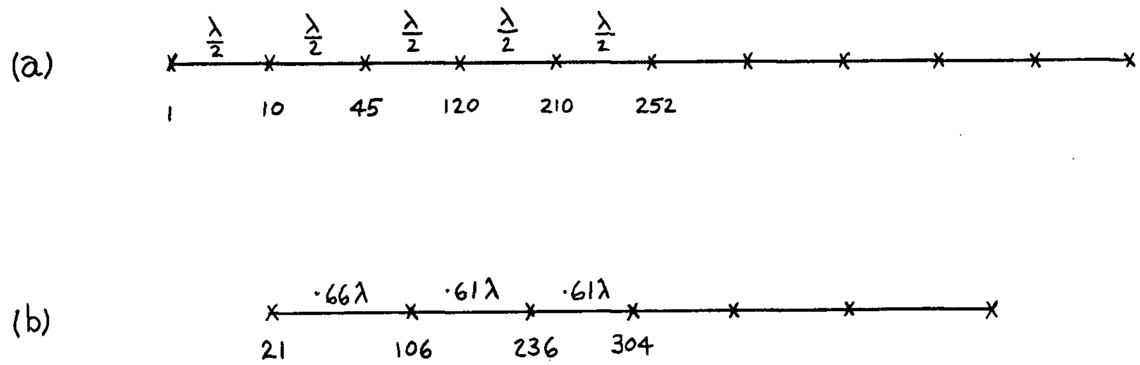


Fig. 4.3 (a) Uniformly-Spaced Binomial Array and

(b) Non-Uniformly-Spaced Synthesized Array.

For each pattern, the magnitude of the array factor is plotted against $u = \cos \theta - \cos \theta_o$, for u in the range $0 \leqslant u \leqslant 2$, which represents the complete visible and scanning regions.

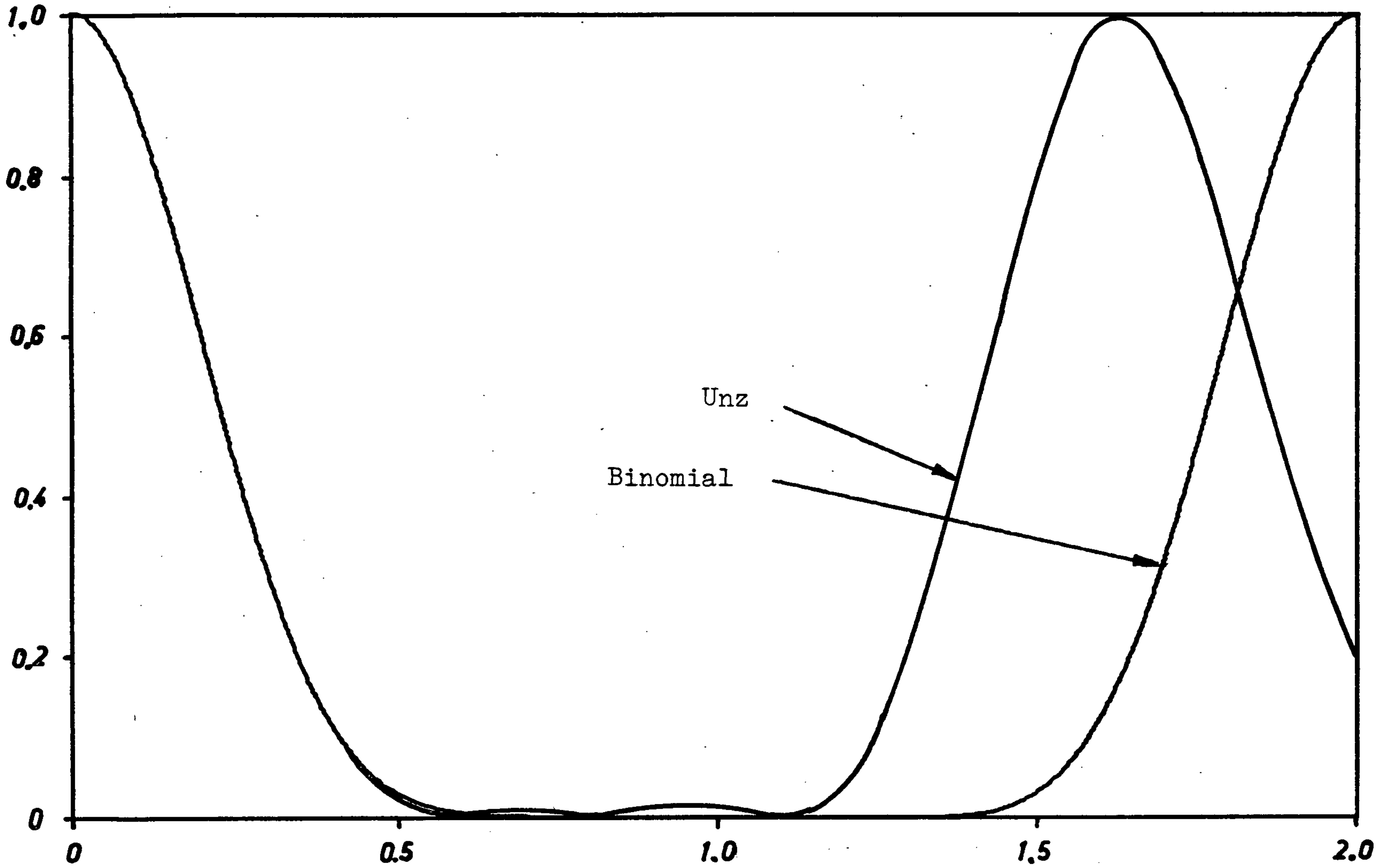


Fig. 4.4 Unz Binomial Pattern Synthesis

The correspondence of the two patterns is excellent throughout the visible region (within 1%), but deteriorates rapidly in the scanning region.

Thus the pattern due to the array of Fig. 4.3a is an excellent approximation to that of the binomial array of Fig. 4.3a if no beam scanning is required and also saves four elements. However the computation involved in producing this array is very lengthy and becomes prohibitive for larger arrays.

4.4 Ishimaru's Method

As indicated in equation (2.16) this method assumes that the excitations are uniform and requires the solution for x of an equation of the form

$$y = f(x), \text{ for each of the elements.}$$

The solution could be obtained by inverting this equation, but a Newton-Raphson iteration procedure is used here. Convergence is good and results are readily obtained for small arrays. For arrays of more than about thirty elements convergence is much slower.

Arrays were synthesized to produce 15 dB sidelobes using from 11 to 21 elements. Usually only the sidelobe envelope is of importance when considering the sidelobe region and so only the envelopes of the six synthesized arrays are shown in Fig. 4.5, which shows the "visible region" for an array of aperture 10λ. The full patterns for 15 and 21 elements are shown in Figs. 4.6 and 4.7, for the complete visible region and scanning region of an array of aperture 10λ. The element

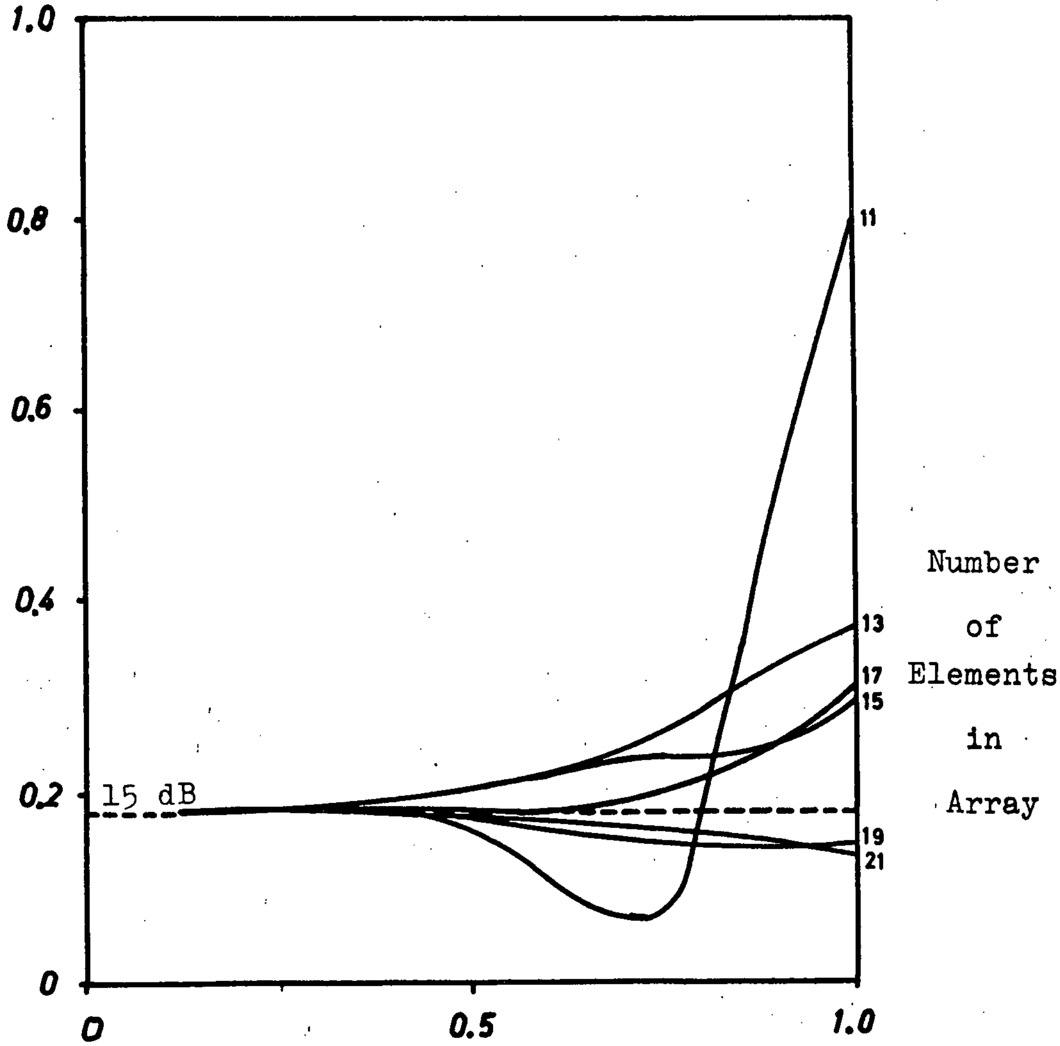


Fig. 4.5 Ishimaru's Synthesis of 15 dB Equal-Sidelobe Pattern

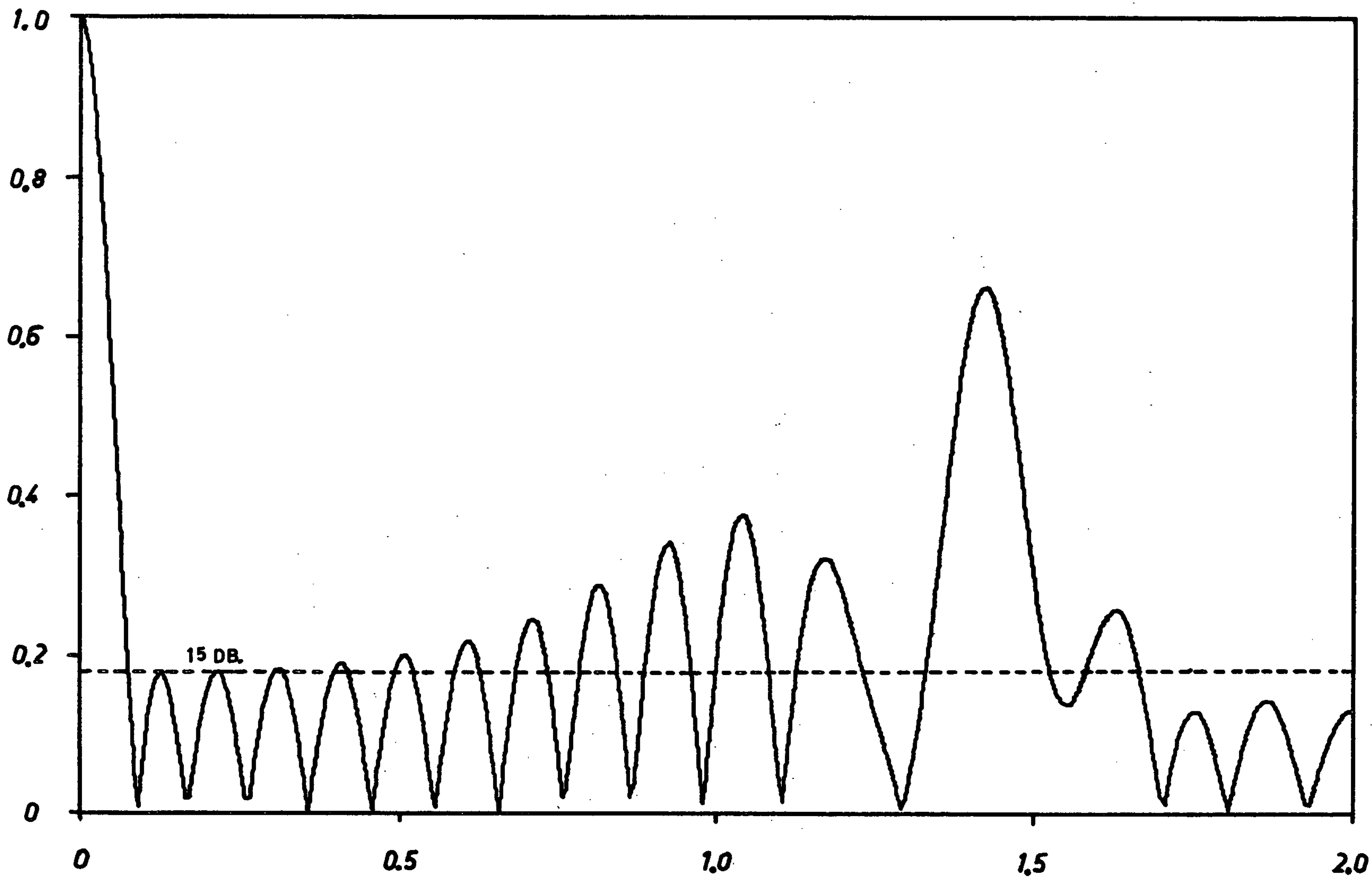


Fig. 4.6 15 element Array by Ishimaru's Method

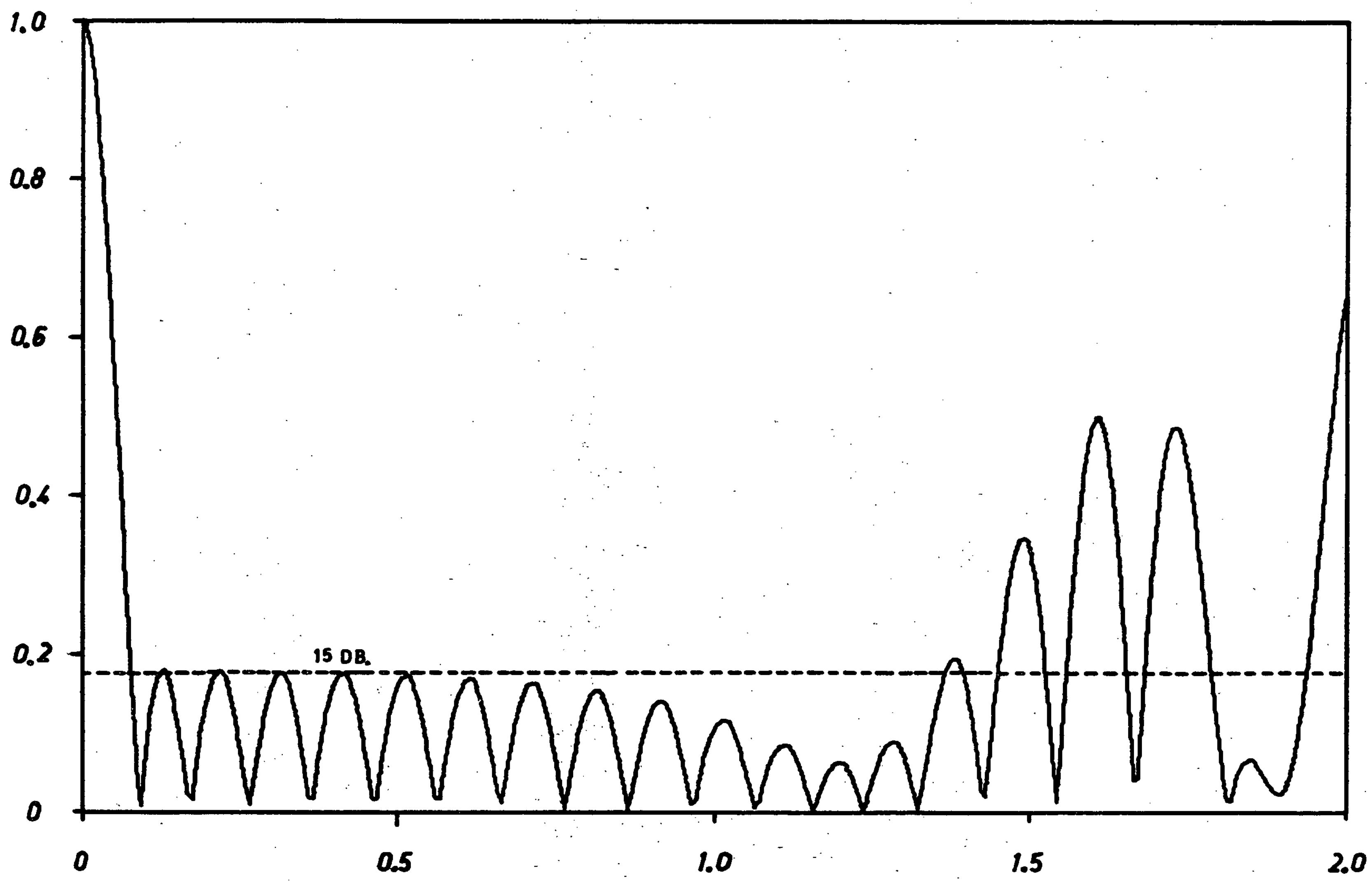


Fig. 4.7 21 element Array by Ishimaru's Method

arrangements in these two arrays are shown in Fig. 4.8.

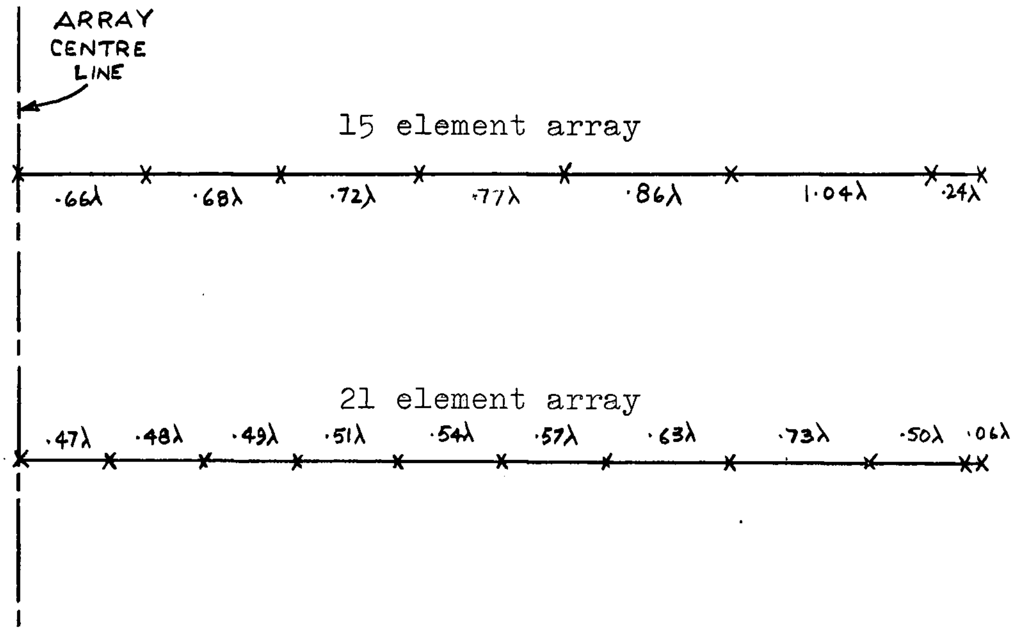


Fig. 4.8 Array Configurations for Ishimaru's Method

The beamwidths of the resulting patterns are in very close agreement with the Taylor pattern beamwidth and the close-in sidelobes are at the designed level. In the case of the 15 element array, deterioration of the pattern sets in after the first three or four sidelobes; the increased sidelobes thereafter are unacceptable. As would be expected, the approximation improves as the number of elements increase such that with 21 elements in the broadside configuration, the array pattern is within a few percent of the desired pattern out to 61° from broadside. However, if the only requirement is that all sidelobes be below the designed level, then the main beam produced by this array may be scanned to $\pm 48^{\circ}$ from the broadside direction without the appearance of sidelobes larger than 15 dB.

A further interpretation of Fig. 4.7 is that by reducing the size of the array aperture from 10λ to 7.5λ, an array can be obtained whose main beam may be scanned to within $\pm$ 90° of the broadside direction without the appearance of lobes larger than 15 dB. The cost of this, however, is an increase in the width of the main beam. This use of aperture reduction, for extending the constant sidelobe region at the expense of the main beam width, can be applied to all the patterns that follow.

An attempt was made to reduce the secondary maximum which appears in the scanning region of the preceding patterns by considering the second term in the series expansion of equation 2.14. However this gave only a very slight improvement.

4.5 Maffett's Method

The procedure is similar to the previous method but the convergence of the equations for the x values is somewhat slower. The pattern envelopes for arrays of from 11 to 21 elements are shown in Fig. 4.9, over the visible region corresponding to a 10λ aperture, and two full patterns are shown in Figs. 4.10 and 4.11. The improvement with increasing number of elements is easily seen. The results obtained by this method show little, if any, improvement over the previous method.

It can be seen that both of the preceding methods fail to give a good approximation to the desired pattern throughout the visible and scanning regions even when using 21 elements, as in the corresponding uniformly-spaced array. Thus there is no element saving in these cases.

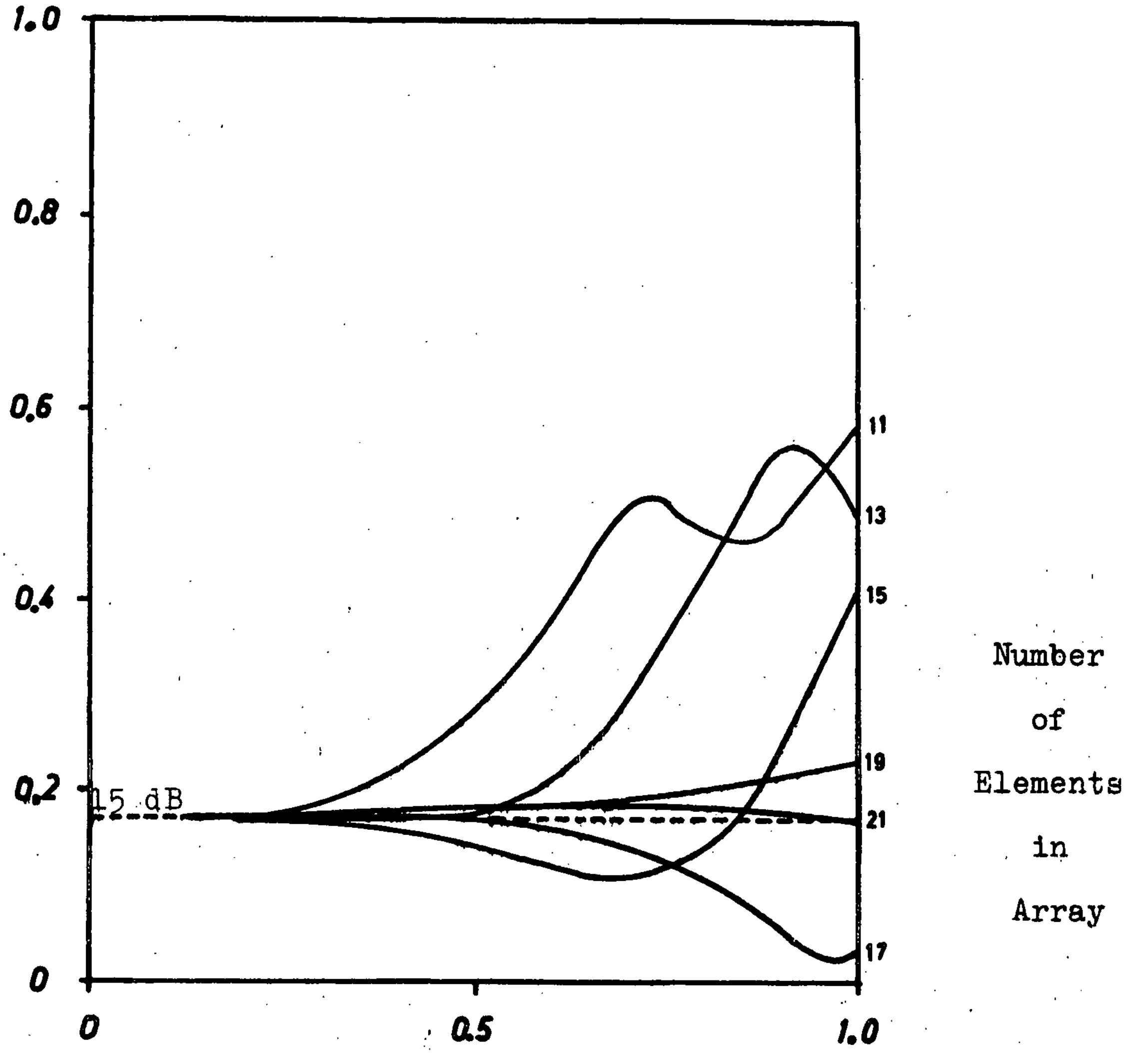


Fig. 4.9 Maffett's Synthesis of 15 dB Sidelobe Pattern

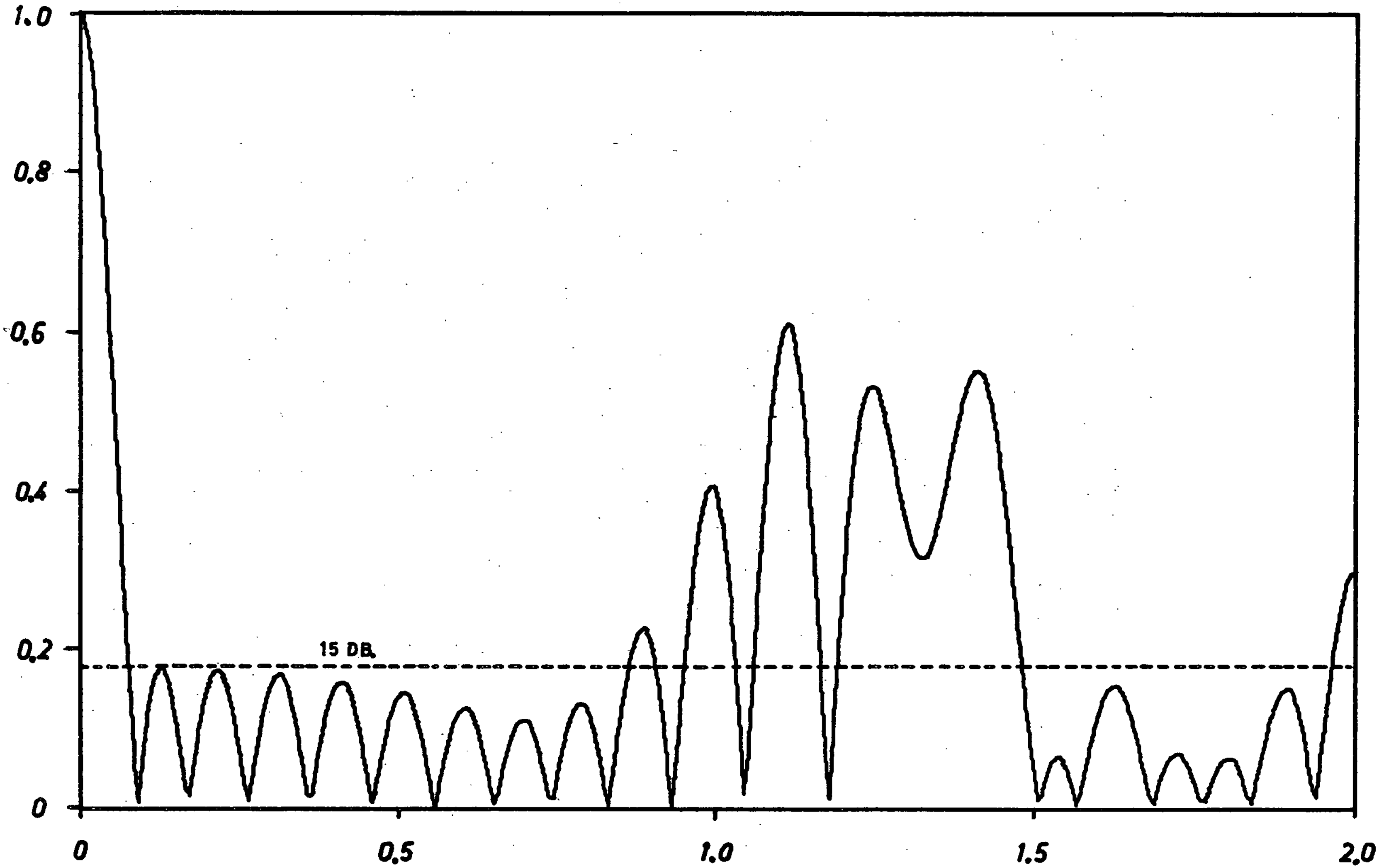


Fig. 4.10 15 element Array by Maffett's Method

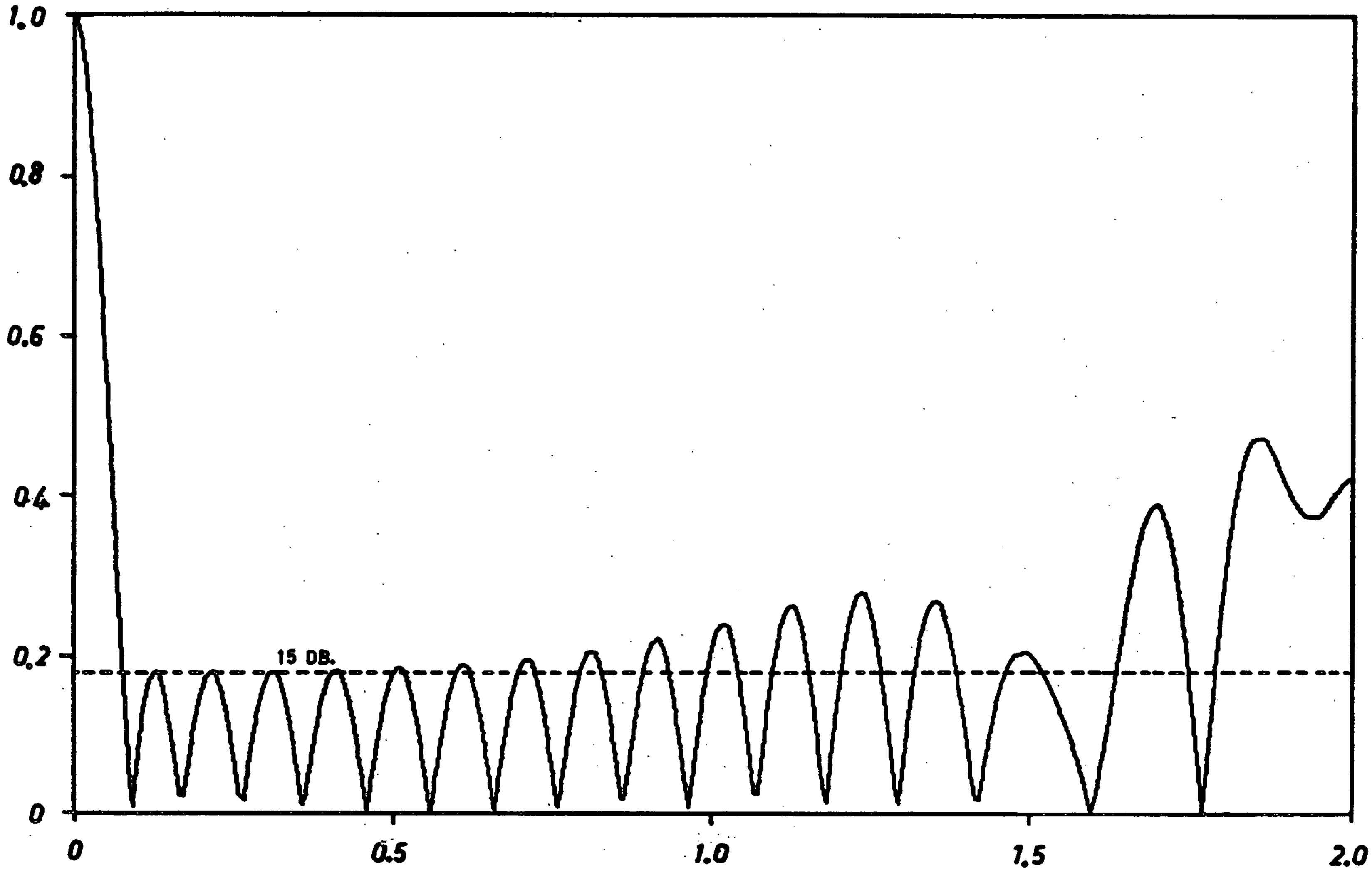


Fig. 4.11 21 element Array by Maffett's Method

4.6 Harrington's Method

The underlying assumption of this method is that the inter-element spacings deviate only slightly from the uniform case; hence the values of ε_n calculated from equation (2.23) should be such that $\varepsilon_n \ll d$. A 15 element array was synthesized to give 15 dB equal sidelobes and the resulting pattern is shown in Fig. 4.12. The values of ε_n corresponding to this array do not all satisfy the above criterion, $\varepsilon_n \ll d$; in fact for some values of n, $\varepsilon_n > d$.

The restriction on the ε_n implies a restriction on the pattern to be synthesized. For small ε_n the difference between the pattern to be synthesized and the unperturbed pattern, i.e. the pattern of the uniformly-spaced, uniformly-excited array, must not be too large. It follows from this that sidelobe levels significantly lower than the 13.2 dB of the uniform array cannot be obtained by this method.

A problem more suited to this method is the reduction of the first few sidelobes of the uniform array, rather than a complete pattern synthesis. The term $\frac{E_u - F(u)}{u}$ in equation (2.23) represents the normalized difference between the desired pattern and the pattern of the uniform array. For reducing a number of sidelobes, this function may be regarded as the sum of a series of impulses positioned at the corresponding sidelobes. This is shown in Fig. 4.13 for the reduction of the first two sidelobes.

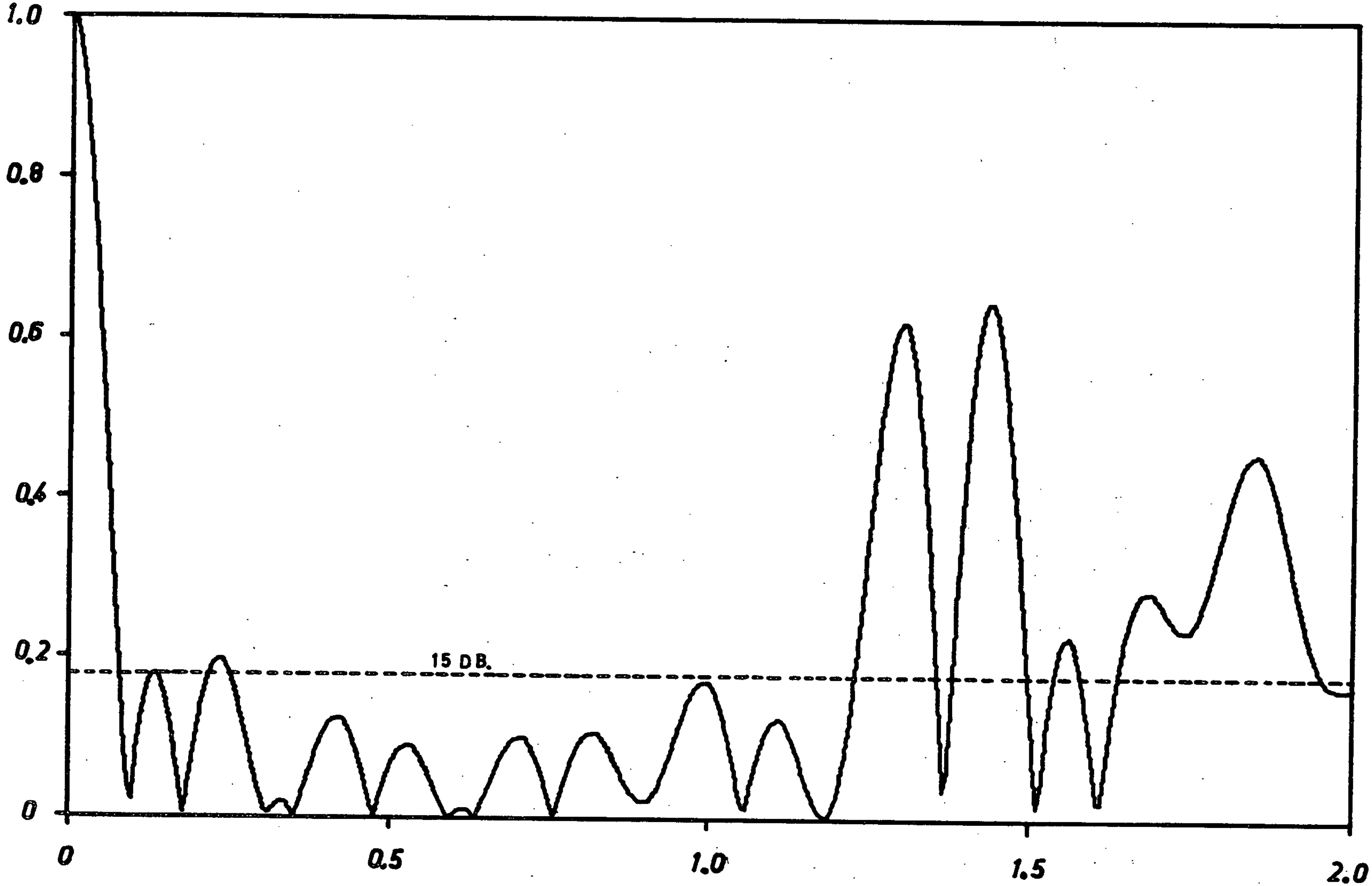


Fig. 4.12 Harrington's Synthesis of 15 element Array

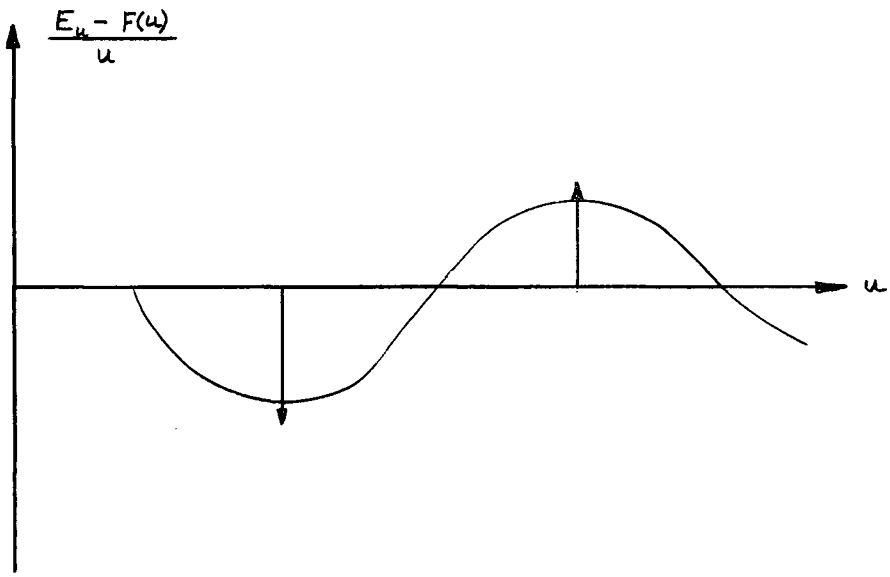


Fig. 4.13 Impulse Functions for Sidelobe Reduction

If there are K impulses and the k^{th} has a height of a_k and is situated at $u=u_k$, then $\frac{E_n - F(n)}{u}$ may be approximated by

$$\frac{E_u - F}{u} - \frac{1}{u} \sum_{k=1}^{k} a_k \ \delta(u-u_k), \text{ where } \delta \text{ is}$$

the unit impulse function.

Applying this to equation (2.23) we obtain

$$\varepsilon_n = \frac{N}{\pi} \sum_{k=1}^{K} a_k \frac{\sin nu_k}{u_k}$$

An example of the reduction of the first four sidelobes is given. The values of ε_n are tabulated in Fig. 4.14 and the complete pattern is shown in Fig. 4.15. A further use of this method is for the elimination of the grating lobes

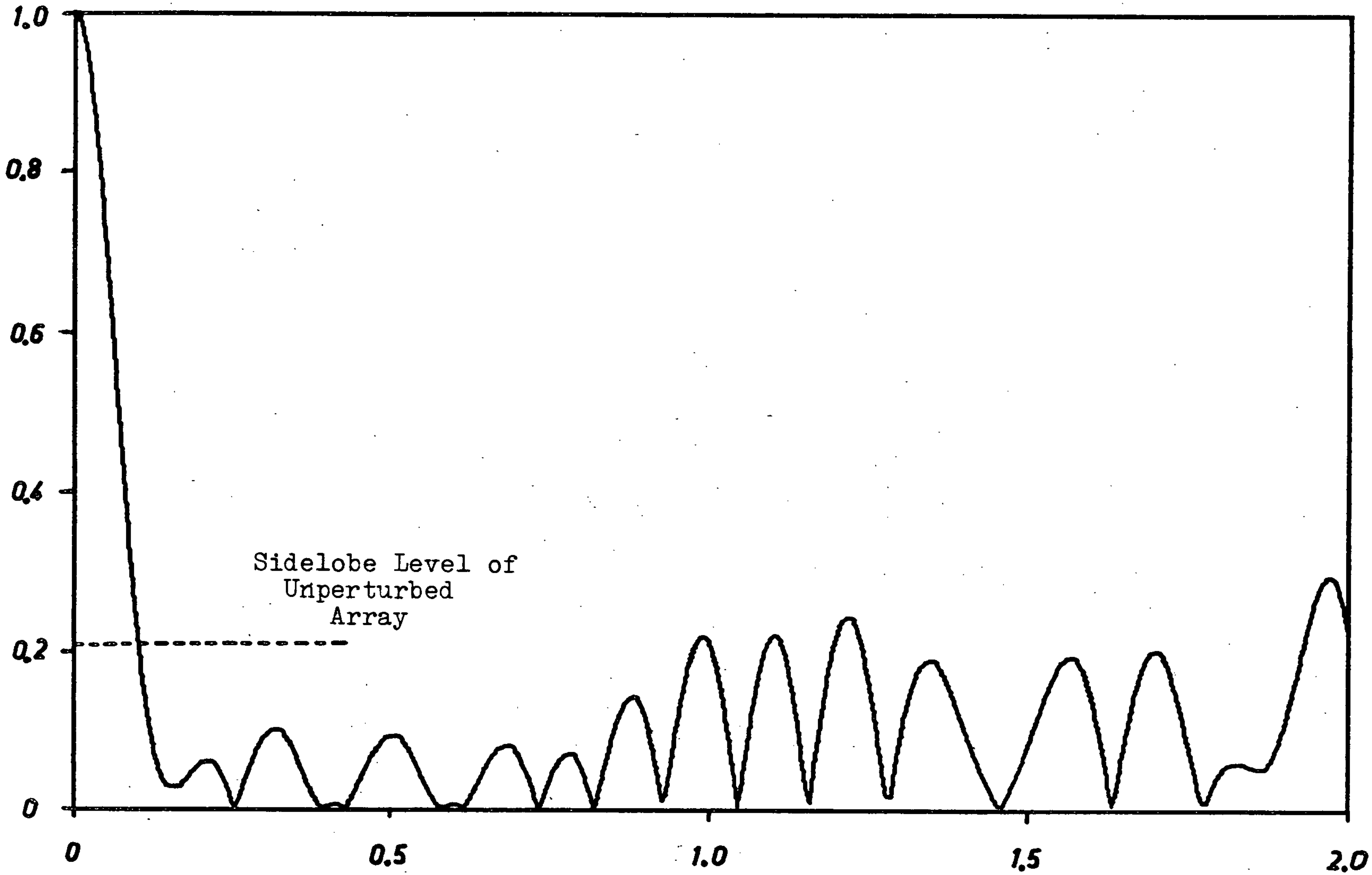


Fig. 4.15 Reduction of Sidelobes by Harrington's Method

associated with uniformly spaced arrays.

n	2	4	6	8	10	12	14	16	18	20
ε_n	-0.048	-0.176	-0.354	-0.518	-0.603	-0.616	-0.541	-0.328	0.153	0.780

Fig. 4.14 Perturbation Parameters

4.7 Willey's Method

In this method the excitations are assumed uniform and the set of equations represented by equation (2.26) is solved for the element positions. As with Unz's method the solutions are very difficult to obtain. The pattern of an 11 element binomial array was synthesized using 7 elements in a 3.5λ aperture. The resulting pattern is shown in Fig. 4.16. Arrays of more than 7 elements could not be synthesized by this method due to the difficulty of solving equation (2.26).

4.8 The Quadrature Method

Synthesis by this procedure is very convenient numerically since the element positions are given by the roots of

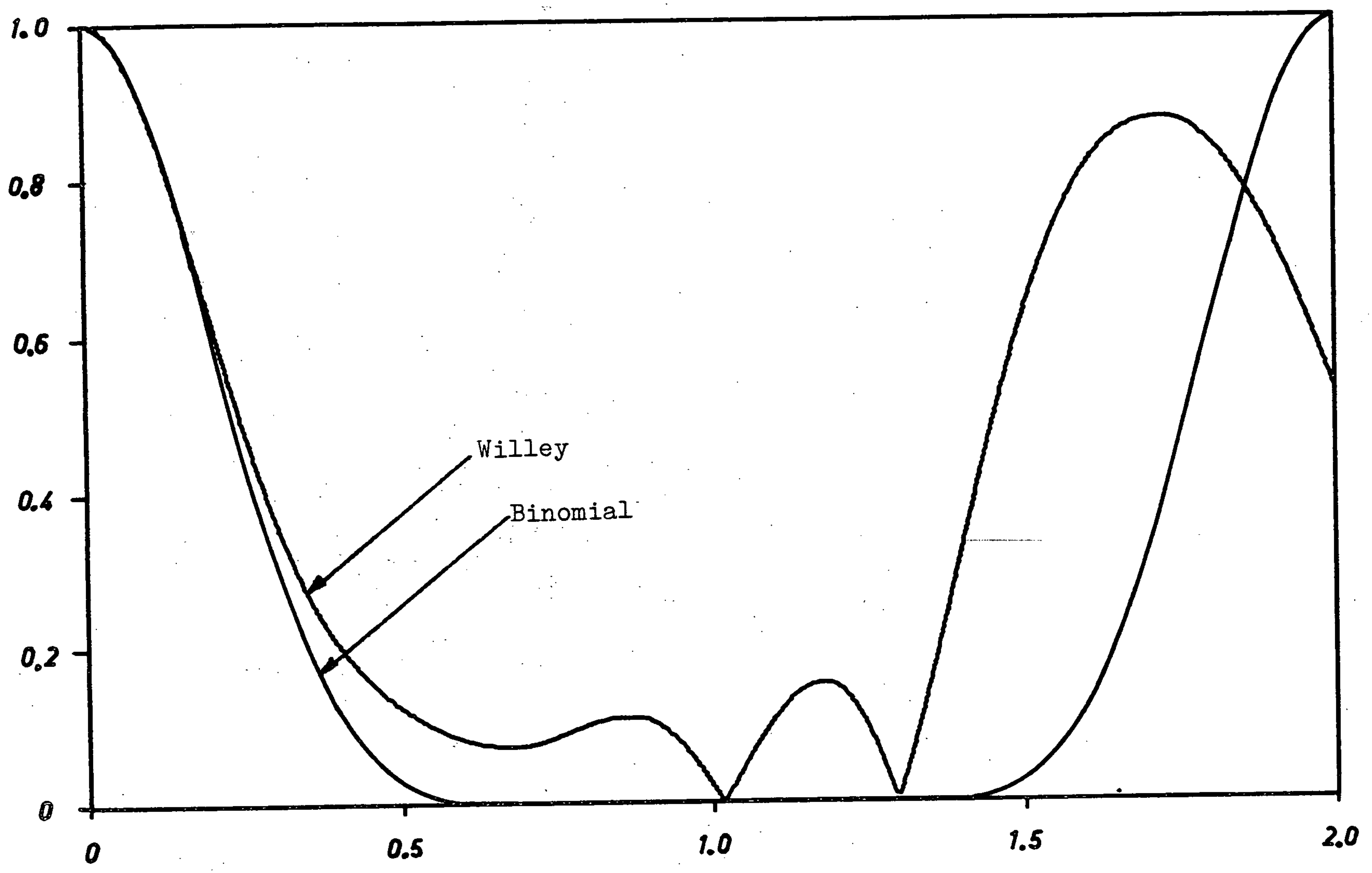


Fig. 4.16 Willey's Synthesis of Binomial Pattern

the Chebychev polynomial and the current excitations are obtained by evaluating a function at these roots. Thus there are no iterations for solutions of equations required. The properties of the Chebychev polynomials give the synthesized arrays an inverse space taper, i.e. the elements tend to be closer together at the ends of the array than they are in the centre. This appears to be a characteristic of methods that start from a line-source pattern.

Patterns have been obtained for arrays of 11 to 21 elements; their sidelobe envelopes are shown in Fig. 4.17 and the actual patterns of the 15 and 21 element arrays are shown in Figs. 4.18 and 4.19 respectively.

4.9 Summary of Results

A table demonstrating the properties of the different methods is shown in Fig. 4.20 and an evaluation of their results follows.

Unz's method, starting from the Fourier series formulation of the desired pattern, has given good results for small arrays, though computationally the method is tedious. For arrays of more than about seven elements the solution of the set of equations (2.6) becomes exceedingly difficult. The two methods based on the uniform-array representation of the desired pattern are generally unsatisfactory. Willey's method becomes increasingly difficult for larger arrays and Harrington's method fails in synthesizing patterns far removed from the pattern of a uniformly-excited, uniformly-spaced array.

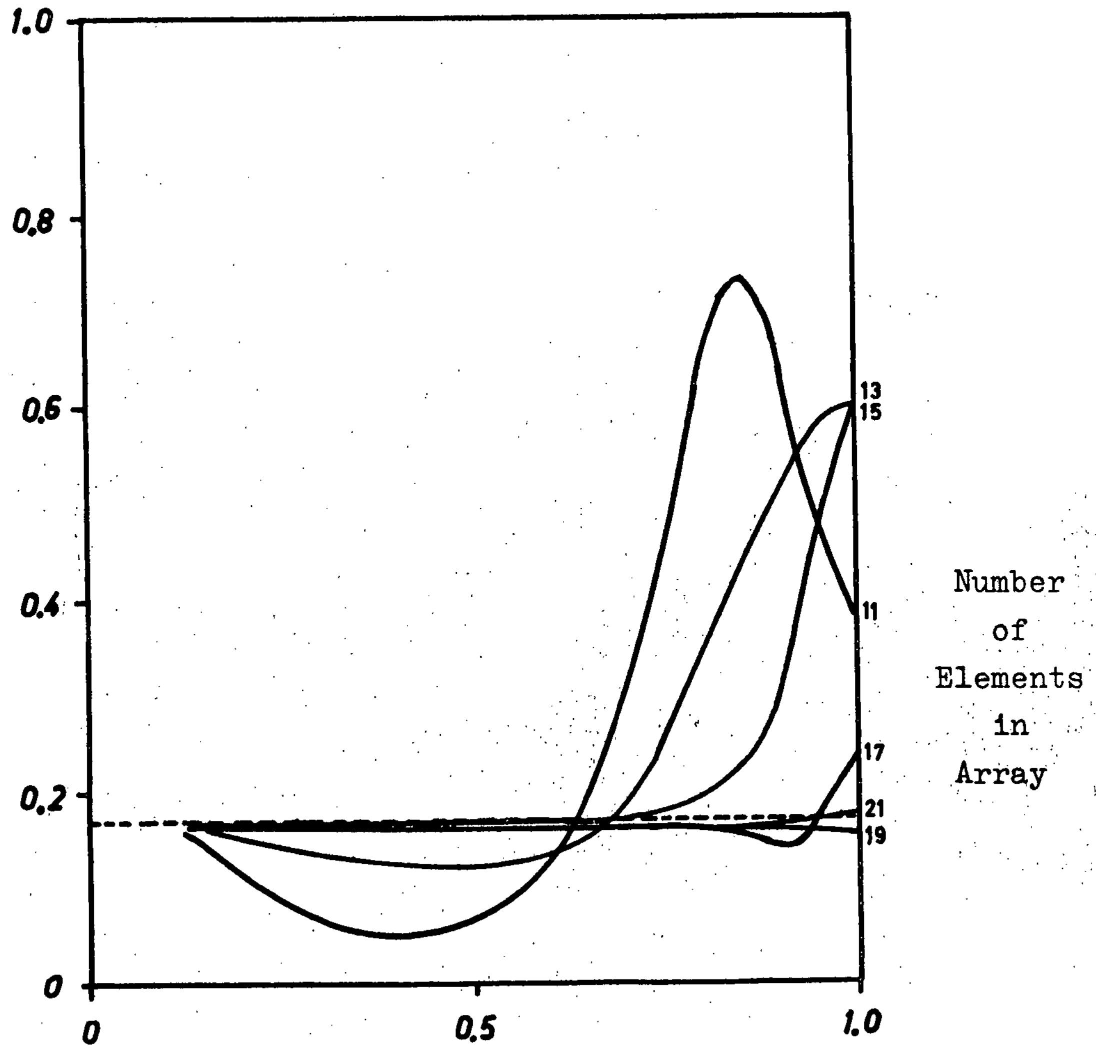


Fig. 4.17 Quadrature Method Synthesis of 15 dB Sidelobe Pattern

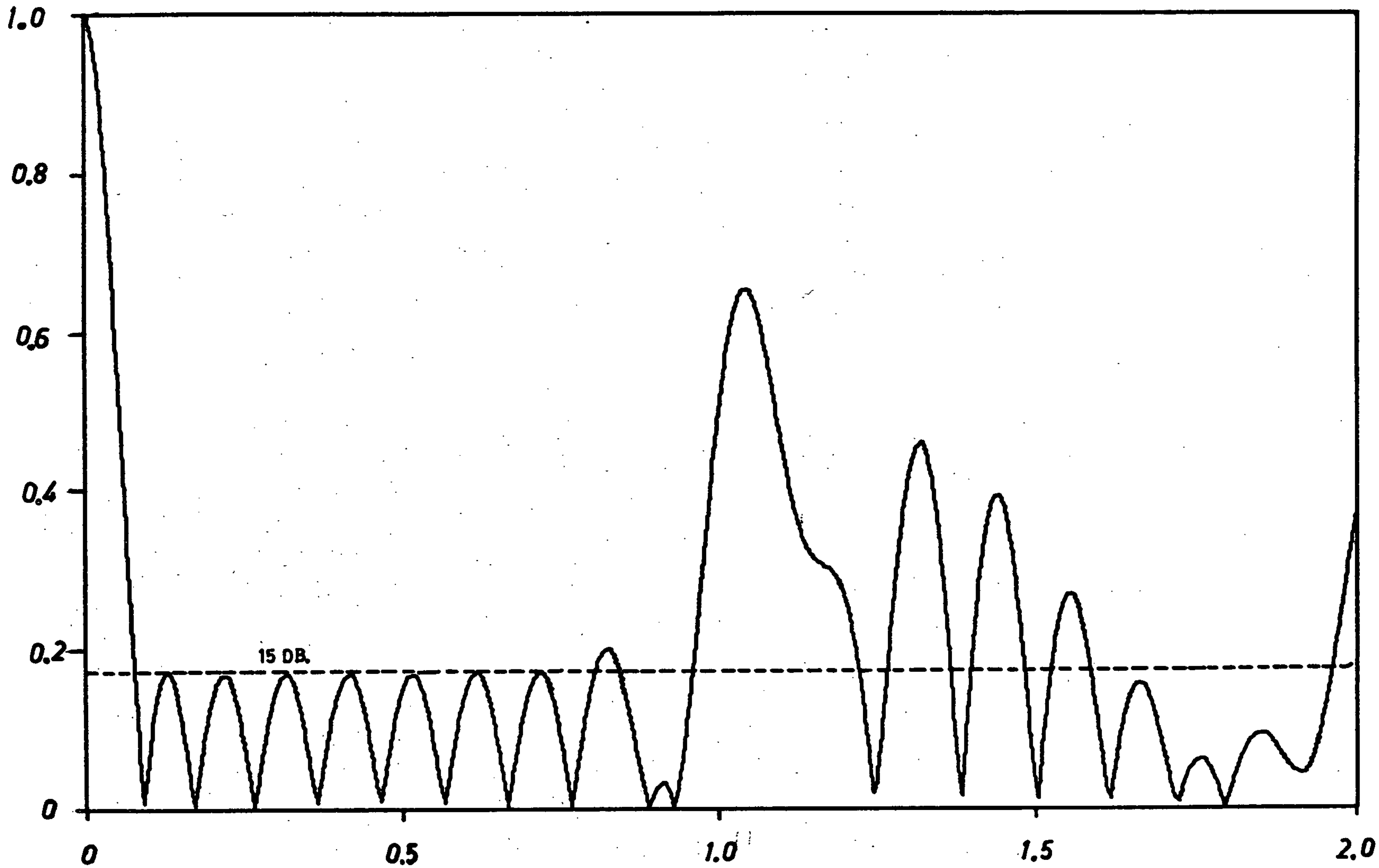


Fig. 4.18 15 element Array by Quadrature Method

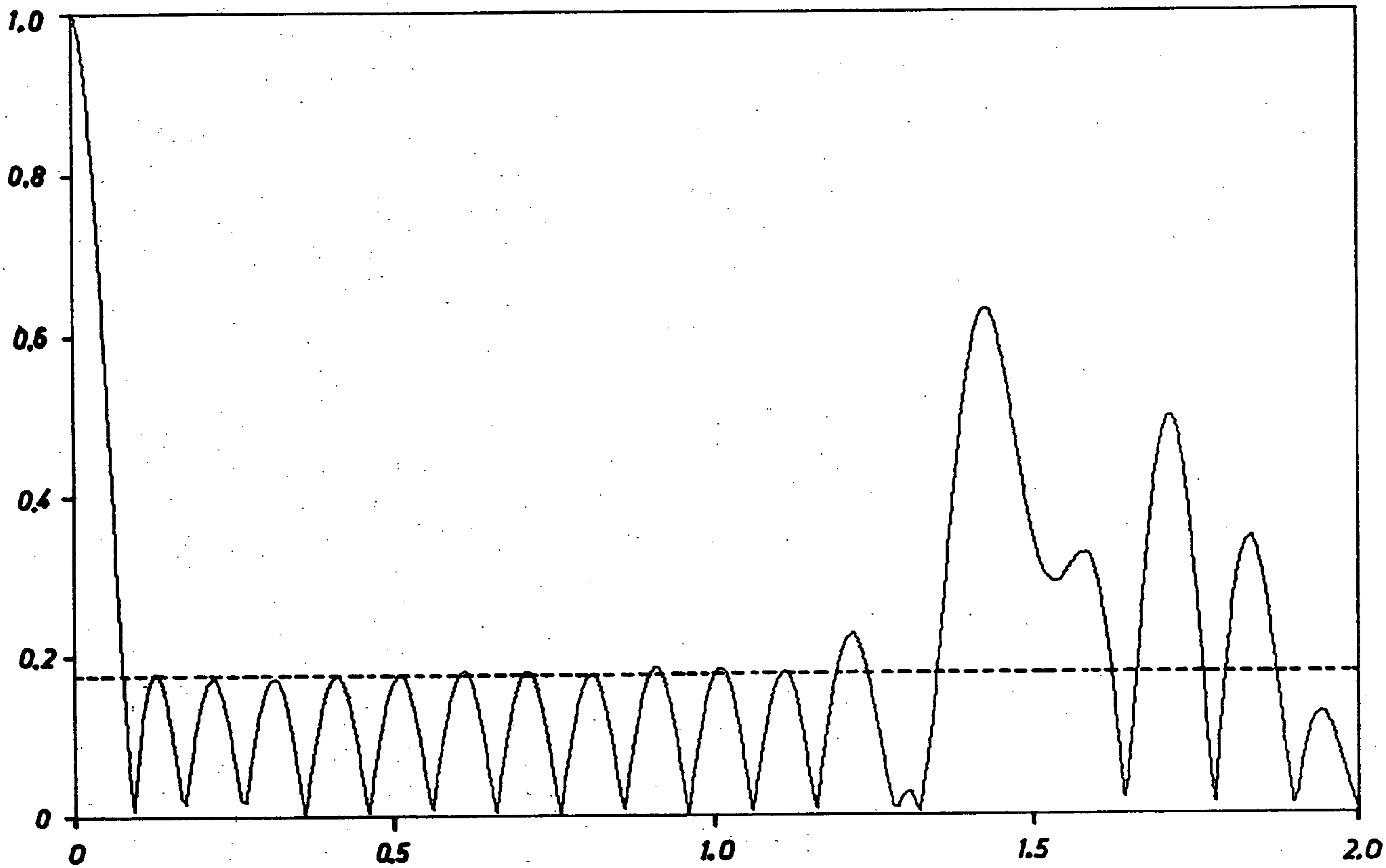


Fig. 4.19 21 element Array by Quadrature Method

PROPERTY / METHOD	ELEMENT NUMBER RESTRICTION	REFERENCE PATTERN FORMULATION	SYNTHESIZED PARAMETERS	RESULTING ELEMENT EXCITATION	MATHEMATICAL TECHNIQUE FOR SOLUTION	COMPUTING FEASABILITY FOR LARGE ARRAYS	RANGE OF U FOR WHICH SIDELOBES ARE WITHIN 2% OF DESIGN LEVEL FOR 21 ELEMENT ARRAY
UNZ	NONE	FOURIER SERIES SUM	POSITIONS AND EXCITATIONS	NON-UNIFORM	SOLUTION OF SET OF SIMULTANEOUS EQUATIONS	VERY POOR	IMPRACTICAL TO SYNTHESIZE ARRAYS OF MORE THAN ABOUT SEVEN ELEMENTS
HARRINGTON	NUMBER OF ELEMENTS IN UNPERTURBED ARRAY	UNIFORM ARRAY PATTERN	POSITIONS	UNIFORM	NUMERICAL EVALUATION OF AN INTEGRAL	FAIR	
WILLEY	NONE	UNIFORM ARRAY PATTERN	POSITIONS	UNIFORM	SOLUTION OF SET OF SIMULTANEOUS EQUATIONS	VERY POOR	
ISHIMARU	NONE	SOURCE	POSITIONS	UNIFORM	ITERATION OF SINGLE EQUATION	GOOD	$0 \leq U \leq 0{\cdot}75$
MAFFETT	NONE	LINE SOURCE PATTERN	POSITIONS	QUASI-UNIFORM	ITERATION OF SINGLE EQUATION	GOOD	$0 \leq U \leq 0{\cdot}80$
QUADRATURE	NONE	LINE SOURCE PATTERN	POSITIONS AND EXCITATIONS	NON-UNIFORM	EVALUATION OF A FUNCTION	VERY GOOD	$0 \leq U \leq 1{\cdot}20$

Fig. 4.20 Properties of the Methods

The methods starting from the line-source fomulation, i.e. with the desired pattern in the form of an integral, are much easier to apply to larger arrays. The methods yield qualitatively similar results in that the approximation is excellent from the main beam out to a point, after which the side lobes depart rapidly from the equal-sidelobe design. Of these three methods the quadrature method gives the largest region of good approximation, while still offering no element saving over the uniformly-spaced array.

4.10 Comparison with Uniformly-Spaced Array

The question has been posed as to whether it might be advantageous to use a non-uniformly-spaced array in place of a uniformly-spaced array. The pattern of a uniformly-spaced array, a 21 element Dolph-Chebychev array in a 10λ aperture, is shown in Fig. 4.21 for a sidelobe level of 15 dB. The main beam produced by this array may be scanned to within half a beamwidth of end-fire without an increase in the sidelobe level. None of the synthesis techniques have produced arrays with the same beamwidth and scanning capabilities. If beam scanning is not required, i.e. only the visible region of the pattern need be considered, then the quadrature method will produce a 17 element array in a 10λ aperture with equal-level sidelobes. However, if no scanning is required, the number of elements in the Dolph-Chebychev array can be decreased until the second main beam has moved into the edge of the visible region. The pattern of an 11 element array (one wavelength spacing) is shown in Fig. 4.22. Thus it would appear that no element

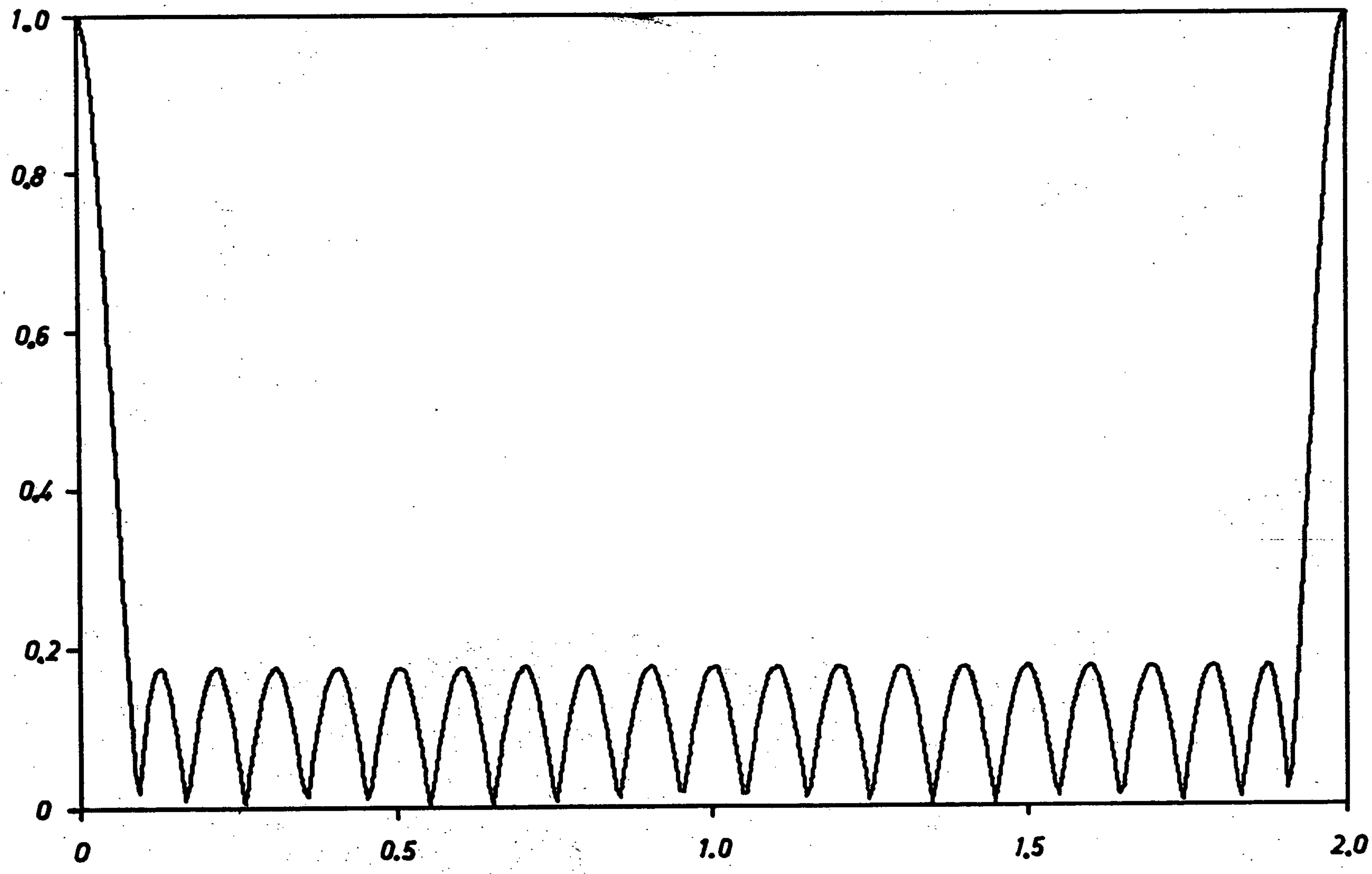


Fig. 4.21 Dolph-Chebychev Array Pattern, 21 elements with 15 dB Sidelobe Level

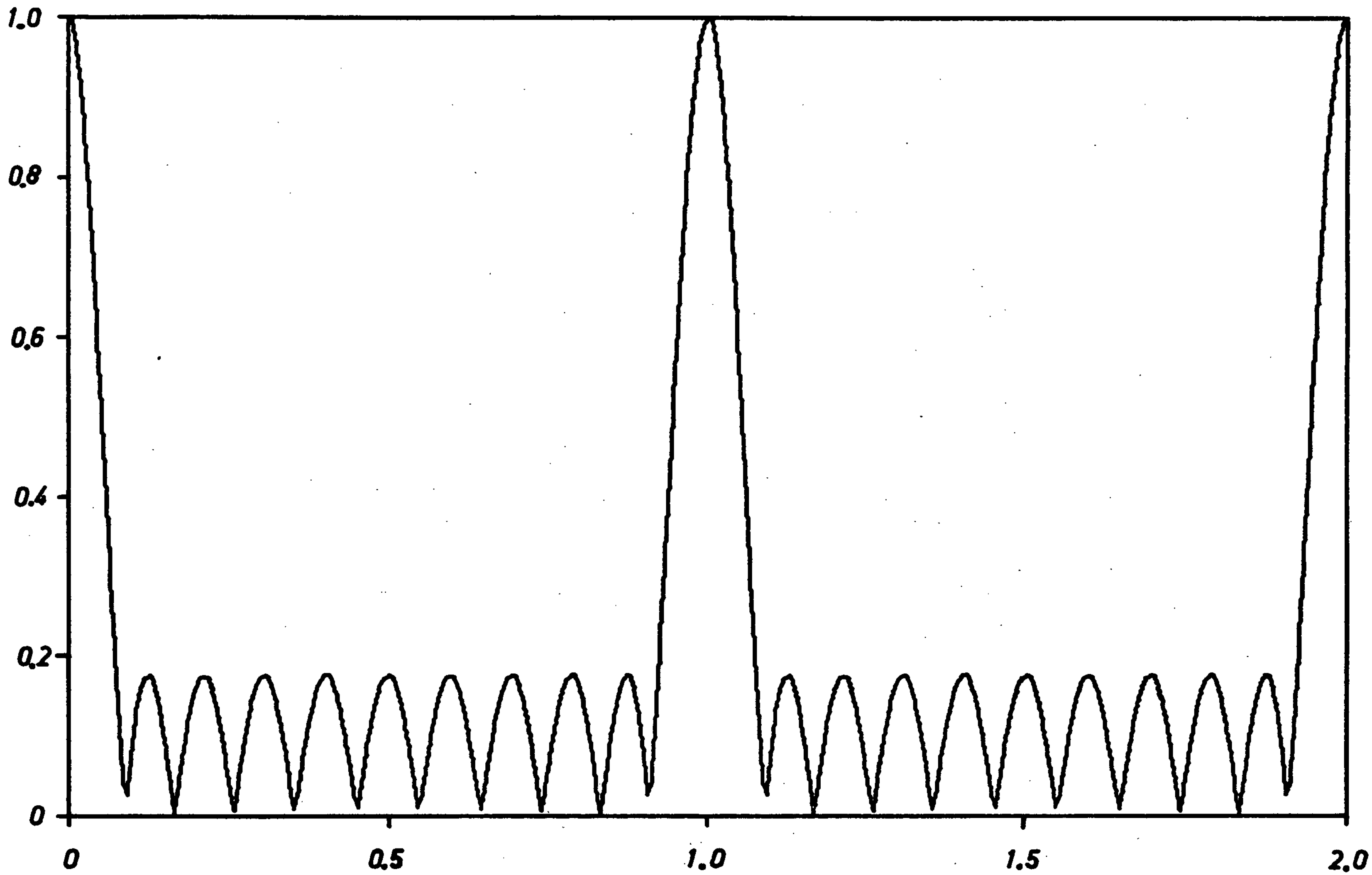


Fig. 4.22 Dolph-Chebychev Array Pattern, 11 elements with 15 dB Sidelobes

saving can be made using these synthesis methods.

Snover et al.[27] have designed arrays by a trial and error method which they claim to be an improvement on the Dolph-Chebychev array for a given number of elements and a given sidelobe level. However, the sidelobe levels which they quote for their arrays hold only in the visible region and therefore should be compared with a Dolph-Chebychev array of one wavelength spacing, not half-wavelength spacing as they have done. If this is done than there is no saving of elements.

5. CONCLUSIONS

The preceding chapters have examined the theory of five existing methods for the synthesis of non-uniformly-spaced arrays. The methods were categorized according to the mathematical formulation that the required pattern takes i.e. either a Fourier series sum, the pattern of a uniformly-spaced array or the pattern of a line source. In the course of this examination a new synthesis technique falling into the last of the given categories was devised.

Previous workers have made little study of the usefulness of the existing methods. As a result, the practical value of non-uniformly-spaced arrays is in some doubt. Furthermore, some invalid comparisons with uniformly-spaced arrays have resulted in misleading claims.

A comparison of all these methods has demonstrated that the new technique based on numerical quadrature is superior in two respects. Firstly it produces a more accurate synthesis of the equal-sidelobe type of pattern, and secondly it is the simplest method to use. This second characteristic makes the method especially useful for handling arrays with large numbers of elements.

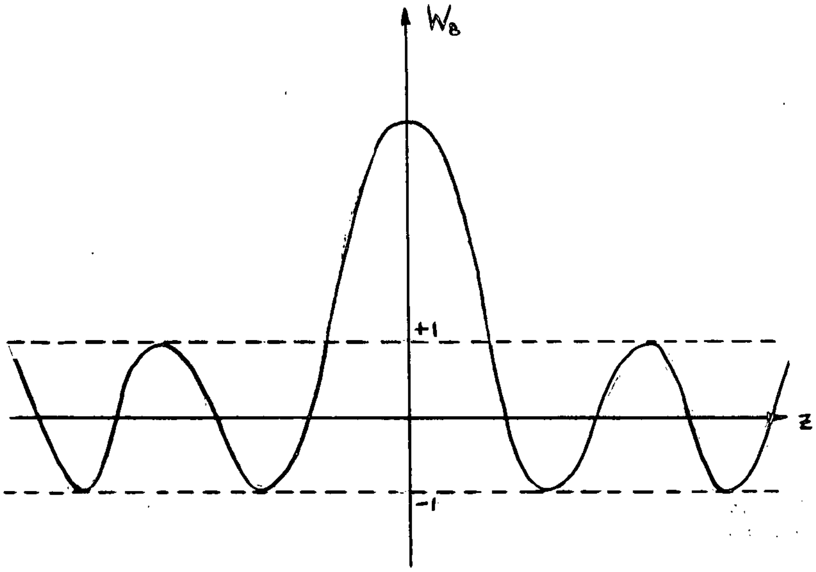
Despite this, no improvement over the uniformly-spaced array was demonstrated since Dolph-Chebychev arrays could always be found which used fewer elements for the same pattern. This might be due in part to the particular choice of the pattern to be synthesized, since a uniformly-spaced array can produce an equal-sidelobe pattern exactly, whereas a non-uniformly spaced array cannot. For larger arrays synthesized by the quadrature method, it was found that their performance improved as the number of elements increased. This may indicate that an improvement over the uniformly spaced array might be achieved when extremely large arrays are being considered.

The value of non-uniformly-spaced arrays is clearly in doubt. However no final verdict is possible until a true optimization procedure is found: one which adjusts the positions, the amplitudes and the phases of the elements to achieve the best combination of beamwidth and sidelobe level. This problem has resisted the efforts of many workers, and earlier attempts by this author were no more successful.

APPENDIX I

1. The Taylor Line Source

This is based on the equal-ripple property of the Chebychev polynomials (as is its discrete source counterpart, the Dolph-Chebychev array). Taylor combined two polynomials to give the form of Fig. A.1.



The function is

$$W_{2N} = T_N(B - a^2 z^2)$$

where a is a constant, $B = \cosh \frac{\pi A}{N}$, $A = \frac{1}{\pi}$ arc cosh η , η being the height of the main beam.

Since $T_N(z) = \cos(N \text{ arc cos } z)$ the zeros of W_{2N} are given by

$$z_n = \pm \frac{1}{a} \left[B - \cos\left(\frac{n\pi}{N} - \frac{\pi}{2N}\right)\right]^{\frac{1}{2}}$$

Letting the order, N, tend to infinity but keeping the position of the first zero fixed, the zeros become

$$z_n = \pm \left[A^2 + (n - \tfrac{1}{2})^2\right]^{\frac{1}{2}}$$

The corresponding space factor has unity amplitude sidelobes and a main beam amplitude, η, and is

$$F(z) = C \frac{\cos \pi (z^2 - A^2)^{\frac{1}{2}}}{\cosh \pi a} \quad ,$$

or, putting $C = \cosh \pi a$,

$$F(z) = \cos \left[\pi (z^2 - A^2)^{\frac{1}{2}}\right]$$

Since the remote sidelobes do not decay, this 'ideal' space factor is unrealizable. If the z scale is stretched by a factor σ, so that the close in zeros of the ideal space factor are approximated closely, then at some point, $\bar{n}$, a zero occurs due to the stretching. If then, from this transition point on, the zeros occur at $\pm$ n, the approximate space factor has zeros

$$z_n = \pm \sigma \left[A^2 + (n - \tfrac{1}{2})^2 \right]^{\frac{1}{2}} \qquad 1 \leqslant n < \bar{n}$$

$$z_n = \pm n \quad , \qquad \bar{n} \leqslant n < \infty$$

The approximate space factor is then

$$F(z) = \frac{\sin \pi z}{\pi z} \prod_{n=1}^{\bar{n}-1} \frac{1 - \left(\frac{z}{z_n}\right)^2}{1 - \left(\frac{z}{n}\right)^2}$$

As $\bar{n} \to \infty$ the approximate space factor approaches the ideal space factor. The stretchout parameter, σ, is

$$\sigma = \frac{\bar{n}}{\left[A^2 + (\bar{n} - \frac{1}{2})^2\right]^{\frac{1}{2}}}$$

The sidelobes are nearly constant (and smaller than the main beam by a factor η) up to $\bar{n}$ and then decay as $\frac{1}{z}$.

The line source producing this approximate space factor is found by the Woodward synthesis technique, and as a Fourier series is given by

$$g(p) = 1 + 2 \sum_{n=1}^{\bar{n}-1} F(n) \cos np \qquad \left(p = \frac{2\pi x}{L}, \text{ and } L \text{ is the aperture length}\right),$$

where

$$F(n) = \frac{\left[(\bar{n}-1)!\right]^2}{(\bar{n}-1+n)!(\bar{n}-1-n)!} \prod_{m=1}^{\bar{n}-1} \left(1 - \frac{n^2}{z_m^2}\right).$$

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