DYNAMIC BUCKLING OF PLATES
UNDER IMPACT LOADING

by

SEE-KOK LOH
B.Sc.(Eng.) National Taiwan University
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See-Kok Loh

Department of Mechanical Engineering
The University of British Columbia
Vancouver 8, British Columbia

Date June 24, 1970
ABSTRACT

A theory is presented to examine the formation of wrinkles in plates when subjected to high rates of loading in the axial direction. The type of instability examined occurs in metals when the strains are well beyond the elastic range. For this reason the metals are assumed to be governed by the equations of a rigid plastic material. In particular, the von-Mises yield criterion is used in conjunction with the Levy-Mises flow rule.

A parameter is introduced which measures the lateral restraint of the plate. By giving this parameter different values, all plate widths can be examined. The theory predicts wavelengths of the buckled plates which are compared with some experimental results obtained in 1968 by Goodier.
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LIST OF SYMBOLS

\(a_n\) Initial displacement amplitude corresponding to \(n\)th mode.

\(A_n\) Amplification of initial displacement imperfections.

\(b\) Width of plate.

\(b_n\) Initial velocity amplitude corresponding to \(n\)th mode.

\(B_n\) Amplification of initial velocity imperfections.

\(E_h\) Strain-hardening modulus.

\(h\) Thickness of plate.

\(J_2\) Second invariant of deviatomic stress.

\(k\) A parameter introduced to measure the lateral restraint of the plate.

\(k_1\) Function of \(k\), \(k_1 = \sqrt{k^2 - k + 1}\)

\(k_2\) Function of \(k\), \(k_2 = 2 - k\).

\(k_3\) Function of \(k\), \(k_3 = 2k - 1\).

\(L\) Length of plate.

\(M\) Mass attached to the plate.

\(m\) Mass of the plate.

\(M_x, M_y\) Moments in the \(x\), \(y\) directions respectively.

\(N_x, N_y\) Membrane forces in the \(x\), \(y\) directions respectively.
n  Number of half-waves along the length of the plate.

p  Mean stress, \( p = -\left(\sigma_x + \sigma_y + \sigma_z\right)/3 \)

q  Sum of components of forces in the z-direction.

r  Positive integer used in summation sign.

t  Time in microseconds.

t_f  Duration of flow motion in microseconds.

V  Constant velocity of impact.

W  Weight of the block in the dynamic representation of stress-strain relation.

w  Deflection of plate.

x-y-z  Cartesian coordinates used to describe the plate.

\( \alpha \)  Function of \( k \), \( \alpha = k_2/k_1 \)

\( \beta \)  Function of \( k \), \( \beta = (4 - \alpha^2)/k_1 \)

\( \gamma \)  Function of \( k \), \( \gamma = k_3/k_1 \)

\( \eta \)  Function of \( k \), \( \eta = (2 + 2\gamma)/k_1 \)

\( \rho \)  Density of plate material

\( \zeta \)  Dimensionless time variable, \( \zeta = 1-t/t_f \)

\( \nu \)  Order of modified Bessel's functions, defined as \( \nu = (1+P_n)/2 \)

\( \sigma \)  Generalized stress.
\[ \sigma_0 \] Yield stress of the plate material in uniaxial tensile test.

\[ \sigma^0 \] Approximate value of \( \sigma \).

\[ \sigma_x \] Real stress component in x-direction.

\[ \sigma_y \] Real stress component in y-direction.

\[ \sigma_z \] Real stress component in z-direction.

\[ \sigma'_x \] Deviatoric stress component in x-direction.

\[ \sigma'_y \] Deviatoric stress component in y-direction.

\[ \sigma'_z \] Deviatoric stress component in z-direction.

\[ \dot{\varepsilon} \] Generalized strain rate.

\[ \dot{\varepsilon}_x \] Strain rate in x-direction.

\[ \dot{\varepsilon}_y \] Strain rate in y-direction.

\[ \dot{\varepsilon}_z \] Strain rate in z-direction.

\[ \dot{\varepsilon}_{xy} \] Shear strain rate on xy plane.

\[ \dot{\varepsilon}_{xz} \] Shear strain rate on xz plane.

\[ \dot{\varepsilon}_{yz} \] Shear strain rate on yz plane.

\[ \lambda \] Proportionality factor, \( \lambda = 3 \dot{\varepsilon}/2\sigma \)

\[ \lambda^n^+ \quad \lambda^n^- \] Roots of the equation \( \lambda^n^2 + p^n \lambda^n + q^n = 0 \)
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CHAPTER I

INTRODUCTION

1.1 Preliminary Remarks

The buckling of simple structural forms such as plates, shells and columns provides the foundation for the more general theory of stability of composite structures. When a structure is loaded beyond its load carrying capacity, either dynamically or statically, it very often happens that the failure of the structure occurs, not because of the high stress levels present but because of the low stability of the structure. Study of structural stability under static loads has developed rapidly in the last fifty years to meet the technological advancement in building ships, bridges, aircraft and machinery. The modern trend in industry further requires knowledge of dynamic stability in order to develop high speed machines and vehicles. Therefore, the problem of dynamic stability of structures becomes more and more important and involved in modern design.

The elastic stability of structures has been extensively discussed and many rather complete literatures in theory and practice [1] [2] are available. In contrast, the theory of dynamic plastic stability of structures is still in its early stage of development. This is mainly because of the later development of the theory of plasticity in relation to the theory of elasticity.
1.2 Purpose and Scope of Investigation

In this thesis, a theory is presented to examine the dynamic plastic buckling of plates under in-plane impact loading. This theory provides a method of obtaining the wavelength of the buckled form. In 1968, J. N. Goodier proposed a theory [3] to serve this purpose for the case of plane stress in the lateral direction. The experimental results recorded were in fairly good agreement with his theory. In our work, a parameter k is introduced to measure the lateral restraint of the plate. By giving this parameter different values which correspond to different sizes of plate, all plate widths can be examined. The parameter k depends on the plate dimensions. It will be shown that when k = 1/2, we recover the case that was considered in [3]. Thus, our work extends that developed by Goodier.

By varying the parameter k and other influential factors such as impact velocity, strain hardening modulus, etc., we examine many cases theoretically. The time dependence is described by a function f(t). We study two different functions of time t, namely f(t) = 1 and f(t) = 1-t/t_f which describe the flow motion. The former was used in [3] and [4] while the latter was considered in [5] for cylindrical shells. The results obtained from the present theory are compared with the experimental results recorded in [3].

1.3 Method of Investigation

The analysis is based on the coupling of classical plate theory and the theory of plasticity. Since emphasis is placed on obtaining an approximate theory for application, a number of assumptions and
approximations are made throughout the analysis. When the plate is projected towards a rigid target at a constant velocity, the resulting distortion is considered as the composition of a uniform shortening and thickening of the plate together with a small superimposed non-uniform bending or perturbed motion. In this analysis, the uniform motion is examined first and then the bending motion describing the buckling is treated as a perturbation.
CHAPTER II

THEORY

The analysis of the phenomenon of dynamic plastic buckling of plates under impulsive loading involves both the theory of plasticity and the well-developed theory of plates. Previous investigations have been made involving many assumptions and simplifications. Usually these assumptions are the outcome of careful experiments and the simplifications are for the sake of mathematical simplicity. However, in some cases these simplifications impose undesirable limitations on the applicability of the theory. On the other hand the assumptions do enable a complicated physical process to be described by a fairly simple theoretical model which gives satisfactory results.

In our analysis, a number of assumptions are made, some of which have been proposed by earlier workers and some of which are original and necessary to describe the particular problem being considered.

2.1 The General Theory

Consider a plate with a mass M attached at one end travelling at a constant velocity towards a rigid target (Figure 1). The strain rates introduced by the impact of the plate-mass system on the rigid target are denoted by $\dot{e}_x$, $\dot{e}_y$, and $\dot{e}_z$ where the subscripts $x$, $y$, and $z$ refer to the coordinate axes used to describe the event (Figure 2). The distortional strain rates $\dot{e}_{xy}$, $\dot{e}_{yz}$ and $\dot{e}_{xz}$ are small for small
Figure 1 Plate-mass system approaching the target at a constant velocity $V$
Figure 2 Co-ordinate system used for the plate
amplitude flexural motions and are taken as zero in the theory. The
plate material is assumed to be incompressible, and thus, the following
equation of incompressibility, relating the principal strain rates holds
\[ \dot{\varepsilon}_x + \dot{\varepsilon}_y + \dot{\varepsilon}_z = 0 \]  
(2.1.1)

The generalized strain rate is defined as
\[ \dot{\varepsilon}^2 = \frac{2}{3}(\dot{\varepsilon}_x^2 + \dot{\varepsilon}_y^2 + \dot{\varepsilon}_z^2) \]  
(2.1.2)

The von-Mises yield criterion is used and may be expressed in the form
\[ \sigma^2 = 3J_2 = \frac{3}{2}(\sigma_x'^2 + \sigma_y'^2 + \sigma_z'^2) \]  
(2.1.3)

where \( \sigma_x \), the stress at which the material first yields in a uniaxial
tension or compression test, has been replaced by \( \sigma \), the generalized
stress which defines the subsequent yield stress during loading. Also,
\( J_2 \) is the second invariant of deviatoric stress and \( \sigma_x', \sigma_y' \) and \( \sigma_z' \) are
the deviatoric stress components in the \( x, y, \) and \( z \) directions
respectively. The deviatoric stresses are related to the real stress
components \( \sigma_x, \sigma_y \) and \( \sigma_z \) by the following equations
\[ \sigma_x' = \sigma_x + p, \quad \sigma_y' = \sigma_y + p, \quad \sigma_z' = \sigma_z + p \]  
(2.1.4)

where \( p \) is the mean stress which is defined as \( p = -1/3(\sigma_x + \sigma_y + \sigma_z) \).
In thin plate theory (thickness of plate small compared to other
dimensions), the stress component normal to the middle plane is
egligible when compared to other stresses and may be taken as zero.
Putting \( \sigma_z = 0 \) into equation (2.1.4) and solving for the stresses, the
following expressions are obtained.

\[
\begin{align*}
\sigma_x &= 2 \sigma_x' + \sigma_y' \\
\sigma_y &= 2 \sigma_y' + \sigma_x' \\
\sigma_z &= 0
\end{align*}
\] (2.1.5)

Having introduced the incompressibility condition (2.1.1) and the yield criterion (2.1.3) we now introduce the flow law. Here we use the Levy-Mises flow law of incremental plasticity which may be expressed in the form

\[
\frac{d\varepsilon_x}{\sigma_x'} = \frac{d\varepsilon_y}{\sigma_y'} = \frac{d\varepsilon_z}{\sigma_z'} = d\lambda
\] (2.1.6)

where \(d\varepsilon_x, d\varepsilon_y, \) and \(d\varepsilon_z\) are the incremental strains and \(d\lambda\) is the proportionality factor defined by \(d\lambda = 3\varepsilon/2\sigma\). Upon dividing by \(dt\), equation (2.1.6) becomes

\[
\frac{\dot{\varepsilon}_x}{\sigma_x'} = \frac{\dot{\varepsilon}_y}{\sigma_y'} = \frac{\dot{\varepsilon}_z}{\sigma_z'} = \dot{\lambda}
\] (2.1.7)

where \(\dot{\lambda}\) is given by \(\dot{\lambda} = 3\dot{\varepsilon}/2\sigma\). Eliminating \(\sigma_x'\) and \(\sigma_y'\) from equations (2.1.5) by using (2.1.7), and using the above expression for \(\dot{\lambda}\) it follows that the stresses are related to the strain rates through the following relations:

\[
\begin{align*}
\sigma_x &= 2 (2\dot{\varepsilon}_x + \dot{\varepsilon}_y) \sigma/3 \dot{\varepsilon} \\
\sigma_y &= 2 (2\dot{\varepsilon}_y + \dot{\varepsilon}_x) \sigma/3 \dot{\varepsilon}
\end{align*}
\] (2.1.8)
The general procedure to be followed is now outlined. A relation must be obtained between the generalized stress $\sigma$ and the generalized strain $\varepsilon$. Concurrently the strain rate components must be found from kinematical considerations and the generalized strain rate $\dot{\varepsilon}$ found from (2.1.2). The stresses can then be calculated from (2.1.8) which in turn yield the membrane forces and bending moments by integration. The equations of motion of the plate then yield differential equations in the displacements. Buckling is then considered to be present if non-uniform solutions can be found. Since the buckling of the plate is considered as a perturbation on a uniform motion the analysis is composed of two parts. First the uniform compression of the plate is examined following the above procedure. Then a small displacement perturbation or velocity perturbation is superimposed on this known uniform flow. Buckling is then detected if the imperfections grow rapidly with increasing time.

We now proceed accordingly. Figure 3 shows a typical one-dimensional true stress-strain curve in an uniaxial tension or compression test for aluminum. The straight line is the corresponding stress-strain curve used in this analysis. In particular we ignore completely the elastic region of the curve and apply the approximate straight line relation to a rigid plastic theory. This is reasonable since we shall examine impacts sufficiently large to produce total strains much greater than the elastic strain. For many materials, such as aluminum and magnesium alloys, the straight line approximation of the strain hardening portion of the $\sigma$-$\varepsilon$ curve is satisfactory [6].
The main advantage of this approximation is in the resulting mathematical simplifications, simplifications which do not affect the validity of the analysis to any great extent. More specifically we shall show that buckling is fairly insensitive to the magnitude of the slope of the above \( \sigma-\varepsilon \) curve. This is not so for quasi-static buckling.

With the assumption of linear hardening, the generalized stress and strain increment are related by \( d\sigma = E_h d\varepsilon \). Integration of this equation leads to the following stress-strain relation

\[
\sigma = \sigma_o + E_h \varepsilon \quad (2.1.9)
\]

where \( \sigma_o \) is the yield stress in an uniaxial tension or compression test and \( E_h \) is the slope of the line which describes strain hardening. A spring-mass dynamic model representation of (2.1.9) is also shown in Figure 4 in which the stress is replaced by the force \( P \) and strain is replaced by the displacement of weight \( W \). We can see from Figure 4 that the block \( W \) can only be moved along the plane when the force \( P \) reaches a certain value, say \( P_0 \). Then starting from \( P_0 \) the displacement of \( W \) varies linearly with the applied force \( P \) because of the restraint of the spring. The first part corresponds to the vertical line portion up to elastic limit in the approximate \( \sigma-\varepsilon \) diagram (Figure 3) whereas the latter part corresponds to the straight line beyond the elastic limit shown in Figure 3. It is important to note that \( \sigma_o \) is also the initial value of the generalized stress which fixes the initial size of the von-Mises yield ellipse in the \( \sigma_x-\sigma_y \) plane as shown in Figure 5.
Figure 3  True stress versus logarithmic strain and the assumed stress-strain curve

Figure 4  The dynamic model representation of a rigid linear strain-hardening stress-strain relation
Figure 5 The initial and subsequent von-Mises yield ellipses
The subsequent yield loci (Figure 5) are determined by the subsequent values of $\sigma$, which in general, is a function of the history of loading and of position. It is assumed that, for isotropic hardening, the yield surface will expand with stress and strain history but will retain the original shape as it first yields. However this assumption does not take into account the Bauchinger effect which would tend to distort the shape of the yield surface as the yielding progresses [7].

It has been mentioned that $\sigma$ is a function of time and position. Its variation throughout the motion is found through its dependence on $\varepsilon$.

The membrane forces can be found by evaluating the integrals of the stresses over the plate thickness. In particular we have

$$N_x = \int_{-h/2}^{h/2} \sigma_x \, d z$$

$$N_y = \int_{-h/2}^{h/2} \sigma_y \, d z \tag{2.1.10}$$

Similarly, the bending moments are

$$M_x = -\int_{-h/2}^{h/2} \sigma_x \, z \, d z$$

$$M_y = -\int_{-h/2}^{h/2} \sigma_y \, z \, d z \tag{2.1.11}$$

The general equation of motion of a plate under the in-plane forces $N_x$ and $N_y$ given by (2.1.10) and bending moments $M_x$ and $M_y$ given by (2.1.11) is given in [8], as follows

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} - 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} = q \tag{2.1.12}$$
where \( q \) is the sum of the \( z \)-components of the membrane forces (Figures 6a and 6b) and the inertia force due to deflection of the plate (Figure 7) and is given by

\[
q = \rho h \ddot{w} - \left( N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2 N_{xy} \frac{\partial^2 w}{\partial x \partial y} \right) \quad (2.1.13)
\]

Since the distortional strain rates are taken as zero, it follows from the Levy-Mises flow law that the shear stresses are also zero. Combining equations (2.1.12) and (2.1.13) and putting \( M_{xy} = N_{xy} = 0 \), we get

\[
\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} = \rho h \ddot{w} - N_x \frac{\partial^2 w}{\partial x^2} - N_y \frac{\partial^2 w}{\partial y^2} \quad (2.1.14)
\]

This is the fundamental equation of motion which must be examined in terms of displacements for the detection of flexural motions. We eliminate \( N_x, N_y \) and \( M_x, M_y \) immediately from (2.1.14) by using (2.1.10) and (2.1.11) respectively.

### 2.2 Unperturbed Motion

Figure 2 shows a plate with a mass \( M \) attached at one end approaching a rigid target at a constant velocity \( V \). The combination of \( V \) and \( M \) is such that plastic flow will occur. As mentioned previously, the ensuing motion is considered as a combination of an unperturbed and perturbed motion. In this section we examine the regular unperturbed motion.

After striking the rigid target, the strain rate \( \dot{\epsilon}_x \) in the plate is

\[
\dot{\epsilon}_x = -\frac{V}{L} f(t) \quad (2.2.1)
\]
Figure 6 The force equilibrium of an element cut from the plate
Figure 7 The moment equilibrium of an element cut from the plate
The minus sign introduced in (2.2.1) indicates that the strain rate is compressive. The function \( f(t) \) describes the flow from the instant of impact until the completion of the motion. Thus, \( t \) is defined in the region \( 0 \leq t \leq t_f \) where \( t_f \) is the duration of the motion and \( f(0) = 1 \). Due to the compression in the \( x \)-direction we also expect flow to occur in the \( y \) and \( z \) directions. We assume that, at the midsurface \( \dot{e}_y \) can be represented by

\[
\dot{e}_y = -k \dot{e}_x = \frac{kV}{L} f(t) \tag{2.2.2}
\]

where \( k \) is a constant which varies between 0 and 0.5. When \( k = 0 \), we have \( \dot{e}_y = 0 \) for all time corresponding to a state of plane strain. This state of plane strain may be due to constraints acting at the edges \( y = \pm b/2 \) or it may correspond to a very wide plate. When \( k = 1/2 \), we see from equation (2.1.8) that \( \sigma_y = 0 \). Hence \( k = 1/2 \) corresponds to the case of plane stress, or a very narrow plate. We expect plates of all widths to have a corresponding value of \( k \) between 0 and 0.5. The value of \( k \) for a particular plate may be determined experimentally from surface measurements. Since we are considering a uniform compression, equation (2.2.2) can be extended to give good approximation over the thickness. Equation (2.1.1) now yields

\[
\dot{e}_z = \frac{V f(t)}{L} (1 - k) \tag{2.2.3}
\]

The generalized strain given by (2.1.2) becomes

\[
\dot{\varepsilon} = \frac{2Vf(t)}{\sqrt[3]{3}L} \sqrt{k^2 - k + 1} \tag{2.2.4}
\]
Denoting \( \int_0^t f(t) \, dt \) by \( F(t) \) and noting that \( \varepsilon = 0 \) at \( t = 0 \), equation (2.2.4) can be integrated with respect to \( t \) to give

\[
\varepsilon = \frac{2 \sqrt{\frac{V F(t)}{L}}}{\sqrt{3 K}} \sqrt{K^2 - K + 1} \tag{2.2.5}
\]

With this result, the generalized stress can be found from (2.1.9) as follows

\[
\sigma = \sigma_0 + \frac{2 E_h V F(t)}{\sqrt{3 L}} \sqrt{K^2 - K + 1} = \sigma^0 \tag{2.2.6}
\]

where \( \sigma^0 \) is current value of the generalized stress and determines the size of the current yield surface. The true stresses \( \sigma_x \) and \( \sigma_y \), the membrane forces \( N_x \) and \( N_y \), the bending moments \( M_x \) and \( M_y \) are calculated from (2.1.8), (2.1.10) and (2.1.11) respectively and are found to be

\[
\sigma_x = -K_2 \sigma^0 \sqrt{3 K}, \quad \sigma_y = K_3 \sigma^0 \sqrt{3 K}, \tag{2.2.7}
\]

\[
N_x = -K_2 h \sigma^0 \sqrt{3 K}, \quad N_y = K_3 h \sigma^0 \sqrt{3 K}, \tag{2.2.8}
\]

\[
M_x = M_y = 0 \tag{2.2.9}
\]

The constants \( k_1, k_2 \) and \( k_3 \) used in the above equations are defined by

\[
k_1 = \sqrt{K^2 - K + 1}, \quad k_2 = 2 - K, \quad k_3 = 2K - 1 \tag{2.2.10}
\]

As we expect for uniform compression, the bending moments \( M_x \) and \( M_y \) are both zero.
2.3 Perturbed Motion

As mentioned earlier experimental results reported in [3] showed that no curvature develops in the y-direction for a rectangular strip under impact in the x-direction. This result indicates that the perturbational deflection $w$ is independent of $y$ and is only a function of $x$ and $t$. Due to this cylindrical bending the strain rate $\dot{\epsilon}_x$ is given by

$$\dot{\epsilon}_x = -\frac{V}{L}f(t) + zw''$$  \hspace{1cm} (2.3.1)

The additional strain rate $zw''$ in (2.3.1) is the perturbational curvature rate introduced by the bending displacement $w(x,t)$ and is small compared with the unperturbed strain rate. Thus we may be sure that no strain reversal occurs whilst $w(x,t)$ is small. It is possible that the combination of $V$ and $M$ is such that very large amplitude buckling occurs. The theory cannot describe such a situation since then unloading in the plastic sense must have taken place. However, it is possible that the theory will still predict correct wavelengths since wavelength is determined early in the motion when amplitudes are still small, that is, when $w(x,t)$ is small. The prime in (2.3.1) denotes partial differentiation with respect to $x$. For cylindrical bending, $\dot{\epsilon}_y$ is unaffected by the perturbation and hence $\dot{\epsilon}_y$ is again given by

$$\dot{\epsilon}_y = \frac{kV}{L}f(t)$$  \hspace{1cm} (2.3.2)

The strain rate in the $z$ direction can be found from the incompressibility
requirement (2.1.1), that is \( \dot{\varepsilon}_z = -(\dot{\varepsilon}_x + \dot{\varepsilon}_y) \). Substituting these new values of \( \dot{\varepsilon}_x, \dot{\varepsilon}_y \) and \( \dot{\varepsilon}_z \) into (2.1.2) and using (2.2.10) the generalized strain rate is found to be

\[
\dot{\varepsilon} = \frac{2}{3} \left[ \frac{V^2 f(t)}{L^2} K_i^2 - K_2 \frac{V f(t)}{L} Z \ddot{w}'' + \frac{2}{3} \dddot{w}''^2 \right]^{\frac{1}{2}} (2.3.3)
\]

Taking the first term of (2.3.3) out of the root sign and assuming

\[
\left| \frac{L Z \ddot{w}''}{V f(t) K_i^2} \right| < 1 \quad (2.3.4)
\]

the last term of (2.3.3) becomes negligibly small when compared with unity. Applying the Binomial Expansion to equation (2.3.3) and neglecting higher order terms, the following simple form of (2.3.3) is obtained:

\[
\dot{\varepsilon} = \frac{2}{3} \frac{V f(t)}{L} K_i - K_2 \frac{1}{\sqrt{3} K_i} Z \ddot{w}'' \quad (2.3.5)
\]

It is apparent that if \( f(t) \) is such that the integration of (2.3.5) is possible, \( \varepsilon \) can be found without difficulty. Recalling from the last section that the indefinite integral of \( f(t) \) is denoted by \( F(t) \), we see that integration of (2.3.5) leads to

\[
\varepsilon = \frac{2}{3} \frac{V F(t)}{L} K_i - \frac{K_2}{\sqrt{3} K_i} Z (\dot{w} - \ddot{w})'' \quad (2.3.6)
\]

where \( \ddot{w} = w(x,0) \) is the initial deflection of the plate before impact. The generalized stress is thus given by (2.1.9) as

\[
\sigma = \sigma^o - \frac{E_b K_2}{\sqrt{3} K_i} Z (\dot{w} - \ddot{w})'' \quad (2.3.7)
\]
where \( \sigma^0 \) is interpreted as in (2.2.6). Substitution of (2.3.1) and (2.3.2) into (2.1.8) leads to

\[
\sigma_x = \frac{2 \sigma}{3 \varepsilon} \left[ 2z \dot{w}'' - \frac{Vf(t)K_2}{L} \right]
\]
\[
\sigma_y = \frac{2 \sigma}{3 \varepsilon} \left[ z \dot{w}'' + \frac{Vf(t)K_2}{L} \right]
\]

(2.3.8)

To evaluate \( \sigma/\varepsilon \) it is necessary to approximate \( (\varepsilon)^{-1} \) by using the Binomial expansion and neglecting products of \( \dot{w}'' \) and \( (w - \bar{w})'' \). In this way we find that

\[
\frac{\sigma}{\varepsilon} = \frac{E_h}{\delta} \left[ \sigma^0 + \frac{\sigma^0 E_h K_2}{3 K_1 \delta} z \dot{w}'' - \frac{E_h K_2}{3 K_1} z (w - \bar{w})'' \right]
\]

where \( \delta = \frac{2E_h K_1 Vf(t)}{\sqrt{3} L} \). Substituting this result into (2.3.8), the true stress components are found to be

\[
\sigma_x = \left[ \frac{L \beta \sigma^0}{2 \sqrt{3} Vf(t)} \dot{w}'' + \frac{E_h \alpha^2 (w - \bar{w})''}{3} \right] z - \frac{\sigma^0 \alpha}{\sqrt{3}}
\]
\[
\sigma_y = \left[ \frac{L \gamma \sigma^0}{2 \sqrt{3} Vf(t)} \dot{w}'' - \frac{E_h \alpha \gamma (w - \bar{w})''}{3} \right] z - \frac{\sigma^0 \gamma}{\sqrt{3}}
\]

(2.3.9)

where \( \alpha, \beta, \gamma \) and \( \eta \) are defined as

\[
\alpha = \frac{K_2}{K_1}, \quad \beta = (4 - \alpha^2)/K_1,
\]
\[
\gamma = \frac{K_3}{K_1}, \quad \eta = (2 + 2 \gamma)/K_1
\]

The membrane forces and bending moments are found by using equations (2.1.10) and (2.1.11). They are

\[
N_x = -\frac{\sigma^0 h \alpha}{\sqrt{3}}
\]
\[
N_y = \frac{\sigma^0 h \gamma}{\sqrt{3}}
\]

(2.3.10)
Equations (2.3.11) may be partitioned into two groups of terms. The first group in each equation depends linearly on \( E^h \)--the strain hardening modulus. These contributions to the bending moments are termed the "strain-hardening moments". When \( E^h = 0 \) the bending moments do not vanish since the second group of terms are independent of \( E^h \). These parts of the bending moments have been termed the "directional moments" by Goodier in his work on plastic plates [3]. They arise through the fact that the yield surface is locally curved and because different points through the plate thickness correspond to different local points on this yield surface (Figure 5). The normality condition governing the strain rate increment vector then ensures that the strain rate is not uniform through the plate thickness thereby causing the "directional" bending moments. It also follows that if we use the Tresca yield criterion which is piecewise linear then no directional moment arises.

For cylindrical bending \( w \) is independent of \( y \). Hence \( \frac{\partial^2 M_y}{\partial y^2} \) and \( \frac{\partial^2 w}{\partial y^2} \) are both zero. The equation of motion (2.1.14) reduces to the form

\[
M_x'' + N_x w'' = \rho h \ddot{w}
\]  

Differentiating twice the first of (2.3.11) and substituting this result together with the value of the first of (2.3.10) into (2.3.12),
we get
\[
\frac{\sqrt{3} L \beta h^2 \sigma_0}{72 V f(t)} \dddot{W} + \frac{E_h a^2 h^2}{36} (W - \bar{W})'''' + \frac{\sigma_0 a}{\sqrt{3}} W'' + \rho \dddot{W} = 0 \tag{2.3.13}
\]

For \( k = 0.5 \) and \( f(t) = 1 \), equation (2.3.13) reduces to
\[
\frac{L h^2 \sigma_0}{36 \rho V} \dddot{W}'''' + \frac{E_h h^2}{12 \rho} (W - \bar{W})'''' + \frac{\sigma_0 W''}{\rho} + \dddot{W} = 0 \tag{2.3.14}
\]

Equation (2.3.14) is exactly the same as the one obtained in [3] and [4].

2.4 Solution of The Equation of Motion

The differential equation governing the flexural motion of the plate is given by (2.3.13). We seek a solution of the form
\[
W(x, t) = \sum_{n=1}^{\infty} W_n(t) \sin \frac{n \pi x}{L} \tag{2.4.1}
\]

The initial deflections and velocities are taken to be
\[
\begin{align*}
W(x, 0) &= \bar{W} = \sum_{n=1}^{\infty} A_n \sin \frac{n \pi x}{L} \\
\dot{W}(x, 0) &= \sum_{n=1}^{\infty} B_n \sin \frac{n \pi x}{L}
\end{align*} \tag{2.4.2}
\]

where \( a_n \) and \( b_n \) are constants. The substitution of (2.4.1) and the first of (2.4.2) into (2.3.13) leads to the following differential equation for each \( n \),
\[
\dddot{W}_n + \frac{P_n}{f(t)} \dot{W}_n + Q_n^2 W_n = R_n a_n \tag{2.4.3}
\]

where \( P_n, Q_n^2 \) and \( R_n \) are given by
The solution of (2.4.3) leads to a complete solution of (2.3.13). It can be expected that the solution of equation (2.4.3) depends on the choice of the function \( f(t) \). The choice of \( f(t) \) should be such that it can adequately describe the real buckling motion. Some idealized assumptions are employed here to determine \( f(t) \). First the plastic flow motion is considered as a uniform motion in which the plate deforms uniformly with constant velocity from the instant of impact and this motion is abruptly curtailed when \( t = t_f \), where \( t_f \) is the duration of this motion and is to be determined separately. Secondly, we consider a constant deceleration of the free end of the plate so that the velocity decreases linearly with time from \( V = V_0 \) at \( t = 0 \) to \( V = 0 \) at \( t = t_f \). The former case was considered in [3] and [4] whereas the latter case was considered in [5]. In the following analysis, both of the cases will be considered separately.

### 2.4.1 Solution For The Case \( f(t) = 1 \)

As described above this case corresponds to the plate being compressed at a constant velocity. For this case equation (2.4.3) becomes

\[
\ddot{w}_n + P_n \dot{w}_n + Q_n^2 w_n = R_n a_n \tag{2.4.1.1}
\]
and the initial conditions become

\[ W_n(0) = a_n \quad \text{and} \quad \dot{W}_n(0) = b_n \]

The solution of (2.4.1.1) is

\[ W_n(t) = A_n(t) a_n + B_n(t) b_n \quad (2.4.1.2) \]

where

\[
A_n(t) = \frac{(1 - R_n/Q_n^2)(\lambda_n^+ e^{\lambda_n^+ t} - \lambda_n^- e^{\lambda_n^- t}) + R_n/Q_n^2}{\lambda_n^+ - \lambda_n^-}
\]

\[
B_n(t) = \frac{e^{\lambda_n^+ t} - e^{\lambda_n^- t}}{\lambda_n^+ - \lambda_n^-} \quad (2.4.1.3)
\]

where \( \lambda_n^+, \lambda_n^- \) are the roots of the equation \( \lambda_n^2 + P_n \lambda_n + Q_n^2 = 0 \), that is,

\[
\lambda_n^+ = -\frac{1}{2} \left( P_n - \sqrt{P_n^2 - 4 Q_n^2} \right)
\]

\[
\lambda_n^- = -\frac{1}{2} \left( P_n + \sqrt{P_n^2 - 4 Q_n^2} \right) \quad (2.4.1.4)
\]

Therefore, the complete solution of equation (2.3.13) for \( f(t) = 1 \) is given by

\[
W(x, t) = \sum_{n=1}^{\infty} \left( A_n(t) a_n + B_n(t) b_n \right) \sin \frac{n \pi x}{L} \quad (2.4.1.5)
\]

Equation (2.4.1.5) gives the perturbational deflection of the plate in terms of the initial imperfections. It also shows that \( A_n(t) \) and \( B_n(t) \) are the amplifications of the initial deflection amplitudes and velocity amplitudes respectively. We see that the initial displacement imperfections are expressed by using Fourier sine series with coefficients \( a_n \). The values of \( a_n \), in general, are distinct, and could be determined
in the usual way if the initial forms of displacements were known. The
displacement at any later time could then be found by the appropriate
Fourier type summation

\[ W(x, t) = \sum_{n=1}^{\infty} A_n(t) A_n \sin \frac{n\pi x}{L} \]

Therefore, we could examine plate buckling when known initial displace-
ments are given their appropriate Fourier series representation. This
result is equally true for initial velocity imperfections \( b_n \). In our
case, we let \( a_n = 1 \) so that \( A_n \) describes the motion of the plate
deflection and the mode number determined by \( A_n \) is taken as the number
of half-waves into which the plate will buckle. Similarly, we let
\( b_n = 1 \) for the initial velocity imperfections and the magnification
factor \( B_n \) determines the magnitude of deflection and the number of
half-waves.

2.4.2 Solution For The Case \( f(t) = 1-t/t_f \)

We now consider the case in which \( f(t) = 1-t/t_f \). The buckling
motion so described implies that the rate of deformation of the plate
decreases as time increases and this rate is inversely proportional
to the duration of the motion. Now equation (2.4.3) leads to

\[ \ddot{w}_n + \frac{P_n}{1-t/t_f} \dot{w}_n + Q_n^2 w_n = R_n a_n \]

(2.4.2.1)

Introducing the new variable \( \zeta \) where

\[ \zeta = 1 - t/t_f \]

(2.4.2.2)
equation (2.4.2.1) becomes

\[ \dddot{W}_n - \frac{P_n'}{\xi} \dot{W}_n + Q_n^2 W_n = R_n' a_n \]  (2.4.2.3)

with the initial conditions

\[ W_n(1) = a_n \]  (2.4.2.4)
\[ \dot{W}_n(1) = -t_f b_n \]

and \( P_n', Q_n^2 \) and \( R_n' \) are defined as follows

\[ P_n' = t_f P_n, \quad Q_n^2 = t_f^2 Q_n^2, \quad R_n' = t_f^2 R_n \]  (2.4.2.5)

Note that \((\cdot)\) used in (2.4.2.3) denotes differentiation with respect to \(\xi\). This notation will be used onward from (2.4.2.3). The solution of (2.4.2.3) depends on the sign of \(Q_n^2\). However, the solution takes the form below regardless of the sign of \(Q_n^2\)

\[ W_n(\xi) = A_n(\xi) a_n + B_n(\xi) b_n \]  (2.4.2.6)

For \(Q_n^2 > 0\), \(A_n\) and \(B_n\) are given by

\[ A_n(\xi) = \frac{\xi \nu Q_n' (1 - R_n'/Q_n^2)}{2} \left[ Y_{\nu-1}(Q_n') J_{\nu}(Q_n' \xi) - J_{\nu-1}(Q_n') Y_{\nu}(Q_n' \xi) \right] + R_n'/Q_n^2 \]  (2.4.2.7)
\[ B_n(\xi) = \frac{\xi \nu t_f}{2} \left[ Y_{\nu}(Q_n') J_{\nu}(Q_n' \xi) - J_{\nu}(Q_n') Y_{\nu}(Q_n' \xi) \right] \]

where \(J_{\nu}(Q_n' \xi)\) and \(Y_{\nu}(Q_n' \xi)\) are the Bessel functions of the first and second kind of order \(\nu\).
For $Q_n^2 < 0$, $A_n$ and $B_n$ are alternatively given by

$$A_n(\xi) = \mathcal{I}_\nu(Q_n^\prime(1-R_n/Q_n^2)) \left[ K_{\nu'}(Q_n^\prime) I_{\nu}(Q_n^\prime \xi) + I_{\nu'}(Q_n^\prime) K_{\nu}(Q_n^\prime \xi) \right] + R_n/Q_n^2$$

$$B_n(\xi) = \mathcal{I}_\nu t_n \left[ I_{\nu}(Q_n^\prime) K_{\nu}(Q_n^\prime \xi) - K_{\nu}(Q_n^\prime) I_{\nu}(Q_n^\prime \xi) \right]$$

(2.4.2.8)

where $I_{\nu}(Q_n^\prime \xi)$ and $K_{\nu}(Q_n^\prime \xi)$ are the modified Bessel functions of the first and second kind of order $\nu$ and the value of $\nu$ is given by

$$\nu = \frac{1}{2} \left( 1 + \mathcal{P}_n^\prime \right)$$

(2.4.2.9)

The complete solution of (2.3.1.13) for $f(t) = 1-t/t_f$ is

$$W(x, \xi) = \sum_{n=1}^{\infty} (A_n(\xi) Q_n + B_n(\xi) b_n) \sin \frac{n\pi x}{L}$$

(2.4.2.10)

where $A_n$ and $B_n$ are given by either (2.4.2.7) or (2.4.2.8) depending on whether $Q_n^2$ in (2.4.2.3) is positive or negative. We also let $a_n = b_n = 1$ as for the case $f(t) = 1$.

To detect the existence of buckling, it is only necessary to examine the amplitudes $A_n(\xi)$ and $B_n(\xi)$ at the end of the flow motion.

As $t \to t_f$, that is as $\xi \to 0$, the functions $J_{\nu}$ and $Y_{\nu}$ approach zero, hence for $Q_n^2 > 0$, there is no buckling. Consequently only the case for which $Q_n^2 < 0$ is considered and the corresponding asymptotic form is now obtained.

As $\xi \to 0$ $I_{\nu}(Q_n^\prime \xi) \to 0$ and $\xi^\nu K_{\nu}(Q_n^\prime \xi) \to 2^{\nu-1} \Gamma(\nu)/Q_n^\nu \nu$ where $\Gamma(\nu)$ is the Gamma function. Expressing $I_{\nu}(Q_n^\prime)$ and $I_{\nu-1}(Q_n^\prime)$ in power series, we obtain
Substituting (2.4.2.11) into (2.4.2.8) and noting that \( r(v) = (v-1)! \), the following asymptotic expansion of \( A_n(o) \) and \( B_n(o) \) are obtained:

\[
A_n(o) \sim (1 - S_n) \left[ 1 + \frac{\lambda}{2} \left( \frac{\theta_n'}{2} \right)^2 + \frac{1}{2!} \frac{1}{\nu(\nu+1)} \left( \frac{\theta_n'}{2} \right)^4 + \cdots \right] + S_n
\]

\[
B_n(o) \sim \frac{t_f}{1 + \frac{P}{p_n'}} \left[ 1 + \frac{\lambda}{\nu+1} \left( \frac{\theta_n'}{2} \right)^2 + \frac{1}{2!\nu(\nu+1)(\nu+2)} \left( \frac{\theta_n'}{2} \right)^4 + \cdots \right]
\]

when \( S_n \) is defined as

\[
S_n = \frac{R_n'}{Q_n'^2} \quad (2.4.2.13)
\]

2.5 Determination of \( t_f \)

For the case shown in Figure 1 it is reasonable to describe the time dependence by the function \( f(t) = 1 - t/t_f \). This implies that the force acting on the mass \( M \) is constant during deformation. The actual force is of course

\[
\int \sigma_x d\text{Area}
\]

Area of plate

Since \( \sigma_x \) does not change significantly (\( E_n \) not steep in Figure 3) it follows that the force acting on \( M \) is indeed sensibly constant. A simple conservation of energy calculation then gives

\[
t_f = \frac{\sqrt{3}}{2} \frac{p VL}{K \sigma^0} \left( 1 + \frac{M/m}{\nu} \right)
\]

(2.5.1)
CHAPTER III

RESULTS AND DISCUSSION

The solution of (2.3.13) for both cases of \( f(t) \) takes the general form given by (2.4.15). However, the amplification factors \( A_n \) and \( B_n \) are different for the different functions \( f(t) \). Equations (2.4.1.3) give the value of these factors for \( f(t)=1 \) and (2.4.2.12) yield the corresponding factors for \( f(t)=1-t/t_f \). If buckling of the plate under the given impact loading condition does exist, the graphical plots of \( A_n \) versus \( n \) and \( B_n \) versus \( n \) at any \( t \) for \( 0<t<t_f \) will exhibit sharp peaks. These peaks indicate the wavelength into which the plate will deform.

Figures 8 to 11 are plots for \( A_n(t) \) versus \( n \) and Figures 12 to 15 are for \( B_n(t) \) versus \( n \) at different values of the parameter \( k \) for \( f(t)=1 \). The values of \( k \), the durations of buckling motion \( t_f \) and the axial strains \( \varepsilon_x \) are given on each graph. The static values used in the calculation to obtain the results for Figures 8 to 15 are listed below:

Yield stress \( \sigma_0 = 45,000 \) psi. Impact velocity \( V=1,500 \) in/sec.
Strain hardening modulus \( E_h=150,000 \) psi.
Plate length \( L=3 \) inches.
Density of the material \( \rho = 0.00025 \dfrac{1\text{b}-\text{sec}^2}{\text{in}^4} \)
Plate thickness \( h = 1/8 \) inch.
Figure 8 $A_n$ versus $n$ for $k = 0.20$

$K = 0.20$

$t_f = 66.92 \mu\text{sec.}$

Strain = 3.34%
$K = 0.30$

$t_f = 69.00 \, \mu\text{sec.}$

Strain $= 3.45\%$

Figure 9 $A_n$ versus $n$ for $k = 0.30$
K = 0.40

$ t_f = 70.35 \mu \text{sec.} $

Strain = 3.52%

Figure 10 $ A_n $ versus $ n $ for $ k = 0.40 $
Figure 11 $A_n$ versus $n$ for $k = 0.50$

$K = 0.50$

$t_f = 70.82 \mu\text{sec.}$

Strain $= 3.54\%$
Figure 12  $B_n$ versus $n$ for $k = 0.20$

$K = 0.20$
$\tau_f = 66.92 \mu\text{sec.}$
Strain = 3.34%
Figure 13 $B_n$ versus $n$ for $k = 0.30$

$$K = 0.30$$

$$t_f = 69.00 \mu \text{sec.}$$

Strain = $3.45\%$
Figure 14 $B_n$ versus $n$ for $k = 0.40$

\begin{align*}
    K &= 0.40 \\
    t_f &= 70.35 \mu\text{sec.} \\
    \text{Strain} &= 3.52\% 
\end{align*}
$K = 0.50$

$t_f = 70.82 \mu sec.$

Strain = 3.54%

Figure 15 $B_n$ versus $n$ for $k = 0.50$
3.1 General Results Obtained for \( f(t) = 1 \)

Figures 8 to 11 show \( A_n(t) \) versus \( n \) for various \( k \) ranging from 0.20 to 0.50 as indicated. The family of curves drawn in the first figure is representative of all the other curves. The only difference being in the position of the peaks (the degree of amplification and the associated values of the mode number). Therefore, any interpretation of one such family of curves will be equally applied to others without loss of generality. For example in Figure 8 the five curves show the continuous change of the buckling amplitude and the mode number from \( t/t_f = 0.20 \) to \( t/t_f = 1.0 \). However, the mode number corresponding to the peak at various times does not change by more than one in the above mentioned time interval. That is, the predominant buckling mode, as found theoretically, is selected early in the motion. This has also been observed experimentally by examining film sequences taken of some of the experiments conducted in the laboratory using the impact machine shown in Figure 21.

Comparing Figures 8, 9, 10 and 11, we observe the effect of the changes of \( k \) on the number of half waves and the magnitudes of amplifications. The increase in the value of \( k \) controls the decreases of mode number as well as the magnitude of amplification. The variation of strain is small. If we arrange the mass \( M \) such that the strains in these figures are the same, the effect of \( k \) on the mode number and magnitudes of amplifications would be similar to those shown in Figures 8 to 11.
Figures 12 to 15 show $B_n(t)$ versus $n$. The explanation and results given above are equally applicable to these curves.

Figures 8 and 12 show the displacement and velocity magnifications for $k = 0.20$. It is evident that the number of half-waves predicted by displacement perturbation is approximately one higher than that given by velocity perturbation. The comparison of any other group for a particular value of $k$ shows the similar result.

Figure 16 shows the effect of the impact velocity on the mode number for 2 values of $k$ namely $k = 0.25$ and $k = 0.5$. It is clear that the slope (the change of $n$ with respect to velocity) of the $k = 0.25$ curve being larger, implies that the effect of impact velocity on mode number is more significant when $k$ is small, that is, when the plate in question has a larger $b/L$ ratio where $b$ is the width of the plate. The mode numbers shown in this figure are obtained from the plots of $A_n(t)$ versus $n$ at the various velocities and they are the values (not nearest integers) corresponding to the final stage of the flow motion ($t = t_f$). Any other curve for $k$ between 0.25 and 0.5 will be between the two curves shown in Figure 16 with slope decreases as $k$ increases. It is also clear that the mode number increases with increasing impact velocity. However, when impact velocity is very high, fracture of the material may occur and also wave propagation becomes important. Moreover, Figure 16 is also in agreement with the previous result, that the mode number decreases as $k$ increases, obtained in considering the amplification curves. It is expected that a set of results similar to those shown in Figure 16 would be obtained if velocity perturbation mode numbers are taken.
Figure 16 The effect of impact velocity on buckling mode number
Figure 17 shows the effect of $E_h$ -- the strain hardening modulus -- on the displacement perturbation mode number for various values of $k$. The impact velocity, the thickness and the length of the plate are all kept constant at 1500 in/sec., 1/8 and 3 inches respectively. From this figure, we see that the number of half-waves is quite insensitive to the change of the strain-hardening modulus, especially for large value of $k$. Even in the case for $k = 0.20$, the reduction in $n$ is only 2 when $E_h$ changes from 0 psi to 300,000 psi. The result leads to the conclusion that the strain-hardening effect (bending moment containing $E_h$ term) on mode number is secondary when compared to the directional effect. A similar result was concluded in [3] and [4]. Again, the previous conclusion that the mode number decreases as $k$ increase for constant $E_h$ with all other parameters fixed is evident from Figure 17. It should be noted that the insensitivity of mode number to strain-hardening modulus only holds for reasonably small values of $E_h/\sigma$, say $E_h/\sigma<5$.

Figure 18 shows the effect of plate thickness on the mode number. As the plate thickness $h$ increases, the mode number decreases for all values of $k$. The curves shown in Figure 18 are for $k = 0.1$, 0.25 and 0.50 with velocity of impact, plate length, etc., being held fixed. For the curve shown for $k = 0.1$, mode number decreases rapidly from 11 for $h = 0.1$ inch to 4.6 for $h = 0.25$ inch. The other two curves indicate a similar result.
Figure 17 The effect of strain-hardening modulus on buckling mode number
Figure 18 The effect of plate thickness on buckling mode number
3.2 General Results Obtained for $f(t) = 1 - \frac{t}{t_f}$

In general, results similar to those obtained for $f(t) = 1$ are obtained for $f(t) = 1 - \frac{t}{t_f}$. Figures 19 and 20 show the results of $A_n(0)$ and $B_n(0)$ obtained by using (2.4.2.12). Again, the curves indicate sharp peaks for some mode numbers indicating the physical wavelength. The general results presented in the last section are generally applicable to this case.

The effect of the mass ratio on the mode number has not yet been shown. In general, the mode number decreases as the mass ratio increases, other parameters being kept constant. Moreover, higher mass ratio contributes to larger magnifications.

3.3 Comparison of Theoretical and Experimental Results

Table 1 shows the data corresponding to 6 plates reported in [3]. In Table 2, the theoretical results for $f(t) = 1$ and $f(t) = 1 - \frac{t}{t_f}$ are given in the last two columns. It should be noted that the axial strains are the same for both cases. The comparison of the last two columns shows that the average half-wavelengths obtained for $f(t) = 1 - \frac{t}{t_f}$ are somewhat larger than that predicted for $f(t) = 1$. A comparison of theoretical and experimental results is also shown in Table 2. A general good agreement is obtained. It is expected that the predicted half-wavelengths using the present theory ($f(t) = 1$) are less than those predicted in [3] because smaller values of $k$ are used in the present theory. Bearing this in mind, we see that in most cases in Table 3 which is reproduced from [3], the present theory would predict more satisfactory results.
Figure 19 $A_n(0)$ versus $n$ for plate LAC-1

$k = 0.395$
$t_f = 312 \mu\text{sec.}$
Strain $= 7\%$
Figure 20  $B_n(0)$ versus $n$ for plate LAC-1

$k = 0.395$

$t_f = 312 \mu\text{sec}$

Strain = 7%
TABLE 1
DATA OF SPECIMENSRecordED FROM [3]

<table>
<thead>
<tr>
<th>Specimen Designation</th>
<th>Impact Velocity ft/sec.</th>
<th>Plate Size in x in x in</th>
<th>$\sigma_0$ $10^3$psi</th>
<th>$E_h$ $10^3$psi</th>
<th>$1 + M/m$</th>
<th>$k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>SAC-1</td>
<td>400</td>
<td>5x1/2x1/16</td>
<td>30</td>
<td>48.3</td>
<td>5.25</td>
<td>0.427</td>
</tr>
<tr>
<td>SAC-2</td>
<td>300</td>
<td>5x1/2x1/16</td>
<td>30</td>
<td>48.3</td>
<td>5.25</td>
<td>0.427</td>
</tr>
<tr>
<td>SAC-3</td>
<td>199</td>
<td>5x1/2x1/16</td>
<td>30</td>
<td>48.3</td>
<td>5.25</td>
<td>0.427</td>
</tr>
<tr>
<td>LAC-1</td>
<td>184</td>
<td>5x3/4x1/16</td>
<td>28.7</td>
<td>58.6</td>
<td>3.50</td>
<td>0.395</td>
</tr>
<tr>
<td>LAC-2</td>
<td>310</td>
<td>5x3/4x1/16</td>
<td>30</td>
<td>48.3</td>
<td>3.50</td>
<td>0.395</td>
</tr>
<tr>
<td>LAC-3</td>
<td>344</td>
<td>5x3/4x1/16</td>
<td>30</td>
<td>48.3</td>
<td>6.00</td>
<td>0.395</td>
</tr>
</tbody>
</table>
### TABLE 2
**COMPARISON OF THEORETICAL AND EXPERIMENTAL RESULTS**

<table>
<thead>
<tr>
<th>Specimen Designation</th>
<th>Impact Velocity ft/sec</th>
<th>Axial Strain %</th>
<th>Experimental half-wavelength in</th>
<th>Axial Strain %</th>
<th>$f(t)=1$ Average half-wavelength in</th>
<th>$f(t)=1-t/t_f$ Average half-wavelength in</th>
</tr>
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<tr>
<td>SAC-1</td>
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<td>0.31</td>
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<td>0.23</td>
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<td>0.37</td>
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<td>11</td>
<td>0.39</td>
<td>0.48</td>
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<td>7</td>
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<td>0.47</td>
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<td>0.39</td>
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<td>0.25</td>
<td>0.31</td>
</tr>
<tr>
<td>Specimen Designation</td>
<td>Impact Velocity Ft/sec</td>
<td>Axial Strain %</td>
<td>$E_h$ $10^3$ psi</td>
<td>$\sigma_0$ $10^3$ psi</td>
<td>Calculated Average Half-wavelength in</td>
<td>Experimental Average Half-wavelength in</td>
</tr>
<tr>
<td>----------------------</td>
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<td>48.3</td>
<td>30</td>
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<tr>
<td>SAC-1</td>
<td>400</td>
<td>36</td>
<td>48.3</td>
<td>30</td>
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<tr>
<td>LAC-1</td>
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<td>7</td>
<td>58.6</td>
<td>28.7</td>
<td>0.49</td>
<td>0.38</td>
</tr>
<tr>
<td>LAC-2</td>
<td>310</td>
<td>16</td>
<td>48.3</td>
<td>30</td>
<td>0.37</td>
<td>0.41</td>
</tr>
<tr>
<td>LAC-3</td>
<td>344</td>
<td>30</td>
<td>48.3</td>
<td>30</td>
<td>0.29</td>
<td>0.36</td>
</tr>
<tr>
<td>4 CSC-3</td>
<td>59</td>
<td>1</td>
<td>260.0</td>
<td>23.8</td>
<td>0.81</td>
<td>0.61</td>
</tr>
<tr>
<td>4 CSC-3</td>
<td>100</td>
<td>3</td>
<td>121.7</td>
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<td>0.51</td>
</tr>
<tr>
<td>4 CSC-3</td>
<td>115</td>
<td>3</td>
<td>121.7</td>
<td>25.7</td>
<td>0.58</td>
<td>0.48</td>
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<td>48.3</td>
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<td>15</td>
<td>48.3</td>
<td>30</td>
<td>0.57</td>
<td>0.37</td>
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<td>LAC-5</td>
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<td>20</td>
<td>48.3</td>
<td>30</td>
<td>0.47</td>
<td>0.49</td>
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</tbody>
</table>
CHAPTER IV

EXPERIMENTAL WORK REVIEW

In order to study the buckling phenomenon of plates under impact loading, an impact machine has been purchased by the department. The impact machine is a "Varipulse 1500 shock machine" so called because of its ability to generate sinusoidal and square pulses which are useful in vibration analysis. The machine has a heavy table which is free to move up and down along two vertical guides. The table can fall freely under gravity and in this way imparts an impact to the specimen which is placed on the lower table of the machine. Figure 21 shows a photograph of the machine.

Some tests were carried out using this machine as follows: Rectangular shells, considered as four joined plates, were grided in both x and y directions. Grid measurements were made before and after impact and resulting strain obtained. A travelling microscope was used for this purpose.

Some qualitative tests were obtained but are not included here. In general, it was felt that the impact velocity was too low and in order to produce large strains the duration was relatively high. Consequently lateral inertia effects were also quite small. One feature of dynamic buckling is the important role that lateral inertia plays in controlling the amplitudes of the flexural motions. The experimental results obtained by Goodier for high impact velocities
Figure 21  Photograph of impact machine
resulted in narrow plates (struts) being compressed by axial loads four or five times the corresponding Euler buckling load. With the impact machine available here the velocities were so low that only Euler buckling could be obtained and for this reason the results are not included.

One other difficulty encountered was that of plate alignment. The initial well-adjusted alignment could be disturbed by small vibrations occurring during free fall of the table and eccentric impact loading then resulted. The effect of this eccentricity then causes further bending rather than buckling of the plate.

Another possible factor was the heavy mass of the impact table. Large amounts of plastic strains could be obtained even at a low velocity of impact. Therefore, the combined effect of low velocity and large M/m ratio could possibly produce large total strain but a small mode number.

It is rather difficult, at this point, to judge which of the above causes is most important. It is believed that all the factors have contributed to make the experimental work disappointing.
CHAPTER V

SUMMARY OF CONCLUSIONS

The following conclusions are drawn from the above results given in Chapter III.

1. The buckling of plates at sufficiently high velocity exists and the most responsive mode number is detectable.

2. The lateral restraint parameter \( k \) affects the buckling mode number. The mode number decreases as \( k \) increases.

3. The effect of the strain-hardening modulus \( E_h \) on the mode number is small.

4. In general, the mode number increases as impact velocity increases. However, for very high velocity fracture of the material may occur.

5. The mode number decreases as the plate thickness increases. This effect is more significant for smaller value of \( k \).

6. The comparison of the present theoretical results to the experimental ones obtained in [3] is in general agreement.
Bibliography


General References
