STEADY STATE SINGLE CHANNEL QUEUES

by

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ABSTRACT

This thesis extends the application of waiting line theory to situations where both arrival rate and service rate distributions are arbitrary or non-random. It does so only for single channel, single phase, steady state, infinite queues with no feed-back.

Previous work by A.K. Erlang had shown that queueing characteristics could be predicted for one case of an arbitrary service rate distribution, the constant service time. Also, F. Pollaczek had shown that, where arrival rates are random, queue lengths and waiting times were independent of the form of the service rate distribution, being functions of the coefficient of variance squared. But all of the works assumed random arrivals around a stable mean arrival rate and, except for the constant service time case, most applications were limited to cases where both arrival and service rates were random. This restriction has limited applications severely and has required that most analysis of queueing characteristics be done by simulation.

This study develops and proves by inference the hypothesis that system length is dependent on these factors only: the square of the coefficient of variance of the inter-arrival time distribution, C_a^2, the square of the coefficient of variance of the service time distribution, C_s^2, and the ratio of mean arrival rate to mean service rate, ρ. Through a combination of calculation and simulation a set of curves has

been developed covering values, C_a^2 from 0 to 6, C_s^2 from 0 to 6 and of ρ from 0.1 to 0.9. These curves permit the prediction of system length, and then of average queue length and waiting time, for any case where only the mean and variance of the arrival and service time distributons are known, even though nothing is known about the form of the distributions.

In the usage of the set of graphs (figures 10-29), the following steps are all that is required to obtain the necessary characteristics:

a) Calculate the average interarrival time,

$\frac{1}{\lambda}$ = (Total time of observation/Total number of arrivals)

b) Calculate the variance for interarrival times,

$$\mathrm{Var}(t_a) = \sum_{i=1}^{n} (t_{a_i} - \frac{1}{\lambda})^2 / \text{Total number of arrivals}$$

c) Calculate the fractional coefficient of variance squared for interarrival time distribution,

$$C_a^2 = \mathrm{Var}(t_a)/(1/\lambda)^2$$

d) Calculate the average service time,

$\frac{1}{\mu}$ = (Total time service facility is in operation/ Total number serviced)

e) Calculate the variance for service times,

$$\mathrm{Var}(t_s) = \sum_{j=1}^{m} (t_{s_j} - \frac{1}{\mu})^2 / \text{Total number serviced}$$

f) Calculate the fractional coefficient of variance squared for service time distribution,

$C_s^2 = Var(t_s)/(1/\mu)^2$

g) Calculate the utilization factor,

ρ = (Average service time/Average interarrival time)

h) With the values ρ, C_a^2, and C_s^2, read from the set of graphs (figures 10-29) the verticle axis, L.

i) Compute Lq, W and Wq.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

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CHAPTER I

INTRODUCTION

Queues for service of one kind or another are present in many different fields of activity. The mechanism of the queueing processes is very simple and can be broken down into elements, each of which has the following basic behaviour. A sequence of units or customers arrives at some facility or counter, which services each unit and eventually discharges it.

Decisions regarding the amount of service capacity must be made frequently in industry. Unfortunately, the amount of capacity to provide usually cannot be determined by estimating the average flow. If the mean capacity is less than the average flow, a "traffic jam" will build up until either flow is reduced or capacity is increased. Even if mean capacity is greater than average flow, transient and, in certain cases, permanent "traffic jams" can occur because the actual flow or the actual capacity to handle the flow fluctuates, being sometimes larger and sometimes smaller than its mean value. Hence, congestion will occur from time to time if there are sufficient irregularities in the system even though average capacity is more than adequate. Generally the theory of queues deals with the investigation of the stochastic law of different processes arising in connection with mass servicing in cases when random fluctuations occur. However, the practical aim in investigating a system with congestion is usually to improve the system.

In the study of queues, the important characteristics are:

a) Wq , the mean queueing time of a customer. The queueing time is defined as the time a customer has to wait before being attended to.

b) W , the mean waiting time of a customer. The waiting time is defined as the queueing time and the service time of a customer.

c) Lq , the mean queue length of the system. The mean queue length is defined as the average number of customers in the queue excluding the one in service.

d) L , the mean waiting length of the system. The mean waiting length is defined as the average number of customers in the system including the one in service.

e) ρ , the utilization factor of the service mechanism.

In order to predict one or more of these characteristics, it is necessary to specify the system sufficiently fully.

In order to specify a system completely, it is necessary to describe the elements of the queue sufficiently. A system has four elements:

a) The input element is described by the following factors;

(1) The initial input population. It is required to know whether the customers come from a finite or infinite population. It is obvious that no population can be infinite. However, when it is sufficiently large, the error in assuming infinite population is negligible. Messrs. L.G. Peck and

R.N. Hazelwood in their presentation, Finite Queueing Tables, for the Operations Research Society of America in 1958 presented a maximum of 250 customers in his tables for finite population.

(2) The input variation. Customers can arrive singly or in batches according to a certain distribution.

(3) The time factor. The input rate can be transient or stationary.

> "The theory of stationary queues is very important because most of the queueing processes are ergodic; i.e., starting from any initial state, the process tends towards equilibrium irrespective of the initial state. In state of equilibrium the process shows only statistical fluctuation with no tendency to a certain state. Many queueing processes rapidly approach equilibrium and this explains why one can apply with success the stationary approximation. However, the investigation of the transient behaviour of queueing processes is also important, not only from the point of view of the theory but also in the applications."[1]

(4) The interarrival time distribution. It is necessary to know the type of input distribution or interarrival time distribution. One is dependent on the other. The interarrival time distribution may either be arbitrary or it may follow an algebraic expression. The most important algebraic expressions in queueing theory are the exponential interarrival time distribution and its hybrids, the Erlangian and hyper-exponential distributions. The exponential interarrival time distribution is obtained from the assumption that the arrival is "pure random". The Erlangian

[1] Lajos Takacs, Introduction to the Theory of Queues (Oxford University Press, Inc., New York, 1962) p. 6.

distributions provide a family of distributions which ranges from the "pure random" to the completely regular constant interarrival time, while the hyper-exponential interarrival time distributions provide another family which is "more random" than the "pure random". Figures 3 and 4 give the cumulative graphical representation of the distributions and the frequency graphical representation of the distributions respectively.

> "The simplest arrival pattern, mathematically, and the most commonly useful one in applications, is when the arrivals are completely random."[1]

(5) The customers behaviour before joining the queue. The following are various possibilites:

(i) Lost calls. Customers may not be able to wait and hence the presence of any queue will result in lost calls.

(ii) Balking. Arriving customers may not join the queue because of the length of the existing queue. This is considered as balking.

(iii) Delayed calls. It may be possible to delay the customers so that when they arrive there will be no queue.

(iv) Imperfect information about the line to be joined. In a multiple queue operation there may be insufficient or imperfect information about the various queues and/or channel.

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[1] Cox, D.R. and W.L. Smith, Queues (Methuen and Co. Ltd., London, 1965) p. 5.

(v) Join the nearest queue. The customer arriving at the service mechanism will join the nearest queue regardless of its relative length.

(vi) Collusion of customers. Several customers may be in collusion whereby one person may wait in line while the rest are then free to attend to other things.

(6) The control of customers before arrivals. There can be control or no control.

b) The following factors describe the waiting line element:

(1) The type of queue discipline. The queue discipline can be on

(i) First come-first served.

(ii) Allocation to definite channels.

(iii) Ordered queues.

(iv) Last come-first served.

(v) Random selection for service.

(vi) Priorities, preemptive or nonpreemptive.

(2) The properties of the line and customers after joining the queue. The queue or queues can be of

(i) A specialized type where a particular need of customers are met or general type where all the needs of customers are met.

(ii) Finite or infinite maximum length.

and the customers can be described as reneging or non-reneging.

c) The components to describe the service element are:

(1) The service time distribution. The distribution can be arbitrary or follow an algebraic expression.

> "For many applications the assumption of random arrivals is reasonable; however, the assumption of exponential distributions of service time is often unsatisfactory."[1]

> "Nelson's[2] study indicated that other mathematical distributions such as the hyperexponential and the Erlangian distributions are better descriptions of the actual distribution."[3]

(2) The servicing variation. Servicing can be done singly or in batches, according to a probability distribution.

(3) The number of channels. This can be fixed or varying.

(4) The number of phases in the channels.

(5) The type of channels. The channels can be of a special or general type.

d) The output element. The output can either go off or cycle.

Historical Brief

The origin of queueing theory is to be found in telephone-network congestion problems. Erlang[4] in 1917 published his paper where he assumes that input is Poisson and service time distribution either exponential or constant. His models

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[1] _Ibid._, p. 50.

[2] Nelson, R.T., _An Empirical Study of Arrival, Service Time, and Waiting Time Distributions of a Job Shop Production Process_ (Research Report No. 6, Management Sciences Research Project, U.C.L.A., 1959).

[3] Buffa, E.S., _Models for Production and Operations Management_ (John Wiley and Sons, Inc., New York, 1966) p. 252.

[4] Brockmeyer, E., H.L. Halstrom, and A. Jensen, _The Life and Works of A.K. Erlang_ (Copenhagen Telephone Company, Copenhagen, 1948).

are for a delay system with an arbitrary number of channels for exponential service time distribution and one, two, or three channels for constant service time distribution.

Erlang's work simulated other works in the field by T.C. Fry[1], E.C. Molina[2], and G.F. O'Dell[3]. This, as described by Syski[4], is the "Erlang-O'Dell" period, and the works were basically concerned with proving or disproving Erlang's results.

In the early thirties, F. Pollaczek developed his well known formula for a single channel with Poisson input and arbitrary holding time. This formula is known as the Pollaczek-Khintchine formula.

From then on numerous works had been done on queueing theory. T.L. Saaty[5] provided a bibliography of about 900 articles in 1961. This quantity has grown tremendously from then.

Queues of all types are being studied, but it is regretted that most of the results are extremely complex and hence do not provide a very good base for business applications:

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[1]Fry, T.C., The Theory of Probability as Applied to Problems of Congestion, in "Probability and Its Engineering Uses" (D. Van Nostrand, Inc., Princeton, N.J., 1928).

[2]Molina, E.C., Application of the Theory of Probabilities Applied to Telephone Trunking Problems (Bell System Tech. Journal, vol. 6, 1927) p. 461-494.

[3]O'Dell, G.F., Theoritical Principles of the Traffic Capacity of Automatic Switches (P.O. Elec. Congrs. J., vol. 13, 1920) p. 209-223.

[4]Syski, R., The Theory of Congestion in Lost-call Systems (A.T.E. J., vol. 9, 1953) p. 182-215.

[5]Saaty, T.L., Elements of Queueing Theory with Applications (McGraw-Hill Book Co., Inc., New York, 1961).

"We did not cover any of the multiple-phase situations involving arrival and/or service distributions other than Poisson, and queue disciplines other than first come-first served. It is felt that the more complex structures are best handled through the technique of computer simulation."[1]

It is the objective of this thesis to provide a set of graphs to enable the businessman to predict the necessary characteristics of a single channel queue with arbitrary inter-arrival time and service time distributions, and with customers arriving from an infinite population. It is assumed that the queue and service facility are in statistical equilibrium.

[1]Buffa, E.S., Models for Production and Operations Management (John Wiley and Sons, Inc., New York, 1966) p.266.

CHAPTER 2

DERIVATIONS OF DISTRIBUTIONS AND THEIR PROPERTIES

Derivations of Distributions

This section shall deal with the derivation of the Poisson and exponential distributions from random arrivals and thence to derive the Erlangian and hyperexponential distributions via simulation in which only the exponential element is used.

It is necessary to specify the mechanism by which the Erlangian and hyperexponential distributions are obtained so that we may be able to obtain the characteristics of queues with these distributions for interarrival time and service time. Figure 1 shows the utilization of the exponential distribution to simulate the Erlangian distribution while Figure 2 is used to simulate the hyperexponential distribution.

Appendix II shows how this property is utilized to simulate the queues with Erlangian/hyperexponential interarrival time distribution and Erlangian/hyperexponential service time distribution.

Poisson and Exponential Distribution

In the study of queues, the Poisson and its counterpart, the exponential, distributions are of prime importance. This is due to the properties of these distributions. It is only the exponential facilities and Poisson arrivals that give rise to simple linear equations for detailed balance of transitions between states, independent of time.

Assuming that the arrivals are random with an average rate of λ from an infinite population.

In a short interval of time Δt,

Probability of 1 arrival, $P_1 = \lambda \Delta t$

Probability of 0 arrival, $P_0 = 1-\lambda.\Delta t - 0.\Delta t$ where $0.\Delta t$ is such that $\lim\limits_{\Delta t \to 0} \frac{0\Delta t}{\Delta t} = 0$

Probability of n arrivals, $P_n = 0.\Delta t \to 0$ as $\Delta t \to 0$

In the period t, where $t = m.\Delta t$,

By the use of the binomial probability law,

$$P_r = \frac{m!}{(m-r)!}(\lambda.\Delta t)^r (1-\lambda.\Delta t)^{m-r}$$

Consider $\Delta t \to 0$ and $m = t/\Delta t$, $m \to \infty$.

Therefore $$P_r = \lim_{m\to\infty} \frac{m!}{r!(m-r)!} \left(\frac{\lambda t}{m}\right)^r \left(1-\frac{\lambda t}{m}\right)^{m-r}$$

$$= \left[\frac{(\lambda t)^r}{r!}\right] \lim_{m\to\infty} \frac{m!}{m^r (m-r)!} \lim_{m\to\infty}\left(1-\frac{\lambda t}{m}\right)^m \lim_{m\to\infty}\left(1-\frac{\lambda t}{m}\right)^{-r}$$

Also $$\lim_{m\to\infty} \frac{m!}{m^r(m-r)!} = \lim_{m\to\infty} \frac{m.(m-1).(m-2)\ldots.(m-r+1)}{m.\ m.m.\ldots\ldots\ldots.m}$$

$$= \lim_{m\to\infty} \frac{m}{m}.\lim_{m\to\infty} \frac{(m-1)}{m}.\lim_{m\to\infty} \frac{(m-2)}{m}\ldots.\lim_{m\to\infty} \frac{(m-r+1)}{m}$$

$$= 1 \quad . \quad 1 \quad . \quad 1 \ldots\ldots\ldots.1$$

$$= 1$$

$$\lim_{m\to\infty}.\left(1-\frac{\lambda t}{m}\right)^{-r} = 1^{-r} = 1$$

$$\lim_{m\to\infty}.\left(1-\frac{\lambda t}{m}\right)^{m} = \lim_{y\to 0}.(1+y)^{\frac{-\lambda t}{y}} \quad \text{where} \quad y = \frac{-\lambda t}{m}$$

$$=[\lim_{y\to 0}(1+y)^{\frac{1}{y}}]^{-\lambda t}$$

But $\lim_{y\to 0}(1+y)^{\frac{1}{y}}$ is the definition of e .

Therefore $\lim_{m\to\infty}(1+\frac{\lambda t}{m})^{m} = e^{-\lambda t}$

Therefore $P_r = \frac{(\lambda t)^r}{r!}e^{-\lambda t}$

This is the POISSON distribution.

$$P_0 = \frac{(\lambda t)^0}{0!} e^{-\lambda t} = e^{-\lambda t}$$

This is the probability that interarrival time is greater than t
This is so because the probability of no arrivals in time, t is the probability that interarrival time is greater than t .

Hence $P_0 = \int_t^{\infty} a(t).dt$ where $a(t)$ is the interarrival time distribution.

$$P_0 = \int_0^{\infty} a(t).dt - \int_0^{t} a(t)dt$$

But

$$\int_0^{\infty} a(t).dt = 1$$ as this is the total probability.

Therefore

$$a(t) = \frac{d(1-P_0)}{dt}$$

Hence $a(t) = \frac{d(1-e^{-\lambda t})}{dt}$

Therefore $a(t) = \lambda e^{-\lambda t}$

The probability that interarrival time is greater than t,

$$A(t) = \int_t^{\infty} a(t)dt = P_0 = e^{-\lambda t}$$

$a(t)$ is the EXPONENTIAL distribution

and $A(t)$ is the cumulative EXPONENTIAL distribution.

Erlangian Distribution

With a chosen set of exponential phases and appropriate rules for transition as represented by Figure 1, the Erlangian distribution can be simulated.

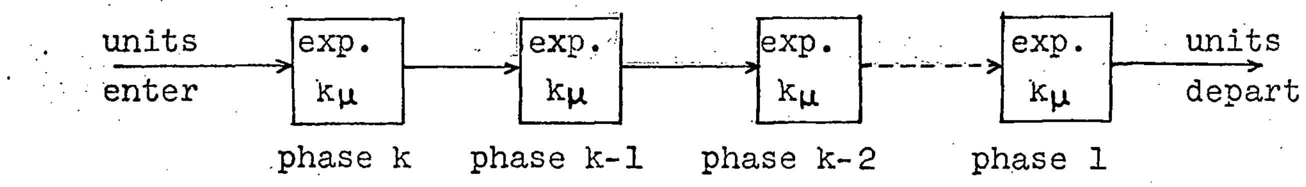


Figure 1: Erlangian distribution facility of k phases.

Rules: a) There are k phases, with each phase having an exponential distribution of rate $k\mu$.

b) No units are to be introduced into the facility until the previous unit has completed all k phases.

Probability that unit passes through all k phases between time t and $t+dt$

= Probability of unit in phase one starting in time x_2 and finishing in time t.

x Probability of unit in phase two starting in time x_3 and finishing in time x_2

x ....

x ....

× Probability of unit in phase k starting in time 0 and finishing in time x_k

$$= [\int_0^t k\mu e^{-k\mu(t-x_2)} dx_2 \int_0^{x_2} k\mu e^{-k\mu(x_2-x_3)} dx_3 \ldots \int_0^{x_{k-1}} k\mu e^{(x_{k-1}-x_k+x_k)} dx_k]$$

By successive integration,

$$s(t) = (k\mu t)^{k-1}[e^{-k\mu t}/(k-1)!]k\mu$$

This is the ERLANGIAN distribution.

The cumulative ERLANGIAN distribution

$$S(t) = \int_t^\infty s(t).dt$$

$$= .e^{-k\mu t} \sum_{n=0}^{k-1} (k\mu t)^n/n!$$

The following situation generates an Erlangian arrival rate. Consider a depot with a constant queue of customers entering one at a time into a constantly busy mechanism of ℓ phases, each with an exponential time distribution and mean service rate $\ell\lambda$ from the mechanism after completing all ℓ phases and join another queue. No customer enters the first phase until the previous customer has completed all ℓ phases. Customers leaving this mechanism join a second queue. The interarrival time distribution for this second service facility is an ℓ phase Erlangian.

With the same form of analysis used previously for the service time distribution, the inter arrival time distribution function, $a(t)$, can be determined:

$$a(t) = (\ell\lambda t)^{\ell-1}[e^{-\ell\lambda t}/(\ell-1)!]\ell\lambda$$

hence the cumulative interarrival time distribution,

$$A(t) = e^{-\ell\lambda t}\sum_{n=0}^{\ell}(\ell\lambda t)^n/n!$$

Hyperexponential Distribution

The hyperexponential distribution can be simulated with exponential distribution by assuming that the service channel is made up of two independent branches, one of rate $2\sigma\mu$, the other of rate $2(1-\sigma)\mu$ (where $0 \leq \sigma \leq \frac{1}{2}$) . When a unit enters for service, it is assigned to one of the two branches at random, the choice going to the $2\sigma\mu$ branch, on the average, σ of the time, and going to the $2(1-\sigma)\mu$ branch, $(1-\sigma)$ of the times, on the average. When it enters one of the branches no units are to be introduced into the facility until the previous unit has completed its service. Figure 2 gives a diagramatic representation of the facility.

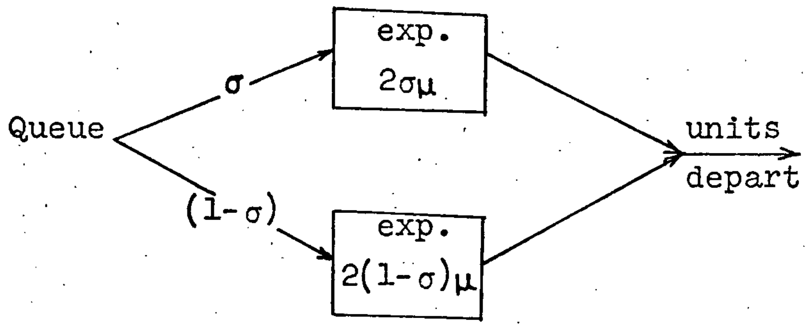


Figure 2: Hyperexponential distribution facility.

The probability that a unit will complete service in time between t and $t+\Delta t$

= σ(Probability that a unit will complete service in time between t and $t+\Delta t$ in the $2\sigma\mu$ branch)
+ $(1-\sigma)$(Probability that a unit will complete service in time between t and $t+\Delta t$ in the $2(1-\sigma)\mu$ branch).

Therefore
$$s(t).dt = (\sigma)(2\sigma\mu)e^{-(2\mu\sigma)t}.dt + (1-\sigma)[2(1-\sigma)\mu]e^{-[2(1-\sigma)\mu]t}.dt$$
$$= [2\sigma^2\mu e^{-2\mu\sigma t} + 2(1-\sigma)^2\mu e^{-2(1-\sigma)\mu t}]dt$$

Let $j = [1 + \frac{(1-2\sigma)^2}{2\sigma(1-\sigma)}]$

Here j plays the same role as integers k and ℓ play with the Erlangian distribution. The quantity measures the departure from pure random.

Therefore $\sigma = \frac{1}{2} - \sqrt{[\frac{1}{4} - \frac{1}{2(1+j)}]}$ since $0 \leq \sigma \leq \frac{1}{2}$

$$S(t) = \int_t^\infty s(t)dt$$
$$= \sigma e^{-2\sigma\mu t} + (1-\sigma)e^{-2(1-\sigma)\mu t}$$

A depot with a constant queue of customers entering one at a time into a mechanism, with two branches, one with an exponential time distribution of rate $2\sigma\lambda$ and the other with an exponential distribution of rate $2(1-\sigma)\lambda$. When a unit enters a mechanism it is assigned to one of the two branches at random, the choice going to the $2\sigma\lambda$ branch, on the average σ of the time, and to the $2(1-\sigma)\lambda$ branch $(1-\sigma)$ of the time.

Only one unit may be in the system at a time. The unit leaves the mechanism to join a second queue. The interarrival time distribution for this second queue is hyper-exponential.

With the same form of analysis as for the service time distribution above, the interarrival time distribution, $a(t)$, may be determined:

$$a(t) = [2\sigma^2\lambda e^{-2\lambda\sigma t} + 2(1-\sigma)^2\lambda e^{-2(1-\sigma)^2\lambda t}]$$

and the cumulative interarrival time distribution,

$$A(t) = \sigma e^{-2\sigma\lambda t} + (1-\sigma)e^{-2(1-\sigma)\lambda t}$$

$$j = [1 + \frac{(1-2\sigma)^2}{2\sigma(1.-\sigma)}]$$

where $\sigma = \frac{1}{2} - \sqrt{[\frac{1}{4} - \frac{1}{2(1+j)}]}$ since $0 \leq \sigma \leq \frac{1}{2}$

Properties of Distributions

Since the fractional coefficients of variance squared of interarrival time distribution and service time distribution will be of prime importance in the construction of the graphs, this property will be derived for the exponential, Erlangian, and hyperexponential distributions.

Let $f(t)$ be the continuous probability distribution of time t.

α be the average rate.

$Av(t)$ be the average time.

$Var(t)$ be the variance of distribution $f(t)$.

C^2 be the fractional coefficient of variance squared.

$$Av(t) = \int_0^\infty t.f(t).dt$$

$$Var(t) = \int_0^\infty [t - Av(t)]^2 f(t).dt$$

$$= \int_0^\infty t^2 f(t)dt - 2Av(t) \int_0^\infty tf(t).dt + [Av(t)]^2 \int_0^\infty f(t).dt$$

$$= \int_0^\infty t^2 f(t)dt - 2[Av(t)]^2 + [Av(t)]^2$$

$$= \int_0^\infty t^2 f(t)dt - [Av(t)]^2$$

$$C^2 = Var(t)/[Av(t)]^2$$

Exponential Distribution

$$f(t) = \alpha e^{-\alpha t}$$

$$Av(t) = \int_0^\infty t\alpha e^{-\alpha t} = \alpha[-\frac{te^{-\alpha t}}{\alpha}]_0^\infty - \alpha \int_0^\infty - \frac{e^{-\alpha t}}{\alpha}.dt$$

$$= \int_0^\infty e^{-\alpha t}dt = - \frac{1}{\alpha}[e^{-\alpha t}]_0$$

$$= \frac{1}{\alpha}$$

$$Var(t) = \int_0^\infty t^2 \alpha e^{-\alpha t}.dt - (\frac{1}{\alpha})^2 = \alpha[\frac{t^2 e^{-\alpha t}}{-\alpha}]_0^\infty - \alpha \int_0^\infty - \frac{2te^{-\alpha t}}{\alpha}.dt - (\frac{1}{\alpha})^2$$

$$= 2 \int_0^\infty te^{-\alpha t}.dt - (\frac{1}{\alpha})^2 = \frac{2}{\alpha}(\frac{1}{\alpha}) - (\frac{1}{\alpha})^2$$

$$= (\frac{1}{\alpha})^2$$

$$C^2 = \frac{(\frac{1}{\alpha})^2}{(\frac{1}{\alpha})^2} = 1$$

Erlangian Distributions

$$f(t) = (\ell\alpha t)^{\ell-1}[e^{-\ell\alpha t}/(\ell-1)!]\ell\alpha$$

$$Av(t) = \int_0^\infty t(\ell\alpha t)^{\ell-1}[e^{-\ell\alpha t}/(\ell-1)!]\ell\alpha.dt$$

$$= \int_0^\infty(\ell\alpha t)^{\ell}[e^{\ell\alpha t}/(\ell-1)!]dt$$

$$= \frac{(\ell\alpha)^\ell}{(\ell-1)!}\int_0^\infty t^\ell e^{-\ell\alpha t}dt$$

By successive integration by parts,

$$\int_0^\infty t^\ell e^{-\ell\alpha t}.dt = \frac{\ell!}{(\ell\alpha)^{\ell+1}} .$$

Therefore $Av(t) = \frac{(\ell d)^\ell}{(\ell-1)!}\ \frac{(\ell)!}{(\ell\alpha)^{\ell+1}} = \frac{\ell}{\ell\alpha}$

$$= \frac{1}{\alpha}$$

$$Var(t) = \int_0^\infty t^2(\ell\alpha t)^{\ell-1}[e^{-\ell\alpha t}/(\ell-1)!]\ell\alpha.dt - (\frac{1}{\alpha})^2$$

$$= \int_0^\infty t(\ell\alpha t)^{\ell}[e^{-\ell\alpha t}/(\ell-1)!]dt - (\frac{1}{\alpha})^2$$

$$= \frac{(\ell\alpha)^\ell}{(\ell-1)!}\int_0^\infty t.t^\ell e^{-\ell\alpha t}.dt - (\frac{1}{\alpha})^2$$

$$= \frac{(\ell\alpha)^{\ell}}{(\ell-1)!}\left[\frac{(\ell+1)!}{(\ell\alpha)^{\ell+2}}\right] - \left(\frac{1}{\alpha}\right)^2$$

$$= \frac{(\ell+1)\ell}{(\ell\alpha)^2} - \left(\frac{1}{\alpha}\right)^2$$

$$= \frac{(\ell+1)}{\ell}\left(\frac{1}{\alpha}\right)^2 - \left(\frac{1}{\alpha}\right)^2 = \frac{1}{\alpha^2}\left(\frac{\ell+1}{\ell} - 1\right)$$

$$= \frac{1}{\alpha^2}\left[\frac{1}{\ell}\right]$$

$$\frac{\left(\frac{1}{\alpha^2 \ell}\right)}{\frac{1}{\alpha^2}} = \frac{1}{\ell}$$

Hyperexponential Distribution

$$f(t) = 2\sigma^2\alpha e^{-2\alpha\sigma t} + 2(1-\sigma)^2\alpha e^{-2\sigma(1-\sigma)t}$$

$$Av(t) = \int_0^\infty t[2\sigma^2\alpha e^{-2\alpha\sigma t} + 2(1-\sigma)^2\alpha e^{-2\alpha(1-\sigma)t}]dt$$

$$= \int_0^\infty t[2\sigma^2\alpha e^{-2\alpha\sigma t}]dt + \int_0^\infty t[2(1-\sigma)^2\alpha e^{-2\alpha(1-\sigma)t}]dt$$

Integrating by parts

$$Av(t) = \sigma\left(\frac{1}{\alpha}\right) + (1-\sigma)\left(\frac{1}{\alpha}\right)$$

$$= \left(\frac{1}{\alpha}\right)$$

$$Var(t) = \int_0^\infty t^2[2\sigma^2\alpha e^{-2\alpha\sigma t} + 2(1-\sigma)^2\alpha e^{-2\alpha(1-\sigma)t}]dt - \left(\frac{1}{\alpha}\right)^2$$

$$= \int_0^\infty t^2[2\sigma^2\alpha e^{-2\alpha\sigma t}]dt + \int_0^\infty t^2[2(1-\sigma)^2\alpha e^{-2\alpha(1-\sigma)t}]dt - (\frac{1}{\alpha})^2$$

$$= \frac{2\sigma}{(2\alpha\sigma)^2} + \frac{2(1-\sigma)}{[2(1-\sigma)\alpha]^2} - (\frac{1}{\alpha})^2$$

$$= \frac{1}{\alpha^2}[\frac{2\sigma}{4\sigma^2} + \frac{2(1-\sigma)}{4(1-\sigma)^2} - 1]$$

$$= \frac{1}{\alpha^2}[\frac{2-2\sigma+2\sigma-4\sigma+4\sigma^2}{4\sigma(1-\sigma)}]$$

$$= \frac{1}{\alpha^2}[\frac{2-4\sigma+4\sigma^2}{4\sigma(1-\sigma)}]$$

$$= \frac{1}{\alpha^2}[\frac{1-2\sigma+2\sigma^2}{2\sigma(1-\sigma)}]$$

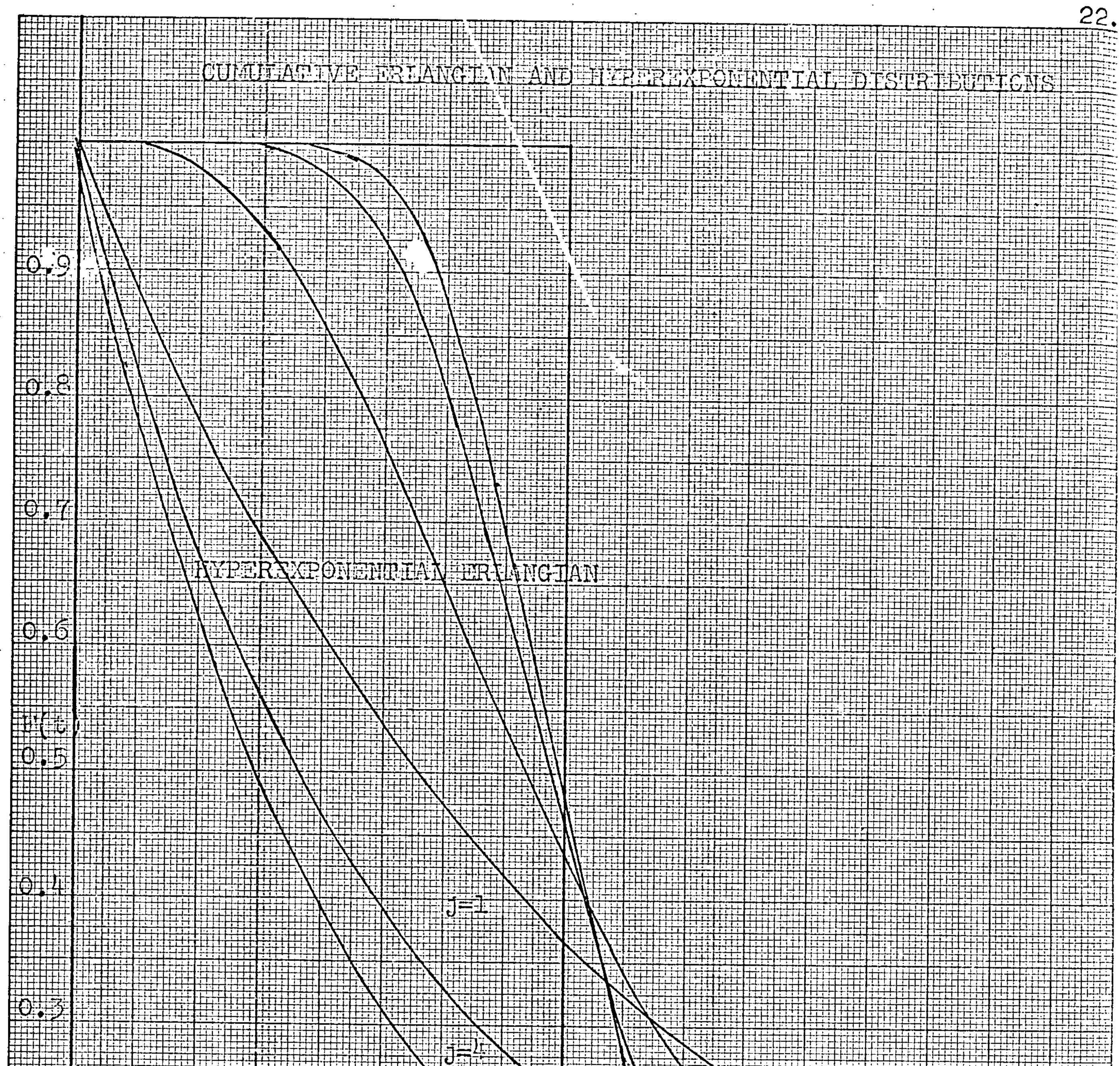
$$= \frac{1}{\alpha^2} j$$

where $j = [\frac{1-2\sigma+2\sigma^2}{2\sigma(1-\sigma)}]$ or $\sigma = \frac{1}{2} - \sqrt{[\frac{1}{4} - \frac{1}{2(1+j)}]}$ since $0 \leq \sigma \leq \frac{1}{2}$

$$C^2 = \frac{(\frac{1}{\alpha^2})j}{(\frac{1}{\alpha^2})} = j$$

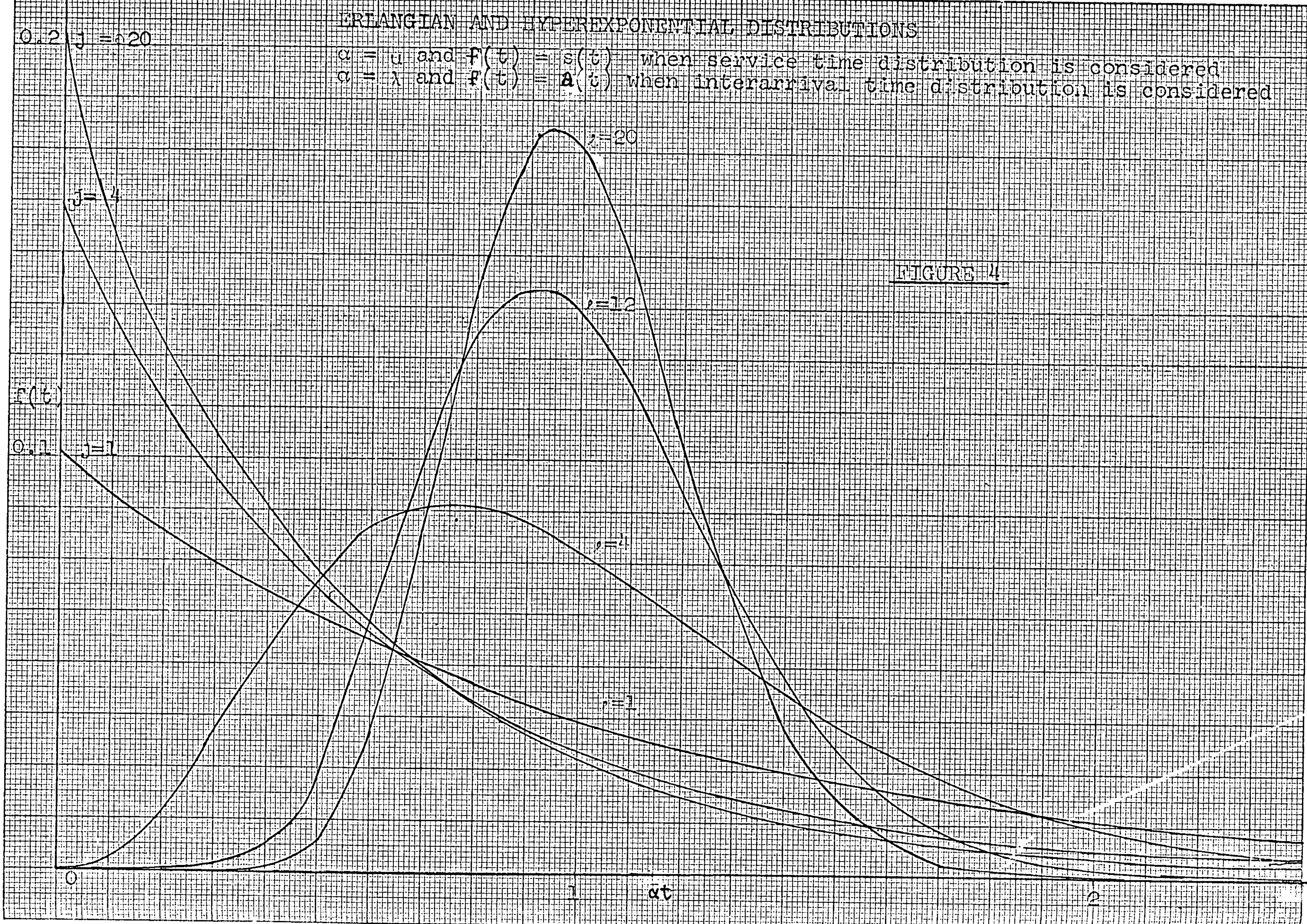
	Av(t)	Var(t)	C^2	Comments
Exponential	$\frac{1}{\alpha}$	$\frac{1}{\alpha^2}$	1	
Erlangian	$\frac{1}{\alpha}$	$\frac{1}{\alpha^2 \ell}$	$\frac{1}{\ell}$	
Hyperexponential	$\frac{1}{\alpha}$	$\frac{j}{\alpha^2}$	j	$\sigma = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{1}{2(1+j}}$

Table I. Properties of Distributions



$\alpha = \mu$ and $F(t) = S(t)$ when service time distribution is considered.
$\alpha = \lambda$ and $F(t) = A(t)$ when interarrival time distribution is considered.

FIGURE 3


ERLANGIAN AND HYPEREXPONENTIAL DISTRIBUTIONS
α = u and f(t) = s(t) when service time distribution is considered
α = λ and f(t) = a(t) when interarrival time distribution is considered
FIGURE 4
0.2
J = 20
J = 4
f(t)
0.1
J=1
ρ=20
ρ=12
ρ=4
ρ=1
0
1
αt
2

CHAPTER 3

EXPONENTIAL INTERARRIVAL TIME DISTRIBUTION: EXPONENTIAL SERVICE TIME DISTRIBUTON

Erlang's Model

The model is for steady state in which the customers come from an infinite population with an exponential inter-arrival time distribution of rate λ to a service channel where the service time distribution is exponential with mean rate μ. The utilization factor $\rho = \frac{\lambda}{\mu} < 1$.

$P_n(t+\Delta t)$: Probability of n customers in the waiting line in the system at the time $t+\Delta t$.

$P_n(t)$: Probability of n customers in the waiting line in the system at the time t.

The state probabilities equations are:

Initial equation:

$$P_0(t+\Delta t) = P_0(t)[1-\lambda\Delta] + P_1(t)\mu\Delta t \qquad \ldots\ldots 1$$

Queue equation:

$$P_n(t+\Delta t) = P_n(t)[1-(\lambda+\mu)\Delta t] + P_{n-1}(t)\lambda\Delta t + P_{n+1}(t)\mu\Delta t \qquad \ldots\ldots 2$$

Therefore $\frac{dP_0}{dt} = -\lambda P_0 + \mu P_1$

and $\frac{dP_n}{dt} = -(\lambda+\mu)P_n + \lambda P_{n-1} + \mu P_{n+1}$

Since the system is in steady state:

$$\frac{dP_0}{dt} = 0 \quad \text{and} \quad \frac{dP_n}{dt} = 0$$

Hence $-\lambda P_0 + \mu P_1 = 0$

Therefore $P_1 = \frac{\lambda}{\mu} P_0$

$$= \rho P_0 \qquad \ldots\ldots 3$$

Also $-(\lambda+\mu)P_n + \lambda P_{n-1} + \mu P_{n+1} = 0$

Dividing by μ ,

$$-(\rho+1)P_n + \rho P_{n-1} + P_{n+1} = 0$$

Let $n = 1$, $P_2 = (\rho+1)P_1 - \rho P_0$

But from (3) , $P_1 = \rho P_0$

Therefore $P_2 = \rho^2 P_0$ $\ldots\ldots 4$

Let $n = 2$, $P_3 = (\rho+1)P_2 - \rho P_1$

But from (3), $P_1 = \rho P_0$ and (4), $P_2 = \rho^2 P_0$

Therefore $P_3 = \rho^3 P_0$ $\ldots\ldots 5$

Let $n = 3$, $P_4 = (\rho+1)P_3 - \rho P_2$

But from (4), $P_2 = \rho^2 P_0$ and (5) $P_3 = \rho^3 P_0$

Therefore $P_4 = \rho^4 P_0$ $\ldots\ldots 6$

By deduction $P_n = \rho^n P_0$ $\ldots\ldots 7$

But $P_0 + P_1 + P_2 + \ldots + P_n + \ldots + P_\infty = 1$

i.e. $\sum_{n=0}^{\infty} P_n = 1$ or $\sum_{n=0}^{\infty} \rho^n P_0 = 1$

Therefore $P_0 = \dfrac{1}{\sum_{n=0}^{\infty} \rho^n} = \dfrac{1}{\frac{1}{(1-\rho)}}$

Therefore $P_0 = (1-\rho)$ 8

$L = \sum_{n=0}^{\infty} nP_n = \sum_{n=0}^{\infty} n(1-\rho)\rho^n = \dfrac{(1-\rho)\rho}{((1-\rho)^2)}$

Therefore $L = \dfrac{\rho}{(1-\rho)}$ 9

$L_q = \sum_{n=1}^{\infty} (n-1)P_n = \sum_{n=1}^{\infty} nP_n - \sum_{n=1}^{\infty} P(n) = \dfrac{\rho}{(1-\rho)} - \rho =$

$= \dfrac{\rho^2}{(1-\rho)}$ 10

$W = \dfrac{L}{\lambda}$

$W_q = \dfrac{Lq}{\lambda}$

The main limitation of Erlang's model for steady state single channel analysis is its restriction to exponential interarrival time distribution and exponential service time distribution.

CHAPTER 4

EXPONENTIAL INTERARRIVAL TIME DISTRIBUTION: ARBITRARY SERVICE TIME DISTRIBUTION

The Pollaczek - Khintchine Formula

The Pollaczek - Khintchine formula also deals with steady state single channel with exponential interarrival time distribution. However, it goes one step further by considering service time distribution as arbitrary. Arrivals occur at random by a Poisson process with the rate λ per unit time, to a waiting line, in statistical equilibrium before a single service facility. The arrivals are to be served by an arbitrary service time distribution facility at the rate μ per unit time first come-first served basis. The utilization factor, $\rho = \frac{\lambda}{\mu} < 1$.

Suppose that the departing customer leaves q in line, including the one in service, whose service time is t . Let r customers arrive during this time t . If the next departing customer leaves q' customers behind,

then $q' = q - 1 + r + \delta$ where $\delta = \begin{cases} 0 & \text{if } q > 0 \\ 1 & \text{if } q = 0 \end{cases}$

Therefore $E(q') = E(q) - E(1) + E(r) + E(\delta)$

But since it is in steady state or statistical equilibrium, $E(q') = E(q)$

Therefore $E(\delta) = E(1) - E(r)$

$$= 1 - E(r)$$

But during the service time of length t , we have $E(r) = (\lambda t)$

$$E(r^2) = \sum_{r=0}^{\infty} r^2 \frac{(\lambda t)^r}{r!} e^{-\lambda t} = [E(r)]^2 + E(r) = (\lambda t)^2 + (\lambda t)$$

Let $s(t)$ be the service time distribution with mean $\frac{1}{\mu}$.

$$E(r) = \sum_{t=0}^{\infty} \lambda t s(t) = \lambda \sum_{t=0}^{\infty} t s(t)$$

But $\sum_{t=0}^{\infty} t s(t)$ is the mean and is equal to $\frac{1}{\mu}$.

Therefore $E(r) = \frac{\lambda}{\mu} = \rho$

$$\text{Also} \quad E(r^2) = \sum_{t=0}^{\infty} [\lambda t)^2 + (\lambda t)]s(t)$$

$$= \lambda^2 \sum_{t=0}^{\infty} t^2 s(t) + \lambda \sum_{t=0}^{\infty} t s(t)$$

$$= \lambda^2 \sum_{t=0}^{\infty} (t-\frac{1}{\mu})^2 s(t) + \frac{2\lambda^2}{\mu} \sum_{t=0}^{\infty} t s(t) - \frac{\lambda^2}{\mu^2} \sum_{t=0}^{\infty} s(t) + \rho$$

$$= \lambda^2 \mathrm{Var}(t) + 2\rho^2 - \rho^2 + \rho$$

$$= \lambda^2 \mathrm{Var}(t) + \rho^2 + \rho$$

Squaring both sides of the equation $q' = q - 1 + r + \delta$

$$q'^2 = (q-1+r+\delta)^2$$

But since $\delta = 1$ when $q = 0$

and $\delta = 0$ when $q \geq 1$

Therefore $\delta^2 = \delta$

and $q(1-\delta) = q$

Hence $q'^2 = q^2 - 2q(1-r) + (r-1)^2 + \delta(2r-1)$

But since it is statistical equilibrium,

$$E(q'^2) = E(q^2)$$

Hence $E(q'^2) - E(q^2) = 2E(q)E(r-1) + E[(r-1)^2] + E(\delta)E(2r-1) = 0$

i.e. $$E(q) = \frac{E[(r-1)^2]+E(\delta)E(2r-1)}{2E(1-r)}$$

$$= \frac{E(r^2)-2E(r)+E(1)+E(\delta)[2E(r)-E(1)]}{2[E(1)-E(r)]}$$

$$= \frac{\lambda^2 Var(t)+\rho^2+\rho-2\rho+1+(1-\rho)(2\rho-1)}{2(1-\rho)}$$

$$= \rho + \rho^2 + \frac{\lambda^2 Var(t)}{2(1-\rho)}$$

But the fractional coefficient of variance squared of service time,

$$C_s^2 = \frac{Var(t)}{(\frac{1}{\mu})^2}$$

Therefore $$L = E(q) = \rho + \frac{\rho^2(1+C_s^2)}{2(1-\rho)}$$

$$Lq = L - \rho = \frac{\rho^2(1+C_s^2)}{2(1-\rho)}$$

Therefore L and Lq are functions of ρ and C_s^2 only.

Since L is a function of the utilization factor, ρ, and the square of the coefficient of variance of service rate distribution, C_s^2, only, it is necessary only to vary ρ and C_s^2 to compute the different values of L in the equation

$$L = \rho + \frac{\rho^2(1+C_s^2)}{2(1-\rho)}$$

Table II was obtained by substituting values of ρ from 0.1 to 0.9 and C_s^2 from 0 to 6 in the above equation.

Using C_s^2 as the ordinate and L as the abscissa, and varying ρ, Figure 5 was constructed.

Knowing that the interarrival time distribution is exponential, i.e. arrivals are random, and also the coefficient of variance squared of service time, C_s^2, and the utilization factor, ρ, the average length of the system, L, can be read from the verticle axis.

The values of the average queue length, Lq, the average waiting time in the system, W, and the average queue time, Wq, can be easily computed once L is known, as they are all functions of L, ρ and the average arrival rate, λ, only. The relationships are listed below.

$$Lq = (L-\rho)$$

$$W = \frac{L}{\lambda}$$

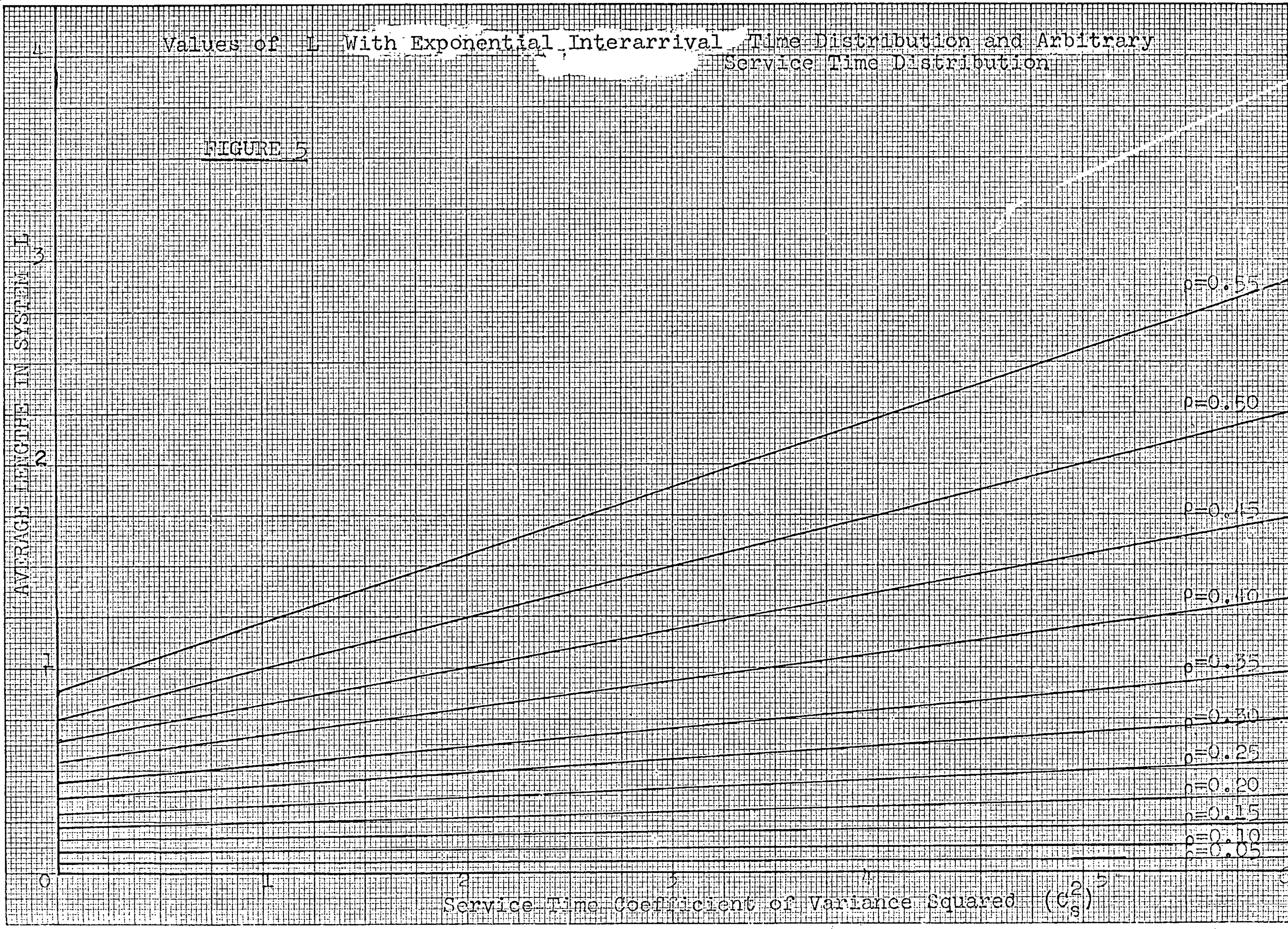
$$Wq = \frac{(L-\rho)}{\lambda}$$

Values of L with given ρ for exponential interarrival time distribution and arbitrary service time distribution with a fractional coefficient of variance squared, C_s^2

C^2_s

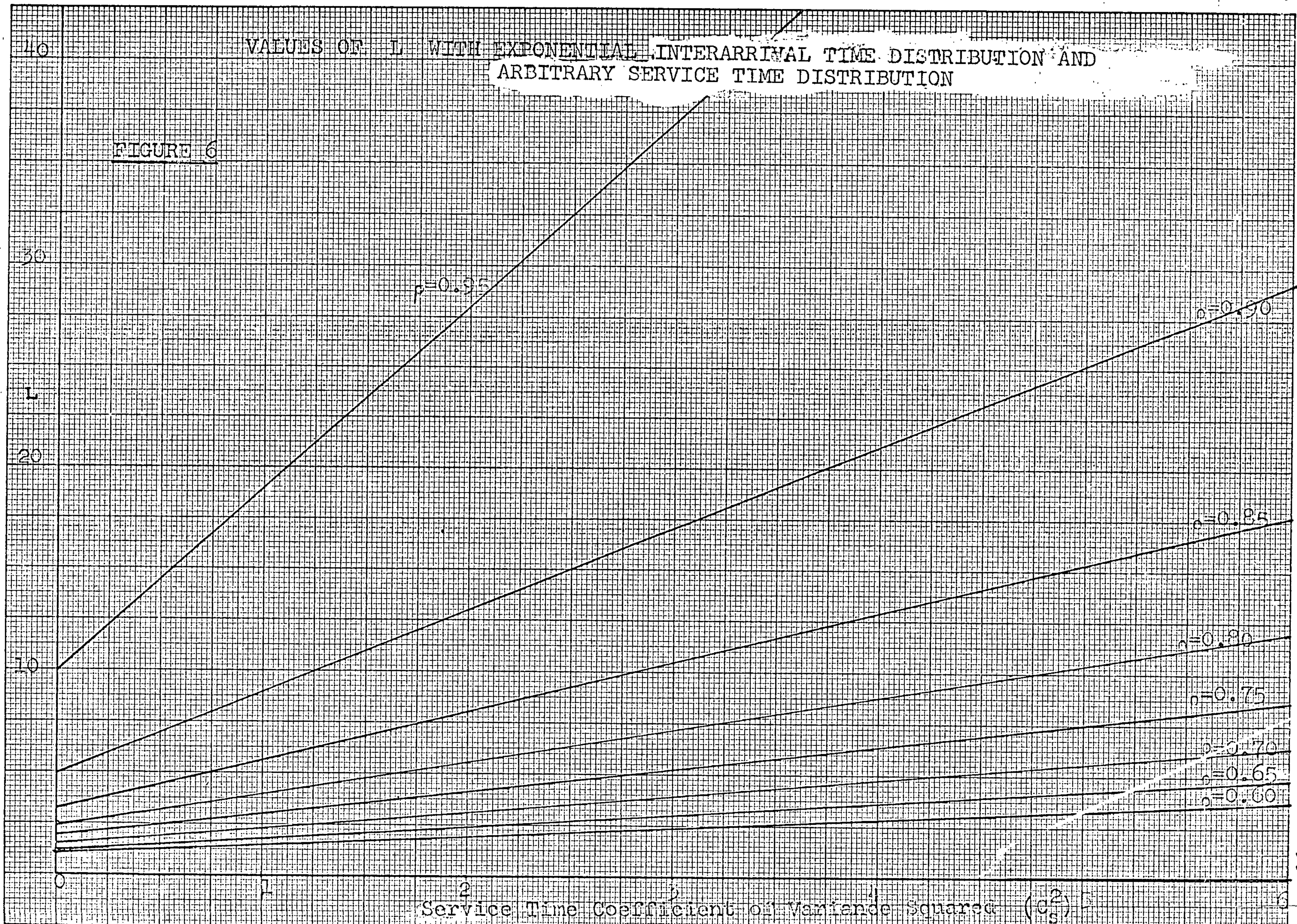
ρ	0	1	2	3	4	5	6
0.05	0.051316	0.052632	0.053948	0.055264	0.056580	0.057896	0.059212
0.10	0.105556	0.111111	0.116667	0.122222	0.127777	0.133333	0.138888
0.15	0.163235	0.176471	0.189706	0.202941	0.216176	0.229411	0.242646
0.20	0.225000	0.250000	0.275000	0.300000	0.325000	0.350000	0.375000
0.25	0.291667	0.333333	0.375000	0.416667	0.458333	0.500000	0.541667
0.30	0.364286	0.428572	0.492858	0.557144	0.621430	0.685716	0.750000
0.35	0.438846	0.527692	0.616538	0.705384	0.794230	0.883076	0.971922
0.40	0.533333	0.666667	0.800000	0.933333	1.066667	1.200000	1.333333
0.45	0.634091	0.818182	1.002273	1.186364	1.370455	1.554546	1.738637
0.50	0.750000	1.000000	1.250000	1.500000	1.750000	2.000000	2.250000
0.55	0.886111	1.222222	1.558333	1.894444	2.230555	2.566666	2.902777
0.60	1.050000	1.500000	1.950000	2.400000	2.850000	3.300000	3.750000
0.65	1.253571	1.857142	2.560713	3.064284	3.667855	4.271426	4.874997
0.70	1.516667	2.333333	3.150000	3.966667	4.783333	5.600000	6.416667
0.75	1.875000	3.000000	4.125000	5.250000	6.375000	7.500000	8.625000
0.80	2.400000	4.000000	5.600000	7.200000	8.800000	10.40000	12.00000
0.85	3.258333	5.666667	8.075000	10.48333	12.89166	15.30000	17.70833
0.90	4.950000	9.000000	13.05000	17.10000	21.15000	25.20000	29.25000
0.95	9.975000	19.00000	28.02500	37.05000	46.07500	55.10000	64.12500

Table II Values of L for Exponential Interarrival Time Distribution and Arbitrary Service Time Distribution



Values of L With Exponential Interarrival Time Distribution and Arbitrary Service Time Distribution

FIGURE 5



VALUES OF L WITH EXPONENTIAL INTERARRIVAL TIME DISTRIBUTION AND ARBITRARY SERVICE TIME DISTRIBUTION
FIGURE 6
40
30
20
10
L
ρ=0.95
ρ=0.90
ρ=0.85
ρ=0.80
ρ=0.75
ρ=0.70
ρ=0.65
ρ=0.60
0
1
2
3
4
5
6
Service Time Coefficient of Variance Squared (C_s^2)

CHAPTER 5

ARBITRARY INTERARRIVAL TIME DISTRIBUTION: EXPONENTIAL SERVICE TIME DISTRIBUTION

Relationship between L and ρ and C_a^2

The Pollaczek - Khintchine formula provides a relationship between L , ρ and C_s^2 for the exponential interarrival time distribution and arbitrary service time distribution. The following analysis will attempt to show that, for the arbitrary interarrival time distribution and exponential time distribution, L is a function of ρ and C_a^2 only.

The interarrival time distribution, a(t), is arbitrary with a mean rate of λ per unit, to a waiting line, in statistical equilibrium before a single facility. The arrivals are then served by an exponential service time distribution channel at the rate of μ per unit time first come-first served discipline. The utilization factor, $\rho = \frac{\lambda}{\mu} < 1$.

Suppose that an arriving customer finds q in line, including the one in service and himself. The time before the arrival of the next customer is t . Let r customers depart during this time t . If the next arriving customer finds q' customers, including the one in service and himself,

then $q' = q + 1 - r + \delta$ where $\delta = \begin{cases} 0 & \text{if } q \geq r \\ r-q & \text{if } q < r \end{cases}$

$$E(q') = E(q) + E(1) - E(r) + E(\delta)$$

But since it is in statistical equilibrium or steady state,

$E(q') = E(q)$

Therefore $E(\delta) = 1 - E(r)$

But during an arrival time of length t,

we have $E(r) = \sum_{r=0}^{\infty} \frac{r(\mu t)^r}{r!} e^{-\mu t} = \mu t$

where $\frac{(\mu t)^r e^{-\mu t}}{r!}$ is the Poisson distribution.

Therefore $E(r) = \mu t$

$$E(r^2) = \sum_{r=0}^{\infty} r^2 \frac{(\mu t)^r}{r!} e^{-\mu t} = (\mu t)^2 + (\mu t)$$

Let $a(t)$ be the interarrival time distribution with mean

i.e. $\int_0^\infty ta(t)dt = \frac{1}{\lambda}$

$$E(r) = \int_0^\infty \mu ta(t)dt = \mu \int_0^\infty ta(t)dt$$

$$= \frac{\mu}{\lambda} = \rho$$

Also
$$E(r^2) = \int_0^\infty [(\mu t)^2 + (\mu t)]a(t)dt$$

$$= \mu^2 \int_0^\infty t^2 a(t)dt + \mu \int_0^\infty ta(t)dt$$

$$= \mu^2 \int_0^\infty (t-\tfrac{1}{\lambda})^2 a(t)dt + \mu^2 \int_0^\infty \tfrac{2}{\lambda} ta(t)dt -$$

$$- \mu^2 \int_0^\infty \frac{1}{\lambda^2} a(t) + \frac{\mu}{\lambda}$$

$$= \mu^2 \mathrm{Var}(t) + \frac{2\mu^2}{\lambda^2} - \frac{\mu^2}{\lambda^2} + \frac{\mu}{\lambda}$$

$$= \mu^2 \mathrm{Var}(t) + \frac{1}{\rho^2} + \frac{1}{\rho}$$

Squaring $q' = q - 1 + r + \delta$ on both sides.

$$q'^2 = (q-1+r+\delta)^2$$

$$q'^2 = q^2 + 1 + r^2 + \delta^2 - 2q + 2rq + 2\delta q - 2r - 2\delta + 2r\delta$$

Hence,

$$E(q'^2) = E(q^2) + E(1) + E(r^2) + E(\delta)^2 - 2E(q) + 2E(r)E(q) + \\ + 2E(\delta)E(q) - 2E(r) - 2E(\delta) + 2E(r)E(\delta)$$

But since it is in statistical equilibrium,

$$E(q'^2) = E(q^2)$$

Therefore $0 = 1 + E(r^2) + E(\delta^2) - 2E(q) + 2E(r)E(q) + 2E(\delta)E(q) - - 2E(\delta) + 2E(r)E(\delta)$

Therefore $E(q) = \dfrac{1+E(r^2)+E(\delta^2)-2E(r)-2E(\delta)+2E(r)E(\delta)}{2-E(r)+2E(\delta)}$

Therefore $L = E(q)$ is a function of $E(r^2)$, $E(\delta^2)$, $E(r)$ and $E(\delta)$ only.

Also $E(r) = \rho$,

$E(\delta) = 1 - E(r) = 1 - \rho$;

$E(r^2) = \mu^2 \mathrm{Var}(t) + \dfrac{1}{\rho^2} + \dfrac{1}{\rho}$

Therefore L is a function of ρ, C_a^2 and $E(\delta^2)$ only.

Since $\delta = 0$ if $q \geq r$

and $\delta = r - q$ if $q < r$

Therefore $\delta^2 = 0$ if $q \geq r$

$\delta^2 = (r-q)^2$ if $q < r$

Hence δ^2 is a function of r^2 , q^2 , r and q only,

or $E(\delta^2)$ is a function of $E(r^2)$, $E(q^2)$, $E(r)$ and $E(q)$ only.

But $(q'-q) = (1-r+\delta)$

Squaring both sides,

$$q'^2 - 2qq' + q^2 = 1 + r^2 + \delta^2 - 2r + 2\delta - 2r\delta$$

Therefore $E(q'^2)$ is a function of $E(r)$, $E(\delta)$, $E(q)$ and $E(r^2)$ only.

Since $E(q'^2) = E(q^2)$ and $E(q') = E(q)$

Therefore $E(\delta^2)$ is a function of $E(r)$, $E(\delta)$, $E(q)$ and $E(r^2)$ only.

Therefore $E(\delta^2)$ is a function of $E(r)$, $E(q)$ and $E(r^2)$ only as $E(\delta) = 1 - E(r)$, and since $E(r)$, $E(q)$ and $E(r^2)$ are functions of ρ and C_a^2 only,

Therefore L is a function of ρ and C_a^2 only.

It is unfortunate that no simple equation can be obtained directly, but a set of tables or graphs can be set up through two basic approaches:

(i) Morse's method: Simulation of non-exponential distributions, Erlangian and hyperexponential distributions, via the use of exponential phases and branches.

(ii) Monte Carlo or General Purpose Simulation System III to obtain various L with different ρ's and C_a^2's .

Analytical Approaches for Exponential Service Time Distribution and Erlangian or Hyperexponential Interarrival Time Distributions

As previously mentioned, there are two main techniques via which a set of tables or graphs can be set up to enable one to obtain the characteristics of a queue, given that the service time distribution is exponential, an utilization factor of ρ, and an interarrival time distribution which is arbitrary and having a fractional coefficient of variance squared, C_a^2.

As P.M. Morse provided an analytical approach, I shall use the former technique for the following reasons:

(i) The values obtained are definite and not an approximate.

(ii) Much less computer time is needed for computation.

Erlangian Interarrival Time Distribution

Units, from a depot with an infinite population, passes through ℓ phases to join a queue for service. The service time distribution is exponential. No units are to enter the phases until the previous one has completed all ℓ phases. The units stay in each phase for an interval of time. This interval of time is exponentially distributed. The average rate for each phase is $\ell\lambda$. The average rate for service is μ.

As had previously been shown, the result of using a channel with ℓ phases, each with time interval exponentially distributed and average rate $\ell\lambda$, is an Erlangian interarrival time distribution.

Let $P_{s,n}(t)$ be the probability of the state where a unit is in phase s and there are n units in the queue and service facility at the time t .

Since at least one unit must be in the channel, $1 \leq s \leq \ell$.

In the time interval t to $(t+\Delta t)$ where $\Delta t \to 0$;

Initial equations;

$$P_{1,0}(t+\Delta t) = [1-(\ell\lambda)\Delta t)]P_{1,0}(t) + \mu\Delta t P_{1,1}(t) \qquad \ldots\ldots 1$$

$$P_{s,0}(t+\mu t) = [1-(\ell\lambda)\Delta t]P_{s,0}(t) + \mu\Delta t P_{s,1}(t) + \\ + \ell\lambda P_{s,-1,0}(t) \qquad \ldots\ldots 2$$

Queue equations;

$$P_{1,n}(t+\Delta t) = [1-(\ell\lambda+\mu)\Delta t]P_{1,n}(t) + \ell\lambda\Delta t P_{\ell,n-1}(t) + \\ + \mu\Delta t P_{1,n+1}(t) \qquad \ldots\ldots 3$$

$$P_{s,n}(t+\Delta t) = [1-(\ell\lambda+\mu)\Delta t]P_{s,n}(t) + \ell\lambda\Delta t P_{s-1,n}(t) + \\ + \mu\Delta t P_{s,n+1}(t) \qquad \ldots\ldots 4$$

Therefore $$\frac{dP_{1,0}}{dt} = \ell\lambda P_{1,0} + \mu P_{1,1} \qquad \ldots\ldots 5$$

$$\frac{dP_{s,0}}{dt} = \ell\lambda P_{s,0} + \mu P_{s,1} - \ell\lambda P_{s-1,0} \qquad \ldots\ldots 6$$

$$\frac{dP_{1,n}}{dt} = \ell\lambda P_{s,n-1} + \mu P_{1,n+1} - (\ell\lambda+\mu)P_{1,n} \qquad \ldots\ldots 7$$

$$\frac{dP_{s,n}}{dt} = \ell\lambda P_{s-1,n} + \mu P_{s,n+1} - (\ell\lambda+\mu)P_{s,n} \qquad \ldots\ldots 8$$

Since the queue is in statistical equilibrium or steady state, the left hand side of equations 5 to 8 are zero.

Hence $\ell\lambda P_{1,0} + \mu P_{1,1} = 0$ 9

$\ell\lambda P_{s,0} + \mu P_{s,1} - \ell\lambda P_{s-1,0} = 0$ 10

$\ell\lambda P_{\ell,n-1} + \mu P_{1,n+1} - (\ell\lambda+\mu)P_{1,n} = 0$ 11

$\ell\lambda P_{s-1,n} + \mu P_{s,n+1} - (\ell\lambda+\mu)P_{s,n} = 0$ 12

Let $P_{s,n} = B_s\omega^n$

Substituting in

(9) $(\mu\omega+\ell\lambda)B_1 = 0$ 13

(10) $(\ell\rho-\omega)B_s = \ell\rho B_{s-1}$ 14

(11) $\omega(\ell\rho+1-\omega)B_1 = \ell\rho B_\ell$ 15

(12) $(\ell\rho+1-\omega)B_s = \ell\rho B_{s-1}$ 16

From (12) $B_s = \frac{\ell\rho}{(\ell\rho+1-\omega)} B_{s-1}$

$$B_{s-1} = \frac{\ell\rho}{(\ell\rho+1-\omega)} B_{s-2}$$

$$\vdots \qquad \vdots$$

$$B_2 = \frac{\ell\rho}{(\ell\rho+1-\omega)} B_1$$

Hence $B_s = [\frac{\ell\rho}{(\ell\rho+1-\omega)}]^{s-1} B_1$

Therefore $B_\ell = [\frac{\ell\rho}{(\ell\rho+1-\omega)}]^{\ell-1} B_1$ 17

Dividing (15) by (17),

$$\omega = [\frac{\ell\rho}{\ell\rho+1-\omega}]^{\ell}$$

Let $u = [\frac{\ell\rho}{(\ell\rho+1-\omega)}]$

Therefore $\omega = u^{\ell}$ and $B_s = u^{s-1}B_1$

Substituting $B_s = u^{s-1}B_1$ in $P_{s,n} = B_s\omega^n$

$P_{s,n} = B_1 u^{\ell n+s-1}$

Substituting in (12)

$\ell\rho P_{s-1,n} + P_{s,n+1} - (\ell\rho+1)P_{s,n} = 0$

becomes $\ell\rho B_1 u^{\ell n+s-2} + B_1 u^{\ell n+\ell+s-1} - (\ell\rho+1)B_1 u^{\ell n+s-1} = 0$

Therefore $\ell\rho + u^{\ell+1} - (\ell\rho+1)u = 0 = (u-1)(u^{\ell}+u^{\ell+1}+\ldots+u^2+u-\ell p)$

....18

One root is $u = 1$ and is discarded.

Since by Descarte's rule of sign, there are one or two roots left in which one root is less than one but greater than zero and possibly another root which is less than zero.

Discard the negative root.

Therefore $0 \leq u < 1$

Multiplying (18) by $\frac{u^m}{u^{\ell+1}}$

$u^m = (1+\ell\rho)u^{m-\ell} - \ell\rho u^{m-\ell-1}$

Also $u^{\ell} = \ell\rho - u - u^2 - \ldots - u^{\ell-1}$

Using the above relations and the initial equations,

$$P_{\ell,0} = B_1 u^{\ell-1}$$

$$P_{\ell-1,0} = B_1 u^{\ell-2} - \frac{B_1}{\ell\rho}(u^{\ell-1})$$

$$P_{\ell-2,0} = B_1 u^{\ell-3} - \frac{B_1}{\ell\rho}(u^{\ell-1}+u^{\ell-2})$$

$$P_{\ell-3,0} = B_1 u^{\ell-4} - \frac{B_1}{\ell\rho}(u^{\ell-2}+u^{\ell-2}+u^{\ell-3})$$

$$\begin{array}{ccc} \vdots & \vdots & \vdots \end{array}$$

$$P_{2,0} = B_1 u^1 - \frac{B_1}{\ell\rho}(u^{\ell-1}+u^{\ell-2}+u^{\ell-s}+\ldots+u^2)$$

$$P_{1,0} = B_1 u^0 - \frac{B_1}{\ell\rho}(u^{\ell-1}+u^{\ell-2}+u^{\ell-3}+\ldots+u^2+u)$$

Adding $P_0 = B_1 \sum_{m=0}^{\ell-1} u^m - \frac{B_1}{\ell\rho}[(\ell-1)u^{\ell-1}+(\ell-2)u^{\ell-2}+\ldots+2u^2+u)$

$$= B_1 \sum_{m=0}^{\ell-1} u^m - \frac{B_1}{\ell\rho} \sum_{m=1}^{\ell-1} mu^m$$

$$= B_1[\frac{1-u^\ell}{1-u}] - \frac{B_1 u}{\ell\rho}[\frac{1-\ell u^{\ell-1}+(\ell-1)u^\ell}{(1-u)^2}]$$

$$= B_1\frac{(1-\rho)u^\ell}{\rho(1-u)}$$

$$P_n = \sum_{s=1}^{\ell} P_{s,n} = B_1\frac{(1-u^\ell)u^{n\ell}}{(1-u)}$$

$$\sum_{n=0}^{\infty} P_n = 1 = P_o + \sum_{n=1}^{\infty} P_n = B_1\frac{(1-\rho)u^\ell}{\rho(1-u)} + \frac{B_1 u^\ell}{1-u} = \frac{B_1 u^\ell}{\rho(1-u)}$$

Hence $P_0 = 1 - \rho$; $P_n = \rho(1-u^\ell)u^{(n-1)\ell}$

$$L = \sum_{n=0}^{\infty} nP_n = \frac{\rho}{(1-u^\ell)}$$

$$Lq = \sum_{n=1} (n-1)P_n = \frac{\rho u^\ell}{(1-u^\ell)}$$

Hyperexponential Interarrival Time Distribution

Units from a depot, with an infinite population, pass through a channel, with two branches, to join a queue. The service time distribution is exponential. The units enter one branch σ of the time. Its period in this branch is exponentially distributed with a mean of $\frac{1}{2\sigma\lambda}$. They enter the other branch $(1-\sigma)$ of the time. Its period in this branch is exponentially distributed with a mean of $\frac{1}{2(1-\sigma)\lambda}$. No units are to enter the channel if one of the branches is occupied.

As has previously been shown, the result of using this mechanism is a hyperexponential interarrival time distribution.

Let $P_{1,n}$ be the probability of the state where a unit is in branch 1 and there are n units in the queue and service facility, and $P_{2,n}$ be the probability of the state where a unit is a branch 2 and there are n units in queue and service facility.

In the time interval t to $t+\Delta t$ where $\Delta t \to 0$

Initial equations;

$$P_{1,0}(t+\Delta t) = [1-(2\sigma\lambda)\Delta t]P_{1,0}(t) + \mu\Delta t P_{1,1}(t) \qquad .....1$$

$$P_{2,0}(t+\Delta t) = [1-2(1-\sigma)\lambda\Delta t]P_{2,0}(t) + \mu\Delta t P_{2,1}(t) \qquad .....2$$

Queue equations;

$$P_{1,n}(t+\Delta t) = \sigma(2\sigma\lambda)\Delta t P_{1,n-1}(t) + 2\sigma(1-\sigma)\lambda\Delta t P_{2,n-1}(t) + \mu\Delta t P_{1,n+1}(t) + [1-(\mu+2\sigma\lambda)\Delta t]P_{1,n} \qquad .....3$$

$$P_{2,n}(t+\Delta t) = (1-\sigma)(2\sigma\lambda)\Delta + P_{1,n-1}(t) + (1-\sigma^2)2\lambda\Delta t P_{2,n-1}(t) +$$

$$\mu\Delta t P_{2,n+1}(t) + [1-\{2(1-\sigma)\lambda+\mu\}\Delta t]P_{2,n}(t) \quad \ldots\ldots 4$$

Therefore $$\frac{dP_{1,0}}{dt} = \mu P_{1,1} - 2\sigma\lambda P_{1,0} \quad \ldots\ldots 5$$

$$\frac{dP_{2,0}}{dt} = \mu P_{2,1} - 2(1-\sigma)\lambda P_{2,0} \quad \ldots\ldots 6$$

$$\frac{dP_{1,n}}{dt} = 2\sigma^2\lambda P_{1,n-1} + 2\sigma(1-\sigma)\lambda P_{2,n-1} +$$

$$+ \mu P_{1,n+1} - (\mu+2\sigma\lambda)P_{1,n} \quad \ldots\ldots 7$$

$$\frac{dP_{2,n}}{dt} = 2\sigma(1-\sigma)\lambda P_{1,n-1} + 2(1-\sigma)^2\lambda P_{2,n-1} +$$

$$+ \mu P_{2,n+1} - [\mu+2(1-\sigma)\lambda]P_{2,n} \quad \ldots\ldots 8$$

Since the queue is in statistical equilibrium or steady state, the left hand sides of equations (5) to (8) are zero.

$$\mu P_{1,1} = 2\sigma\lambda P_{1,0} = 0 \quad \ldots\ldots 9$$

$$\mu P_{2,1} = 2(1-\sigma)\lambda P_{2,0} = 0 \quad \ldots\ldots 10$$

$$2\sigma^2\lambda P_{1,n-1} + 2\sigma(1-\sigma)\lambda P_{2,n-1} + \mu P_{1,n+1} - (\mu+2\sigma\lambda)P_{1,n} = 0 \ldots 11$$

$$2\sigma(1-\sigma)\lambda P_{1,n-1} + 2(1-\sigma)^2\lambda P_{2,n-1} + \mu P_{2,n+1} -$$

$$- [\mu+2(1-\sigma)\lambda]P_{2,n} = 0 \quad \ldots\ldots 12$$

Let $P_{s,n} = B_s\omega^{n-1}$

Substituting in

(11) $$2\sigma^2\lambda\omega^{n-2}B_1 + 2\sigma(1-\sigma)\lambda\omega^{n-2}B_2 + \mu\omega^n B_1 - (\mu+2\sigma\lambda)\omega^{n-1}B_1 = 0$$

$$\ldots\ldots 13$$

$$(12) \quad 2\sigma(1-\sigma)\lambda\omega^{n-2}B_1+2(1-\sigma)^2\lambda\omega^{n-2}B_2 + \mu\omega^n B_2 - [\mu+2(1-\sigma)\lambda]\omega^{n-1}B_2 = 0$$

.....14

and the secular equation from (13) and (14) is

$$\omega(\omega-1)[\omega^2-(1+2\rho)\omega+2^{\rho}-4\sigma(1-\sigma)\rho(1-\rho)] = 0$$

The roots $\omega = 0$ and $\omega = 1$ are discarded.

The other roots are

$$\omega = \frac{1}{2} + \rho \pm \sqrt{[\tfrac{1}{4} - (2\sigma-1)^2\rho(1-\rho)]}$$

There is only one root which is greater than 0 and less than 1 .

This is $\omega = \frac{1}{2} + \rho - \sqrt{[\tfrac{1}{4} - (2\sigma-1)^2\rho(1-\rho)]}$

Discard the other root which is greater than one.

Substituting in (11) and (12),

we obtain $B_2 = \frac{\omega(1-\omega)+2\sigma\rho(\omega-\sigma)}{2\sigma(1-\sigma)}B_1 = \frac{1-\sigma+2\sigma\rho-\omega}{\sigma}B_1$ 15

Substituting (15) and $P_{s,n} = B_s\omega^{n-1}$ in

(11) $P_{1,n} = B_1\omega^{n-1}$

(12) $P_{2,n} = B_2\omega^{n-2}$

Substituting $P_{s,n} = B_s\omega^{n-1}$ in

(9) $P_{1,0} = [\frac{B_1}{2\sigma\rho}]$

(10) $P_{2,0} = [\frac{B_2}{2(1-\sigma)\rho}]$

$P_0 = P_{1,0} + P_{2,0}$

$$= \frac{B_1}{2\sigma\rho} + \frac{B_2}{2(1-\sigma)\rho}$$

$$= \frac{B_1(1-\sigma)+B_2\sigma}{2\sigma(1-\sigma)\rho} = [\frac{2-2\sigma+2\sigma\rho-\omega}{2\sigma(1-\sigma)\rho}]B_1$$

$$P_n = P_{1,n} + P_{2,n}$$

$$= (B_1+B_2)\omega^{n-1}$$

$$= [\frac{1+2\sigma\rho-\omega}{\sigma}]B_1\omega^{n-1}$$

$$P_0 + P_1 + P_2 + \dots + P_n + \dots + P_\infty = 1$$

i.e. $P_0 + \sum_{n=1}^{\infty} P_n = 1$

$$[\frac{2-2\sigma+2\sigma\rho-\omega}{2\sigma(1-\sigma)\rho}]B_1 + [\frac{1+2\sigma\rho-\omega}{\sigma}]B_1 \sum_{n=0} \omega^n = 1$$

Since $\sum_{n=0}^{\infty} \omega^n = \frac{1}{1-\omega}$ and $(1-\omega)(2\rho-\omega) = 4\sigma(1-\sigma)\rho(1-\rho)$

Therefore $B_1 = \frac{2\sigma(1-\sigma)\rho(1-\rho)}{(2-2\sigma+2\sigma\rho-\omega)}$

Hence $P_0 = 1 - \rho$

and $P_n = \frac{(1+2\sigma\rho-\omega)(1-\omega)(2\rho-\omega)}{2\sigma(2-2\sigma+2\sigma\rho-\omega)}\omega^{n-1}$

$$= (1-\omega)\rho\omega^{n-1}$$

$$L = \sum_{n=0}^{\infty} nP_n = \sum_{n=0}^{\infty} nP_n = \frac{\rho}{1-\omega}$$

$$Lq = \sum_{n=1}^{\infty} (n-1)P_n = \frac{\rho\omega}{1-\omega}$$

It should be noted from the derivation of the hyperexponential that $j = [1/2\sigma(1-\sigma)] - 1$

or $\sigma = \frac{1}{2} - \sqrt{[\frac{1}{4} - \frac{1}{2(1+j)}]}$

Construction of Tables and Graphs for Single Channel with Arbitrary Interarrival Time Distribution and Exponential Service Time Distribution

Since the average waiting length of a queue with arbitrary interarrival time distribution and exponential service time distribution is only dependent on the utilization factor, ρ, and the fractional coefficient of variance squared, C_a^2, the plot will have an ordinate in term of the fractional coefficient of variance squared and the abscissca will be the average waiting length, for varying values of utilization factor.

With the aid of the University of British Columbia I.B.M. 7040, L was computed for varying ρ and ℓ for the Erlangian interarrival time distributions and ρ and j for the hyperexponential interarrival time distributions.
For the Erlangian distributions, the formulae

$$L = \frac{\rho}{(1-u^{\ell})}$$

$$C_a^2 = \frac{1}{\ell}$$

where $0 \leq u < 1$ in $u^{\ell} + u^{\ell-1} + \ldots + u^2 + u = \rho\ell$
and for the hyperexponential distributions, the formulae

$$L = \frac{\rho}{(1-\omega)}$$

$$C_a^2 = j$$

where $0 < \omega < 1$ in $\omega = \frac{1}{2} + \rho - \sqrt{[\frac{1}{4} - (2\sigma-1)^2\rho(1-\rho)]}$

$$\text{and } \sigma = \frac{1}{2} - \sqrt{[\frac{1}{4} - \frac{1}{2(1+j)}]}$$

were used to compute L. Table II and Figure 7 give the computed values of L against C_a^2 for various values of ρ.

It is not necessary to provide graphs for the other characteristics as the relationship between L and Lq, W, Wq and the busy period can be easily computed.

$Lq = (L-\rho)$

$W = \frac{L}{\lambda}$

$Wq = \frac{(L-\rho)}{\lambda}$

Busy period = utilization factor = ρ

c_a^2

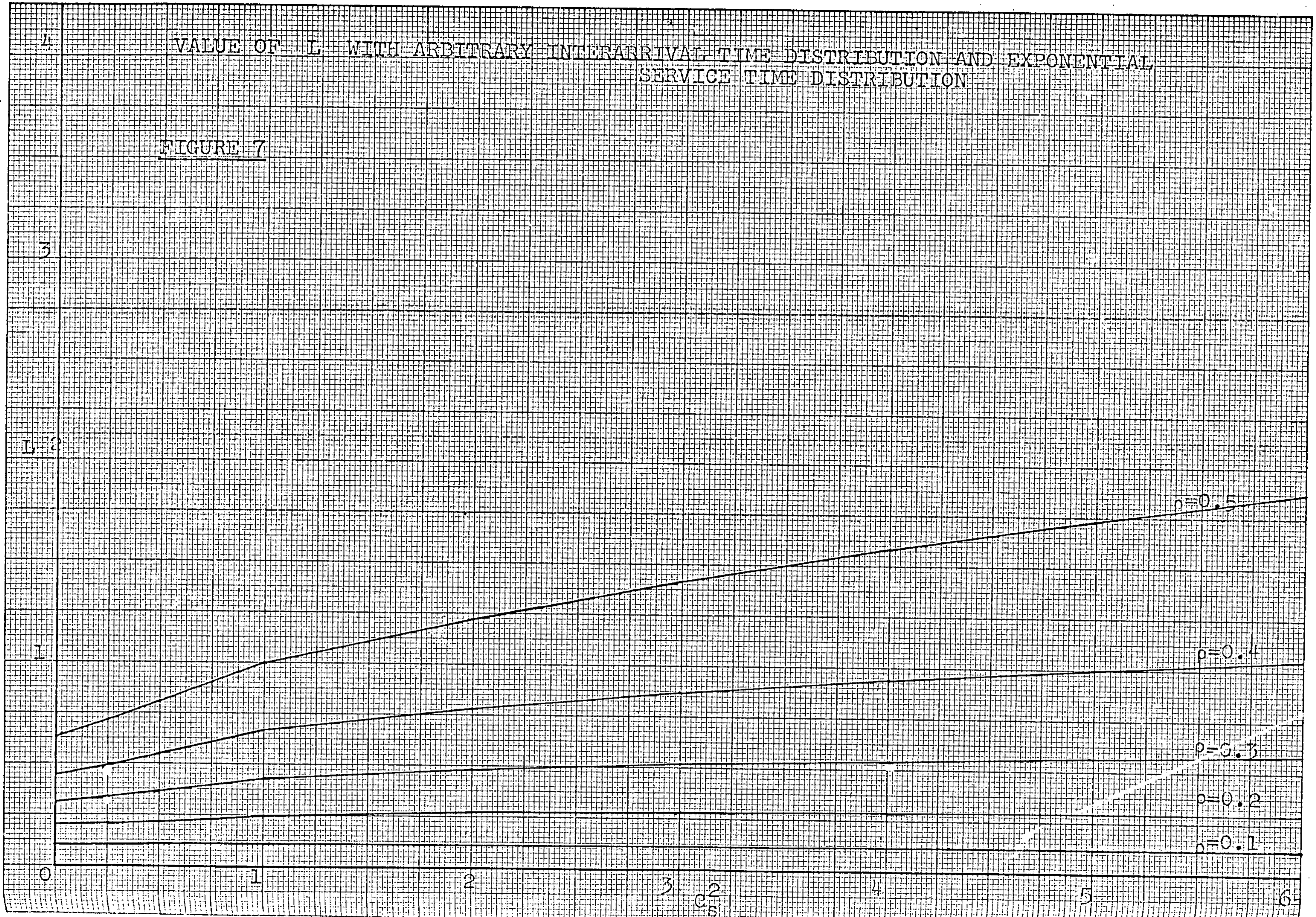
ρ	0	1/20	1/10	1/5	1/4	1/3	1/2
0.05		0.050000	0.050001	0.050016	0.050039	0.050112	0.050423
0.10		0.100030	0.100098	0.100419	0.100684	0.101282	0.103006
0.15		0.150485	0.150936	0.152299	0.153193	0.154952	0.159300
0.20	0.201409	0.202450	0.203758	0.207050	0.208982	0.212547	0.220696
0.25		0.257390	0.260045	0.266148	0.260524	0.275535	0.288675
0.30	0.312793	0.316926	0.321409	0.331210	0.336448	0.345576	0.364984
0.35		0.382884	0.389692	0.404137	0.411694	0.424684	0.451800
0.40	0.448129	0.457463	0.467149	0.487312	0.497714	0.515430	0.551956
0.45		0.543486	0.556699	0.583866	0.597749	0.621244	0.669264
0.50	0.627410	0.644771	0.662321	0.698098	0.716262	0.746863	0.809017
0.55		0.766733	0.789668	0.836146	0.859635	0.899078	0.978831
0.60	0.887837	0.917412	0.947151	1.007165	1.037394	1.088038	1.190100
0.65		1.109375	1.147924	1.225490	1.264465	1.329653	1.460707
0.70	1.313074	1.363507	1.413863	1.514969	1.565685	1.650409	1.820436
0.75		1.717353	1.784292	1.918494	1.985729	2.097951	2.322876
0.80	2.153432	2.245948	2.337815	2.521805	2.613907	2.767538	3.075184
0.85		3.124321	3.257793	3.524927	3.658572	3.881411	4.327373
0.90	4.651163	4.877497	5.094249	5.527877	5.744745	6.106251	6.829455
0.95		10.13049	10.59720	11.53067	11.99744	12.77540	14.33144

TABLE III A Values of L for Arbitrary Interarrival Time Distribution

and Exponential Service Time Distribution

ρ	c_a^2 = 1	2	3	4	5	6	7
0.05	0.052632	0.053538	0.054016	0.054310	0.054510	0.054655	0.054764
0.10	0.111111	0.115069	0.117265	0.118664	0.119633	0.120345	0.120890
0.15	0.176471	0.186215	0.191922	0.195686	0.198360	0.200358	0.201910
0.20	0.250000	0.269008	0.280776	0.288839	0.294727	0.299225	0.302776
0.25	0.333333	0.366025	0.387426	0.402700	0.414421	0.423232	0.430501
0.30	0.428571	0.480566	0.516539	0.543344	0.564268	0.581139	0.595075
0.35	0.538462	0.616922	0.674217	0.718837	0.754991	0.785098	0.810681
0.40	0.666667	0.780776	0.868517	0.939902	1.000000	1.051785	1.097168
0.45	0.818182	0.979821	1.110243	1.220867	1.317532	1.403700	1.481617
0.50	1.000000	1.224745	1.414214	1.581139	1.732051	1.870829	2.000000
0.55	1.222222	1.530892	1.801408	2.047726	2.276984	2.493411	2.699754
0.60	1.500000	1.921165	2.302776	2.659852	3.000000	3.327677	3.645751
0.65	1.857143	2.431426	2.966403	3.477840	3.973555	4.458039	4.934122
0.70	2.333333	3.121320	3.871924	4.601136	5.316624	6.022657	6.721841
0.75	3.000000	4.098076	5.162278	6.208099	7.242641	8.269696	9.291503
0.80	4.000000	5.576034	7.123105	8.655354	10.17891	11.69690	13.21110
0.85	5.666667	8.055216	10.42089	12.77555	15.12404	17.46870	19.81082
0.90	9.000000	13.03562	17.05538	21.06797	25.07669	29.08309	33.08799
0.95	18.99999	28.01722	37.02628	46.03187	55.03566	64.03839	73.04044

TABLE III B Values of L for Arbitrary Interarrival Time Distribution and Exponential Service Time Distribution



VALUE OF L WITH ARBITRARY INTERARRIVAL TIME DISTRIBUTION AND EXPONENTIAL SERVICE TIME DISTRIBUTION

FIGURE 7

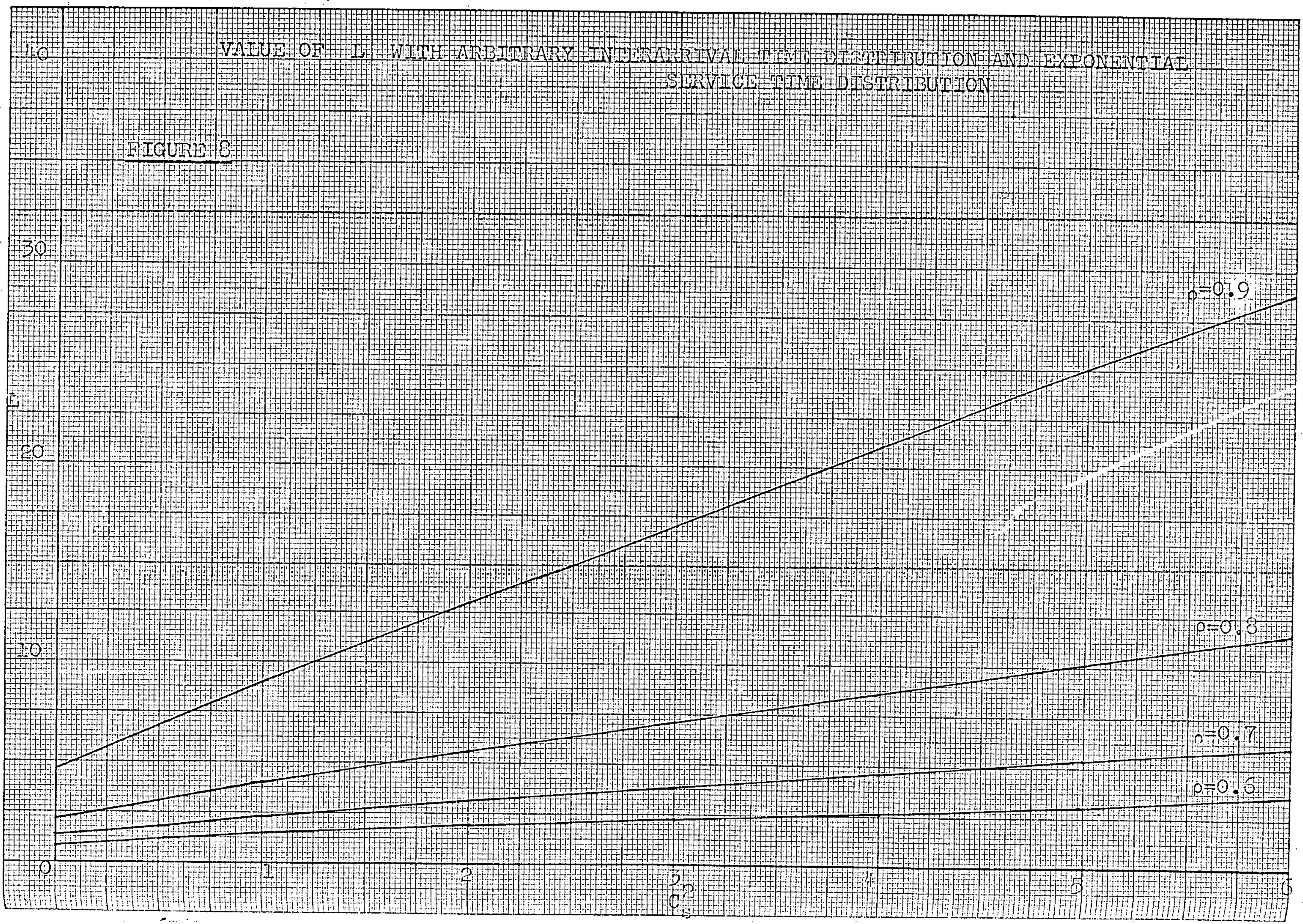


FIGURE 8

VALUE OF L WITH ARBITRARY INTERARRIVAL TIME DISTRIBUTION AND EXPONENTIAL SERVICE TIME DISTRIBUTION

Testing of Graph

The validity of the graph will now be tested with various arbitrary interarrival time distributions, and exponential service time distribution with the aid of the University of British Columbia I.B.M. 7040 computer. A ρ of 0.8 were used for the different combinations of arbitrary interarrival time distribution and service time distribution. The program G.P.S.S. III was used to simulate the queue. A run of ten thousand transactions were simulated for each of the combination. The flow chart for the program is as shown in Appendix III.

The distributions used and the results from G.P.S.S. III and from the graph are shown in Table IV.

Appendix I/x : The distribution as given in Appendix I, section x , e.g. Appendix I/a is the distribution as depicted in Appendix I, section a .

ρ : The utilization factor used.

ρc : The utilization of the facility as obtained from G.P.S.S. II

Lqc : The average length of queue as obtained from G.P.S.S. III.

Lc : The average waiting length as obtained from G.P.S.S. III = Lqc + ρc .

$L\rho$: The average waiting length in the system as obtained from graph with utilization factor = 0.8 = ρ .

$L\rho c$: The average waiting length in the system as obtained from graph with utilization factor = ρc .

	Interarrival time distribution	C_a^2	Service time distribution	C_s^2	ρ	ρc	Lqc	Lc	$L\rho$	$L\rho$	$\lvert L\rho c - Lc \rvert$
(1)	Appendix I/a	0.4046	Exponential	1	0.8	0.8330	3.39	4.22	2.90	3.68	.54
(2)	Appendix I/b	1	Exponential	1	0.8	0.7412	2.47	3.21	4.00	3.52	.31
(3)	Appendix I/c	5.8	Exponential	1	0.8	0.7054	7.33	8.04	11.40	7.47	.57
(4)	Appendix I/d	1.6	Exponential	1	0.8	0.7420	3.39	4.13	4.85	3.55	.58
(5)	Appendix I/e	0.3333	Exponential	1	0.8	0.8087	2.61	3.42	2.77	2.99	.43
(6)	Exponential	1	Exponential	1	0.8	0.8032	3.68	4.48	4.00	4.17	.31

TABLE IV Test Results for Arbitrary Interarrival Time Distributions and Exponential Service Time Distributions

The following factors will account for the difference between Lc , $L\rho$ and $L\rho c$

a) The C_a^2 of the actual run of 10,000 transactions is not equal to the C_a^2 of an infinite number of runs as used in the construction of the graph.

b) ρc and ρ are not equal.

c) The C_s^2 of the actual run of 10,000 transactions is not equal to the C_s^2 of an infinite number of runs as used in the construction of the graph.

d) The accuracy of the random number generator.

It can be noted that in the case where the interarrival time and service time distributions are exponential, the difference of the simulated L and that of the computed L is 12.08 percent.

$$\text{Average } \rho c = (4.6335/6) = 0.7723.$$

Since average $\rho c < \rho$ it is probable that $\Sigma L\rho c < \Sigma Lc$ and $\Sigma Lc < \Sigma L\rho$ if the assumptions in the usage of the graph are correct. This is borne out by the facts: $\Sigma Lc = 27.50$, $\Sigma L\rho = 29.92$ and $\Sigma L\rho c = 25.38$, then $\Sigma L\rho c < \Sigma Lc < \Sigma L\rho$.

Furthermore we shall examine the differences between Lc and $L\rho c$. We use $L\rho c$ instead of $L\rho$ for comparison because ρc is the actual utilization factor of the simulation. The differences between Lc and $L\rho c$ for the six sets of simulation are shown in the final column of Table IV. The average difference is 0.46. When we use exponential interarrival and service time distributions and where we were sure of $L\rho c$ because it was calculated from the classical Erlang's formula there was a difference between Lc and $L\rho c$ of 0.31. Hence, the value of 0.46 average difference is not disturbing.

It is necessary for further works to be done on the testing of the graphs. This, unfortunately, is outside the scope of the limited time available in the presentation of this thesis.

CHAPTER 6

ARBITRARY INTERARRIVAL TIME DISTRIBUTION: ARBITRARY SERVICE TIME DISTRIBUTION

When interarrival time distribution is exponential and service time distribution is arbitrary, the average length in the system is a function of ρ and C_s^2 only as proved by the Pollaczek - Khintchine formula; and when interarrival time distribution is arbitrary and service time distribution is exponential, the average length in the system is a function of ρ and C_a^2 only as proved in Chapter 5. If, then, the interarrival time distribution is arbitrary and the service time distribution is arbitrary, is the average length of the system, L, a function of ρ, C_a^2 and C_s^2 only? Our purpose is to show, empirically, that it is.

The problem is to determine the value of L for each combination of C_a^2, C_s^2 and ρ. This problem is resolved through four separate techniques. These, related to the values of C_a^2 and C_s^2, are illustrated in figure 9 and explained below.

i) For Poisson distribution arrival rates and arbitrary service times $(C_a^2 = 1,\ C_s^2 \geq 0)$ the Pollaczek - Khiatchine approach of Chapter 4 can be used. Thus values of L may be taken directly from Table II.

ii) For arbitrary arrival rates and exponential distributed service times $(C_a^2 \geq 0,\ C_s^2 = 1)$

the work of Chapter 5 is pertinent. Values of L can be taken from Table III.

iii) Approaches i and ii are indicated in figure 5 by lines 1 and 2 respectively. For all other areas of figure 5, L must be determined by simulation, except for one point ($C_a^2 = 0$, $C_s^2 = 0$)

The sampling distributions must be hyper-exponential for values of C_a^2 or $C_s^2 > 1$; must be Erlangian for values of C_a^2 or $C_s^2 < 1$; and must be constant for values of C_a^2 or $C_s^2 = 0$. Thus the actual simulation technique depends on the particular combination of values of C_a^2 and C_s^2 being tested. The flow charts of these techniques are detailed in Appendix III and are referenced below to the relevant areas of the chart in figure 9.

Fig. 9 Area	C_a^2	C_s^2	Flow Chart
3	$C_a^2>1$	$C_s^2>1$	1
4	$0<C_a^2<1$	$C_s^2>1$	2
5	$C_a^2>1$	$0<C_s^2<1$	3
6	$C_a^2 = 0$	$C_s^2>1$	4
7	$C_a^2>1$	$C_s^2 = 0$	5
8	$0<C_a^2<1$	$0<C_s^2<1$	6
9	$C_a^2 = 0$	$0<C_s^2<1$	7
10	$0<C_a^2<1$	$C_s^2 = 0$	8

For $C_a^2 = 0$ and $C_s^2 = 0$, we know that $Lq = 0$. Since $L = Lq + \rho$, then $L = \rho$. This applies to point 11 in figure 9.

Each simulation was carried through 10,000 transactions and total waiting time in the queue (TWT) was accumulated. Thus for each set of values Wq, the average waiting time, could be determined. ($Wq = TWT \div 10{,}000$)

The value of L, average system length, was determined from the equation[1]

$$L = Wq\lambda + \rho$$

The results of these calculations and simulations are shown in a set of graphs, figures 10 - 29.

For each of 10 values of C_a^2, values of L are plotted against ten values of C_s^2 for values of ρ from 0.1 to 0.9. There are two graphs for each value of C_a^2 because the L scale for values of ρ between 0 and 0.5 was inadequate for higher ρ values.

From the plotted points, curves have been constructed by inspection representing each value of ρ. For values of ρ below 0.6 there appears to be a linear relationship between C_s^2 and L.

---

[1]Buffa, E.S., Models in Production and Operations Management, John Wiley and Sons, Inc., New York, 1966.

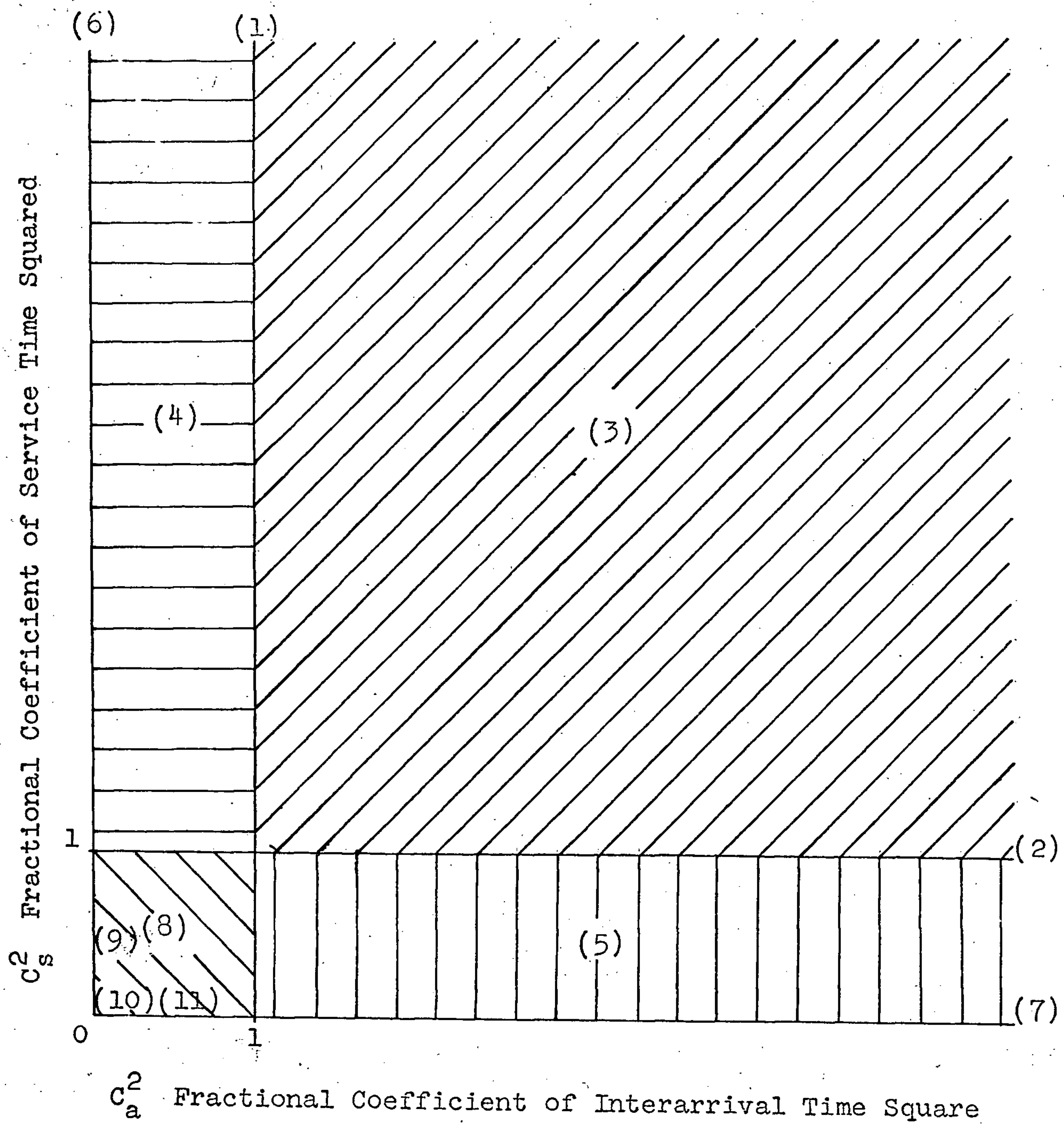


Figure 9

Illustration of Derivations for Arbitrary Interarrival and Service Time Distributions

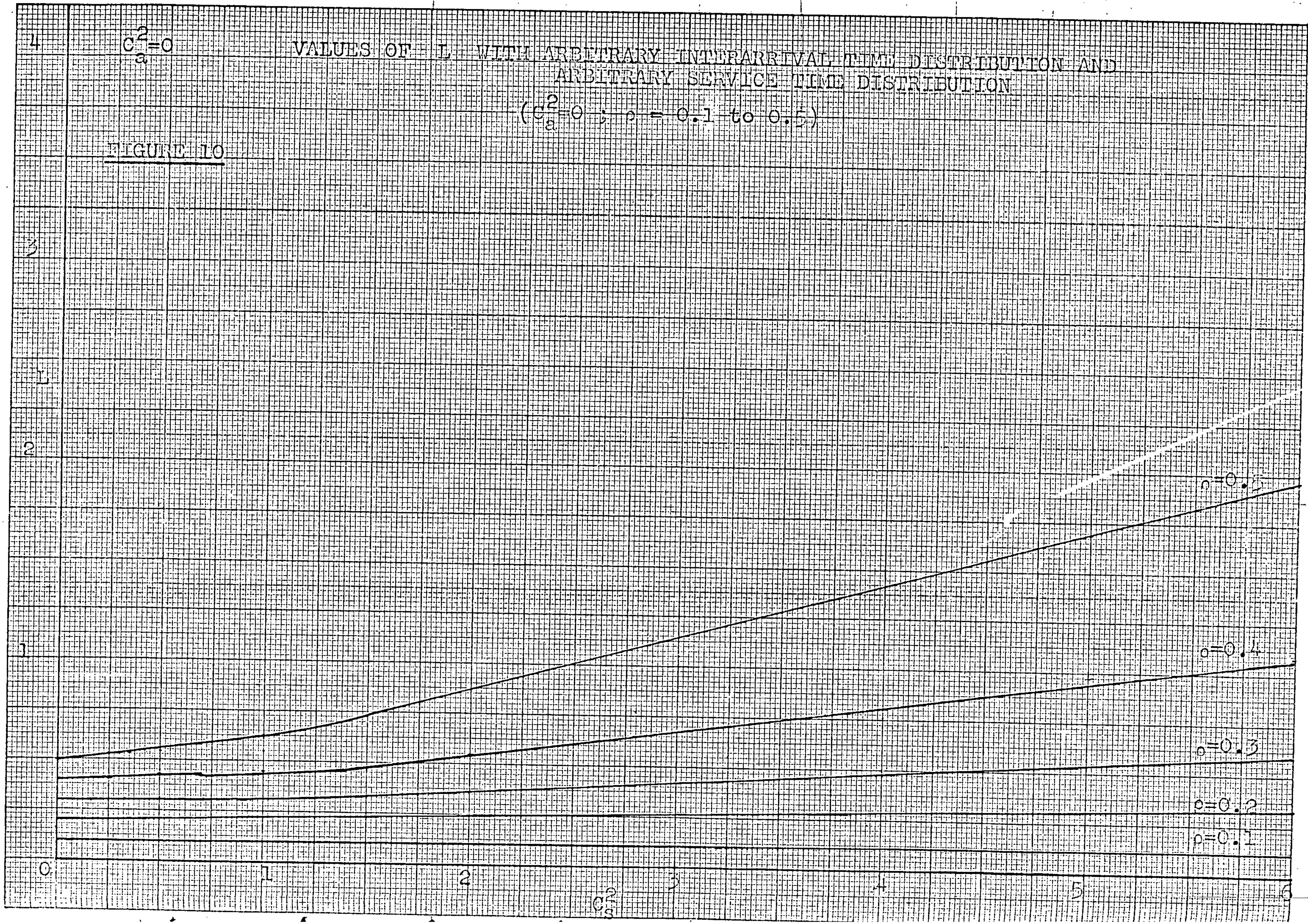


FIGURE 10

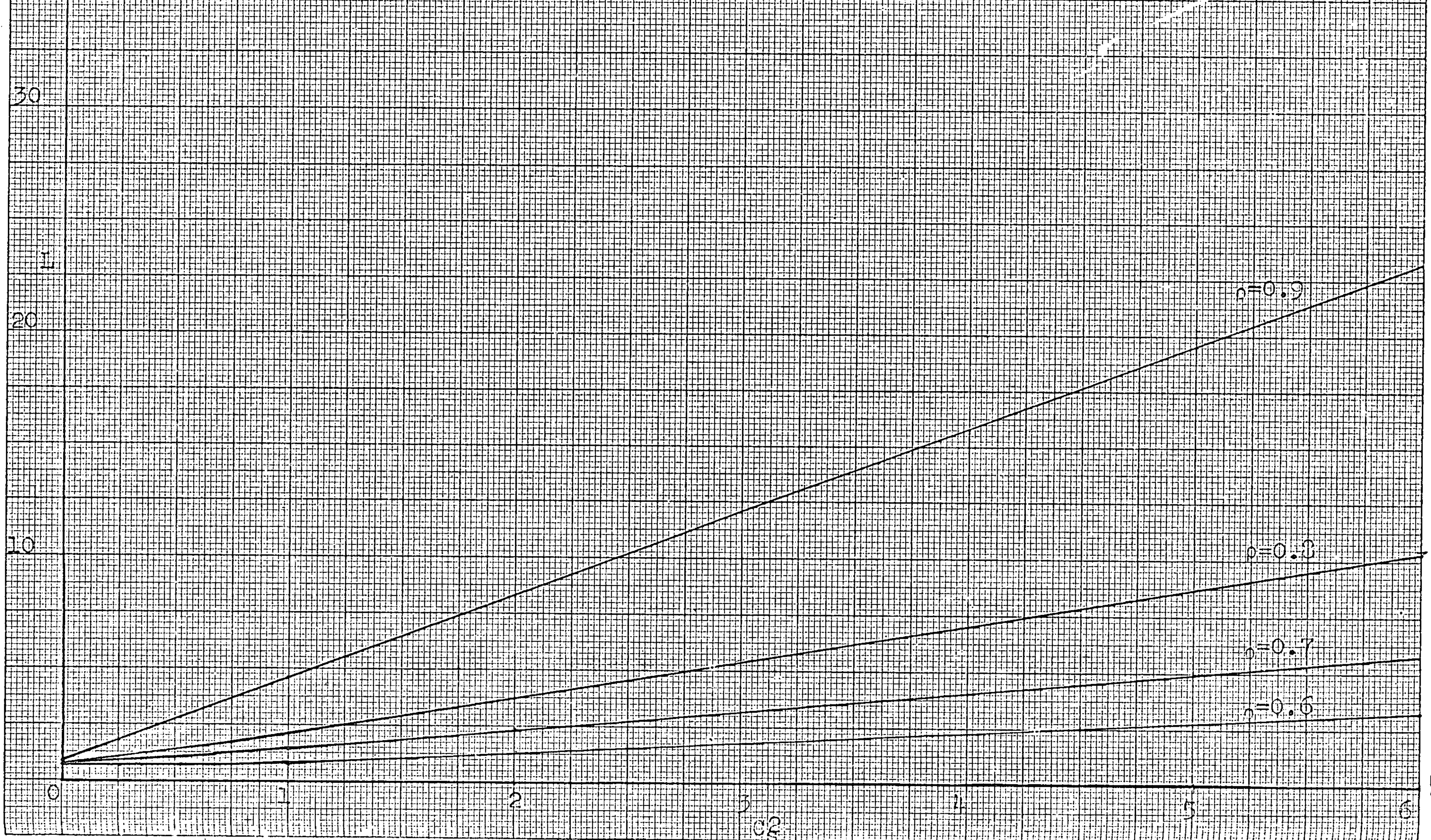
VALUES OF L WITH ARBITRARY INTERARRIVAL TIME DISTRIBUTION AND ARBITRARY SERVICE TIME DISTRIBUTION

($C_a^2=0$; ρ = 0.1 to 0.5)

FIGURE 11

VALUES OF L WITH ARBITRARY INTERARRIVAL TIME DISTRIBUTION AND ARBITRARY SERVICE TIME DISTRIBUTION

($c_a^2 = 0$; ρ = 0.6 to 0.9)



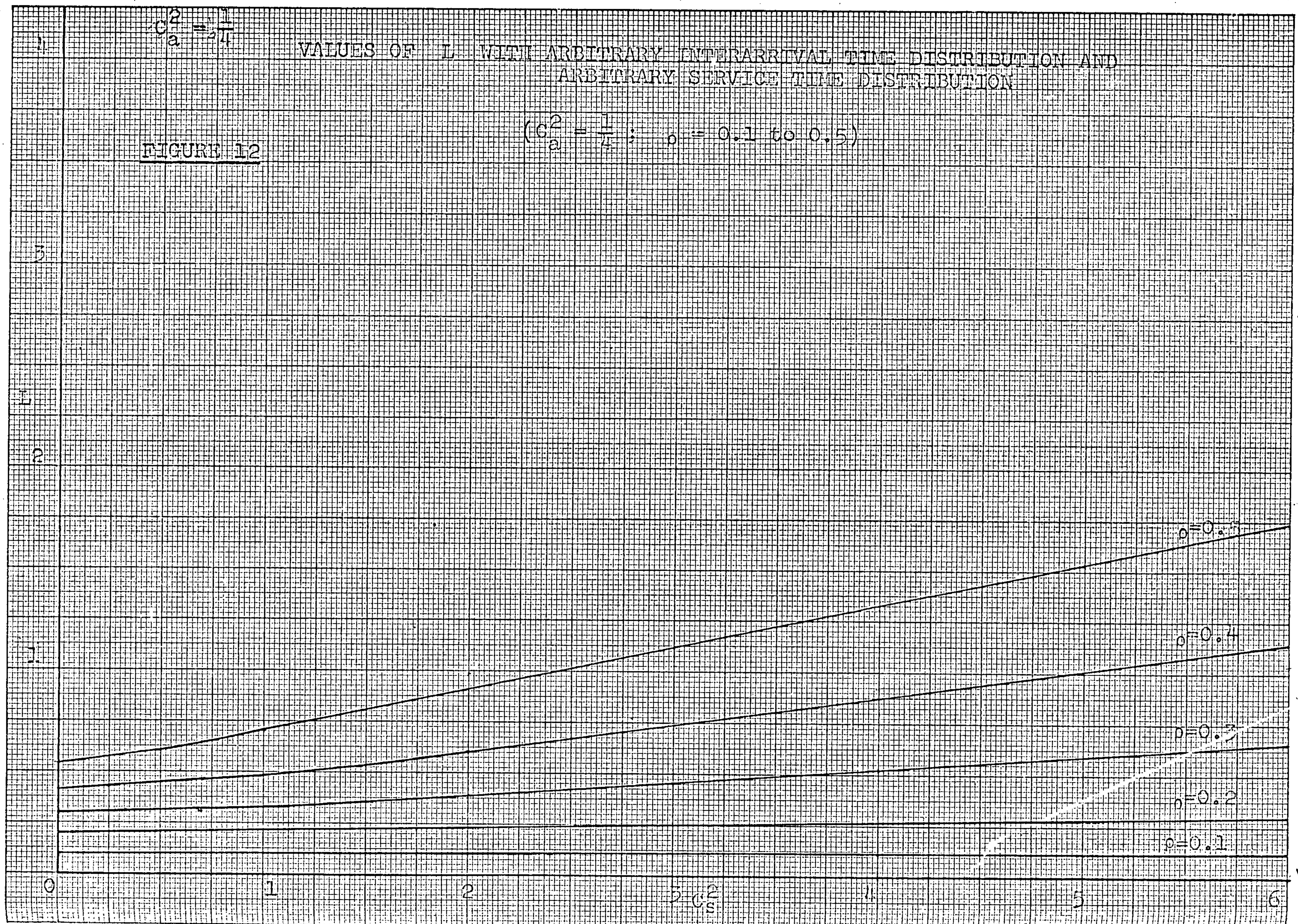


FIGURE 12

VALUES OF L WITH ARBITRARY INTERARRIVAL TIME DISTRIBUTION AND ARBITRARY SERVICE TIME DISTRIBUTION

$(C_a^2 = \frac{1}{4}; \ \rho = 0.1 \text{ to } 0.5)$

FIGURE 13

VALUES OF L WITH ARBITRARY INTERARRIVAL TIME DISTRIBUTION AND ARBITRARY SERVICE TIME DISTRIBUTION

$(C_a^2 = \frac{1}{4};\ \rho = 0.6 \text{ to } 0.9)$

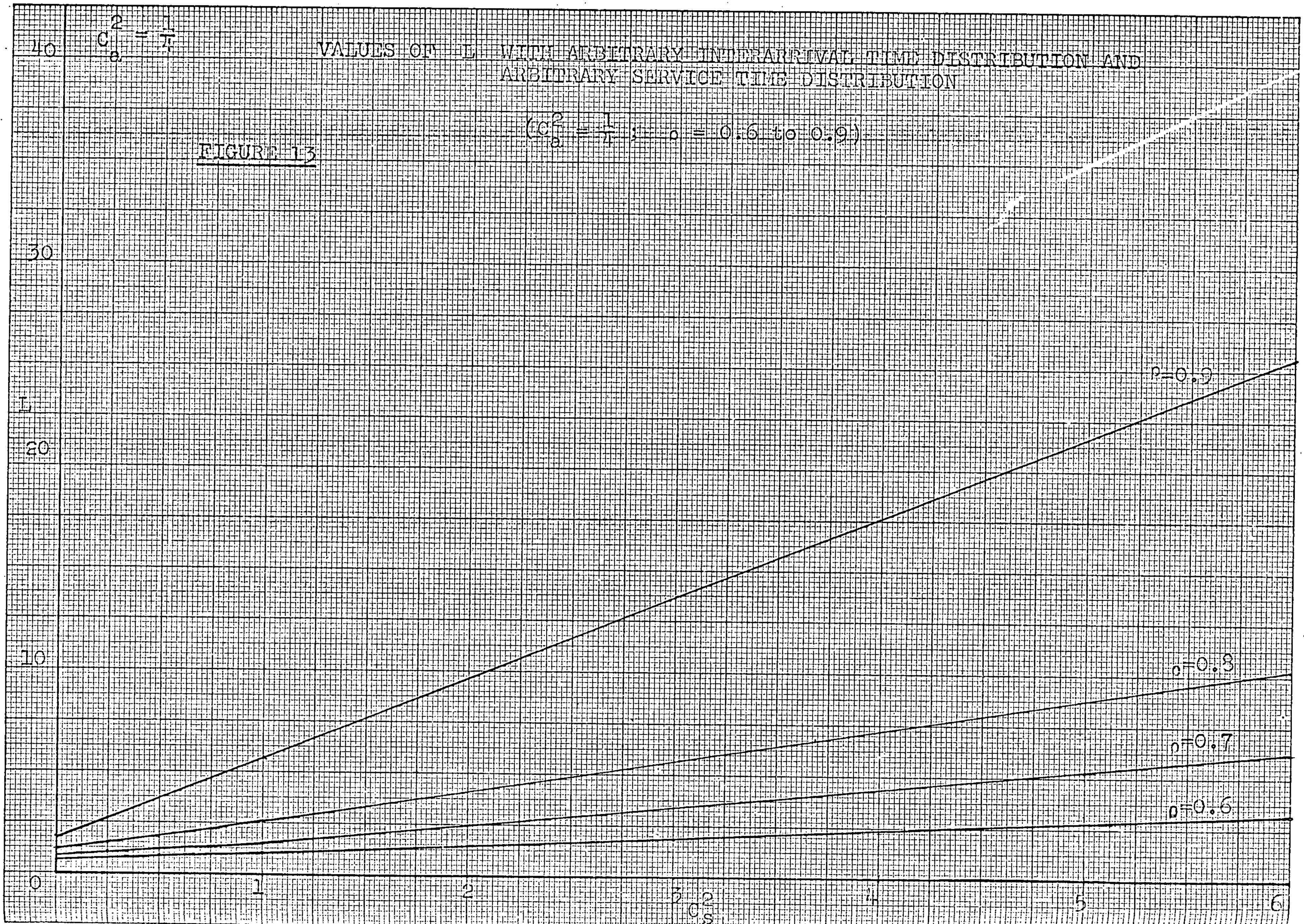
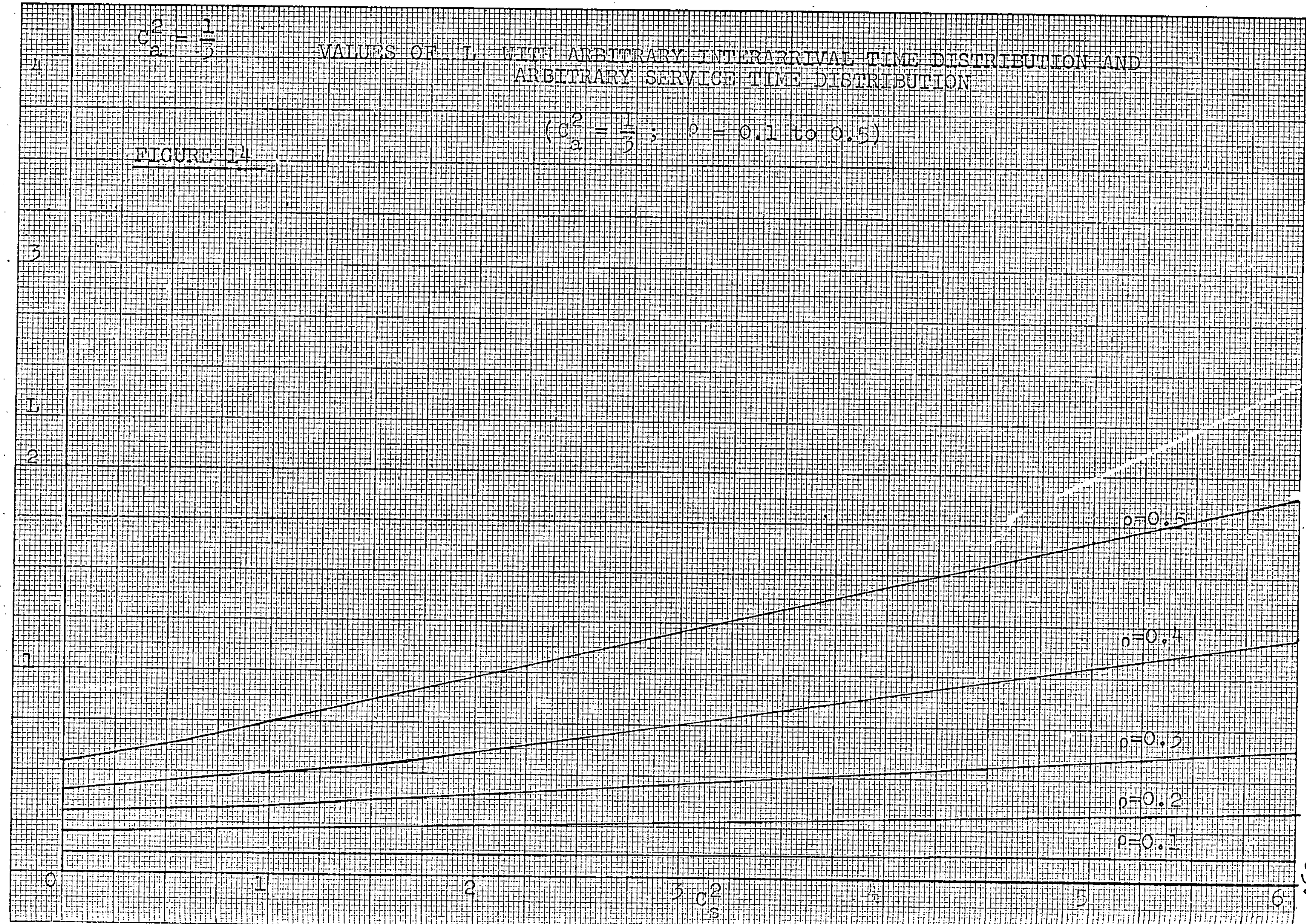
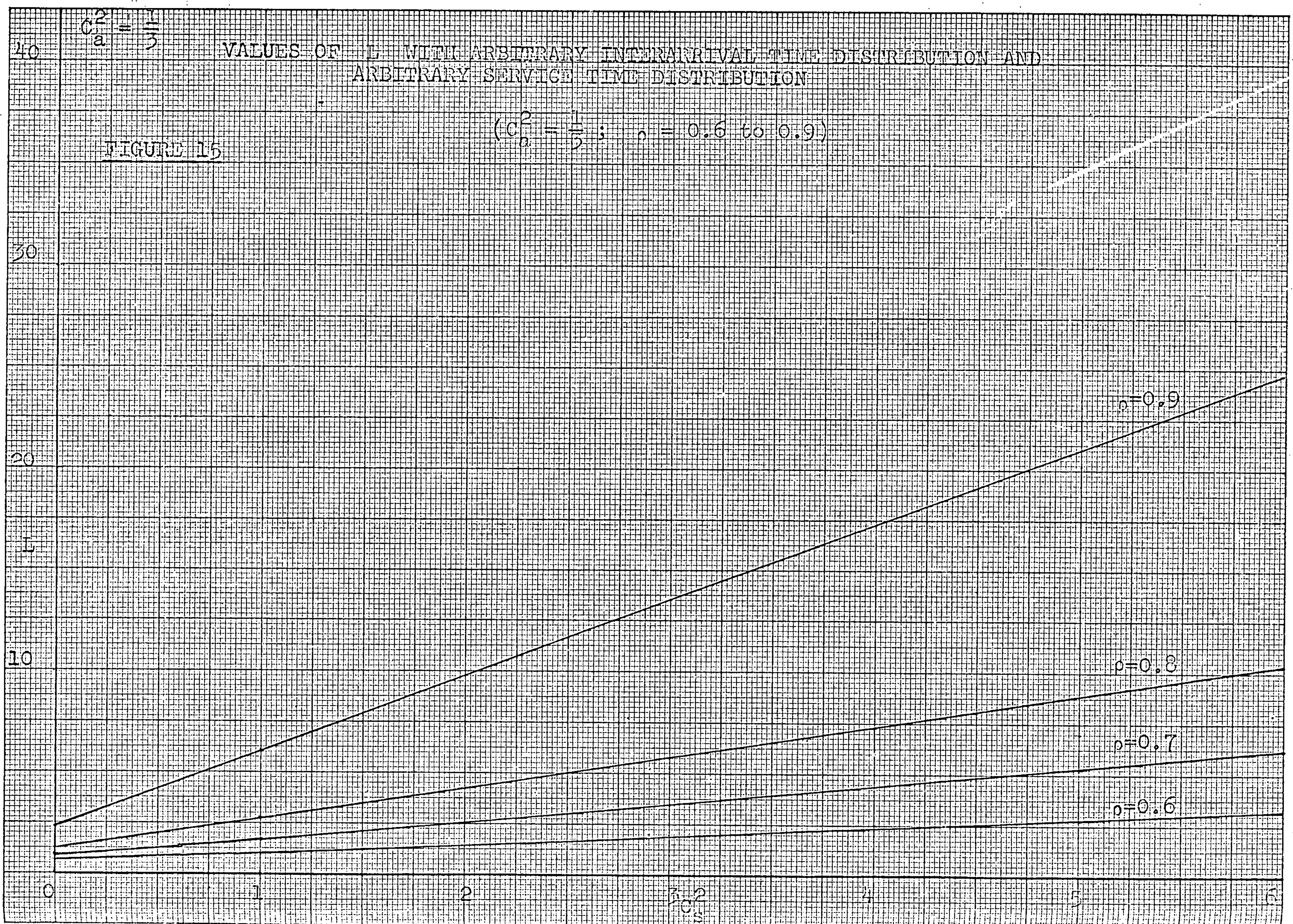


FIGURE 14

VALUES OF L WITH ARBITRARY INTERARRIVAL TIME DISTRIBUTION AND ARBITRARY SERVICE TIME DISTRIBUTION

$(C_a^2 = \frac{1}{3}\ ;\ \rho = 0.1 \text{ to } 0.5)$



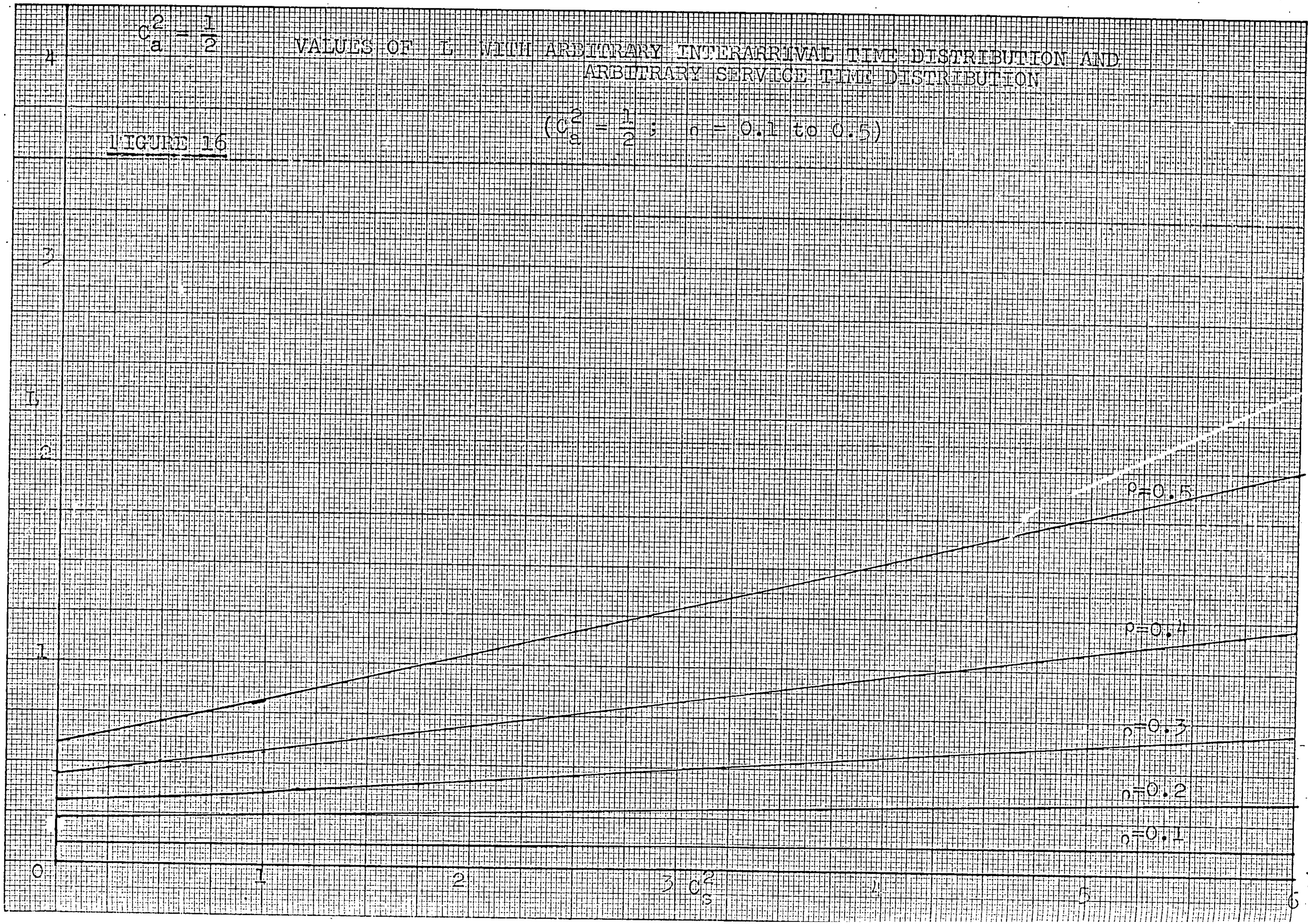


$C_a^2 = \frac{1}{3}$
VALUES OF L WITH ARBITRARY INTERARRIVAL TIME DISTRIBUTION AND ARBITRARY SERVICE TIME DISTRIBUTION
$(C_b^2 = \frac{1}{3}$; $\rho = 0.6$ to $0.9)$
FIGURE 15
40
30
20
L
10
$\rho=0.9$
$\rho=0.8$
$\rho=0.7$
$\rho=0.6$
0
1
2
3
4
5
6
C_s^2

FIGURE 16

VALUES OF L WITH ARBITRARY INTERARRIVAL TIME DISTRIBUTION AND ARBITRARY SERVICE TIME DISTRIBUTION

$(C_a^2 = \frac{1}{2}$; $\rho = 0.1$ to $0.5)$



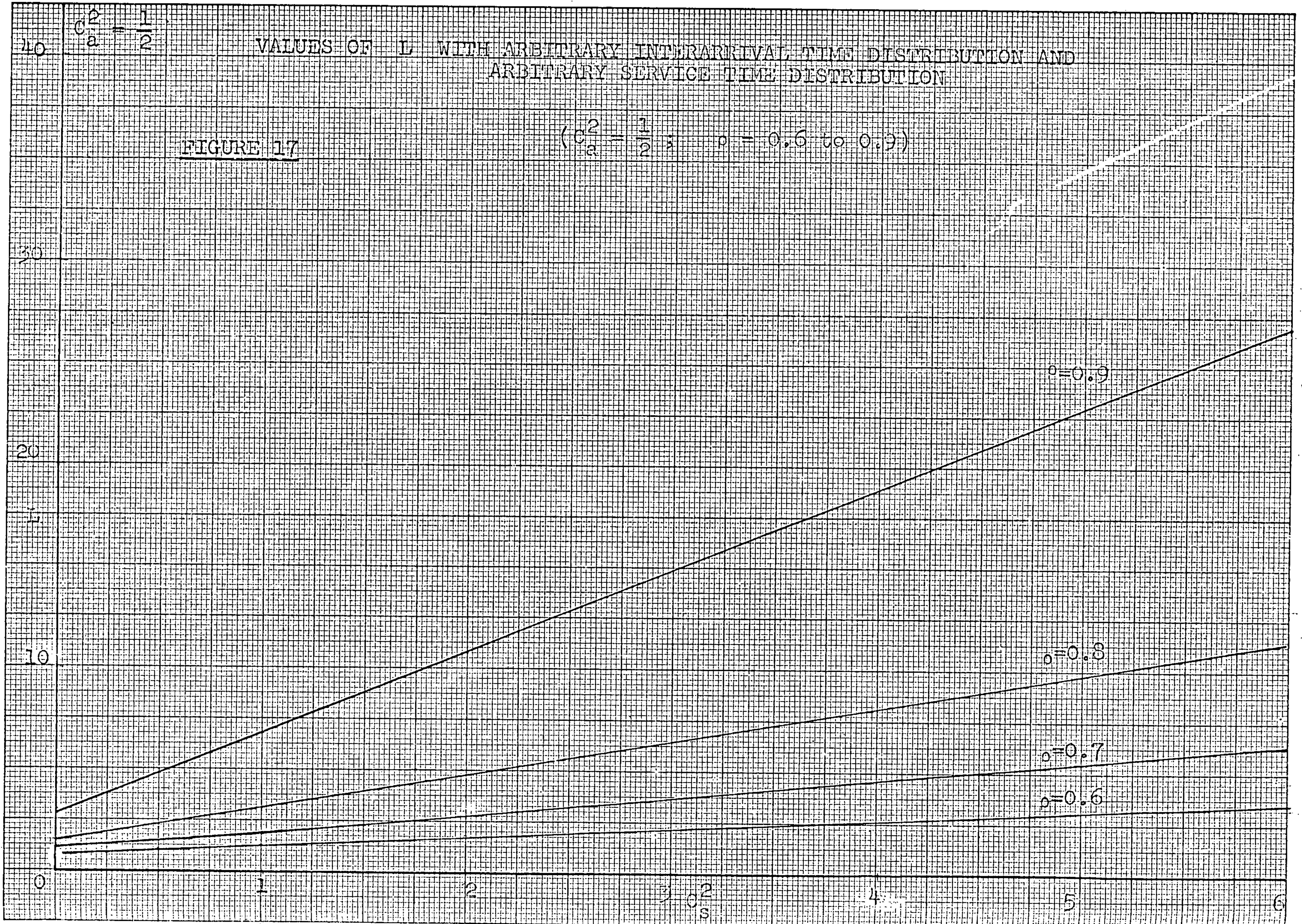


FIGURE 17

VALUES OF L WITH ARBITRARY INTERARRIVAL TIME DISTRIBUTION AND ARBITRARY SERVICE TIME DISTRIBUTION

$(C_a^2 = \frac{1}{2};\ \rho = 0.6 \text{ to } 0.9)$

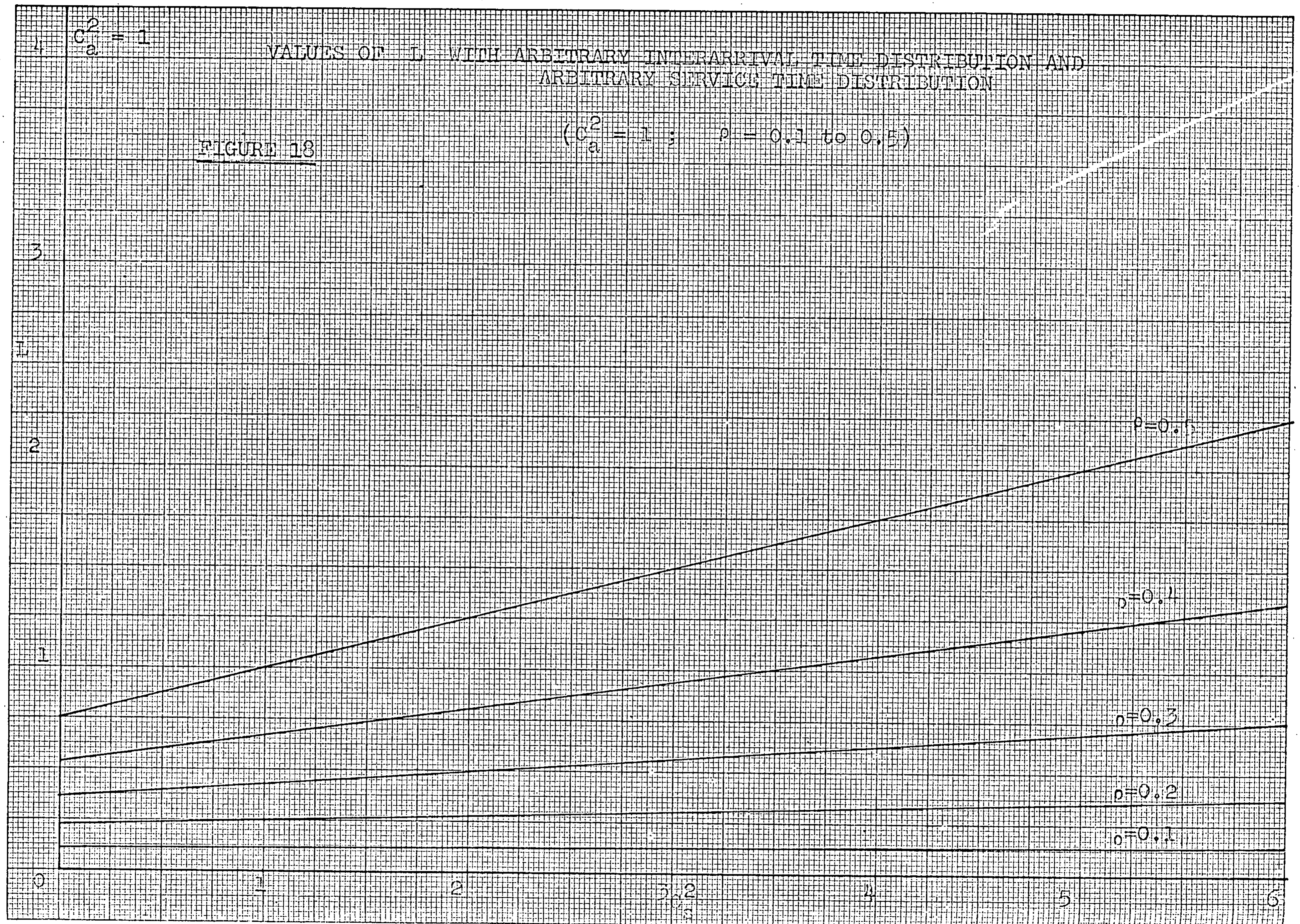
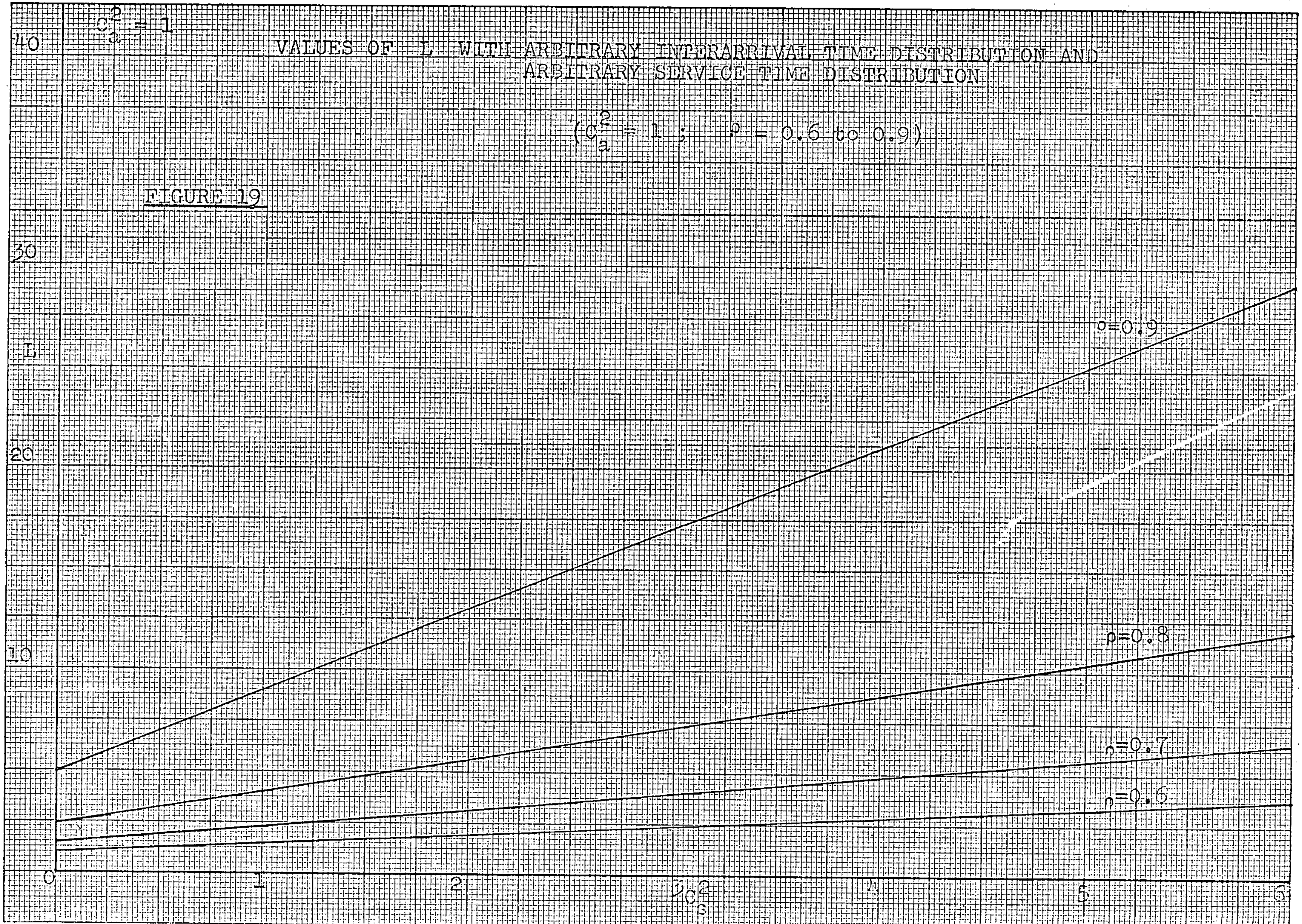
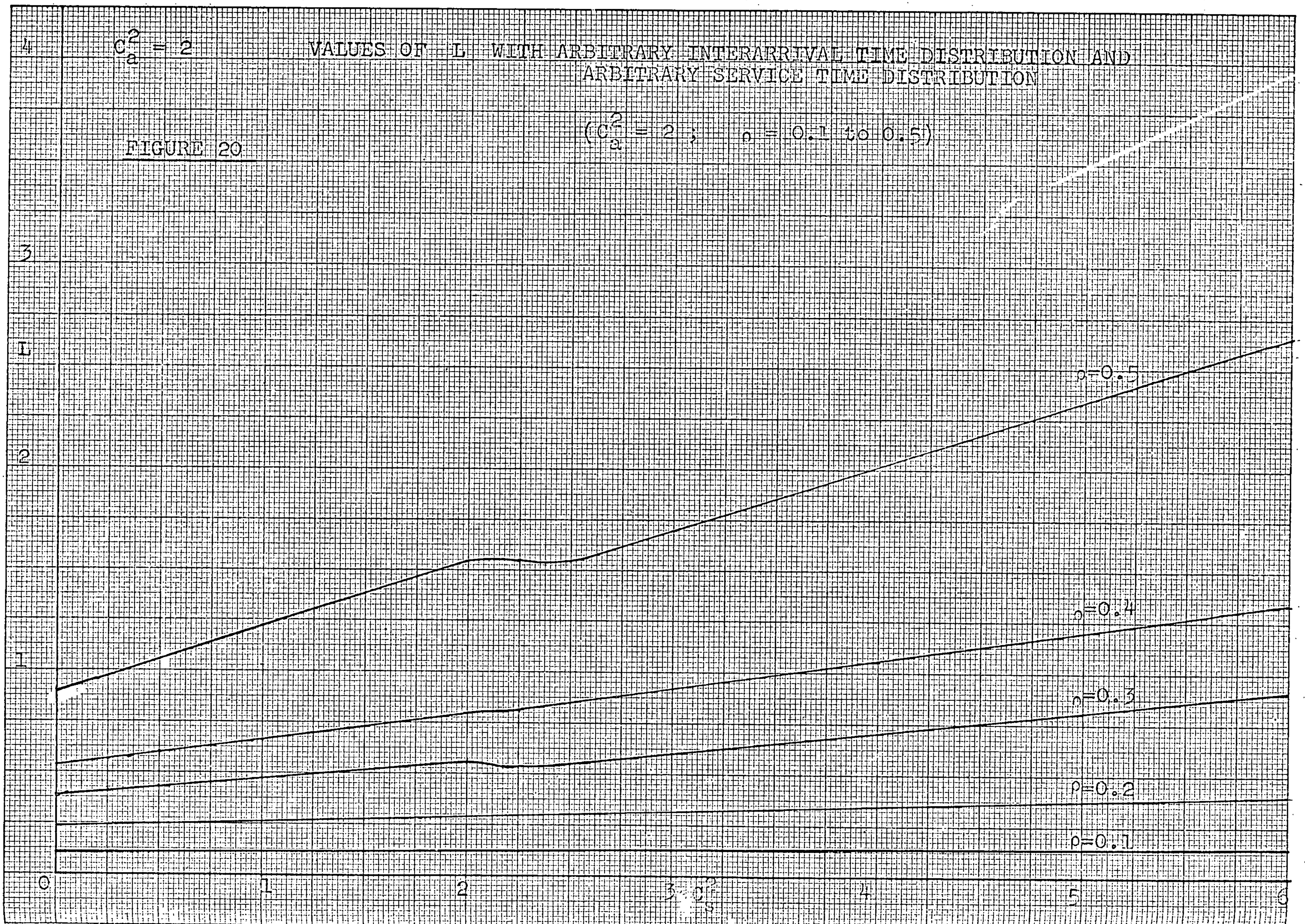


FIGURE 18

VALUES OF L WITH ARBITRARY INTERARRIVAL TIME DISTRIBUTION AND ARBITRARY SERVICE TIME DISTRIBUTION

($C_a^2 = 1$; $\rho = 0.1$ to 0.5)


$C_a^2 = 1$
VALUES OF L WITH ARBITRARY INTERARRIVAL TIME DISTRIBUTION AND ARBITRARY SERVICE TIME DISTRIBUTION
($C_a^2 = 1$; $\rho = 0.6$ to 0.9)
FIGURE 19
40
30
L
20
10
0
1
2
C_s^2
5
6
$\rho=0.9$
$\rho=0.8$
$\rho=0.7$
$\rho=0.6$


$c_a^2 = 2$
VALUES OF L WITH ARBITRARY INTERARRIVAL TIME DISTRIBUTION AND ARBITRARY SERVICE TIME DISTRIBUTION
FIGURE 20
$(c_a^2 = 2 ; \rho = 0.1 \text{ to } 0.5)$
L
$\rho = 0.5$
$\rho = 0.4$
$\rho = 0.3$
$\rho = 0.2$
$\rho = 0.1$
c_s^2
0
1
2
3
4
5
6

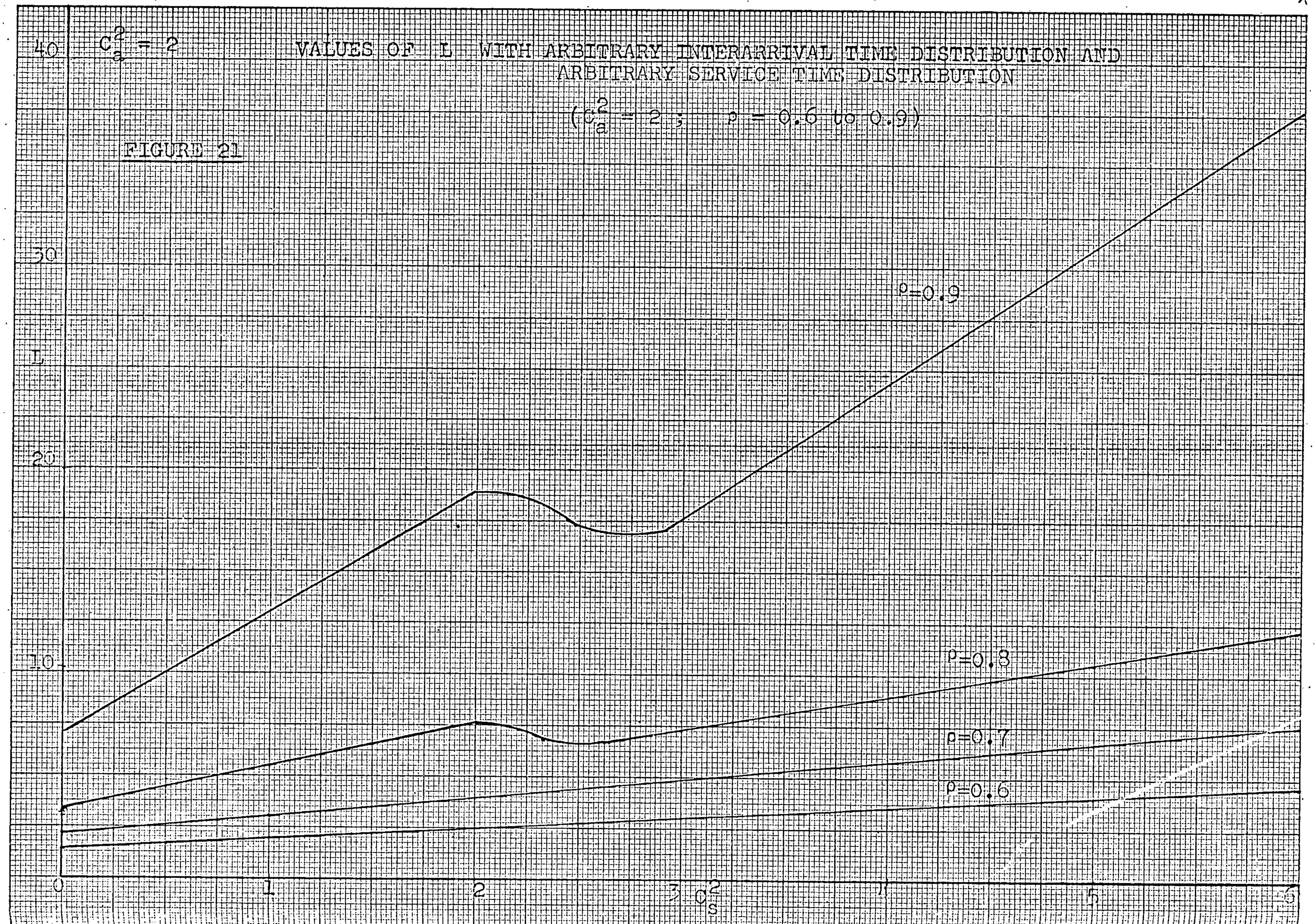


FIGURE 21

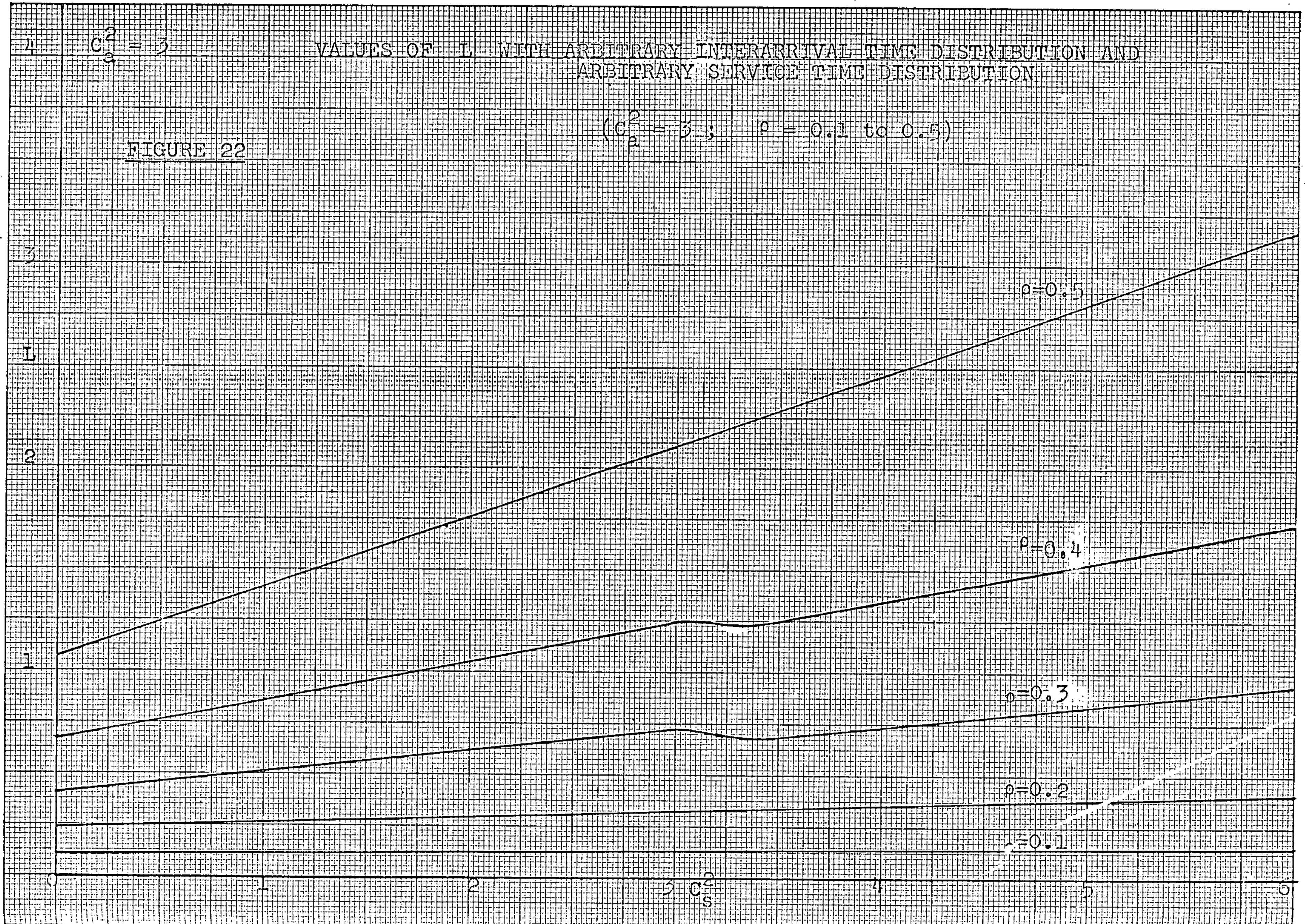
VALUES OF L WITH ARBITRARY INTERARRIVAL TIME DISTRIBUTION AND ARBITRARY SERVICE TIME DISTRIBUTION

($c_a^2 = 2$; ρ = 0.6 to 0.9)

FIGURE 22

VALUES OF L WITH ARBITRARY INTERARRIVAL TIME DISTRIBUTION AND ARBITRARY SERVICE TIME DISTRIBUTION

$(C_a^2 = 3;\ \rho = 0.1 \text{ to } 0.5)$



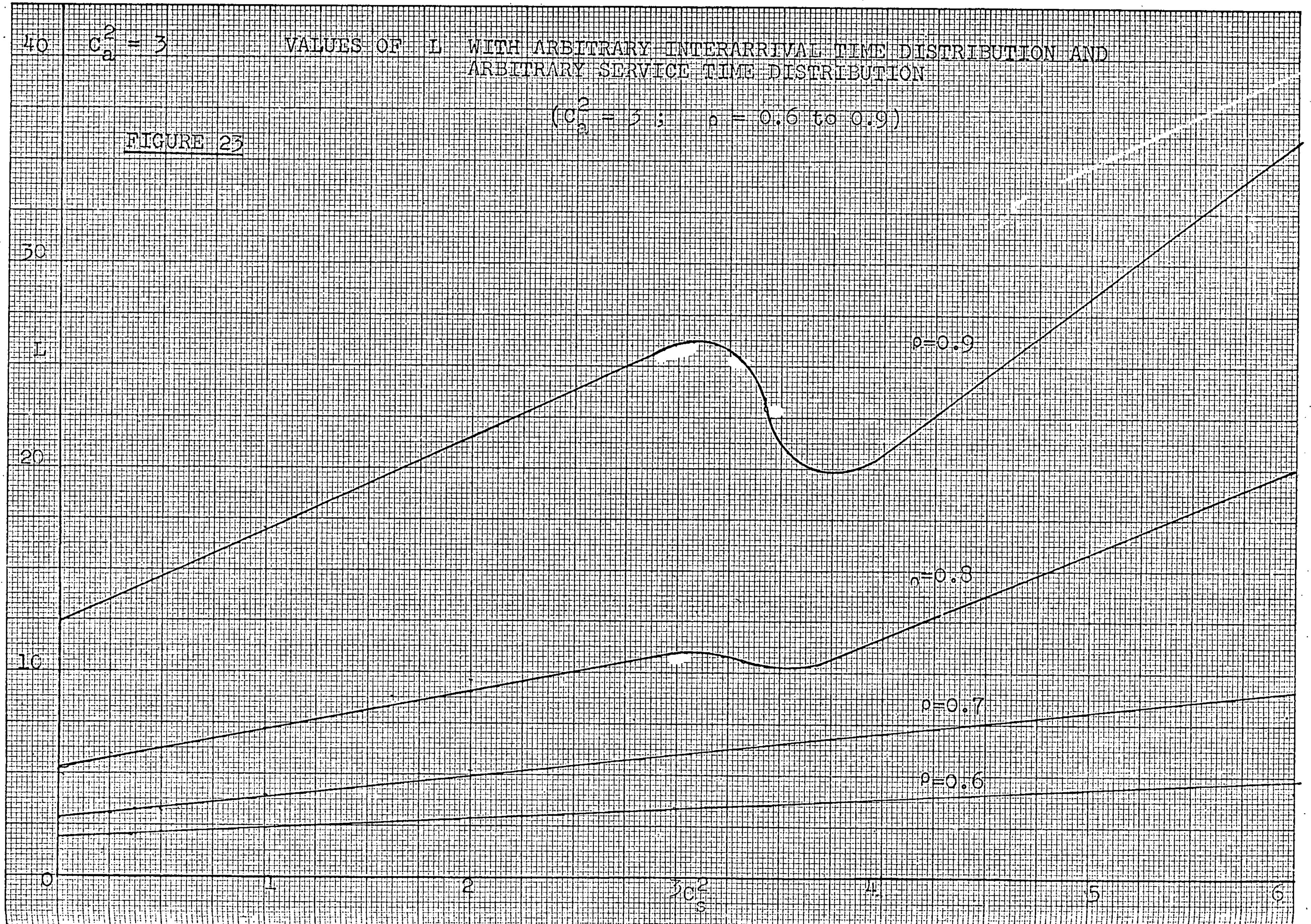


FIGURE 23

VALUES OF L WITH ARBITRARY INTERARRIVAL TIME DISTRIBUTION AND ARBITRARY SERVICE TIME DISTRIBUTION

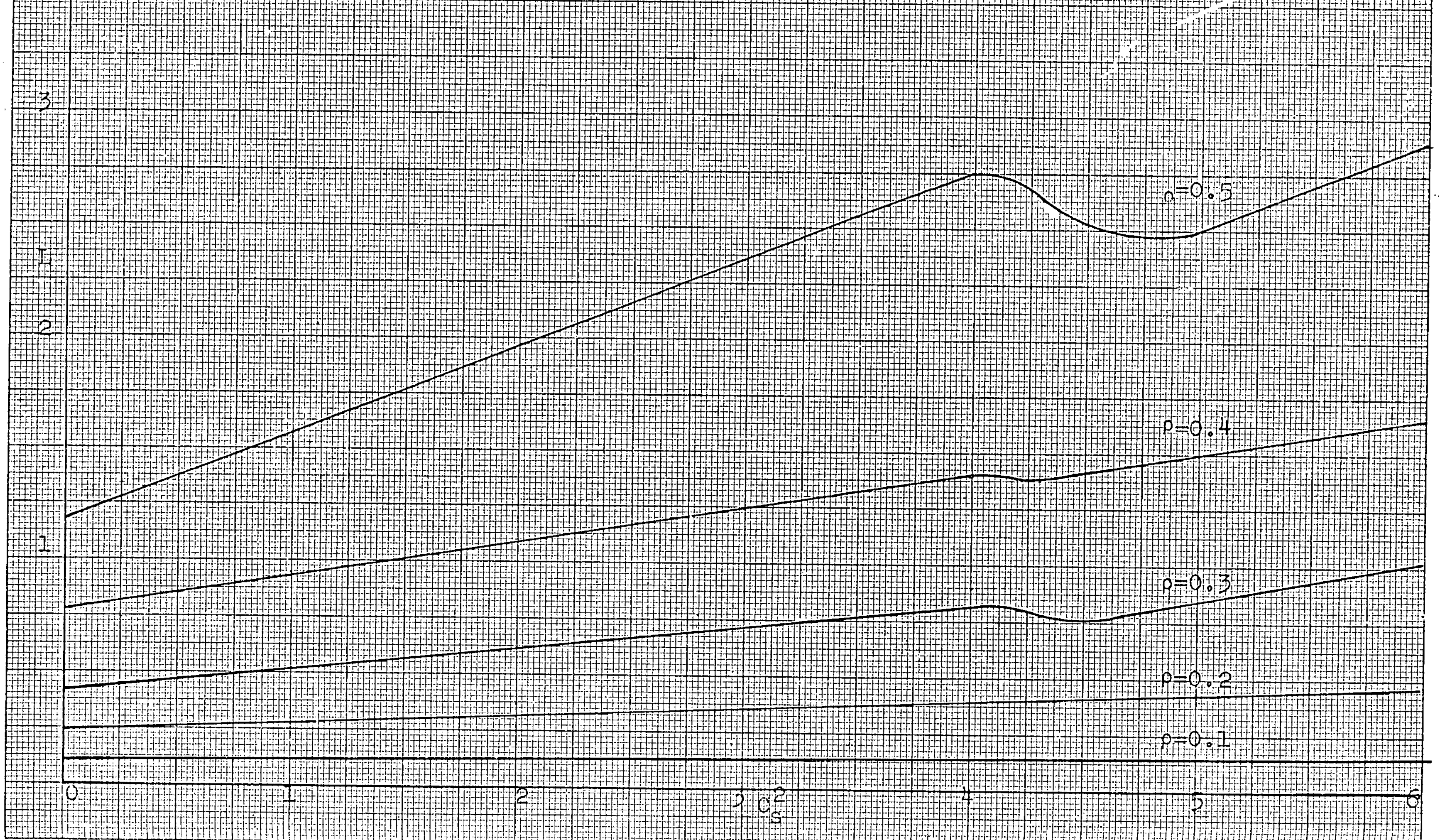
($C_a^2 = 3$; $\rho = 0.6$ to 0.9)

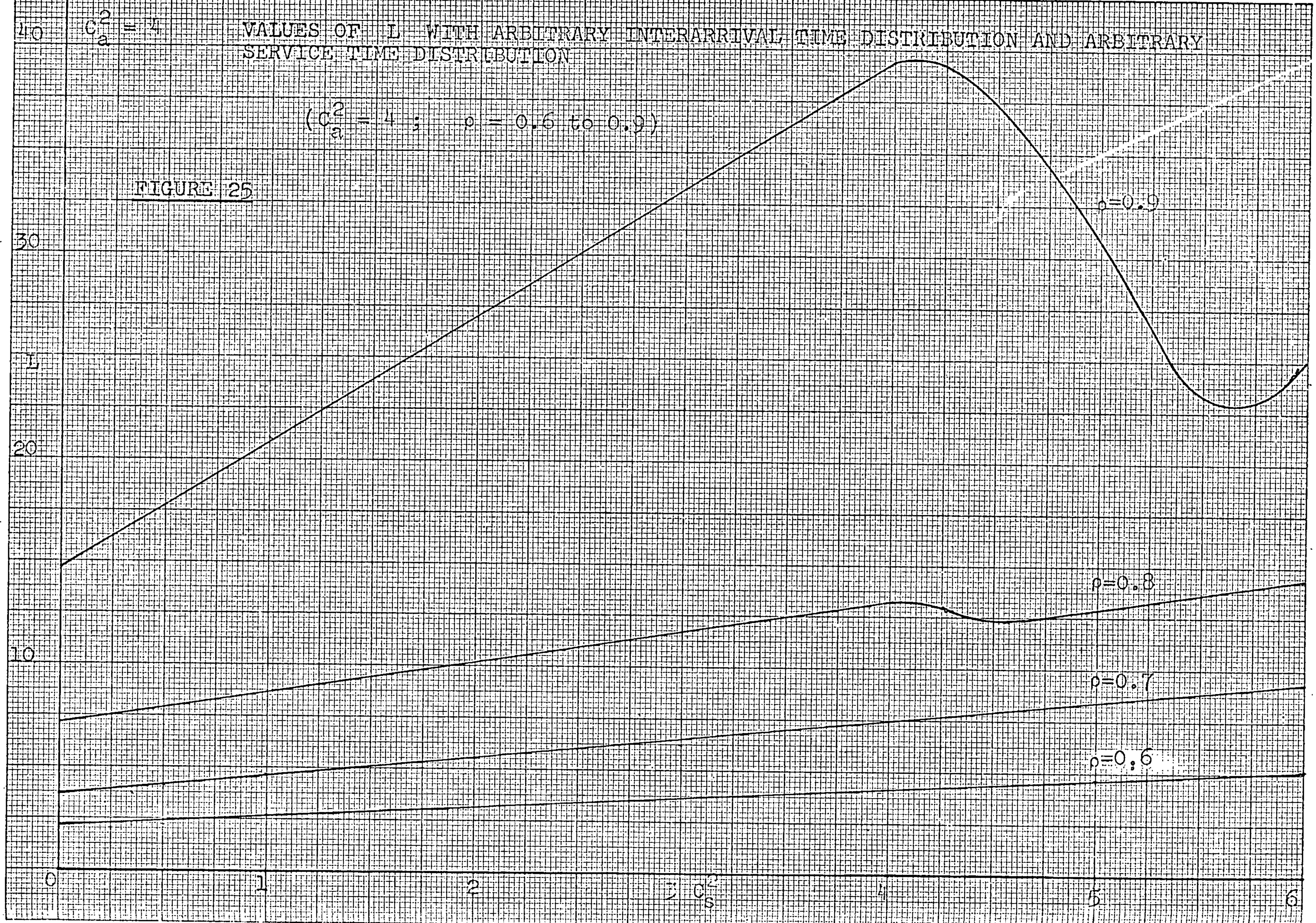
$C_a^2 = 4$

VALUES OF L WITH ARBITRARY INTERARRIVAL TIME DISTRIBUTION AND ARBITRARY SERVICE TIME DISTRIBUTION

$(C_a^2 = 4; \quad \rho = 0.1 \text{ to } 0.5)$

FIGURE 24





$C_a^2 = 4$

VALUES OF L WITH ARBITRARY INTERARRIVAL TIME DISTRIBUTION AND ARBITRARY SERVICE TIME DISTRIBUTION

($C_a^2 = 4$; ρ = 0.6 to 0.9)

FIGURE 25

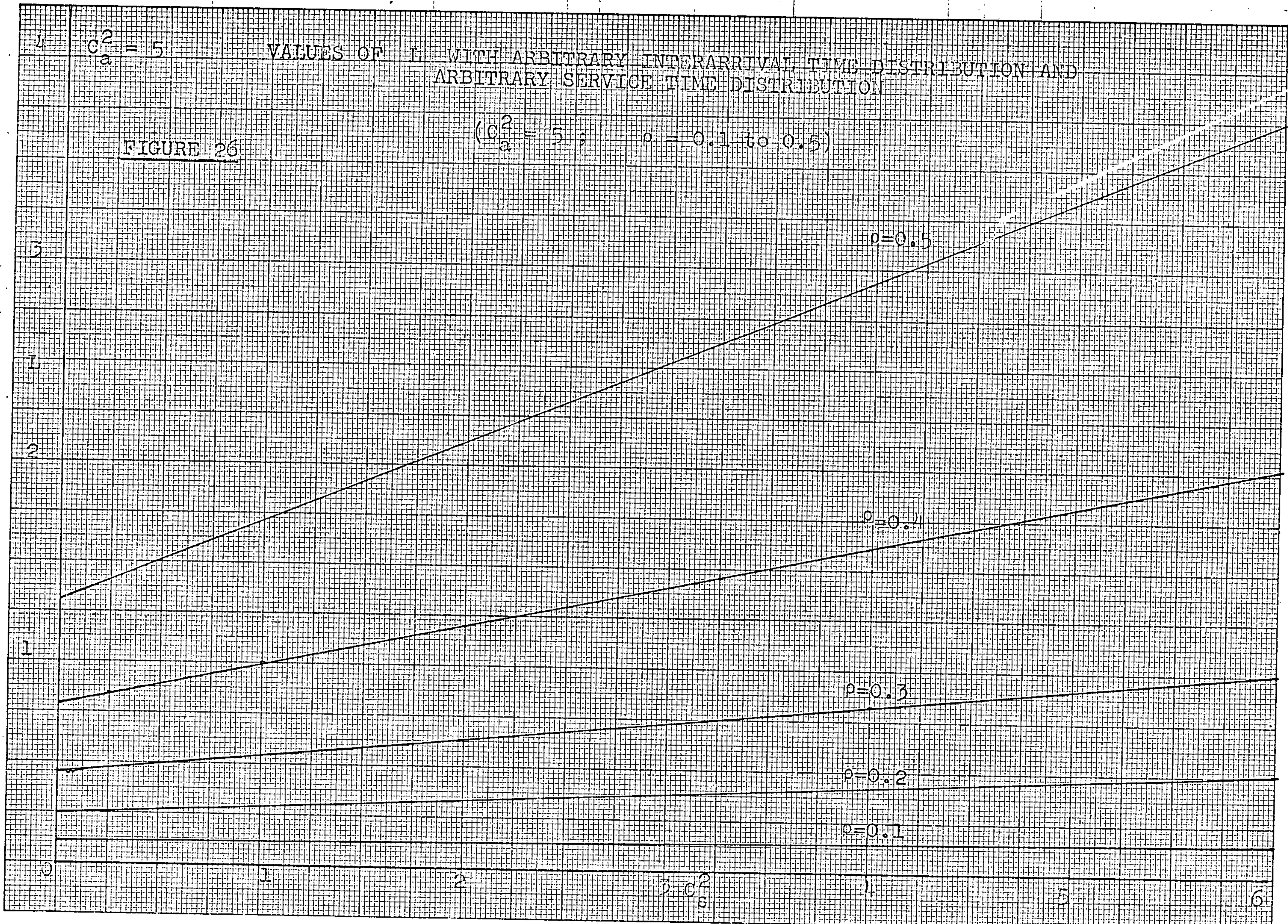
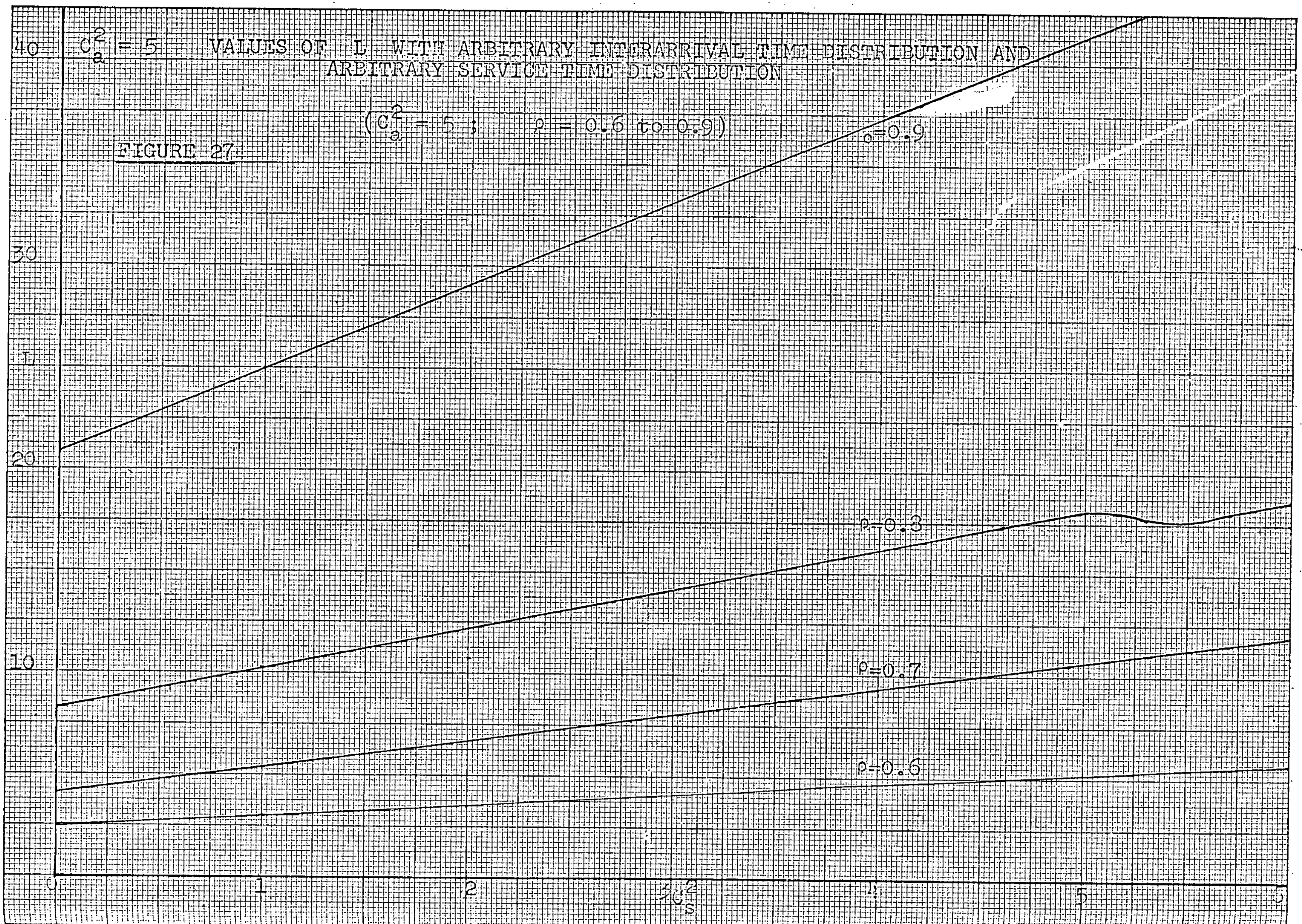


FIGURE 26

VALUES OF L WITH ARBITRARY INTERARRIVAL TIME DISTRIBUTION AND ARBITRARY SERVICE TIME DISTRIBUTION

($c_a^2 = 5$; $\rho = 0.1$ to 0.5)



$C_a^2 = 5$ VALUES OF L WITH ARBITRARY INTERARRIVAL TIME DISTRIBUTION AND ARBITRARY SERVICE TIME DISTRIBUTION

($C_a^2 = 5$; ρ = 0.6 to 0.9)

FIGURE 27

$C_a^2 = 6$ VALUES OF L WITH ARBITRARY INTERARRIVAL TIME DISTRIBUTION AND ARBITRARY SERVICE TIME DISTRIBUTION

($C_a^2 = 6$; $\rho = 0.1$ to 0.5)

FIGURE 28

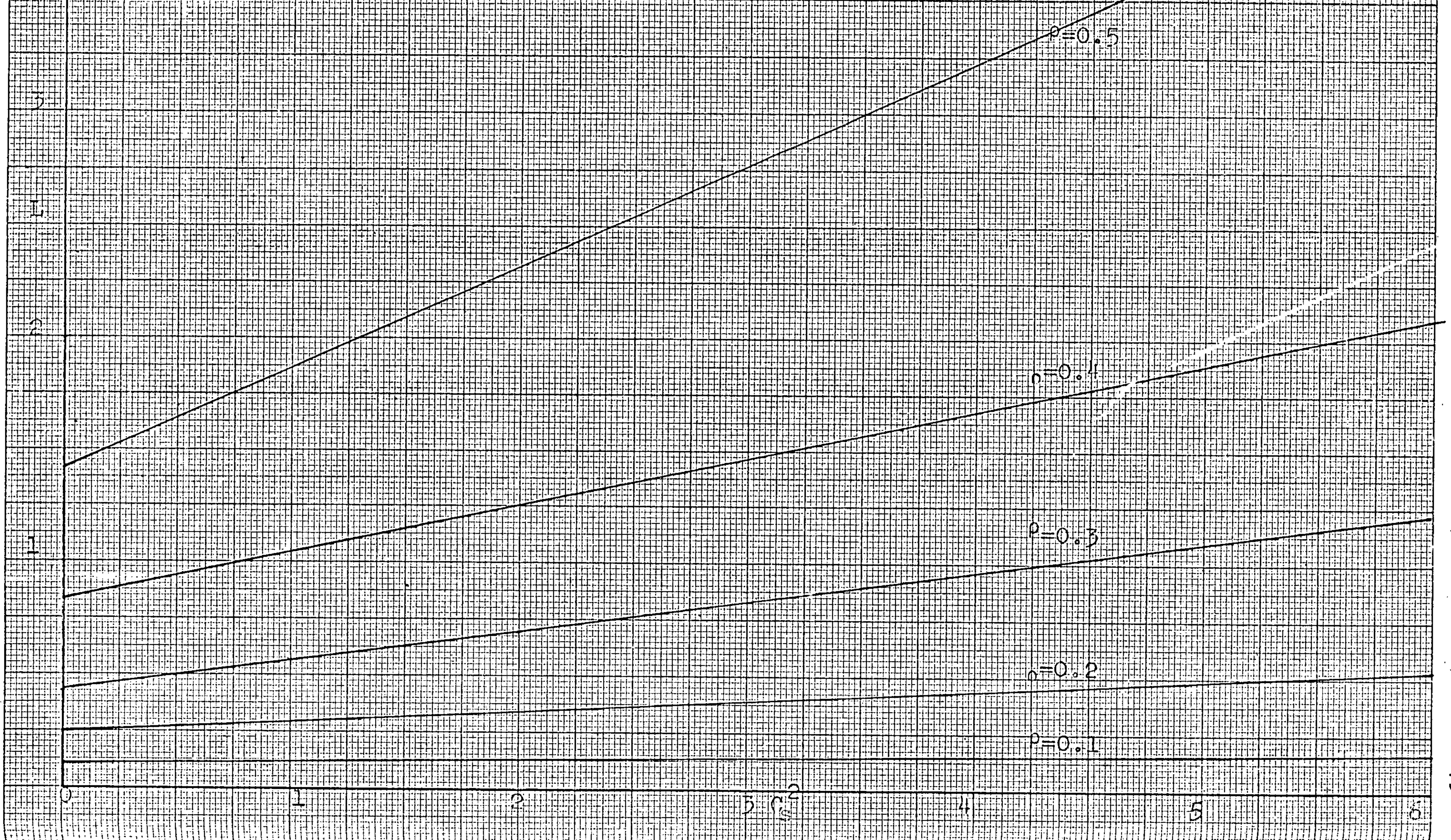
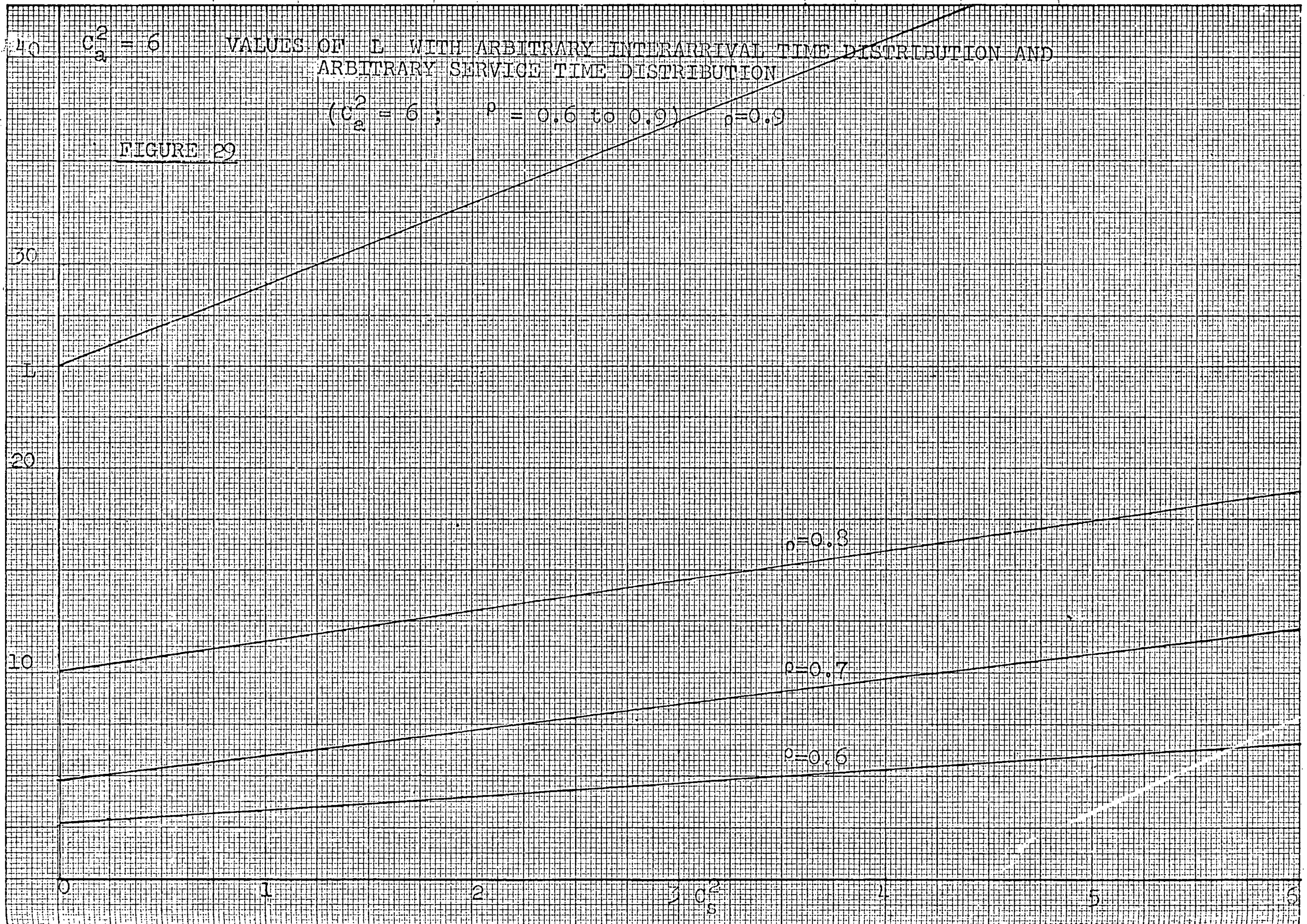


FIGURE 29

VALUES OF L WITH ARBITRARY INTERARRIVAL TIME DISTRIBUTION AND ARBITRARY SERVICE TIME DISTRIBUTION

($c_a^2 = 6$; $\rho = 0.6$ to 0.9)



Testing of Graphs

The validity of the graphs will now be tested with various arbitrary interarrival time distributions and arbitrary service time distributions with the aid of the University of British Columbia I.B.M. 7040 computer. A ρ of 0.8 were used for the different combinations of arbitrary interarrival time distribution and service time distribution. A second set with $\rho = 0.5$ was also tested. The program G.P.S.S. III was used to simulate the queue. A run of 10,000 transactions were simulated for each of the combination. The flow chart for the program is as shown in Appendix III.

The distributions used and the results from G.P.S.S. III and from the graphs (figures 10-29) are shown in Table 5.

Appendix I/x : The distribution used as given in Appendix I, section x . e.g. Appendix I/c is the distribution in section c of Appendix I.

ρ : The utilization factor used.

ρc : The utilization of the facility as obtained from G.P.S.S. III.

Lqc : The average length of queue as obtained from G.P.S.S. III.

Lc : The average waiting length as obtained from G.P.S.S. III $= Lqc + \rho c$.

$L\rho$: The average waiting length in the system as obtained from graphs with the utilization factor used.

$L\rho c$: The average waiting length in the system as obtained from the graphs with utilization factor ρc .

	Interarrival time distribution	C_a^2	Service time distribution	C_s^2	ρ	ρc	Lqc	Lc	$L\rho$	$L\rho c$	$\lvert L\rho c - Lc \rvert$
(1)	Appendix I/d	1.6	Appendix I/c	5.8	0.8	0.8319	14.43	15.26	12.20	16.00	.74
(2)	Appendix I/a	0.4046	Appendix I/b	1	0.8	0.8870	4.89	3.75	2.90	5.80	.04
(3)	Appendix I/b	1	Appendix I/c	5.8	0.8	0.8153	9.85	10.67	11.67	12.80	.13
(4)	Appendix I/d	1.6	Appendix I/a	0.4046	0.8	0.7450	2.84	3.59	4.06	3.61	0.02
(5)	Appendix I/c	5.8	Appendix I/e	0.3333	0.8	0.7432	8.71	9.45	10.50	9.01	0.44
(6)	Appendix I/e	0.3333	Appendix I/d	1.6	0.8	0.8624	3.33	6.19	4.05	6.25	0.06
(7)	Appendix I/b	1	Appendix I/c	5.8	0.5	0.5201	1.88	2.40	2.20	2.50	0.10
(8)	Appendix I/d	1.6	Appendix I/c	5.8	0.5	0.5017	1.95	2.45	2.50	2.52	0.07
(9)	Appendix I/a	0.4046	Appendix I/a	0.4046	0.5	0.4722	0.25	0.72	0.64	0.61	0.09
(10)	Appendix I/d	1.6	Appendix I/b	1	0.5	0.5011	1.11	1.61	1.13	1.20	0.41
(11)	Appendix I/e	0.3333	Appendix I/e	0.3333	0.5	0.4547	0.16	0.61	0.62	0.55	0.06
(12)	Appendix I/c	5.8	Appendix I/d	1.6	0.5	0.4901	3.35	3.84	2.18	2.15	1.69

TABLE V Test Results for Arbitrary Interarrival Time Distributions

and Arbitrary Service Time Distributions

The following factors will account for the difference between Lc and $L\rho$:

a) The C_a^2 of the actual number of transactions of 10,000 is not equal to the C_a^2 of an infinite number of transactions, so error is incurred in both, The G.P.S.S. III program and the construction of the graphs. This increases the maximum possible error.

b) ρc and ρ are not equal.

c) The C_s^2 of the actual number of transactions of 10,000 is not equal to the C_a^2 of an infinite number of transactions, so error is incurred in both, the G.P.S.S. III program and the construction of the graphs. This increases the maximum possible error.

d) The accuracy of the random number generator.

$$\text{Average } \rho c = (7.7950/12) = 0.6625$$
$$\text{Average } \rho = (7.8/12) = 0.65$$

Since average $\rho c > \rho$, it is possible that $L\rho c > Lc$ and $Lc > L\rho$ if the assumptions in the usage of the graphs are correct. Since this is borne out to the facts: $Lc = 60.55$ $L\rho = 57.65$ and $L\rho c = 63.02$, then $L\rho c > Lc > L\rho$.

Furthermore we shall examine the differences between Lc and $L\rho c$. We use $L\rho c$ instead of $L\rho$ for comparison because ρc is the actual utilization factor of the simulation The differences between Lc and $L\rho c$ for the twelve sets of simulation are shown in the final column of Table V. The average difference is 0.49. Where previously we used exponential interarrival and service time distributions and where we were sure of $L_{\rho c}$ because it was calculated from the Classical Erlang's formula, (Table IV) there was a difference between L_c and $L_{\rho c}$ of 0.31 . Hence the value of 0.49 average difference is not disturbing.

It is necessary for further works to be done on the testing of the graphs (figures 10-29). This, unfortunately, is outside the scope of the limited time available in the presentation of this thesis.

CHAPTER 7

CONCLUSION

The usage of the graphs will enable one to compute the required characteristics to make the necessary decisions without going through the process of simulation or curve fitting for each particular case of single channel queue with customers coming from an infinite population. It is not necessary to know the forms of distribution of interarrival time and service time to obtain the required characteristics.

Since the average waiting length of the system is dependent only on utilization factor, ρ, square of the fractional coefficient of interarrival time, C_a^2 and square of the fractional coefficient of service time C_s^2. The following process will enable one to compute the necessary characteristics.

The possible error is relatively small and as shown by comparison with simulation of 10,000 transactions via G.P.S.S. III for twelve combinations of arbitrary distributions, the average differences in waiting length of the system is only 0.49 customer. This gives a percentage difference of 10°/o.

a) Calculate the average interarrival time.
Average interarrival time $= \frac{1}{\lambda} =$ (Total time of observation/Total number of arrivals) $= \frac{I}{n}$.

b) Calculate the variance for interarrival time.
Variance $= Var(t_a) = \sum_{i=1}^{n} (t_{ai} - \frac{1}{\lambda})^2$ /Total number of arrival $= \Sigma\, f_i(t_{ai} - \frac{1}{\lambda})^2$

c) Calculate the fractional coefficient of variance squared for interarrival time distribution, C_a^2 .

$$C_a^2 = \frac{Var(t_a)}{(\frac{1}{\lambda})^2}$$

d) Calculate the average service time.

Average service time $= \frac{1}{\mu}$ = (Total time facility is in operation/Total number serviced) $= \frac{S}{m}$

e) Calculate the variance for service times.

Variance $= Var(t_s) = \sum_{j=1}^{m}(t_{sj}-\frac{1}{\mu})^2$ /Total number serviced $= \Sigma f_i(t_{si}-\frac{1}{\mu})^2$

f) Calculate the fractional coefficient of variance squared for service time distribution, C_s^2 .

$$C_s^2 = \frac{Var(t_s)}{(\frac{1}{\mu})^2}$$

g) Utilization factor, ρ = (Average service time/ average interarrival time)

h) With the values ρ , C_a^2 , and C_s^2 , read from the graphs (figures 10-29) the verticle axis, L .

i) Compute;

The average queue length $Lq = (L-\rho)$

The average queueing time, $Wq = (L-\rho)/\lambda$

The average waiting time in the system, $W = L/\lambda$

The busy period $= \rho$.

For an example of the usage of the graphs (figures 10-29), we shall assume that a saw receives logs from two sources. During the entire period of observation, it is noted that a total of 1000 logs arrive at the saw. The total period of observation is 5000 minutes. It is noted that 50°/o of the interarrival time is 3 minutes and 50°/o is 7 minutes. It is also known that 30°/o of the logs require 3 minutes to saw each log, 40°/o requires 4 minutes and 30°/o requires 5 minutes. It is required to compute the average waiting length, the average queue length, the average waiting time, the average queue time and the utilization factor

a) Average interarrival time = $\frac{5000}{1000}$ = 5 minutes/log.

b) Variance of interarrival time = $[(0.50)(3-5]^2 + (0.50)(7-5)^2] = 4$

c) $C_a^2 = \frac{4}{5^2} = \frac{4}{25} = 0.16$

d) Average service time = $(.30\times3 + .40\times4 + .30\times5)$
$= 4$ minutes/log

e) Variance of service time = $[0.30(3-4)^2 + 0.40(4-4)^2 + 0.50(5-4)^2]$
$= 0.80$

f) $C_5^2 = \frac{0.8}{4^2} = 0.05$

g) $\rho = \frac{4}{5} = 0.8$

h) From figures 10-29, and using interpolation:

From figure 11 with $C_a^2 = 0$, $\rho = 0.8$ and $C_s^2 = 0.05$

$L = 0.85$

From figure 13 with $C_a^2 = 0.25$, $\rho = 0.8$ and $C_s^2 = 0.05$

By interpolation:

$$L = 0.85 + \frac{0.15\times 0.16}{0.25}$$

$$= 0.95 \text{ logs}$$

i) $Lq = (L-\rho) = (0.95-0.80) = 0.15$ logs.

$$W = \frac{L}{\lambda} = 0.95\times 5$$

$$= 4.75 \text{ minutes/log.}$$

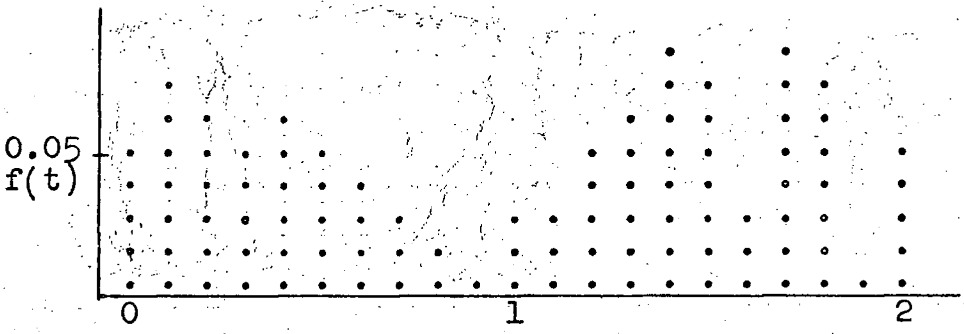
$$Wq = Lq = 0.15\times 5 = 0.75 \text{ minutes/log.}$$

APPENDIX I

DISTRIBUTIONS FOR TESTING OF GRAPHS

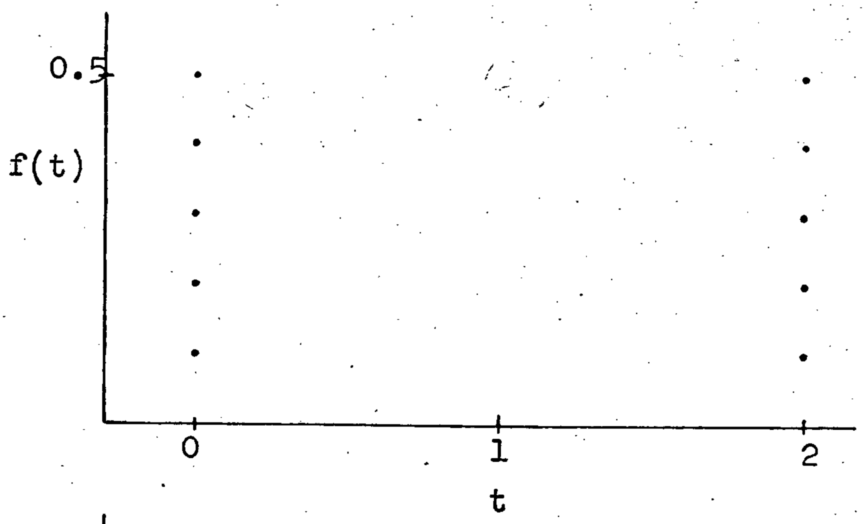
a)

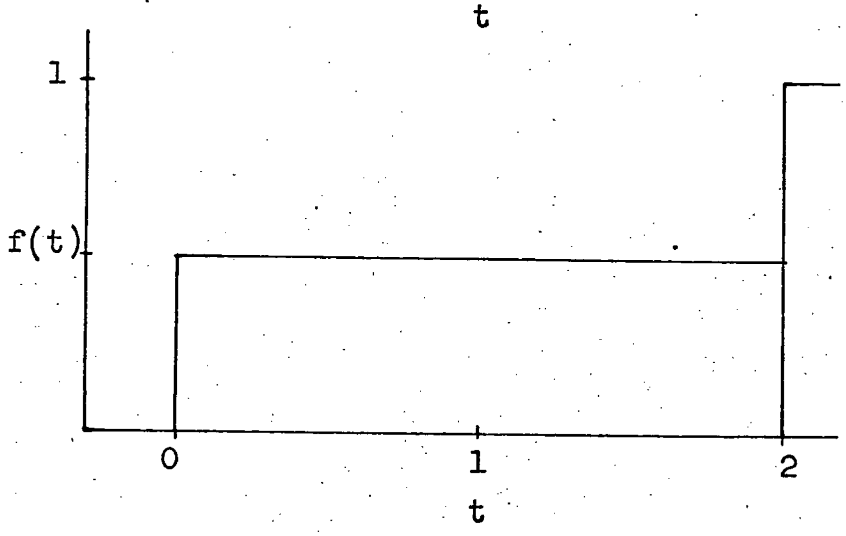
t	f(t)	F(t)	t.f(t)	(t-1)	$(t-1)^2$	Variance (t)	c^2
0.0	0.05	0.05	0.000	-1.0	1.00	0.0500	
0.1	0.07	0.12	0.007	-0.9	0.81	0.0567	
0.2	0.06	0.18	0.012	-0.8	0.64	0.0384	
0.3	0.05	0.23	0.015	-0.7	0.49	0.0245	
0.4	0.06	0.29	0.024	-0.6	0.36	0.0216	
0.5	0.05	0.34	0.025	-0.5	0.25	0.0125	
0.6	0.04	0.38	0.024	-0.4	0.16	0.0064	
0.7	0.03	0.41	0.021	-0.3	0.09	0.0027	
0.8	0.02	0.43	0.016	-0.2	0.04	0.0008	
0.9	0.01	0.44	0.009	-0.1	0.01	0.0001	
1.0	0.03	0.47	0.030	0.0	0.00	0.0000	
1.1	0.03	0.50	0.033	0.1	0.01	0.0003	
1.2	0.05	0.55	0.060	0.2	0.04	0.0020	
1.3	0.06	0.61	0.078	0.3	0.09	0.0054	
1.4	0.08	0.69	0.112	0.4	0.16	0.0128	
1.5	0.07	0.76	0.105	0.5	0.25	0.0175	
1.6	0.03	0.79	0.048	0.6	0.36	0.0108	
1.7	0.08	0.87	0.136	0.7	0.49	0.0392	
1.8	0.07	0.94	0.126	0.8	0.64	0.0448	
1.9	0.01	0.95	0.019	0.9	0.81	0.0081	
2.0	0.05	1.00	0.100	1.0	1.00	0.0500	
			1.000			0.4046	0.4046



b)

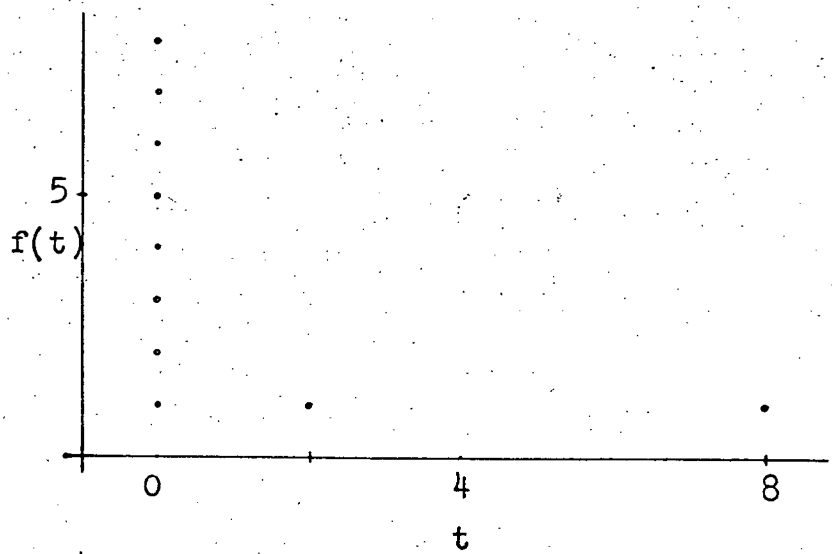
t	f(t)	F(t)	t.f(t)	(t-1)	$(t-1)^2$	Variance (t)	C^2
0.0	0.5	0.5	0.0	-1.0	1.00	0.50	
2.0	0.5	1.0	1.0	1.0	1.00	0.50	
			1.0			1.00	1.00

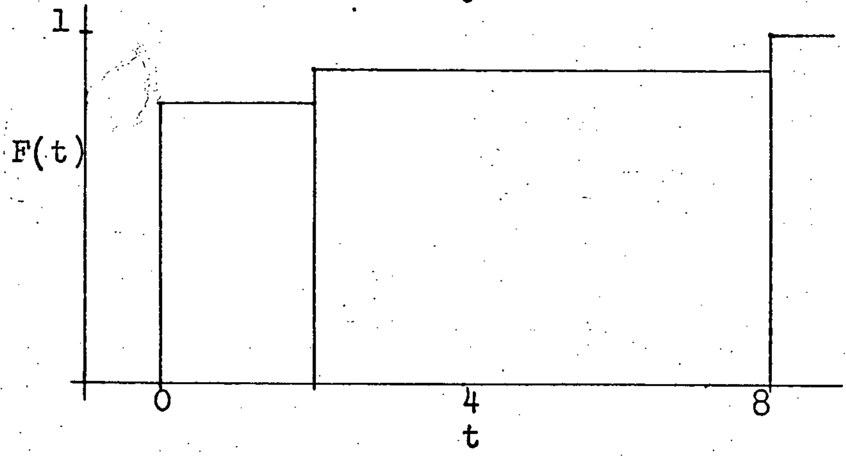




c)

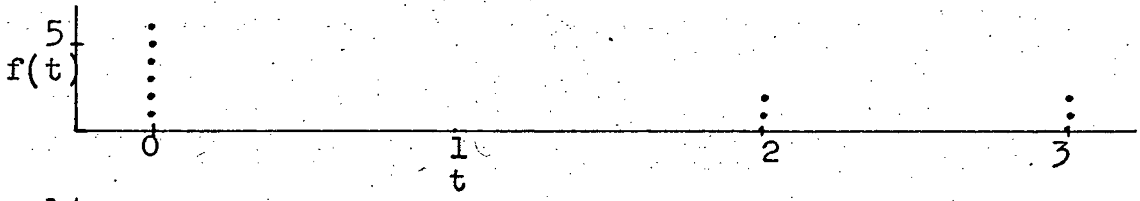
t	f(t)	F(t)	t.f(t)	(t-1)	$(t-1)^2$	Variance (t)	c^2
0.0	0.8	0.8	0.0	-1.0	1.00	0.80	
2.0	0.1	0.9	0.2	1.0	1.00	0.10	
8.0	0.1	1.0	0.8	7.0	49.00	4.90	
			1.0			5.80	5.80

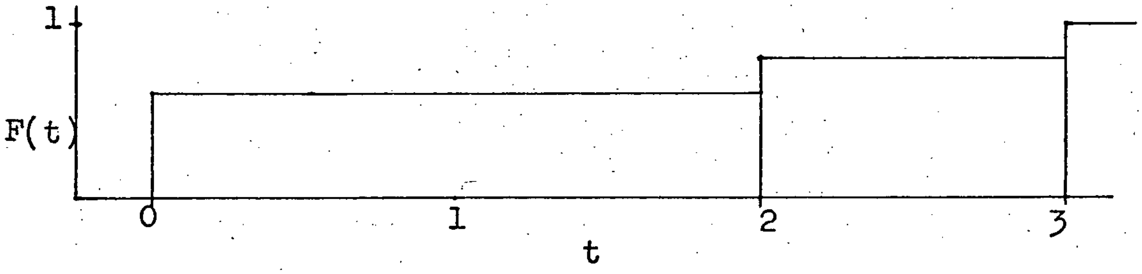




d)

t	f(t)	F(t)	t.f(t)	(t-1)	$(t-1)^2$	Variance (t)	C^2
0.0	0.6	0.6	0.0	-1.0	1.00	0.6	
2.0	0.2	0.8	0.4	1.0	1.00	0.2	
3.0	0.2	1.0	0.6	2.0	4.00	0.8	
			1.0			1.6	1.5





e) $f(t) = 0.5$ when $0 \leq t \leq 2$

$= 0.0$ when $0 > t > 2$

$$F(t) = \int_0^2 0.5\, dt = 1$$

$$\text{Average }(t) = \int_0^2 0.5\, t.dt = \left[\frac{0.5t^2}{2}\right]_0^2 = 1$$

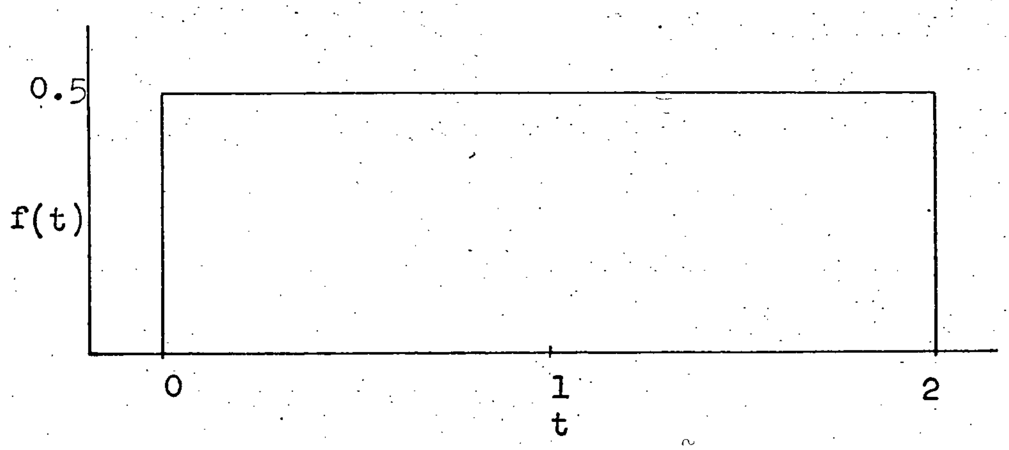
$$\text{Variance }(t) = \int_0^2 (t-1)^2 0.5dt = \int_0^2 t^2 0.5dt - 2\int_0^2 t.0.5dt + \int_0^2 0.5dt$$

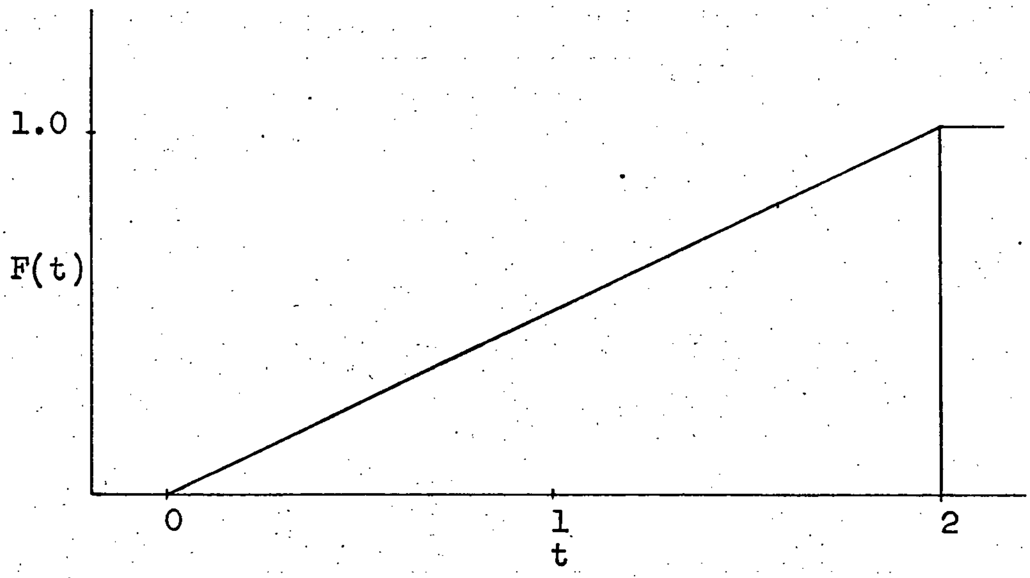
$$= 0.5\left[\frac{t^3}{3}\right]_0^2 - 2 \qquad + 1$$

$$= \frac{8}{6} \quad - \quad 2 \quad + \quad 1$$

$$= 0.3333$$

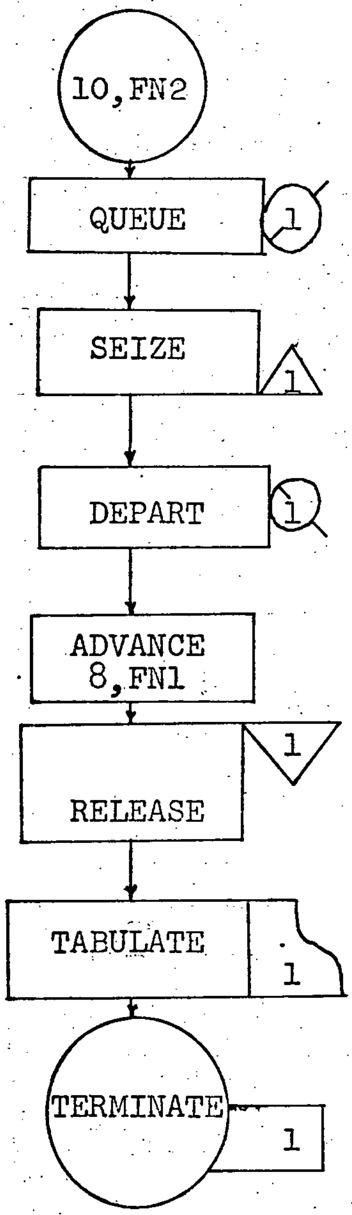
Fractional coefficient of variance squared, $C^2 = 0.3333$





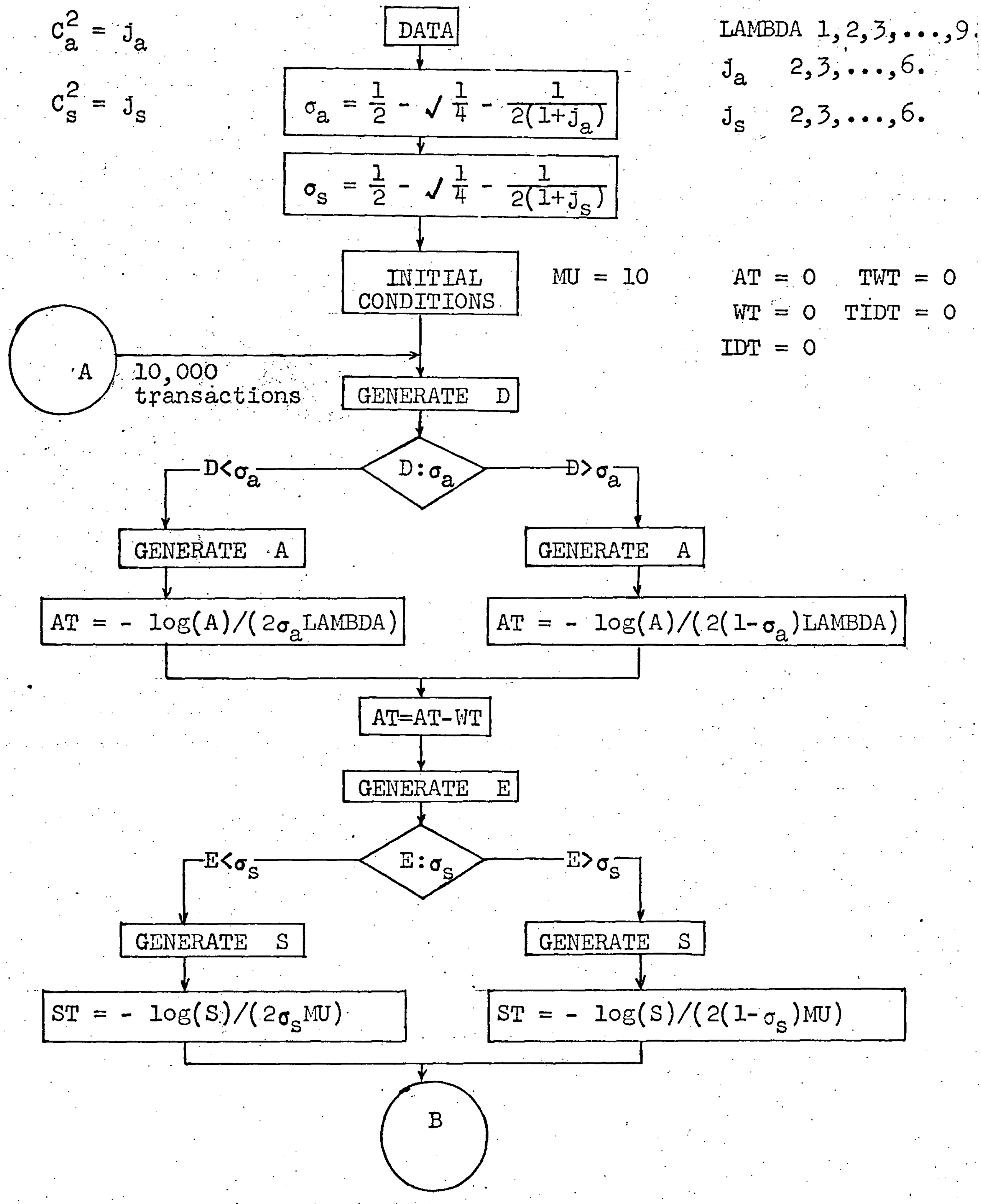
APPENDIX II

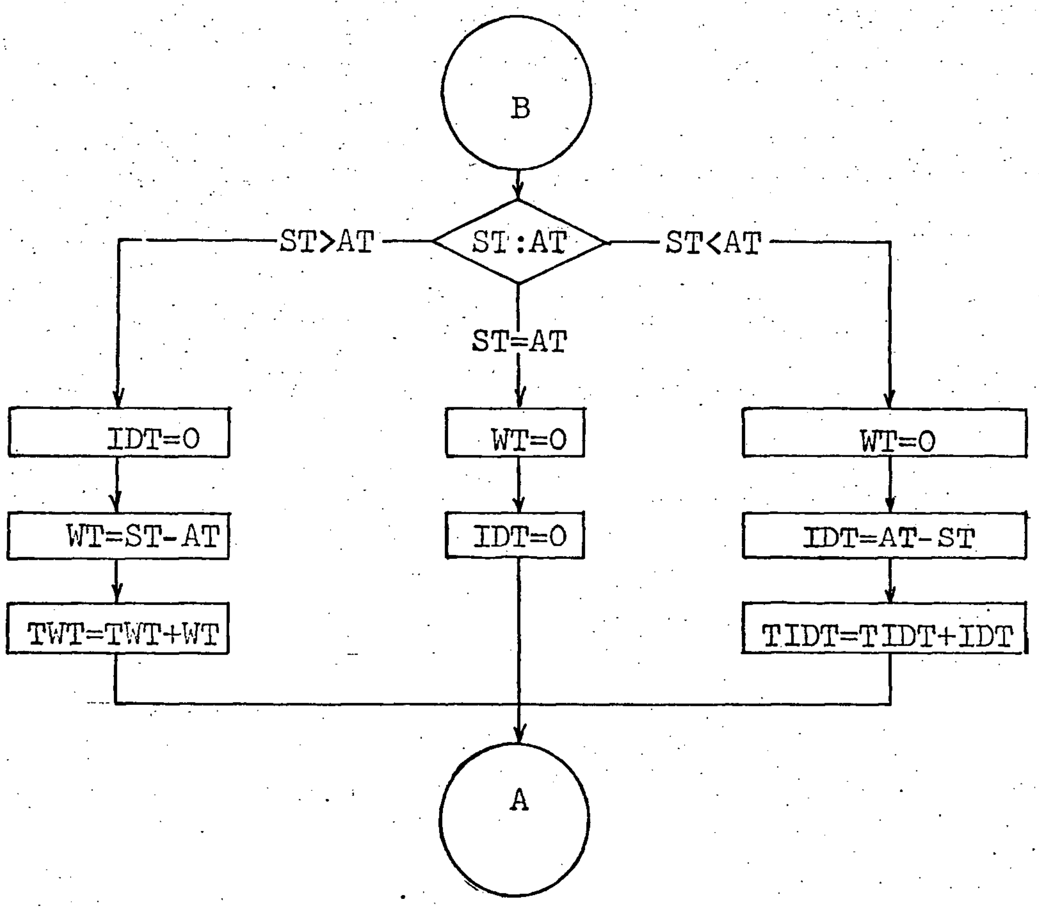
Flow Chart; G.P.S.S. III: Single Channel Queue



APPENDIX III

Flow Chart 1




B
ST:AT
ST>AT
ST=AT
ST<AT
IDT=0
WT=0
WT=0
WT=ST-AT
IDT=0
IDT=AT-ST
TWT=TWT+WT
TIDT=TIDT+IDT
A

Flow Chart 2

$C_a^2 = 1/k$

$C_s^2 = j_s$

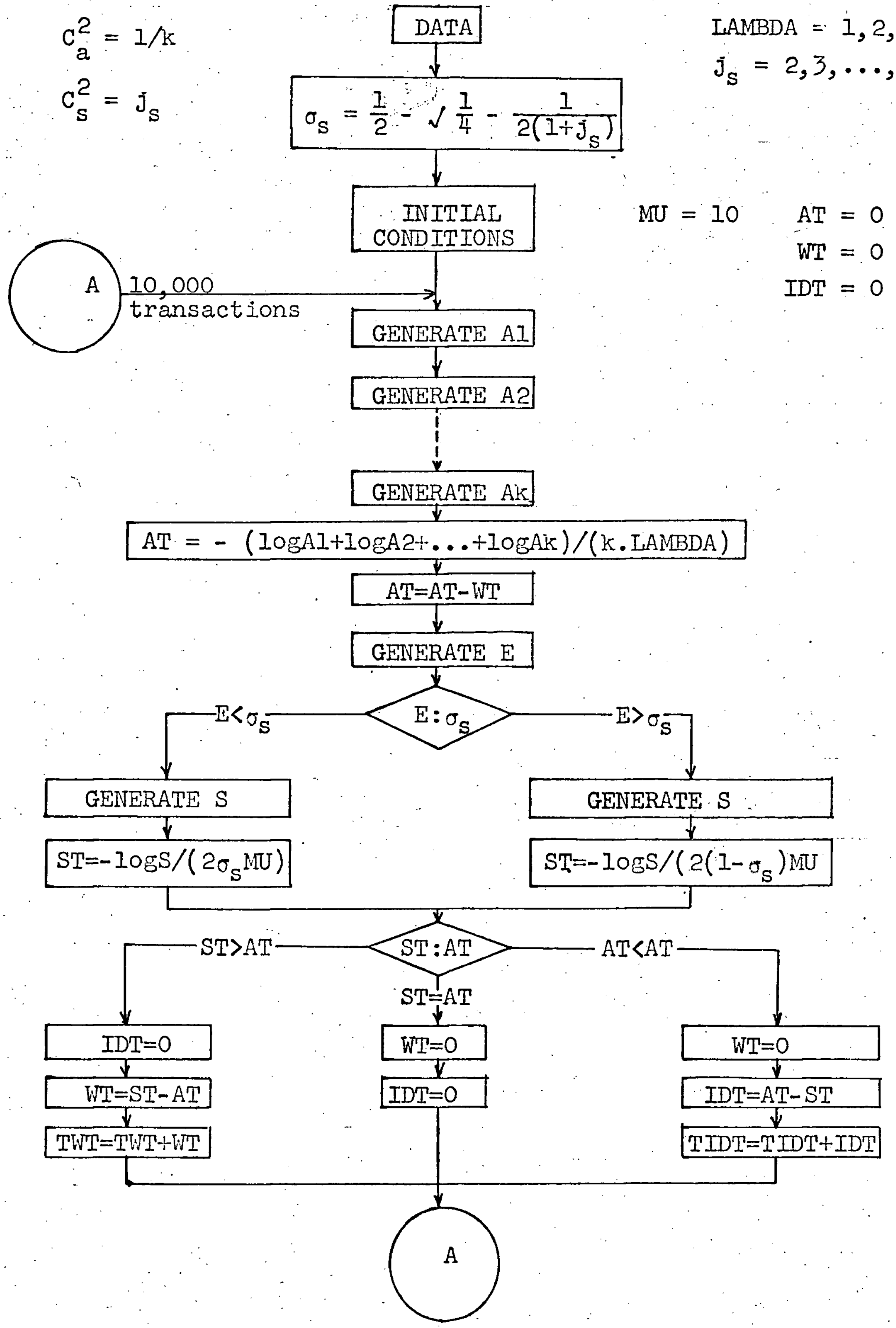
LAMBDA = 1,2,3,...,9.

j_s = 2,3,...,6.

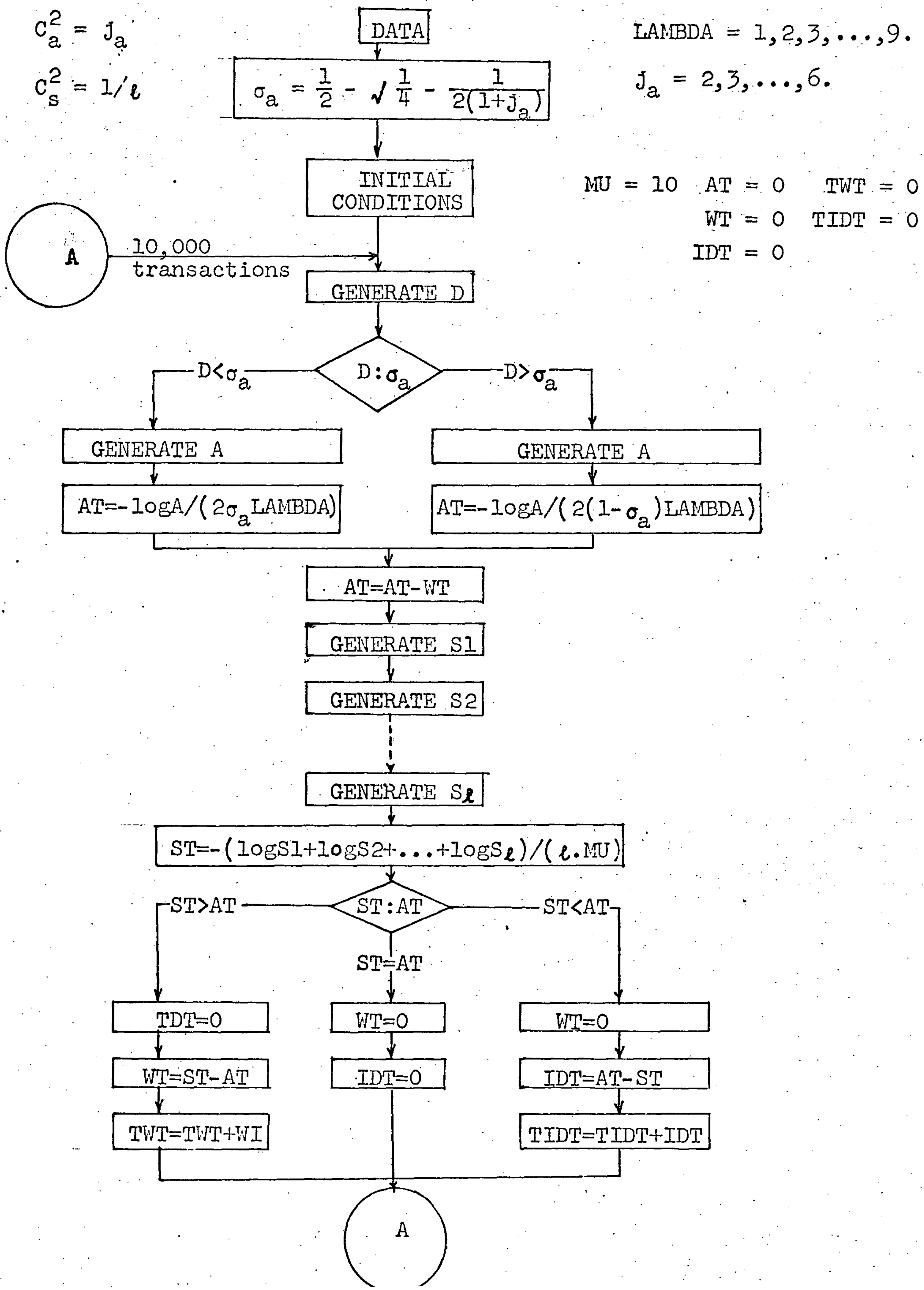
MU = 10 AT = 0 TWT = 0

WT = 0 TIDT = 0

IDT = 0

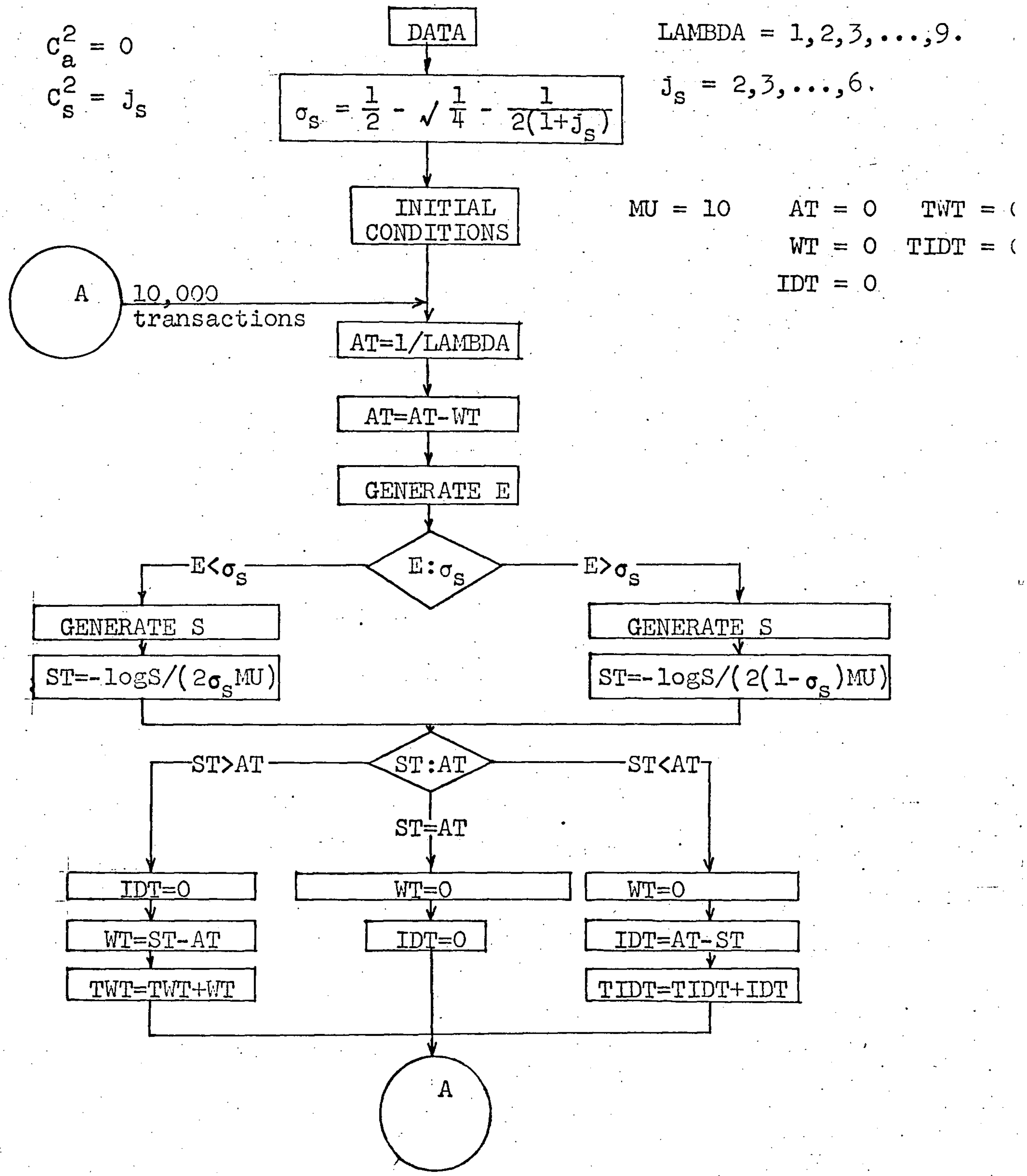


Flow Chart 3



$c_a^2 = j_a$
$c_s^2 = 1/\ell$
DATA
$\sigma_a = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{1}{2(1+j_a)}}$
LAMBDA = 1,2,3,...,9.
j_a = 2,3,...,6.
INITIAL CONDITIONS
MU = 10 AT = 0 TWT = 0
WT = 0 TIDT = 0
IDT = 0
A
10,000 transactions
GENERATE D
$D:\sigma_a$
$D<\sigma_a$
$D>\sigma_a$
GENERATE A
$AT=-\log A/(2\sigma_a LAMBDA)$
GENERATE A
$AT=-\log A/(2(1-\sigma_a)LAMBDA)$
AT=AT-WT
GENERATE S1
GENERATE S2
GENERATE S_ℓ
$ST=-(\log S1+\log S2+\ldots+\log S_\ell)/(\ell.MU)$
ST:AT
ST>AT
ST=AT
ST<AT
TDT=0
WT=ST-AT
TWT=TWT+WI
WT=0
IDT=0
WT=0
IDT=AT-ST
TIDT=TIDT+IDT
A

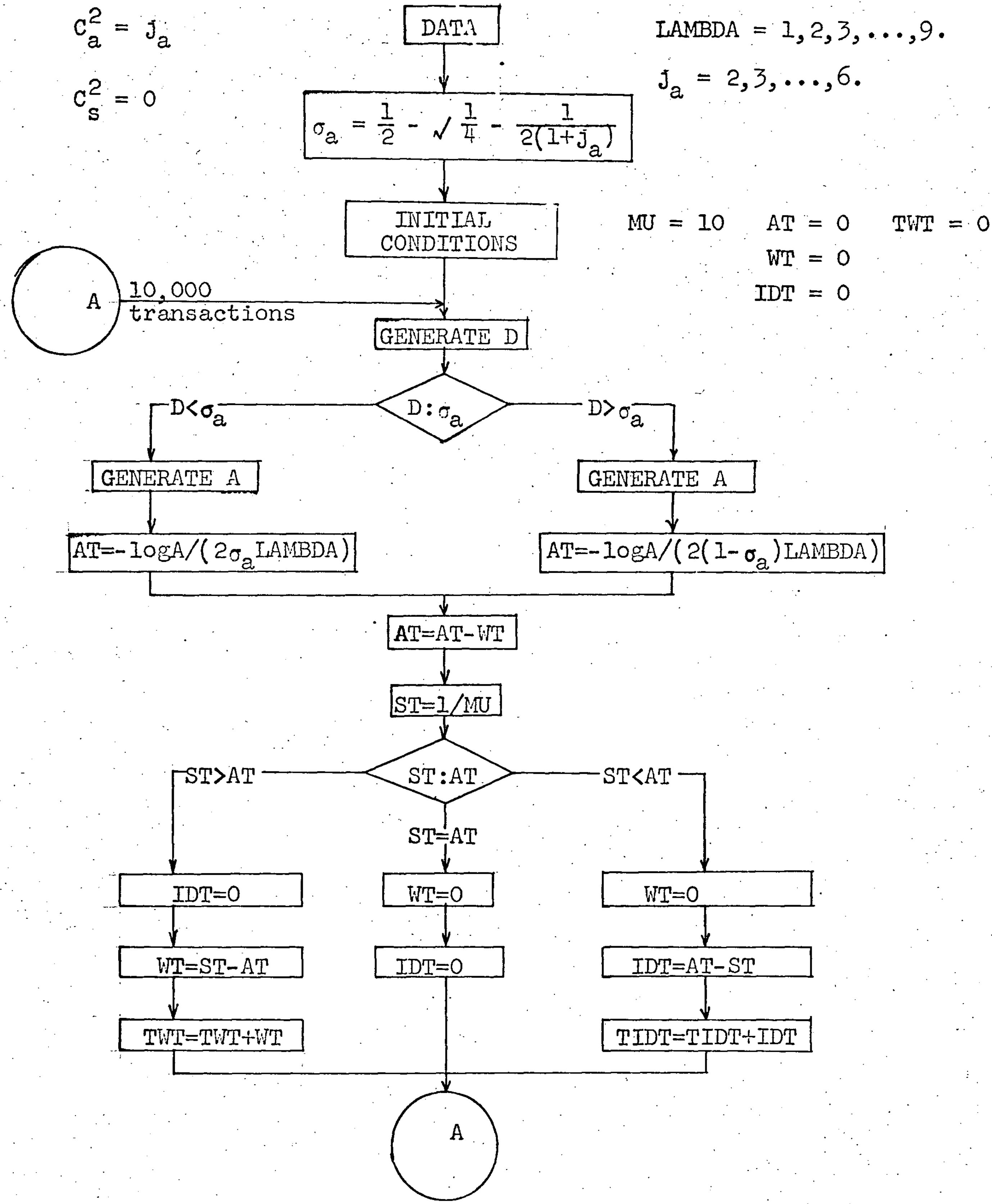
Flow Chart 4


$C_a^2 = 0$
$C_s^2 = j_s$
LAMBDA = 1,2,3,...,9.
j_s = 2,3,...,6.
DATA
$\sigma_s = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{1}{2(1+j_s)}}$
INITIAL CONDITIONS
MU = 10
AT = 0
TWT = (
WT = 0
TIDT = (
IDT = 0
A
10,000 transactions
AT=1/LAMBDA
AT=AT-WT
GENERATE E
E:σ_s
E<σ_s
E>σ_s
GENERATE S
ST=-logS/(2σ_sMU)
GENERATE S
ST=-logS/(2(1-σ_s)MU)
ST:AT
ST>AT
ST=AT
ST<AT
IDT=0
WT=ST-AT
TWT=TWT+WT
WT=0
IDT=0
WT=0
IDT=AT-ST
TIDT=TIDT+IDT
A

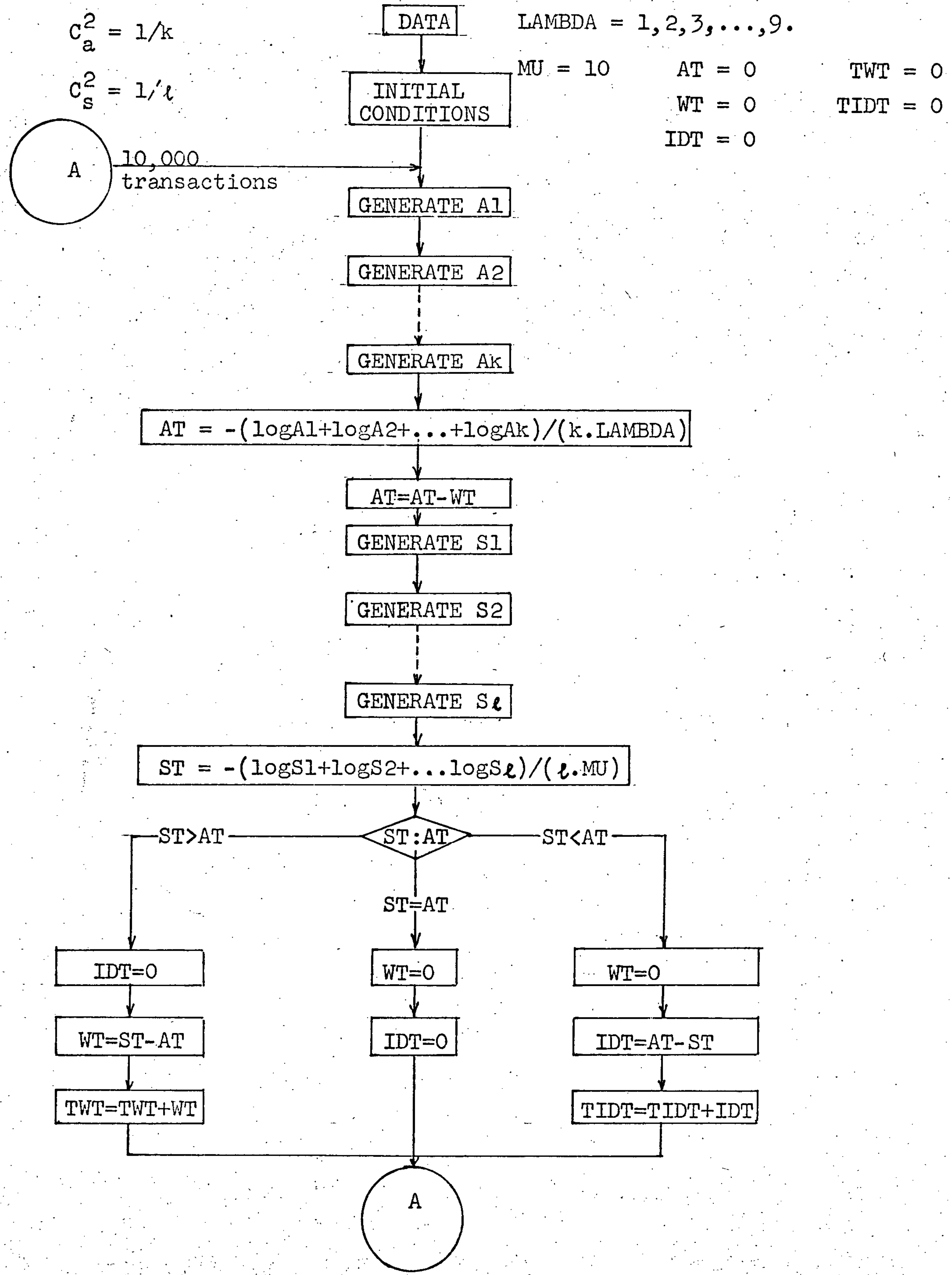
Flow Chart 5

$c_a^2 = j_a$

$c_s^2 = 0$



Flow Chart 6



$C_a^2 = 1/k$
$C_s^2 = 1/\ell$
DATA
LAMBDA = 1,2,3,...,9.
MU = 10
AT = 0
WT = 0
IDT = 0
TWT = 0
TIDT = 0
INITIAL CONDITIONS
A
10,000 transactions
GENERATE A1
GENERATE A2
GENERATE Ak
AT = -(logA1+logA2+...+logAk)/(k.LAMBDA)
AT=AT-WT
GENERATE S1
GENERATE S2
GENERATE Sℓ
ST = -(logS1+logS2+...logSℓ)/(ℓ.MU)
ST:AT
ST>AT
ST=AT
ST<AT
IDT=0
WT=ST-AT
TWT=TWT+WT
WT=0
IDT=0
WT=0
IDT=AT-ST
TIDT=TIDT+IDT
A

Flow Chart 7

$C_a^2 = 1/k$

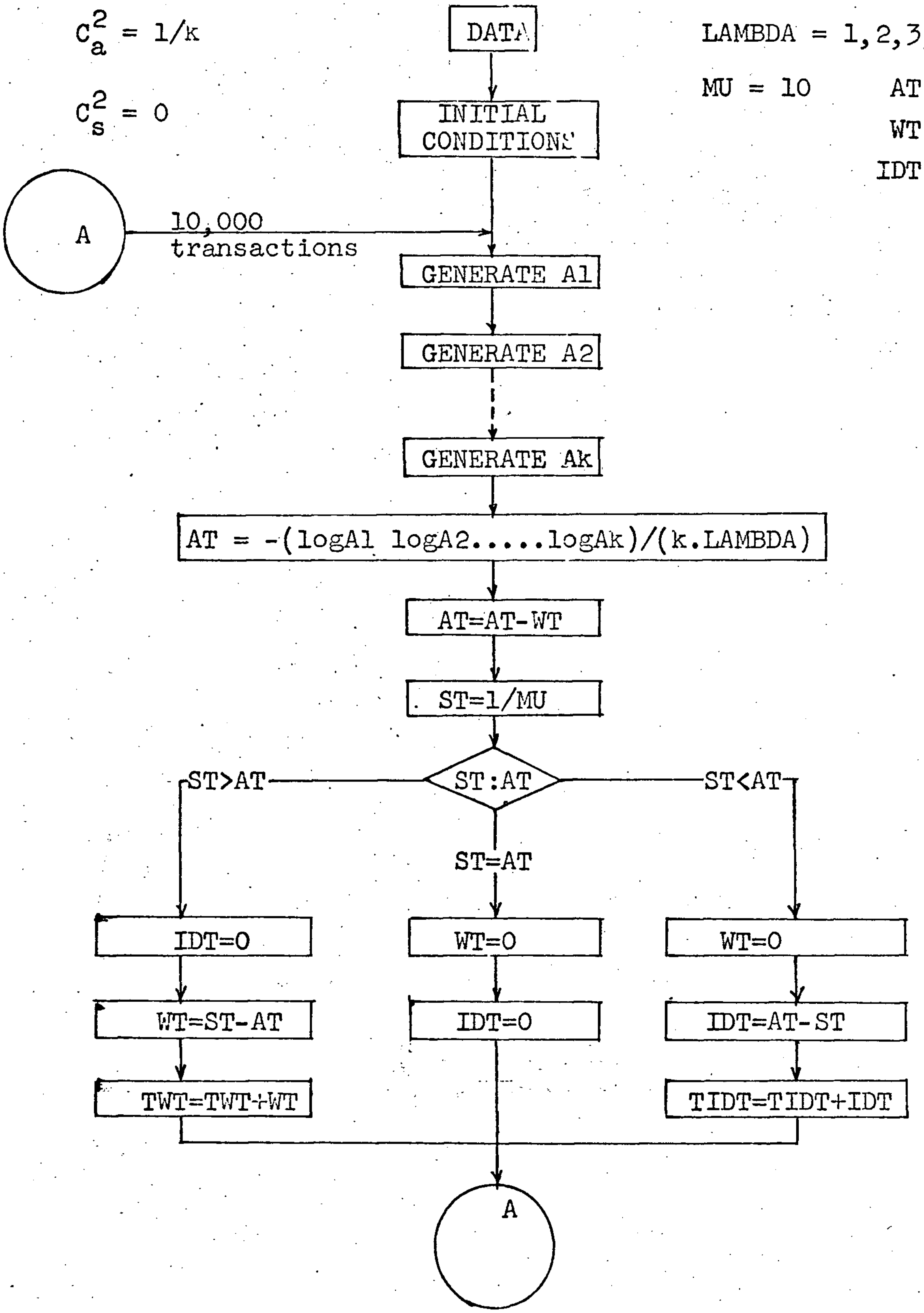
$C_s^2 = 0$

LAMBDA = 1,2,3,...,9.

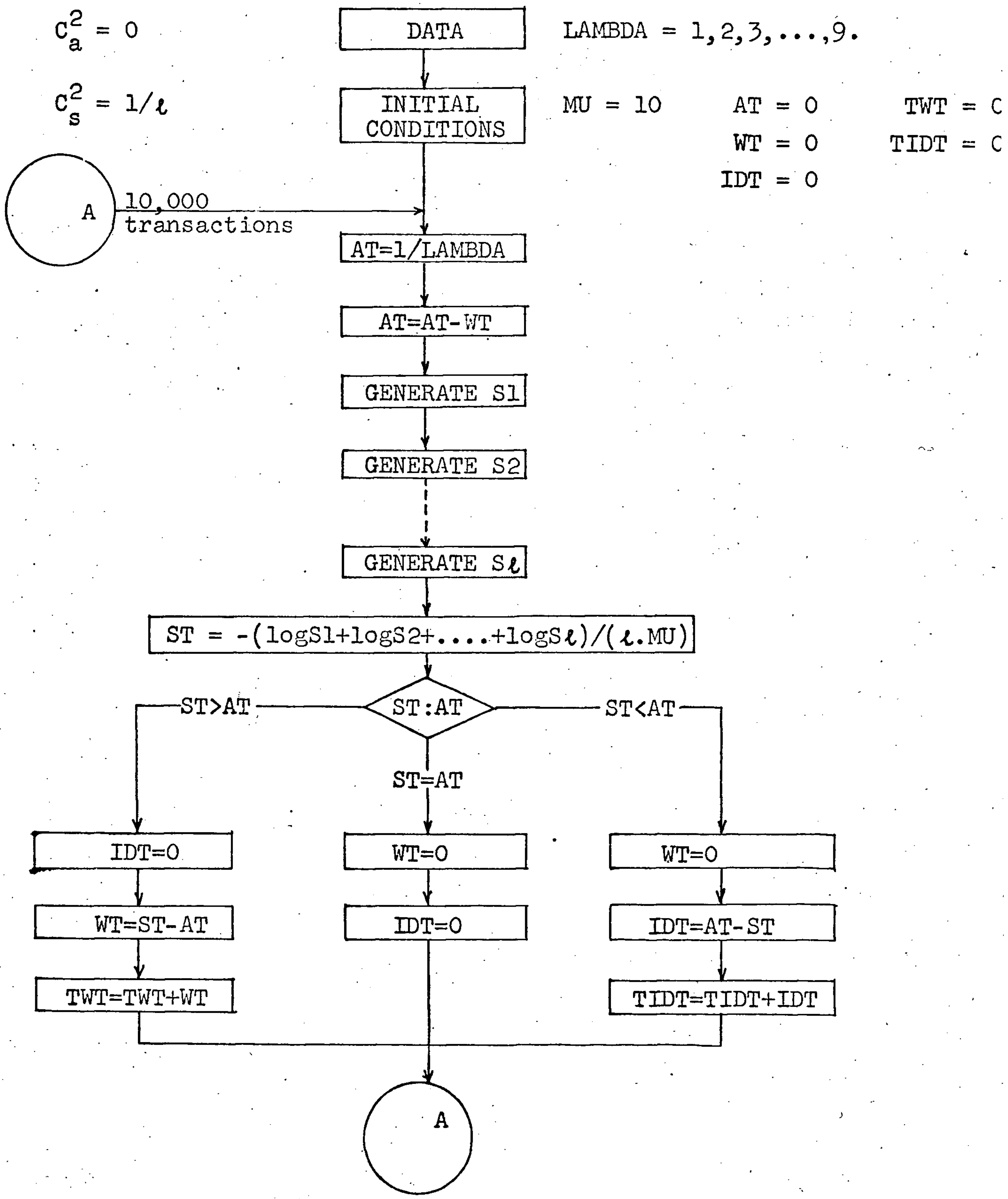
MU = 10 AT = 0 TWT = (

WT = 0 TIDT = (

IDT = 0



Flow Chart 8


$C_a^2 = 0$
$C_s^2 = 1/\ell$
DATA
LAMBDA = 1,2,3,...,9.
INITIAL CONDITIONS
MU = 10
AT = 0
WT = 0
IDT = 0
TWT = C
TIDT = C
A
10,000 transactions
AT=1/LAMBDA
AT=AT-WT
GENERATE S1
GENERATE S2
GENERATE Sℓ
ST = -(logS1+logS2+....+logSℓ)/(ℓ.MU)
ST:AT
ST>AT
ST=AT
ST<AT
IDT=0
WT=ST-AT
TWT=TWT+WT
WT=0
IDT=0
WT=0
IDT=AT-ST
TIDT=TIDT+IDT
A

APPENDIX IV

The following tables contain the values of the average length of the system, L , with given coefficient of service time variance squared, C_s^2 , coefficient of interarrival time variance squared, C_a^2 , and utilization factor, ρ, as obtained from the application of the various techniques listed in Chapter 6.

The values are used to obtain the graphs (figures 10-29) for arbitrary service and interarrival time distributions.

C_a^2	C_s^2	ρ = 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
	0	0.10000	0.20000	0.30000	0.40000	0.50000	0.60000	0.70000	0.80000	0.90000
	0.25	0.10000	0.30000	0.40171	0.50964	0.63715	0.63715	0.79283	1.05906	1.63273
	0.333	0.10000	0.20000	0.30397	0.40401	0.51843	0.64974	0.86288	1.22588	1.99542
	0.5	0.10000	0.20018	0.30292	0.41179	0.53650	0.70786	0.96359	1.38123	2.59995
0	1	0.10000	0.20141	0.31279	0.44813	0.62757	0.88784	1.31307	2.15343	4.65116
	2	0.10159	0.21481	0.36150	0.54731	0.97988	1.37234	2.30510	4.72062	11.0258
	3	0.10319	0.23285	0.43432	0.70498	1.06953	2.01297	3.34897	5.40883	10.7971
	4	0.10682	0.25517	0.47471	0.76565	1.46049	2.71278	4.06366	8.22504	13.7601
	5	0.11079	0.26836	0.56204	0.96452	1.68314	3.04609	4.21116	6.23288	20.2304
	6	0.11261	0.33056	0.56176	1.08654	1.87173	2.79277	5.22927	10.6126	20.2308
	0	0.10008	0.20112	3.30217	0.41327	0.53172	0.67211	0.85272	1.02341	1.87272
	0.25	0.10016	0.20190	0.30769	0.42557	0.56871	0.74431	1.04190	1.52993	3.12692
	0.333	0.10024	0.20237	0.31090	0.43293	0.57821	0.78070	1.06483	1.62454	2.95744
	0.5	0.10021	0.20385	0.31793	0.44485	0.61133	0.81837	1.21688	1.82870	3.37091
0.25	1	0.10068	0.20898	0.33645	0.49771	0.71626	1.03739	1.56569	2.61391	5.74475
	2	0.10313	0.22119	0.39443	0.60567	0.98002	1.47787	2.42298	4.61372	11.7321
	3	0.10710	0.25373	0.44918	0.70385	1.12102	1.88185	4.07435	5.20162	18.9732
	4	0.11014	0.27090	0.52167	0.92615	1.29407	2.25240	4.72930	8.61731	21.3472
	5	0.11375	0.29479	0.57640	1.00238	1.53755	3.18722	5.22968	6.91972	20.5841
	6	0.11875	0.33523	0.65790	1.03617	1.85463	3.28373	6.19331	9.47219	22.7431

APPENDIX IV (Continued)

C_a^2	C_s^2	ρ 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
	0	0.10010	0.20143	0.30607	0.41932	0.54524	0.69343	0.90245	1.19660	2.33076
	0.25	0.10030	0.20376	0.31460	0.44034	0.58641	0.79296	1.08486	1.62271	3.05330
	0.333	0.10030	0.20453	0.31663	0.44776	0.59865	0.80339	1.11857	1.65803	3.46527
	0.5	0.10074	0.20573	0.32521	0.46231	0.64446	0.89042	1.27285	1.83528	4.10961
0.333	1	0.10128	0.21255	0.34558	0.51543	0.74686	1.08804	1.65041	2.76754	6.10625
	2	0.10517	0.23153	0.40191	0.64413	1.02115	1.58734	2.43890	4.32828	10.1476
	3	0.10768	0.24256	0.44861	0.76608	1.21298	1.89224	2.98250	4.82765	12.4942
	4	0.11424	0.27457	0.52333	0.82266	1.35042	2.50724	4.68592	8.89511	19.5802
	5	0.11646	0.28981	0.60960	0.97263	1.67285	2.67403	5.23712	7.07702	2.17321
	6	0.12517	0.31052	0.62166	1.21860	1.76441	2.68410	5.74852	9.13246	2.29763
	0	0.10066	0.20505	0.31643	0.44242	0.59126	0.78419	1.04262	1.53829	2.98837
	0.25	0.10120	0.20906	0.32725	0.47147	0.64001	0.86394	1.21500	1.80949	3.60442
	0.333	0.10155	0.20935	0.33605	0.47274	0.66756	0.91195	1.28552	2.06145	4.35988
	0.5	0.10152	0.21327	0.33541	0.50288	0.70519	1.03031	1.40874	2.28205	5.02268
0.5	1	0.10301	0.22070	0.36498	0.55196	0.80902	1.19010	1.82044	3.07518	6.82946
	2	0.10610	0.24118	0.42510	0.68397	1.08013	1.70654	3.02803	5.32465	10.0644
	3	0.11146	0.26572	0.47094	0.78140	1.25267	2.24019	3.73864	5.83091	11.9469
	4	0.11489	0.28547	0.55945	1.02071	1.59718	2.47496	3.87322	8.86966	26.9729
	5	0.12509	0.29889	0.68101	1.16238	1.62487	2.57067	5.23566	9.88287	15.2311
	6	0.13079	0.32216	0.60497	1.16016	2.26760	3.20692	4.64315	12.[illegible]520	23.0245

APPENDIX IV (Continued)

C_a^2	C_s^2	ρ = 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
	0	0.10556	0.22500	0.36429	0.53333	0.75000	1.05000	1.51667	2.40000	4.95000
	0.25	0.10695	0.23125	0.38036	0.56667	0.81250	1.16250	1.72083	2.80000	5.96250
	0.333	0.10741	0.23333	0.38572	0.57778	0.83333	1.20000	1.78889	2.93333	6.30000
	0.5	0.10833	0.23750	0.39643	0.60000	0.87500	1.27500	1.92500	3.2000	6.97500
1	1	0.11111	0.25000	0.42857	0.66667	1.00000	1.50000	2.33333	4.00000	9.00000
	2	0.11667	0.27500	0.49286	0.80000	1.25000	1.95000	3.15000	5.60000	13.0500
	3	0.12222	0.30000	0.55714	0.93333	1.50000	2.46000	3.96667	7.20000	17.1000
	4	0.12778	0.32500	0.02143	1.06667	1.75000	2.85000	4.78333	8.80000	21.1500
	5	0.13333	0.35000	0.68572	1.20000	2.00000	3.30000	5.60000	10.4000	25.2000
	6	0.13889	0.37500	0.75000	1.33333	2.25000	3.75000	6.41667	12.0000	29.2500
	0	0.10757	0.23648	0.39029	0.61083	0.89884	1.32682	2.21766	3.37399	10.2147
	0.25	0.10895	0.24181	0.42221	0.63925	0.96764	1.42819	2.45896	4.27245	9.60631
	0.333	0.11127	0.24837	0.42084	0.65172	0.98313	1.56852	2.55702	9.20991	11.0309
	0.5	0.11049	0.25417	0.43535	0.61842	1.11019	1.56787	2.70784	4.87855	9.65153
2	1	0.11507	0.26901	0.48051	0.78078	1.22247	1.92117	3.12132	5.57604	13.0356
	2	0.12146	0.30195	0.56395	0.90049	1.61385	2.11494	3.54559	7.64930	19.0226
	3	0.12948	0.35483	0.60179	1.09154	1.70178	3.31551	5.52696	6.05876	20.1160
	4	0.13312	0.35208	0.73694	1.14807	1.97231	3.73966	5.40173	9.87430	23.2066
	5	0.14174	0.36358	0.77948	1.33503	2.42614	3.32679	6.53788	10.4964	23.2675
	6	0.15407	0.39614	0.92348	1.48957	2.82510	4.20151	7.37259	9.66253	44.0174

APPENDIX IV (Continued)

C_a^2	C_s^2	ρ = 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
	0	0.10855	0.24247	0.41111	0.67646	1.03737	1.61582	2.81606	4.95396	14.1672
	0.25	0.11128	0.25371	0.44517	0.69227	1.14738	1.90455	3.03737	6.03616	14.2494
	0.333	0.11087	0.25394	0.45416	0.71227	1.19124	1.97946	3.17864	4.97193	14.90190
	0.5	0.11325	0.26437	0.46671	0.67138	1.18953	2.09178	3.15166	5.78786	11.0156
3	1	0.11727	0.28078	0.51654	0.86852	1.41421	2.30277	3.87192	7.12311	17.0554
	2	0.12453	0.32619	0.62003	0.97645	1.76824	2.82192	5.07187	9.70364	22.7822
	3	0.13425	0.34646	0.64902	1.26669	2.10237	3.41783	5.46346	12.2220	24.8373
	4	0.13857	0.38195	0.89872	1.47674	2.26911	3.67156	6.98835	10.7047	21.1832
	5	0.14537	0.42131	0.81541	1.46787	3.00509	4.88489	6.44592	16.6080	26.3278
	6	0.15013	0.40890	0.81627	1.68992	3.28547	4.24825	8.77385	10.3973	50.1982
	0	0.10863	0.24715	0.42996	0.72603	1.12429	2.05363	3.33252	6.12272	13.6467
	0.25	0.11165	0.25259	0.45557	0.74655	1.34374	2.07006	3.83790	7.18536	15.5862
	0.333	0.11438	0.25736	0.47436	0.77544	1.28241	2.22168	3.66689	7.37606	15.3655
	0.5	0.11357	0.26955	0.49510	0.82489	1.35884	2.22716	4.38901	10.7399	14.2913
4	1	0.11866	0.28884	0.54334	0.93990	1.58114	2.60599	4.60114	8.65535	21.0680
	2	0.12743	0.32294	0.64825	1.09694	1.86501	2.96871	5.35597	11.2416	24.6432
	3	0.13452	0.35704	0.76677	1.24433	2.54814	3.23842	6.11950	9.16504	30.9026
	4	0.14352	0.41299	0.82828	1.39389	2.42111	3.66788	6.66366	13.3980	39.5400
	5	0.16338	0.41527	0.87287	1.67807	2.49200	4.59061	7.53255	10.3102	28.0481
	6	0.15050	0.48738	1.04915	1.61502	2.88779	5.05676	8.69011	16.1096	24.2348

APPENDIX IV (Continued)

C_a^2	C_s^2	ρ								
		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
	0	0.10974	0.24976	0.45469	0.74198	1.18959	2.34047	4.15988	9.95129	17.7038
	0.25	0.11213	0.25984	0.47350	0.84765	1.39529	2.28061	5.25828	9.42372	14.6508
	0.333	0.11259	0.26807	0.47861	0.82741	1.49704	2.42461	4.306053	10.1099	19.8948
	0.5	0.11602	0.27471	0.51010	0.89675	1.45440	2.38952	4.88110	11.411	23.1195
5	1	0.11963	0.294727	0.56427	1.00000	1.73205	3.0000	5.31662	10.1789	25.0767
	2	0.12778	0.31860	0.66398	1.27566	2.05450	3.36538	5.72102	10.7051	21.0272
	3	0.14059	0.36315	0.73894	1.34695	2.44168	3.90672	6.92747	11.9538	30.1044
	4	0.14772	0.42538	0.81759	1.64341	2.39534	4.29775	9.47411	12.6639	27.8236
	5	0.15710	0.42184	0.83871	1.67065	3.07552	4.16742	12.4958	12.5821	34.4966
	6	0.15621	0.45848	1.00927	2.12429	3.77178	5.05736	9.79020	26.0799	31.3911
	0	0.11006	0.25212	0.45915	0.76373	1.38453	2.51338	4.95082	10.9494	28.3914
	0.25	0.11337	0.26409	0.47717	0.89527	1.46161	2.87657	5.47000	7.56922	26.1561
	0.333	0.11484	0.26825	0.49979	0.84871	1.58220	2.75695	4.77620	9.63092	50.0923
	0.5	0.11611	0.27616	0.50740	0.88359	1.69439q	2.38952	5.52528	12.1382	26.2817
6	1	0.12035	0.29923	0.58114	1.05179	1.87083	3.32768	6.00227	11.6969	29.0831
	2	0.13035	0.34015	0.69221	1.34757	2.03004	3.44372	6.80750	12.4843	36.0844
	3	0.13592	0.38491	0.82157	1.41942	3.17138	4.55609	6.21833	14.0377	31.4323
	4	0.14544	0.44481	0.99054	1.70706	2.40029	4.79811	13.3445	13.4099	27.5852
	5	0.15312	0.48633	0.98696	1.63391	2.84033	5.38700	11.0103	16.7443	63.1330
	6	0.16953	0.48806	0.91924	1.80825	3.91614	5.33443	10.6469	19.6618	29.4355

APPENDIX IV (Continued)

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