DIVERSIFICATION OF PORTFOLIOS IN CANADIAN INVESTMENT FUNDS

by

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Date Sept. 25, 1963.
Abstract

Diversification is a desirable characteristic of a well balanced investment portfolio. By diversifying investments over a number of assets, one has a better chance of maintaining overall stability of capital and earning power. Prior to the development of the Markowitz theory, diversification had always been an intuitive process for most investment analysts. Using the Markowitz criteria of efficient diversification, the analyst now has a means of evaluating diversification in an investment portfolio.

In his theory of diversification, Markowitz showed that the selection of securities could be made in a scientific manner so as to give a minimum of variation in an investment portfolio. His theory was based on the fact that all securities were not closely correlated in their price fluctuations, and each security had its own characteristics of expected return and variations in expected return. Using a mathematical procedure called quadratic programming, Markowitz suggested that a series of portfolios which had a minimum of variation for a given return could be selected from any list of investment alternatives. The only information needed on each investment alternative was its return based on price fluctuations and dividends, its statistical variance of return, and the amount of correlation between it and the other investment alternatives.

The purpose of this thesis is to investigate the diversification in Canadian investment funds. In order to do this, it was necessary to divide the Canadian securities market into a workable number of investment alternatives, and to generate a series of optimum portfolios for each given rate of return. The optimum portfolios, based on the Markowitz criteria for diversification, could then be compared with the actual portfolios of Canadian investment funds. As a further basis of comparison, a series of random portfolios were also generated.
The results of the comparisons revealed that Canadian investment funds did not exhibit properties of diversification as defined by the Markowitz criteria. The results also revealed that the investment funds were not significantly better than a random selector.

These results could possibly be attributed to the size and structure of the Canadian securities market. A large investment fund, for example, could not concentrate on the few issues needed for proper diversification without affecting prices in the securities market. Another reason for lack of diversification is that the Canadian market is possibly not independent enough to present sufficient variety to the investor who wishes to properly diversify. Finally, there is reason to suspect that the basic philosophy behind the Markowitz approach, may not apply to many of the Canadian investment funds.
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Chapter I  Introduction

Diversification

Diversification of investments means the selection and inclusion of different, or diverse, assets and securities in an investment portfolio. The standard texts on security analysis usually explain several different methods of diversification. These include diversification by types of assets, by different industries, by different companies within an industry, by geographic location both within a country and internationally, by income, and by maturity dates of the securities.

The purpose of diversification, as described in the texts, is to guard against risk and uncertainty of future events. In general terms, risk may be subdivided to include secular risk as evidenced by long term decline in certain industries, cyclical risk as is caused by business fluctuations, valuation risk which includes uncertainties such as war and disaster, and functional risk covering the need for recovery of capital, growth, current income, and preservation of purchasing power.

The standard textbook approach to diversification remains, however, an intuitive method of safeguarding an investment portfolio from the effects of unforeseen future events. No attempt is made to measure diversification, or to suggest criteria to decide in what way an investment portfolio can be best diversified.

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1 See, for example, James C. Dolley, Principles of Investment, New York, Harper Bros., 1940.

2 For a discussion of the meaning of risk and uncertainty see Ch.II.

Terms used in Markowitz Approach to Diversification

In 1952, Harry M. Markowitz, published an article outlining a method of obtaining optimal diversification in an investment portfolio. Before one can discuss the Markowitz contribution, however, it is necessary to define some of the terms to be used.

**Investment Portfolio**: A number of securities or other assets which are grouped together as one in terms of investment objectives.

**Return of a Security**: Return is made up of two components. The first is the interest or dividends paid to the security holder, and the second is the capital gain or loss to the holder caused by market price changes.

**Return of a Portfolio**: The return of the portfolio is the sum total of the returns on the individual securities within the portfolio. Mathematically, this can be expressed as:

\[ R = \sum_{i=1}^{n} \gamma_i x_i \]

where \( R \) = return of a portfolio

\( \gamma_i \) = fraction of portfolio invested in security \( i \)

\( x_i \) = return on security \( i \)

\( n \) = number of securities in the portfolio.

**Expected Return of a Security**: Over a given number of fixed time periods, a security will exhibit different rates of return. The expected return of the security is the arithmetic mean of the actual returns. Expressed mathematically:

\[ \bar{x} = \frac{\sum_{i=1}^{k} x_i}{K} \]

where \( \bar{x} \) = expected return of the security

\( K \) = no. of fixed time periods.

---

Variance of Return of a Security: Variance of return is the average of the squared deviations from the average. Variance is in effect a measure of dispersion from the average value of return and measures the size of price fluctuations of a security. Mathematically, variance \( v \) is expressed as:

\[
v = \frac{\sum_{i=1}^{k} (x_i - \bar{x})^2}{k}
\]

The variance is the square of the more commonly used statistical term standard deviation.\(^5\)

Covariance between two Securities: Covariance is a measure of statistical correlation between two securities. Two securities which fluctuate in the same direction over time will show a positive correlation. Similarly, two securities which tend to move in opposite directions will have a negative covariance. The mathematical expression for covariance is:

\[
\sigma_{ij} = \text{expected value of } (x_i - \bar{x}_i)(x_j - \bar{x}_j)
\]

also \( = \rho_{ij} \sigma_i \sigma_j \)

where \( \sigma_{ij} \) = covariance between securities \( i \) and \( j \)

\( \bar{x}_j \) = expected value of return of security \( j \)

\( \rho_{ij} \) = correlation coefficient of securities \( i \) and \( j \)

Correlation Coefficient of two Securities: The correlation coefficient expresses the degree of market price association between two securities. In this sense it is similar to covariance but is a more commonly used term than covariance. The correlation coefficient is mostly used in explanatory work while covariance is more useful in calculations.

The range of the correlation coefficient is from +1 to -1. When there is no statistical association between the movements of two securities, their coefficient will be zero. A graphical illustration of correlation is shown in exhibit II.

**Variance of Return of a Portfolio:** For the portfolio the variance of return is the summation of the variances and covariances of the securities that make up the portfolio. Variance is a measure of stability of the portfolio in terms of return. In mathematical terms, variance is equal to:

\[ V = \sum_{i=1}^{n} \sum_{j=1}^{n} y_i \cdot \sigma_{ij} \cdot y_j \]

where \( V \) = variance of return of the portfolio

\( y_j \) = fraction of portfolio invested in security \( j \)

It should be noted that in the above formula, the variance of the portfolio may be less than the variance of any individual security, providing some of the covariances are negative.

**Markowitz Theory of Diversification**

In looking at the market performance of a security over a number of years, it becomes apparent that the return from the security, made up of dividends and capital gains and losses, will vary over a certain range. For some securities the variation will be small and the annual returns will tend to cluster around the average return. For another security, the variation may be high and the annual returns will show widespread differences. Usually, as the rate of return of a security increases, the variation in the return of the security will also increase.

By plotting the rates of return of a security over a number of years, it is possible to produce a frequency distribution exhibiting the usual
properties of a mean and standard deviation. Exhibit I shows the distribution of three securities, "A", "B" and "C", plotted on return-frequency of return axes. Security "A" shows an expected return of 6% and a standard deviation of 2%. Similarly, security "B" shows an expected return of 10% and a standard deviation of 2% while security "C" shows an expected return of 10% and a standard deviation of 6%.

The first step in developing the Markowitz theory, then, is acknowledging the fact that each security will exhibit properties of a given return within boundaries as measured by the standard deviation. One must then also agree that a high rate of return is desirable while a high standard deviation is undesirable. Because of its advantages in computations, Markowitz uses variance, which is the square of standard deviations, as his measure of the undesirable risk property of a security. Referring again to exhibit I, securities "A", "B" and "C", show expected returns of 6, 10, and 10, and variance of returns or risk factors of 4, 4, and 36 respectively.

The next property of a security which ranks in importance with return and variance of return, is covariance. Covariance is a measure of statistical correlation between price movements of one security and another. As was explained earlier, securities which tend to move together in the same direction will have a high correlation. Exhibit II shows a graphical record of the price movements of three sets of securities with different correlation coefficients. For the purposes of the Markowitz theory, then, one must know the return of a security, the variance of return, and the correlation of the security with other securities.

The whole idea behind the Markowitz theory of diversification is that risk or variance, is undesirable and should be reduced to a minimum in a portfolio of securities. This can be done in either of two ways. The first,
RELATIVE RISK OF "A" IS \((2)^2 = 4\)

THE SQUARE OF STANDARD DEVIATION

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Expected Return: 6%
Standard Deviation: 2%

SECURITY "A"

RELATIVE RISK OF "B" IS \((2)^2 = 4\)

Greater expected return than "A" at same level of risk

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Expected Return: 10%
Standard Deviation: 2%

SECURITY "B"

RELATIVE RISK OF "C" IS \((6)^2 = 36\)

Higher risk than "B" at same expected return

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Expected Return: 10%
Standard Deviation: 6%

SECURITY "C"
EXHIBIT II

TWO STOCKS WITH .77 CORRELATION

TWO STOCKS WITH .49 CORRELATION

TWO STOCKS WITH .18 CORRELATION

EXAMPLES OF PRICE CORRELATION

Data taken from Markowitz (6) for 24 year record of year-end closing prices.
and most obvious way, is to select securities in a portfolio which in themselves show very little variance of return. Unfortunately, securities such as bonds and preferred shares which show very little variance of return, also show very little expected return. It is therefore difficult to assemble a portfolio with a high rate of return and a low variance by this method alone.

The second method of reducing variance in a portfolio is to select securities whose prices traditionally move in different directions. By doing this, variations in one security are balanced by opposite variations in the other security. One can visualize this by imagining a portfolio made up of the two securities of .18 correlation shown in exhibit II. Obviously, the variance of a portfolio made up of these two securities will be of a composite value somewhere between the values of the variances of the two securities. If two securities have a negative correlation, the variance of a portfolio made up of the two will be less than the variance of either one of the securities.

The Markowitz theory provides a method of diversifying securities so that a portfolio will have a minimum variance for a given rate of return. As the expected return on a portfolio rises, the variance of return will also rise. Each investor will have different ideas of a suitable rate of return for his requirements, but if he acts in a rational manner, he should select a portfolio which will yield his desired return with a minimum of risk or variance. The Markowitz theory provides him with a method of selecting securities in their correct proportions for any given rate of return.

A series of optimum portfolios can be generated for all feasible rates of return and plotted on return-variance axes to form a line. This line, as
shown in exhibit VII, is known as the efficiency frontier. Any point above the line will represent a portfolio that does not exhibit the optimal characteristic of a minimum variance for a given return. Any point below the line will be in the non-feasible region and will be unattainable. The efficiency frontier will then become the basis of comparison between actual portfolios and optimum portfolios, as defined by the Markowitz theory.

Canadian Investment Funds

The purpose of this thesis is to investigate the diversification in Canadian investment funds in terms of the Markowitz criteria. Before the procedure for doing this is outlined, it will be necessary to discuss the various types of investment funds.

Basically, investment funds can be divided up into two categories; the open-end or mutual funds, and the closed-end investment companies. The closed-end investment companies are incorporated under a Federal or Provincial Companies Act, and their stock is traded on the various stock exchanges in the same way as an industrial company's. The capitalization of a closed-end company may include senior securities such as bonds, debentures, and preferred shares. Most of the assets of the closed-end company are invested in securities of other companies, in which the investment company may or may not take an active management role. In general, however, a closed-end investment company is not organized as a means of obtaining a diversified investment portfolio.

Open-end mutual funds, on the other hand, are set up to service the small investor. The advantage claimed by mutual funds is that the small investor is provided with a well-rounded diversified investment portfolio administered by professional management. He can usually participate in an
investment plan which allows him to purchase a small number of shares over a given time period, which in effect gives him the advantages of dollar averaging.  

To provide daily price quotations, the bid prices of the fund assets are totalled and divided by the number of shares outstanding. Usually, if a person wishes to purchase shares in a fund, he must pay a loading charge in addition to the bid price of the share. The loading charge provides the difference between bid and asked prices for the share. In addition to a loading charge, an investor also pays a management fee which is usually less than 1% of his investment.

There are several main types of mutual funds, each with slightly different objectives. The first type, a balanced fund, attempts to maintain stability of investment by holding bonds and preferred shares as well as common shares. It is expected that the interest and dividend income in this fund will be higher than from common stocks, and that market fluctuations of the common stocks will be tempered by the inclusion of the more senior issues. The result is a portfolio which maintains fairly good income and stability in addition to a limited amount of growth.

The next main category, the fully managed fund, also invests in all types of securities. The proportions invested in each, however, varies with market conditions as assessed by the fund management. If the management feels that common stocks are not a good investment at any one time, it has the freedom to shift, in total, to bonds or cash.

By purchasing securities on a continuing basis regardless of price, the investor averages out his costs and eliminates the effects of short term market fluctuations. This technique, known as dollar averaging, allows the investor to benefit from the long-term upward trend of stock prices.
Common stock or growth funds are another main type of mutual fund. In these funds, the portfolio is almost entirely made up of common stocks. Depending on the individual fund, the proportion of the more speculative stocks will vary.

The remainder of the mutual funds could be roughly classified as specialty funds. Investments in these funds may be limited to bonds, or bank stocks, or foreign stocks, depending on the particular function of the fund. In some of these funds, there will be little meaningful diversification.

The income tax position of investment funds divides them into two main categories. The closed-end funds, and most of the mutual funds are incorporated as companies. The majority of these companies are classified as investment companies under the Income Tax Act. In order to qualify as an investment company, the company must satisfy requirements of type of assets held, amount and source of income, number of shareholders and disposal of income. If the company satisfies the requirements, it is eligible for a special tax of 21% of its taxable income.

The remainder of the mutual funds are organized as investment trusts. They are governed by the trust deed under which they are set up and by the Provincial Securities Acts in the provinces in which the shares are sold.

7 Income Tax Act, R.S.C., 1952, c.148, s.69(2).
8 The requirements and tax rate are specified in section 69(2) of the Income Tax Act.
Since the funds are unincorporated, they pay no taxes, but rather, the individual shareholders pay personal income taxes on distributed dividends.

**Application of the Markowitz Theory to Canadian Investment Funds**

In order to investigate the diversification properties of Canadian investment funds, it will first be necessary to generate a series of efficient portfolios for various rates of return. To be able to do this, one must select and aggregate the security input data in terms of return and variance of return.\(^9\) Variance for each given rate of return is then minimized using a mathematical procedure called quadratic programming.\(^10\) This will give the efficiency frontier line for all feasible portfolios. (Shown in exhibit VII).

The next step will be to evaluate each actual investment fund in terms of return and variance of return. In order to do this, it is necessary to divide all the investments of the fund into the aggregated categories used in deriving the efficient portfolios. The return and variance of each category is then summed according to the proportions of each category to give an overall return and variance for the portfolio. This will allow a point to be plotted for each fund on a graph of return and variance.

As another basis of comparison, a number of random portfolios will be generated, by use of a table of random numbers. These portfolios will also be evaluated in terms of return and variance of return, and plotted on the graph.

The final graph, included as exhibit VII, will show a line representing the efficiency frontier, points representing the position of each actual

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9 An explanation of how the data was selected is given in Chapter III.

10 Quadratic programming will be discussed in more detail in Chapter II.
investment fund in terms of return and variance, and similar points representing the position of the random portfolios.

Once the position of the optimal, actual, and random portfolios has been determined, a hypothesis will be advanced that Canadian investment funds do in fact exhibit properties of efficient diversification. The basis of the decision will be statistical significance tests between optimal, actual and random portfolios. From the results of these tests, conclusions will be drawn as to whether Canadian investment fund management is significantly different than a random selector, and also whether Canadian investment funds exhibit properties of efficient diversification as defined by the Markowitz criteria.
Chapter II Historical and Theoretical Background

Risk and Uncertainty

It is useful to classify future events in terms of risk or uncertainty situations. Frank Knight, in his book on risk, uncertainty and profit, explains the difference in the following way. "To preserve the distinction between the measurable uncertainty and an unmeasurable one we may use the term 'risk' to designate the former and the term 'uncertainty' for the latter."

"The practical difference between the two categories, risk and uncertainty, is that in the former the distribution of the outcome in a group of instances is known (either through calculation, a priori or from statistics of past experience), while in the case of uncertainty this is not true, the reason being in general that it is impossible to form a group of instances, because the situation dealt with is in a high degree unique."

Farrar, in his discussion of risk and uncertainty, says the following. "To qualify as a risk situation, then, an experiment must be repetitive in nature and must possess a frequency distribution from which observations can be drawn and about which inferences can be made by objective, statistical procedures. Virtually any of the contingencies against which commercial insurance can be drawn for example, may be classified as risks."

"Uncertainty, in contrast, is said to be present when the experiment in question cannot be carefully replicated by (or upon) other persons or at other times or places; that is, when the situation is unique. Its frequency distribution, therefore, cannot be objectively specified."


Investment, therefore, is an uncertainty situation since a probability distribution based solely on objective evidence cannot be assigned to the process. However, the investor may have a great deal of data on past performance of investments as well as evidence of probable future performance. He can combine this information and assemble it into what could be considered a subjective probability distribution of expected future performance. If the investor follows this course of action, his solution to the investment decision will be in the area of statistical decision theory. If the investor feels he cannot use the formal approach of setting up a subjective probability distribution, his approach to the problem of investment will be by intuitive means.

Markowitz, in his investment theory is relying on the acceptability of using subjective probability distributions as a basis for investment decisions. Whether one uses past records, or future estimates, or any other means of assigning probability distributions, however, this remains the weakest area in the whole Markowitz approach to investment.

Development of Markowitz Theory of Investment

Prior to Markowitz, very little was written on procedures and rationale of diversification. Although the principle was recognized, not too much emphasis was placed on its effective use. However, there are some examples in the early literature. In 1907, Henry Lowenfeld\(^\text{14}\) developed a theory of geographical distribution of investment as a means of protecting capital. He also outlined three tests for measuring a sound investment position. These included the ability to realize a considerable portion of the invested capital at any time without loss, the maintenance of the total realizeable value of all the securities held at a fairly permanent level, and the

regularity of income, which he maintained indicated that the various securities had been properly selected. In other words, in order for a portfolio to exhibit properties of liquidity, stability, and earning power, it was desirable to have diversified investments.

On the other hand, J.M. Keynes, one of the most well-known writers in economics and finance, did not mention the principle of diversification in his writings on investment and the speculative demand for money.¹⁵

In discussing the speculative demand for money, Keynes maintained that the form in which an investor kept his assets depended upon the rate of return he would receive for his investment. The rate of return is dependent upon the current interest rate and the expected capital gain or loss which would result from a change in current interest rates. When the rate of interest is very low, security prices are high. If the investor felt that interest rates would rise in the future, he would probably be better off to keep his assets in cash rather than securities. In this way, although losing the current interest on his cash, he would avoid the capital loss resulting from the falling security prices.

The implications of this theory are that an investor should be fully invested in either cash or securities, depending upon his expectations of future interest rates. The reason why there would not be wide swings from cash to securities, and vice versa, as interest rates changed, was because different investors had different ideas of what future interest rates would be. However, the investor following the Keynes approach was either invested all in cash or all in securities, and his investments were therefore not diversified.

The more modern form\textsuperscript{16} of the liquidity preference argument recognizes the fact that there is more than one form of security, and that each have different characteristics. Under the newer theory, the investor will diversify his holdings with cash, short term, and long term securities. The investor will adjust his portfolio as interest rates vary, and the tendency will be towards increased liquidity at lower interest rates.

Markowitz,\textsuperscript{17} in developing his theory, begins by investigating the theory that a portfolio is selected by maximizing the discounted value of all future returns. This theory, put forward by Williams,\textsuperscript{18} can be expressed in the following way:

\[ R_n = \sum_{t=1}^{T} \frac{r_t}{(1+i)^t} = \frac{r_1}{1+i} + \frac{r_2}{(1+i)^2} + \ldots + \frac{r_t}{(1+i)^t} \]

where

- \( R_n \) = investment value or discounted return of security \( n \)
- \( t \) = time in years
- \( r_t \) = dollar return in year \( t \)
- \( i \) = interest rate required by investor

Markowitz proceeds to show that in the \( n \) security case, this theory does not imply diversification, but rather, requires the investor to invest in the security with the highest discounted return. For a portfolio with \( n \) securities:


\textsuperscript{17} H.M. Markowitz, 1952, \textit{op.cit.}

\textsuperscript{18} J.B. Williams, \textit{The Theory of Investment Value}, North Holland Publishing Co., Amsterdam, 1956, p.56.
\[ R = \sum_{t=1}^{T} \sum_{n=1}^{N} \left[ \frac{r_t}{(1+i)^t} \right] X_n \]

\[ = \sum_{n=1}^{N} R_n X_n \]

where \( R = \) discounted return of portfolio
\( X_n = \) fraction of portfolio invested in security \( n \)

Since \( X_n = 1 \) for full investment, and \( X_n \geq 0 \), \( R \) can only be maximized by selecting the greatest fraction allowable of security \( n \), with the highest discounted return \( R_n \). If several securities have the same return \( R_n \), then a portfolio with any fraction of either would be equally suitable. This is incompatible with Markowitz' expected return - variance of return theory.

Markowitz, in developing his theory, starts by considering return on a security as a random variable \( R_i \). He shows that the return on a portfolio \( (R) \), is the weighted sum of the returns on the individual securities.

\[ R = \sum_{i=1}^{N} X_i R_i \]

where \( X_i = \) fraction of portfolio invested in security \( i \)
\( N = \) total number of securities in the portfolio

defining covariance between two securities \( i \) and \( j \) \( (\sigma_{ij}) \) as \( \sigma_{ij} = \) expected value of \((R_i - \) expected value of \( R_i)(R_j - \) expected value of \( R_j)\) and variance of return \( (V) \) as

\[ V = \sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_{ij} X_i X_j \]

Markowitz now states that an investor should maximize the return on his portfolio subject to a minimum of variance. This then becomes a problem in quadratic programming and is expressed as follows:
Minimize $V = \sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_{ij} x_i x_j$

Subject to

$\sum x_i r_i = M$

$\sum x_i = 1$

$x_i \geq 0$

where $M$ = minimum acceptable expected return.

By varying the parameter $M$ and solving the above problem, one can generate a whole series of efficient portfolios, each with a minimum of variance for the given return.

**Method of Solution**

Quadratic programming is an extension of the more commonly used linear programming. In order to explain quadratic programming, it is easier to start with the linear case. Linear programming was developed in 1947 by G.B. Dantzig in connection with problems of the U.S. Air Force. The technique involves optimizing (maximize or minimize) an objective function subject to a set of linear constraints. A typical application of linear programming is in deciding which products to produce from crude oil in an oil refinery. Depending on the fractioning process used, different amounts of the different products can be produced from a given amount of crude oil. In this case, the objective function will be the cost function, which will be minimized, and the constraints will be the various sales and output capacities for the various products.

To present the idea of linear programming in a clearer manner, it will be easier to solve a theoretical problem using a graphical solution. In this way, the objective function to be optimized can be seen along with the various constraints on the solution. Consider the following problem:

An automobile plant can build autos and trucks. Its metal stamping capacity limits production to 25000 autos or 35000 trucks. Engine assembly limits it to 33,333 autos or 16,667 trucks. Assembly capacity is either 22,500 autos or 15,000 trucks. Profit on an auto is $300 and on a truck is $250. How many of each product should the plant produce?

Solution: Let \( X \) be the optimum production of autos, and \( Y \) be the optimum production of trucks.

In order to produce one auto, we require:

\[
\frac{1}{25000} = .0040\% \text{ of metal stamping capacity}
\]

\[
\frac{1}{33,333} = .0030\% \text{ of engine assembly capacity}
\]

Similarly, to produce one truck we require:

\[
.0029\% \text{ of metal stamping capacity}
\]

\[
.0060\% \text{ of engine assembly capacity}
\]

Expressing our capacity constraints as equations:

\[
.0040X + .0029Y = 100\%
\]

or

\[
40X + 29Y = 1,000,000
\]

also

\[
.0030X + .0060Y = 100\%
\]

or

\[
30X + 60Y = 1,000,000.
\]

Our problem then becomes one of maximizing:

\[
\text{Profit} = 300X + 250Y
\]

Subject to the following constraints:

\[
40X + 29Y = 1,000,000 \text{ (Metal stamping capacity)}
\]

\[
30X + 29Y = 1,000,000 \text{ (Engine assembly capacity)}
\]

\[
X \leq 22,500 \text{ (Auto assembly capacity)}
\]
\[ Y \leq 15,000 \text{ (Truck assembly capacity)} \]
\[ X,Y \geq 0 \text{ (No negative production)} \]

Referring now to exhibit III, we can see the above four constraints plotted. The positive area within the constraints (shaded on the edges) shows us the feasible region within which we can find a solution. We next have to select a production which allows us to maximize our profit and still be in the feasible region. We do this by letting our profit first be \( \$3,000,000 \) and next \( \$6,000,000 \). The plots of these two profit lines show that they are both in the feasible region and both have the same slopes. The solution to the problem then is to move the profit line away from the centre, as far as feasible, keeping the slope constant. This gives the solution as shown of 20,200 autos and 6600 trucks.

The above example shows a graphical solution to a linear programming problem. The analytical method of solution would be a trial and error procedure which begins at the origin and tests the profit function. Moving out along the constraint lines, the profit would be tested at each point of intersection of two constraint lines. When the profit no longer increases, with a movement along a line, a solution has been reached. One of the most commonly used methods of solution of linear programming problems is the Simplex Method\(^{20} \) developed by George Dantzig. The Simplex Method has been programmed for use on computers and now is readily available for most computers.

EXHIBIT III

\[ 40X + 29Y = 1,000,000 \]

**MAXIMUM PROFIT LINE**
\[ P_{\text{max}} = 300X + 250Y \]

**POINT OF MAXIMUM PROFIT IN FEASIBLE REGION**

\[ 30X + 60Y = 1,000,000 \]

\[ \text{PROFIT} = 3,000,000 = 300X + 250Y \]

\[ \text{PROFIT} = 6,000,000 = 300X + 250Y \]

\[ X = 22,500 \]
\[ Y = 15,000 \]
Quadratic programming is considerably more complicated than linear programming, in that the objective function and the constraints may be curved. This makes the finding of an optimum solution much more difficult because as one moves along the constraint lines, the objective function may first decrease and then increase. A decrease in the objective function therefore does not signal that an optimum solution has been found.

As with the linear case, quadratic programming is best explained using a graphical analysis with only a few variables. This time, the example will develop a solution for an investment problem with only three possible securities in the portfolio.

Problem: Find the set of optimal portfolios exhibiting the least variance for three possible securities. The fractions of the portfolio invested in each are $X_1$, $X_2$, and $X_3$.

Solution: The constraints in this case are

$$X_1 \geq 0, \quad X_2 \geq 0, \quad X_3 \geq 0 \quad \text{(No short sales or negative investment)}$$

$$X_1 + X_2 + X_3 = 1 \quad \text{(full investment)}$$

Referring to exhibit IV, the feasible region is therefore bounded by the positive quadrant and the equation $X_3 = 1 - X_1 - X_2$

The expected return from the portfolio is given by the equation

$$E = X_1 R_1 + X_2 R_2 + X_3 R_3$$

where $E = \text{expected return of portfolio}$

$R_1 = \text{return on security 1}$

Using the fact that $X_3 = 1 - X_2 - X_1$ and substituting in the above

$$E = X_1 (R_1 - R_3) + X_2 (R_2 - R_3) + R_3$$

21 Three security problem is condensed from Markowitz, op.cit., 1959, pp.130-139.
Assigning various values to $R_1$, $R_2$ and $R_3$ gives us the series of return lines shown in Exhibit IV as $E_1$, $E_2$, $E_3$. It is noticed that the direction of increasing return is downward to the right.

Next, the objective function, variance, must be plotted. The variance of a three security portfolio is expressed as:

$$V = X_1 \sigma_{11} + X_2 \sigma_{22} + X_3 \sigma_{33} + 2X_1X_2 \sigma_{12} + 2X_1X_3 \sigma_{13} + 2X_2X_3 \sigma_{23}$$

Assigning various values to the covariances given in the above equation yields a series of constant variance or iso-variance curves which turn out to be elliptical for the three security case. These are labelled $V_1$, $V_2$, $V_3$ in Exhibit IV.

The problem is to determine the proper proportions, $X_1$, $X_2$ and $X_3$ which yield a minimum variance for a given return. This is shown to be anywhere on the line $l-l$ in the diagram. As one moves down to the right along $l-l$, it will be noticed that up to the point $c$, return is increasing and variance is decreasing. Past point $c$, return is further increasing while variance is also increasing. Any combination of securities on line $l-l$, however, exhibit a minimum of variance for the given return. Markowitz calls this type of line a "critical line" and it is the function of the quadratic programming procedure to yield a series of critical lines.

This, generally, is the method of solution for the portfolio selection problem. In a practical case, there will be a great deal of investment possibilities. As the number of variables increase, the complexity of the calculations increase. In order to solve a practical problem it is necessary to use an electronic computer, and even using a computer, it is necessary to group similar securities so as to limit the number of variables handled by the computer. Only in this way is one able to solve an actual problem.
EXHIBIT IV

Increasing \( E \)

\( X_3 = 1 - X_1 - X_2 \)

Critical Line

Increasing \( V \)
Efficient Diversification and Maximization of Investors Utility Function:

Each investor makes individual investment decisions according to his own background, present position, and future goals. If his ideas could be expressed in terms of utility, it would be possible to draw a utility function for each individual. Assuming a traditional, concave, positively-sloped function, the curve would appear as in Exhibit V. This shape is justified by the supposition that doubling the return to the investor will not double his utility. It is possible to approximate this curve by an equation of the form:

\[ U = R - AR^2 \]

where

- \( U \) = investors total utility
- \( R \) = return on investment
- \( A \) = constant

Since variance of return (\( V \)) is a function of \( R^2 \), it is possible to rewrite the utility equation in the form of:

\[ U = R - AV \]

If \( V \) is considered as risk, the \( A \) constant becomes a coefficient of risk aversion for the investor. In other words, the \( A \) constant provides a measure of the effect of increased variance on the total utility of the investor.

Turning our attention now to a particular investor with a given coefficient of risk aversion, it is possible to plot a set of indifference curves on return-variance of return axes. By letting the \( V \) in the above equation

---


23 This is an application of the theory of diminishing return. For a discussion of the theory see Leftwich, R.H., ibid.
equal values of $V_1$, $V_2$, $V_3$ etc., a set of linear indifference curves are formed as shown in Exhibit VI. It will be noticed that the direction of increasing utility is downward to the right, and an investor maximizes his utility at the point of tangency between his indifference curve and the efficiency frontier. This leads to the conclusion that the investor will maximize his utility only by the selection of a properly diversified investment portfolio.
Selection of Input Data

Investing in the security markets involves decision making under uncertainty conditions. As was discussed in the last chapter, one method of treating uncertainty is to collect all the available data on a security, and from this, arrive at a probability distribution for the expected future performance of the security. Using this approach, one solves the problem using the same techniques as would be used in a risk situation where there are objective probability distributions available.

The main difficulty with this method of allowing for uncertainty is the selection of expected probability distributions for each of the investment alternatives. There are several approaches to this problem. One is to rely solely on past records as an indication of future performance. Another approach is to use the techniques of security analysis to predict future performance of securities. In the former case, rate of return, variance of return and covariances can be directly calculated.

In the latter case, return, variance and covariance can be implicitly calculated by relating the future performance of a security to one of the security price indexes such as the Toronto stock exchange industrial average. What is required is a subjective evaluation of the limits within which the price of a security will move in relation to the security price index. From the information on the expected performance of the security in relation to the index, it is possible to calculate the expected return, variance, and covariance for the individual security.

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For a discussion of this approach, see H. Markowitz, op.cit., 1959, pp.96-101.
Because of the large amount of subjective evaluation required with the second approach, it was decided to calculate means, variances, and covariances directly from past records. With this decision made, the problem became one of collecting the necessary information.

The next major problem was one of reducing the number of investment possibilities to the point where the data could be handled by a present day computer. In this regard, it should be pointed out that with 10 possible investments, there are $\binom{10}{2} = 45$ covariances. When the number of investment possibilities increases to 100, there are $\binom{100}{2} = 4950$ covariances required. It is easy to see, therefore, that it is desirable to keep the number of securities down, both in terms of collection of input data, and of computer time required to solve the problem. Investigation into this problem by Farrar,\textsuperscript{25} revealed that in terms of covariance, the 47 Standard and Poor industrial classifications in the United States could be effectively reduced to five.

With this in mind, it was decided to look for Canadian groupings which could be used to supply the necessary data. The Investors Index of Common Stocks, compiled by the Dominion Bureau of Statistics, appeared to be the only source with a representative number of security groupings. The Investors Index is made up of 94 leading common stock securities, and 13 industrial classifications. The weighting factor for each security is based on the number of shares outstanding. The price index is calculated by multiplying the outstanding number of shares of each stock by its closing quotation each Thursday of the week. The calculations are then totalled and expressed as a percentage of the 1935 base year.

\textsuperscript{25} D.E. Farrar, op.cit., p.44.
The problem in using the index was to decide whether weekly, monthly or annual values should be used, and over what time period. On the basis of availability of information, it was decided to use annual quotations for the first week in January from the years 1945 to 1963. Price quotations and number of shares outstanding were then obtained from the Dominion Bureau of Statistics for the 94 stocks during the 18 years in question. Each price was multiplied by the number of shares outstanding and totalled on the basis of the 13 industry groupings. This was done for each of the 18 years giving a 13 by 18 matrix of aggregate totals. Annual percentage change was then calculated for each year in each grouping. This then gave a distribution of eighteen annual returns in terms of price fluctuations for each of the industry groupings.

To round out the list of investment possibilities, one grouping for each of bonds and preferred stocks was added. A theoretical Government of Canada 4% Bond was used as a basis of the bond classification. Price fluctuations were calculated from theoretical yields given in the Bank of Canada Statistical Supplement. The preferred classification was taken from the record of a 5% Calgary Power Co. preferred issue given in Financial Post records. The final security groupings consisted of industrial mines, beverages, power and traction, transportation, telephone and telegraph, building materials, food and allied products, oils, textiles and clothing, pulp and paper, milling, machinery and equipment, banks, preferred stocks, and bonds.

This then gave 18 years of price fluctuations for the fifteen security classifications. The dividend and interest portion of return was then calculated assuming the 1963 dividends would be of the same order as 1962. Dividends and interest in 1962 for the 96 issues were then multiplied by their proper weighting to give a percentage return for each of the 15 classifications.
A library program\textsuperscript{26} for the IBM 1620 computer was then used to calculate means, variances, covariances, standard deviations and correlation coefficients for the 15 classifications. The input to the computer was the 15 by 18 matrix of price fluctuations in terms of percentage returns. The output then became the data for calculating efficient portfolios and for evaluating means and variance for random portfolios and actual portfolios.

**Selection of Mutual Funds**

Selection of the funds to be tested was based mainly on the size of the fund, and the availability of information. Most of the information was obtained from the Financial Post Survey of Investment Funds. In the case of several large funds, the information was obtained directly from annual financial statements.

An attempt was made to include all the different types of open and funds. At least one of each of fully managed, balanced, common stock, specialty, bond, Trust Company, and non-resident owned funds was included. Several closed-end companies were also evaluated. A list of the companies selected, and their respective positions on a mean variance plot is given in Exhibit VII.

Generally, if a fund had assets of over ten million dollars, and if the information on its portfolio was readily available, it was included in the study. These criteria resulted in the selection of 18 funds. In order to evaluate differences in management between large and small funds, an additional ten smaller funds were evaluated. This gave a total of 28 funds, with assets ranging from one million to 319 million dollars.

\textsuperscript{26} U.B.C. Computing Centre, no. S3-4A
Many Canadian Funds invest in foreign securities. If the amount invested was very large, the fund was not selected as there was no information available on returns, variances and covariances of foreign securities.

**Generation of Random Portfolios**

As a further test of investment fund management, a comparison between actual and random portfolios was felt desirable. It was therefore decided to generate a series of random portfolios. Using a table of random numbers, the investment alternatives included in each portfolio were selected on a random basis. In a similar manner, the amount invested in each of the alternatives was also selected. In all, sixteen random portfolios were generated.

The procedure followed was to assign a number from one to fifteen to each of the fifteen alternatives. Using a four number column of random digits, the first two numbers would then correspond to the investment alternative selected. The second two numbers were used as a weighting factor to determine the amount of each investment alternative to include in the portfolio. By running down the column of digits until a member between one and fifteen occurred, and recording the weighting factor, it was possible to select both the type and amount of investment on a random basis. In this way, by stopping the selection after fifteen valid numbers had been recorded, all of the alternatives had an equal chance of being selected. In fact, some of the alternatives were selected two or three times, while others were not selected at all. By totalling the weighting factors, the percentage of each investment was determined.

**Calculation of Return and Variance for the Funds**

In order to plot the actual investment funds and the random portfolios on return-variance axes, it was necessary to determine an overall return and variance for each of the funds. In chapter I, return and variance for a
The \( R \) return on security \( i \), was determined from the first processing of the input data. The \( \chi_i \), fraction of assets invested in security \( i \), however, was more difficult to obtain. For the investment funds, each individual asset had to be categorized into one of the fifteen investment alternatives. Once all the assets had been placed into the category to which it most closely fit, totals were obtained, and the fraction in each security \( \chi_i \), was determined. As explained previously, the covariances, \( \sigma_{ij} \), were determined by processing the original data through the computer. For the random portfolios, the \( \chi_i \) and \( X_i \) elements were explicitly calculated from the random selection process.

Because of the enormous amount of calculations needed to calculate the return and variance for the 28 actual and 16 random portfolios, it was necessary to write a computer program for the process. (A program listing is given in appendix I). The input to the program consisted of individual returns and covariances as fixed data, and fractional breakdown of portfolio assets as variable data. The output of the program was return and variance for each portfolio.

Once the program had been written and tested on the computer, the returns and variances were computed and plotted as shown in Exhibit VII.

**Generation of Optimum Portfolios**

There are two well known methods of solving quadratic programming problems.
One, by Philip Wolfe\textsuperscript{27} of the Rand Corporation, is an extension of the commonly used Simplex method of linear programming. The other method, by H.S. Houthakker\textsuperscript{28} of Harvard, is called the Variable Capacity method of quadratic programming. Both of these men were contacted to enquire whether or not computer programs were available for their routines. From these inquiries it was found that I.B.M. had just completed a program using the Simplex method, and the program was available for an I.B.M. 7090 computer. Since an I.B.M. 7090 is roughly 300 times faster than the U.B.C. computer, it was decided to use an available I.B.M. 7090 rather than modify the program to fit the UBC 1620 computer. Because 12 tape units were required for the program, it was necessary to request use of the computing facilities of the Western Data Processing Centre in Los Angeles. After permission had been obtained, the input data were gathered and submitted.

Input data required for the I.B.M. program\textsuperscript{29} included means, standard deviations, correlation coefficients, dividend and interest rates, tax rates and constraints on solution. All of this information was required for each of the investment alternatives. The constraints submitted included one with no limitation on the amount which could be invested in each investment alternative, and secondly, a twenty percent maximum investment in any one alternative.

The no limitation section of the data ran through the program without any problem and gave satisfactory results. The twenty percent limitation section, however, would not give a completed run even after several attempts. At this point there were five possible reasons why the program would not work, and each of them could only be checked out on a trial and error basis. After


\textsuperscript{29} IBM Portfolio Selection Program (IB PS 90) 7090-FI-03X.
considering the time and expense required to compute this section, it was decided that the results of the first section would have to be used as the basis for comparison of investment portfolio management. It is difficult to decide what affect the twenty percent limitation would have in shifting the efficiency frontier in the region under study. The partially completed results, however, are shown on Exhibit VII, and seem to indicate a slight shift of the efficiency frontier to the left.

For a properly completed run, the output data consisted of a series of critical lines, and series of variance figures for each requested rate of return. The critical lines output showed the percentage of each alternative invested for each given return and variance of return. Whenever a given investment alternative was either added to or removed from the portfolio, a new critical line was generated. From the results of the critical line output, it was therefore possible to calculate all possible efficient portfolios by interpolating between points on the critical line. The efficiency frontier shown in Exhibit VII is a graphical interpolation of the critical lines output showing return and variance for all possible efficient portfolios.

Selection of Statistical Significance Tests

With only a limited number of random and actual portfolios from the total possible population evaluated, it was necessary to statistically determine whether or not any significant differences existed between the actual and random portfolios. It was decided that a chi-square test would indicate differences in the positions on the graph of the 16 random portfolios and the 28 actual portfolios. The graph was divided up into three sections (rate of return from 10 to 12.5%, 12.5 to 14.0%, and 14.0% and up) and the number of each type of portfolio in each section was counted. A chi-square test was
then made to see whether there were any significant differences between the number of actual and the number of random portfolios in each section.

To determine whether a difference between actual and optimal portfolios existed, a difference between means test was selected. The reason this test was used was because a comparison was being made between 28 actual portfolios and an infinite number of optimal portfolios, as represented by a line on the graph. The procedure followed was to record the variance of the actual and the optimal for each of the 28 actual portfolios, holding the return constant. This gave two distributions of variance for the actual and optimal portfolios. The means of the two variance distributions were then compared in a difference of means test. Since the sample size was less than 30, a two-tail Student-t distribution test at a significance level of .025 was used as the basis for the decision.

Finally, as a means of comparison between random, small, and large funds, the returns and variances were averaged for each group. A significance test was not used on the small funds as the number evaluated was too small.
Comparisons of Optimum, Actual and Random Portfolios

As was explained in the last chapter, the optimum portfolios are made up of different proportions of the fifteen possible security groups. The actual results showed that only eight classifications were used to make up all the possible optimum portfolios. In the high return, high variance area, only two groups, telephone and telegraph and pulp and paper made up the portfolios. In the mid regions, preferred shares and banks became more prominent, while in the lower regions, oils and textiles were brought into the portfolios. The maximum number of groups included in any of the optimum securities was six, and most often was only five.

Turning now to Exhibit VII, a visual inspection seems to indicate that random and actual portfolios are not too different in terms of return and variance. It also seems apparent that the line of optimum portfolios is not too close to the actual and random portfolios.

In order to verify these conclusions, statistical significance tests were made. For the first test, a null hypothesis was made that the positions on the graph of the actual random portfolios were not significantly different. The results of the chi-square test indicated very little difference, and certainly not enough to be significantly different at a .05 significance level. The value of \( \chi^2 \) was equal to .496 which is much less than \( \chi^2 .05 = 5.991 \). The hypothesis could not be rejected and it was therefore concluded that there was no significant difference between the actual and random portfolios.

Calculations are shown in Appendix II.
Since the actual and random portfolios were very similar, it was only felt necessary to test the difference between one of them and the optimum portfolios. In this case, a difference between means for a given variance was selected as the test to be used, and two distributions of actual and optimal means were calculated. The results of this test showed overwhelming differences between the means of the two distributions. It was therefore concluded that there were significant differences between the optimum portfolios and random and actual portfolios.

Comparison of Large and Small Funds and Random Portfolios

Statistical significance tests could not be used to determine differences between large, small, and random portfolios because of the small samples involved. However, as a matter of interest, returns and variances were averaged for the 18 large funds (over ten million dollars assets) the 10 small funds (over one million dollars assets) and the 16 random funds. The results were as follows:

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>Return</th>
<th>Variance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Small Funds</td>
<td>10.6%</td>
<td>.00201</td>
</tr>
<tr>
<td>Large Funds</td>
<td>13.1</td>
<td>.00235</td>
</tr>
<tr>
<td>Random Funds</td>
<td>13.2</td>
<td>.00263</td>
</tr>
</tbody>
</table>

For the small funds, the fact that several were invested very heavily in bonds explains the relatively low return. For the large and random funds, it is interesting to note that for approximately the same return, the large funds show much less variance of return than the random portfolios. This would indicate, therefore, that investment fund management is better than a random selector. However, it must be kept in mind that the small statistical sample involved makes it difficult to come to any definite conclusions.
Conclusions

From the results of the graphical analysis and statistical tests, it appears that Canadian investment funds do not exhibit qualities of efficient diversification as defined by the Markowitz criteria. It also appears that the investment company management is not too much better than a random selector in its investment practices. In interpreting these results, one should look first at their validity.

Probably one of the weakest areas in the whole process is in selection of the input data. As was previously explained, input data for this process were obtained from past records. Specifically, the past records were based on annual records for the past 18 year period. It can be argued that since investment is a dynamic, continually changing process, the investment decision should be based on future estimates rather than past records. Providing that a set of future estimates which are better than past records are available, this argument can be considered valid. As investment companies are using future estimates which may be significantly different from past records, it can be expected that their portfolios would differ from the optimum portfolios produced from past records. This may be one of the reasons for the differences.

Another weakness in the input data results from the aggregation process. The means, variances, and covariances used in the analysis were obtained from the behaviour of a group of weighted averages. The weighting factors were the number of shares outstanding, which over the years increased as the companies grew. The index was therefore not solely a price index, and the growth in the index was not solely due to price increases. This effect would increase the return recorded in the growth industries. However, it is doubtful whether the overall effect on the analysis would be of any significance.
The makeup of the security index could also introduce errors. The investments of the funds were obviously not identical and not in the same proportions as the security index. This, of course, would introduce errors into the analysis. Counterbalancing this, however, was the fact that larger better known stocks made up the majority of both the price index and investment fund portfolios.

Once the input data has been decided upon as being correct, any other errors in the system are purely mathematical in nature. Since the computer program used has been checked out, this type of error can then be ruled out. The next step in interpreting the results of this study would then be to assume that the results are correct and to try to determine why Canadian investment funds do not practice efficient diversification.

Probably the first area one should look at in this regard is the Canadian securities market. Compared to the United States, Canada's market is much smaller in terms of both number of securities listed, and dollar volume of transactions. It may be that some of the larger funds in Canada would have trouble diversifying in a few industry groupings without jeopardizing their liquidity and flexibility. The portfolio of Investors Mutual of Canada, for instance, contains over 100 common stocks, 40 preferred stocks, and 50 bond issues. With assets of approximately $320 million, it is not hard to imagine the effect this fund would have on the market if it concentrated on only a few stocks. This argument can be used to explain the lack of efficient diversification in the larger funds, but cannot, of course, be applied to the smaller funds.

Another reason why investment funds do not diversify effectively is that the market does not seem to have enough variety in terms of covariance. The fact that all the efficient portfolios were made up of six or fewer groupings
indicates that many of the other groups behaved too similarly to qualify under any investment philosophy. This market property could possibly be explained in terms of the dependence of the Canadian market on the performance of the United States market. Again, this argument applies mainly to the larger investment funds, although the smaller funds, for other reasons, may also desire a wider diversification.

Other reasons for inefficient diversification, which apply to both large and small funds, are found in their basic investment policy. The common stock funds, for instance, are not trying to build stability into their portfolios, but are looking for long term growth. Without a good proportion of bonds and preferred shares in their portfolios, a fund cannot exhibit properties of stability. Common stock funds, and other specialty funds which have been included in this study, cannot therefore be expected to be efficiently diversified.

Many of the funds have limitations on the amount of their assets which they may hold in any one particular security or industry. These constraints may be explicitly stated in the company's articles of organization, or they may be implicitly followed by the management of even the so called fully-managed funds. Canadian Investment Fund, Commonwealth International Corporation, and many of the other funds, for example, are limited to investing less than 5% of their assets in the securities of any one issuer. Investors Mutual of Canada, the largest Canadian mutual fund, may not have more than 40% of its assets invested in securities which are not authorized for the investment in the funds of companies registered under the Canadian and British Insurance Companies Act, and in addition has a 10% limitation on the amount it may invest with any one issuer. In actual fact, the results of the study showed that none of the funds investigated had more than 20% of their assets invested in any one of the common stock industry classifications.
Since this is the case, the average investment fund would probably show better results if compared to a series of efficient portfolios based on a maximum of twenty percent investment in any one security group. Unfortunately, limitations on the computer time available made it impossible to follow up this approach.

Finally, one must look at the basic philosophy behind the Markowitz approach, and try to determine whether or not it really applies to investment fund management. Briefly, the Markowitz approach takes care of uncertainty by assigning a subjective probability distribution to the expected behaviour of an investment. Using this approach in this study, one ends up selecting six or fewer investment alternatives. The other approach to investment is to evaluate the different investment alternatives in an intuitive manner. With this second approach, one ends up with any number of investment alternatives. The only limitation on the number of investments is the cost of the routine security analysis done on each investment. In both cases diversification practices are followed; in the first efficient diversification, and in the second intuitive diversification. In any event, if investment companies are following the second practice, one would not expect to find evidence of efficient diversification in their portfolios. From the results of the study, it appears that either for this or one of the other reasons outlined, the investment companies are not following the Markowitz principles of efficient diversification.

Other Possible Applications of Markowitz Approach to Diversification

The market for stocks, bonds and short term notes is generally very active. Because an investment company can very quickly change its asset position, and because of the relative ease with which probability distributions can be assigned to its investment alternatives, the investment company seems most likely to be the institution which would follow the Markowitz approach to diversification.
However, there are other institutions which could probably benefit from this type of investigation.

Insurance companies present some interesting possibilities for applications in this area. With an insurance company, the investigations could be extended to its liabilities which in most cases are risk situations. An attempt could probably be made to more evenly balance assets and liabilities and thus stabilize company profits. In any event, the investigations could certainly be applied to a company's assets. Similarly, the assets of banks, pension funds, and other financial organizations would have use for this type of analysis.

For industrial companies, an interesting area to which this approach would apply is in capital budgeting. Many companies have improved their capital budgeting procedures to the point now where they maximize discounted expected returns. This, of course, is an improvement over payback and some of the other methods used in some industries. However, as was pointed out earlier, maximizing discounted expected return ignores the principle of diversification. It may be that what companies should be doing is to correlate their various capital expenditure proposals to some index and from this calculate variances and covariances. In actual practice, it would not be much more difficult to do this than to follow their present practice of estimating future returns. By diversifying their capital expenditures, an industrial company would be better prepared for unexpected future contingencies. One of the main limitations of this approach, of course, is the fact that capital expenditure proposals are not divisible and usually must be done in total or not at all. Because of this, effective diversification could only then be approximated.

Recommendations for Future Research

The previous section dealt with research possibilities in areas other than straight security investment. However, the results of the present study point to a need for further research in the area of investment funds and the Canadian securities market.
There are several variations which can be made to the input data for this problem. It may be interesting to use past records for only the last five or ten years and to use monthly price fluctuations rather than annual fluctuations as a basis for calculating variances and covariances. Another possibility is to use an analyst's future expectations rather than past records as a basis for the input data. This would involve a great deal more work but would probably give more valid results. Factors such as purchasing power risk, which was not directly included in the present study, could be taken into consideration.

In this regard, it would be interesting to evaluate one investment fund using the company's own estimates of future expectations. An optimum portfolio could be worked out on the basis of the company's present investments and compared to the actual portfolio of the company.

A final area for research could be an investigation into the inter-relationships within the Canadian security market. A factor analysis study, such as the one done by Farrar on the United States market, may reveal a high correlation between all Canadian securities in terms of price fluctuations. If this is the case, it certainly would have an important effect on any future work in this area.

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FORTRAN 2 COMPIL.
C PROGRAM FOR COMPUTING VARIANCE OF A PORTFOLIO
C C. ROSEN MAY 1963
DIMENSION C(15,15),F(15),R(15)
READ 2,((C(I,J),J=1,15),I=1,15)
2 FORMAT (5F6.1)
READ 8,(R(I),I=1,15)
8 FORMAT(12F6.4)
READ 3,(F(I),I=1,15)
3 FORMAT (12F6.3)
VAR=0.0
RET=0.0
DO 4 I=1,15
DO 4 J=1,15
4 VAR=VAR+F(I)*F(J)*C(I,J)
DO 9 I=1,15
9 RET=RET  +F(I)*R(I)
PRINT 3, (F(I),I=1,15)
PRINT 6,RET,VAR
6 FORMAT (F6.4,4X,F12.4//)
GO TO 7

Note: C Matrix is 15x15 table of Covariances
R " 1x15 " Returns
F " 1x15 " Fractions of
    Investment in each alternative

TABLE OF MEMORY ALLOCATIONS.
  PROGRAM STARTS AT 14696
  PROGRAM ENDS AT  16114
  LOWER END OF COMMON AT 39999
NO SUBPROGRAMS CALLED
NO LIBRARY FUNCTIONS CALLED.
1. Chi-Square Test

Null Hypothesis: The distributions of proportions are the same in all sections for the actual and random portfolios.

<table>
<thead>
<tr>
<th>Range of Return</th>
<th>Random</th>
<th>Actual</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 - 12.5 %</td>
<td>6</td>
<td>7</td>
<td>.13</td>
</tr>
<tr>
<td></td>
<td>(5.2)</td>
<td>(7.8)</td>
<td></td>
</tr>
<tr>
<td>12.5 - 14.0 %</td>
<td>5</td>
<td>10</td>
<td>15</td>
</tr>
<tr>
<td></td>
<td>(6.0)</td>
<td>(9.0)</td>
<td></td>
</tr>
<tr>
<td>14.0 % up</td>
<td>5</td>
<td>7</td>
<td>12</td>
</tr>
<tr>
<td></td>
<td>(4.8)</td>
<td>(7.2)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>16</td>
<td>24</td>
<td>40</td>
</tr>
</tbody>
</table>

\[
\chi^2 = \frac{(6-5.2)^2}{5.2} + \frac{(7-7.8)^2}{7.8} + \frac{(5-6)^2}{6} + \frac{(10-9)^2}{9} + \frac{(5-4.8)^2}{4.8} + \frac{(7-7.2)^2}{7.2}
\]

\[
= .123 + .082 + .167 + .111 + .008 + .005
\]

\[
= .496
\]

No. degrees of freedom = (3-1)(2-1) = 2

\[
\chi^2.05 = 5.991
\]

\[
\therefore \chi^2 < \chi^2.05
\]

\[
\therefore \text{Cannot reject hypothesis that proportions are the same}
\]

\[
\therefore \text{Conclude that there is no significant difference between random and actual portfolios.}
\]
2. Difference Between Means Test

<table>
<thead>
<tr>
<th>Fund No.</th>
<th>Opt</th>
<th>Opt²</th>
<th>Actual</th>
<th>Actual²</th>
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<tr>
<td>Variances</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>23</td>
<td>.03</td>
<td>.001</td>
<td>.17</td>
<td>.03</td>
</tr>
<tr>
<td>21</td>
<td>.08</td>
<td>.006</td>
<td>.25</td>
<td>.10</td>
</tr>
<tr>
<td>19</td>
<td>.17</td>
<td>.03</td>
<td>.55</td>
<td>.30</td>
</tr>
<tr>
<td>26</td>
<td>.21</td>
<td>.04</td>
<td>1.50</td>
<td>2.2</td>
</tr>
<tr>
<td>27</td>
<td>.26</td>
<td>.07</td>
<td>1.32</td>
<td>1.7</td>
</tr>
<tr>
<td>11</td>
<td>.35</td>
<td>.12</td>
<td>1.45</td>
<td>2.1</td>
</tr>
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<td>.32</td>
<td>.11</td>
<td>1.60</td>
<td>2.5</td>
</tr>
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<td>22</td>
<td>.35</td>
<td>.12</td>
<td>1.75</td>
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</tr>
<tr>
<td>2</td>
<td>.36</td>
<td>.13</td>
<td>1.95</td>
<td>3.8</td>
</tr>
<tr>
<td>10</td>
<td>.38</td>
<td>.14</td>
<td>1.82</td>
<td>3.3</td>
</tr>
<tr>
<td>12</td>
<td>.40</td>
<td>.16</td>
<td>2.05</td>
<td>4.2</td>
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<td>2.23</td>
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<td>.38</td>
<td>.14</td>
<td>2.32</td>
<td>5.4</td>
</tr>
<tr>
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<td>.45</td>
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<td>2.92</td>
<td>8.5</td>
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<td>2.45</td>
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<td>.45</td>
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<td>2.52</td>
<td>6.4</td>
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<td>.48</td>
<td>.23</td>
<td>2.70</td>
<td>7.3</td>
</tr>
<tr>
<td>1</td>
<td>.50</td>
<td>.25</td>
<td>2.60</td>
<td>6.7</td>
</tr>
<tr>
<td>24</td>
<td>.52</td>
<td>.27</td>
<td>2.82</td>
<td>8.0</td>
</tr>
<tr>
<td>5</td>
<td>.52</td>
<td>.27</td>
<td>2.32</td>
<td>5.4</td>
</tr>
<tr>
<td>16</td>
<td>.53</td>
<td>.28</td>
<td>2.48</td>
<td>6.1</td>
</tr>
<tr>
<td>4</td>
<td>.53</td>
<td>.28</td>
<td>2.85</td>
<td>8.1</td>
</tr>
<tr>
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<td>.60</td>
<td>.36</td>
<td>2.72</td>
<td>7.4</td>
</tr>
<tr>
<td>14</td>
<td>.60</td>
<td>.36</td>
<td>2.83</td>
<td>8.0</td>
</tr>
<tr>
<td>9</td>
<td>.60</td>
<td>.36</td>
<td>3.25</td>
<td>10.6</td>
</tr>
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<td>.62</td>
<td>.39</td>
<td>3.55</td>
<td>12.6</td>
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<tr>
<td>18</td>
<td>.62</td>
<td>.39</td>
<td>2.30</td>
<td>5.3</td>
</tr>
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<td>.70</td>
<td>.49</td>
<td>2.42</td>
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<tr>
<td></td>
<td>11.86</td>
<td>5.76</td>
<td>59.69</td>
<td>203.6</td>
</tr>
</tbody>
</table>

Null Hypothesis $J_{act} = J_{opt}$

\[
S_{opt} = \frac{1}{n} \left[ n(\bar{x}_1^2 - \bar{x}_1^2) \right]^{\frac{1}{2}}
\]

\[
= \frac{1}{28} \left[ 28(5.76) - (11.86)^2 \right]^{\frac{1}{2}}
\]

\[
= \frac{1}{28} \sqrt{20} = \frac{4.47}{28} = .16
\]

\[
S_{act} = \frac{1}{28} \left[ 28(203.6) - (59.7)^2 \right]^{\frac{1}{2}}
\]

\[
= \frac{1}{28} \sqrt{1148} = \frac{33.2}{28} = 1.21
\]
\[ t = \frac{x_{\text{act}} - x_{\text{opt}}}{\sqrt{\frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1 + n_2 - 2}} \left( \frac{1}{n_1} + \frac{1}{n_2} \right)} \]

\[ \bar{x}_{\text{opt}} = \frac{11.86}{28} = 0.42 \]

\[ x = \frac{59.69}{28} = 2.13 \]

\[ t = \frac{2.13 - 0.42}{\sqrt{\frac{27(1.16)^2 + 27(1.21)^2}{54} \cdot \frac{2.13}{28}}} \]

\[ \frac{1.71}{\sqrt{0.075 \cdot 0.07}} = 1.71 \]

\[ \sqrt{0.052} \cdot 0.228 = 1.96 \]

\[ t_{0.025} = 1.96 \]

\[ \therefore t > t_{0.025} \]

\[ \therefore \mu_{\text{act}} \neq \mu_{\text{opt}} \]

\[ \therefore \text{Samples do not come from same populations} \]

\[ \therefore \text{Actual and Optimum portfolios are significantly different.} \]
Bibliography


