THE ABSOLUTE STABILITY OF NONLINEAR SYSTEMS

by

TE-LUNG CHANG

B.A.Sc. in E.E., Cheng Kung University, 1966

A THESIS SUBMITTED IN PARTIAL FULFILMENT OF THE

REQUIREMENTS FOR THE DEGREE OF

MASTER OF APPLIED SCIENCE

in the Department of

Electrical Engineering

We accept this thesis as conforming to the

required standard

Research Supervisor....................................

Members of Committee..................................

....

Acting Head of Department............................

Members of the Department

of Electrical Engineering

THE UNIVERSITY OF BRITISH COLUMBIA

February, 1970

ABSTRACT

This thesis is in two parts, both considering the absolute stability of nonlinear systems. In the first two chapters the stability of certain classes of nonlinear time invariant systems involving several nonlinearities is considered. A number of graphical methods are given for testing the stability of these systems. The graphical tests are equivalent to a weakened form of the Popov criterion. The third chapter derives a stability condition for nonlinear systems involving a linear time-varying gain. The time-varying gain is assumed to satisfy conditions on its magnitude and rate of change.

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ACKNOWLEDGEMENT

I wish to express my sincere gratitude to Dr. M.S. Davies, the supervisor of this project, for his invaluable guidance, and Dr. E.V. Bohn for reading the manuscript. Grateful acknowledgement is given to the National Research Council for financial support received under NRC Grant A-4148, and to the University of British Columbia for U.B.C. Fellowship awarded since 1968. I would also like to thank Miss Beverly Harasymchuk for typing the manuscript.

CHAPTER 1 TIME INVARIANT NONLINEAR FEEDBACK SYSTEM AND TIME VARYING NONLINEAR FEEDBACK SYSTEM

§1.1 Introduction

This thesis considers the absolute stability of the equilibrium position, $X \equiv 0$, of feedback systems defined by

$$\begin{aligned} \dot{X} &= AX + BY \\ Y &= \emptyset(\sigma) \\ \sigma &= C^T X, \end{aligned} \tag{1.1.1}$$

or, in the time varying case,

$$\begin{aligned} \dot{X} &= AX + BY \\ Y &= \emptyset(\sigma,t) \\ \sigma &= C^T X, \end{aligned} \tag{1.1.2}$$

where X is an n-vector, Y is an m-vector, $\emptyset$ is an m-vector, A is an nxn constant matrix, B is an nxm constant matrix, and C^T is an mxn constant matrix.

In (1.1.1), each element of $\emptyset(\sigma)$ is a nonlinear function of σ alone, so that the system (1) is time invariant. In (1.1.2), each element of $\emptyset(\sigma,t)$ is a time varying nonlinear function of both σ and t. It is assumed that the i-th element can be separated into a nonlinear part $\emptyset_i(\sigma_i)$ and the time varying gain $k_i(t)$, where $\emptyset_i(\sigma_i,t) = \emptyset_i(\sigma_i)k_i(t)$. The system (2) is thus a time varying nonlinear feedback system.

The transfer matrix $\Gamma(s)$ of the linear part is

$$\Gamma(s) = C^T(sI-A)^{-1} B, \tag{1.1.3}$$

where I is an nxn unit matrix. The feedback system (1) may be depicted as shown in Fig. 1.1. The forward path consists of an mxm linear time invariant matrix $\Gamma(s)$ and the nonlinearity matrix N.L. In Fig. 1.2, the forward path of the time varying nonlinear feedback system (2) consists of the transfer

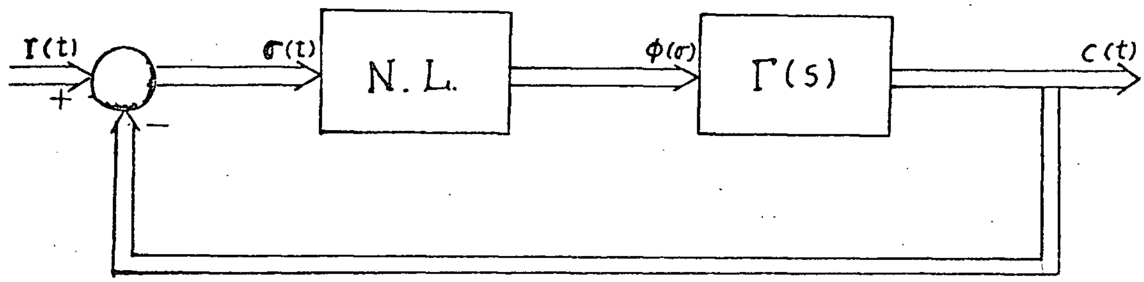


Fig. 1.1 General time invariant nonlinear feedback system

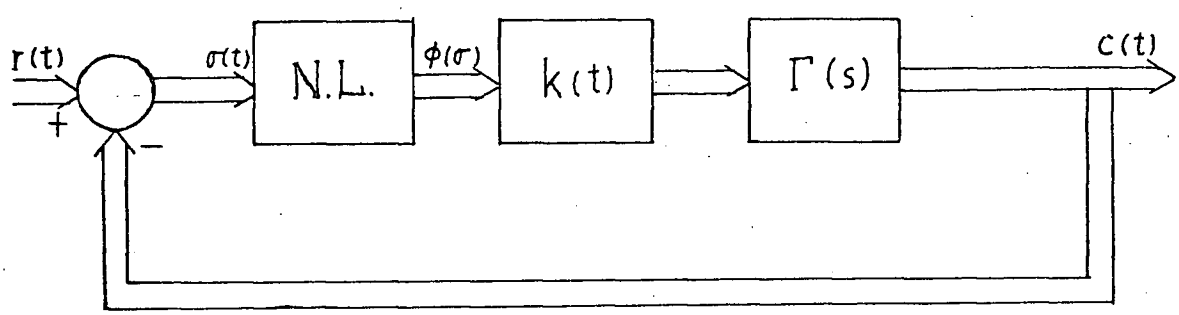


Fig. 1.2 General time varying nonlinear feedback system

matrix $\Gamma(s)$, the mxm diagonal nonlinearity matrix N.L. with elements $\emptyset_1(\sigma_1)$, $\emptyset_2(\sigma_2)$,...$\emptyset_m(\sigma_m)$ and the mxm diagonal time-varying gain matrix k(t) with elements $k_1(t), k_2(t), \ldots k_m(t)$.

A wide variety of systems may be treated by choice of $\Gamma(s)$, some forms of particular interest are:

(1) A series system. The forward path consists of the single-input single-output linear time invariant transfer functions separated by amnesic nonlinearities. This is shown in Fig. 1.3.

(2) A parallel system. The forward path consists of m-parallel branches, each of which has one nonlinearity in series with one linear time invariant transfer function. This is shown in Fig. 1.4.

(3) An internal feedback system. The forward path consists of m-single nonlinear feedback loops. This is shown in Fig. 1.5.

(4) A multi-circuit system. Such systems do not fall into the previous classes. Such a system is shown in Fig. 1.6.

The elements of the nonlinearity matrix are to be considered in 4 classes. In each it is assumed that $\emptyset(\sigma)$ is a piece-wise continuous, angle valued function of σ.

(1) A sector nonlinearity; any function which satisfies the condition

$$k_1 < \frac{\emptyset(\sigma)}{\sigma} < k_2, \qquad \sigma \neq 0$$

$$\emptyset(\sigma) = 0, \qquad \sigma = 0$$

where $k_2 > 0$ and k_1 may be positive, negative or zero.

(2) A first and third quadrant nonlinearity; this is a special case of (1).

$$0 < \frac{\emptyset(\sigma)}{\sigma} \leq k < \infty \quad , \qquad k > 0,$$

for all finite nonzero values of σ.

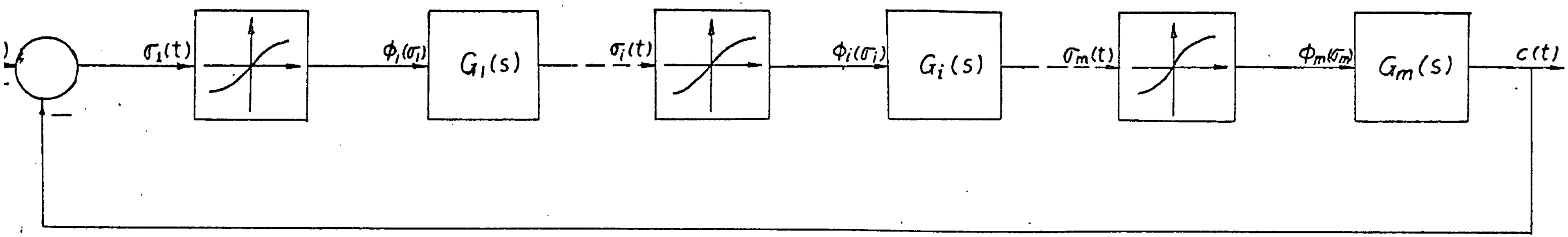


Fig. 1.3 Series system with m nonlinearities

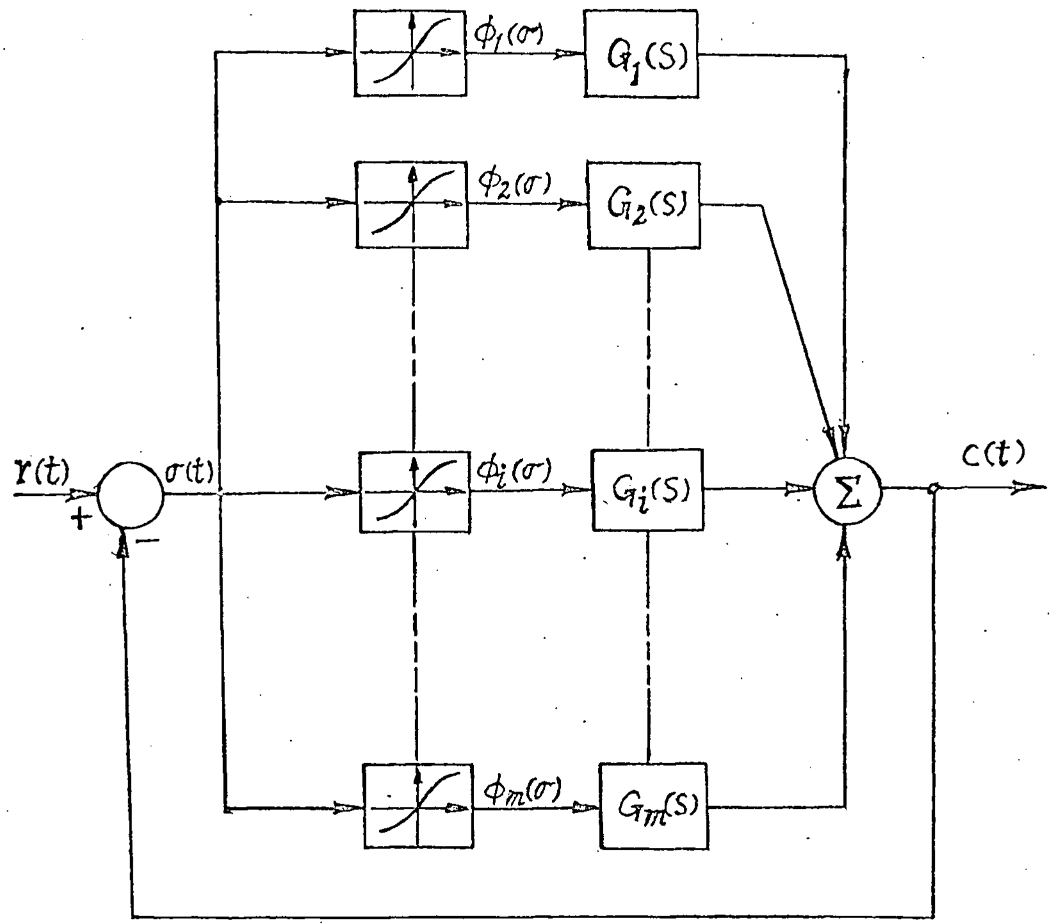


Fig. 1.4 Parallel system with m nonlinearities

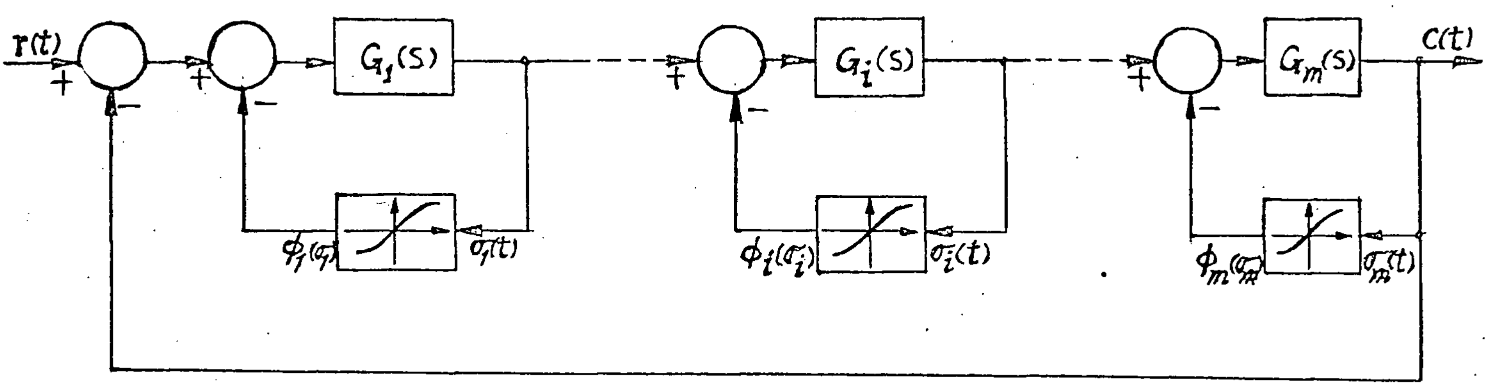


Fig. 1.5 Internal feedback system with m nonlinearities

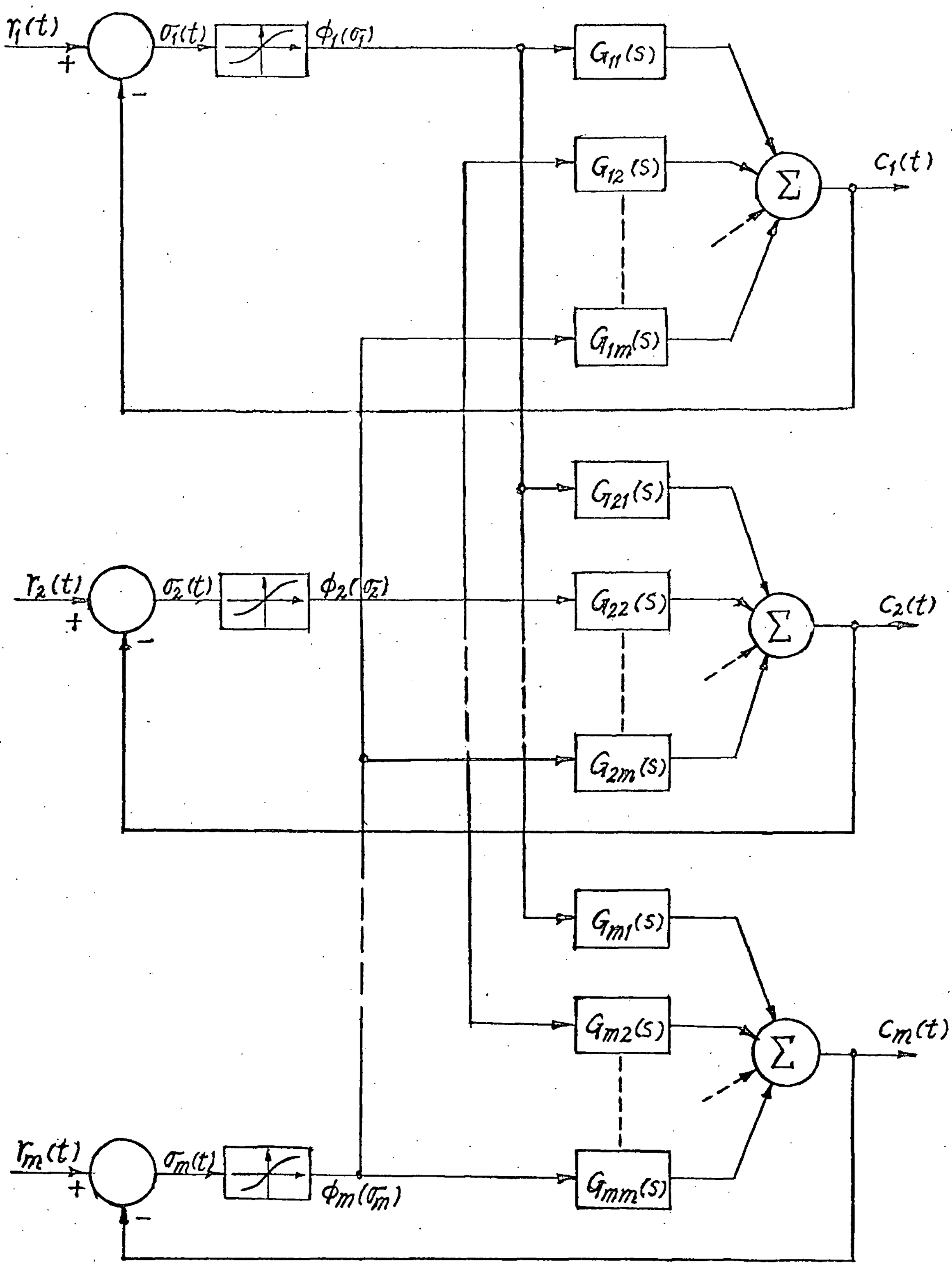


Fig. 1.6 Multi-circuit system with m nonlinearities

(3) A monotonic nonlinearity; this belongs to a subclass of (2), in which it is assumed that

$$\frac{d\emptyset(\sigma)}{d\sigma} > 0.$$

(4) A monotonic odd nonlinearity; this belongs to a subclass of (3), in which it is further assumed that $\emptyset(-\sigma) = -\emptyset(\sigma)$ for all σ.

Furthermore, the elements of the time varying gain matrix to be considered are assumed positive, bounded and continuous.

§1.2 Lyapunov Method

Investigations of the stability of such systems were initiated by Luré who proposed a Lyapunov function of the form:

$$V(X) = X^T PX + \sum_{i=1}^{m} \beta_i \int_0^{\sigma_i} \emptyset_i(z)dz, \tag{1.2.1}$$

where P is an nxn symmetrical positive definite matrix ($P = P^T > 0$), all $\beta_i \geq 0$ and the upper limits of the integral terms are the elements of the matrix

$$\sigma = C^T X. \tag{1.2.2}$$

It is obvious that V(X) is a positive function since

$$0 < \emptyset(\sigma)\sigma < K\sigma^2. \tag{1.2.3}$$

From (1.1.1), the derivative of (1.2.1) is

$$\dot{V}(X) = X^T(AP^T+PA) + B^T PX\emptyset(\sigma) + X^T PB\emptyset(\sigma)$$

$$- \beta AX\emptyset(\sigma) + \beta C\emptyset(\sigma), \tag{1.2.4}$$

where β is a diagonal constant matrix with elements $\beta_1, \beta_2, \ldots \beta_m$.

From the Lyapunov second method, in order to find the sufficient condition of absolute stabiltiy, it is necessary to determine the conditions under which $\dot{V}(X)$ is negative definite except at the null state, $X \equiv 0$, where $\dot{V}(X) \equiv 0$.

In [5], it was shown that the conditions of absolute stability of a time varying nonlinear system may be found from using the Lyapunov function (1.2.1) with all $\beta_i = 0$. This method of using Lyapunov functions was further developed by Narendra and Taylor [6] using the modified Lyapunov function, viz.,

$$V(X,t) = X^T PX + \sum_{i=1}^{m} \beta_i k_i(t) \int_0^{\sigma_i} \phi_i(z)dz, \tag{1.2.5}$$

where $k_i(t)$ is a time varying gain.

In [4,5], it is also proved that the sufficient condition under which the Lyapunov function is valid is similar to the Popov criterion, discussed below.

§1.3 The Popov Criterion

The sufficient condition of absolute stability for a controllable and observable time invariant system with one nonlinearity satisfying (1.2.3) was established by V.M. Popov [1]. The Popov criterion takes the form

$$\mathrm{Re}[(1+qj\omega)\ G(j\omega)] + \frac{1}{K} > 0 \tag{1.3.1}$$

for all ω, where q is a nonnegative number.

A convenient graphical method exists for testing (1.3.1).

A sufficient condition for the absolute stability of a system with many nonlinearities was given by Jury and Lee [2]. The condition requires the Hermitian matrix:

$$2K^{-1} + H(j\omega) + H^T(-j\omega) \tag{1.3.2}$$

to be positive definite for all ω. Here K is a constant diagonal matrix of elements $K_1, K_2, \ldots K_m$, all of which are positive numbers such that the inequality

$$0 < \phi_i(\sigma_i)\sigma_i < K_i\sigma_i^2 \tag{1.3.3}$$

is satisfied for $i = 1,2,\ldots m$, and where

$$H(j\omega) = (I + j\omega Q)\ \Gamma(j\omega), \tag{1.3.4}$$

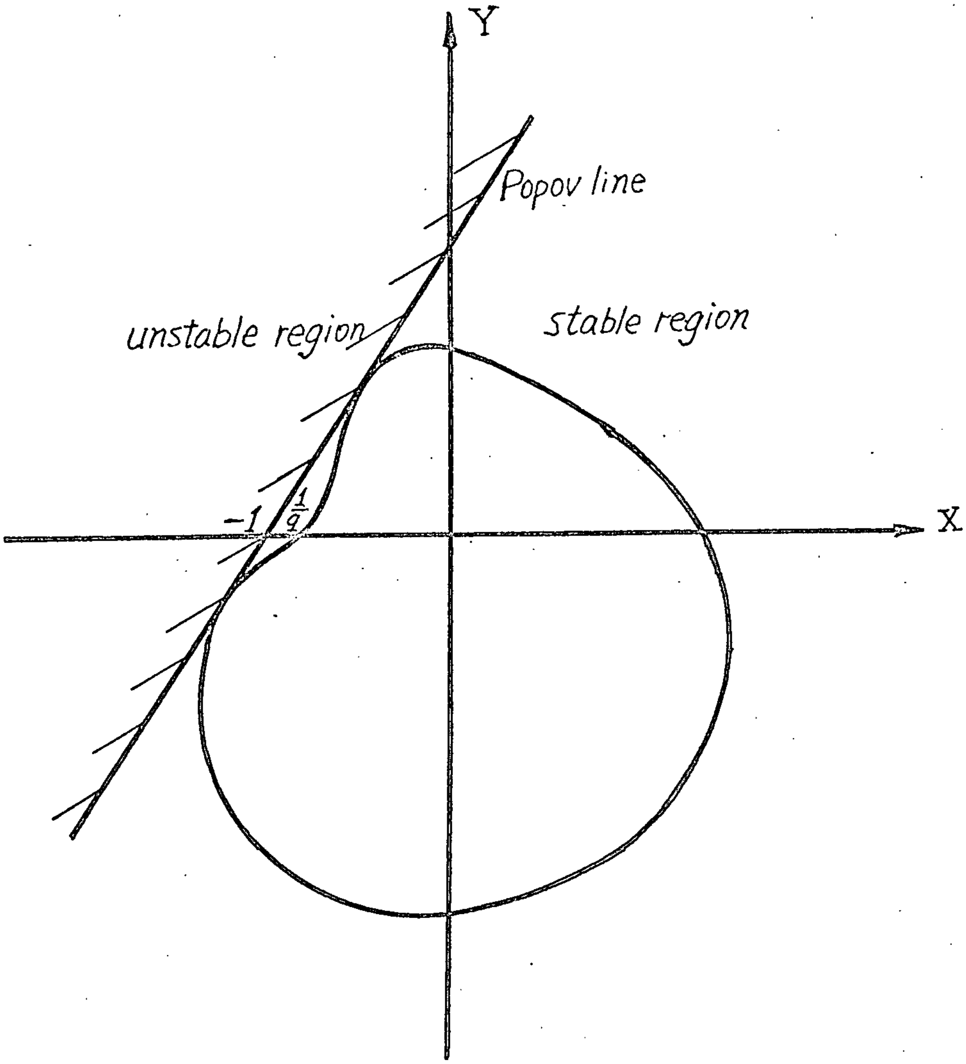


Fig. 1.7 Popov criterion

where Q is a constant diagonal matrix with elements $q_1, q_2 \dots q_m$ which are nonnegative numbers. This forms a generalization of the Popov result.

For the time varying nonlinear feedback system, the sufficient condition of absolute stability established by Rozenvasser is as below,

$$\mathrm{Re}\, G(j\omega) + \frac{1}{K} > 0 \tag{1.3.5}$$

for all ω. It is but the Popov criterion with $q = 0$.

Other new criteria, similar to the Popov's, were introduced by Zames and Falb [8,9], Yakubovich [4], Narendra and Taylor [6], Baker and Desoer [10,11], Bergen and Rault [12], and Anderson [13].

§1.4 L_2 Stability

The concept of the L_2 stability has been introduced by Sandberg [14]. It is closely related to asymptotic stability.

L_2 is the space of square integrable, valued functions on $[t_o, \infty)$, it is assumed that L_2 is a linear, inner-product, Vformed space; the inner product of x and y in L_2 is

$$\langle x,y \rangle = \int_{t_o}^{\infty} x(t) \cdot y(t)\,dt < \infty, \tag{1.4.1}$$

and the norm of x is $\| x \|_2 = \sqrt{\langle x,x \rangle}$.

Suppose $\sigma(x)$ is in $L_2[0,\infty]$, $\sigma(t)$ is uniformly continuous, and $\dot{\sigma}(t)$ is bounded, then the state $\underset{t\to\infty}{\sigma(t)}$ approaches null state if the sufficient condition of absolute stability of a time invariant nonlinear system, the Popov criterion, is satisfied. A further L_2 bounded condition was introduced by Zames [8,9].

CHAPTER 2 THE ABSOLUTE STABILITY OF A TIME INVARIANT NONLINEAR FEEDBACK SYSTEM

§2.1 Introduction

In the previous chapter, it was mentioned that the nonlinear systems can be considered in 4 classes according to the form of the transfer matrix $\Gamma(s)$.

Testing absolute stability of a single-loop time invariant nonlinear system using the modified Nyquist diagram was first initiated by Popov, basing the method on his criterion. Further developments using a graphical method to test the absolute stability of a nonlinear system have been furnished by Naumov [15], Meyer and Hsu [16], and Murphy [17].

A graphical method of testing the absolute stability of a time invariant series system with m-nonlinearities and m-identical linear transfer functions was introduced by Davies [18].

§2.2 A Graphical Test of the Absolute Stability of a Series System with Nonlinearities and Identical Transfer Functions

Consider a series system with the linear time invariant transfer function matrix

$$\Gamma(s) = \begin{bmatrix} 0 & 0 & \cdots & 0 & G(s) \\ -G(s) & 0 & \ddots & & 0 \\ & -G(s) & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & 0 \\ & 0 & & -G(s) & 0 \end{bmatrix} . \qquad (2.2.1)$$

The input, $\sigma_i(t)$, and output, $\phi_i(\sigma_i)$, of the i-th nonlinear element satisfy the inequality

$$0 < \sigma_i \phi_i(\sigma_i) < \sigma_i^{\,2}. \qquad (2.2.2)$$

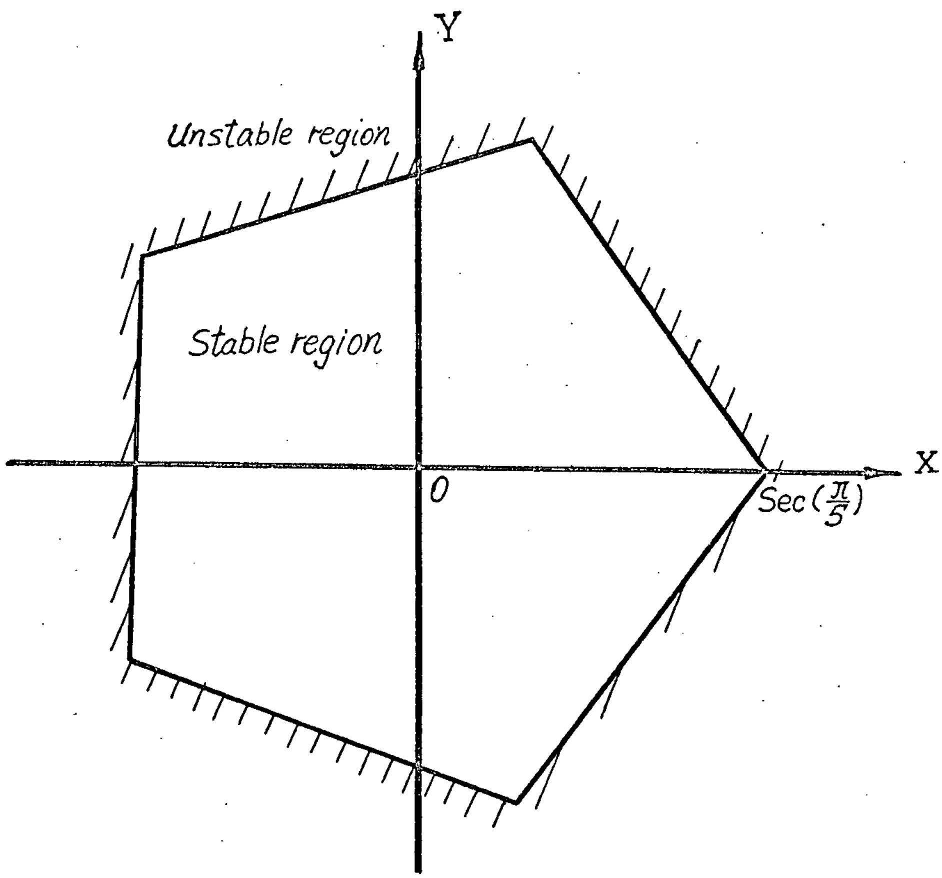


Fig. 2.1 Graphical criterion for the time invariant series system with 5 ideal nonlinearities and 5 identical linear transfer functions.

Such a system falls within the class for which a Popov-like stability criterion has been established. In applying this result, the matrix

$$H(j\omega) = (I+j\omega Q)\Gamma(j\omega), \tag{2.2.3}$$

where Q is an arbitrary, semi-positive, diagonal matrix of constants, is considered. A sufficient condition to establish the absolute stability of a nonlinear system is that the Hermitian matrix (1.3.2) must be positive definite for all ω.

If all nonlinear elements are assumed to have the same upper bound; that is, $K_i = K$ for $i = 1,2,\ldots m$, then, without loss of generality, K may be taken as the indentity matrix. If Q=0, the stability criterion is equivalent to the Nyquist plot of G(s) lying within a symmetric m-sided polygon. In the subsequent development it is not required that Q=0, but rather that all elements of Q are equal; that is, Q=qI, where q is a positive scalar constant.

If Q is restricted in this manner, then

$$(I+j\omega Q)\Gamma(j\omega) = \Gamma'(j\omega), \tag{2.2.4}$$

where $\Gamma'(j\omega)$ is identical to $\Gamma(j\omega)$ except that $G(j\omega)$ has been replaced by $G'(j\omega) = (1+j\omega q)G(j\omega)$. Thus it is possible to consider the case Q=0 by applying the earlier results for Q=0 to $G'(j\omega)$ instead of $G(j\omega)$ itself. If it can be shown that $G'(j\omega)$ lies within the appropriate polygon, for any positive q, then stability has been established.

§2.2.1 Main Method-Common Popov Line

Let $|G'(j\omega)| = \gamma$ and $G'(j\omega) = \theta$. If $G'(j\omega)$ lies within a polygon, then

$$\gamma \cos (\theta - \alpha) < 1, \tag{2.2.5}$$

where $\alpha = \frac{2i\pi}{m} + \frac{\pi}{m}$

and $i = 1,2,\ldots m$,

and where i is one of m values each corresponding to the m sides of the polygon.

Now

$$G'(j\omega) = (1+j\omega q)G(j\omega) = (1+j\omega q)(R+jI), \tag{2.2.6}$$

where

$$G(j\omega) = R(\omega) + jI(\omega).$$

Thus

$$\text{Re } G'(j\omega) = R - j\omega I = \gamma\cos\theta, \tag{2.2.7A}$$

$$\text{Im } G'(j\omega) = \omega qR + I = \gamma \sin\theta, \tag{2.2.7B}$$

and

$$(R-\omega qI)\cos\alpha + (\omega qR+I)\sin\alpha < 1, \tag{2.2.8A}$$

$$(R\cos\alpha+I\sin\alpha) + q\omega(-I\cos\alpha+R\sin\alpha) < 1. \tag{2.2.8B}$$

Define

$$-X(\omega) = R\cos\alpha + I\sin\alpha, \tag{2.2.9A}$$

$$Y(\omega) = \omega(-I\cos\alpha + R\sin\alpha). \tag{2.2.9B}$$

(2.2.8B) gives

$$-X + qY < 1. \tag{2.2.10}$$

From (2.2.9A) and (2.2.9B), the m different modified Nyquist lcci are plotted each corresponding to one of the m sides of the polygon. To satisfy inequality (2.2.10), all these loci must be to the right side of a straight line, the Popov line, which passes the point (-1,0) having slope 1/q. If such a straight line exists, then the absolute stability of the system is established.

Example 2.1

Consider a feedback system of the type shown in Fig. 1.3 (m=3). Let every nonlinearity satisfy the inequality

$$0 < \phi_i\sigma_i < \sigma_i^2, \tag{2.2.11}$$

where i = 1,2,3 and let every linear block be represented by

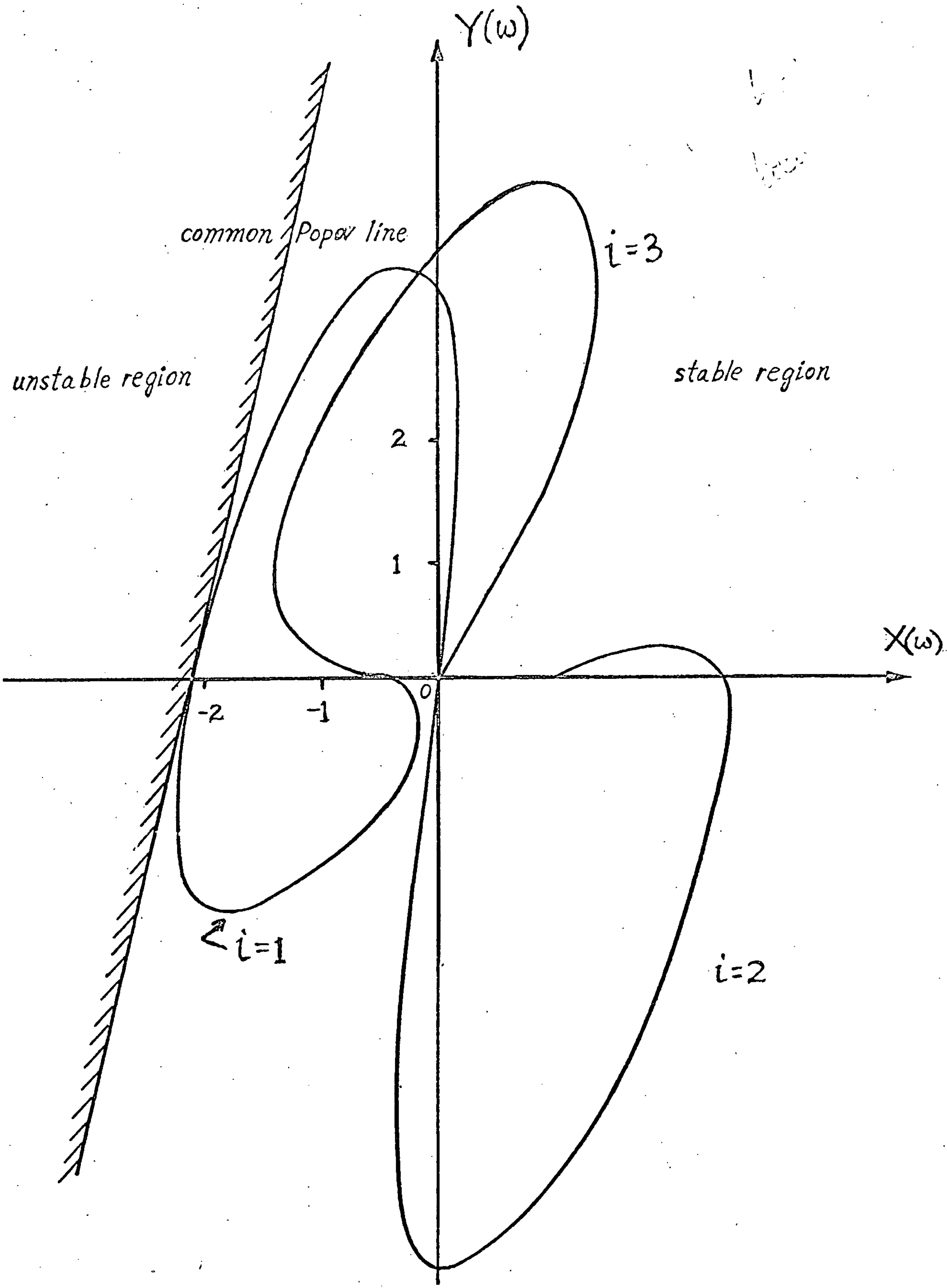


Fig. 2.2 Modified Nyquist loci and the common Popov line for the series system, $m=3$, $G(s) = \dfrac{32(s+0.25)}{(s+1)(s+2)(s+4)}$, for Example 2.1.

$$G_i(s) = \frac{32K(s+0.25)}{(s+1)(s+2)(s+4)} \, . \tag{2.2.12}$$

The m different modified Nyquist loci and common Popov line are shown in Fig. 2.2. Then, by setting q = 0.23, the absolute stability condition

$$K < 0.476 \tag{2.2.13}$$

is obtained.

§2.2.2 Simplification in Particular Cases

Plotting the m different modified Nyquist loci is tedious if $m \geq 4$. A simpler and more direct approach is possible if G(s) is of the form

$$G(s) = \frac{k}{\prod_{j=1}^{p} (s+D_j)} \tag{2.2.14}$$

or

$$G(s) = \frac{k \prod_{i=1}^{n} (s+N_i)}{\prod_{j=1}^{p} (s+D_j)} \, , \tag{2.2.15}$$

where $p > 1+m$,

N_i and D_j are real positive constants, and $N_i > D_j$ for i = j.

If the transfer function $G'(j\omega) = (1+j\omega q)G(j\omega)$ having q_1 and k_1 which are found from the modified Nyquist locus for i = 1, satisfies the above conditions, then,

(1) The phase of $G'(j\omega)$ is decreasing as ω increases.

(2) For some $\omega = \omega_c$, giving $\angle G'(j\omega_c) \geq -\frac{2\Pi}{m}$ and $|G'(j\omega_c)| < 1$, $|G'(j\omega)|$ decreases when ω increases for $\omega > \omega_c$.

And k_1 is the maximum value satisfying the absolute stability conditon.

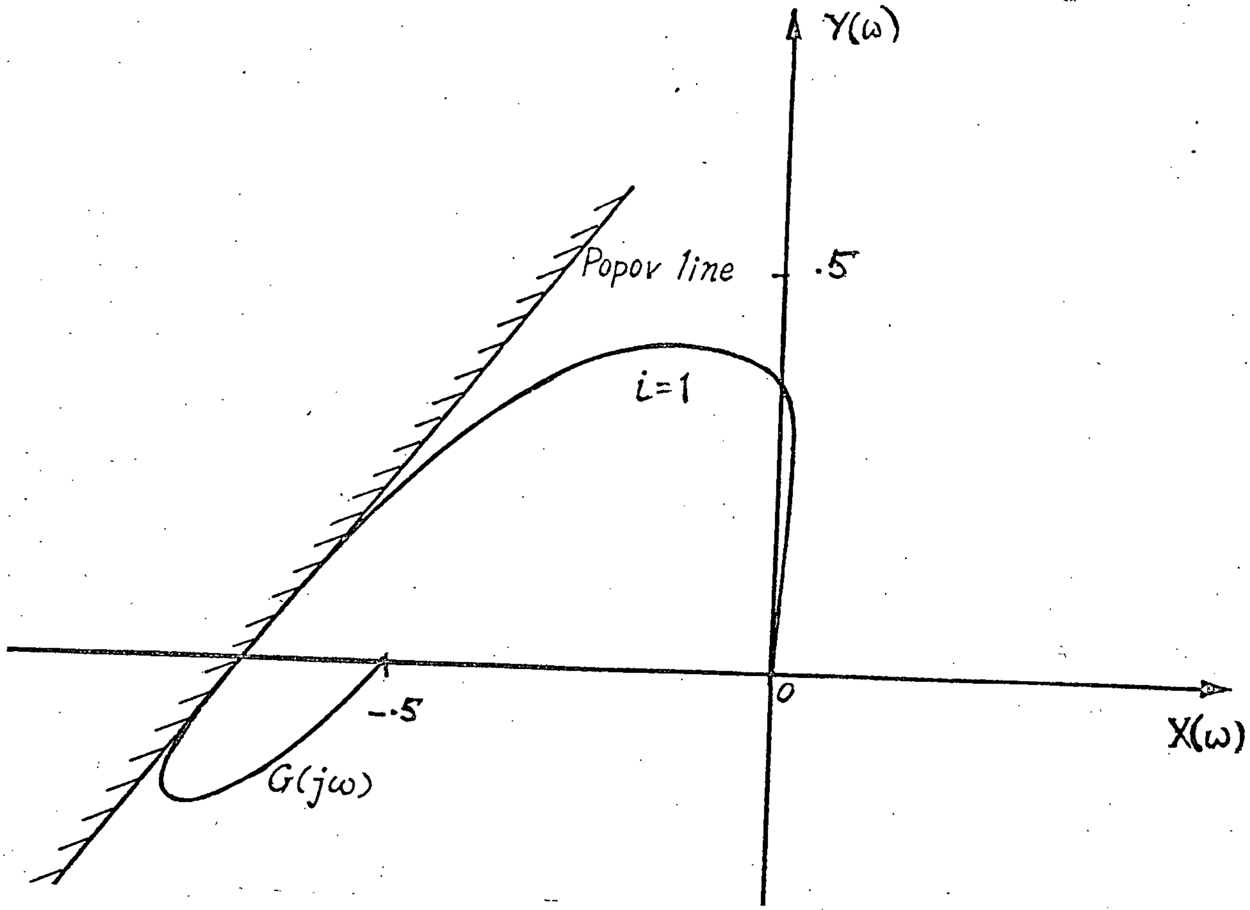


Fig. 2.3 Modified Nyquist locus for i=1, and the corresponding Popov line for the series system, m=3, $G(s) = \dfrac{100}{(s+1)(s+5)(s+20)}$, for Example 2.2.1.

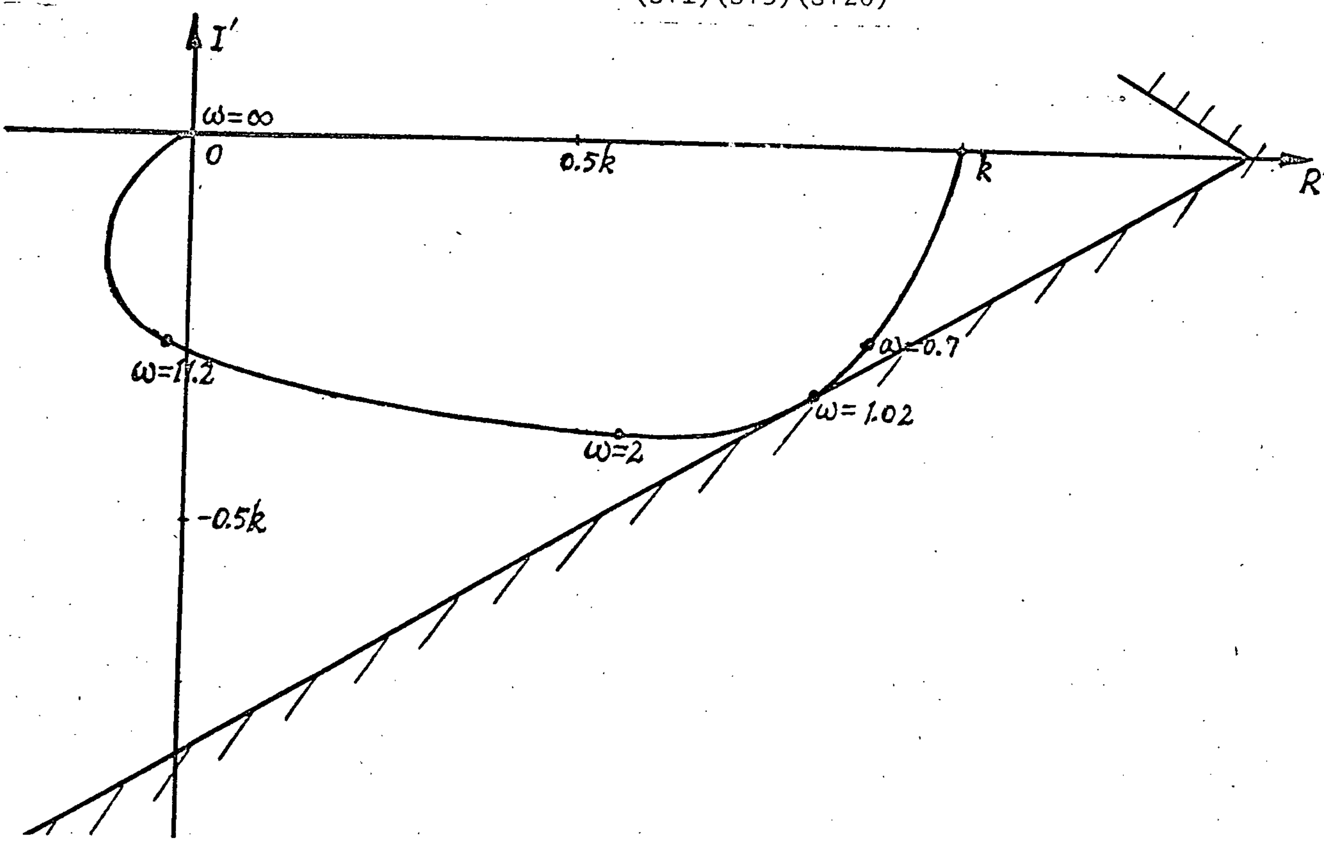


Fig. 2.4 Nyquist plot of (1+0.788S)G(s), where $G(s) = \dfrac{100}{(s+1)(s+5)(s+20)}$, and a graphical testing the stability of the series system, m=3, for Example 2.2.

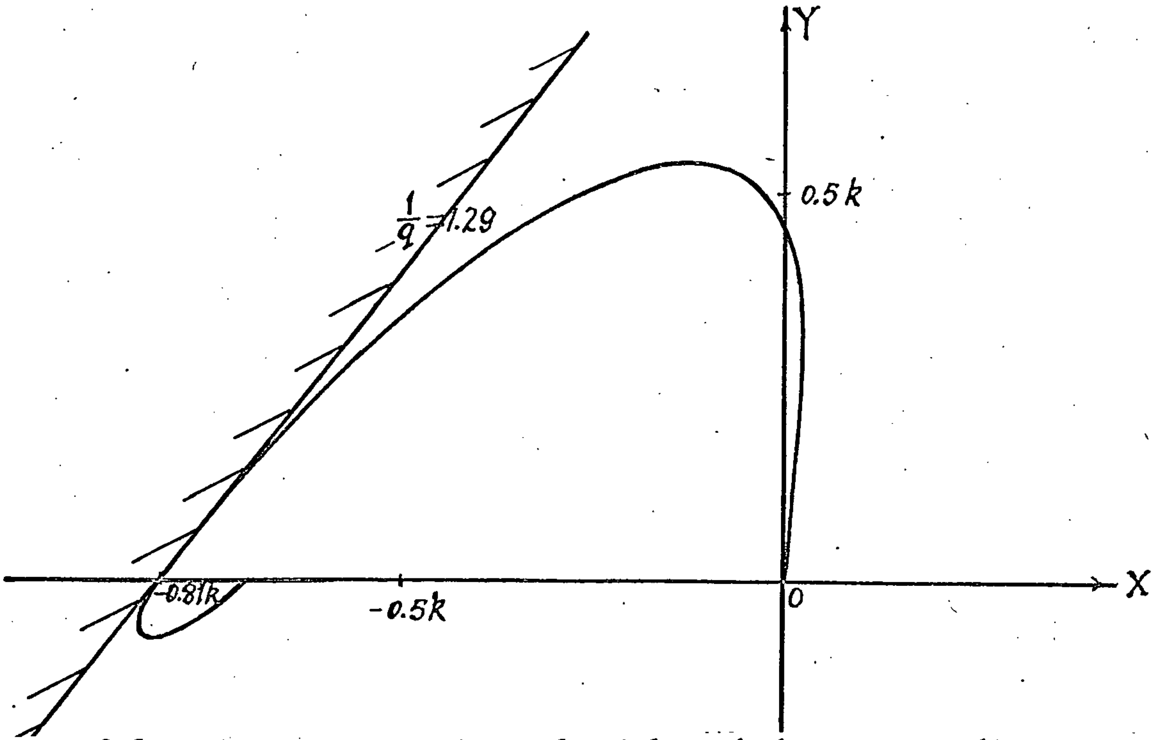


Fig. 2.5 Modified Nyquist locus for i=1, and the corresponding Popov line for the series system, m=4, $G(s) = \frac{100}{(s+1)(s+5)(s+20)}$, for Example Example 2.2.2.

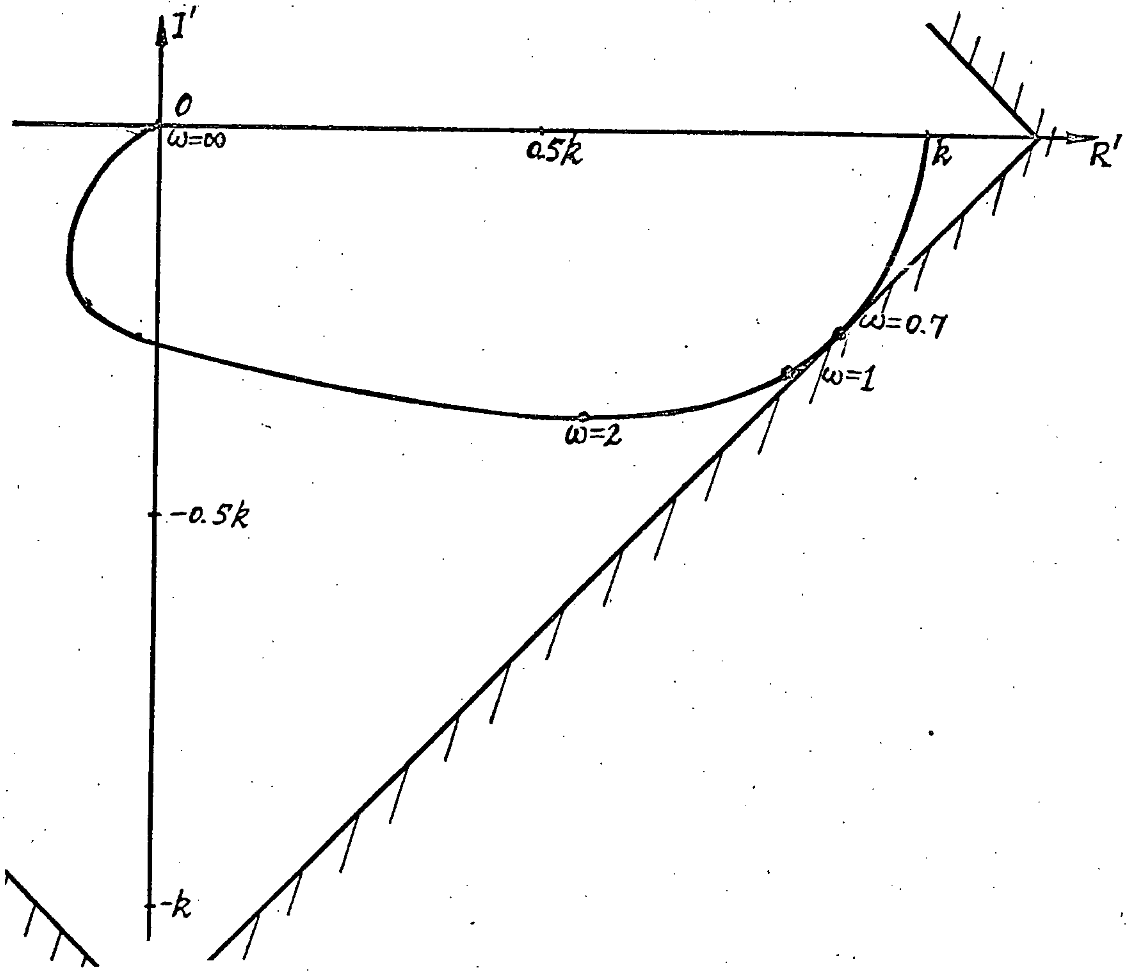


Fig. 2.6 Nyquist plot of (1+0.766s)G(s), where $G(s) = \frac{100}{(s+1)(s+5)(s+20)}$, and a graphical testing the stability of the series system, m=4, for Example 2.2.2.

Example 2.2

(1) Consider $m = 3$ with every linear element having a transfer function

$$G(s) = \frac{100k}{(s+1)(s+5)(s+20)}. \tag{2.2.16}$$

From Fig. 2.3,

$$q_1 = 0.788, \tag{2.2.17}$$

and

$$k_1 = 1.46. \tag{2.2.18}$$

Setting $q = 0.788$, $k < 1.46$ is the absolute stability condition since, as shown in Fig. 2.4, $G'(s)$ staisfies the above conditions.

(2) Using the same approach the absolute stability condition of the system with 4 nonlinearities is obtained as:

$$k < 1.234 \tag{2.2.19}$$

by setting $q = 0.766$.

The information required for the previous method may be obtained directly from the Nyquist Plot of G(s) by noting that the critical point is that having phase α_1, and the corresponding value of q is given by $\tan(\alpha_1 - \beta)$ where $\tan\beta$ is the slope of G(s) at $\alpha = \alpha_1$.

If the transfer function G(s) does not satisfy the special form of (2.2.14) or (2.2.15), the Nyquist locus of $G'(s)$ haivng q_1 and k_1 found from inequality (2.2.10) for $i=1$ may be tested by the polygon criterion. If this fails, the general approach must be adopted.

§2.2.3 Use of the Modified Nichols Chart to Obtain Q

The use of the modified Nichols chart to obtain q and to test the absolute stability of the system with many nonlinearities is also possible.

A polygon will be described by some relationship between the log amplitude and the phase of the form

$$M = 20 \log_{10} \sec\left(-\frac{2i\pi}{m} + \frac{\pi}{m} - \theta\right) db, \tag{2.2.20}$$

where $-\frac{2\pi}{m}(i-1) \leq \theta \leq -\frac{2\pi}{m} i$,

and $i = 1,2,\ldots m$.

The stable boundary G' can be represented in the modified Nichols chart. Let us consider

$$M_j = M - 20 \log_{10}(1+N_j^2)^{\frac{1}{2}}, \tag{2.2.21A}$$

and

$$\theta_j = \theta - \tan^{-1} N_j, \tag{2.2.21B}$$

where $j = 0,\ldots\infty$

and N_j is an arbitrary, positive constant. (2.2.21A) and (2.2.21B) give a family of the stable boundaries as shown in Fig. 2.7, each corresponding to one of the constants N. Note that these curves are all of the same form and may easily be sketched.

If the locus of $G(j\omega)$ is sketched and q is chosen, such that each point ω_j on the locus of $G(j\omega)$ on the modified Nichols chart is beneath the corresponding stable boundary $N_j = \omega_j q$, then the absolute stability of the system is established.

Also, if the position of the point ω_j is known, and is beneath a family of stable boundaries N, then the corresponding values of $q_j = \frac{N_j}{\omega_j}$, $j = 0,\ldots\infty$, is known, so that every point has one corresponding stable q range. Then, any q in the interior q range is permitted to be chosen for establishing the stability of the system. This seemingly complex procedure is, in fact, quite straight-forward. If a range of q's is permissible, that value of q is chosen which gives the greatest possible value of k.

Example 2.3

Consider the system in the Example 2.2 with every linear element G(s) where

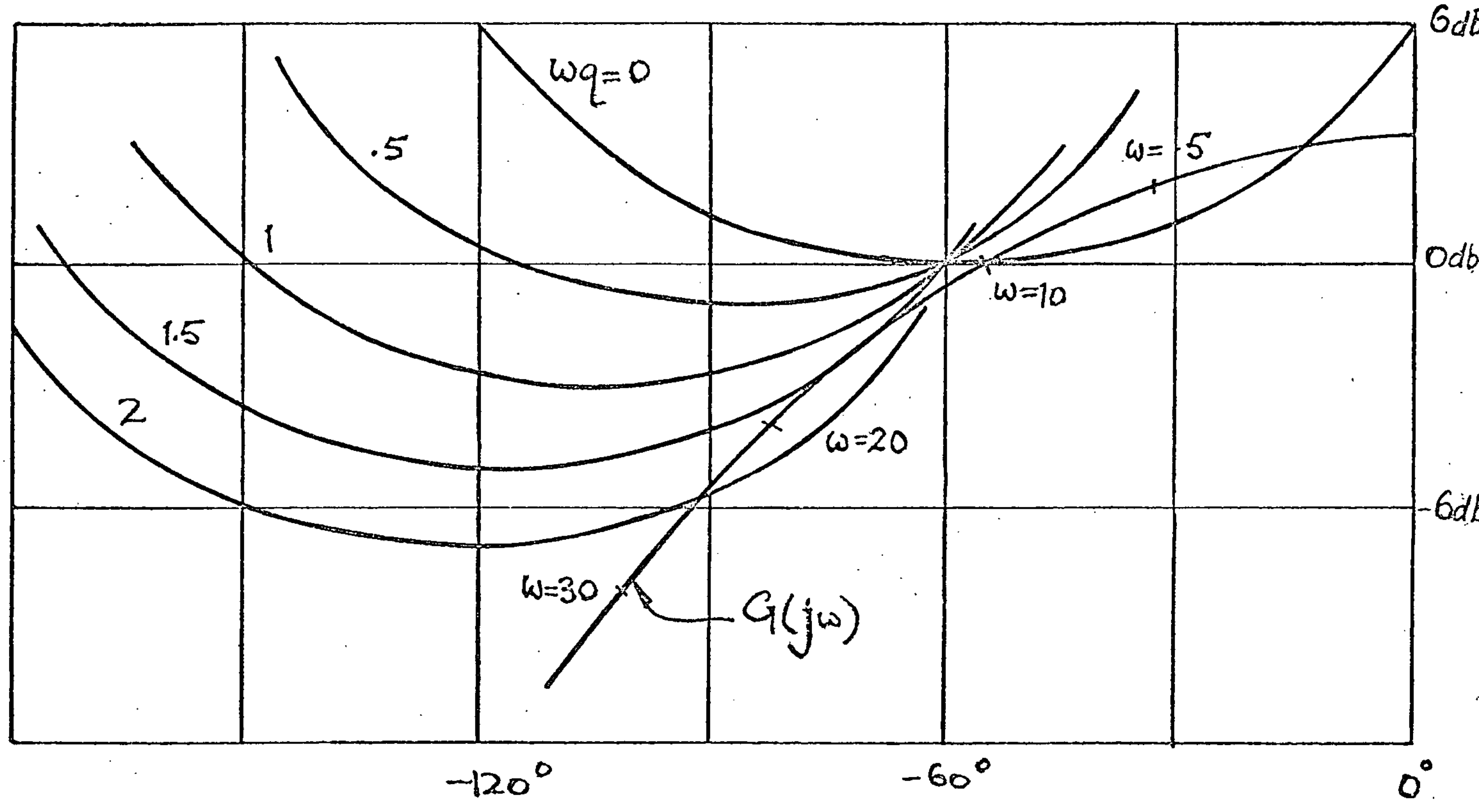


Fig. 2.7 Gain-phase plot for $G(s) = \frac{700}{(s+10)(s+50)}$ and a family of boundaries for the series system, m=3, for Example 2.3.

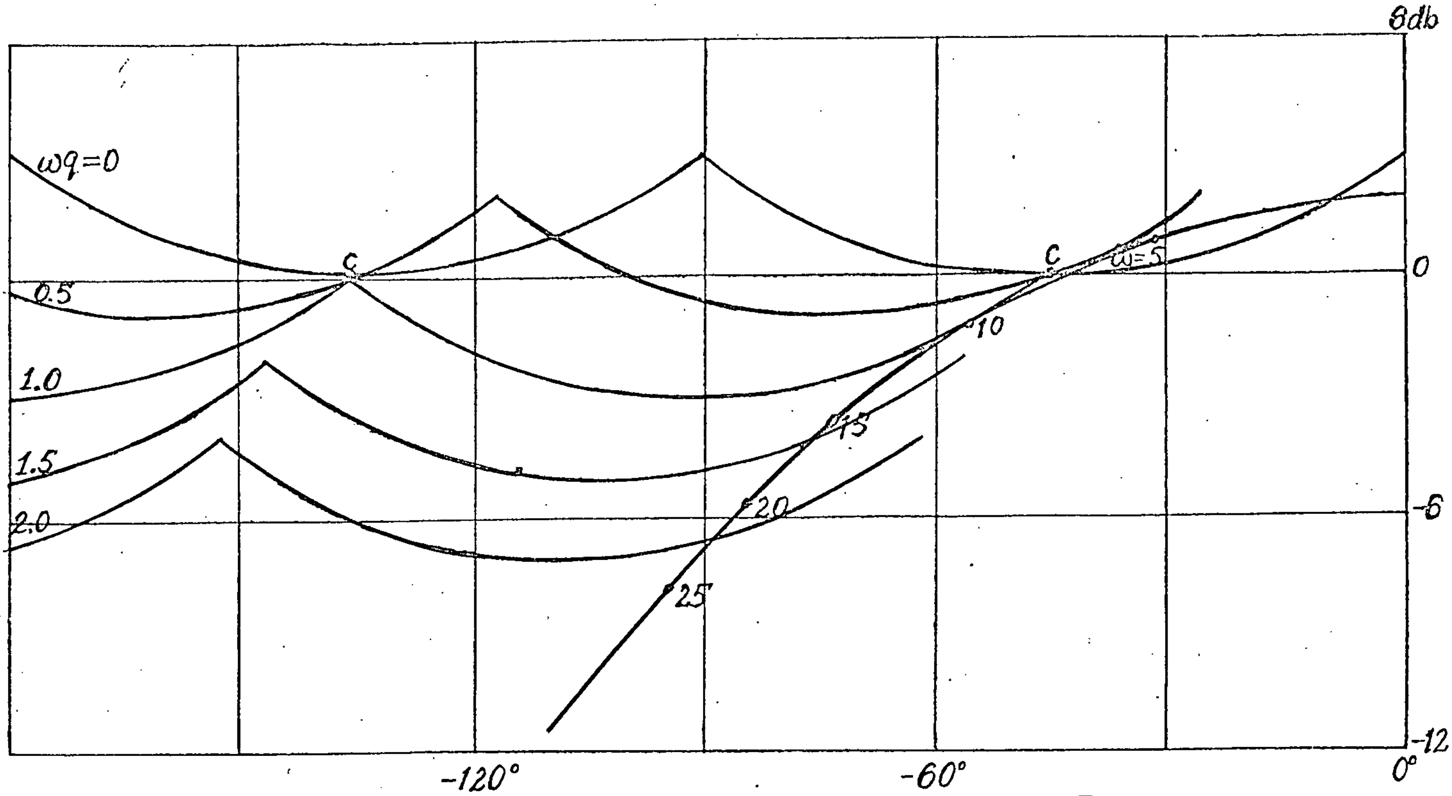


Fig. 2.8 Gain-phase plot for $G(s) = \frac{630}{(s+10)(s+50)}$ and a family of boundaries for the series system, m=4, for Example 2.3.

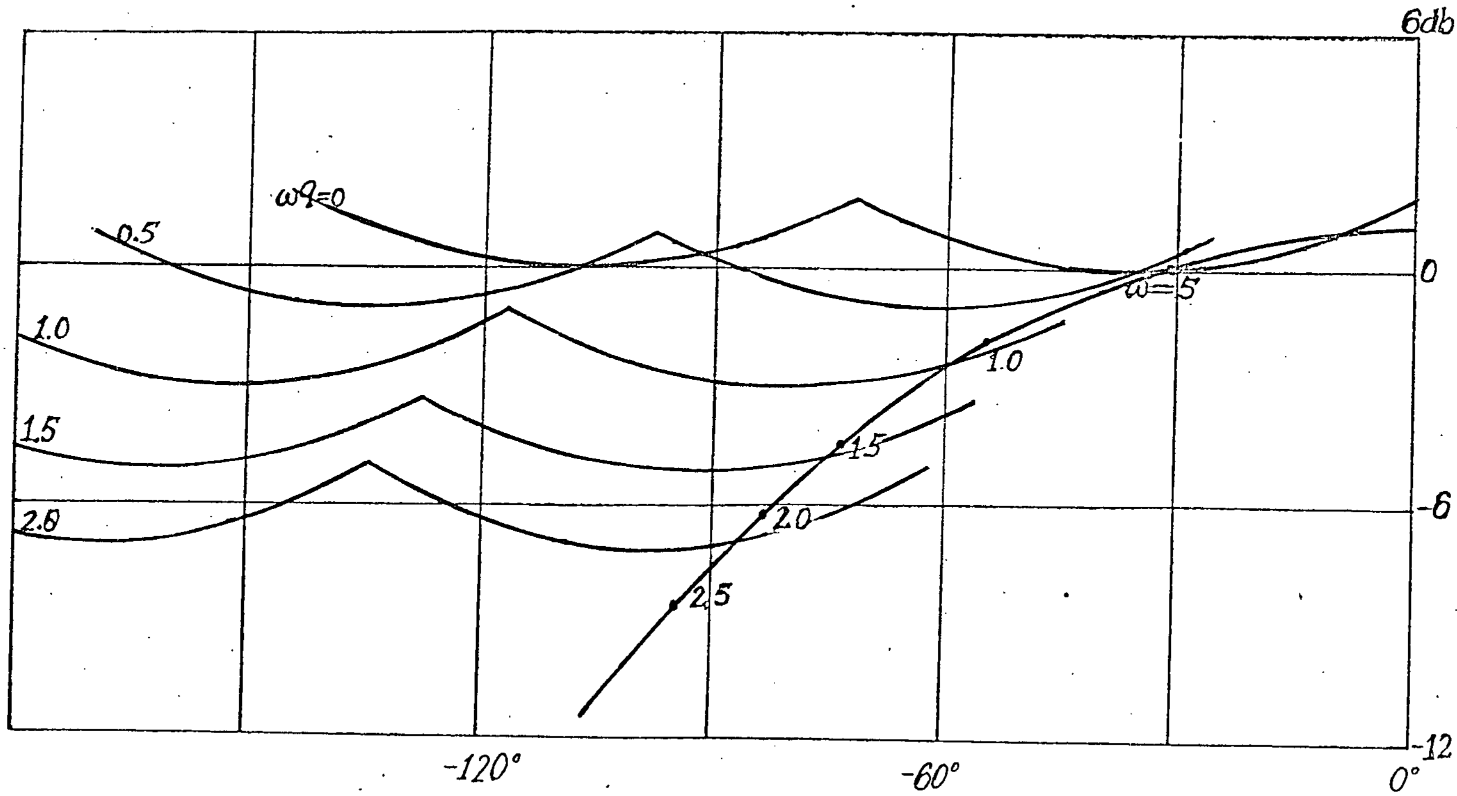


Fig. 2.9 Gain-phase plot for $G(s) = \frac{580}{(s+10)(s+50)}$ and a family of boundaries for the series system, m=5, for Example 2.3.

$$G(s) = \frac{k}{(s+10)(s+20)} \tag{2.2.22}$$

and m = 3. Comparing the frequency response curve of $G(j\omega)$ with a family of the stable boundaries as in Fig. 2.7, the value of q and stable condition are

$$q = 0.067 \tag{2.2.23A}$$

$$k \lesssim 700 \tag{2.2.23B}$$

Similarly, from Fig. 2.8 and Fig. 2.9, by setting q = 0.067, the following absolute stability conditions

$$k \leq 630 \quad \text{for } m = 4, \tag{2.2.24}$$

and

$$k \leq 580 \quad \text{for } m = 5, \tag{2.2.25}$$

are obtained.

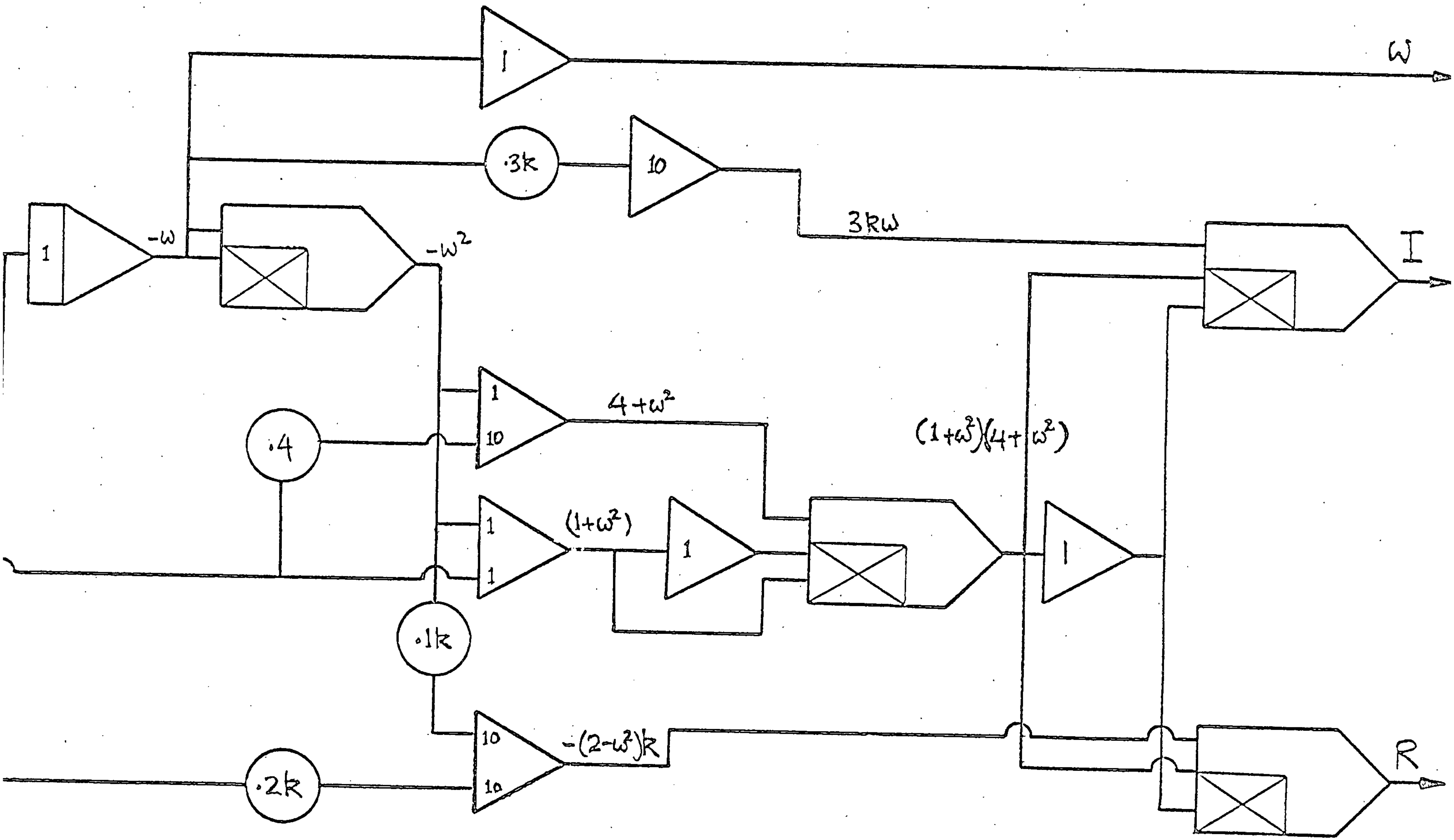
§2.2.4 An Analogue-Computer Technique

This last method makes use of the analogue-computer to test the absolute stability of a nonlinear system.

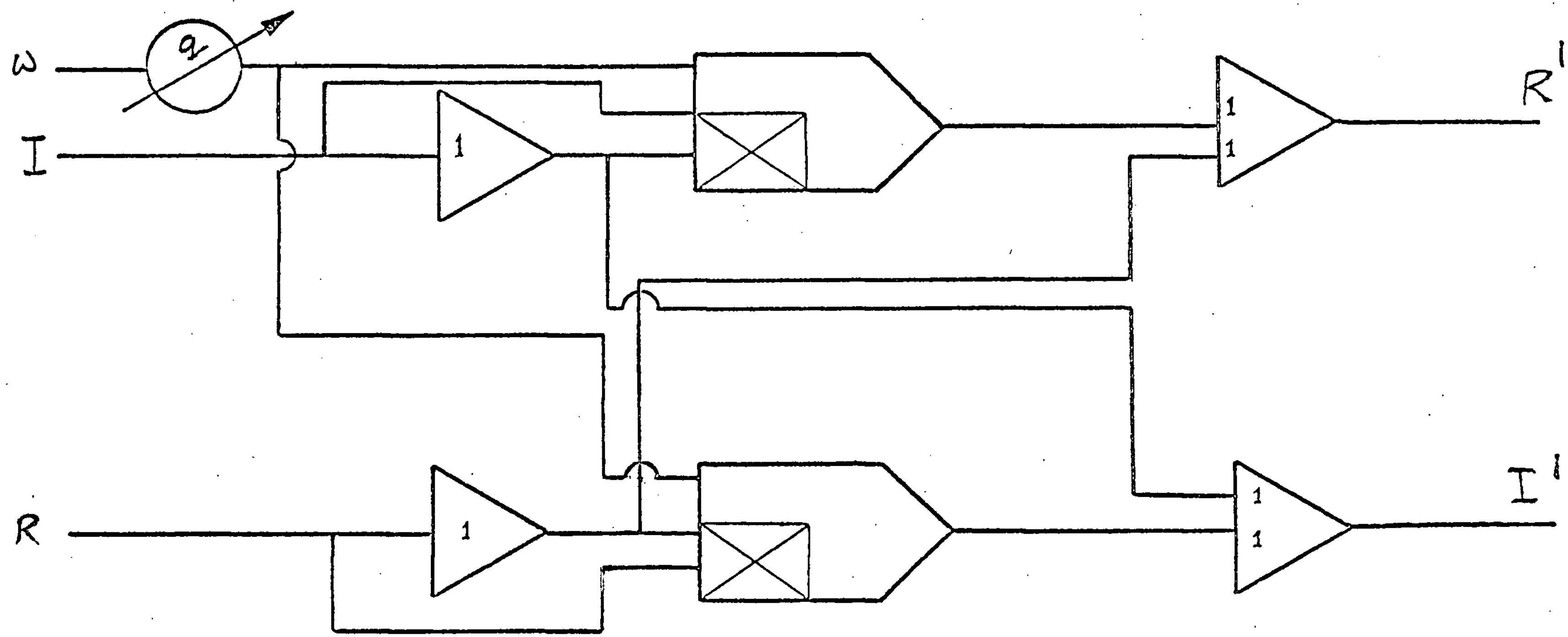
The computer arrangement, which is shown in Fig. 2.10A and Fig. 2.10B, is divided into two main parts. The first generates R and I, and depends on the particular transfer function being considered while the second gives the components of $G(j\omega)$ which are set to remain unchanged for differing systems. The effect of varying q on the $G'(j\omega)$ locus is easily obtained by adjusting a potentiometer.

This method, however, suffers from difficulties in amplitude scaling. Besides, the analogue-computer set-up becomes more complex with increase in system order.

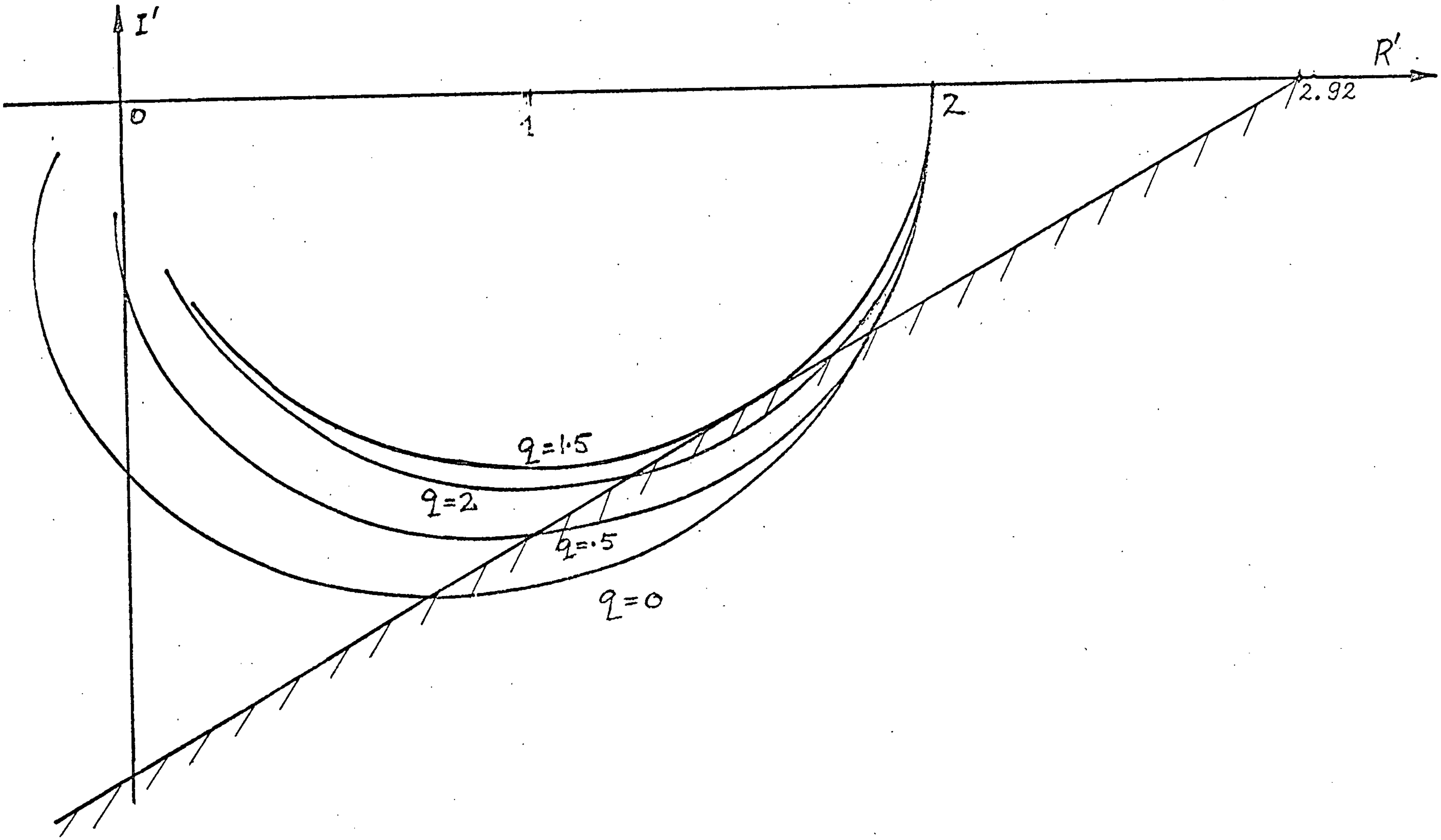
It is also noted that to sketch the complete locus of $G'(j\omega)$ from $\omega_o = 0$ to $\omega_M = \infty$ by the computer is impossible because the describing time is proportional to ω_M. To improve the accuracy of the output it is necessary to use a three- (or two-) stage programming and rescaling technique, and also to limit the ω range which is dependent on the computer characteristics.



2.10A Computer program for R-I generator



2.10B Computer program for X-Y producer



2.11 The loci plotted by analogue computer

Example 2.4

The system (m=3) has linear elements $G_i(s)$ where

$$G_i(s) = \frac{k}{(s+0.5)(s+1)}. \tag{2.2.26}$$

The circuits of the two parts of the analogue computer set-up are shown in Fig. 2.10A and Fig. 2.10B, and the typical resultant loci for various values of q shown in Fig. 2.11 lead to the choice q = 1.5. This therefore permits the choice of $k_{max} = 0.685$.

§2.3 Absolute Stability of the Series Nonlinear System with Different Transfer Functions

Let us consider the case where the identical linear transfer functions in the previous section have been replaced by different linear transfer functions. Now the transfer matrix $\Gamma(s)$ may be written

$$\Gamma(s) = \begin{pmatrix} 0 & 0 & \cdots & 0 & G_m(s) \\ -G_1(s) & 0 & \ddots & & 0 \\ & -G_2(s) & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & 0 \\ 0 & & & -G_{m-1}(s) & 0 \end{pmatrix}. \tag{2.3.1}$$

Let us consider

$$H(j\omega) = (I+j\omega Q)\Gamma(j\omega),$$

where I is a unit matrix and Q is a diagonal constant matrix with elements $q_1, q_2, \ldots q_m$; thus H(s) may be considered as below,

$$H(s) = \begin{pmatrix} 0 & 0 & \cdots & 0 & G'_m(s) \\ G'_1(s) & 0 & \ddots & & 0 \\ & -G'_2(s) & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & 0 \\ 0 & & & -G'_{m-1}(s) & 0 \end{pmatrix}, \tag{2.3.2}$$

where $G_i'(s) = (1+j\omega q_i)G_i(s)$, $i=1,2,\ldots m$. (2.3.3)

Let $|G_i'(j\omega)| = \gamma_i$ and $\angle G_i'(j\omega) = \theta_i$; then the Hermitian matrix $2I + H(j\omega) + H^T(-j\omega)$ becomes

$$\begin{bmatrix} 2 & -\gamma_1 e^{-j\theta_1} & 0 & \cdots & \gamma_m e^{j\theta_m} \\ -\gamma_1 e^{j\theta_1} & 2 & -\gamma_2 e^{-j\theta_2} & \ddots & 0 \\ 0 & -\gamma_2 r^{j\theta_2} & 2 & \ddots & -\gamma_{m-1} e^{-j\theta_{m-1}} \\ \vdots & \ddots & \ddots & \ddots & \\ \gamma_m e^{-j\theta_m} & \cdots\ 0 & & -\gamma_{m-1} e^{j\theta_{m-1}} & 2 \end{bmatrix} \tag{2.3.4}$$

and must be positive definite for all ω to satisfy the stability condition.

The first (m-1) principal minors of Δ_m are generated by the recurrence relation $\Delta_i = 2\Delta_{i-1} - \gamma_{i-1}^2 \Delta_{i-2} > 0$, $i = 3,4,\ldots m-1$, with $\Delta_1 = 2$ and $\Delta_2 = 4 - \gamma_1^2$.

The last condition is

$$\Delta_m = \begin{vmatrix} 2 & -\gamma_1 e^{-j\theta_1} & 0 & \cdots & \gamma_m e^{j\theta_m} \\ -\gamma_1 e^{j\theta_1} & 2 & -\gamma_2 e^{-j\theta_2} & \ddots & \vdots \\ 0 & -\gamma_2 e^{j\theta_2} & 2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -\gamma_{m-1} e^{-j\theta_{m-2}} \\ \gamma_m e^{-j\theta_m} & \cdots\ 0 & & -\gamma_{m-1} e^{j\theta_{m-1}} & 2 \end{vmatrix} > 0, \tag{2.3.5}$$

that is,

$$\Delta_m = 2\Delta_{m-1} - (\gamma_{m-1}^2 + \gamma_m^2)\Delta_{m-2} + (-1)^m (\prod_{i=1}^m \gamma_i) e(\sum_{i=1}^m \theta_i)$$

$$+ (-1)^m (\prod_{i=1}^m \gamma_i) e^{(-\sum_{i=1}^m \theta_i)}$$

$$= 2\Delta_{m-1} - (\gamma_{m-1}^2 + \gamma_m^2)\Delta_{m-2} + (-1)^m (\prod_{i=1}^m \gamma_i) 2 \cos(\sum_{i=1}^m \theta_i) > 0. \tag{2.3.6}$$

If all $\gamma_i < 1$ for all ω, $i=1,2,\ldots m$, then

$$\Delta_2 = 4-\gamma_1^2 > 3,$$

$$\Delta_3-\Delta_2 = \Delta_2-\gamma_2^2\Delta_1 > \Delta_2-\Delta_1 = (4-\gamma_1^2) - 2 > 1,$$

$$\Delta_3 > \Delta_2+1 = 4-\gamma_1^2+1 > 4,$$

$$\Delta_4-\Delta_3 = \Delta_3-\gamma_3^2\Delta_2 > \Delta_3-\Delta_2 > 1,$$

$$\Delta_4 > \Delta_3+1 > 5,$$

$$\vdots$$

$$\Delta_i-\Delta_{i-1} = \Delta_{i-1}-\gamma_{i-1}^2\Delta_{i-2} > \Delta_{i-1}-\Delta_{i-2} > 1,$$

$$\Delta_i > \Delta_{i-1}+1 > i+1,$$

$$\vdots$$

$$\Delta_{m-1}-\Delta_{m-2} = \Delta_{m-2}-\gamma_{m-2}^2\Delta_{m-3} > \Delta_{m-2}-\Delta_{m-3} > 1,$$

$$\Delta_{m-1} > \Delta_{m-2}+1 > m,$$

and

$$\Delta_m > 2\Delta_{m-1}-2\Delta_{m-2}+(-1)^m 2\cos\left(\sum_{i=1}^{m} \theta_i\right)$$

$$> 2+(-1)^m 2\cos\left(\sum_{i=1}^{m} \theta_i\right) \geq 0. \qquad (2.3.8)$$

Therefore, the absolute stability of a nonlinear system is assured if every locus of $G_i'(j\omega)$ lies within a circle of unit radius. Obviously, q_i must be chosen zero so that

$$G_i'(j\omega) = G_i(j\omega).$$

In order to test absolute stability of nonlinear system it is thus necessary to sketch the loci of $G_i(j\omega)$ and to observe whether all of the loci lie within the unit circle.

As $m \to \infty$, this result coincides with the previous result in §2.2, but here the $G_i(j\omega)$'s are not necessarily the same.

§2.4 Absolute Stability of the Parallel Nonlinear System

The transfer matrix of the parallel nonlinear system shown in Fig. 1.4 may be written

$$\Gamma(s) = \begin{bmatrix} G_1(s) & G_2(s) & \cdots & G_m(s) \\ G_1(s) & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ G_1(s) & G_2(s) & \cdots & G_m(s) \end{bmatrix}. \tag{2.4.1}$$

Suppose that $G_i(s) = G(s)$, $i=1,2,\ldots m$, and

$$G'(j\omega) = (1+j\omega q)G(j\omega),$$

then

$$H(j\omega) = \begin{bmatrix} G_1'(j\omega) & G_2'(j\omega) & \cdots & G_m'(j\omega) \\ G_1'(j\omega) & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ G_1'(j\omega) & G_2'(j\omega) & \cdots & G_m'(j\omega) \end{bmatrix}$$

$$= G'(j\omega)\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}. \tag{2.4.2}$$

Define

$$G(j\omega) = R(\omega) + jI(\omega), \tag{2.4.3A}$$

$$R'(j\omega) = R(\omega) + qI(\omega), \tag{2.4.3B}$$

and

$$I'(j\omega) = q\omega R(\omega) + I(\omega). \tag{2.4.3C}$$

Let us suppose the i-th nonlinearity satisfies

$$0 < \sigma_i \emptyset_i(\sigma_i) < \sigma_i^2. \qquad (2.2.2)$$

The Hermitian matrix is

$$2I + 2R' \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & & & \vdots \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2(1+R') & 2R' & \cdots & 2R' \\ 2R' & 2(1+R') & \ddots & \vdots \\ \vdots & \ddots & \ddots & 2R' \\ 2R' & \cdots & 2R' & 2(1+R') \end{pmatrix}. \qquad (2.4.4)$$

The sufficient condition of absolute stability is that the Hermitian matrix must be positive definite for all ω; consequently, $\Delta_i > 0$. Now

$$\Delta_i = \begin{vmatrix} 2(1+R') & 2R' & \cdots & 2R' \\ 2R' & 2(1+R') & \ddots & \vdots \\ \vdots & \ddots & \ddots & 2R' \\ 2R' & \cdots & 2R' & 2(1+R') \end{vmatrix}$$

$$= \begin{vmatrix} 2(1+R') & -2 & -2 & \cdots & -2 \\ 2R' & 2 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 2R' & 0 & \cdots & 0 & 2 \end{vmatrix}$$

$$= 2\Delta_{i-1} + 2^i R'$$

$$= 2(2\Delta_{i-2} + 2^{i-1}R') + 2^i R'$$

$$= 2^{i-1}\Delta_1 + (i-1)2^i R'$$

$$= 2^i(1+iR') > 0. \tag{2.4.5}$$

Substituting (2.4.3B) in (2.4.5),

$$R(\omega) - q\omega I(\omega) + \frac{1}{i} > 0. \tag{2.4.6}$$

Define

$$X(\omega) = R(\omega), \tag{2.4.7A}$$

and

$$Y(\omega) = \omega I(\omega). \tag{2.4.7B}$$

(2.4.7A) and (2.4.7B) give

$$X(\omega) - qY(\omega) + \frac{1}{i} > 0. \tag{2.4.8}$$

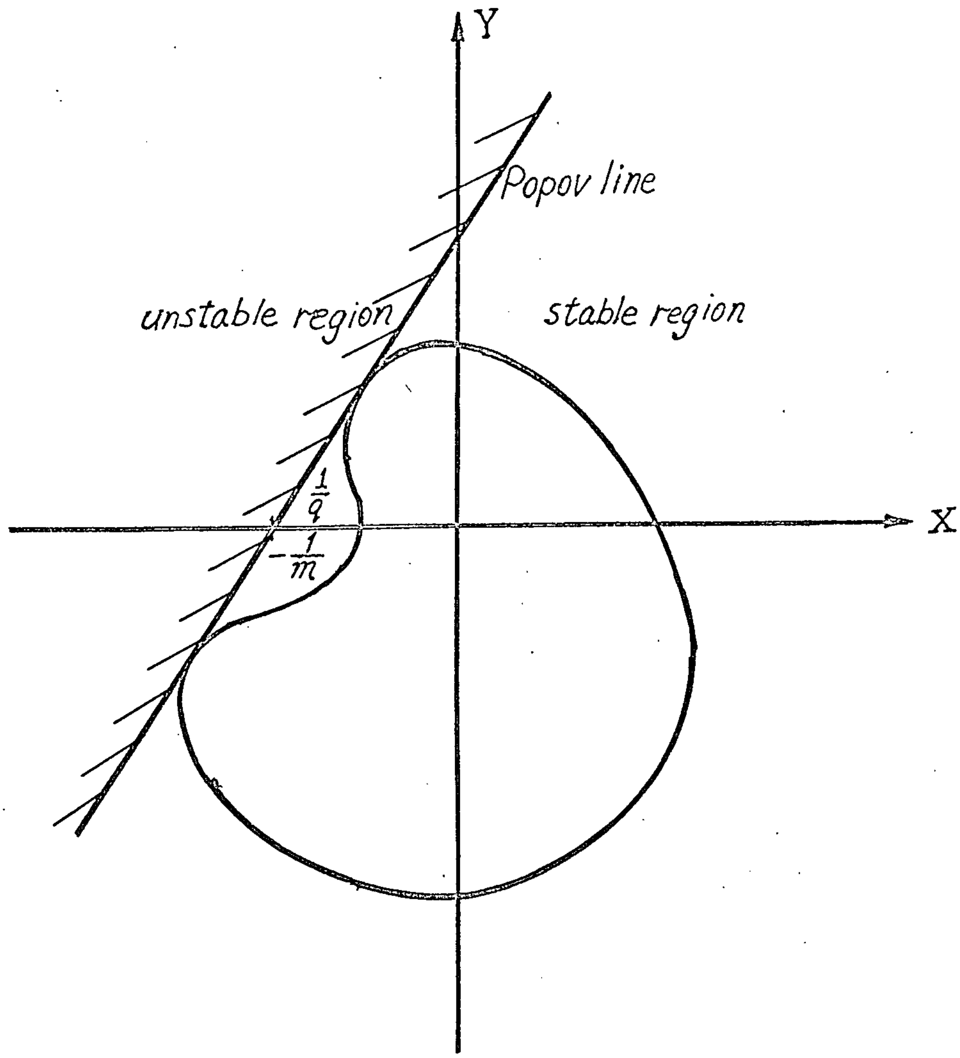
The condition

$$\Delta_m = X(\omega) - qY(\omega) + \frac{1}{m} > 0. \tag{2.4.9}$$

implies that

$$\Delta_i > 0, \quad i = 1,2,...m-1. \qquad (2.4.5)$$

Hence, the new Popov line, shown in Fig. 2.12, passes through the point $(-\frac{1}{m}, 0)$ with slope $\frac{1}{q}$.



2.12 Extension of Popov criterion for the parallel system with m identical nonlinearities and m linear transfer functions

Chapter 3 ABSOLUTE STABILITY OF A TIME VARYING FEEDBACK SYSTEM WITH MONOTONIC NONLINEARITIES

§3.1 The Absolute Stability of the Single-Loop Time Varying Nonlinear Feedback System

In recent years some results concerning the absolute stability of a single-loop nonlinear system with a time varying gain have been obtained by Rozenvasser [5], Zames [22], Bergen and Rault [12]. The results to be presented here extend this previous work.

Let us consider single-loop time varying nonlinear system shown in Fig. 3.1.

In that system,

$$\begin{aligned} \sigma_e(t) &= \int_o^t g(t-\tau)\ e(\tau)d\tau \quad \text{for } t \geq 0 \\ &= 0 \qquad\qquad\qquad\qquad\quad \text{for } t < 0 \end{aligned} \tag{3.1.1}$$

is the zero-state response of the linear time invariant part with transfer function $G(j\omega) = \mathfrak{F}[g(t)]$. The input $\eta(t)$ represents the zero-input response of $g(t)$. The complete response of $g(t)$ is thus

$$c(t) = \sigma_e(t) + \eta(t). \tag{3.1.2}$$

The input, $\sigma(t) = -c(t)$, and the output $\emptyset(\sigma)$ of the amnesic nonlinearity N.L. are related in the following manner:

$$\begin{aligned} &(1)\quad 0 < \sigma\emptyset(\sigma) < \sigma^2 \quad \text{for } \sigma \neq 0,\ \emptyset(0) = 0, \\ &(2)\quad 0 < \frac{\emptyset(\sigma_1) - \emptyset(\sigma_2)}{\sigma_1 - \sigma_2} \leq 1 \quad \text{for } \sigma_1 \neq \sigma_2. \end{aligned} \tag{3.1.3}$$

The block, $k(t)$, represents a linear time varying gain, thus

$$e(t) = k(t)\emptyset[\sigma(t)].$$

The instantaneous value of this gain is constrained so that

(1) $K_1 < k(t) < K_2$, where $K_2 > K_1 > 0$,

(2) $bk(t) \leq \dot{k}(t) < ak(t)$, where the number $a > 0$, and the number b

is finite.

It is assumed that the linear part is stable, more specifically,

(1) $g(t) \varepsilon L_2(0,\infty)$, $\dot{g}(t) \varepsilon L_1(0,\infty)$,

(2) $\eta(t) \varepsilon L_1(0,\infty)$,

(3) $\eta(t)$ is differentiable and $\dot{\eta}(t) \varepsilon L_1(0,\infty)$.

Condition (1) above ensures that $g(t)$ is bounded on $(0,\infty)$ and that $g(t) \to 0$ as $t \to \infty$; besides, conditions (2) and (3) ensure that $\eta(t)$ behaves in the same manner.

Denote

$$\eta(t)_M = \sup_{t \geq 0} |\eta(t)|,$$

$$g(t)_M = \sup_{t \geq 0} |g(t)|.$$

The Fourier transforms of $g(t)$, $e(t)$, etc., are denoted by $G(j\omega)$ $E(j\omega)$, etc. The notation $\| \cdot \|$ denotes norms in the space $L_1(0,\infty)$. Thus

$$\|\eta(t)\| = \int_0^\infty |\eta(t)| dt.$$

§3.2 The Main Result

The main result is the following theorem.

§3.2.1 Theorem 1

Consider the system shown in Fig. 3.1 to which the assumptions made above apply. Let $y(t)$ be any real function such that

(1) $y(t) = 0$ for $t < 0$,

(2) $y(t) \leq 0$ for $t \geq 0$,

(3) $\|y(t)\| < \frac{K_1}{K_2}$,

and let q be any nonnegative number. If

$$\mathrm{Re}\{[1+qj\omega+Y(j\omega)][G(j\omega)+\frac{1}{K_2}]+aqG(j\omega)\}$$
$$-\|y(t)\| (\frac{1}{K_1} - \frac{1}{K_2}) \geq 0 \qquad \text{(Q1)}$$

for all ω, then

(1) $\sup_{t \geq 0} |\sigma(t)| < \infty$,

(2) $\sigma(t) \to 0$ as $t \to \infty$,

(3) as $\|\eta(t)\| + \|\dot{\eta}(t)\| \to 0$, the corresponding $\sigma(t)$ has the property that $\sup_{t \geq 0} |\sigma(t)| \to 0$.

§3.2.2 A Special Case of the Theorem

It should be noted that if the time varying gain k(t) is monotonically non-increasing and $a \leq 0$ for all $\forall t \geq 0$, then the condition (Q1) for absolute stability may be replaced by

$$\mathrm{Re}[1+qj\omega+Y(j\omega)][G(j\omega)+\frac{1}{K_2}] - \|y(t)\| (\frac{1}{K_1} - \frac{1}{K_2}) - \frac{a}{2K_2} \geq 0 \qquad \text{(Q1')}$$

for all ω,

where again, q is any nonnegative real number.

§3.3 Proof of Main Result

The body of the proof of Theorem 1 will be given in a series of appendices; a brief summary is given below in this section.

Define

$$\sigma_T(t) = \begin{cases} \sigma(t) & \text{for} \quad t \leq T \\ 0 & \text{for} \quad t \geq T, \end{cases}$$

$$e_T(t) = k(t)\emptyset[\sigma_T(t)],$$

and

$$\sigma_{eT}(t) = \int_0^t g(t-\tau)\, e_T(\tau)d\tau. \qquad (3.3.1)$$

Thus

$$\sigma_{eT}(t) = \sigma_e(t) \quad \text{for} \quad t \leq T,$$

and

$$\sigma_{eT}(t) \in L_1(T,\infty).$$

The notation $(x*y)(t)$ denotes convolution between $x(t)$ and $y(t)$;

$$(x*y)(t) = \int_0^t x(\tau)y(t-\tau)d\tau. \tag{3.3.2}$$

Define

$$\sigma_m = \sigma + \sigma*y, \quad c_m = c + c*y.$$

Then

$$\int_0^T [\sigma_m(t) - \frac{e_m(t)}{K_2}]e(t)dt$$

$$= \int_0^T[\sigma(t) - \frac{e(t)}{K_2}]e(t)dt + \int_0^T[y*(\sigma - \frac{e}{K_2})(t)e(t)dt. \tag{3.3.3}$$

Define

$$R(\tau) = \int_0^T[\sigma_T(t-\tau) - \frac{k(t-\tau)\emptyset[\sigma_T(t-\tau)]}{K_2}]k(t)\emptyset[\sigma_T(t)]dt. \tag{3.3.4}$$

Now

$$R(\tau) = R_1(\tau) + R_2(\tau), \tag{3.3.5}$$

where

$$R_1(\tau) = \int_0^\infty [\sigma_T(t-\tau)-\emptyset[\sigma_T(t-\tau)]]k(t)\emptyset[\sigma_T(t)]dt, \tag{3.3.6}$$

and

$$R_2(\tau) = \int_0^\infty[1 - \frac{k(t-\tau)}{K_2}]k(t)\emptyset[\sigma_T(t-\tau)]\emptyset[\sigma_T(t)]dt. \tag{3.3.7}$$

Frem Appendix 1,

$$R_1(\tau) \leq \int_0^\infty \frac{K_2}{K_1}[\sigma_T(t) - \frac{k(t)\emptyset[\sigma_T(t)]}{K_2}]k(t)\emptyset[\sigma_T(t)]dt, \tag{3.3.8}$$

and from Appendix 2,

$$R_2(\tau) \leq \int_0^\infty(\frac{1}{K_1} - \frac{1}{K_2})e_T(t)^2dt. \tag{3.3.9}$$

Thus

$$R(\tau) \leq \int_0^\infty[\frac{K_2}{K_1}\sigma_T(t) - \frac{e_T(t)}{K_2}]e_T(t)dt. \tag{3.3.10}$$

The first term of the right side of (3.3.3) is always positive. Let us now consider the second term of the right side of (3.3.3)

$$\int_0^T[y*(\sigma - \frac{e}{K_2})](t)dt$$

$$= \int_0^T \int_0^t y(\tau)[\sigma(t-\tau) - \frac{e(t-\tau)}{K_2}]e(t)d\tau\, dt$$

$$= \int_0^\infty \int_0^\infty y(\tau)[\sigma_T(t-\tau) - \frac{e_T(t-\tau)}{K_2}]e_T(t)d\tau\, dt$$

$$= \int_0^\infty y(\tau)R(\tau)d\tau. \tag{3.3.11}$$

But, employing (3.3. 10), it may be shown that

$$\int_0^T [y*(\sigma - \frac{e}{K_2})](t)e(t)dt$$

$$\geq \int_0^\infty y(\tau)d\tau \int_0^\infty [\frac{K_2}{K_1}\sigma_T(t) - \frac{e_T(t)}{K_2}]e_T(t)dt \tag{3.3.12}$$

$$\geq -\|y\| \int_0^\infty [\frac{K_2}{K_1}\sigma_T(t) - \frac{e_T(t)}{K_2}]e_T(t)dt. \tag{3.3.13}$$

Substitution of (3.3.13) in (3,3.3) yields

$$\int_0^T [\sigma_m(t) - \frac{e_m(t)}{K_2}]e(t)dt + (\frac{1}{K_1} - \frac{1}{K_2})\|y\| \int_0^T e_T(t)^2 dt$$

$$\geq (1 - \frac{K_2}{K_1}\|y\| \int_0^T [\sigma_T(t) - \frac{e_T(t)}{K_2}]e_T(t)dt \tag{3.3.14}$$

and, from the assumptions of the Theorem 1, the right side of (3.3.14) is non-negative. Hence,

$$\int_0^T [\sigma_m(t) - \frac{e_m(t)}{K_2}]e(t)dt + (\frac{1}{K_1} - \frac{1}{K_2})\|y\| \int_0^T e_T(t)^2 dt \geq 0. \tag{3.3.15}$$

Consider the following integral

$$I = \int_0^T [-\sigma_{em}(t) - q\dot{\sigma}_e(t) - \frac{e_m(t)}{K_2} - aq\sigma_e(t) + (\frac{1}{K_1} - \frac{1}{K_2})\|y\| e(t)]e(t)dt$$

$$= I_1 + I_2, \tag{3.3.16}$$

where

$$I_1 = \int_0^T [-\sigma_{em}(t) - \frac{e_m(t)}{K_2} + (\frac{1}{K_1} - \frac{1}{K_2})\|y\| e(t)]e(t)dt, \tag{3.3.17}$$

and

$$I_2 = \int_0^T - q[\dot{\sigma}_e(t) + a\sigma_e(t)]e(t)dt. \tag{3.3.18}$$

Now

$$I_1 = \int_0^T [\sigma_m(t) - \frac{e_m(t)}{K_2} + (\frac{1}{K_1} - \frac{1}{K_2}) \|y\| e(t)]e(t)dt$$

$$+ \int_0^T \eta_m(t)e(t)dt \tag{3.3.19}$$

giving, due to (3.3.15)

$$I_1 \geq \int_0^T \eta_m(t)e(t)dt. \tag{3.3.20}$$

In a similar manner,

$$I_2 = \int_0^T q[\dot{\sigma}(t)+a\sigma(t)]e(t)dt+\int_0^T q[\dot{\eta}(t)+a\eta(t)]e(t)dt. \tag{3.3.21}$$

Invoking the result of Appendix 3,

$$I_2 \geq q[k(T)\Phi(T)-k(0)\Phi(0)+\int_0^T q[\dot{\eta}(t)+a\eta(t)]e(t)dt. \tag{3.3.22}$$

Recall that

$$\int_0^T f(t)dt = \int_0^\infty f_T(t)dt,$$

where $f_T(t)$ is the truncated version of $f(t)$ to between 0 and T. Hence from (3.3.16),

$$I = \int_0^\infty [-\sigma_{eTm}(t) - q\dot{\sigma}_{eT}(t) - \frac{e_{Tm}(t)}{K_2} - qa\sigma_{eT}(t)$$

$$+ (\frac{1}{K_1} - \frac{1}{K_2}) \|y\| e_T(t)]e_T(t)dt \tag{3.3.23}$$

and, from the conditions of Theorem 1 and since

$$I = -\frac{1}{2\pi} \int_{-\infty}^\infty \mathrm{Re}\{[1+qj\omega+Y(j\omega)][G(j\omega) + \frac{1}{K_2}] + aqG(j\omega)$$

$$-(\frac{1}{K_1} - \frac{1}{K_2}) \|y\| \}E_T(j\omega)E_T^*(j\omega)d\omega,$$

it follows that

$$I \leq 0. \tag{3.3.24}$$

Since $I = I_1 + I_2$, from Appendix 3 and after substituting (3.3.20) in (3.3.24),

$$q[k(T)\Phi(T)-k(0)\Phi(0)] \leq -\int_0^T [aq\eta(t)+\eta_m(t)+q\eta(t)]e(t)dt. \tag{3.3.25}$$

Define

$$e_M = \sup_{0 < t < T} |e(t)|$$

and invoking the conditions already imposed upon $\eta(t)$, $\dot{\eta}(t)$ and $y(t)$, the right side of (3.3.25) must be less than the quantity

$$e_M[(1+\|y\|+aq)\|\eta\|+q\|\dot{\eta}\|] = Me_M. \tag{3.3.26}$$

Furthermore, considering the first term of the left side of (3.3.25), since $\emptyset$ is monotonic,

$$\Phi(t) \geq \frac{1}{2}\{\emptyset[\sigma(t)]\}^2. \tag{3.3.27}$$

Using (3.3.26) and (3.3.27) in (3.3.25) yields

$$\frac{q}{2}k(T)\{\emptyset[\sigma(T)]\}^2 \leq qk(0)\Phi(0) + Me_M. \tag{3.3.28}$$

From (3.3.28), since $\frac{K(T)}{K_2} \leq 1$,

$$\frac{q}{2K_2}e(T)^2 \leq qk(0)\Phi(0) + Me_M. \tag{3.3.29}$$

The inequality (3.3.29) holds for any $T \geq 0$ and implies

$$\sup_{t\geq 0}|e(t)| \leq \frac{K_2M}{q} + [(\frac{K_2M}{q})^2 + 2K_2k(0)\Phi(0)]^{1/2}. \tag{3.3.30}$$

Furthermore, since $\sigma_e(0) = 0$ by (3.3.1), this bound on $e(t)$ tends to zero with $\|\eta\| + \|\dot{\eta}\|$.

It remains to be shown that $|e(t)| \to 0$ as $t \to \infty$. After substituting in (3.3.14) of (A 3.3) of Appendix 3 and using (3.3.24),

$$(1 - \frac{K_2}{K_1}\|y\|)\int_0^T[\sigma(t) - \frac{e(t)}{K_2}]e(t)dt + \int_0^T\eta_m(t)e(t)dt$$

$$-qk(0)\Phi(0) + q\int_0^T[\dot{\eta}(t)+a\eta(t)]e(t)dt \leq 0. \tag{3.3.31}$$

Thus

$$(1 - \frac{K_2}{K_1}\|y\|)\int_0^T[\sigma(t) - \frac{e(t)}{K_2}]e(t)dt \leq qk(0)\Phi(0) + Me_M. \tag{3.3.32}$$

From which it follows that

$$\int_0^T[\sigma(t) - \frac{e(t)}{K_2}]e(t)dt \leq \frac{qk(0)\Phi(0) + Me_M}{1 - \frac{K_2}{K_1}\|y\|}. \tag{3.3.33}$$

Since right side of (3.3.33) is independent of T, then letting $T \to \infty$,

$$\int_0^\infty [\sigma(t) - \frac{e(t)}{K_2}] e(t)dt \le \frac{qk(0)\Phi(0) + Me_M}{1 - \frac{K_2}{K_1} \quad \|y\|} , \qquad (3.3.34)$$

and this bound on the integral tends to zero with $\|\eta\| + \|\dot{\eta}\|$.

However, the bound already placed on e together with the conditions demanded of $\dot{g}(t)$ require $\dot{\sigma}_e(t)$ to be bounded and to tend to zero. It may now be shown that the integral of (3.3.34) is infinite unless $\sigma_e(t) \to 0$ as $t \to \infty$, thus contradicting (3.3.34). Therefore it can be concluded that $\sigma(t) \to 0$ as $t \to \infty$.

Since $\sigma_e(t)$ is uniformly continuous and if $\sup_{t>0} |\sigma(t)|$ does not go to zero as $\|\eta\| + \|\dot{\eta}\| \to 0$, then (3.3.34) does not tend to zero either. This is a contradiction. Hence, as $\|\eta\| + \|\dot{\eta}\| \to 0$, $\sup_{t\geq 0} |\sigma(t)| \to 0$. The proof is therefore completed.

§3.4 Examples

Example 3.1

In the single-loop nonlinear system with a linear time varying gain shown in Fig. 3.1, the linear part has a transfer function

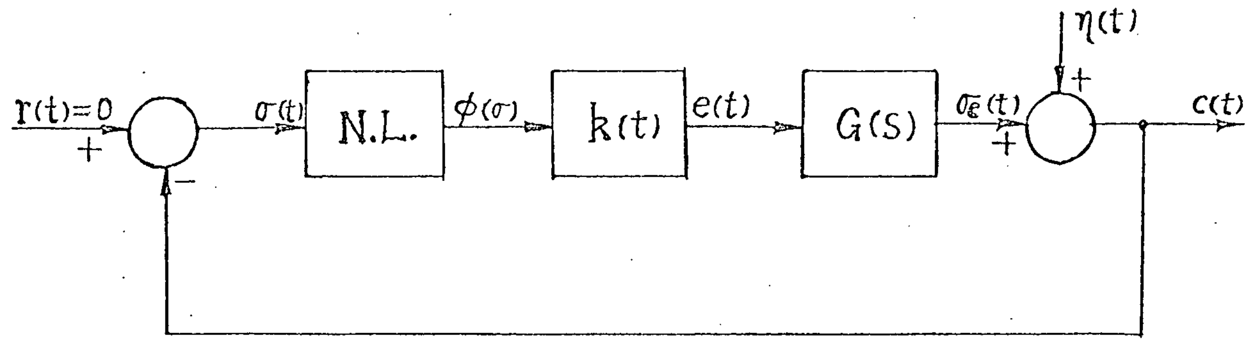
$$G(s) = \frac{K}{(s+1)(s+2)} ,$$

the input and output of the nonlinear part satisfy (3.1.2), and the time varying gain is such that

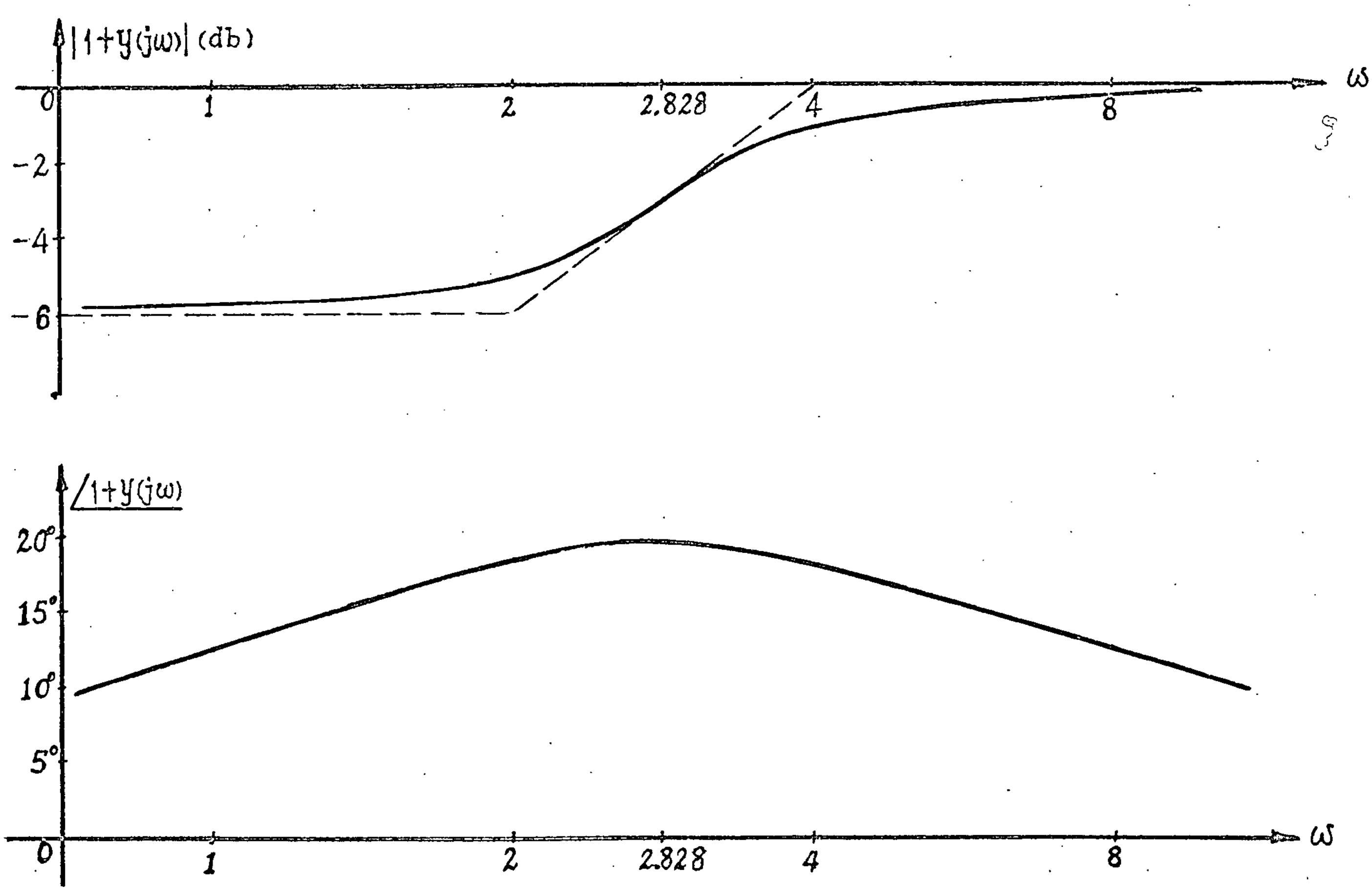
(1) $1 \le k(t) \le 1.2$, i.e., $K_1 = 1$, $K_2 = 1.2$,

(2) $bk(t) \le \dot{k}(t) \le ak(t)$, where a,b are real numbers such that $a > 0$, and b is finite.

Suppose that a is large enough, q must be chosen zero. Let us assume that y(t) is an exponential function such that



3.1 Single loop time varying nonlinear feedback system with zero-input



3.2 Bode diagram for a compensator $1+y(s) = \frac{2+s}{4+s}$

$$y(t) = \begin{cases} -\frac{1}{\gamma} e^{-\frac{\beta}{\gamma}t} & \text{as} \quad t \geq 0 \\ 0 & \text{as} \quad t > 0, \end{cases}$$

where $\gamma > 0$, and $\beta > 1.2 = \frac{K_2}{K_1}$. Then

$$||y(t)|| = \int_0^\infty \left| -\frac{1}{\gamma} e^{-\frac{\beta}{\gamma}t} \right| dt = \frac{1}{\beta} < \frac{K_1}{K_2} .$$

Thus the conditions on $y(t)$ are satisfied. After taking the Laplace transform of $y(t)$,

$$1 + Y(s) = 1 - \frac{1}{\gamma} \frac{1}{s + \frac{\beta}{\gamma}} = \frac{(\beta-1) + \gamma s}{\beta + \gamma s} . \tag{3.4.1}$$

The sufficient condition of absolute stability is

$$\mathrm{Re}\{ [\frac{(\beta-1) + \gamma j\omega}{\beta + \gamma j\omega}][G(j\omega) + \frac{1}{1.2}] - \frac{1}{\beta} \frac{0.2}{1.2}\} \geq 0 \tag{3.4.2}$$

for all ω.

Let us define

$$G_1(j\omega) = G(j\omega) + \frac{1}{1.2}$$

and

$$G_1'(j\omega) = [\frac{(\beta-1) + \gamma j\omega}{\beta + \gamma j\omega}] G_1(j\omega).$$

From the relation (3.4.2), the locus of $G_1'(j\omega)$ must lie on the right side of the vertical line passing through the point $(\frac{0.167}{\beta}, 0)$. The multiplier of $G_1(j\omega)$ in $G_1'(j\omega)$ may be considered as a compensator which is shown on the Bode diagram Fig. 3.2. The function of the compensator is to improve the characteristics of $G_1(j\omega)$. From the plot of $G_1(j\omega)$ in Fig. 3.3, it is a simple matter to choose the proper values of β and γ. From the fact that the left-most point $\omega_1 = 2.5$ on the locus of $G_1(j\omega)$ lies between $\frac{\beta-1}{\gamma}$ and $\frac{\beta}{\gamma}$ and that β satisfied the relation $\frac{1}{\beta}[\frac{1}{K_1} - \frac{1}{K_2}] << \frac{1}{K_2}$, the stability condition

$$K \leq 23.75 \tag{3.4.3}$$

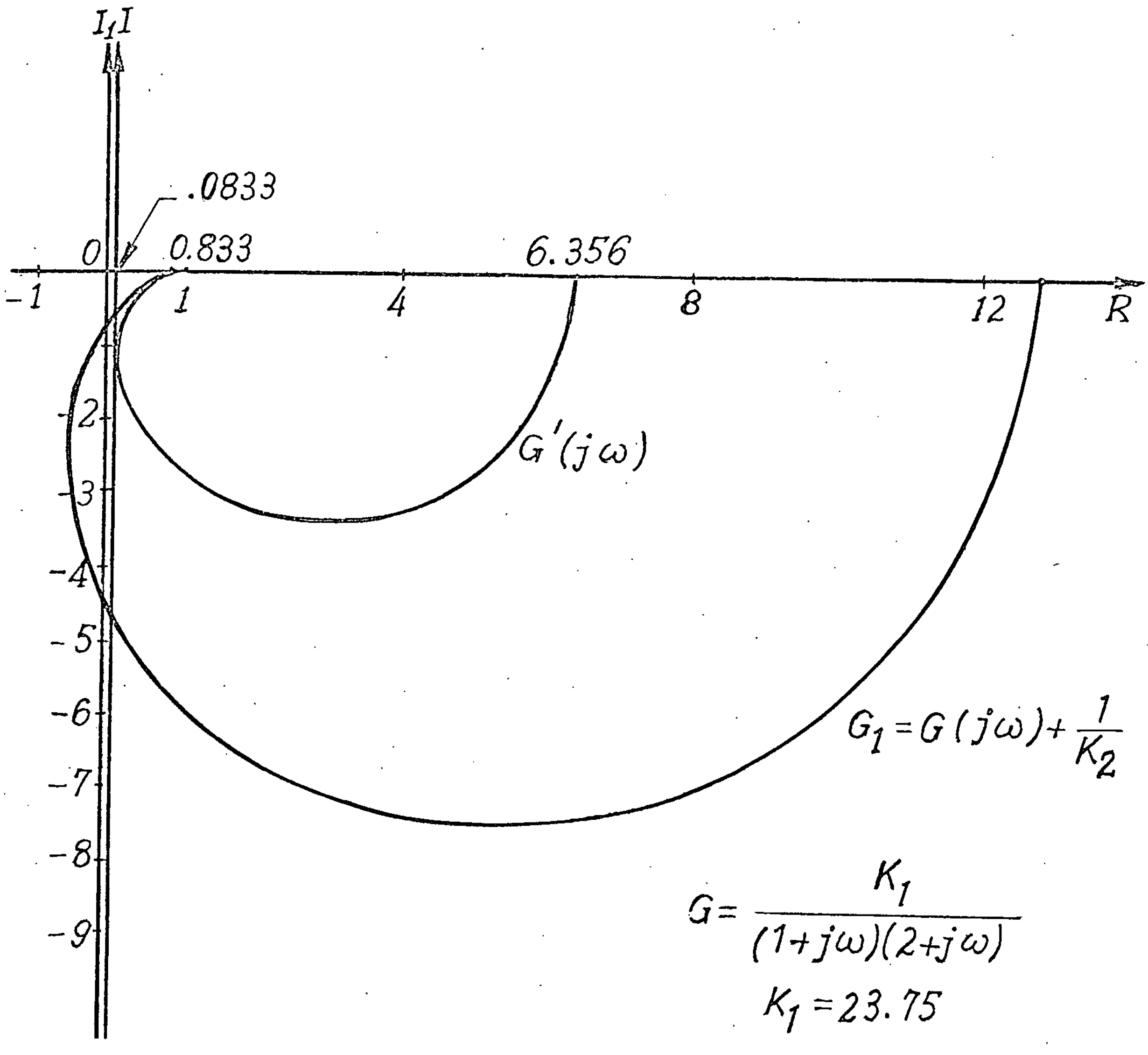


Fig. 3.3 The plots of $G(s) = \frac{23.75}{(s+1)(s+2)}$ and $G'(s) = (\frac{2+s}{4+s})G(s)$ for Example 3.1.

has been found by running a suitably written programme on the digital computer.

It is noted that the combination $\beta = 2$ and $\gamma = 0.5$ is not the best one because of the particular choice of y(t). That optimum y(t) which gives the best combination of β and γ may be determined by a digital computer technique.

Example 3.2

Consider

$$G(s) = \frac{K}{(s+1)(s+D)},$$

where D is any positive real constant, instead of the G(s) in the previous problem and let

$$0 < k(t) \leq 1,$$

and

$$bk(t) \leq \dot{k}(t) \leq ak(t).$$

In this case, let us suppose $y(t) \equiv 0$, then the condition of absolute stability is

$$\mathrm{Re}\{(1+qj\omega)[G(j\omega)+1]+aqG(j\omega)\} \geq 0 \qquad (3.4.4)$$

for all ω. Rewritting,

$$R(\omega) - q\omega I(\omega) + 1 + aqR(\omega) \geq 0 \qquad (3.4.5)$$

for all ω, since

$$G(j\omega) = R(\omega) + jI(\omega). \qquad (3.4.6)$$

Here

$$R(\omega) = \frac{K_1(D-\omega^2)}{(1+\omega^2)(D^2+\omega^2)}, \qquad (3.4.7A)$$

and

$$I(\omega) = \frac{-K_1(1+D)\omega}{(1+\omega^2)(D^2+\omega^2)} \qquad (3.4.7B)$$

Define

$$X(\omega) = R(\omega) \qquad (3.4.8A)$$

and

$$Y(\omega) = \omega I(\omega) - aR(\omega). \qquad (3.4.8B)$$

Substituting (3.4.8A) and (3.4.8B) in (3.4.5),

$$X(\omega) - qY(\omega) + 1 \geq 0. \qquad (3.4.9)$$

To satisfy inequality (3.4.9), the locus must be on the right side of the straight line passing through the point (-1,0) having positive slope $\frac{1}{q}$.

From (3.4.8A), (3.4.8B) and (3.4.9), if $a \leq 1+D-\varepsilon$, where the small number $\varepsilon > 0$ is arbitrarily chosen, and q is $\frac{1}{\varepsilon}$, then the stability condition (Q1) is satisfied for any nonnegative real constant k. Besides, the modified Nyquist plot is on the right side of the Popov line. This is shown in Fig. 3.4.

If $a \lessgtr 0$, the sufficient condition of absolute stability is

$$R(\omega) - q\omega I(\omega) + 1 \geq 0. \qquad (3.4.10)$$

This is the Popov criterion and is satisfied for any nonnegative real constant k by choosing $q = \frac{1}{1+D}$.

§3.5 Absolute Stability of a System with Many Nonlinearities and Many Time Varying Gains

In the previous sections, the absolute stability of the system with one nonlinearity and one time varying gain is established. Now, let us consider a system with many nonlinearities and many time varying gains. Such a system is shown in Fig. 3.5. The input σ_i and the output $\emptyset_i(\sigma_i)$ of the i-th nonlinearity are related by the following:

$$\begin{aligned} &(1)\quad 0 < \sigma_i \emptyset_i(\sigma_i) \leq \sigma_i^2 \quad \text{for} \quad \sigma_i \neq 0, \quad \emptyset_i(0) = 0, \\ &(2)\quad 0 < \frac{d\emptyset_i(\sigma_i)}{d\sigma_i} \leq 1, \end{aligned} \qquad (3.5.1)$$

and the instantaneous value of the i-th time varying gain is constrained so that

(1) $K_{1i} \leq k_i(t) \leq K_{2i}$, where $K_{2i} > K_{1i} > 0$,

(2) $b_i k(t) \leq \dot{k}_i(t) \leq a_i k_i(t)$, where the number $a_i > 0$ and the number b_i is finite.

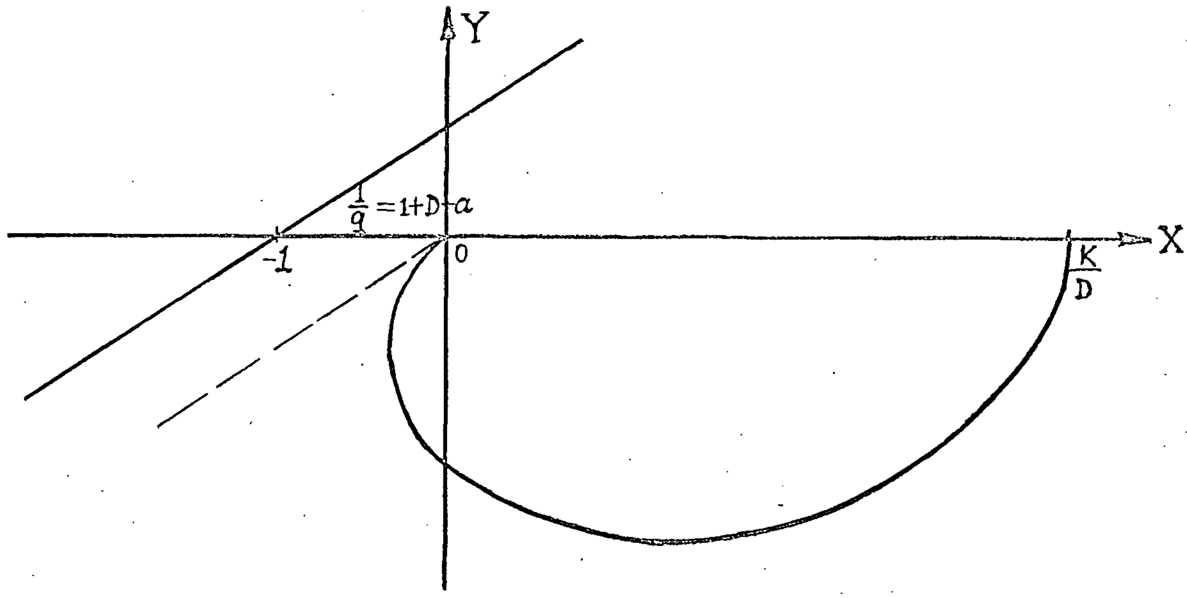


Fig. 3.4 Modified Nyquist plot of $G(s) = \frac{K}{(s+1)(s+D)}$ for Example 3.2.

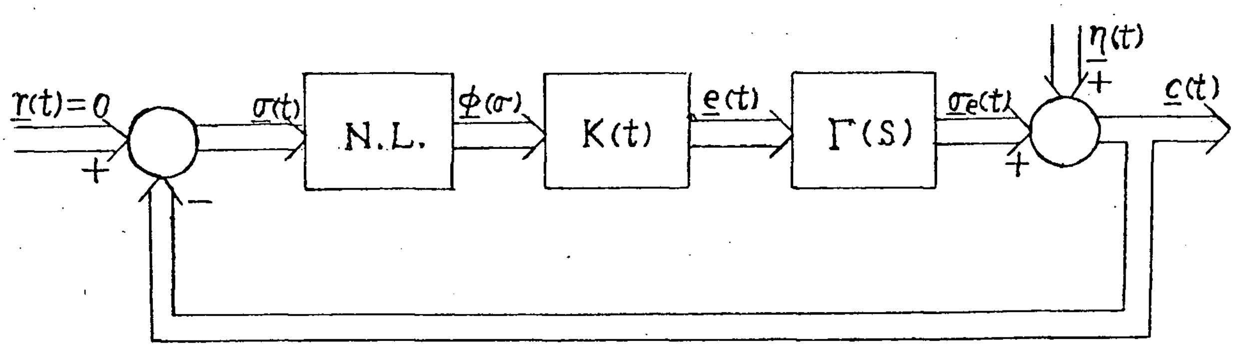


Fig. 3.5 General time varying nonlinear feedback system with zero-input

Besides,

$$\sigma_{\underline{e}}(t) = \int_0^t \Gamma(t-\tau)\underline{e}(\tau)d\tau$$

is the zero state response of the linear time invariant transfer matrix $\Gamma(j\omega) = \mathcal{F}[\Gamma(t)]$, where

$$\Gamma(j\omega) = \begin{bmatrix} G_{11}(j\omega) & G_{12}(j\omega) & \cdots & G_{1m}(j\omega) \\ G_{21}(j\omega) & G_{22}(j\omega) & \cdots & G_{2m}(j\omega) \\ \vdots & & & \vdots \\ G_{m1}(j\omega) & \cdots & \cdots & G_{mm}(j\omega) \end{bmatrix} \qquad (3.5.2)$$

The input vector $\underline{\eta}(t)$ represents the zero-input response of $\Gamma(t)$. The complete response of $\Gamma(t)$ is thus

$$\underline{c}(t) = \underline{\sigma}_e(t) + \underline{\eta}(t).$$

It is assumed that all elements of the linear transfer matrix $\Gamma(t)$ are stable, more specifically,

(1) $g_{ij}(t) \in L_2(0,\infty)$, $\dot{g}_{ij}(t) \in L_1(0,\infty)$, $i=1,2,\ldots m$, $j=1,2,\ldots m$,

(2) $\eta_i(t) \in L_1(0,\infty)$,

(3) $\eta_i(t)$ is differentiable and $\dot{\eta}_i(t) \in L_1(0,\infty)$.

Condition (1) above ensures that each element of $\Gamma(t)$ is bounded on $(0,\infty)$ and that $g_{ij} \to 0$, and conditions (2) and (3) ensure that $\eta_i(t)$ behaves in the same manner as $g_{ij}(t)$.

Denote

$$\underline{\eta}_M = \sum_{i=1}^m \eta_{iM} = \sum_{i=1}^m \sup_{t \geq 0} |\eta_i(t)|,$$

$$\Gamma_M = \sum_{i=1}^m \sum_{j=1}^m g_{ijM} = \sum_{i=1}^m \sum_{j=1}^m \sup_{t \geq 0} |g_{ij}(t)|.$$

The notation $\|\cdot\|$ denotes norms in the space $L_1(0,\infty)$ such that, for example,

$$\| \underline{\eta} \| = \int_0^\infty \sum_{i=1}^m |\eta_i(t)| dt$$

Define

$$K_1^{-1} = \text{diagonal matrix } (K_{11}^{-1}, K_{12}^{-1}, \ldots . K_{1m}^{-1})$$

$$K_2^{-1} = \text{diagonal matrix } (K_{21}^{-1}, K_{22}^{-1}, \ldots . K_{2m}^{-1}),$$

and

$$A = \text{diagonal matrix } (a_1, a_2, \ldots . a_m).$$

§3.6 Theorem 2

Consider the system shown in Fig. 3.5 to which assumptions given above apply. Let $Y(t) = \text{diag}\{y_1(t), y_2(t) \ldots . y_m(t)\}$ be such that each element is a real function and that

(1) $y_i(t) = 0$ for $t < 0$, $i = 1,2,\ldots m$,

(2) $y_i(t) \leq 0$ for $t \geq 0$,

(3) $\| y_i(t) \| < \dfrac{K_{1i}}{K_{21}}$.

Let Q be any positive semi-definite constant diagonal matrix. If there exists an mxm matrix $H(j\omega)$ such that

$$H(j\omega) = [I + j\omega Q + Y(j\omega)][\Gamma(j\omega) + K_2^{-1}] + AQ\Gamma(j\omega)$$

$$- \| Y \| \, [K_1^{-1} - K_2^{-1}], \tag{Q2}$$

and

(1) $H(j\omega) + H^T(-j\omega)$ is a positive semi-definite Hermitian matrix for all ω,

(2) $H^*(j\omega) = H(-j\omega)$

(3) Every element of $H(j\omega)$ is analytic for all ω,

then

(1) $\sup\limits_{t \geq 0} |\underline{\sigma}(t)| < \infty$, and $\sup\limits_{t \geq 0} |\sigma_i(t)| < \infty$, $i=1,2,\ldots m$,

(2) $\underline{\sigma}(t) \to 0$ as $t \to \infty$, and $\sigma_i \to 0$ as $t \to \infty$,

(3) as $\|\underline{\eta}\| + \|\dot{\underline{\eta}}\| \to 0$, the corresponding $\underline{\sigma}$ has the property $\sup_{t \geq 0} |\underline{\sigma}(t)| \to 0$, and $\sup_{t \geq 0} |\sigma_i(t)| \to 0$.

§3.7 Proof of Theorem 2

This proof follows the same vein as that of the previous Theorem. The only difference here is that all vectors such as $\underline{e}(t)$, $\underline{\sigma}_e(t)$, $\underline{\eta}(t)$, $\dot{\underline{\eta}}(t)$, $\underline{c}(t)$, $\underline{\emptyset}(t)$, $\underline{\sigma}(t)$ are m-vectors, and all matrices such as Q,Y are mxm matrices. All formulae and the proofs of which developed in §3.3 still hold, except that the proof of (3.3.24) must be performed in the following manner.

From (3.3.23),

$$
\begin{aligned}
I' &= \int_0^\infty [-\underline{\sigma}_{eTm}(t) - Q\dot{\underline{\sigma}}_{eT}(t) - K_2^{-1}\underline{e}_{Tm}(t) - QA\underline{\sigma}_{eT}(t) \\
&\quad + (K_1^{-1} - K_2^{-1}) \, \|Y\| \, \underline{e}_T] \underline{e}_T(t)dt \qquad (3.3.23') \\
&= -\int_0^\infty \underline{e}_T^T(t) \int_0^t h(t-\tau)\underline{e}_T(\tau)d\tau \; dt,
\end{aligned}
$$

where h(t) is the inverse Fourier transform of $H(j\omega)$. From the condition of Theorem 2 and Newcomb's result [25],

$$I' \leq 0. \qquad (3.3.24')$$

This, however, is the same as (3.3.24).

Hence, by the same argument used in the proof of Theorem 1,

(1) $\sup_{t \geq 0} |\underline{\sigma}(t)| < \infty$,

(2) $\underline{\sigma}(t) \to 0$ as $t \to \infty$,

(3) as $\|\underline{\eta}\| + \|\dot{\underline{\eta}}\| \to 0$, the corresponding $\underline{\sigma}$ has the property $\sup_{t \geq 0} |\underline{\sigma}(t)| \to 0$.

Now, $\sup_{t \geq 0} |\underline{\sigma}(t)| < \infty$ if and only if every component $\sigma_i(t)$ satisfies

$$\sup_{t\geq 0} |\sigma_i(t)| < \infty.$$

Similarly, $\underline{\sigma}(t) \to 0$ as $t \to \infty$ if and only if every component $\sigma_i(t)$ satisfies

$$\sigma_i(t) \to 0 \quad \text{as} \quad t \to \infty,$$

and $\sup_{t\geq 0} |\underline{\sigma}(t)| \to 0$ if and only if every component $\sigma_i(t)$ satisfies

$$\sup_{t\geq 0} |\sigma_i(t)| \to 0.$$

§3.8 Example

Example 3.3

Let us consider the parallel system, where m = 3, each branch of which has one nonlinearity and one time varying gain in series with one linear time invariant transfer function. Here these three parts of each the three branches are identical to the corresponding ones used in Example 3.2. Suppose that the matrix $Y(t) \equiv 0$. Let us consider the matrix

$$H(j\omega) = (I+j\omega Q)(\Gamma(j\omega)+I) + a\, Q\Gamma(j\omega), \tag{3.8.1}$$

where

$$\Gamma(j\omega) = G(j\omega)\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

and

$$Q = qI.$$

Rewritting,

$$H(j\omega) = (1+j\omega q+aq)G(j\omega)\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} + (1+j\omega q)I. \tag{3.8.2}$$

Obviously, $H^*(j\omega) = H(-j\omega)$, and the elements of $H(j\omega)$ are all analytic for all ω.

Invoking the proof of §2.4, the Hermatian matrix

$$H(j\omega) + H^T(-j\omega) = \mathrm{Re}(1+j\omega q+aq)G(j\omega) + \frac{1}{3}. \tag{3.8.3}$$

Following the argument used in Example 3.2, if $a \leq 1+D-\varepsilon$. and q is $\frac{1}{\varepsilon}$, the sufficient condition Q2 of absolute stability is satisfied for any positive real constant K.

Chapter 4 CONCLUSIONS

A graphical method using the Popov line is possible for a particular class of time invariant nonlinear system. The method may be simplified in a number of cases.

Two alternative approaches, one using the Nichols chart, the other the analogue computer, are mentioned briefly and illustrated.

No simple graphical method exists to test the absolute stability of the parallel system with many <u>different</u> linear transfer functions, although a graphical method using the Popov line to test the absolute stability of the parallel system with many <u>identical</u> nonlinear transfer functions is possible.

Neither is there any simple graphical method available to establish the criterion of absolute stability of a multi-circuit or an internal feedback system. However, work on the determination of the criterion of absolute stability for any one of the four classes mentioned in §1.1 by digital technique is underway. The digital technique [23] is in essence concerned with location of the optimum combination of matrices Q and K which will define the boundary of absolute stability region. It must be pointed out, however, that with the systems that have been discussed so far in this thesis the graphical method is so far simpler and less cumbersome in obtaining the requisite conditions for absolute stability.

In chapter 3, Theorems 1 and 2 provide the sufficient, but not necessary, conditions for the absolute stability of a time varying nonlinear system in which the nonlinear part must be monotonically nonlinear.

Of great importance in establishing the sufficient condition of absolute stability is the appropriate choice of A and y(t). 1 + Y(s) is identical to the function describing some RC passive network if y(t) is an exponential function. The optimum region of absolute stability may be found by a digital technique.

If the time varying gain is frozen, that is, $k(t) = 1$, the sufficient condition (Q1) may be rewritten

$$\mathrm{Re}[1+qj\omega+Y(j\omega)][G(j\omega) + \frac{1}{K_2}] \geq 0.$$

This is the result of Baker and Desoer's [11].

For some classes of time varying nonlinear systems, it is possible to use the graphical method discussed in chapter 2.

APPENDIX 1

From (3.3.6)

$$R_1(0)-R_1(\tau) = \int_0^\infty \{[\sigma_T(t)-\emptyset[\sigma_T(t)]] - [\sigma_T(t-\tau)-\emptyset[\sigma_T(t-\tau)]]\}$$

$$\emptyset[\sigma_T(t)]k(t)dt \tag{A1.1}$$

Noting that $\emptyset(t)$ and $[\sigma(t)-\emptyset(t)]$ are monotonic, and

$$0 < \frac{\emptyset_1-\emptyset_2}{\sigma_1-\sigma_2} < 1,$$

$$\frac{\sigma_1-\sigma_2}{\emptyset_1-\emptyset_2} > 1,$$

or

$$(\sigma_1-\sigma_2)(\emptyset_1-\emptyset_2) - (\emptyset_1-\emptyset_2)^2 > 0, \tag{A1.2}$$

or

$$[(\sigma_1-\emptyset_1)-(\sigma_2-\emptyset_2)](\emptyset_1-\emptyset_2) > 0, \tag{A1.3}$$

thus $\emptyset(t)$ is monotonic increasing in $[\sigma(t) - \emptyset(t)]$.

Let us define

$$A = \{[\sigma_T(t)-\emptyset[\sigma_T(t)]-[\sigma_T(t-\tau)-\emptyset[\sigma_T(t-\tau)]]\}\emptyset[\sigma_T(t)], \tag{A1.4}$$

and observe that

$$A \geq P(t) - P(t-\tau),$$

where

$$P(t) = \int_0^{\sigma_T(t)} \emptyset[\sigma_T(\tau)]d[\sigma_T(\tau)-\emptyset[\sigma_T(\tau)]]. \tag{A1.5}$$

From (A1.1), (A1.4) and (A1.5),

$$R_1(0)-R_1(\tau) \geq \int_0^\infty [P(t)-P(t-\tau)]k(t)dt \tag{A1.6}$$

and

$$R_1(\tau) \leq R_1(0) + \int_0^\infty [k(t+\tau)-k(t)]P(t)dt. \tag{A1.7}$$

But

$$R_1(0) \leq \int_o^\infty [\sigma_T(t) - \frac{k(t)}{K_2} \emptyset[\sigma_T(t)]]k(t)\emptyset[\sigma_T(t)]dt, \tag{A1.8}$$

and, since,

$$P(t) \leq [\sigma_T(t)-\emptyset[\sigma_T(t)]]\emptyset[\sigma_T(t)]$$

$$\leq [\sigma_T(t) - \frac{k(t)}{K_2} \emptyset[\sigma_T(t)]] \frac{k(t)}{K_1} \emptyset[\sigma_T(t)], \tag{A1.9}$$

then

$$\int_0^\infty [k(t+\tau)-k(t)]P(t)dt \leq (K_2-K_1) \int_0^\infty P(t)dt$$

$$\leq \frac{K_2-K_1}{K_1} \int_0^\infty [\sigma_T(t) - \frac{k(t)}{K_2} \emptyset[\sigma_T(t)]]k(t)\emptyset[\sigma_T(t)]dt. \tag{A1.10}$$

Substituting (A1.9) and (A1.10) in (A1.8) yields

$$R_1(\tau) \leq \int_0^\infty \frac{K_2}{K_1}[\sigma_T(t) - \frac{k(t)\emptyset[\sigma_T(t)]}{K_2}]k(t)\emptyset[\sigma_T(t)]dt. \tag{A1.11}$$

APPENDIX 2

From (3.3.7)

$$R_2(\tau) = \int_0^\infty [1 - \frac{k(t-\tau)}{K_2}]k(t)\emptyset[\sigma_T(t-\tau)]\emptyset[\sigma_T(t)]dt. \tag{A2.1}$$

Therefore,

$$R_2(\tau) \leq \int_0^\infty \left|\frac{1}{k(t-\tau)} - \frac{1}{K_2}\right| |k(t-\tau)\emptyset[\sigma_T(t-\tau)]| \, |k(t)\emptyset[\sigma_T(t)]|dt$$

$$\leq \int_0^\infty \left|\frac{1}{K_1} - \frac{1}{K_2}\right| |k(t-\tau)\emptyset[\sigma_T(t-\tau)]| \, |k(t)\emptyset[\sigma_T(t)]|dt \tag{A2.2}$$

giving

$$R_2(\tau) \leq \frac{1}{2}(\frac{1}{K_1} - \frac{1}{K_2})\{\int_0^\infty [k(t-\tau)\emptyset[\sigma_T(t-\tau)]]^2 dt$$

$$+ \int_0^\infty [k(t)\emptyset[\sigma_T(t)]]^2 dt\} \tag{A2.3}$$

from which (3.3.9) follows.

APPENDIX 3

Since $a k(t) \geq \dot{k}(t)$, $ak(t) > 0$, and

$$\sigma(t)\emptyset[\sigma(t)] \geq \int_0^{\sigma(t)} \emptyset[\sigma]d\sigma \triangleq \Phi(t) > 0,$$

clearly,

$$\int_0^T ak(t)\sigma(t)\emptyset[\sigma(t)]dt \geq \int_0^T \dot{k}(t)\Phi(t)dt, \tag{A3.1}$$

$$\mathcal{J} = q\int_0^T(\dot{\sigma}(t)a\sigma(t))k'(t)\emptyset(t) \geq q \int_0^T[\dot{k}(t)\Phi(t)+k(t)\emptyset[\sigma(t)]\ \dot{\sigma}(t)]dt$$

$$\geq q \int_0^T \frac{d}{dt} [k(t)\Phi(t)]dt$$

$$\geq q[k(T)\Phi(T) - k(0)\Phi(0)]. \tag{A3.2}$$

Therefore,

$$I_2 = \mathcal{J} + q \int_0^T[\dot{\eta}(t)+a\eta(t)]e(t)dt$$

$$\geqslant q[k(T)\Phi(T)-k(0)\Phi(0) + \int_0^T[\dot{\eta}(t)+a\eta(t)]e(t)dt. \tag{A3.3}$$

APPENDIX 4

Since $\dot{k}(t) \leq ak(t) \leq 0$ and $0 \leq \frac{k(t)\emptyset[\sigma(t)]^2}{2k_2} \leq \Phi(t)$, clearly,

$$\int_0^T \frac{a}{2K_2} e(t)^2 dt = \int_0^T ak(t) \cdot \frac{k(t)\emptyset[\sigma(t)]^2}{2K_2} \geq \int_0^T \dot{k}(t)\Phi(t)dt, \tag{A4.1}$$

$$J' = q\int_0^T[\dot{\sigma}(t) + \frac{a}{2K_2} e(t)]e(t)dt \geq q\int_0^T [\dot{k}\Phi(t)+k(t)\emptyset[\sigma(t)]\dot{\sigma}(t)]dt$$

$$\geq q[k(T)\Phi(T) - k(0)\Phi(0)]. \tag{A4.2}$$

Therefore,

$$I_2' \geqslant q[k(T)\Phi(T) - k(0)\Phi(0) + \int_0^T \dot{\eta}(t)e(t)dt]. \tag{A4.3}$$

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