AN ANALYSIS
OF THE
FRASER RIVER MODEL TIDAL CONTROL

by

DONALD HUGHIE JAMES KAY

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Abstract

The development of an automatic control system involves a consideration of the problems of the stability and response of such a system. The purpose of this thesis is to outline these problems as they appear in the automatic control of water level in a tidal basin model.

A synopsis of general servomechanism theory is briefly outlined stressing three points (1) Design the system so that oscillatory conditions prevail. (2) Design the natural frequency well above the operating frequency. (3) Where necessary introduce stabilizing networks.

The results obtained by tests revealed one thing; the introduction of external circuits had very little effect on the continuous operation of the tidal model. An examination of the theoretical analysis of the control system however brought out three reasons for limiting the value of the constant K_4. The constant K_4 corresponds to the moment of inertia of a mechanical system. The constant K_4 is related to the parameters of the system, which are the area of the basin, the length of the weirs, and the pump discharge.

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Introduction

The development of an automatic control system involves a consideration of the problems of the stability and response of such a system. Under stability, the system must neither induce nor support any harmonic oscillations. Further, any oscillations that are induced by outside disturbances must be attenuated to a negligible value in a very few cycles. Under response, the system must follow the input signal as accurately as possible and with a minimum time delay.

The purpose of this thesis is to outline these problems as they appear in the automatic control of water level in a tidal basin model. The servo-mechanism itself is not unlike that used in many other systems but the equations introduced by the hydraulics of the system make the problem unique. One of the factors involved is the non-linearity of the weir discharge equation. Another is the lag in the water level response to weir movement caused by the finite velocity of a wave in water.

General Theory

Control systems may be either of the open cycle or of the closed cycle types. In an open cycle system the signal that operates the controller is independent of the output. In a closed cycle system a percentage of the output is fed back and compared with the input signal. The difference or error then operates the controller so as to reduce the error. The open cycle system is fairly simple and will not be considered in this thesis.

The closed cycle system can be further subdivided into automatic control or regulator systems and servomechanisms. The fundamental difference in the two systems is in their application, rather than the principles involved. The automatic regulator is designed to maintain the output close to some fixed input, as for example, in a voltage regulator. The servomechanism is designed to maintain the output arbitrarily close to some input which varies with time, as for example, an automatic position controller. The input in the latter case may be continuous or discontinuous.

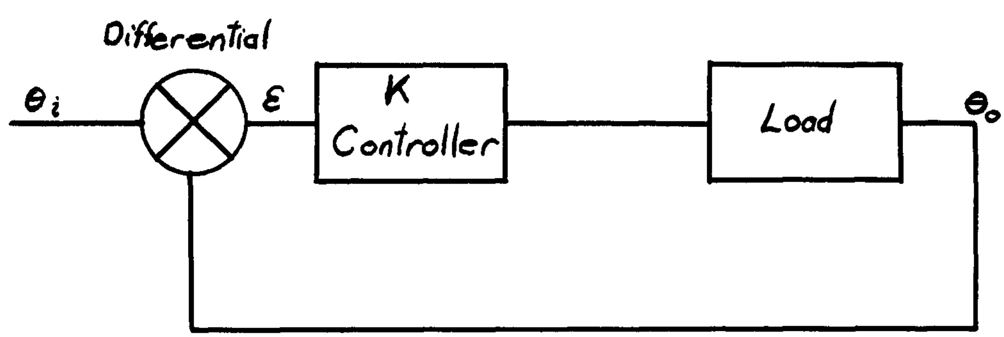

Fig. 1 Block Diagram of a Simple Servomechanism

Fig. 1 is a diagram of an elementary closed cycle control system. θ_i represents the input signal, θ_o the output and ε the difference or error. The differential compares the input and output signal and injects the difference into the controller. The controller will amplify the error and will produce such a change in the output as will tend to reduce the error.

Consider for example a simple mechanical positioning system with θ_i and θ_o angular positions. The load in this case will be a moment of inertia "J" and a friction component "F". Let the torque applied to the load be proportional to . Then the basic equations will be

$$\theta_i - \theta_o = \varepsilon$$

$$T = K\varepsilon$$

$$T = J\frac{d^2\theta_o}{dt^2} + F\frac{d\theta_o}{dt}$$

Where J = moment of inertia of system

F = viscous friction

T = output torque of controller

Combining the three equations

$$K\theta_i = J\frac{d^2\theta_o}{dt^2} + F\frac{d\theta_o}{dt} + K\theta_o$$

$$\text{or} \quad \theta_o = \frac{K\theta_i}{Jp^2 + Fp + K} \qquad p = \frac{d}{dt}$$

This yields a Characteristic linear equation that is a second order differential equation $Jp^2 + Fp + K$. The three constants are in general independent and can be individually adjusted in the design.

There is one basic fault with this simple servo system. An error must exist before any correction is applied. This will cause a velocity error in the system, or the error will be proportional to the rate of change of position. This velocity error plus the inertia of the system will tend to cause oscillation. Increasing the friction factor will decrease the time of oscillation but it will increase the magnitude of the velocity error.

Where more rigid design requirements have to be met, stabilizing circuits must be introduced to reduce the magnitude of the velocity error and reduce the time of oscillation. In the controller of the elementary system the output is proportional to the error, that is the controller transfer function equals a constant.

$$T.F. = K$$

The stabilizing circuits may be introduced into the controller so the transfer function may be a differential equation of the type

$$T.F. = \frac{A}{p} + B + Cp + Dp^2 + \ldots \qquad p = \frac{d}{dt}$$

The circuits can be adjusted to make any of the constants of any desired value. The design requirements will determine the various values.

The term $\frac{A}{p}$ represents the integral factor or reset component. Increase in the value of this term decreases the amplitude of the velocity error. The disadvantage of this is that as the velocity error approaches zero the system approaches instability. If the velocity error equals zero any induced oscillation will be sustained. If the velocity error becomes negative then the oscillations will build up indefinitely.

The terms involving p, p^2, etc. represent the various powers of the derivative of the error. The primary reason for introducing these terms is to improve the frequency response. In most practical systems the reponse will fall off as the frequency increases. The derivative factors can be adjusted to increase the cut off frequency. In most practical applications the first or second order of derivative is all that is used.

Referring to the simple servo system again the characteristic equation has its roots as

$$p = -\frac{F}{2J} \pm \sqrt{\left(\frac{F}{2J}\right)^2 - \frac{K}{J}}$$

This will permit three solutions depending on the magnitude of the parameters.

(1) $\left(\frac{F}{2J}\right)^2 > \frac{K}{J}$ overdamped

(2) $\left(\frac{F}{2J}\right)^2 = \frac{K}{J}$ critically damped

(3) $\left(\frac{F}{2J}\right)^2 < \frac{K}{J}$ under damped

In the first case p is real with two surd roots. The transient solution of differential equations of this form may be expressed in hyperbolic functions with an exponential decay term. This case is termed as the overdamped case. There is no tendency for oscillation at all but the time of response is too long to be practical. In the second case the equation has two identical real roots. This equation has a solution of an exponential decay term multiplying a constant term plus a constant times time. This case is referred to as the critically damped case. The damping factor F_c is known as the critical damping and equals $2\sqrt{KJ}$. Again there is no tendency for oscillation but the time of response, although shorter than the first case, is still too long to be practical.

In the third and most important case the equation has two roots that are complex conjugates. This yields the oscillatory solution, an exponential factor multiplying a sine and cosine term. This case is called the underdamped case. The sine and cosine terms will have a frequency referred to as the natural frequency of the system

$$f_b = \frac{\omega_b}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{K}{J} - \left(\frac{F}{2J}\right)^2}$$

Fig. 2 is a graph of servomechanism response to a step function input. The various curves correspond to different damping ratios "C". The damping ratio "C" is defined as the ratio of the actual damping to the critical damping of the system. ω_n is the natural frequency of the system without damping and ω_1 is the applied frequency. The curves are made dimensionless to permit comparison of different systems.

If instead of a step function input a sinusoidal input is applied, the output will also be sinusoidal with either positive or negative amplification and some time phase displacement. The degree of amplification and phase displacement will depend on the applied frequency. If the applied

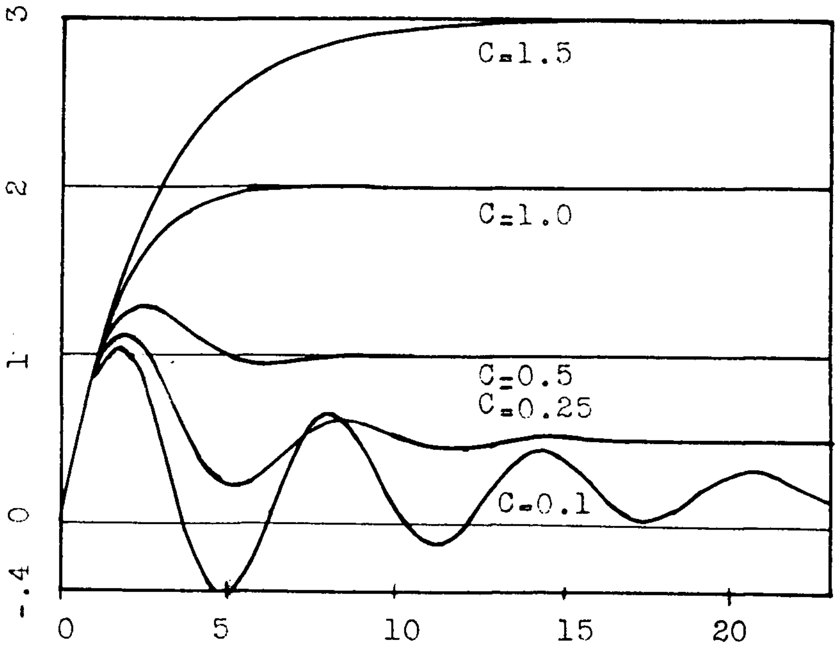

FIG. 2

Error-time Curves for Viscous-damped Servomechanism

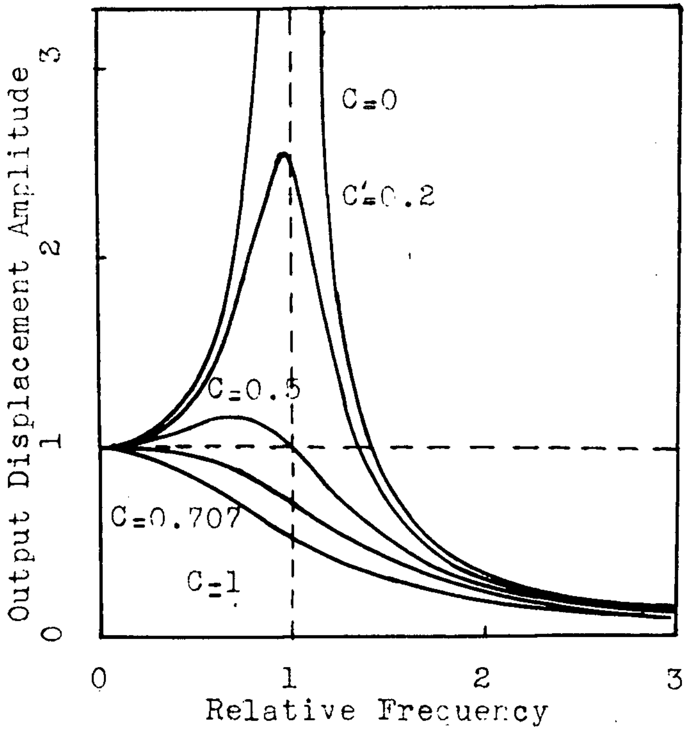

FIG. 3

Resonance Curves of Servomechanism With Viscous Output Damping

to follow page 6

frequency approaches the natural frequency a condition exists very similar to that encountered in an oscillator tank circuit. A very small input signal will result in a large output signal, the only limiting factor is the damping or friction coefficient. This is very undesirable in most servo systems. The ideal situation exists when the output equals the input and the phase displacement is a minimum. Fig. 3 is a graph of the response of a simple servo subject to various frequency inputs.

This review of the servomechanism theory has revealed three points. (1) Design the system so that oscillatory conditions will prevail, that is have it satisfy case III, the underdamped case. (2) Design the natural frequency well above the operating frequency especially if the input should contain predominant harmonics higher than the fundamental. (3) Where more rigid design requirements have to be met, introduce networks so as to reduce the velocity error, or to increase the frequency response.

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Photo #1 West Portion of River Basin

Photo #2 Centre Portion of River Basin

to follow page 7

Photo #3 East Portion of River Basin

Photo #4 Tidal Basin with Recorder in Foreground

Description of the Tidal Control Equipment

The model covers an area of three acres. It includes the Fraser River from Mission to the mouth, Pitt River and Pitt Lake, and a portion of the Gulf of Georgia in the vicinity of the tide flats. The horizontal scale is one in six hundred, the time and vertical scale is one in seventy, and the velocity scale is one in eight point five. Various views of the model are illustrated in photographs 1,2,3 and 4. Photographs 1,2 and 3, are views of the west, centre, and east portions of the river basin. Photograph 4 is a view of the tidal basin.

The water level control equipment consists of six components illustrated in photographs 5,6,7 and 8.

(1) A photo cell reader that provides a voltage proportional to the desired tide level.

(2) a pair of load resistors that acts as the differential.

(3) An amplifier that consists of a D.C. electronic amplifier and a hydraulic amplifier.

(4) The hydraulic jack that raises and lowers the weirs.

(5) The set of weirs that control the water level.

(6) The tide float potentiometer that produces a voltage proportional to the actual tide level.

There are other pieces of equipment, such as the 20 cubic foot per second pump that supplies water to the basin; The power supplies for the float potentiometer, the photo cell

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Photo #5 Electronic Control Equipment Including

(1) Photo Cell Reader

(2) D.C. Amplifier

(3) Junction box containing power supplies
and differential

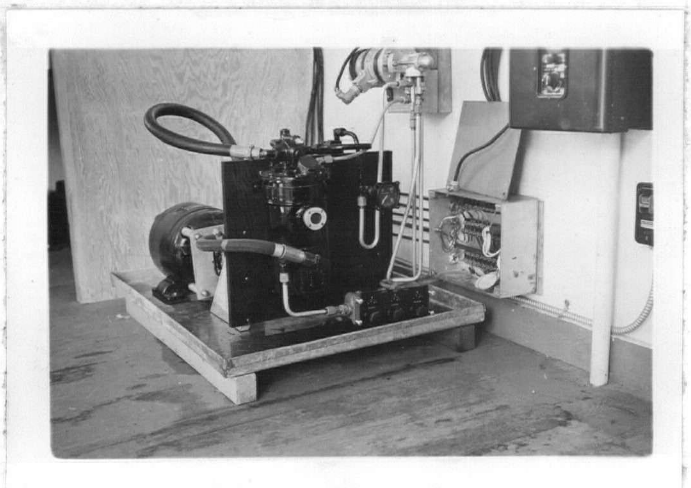

Photo #6 Hydraulic Amplifier

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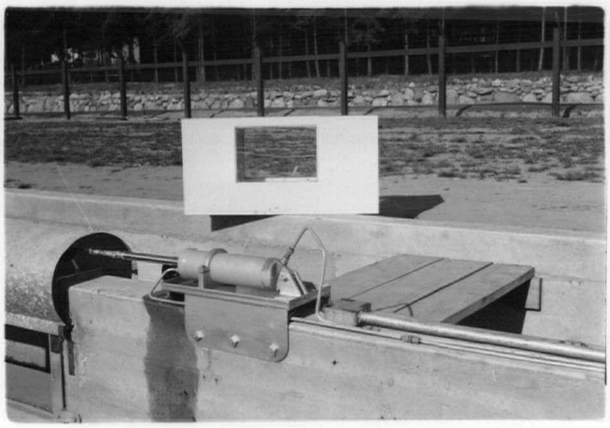

Photo #7 Hydraulic Jack

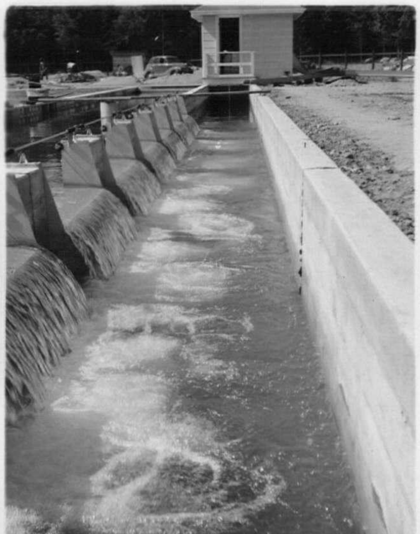

Photo #8 The Set of Weirs

reader, and the D.C. amplifier; and the pump and regulating system that supplies the pressurized oil for the hydraulic amplifier, that are not actually part of the closed cycle control system.

The photo cell reader, which is encased in a light tight box, is shown schematically in Fig. 4. The light source is reflected off the galvanometer mirror through a piece of transparent paper and a pair of convex lenses onto a photo multiplier tube. The output of the tube is connected in series with the galvanometer and one of the load resistors in the differential. The galvanometer will deflect until the light is interrupted by the tide cycle, which is a black line painted on the transparent paper. Time scale extends along the paper and water level scale across. The tide chart is driven past the light by synchronous motor. Thus the output voltage of the photo cell reader is proportional to the desired tide level.

The differential receives the voltage proportional to the desired tide level from the photo cell reader and that proportional to actual tide level from the tide float potentiometer and has an output voltage equal to the difference. The circuit consists of two load resistors connected in series with the input voltages impressed across the resistors in opposite polarity.

The amplifier system consists of a D.C. electronic amplifier that actuates a balanced oil valve in the hydraulic system. The displacement of the balanced oil valve is proportional to the magnitude and polarity of the error or difference voltage. The output of the hydraulic amplifier is

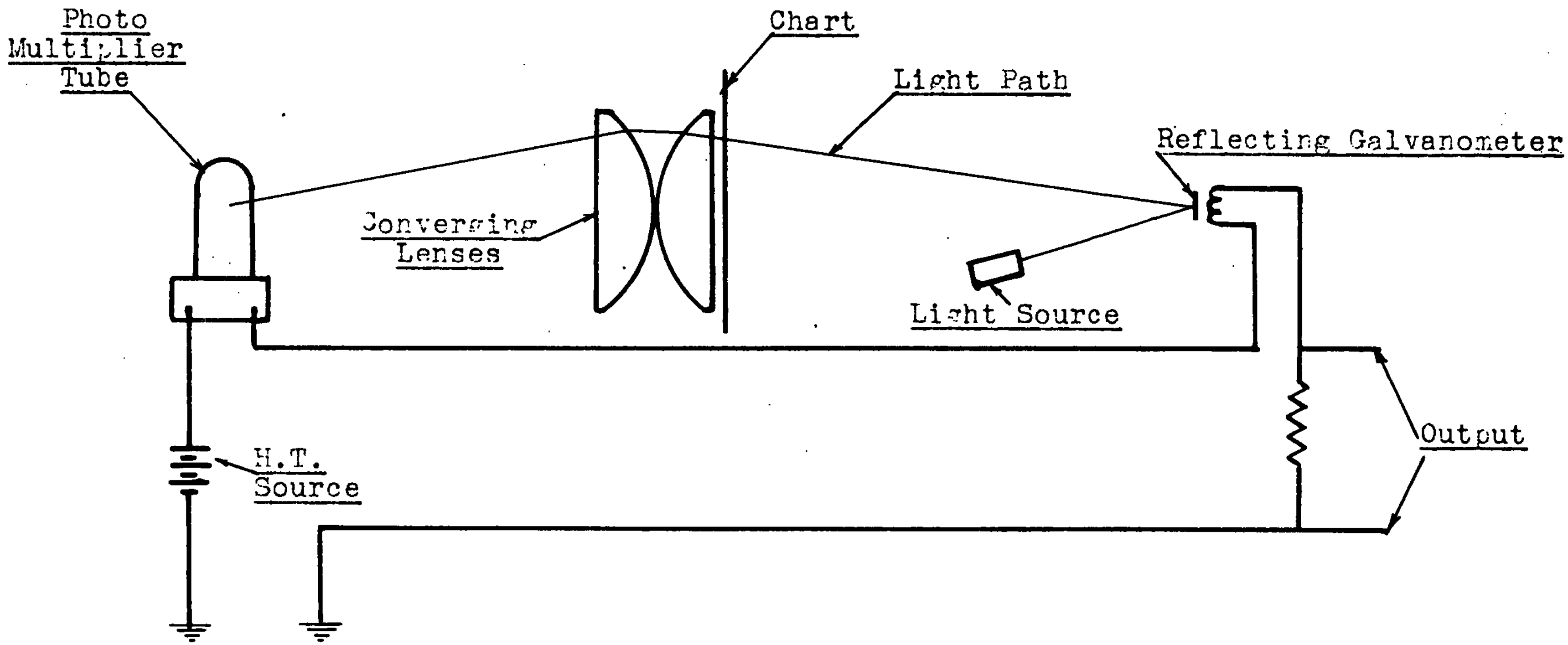

FIG. 4 BLOCK DIAGRAM OF THE CHART READER

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in turn proportional to the displacement of the balanced oil valve.

The output of the hydraulic amplifier moves the hydraulic jack which is mechanically connected to the weirs. The hydraulic jack speed and hence the weir speed is proportional to the hydraulic amplifier output. Water is pumped continuously at a rate of 20 cubic feet per second from a sump into the tidal basin. The surplus water is spilled over the weirs back into the sump. Any variation in weir level will thus result in a similar variation in the water level.

A voltage proportional to the actual tide level is obtained by means of the tide float potentiometer. The potentiometer is mechanically connected to a float. The output of the potentiometer is fed into the other load resistor in the differential.

During operation the tide chart, which has the sequence of tides for an entire year plotted on it, is driven continuously past the light source. This calls for a continuous sequence of operations for five days. It is highly desirable to have the system operate without any break and to have the error in the output less than 5%, preferably less than 3%. This error only refers to the amplitude, the phase displacement is relatively unimportant as long as it is not unreasonably large and remains fairly constant.

Theoretical Analysis

In designing a servo system more than one approach can be taken. The input can be assumed to be a series of steps. This approach imposes very severe requirements on the system and in certain cases may not be justified. Another approach can be to assume the input is a sinusoidal function. The system can then be designed to have the amplification less than some predetermined value and the phase displacement at the operating frequencies less than the allowable value. If the function is more complex a Fourier analysis of the function can be made and the highest appreciable harmonic can be made to comply with the amplitude and phase displacement requirements.

In analyzing the tidal control system it is obvious that requiring a good step function response is imposing conditions that are more severe than necessary. The non repetetive nature of the tide cycle does not permit the usual Fourier analysis of the cycle. Harmonic analyses of tide cycles have been made to facilitate tide prediction. This analysis relates the various components of the tide cycle to the behaviour of different celestial bodies. Some 170 components exist in a detailed analysis but the four major components are the pull of the moon, the pull of the sun, and two components due to moons declination. Table #1 gives the relative magnitude and period of the various components. The magnitude of the moons component is taken as unity.

These four major components are near or less than the fundamental frequency, which is taken as the frequency

Component	Relative Amplitude	Period
Moons pull	1.0	12.4 hrs.
Suns pull	0.280	12.0 hrs.
Moons declination #1	0.415	24 hrs.
Moons declination #2	0.258	25.9 hrs.

Table #1

Table of periods and relative amplitudes of major components of the harmonic constituents of the tidal cycle

of the moons attractive component. There are components of higher frequency, in shallow water tides periods may be as low as three hours, but the relative magnitudes of these components are less than 0.01. With these factors in mind the design frequency was taken to be the frequency of the moons attractive component.

The step function response although of secondary importance can not be completely ignored. A reasonable response to a step function is desirable for two reasons. One, when starting the system it is desirable to have the water level settle to the starting tide level in a reasonable length of time. Two, if in the middle of the run the water level is either intentionally or accidentally shifted off the curve it will be desirable to have it return as quickly as possible and with the minimum amount of oscillation.

After considering these properties of the tidal cycle, it was decided to analyze the system using a design frequency equal to the frequency of the moons attractive component. The step function analysis will be used as a check.

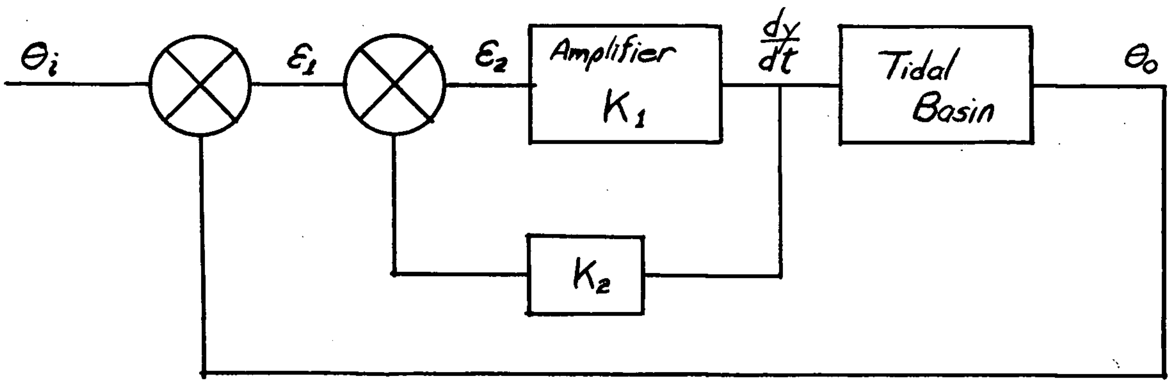


Fig. 5 Block Diagram of Tidal Control System

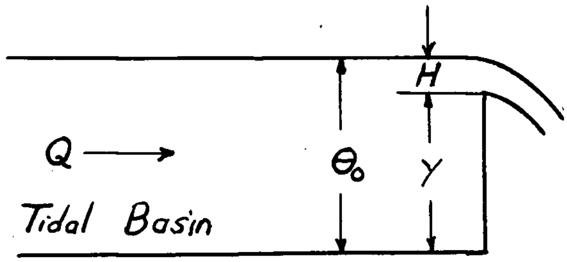


Fig. 6 Schematic Diagram of Tidal Basin

Definitions

θ_i = input signal

θ_o = output signal

ε_1 = error between θ_i and θ_o

ε_2 = error between ε_1 and $K_2 \frac{dy}{dt}$

K_1 = amplifier constant

y = weir level

A = area of tidal basin = 12,000 sq. ft.

H = head over weir

K_2 = feed back constant

Q = pump discharge = 20 c.f.s.

b = weir length = 40 ft.

A block diagram of the system as it originally existed is illustrated in Fig. 5 and a diagram of the tidal basin at the weirs is shown in Fig. 6. The basic relationships taken off these diagrams are

$$\theta_i - \theta_o = \mathcal{E}_1 \tag{1}$$

$$\mathcal{E}_1 - K_2 \frac{dy}{dt} = \mathcal{E}_2 \tag{2}$$

$$\frac{dy}{dt} = K_1 \mathcal{E}_2 \tag{3}$$

$$\theta_o = H + y \tag{4}$$

The water flowing into the basin minus the water flowing out over the weirs will equal the change in volume. The water flowing over the weir can be calculated by the weir discharge formula.

Weir discharge = $3.33\, bH^{3/2} = K_3 H^{3/2}$

with $K_3 = 3.33\, b$

$$\therefore\ Q - K_3 H^{3/2} = \frac{d\,Vol}{dt}$$

or
$$K_3 H^{3/2} = Q - A\frac{d\theta_o}{dt} \tag{5}$$

Equation five is a non-linear equation but it can be expressed by a MacLaurens Series. Therefore by power expansion

$$H = \left(\frac{Q}{K_3}\right)^{2/3}\left(1 - \frac{A}{Q}\frac{d\theta_o}{dt}\right)^{2/3}$$

$$= \left(\frac{Q}{K_3}\right)^{2/3}\left(1 - \frac{2}{3}\frac{A}{Q}\frac{d\theta_o}{dt} - \frac{1}{9}\left(\frac{A}{Q}\frac{d\theta_o}{dt}\right)^2 - \frac{4}{81}\left(\frac{A}{Q}\frac{d\theta_o}{dt}\right)^3 - \dots\right)$$

If $\frac{A}{Q}\frac{d\theta_o}{dt}$ is small, all but the first two terms may be neglected. Here the maximum value is of the order of 0.5 so it reduces the validity of the approximation but the third term is less than one tenth of the second term. so

$$H \approx \left(\frac{Q}{K_3}\right)^{\frac{2}{3}}\left(1 - \frac{2}{3}\frac{A}{Q}\frac{d\theta_o}{dt}\right) \tag{6}$$

From equations 3,4 and 6

$$K_1 \mathcal{E}_2 = \frac{d\theta_o}{dt} + K_4 \frac{d^2\theta_o}{dt^2} \tag{7}$$

where $K_4 = \frac{2}{3}\frac{A}{(3.3b)^{2/3} Q^{\frac{1}{3}}}$

also from equations 2 and 3

$$\mathcal{E}_2 = \frac{\mathcal{E}_1}{1 + K_1 K_2} \tag{8}$$

from equations 1,7 and 8

$$\frac{K_1}{1 + K_1 K_2}\mathcal{E}_1 = \frac{d\theta_o}{dt} + K_4 \frac{d^2\theta_o}{dt^2} \tag{9}$$

$$\frac{K_1}{1 + K_1 K_2}\mathcal{E}_1 + \frac{d\mathcal{E}_1}{dt} + K_4 \frac{d^2\mathcal{E}_1}{dt^2} = \frac{d\theta_i}{dt} + K_4 \frac{d^2\theta_i}{dt^2} \tag{10}$$

Equation (9) and (10) relate the error, input, and output of a servo system. The equations could be solved for any given input conditions. Initially the investigation will be carried out without the internal feedback loop, see Fig. 7.

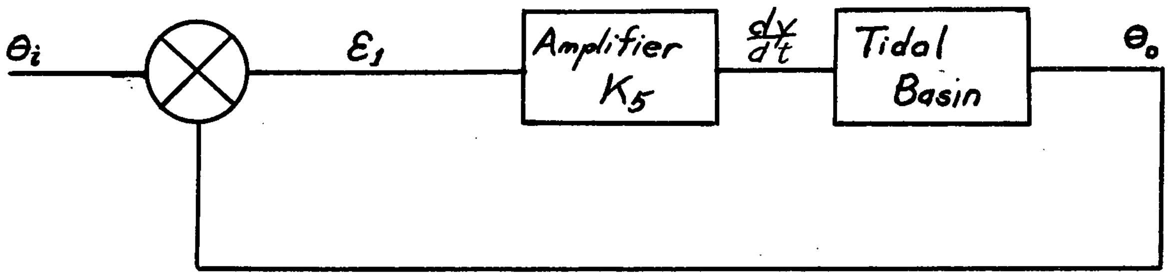


Fig. 7 Block Diagram of Modified Tidal Control System

The basic equations now are

$$\theta_i - \theta_o = \varepsilon_1 \tag{1}$$

$$\frac{dy}{dt} = K_5 \varepsilon_1 \tag{11}$$

$$\theta_o = H + y \tag{4}$$

$$H = \left(\frac{Q}{K_3}\right)^{2/3}\left(1 - \frac{2}{3}\frac{A}{Q}\frac{d\theta_o}{dt}\right) \tag{6}$$

Combining these equations give

$$K_5 \varepsilon_1 = \frac{d\theta_o}{dt} + K_4 \frac{d^2\theta_o}{dt^2} \tag{12}$$

$$K_5 \varepsilon_1 + \frac{d\varepsilon_1}{dt} + K_4 \frac{d^2\varepsilon_1}{dt^2} = \frac{d\theta_i}{dt} + K_4 \frac{d^2\theta_i}{dt^2} \tag{13}$$

Comparing equations (9) and (12) one can see that the two equations are identical if $K_5 = \frac{K_1}{1\ K_1 K_2}$. In other words the only effect of this internal feedback loop is the reduction of the apparent amplification of the amplifier. This effect can be more simply accomplished by the use of an amplifier of lower gain. Since it is desirable to have the system work as accurately as possible and yet be as simple and trouble free, so the first modification suggested is removal of the internal feedback loop.

The solution of equations (12) and (13) will now be considered for a frequency input and for a step input. First a transfer function analysis of equation (12) will be considered.

If the error is applied arbitrarily as a cosine function of time of constant amplitude, say unity, and varying frequency then the output will also be a cosine function of time of the same frequency displaced by some phase angle of a different amplitude. The input and output can then be represented by

$$\mathcal{E}_1 = e^{j\omega t}$$

$$\theta_o = C\,e^{j(\omega t + \lambda)}$$

Then equation (12) becomes

$$K_5 e^{j\omega t} = j C\omega e^{j(\omega t+\lambda)} - C\omega^2 e^{j(\omega t+\lambda)}$$

$$C e^{j\lambda} = \frac{K_5}{-\omega^2 K_4 + j\omega} \tag{14}$$

which may be represented as a vector of magnitude

$$C = \frac{K_5}{\sqrt{(\omega^2 K_4)^2 + \omega^2}} \tag{15}$$

and of phase displacement

$$\lambda = 180° + \arctan\frac{1}{\omega K_4} \tag{16}$$

The locus or Nyquist plot of these vectors for existing conditions is shown in graph #1. All the necessary information for a frequency study is contained in this graph. A vector from the origin to the curve is the output of the system, a vector from the origin to (-1,0) is the error, and from (-1,0) to the curve is the input signal. The angle between the input and output vectors is the phase displacement.

As has already been mentioned the aim is to have the input and output vectors of equal magnitude. This could be accomplished if the locus could be swung counter-clockwise until the point on the locus corresponding to the operating frequency is on the line, reals equal minus one half. To accomplish this by adjusting K_5 would not be practical. Adjustment of K_5 will only change the length of the output vector for a given error vector, it will not affect the phase angle. Refer to graph #1. If the vector $(0;\omega_1)$ were reduced to the point where it intersects the line reals equal minus one half, then it would be shorter than the error vector and the angle between $(0;\omega_2)$ and $(-1,0;\omega_2)$ would be greater than 90 degrees.

Consider now equation (14), if the j term had a multiplying constant greater than unity, the curve would be moved closer to the negative real axis. This j term originates from the first derivative of the output. Here it can be seen how desirable it is to have a flexible constant multiplying this term.

Now a step function analysis of equation (13) will be carried out. The conditions will be

$$\theta_i = 0 \qquad t < 0$$

$$\theta_i = \Theta \qquad t \geq 0$$

$$p\theta_i = p^2\theta_i = 0 \qquad t > 0$$

Equation (13) becomes

$$(K_4 p^2 + p + K_5)\mathcal{E} = (K_4 p^2 + p)\theta_i = 0$$

$$\mathcal{E} = \left(\frac{1}{K_4 p^2 + p + K_5}\right)[0]$$

$$\mathcal{E} = \Theta e^{-\frac{t}{2K_4}} \cos\sqrt{\frac{K_5}{K_4} - \left(\frac{1}{2K_4}\right)^2}\, t \qquad (17)$$

Equation (17) is an expression for the error of the servo system subject to a step input. This is a transient term that approaches zero as time becomes large. It is desirable to keep this term small and have it approach zero as soon as possible. This may be accomplished by making K_4 small or K_5 large. Making K_4 small will cause the exponential term to die away faster and in the range of values considered tend to increase the value of the radical. Unfortunately K_4 is for all practical purposes fixed, it represents the configuration of the model already constructed. This leaves K_5 which is the amplifier gain. Any manipulation of this quantity does not reduce the time of die away.

Since the time scale is 1:70, the period of the component of the moon is 10.63 minutes. This component then has an angular velocity;

$$\omega_1 = \frac{2\pi}{T} \cong \frac{2\pi}{600} = 1.04 \times 10^{-2} \text{ rads/sec}$$

The natural period of the system;

$$\omega_6 = \sqrt{\frac{K_5}{K_4} - \left(\frac{1}{2K_4}\right)^2} = 2.96 \times 10^{-2} \text{ rads/sec}$$

$$K_5 = \text{Amplifier Constant}$$

$$= 0.1 \text{ of a foot/sec/foot of error}$$

$$K_4 = \frac{2}{3} \frac{A}{(3.33\, b)^{\frac{2}{3}} Q^{\frac{1}{3}}} = 112$$

Taking

$$\omega_b \approx \omega_n = \sqrt{\frac{K_5}{K_4}} = 2.98 \times 10^{-2} \text{ rads/sec}$$

$$\omega_n = \sqrt{\frac{3.36\ Q^{\frac{1}{3}}\ b^{\frac{2}{3}}\ K_5}{A}} \qquad (18)$$

Equation (18) indicates the relationship between the parameters and the natural frequency of the system.

The next point of interest on these equations is the coefficient of the first derivative of the output. This coefficient of the first derivative is the friction or damping coefficient. If it were zero any induced oscillation would be sustained until some external force appeared to alter it. The system operation may be improved by increasing this factor but this is impossible because of hydraulic relationships. One alternative is to reduce the magnitude of K_4 and K_5. Although K_5 is flexible any adjustment of K_4 would involve extensive modifications of the system. Another alternative would be to look for some stabilizing circuit that would improve the operation.

From the consideration of equations (14),(15),(16) and (17) it has been seen how desirable it is to have something else to increase the flexibility of the system and improve the response. To this end a consideration of derivative control will be studied. The only difference in the equations of the system will be in the transfer function of the amplifier. It will now be

$$\left(K_5 + K_6 \frac{d}{dt}\right)$$

The basic equations now become

$$\theta_i - \theta_o = \mathcal{E} \tag{1}$$

$$\frac{dy}{dt} = \left(K_5 + K_6 \frac{d}{dt}\right)\mathcal{E} \tag{19}$$

$$\theta_o = H + y \tag{4}$$

$$H = \left(\frac{Q}{K_3}\right)^{\frac{2}{3}}\left(1 - \frac{2}{3}\frac{A}{Q}\frac{d\theta_o}{dt}\right) \tag{6}$$

Combining these equations

$$K_5 \mathcal{E} + K_6 \frac{d\mathcal{E}}{dt} = \frac{d\theta_o}{dt} + K_4 \frac{d^2\theta_o}{dt^2} \tag{20}$$

$$K_5 \mathcal{E} + (1 + K_6)\frac{d\mathcal{E}}{dt} + K_4 \frac{d^2\mathcal{E}}{dt^2} = \frac{d\theta_i}{dt} + K_4 \frac{d^2\theta_i}{dt^2} \tag{21}$$

A study of this system with a sinusoidal input will now be carried out. Equation (20) becomes

$$K_5 e^{j\omega t} + j\omega K_6 e^{j\omega t} = jC\omega e^{j(\omega t + \lambda)} - C\omega^2 K_4 e^{j(\omega t + \lambda)}$$

$$Ce^{j\lambda} = \frac{K_5 + j\omega K_6}{-\omega^2 K_4 + j\omega} \tag{22}$$

which again may be represented as a vector of magnitude

$$C = \sqrt{\frac{K_5^2 + \omega^2 K_6^2}{(\omega^2 K_4)^2 + \omega^2}} \tag{23}$$

and of phase displacement

$$\lambda = 180^\circ + \text{Arctan}\frac{1}{\omega K_4} + \text{Arc tan}\frac{\omega K_6}{K_5} \tag{24}$$

The extra term arctan $\frac{\omega K_6}{K_5}$ results in an increase in the angle λ. Curve (2) in graph #1 is a plot of this result. Increasing K_6 improves the amplification ratio.

Now consider a step function analysis of equation (21)

$$\mathcal{E} = \left(\frac{1}{K_5 + (1 + K_6)p + K_4 p^2}\right)\left[\Theta\right] \tag{25}$$

which has as a solution

$$\mathcal{E} = \Theta\, e^{-\frac{K_6+1}{2K_4}t} \cos\sqrt{\frac{K_5}{K_4} - \left(\frac{K_6+1}{2K_4}\right)^2}\; t \tag{26}$$

Equation (26) is the solution of the equation to a step function input. Here the advantage of the introduction of the K_6 factor is revealed. It increases the attenuation constant of the circuit. The introduction of this factor is quite practical and does not involve any extensive modification of the equipment.

However the introduction of the K_6 factor reduces the natural frequency of the system. Either this condition must be accepted or one of the other parameters must be adjusted to offset the change. A probable adjustment would be to increase K_5.

The possibility of introducing an integral component was considered and discarded. Integral control reduces the velocity error but does not improve the amplification ratio.

Summarizing the theory there are two things to be done.

One, obtain the best results by manipulation of K_1 and K_2 and then do the same by manipulation of K_5 and show experimentally that the response of the latter will equal or better the former. Two, determine the value of K_6 that will improve the response without causing overdamping.

Tests and Results

To impose the most severe conditions possible a curve was plotted that included the highest water on record, the lowest water on record, and the greatest rate of change of water level. This was actually more severe than any recorded tide because the periods of highest and lowest waters do not occur on the same day. The early tests were run using this hypothetical curve as a standard for comparing the response to the various servomechanism conditions. Later tests were run using the record of January 4 to 12 1947 which is the week of most severe conditions.

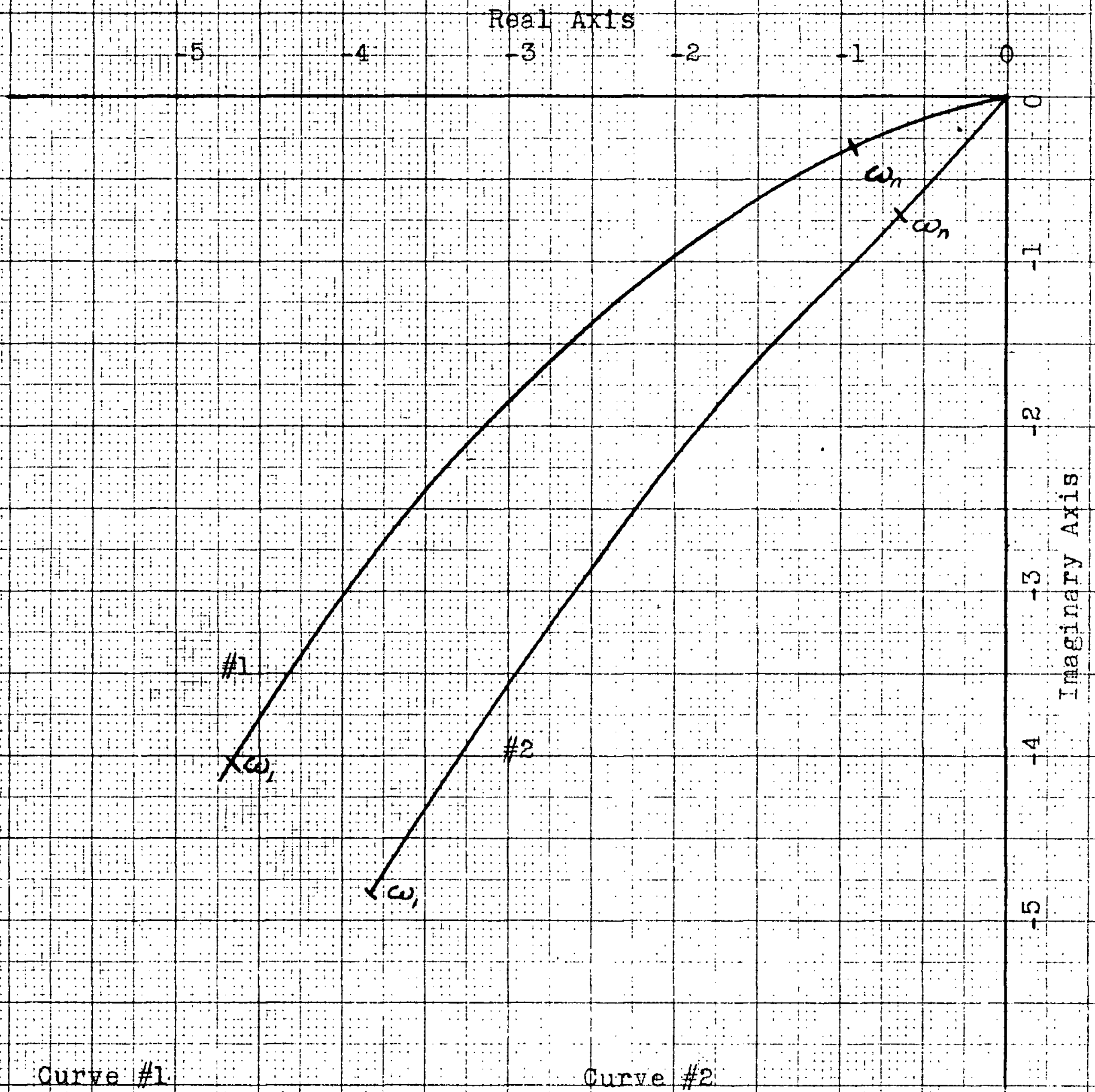
Graph #2 is a graph of one of the better curves obtained with the internal feedback circuit. Graph #3 is a plot of one of the curves obtained without the circuit. The curves are compared with the input signal. Although graph #3 has better amplitude characteristics than graph #2, an oscillation has appeared at the low tide.

The curve drawn up from most severe conditions is made up of two major components. One is at the fundamental frequency and the other is at the second harmonic frequency. The oscillation appearing at the low tide is caused by the presence of the second harmonic. The frequency of the second harmonic is close to the natural frequency of the system. The second harmonic is not as predominant in the tide cycle of the most severe recorded week nor was it considered in the theoretical analysis.

Graphs #4 and #5 compare the results of tests with and without the differentiating circuit. The week of January 4 to 12 was used for these tests with the day of January 5 used here for comparison. Some tendency for oscillation is still apparent but not nearly as severe as for the hypothetical curve. In comparing the response with and without the differentiating circuit very little improvement is obtained by the addition.

Graphs #6 and #7 show the response of the system to a one and one half inch step function with and without the differentiating circuit. Here, as in the analysis of the system, the advantage of the differentiating circuit is brought out. Both the amplitude of the overshoot and the time for the oscillations to die away are reduced considerably. It is on the strength of this latter result that the differentiating circuit is suggested as a modification to the control system.

to follow page 25

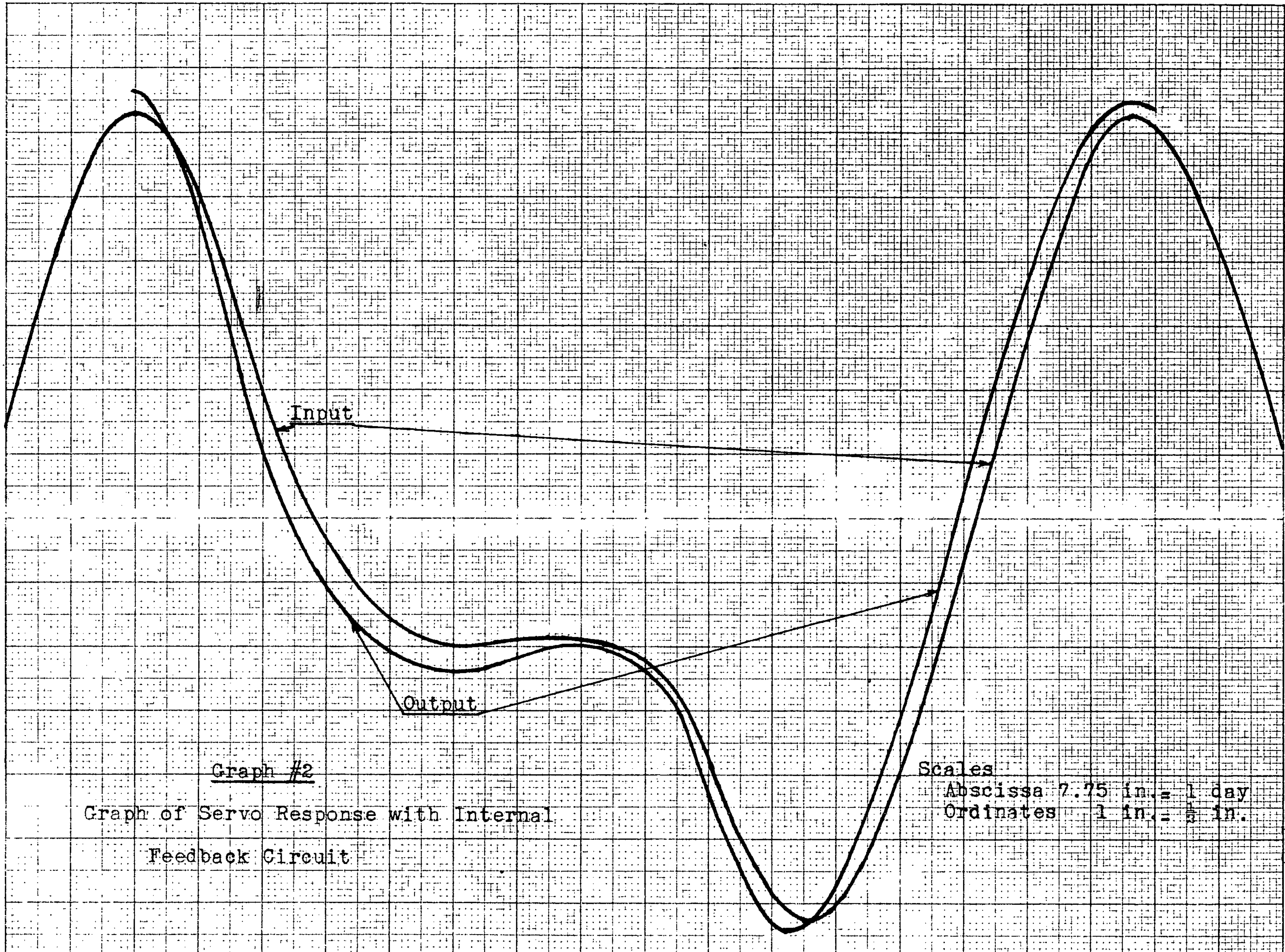


Curve #1
K_5 = 0.1 ft./sec./ft.
A = 12,000 sq. ft.
Q = 20 cubic ft./sec.
b = 40 ft.

Curve #2
K_6 = 2 ft./sec./ft./sec.

Graph #1

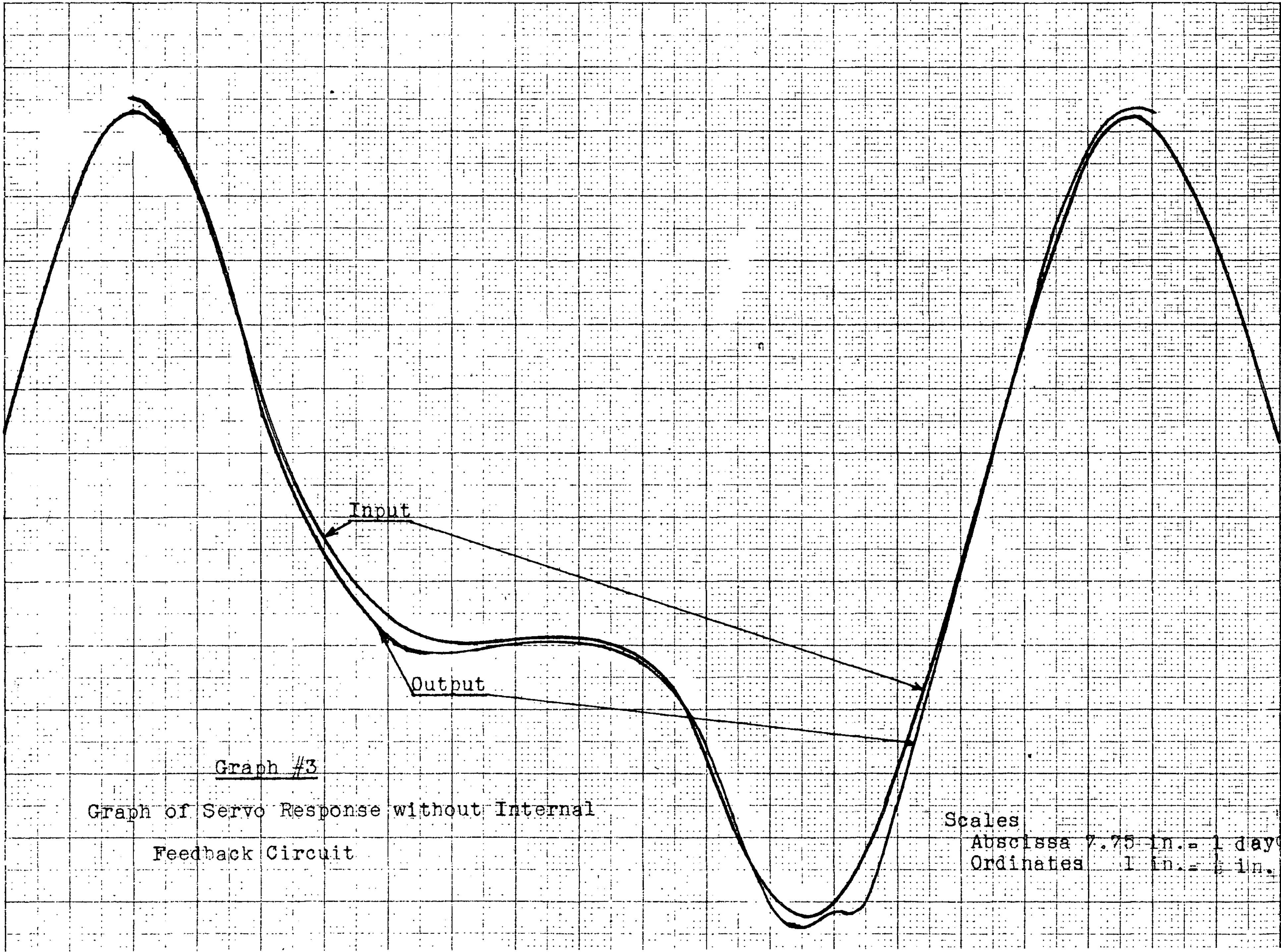
Graph of Output Vector Locus with Proportional Control, Curve #1 and Proportional plus Derivative Control, Curve #2.



Graph #2

Graph of Servo Response with Internal Feedback Circuit

Scales
Abscissa 7.75 in. = 1 day
Ordinates 1 in. = $\frac{1}{3}$ in.



Graph #3

Graph of Servo Response without Internal Feedback Circuit

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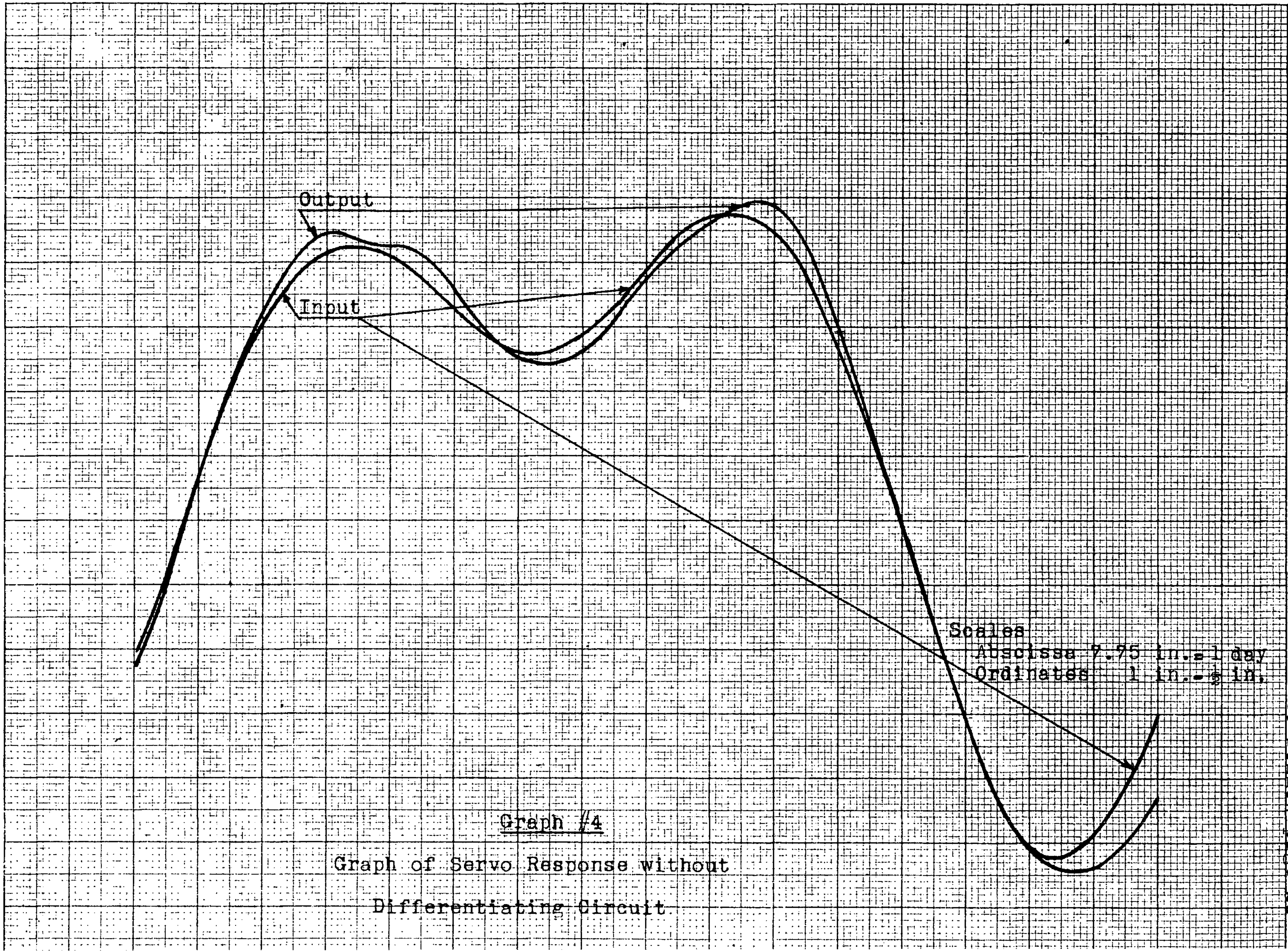

Graph #4

Graph of Servo Response without

Differentiating Circuit

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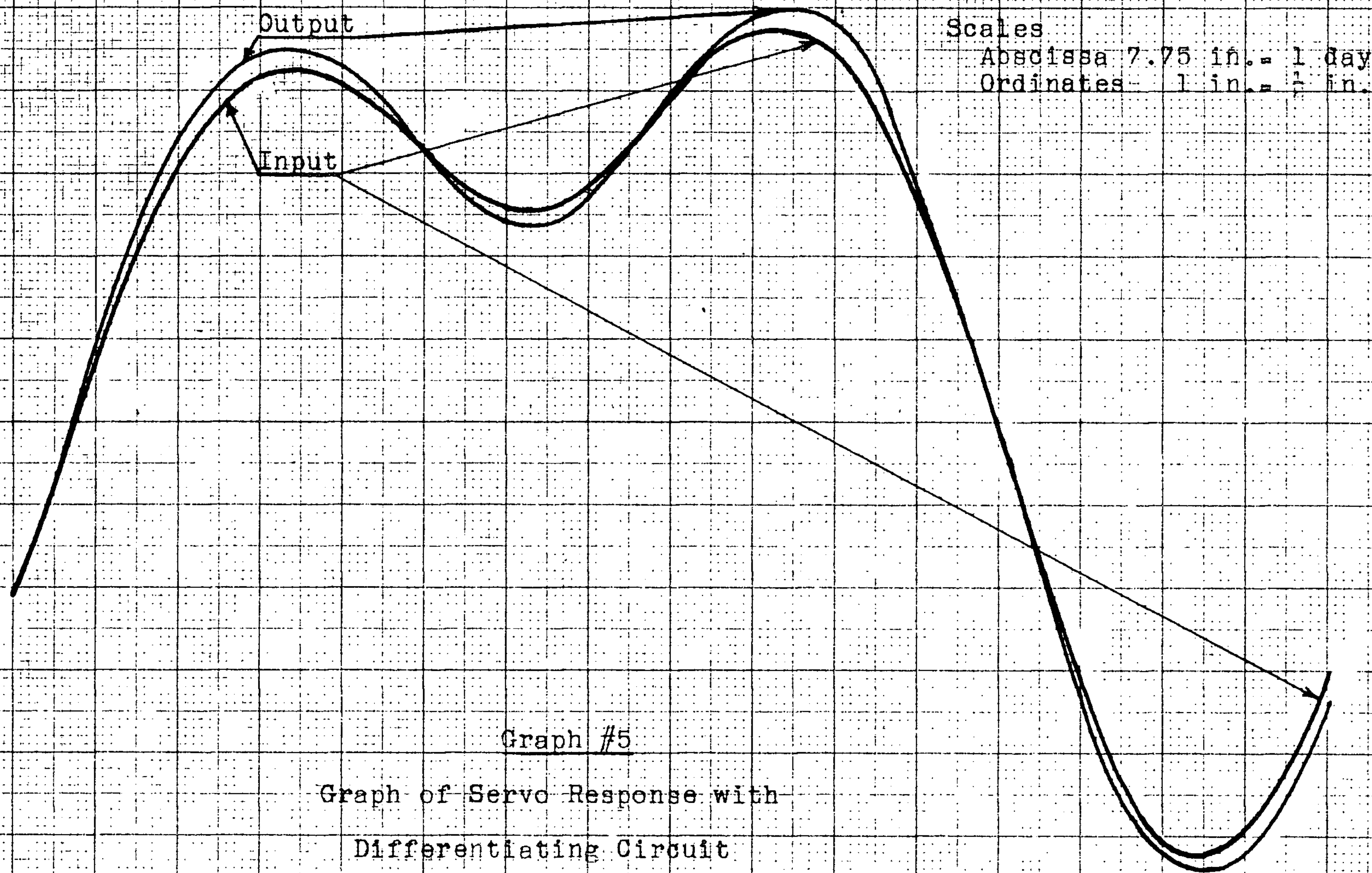

Graph #5

Graph of Servo Response with

Differentiating Circuit

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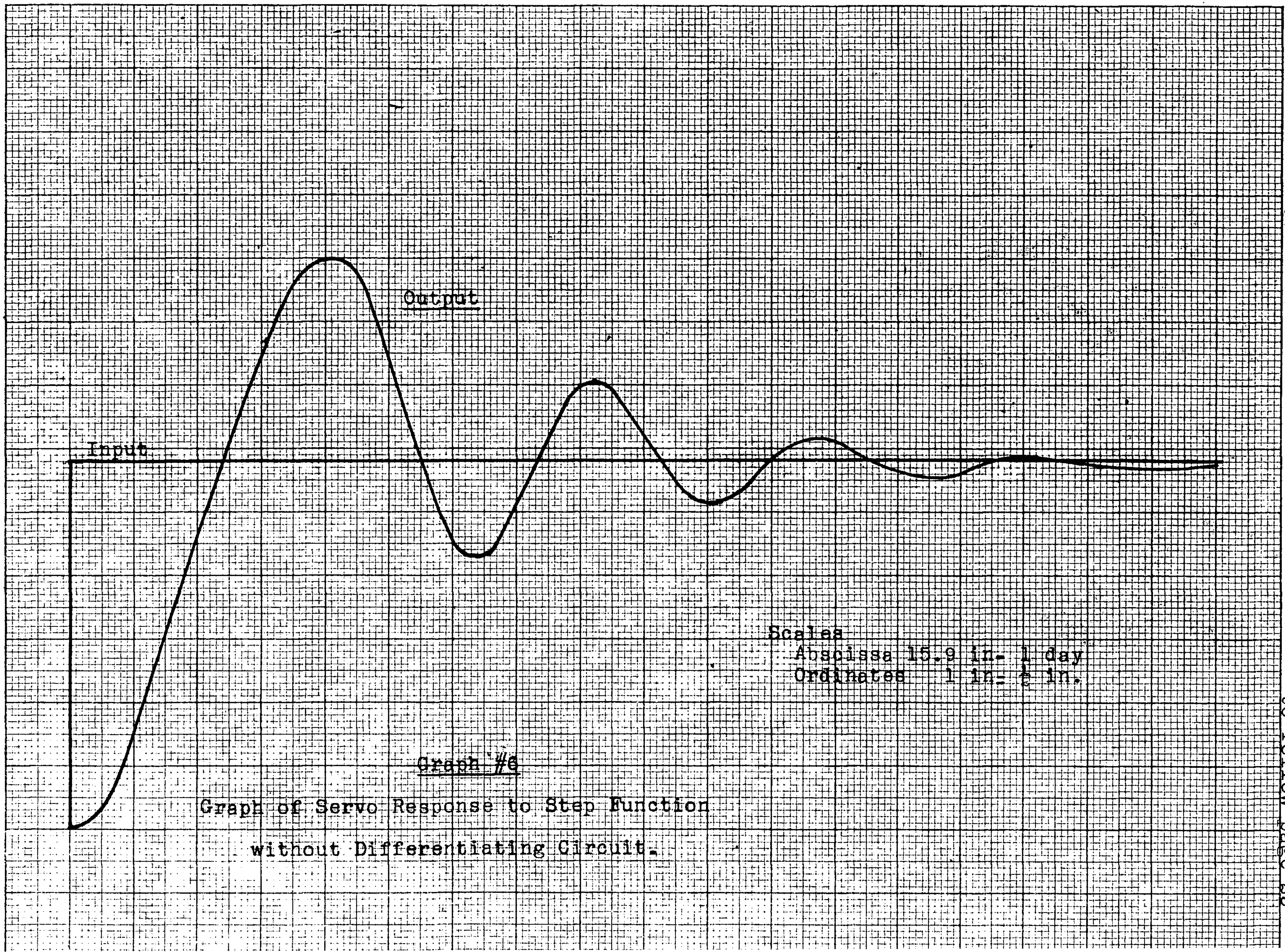

Graph #6

Graph of Servo Response to Step Function

without Differentiating Circuit.

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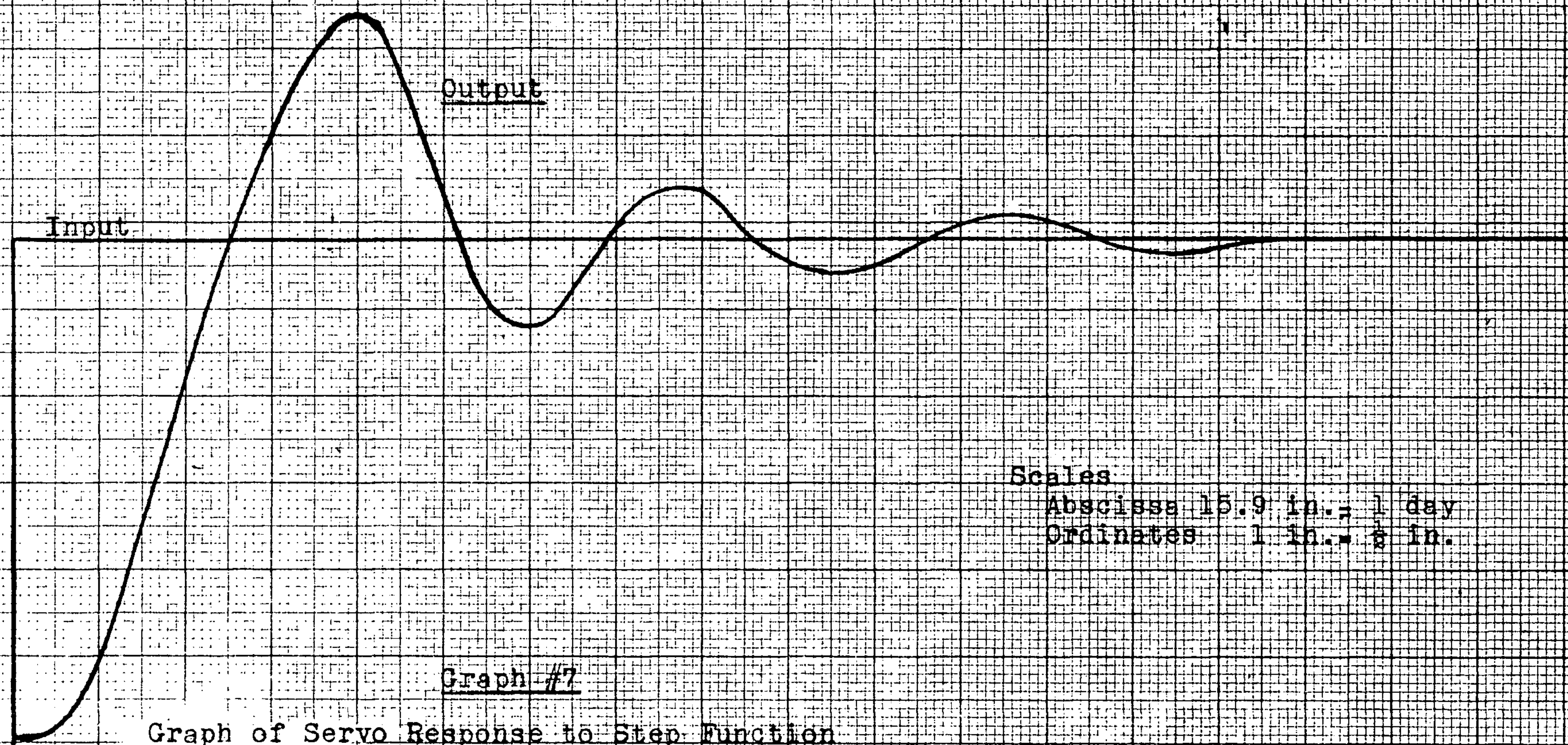

Graph #7

Graph of Servo Response to Step Function

with Differentiating Circuit.

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Discussion and Conclusions

The results obtained by tests have revealed one thing; the introduction of external circuits has very little effect on the continuous operation of the tidal model. Tests were carried out on several stabilizing circuits that have not been mentioned in this thesis, but in all cases the results were poorer than any obtained by methods discussed herein. Some improvement in the step function response could be obtained in some cases but the best overall results were obtained using the differentiating circuit that has already been discussed.

Repeating equation (12) the transfer function of the system is

$$\frac{\Theta_o}{\mathcal{E}} = \frac{K_5}{p(K_4 p + 1)} \qquad (12)$$

K_5 is theoretically arbitrary but practically it is limited to one tenth of a foot per second per foot of error. If K_5 is larger than 0.1 the weirs would rise faster than the pump can bring up the water level which means the weirs will break through the water surface and the water level will no longer follow the weir movement.

The constant K_4 is representative of the time constant of the system. The smaller this time constant is the better will be the response of the system. Repeating equation (17)

$$\mathcal{E} = \Theta\, e^{-\frac{t}{2K_4}} \cos\sqrt{\frac{K_5}{K_4} - \left(\frac{1}{2K_4}\right)^2}\; t \qquad (17)$$

$T = 2K_4 = 224$ seconds

This equation reveals the importance of the time constant. Any reduction in K_4 will reduce the time for the exponential term to approach zero. If the coefficient of the first derivative had a constant greater than unity it also would decrease the time constant of the system.

Refer now to Fig. (3) in the general theory. This figure is a graph of input, output amplification to relative frequency. It is highly desirable to have the amplification ratio equal to unity. The general shape of the curve is dependent on the damping ratio "C". In this case the critical damping

$$F_c = 2\sqrt{K_4 K_5} = 6.7 \tag{27}$$

$$C = \frac{1}{6.7} = 0.149$$

The resonance curve when C = 0.149 rises very sharply as the applied frequency approaches the natural frequency. The actual value of "F" is fixed at unity, therefore "C" can only be increased by reducing F_c which again calls for a reduction in K_4. A good value for "C" Would be 0.6.

To keep away from the resonant point the natural frequency should be well above the applied frequency. Since the applied frequency is fixed by design requirements the natural frequency should then be made as large as possible.

$$\omega_n = \sqrt{\frac{K_5}{K_4}} \tag{18}$$

Here again the constant K_4 should be made as small as possible. This investigation has brought out three reasons why the constant K_4 should be kept as low as possible.

$$K_4 = \frac{2}{3} \frac{A}{(3.33\, b)^{2/3}\ Q^{1/3}} \tag{28}$$

Equation (28) shows the relationship between K_4 and the components of the tidal basin. "A" is the area of the basin which is approximately 12,000 square feet, b is the length of the weirs which is 40 feet, and Q is the pump discharge which is 20 cubic feet per second.

In this particular problem an increase in b and Q is practical. In fact facilities are available to double both components. K_4 will then be halved. This modification will increase the natural frequency from 2.96 x 10^{-2} radians per second to 4.2 x 10^{-2} radians per second, it will increase "C" from 0.149 to 0.21 and it will reduce the time constant from 224 seconds to 112 seconds.

In the design of any future model basins of this type formula (28) may well be used as a guide. The area should be made as small as possible and the weir length and pump discharge should be as large as possible. When "C" is 0.3 or less, then ω_n should be at least five times ω_j.

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D.H.J.Kay

The servo-control mechanism of the Fraser River Model discussed in this thesis was designed, tested and fabricated in the National Research Council laboratories, Ottawa.

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