A CRITICAL EVALUATION OF THE ROLE OF THE COST OF CAPITAL AS A RISK-ADJUSTED DISCOUNT RATE IN THE ECONOMIC ANALYSIS OF CAPITAL INVESTMENTS

by

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ABSTRACT

This study consists of a critical evaluation of the role of the cost of capital as a "risk-adjusted" discount rate in the economic analysis of capital investments.

In conventional theory, the cost of capital is formulated as a discount rate which serves as a financial standard, in accordance with one variation or another of the following definition: The cost of capital is the minimum acceptable rate of return that a proposed investment in real assets must offer in order to be worthwhile undertaking from the stand-point of the current owners of the firm. Unfortunately, theorists have found it difficult to incorporate a proper measure of risk into the specification of the cost of capital as a single-valued rate of discount. Ezra Solomon, among others, has avoided much of the difficulty by assuming that all projects to be evaluated are of a quality, in respect to uncertainty of future earnings, which is "homogeneous" with the quality of earnings attributed to existing operations. The problem of dealing with investments of a quality significantly different from earnings from existing assets is largely unresolved. This study consists of an analysis of the relationship which should exist between a project's risk and the cost of capital appropriate to its evaluation. The analysis rests upon several simplifying assumptions regarding

the behavior of investors and capital markets; and employs for its investigation two models of risk and valuation: The classical certainty-equivalence model and John Lintner's recently derived risk asset valuation and portfolio selection model.

In recognition of certain weaknesses in the conventional discounted cash flow approaches to capital project evaluation, several theorists including David B. Hertz and Frederick S. Hillier, have proposed that Monte Carlo Simulation and analytical-statistics methods be employed to account for risk by generating stochastic expressions for valuation indices. To the extent that the expression of probabilistic valuation indices depends upon a "risk-adjusted" cost of capital discount rate, there exists the inconsistency of "double accounting for risk;" once in the cost of capital, and once again in the stochastic expression of the indices themselves. This study assesses the relevance of the cost of capital as a discount rate in the generation of stochastic discounted cash flow indices.

The investigation disclosed that: (1) the cost of capital is a derived variable consisting of a complex function of the risk-free rate of interest, and the expected values and risk parameters of earnings expectations of the firm, the project concerned, and securities comprising the market as a whole; that (2) the cost of capital is essentially inefficient as a means of accounting for risk because its correct derivation depends upon the employment of a valuation model which is of itself both sufficient and more direct as a means of evaluation; and (3) that the cost of capital, as a "risk-adjusted" rate of discount is both inappropriate and irrelevant for employment in the generation of stochastic expressions of valuation indices.

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CHAPTER I

INTRODUCTION AND JUSTIFICATION OF THE PROBLEM

Capital investments require the commitment of resources into the uncertain future and hence the acceptance of risk. The evaluation of the riskiness associated with capital investment proposals is therefore an important aspect of capital budgeting.

Unfortunately, financial theorists have found it difficult to incorporate a proper measure of risk into the cost of capital discount rate which has become an essential element of conventional discounted cash flow approaches to capital investment evaluation. According to the conventional theory, in the absence of capital rationing a project is deemed acceptable by discounted cash flow criteria if either (1) the net present value of its expected cash flows is positive when discounted at the cost of capital; or (2) the cost of capital is less than the project's internal rate of return on expected cash flows. In this its application as a "risk-adjusted" rate of discount, the cost of capital is formulated as a financial standard having one variation or another of the following definition: The cost of capital is the minimum acceptable rate of return that a proposed investment in real assets must offer in order to be worthwhile undertaking from the standpoint of the current owners of the firm.[1]

Bierman and Smidt,[2] Porterfield,[3] and Van Horne,[4] have argued that the problem of combining the time value of money and a measure of compensation for risk in a single valued rate of discount such as the cost of capital is not only difficult but may be inefficient as well. But their arguments are of a descriptive, rather than quantitatively analytical persuasion. In essence, they claim that since a single discount rate is employed to account for an investment's total risk, the impact will be much greater upon returns in distant years than upon returns in earlier years, resulting in either an underadjustment for risk in early years or an overadjustment for later returns (unless, of course, risk is expected to increase in a very special pattern with time).

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[1]This definition is adapted from Franco Modigliani and Merton Miller, "Estimates of the Cost of Capital Relevant for Investment Decisions Under Uncertainty," Determinants of Investor Behavior: A Conference of the Universities-National Bureau for Economic Research, p. 182.

[2]Harold Bierman, Jr. and Seymour Smidt, The Capital Budgeting Decision (Second edition), Part III, pp. 281-357.

[3]James T.S. Porterfield, Investment Decisions and Capital Costs, Chap. VII, pp. 107-133.

[4]James C. Van Horne, Financial Management and Policy, Chap. VI, p. 66.

Robichek and Myers,[5] and Chen[6] after them, have shown by quantitative analysis that the "risk-adjusted" rate of discount approach to the market valuation of equity capital involves a complex expression of investor's attitude to risk. In its conventional formulation as a weighted average of costs of sources of funds, the cost of capital includes a measure of the required rate of return on equity and hence its faults and weaknesses as a risk-adjusted rate of discount.

But whatever the difficulties, the cost of capital is a concept fundamental to contemporary financial theory. There is a need, therefore, for (1) a comprehensible quantitative analysis of the relationship which should exist between project risk and the cost of capital discount rate appropriate to its evaluation; and (2) an assessment of the efficiency of the cost of capital in relation to other methods of accounting for risk.

In the traditional approach to capital investment evaluation, the cost of capital is used to discount the

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[5]Alexander A. Robichek and Stewart C. Myers, Optimal Financing Decisions, Chap. V, pp. 313-326.

[6]Houng-Yhi Chen, "Valuation Under Uncertainty," Journal of Financial and Quantitative Analysis, Vol. XI, No. 3, (September, 1967), pp. 313-326. The features of Chen's (and hence Robichek and Myer's) analysis which are essential to this study are presented in Chapter V.

expected values of cash flows to their net present value, or alternatively, is used as a standard of comparison for the internal rate of return on the expected values of the cash flows. In both circumstances the conventional discounted cash flow technique ". . . summarizes into a single figure the quantifiable factors affecting the economic desirability of the project under consideration."[7]

It is argued by Hertz,[8] Hillier,[9] and Hess and Quigley,[10] that a valid criterion for decision where risk is involved must be based on not only a single measure such as the expected value of net present value or internal rate of return, but also upon the variance and other risk parameters of the decision variable. In order to provide measures of risk in addition to the expected value of discounted cash flow indices, these authors' recent advances in the techniques of risk analysis have involved the deter-

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[7] Sidney W. Hess and Harry A. Quigley, "Analysis of Risk in Investments Using Monte Carlo Simulation," <u>Chemical Engineering Symposium Series 42: Statistics and Numerical Methods in Chemical Engineering</u>, p. 55.

[8] David B. Hertz, "Risk Analysis in Capital Investment," <u>Harvard Business Review</u>, Vol. XLI, No. 1, (January-February, 1964), pp. 95-106.

[9] Frederick S. Hillier, "The Derivation of Probabilistic Information for the Evaluation of Risky Investments," <u>Management Science</u>, Vol. IX, No. 3, (April, 1963), pp. 443-457.

[10] Hess and Quigley, <u>op</u>. <u>cit</u>.

mination of probability distributions of net present value, internal rate of return, and other financial indices of valuation, by means of statistical analysis and Monte Carlo simulation.

To the extent that the expression of probabilistic valuation indices depend upon a "risk-adjusted" cost of capital discount rate, there exists the inconsistency of "double accounting for risk"; once in the cost of capital, and once again in the stochastic expression of the indices themselves. Whether or not the cost of capital is relevant to risk analysis by simulation or statistical analysis is worthy, therefore, of assessment.

1. THE PROBLEM

Statement of the problem. It is the objective of the research reported in this thesis (1) to analyze the relationship which should exist between a project's risk and the cost of capital appropriate to its evaluation, given certain simplifying assumptions regarding the behavior of investors and capital markets; (2) to assess the relative efficiency of using the cost of capital, rather than alternative methods, for accounting for risk, and (3) to assess the relevance of the cost of capital as a discount rate in the Monte Carlo simulation and analytical-statistics approaches to the derivation of stochastic discounted cash flow indices.

Limitations of the problem. The objective of this study is not to analyze or recommend algorithms for attaining optimal capital budgets from the total opportunity set of feasible combinations of proposals under consideration. It is solely directed to the clarification of the role of the cost of capital in the evaluation of risky investments.

II. THE APPROACH TO RESEARCH

The research methodology involves two facets; the first consisting of a survey, summary and critical analysis of some current investment and cost of capital theories as they relate to risk evaluation, and the second involving the development and extension of two theoretical models of risk and valuation.

The first aspect of the research procedure establishes the state of the art of current theory, thereby deliniating both the conceptual problems and the theoretical inconsistencies which are relevant to the analysis of risk. Reference is made to the von Neumann-Morgenstern theories of utility and subjective probability in the construction of a model of economic man. The current financial literature is surveyed to derive the normative objective for capital investment management. Relevant aspects of conventional cost of capital theory are elaborated, using as a basis, Ezra Solomon's classic text, The Theory of Financial Management.[11]

The second aspect of the research procedure relates risk to valuation and the cost of capital by the reconstruction and extension of two models of investment behavior. The first model is basically the classical certainty-equivalence model of economic behavior which was recently analyzed by Chen in his critique of the "risk-adjusted" discount rate approach to valuation.[12] The second model is a simple extension of Lintner's portfolio selection and risk-asset valuation model.[13] Both models incorporate the concepts of economic man, the financial objective of management, and the cost of capital, which were established in the first facet of the research.

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[11]Ezra Solomon, The Theory of Financial Management.

[12]Chen, op. cit.

[13]John Lintner, "The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets," The Review of Economics and Statistics, Vol. XLVII, No. 1, (February, 1965), pp. 13-37.

The models are designed to express the relationship which should exist between a project's risk and the cost of capital appropriate to its evaluation, in accordance with the first objective of the research. The models are also used to assess the relative efficiency of the cost of capital as a means of accounting for risk, in accordance with the second objective. The third objective, to analyze the problem of "double accounting for risk" in probabilistic valuation indices, is achieved through conceptual argument founded upon the theory and concepts summarized in the first facet of the research.

Organization of chapters. The approach to the research is reflected in the organization of subsequent chapters. Chapter II presents a model of economic man which incorporates a concept of subjectively measurable risk and a theory of choice under uncertainty. The purpose of the model is to provide an explicit means by which to explore the effect of changes in the "quality" of earnings expectations of investors upon valuation and the cost of capital appropriate to an enterprise and a project proposal. From the concepts of subjective probability in expectations and the von Neumann-Morgenstern utility and axioms of rational decision, the basis for the mean-variance and certainty-equivalence approach to quantifying risk is established.

Chapter III briefly summarizes four different normative objectives for financial management, namely; profit maximization, utility maximization, net present worth maximization, and market value maximization. The chapter serves as an introduction to conventional cost of capital theory by establishing a framework for its elaboration. The irrelevance of profit maximization to conditions of uncertainty is established; and the equivalence of the three other objectives under the idealized conditions of perfect capital markets and rational investors is explained.

Chapter IV summarizes certain contemporary approaches to the theory of the cost of capital, with concentration given to the work of Ezra Solomon. The relevance of the risk-free rate of interest as the cost of capital under the idealized circumstances of perfect certainty, rational behavior, and perfect capital markets is established. The assumption of perfect certainty is then relaxed and the concept of investors' "risk-adjusted" required rate of discount is introduced. Solomon's theory of the cost of capital for conditions of "homogeneity of quality or uncertainty of earnings" is summarized for situations of non-growth and growth in earnings, and simple and complex capital structure. The problem of defining the cost of capital for projects which contravene the "homogeneity of quality" assumption is introduced as a prelude to the second part of the research which involves the analysis of the two models which incorporate subjective risk as a component of value.

Chapter V consists of the development and analysis of a simple reconstruction of the classical certainty-equivalence model of valuation. The model is used to define an expression for the cost of capital appropriate to the evaluation of a project which changes the quality of the earnings of the firm. In this way the cost of capital is shown to be a complex derived function of (1) the risk-free rate of interest, and (2) the means and risk parameters of earnings expectations of the project and existing assets of the firm.

Chapter VI serves two purposes; the first, to summarize relevant aspects of John Lintner's model of risk-asset valuation and portfolio selection, and secondly, by simple extension, to adapt Lintner's sophisticated model to the role served by the simpler certainty-equivalence model of Chapter V. Lintner's model takes into account the observed behavior of diversification of investment portfolios, a characteristic ignored by the simple certainty-equivalence approach. In this respect, Lintner's formulation may be a better reflection of conditions under reality.

Lintner's argument that

> . . . the "cost of capital" (as defined for uncertainty anywhere in the literature) is not the appropriate discount rate to use in accept-reject decisions on individual projects in capital budgeting [all italics in the original] . . . .[14]

prompted this study. Furthermore, the conclusions reached by this research have, for the most part, been previously established by Lintner. This research serves then, only to clarify what in Lintner's elaboration may be so complex as to defy comprehension. The summary and simple extension of Lintner's model shows that the derived value of the cost of equity capital is not only a complex function of (1) the risk-free rate of interest and (2) the means, variances, and covariances of expected earnings of project and firm, but also of (3) the covariances of expected earnings between project, firm, and securities comprising the whole market available to investors. Since the "correct value" of the cost of capital is found by analysis of all the elements required to determine the sign and magnitude of a project's incremental contribution to the value of the firm, the cost of capital is not at all essential to the theoretically correct, and more direct, valuation process.

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[14] Lintner, *op. cit.*, p.15.

Chapter VII describes Monte Carlo simulation and the analytical-statistics approaches to the generation of stochastic expressions of valuation indices, and assesses the relevance of the cost of capital to their processes.

Chapter VIII summarizes the major findings and concepts, states the conclusions, and suggests areas for further research.

III. DEFINITION OF TERMS AND CONCEPTS

Project. The term project refers to any given feasible decision alternative--whether an individual investment or a set of sub-projects--which entails the commitment of capital in expectation of returns.

A risky project is characterized by uncertain returns. Uncertain returns are conceived of as random variables which may be described by subjective probability distributions characterized by expected values, variances, and covariances with the returns of other earning assets, whether existing or envisioned.

The expected value and variance of a random variable return for a project which consists of various sub-projects can be calculated by the appropriate combination of expected values, variances, and covariances for the sub-projects.

In this context a project and a capital budget are synonymous.

<u>The cost of capital</u>. The term cost of capital is defined as the minimum prospective rate of yield, or alternatively, the minimum acceptable rate of return, that a proposed investment in real assets must offer in order to be worthwhile undertaking from the standpoint of the current owners of the firm.

<u>Valuation index</u>. A valuation index, sometimes referred to as a criterion of profitability, summarizes the economic desirability of a proposed project into a single index which serves the purpose of providing a common basis for comparing alternatives. Four common valuation indices are Payback, Average or Accountant's Rate of Return, Net Present Value, and Internal Rate of Return.

<u>Payback</u>. Payback is a valuation index which represents the number of years or periods required to return an original investment by net returns before depreciation but after taxes. Payback favours projects which promise early returns. In this respect it may favour high risk projects of short life over long lived projects which may be much less risky. Often, projects which do not yield their highest returns for a number of years are those strategic to the firm's long term viability and success. Thus, payback may be biased against the very investments which are most critical to the firm's true value as an economic enterprize.

Accountant's rate of return. The accountant's rate of return is usually defined as the ratio of average annual net returns before depreciation but after taxes to the average investment over the life of the project after deducting salvage value. There are several different procedures for calculating variations of this valuation index, i.e., depreciation may be excluded from the numerator of the ratio. The average rate of return is an index superior to payback because it takes into account benefits over the entire economic life of the project. It contains, however, one fundamental weakness which also afflicts payback, and that is its disregard for the time value of money. It treats a dollar to be received at the end of the project's life as equivalent in value to a dollar already in the owner's hands, thereby ignoring the effect of interest on the value of future funds.

Net present value. Net present value, or equivalently, net present worth, is the difference between the present values of the stream of net benefits and the stream of capital costs, both discounted at the cost of capital.[15]

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[15]In the presence of capital rationing the cost of capital is not appropriate to the derivation of net present worth. See James C.T. Mao, Quantitative Analysis of Financial Decisions, Chap. X, pp. 5-9.

As a valuation index it accounts for returns over the full economic life of a project, and as well, it accounts for the time value of money in the discounting process. It is represented symbolically as follows:

$$NPV = \sum_t \bar{R}t(1 - k)^{-t} - \sum_t \bar{C}_t(1 - k)^{-t},$$

where k is the cost of capital, and $\bar{R}_t$ and $\bar{C}_t$ are respectively the expected values of returns and costs in period t.

<u>Internal rate of return</u>. The internal rate of return of a project is the interest rate that equates the present value of the expected future receipts to the present value of its investment outlays, i.e., it is that discount rate, "irr," for which,

$$\sum_t \bar{R}_t(1 - irr)^{-t} - \sum_t \bar{C}_t(1 - irr)^{-t} = 0.$$

Like the net present value index, the internal rate of return accounts for the full economic life of the project. But unlike net present value, which includes the cost of capital discount rate in its derivation, the internal rate of return is uniquely determined by the "shape" of benefit and investment streams. It is compared with the cost of capital after its derivation in the accounting for the time and risk value of the funds committed. For certain patterns of cash flows the internal rate of return may be ambiguous since more than one interest rate may serve to equate the streams of benefits and costs. Teichrow, Robichek and Montalbano give an analytical treatment of the multiple rate of return problem.[16]

Business risk. Business risk is the risk inherent in the physical operations of the firm; it arises simply from the inability to insure absolutely stable sales, costs and profits. The corporation cannot be entirely protected from the vicissitudes of the market. Business risk exists independently of the means by which the firm is financed.[17]

Financial risk. Financial risk is added to business risk when a corporation, instead of meeting all capital requirements with equity funds, borrows a portion of its

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[16] Daniel Teichrow, Alexander Robichek and Michael Montelbano, "An Analysis of Criteria for Investment and Financing Decisions Under Certainty," Management Science, Vol. XII, No. 3, (November, 1965), pp. 151-179; and by the same authors, "Mathematical Analysis of Rates of Return Under Certainty," Management Science, Vol. XI, No. 3, (January, 1965), pp. 395-403. Also see, Mao, op. cit., Chap. VI, pp. 22-41.

[17] The definitions of Business and Financial Risk are taken directly from, Robichek and Myers, op. cit., pp. 17-18.

needs. Borrowing increases risk in two ways. First, borrowing means that the company must meet fixed interest charges and principal repayment schedules or face bankruptcy. Second, to the extent that borrowing is used, the fluctuations of the annual net cash flow available for payment of dividends or for reinvestment will be greater as a proportion of the stockholders' investment.

CHAPTER II

ECONOMIC MAN

The model which underlies the following analysis of the relationship between project risk and the cost of capital describes the behavior of an individual investor who is faced with an investment problem. His problem is to commit a certain amount of his wealth to the acquisition of financial assets in the form of common stock securities. The investor does not know with certainty what return each of the available securities will yield, and he is therefore confronted with the task of making his investment decision under uncertainty.

The model incorporates two conceptual mechanisms which are essential to the decision-making process under uncertainty; a mechanism which establishes the form of the investor's expectations as to the respective returns from alternative securities, and a mechansim which establishes the investor's preferences amoung the available securities once his expectations are fixed.

I. THE RISK-EXPECTATIONS MECHANSIM

Under the terms of the model the investor is required to invest in specific securities under uncertainty as to the outcome of his actions. Nevertheless, prior to making his investments, the investor is presumed to make judgements as to the range and likelihood of the future performance of each of the investment opportunities which confront him. Whether the judgements are made on the basis of analytical projections of past trends and events, pure intuition, or a mixture of both will depend, of course, on the man and his circumstances. In any case, such judgements constitute the investor's expectations.

Subjective probability. An important step in the development of a theory of behavior under uncertainty involved the introduction of the concept that subjective probability distributions could be used to describe an individual's expectations as to the range and likelihood of possible outcomes of his decision activities.[1] For purposes of this analysis it is assumed that investors form their expectations by assigning subjective probabilities to the

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[1] A classic work explaining the concept of subjective probabilities is Leonard Savage's, The Foundations of Statistics. Savage develops a system of postulates which relates "degree of confidence," interpreted in a behavioral sense, to mathematical probabilities. For a critical review of the subject and relevant experiments, see Ward Edward's "Behavioral Decision Theory," in the Annual Review of Psychology, Vol. XII, 1961, pp. 473-498; or alternatively, "The Theory of Decision Making," by the same author, in the Psychological Bulletin, Vol. XLI, No. 4, (July, 1954), pp. 380-417.

uncertain returns of the securities that they may buy.

Investor's expectations. In respect to investor's expectations, and in particular the form of their subjective probability distributions, there is no reason to assume that expectations must necessarily be equal across a population, or for that matter equivalent to those held by a given firm's management. To quote Schlaifer:

> We emphasized that a subjective probability is necessarily an expression of a personal judgement and is therefore necessarily subjective [italics in the original] in the sense that two reasonable men may assign different probabilities to the same event. This by no means implies, hoever, that a reasonable man will assign probabilities arbitrarily.[2]

Variation in expectations is likely in a real economy because (1) the information available to different investors varies greatly in quantity and quality, and from one time to the next, and (2) human character, by its very nature, tends to create divergencies in viewpoint even in regards to equal information.

The formulation of expectations. It is beyond the scope of this work to establish a theory of how investors' expectations are derived. But it is reasonable to assume

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[2]Robert Schlaifer, Probability and Statistics for Business Decisions, p. 15.

that investors' views are in some part extensions of historical patterns.

In Schlaifer's words:

> Reasonable men base the probabilities which they assign to events in the real world on their experience with events in the real world, and when two reasonable men have roughly the same experience with a certain kind of event they assign it roughly the same probability.[3]

Richard Mattessich expresses much the same philosophy in his definition of the "principle of insufficient reason":

> This "principle of insufficient reason," well testable by observing human (and even animal) behavior asserts that <u>in the absence of better evidence about the future one assumes continuation of the present state of an object or the past trend of an event</u> [italics in the original].[4]

<u>Investors' expectations and corporate forecasts</u>. In a market characterized by both uncertainty and some degree of irrational behavior, management's estimates of investors' expectations may not match management's "informed" forecasts of corporate profitability. Nevertheless it is by investors' personal evaluation of their own expectations that the market value of a firm's shares is established.

Disparity between management's relatively knowledgable predictions and investors' speculation is nurtured by

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[3] <u>Ibid.</u>, p. 15.

[4] Richard Mattessich, <u>Accounting and Analytical Methods</u>, p. 25.

(1) the real market necessity of maintaining corporate secrets for competitive reasons, and (2) the general practice of historical rather than "present value" disclosure in financial reports.

Nevertheless, to the extent that investors compensate for weaknesses in their own predictive performance in the past, it is assumed that over the medium to long term the expectations of investors and management will tend to converge. Consequently it is assumed that management is justified, in a normative context, in the employment of corporate forecasts as proxies for the expectations of investors.

Expectations and risk. If the assumption that investors ascribe subjective probability distributions to future uncertain events in the formulation of their expectations is accepted, it is possible to speak of "riskiness" in terms of certain qualities of those distributions. In this manner, Robichek and Myers specify three broad factors which determine the riskiness of a stock to an investor.[5] They are: (1) the dispersion of the subjective probability distributions, (2) the form of the distributions, and (3) the extent to which random fluctuations in the dividends are correlated with the variation in returns of other investment opportunities.

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[5] Alexander A. Robichek and Stewart C. Myers, Optimal Financing Decisions, p. 79.

Relative riskiness. A convenient index of the riskiness of an asset is given by its "relative riskiness," which is defined as the quantity of risk per dollar of expected return. If risk is totally described by the variance of a subjective probability distribution, the relative riskiness of the distribution is given by the ratio of the variance to the expected value.[6]

Risk aversion. However expectations are determined, and whatever form they take, each individual is deemed to act upon his expectations in accordance with his personal preferences. Financial theorists frequently ascribe preference against risk, or "risk-aversion" to investors in the aggregate. Risk-aversion, as a generalized behavioral trait, is given the following definition by Robert Wayne White:

> An individual is aversed to risk in a given situation if (a) given the choice between two investments with the same expected returns, he chooses the alternative with the less risk or (b) given the choice between two investments of the same risk, he choses [sic] the alternative with the largest expected return.[7]

For purposes of this analysis it will be assumed that investors are universally risk-averse.

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[6]James C.T. Mao defines one measure of relative riskiness as the "coefficient of variation" which is the ratio of the standard deviation to the expected value of the random variable; Quantitative Analysis of Financial Decisions, Chap. X, p. 45.

[7]Robert Wayne White, "Risk Aversion in Open-End Investment Companies" (unpublished Master's thesis, The University of British Columbia, 1968), p. 17.

II. THE MECHANISM OF RATIONAL CHOICE

It is assumed that the investor will make a "rational choice" between the subjective probability distributions which make up his expectations of investment opportunities. Rationality under uncertainty is assumed to exist if the investor's choice is motivated by a desire to maximize the expected value of a function which assigns utilities to the possible outcomes of investments. Rational behavior is, therefore, to be in accordance with Savage,[8] von Neumann-Morgenstern,[9] or equivalent axiom systems. In other words, the investor is assumed to act in accordance with "The Expected Utility Maxim" which states that the rational investor should behave as if (1) he holds a consistent set of preferences, (2) he attaches numbers called "utilities" to each of the possible outcomes to the alternative acts open to him, and (3) he selects the one alternative course of action from the set available which exhibits the greatest expected value of utility.[10]

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[8]Savage, op. cit., especially pp. 86-87. Also see, Milton Friedman and Leonard J. Savage, "The Utility Analysis of Choices Involving Risk," Journal of Political Economy, Vol. XLVI, (August, 1948), pp. 279-304; and William J. Baumol, Economic Theory and Operations Analysis, pp. 331-346.

[9]J. von Neumann and O. Morgenstern, Theory of Games and Economic Behavior.

[10]Adapted from, Harry M. Markowitz, Portfolio Selection: Efficient Diversification of Investments, p. 208.

Whether or not "real" investors behave in general accordance with such assumptions about "rational man" has been the subject of considerable debate.[11] Furthermore, experimental evidence in support of the descriptive reliability of the model of rational man and utility theory cannot yet justify generalized acceptance of the axioms beyond a normative context. Markowitz, for example, cites observations which show inconsistencies in behavior which seem to invalidate the axioms as descriptive principles.[12] Ward Edwards has concluded that it is fairly easy to construct examples of behavior that violate the axioms, especially when the amounts of money involved are very large, or when the probabilities or probability differences are extremely small.[13] Nevertheless, for purposes of the following analysis, the assumption of rational economic man will be taken since it serves to illuminate certain problems inherent in the relationship between risk and the cost of capital which do not disappear if behavior is subsequently assumed to deviate somewhat from the axiomatic norm.

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[11]See, for example, Ward Edwards, "Behavioral Decision Theory," op. cit., or "The Theory of Decision Making," op. cit.

[12]Markowitz, op. cit., pp. 218-228.

[13]Ward Edwards, "The Theory of Decision Making," op. cit.

The form of the utility of returns function.[14] The essential element in the investor's mechanism of rational choice consists of a utility of returns function that ascribes to any given return from investment a numerical measure which reflects the desirability of the return to the investor.

The utility of returns function is related to the utility of wealth function, since returns are simply incremental additions to wealth. The concept of utility of wealth is fundamental to utility theory. In accordance with the concept, investors are assumed to prefer higher expected future wealth to lower expected future wealth, ceteris paribus, implying that

$$(dU/dW) > 0 \; ;$$

U being a total utility function of the form,

$$U = f(\bar{W}, w_1, w_2, w_3, \ldots, w_n) \; ,$$

where $\bar{W}$ is the expected value of uncertain future wealth, and w_i is the $i^{th.}$ moment of the subjective probability distribution which describes the wealth expectation.

On the assumption that investors are risk-averse, choosing an investment having a lower risk over one with a

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[14]The theoretical content of this section owes much to, Susan J. Lepper, "Effects of Alternative Tax Structures on Individual's Holdings of Financial Assets" in Risk Aversion and Portfolio Choice, Cowles Foundation Monograph No. 19, pp. 51-109.

greater risk, ceteris paribus, then

$$(dU/dw_i) < 0 ,$$

for all i relevant moments of the subjective probability distribution of expected wealth.

Given the specification of the utility of wealth function, U(W), the utility of returns function, U(r), is derived as follows: Assume that the investor decides to commit a given amount W_i of his present wealth to investment. Let W_t be his expectation of terminal wealth, and let r be the rate of return expected on his investment. Then,

$$r \equiv (W_t - W_i)/W_i$$

and hence, $W_t = rW_i + W_i = W_i(1 + r)$.

Since terminal wealth is shown to be directly related to the expectation of rate of return r, it is possible to express the investor's utility in terms of returns rather than wealth, i.e.,

$$U(r) = U(\bar{r}, r_1, r_2, r_3, \ldots, r_n) ,$$

where $\bar{r}$ is the expected value of return, and r_i is the $i^{th.}$ moment of the subjective probability distribution which describes returns expectations.[15]

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[15]The relationship between utility of wealth and returns is adapted from, William F. Sharpe, "Capital Asset Prices: A Theory of Market Equilibrium Under Conditions of Risk," The Journal of Finance, Vol. XIX, No. 3, (September, 1964), pp. 425-442.

The shape of the utility of returns function, hereafter referred to simply as the utility function, determines which parameters of the subjective probability distributions of the investor's expectations are pertinent to the decision process. The reasoning proceeds as follows: A rational investor is presumed to always act so as to maximize the expected value of utility; that is, he acts so as to

$$\text{maximize } E(U_r) = \int U(r)f(r)dr \ ,$$

where U(r) is the utility function of r, and f(r) is the perceived likelihood that the value r will occur. If U(r) is a polynomial function of r, $E(U_r)$ consists of a sum of integrals. Each term in the sum of integrals will contain one of the powers of r which constitute U(r). Since,

$$\int r^k f(r)dr$$

is by definition the $k^{th.}$ moment of f(r), $E(U_r)$ will contain one moment of f(r) for each power of r appearing in the polynomial expression for U(r). Therefore, the number of parameters of the probability distribution f(r) which are pertinent to the investment decision depends upon the degree of the polynomial which defines the utility function.

Consider, for example, a utility function of the form

$$U(r) = \sum_{i=0}^{n} B_i r^i .$$

If $U(r)$ is a quadratic, $E(U_r)$ is simply,

$$E(U_r) = B_o + B_1\bar{r} + B_2\bar{r}^2 + B_2\overset{v}{r} ,$$

where $\overset{v}{r}$ signifies the variance of $f(r)$. The expression shows that the decision-maker's choice would be dependent upon the mean and variance of $f(r)$ but not upon its higher order moments, such as skewness and kurtosis.[16]

Furthermore, if r is certain; that is, if $f(r)$ degenerates so that the whole mass of probability is concentrated at one point $\bar{r}$, $f(r)$ has no second moment, and even though $U(r)$ may be an $n^{th.}$ order quadratic, only the mean of $f(r)$ would be relevant to rational choice. It therefore follows that the number of moments of the probability distribution of r which are relevant to a rational decision will be equal to which ever is the lessor or (1) the degree of the expression of $U(r)$, or (2) the number of moments which exist for the probability distribution, $f(r)$.

Given specifications for the shape of a utility function $U(r)$, $E(U_r)$ can be found in terms of the moments of $f(r)$. By differentiating $E(U_r)$ with respect to each of the relevant moments of $f(r)$ it is possible to determine whether the investor has a preference or an aversion for

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[16]Fred D. Arditti found that the third moment, skewness, and the fourth moment, kutosis, were both reasonable risk measures, by an empirical investigation; "Risk and the Required Rate of Return on Equity," The Journal of Finance, Vol. XXII, No. 1, (March, 1967), pp. 19-36.

risk parameters. If a particular derivative is positive, a preference exists; if negative, an aversion exists.

The function, $U(r) = r - Br^2$, for which

$$E(U_r) = \int_r (r - Br^2)f(r)dr = \bar{r} - B\bar{r}^2 - B\check{r} ,$$

has, for example,

$$(dE(U_r)/\, d\check{r}) = -B ,$$

which for a positive value of B indicates risk aversity. Note that for increasing variance, $\check{r}$, ceteris paribus, the expected value of utility declines.

<u>Indifference curves</u>. Indifference maps are implicitly contained in utility function-probability distribution systems. An indifference curve (surface) is simply a locus of points representing sets of values of moments of $f(r)$ for which expected utility is constant. From the equation,

$$E(U_r) = \sum_{\text{all } i} \int_r B_i r^i f(r)dr ,$$

each relevant moment can be expressed as a function of every other moment and $E(U_r)$. By setting $E(U_r)$ at various constant values, a family of indifference curves (surfaces) can be derived.

For the simple example of $U(r) = r - Br^2$, for which $E(U_r) = \bar{r} - B\bar{r}^2 - B\check{r}$, the equation for an indifference curve of the form $(\bar{r}, \check{r})$ is given by

$$\check{r} = A + B\bar{r}^2 - \bar{r} \,.$$ [17]

Figure 1 shows that the shape of such a family of indifference curves is concave to the axis of expected return, $\bar{r}$, with utility increasing from curve to curve in the North-West direction, i.e., curve I_2 is of lower utility than curve I_3.

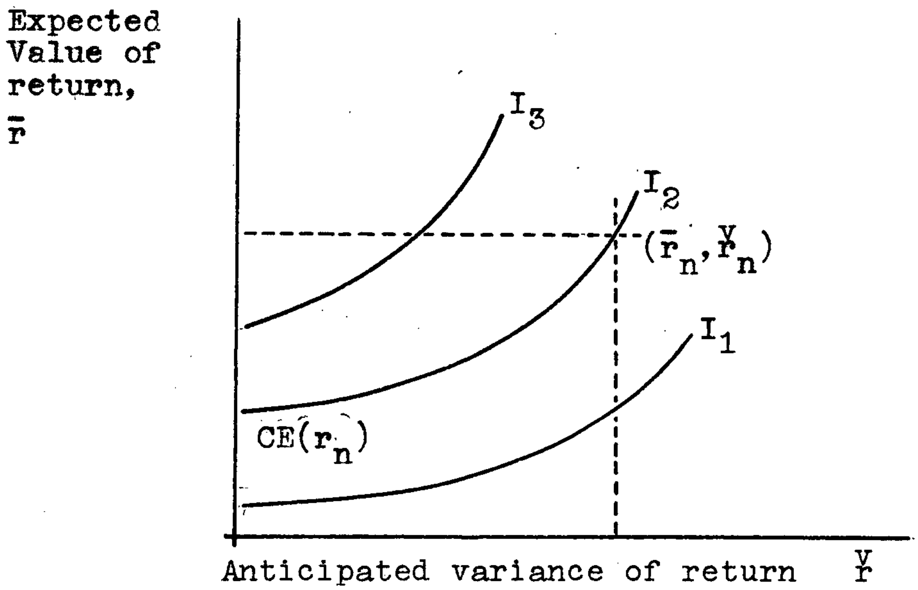

FIGURE I

A FAMILY OF INDIFFERENCE CURVES APPROPRIATE TO THE UTILITY FUNCTION-PROBABILITY FUNCTION SYSTEM, U(r) - f(r).

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[17] According to Karl Borch, (1) if investor expectations are formulated as n-parameter subjective probability distributions, where n 2, and (2) if investors' attitudes to risk are to be completely described in terms of a mean-variance system, then the only form the function U(r) can have if the consistency requirements of von Neumann and Morgenstern

Once the form of the indifference curve has been established it is possible to specify a certainty-equivalent, CE(r) for any mean-variance pair. In figure 1, the certainty-equivalent for the pair ($\bar{r}_n$, $\bar{Y}_n$) is defined by the intersection of the indifference curve appropriate to the pair to the ordinate, i.e., CE(r_n).

It is not necessary to formulate indifference curves in order to obtain the certainty-equivalent, or cash-equivalent as it is sometimes termed, of an uncertain or risky return expectation. Once the utility function of the investor is known it is possible to cpmpute the expected utility of the subjective distribution of his returns expectations. This simply requires that the utility of each possible outcome be multiplied by its assigned probability. The resulting figures are added to obtain the expected utility of the distribution. This expected utility is then converted into its certainty or cash-equivalent by reference to the utility function of the

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shall be fulfilled is

$$U(r) = A + B_1r - B_2r^2,$$

that is, a quadratic in r. Only so long as r takes on values in the interval, -oo = r = (1/2b), will marginal utility be increasing with incremental increases in the expected value of returns. See, Karl Borch, "A Note on Utility and Attitudes to Risk," Management Science, Vol. LX, (July, 1963), pp. 697-700.

investor. Nevertheless, the concept of indifference curves is pedagogically effective, and serves to clarify a relatively complex conception.

<u>The certainty-equivalence factor</u>. The certainty-equivalence factor, a_i , is defined by the relationship

$$CE(r_i) = a_i\bar{r}_i \ .$$

The value of the certainty-equivalence factor for the utility function-probability distribution system, $U(r) = r + Br^2$ and $f(r)$ is found as follows: Since by definition, $CE(r_i) = E(U_{r_i})$, then

$$a_i = E(U_{r_i})(\bar{r}_i)^{-1} = \frac{r_i - B\bar{r}_i^2 - BV_i}{\bar{r}_i} = 1 - B\bar{r}_i - B(V_i/\bar{r}_i).$$

<u>Mean-variance indifference curves</u>. Indifference curves relating expected value to variance (or equivalently to standard deviation) as shown in Figure 1, page 31, are perhaps the most familiar formulations of investor attitude toward risk in economic and financial theory. It has been shown, however, that the mean-variance indifference curves can only be derived from the von Neumann-Morgenstern axioms if an arbitrary restriction is placed upon either the subjective probability distributions, or upon the form of the investor's utility function. The arbitrary restriction is that (1) the utility of return fuction is quadratic, or alternatively, that (2) the investor's subjective

expectation are all formulated as probability distributions of a two-parameter (mean-variance) family, such as normal distributions.[18]

If the mean-variance formulation is taken to be representative of the simplest conceivable case for a risky decision system, it may be justifiably employed to investigate relationships which would not be invalidated under more complex circumstances. Therefore, in the analysis which follows, riskiness will be considered in the context of variance alone. Nevertheless, it is significant that the results which apply to the variance-only case will be suitable for generalization to less restricted situations for which higher moments have a bearing on investors' perceptions of relative riskiness.

III. SUMMARY

The value of a security to an investor will be a function of (1) his expectations as to the monetary returns which he anticipates will accrue from his investment, and (2) his personal utility function, which ascribes a worth

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[18] Borch, op. cit. Also see Jack Hirshleifer's, "Efficient Allocation of Capital in an Uncertain World," The American Economic Review, Vol. LIV, No. 3, (May, 1964), pp. 77-85, especially, p. 80; and also J. Tobin, "Liquidity Preference as a Behavior Towards Risk," The Review of Economic Studies, Vol. XXVI, No. 1, (February, 1958), pp. 65-86.

to his expectations that is dependent upon expected values of anticipated returns and their inherent subjective riskiness.

Investors are assumed to be universally rational in accordance with Savage or von Neumann-Morgenstern axioms; and as well, are assumed to be risk-averse. In order to deal with risk in terms of a two-parameter, or dual co-ordinate system, involving only means and variances of returns, it is assumed that either (1) investors' utility of returns functions are quadratic without limitation on the form of their subjective probability distributions for uncertain future events, or (2) that investors expectations are formed as two-parameter subjective probability distributions, with no restriction on the form of their utility of returns functions.

In establishing the terms for corporate capital budgeting functions, it is taken that the management of enterprises founded upon the issue of securities is justified, in a normative context, in the employment of corporate forecasts as proxies for the expectations of investors.

CHAPTER III

THE OBJECTIVE OF CAPITAL BUDGETING

It is a basic trait of economic man and economic entities, whether industrial concerns or even nations, that wealth be employed for productive gain. Such is the problem of capital allocation or capital budgeting. Available funds, whether currently held or available for utilization by other means, must be allocated to their most satisfying employment.

It is convenient to conceive of three essential elements to any capital allocation program; an economic objective, a method of measuring and comparing alternative employments of funds, and a criterion of choice or a financial standard, that when applied, will lead the economic unit to its objective. These three elements; the objective, the measurement method, and the financial standard, are concisely represented in Ezra Solomon's statement that:

> Both . . . the Net Present Value and Internal Rate of Return approaches will identify all available proposals that promise to increase net present worth. . . . Both depend heavily on a correct measure of k, the cost of capital. This serves in either formulation as a fundamental standard of financial performance that determines the acceptability of all uses of funds.[1]

This chapter is concerned with the first element of capital budgeting theory; that is, the establishment of a

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[1]Ezra Solomon, The Theory of Financial Management, p. 20.

normative objective for financial management of capital investments. The chapter briefly summarizes four different approaches which have been proposed in the literature, namely; profit maximization, utility maximization, net present worth maximization, and market value maximization. This chapter serves as an introduction to conventional contemporary cost of capital theory by establishing a framework for the elaboration of that controversial matter. The mechanics of the second element, the discounted cash flow approach to the measurement and comparison of uses of funds, has been summarized in Chapter I.

I. NET PRESENT WORTH MAXIMIZATION

According to Ezra Solomon, the prime objective of capital investment management should be to maximize shareholder's net present worth.[2] But as will be shown, this formulation does not appear to be a universal precept among contemporary financial theorists. Nevertheless, its construction is in close accord with the familiar and widely accepted discounted cash flow methods of Net Present Value and Internal Rate of Return, as can be readily perceived in Solomon's definition:

> The gross present worth of a course of action is equal to the capitalized value of the flow of future expected benefits, discounted (or capitalized) at a rate which reflects their certainty or uncertainty.

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[2] *Ibid.*, Chap. II, pp. 15-26.

Wealth or net present worth is the difference between gross present worth and the amount of capital investment required to achieve the benefits being discussed.[3]

In algebraic symbolism, net present worth is defined as

$$NPW = V - I,$$

where I is the capital required to pursue the course of action, and V is the gross present worth of the course of action.

On the assumption that the capital invested will return a perpetual, growthless, stream of net dollar returns of expected value $\bar{R}$ for each period,

$$V = \sum_{t=1}^{\infty} \frac{\bar{R}}{(1 + k)^t} = \frac{\bar{R}}{k}\ ,^{4}$$

where k is the rate of discount which reflects both the time value of money and the appropriate measure of compensation for the uncertainty surrounding $\bar{R}$.

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[3] Ibid., p. 20.

[4] From Solomon, op. cit., p. 24:

$$V = E\left[\frac{1}{1+k} + \left(\frac{1}{1+k}\right)^2 + \left(\frac{1}{1+k}\right)^3 \ . \ . \ . + \left(\frac{1}{1+k}\right)^{\infty}\right]$$

$$= E\left(\frac{1}{1+k}\right)\left[\frac{1}{1+k} + \left(\frac{1}{1+k}\right)^2 . \ . \ . + \left(\frac{1}{1+k}\right)^{\infty}\right]$$

The sum of the geometric progression inside the brackets is given by the formula (1 + k)/k. Thus,

$$V = E\ (1+k)^{-1}(1+k)(k)^{-1} = E/k\ .$$

Hence, net present worth is given by

$$NPW = \sum_{t=1}^{\infty} \frac{\bar{R}}{(1 + k)^t} - I.$$

Consider then, an expansion of existing assets which promises risky returns of R from an investment of I. Net present worth will be increased only if the change in net present worth, ΔNPW, attributable to the new assets is positive, i.e., if

$$\Delta NPW = \sum_{t=1}^{\infty} \frac{\bar{R}}{(1 + k)^t} - I > 0 .$$

As can be clearly seen, the expression for the change in net present worth is equivalent to the conventional formulation for net present value. It is explicit in Solomon's approach to the measurement of uses of funds, and the definition of the financial standard, k, the cost of capital, that a positive net present value is equivalent to a positive contribution to net present worth, and hence a step toward the fulfilment of the primary objective of financial management.

In order to place Solomon's thesis in its proper perspective, three other theories as to the proper objective of financial management will also be discussed.

III. PROFIT MAXIMIZATION

Net present worth maximization is not the same as profit maximization, which has been and still is the valid goal of an economic enterprise which functions in accordance with micro-economic theory. In the idealized world of perfect certainty, wherein funds are available in unlimited supply at a "pure" rate of interest, profit maximization involves setting output at that level for which marginal revenue equals marginal cost. In this its proper context, profit maximization is the proper means of achieving the most efficient use of society's economic resources. But the idealized world of economic theory is not the real world, which writhes; perhaps fortunately for those of an enterprising and adventurous spirit; in a morass of uncertainty. Consequently, profit maximization has been found wanting as a normative objective for real-world economic activity. To quote Modigliani and Miller:

> Under uncertainty there corresponds to each decision of the firm not a unique profit outcome, but a plurality of mutually exclusive outcomes which can best be described by a subjective probability distribution. The profit outcome, in short, has become a random variable and as such its maximization no longer has an operational meaning. Nor can this difficulty generally be disposed of by using the mathematical expectation of profits as the variable to be maximized. For decisions which affect the expected value will also tend to affect the dispersion and other characteristics of the distribution of outcomes.[5]

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[5]Franco Modigliani and Merton Miller, "The Cost of Capital, Corporation Finance and the Theory of Investment," in The Management of Corporate Capital, edited by Ezra Solomon, p. 152.

III. UTILITY MAXIMIZATION

Under conditions of uncertainty, wherein outcomes are described in terms of subjective probability distributions, cardinal utility (in the von Neumann-Morgenstern sense)[6] has been developed to explain how individuals should make choices or decisions, and maximization of utility has been considered as an alternative goal for the firm. But utility maximization requires that alternative outcomes of a course of action be valued in accordance with a utility function. Although it has been shown that the form of an individual's cardinal utility function can be found empirically, the task is time-consuming and expensive, and the results may be justifiably viewed with distrust. Experiments to date have involved laboratory subjects in games using insignificant sums of real money; i.e., bets of pennies, nickles and dimes, and payoffs in the tens of dollars;[7] or alternatively, have employed practicing management in imaginary situations involving large imaginary investment

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[6]For a distinguishment between classical and von Neumann-Morgenstern "cardinal" utility see, James C. T. Mao, Quantitative Analysis of Financial Decisions, Chap. II, p. 39; and William J. Baumol, Economic Theory and Operations Analysis, pp.

[7]F. Mosteller and P. Nogee, "An Ecperimental Measure of Utility, "Journal of Political Economy, Vol. XIX, (1951), pp. 371-404.

decisions under contrived field conditions.[8] There is no ready evidence that utility functions so derived remain stable over time. Furthermore, a cardinal utility function ". . . cannot be said to have any measurable relation with satisfaction . . . ."[9] This is equivalent to saying that while relative utilities are measurable, absolute utility is not, the reason being that the scale on which utility is measured has no natural origin. Consequently, the aggregation of utility over a population is patently impossible, making the maximization of utility across a body of shareholders a problem of definition at the very outset. To quote Ezra Solomon,

> Whose utility scales do we use--the owner's management's, or society's? And how do we measure utility preferences so that this criterion can lead to decisions? The approach does not provide a solution to these difficulties. To use an analogy, the utility approach takes the swimmer some distance from the shore and leaves him there, out of his depth.[10]

Having so neatly disposed of utility maximization as a contender for the operational goal, Solomon offers ". . . an alternative and useful solution . . . provided by the concept

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[8] Ralph O. Swalm, "Utility Theory--Insights into Risk Taking," Harvard Business Review, Vol. 44, No. 6, (November-December, 1966), pp. 123-136.

[9] Alexander A. Robichek and Stewart C. Myers, Optimal Financing Decisions, p. 75.

[10] Solomon, op. cit., p. 20.

of wealth-maximization or net present worth maximization."[11] Although Solomon's alternative seems weakened by the stigma of acceptance by default, he offers this justification:

> The basic rational for the objective of wealth-maximization . . . is that it reflects the most efficient use of society's economic resources and thus leads to a maximization of society's economic wealth.[12]

As intuitively acceptable as Solomon's proposition may seem; at least to those who profess a concern for the welfare of society as a whole, as is popular today; other theorists react to the apparent weaknessess and difficulties inherent in utility maximization by proposing what appear to be alternative goals.

IV. MARKET VALUE MAXIMIZATION

Modigliani and Miller advocate market value maximization as a basis for an operation definition of the cost of capital and a workable theory of investment.

> Under this approach any investment project and its concommitant financing plan must pass only the following test: Will the project, as financed, raise the market value of the firm's shares? If so, it is worth cost of capital to the firm. Note that such a text is entirely independent of the tastes of the current owners, since market prices will reflect not only their preferences but those of all potential owners as well. If any current stockholder disagrees with management

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[11] Solomon, _op_. _cit_., p. 20.

[12] Solomon, _op_. _cit_., p. 22.

> and the market over the valuation of the project, he is free to sell out and reinvest elsewhere, but will still benefit from the capital appreciation resulting from management's decision. 13

Since Modigliani and Miller stress the role of the cost of capital, it is implicit in their argument that its employment as a financial standard should involve net present valuation in one form or another. Consequently there exists a resemblance to Solomon's net present worth maximization criterion.

Robichek and Myers also advocate market value maximization, at least when the common shares of the firm are widely traded:

> Even though a decision decreases the "value" of the stock to an investor, he will be better off if, as a result of the decision, the stock price rises above the original "value" of the stock to this investor.
>
> Given perfect capital markets and equilibrium, this condition will hold whenever market price rises, since at equilibrium every investor's valuation of a marginal share of the stock will be equal to the market price. In this case, any increase in market price benefits *every* [italics in the original] stockholder, regardless of how any individual investor's estimate of the stock's value changes. . . .14

It must be recognized that Robichek and Myers appeal to "perfect capital markets" in their arguement for market

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[13] Modigliani and Miller, *op. cit.*, p. 152.

[14] Robichek and Myers, *op. cit.*, p. 75.

value maximization. In a perfect capital market,

> . . . no buyer or seller (or issuer) of securities is large enough for his transactions to have an appreciable impact on the then ruling price. All traders have equal and costless access to information about the ruling price and about other relevant characteristics of shares. . . . No brokerage fees, transfer taxes or other transaction costs are incurred when securities are bought, sold, or issued. [15]

Modigliani and Miller argue that under perfect capital markets there is an equivalence between market value maximization, utility maximization, and maximizing economic welfare:

> Under perfect capital markets there is a one-for-one correspondence between "worthwhileness" in the above sense and the current market value of the owners' interest. If the management of the firm takes as its working criterion for investment (and other) decisions "maximize the market value of the shares held by current owners of the firm," then it can be shown . . . that this policy is also equivalent to maximizing the economic welfare or utility of the owners. Thus under the assumptions, valuation and the cost of capital are intimately related.[16]

James T.S. Porterfield gives a more general authority to the equivalence between maximizing market value and maximizing wealth, subject however to the single qualification that the value of the firm is independent of the value of

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[15] Robichek and Myers, op. cit., p. 8

[16] Franco Modigliani and Merton Miller, "Estimates of the Cost of Capital Relevant for Investment Decisions Under Uncertainty," in Determinants of Investor Behavior: A Conference of the Universities-National Bureau for Economic Research, p. 182.

other assets, and vice versa.

> No matter what valuation formula is assumed maximizing market value is a necessary condition to maximization of the owner's wealth. This is a truism since an owner's total wealth is equal to the value of his holdings of the shares of the firm, plus the value of his outside holdings, and we have assumed that these two values are independent of each other. No matter what the road to maximum share value, it leads in the direction of maximum wealth. [17]

Summary. Each of the authors cited have, explicitly or implicitly, offered one and the same objective for financial management--the greatest satisfaction of the common shareholders' preferences. Furthermore, since increased current share valuation, ceteris paribus, obviously increases shareholders' current wealth, which in turn implies increased utility, this objective of optimizing shareholders' utility has in practice been identified with the maximization of the current value of the common stock. It is clear that the objective of maximizing utility directly has been rejected as a working objective because of the difficulty of its application. Instead, market value maximization and its equivalents have been set up as proxy objectives which are assumed to make application feasible.

In recent publications dealing with the theory of portfolio selection, Lintner and Sharpe propose normative

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[17] James T.S. Porterfield, Investment Decisions and Capital Costs, p. 69.

theories of market equilibrium under conditions of risk which go far in explaining both the relevance and the applicability of utility theory to capital budgeting.[18] Lintner extends the theory to include normative aspects of the capital budgeting decisions of a company whose stock is traded in the market. That his conclusions hold great portent for cost of capital theory is evident in the extraction:

> There can be no "risk-discount" rate to be used in computing present values to accept or reject individual projects. In particular, the "cost of capital" as defined (for uncertainty) anywhere in the literature is not the appropriate rate to use in these decisions even if all new projects have the same "risk" as existing assets [all italics in the original]. 19

Lintner's theory will be analyzed following a summarization of the conventional theory of the cost of capital. The summary is intended to reflect how the conventional theory of the cost of capital deals with the problem of the risk inherent in capital projects. The summary has as its foundation the works of Ezra Solomon, and in particular reflects the content of his classic text, The Theory of Financial Management.

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[18] John Lintner, "The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets," The Review of Economics and Statistics, Vol. XLVII, No. 1, (February, 1965), pp. 13-37; and William F. Sharpe, "Capital Asset Prices: A Theory of Market Equilibrium Under Conditions of Risk," The Journal of Finance, Vol. XIX, No. 3, (September, 1964), pp. 425-442.

[19] Lintner, op. cit., p. 15.

CHAPTER IV

THE CONVENTIONAL THEORY OF THE COST OF CAPITAL

In order to introduce theories of valuation and the cost of capital without the distracting complexities of reality, it is convenient to assume an ideal economy. Once the groundwork has been laid for the idealized state, removal of simplifying restrictions may then gainfully begin.

I. PERFECT CAPITAL MARKETS, RATIONAL BEHAVIOR AND PERFECT CERTAINTY

Modigliani and Miller define an "idealized economy" in terms of three basic assumptions of "perfect capital markets, rational behavior, and perfect certainty."[1]

> 1. In perfect capital markets no buyer or seller (or issuer) of securities is large enough for his transactions to have an appreciable impact on the then ruling price. All traders have equal and costless access to information about the ruling price and about all other relevant shares. . . . No brokerage fees, transfer taxes, or other transactions costs are incurred when securities are bought, sold, or issued, and there are no tax differentials either between distributed and undistributed profits or between dividends and capital gains.

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[1]Franco Modigliani and Merton Miller, "Dividend Policy, Growth and the Valuation of Shares," The Journal of Business of the University of Chicago, Vol. XXXIV, No. 4, (October, 1961), pp. 411-433.

2. Rational Behavior means that investors always prefer more wealth to less and are indifferent as to whether a given increment to their wealth takes the form of cash payments or an increase in the market value of their holdings of shares.

3. Perfect certainty implies complete assurance on the part of every investor as to the future investment program and the future profits of every corporation. Because of this assurance, there is, among other things, no need to distinguish between stocks and bonds as sources of funds. . . . We can, therefore, proceed as if there were only a single type of financial instrument which, for convenience, we shall refer to as a share of stock.

Given these assumptions, the capital market in equilibrium will have some unique rate of interest, r^*, and one would always be able to invest, or borrow against future value, at the market rate.

Disequilibrium in the capital market could not persist under the assumptions, since owners of high-priced (low-return) stocks would be motivated to sell, in order to invest the proceeds in low-priced (high-return) shares. Consequently, any differential in rates of return between shares would be eliminated, so that on the average, a constant rate of interest would apply, thereby establishing the "time value of money."

Given a unique, all-embrassive interest rate, rational behavior dictates that the value of a share, that is, the market price in equilibrium, would equal the present value of future dividends, i.e.,

$$V = \sum_{t=1}^{\infty} \frac{\bar{D}_t}{(1 + r^*)^t} .$$

At equilibrium, any change in the dividend flow attributed to a given stock would result in a "windfall" gain (or loss) to holders of the stock. Such a windfall gain could only be precipitated by a firm's commitment to a project that promised to be more profitable than the standard market rate.

Even in an idealized economy, firms are assumed to find opportunities to invest in independent projects which are recognized and become feasible through changes in the technical and demographic environment. Thus, in order to maximize the present value of a share of stock, the firm should invest in all projects which promise a rate of return on investment which exceeds, or at the margin equals, the riskless rate r^*. Since a firm, like an individual, can borrow or lend without restraint at the market rate, investment should continue until the rate of return promised by the next marginal project is less than the market rate. This is equivalent, of course, to setting output at the level where marginal revenue equals marginal cost, in accordance with the classical micro-economic theory of the firm.

To summarize; rational behavior in perfect capital markets under conditions of perfect certainty implies that economic entities ascribe value to investment opportunities by discounting sure future cash flows at a unique market interest rate which represents the pure, riskless, time value of money. This unique interest rate is <u>the</u> financial standard for investment decision-making under the idealized circumstances.

II. PERFECT MARKETS, RATIONAL BEHAVIOR, AND UNCERTAINTY

The consequence of introducing uncertainty to the idealized economy is that the future outcomes attributable to economic events must be based upon intuitive judgements, or expectations, as to the probable level and range of future performance of the various securities available to the investor.

Given uncertainty in perfect capital markets, wherein rational investors are in general risk averse, the "pure" risk-free discount rate will no longer apply as the unique standard for financial decision making. Given aversion to risk, the higher the risk of a stream of expected cash flows the lower will be the "value" of the stream to investors. This is equivalent to saying that investors require a rate of return which is greater than the riskless rate r^* for risky investments.

In this context the present value of a risky stream of cash flows is found by discounting expected values at a rate which reflects not only the time value of money, but also a sufficient measure of compensation for the risk inherent in the stream.

The concept of a "required rate of discount" is common in the financial literature. Much of its intuitive appeal stems from its suitability for employment in the standard present value formulation which was shown appropriate for the perfect certainty case; i.e.,

$$V = \sum_{t=1}^{\infty} \frac{\bar{D}_t}{(1 + ke)^t},$$

where V is the "value" of the common share, $\bar{D}_t$ is the expected value of the uncertain dividend for period t, and ke is the required rate of discount.

That the "required rate of discount" concept has grave disadvantages for risk analysis will become clearly evident as the discourse proceeds. Nevertheless, the idea is basic to the theory of the cost of capital, and for the time being it will be accepted as a useful model of investor behavior in reaction to risk and uncertainty.

<u>The firm's market rate of discount</u>. According to Porterfield,

The firm's market rate of discount is the rate at

> which the market discounts the expected future dividends to be paid by the firm, in order to arrive at the market price of a share of the firm's stock. 2

It is implicit in this theory of valuation that the market price of a stock can be influenced by financial undertakings which change (1) investors' subjective estimates of the stream of expected values of dividends to be paid on the shares, and (2) the discount rate by which the market discounts the stream. Consequently, in accordance with the "required rate of return" approach to valuation, proposed investments by the firm should be appraised in terms of their effect upon the stream of future dividends and the market rate of discount.

The cost of equity capital. Although the "cost of capital" is most precisely defined as ". . .the price which a firm pays for acquiring funds from its capital suppliers,"[3] its employment as a financial standard for investment decisions is better described by the alternative definition, which is accepted for this analysis, that the cost of capital is

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[2]James T.S. Porterfield, Investment Decisions and Capital Costs, p. 75.

[3]James C.T. Mao, Quantitative Analysis of Financial Decisions, Chap. X. p. 1.

". . . the minimum prospective rate of yield that a proposed investment in real assets must offer to be worthwhile undertaking from the standpoint of the current owners of the firm."[4] In any case, the central concept behind the notion is that projects should not be undertaken that do not earn the firm's cost of money, including both implicit and explicit costs.

As will be subsequently shown, under several severely limiting restrictions it may be validly argued that the cost of capital is equal to the market rate of discount on the common shares. As the restrictions are dropped however, the theoretically correct expression for the cost of capital becomes a complex function of many other factors as well, and therefore its usefulness as a practical financial standard becomes ever more tenuous and difficult to justify. Consequently it is necessary to summarize conventional cost of capital theory in order to properly assess its strengths and weaknesses as a financial standard. In the sections which follow, the essence of cost of capital theory is scrutinized from its development in the simplest case, up to its relative maturity as a "weighted average cost of source of

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[4] Franco Modigliani and Merton Miller, "Estimates of the Cost of Capital Relevant for Investment Decision Under Uncertainty," in Determinants of Investor Behavior: A Conference of the Universities-National Bureau for Economic Research, p. 182.

funds." The emphasis throughout is given to the relationship between riskiness and the cost of capital, the intention being to discover how well risk is accounted for in the conventional formulation.

The cost of equity capital. The simplest case.

According to Ezra Solomon:

> We want a correct basis for setting the minimum rate of return required to justify the use of equity funds, correct in that it can always be expected to lead to that set of investment decisions which will maximize net present worth. [5]

In order to define the minimum required rate of return for the simplest case, Solomon analyzes a hypothetical project investment decision, and adopts for the purpose the following four assumptions:[6]

1. The company is, and will be, financed entirely by externally derived equity funds.

2. True earnings are equal to book earnings, i.e., the amount of depreciation deducted from the cash flow generated by operations is exactly enough to maintain earnings at the anticipated level.

3. The anticipated stream of earnings contains no upward or downward trend, i.e., growth is nonexistent.

4. The quality, or degree of certainty, of future expected earnings with the project is identical to the quality of future expected earnings without the project.

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[5] Ezra Solomon, The Theory of Financial Management, p. 37.

[6] Ibid., p. 38.

The fourth assumption is crucial to this discourse. Its implication is that adoption of the project will not change the market rate of discount appropriate to the firm's shares. Here Solomon assumes that investment proposals that promise returns which are of like quality, as far as degree of certainty or uncertainty is concerned, as those expected from existing assets, will not change the riskiness, and hence the market's capitalization rate for the firm.

Given the assumptions, Solomon argues that the minimum required rate of return on new equity is simply the ratio of expected earnings per share from existing investments, Ea, to the net proceeds per share of the new issue, P, i.e.,

$$ke = Ea/\ P\ .$$

Solomon notes that if P equals the going market price per share, that is, if there are no flotation costs,[7] the cost of equity capital is simply "the market capitalization rate at which the market values an expected stream of earnings of this quality," i.e.,

$$ke = Ea/\ M\ ,$$

where M is the going market price per share.

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[7]Floatation costs may exceed ten percent of the value of the offering. See, for example, C.C. Potter, Finance and Business Administration in Canada, p. 475.

Solomon justifies his conclusion by arguing,

> . . . the minimum earnings rate required on the investment of new funds follows from the function for which this minimum rate is designed. This is to screen proposals according to whether they do or do not increase net present worth. So long as we are assuming that any added earnings have the same quality as earnings from existing investments we can say that the function of the screening standard is to identify proposals that offer to increase earnings per share. Since Ea measures earnings expected from existing investments and since it costs existing owners one share to raise P dollars of new funds, it follows that new investments must generate an earnings rate of at least Ea/P if present owners are to enjoy an increase in earnings per share. 8

In his arguments Solomon does not state explicitly that earnings are established in the form of subjective probability distributions. However, his various references to "expected earnings" seem to infer that Ea symbolizes the mean of a subjective probability distribution which describes earnings expectations. If this inference is valid, Solomon has concluded that the cost of equity capital is given by the ratio of expected earnings on existing assets to current market price less floatation costs. In such a formulation as Ea/M or Ea/P, account is taken of the riskiness inherent in the earnings expectations because M is set by the market in accordance with the relative riskiness of the stream.

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[8] Solomon, *op. cit.*, p. 41

Generally speaking, one would expect that for a given Ea, the cost of equity, ke, would increase as riskiness increased, since the risk-averse market would be willing to pay less for streams of higher relative uncertainty; i.e., M would fall as uncertainty increased, thereby increasing the magnitude of ke = Ea/ M.

In any case, it is to be remembered that Solomon's formulation holds only for proposals of homogeneous quality, having the same relationship between expected earnings and riskiness as do the anticipated earnings of existing assets. Solomon's model does not explain how the investors derive M. Solomon merely states that a market price exists, and therefore that it should be used in the specification of ke. It is precisely because Solomon has no model of how M is (or should be) set that he is forced (1) to adopt the constraint that projects are of homogeneous risk, and (2) to ignore the direct relationship between project riskiness and the cost of capital.

A critique of the certainty-equivalence cost of equity. An ingenious modification of Solomon's formulation has been proposed by Mao, who employs certainty-equivalents of earnings expectations, rather than expected values, as the numerator in the cost of equity capital model.[9] Mao

[9] Mao, op. cit., Chap. X.

notes that ". . . return on common equity, unlike that on debt and preferred stock, is anything but constant."[10] Having recognized the subjective stochasticism of anticipated returns, Mao argues that ". . . financial management of a firm needs a method for removing uncertainty if it is to measure the cost of common equity."[11] He then concludes that:

> In determining the cost of common equity, the financial management of a firm should first survey its common stockholders to get a consensus as to the position and shape of their certainty-equivalence functions. Once constructed, the financial management can then convert the uncertain earnings of the common stockholders into their certainty-equivalents and proceed to calculate the cost of common equity as a constant, rather than a random variable. [12]

Consequently, Mao's formulation of the cost of capital is given by the expression,

$$ke^* = Ea^* / M,$$

where Ea^* is financial management's aggregated estimate of investors' certainty-equivalent of the uncertain earnings expectation. For risk-averse investors, Ea^* is by definition smaller than Ea, and therefore the value of Mao's cost of equity capital is smaller than the value specified by Solomon. But since the two formulations are not employed in the same evaluation model, it is not immediately apparent

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[10] Mao, <u>op</u>. <u>cit</u>., Chap. X, p.9.

[11] Mao, <u>op</u>. <u>cit</u>., Chap. X, p.8.

[12] Mao, <u>op</u>. <u>cit</u>., Chap. X, p.9.

whether the same results will always be obtained from their employment.

Solomon's cost of capital, for example, is employed to discount expected values of future cash flows to present value in order to obtain a proposal's net present worth. In this application, Solomon's cost of capital serves as a criterion of choice which distinguishes between acceptable and non-acceptable projects according to the sign of their net present worth. Solomon's definition of the cost of capital as a financial standard is based upon the economic rational that projects of positive net present worth should always be accepted without any need for management to resort to subjective judgement.

Mao's cost of capital, on the other hand, is designed to discount whole probability distributions of future cash flows, rather than the series of uniquely defined arithmetic means, and therefore results in the generation of a stochastic, rather than deterministic, net present value index.

Mao's approach provides management with a probability distribution rather than a simple number, and management must therefore rely upon subjective or intuitive judgement in order to reach a decision. Mao's cost of capital does not function as a financial standard that differentiates between projects that further or hinder the owners' interests.

Thus although Mao's approach is formulated in implicit recognition of the objectives of financial management, and embodies a discounted cash flow methodology for measuring and comparing possible uses of funds, it lacks the third essential element of a functional capital allocation mechanism. That element is a "criterion of choice" that is consistent with the method of measuring the prospective commitments, and that when applied will lead to the achievement of the economic objective. In conventional capital investment theory, the cost of capital has been defined and employed as the criterion for choosing from among potential sources and uses of funds. In Mao's approach, the certainty-equivalence cost of capital serves only as a discount rate. Consequently, not only is the function of the criterion of choice ignored, but it is also difficult to rationalize why any other interest rate would not serve just as well as a discount rate for deriving stochastic present values for subjective evaluation.

Mao's argument that ". . . financial management of a firm needs a method of removing uncertainty if it is to measure the cost of common equity,"[13] contains a conceptual flaw. How, for instance, does the certainty equivalence approach remove uncertainty? Is Solomon's approach invalid because it contains uncertainty, that by Mao's reckoning,

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[13]Mao, op. cit., Chap. X, p. 8.

requires removal? To begin with the second question; the uncertainty or subjective riskiness inherent in earnings expectations is explicitly accounted for by the market price, M, in Solomon's formulation. For risk-averse investors, the greater the riskiness for a given earnings expectation, the lower will be the market price. Thus the relationship between price and expected value given by Ea/M is sufficient to define the required rate of return discount rate necessary for the evaluation of expected values of earnings streams of the relevant quality. The rigor and conceptual validity of Solomon's thesis cannot be rationally denied.

What, therefore, does Mao mean when he "removes uncertainty" by resorting to certainty-equivalents of expected values in the specification of his particular brand of the cost of equity capital? First, Mao does not "remove uncertainty" in any sensible respect, for the market price, M, is as much an element of his cost of capital as it is of Solomon's. Nevertheless, Mao re-accounts for uncertainty in the numerator of the expression for ke^*, and may be justly accused of double accounting for uncertainty; once in the denominator, and again in the numerator of Ea^*/M. The economic utility of his formulation is therefore difficult to recognize since there is no direct relationship between his cost of capital and the objective of maximizing net present worth and the owners' welfare; unless, of course, it is fortuitously encapsulated in the judgemental mechansim of the managerial elite.

A counter-argument that the certainty-equivalent approach is valid because it is used to discount probability distributions rather than expected values seems lacking in theoretical foundation. The basis for such a counter-argument rests in the recognition that Solomon's cost of capital is not intended for use as a discount rate for whole probability distributions. But recognition that Solomon's cost of capital is intended for discounting expected values of relevant quality is not, of itself, sufficient to justify the use of the certainty-equivalence discount rate for stochastic derivations. Consequently, on the evidence at hand, the conceptual validity of the certainty-equivalence cost of capital as a financial standard has not been established. Therefore, since in the final analysis it must be conceptual validity rather than academic ingenuity that decides the worthiness of a normative economic tool or technique, this analysis will continue to focus upon Solomon's theory of the cost of capital as the most valid reflection of conventional wisdom.

Summary. According to Solomon, given the assumptions that:

1. Investment proposals promise returns of equal quality to those expected from existing assets.

2. Depreciation is just sufficient to maintain earnings at the existing level without either a positive or negative growth trend.

3. The company is and will be financed entirely by external equity funds.

then the cost of new equity capital is given by the expression,

$$ke = Ea/ P .$$

For circumstances defined by the assumptions, the cost of equity capital serves as a criterion of choice which is deemed consistent with (1) the objective of financial management, which is to maximize net present worth, and (2) the discounted cash flow methods of measuring and comparing possible uses of funds.

In accordance with the objective and the method of measurement, the cost of capital is used to discount expected values of subjective probability distributions describing future cash flows. According to the theory, projects exhibiting positive net present values, or alternatively, internal rates of return greater than the cost of capital, should be accepted. In this regard, risk is explicitly accounted for in the formulation of the cost of capital.

The riskiness of the subjective probability distributions serves only as evidence that the quality of the proposal's cash flows is equivalent to the quality of the cash flow from existing assets. Of course, if the quality of the subjective probability distributions of future cash flows from the project are not of a quality compatable with those of existing assets, it is necessary to account for the economic implications whether in the formulation of the cost of capital appropriate to the particular case, or alternatively, by means of other conceptually valid techniques. The choice of the method to use for accounting for a quality differential will depend upon relative convenience. In Chapters V and VI, the problem of accounting for the quality differential is treated in detail. But in order to complete the summary of conventional cost of capital theory, relaxation of the "growth" and "all-equity" constraints will be discussed as a prelude to that more significant problem of risk and valuation.

Relaxation of the growth limitation. Growth is expressed in a rising or falling level of earnings. In the assumptions adopted for the derivation of the cost of equity capital for the simplest case, the following limitation was implied:

> True earnings are equal to book earnings, i.e., the amount of depreciation deducted from the cash flow generated by operations is exactly enough to maintain earnings at an anticipated constant level

An upward or downward trend in earnings may be precipitated by various circumstances. A common cause of growth in earnings is the reinvestment of a portion of earnings within the firm, on behalf of the current shareholders. Regular reinvestment will bring about an expansion in assets, earnings and dividends which is in accordance with the relative verility of the investments committed.

<u>Retained earnings</u>. When a portion of a given period's earnings are retained for internal investment, that period's dividend payment must be correspondingly reduced. Since rational investors seek to maximize their wealth, they will be indifferent to whether earnings are retained or payed out as cash only if the increase in market value of their shares due to the internal reinvestment equals the value of the corresponding decrement of dividend receipt. In accordance with this maxim, Solomon established the "personal use criterion," which states that earnings should be used for additional internal investment rather than for dividend payments only if the internal investment adds more to stockholders' net present worth than they could add by the personal employment of an equivalent amount received in the form of dividends.[14]

In order to fulfill the personal use criterion, given the absence of income taxes and continuing the assumption of

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[14] Solomon, <u>op</u>. <u>cit</u>., p. 53.

a constant earnings expectation and homogeneity of earnings quality, each dollar of internal investment is justified only if it adds at least one dollar to the present value of the shares. It necessarily follows that the yield on internal reinvestment must be at least equal to ke=Ea/M if the investment is to add the required dollar of present value to the worth of ownership rights.

If personal taxes are introduced by assuming a uniform income tax rate of t, the use of funds for any investment yielding more than (1 - t)ke is justified by the personal use criterion.

If personal taxes are assumed to conform to reality, wherein income tax rates are progressive with income, and capital gains taxes may apply, a rigorous derivation of an explicit expression for the required rate of return on internal reinvestment is confounded by formidable difficulties. Nevertheless, the general form of the expression may be visualized as

$$k_{re} = \frac{f(t)}{s(g)} ke , \quad \text{for } f(t) < 1, \text{ and } s(g) < 1.$$

Both the personal income tax function, f(t) and the capital gains tax function, s(g), are complex expressions of indeterminant structure; the former operating to reduce the cost, and the latter to increase the cost of retained earnings.

Growth and the cost of equity. If the level of earnings is expected to rise, the preceeding expressions for the cost of new equity and the cost of retained earnings which were based upon non-growth trends are no longer relevant. More complex formulations must be derived to suit the particular growth expectations which might apply. Solomon distinguishes two categories of growth models: (1) earnings growth due to either (a) past investments' contributions to earnings, or (b) investment of depreciation provisions which prove sufficient not only to maintain net earnings at current levels, but also to add to the company's stock of assets and hence to earning power; and (2) earnings growth due to retention and reinvestment of a portion of earnings. The second category, of growth through reinvestment, may be of either or both of (a) growth due to internal opportunities to invest capital at above normal yields, and (b) growth through expansion of assets and earnings without recourse to above normal yield opportunities.

According to Solomon, the cost of equity appropriate to the first catagory of growth is that rate of discount which makes the present value of the anticipated stream of earnings equal to its market value. The appropriate rate is found by solving for ke in a polynomial expression of the form

$$M = \sum_{t=1}^{\infty} \frac{\bar{E}a_t}{(1 + ke)^t} ,$$

where

$$\bar{E}a_t \neq \bar{E}a_{t+1}.$$

The second category of growth, from reinvestment, is particularly difficult to represent in a general form in a mathematical model. The task may be greatly simplified however, if the following assumption is made: Opportunities will continue to exist that permit the reinvestment of a constant portion "b" of any period's net earnings Ea_t at a rate of return "m" times greater than the required rate of return on equity of the relevant quality. This assumption may be criticized because it implies that the firm in question may eventually "own the world" through the magical expansion of compounding growth. Setting such implications aside, the assumption leads to a model of the form

$$M = \frac{\bar{E}a}{ke} + \frac{b\bar{E}a}{(ke - bm)} \left(\frac{m}{ke} - 1\right) = \frac{\bar{E}a(1 - b)}{(ke - bm)} .$$

Solving for ke gives,

$$ke = \frac{\bar{E}a}{M} (1 - b) + bm = \frac{\bar{D}}{M} + bm ,$$

which is the form of Gordon and Shapiro's famous model.[15]

[15]Myron J. Gordon and Eli Shapiro, "Capital Equipment Analysis: The Required Rate of Profit," in The Management of Corporate Capital, edited by Ezra Solomon, pp. 141-149.

The significance of the growth models lies not in their particular formulation, but rather in the fact that an expectation of growth affects the required rate of return on equity for a rational investor, and is therefore a determinant of the cost of capital. But earnings may grow not only in magnitude of expected values but also in riskiness. The preceeding growth models ignore this aspect of the growth problem, and their formulations are not suited to an analysis of the complexities of growth in risk. The models of valuation and investor behavior which are presented in Chapters V and VI show that the problem of accounting for the growth of risk lies in the discount rate approach to valuation which is fundamental to the preceeding growth models. But before proceeding with that matter, the last component of the conventional discount rate approach must first be summarized by dropping the constraint that the company be financed entirely by equity funds.

Debt, preferred stock, and the weighted average cost of funds. In adaptation to a market wherein investors are averse to risk, firms have issued a variety of financial instruments, each characterized by a different combination of riskiness and expected return. "By issuing bonds, preferred and common stock, a company is able to breakdown its total income into component parts characterized by varying degrees of uncertainty."[16] Thus the financial market, and usually the firm's financial structure, consists of a range of commodities called securities, each characterized by a "required rate of discount" which reflects the market's valuation of the relative riskiness of their respective returns.

Debt financing commits the firm to a contractual obligation to pay interest and to repay principle at specified points in time. The claim by debt on earnings is prior to the claim of preferred and common stock, and consequently its relative riskiness is generally lower than for other forms of financing. To the risk-averse market, high quality and low yield are related. The market yield, or required rate of return, on debt is therefore generally less than that of equity. And as the quality of a debt issue increases, it is presumed that its yield approaches the risk-free interest rate, thereby reflecting the time value of money. Perhaps the closest approximation to risk-free instruments are government short term notes.[17]

The quality of preferred stock is generally lower than that of debt, which has a prior claim on earnings,

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[16] Mao, _op_. _cit_. , Chap. X, p. 5.

[17] Robert Wayne White, "Risk Aversion in Open-End Investment Companies" (unpublished Master's thesis, The University of British Columbia, 1968), p. 18

but is higher than that of equity which is subordinate in claim. In actual practice there exists a wide variety of forms of preferred stock, ranging over a spectrum from near-debt to near-equity in characteristics. But for the purpose of this analysis, preferred stock financing is considered a variant of debt financing, involving a "quasi-contractual" obligation by the firm. Dividends on preferred stock, like interest on debt, will be treated as a fixed charge to the firm.

The explicit costs of debt financing may be found by solving for kd in the general formula

$$B - \sum_{t=1}^{\infty} \frac{C_t}{(1 + kd)^t} = 0 ,$$

where B is the sum received from the issuance, net of all underwriting costs, and C_t is the cash flow necessary to pay the interest, sinking fund contributions, and repayments of principle (after deduction of tax credits) in the period t. The overall cost of debt financing is not given by the explicit cost kd, since implicit costs having to do with the impact of fixed commitment financing upon the value of the firm are not accounted for in the general formula. Implicit costs exist because fixed commitment financing increases the relative riskiness of residual earnings which accrue to common shareholders. Since the cost of equity increases with increasing riskiness, fixed commitment financing should be held accountable for implicit increases in the cost of equity as well as explicit contractual payments.

That fixed commitment financing increases the relative riskiness of residual earnings can be most simply shown by considering the following example: Two firms, one having an all-equity capital structure while the other includes debt financing, are expected to achieve equal net operating income from their existing assets. Net operating income, O, is defined as total cash earnings less whatever capital consumption allowances are required to maintain the flow of cash earnings at the projected level. Net income, E, is defined as the amount available to shareholders after both service charges on borrowed funds and corporate income taxes have been paid out of net operating income, i.e.,

$$E = (1-t)(O - rB) ,$$

where B is the value of debt in the capital structure, r is the interest rate on debt, and t is the marginal tax rate on corporate income. When uncertainty prevails, net operating income, and hence net income, are considered to be stochastic variables defined by subjective probability distributions. For this example it will be assumed that O_t is independently and normally distributed with mean $\bar{O}$

and variance, $\check{O}$.[18] The expected value of net income is then

$$\bar{E} = (1-t)(\bar{O} - rB),$$

and the variance is $\check{E} = (1-t)^2(\check{O})$.

Taking the ratio of variance of net income to expected value of net income as a measure of relative riskiness, rr, the ratio appropriate to the debt-free firm is

$$rr_1 = \frac{\check{E}_1}{\bar{E}_1} = \frac{\check{O}\,(1-t)^2}{\bar{O}\,(1-t)} = \frac{\check{O}\,(1-t)}{\bar{O}},$$

while the relative riskiness for the firm with debt financing is

$$rr_2 = \frac{\check{E}_2}{\bar{E}_2} = \frac{\check{O}\,(1-t)}{(\bar{O} - rb)}$$

It is evident that the greater the proportion of debt in a firm's capital structure, the greater will be the relative riskiness per dollar of residual earnings out of net operating income. Given a market of risk-averse investors, the equilibrium market price paid for a dollar of expected

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[18]Although it is recognized that $\check{X}$ is less common than V(X), or VAR(X) as a symbol for the variance of the variable X, its brevity is an advantage in long formulations, and it is not ambiguous in its inference. It is employed by John Lintner, "The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets, "The Review of Economics and Statistics, Vol. XLVII, No. 1 (February, 1965), pp. 13-37.

earnings of the quality reflected by rr_1 will be greater than the market price paid for a dollar of expected earnings of quality rr_2. All this is not to say however that debt financing should never be used, for it can be shown that under certain circumstances, fixed commitment financing which takes advantage of relatively low explicit costs may be quite within the owners' interest.

To illustrate, consider the following example: A newly formed corporation requires an investment of C in order to acquire assets which are expected to generate a constant level of expected net operating earnings of O. The founders of the corporation may finance the assets by a combination of debt and equity. Two alternative plans are considered, the first involving an issue of n_1 shares to the existing shareholders at C/n_1 dollars per share, and the second by an issuance of n_2 shares to existing owners at $(C-B)/n_2$ dollars per share, where B is the amount of a long-term loan negotiated with the bank. Expected earnings per share under the first plan are

$$\bar{e}_1 = \frac{\bar{O}}{n_1},$$

and under the second plan are

$$\bar{e}_2 = \frac{\bar{O} - rB}{n_2}$$

Given that the owner-investors are risk-averse wealth maximizers, whose wealth consists of their cash hoards and income from investments, the best of the two plans is the one which contributes the largest incremental addition to the investors' wealth. In a market characterized by risk-aversion, the market price, M, will be some function of expected earnings and variance on the share, i.e., $M(\bar{e}, \check{e})$, assuming, of course, that variance is an adequate representative of riskiness. Thus the incremental addition to wealth according to the first plan is given by

$$W_1 = n_1 M(\bar{e}_1, \check{e}_1) - C.$$

The incremental addition to wealth according to the second plan is

$$W_2 = n_2 M(\bar{e}_2, \check{e}_2) - (C - B).$$

Under such circumstances, debt financing is justifies if, and only if,

$$n_2 M(\bar{e}_2, \check{e}_2) + B > n_1 M(\bar{e}_1, \check{e}_1) \;;$$

which is to say, "if the total market value of the firm with debt exceeds the total market value without debt, debt should be employed."

<u>The optimal capital structure problem</u>. Whether or not the total market value of the firm is affected by the level of debt in its capital structure has been a subject of considerable debate in the literature. One school of thought, which was founded by Modigliani and Miller, argues that <u>in the absence of income taxes</u>, the total value

of the business depends "not at all upon the particular mix of security types that characterize its financial structure."[19] The second, or traditional, school of thought, holds that the total value of the firm first rises and then falls as the proportion of debt increases, and hence that an optimal capital structure exists.[20]

Modigliani and Miller support the "independence of capital structure" proposition by showing through rigorous theoretical argument that the breaking of net operating income into portions paid to bondholders and portions paid to stockholders cannot change the value of the firm <u>so long as personal and corporate leverage (borrowing power) are deemed equivalent</u>. Modigliani and Miller argue that the process of "arbitrage", whereby investors employ personal leverage to purchase and drive up the price

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[19]Franco Modigliani and Merton Miller, "Some Estimates of the Cost of Capital to the Electric Utility Industry, 1954-57," <u>American Economic Review</u>, Vol. LVI (June, 1966), p. 338

[20]For representative writings of traditional theorists, see: David Durand, "Costs of Debt and Equity Funds for Business: Trends and Problems of Measurement," <u>Conference on Research in Business Finance</u> (New York, N.Y.: National Bureau of Economic Research, 1952), pp. 215-261., and Eli Schwartz, "Theory of the Capital Structure of the Firm," <u>Journal of Finance</u>, Vol. XIV, (March, 1959), pp. 18-39.

For an explanation of both positions see Mao, <u>op. cit.</u>, Chap. XI; and the <u>Theory of Business Finance</u> edited by J. Fred Weston and Donald H. Woods, pp. 2-28.

of stocks which are "over-valued", will insure that at equilibrium, the market price per dollar of expected earnings of firms of a given "risk-class" will be equal. The concept of a "homogeneous risk-class" is explained by Mao as follows:

> Two firms i and j are said to be in the same risk class if their returns x_i and x_j ... are perfectly correlated. Perfect correlation implies that x_i and x_j are always proportional to one another and consequently that the ratios $(x_i/\bar{x}_i)$ and $(x_j/\bar{x}_j)$ have identical probability distributions. To illustrate, if x_i is N (10,9) and $x_j = 2x_i$, then $(x/\bar{x}_j)$ and $(x_j/\bar{x}_j)$ will both be N(1,9/100). For the investor who appraises investment returns solely on the basis of their expected value and variance, the earnings stream of any two firms with identical risk ratings and [italics in the original] capital structures are clearly perfect substitutes... 21

Modigliani and Miller's arguments are encapsulated in two propositions which are of great import to cost of capital theory. The first propositon states that

> ... the market value of any firm is independent of its capital structure and is given by capitalizing its expected return and the rate k_j appropriate to its class. 22

That is,

$$V_i = (S_i + B_i) = \bar{O}_i/k_j,$$

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[21] Mao, op. cit., Chap. XI, p. 20

[22] Franco Modigliani and Merton Miller, "The Cost of Capital, Corporation Finance and the Theory of Investment," in The Management of Corporate Capital, edited by Ezra Solomon, p. 152.

where V_i is defined as the total market value of the firm, S_i is the market value of its common shares, B_i is the market value of the debts of the company, and k_j is the capitalization rate appropriate to expected returns $\bar{O}_i$ of the $j^{th.}$ risk class.

That is,

$$k_j = \bar{O}_i / V_i = \bar{O}_i / (S_i + B_i).$$

By restating their first proposition in a different form, Modigliani and Miller derive their second proposition:

> The expected yield of a share of stock is equal to the appropriate capitalization rate k_j for a pure equity stream in the class, plus a premium related to financial risk equal to the debt-to-equity ratio times the spread between k_j and r.

Their derivation is as follows: The expected net earnings $\bar{e}$ to common shareholders is $\bar{O}$ - rB, which from proposition one is also given by $k_jV - rB$. By substituting S + B for V,

$$\bar{e} = k_jS + k_jB - rB.$$

$$= k_jS + (k_j - r)B.$$

Hence, according to Modigliani and Miller, the required rate of return, or cost of equity capital, is given by the expression,

$$ke = \bar{e}/ S = k_j + (k_j - r) \frac{B}{S} .$$

The relationship between the cost of equity, the cost of debt, and the weighted average cost of capital, according to the Modigliani and Miller thesis, is shown in Figure 2.

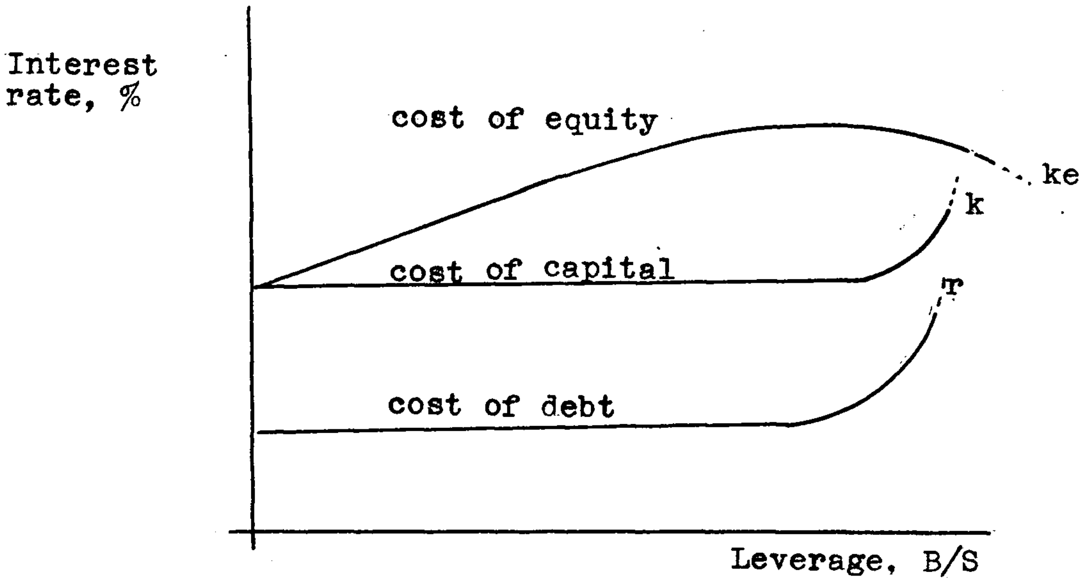

FIGURE 2

THE RELATIONSHIP BETWEEN THE COSTS OF DEBT, EQUITY AND THEIR WEIGHTED AVERAGE WITH RESPECT TO LEVERAGE ACCORDING TO MODIGLIANI AND MILLER (NO TAXES)

According to the Modigliani and Miller theory, the behavior of the cost of equity function is unrelated to the form of investors' utility for income functions. Instead it is a result of investors' ability to undertake personal arbitrage in order to adjust the level of risk and return in their personal portfolios.

It is with the assumption of equivalence of corporate and personal leverate that the traditionalists take issue. Arguing that investors consider that margin trading entails greater risk than corporate leverage, due to the limited liability clause inherent in incorporation, the traditionalists conclude that the value of the firm, and hence the weighted average cost of capital, will vary with capital structure. According to the traditionalist school, the weighted average cost of capital function has a "U" shape, with a minimum at the optimal capital structure, as shown in Figure 3.

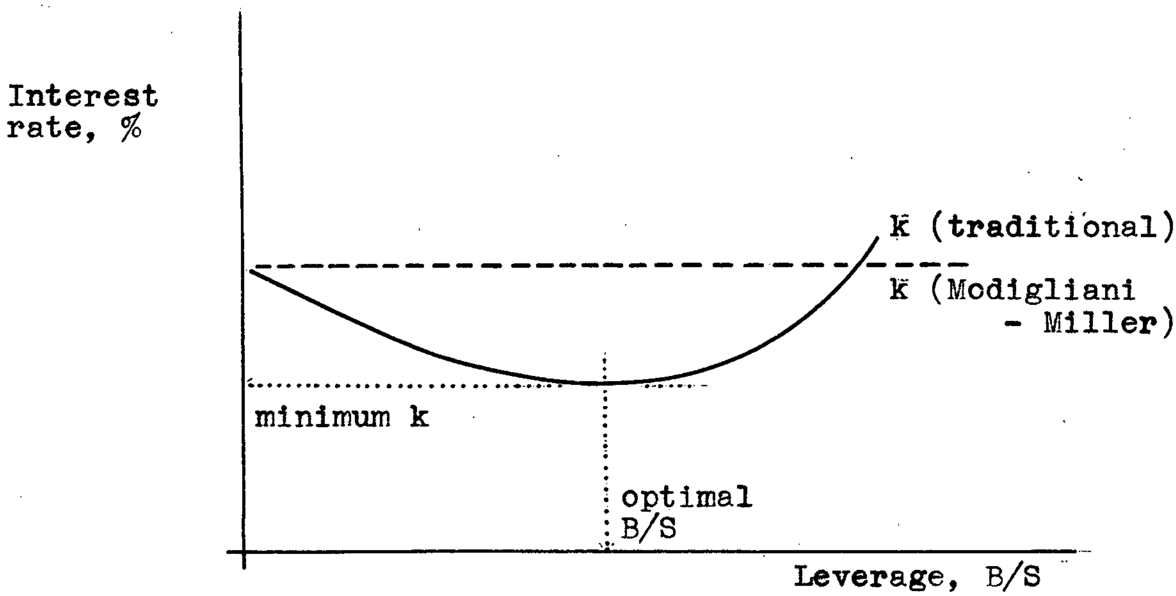

FIGURE 3

THE WEIGHTED AVERAGE COST OF CAPITAL FUNCTION
ACCORDING TO TRADITIONALIST THEORY.

With the incorporation of corporate income taxes into the analysis, Modigliani and Miller also recognize that the total market value of the firm is a function of its capital structure; not because there is an inherent advantage in debt financing, but rather because interest on debt is tax deductable. With corporate income tax, their expression for the cost of equity capital takes the form

$$ke = \bar{e}/S = k(t)_j + (1-t)(k(t)_j - r)\ \frac{B}{S}\ .$$

Instead of rising with leverage by an amount equal to the difference between the firm's overall cost of capital $k(t)_j$ and its cost of debt r, as shown for the tax-free case, the yield on equity rises with leverage according to the weighting (1-t), where t is the marginal tax rate. As before, $k(t)_j$ is the capitalization rate appropriate to the $j^{th.}$ risk class.

Although the matter will not be analyzed in this brief summary, it can be shown that Modigliani and Miller's tax adjusted formulation specifies that the capital structure which maximizes the total value of the firm is not the capital structure which minimizes the after-tax weighted average cost of capital.[23] In any case, given corporate

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[23]Mao, op. cit., Chap. XI, p. 33.

taxation, both the traditionalists and Modigliani and Miller agree that an "optimal" capital structure is a valid concept in the sense that there exists a financing mix which maximizes the value of the firm. Unfortunately, however, empirical evidence to date neither confirms the relevance of a particular normative theory, nor provides practicing management with a precise specification of the correct capital structure for a given firm in a given situation. The dispute between the traditional school and Modigliani and Miller over the validity of the "independence hypothesis" continues to be primarily a matter of academic import.[24]

Consequently, in actual practice, specification of a firm's capital structure will depend upon intuitive managerial judgement, tempered with some consideration of valuation theory, and constrained by the institutionalized habits of the financial community.

Nevertheless, whatever the rational behind the specification of a given firm's capital structure, there exists some evidence that the structure which is "optimal" for the firm will be some function of the market's expectations as

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[24]The subject of empirical testing of the Modigliani-Miller and Traditional hypotheses is summarized in: Robichek and Myers, Optimal Financing Decisions, pp. 45-47.; Mao, op. cit., Chap. XI, pp. 33-45; and Weston and Woods, Theory of Business Finance. pp. 7-18.

to the subjective riskiness of the firm's net operating income. In this regard it is hypothesized that industries characterized by relatively high degree of uncertainty as to the level of future net operating earnings will tend to exhibit low ratio's of debt to equity in their capital structures, and vice versa. The fact that utilities, which are in general characterized by extremely stable sales and net operating income, exhibit lower debt equity ratios, lends credence to the conjecture that optimal leverage decreases with riskiness of net operating income.[25]

Summary. The wealth of common shareholders may be enhanced by resorting to fixed commitment financing, and there will exist an optimal capital structure for the firm which depends upon the subjective riskiness of expected net operating income. According to the conventional wisdom, the over-all cost of capital appropriate to the evaluation of projects which do not change the "quality" of the firm's net operating income is defined as a rate of return consisting of a weighted average of the costs of specific

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[25]The hypothesis is supported by: Eli Schwartz, "Theory of the Capital Structure of the Firm," Journal of Finance, Vol. XIV, (March, 1959), pp. 18-39; and J. Fred Weston and Eugene F. Brigham, Managerial Finance, Second Edition; especially Table 11-4, p. 261.

sources of funds, with the weights being equal to the proportions of the particular sources of capital funds in the (current) optimal capital structure. Unfortunately, the definition of the cost of capital, or required rate of return, for projects which change the "quality" of a firm's total net operating income cannot be so simply defined.

When the quality of a firm's net operating income is changed, so does the quality of net income, ceteris paribus, change. A change in the quality of net operating income will therefore effect a change in the required rates of return on equity capital, given a market of risk-averse investors. Recognizing this fact, Solomon proposes the methodology of "imputed Borrowing power" which balances the "business risk" or a project with the "financial risk" of the funds employed for its investment.[26] Solomon suggests that each individual project be allotted a borrowing quota for fixed commitment financing, which represents the maximum

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[26]See Ezra Solomon, "Measuring a Company's Cost of Capital," Journal of Business, Vol.28, No.4, (October, 1955), pp. 240-252; reprinted in The Management of Corporate Capital, edited by Ezra Solomon, pp. 128-140. This summary is adapted from James C. Van Horne, Financial Management and Policy, p. 137; and G. David Quirin, The Capital Expenditure Decision, p. 215.

borrowing power of the project such that the risk of default is made negligible. The riskier the project, the lower the borrowing quota assigned to it. By this means, in theory at least, the firm may maintain its combined business and financial risk.

Once the "imputed borrowing quota" is established the remainder is equity financing. If the expected return on the equity-financed portion of the project exceeds the cost of equity funds, the project should be accepted. This system is equivalent to the process of using a weighted average of the cost of debt and equity in the conventional approach, except that the overall cost of capital is calculated in accordance with the formula:

$$k = pr + (1-p)ke \ ,$$

where p is the imputed borrowing power as a fraction of the total funds required for the project.

Of course this approach requires the specification of the borrowing power of the project. Solomon does not provide an analytical explanation for the required methodology. Quirin suggests that such estimates might obtain from an examination of lending practices and capital structures.[27] So long as the specification is made on the basis of intuitive judgement,

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[27] G. David Quirin, op. cit., p. 125.

even the most judicious design of the financing mix for a project may not entirely compensate for changes in the firm's operating risk.

Solomon's approach views each individual project as a separate entity with its own individual borrowing power. In this respect the effects of diversification of risk by combining projects of imperfect correlation is ignored. Taking correlation into account it would be more likely than unlikely that the borrowing power of a combination of risky investments would be more than the borrowing power attributable to the sum of the individual projects.

Thus, for the reasons that (1) Solomon does not provide an explicit theory to specify the "imputed borrowing power" of a given project, and (2) he ignores the diversification of risk attainable by combining projects, it is concluded that the problem of accounting for "non-homogeneity" of risk is not fully resolved by Solomon's approach.

In any case, the simple weighted average cost of capital defined for the case where quality is unaffected by project adoption is not appropriate as the minimum prospective rate of yield that a proposed project must offer in order to be worthwhile when the "business risk" of the firm is changed. The difficulty of coping with the problem of defining a uniquely valued but conceptually correct cost of capital for the condition of changing risk is apparent in the following comment by Johnson:

> Upto this point we have been assuming that capital projects selected do not change the risk class of the firm. . . . There are certainly instances where a major investment will change the entire cost of capital function for a firm. . . .
>
> The basic question, then, is whether there is a cost of capital that is unique for each firm in a given risk class and independent of the project to be financed. While most cost-of-capital discussions have adopted this assumption, it does not seem to hold in important instances. While I have no ready-made solution to this problem, it appears possible that its solution must come from a series of successive approximations that will require adjustments in the long-run cost of capital to reflect the basic change in the risk class of the firm brought about by major investments. <u>In turn, the change in cost of capital will influence the desirability of the proposed capital expenditures. As these expenditures are reduced or increased,</u> further adjustments in the cost of capital may be required, with additional refinements to the capital-budgeting plans, and so on. [emphasis is added].[28]

One detects, in Johnson's comment, a hint of a misconception. The phrase, "In turn, the change in cost of capital will influence the desirability of the proposed capital expenditures.", seems to ascribe to the cost of capital deiform powers to determine the worthiness of a given project. But the cost of capital is not a primary variable. The desirability of a project is surely a function of its characteristics in relation to those of the firm and the greater environment. The cost of capital is simply a tool of human

[28]Robert W. Johnson, "An Integration of Cost of Capital Theories," in <u>Theory of Business Finance</u>, edited by J. Fred. Weston and Donald H. Woods, pp. 303-304.

manufacture which serves, perhaps, as a convenient derived parameter for the evaluation of investments. It must be defined properly in order that it might serve effectively. Of course, to the extent that Johnson's comments allude to a trial-and-error technique for defining the cost of capital properly, this critique is too harsh.

Before proceeding to directly confront the issue of defining the cost of capital appropriate to conditions of changing risk it is necessary to precisely define the task. Accordingly, the objective of the exercise is specified as follows:

> To derive an expression for the cost of equity capital in the form of a uniquely defined rate of return required of an investment which will change the "quality" or equivalently, the risk class, of the firm. The expression is to define the cost of capital as a financial standard or "criterion of choice" which will insure that risky investment projects will be accepted only if they increase shareholder's wealth.

In order to formulate the required expression, riskiness must be explicitly considered as a determinant of value. Consequently it is necessary to establish a model of investor behavior which incorporates a concept of subjectively measurable risk and a theory of choice under uncertainty in order to explore the effect of quality change upon valuation and the cost of equity. In Chapter V, entitled "The Certainty-Equivalence Model and the Cost of Equity", an expression for the cost of equity capital is derived on the basis of the assumption of economic man and the classical certainty-equivalence model of valuation under uncertainty.

CHAPTER V

THE CERTAINTY-EQUIVALENCE MODEL AND THE COST OF EQUITY

Given the restricted description of riskiness which is embodies in the mean-variance approach to reflection of investor attitudes, the objective of the following analysis is to investigate the relationship which should exist between the cost of equity capital and the riskiness inherent in capital projects under the idealized circumstances of perfect capital markets and rational investors.

Unfortunately, the state of the art of investment theory is such that foolproof procedures for dealing with risk and uncertainty do not yet exist. Furthermore, a reliable descriptive model of investor behavior cannot yet be properly defined. Consequently, this analysis requires the assumption of a normative model of investor behavior which is neither intended to be, nor is expected to be, a useful description of actual behavior. Nevertheless, the normative model is a convenient vehicle for investigating the effect of risk upon the cost of equity capital under idealized conditions.

Normative models. Broadly speaking, three normative models of investor behavior may be used to describe asset valuation under uncertainty and investor aversion to risk. They are; the certainty-equivalence model, the risk-adjusted discount rate model, and, as an extension of the certainty-equivalence model, the portfolio selection model. Under the certainty-equivalence model each future return is converted to its certainty-equivalent, which is then discounted to present value at the pure, risk-free interest rate. Under the risk-adjusted discount rate model, the expected value of each future return is discounted at an appropriate rate which may be thought to contain two elements; a risk-free interest rate, and a term representing a measure of compensation for the riskiness surrounding the expected value.[1] The portfolio selection model, unlike the other two, recognizes the interrelation of returns on various assets as a component of relative riskiness. The general theory and construction of the portfolio selection model is discussed in detail in Chapter VI. For purposes of the following discussion it is assumed that the returns expected from all assets are considered to be perfectly uncorrelated, so that covariances do not enter the valuation mechansim.

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[1]For a concise analysis and differentiation between the certainty-equivalent model and the risk-adjusted discount rate model, see: Houng-Yhi Chen, "Valuation Under Uncertainty," Journal of Financial and Quantitative Analysis, Vol. II, No. 3, (September, 1967), pp. 313-326.

<u>The certainty-equivalence model.</u> The certainty-equivalence model is expressed as follows:

$$V = \sum_{t=1}^{\infty} \frac{a_t \bar{R}_t}{(1 + r^*)^t} , \qquad (1)$$

where,

V = Present Value. V represents the amount that the investor is willing to pay for the stream of expected returns. Under equilibrium, V is the market value.

a_t = The certainty-equivalence factor for the period t.

$\bar{R}_t$ = The expected value of R_t, where R_t is the uncertain dollar return for period t.

r^* = The pure risk-free rate of interest, which is assumed to be constant through time.

<u>The risk-adjusted rate of return model.</u> The risk-adjusted rate of return model is expressed as follows:

$$V = \sum_{t=1}^{\infty} \frac{\bar{R}_t}{(1 + ke_t)^t} , \qquad (2)$$

where, ke_t is the appropriate risk-adjusted discount rate for period t and the uncertainty surrounding $\bar{R}_t$.

If,

$$ke = \frac{(1 + r^*)}{a_t} - 1 , \qquad (3)$$

the two models are equivalent.

In its conventional formulation, the cost of equity capital is defined as a single valued rate of discount, i.e., say, ten percent. Chen has shown that in order that ke_t will equal ke_{t+1} for all t in equation (3), and hence that ke_t will equal the single valued rate of interest ke, <u>risk of future returns must increase at a constant rate over time</u>. This is equivalent to stating that the certainty-equivalence factors, a_t, must decrease at a constant rate over time.[2]

If the risk of future returns increases at an increasing rate over time, or increases at a decreasing rate over time, the discount rates will be changing with time and the value of the single "representative" cost of equity can only be found by solving for ke in the complex polynomial,

$$\sum_{t=1}^{\infty} \frac{\bar{R}_t}{(1 + ke)^t} = \sum_{t=1}^{\infty} \frac{\bar{R}_t}{(1 + ke_t)^t} \; . \qquad (4)$$

According to Chen, it is plausible that for any investor the risk of future dividends increases into the future, and that the increase of risk from year twenty-nine to thirty is likely to be less than that from two to three years ahead.[3] This leads to the assumption that the risk of future dividends increases at a decreasing rate over time, and therefore that the "representative" single valued cost of equity capital parameter is in fact a complex function

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[2]Ibid.

[3]Ibid.

of subjective risk and income expectations.

Although the risk-adjusted discount model most closely resembles the conventional net present value formulation, for purposes of this investigation the certainty-equivalence model will be used. The reasons will become apparent as the analysis proceeds.

The certainty-equivalence model revisited. Given the certainty-equivalence formulation, and an "aggregated" certainty-equivalence factor for the firm, it is possible to analyze the effect of a risky project upon the value of the enterprise. In the analysis which follows it is assumed that management knows the form of investors' risk taking attitudes, and that management's projection of expected earnings is representative of investor's expectations. Furthermore, it is assumed that the risk preferences of individual investors do not count, except insofar as departures from them are fully compensated by the market, so that it is the risk taking attitudes of the entire investing public which affects the value of the firm. Consequently, the model will be formulated in the aggregate, with total rather than per share cash flows entering the equations.

It is assumed that the firm concerned is debt-free and that future financing will be by new equity.

The return expected from existing operations in each period t is a random variable R_t, having a mean $\bar{R}_t$ and a variance $\check{R}_t$. The risky project under consideration offers an incremental random return of dR_t, with mean $d\bar{R}_t$ and variance $d\check{R}$. The coefficient of correlation between the returns to firm and project, c_t, is also pertinent to the analysis and must be defined for all t.[4]

Given this information it is possible to calculate the total return to the firm <u>with</u> the project, $R_{T,t}$, in terms of expected values and variances of R_t and dR_t (where the subscript "T" in $R_{T,t}$ signifies "total returns" of firm and project.), i.e.,

$$\bar{R}_{T,t} = \bar{R}_t + d\bar{R}_t \ , \qquad (5)$$

$$\check{R}_{T,t} = \check{R}_t + d\check{R}_t + 2(\check{R}_t)^{\frac{1}{2}}(d\check{R}_t)^{\frac{1}{2}}(c_t). \qquad (6)$$

The market value of the firm both with and without the project can now be found by substituting the appropriate parameters in the valuation model. The value of the firm <u>with</u> the project is simply

$$V_T = \sum_{t=1}^{\infty} \frac{a_{T,t}(\bar{R}_{T,t})}{(1 + r^*)^t} \ , \qquad (7)$$

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[4] G. David Quirin discusses the problem of defining correlation coefficients in <u>The Capital Expenditure Decision</u>, pp. 229-230.

while the value of the firm <u>without</u> the project is

$$V = \sum_{t=1}^{\infty} \frac{a_t(\bar{R}_t)}{(1 + r^*)^t} . \qquad (8)$$

The change in market value which is attributable to the project is therefore,

$$\Delta V = V_T - V.$$

Since management's objective is to maximize the wealth of existing shareholders, the project should be undertaken only if the value of the firm <u>with</u> the project exceeds the value of the firm <u>without</u> the project plus the value of the funds invested by the shareholders to finance the project. More simply, the project is acceptable if, and only if, V_T exceeds (V + Io), where Io is the required investment. In order to concentrate entirely upon the question of risk and the cost of capital, it is assumed that the new shares will be issued to existing shareholders at the going market price and in proportion to their existing holdings. According to Mao, if new shares are issued entirely and proportionately to existing shareholders, but at a price different from the market price, the cost of equity capital, ke, is unaffected by the discount at which the new shares are issued.[5] It is implied therefore, that the

[5]Mao, <u>op</u>. <u>cit</u>., Chap. X, pp. 22-28.

following analysis is also appropriate to this case. But if a portion of the issue of new shares is distributed to new shareholders, whether at, or discounted from, the market price, analysis of the cost of capital is complicated. In essence, the issuance to new owners carries with it a right to a proportion of the net earnings of the firm. In order that the market value of a share will remain at least at the going market price, so that wealth is maintained, both new and existing shareholders must be satisfied with the level and quality of their respective returns on investment. Mao shows that the cost of capital appropriate to this case is a linear combination of the cost arrived at for (1) an issue made entirely to existing shareholders and (2) an issue made entirely to new shareholders. The following analysis does not treat this more complex case. By restricting attention to the simpler situation, the analysis is kept manageable. However, the risk-valuation relationship which applies to the simplest case will also apply, albeit in a more complex fashion, to the case of issuance to old and new shareholders. To that extent, the following analysis is not invalidated by the simplifying assumptions.

The investment decision criterion is now restated: Accept the proposal if, and only if, $\Delta V = (V_T - V) > Io$. For the example at hand, the criterion takes the form,

$$\sum_{t=1}^{\infty} \frac{a_{T,t}(\bar{R}_t + d\bar{R}_t)}{(1 + r^*)^t} - \sum_{t=1}^{\infty} \frac{a_t(\bar{R}_t)}{(1 + r^*)^t} > Io \,. \qquad (9)$$

Upon collecting and reorganizing terms, expression (9) becomes,

$$\sum_{t=1}^{\infty} \frac{(a_{T,t} - a_t)\bar{R}_t}{(1 + r^*)^t} - \sum_{t=1}^{\infty} \frac{a_{T,t}(d\bar{R}_t)}{(1 + r^*)^t} - Io > 0 \qquad (10)$$

Expression (10) is of particular significance to the problem of risk evaluation. Note that the first term in (10) expresses the magnitude of the change in value of the stream of earnings from existing assets, which is attributable to the effect of the project upon the overall riskiness of the firm. If acceptance of the project makes the riskiness of earnings increase, then $a_{T,t}$ will be less than a_t and the project will have a detrimental effect on the value of existing operations.

The second and third terms in (10), when taken together, represent the net present value of the project itself, over that particular discount rate system, $ke_{T,t}$, for which

$$ke_{T,t} = \frac{(1 + r^*)}{a_{T,t}} - 1 \,,$$

for all t, i.e.,

$$\sum_{t=1}^{\infty} \frac{a_{T,t}(d\bar{R}_t)}{(1 + r^*)^t} - Io = \sum_{t=1}^{\infty} \frac{d\bar{R}_t}{(1 + ke_{T,t})^t} - Io \;. \qquad (11)$$

The right-hand side of equation (11), which will be symbolized by $NPV(ke_{T,t})$, is a net present value based upon the system of "risk-adjusted discount rates" appropriate to earnings of the firm after adoption of the project. Even if a representative discount rate ke' was calculated so that

$$NPV(ke_{T,t}) = NPV(ke') = \sum_{t=1}^{\infty} \frac{d\bar{R}_t}{(1 + ke')^t} - Io \;, \qquad (12)$$

such a rate would not represent the cost of equity capital in its conventional sense. This is true since it is possible for $NPV(ke_{T,t})$ to be positive while, at the same time, the proposal in question remains unprofitable because its contribution to the revaluation of assets, as represented by the first term in expression (10), is sufficiently negative. Furthermore, a project having a negative $NPV(ke_{T,t})$ may be quite profitable, providing that its contribution to the revaluation of existing assets is sufficiently positive.

The cost of equity capital is defined as the rate of return ke, required of the employment of the funds toward the maximization of the present worth of existing shareholders. In conventional practice, the required rate of return is used to discount the project's cash flows, the $d\bar{R}_t$ in the example, to net present value; or alternatively is employed as a financial yardstick for internal rate of return evaluation. A positive net present value, or an internal rate of return which exceeds the required rate of return is a signal for adoption of the project. But in the preceeding analysis it was shown that an investment is acceptable if it makes a positive contribution to present worth in accordance with expression (10). Then the "true" cost of equity, ke, may be found by equating the conventional formulation for net present value of the project,

$$NPV(ke) = \sum_{t=1}^{\infty} \frac{d\bar{R}_t}{(1 - ke)^t} - Io \text{ ,} \qquad (\ 13\ )$$

to expression (10), as follows;

$$\sum_{t=1}^{\infty} \frac{d\bar{R}_t}{(1+ke)^t} - Io = \sum_{t=1}^{\infty} \frac{a_t(\bar{R}_t)}{(1+r^*)^t} + \sum_{t=1}^{\infty} \frac{a_{T,t}d\bar{R}_t}{(1+r^*)^t} - Io \text{ ,} \qquad (\ 14\ )$$

where, $\Delta a_t = a_{T,t} - a_t$.

Although it is possible to derive an explicit expression for ke in terms of the parameters and variables in equation (14), such a task involves the solution of an extremely complex polynomial equation. But an expression for ke which is appropriate to special circumstances can be easily defined if the following simplifying assumption is made: Assume that $R_{i,t} = R_t$ and $a_{i,t} = a_i$, for all t and all cash flows under consideration; this being parallel to Solomon's non-growth assumption for the derivation of the cost of equity in the simplest case.

Then by cancelling Io from both sides of (14) and rearranging terms,

$$d\bar{R} \sum_{t=1}^{\infty} \frac{1}{(1+ke)^t} = (\Delta a \frac{\bar{R}}{d\bar{R}} + a_T) d\bar{R} \sum_{t=1}^{\infty} \frac{1}{(1+r^*)^t} \qquad (\ 15\ )$$

By cancelling $d\bar{R}$, and noting that $\lim_{t \to \infty} \sum_t \frac{1}{(1+x)^t} = \frac{1}{x}$,

expression (15) reduces to,

$$ke = r^*(a_T + \Delta a \frac{\bar{R}}{d\bar{R}})^{-1} \qquad (\ 16\ )$$

Expression (16) implies that given the simplifying assumptions, any project that does not change the riskiness of the firm, i.e., for which a=0, the cost of equity is simply $ke = r^*/a_T$, where a_T is the certainty-equivalence factor appropriate to the level of riskiness inherent in earnings from both existing assets and project combined. Furthermore, ke will equal r^* if, and only if, the firm is riskless to begin with, i.e., if a = 1.

Secondly, if the riskiness of the firm changes, as evidenced by the existence of the term $\Delta a \neq 0$, expression

(16) indicates that ke may be greater than r^*, less than r^* but positive, or negative.

Before continuing with an analysis of these three possibilities, it is advantageous to define the following variables:

1. The risk modification coefficient, $\Delta a/a_T$, is the ratio of the change in the firm's aggregated certainty-equivalence factor to the certainty-equivalence factor appropriate to the firm after acceptance of the project.

2. The expected return modification coefficient, $d\bar{R}/\bar{R}$, is the project's contribution to expected returns divided by the level of the expected value of returns from existing assets.

The risk modification coefficient will be positive if the project reduces the riskiness of the firm, i.e., if a_T exceeds a. And the expected return modification coefficient will be positive if the project is "normal", that is, if its returns are positive and the investment requires a cash outlay for earning assets.

If the expected return modification coefficient is held constant while the coefficient of risk modification is varied, the cost of equity can be graphed as a function of risk modification as shown in Figure 4.

Figure 4 shows that the cost of equity, ke, will be:

1. Negative for $(\Delta a/a_T) < (-d\bar{R}/\bar{R})$.
2. Undefined for $(\Delta a/a_T) = (-d\bar{R}/\bar{R})$.
3. Positive and greater than r^* for

$$(-dR/R) < (\Delta a/a_T) > \frac{(1-a_T)}{(a_T)}\left(\frac{d\bar{R}}{\bar{R}}\right).$$

4. Equal to r^* if $(\Delta a/a_T) = \frac{(1-a_T)}{(a_T)}\left(\frac{d\bar{R}}{\bar{R}}\right)$.
5. Less than r^* but positive if $(\Delta a/a_T) = \frac{(1-a_T)}{(a_T)}\left(\frac{d\bar{R}}{R}\right)$.

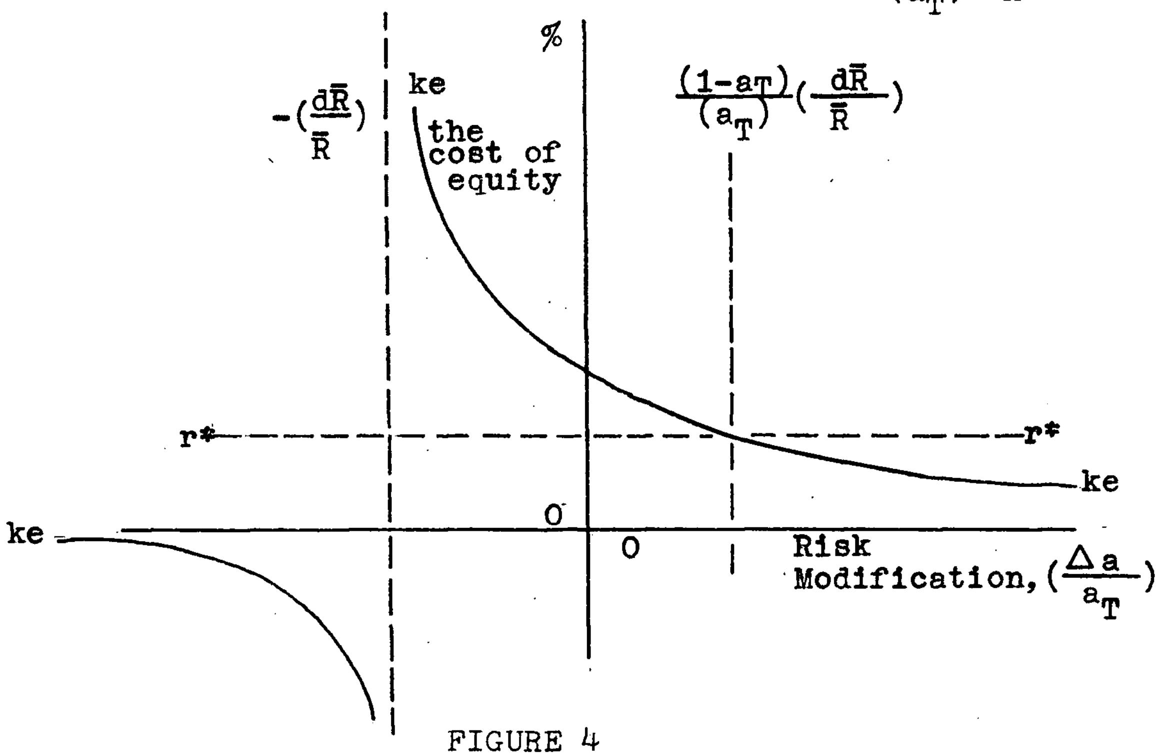

FIGURE 4

THE RELATIONSHIP BETWEEN RISK MODIFICATION AND THE COST OF EQUITY FOR THE CASE OF A CONSTANT LEVEL AND RISK OF RETURNS THROUGH TIME.

Although the values of ke which are positive but less than r^* may at first glance seem unacceptable in theory, reflection upon the definition of risk-aversion will give their existence justification in the light of investors' willingness to pay a premium for variance reduction when all else is held constant.

But on the other hand, a negative value for ke is both confusing and patently irrelevant in the context of risk-averse investors. Figure 4, page 104, shows that ke can be negative only if

$$\left(\frac{\Delta a}{a_T}\right) < -\left(\frac{d\bar{R}}{\bar{R}}\right) , \qquad (17)$$

or equivalently, if

$$\Delta a(\bar{R}) + a_T(d\bar{R}) < 0 \qquad (18)$$

Expression (18) implies that the investment under consideration will reduce the value of existing assets by the amount $\Delta a(\bar{R})$ while contributing a lesser amount of magnitude $a_T(d\bar{R})$ toward the value of the enterprise. Clearly, any such undertaking could never be justified as an activity designed to enhance the wealth of existing shareholders. Yet unfortunately, the expression used for the cost of equity is by its design inherently capable of giving rise to such confusion simply because it is a derived rather than a primary variable.

The confusion is avoided if the fundamental relationship given by expression (10) is used directly to evaluate the project; that is: Accept the project if, and only if,

$$\frac{\Delta a(\bar{R})}{r^*} \quad \frac{a_T(d\bar{R})}{r^*} > Io .$$

For the particular circumstances wherein the value of the risk modification coefficient is less than the negative of the expected return coefficient, it is easily shown that the project should be rejected; i.e., since

$$\left(\frac{\Delta a}{a_T}\right) < - \left(\frac{d\bar{R}}{\bar{R}}\right)$$

then necessarily,

$$\Delta a(\bar{R}) + a_T(d\bar{R}) < 0 < r^* Io ,$$

and rejection of the project is required in accordance with the wealth maximization criterion of expression (10).

<u>Summary and conclusions</u>. That the process of defining the cost of equity capital in order to discount expected values of risky future cash flows is a more round-about approach to evaluation than the direct application of the valuation model cannot be rationally denied. The cost of equity capital approach, in order to be scrupulously correct, requires the precise specification of the "risk-adjusted" discount rate; and such specification requires, of course, the employment of the very valuation model which is of

itself sufficient to decide a project's worth directly.

Furthermore, the more complex the relationships involved, the more complex becomes the "correct" expression for the cost of equity capital, and the more advantage there will be in approaching evaluation directly by means of the valuation model itself.

Unfortunately however, this ideal means of solution, whereby a valuation model exists which can measure directly and exactly the effect of a risky project upon the value of a share does not yet exist. In fact it seems unlikely that such a "perfect" model will ever exist, although it is sure that as time passes, theoretical reasoning and empirical investigation will steadily improve the quality of various imperfect models of valuation and investment behavior.
But even given that the "perfect" model is beyond mans' grasp, the "risk-adjusted", single-valued cost of capital approach to valuation retains its conceptual disadvantage; it must account for the risk of the project as a whole in a specific number. In this respect the certainty-equivalence has great advantage, for it specifies risk period by period and does not confuse accounting for risk with accounting for the time value of money. Of course, the host of simplifying assumptions and the naivete of the certainty-equivalence model employed in this chapter's analysis are far from reflections of reality. Perhaps its

most serious fault is the certainty-equivalence model's failure to account for the normatively valid behavior of diversification of risk through portfolio selection procedures. In the construction of the model it was assumed that the returns expected from all assets are considered to be perfectly uncorrelated, so that co-variances do not enter the valuation mechanism. This assumption is untenable in a world where returns on securities are recognized to move up and down in imperfect correlation so that diversification can reduce the relative riskiness of investment commitments.

In the following chapter, the effect of correlation between returns expectations is included in a sophisticated model of valuation of risk assets created by John Lintner. The inclusion of correlation factors greatly complicates the valuation model, and concommitantly, the "correct" formulation of the cost of equity capital. Yet it remains true that under the assumptions cast, the cost of capital is theoretically redundant for risk analysis and economic evaluation of capital projects.

CHAPTER VI

THE COST OF EQUITY AND LINTNER'S PORTFOLIO SELECTION MODEL

The conclusion that the cost of equity is redundant when riskiness is affected by investment is of singular significance to capital budgeting theory. But the analysis of the preceeding chapter may be justifiably criticized as dependent upon too many restrictive assumptions to permit generalized acceptance. Nevertheless, considerable support is lent to the conclusion by similar conclusions reached by John Lintner in his work "The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets."[1]

> There can be no single "risk discount rate" to use in computing present values for the purpose of deciding on the acceptance or rejection of different individual projects out of a subset of projects even if all projects in the subset have the same degree of "risk". The same conclusion follows a fortiori among projects with different risks. 2

Although Lintner's analysis is more general and sophisticated than the meagre efforts of the preceeding chapter, Lintner also admits to a ". . . rather heroic set of simplifying assumptions which were made at the

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[1]John Lintner, "The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets," The Review of Economics and Statistics, Vol. XLVII, No. 1, (February, 1965), pp. 13-37.

[2]*Ibid.*, p. 32.

beginning. . . ."[3] But he concludes that "a little reflection should convince the reader that . . . the . . . above conclusions will still hold under more realistic (complex) conditions."[4]

Lintner's work is of such importance to capital budgeting and the theory of the cost of capital that it fairly demands a summary of its essential features here. The work is an extension of the theory of selection of efficient portfolios which was originally formulated by Markowitz.[5] Lintner, among others such as Tobin[6] and Sharpe,[7] developed extensions to Markowitz's pioneering effort in order to better explain observed phenomenon of diversification of assets in investment practice.

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[3] Ibid., p. 32. [4] Ibid., p. 32

[5] Harry M. Markowitz, Portfolio Selection: Efficient Diversification of Investments; and by the same author, "Portfolio Selection," The Journal of Finance, Vol. VII, No. 1, (March, 1952), pp. 77-91.

[6] J. Tobin, "Liquidity Preference as Behavior Towards Risk," The Review of Economic Studies, Vol. XXVI, No. 1, (February, 1958), pp. 65-86.

[7] William F. Sharpe, "Capital Asset Prices: A Theory of Market Equilibrium Under Conditions of Risk," The Journal of Finance, Vol. XIX, No. 3, (September, 1964), pp. 425-442.

The theory of efficient portfolios. The theory of selection of efficient portfolios, as originally formulated by Markowitz, was established in accordance with the mean-variability approach to explaining how investors interpret riskiness when making investment decisions in an uncertain world. Therein, investors are assumed to face an opportunity set of alternative investments, each of which is described by a mean-variability pair (u,b), where u is symbolic of expected value, and b symbolizes whichever of variance, standard deviation, relative dispersion, or other risk parameter deemed fitting by the theorist concerned.

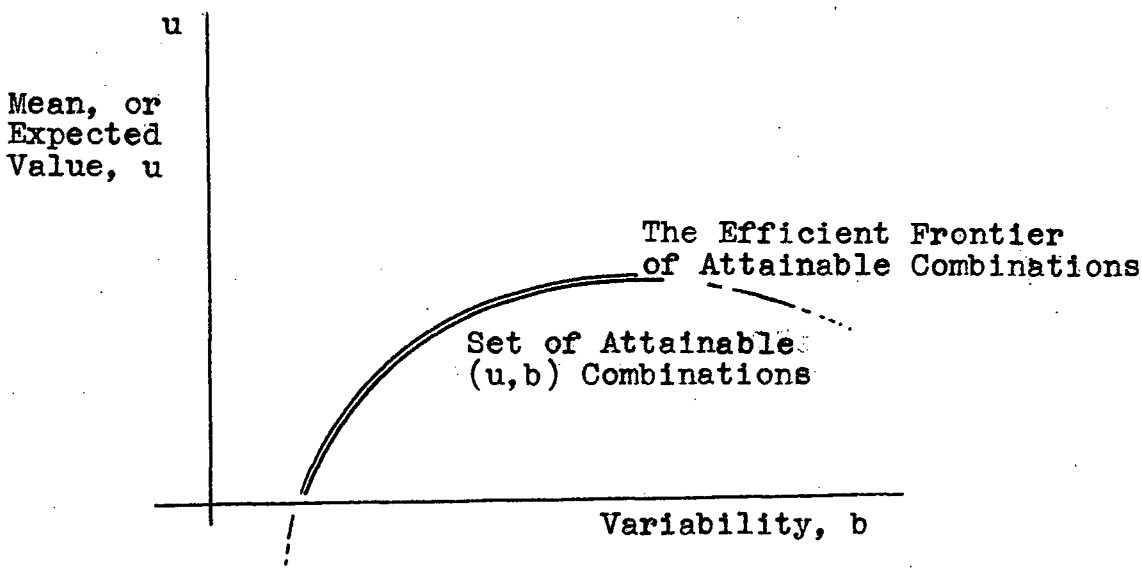

FIGURE 5

MARKOWITZ'S OPPORTUNITY SET OF RISKY INVESTMENTS AND THEIR POSSIBLE COMBINATIONS

The opportunity set is charted by plotting the mean variability pair for each alternative investment, and for every possible combination thereof, on a graph as shown in Figure 5.

Markowitz showed that the opportunity set is bounded by an "efficient frontier" which is comprised of those portfolios having the minimum attainable b for each possible value of u. The efficient frontier is shown heavily shaded in Figure 5. The curvature of the efficient frontier is prescribed by the covariance effect since increasingly higher values of portfolio u progressively reduce the number of securities that can be combined to lower the portfolio b.

When an investor's (u,b) indifference curves, which portray the form of his investment preference functions,

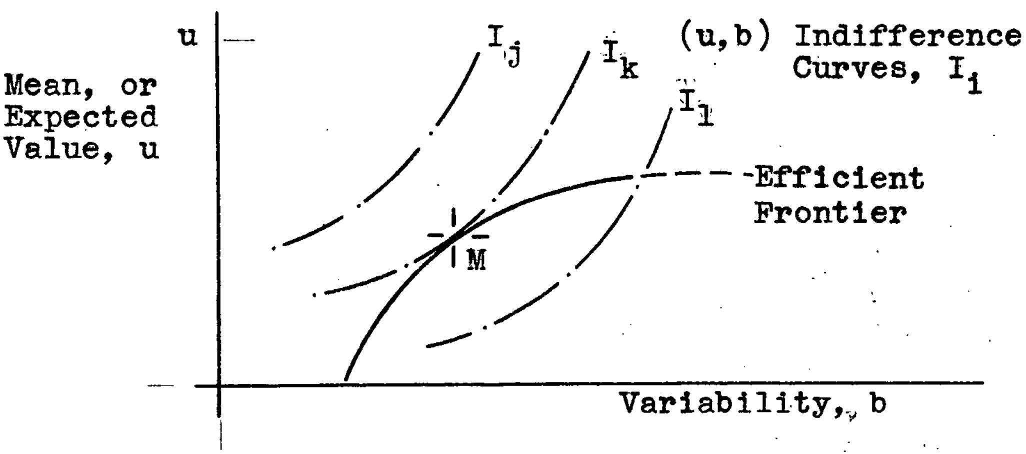

FIGURE 6

THE FORM OF (u,b) PREFERENCE FUNCTIONS AND THE EFFICIENT FRONTIER are superimposed on the efficiency frontier plot, it can be shown that the investor maximizes his utility if he selects that portfolio, denoted as M in Figure 6, which is located at the point of tangency between an indifference curve and the efficient frontier.

James Tobin extended the work of Markowitz by showing that an investor's optimal portfolio of risk and non-risk assets is determined by the tangency of his indifference function to a market opportunity line, rather than to the efficient frontier of Markowitz's opportunity set of risky assets.[8] Tobin's argument, as adapted from Lintner's summary, is paraphrased as follows:[9] Assume that (1) each individual investor can invest any part of his total capital in certain risk-free assets, all of which pay interest at a common positive rate r^*, which is exogenously determined and is constant through time; and that (2) he can invest any fraction w of his capital in any or all of a given finite set of risky securities which are traded in a

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[8] Tobin, op. cit.

[9] This summary is heavily based upon Lintner's summary in "The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets," op. cit., and presents the essential assumptions necessary for the following elaboration of Lintner's Capital Budgeting analysis.

single purely competitive market, free of transaction costs and taxes, at given market prices which do not depend on his investment or transactions. Also assume that (3) any investor may borrow funds to invest in risk assets at the risk-free interest rate r^* without limit to the amount; and that (4) all purchases and sales of securities, and all deposits and loans, are made at discrete points in time, so that in selecting his portfolio at a "transaction point" each investor will consider only (a) the cash throw-off (typically interest payments and dividends received) within the period upto the next transaction point and (b) changes in market prices of securities during the same period. Thus the return on any security or portfolio of securities, is defined to be the sum of the cash throw-off received plus the change in its market price over the period in question.

Assume that (5) the investor assigns at least an expected value-variance pair to every individual security's return, and a covariance or correlation to every pair of returns. Assume that (6) the investor calculates the expected value and variance on any possible portfolio of available securities by the appropriate statistical manipulation of and between individual securities.

Note that assumptions (1) through (4) construct an economy which is equivalent to Modigliani and Miller's "perfect capital market." Under the assumed circumstances the investor's problem is to decide how to allocate his capital between risk-free assets with a certain positive return r^*, and a portfolio of risky securities having an uncertain aggregated return r per dollar invested in the portfolio.

The investor's total net return will be:

$$yA = (1-w)Ar^* + wAr \ , \qquad (19)$$

where y is simply the net return per dollar of total net investment A. Dividing through by A gives:

$$y = (1-w)r^* + wr = r^* + w(r + r^*); \quad 0 = w = \infty, \qquad (20)$$

where a value of w less than unity denotes that the investor holds some of his capital in riskless assets and receives interest amounting to $(1-w)r^*$; while a value of w exceeding unity indicates that the investor borrows to buy risky securities, and pays interest equal in absolute value to $(1-w)r^*$.

The mean and variance of the random variable y are:

$$\bar{y} = r^* + w(\bar{r} - r^*) \ , \text{ and} \qquad (21)$$

$$\check{y} = w^2\check{r} \ . \qquad (22)$$

From equation (21) it is evident that by varying w the investor can obtain any level of expected return $\bar{y}$ from any securities mix. But (22) indicates that the "price" paid for increasing expected return by increasing the "leverage" w is a proportionately greater increase in variance of return on the total investment. Thus the investor must balance the benefits of increasing expected return against the detrements of increasing variance in his selection of an appropriate value for w.

By eliminating w in equations (21) and (22), the expected value of the investor's net return per dollar of his total net investment can be expressed as a function of the risk-free rate of interest and the parameters $\bar{r}$ and $\overset{v}{r}$ of the particular portfolio in question, i.e.,

$$y = r^* + \Theta(\overset{v}{y})^{\frac{1}{2}}, \text{ where} \qquad (23)$$

$$\Theta = (\bar{r} - r^*)/\ (\overset{v}{r})^{\frac{1}{2}}. \qquad (24)$$

Equation (23) is the "investment opportunity line" function which is shown in Figure 7. Note that its intercept of the ordinate is r^*.

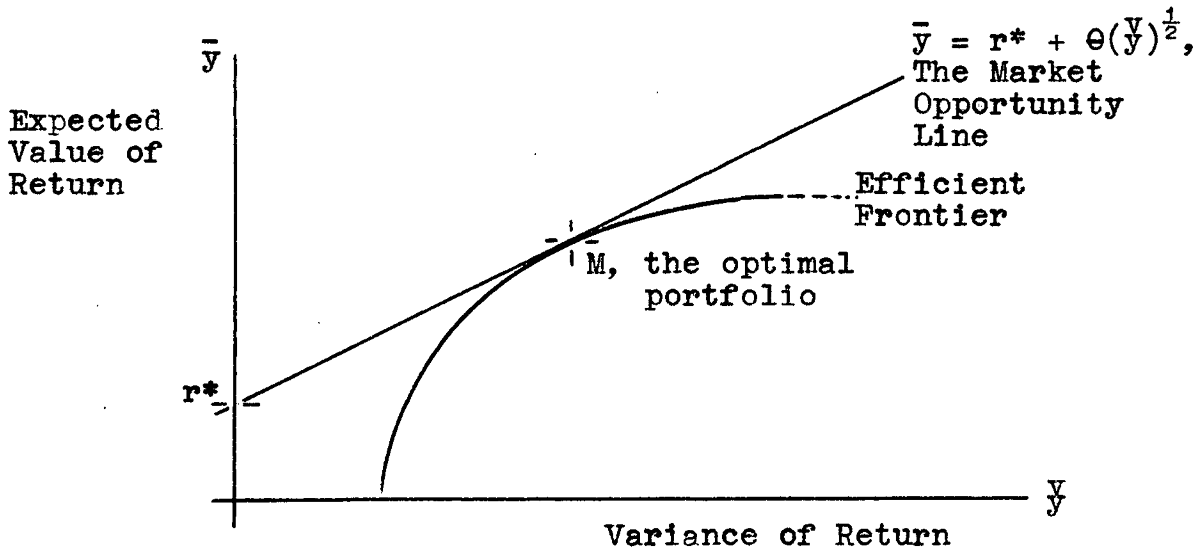

FIGURE 7

THE MARKET OPPORTUNITY LINE FOR PORTFOLIO SELECTION

Selection of the optimal portfolio. In the context of normative economics, the rational risk-averse investor should choose that portfolio of securities which exhibits the maximum value of θ in order to minimize the ratio of $\check{y}$ to $\bar{y}$ on his total investment. The portfolio "M" in Figure 7 is the one for which the slope, θ , of the market opportunity line function is a maximum and thereby minimizes the ratio of $\check{y}$ to $\bar{y}$ on total investment for the circumstances in question.

The truly significant point that Tobin established through his analysis is that θ is independent of w and of $\bar{y}$. This is Tobin's "Separation Theorem";

> Given the assumptions about borrowing, lending, and investor preferences stated earlier, . . . the optimal proportionate composition of the stock (risk-asset) portfolio is independent of the ratio of the gross investment in stocks to the total net investment, w.[10]

Since the indifference curves of risk-averse investors are concave upwards and exhibit increasing utility toward the North-West sector of Figure 7, the preferred ratio of investment in stocks to total net investment, w , is determined by the tangency of the market opportunity locus and an

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[10]Paraphrased from Lintner, *op. cit.*, p. 17.

indifference curve as is shown by point Q in Figure 8.

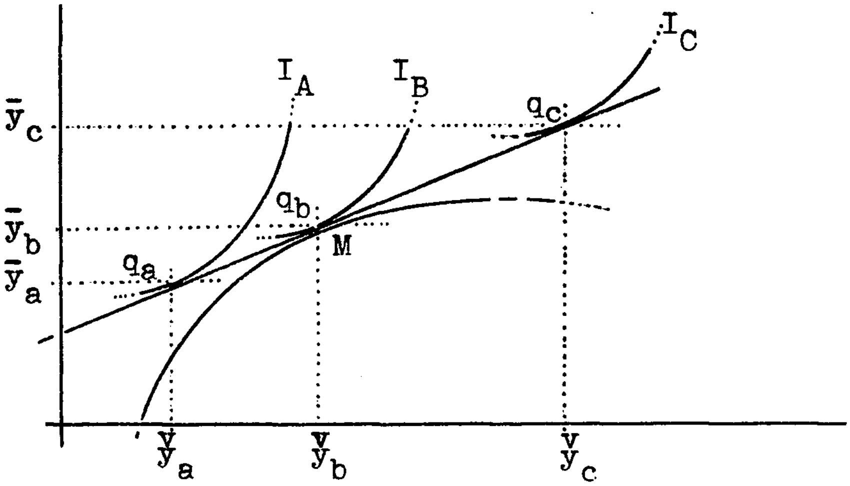

FIGURE 8

THE RELATIONSHIP BETWEEN PREFERENCE FUNCTIONS, THE MARKET OPPORTUNITY LINE, AND THE RATIO OF INVESTMENTS IN STOCKS TO TOTAL NET INVESTMENT

In Figure 8, investor A would lend a portion (1-w) of his capital at the risk-free rate r*, and would commit the rest in the optimum portfolio M in order to obtain the overall achievement $(\bar{y}_a, \check{y}_a)$ in accordance with his own particular preferences as distinguished by the concave-upward indifference functions, I_A. Figure 8 also shows how investors C and B would borrow, and neither borrow nor loan, respectively, on the basis of the relevant indifference functions.

In the preceeding elaboration of the theory of the opportunity locus it was assumed that the efficiency frontier bounding the opportunity set was known, and necessarily known, in order to select the optimum portfolio. Lintner shows, however, that given the expected returns, variances and covariances of all available securities, the securities mix which maximizes θ can be obtained directly by analytical procedures. In other words, the optimal portfolio can be found without resorting to the calculation of each portfolio out of the myriad possible. The expression for the composition of the optimal portfolio is derived by Lintner as follows:

Define:

$|h_i|$ = The ratio of the gross investment in the $i^{th.}$ stock (the market value of the amount bought or sold) to the gross investment in all stocks. A positive value of h_i indicates a purchase, while a negative value indicates a short sale.

r_i = The return per dollar invested in a purchase of the $i^{th.}$ stock (cash dividends plus price appreciation).

The return on any stock will simply be $h_i r_i$, which can be expressed for convenience in the equivalent form:

$$h_i r_i = h_i(r_i - r^*) + |h_i| r^*. \qquad (25)$$

The total return on any stock mix or portfolio is then,

$$r = \sum_i \left[h_i(r_i - r^*) + |h_i| r^*\right] = r^* + \sum_i h_i(r_i - r^*) \qquad (26)$$

because $\sum_i |h_i| = 1$ by its definition.

The expected return and variance of any stock mix are expressed as:

$$\bar{r} = r^* + \sum_i h_i(\bar{r}_i - r^*) = r^* + \sum_i h_i\bar{x}_i \text{ , and} \qquad (27)$$

$$\check{r} = \sum_i h_i h_j \check{r}_{ij} \text{ ,} \qquad (28)$$

where $\check{r}_{ij}$ denotes variances when $i = j$ and covariances otherwise; and $\bar{x}_i = \bar{r}_i - r^*$, serves as a "risk-premium."

The expression for θ is then rewritten in the form:

$$\theta = \frac{\bar{r} - r^*}{(\check{r})^{\frac{1}{2}}} = \frac{\bar{x}}{(\check{x})^{\frac{1}{2}}} = \frac{\sum_i h_i\bar{x}_i}{(\sum_{ij} h_i h_j \check{x}_{ij})^{\frac{1}{2}}} \qquad (29)$$

The problem is to find the value of h_i for all i which maximizes θ. The solution is found by differentiating θ with respect to h_i and proceeding as follows:

Set $u = \sum_i h_i\bar{x}_i$, and $v = (\sum_{ij} h_i h_j \check{x}_{ij})^{\frac{1}{2}}$.

Then $du/dh_i = \bar{x}_i$, and $dv/dh_i = v^{-1}(h_i\check{x}_{ii} + \sum_j h_j\check{x}_{ij})$. Hence,

$$\frac{d\theta}{dh_i} = \frac{v\frac{du}{dh_i} - u\frac{dv}{dh_i}}{v^2} =$$

$$v^{-1}\left[\bar{x}_i - \frac{\sum_i h_i\bar{x}_i(h_i\check{x}_{ii} + \sum_j h_j\check{x}_{ij})}{v}\right]. \qquad (30)$$

Set $Z_i = L(h_i)$.

Define $L = (\bar{x} / \overset{v}{x}) = \dfrac{\sum_i h_i \bar{x}_i}{\sum_{ij} h_i h_j \overset{v}{x}_{ij}} = \dfrac{\sum_i h_i \bar{x}_i}{v^2}$,

By substituting L and z_i into equation (30) and setting it equal to zero at the maximum, a set of m equations of m unknowns (one for each of i = 1,2,3,...m, identifying the m stocks) is obtained:

$$z_i \overset{v}{x}_{ii} + \sum_j z_j \overset{v}{x}_{ij} = \bar{x}_i \text{ , for } i = 1,2,3,\ldots m \text{ .} \qquad (31)$$

This system of equations has a unique solution,

$$z_i^{o} = \sum_j \overset{v}{x}^{ij} \bar{x}_j \text{ ,} \qquad (32)$$

where $\overset{v}{x}^{ij}$ represents the ij$^{th.}$ element of ($\overset{v}{\underline{\underline{x}}}$)$^{-1}$, the inverse of the covariance matrix.

From equation (32), h_i^o for each i$^{th.}$ stock is obtained since,

$h_i^o = z_i^o / L^o$,

where $L^o = (\bar{x}^o \;/\; \overset{v}{x}^o)$ by definition.

But $z_i = L(h_i)$ implies $\sum_i |z_i^o| = L^o \sum_i |h_i^o| = L^o$,

and therefore, $L^o = \sum_i |z_i^o|$.

In summary then, given $\bar{r}_i$, $\overset{v}{r}_{ij}$ for all available stocks, the composition of the optimal portfolio can be found by calculating

$$z_i^o = \sum_j \overset{v}{x}^{ij} \bar{x}_j$$

for all m stocks and then dividing each z_i^o by $\sum_i |z_i^o|$ to obtain h_i^o for all m stocks.

Investor's required rate of return and market value under uncertainty. In order to derive expressions for the equilibrium market values of stocks, and coincidently to derive an expression for the cost of equity under idealized uncertainty, the following assumption is necessary: Assume that for any given set of market prices for all stocks, all investors assign identical sets of means, variances and co-variances to the joint distribution of these dollar returns (and hence for any set of prices, to the vector of means and the variance-covariance matrix of the rates of return r_i of all stocks), and that all correlations are less than unity.[11]

Define:

V_{oi} = The aggregate market value of the $i^{th.}$ stock at time zero.

R_i = The aggregate return on the $i^{th.}$ stock.

$T = \sum_i V_{oi}$.

Then,

$$h_i = V_{oi} / T ,$$

$$r_i = R_i / V_{oi} ,$$

$$x_i = r_i - r^* = (R_i - r^* V_{oi}) / V_{oi} ,$$

$$\check{x}_{ij} = \check{r}_{ij} = \check{R}_{ij} / V_{oi} V_{oj} ,$$

The m expressions,

$$\bar{x}_i = z_i \check{x}_{ii} + \sum_j z_j \check{x}_{ij} , \; i = 1,2,3,\ldots.m ,$$

[11] Both Sharpe and Lintner evoke the assumption of

for the optimum Θ are then expressed in aggregated form as

$$\frac{\bar{R}_i - r^*V_{oi}}{V_{oi}} = \frac{L\ V_{oi}}{T}\ \frac{\check{R}_{ii}}{{V_{oi}}^2}) + L\sum_{j\neq 1}\frac{V_{oj}}{T}\ \frac{\check{R}_{ij}}{V_{oi}V_{oj}}\ , \qquad (33)$$

for $i = 1, 2, 3, \ldots m$.

Multiplying through by V_{oi} and collecting terms gives,

$$R_i - r^*V_{oi} = \frac{L}{T}\left[\check{R}_{ii} + \sum_{j\neq 1}\check{R}_{ij}\right] = \frac{L}{T}\sum_j \check{R}_{ij}\ . \qquad (34)$$

Expression (34) is the theoretical basis for Lintner's theorem:

> Under idealized uncertainty, equilibrium in purely competitive markets of risk-averse investors requires that the values of all stocks will have adjusted themselves so that the ratio of the expected excess aggregate dollar returns of each stock,
>
> $$R_i - r^*V_{oi}$$
>
> to the aggregate dollar risk of holding the stock
>
> $$\sum_j \check{R}_{ij}$$
>
> will be the same for all stocks (and equal to L?T), when

"homogeneity of investor expectations" in their analyses. See, Sharpe, op. cit., p. 433; and Lintner, op. cit., p. 25. To quote Sharpe:

> "Needless to say, these are highly restrictive and undoubtedly unrealistic assumptions. However, since the proper test of a theory is not the realism of its assumptions but the acceptability of its implications, and since these assumptions imply equilibrium conditions which form a major part of classical financial doctrine, it is far from clear that this formulation should be rejected . . . ."

> the risk of each stock is measured by the variance of its own dollar return and its combined covariance with that of all other stocks. 12

In order to derive an expression for V_{oi}, the following procedure is necessary. Sum equation (34) over all stocks other than the $i^{th.}$ to give:

$$\sum_{k\neq i} (R_k - r^* V_{ok}) = \frac{L}{T} \sum_{k\neq i} \sum_{j} \check{R}_{kj}. \qquad (35)$$

Divide corresponding sides of (34) by those of (35), and solve for V_{oi}; obtaining,

$$V_{oi} = \frac{\bar{R}_i}{r^*} - \sum_{j} \frac{\check{R}_{ij}}{r^*} \left[\frac{\sum_{k\neq i} (\bar{R}_k - r^* V_{ok})}{\sum_{k\neq i} \sum_{j} \check{R}_{kj}} \right] \qquad (36)$$

Letting, $W_i = K_i \sum_j \check{R}_{ij}$, where

$$K_i = \frac{\sum_{k\neq i} (\bar{R}_k - r^* V_{ok})}{\sum_{k\neq i} \sum_{j} \check{R}_{kj}}, \text{ then,}$$

$$V_{oi} = (\bar{R}_i - W_i) / r^* . \qquad (37)$$

Therefore,

$$W_i = \bar{R}_i - r^* V_{oi} = \frac{L}{T} \sum_{j} \check{R}_{ij} . \qquad (38)$$

---

[12]Paraphrased from Lintner, *op. cit.*, p. 26.

Solving for $\frac{L}{T}$ gives,

$$\frac{L}{T} = \frac{\bar{R}_i - r^* V_{oi}}{\sum_j \overset{v}{R}_{ij}} . \tag{39}$$

But from expressions (35) and (36),

$$\frac{L}{T} = \frac{\sum_{k \neq i} \bar{R}_k - r^* V_{ok}}{\sum_{k \neq i} \sum_j \overset{v}{R}_{kj}} = K_i , \tag{40}$$

which implies,

$$\frac{L}{T} = \frac{\sum_i (\bar{R}_i - r^* V_{oi})}{\sum_i \sum_j \overset{v}{R}_{ij}} = K_i = K_j = K_m . \tag{41}$$

Thus, $\frac{L}{T} = K$ is a common value for all companies in the market at equilibrium. Therefore,

$$V_{oi} = (\bar{R}_i - \frac{L}{T} \sum_j \overset{v}{R}_{ij})/r^* = (\bar{R}_i - K \sum_j \overset{v}{R}_{ij})/r^* . \tag{42}$$

The preceeding argument permits Lintner to proclaim the following theorem:

> Under idealized uncertainty, in purely competitive markets of risk-averse investors: (1) the total market value of any stock in equilibrium $[V_{oi}]$ is equal to the capitalization at the risk-free interest rate r^* of the certainty equivalent $[(R_i - W_i)]$ of its uncertain aggregate dollar return R_i; and (2) the difference between the expected value $\bar{R}_i$ of these returns and their certainty equivalent is proportional for each company to its aggregate risk represented by the sum
>
> $$\sum_j \overset{v}{R}_{ij}$$

of the variance of these returns and their total covariance with those of all other stocks; and (3) the factor of proportionality [(K = L/T)] is the same for all companies in the market.[13]

The "required rate of return" on risky investments.

An expression for an investors "risk discount rate" k_r with which expected values under uncertainty should be discounted for risk asset valuation can be derived from expression (42) as follows: Let k_r be defined as that interest rate for which

$$V_{oi} = \sum_t \frac{\bar{R}_i}{(1 - k_r)^t} = \frac{\bar{R}_i}{k_r} \quad . \qquad (43)$$

But from expression (34),

$$V_{oi} = (\bar{R}_i - K \sum_j \overset{v}{R}_{ij})/r^* ,$$

and therefore, by rearrangement,

$$V_{oi} = \frac{\bar{R}_i}{r^* \left(1 - \frac{K \sum_j \overset{v}{R}_{ij}}{\bar{R}_i} \right)^{-1}} \qquad (44)$$

so that by equating (43) and (44) and solving for k_r ,

$$k_r = r^* \left(1 - \frac{K \sum_j \overset{v}{R}_{ij}}{\bar{R}_i} \right)^{-1}. \qquad (45)$$

---

[13] Paraphrased from Lintner, op. cit., p. 26.

By means of the preceeding argument, Lintner concludes that: (1) the appropriate "risk" discount rate k_r is unique to each individual company in a competitive equilibrium; (2) that efforts to derive it complicate rather than simplify the analysis, since (3) it is a derived rather than a primary variable; and that (4) it explicitly involves all the elements required for the determination of V_{oi} itself, and, (5) does so in a more complex and non-linear fashion.

The cost of equity capital from Lintner's model. The remainder of this chapter concerns a simple extension of Lintner's model to describe the effect of a risky project upon the valuation of the firm. The responsibility for any faults in logic are in no way attributable to Lintner, whose classic analysis has brought the theory this far.

It must be understood that k_r, the appropriate risk discount rate for investors, is not necessarily the cost of equity capital appropriate to the corporate proposal evaluation mechanism. The cost of equity funds is defined as the rate of return required of their investment to insure that the present worth of existing shareholders is maximized. The cost of equity pertains to the employment of a particular quantity of shareholders' funds for investment in a specific project which is intended to increase their wealth. Thus the value appropriate to the cost of equity may well deviate from the rate of return required on total equity, since the riskiness of the project may be significantly different from that of the firm.

In deriving an expression for the cost of equity it is accepted that since management's objective is to maximize the wealth of existing shareholders, the project should be undertaken if, and only if, the value of the firm <u>with</u> the project, V_{fp}, exceeds the sum of the value of the firm <u>without</u> the project, V_f, plus the value of the funds invested by the existing shareholders to finance the project, Io. For the purpose of deriving an expression for the cost of equity which is in accordance with Lintner's work but is at the same time directly comparable to the expression derived from the certainty-equivalence model, it is assumed that the project in question shall be financed entirely by equity. Although this assumption is in conflict with the overall rational of Lintner's model, whereby the firm can borrow at the risk-free rate and presumably would, the assumption is taken to permit a simple comparison which in no way destroys the fundamental conclusion that the cost of equity is a derived rather than a primary variable.

In accordance with the wealth maximization criterion, the project should be accepted if, and only if,

$dV = V_{fp} - V_f > Io.$

Then from equation (42),

$$V_{oi} = (\bar{R}_i - W_i)/r^* = (\bar{R}_i - K\sum_j \check{R}_{ij})/r^* \qquad (46)$$

By taking total differentials, (46) becomes

$$dVoi = \frac{dR_i - Kd\check{R}_{ii} - \sum_j \check{R}_{ij} dK}{r^*} \qquad (47)$$

The conventional wisdom of discounted cash flow analysis deems a project to be profitable if, and only if its net present value at the cost of capital is positive, i.e., if,

$$NPV(ke) = \sum_t \frac{d\bar{R}_i}{(1 + ke)^t} - Io > 0 . \qquad (48)$$

Therefore, in order to insure that the net present value for the cost of capital will be positive only if dV_{oi}-Io is positive, define ke so that,

$$NPV(ke) = dV_{oi} - Io \qquad (49)$$

or equivalently, that

$$\frac{d\bar{R}_i}{ke} - Io = \frac{d\bar{R}_i - Kd\check{R}_{ii} - dK\sum_j \check{R}_{ij}}{r^*} - Io. \qquad (50)$$

By cancelling Io's and reorganizing, (50) becomes,

$$\frac{dR_i}{ke} = \frac{d\bar{R}_i}{r^* \left(1 - \frac{Kd\check{R}_{ii}}{d\bar{R}_i} - \frac{dK\sum_j \check{R}_{ij}}{d\bar{R}_i}\right)^{-1}} \qquad (51)$$

Hence, ke may be expressed as a function of the risk-free rate of interest and of risk and investor preference parameters:

$$ke = r^*\left(1 - \frac{Kd\bar{R}_{ii}}{d\bar{R}_i} - \frac{dK\sum_j \overset{V}{R}_{ij}}{d\bar{R}_i}\right)^{-1}. \qquad (52)$$

Expression (52) bears some resemblance to expression (16), which was derived from the certainty-equivalence model for similar circumstances, i.e.,

$$ke = r^*\left(a_T + \Delta a \frac{\bar{R}}{d\bar{R}}\right)^{-1}. \qquad (16)$$

If the certainty-equivalence function is of the form,

$$a_i = 1 - B\bar{R}_i - B\frac{\overset{V}{R}}{\bar{R}_i}$$

which is the form appropriate to the utility function

$U(R) = R_i - BR_i^2$,[14] (16) becomes,

$$ke = r^*\left(1 - B(d\bar{R}_i) - B\frac{d\overset{V}{R}_i}{d\bar{R}_i}\right)^{-1} \qquad (53)$$

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[14] Recall from footnote 17, Chapter II, that this form of the utility function is the <u>only</u> form which fulfills the consistency requirements of the von Neumann-Morgenstern axioms when utility is considered to be a function of mean and variance alone in a system for which subjective probability distributions are not considered to be simple two-parameter functions.

Although expression (53) is not equivalent to (52), the difference is concentrated in two terms, BdR_i in the former, and the term $dK \sum_j (\check{R}_{ij}/d\bar{R}_i)$ in the latter.

If the certainty-equivalence factor is of the form $a_i = 1 - B(\check{R}_i / \bar{R}_i)$, expression (16) becomes,

$$ke = r^* \left(1 - B \frac{d\check{R}_i}{d\bar{R}_i} \right)^{-1},$$

which is a closer approximation to the cost of equity appropriate to Lintner's model.[15]

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[15] The certainty-equivalence factor of the form

$$a_i = 1 - B(\check{R}_i / \bar{R}_i)$$

is derived from an expected utility formulation of the construction

$$E(U_{R_i}) = R_i - B\check{R}_i.$$

Unfortunately, except for the special circumstance for which investors' expectations are described <u>only</u> by simple two-parameter subjective probability distributions, this particular form of expected utility function is inconsistent with the von Neumann-Morgenstern axioms.

To quote Sharpe: "That such a transformation from $E(U_R)$ into $E(U_{R_i})$ is not consistent with the axioms can readily be seen: . . . since the first equation implies non-linear indifference curves in the $\bar{R},\check{R}$ . . . . plane while the second implies a linear relationship. . . . Thus the two functions must imply different orderings among alternative choices in at least some instances." From Sharpe, *op. cit.*, p.434.

Of course, if investors' expectations are entirely in

If certain additional assumptions are made in respect to Lintner's model, the expression (52) can be simplified into a form equivalent to that of expression (53).

Assume that the aggregate market value of all stocks other than the firm's, and all covariances between the firm and other stocks, are independent and invariant with respect to the capital marketing decisions of the company.

The consequence of these further limitations is to make the third term in the bracketed expression of (52) equal to zero, and therefore (52) becomes,

$$ke = r^*\left(1 - K \frac{d\check{R}_i}{d\,\bar{\bar{R}}}\right)^{-1}, \qquad (54)$$

which is directly equivalent to expression (53), since both B and K are constants which reflect investor risk-aversiveness.

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the form of two-parameter distributions, the form given to a_i is consistent with the von Neumann-axioms.

In his analysis Lintner does not state explicitly that investors are assumed to form their expectations only in the form of two-parameter subjective probabilities although the inference is left that they do. See for example, Lintner, op. cit., especially Assumption (2) p. 15 and Assumption (2) p. 25.

Unfortunately however, since the certainty-equivalence function for $a_i = 1 - B(\check{R}_i / \bar{R}_i)$ does not generally conform to the von Neumann-Morgenstern axioms of rational behavior, whereas the form $a_i = 1 - B\bar{\bar{R}}_i - B(\check{R}_i / \bar{R}_i)$ does, what remains to be explained is whether or not expression (52) which was derived from Lintner's model, is also contaminated by dissaffiliation from the postulates of rational behavior.

Fortunately it can be shown that Lintner's construct does not imply irrational behavior on the part of investors. Consider, first, the Separation Theorem, which is fundamental to Lintner's model:

> Given the assumptions about borrowing, lending and investor preferences (implied by maximization of a von Neumann-Morgenstern utility function if either (1) the investor's utility function is concave and quadratic or (2) the investors utility function is concave, and he has assigned two-parameter probability distributions to reflect his expectations, the optimal proportionate composition of the stock portfolio is independent of the ratio of the gross investment in stocks to the total net investment. 16

From the theorem, Lintner draws the following corollaries:

> (1) Given the assumptions about borrowing and lending stated above, any investor whose choices maximize the expectation of any particular

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[16] Lintner, op. cit., p. 17

> utility function consistent with these conditions will make <u>identical decisions regarding the proportionate composition of his stock (risk-asset) portfolio. This is true regardless of the particular utility function</u> whose expectation he maximizes [italics in original].
>
> (2) The parameters of the investor's particular utility within the relevant set determine <u>only</u> the ratio of his gross investment in stocks to his total <u>net</u> investment (including riskless assets and borrowing); and . . . the investor's wealth is also, consequently, relevant to determining the <u>absolute size of</u> his investment in individual stocks, but not to the relative distribution of his gross investment in stocks among individual issues. [italics in original]. 17

Although Lintner assumes rationality among investors, his assumption of unlimited borrowing and lending capacity at the risk-free interest rate makes possible the relevance of his model to a market of investor's characterized by utility functions which do not meet the requirements of the von Neumann-Morgenstern axioms. The function, $U(r) = 1\text{-}B\dfrac{V_R}{\bar{R}}$ is an example of a function which <u>may</u> deviate from generalized "rationality", yet it would clearly function in Lintner's model.[18] Since, in the simple certainty-equivalence model

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[17] Lintner, <u>op</u>. <u>cit</u>., pp. 17-18

[18] See footnote 15, this chapter.

unlimited leverage is not presumed, expression (53) is only obtained by assuming both irrational behavior and, implicitly, expectations of zero correlation between returns on securities.

The reconciliation of the portfolio selection and certainty-equivalence forms of the cost of equity capital expression is of little importance to the overall state of the art of capital budgeting. The reconciliation merely serves to relate one model to the other and to explain their differences.

What is important to the theory of capital budgeting is that the explicit inclusion of risk-parameters in the expression for the cost of capital is shown to be redundant (by both models), since the valuation equation necessary to derive the cost of capital is of itself sufficient to decide a project's rejection or acceptance.

Summary. The simple "cost of equity capital" extension to Lintner's classic portfolio selection model supports the contention that the cost of capital, as conventionally defined, is a derived rather than a primary variable. The analysis shows that the cost of equity investment is a complex function of the risk-free rate of interest, and of expected values, variances and covariances between the project, the firm, and other firms in the market. Although the model used for the analysis is admittedly derived under a set of severely limiting and simplifying assumptions, the conclusion reached may be validly extended to more complex economic circumstances which may more closely reflect the realities of existing markets.

That the cost of capital no longer holds preeminence in conceptual validity as a means of economic evaluation is attested to by the increasing number of mathematical programming models for capital budgeting which rely upon the risk-free rate of interest for discounting for futurity. Although the preceeding summary gave little evidence to the fact, Lintner's analysis was largely directed to the establishment of a mathematical programming model for "determination of the optimal corporate capital-budget-portfolio" under the simplified circumstances of uncertainty which were assumed. In Lintner's model, all present values are calculated with the riskless rate r*.

In a somewhat less analytical treatment of the problem of evaluation of risky investments, James van Horne also specifies that the risk-free rate of discount is appropriate when capital budgets involve combinations of risky investments.[19]

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[19]James Van Horne, "Capital-Budgeting Decisions Involving Combinations of Risky Investments," Management Science, Vol. XIII, No. 2, (October, 1966), pp. B 84-92.

In an earlier paper, Neil R. Paine proposes a model for the analysis of combinations of risky investments but does not specify the discount rate which should apply.[20] Cord avoids the difficulty of contending with the problem by resorting to the assumption of a fixed amount of funds and the employment of internal rate of return as the means of measurement and comparison of uses of funds.[21]

It is neither the purpose nor within the capabilities of this research to undertake a critical analysis of these mathematical programming techniques. It is sufficient to say that those of most recent vintage and of greatest promise do not rely upon the conventional formulation of the cost of capital. It is possible therefore that continuing over-emphasis of the "risk-adjusted" cost of capital discount rate approach to valuation may be hindering the advancement of financial theory.

In respect to the practical application of capital budgeting techniques, wherein theoretical validity requires temperence with pragmatic considerations of cost and administrative efficiency, it seems unlikely that the cost of

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[20] Neil R. Paine, "Uncertainty and Capital Budgeting," The Accounting Review, Vol. XXXIX, No. 2, (April, 1964), pp. 330-332.

[21] Joel Cord, "A Method for Allocating Funds to Investment Projects When Returns are Subject to Uncertainty," Management Science, Vol. 10, No. 2, (January, 1964), pp. 335-341.

capital will soon disappear. Neither Lintner's model, nor the certainty-equivalence model which preceeds it, are perfectly applicable to practical decisions due to the host of factors which were assumed away. But the models are not intended for that purpose. To quote Lintner,

> The purpose of these simplifying assumptions has been to permit a rigorous development of theoretical relationships and theorems, which reorient much current theory (especially on capital budgeting) and provide a basis for further work.[22]

The purpose of the analysis is achieved, therefore, if the complications of treating the cost of capital as a "risk-adjusted" discount rate are made clear even in an "idealized" context. Comprehension of relationships in the ideal state leads, a priori, to a fuller understanding of, and a more rational response toward, relationships in the complexities of reality.

In the brief chapter which follows, the role of the cost of capital in the Monte Carlo simulation and analytical-statistic approaches to risk analysis of capital investment projects is discussed. Both Monte Carlo simulation and analytical-statistics involve mathematical models, and therefore, the preceeding analysis and reasoning is pertinent to their discussion.

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[22] Lintner, op. cit., p.

CHAPTER VII

THE ROLE OF THE COST OF CAPITAL IN MONTE CARLO SIMULATION AND THE ANALYTICAL-STATISTICS APPROACHES TO RISK ANALYSIS

Methods of objectively quantifying and analyzing the risk inherent in the commitment of financial resources are relatively recent innovations in the theory of finance. And it is safe to say that the majority of industrial capital investment programs still rely on analytical techniques which do not explicitly include quantitatively objective measures of risk in their process of economic evaluation.[1] As will be made clear, even the moder widely accepted discounted cash flow techniques of net present valuation and the internal rate of return have been accused of failing to properly account for risk. To the extent that the cost of capital is conceptually associated with these techniques, its true relevance to their application is worthy of assessment. To what extent do criticisms of conventional discounted

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[1]See, for example, S.J. Pullara and L.R. Walker, "The Evaluation of Capital Expenditure Proposals: A Survey of Firms in the Chemical Industry," The Journal of Business, Vol. XXXVIII, No. r, (October, 1965), pp. 403-408; and John T. Nicholson and J.D. Ffolliott, "Investment Evaluation in Criteria of Canadian Companies," The Business Quarterly, (Summer, 1966), pp. 54-62.

cash flow techniques cast aspersion on the cost of capital? And perhaps more importantly, what role does the cost of capital have to play in the more recent innovations developed to overcome the weaknesses which are claimed to fault the conventional methodology?

Although economic theorists generally agree that discounted cash flow techniques are of greater conceptual validity than payback and accounting rate of return methods for determining the worth of a project to the firm, it has been widely recognized that the conventional approach, based upon deterministic measures of cash flows, is not infallable.

> Weyerhaeuser's financial experience does corroborate the literature, i.e., we have found that only rarely do investments provide the financial return which is suggested at the time that the investment is recommended or undertaken. Even under the idealized conditions of a consistent, able management group, investment return prognostications will differ from subsequent events. 2

It is argued that the conventional discounted cash flow techniques ignore the riskiness inherent in capital investments since, according to the argument, only the expected values of uncertain future cash flows are considered in the evaluation mechansim. To quote Hertz,

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[2]R.M. Curley, R.L. Schock and B.E. Wynne, Jr., "Simulation Applied to Project Analysis," in Operations Research and the Design of Management Information Systems, edited by John F. Pierce, Jr., p. 230.

> In short, the decision-maker realizes that there is something more he ought to know, something in addition to the expected rate of return. He suspects that what is missing has to do with the nature of the data on which the expected rate of return is calculated, and with the way those data are processed. It has something to do with uncertainty, with possibilities and probabilities extending across a wide range of rewards and risks. 3

Much of the blame for this "weakness" ascribed to the conventional discounted cash flow approach is accredited to the fact that " . . . the main purpose of this criterion is to summarize into a single measure the quantifiable factors affecting the economic desirability of the project under consideration."[4] It is argued that a valid criterion for decision where risk is involved must be based on not only a single measure such as the mean of a profitability index, but also upon the variance and other risk parameters relative to the subjective uncertainty of its description.

In order to provide measures of risk in addition to the "expected value" of discounted cash flow indices, recent

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[3]David B. Hertz, Risk Analysis in Capital Investment," Harvard Business Review, Vol. XLII, No. 1, (January, 1964), p. 96.

[4]Sidney W. Hess and Harry A. Quigley, "Analysis of Risk in Investments Using Monte Carlo Techniques," Chemical Engineering Symposium Series 42: Statistical and Numerical Methods in Chemical Engineering, p. 55.

advances in techniques of risk analysis in capital budgeting have involved the determination of probability distributions of net present value, internal rate of return and other financial criteria. Two distinct lines of attack can be readily distinguished: the Monte Carlo simulation approach, and the analytical statistics approach.

This chapter describes the two approaches and shows their foundation in financial theory. Since both approaches employ stochastic specifications of future cash flows in their respective methodologies, the technique of deriving subjective probability distributions for the uncertain factors involved is discussed. Once the subjective probability description of cash flows is complete, the project's subjective riskiness is quantitatively established. Given the risk of the project in terms of means, variances and co-variances of cash flow expectations in accordance with the theory established in Chapters II through VI, the cost of capital appropriate to the project's evaluation may be defined. This chapter shows that the cost of capital so defined is not relevant to the generation of probabilistic expressions of valuation indices regardless of its conceptual association with the discounted cash flow techniques employed in the Monte Carlo simulation and analytical-statistics approaches.

The Monte Carlo Simulation Approach. Monte Carlo simulation is an experimental procedure used in the evaluation of complicated expressions or models which involve one or more probability distributions defining the variables relevant to the investigation. The procedure of Monte Carlo simulation may be resolved into four distinct steps:

(1) A mathematical model is designed to capture the essence of the relevant features of the experimental subject and its environment in order to reveal the functional relationships between the variables being investigated.

(2) Probability distributions are specified to describe the range and likelihood of the values of each variable making up the problem.

(3) A value for each variable is selected at random from its appropriate probability distribution for substitution into the model. In this manner a single value of the independent variable is computed and recorded.

(4) Steps (2) and (3) are repeated as many times as are necessary to generate a frequency distribution of values for the independent variable. The frequency distribution is taken as an approximation to the "true" probability distribution relevant to the problem involved.

David B. Hertz,[5] and Hess and Quigley,[6] have been instrumental in the promotion of Monte Carlo simulation as

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[5]Hertz, op. cit., pp. 95-106.

[6]Hess and Quigley, op. cit., pp. 55-63

a risk analysis technique for capital budgeting. The general approach they advocate may be summarized as follows: The model to be used for the simulation is an appropriate mathematical expression for net present value, internal rate of return or some other profitability index. Probabilistic estimates of "key" input factors are then made. Hertz, for example suggested that the following "key" input factors or variables might be relevant as stochastic functions for the analysis of a proposed extension to a processing plant:[7]

1. Market size
2. Selling prices
3. Market growth rate
4. Share of market
5. Investment required
6. Residual value of investment
7. Operating costs
8. Fixed costs
9. Useful life of facilities.

Given the stochastic definitions of the "key" factors, the frequency distribution of the appropriate index is generated by the iterative random combination of the factors in the model. The repetitive trial process is particularly suited to digital computer application, since by that means the many tedious reiterations can be made at great speed and accuracy.

The resulting frequency distribution of net present value, internal rate of return, or other index, is considered to be a better measure of the attractiveness of a proposed investment than a simple expression of its "expected value".

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[7] Hertz, _op_. _cit_., p. 102

The analytical-statistics approach. The analytical statistics approach, hereafter termed the analytical approach, involves the direct calculation of the mean, variance and other risk parameters of the profitability index by means of statistical mathematics. Unlike the simulation approach, the analytical approach does not involve an iterative procedure, but instead proceeds to the answer by a single mathematical process which neither depends upon, nor benefits from, the employment of a computer. The final result however, is of the same form as that which ensues from simulation, i.e., a probabilistic expression of the range and likelihood of occurance of the profitability index used to evaluate the project in question. The analytical approach has been advocated by Frederick S. Hillier[8] and B. Wagle.[9] A brief summary of the essence of Hillier's approach follows for illustration.

Let X_j be a random variable which takes on the value of the net cash flow during the j-th. year, where $j = 0,1,2,\ldots n$. Assume that X_j is normally distributed

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[8]Frederick S. Hillier, "The Derivation of Probabilistic Information for the Evaluation of Risky Investments," Management Science, Vol. IX, No. 3, (April, 1963), pp. 443-57.

[9]B. Wagle, "A Statistical Analysis of Risk in Capital Investment Projects," Operational Research Quarterly, (March, 1967), pp. 13-35.

with a known mean, $\bar{X}_j$, and known standard deviation, b_j. Then if the appropriate discount rate is defined as k, the present worth of the series of cash flows is given by

$$W = \sum_{j=o}^{n} \frac{X_j}{(1 + k)^j} \qquad (55)$$

Note that this expression defines present worth as a random variable. The expected value of present worth is given by

$$\bar{W} = \sum_{j=o}^{n} \frac{\bar{X}_j}{(1 + k)^j} . \qquad (56)$$

The conventional approach to capital investment evaluation is based upon the expected value of present worth, rather than the random variate formulation given by the expression (55).

If it is assumed that Xo, X_1, X_2, . . .X_n, are mutually independent, the variance of present worth is

$$b_W^2 = \sum_{j=o}^{n} \frac{b_j^2}{(1+k)^{2j}} \qquad (57)$$

If, on the otherhand, X_o, X_1, X_2, . . .X_n, are assumed perfectly correlated, the variance is

$$b_W^2 = \left[\sum_{j=o}^{n} \frac{b_j}{(1+k)^j} \right]^2 \qquad (58)$$

A more realistic model would be obtained if partial correlation was included in the derivation.

Wagle extends Hillier's simple formulation to account for correlation between cash flows, and as well, for correlation between the "key" factors which combine to determine the cash flows, X_j.[10]

Presumably the drivation of higher moments of the probability distribution of present worth, or for any other profitability index for that matter, would entail a straightforward extension of the preceeding methodology. Of course, for the normal distribution, the mean and variance are sufficient to completely describe the stochastic form so that further parameters are irrelevant.

That there are advantages and disadvantages attributable to either simulation or the analytical approach is not denied. Simulation can handle probabilistic calculations which would be too complex and awkward for analytical derivation, but the analytical method does not depend upon iterative calculations which may require the services of a computer. But it is not the purpose of this discourse to deal with the matter of relative advantage. Instead, the concern is with the theoretical relevance of the two approaches to risk analysis, as particularly expressed in the stochastic expression of valuation indices. In order to assess conceptual relevance,

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[10] B. Wagle, "A Statistical Analysis of Risk in Capital Investment Projects," Operational Research Quarterly, (March, 1967), pp. 13-35

two characteristics, common to both approaches and relevant to the financial theory which underlies their application to capital budgeting, will be discussed; they are, (1) the subjective probability distributions which make-up the stochastic expression of future cash flows, and (2) the cost of capital, which in financial theory serves as a discount rate for net present value derivation and as a hurdle rate for the evaluation of internal rate of return.

Subjective probability and expectations. The definition of current and future cash flows in probabilistic form is essential to the risk analysis of capital investments. But, alas, the specification of such probability distributions is at best an arduous task, the results of which may be of less than acceptable credibility.

Since capital projects relate to future expectations, recourse to historical data may have little or no relevance to the establishment of stochastic forecasts. Proposals of the greatest risk frequently involve the introduction of innovative products for which past experience is non-existent. Consequently forecasts must often take on a highly subjective nature.

The means of specifying subjective probability distributions for each of the variables affecting an investment decision are not yet fully developed. Hess and Quigley suggest that ". . . the probability distributions of individual parameters are best developed by experienced personnel,"[11] but do not suggest a methodology. Norton recommends that

> . . . one approach is to question, directly, persons believed qualified to express judgements. . . . By proceeding in a systematic manner, the questioner can detect and call attention to any inconcistencies which develop and end with a unique set of probability estimates which express the respondent's feeling toward the likelihood of every demand pattern.12

An interesting method suggested by Schlaifer involves the respondent in an imaginary standard lottery.[13] The respondent is offered a choice between the result of the uncertain event with which the forecast is concerned, and a certain number of tickets in the standard lottery. The number of tickets needed to make him feel indifferent in the choice becomes a measure of his subjective probability estimate for the likelihood of the event.

Much of the difficulty of specifying full probability distributions may be avoided if (1) only the mean and variance are estimated, or (2) theoretical probability

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[11] Hess and Quigley, op. cit., p. 60.

[12] John H. Norton, "The Role of Subjective Probability in Evaluating New Product Ventures," in Chemical Engineering Symposium Series 42: Statistical and Numerical Methods in Chemical Engineering, p. 52.

[13] Robert Schlaifer, Probability and Statistics for Business Decisions, p. 11.

distributions are fitted to two or three point estimates of likelihood and value.

The mean-variance approach has been criticized because it may lead to "bell-shaped thinking", and thereby to the habit of neglecting or ignoring the skewness which is inherent in many natural stochastic processes.[14]

An ingenious approach to the fitting of theoretical probability distributions (lognormal, Weibull, normal and triangular) to three level estimates of input variables is presented by Moon Hoe Lee, who tested the suitability of the theoretical distributions as proxies for the "real thing" by employing them in a replication of Hertz's famous Monte Carlo simulation of the chemical plant investment decision.[15] The investigation showed that the lognormal function showed considerable promise, but whether the conclusion can be justifiably generalized to other situations remains to be seen.

Since a critical analysis of techniques for the derivation of respondent's subjective probability forecasts

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[14]William D. Lamb, "A Technique for Subjective Probability Assignment in Risk Analysis Problems," (paper presented to the Institute of Management Sciences, American Meeting, Boston, Massachussetts, April 5-7, 1967) p. 2

[15]Moon Hoe Lee, "Statistical Transformation of Probabilistic Information," (unpublished Master's thesis, The University of British Columbia, 1967)

is beyond the scope of this research, it is assumed that it is at least possible for analysts to define subjective probability distributions for the key variables affecting a capital investment proposal. Furthermore, it is accepted as congruent with normative theory that management should describe uncertain future economic events by stochastic rather than deterministic measures whenever it is financially justified. Note, however, that this is not a blanket endorsement of the practice of expressing valuation indices in stochastic form. Whether or not the generation of probabilistic valuation indices is conceptually justifiable or even necessary requires the more detailed analysis which follows. For the time being therefore, it is accepted that the definition of future uncertain events in terms of subjective probability distributions is a valid means of quantifying their subjective riskiness.

<u>The cost of capital in simulation and analytical approaches to risk analysis</u>. In the application of Monte Carlo simulation and the analytical-statistics approaches to the derivation of stochastic discounted cash flow indices, an interest rate is a necessary component of the generative mechansim. The interest rate serves to discount the randomly selected values of future cash flows. To the extent that proponents of the Monte Carlo simulation and

analytical-statistics approaches to risk analysis vest the final decision of acceptance or rejection in either (1) a formal utility function and the criterion of accepting the investment with the highest expected utility, or (2) an informal subjective assessment of risk surrounding a particular valuation index, the interest rate serves neither as a financial standard nor a criterion of choice. Since the risk inherent in the alternative is accounted for by other means, a measure of compensation for risk in the interest rate is irrelevant to the evaluation process.

In conventional financial theory, the "risk-adjusted" cost of capital (as it is defined for this research) serves both as an interest rate and as a criterion of choice for deciding acceptability. In this respect it is irrelevant to the Monte Carlo simulation or analytical-statistics methodology.

Nevertheless, it is basic to scientific reasoning and the theoretical approach to improving human decision processes that guidelines in the form of decision criteria be established to differentiate between good decisions and bad. Without the cost of capital as a financial standard, the scientific approach requires the application of another conceptually valid criterion of choice, which in theory at least, will provide a concise distinguishment between capital projects which will further the firm's objectives and those that will not.

Although proponents of the Monte Carlo simulation and analytical-statistics approaches to risk analysis have for the large part concentrated upon the development of stochastic approaches to the measurement and comparison of alternative uses of funds, some thought has been given to the definition of an appropriate criterion of choice. But not always, for Hertz, for example, simulated a probabilistic expression for a project's internal rate of return without explaining how management _should_ decide between alternatives.[16] Hillier, on the other hand, suggests that

> . . . Considering the probabilities involved, management would, in effect, implicitly assign utilities to the possible outcomes of the investments and select the investment with the larger expected utility.[17]

Hillier therefore implies that it is the "utility of management", rather than the "interests of the shareholders," which is relevant to the investment decision. Of course, the utility preferences of the company, as set forth by management, are likely to differ from those of the shareholders, and if such is the case, it would be only by chance that investor's wealth would be maximized. Hess and Quigley also leave the final decision to the subjective judgement of

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[16] Hertz, _op. cit._

[17] Hillier, _op. cit._, p. 444

management:

> . . . Developing the profitability distribution. . . allows management to make a quantitative assessment of the risks involved in approving a particular investment. . . . so management knows the size of the risk it is undertaking. 18

Wagle, who follows Hillier, also appeals to utility ranking of probabilistic expressions of valuation indices, but does not define to whom the utility belongs.[19]

Van Horne appeals to ". . . the utility preferences of a company with respect to expected net-present value and variance . . ."[20] in his elaboration of a capital budgeting procedure which evaluates combinations of projects according to their incremental contributions of expected net-present value and variance to the firm as a whole. Van Horne's formulation closely approximates Lintner's model, since the method recognizes covariance between investments and the fact that total variance must take account of existing investment projects as well as proposals under consideration. But Van Horne's model fails in two respects; first, it does not account for covariance between the firm and the market, and secondly, it does not explicitly account

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[18]Hess and Quigley, op. cit., p. 60.

[19]Wagle, op. cit.

[20]James Van Horne, "Capital Budgeting Decisions Involving Combinations of Risky Investments," Management Science, Vol. XIII, No. 2, (October, 1966), p. B-89.

for the welfare of the shareholders. Van Horne does, however, employ the risk-free rate of discount in the derivation of the expected value and variance of net present values for investment alternatives.

Summary. Monte Carlo simulation and analytical-statistics are techniques for deriving probabilistic expressions defining the range and likelihood of dependent variables. The techniques do not, of themselves, decide between profitable and unprofitable investments. They merely provide the methodology for deriving the expected values and risk parameters of stochastic variables for a combination of investments, from the expected values, variance and covariances which characterize the investments individually. Whether or not the expected value and risk parameters so derived constitute the most desirable set will depend upon the particular criterion of choice employed for their evaluation.

To the extent that the decision to accept or reject a project is made on the basis of either (1) a formal utility function and the criterion of accepting the investment combination with the highest utility, or (2) an informal subjective assessment of risk surrounding a particular valuation index, the cost of capital as a "risk-adjusted" financial standard is irrelevant to the Monte Carlo simulation or analytical-statistics methodology. The risk-free rate of interest is the <u>only</u> rate which can be employed without compromising the conceptual validity of the analysis by double accounting for risk.

CHAPTER VIII

SUMMARY AND CONCLUSIONS

This chapter summarizes the more important concepts founding the research and outlines the conclusions reached through the analysis. The chapter concludes by identifying problems deliniated by the study which are worthy of additional research.

I. THE SUMMARY

The concept of the cost of capital as ". . . a discount rate with the property that an investment with a rate of profit above this rate will raise the value of the firm"[1] is fundamental to conventional capital budgeting theory. According to the conventional theory, in the absence of capital rationing, a project is deemed acceptable by discounted cash flow criteria if (1) the net present value of its expected cash flows is positive when discounted at the cost of capital; or (2) the cost of capital is less than the project's internal rate of return on expected cash flows. The cost of capital, in its service as a

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[1] Myron J. Gordon, The Investment, Financing and Valuation of the Corporation, p. 218.

financial standard rate of discount, is formalized as an appropriately weighted average cost of the firm's sources of capital financing, including both explicit and implicit costs of debt, equity and preferred stock in their many and varied forms.

That there is great difficulty in defining the cost of capital as a measure of both the time value of money and a compensation for the riskiness of a project has been recognized by several theorists, including Bierman and Smidt,[2] Robichek and Meyers,[3] and Porterfield.[4] The particular difficulty is found to lie in the specification of the relationship between risk, as a characteristic of earnings expectations, and the shareholders' required return on equity capital. Conventional theorists, in establishing cost of capital theories, have tended to avoid the problem by

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[2]Harold Bierman, Jr. and Seymour Smidt, The Capital Budgeting Decision, Second edition, Part III, pp. 281-357.

[3]Alexander A. Robichek and Stewart C. Myers, Optimal Financing Decisions, Chap. V, pp. 67-93.

[4]James T.S. Porterfield, Investment Decisions and Capital Costs, Chap. VII, 107-133.

assuming explicitly or implicitly that the quality, or degree of certainty, of future expected earnings with the project is identical to the quality of future expected earnings without the project. Costs of capital so derived are simply ratio functions of expected earnings per share from existing investments to the net proceeds per share from the new issue of equity required to finance the project in question. To the extent that the assumption of "homogeneity of quality" is maintained, the overall cost of capital is defined as the weighted average of costs of sources of funds, with the weights being proportional to the current "optimal" capital structure of the firm.

If the "homogeneity of quality" assumption is relaxed, the problem of defining the cost of capital is severely complicated. Nevertheless, the concept of a "risk-adjusted" rate of discount has considerable intuitive appeal, and the methodology of project evaluation based on discounted cash flows and the cost of capital cannot be easily discarded.

In order to analyze the relationship between changing risk and the cost of capital, it was necessary to establish a valuation model which reflected the manner in which investors are assumed to react to risk as a characteristic of investment under uncertainty. For this purpose a normative model of economic man, founded upon the theories of rational choice postulated in the von Neumann-Morgenstern axioms and cardinal utility theory, was employed to relate the desirability of an uncertain investment to both the expected value and risk parameters accorded to subjective probability distributions describing expectations of returns.

The concept of certainty-equivalents of uncertain returns, evolving from the model of economic man, was used to investigate the effect of risk upon the cost of capital under idealized conditions by means of the classical certainty-equivalence model of risk-asset valuation. Within the severely restricting limitations of the simplifying assumptions which were made, the cost of equity capital was shown to be a complex function of (1) the risk-free rate of interest, and (2) the expected values and variances of, and the covariances between, the earnings expected from both the firm and the project. Since the "correct value" of the cost of capital; correct in that it will distinguish between projects which will and will not enhance shareholders' wealth; is a value unique to characteristics of the firm and the particular project considered, it is necessarily a derived rather than a primary variable. In other words, the correct value is found by the analysis of all the elements required to determine the sign and magnitude of the project's incremental addition to the value of the firm, and hence to shareholders' wealth. Since the objective of management is to maximize shareholders' wealth, the cost of capital is not at all essential to the theoretically correct, and direct, valuation process.

In order to give a broader definition to this result, and to acknowledge the source of the idea that the cost of capital is a complex "derived" variable, Lintner's classic model of risk-asset valuation was summarized and adapted to the problem of defining the "risk-adjusted" cost of capital. Lintner's sophisticated model takes into account the investment trait of diversification of risk-asset portfolio's which is ignored in the simple certainty-equivalence model. Diversification compounds the complexity of the problem of defining a "correct value" for the cost of capital by requiring the inclusion of a measure of correlation between expected returns to the project, to the firm, and to all other investment opportunities in the securities market. Although Lintner's thesis points out the need for a new approach to capital budgeting through the development of algorithms for attaining optimal investment sets without resort to the cost of capital as a financial standard, this research did not include an analysis of design of algorithms for that purpose.

Consideration was given, however, to assessment of the relevance of the cost of capital as a "risk-adjusted" discount rate in the Monte Carlo simulation and analytical-statistics approaches to risk analysis in capital budgeting. Both techniques have gained recent popularity as means of quantifying risk inherent in capital projects by expressing discounted cash flow valuation indices such as net present value and internal rate of return in probabilistic rather than deterministic form. The rational behind the approach is that only by considering risk as well as the "expected values" of the indices can management truely make valid economic decisions. To the extent then, that the cost of capital includes a measure of compensation for risk as a valid criterion for assessing the worth of the expected values of the uncertain cash flows, its employment for the stochastic expression of valuation indices patently involves "double accounting for risk"; once in the cost of capital and once again in the probabilistic formulation of the indices. The cost of capital is therefore inappropriate and irrelevant to the stochastic expression of discounted cash flow valuation indices.

II. THE CONCLUSIONS

The cost of capital, when defined as

> . . . the minimum prospective rate of yield that a proposed investment in real assets must offer to be worthwhile undertaking from the standpoint of the current owners of the firm, . . . [5]

[5] Franco Modigliani and Merton Miller, "Estimates

is a derived and complex variable which must be specified as a function of at least (1) the risk-free rate of interest, (2) the expected values of uncertain returns to the projet, (3) the expected values of uncertain returns to existing assets, (4) the variances of, and covariances between, expected returns to the project, to the existing assets of the firm, and to all other securities available to the market of investors, and finally to (5) the aggregated risk aversion of investors in the market. This conclusion is essentially that of Lintner, but the analysis leading upto its foundation is perhaps more simply and comprehensibly established. This research does not claim to have "discovered" the conclusion.

In theory, employment of the cost of capital as a means of accounting for risk is essentially inefficient. This is so because as a derived variable it is a function of all the elements required to determine the sign and magnitude of the project's incremental addition to the

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of the Cost of Capital Relevant for Investment Decisions Under Uncertainty," Determinants of Investor Behavior: A Conference of the Universities-National Bureau for Economic Research. p. 182.

value of the firm. The valuation equation necessary to relate the elements into the "correct" value of the cost of capital is of itself sufficient to determine whether or not the project in question is worthwhile. The risk-free rate of interest is the only discount rate appropriate to the valuation equation.

Finally, in as much as the cost of capital is a "risk-adjusted" discount rate, it is inappropriate and irrelevant to the probablilistic expression of valuation indices by means of either Monte Carlo simulation or analytical-statistics procedures. Employment of the cost of capital in such approaches to risk-analysis involves "double-accounting for risk"; once in the cost of capital, and once again in the stochastic expression of the valuation indices.

III. GENERAL COMMENTS AND RECOMMENDATIONS FOR FURTHER RESEARCH

It is possible that over-emphasis of the "risk-adjusted" discount rate approach to capital budgeting and quantitative financial analysis may hinder the advance of the theory of finance. Nevertheless, that the models of certainty-equivalence and portfolio-selection used in this research contain serious elements of impractability cannot be rationally denied. In fact there is little to indicate that the cost of capital approach to investment appraisal will soon be replaced. Payback, which has felt the impact of theoretically superior discounted cash flow techniques for in excess of a decade, still retains great sway in the councils of corporate decision-making.[6] It may be that payback owes much of its longevity to its simplicity and ease of administrative application in systems of corporate capital budgeting. If history repeats itself, it is safe to expect that the cost of capital concept, and the "risk-adjusted" discount rate approach to accounting for uncertainty, will continue to be employed by practicing analysts long after the theoretical advantages of other more conceptually valid techniques have been firmly established in the halls of academe.

This is not to say however, that further research and synthesis of the concepts and techniques of risk analysis are unjustified. To the extent that they but enlarge the general

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[6] For evidence as to the extent to which Payback (and other non-discounted cash flow criteria) is employed today, see, S.J. Pullara and L.R. Walker, "The Evaluation of Capital Expenditure Proposal: A Survey of Firms in the Chemical Industry," The Journal of Business, Vol. XXXVIII, No. 4, (October, 1965), pp. 403-408; and, John T. Nicholson and J.D. Ffolliott, "Investment Evaluation Criteria of Canadian Companies," The Business Quarterly, (Summer, 1966), pp. 54-62. Also refer to J. Fred Weston and Eugene F. Brigham, Managerial Finance, p. 142.

understanding of the limits and weaknesses of conventional theory and techniques, a purpose is served. And that has been the purpose of this research.

With regards to further research, a difficult but rewarding task awaits those who would modify Lintner's model to account for imperfections in the capital markets; to wit, by imposing restrictions upon personal borrowing. The fact that the freedom of unlimited borrowing capacity at the risk-free rate does not fit well with observed phenomenon in "real" circumstances leaves opportunity for substitution of "traditionalist" conditions for those of Modigliani and Miller's "perfect capital markets" in respect to personal leverage capability in Lintner's model. On a less ambitious scale, the simpler certainty-equivalence model might similarly be adapted to a study of the relationship between valuation and the risk of fixed commitment financing under "traditionalist" market conditions.

The simple certainty-equivalence model might also be employed to study the effect of risk change upon the cost of equity capital appropriate to the issue of new shares among new and existing shareholders in accordance with Mao's analysis.[7]

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[7]James C.T. Mao, Quantitative Analysis of Financial Decisions, Chap. X. pp. 20-29.

The certainty-equivalence and portfolio selection approaches to project evaluation might be fruitfully applied to Hertz's investment-decision criterions testing-model, which employs a simulated economy over time to test the relative advantage of various techniques and criteria of choice.[8]

Needless to say, much remains to be discovered about individual and aggregated utility theory. This is perhaps the area of most critical weakness in the theory of finance; leaving ample scope for the scholar who is intrigued by the relationships between human behavior and economic endeavor.

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[8]David B. Hertz, "Investment Policies That Pay Off," Harvard Business Review, Vol. 46, No. 1, (January-February, 1968), pp. 96-108.

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