AUTONOMOUS QUASI-HARMONIC AND FORCED VIBRATION OF FRIC TIONAL SYSTEMS

by

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We accept this thesis as conforming to the required standard

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ABSTRACT

The behaviour of a system subject to quasi-harmonic type frictional oscillation was investigated. The same frictional system with external excitation was also investigated both experimentally and theoretically.

Various frictional material combinations including steel, polymer, rubber and fibre materials and lubricants were used to provide different forms of friction characteristics. The dynamic friction-velocity curves were obtained by recording simultaneously the acceleration force, damping force, spring force and friction force during one cycle of the quasi-harmonic oscillation. The curves were expressed as a function of sliding velocity and were represented by nth order polynomials as well as by exponential expressions. The first approximation methods by Krylov and Bogoliuboff were used to solve the nonlinear differential equations of motion in both the autonomous and non-autonomous cases. In addition, the method of harmonic balance was also used in the non-autonomous case. In both cases, the Runge-Kutta numerical method was used to investigate the transient state of the oscillations. Theoretical results for the autonomous system indicated that the humped form friction-velocity curve was a necessary condition for the existence of quasi-harmonic oscillation.
Subharmonic entrainment at the frequency of the autoperiodic oscillation or harmonic entrainment at the external excitation frequencies, depending on the magnitude of the external excitation, were also predicted from the analysis.

Experimental results were obtained from a pin on disc type frictional system having a track velocity range of 0.04 in/sec to 13.5 in/sec. External excitation forces were obtained by applying the principle of out-of-balance mass. The frequency range of the external excitation is 0-90 cps. The growth and decay of the quasi-harmonic oscillation was observed. In the non-autonomous case, 'quenching' of the autoperiodic oscillation by the external excitation was recorded. In general, the experimental results substantiate the predictions of the theoretical analyses. The experimental results also showed that the vertical external excitation has the effect of reducing the maximum static friction and subsequently extinguishing stick-slip oscillation.
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\[ \begin{align*}
P_1, P_2, \ldots & \quad \text{Coefficients Derived from } C_0, C_1, \ldots \text{ etc of the Polynomial Expression} \\
Q_1, Q_2, \ldots & \quad \text{Coefficients Derived from } E_1, E_2, \ldots \text{ etc.} \\
R_1, R_2, \ldots & \quad \text{Coefficients Derived from } Q_1, Q_2, \ldots \text{ etc.} \\
R & \quad \text{Damping Factor, } \frac{r}{m\omega} \text{.} \\
U_1, U_2, \ldots & \quad \text{Functions of } a, \psi \text{ and } \alpha \tau \text{.} \\
V & \quad \text{Dimensionless Velocity} \\
X, X, X & \quad \text{Dimensionless Displacement, Velocity and Acceleration} \\
X' & \quad X - \theta \\
Y & \quad X - \left[ \frac{F_0}{(1-a^2)} \right] \sin \alpha \tau \\
Y' & \quad X \\
Z & \quad \text{Function of } a^2 \\
a & \quad \text{Steady State Amplitude of Autoperiodic Oscillation} \\
b_1, b_2 & \quad \text{Steady State Amplitude of Oscillation (Fundamental Resonance)} \\
a_c & \quad \text{Steady State Amplitude of Heteroperiodic Oscillation} \\
c, p, q, s & \quad \text{Constants} \\
e & \quad \text{Eccentricity of Out-Of-Balance Weight } \text{ in} \\
f & \quad \text{External Excitation Force } \text{ lb} \\
f_{\mu k} & \quad \text{Dynamic Friction Force } \text{ lb} \\
g_o, g_{n1}, g_{n2} & \quad \text{Functions of } X \\
h & \quad \text{Linear Parameter } \text{ in} \\
k & \quad \text{Spring Stiffness, also used as } \text{ lb/in} \text{ Variable Integers} \\
m & \quad \text{Mass of Vibratory System } \text{ lb/in/} \text{sec}^2 \end{align*} \]
n  Order of Polynomials
r  Damping Coefficient  lb /in/sec
 t  Time  sec
v  Disc Velocity  in/sec
w  Normal Load  lb
x  Displacement  in
x  Absolute Velocity  in/sec
\dot{x}  Acceleration  in/sec^2
Y_1, Y_2  Functions of V
\alpha  Frequency Ratio, \nu/\omega
\beta  Load Ratio, \eta f/w
\gamma  Value Indicates Smallness
\phi, \psi  Phase
\rho  Mass of Out-of-Balance Weight  lb/in/sec^2
\mu_k, \mu_s  Coefficient of Friction
\omega  Damped Natural Frequency  rad/sec
\omega_n  Natural Frequency  rad/sec
\nu  External Excitation Frequency  rad/sec
\tau  Dimensionless Time
\zeta, \xi, \epsilon, \sigma, \eta, \theta  Constants
\Omega, \phi, \Lambda, \lambda  Functions of Vibration Amplitude, a
ACKNOWLEDGEMENT

The author is grateful for the many helpful suggestions from the faculty and graduate students in the Department of Mechanical Engineering and for the assistance of the technical staff who constructed the experimental apparatus. Sincere appreciation is also expressed to Mr. J.E. Jones, a staff member of the Tribology Laboratory, whose enthusiastic assistance greatly accelerated the research programme. Thanks must also be expressed to the Department of Mechanical Engineering for the use of their facilities. Part of the experimental apparatus used in this investigation was previously constructed by a fellow graduate student, Mr. H.R. Davis, and is gratefully acknowledged.

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INTRODUCTION

The friction characteristics resulting from the motion of one surface over another form an important facet of the behaviour of many physical systems. When two solid bodies are rubbed together, vibration of some type frequently occurs, which may, in general, be called 'friction-induced vibration'. Friction-induced vibration has been observed in a wide variety of systems. In automatic transmissions, the engagement of the clutch depends on the friction characteristic of the fluid and the face materials of the clutch. In positioning systems the accuracy and the sensitivity of response are greatly impaired by vibration induced by the friction of the sliding surfaces. The Froude pendulum and the motion of violin string under the action of a bow are frequently cited as examples of friction-induced vibration.

Two forms of autonomous friction-induced vibration may be classified, namely stick-slip vibration and quasi-harmonic vibration, Fig. 1.1.1. The stick-slip or relaxation oscillation is characterised by the saw tooth displacement-time waveform, whereas the quasi-harmonic vibration has a waveform which is approximately sinusoidal. In the case of stick-slip oscillation the regimes of stick and slip constitute the complete vibration cycle. The stick phase is dependent on static friction forces. At
the end of stick, sudden relaxation occurs and during this movement the system is governed by dynamic friction forces. However, in the case of quasi-harmonic oscillation there is always relative movement between the two sliding surfaces therefore the motion is governed by dynamic friction forces only.

The basic mechanism of friction between unlubricated surfaces can be considered as arising from two main factors, namely adhesion and deformation. When two metal surfaces are placed together only small areas are actually in contact [1]. Under conditions of small contacting areas plastic flow at the contact areas usually occurs under relatively light loads, and a junction is formed between the tips of the asperities. The static friction is related to the junction growth theory. It is generally believed that the static friction is time-dependent and that the kinetic friction is velocity dependent. More recently, the strain rate which is a function of the driving velocity and of the spring stiffness of the system has been considered to be an important aspect of static friction behaviour [2].

The stick-slip type of friction-induced vibration is usually attributed to the kinetic friction force being lower than the static friction force. Under these conditions the friction-velocity characteristic has the effect of introducing negative damping which assists any small disturbance to grow into full vibration cycles. The phenom-
enon of the stick-slip type of friction-induced vibration has received extensive consideration, although satisfactory explanations are lacking in many areas, such as the interesting question as to just how the initial drop from static friction occurs, whether instantaneously or over a finite interval of time, or over a finite distance.

Until the present work very little was known about the quasi-harmonic type of friction-induced vibration other than it was somehow related to the variation of friction force with sliding velocity. Actual friction couples display a variety of forms of dynamic friction curve. It will be shown that the existence of the quasi-harmonic oscillation is critically dependent on the particular shape of the dynamic friction curve.

The subject of controlling or reducing the friction by applying vibration to the friction parts has been used in practice for some time. Vibrators are attached to instrument panels to keep needles free and in motion. 'Dithering' devices were used on the guiding fins of some early air to air missiles to obtain fast response from the servomechanism. It has been shown that sonic vibration reduces the static friction during unlubricated sliding. However, there have been very few attempts to investigate this subject systematically.

The present work is concerned mainly with the quasi-harmonic type of friction-induced vibration. Owing to the large amount of work already involved in the present
studies, no attempt was made to investigate the fundamental aspect of the contacting surfaces, rather, the investigation deals mainly with the dynamic behaviour of a system subject to quasi-harmonic friction-induced vibration. Three systems subjected to friction forces which vary as a function of relative sliding velocity have been investigated. The first system, which consisted of a mass, a spring and a damper but with no external excitation, was used for the investigations of the friction-velocity characteristic curve and of friction-induced vibration. An external harmonic excitation in the direction of the friction force was applied to the basic elements in the second system, whereas in the third a harmonically fluctuating normal force was applied to the loading end of the basic elements (Fig. 1.1.2).

Many past experiments on friction and friction-induced vibration have shown that precise friction measurement is not an easy task and that friction is sensitive to a number of factors which are difficult to control individually. Generally it should be observed that reproducible results are difficult to obtain and in the past the correlation between experimental results and theoretical analysis has been unsatisfactory. Owing to the importance of the friction measurements involved in the present investigation, the design and development of a reliable friction apparatus and its accompanying instrumentation played an important role.
The present investigation constitutes the first complete and thorough study of the quasi-harmonic friction-induced vibration, particularly in the cases of external excitation. The application of nonlinear mechanics revealed the dynamic behaviour of the quasi-harmonic friction-induced vibration both with and without external excitation. The reliable friction apparatus which was subsequently developed together with the technique for the accurate measurement of friction and friction-induced vibration provided a satisfactory correlation between the theoretical analysis and experimental results.
Although the phenomenon of friction-induced vibration has been observed for a long time, it is only during the present century that real advances have been made in gaining some understanding of the mechanics of the processes involved.

The effect of speed on the value of coefficient of friction was first discussed by Coulomb [3], he found that when different metals were rubbed together, the friction force increased with speed. In 1929 Wells [4] experimented with a machine for measuring kinetic friction under conditions of boundary lubrication. The friction force was measured by means of a trifilar suspension of weights arranged in such a way that the restoring force increased with the displacement.

Thomas [5] was possibly the first to demonstrate the possibility of self-excited type oscillations. His graphical analysis showed that in the absence of viscous damping, stable oscillations either of the simple harmonic type or of the stick-slip type could occur depending on whether the coefficient of kinetic friction was equal to or smaller than the coefficient of static friction, with the coefficient of kinetic friction being independent of the sliding velocity. His findings also suggested that the
simple harmonic type oscillations can no longer be sustained in the presence of viscous damping but if the damping is not excessive the stick-slip type oscillations can be maintained. However, he showed no experimental results to substantiate his analysis. It is quite obvious that his conditions for the simple harmonic type oscillation would be equivalent to a mass-spring system with free oscillation, with the static friction force being the initial condition.

In 1933, Kaidanovsky and Haykin [6] made a study of relaxation oscillations as applied to mechanical systems having friction varying with the velocity. They asserted that a necessary condition for such vibrations is that a region must exist for which the friction decreases as the velocity increases. It was Papenhuyzen [7], in his investigation of the mechanics of the skidding of automobile tires, who classified friction-induced vibration into the two general types with relation to the driven surface velocity. He showed the occurrence of the stick-slip type relaxation oscillation and the simple harmonic type oscillation at different stages of the driven surface velocity variation. He also showed that at low driven surface velocities the vibrating member remained stationary at a constant displaced position.

Bowden and Lebon [8] carried out a series of experiments on the friction between sliding metals in the absence of a lubricating film. In their experiments the
recorded sliding velocities were very high in comparison with the driven surface velocity. They suggested that local welding could result from the high temperature flashes during the slip stage of the stick-slip type oscillations. However, Blok [9] indicated that it is unlikely that the welding effect is the main reason for the occurrence of the frictional oscillations. He showed that such vibrations may simply represent a form of relaxation oscillation which may, in turn, depend upon the particular form of friction-velocity curve and the amount of damping in the system. He suggested that the essential condition for the occurrence of frictional oscillations is decreasing frictional force for increasing sliding velocity. He established a quantitative criterion for the onset of stick-slip type oscillation using a relationship plotted between dimensionless parameters of damping and sliding velocity. Further investigation of the temperature flash and the friction behaviour during the slip portion of the stick-slip process was carried out by Morgan, Muskat and Reed [10] and later by Sampson [11].

Various kinds of sliding surfaces and lubricants were used by Bristow [12], [13] to study the friction-velocity relationship and the temperature effects. In his account of the frictional oscillations he stated that the existence of a negative friction-velocity relationship is a necessary condition for relaxation oscillations to be
excited in an elastic system. He showed the existence of micro-slip movement during the stick stage. He also suggested that the condition for the existence of the quasi-sinusoidal oscillation is decreasing of friction with increases of velocity in the high velocity region. The explanation is not convincing. In fact, in the discussion of [13] Swift pointed out that a graphical solution indicated no quasi-sinusoidal oscillations at super-critical speeds in a system with a negative friction-velocity relationship as suggested by Bristow.

Earlier, a simple graphical method for determining the vibration cycle from any friction-velocity curve was developed by Dudley and Swift [14]. The method makes direct use of the experimental friction-velocity curve. They applied the method to study the growth and decay of the self-induced vibration with relation to various friction-velocity curves. They showed that as the speed increased during the stick-slip conditions, the amplitude of the oscillations increased. The results of the increasing amplitudes did not agree with the finding of some other authors. Brockley, Cameron and Potter [15] showed that the amplitude of oscillation decreased as the driving surface velocity was increased. The difference would seem to lie in the fact that in the graphical solution of Dudley and Swift, they have assumed that the static friction was the same as the kinetic friction at zero sliding veloc-
ity; while this may be true in some cases, it certainly cannot be considered as a generalised phenomenon.

In his investigation of automobile brake squeal, Sinclair [16] again showed that frictional vibrations are caused by an inverse variation of coefficient of friction with sliding velocity. He also suggested that the decrease in friction observed at high velocity is caused by the high temperatures developed at high velocity. This behaviour was further discussed by Rabinowicz [17]. He stated that the negative friction-velocity shape at high velocity is connected with thermal softening which produces a low shear surface film on a harder substratum. He also defined the stick-slip oscillations as being time controlled and the quasi-sinusoidal oscillations as being velocity controlled.

Jarvis and Mills [18] investigated the vibration caused by dry friction in a simulated disc brake system. Their investigation shows that unwanted vibration in any system can possibly be avoided by careful choice of dimension in the design. The significance of their work is the demonstration of the 'geometrically induced' instability of two elastic components interacting through the agency of kinetic dry friction. However, it is doubtful that the linearised friction-velocity characteristic used in their theoretical analysis adequately represented the real friction-velocity relation of the system.
Derjaguin, Push and Tolstoi [19], Singh [20], Cook [21] and Brockley, Cameron and Potter [15] all presented theoretical analysis of the stick-slip type friction-induced vibrations. In the analyses the kinetic friction characteristic was assumed to be linear and the static friction characteristic was considered to be time-dependent.

Atsushi Watari and Takanao Sugimoto [22] investigated the self-excited vibrations caused by dry friction with a decreasing friction-velocity relationship. In the theoretical investigation an attempt was made to analyse the friction system using a nonlinear method as well as linear methods. However, many assumptions were made to simplify the nonlinear friction-velocity function, thus restricted the usefulness of the analysis. They showed that the vibration amplitudes increase with the increase of velocity and that the frequencies are nearly equal to the system's natural frequency. The vibration died out at a certain velocity. Similar results were observed by Shizuo Doi and Shinobu Kato [23], in their investigation of chatter vibration of flexible lathe tools. They also attempted to correlate the growth and decay of the vibration with the shape of the friction-velocity characteristic.

The study of machine tool slideways has been the subject of considerable effort recently. These investi-
gations have usually involved a systematic study of slide-way friction and the study of the stick-slip type oscillations. Some important contributions regarding the investigation of the friction-velocity characteristic have been reported.

In his study of the stability of sliding motion Stepanek [24] clarified the difference between friction tests carried out under steady-state (zero acceleration) condition which yielded values of kinetic friction and those carried out under dynamic conditions where the role of acceleration was recognised. However, in the subsequent theoretical analysis, only a simplified linear friction-velocity characteristic with a correction term for acceleration was used. The paper fails to show the methods for obtaining the two different types of friction-velocity characteristic. No experimental results were presented to verify the theoretical analysis.

Hunt, Torbe and Spencer [25] applied the phase-plane trajectory analysis to investigate the stick-slip motion arising from machine tool practice. In their analysis they recognised the role of the acceleration in the frictional vibration system, but failed to obtain a unique friction curve. An ingenious experimental technique to determine the friction-velocity characteristic was developed by Bell and Burdekin [26] for examining the dynamic aspects of slideway friction. In determining the friction character-
istic, the acceleration of the vibratory movement as well as the displacement were recorded. The foregoing measurements permitted the determination of the dynamic friction-velocity curve during one cycle of vibration. The method appears to give a realistic assessment of dynamic friction-velocity curve.

The fundamental of the friction mechanism relating the static friction characteristic has received a wide range of investigation. Rabinowicz [27] explained the high static coefficient as being due to the metallic junction becoming stronger after the surfaces have been in stationary contact for some time. He also suggested that the static coefficient $\mu_s$ and the kinetic coefficient $\mu_k$ are functionally related, since $\mu_s$ for a contact time $t$ is the same as $\mu_k$ for a sliding velocity $v$, provided that $t = d/v$ where $d$ is the mean size of the junction formed at the interface [28].

A detailed study of friction mechanism of steel on steel was presented by Vinogradov, Korepova and Yu. Ya. Podolsky [29]. The study covered a wide range of velocities. At high sliding speeds the friction force reaches a stable value in a very short interval of time. At very low sliding speeds the friction force increases slowly with time, and the greater the load, the longer it takes to reach a steady friction value. They suggested that at high sliding speeds, seizure resulted from the simultaneous
formation of bridges of intensive adhesion between the majoritiy of the micro-areas of the virgin metallic surface; whereas at low sliding speeds the rheological properties of the contacting solids play an important part.

Simkins [30] showed that during the stick period of a stick-slip process there were microslips preceding the gross slip. He suggested that the so called 'static friction force' is merely the 'local maximum' of the values available. The phenomenon of micro-slip during stick is not new, in fact, Papenhuyzen [7] had discussed the micro-slip movement in the stick stage. However, the more advanced experimental technique permitted Simkins to produce more substantial evidence of the phenomenon.

A survey of work in friction, lubrication and wear during the last decade was presented by Bowden and Tabor [31]. The report provided some useful references for work in friction studies.

The subject of controlling or reducing the friction by applying vibration to the friction parts has been received some consideration for sometime. Houch [32] suggested that a properly applied vibration can be used to eliminate or greatly relieve many of the problems involving friction. Godfrey [33] found that the vibration periodically reduced metal-to-metal contract due to reduced load. Thus, an apparent reduction of the coefficient of friction was observed. Gaylord and Shu [34] observed
that static coefficient of friction was lower under dynamically applied loads than under statically applied loads, although no explanation for this observation was given.

A more systematic study of the reduction of static friction by sonic vibration was reported by Fridman and Levesque [35]. The effect of sonic vibrations on the static coefficient of friction was measured for highly polished and for ground and sand blasted steel surfaces. They suggested that the static coefficient of friction can virtually be reduced to zero as a result of increased vibration at frequencies between 6-42 kHz, with a peak to peak vibration amplitude of $7.5 \times 10^{-6}$ cm. Studies of vibration in metal working are reported by Wheeler [36]. The report showed that substantial reduction in yield stress was observed with the application of oscillation at 800 kHz. In another case, as the result of applying low frequency oscillation, 16-40 Hz, to the ram of a hydraulic forging press an apparent reduction in friction of up to 60 percent and close to 50 percent reduction in force required to produce the deformation were achieved. In addition, the metal was deformed more uniformly. However, more systematic investigations as well as theoretical analyses are needed in order to gain a more generalized understanding of the underlying physical realities of the problem.

The problem of friction-induced vibration due to a nonlinear friction-velocity relationship has similarity
to many other engineering problems. Thus the analytical methods developed for the investigation of friction-induced vibration can be applied to other analogue problems. Lempriere [37], in a study of tensile testing, showed that a nonlinear stress-strain rate curve would cause auto-oscillation. The problem was analysed by applying the phase-plane trajectory method. Clauser [38] presented a review of nonlinear systems and showed the similarity of some aeronautical problems with the friction-induced vibration phenomenon. The effect of static and sliding friction in feedback systems was reported by Tou and Schultheiss [39]. A mathematical method was developed to correct the non-linearity in the feedback system due to friction.

In all the previous work on friction-induced vibration, very little consideration has been given to the quasi-harmonic type oscillation. It would appear that the so-called quasi-sinusoidal vibration observed in the high velocity region in a system with decreasing friction-velocity relationship was in fact stick-slip oscillation with a very short period of stick so that the x-x phase plane appeared to be almost circular (Ref. Fig. 5.1.9). The above papers would seem to indicate that an important question for the understanding of friction-induced vibration and the practical treatment of mechanical systems involving friction is its dependence upon velocity. However, little was known about this question other than the general rule that the
condition for the existence of friction-induced vibration was that the static friction should be greater than the kinetic friction and/or a decreasing friction-velocity relationship existed. The friction-velocity curves as described in the above papers were mostly of the linear form with decreasing value as the sliding velocity was increased. However, actual friction couples display a variety of forms of friction curve. The linear form with negative slope is only one possible form. The present work, which relates to the theoretical and experimental investigation of the mechanics of the quasi-harmonic oscillation, has produced an understanding of the phenomenon.
III THEORETICAL

3.1 **Introduction**

Fig. 1.1.2 shows schematically the configurations of the three systems. Basically the system consists of a slider of mass \( m \) with a normal force \( w \) acting to impress the slider against a lower surface which is moving with a constant velocity \( v \). The slider is restrained by a bond of elasticity \( k \) and damper of coefficient \( r \). Let the coefficient of friction between the slider and the lower surface be \( \mu_k \).

The equation of motion for the autonomous system can be written as

\[
mx + rx + kx = wu_k
\]

(3.1.1)

The equation for the system subjected to transverse forcing is

\[
mx + rx + kx = wu_k + f \sin vt
\]

(3.1.2)

In the case of normal vibration the equation is

\[
mx + rx + kx = wu_k(1 + \beta \sin vt)
\]

(3.1.3)

where \( f = \rho v^2 e \); \( \rho \) is the mass of an out-of-balance weight.
and \( e \) is the eccentricity.

\[ \beta = \eta \frac{f}{w}; \eta \text{ is some constant.} \]

Eq. (3.1.2) represents a system having an external excitation force acting in the direction of the friction-induced vibration, whereas eq. (3.1.3) represents a system with the external excitation being applied vertically at the loading end of the beam thus simulating the effect of dynamically applied load.

Equations (3.1.1), (3.1.2) and (3.1.3) can be non-dimensionalized by introducing a displacement parameter \( h \) and letting

\[ \omega^2 = \frac{k}{m}; \quad x = \frac{x}{h}; \quad v = \frac{v}{\omega h} \quad \text{and} \quad \tau = \omega t. \]

Then we have

\[ \ddot{x} = \frac{d^2x}{dt^2} = \frac{x}{\omega h} \quad \text{and} \quad \dddot{x} = \frac{\ddot{x}}{\omega^2 h} \]

Multiplying equations (3.1.1), (3.1.2) and (3.1.3) by \( 1/(m \omega^2 h) \) and substituting the expressions for \( X, \dot{X} \) etc, we have the non-dimensionalized equations for the three systems.

\[ \dddot{X} + R\ddot{X} + X = \frac{1}{E} F(V-\dot{X}) \quad (3.1.4) \]

\[ \dddot{X} + R\ddot{X} + X = \frac{1}{E} F(V-\dot{X}) + F_0 \sin \alpha \tau \quad (3.1.5) \]

\[ \dddot{X} + R\ddot{X} + X = \frac{1}{E} F(V-\dot{X}) \left[ 1 + \beta \sin \alpha \tau \right] \quad (3.1.6) \]
where \( R = r/(m\omega) \); \( E = (m\omega^2 h) \); \( \alpha = \nu/\omega \) and \( F_0 = f/E \).

The friction force function \( f \) has been converted into a function of the dimensionless sliding velocity \( F(V - \dot{x}) \).

Equations (3.1.4), (3.1.5) and (3.1.6) can be investigated by the methods to be described.

Owing to the nonlinearity of the friction characteristic, friction-induced vibration has long been considered one of the classical nonlinear problems of mechanical systems. Although a variety of techniques have been developed for the approximate solution of nonlinear differential equations, very little has been done in applying these methods to a thorough analysis of friction-induced vibration. In general, most of these analytical methods are simple in the sense of qualitative analysis, but usually become complicated once a quantitative solution is sought.

In the present investigation a complete and thorough theoretical analysis of the friction-induced vibration both with and without external excitation will be carried out by applying the available methods. Sometimes it is necessary to apply more than one method in order to investigate a problem, with each method providing part of the solution.

The topological method of analysis is one of the important means of investigating various phenomena of nonlinear oscillations, and it is applicable to the study of autonomous systems. By this method solutions are sought as integral curves in a phase plane. Poincaré [40] has shown
that limit cycles and singular points form certain topological configurations. A simplified statement of his theorem is that every limit cycle contains at least one singular point in its interior of stability opposite to that of the cycle. Thus an unstable singular point is surrounded by a stable cycle and vice versa, as shown in Fig. 3.1.1a and Fig. 3.1.1b. Soft self-excitation corresponds to the case in which a system departs from an unstable singularity as in Fig. 3.1.1a and arrives at the stationary state of the limit cycle $C_a$. Hard self-excitation corresponds to the case as shown in Fig. 3.1.1b in which an impulse is required to enable the system to cross the barrier represented by the unstable limit cycle $C_b$.

In most applied problems of the autonomous type the qualitative features of the oscillatory processes are, generally, completely revealed by the first approximation. For this reason, many methods based on the averaging method are developed. All these methods lead to the solution by the first approximation. The methods of van der Pol [41] and of Krylov-Bogoliubov [42] are very close to each other. In both methods a simple harmonic solution is 'fitted' into the nearly linear equation. The main difference between the two methods is that van der Pol takes the harmonic solution in the form $(b_1 \sin \omega t + b_2 \cos \omega t)$, whereas in the Krylov-Bogoliubov method this solution is...
taken in the form, \( A \cos(\omega t + \phi) \), where \( A, b_1, b_2 \) and \( \phi \) are slowly varying functions of \( t \).

For a nonautonomous system, that is the time \( t \) appears explicitly in the nonlinear term of the differential equation, there are two principal cases to be investigated according to whether the parameter \( \alpha \), the ratio between the heteroperiodic and the autoperiodic frequencies, is a non-integer or an integer. Here the autoperiodic oscillation is defined as the free oscillation with frequency equal to that of the self-excitation oscillation, and the heteroperiodic oscillation is defined as the forced oscillation with frequencies equal to that of the external excitation and/or multiples of it [43]. The first case is relatively simple and leads to the so-called non-resonance oscillation. The second case corresponds to the resonance oscillation.

One could also investigate a more general problem by considering \( \alpha \) as a variable parameter. When \( \alpha \) is sufficiently far away from a rational number, there may appear the so-called asynchronous action, an action in which the heteroperiodic oscillation sometimes manifests itself in the appearance of an autoperiodic oscillation and sometimes in the extinction of an existing autoperiodic oscillation [44]. When \( \alpha \) has integer values such as \( \alpha = 2, 3, 4 \) there may be occasionally subharmonic solutions with frequency ratio 1. Furthermore, there is the problem of synchronization when \( \alpha \) is in the neighbourhood of 1. Thus the problem of inves-
tigating a nonautonomous system may become complicated.

Several qualitative methods had been developed recently for the investigation of the nonautonomous systems. These methods are relatively simple to apply when only a qualitative analysis is required. However, when quantitative results are sought for an applied problem, the calculation is almost always long and complicated, particularly when the nonlinear function involves an extended expression such as a high order polynomial. Under these circumstances numerical methods may have to be used.

In the present investigation, the first approximation of the method of Krylov and Bogoliuboff was used in the autonomous and nonautonomous cases. However, in the nonautonomous case, in addition to the K and B method the method of harmonic balance was used. For the investigation of the transient state a numerical method was employed.

3.2 Form of Friction-Velocity Function Required for the Existence of Quasi-Harmonic Friction-Induced Vibration

In the case of the autonomous quasi-harmonic friction-induced vibration, there is no stationary contact between the sliding surfaces, therefore the motion is governed by the variation in friction force with sliding speed. In particular, it will be shown that the existence
of the quasi-harmonic oscillation is critically dependent on the particular shape of the dynamic friction curve.

The decreasing friction-velocity characteristic curve as assumed in much of the previous work is not the only possible one, in fact, actual friction couples display a variety of forms of dynamic friction curve. Fig. 3.2.1 illustrates two forms of the curve commonly suggested. Alternatively, there is evidence to support the type of curve illustrated by Fig. 3.2.2. Kragelskii [45] demonstrates that this form of curve is found for metals in dry sliding contact. Recently, the study of the friction behaviour of rubber [46] and of polymers [47] has shown that similar humped friction-velocity curves exist for non-metallic materials. Experimental results [48] for the combined rolling and sliding contact of lubricated rotating discs reveal the existence of friction torque versus sliding velocity curves of the form of Fig. 3.2.2. A humped form of friction force-velocity curve was also reported by Muskat and Morgan [49], in their investigation of bearings with various type of lubricants. In another study of bearing lubricants, Hagg [50] showed a humped form for the shearing stress versus journal velocity curve. The study of friction characteristics of automatic transmission fluid components [51] has shown that humped friction-velocity characteristic exists for certain types of additive combinations. In their investigation of slideway friction,
Bell and Burdekin [26] observed that the humped type friction-velocity curve exists with polar lubricants. Materials such as long chain naphthenic acids, fatty acids, long chain alkylphosphates etc., generally possess high polar activity level [52] and are usually used as friction modifiers for automatic transmission fluids [53]. Thus it is of interest to note that the humped friction-velocity curve exists in a wide variety of frictional situations and that in many cases vibration of the quasi-harmonic form may occur. It is probable that the humped form is associated with the viscous component whose viscosity is sensitive to temperature.

For an analytical study, an equation for the friction curve is necessary. Various mathematical expressions have been used to represent the friction force function [54]. From some early work on railway transportation, the friction force was represented by the empirical expression

$$ f_{\mu_k} = \eta + \frac{p-\eta}{1+qV} $$

Later, the relationship was expressed in the form

$$ f_{\mu_k} = f_0 e^{-sV} $$

where \( \eta, p, q \) and \( s \) are some constants, \( f_{\mu_k} \) and \( V \) are the friction force and velocity respectively. Owing to the simplicity of these expressions, their accuracy for fitting
experimental curves is limited.

In the present investigation, the friction force function was expressed in the form of an exponential function as well as in the form of a \(^n\)th order polynomial.

\[
f_{\mu_k} = [C_1 + C_2(v-\dot{x})]e^{-C_3(v-\dot{x})} + C_4(v-\dot{x}) + C_5 \quad (3.2.1)
\]

\[
f_{\mu_k} = C_n(v-\dot{x})^n + C_{n-1}(v-\dot{x})^{n-1} + \ldots + C_0 \quad (3.2.2)
\]

where \((v-\dot{x})\) is the sliding velocity and \(C_0, C_1\) etc. are constants which may be adjusted to fit the equation to measured friction values.

When carrying out the theoretical analysis, the friction force function expressed in the polynomial form has certain advantages over the exponential form. In particular, the polynomial expression reduces to a power series in \(\dot{x}\) which upon integration yields another power series of the amplitude of vibration, thus the stationary state amplitude can be readily obtained by solving the amplitude polynomial. On the other hand, the exponential expression yields a transcendental equation for the amplitude of vibration. In order to solve the equation some initial estimations had to be supplied. Another advantage is that the complete process of the theoretical analysis can be generalised for a \(n\)th order polynomial.

However, the exponential expression has the advantage of being able to provide a more accurate representation
of the experimental friction force curve, particularly of the humped form. This is particularly useful when applied to the numerical method. The hump in the friction curve usually appears very close to the zero velocity end of the curve and under these circumstances the curve fitted by polynomials often presents difficulties (ref. Fig. 5.1.4). In addition, the polynomial fitted curve may show drastic change once outside the range of fit, thus greatly limiting its usefulness.

The existence or non-existence of self-excited vibration of the quasi-harmonic form may be investigated for the various dynamic friction curves proposed. The phase-plane graphical method of Liénard [55] provides a useful technique for a semi-qualitative investigation. In order to apply the method, eq. (3.1.4) is modified by letting $\dot{X} = Y'$. After manipulation, these modifications yield,

$$\frac{dY'}{dX} = \frac{F(V-Y')/E - RY' - X}{Y'}$$  \hspace{1cm} (3.2.3)

If $\frac{dY'}{dX}$ is set to zero in eq. (3.2.3) then it is found that

$$X = \frac{F(V-Y')}{E - RY'}$$  \hspace{1cm} (3.2.4)

Eq. (3.2.4) describes the locus of all points of maximum velocity on a phase plane diagram. It is evident
that the locus is simply a modified friction-velocity characteristic curve, where the $RV'$ term is the system viscous damping which is usually very small. Accordingly, employing this equation and Liénard method, the diagrams of Fig. 3.2.3 were prepared using Fig. 3.2.1a and Fig. 3.2.3a illustrates the case of a limit cycle of the stick-slip type produced by entrainment of the phase trajectory into the static friction axis. Fig. 3.2.3b shows a situation whereby the mass achieves a position of stable dynamic equilibrium and limit cycle motion does not occur. In general, it may be observed that the friction characteristic of Fig. 3.2.1a will give rise to stick-slip vibration or stable displacement, depending on the system parameters. In any event, oscillations of quasi-harmonic form do not occur for this particular friction-velocity relationship. However, limit cycle motion is possible in the case of the humped friction-velocity curve of Fig. 3.2.2 and the phase plane solution of Fig. 3.2.3c illustrates that near-harmonic oscillation occurs. Hence, the hump in the friction-velocity curve appears to be one of the conditions necessary for the existence of this form of oscillation.

3.3 **Autonomous System**

3.3.1 **Method of First Approximation by Krylov-Bogoliuboff**

For an autonomous system, eq. (3.1.4) can be written in the form
\[ \ddot{X} + X + \gamma G(\dot{X}) = 0 \quad (3.3.1) \]

If \( \gamma = 0 \), eq. (3.3.1) reduces to a simple linear d.e. with solution

\[ X = a \sin (\tau + \phi) \quad (3.3.2) \]

where \( a \) and \( \phi \) are constants.

For \( \gamma \neq 0 \) but small, eq. (3.3.2) can be used as a generating solution for the first approximation, provided the quantities \( a \) and \( \phi \) are considered not as constants but as certain functions of time to be determined. Thus we have

\[ X = a(\tau) \sin [\tau + \phi(\tau)] \quad (3.3.3) \]

This is the basic idea of the method for the first approximation of the solution of the d.e. by Krylov-Bogoliubov [56](K.& B. Method). In developing the method an additional condition was imposed that \( \dot{X} \) should be of the form

\[ \dot{X} = a \cos (\tau + \phi) \quad (3.3.4) \]

From equations (3.3.3) and (3.3.4) one obtains the d.e.
\[ \dot{a} \sin (\tau + \phi) + a \dot{\phi} \cos (\tau + \phi) = 0 \]  \hspace{1cm} (3.3.5)

A second d.e. is obtained by substituting the expressions for \( X, \dot{X} \) and \( \ddot{X} \) (differentiating with respect to \( a \) and \( \phi \)) in eq. (3.3.1). Solving these two equations gives expressions for \( \dot{a} \) and \( \dot{\phi} \) as periodic functions of time. Owing to the smallness of \( \gamma \), \( a \) and \( \phi \) can be considered as slowly varying functions of \( \tau \). Hence we can assume \( a \) and \( \phi \) remain constant over the interval \( \tau \) to \( \tau + 2\pi \).

Integration of the equations for \( \dot{a} \) and \( \dot{\phi} \) between the limits \( \tau \) to \( \tau + 2\pi \), shows that all trigonometric terms drop out [57], and only the constant terms \( K_0(a) \) and \( H_0(a) \) remain. We have

\[ \frac{d\dot{a}}{d\tau} = -\gamma K_0(a) \quad ; \quad \frac{d\dot{\phi}}{d\tau} = \frac{\gamma}{a} H_0(a) \]  \hspace{1cm} (3.3.6)

Replacing \( K_0(a) \) and \( H_0(a) \) by their Fourier expansions [58] gives the usual form of the first approximation by Krylov-Bogoliubov.

\[ \frac{d\dot{a}}{d\tau} = -\frac{\gamma}{2\pi} \int_0^{2\pi} G(a \cos \psi) \cos \psi d\psi = \dot{\phi}(a) \]  \hspace{1cm} (3.3.7)
\[
\frac{d\psi}{d\tau} = 1 + \frac{\gamma}{2a^2 \pi} \int_0^{2\pi} G(a \cos \psi) \sin \psi d\psi = \Omega(a) \quad (3.3.8)
\]

The condition for a stationary oscillation or a limit cycle is \( \phi(a) = 0 \) [59]. That is \( \phi(a_1) = 0 \) is the condition for a limit cycle with amplitude \( a_1 \).

To investigate the stability of the first approximation, we consider a slightly perturbed amplitude \((a_1 + \delta a)\) where \( \delta a \) is an absolute value of departure. It can be shown by the variational equations that to the first order

\[
\frac{d}{d\tau} (\delta a) = \phi_a(a_1) \delta a
\]

Thus if \( \phi_a(a_1) < 0 \) we have \( \frac{d}{d\tau} (\delta a) < 0 \), in other words the initial departure \( \delta a \) has a tendency to disappear for \( \phi_a(a_1) < 0 \). Thus \( \phi_a(a_1) < 0 \) is the condition for a stable limit cycle and the condition \( \phi_a(0) > 0 \) is equivalent to the existence of an unstable singularity, i.e. the existence of soft self-excitation.

For higher approximations the Asymptotic method developed by Bogoliubov, Krylov and Mitropolsky [60] can be applied. However, the small amount of accuracy gained from the higher approximations usually does not justify the very long calculations involved in carrying out the higher approximations. The first approximation usually gives sufficient information in a practical problem.
3.3.2 Application of the Exponential Expression to the K and B Method

It has been indicated earlier that the friction force function could be expressed in the form of an exponential.

Substitution of expression (3.2.1) into eq. (3.1.4) gives after manipulation

\[
\ddot{x} + \left( \frac{r\omega h + C_4}{E} \right) \dot{x} - \frac{C_1 + C_2 V}{EeC_3 V} e^{C_3 \dot{x}} + \frac{C_2}{EeC_3 V} \dot{x}e^{C_3 \dot{x}} + x = \frac{C_4 V + C_5}{E}
\]

(3.3.9)

The constant term on the right hand side of eq. (3.3.9) constitutes the static displacement of the vibration, since we are interested only in the amplitude of the oscillation therefore this constant term is omitted in the analysis. Thus from eq. (3.3.1) we have

\[
\gamma G(\dot{x}) = \frac{D_1}{E} (\dot{x} - D_2 e^{C_3 \dot{x}} + D_3 \dot{x} e^{C_3 \dot{x}})
\]

(3.3.10)

where \( D_1 = r\omega h + C_4 \), \( D_2 = \frac{C_1 + C_2 V}{D_1 e^{C_3 V}} \), \( D_3 = \frac{C_2}{D_1 e^{C_3 V}} \)

Substitution of eq. (3.3.10) into eq. (3.3.7) and eq. (3.3.8) yields expressions for \( \ddot{a} \) and \( \dot{\psi} \). (Appendix I)

\[
\ddot{a} = -\frac{D_1}{2E} \{ a - 2D_2 I_1(C_3 a) + D_3 a[I_2(C_3 a) + I_0(C_3 a)] \} = \phi(a)
\]

(3.3.11a)
or 

\[ a = - \frac{D_1}{2E} \{ a - 2(D_2 + \frac{D_3}{C_3})I_1(C_3a) + 2D_3aI_0(C_3a) \} = \phi(a) \]  

(3.3.11b)

where \( I_0(C_3a) \), \( I_1(C_3a) \) and \( I_2(C_3a) \) are modified Bessel functions of the first kind.

and \( \psi = 1 \);  

(3.3.12)

since the integral of eq. (3.3.8) vanishes.

Generally, interest centres on stationary values for \( a \). If \( \phi(a) = 0 \) in eq. (3.3.11b), then

\[ 2(D_2 + \frac{D_3}{C_3})I_1(C_3a) - 2D_3aI_0(C_3a) - a = 0 \]  

(3.3.13)

Eq. (3.3.13) can be solved by a computer for discrete \( V \) values if the system constants are known.

Carrying out the differentiation of eq. (3.3.11b), we have

\[ \phi_a(a) = -\frac{D_1}{2E} \{ 1 - 2C_3D_2I_0(C_3a) + \]

\[ \frac{2}{C_3a} [D_3 + C_3D_2 + (C_3a)^2D_3] I_1(C_3a) \} \]

(3.3.14)

Thus the stability of the amplitude obtained from eq. (3.3.13) can be investigated by substituting it into eq. (3.3.14) and
Following the conditions for stability described earlier.

Eq. (3.3.9) can be further analysed by omitting the viscous damping term \( D_1 \dot{x}/E \). This represents a system with a friction characteristic as shown in Fig. 3.3.1 and with negligible damping. For this condition eq. (3.3.11b) reduces to

\[
\dot{a} = \frac{1}{2E} \left\{ 2 \frac{C_1 + C_2V}{e^{C_3V}} I_1(C_3a) - \frac{C_2a}{e^{C_3V}} [I_2(C_3a) + I_0(C_3a)] \right\}
\]

For the stationary state value of 'a' we set \( \dot{a} = 0 \) and have

\[
C_3V + \frac{C_3C_1}{C_2} = C_3a \frac{I_2(C_3a) + I_0(C_3a)}{2I_1(C_3a)} \tag{3.3.15}
\]

Application of values for \( I_1(C_3a), I_2(C_3a) \) and \( I_0(C_3a) \) permits the construction of the solution displayed by Fig. 3.3.2.

This solution indicates that vibration will commence at a velocity corresponding to the peak of the friction-velocity curve (See Fig. 3.3.1). In the undamped case the vibration amplitude increases without limit as the lower surface velocity increases. However actual systems possess some damping which suggests that amplitude limitation will exist. Indeed, the stability analysis to be discussed in section 3.3.4 will show that with damping present the vibration will be limited at some upper velocity boundary.
3.3.3 Application of the Polynomial Expression to the K and B Method

Substituting expression (3.2.2) in eq. (3.1.4) we have after some manipulation,

\[ \ddot{x} + \frac{1}{E} \left[ (r\omega h + p_1) \dot{x} - p_2 \dot{x}^2 + \ldots \ldots - (-1)^n p_n \dot{x}^n \right] + x = \frac{p_0}{E} \]

(3.3.16)

where \( E = m\omega^2 h \); and

\[ p_0 = c_0 + c_1 v + \ldots \ldots + c_n v^n \]

\[ p_k = c_k + \ldots \ldots + n! \binom{n}{k} c_n v^{n-k} \quad \text{for} \quad k = 0, 1, 2, \ldots, n \]

where the binomial coefficients are given by

\[ \binom{n}{k} = \frac{n!}{(n-k)!k!} \]

The constant term on the right hand side of eq. (3.3.16) constitutes only the static displacement therefore it can be omitted in the amplitude analysis.

Eq. (3.3.16) can be further simplified by letting

\[ Q_1 = r\omega h + p_1 \quad \text{and} \quad Q_k = \frac{p_k}{Q_1}, \quad k = 2, 3, \ldots, n \]

\[ \ddot{x} + \frac{Q_1}{E} \left[ \dot{x} - Q_2 \dot{x}^2 + \ldots \ldots - (-1)^n Q_n \dot{x}^n \right] + x = 0 \]

(3.3.17)

The application of the method specified in [56] gives the integrals
\[ a = - \frac{Q_1}{2E} \sum_{k=1}^{k<n+1} \frac{2k-1\mathcal{J}_k}{2(2k-2)} R_{2k-1} a^{2k-1} = \phi(a) \quad (3.3.18) \]

where \( R_1 = 1 \) and \( R_k = Q_k, \quad K = 2, 3, \ldots, n \)

and \( \psi = 1 \)

The condition for a stationary state amplitude is \( \phi(a) = 0 \). From eq. (3.3.18) we have

\[ \frac{k<n+1}{\sum_{k=1}^{k>n} \frac{2k-1\mathcal{J}_k}{2(2k-2)} R_{2k-1} a^{2k-2} = 0} \quad (3.3.19) \]

Differentiating eq. (3.3.18) with respect to \( a \) we have

\[ \phi_a(a) = - \frac{Q_1}{2E} \sum_{k=1}^{k<n+1} \frac{(2k-1) \mathcal{J}_k}{2(2k-2)} R_{2k-1} a^{2k-2} \quad (3.3.20) \]

Eq. (3.3.19) is a polynomial in \( a \) which can be easily solved with the aid of a computer. The regularity of the coefficients, which are all formed by some series in \( n \) and are related to the power of the variable \( \dot{x} \), permitted a completely generalized computer programme to be set up for carrying out the analysis with the friction force function represented by a \( n \)th order polynomial.
3.3.4 Stability by Singular Point Analysis

A singular point analysis provides further information concerning the conditions necessary for the existence of the oscillation and its stability [61]. For the case where the friction force function is represented by expression (3.2.1), the condition for a stable system can be obtained by substituting the expression for \( F(V-Y') \) into eq. (3.2.3)

\[
\frac{dy'}{dx} = \frac{[C_1+C_2(V-Y')]e^{-C_3(V-Y')} + (C_4V+C_5) - (C_4+R)Y' - X}{Y'}
\]

(3.3.21)

For small perturbations of \( Y' \) about a singular point, expanding \( e^{C_3Y'} \) and deleting terms in \( Y'^2 \) and higher powers gives

\[
\frac{dy'}{dx} = \frac{[(C_1C_3+C_2C_3V-C_2)e^{-C_3V} - C_4 - R] Y'}{Y'}
\]

(3.3.22)

\[
+ \frac{(C_1+C_2V) e^{-C_3V} + C_4V + C_5 - X}{Y'}
\]

Assessment of this equation shows that the singular point occurs at \( Y' = 0 \) and \( X = (C_1+C_2V) e^{-C_3V} + (C_4V+C_5) = 0 \)

The analysis proceeds by transferring the coordinate system to \( X = X' + \theta \), yielding
\[
\frac{dy'}{dx'} = \frac{[C_1C_3 + C_2C_3V - C_2) e^{-C_3V} - C_4 - R] y' - x'}{y'}
\] (3.3.23)

This last equation is of the form

\[
\frac{dy'}{dx'} = \frac{\xi x' + \sigma y'}{\epsilon x' + \xi y'}
\] (3.3.24)

where \(\xi = -1\); \(\sigma = (C_1C_3 + C_2C_3V - C_2) e^{-C_3V} - C_4 - R\);
\(\epsilon = 0\); \(\xi = 1\).

Following the criteria for discriminating between different types of singularities [61] which was derived from the characteristic equation for eq. (3.3.24). The characteristic equation has the form

\[
x^2 - (\sigma + \epsilon) x - (\xi \epsilon - \sigma \epsilon) = 0
\] (3.3.25)

Eq. (3.3.25) has solutions

\[
x_1, x_2 = \frac{1}{2} \left\{ (\sigma + \epsilon) \pm \sqrt{(\sigma - \epsilon)^2 + 4\xi \epsilon} \right\}^{\frac{1}{2}}
\]

Letting \(N = (\sigma - \epsilon)^2 + 4\xi \epsilon\), the criteria shows that for \(N < 0\), then \(\sqrt{N}\) is imaginary and if \((\sigma + \epsilon) < 0\), \(x_1, x_2\) are both complex conjugate, both having negative real parts, and the singularity is a stable spiral point. Thus from
eq. (3.3.24) we have \( (C_1C_3 + C_2C_3 V - C_2) < (R+C_4) e^{C_3 V} \) as the condition for a stable system or non-vibrating system.

At the boundary between stability and instability we have

\[
C_1C_3 + C_2C_3 V - C_2 = (R+C_4) e^{C_3 V} \tag{3.3.26}
\]

Letting \( y_1 = C_1C_3 + C_2C_3 V - C_2 \) and \( y_2 = (R+C_4) e^{C_3 V} \), a plot of \( y_1 \) and \( y_2 \) as functions of \( V \) is shown in Fig. 3.3.3.

It is quite apparent that there is instability for \( V_1 < V < V_2 \) and that eq. (3.3.26) has two roots. Vibration can be avoided by adjusting \( R \) in order that \( y_1 \) becomes tangent to \( y_2 \). At this point we have \( dy_1/dV = dy_2/dV \), therefore it can be shown that the critical velocity at the tangent point is

\[
V_c = \frac{1}{C_3} \ln \left( \frac{C_2}{C_4 + R_c} \right) \tag{3.3.27}
\]

Substituting \( V_c \) in eq. (3.3.26) we obtain the damping \( R_c \) required for a completely stable system, thus

\[
R_c = C_2 e^{C_3} \frac{C_2}{C_4} - C_4 \tag{3.3.28}
\]
3.4 Non-Autonomous Systems

In the presence of the transverse external excitation, the equation of motion takes the form of eq. (3.1.5).

If the friction force function is absent then we have a linear second order d.e. subjected to harmonic forcing. The steady state solution of the equation is [62]

\[ X = \frac{F_0}{\left[ (1 - \alpha^2)^2 + (R\alpha)^2 \right]^2} \]  

(3.4.1)

where \( F_0 = \frac{\rho v^2 e}{m \omega^2 h} \), \( \rho \) is the mass of an out-of-balance weight, \( e \) the eccentricity, \( v \) the frequency of the external excitation, \( \alpha \) the frequency ratio and \( R = r/m\omega \).

A continuous curve of \( X \) vs. \( \alpha \) can be obtained by varying the frequency ratio.

However, when the friction force function is present we have a nonlinear non-autonomous system. Under these conditions the existence of stable amplitude values becomes of interest.

In the analysis of the non-autonomous system the friction force function is represented by a \( n^{th} \) order polynomial. Following the same procedure as carried out for the autonomous system, we have
\[ \ddot{X} + X = \gamma G(\dot{X}) + F_0 \sin \alpha \tau \quad (3.4.2) \]

where \( \gamma G(\dot{X}) = -\frac{Q_1}{E} [\dot{X} - Q_2 \dot{X}^2 + \ldots + (-1)^n Q_n \dot{X}^n] \quad (3.4.3) \)

where \( Q_k \) and \( E \) have the same meaning as in the autonomous case (See p. 35).

Eq. (3.4.2) can be transformed by introducing a new variable \[ X = Y + \frac{F_0}{\alpha^2} \sin \alpha \tau \quad (3.4.4) \]

Substituting the above expression and its corresponding expressions for \( \dot{X} \) and \( \ddot{X} \) into eq. (3.4.2), we have

\[ \ddot{Y} + Y = \gamma G(\dot{Y}, \alpha \tau) \quad (3.4.5) \]

where \( \gamma G(\dot{Y}, \alpha \tau) \) has the form as eq. (3.4.3) except that the variable \( \dot{X} \) is replaced by \( (\dot{Y} + \frac{F_0}{\alpha^2} \cos \alpha \tau) \).

In general, the nonlinear function in eq. (3.4.5) is considered to have the form [64]

\[ G(\dot{Y}, \alpha \tau) = g_0(\dot{Y}) + \sum_{n=0}^{\infty} [g_{n1}(\dot{Y}) \cos n \alpha \tau + g_{n2}(\dot{Y}) \sin n \alpha \tau] \quad (3.4.6) \]

where \( g_0, g_{n1} \) and \( g_{n2} \) are normally expressed as polynomials in \( \dot{Y} \).
When \( \gamma = 0 \), we have as solution

\[
Y = a \sin(\tau + \phi) ; \dot{Y} = a \cos(\tau + \phi), \text{ and } \psi = \tau + \phi
\]

when \( \gamma \neq 0 \) but small, solution for eq. (3.4.5) can be sought in the form

\[
Y = a \sin \psi + \gamma U_1(a, \psi, \alpha \tau) + \ldots (3.4.7)
\]

The function \( U_1(a, \psi, \alpha \tau) \) are periodic in both the angular variables \( \psi \) and \( \alpha \tau \) with a period \( 2\pi \), and that \( \psi = \alpha \tau + \phi \).

### 3.4.1 Non-Resonance Case

In the absence of resonance, there is no stationary phase relationship between the external frequency and the frequency of the self-induced vibration, therefore the phase does not exert any influence either on the amplitude or on the full phase of the oscillation. Thus the quantities \( \dot{a} \) and \( \dot{\psi} \) can be defined as [65]

\[
\dot{a} = \gamma A_1(a) + \gamma^2 A_2(a) + \ldots (3.4.8a)
\]

\[
\dot{\psi} = 1 + \gamma B_1(a) + \ldots (3.4.8b)
\]

The procedure of finding expressions for \( U_1, A_1 \) and \( B_1 \) are similar to that of the autonomous system [66],...
except that function $U_1$ depends on $\alpha \tau$ as well as $a$ and $\psi$.

Accordingly, the expressions for $A_1$, $B_1$ and $U_1$ are:

$$A_1 = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} G(a, \psi, \alpha \tau) \cos \psi \, d(\alpha \tau) \, d\psi \quad (3.4.9a)$$

$$B_1 = -\frac{1}{4a\pi^2} \int_0^{2\pi} \int_0^{2\pi} G(a, \psi, \alpha \tau) \sin \psi \, d(\alpha \tau) \, d\psi \quad (3.4.9b)$$

$$U_1 = \frac{1}{2\pi^2} \sum_{p,q} \frac{\cos(p \alpha \tau + q \psi)}{1 - (p \alpha + q)^2}$$

$$\left[p^2 + (q^2 - 1)^2 \neq 0\right]$$

$$\int_0^{2\pi} \int_0^{2\pi} G(a, \psi, \alpha \tau) \cos(p \alpha \tau + q \psi) \, d(\alpha \tau) \, d\psi$$

$$+ \frac{\sin(p \alpha \tau + q \psi)}{1 - (p \alpha + q)^2}$$

$$\int_0^{2\pi} \int_0^{2\pi} G(a, \psi, \alpha \tau) \sin(p \alpha \tau + q \psi) \, d(\alpha \tau) \, d\psi \quad (3.4.9c)$$

where $G(a, \psi, \alpha \tau) = G(\alpha \cos \psi, \alpha \tau)$

Substitution of eq. (3.4.6) into equations (3.4.9a), (3.4.9b) for $A_1$ and $B_1$, all the terms behind the summation sign of eq. (3.4.6) will disappear upon integration between the interval $2\pi$. Thus $A_1$ and $B_1$ can be written as
\[ A_1 = \frac{1}{2\pi} \int_0^{2\pi} g_0(a \cos\psi) \cos\psi \, d\psi \]  
(3.4.10a)

\[ B_1 = -\frac{1}{2a\pi} \int_0^{2\pi} g_0(a \cos\psi) \sin\psi \, d\psi \]  
(3.4.10b)

It follows that in the equations of the first approximation there appears only the free term \( g_0(a \cos\psi) \) of the expansion of the perturbing force \( G(a \cos\psi, \alpha\tau) \).

The effect of the external periodic excitation is felt only in the second approximation. The first approximation determines the existence and the stability of the auto-periodic oscillation.

The function \( g_0(\dot{Y}) \) of eq. (3.4.6) can be obtained by developing the function \( G(\dot{Y}, \alpha\tau) \) and collecting terms not depending explicitly on \( \tau \). Thus, we have

\[
\gamma g_0(\dot{Y}) = -E_0 + E_1 \dot{Y} - E_2 \dot{Y}^2 + \ldots - (-1)^n E_n \dot{Y}^n
\]  
(3.4.11)

where

\[
E_k = R_k + \sum_{m=1}^{k+2m\leq n} \frac{2m-1}{2m} \frac{\alpha}{2} \frac{R_{k+2m} L^{2m} }{2(2m-1)}
\]  
(3.4.11a)

\[ k = 0, 1, 2, \ldots, n \]

and \( R_0 = 0, \ R_1 = 1, \ R_k = 0 \) for \( k = 2, 3, \ldots, n \);

\[
L = \frac{F_0}{1 - \alpha^2}
\]
where the binomial coefficients are given by $k^m = \frac{k!}{(k-m)!m!}$

Substitution of eq. (3.4.11) in eq. (3.4.10a) yields

$$a = \frac{Q_1}{2E} \sum_{k=1}^{k<n+1} \frac{2k-1}{(2k-2)} \frac{\gamma}{E_{2k-1}} a^{2k-1} = \Lambda(a) \quad (3.4.12)$$

and $\psi = 1$

The stationary state is reached when

$$\lambda(a) = \frac{\Lambda(a)}{a} = 0 \quad (3.4.13)$$

Differentiating eq. (3.4.13) with respect to $'a'$ yields

$$\lambda_a(a) = \frac{Q_1}{2E} \sum_{k=1}^{k<n+1} \frac{2k-1}{(2k-2)} \frac{\gamma}{E_{2k-1}} (2k-2)a^{2k-3} \quad (3.4.14)$$

The condition for the stability of the amplitude $a_1$ according to Krylov-Bogoliuboff [67] is that the stationary state is stable if $\lambda_a(a_1) > 0$; and that $\lambda(0) < 0$ is the condition for self-excitation.

When the autoperiodic oscillation is absent, that is when $a = 0$, in order to investigate the effect of the external periodic excitation the second approximation must be employed. We have from eq. (3.4.9c)
\[ U_1(0, 0, \alpha \tau) = \frac{1}{\pi} \left[ \frac{\cos \alpha \tau}{1 - \alpha^2} \right] \int_0^{2\pi} G(0, 0, \alpha \tau) \cos \alpha \tau \, d(\alpha \tau) \]  \quad (3.4.15) \]

and from eq. (3.4.3)

\[ \gamma G(0, \alpha \tau) = -\frac{Q_1}{E} \left[ L \cos \alpha \tau - Q_2 L^2 \cos^2 \alpha \tau + \ldots + (-1)^n Q_n L^n \cos^n \alpha \tau \right] \]

Substitution of the expression for \( \gamma G(0, \alpha \tau) \) in eq. (3.4.15) yields

\[ \gamma U_1(0, 0, \alpha \tau) = -\frac{Q_1 \cos \alpha \tau}{E(1-\alpha^2)} \sum_{k=1}^{n+1} \frac{2k-1}{2} \frac{J_k}{2} R_{2k-1} L^{2k-1} \]

\[ \quad (3.4.16) \]

where \( Q_1, R_k \) and \( L \) have the same meaning as in the previous cases. The foregoing equation represents a purely heteroperiodic oscillation with frequencies equal to those of the external excitation.

Accordingly, for a system with only heteroperiodic oscillation, we have

\[ Y = -\frac{Q_1}{E} \sum_{k=1}^{n+1} \frac{2k-1}{2} \frac{J_k}{2} R_{2k-1} L^{2k-1} \cos \alpha \tau \]

\[ \quad (3.4.17) \]
3.4.2 Fundamental Resonance Case

The fundamental resonance case occurs when the external excitation is applied at frequencies close to the frequency of the autoperiodic oscillation, that is when $\alpha = 1$. The solution can still be sought in the form of eq. (3.4.7). However, owing to the presence of resonance, the phase difference between the free oscillation and the external excitation may exert a vital influence on the change in the amplitude and the frequency of the oscillation. Therefore the quantities $\dot{a}$ and $\dot{\psi}$ are defined as functions of $\phi$ as well as 'a'.

$$\dot{a} = \gamma A_1(a, \phi) + \gamma^2 A_2(a, \phi) + \ldots$$

$$\dot{\psi} = 1 + \gamma B_1(a, \phi) + \ldots$$

Following the asymptotic method by K.B.M. [68], expressions are obtained for finding the quantities $U_i$, $A_i$ and $B_i$.

The method of equivalent linearization [69] provides a simpler procedure for deriving the expressions of the first approximation which will reveal sufficient information for the fundamental resonance. The solution is sought in the form

$$X = a \sin \psi ; \quad \psi = \alpha t + \phi$$

The nonlinear exciting force of eq. (3.4.5) can be replaced
by the equivalent linear one, thus

\[ YG(\alpha T, \dot{x}) = -k_1 x - \lambda_1 \dot{x} \]

The d.e. in the equivalent linearized form becomes

\[ \ddot{x} + \lambda_1 \dot{x} + (1 + k_1) x = 0 \]

for the first approximation, according to Minorsky [69], we have

\[ \dot{a} = -\frac{a}{2} \lambda_1 \quad ; \quad (3.4.18a) \]

and

\[ \phi = (1 + k_1)^{\frac{1}{2}} - a = \frac{1}{2a} (1 + k_1 - a^2) \]

for \( k_1 \ll 1 \); \( a = 1 \) \quad (3.4.18b)

Expressions for the equivalent parameters \( k_1 \) and \( \lambda_1 \) are obtained by equating the fundamental harmonic of the non-linear force term to the linearized terms [69]. Accordingly, we have from eq. (3.4.2)

\[ \lambda_1 = -\frac{1}{a \pi \alpha} \int_0^{2\pi} \left( G(\dot{x}) + F_0 \sin(\psi - \phi) \right) \cos \psi d\psi \]

\[ = -\frac{1}{a \pi \alpha} \int_0^{2\pi} G(a \cos \psi) \cos \psi d\psi + \frac{F_0}{a \alpha} \sin \phi \quad (3.4.19) \]
\[ k_1 = - \frac{1}{a \pi} \int_0^{2\pi} [G(a \cos \psi) + F_0 \sin(\psi - \phi)] \sin \psi \, d\psi \]

\[ = - \left( \frac{F_0}{a} \right) \cos \phi \quad (3.4.19b) \]

Substituting \( k_1 \) and \( \lambda_1 \) to the expressions for \( \dot{a} \) and \( \dot{\phi} \), eq. (3.4.18), and letting

\[ \delta_\epsilon = - \frac{1}{2a \pi a} \int_0^{2\pi} G(a \cos \psi) \cos \psi \, d\psi \]

we have the expressions of the first approximation:

\[ \dot{a} = - \delta_\epsilon a - \frac{F_0}{2a} \sin \phi \quad (3.4.20a) \]

\[ 2a \dot{\phi} = 1 - a^2 - \left( \frac{F_0}{a} \right) \cos \phi \quad (3.4.20b) \]

or \( K_1(a, \phi) = -2aa \delta_\epsilon - F_0 \sin \phi = 2a \dot{a} \quad (3.4.21a) \)

\( K_2(a, \phi) = (1 - a^2) a - F_0 \cos \phi = 2a \dot{\phi} \quad (3.4.21b) \)

The stationary state is given by the equations

\[ K_1(a, \phi) = 0 \; ; \; K_2(a, \phi) = 0 \quad (3.4.22) \]

Thus, equating equations (3.4.21a) and (3.4.21b) to zero, squaring and adding we have
\[ K(a, \alpha) = 4a^2\alpha^2\delta_e^2 + (1 - \alpha^2)^2 a^2 - F_0^2 = 0 \quad (3.4.23) \]

Substituting eq. (3.4.3), the expression for \( G(\dot{X}) \), into the expression for \( \delta_e \) and carrying out the integration, yields

\[
\delta_e = \frac{Q_1}{2aE\alpha} \sum_{k=1}^{n+1} \frac{2k-1}{2(2k-2)} R_{2k-1} a^{2k-1} a^{2k-1} \]

In order to obtain an expression for \( \delta_e^2 \), we let

\[
z_{k,k} = \frac{(2k-1)^{2}}{2(2k-2)} R_{2k-1} a^{2k-1} a^{2k-1} \]

where \( k = 1, 2, \ldots, n+1 \)

and \( z_{i,j} = \frac{2^{2i-1}}{2(2i-2)} R_{2i-1} a^{2i-1} a^{2i-1} \)

\[
= \frac{(2j-1)^{2}}{2(2j-2)} R_{2j-1} a^{2j-1} a^{2j-1} \]

where \( i = 1, 2, \ldots, n+1 \) and \( j = i+1, i+2, \ldots, n+1 \)

then let \( S_m = \sum_{i=1}^{i \leq \frac{n+1}{2}} z_{i,j} \)

\[
i \leq j \leq \frac{n+1}{2} \quad \text{and} \quad i+j = m \]

where \( m = 2, 3, \ldots, n+1 \); and \( n+1 \) is an even number and \( \leq (n+1) \)
finally we have

\[ 4a^2 \delta_e^2 = \left( \frac{Q_1}{aE} \right)^2 \sum_{n=0}^{\infty} S_m = \left( \frac{Q_1}{aE} \right)^2 \sum_{m=2}^{\infty} T_m a^{2m-2} a^{2m-2} \]

where \( T_m \) are the coefficient terms of the expression \( S_m \).

Substitution of the expression for \( 4a^2 \delta_e^2 \) in eq. (3.4.23) yields

\[ K(a, \alpha) = \left( \frac{Q_1}{E} \right)^2 \left[ \sum_{m=2}^{\infty} S_m \right] + (1 - \alpha^2)^2 a^2 - F_0^2 = 0 \quad (3.4.24) \]

Eq. (3.4.24) gives the condition for the existence of stationary state amplitude 'a' for discrete values of \( \alpha \) and \( F_0 \).

The stability of the stationary state can be investigated from the variational equation [70].

\[ 2a \frac{d\delta a}{d\tau} = K_{1a} \delta a + K_{1\phi} \delta \phi \]

\[ 2a \frac{d\delta \phi}{d\tau} = K_{2a} \delta a + K_{2\phi} \delta \phi \]

where \( \delta a \) and \( \delta \phi \) are perturbations in a and \( \phi \).

The characteristic equation of the system is

\[ aS^2 - (a K_{1a} + K_{2\phi}) S + (K_{1a} K_{2\phi} - K_{1\phi} K_{2a}) = 0 \]
The conditions for stability are

\[ aK_1 + K_2 < 0 ; K_1K_2 - K_1K_2 > 0 \]

The above conditions finally reduce to [70]

\[ \frac{da}{da} > 0 \text{ if } a < 1 \]

\[ \frac{da}{da} < 0 \text{ if } a > 1 \]

From \( \frac{dK}{da} = \frac{da}{da} \frac{dK}{\partial a} + \frac{dK}{\partial a} = 0 \)

we have

\[
\frac{da}{da} = -\frac{Q_1^2}{E} \left[ \sum_{m=2}^{n} 2(m-1)T_m \alpha^{2m-3} \alpha^{2m-2} \right] - 4\alpha(1-\alpha^2) a^2
\]

\[
\frac{Q_1^2}{E} \left[ \sum_{m=2}^{n} 2(m-1)T_m \alpha^{2m-2} \alpha^{2m-3} \right] + 2a(1-\alpha^2)^2
\]

(3.4.26)

Thus the stationary state amplitude \( a_1 \) is stable if \( \frac{da}{da} < 0 \) for \( a > 1 \) or if \( \frac{da}{da} > 0 \) for \( a < 1 \). The stability analysis determines whether or not the vibration is at frequency \( \alpha \) or at other frequencies.
3.4.3 **Entrainment of Frequencies**

The phenomenon of frequency entrainment occurs when a periodic force is applied to a system whose free oscillation is of the self-excited type [71]. If \( \alpha \), the frequency ratio of the external excitation to that of the free oscillation, is sufficiently far away from unity, there is usually the phenomenon of interference or 'beat' of the two frequencies. If however, \( \alpha \) approaches sufficiently near to unity the beats disappear suddenly and there remains only one frequency thus suggesting that the frequency of the auto-periodic oscillation has been entrained by the external frequency.

The entrainment of frequency may also occur when the frequency ratio \( \alpha \) is in the neighbourhood of an integer or a fraction. Under these conditions, the frequency of free oscillation is entrained by a frequency which is an integral multiple or submultiple of the external frequency.

Van der Pol gave a theory for this phenomenon [72] by assuming a solution of the form

\[
X(\tau) = b_1(\tau) \sin \alpha \tau + b_2(\tau) \cos \alpha \tau \quad (3.4.27)
\]

in which the function \( b_1(\tau) \) are assumed to be 'slowly varying functions' of time. From eq. (3.4.27) it is possible to reduce the original d.e. to a system of the form
\[ \dot{b}_1 = M_1(b_1, b_2) \quad \text{and} \quad \dot{b}_2 = M_2(b_1, b_2) \quad (3.4.28) \]

thus the conditions for a stationary oscillation, \( b_1 = \text{const.} \) and \( b_2 = \text{const.} \), reduce to \( M_1(b_1, b_2) = 0 \), and \( M_2(b_1, b_2) = 0 \).

A purely topological theory of synchronization based on the above theory was developed by Andronov and Witt [73], in which the system of equations (3.4.28) is transformed into the form

\[ \frac{db_2}{db_1} = \frac{M_1(b_1, b_2)}{M_2(b_1, b_2)} \]

In the present investigation, owing to the complexity of the non-linear function, the study is limited to the cases where the amplitude and frequency of the external excitation are either inside or outside the regions of entrainment but not near the boundary of the region of entrainment. Under such circumstances a simpler method could be employed by applying the principle of harmonic balance. The method gave sufficient information for the steady state condition.

In applying the method, the harmonic solution of eq. (3.4.2), when \( a=1 \), may be written to the first approximation, as
\[ x(\tau) = b_1 \sin \alpha \tau + b_2 \cos \alpha \tau \]  
\hspace{1cm} (3.4.29)

where the amplitudes \( b_1 \) and \( b_2 \) are expressed as constants since we are interested only with the periodic solution. If the external force is prescribed outside the regions of entrainment, one may expect the occurrence of an almost periodic oscillation. Under these conditions the method would still furnish a fairly good description of the almost periodic oscillation [74]. However, if the external force is prescribed close to the boundary of the regions of entrainment, the analysis does not adequately account for it since the waveform of the almost periodic oscillation differs considerably from that obtained as a sum of two simple harmonic oscillations. The amplitudes \( b_1 \) and \( b_2 \) in eq. (3.4.29) have to be considered as functions varying slowly with time, as in eq. (3.4.27).

Substituting eq. (3.4.2) and equating the coefficients of \( \sin \alpha \tau \) and \( \cos \alpha \tau \) from both sides of the equation, we have, after long algebraic manipulation, two equations in terms of \( b_1 \) and \( b_2 \).

\[ b_1 (1 - \alpha^2) - b_2 \alpha Z(a^2) = F_0 \]  
\hspace{1cm} (3.4.30)

\[ b_2 (1 - \alpha^2) + b_1 \alpha Z(a^2) = 0 \]
where $a^2 = b_1^2 + b_2^2$ and $Z(a^2)$ is a function of $a^2$.

Dividing eq. (3.4.30) through by $a$ and letting $\alpha' = (1 - \alpha^2)/a$ we have from eq. (3.4.30)

1. $b_1\alpha' - b_2Z(a^2) = F_0/\alpha \quad (3.4.31)$
2. $b_2\alpha' + b_1Z(a^2) = 0$

Squaring both equations of (3.4.31) and adding we finally have an equation in terms of $a$, $\alpha'$, and $F_0$

$$a^2[\alpha'^2 + Z^2(a^2)] = (F_0/\alpha)^2 \quad (3.4.32)$$

The function $Z(a^2)$ of eq. (3.4.32) has the generalized form for a friction force function expressed in the form of a $n^{th}$ order polynomial,

$$Z(a^2) = \frac{Q_1}{E} \sum_{k=1}^{n+1} \frac{2k-1}{2(2k-2)} R_{2k-1} \alpha^{2k-2} a^{2k-2} \quad (3.4.33)$$

where $Q_1 = rwh + P_1$, $R_1 = 1$, $R_k = Q_k$ for $k = 2, 3, \ldots$, $< \frac{n+1}{2}$, same as in the previous cases.

Substituting eq. (3.4.33) into eq. (3.4.32) we have

$$a^2 \alpha^2 Z^2(a^2) + (1 - \alpha^2)^2 a^2 - F_0^2 = 0 \quad (3.4.34)$$
Eq. (3.4.34) has a similar form as eq. (3.4.23) of the fundamental resonance case by the first approximation method of Krylov-Bogoliuboff. In fact the method of equivalent linearization is based on the principle of harmonic balance.

\( a^2 \) values can be obtained from eq. (3.4.34) for discrete values of \( a \) and \( F_0 \). The amplitudes \( b_1 \) and \( b_2 \) of eq. (3.4.29) can be obtained by solving the simultaneous equations of eq. (3.4.31) with the \( a^2 \) values obtained from eq. (3.4.34), which gives

\[
b_1 = \frac{F_0 (1-a^2)}{a^2 z^2 + (1-a^2)^2} ; \quad b_2 = -\frac{F_0 z a}{a^2 z^2 + (1-a^2)^2}
\]

(3.4.35)

When \( a \) is in the neighbourhood of 2, 3 etc. sub-harmonic entrainment may exist in the system. An approximation solution for eq. (3.4.2) may have the form [75].

\[
X(\tau) = b_1 \sin c a\tau + b_2 \cos c a\tau + \frac{F_0}{1-a^2} \sin a\tau
\]

(3.4.36)

where \( c = \frac{1}{2}, \frac{1}{3} \) etc.

Again, two equations in terms of \( b_1 \) and \( b_2 \) can be obtained by substituting eq. (3.4.36) and expressions for \( \dot{x} \) and \( \ddot{x} \) into eq. (3.4.2) and equating the coefficients of \( \sin c a\tau \) and \( \cos c a\tau \) separately to zero. However, due to the extremely complicated form of the equations, a generalized equation such as eq. (3.4.32) cannot be obtained for this case.
Alternatively, an approximate solution for eq. (3.4.2) may be obtained from a slightly different approach, in which the solution is sought in the form [76].

\[ X(t) = a \sin t + a_e \sin \alpha t \] (3.4.37)

In addition, a phase angle is introduced to the external forcing term of eq. (3.4.2). Thus eq. (3.4.2) can be written as

\[ \ddot{X} + X = \gamma G(\dot{X}) + F_0 \sin(\alpha t + \phi) \] (3.4.38)

Three equations can be obtained by substituting the solution of eq. (3.4.37) into eq. (3.4.38) and equating the coefficients of \( \sin \alpha t \) and \( \cos \alpha t \) from both sides of the equation, also equating the coefficients of \( \cos t \) to zero. Thus we have

\[ (1 - \alpha^2)a_e \sin t + H_1(a, a_e) \cos t + H_2(a, a_e) \cos \alpha t \]

\[ = F_0 \sin \alpha t \cos \phi + F_0 \cos \alpha t \sin \phi \] (3.4.39)

and

\[ (1 - \alpha^2)a_e = F_0 \cos \phi \] (3.4.40a)

\[ H_2(a, a_e) = F_0 \sin \phi \] (3.4.40b)

\[ H_1(a, a_e) = 0 \] (3.4.40c)
where \( H_1(a,e_a) \) and \( H_2(a,e_e) \) are functions of \( a \) and \( e_e \), the amplitudes of the autoperiodic and heteroperiodic oscillation respectively. The functions are obtained by collecting terms of \( \cos \tau \) and \( \cos \omega \tau \) respectively from the expansion of the polynomial \( G(\dot{\theta}) \) in terms of \( \dot{\theta} = a \cos \tau + a e_e \cos \omega \tau \). Solving the three equations of eq. (3.4.40) gives values of \( a, e_e \) and \( \phi \).

Owing to the presence of the \( e_e \sin \omega \tau \) term, a generalized solution equation as given in equations (3.4.33) and (3.4.34) cannot be obtained for the equations of eq. (3.4.40). For different orders of polynomial, different sets of equations have to be derived. Appendix (II) shows an expansion of a friction force function in terms of \( \dot{\theta} = a \sin \tau + e_e \sin \omega \tau \) for a 7th order polynomial.

The present investigation is mainly concerned with finding the effect of the external excitation upon the amplitude of the friction-induced vibration in the steady state condition; whereas the phase relationship among these amplitudes is not of particular interest. For this reason, the approximate solution of eq. (3.4.37) provides a simple but effective tool for gaining some insight of the non-autonomous system with friction-induced vibration. Approximate solutions for \( \alpha \neq 1 \) and for \( \alpha = 2, 3, \) etc. can be obtained by applying the approximate solution of eq. (3.4.37) to eq. (3.4.38). When \( \alpha=1 \), either the first approximation of the K and B method or the harmonic balance method using
eq. (3.4.29) can be applied. The K and B method for the non-resonance case provides a useful technique for preliminary investigation of the system with particular reference to the external excitation parameters. Needless to say, all these methods involve trigonometric manipulation, and the final numerical solution can only be obtained with the aid of a computer.

3.4.4 Dynamically Loaded System

The equation of motion for the third system is shown by eq. (3.1.6) in the non-dimensionalized form. In establishing this equation the friction force versus velocity characteristic was considered unchanged during dynamic loading, i.e. the same friction-velocity curve as in the autonomous case was applied. This may not be the case in practice; the presence of a normal periodic excitation may cause early breakdown of the asperity contact. The external excitation was applied at the loading end which was connected to the elastic beam system by a massive steel clamp and two heavy steel beams. Therefore normal oscillation at the slider end due to the external excitation would be negligibly small and was not considered in the equation.

Eq. (3.1.6) can be re-written to give

$$\ddot{X} + \gamma (1 + \beta \sin \omega t) G(\dot{X}) + X = C_0 + \beta C_0 \sin \omega t$$  \hspace{1cm} (3.4.41)$$

where $C_0$, a constant, is the static displacement of the
friction-velocity curve, and $G(\dot{x})$ is a nonlinear function in $\dot{x}$.

Eq. (3.4.41) is a nonlinear d.e. with periodic coefficients. There are methods for solving this type of equations such as the Stroboscopic method [77] or the WKBJ method [78]. However, the analysis involves tedious algebraic manipulation even for a linear system or for a system with only one nonlinear term. In the present investigation, eq. (3.4.41) was investigated by numerical methods using a computer.

3.5 Summary

In applying the theoretical analysis, experimental friction force functions were computer fitted by a $n^{th}$ order polynomial with coefficients $C_0, C_1, \text{etc.}$ as shown in eq. (3.2.2). The coefficients $C_k$ were transformed into coefficients $P_k, Q_k$ and $R_k$ respectively as shown in equations (3.3.16), (3.3.17) and (3.3.18). Finally a solution equation for the autonomous case was obtained as shown in eq. (3.3.19). Solving eq. (3.3.19), the steady state amplitude 'a' was obtained. The stability of the amplitude was investigated by applying eq. (3.3.20).

The coefficients $R_k$ were further transformed to give coefficients $E_k$ according to the expression of eq. (3.4.11a). An equation in terms of amplitude 'a' for the non-autonomous non-resonance case was obtained as shown
in eq. (3.4.12). The stationary state amplitude and its stability could be investigated by applying equations (3.4.13) and (3.4.14).

For the fundamental resonance case, the coefficients $R_k$ were applied to the expressions for $Z_{k,k'}$ and $Z_{i,j}$. The coefficients $T_m$ were obtained by collecting terms of $Z_{i,j}$ for $i+j = m$. Finally eq. (3.4.24) and eq. (3.4.26) for the amplitude and frequency of the fundamental resonance oscillation and the stability of the amplitude with frequency $\omega$ were obtained. In the subharmonic entrainment cases, equations were derived for a friction force function expressed by a 7th order polynomial (Appendix II).

The exponential expression of eq. (3.2.1) was used in the theoretical analysis of the autonomous case as well as in all transient state investigations by numerical methods.

In the autonomous case, the existence and non-existence of the self-excited vibration as well as the growth and decay of the stable oscillation is predicted for various forms of friction-velocity characteristic curve. The quasi-harmonic type friction-induced vibration may be extinguished by the application of external viscous damping its magnitude is related to the coefficients of the exponential expression representing the form of the friction-velocity characteristic curve.
When the quasi-harmonic type friction-induced vibration is under the influence of external transverse excitation, there exists two distinctive cases of particular interest, depending on whether the frequency ratio $\alpha$ is close to an integer or far away from it. When $\alpha$ is not equal or close to an integer, the amplitude of the autoperiodic oscillation is decreased and is eventually completely extinguished as the external force magnitude is increased. When the frequency of the external excitation is equal or close to a multiple of the frequency of the autonomous case, subharmonic entrainment is predicted, whereby the system vibrates at a frequency equal or close to the frequency of the autonomous case. The amplitude of vibration during subharmonic entrainment, particularly in the case of $\frac{1}{2}$ harmonic, is increased as the magnitude of the external excitation is increased, until a critical value is reached whereby the amplitude of the vibration drops off rapidly and only the heteroperiodic oscillation remains at the frequency of the external excitation, usually of small amplitude due to the comparatively high frequency.
IV EXPERIMENTAL

4.1 Introduction

The variation of friction force with sliding velocity has been observed and discussed extensively in the literature. Various types of apparatus have been employed for the investigation of friction-velocity relationships. Pin on disc machines, crossed cylinders, four ball tribometers and other configurations have been used. In general, friction force is related to some form of displacement measurement. However, the assessment of dynamic friction force by recording the displacement alone does not always give a true representation of the friction values and caution must be exercised in the interpretation of records.

Dynamic friction measurements are subject to various errors which may be classified as follows:

(i) External vibration: Vibration from the drive mechanism of the friction machine or from external sources can disturb friction measurements. Godfrey [33] observed variations of the measured coefficient of friction produced by external vibrations. Accordingly, the accurate measurement of friction would appear to demand that extraneous vibration be removed from the apparatus and the surroundings.
(ii) Self-Induced Vibration: Many measuring systems are subject to friction-induced vibration of the stick-slip type. At low sliding velocities in particular, stick-slip vibration is liable to occur and the dynamic friction forces obtained by averaging displacements or forces during vibration may be grossly in error. In some systems, quasi-harmonic friction-induced vibration may exist. The vibration waveform is near-sinusoidal and friction forces obtained by averaging displacements are usually reasonably accurate. In some cases both stick-slip and quasi-harmonic vibration may be prevented or limited by the use of suitable damping devices in the measuring system. However, it will be shown that dynamic friction forces can be accurately measured even in the presence of friction excited vibration.

(iii) Surface, Lubrication and Environmental Factors: In many cases fluctuations in friction forces are produced by variations in the physical quality of surfaces, lubrication conditions, chemical and environmental factors. However, many of these fluctuations in friction can be removed by careful surface preparation, control of lubrication conditions and chemical surface factors and by the use of inert or high vacuum environments.
4.2 Apparatus

As it had been indicated earlier that the primary objective of the present investigation was to study the quasi-harmonic friction-induced vibration and its relationship with the dynamic friction characteristic. A secondary objective was to study the effect on the friction-induced vibration produced by external harmonic excitations, either transversely or normally. The apparatus was designed to meet these objectives.

Some early investigations in the present research showed that for certain combinations of friction materials a run-in period was required for the quasi-harmonic friction-induced vibration to occur. The choice of a rotating disc as the lower surface of the friction apparatus provided a convenient means when run-in is required. The ratio of the speed reducer and consequently the speed range of the rotating disc was determined to provide a suitable compromise of maximum and minimum speeds such that it would be low enough for the appearance of some stick-slip type friction-induced vibration and still high enough to show the decay of the quasi-harmonic vibration in some cases.

Consideration was given also to the natural frequency of the friction system which was basically determined by the vibrating mass and the stiffness of the supporting system. It was intended to study the sub-harmonic resonances of order $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$ in the external excit-
ation cases, therefore the natural frequency of the friction system should not exceed a certain limit so that its 4th harmonic would still be within the range of the external excitation generator.

Fig. 4.2.1 and Fig. 4.2.2 show two views of the experimental set-up which includes the friction apparatus and recording instruments.

The apparatus shown in Fig. 4.2.3 was designed for the study of friction and friction-induced vibration between two sliding surfaces. Three major variables were involved in the investigation, namely; surface velocity, stiffness of the supporting system and the normal load. Thus, the essential parts of the apparatus consisted of a rotating disc driven by a variable speed driving unit, a cantilever beam and a load system which did not alter the magnitude of the vibrating mass.

The friction couple was formed by a 4 inch diameter steel disc as the lower surface and a slider as the upper surface. The slider was attached to a hemispherical shaped slider mount (Fig. 4.2.3b) which provided the self-aligning action for the slider thus ensuring uniform contact. The slider mount was made from a steel ball bearing which rode in a hemispherical shaped retaining cup in the specimen holder. The slider, the slider mount and the specimen holder together with a proportion of the supporting beam formed the vibrating mass. The centre of the slider is
within \( \frac{1}{4} \) in. the C.G. of the vibrating mass. The beam was arranged such that its neutral axis is almost in line with the plane of the contacting surfaces, so that the torsional effect on the beam of the friction torque was negligible.

A cantilever supporting beam provided the elasticity of the system. The other end of the steel beam was fixed solidly in a steel clamp and shaft assembly which was pivoted by two low friction bearings. The length of the cantilever beam was adjustable and provision for interchangeability of beams was made so that the spring stiffness of the system could be varied. Load was applied by means of weights through a pulley arrangement which imposed a moment on the system. The system permitted the normal load to be varied without changing the vibrating mass. In order to minimise the curvature effect during vibration, the specimen holder location was arranged such that the radius of curvature for the slider vibration was in the same sense as the running track on the rotating disc.

The driving unit consisted of a variable speed \( \frac{3}{16} \) h.p. d.c. motor and a 100:1 double worm speed reducer. In the early stages of the development of the drive system a \( \frac{1}{20} \) h.p. servo motor driving a chain of spur gears were used to drive the rotating disc through a set of bevel gears. However it was found these elements imparted unwanted vibration to the disc and slider and they were sub-
sequently abandoned. In order to eliminate the effects of motor vibration on friction processes careful isolation techniques were employed. Power was transmitted by grooved aluminum sheaves and a soft rubber O-ring. This type of transmission reduced alignment problems and provided good vibration insulation between the motor and the speed reducer. Finally, in order to give good vibration isolation the friction apparatus was placed on a massive table which rested on a rigid foundation.

The $\frac{3}{16}$ h.p. d.c. motor had an effective speed range from 50 rpm to 3,000 rpm. Two two-step sheaves were used, these gave four speed ratios between the motor shaft and the speed reducer. The ratios were .41, .65, 1.55 and 2.45. These together with the 100:1 speed reducer provided a reasonable range of disc velocity. Thus, on a $3\frac{1}{2}$ in. dia. running track, the track velocity had a range from 0.038 in/sec to 13.5 in/sec. The d.c. motor was so oriented such that a minimum amount of magnetic field influence would be picked up by the measuring instruments.

During low speed tests a constant torque was applied to the rotating disc through a cord and pulley arrangement. This technique permitted the cancellation of the small amount of residual backlash in the worm gear speed reducer.

4.2.1 External Harmonic Vibration Generator

The second and third phases of the investigation required the provision of external excitation, thus harmonic
vibration generators were added to the friction apparatus. The very nature of the investigation, which involved friction-induced vibration, prohibited the application of tranverse excitation through linkages or any other mechanism hinged to a fixed location.

The exciting force could have been applied either by electromagnets or by unbalanced rotating masses. Although a device using electromagnets would have given a better frequency and amplitude control, it unavoidably required more electronic instruments thus the simpler concept using unbalanced rotating masses was adopted. This system consisted of one or two rotating unbalanced masses. The bearings supporting these masses were fixed to the vibrating mass.

In Fig. 4.2.4a was shown a vibration generator developed in the early stage of the investigation. The generator consisted of a miniature permanent magnet d.c. motor and two 1 in. dia. discs rotating in opposite directions. The motor drove the upper disc through miniature couplings while the lower disc was driven by the upper disc shaft through a chain of precision spur gears. The unbalanced masses were arranged such that the force was obtained in the same direction as the self-induced vibration. The motor was housed in a cylindrical shaped motor housing which permitted the motor to move freely in the vertical direction. The complete device weighed only 14 oz
was attached to the slider holder of the friction apparatus. Various out-of-balance masses were prepared which provided variable force amplitudes at a set frequency.

However, the unwanted high frequency oscillations which were associated with gears and ball bearings in the unit showed up in significantly high amplitudes in the acceleration signal, thus rendering the generator unusable when acceleration and velocity were to be recorded. The vibration generator was subsequently redesigned. The ball bearings and spur gears were eliminated and replaced by bearing felt bushes and a rubber O-ring.

Fig. 4.2.4b shows the redesigned harmonic vibration generator. It consisted of a \( \frac{1}{20} \) h.p. servo motor and a 2 in. dia. disc with 4 holes drilled at different radii. The motor was located away from the measuring instruments so that the field interference would not be picked up by the instruments. A soft rubber O-ring and a set of aluminum sheaves were used to drive the disc which ran on a pair of felt bushes. This technique permitted smooth and quiet operation even at rotating speeds exceeding 5,000 rpm. The soft O-ring showed no measurable effects on friction recording during friction-induced vibration. The 4 holes at different radii together with several different out-of-balance masses provided a wide range of force amplitudes over a range of frequencies. Unlike the earlier design, only one rotating disc was used in the final design,
therefore no provision was made to restrict the resultant force to one direction. However, the effect of the component on the longitudinal direction was negligible due to the rigidity of the beam system along this direction.

A similar arrangement (Ref. Fig. 4.2.2) was used for the investigation of the effect of harmonically excited loading. In this case the vibration generator was attached at the loading end of the cantilever beam arrangement.

4.3 Instrumentation

The investigation required the determination of the amplitude of friction-induced vibration over a range of disc speeds under the influence of various types of friction characteristic. The effect of external excitations on the system was also studied. The instrumentation was designed to meet the above requirements. In particular, the determination of the friction characteristic during vibration which was considered to be velocity dependent, was of major importance.

The conventional method of determining the friction characteristic by averaging displacements or forces during vibration does not give accurate information. In fact, the information thus obtained is usually grossly in error if the vibration is of the stick-slip type. During friction-induced vibration the relative speed of the friction couple
was varied and the acceleration had to be considered as well.

Considering the d.e. for a damped free vibration of a mass-spring-damper system:

\[ m\ddot{x} + r\dot{x} + kx = 0 \]

this could be written as \( \dot{x} = -(m\ddot{x} + kx) \). If \( (m\ddot{x} + kx) \) was plotted against the absolute velocity \( \dot{x} \), a straight line was obtained which had slope \( r \).

During frictional vibration the following d.e. applies

\[ m\ddot{x} + r\dot{x} + kx = f \]

thus a plot of \( (m\ddot{x} + kx) \) against \( \dot{x} \) represented the dynamic friction-velocity characteristic which included the viscous damping force. In practice scaled accelerometer and displacement signals were fed to the differential amplifier of an oscilloscope, and the velocity signal was introduced to the horizontal display amplifier. The method gave a realistic assessment of dynamic friction force.

The general arrangement of the circuitry is shown in Fig. 4.3.1. The details of the various measurements are as follows:
a. **Displacement**

In the early stages of the development of the measuring system a light discriminating resistor (LDR) system was used. The device consisted of a light strip with a slot for the light beam to pass through, a light source and a LDR. The strip was attached to one side of the steel cantilever beam. The light beam and the LDR were arranged such that a proportional amount of light which depended on the deflection of the steel beam was transmitted through the slot and impinged on the face of the LDR. The output from the LDR was displayed on the oscilloscope. The response of the LDR was fairly linear but there was a considerable phase lag which did not pose any problems when displacement alone was recorded. However, if velocity and/or acceleration were involved at the same time phase correction had to be applied which was not always a satisfactory solution. The LDR system was subsequently replaced by strain gauges.

At the root of the steel beam 350 ohm strain gauges were cemented, one on each side of the beam. Two identical dummy gauges were used for compensation purposes. The strain gauges formed a four arm bridge circuit and were connected to a bridge amplifier. The output from the amplifier was channeled to various recording instruments.

b. **Velocity**

An electromagnetic type transducer was used to record the velocity of the vibrating mass Fig. 4.3.2a.
The moving part of the transducer consisted of 1,350 turns of fine enamelled wire and was attached to the steel beam. The coil which had a resistance of 630 ohms and an inductance of 0.1 henrys was designed for low mass and high electrical output. Two horseshoe-shaped magnets housed in an aluminum box were positioned such that during beam vibration the conductors cut across the magnetic field between the poles of the magnets. The voltage generated was proportional to the velocity of oscillation of the coil. The output of this transducer was fed to a single stage d.c. amplifier Fig. 4.3.2b before it was channelled to various recording instruments. Capacitor coupling had to be avoided in the d.c. amplifier so as to ensure that no phase shift would occur after the amplification stage. The velocity amplifier had an output of 0.875 volt/in/sec.

c. **Acceleration**

A Model 305A Kistler servo accelerometer was attached at the top of the specimen holder. The accelerometer weighed 3 oz and had a dimension of 1 in. dia. x 2 in. The accelerometer was a self-contained unit; no amplifier was required and the output was fed directly to the recording instruments. Full scale output of the accelerometer was 5 volts. The full scale range was normally set at ± 50 g., thus giving a voltage sensitivity of 0.1 volt/g. and a resolution of less than 5 micro. g. The full scale range could be varied by varying an external resistor. The
displacement and the acceleration signals were within 1° of the expected 180° phase shift.

4.3.1 Recording System

The recording instruments included a Model 13-6624-00, Mark 842 Brush dual channel rectilinear oscillograph, a model 564 Tektronix dual beam storage oscilloscope and a model 502A Tektronix dual beam oscilloscope. The displacement and accelerometer signals were fed to one differential amplifier of the storage oscilloscope, and the velocity signal was introduced to the horizontal display amplifier. The calibration of these signals is shown in Appendix IV. The combined trace from the oscilloscope represented the dynamic friction force versus velocity. Similar friction-displacement and friction-acceleration traces could be obtained by introducing the displacement or acceleration signals to the horizontal display amplifier. The oscillograph was used for recording time based signals.

In addition to recording of various signals the recording system of Fig. 4.3.1 was designed to achieve two main objectives. The first objective was to record data at the same point on the disc running track throughout a test series. This method minimised the inconsistency arising from possible non-uniformity of the disc surface. The second objective was to obtain a single phase plane plot on the storage oscilloscope. The storage oscilloscope did not provide a 'built-in' means of obtaining this type of
display and an undesired continuous trace appeared on the screen. In order to overcome this difficulty the one cycle sequency triggering system of Fig. 4.3.3 was developed. In addition, a modification to the tube beam blanking circuit was necessary. The purpose of the triggering system was to unblank and blank the storage tube beam at the desired instants of time.

4.3.2 Spot Triggering Unit

A light discriminating resistor (LDR) was used for the spot triggering. Each time a shield, which could be attached at any desired point around the disc circumference, moved between the light beam and the LDR it activated a relay which simultaneously triggered the event marker of the oscillograph, the single sweep trigger of the time base amplifier of the oscilloscope and the specially designed sequence circuit of Fig. 4.3.3. The sequence trigger permitted the display of a single \( x - \dot{x} \) phase plane display on the oscilloscope screen. Provision was made also to utilise the sequence circuit for activating the event marker and single sweep trigger thus ensuring synchronous recording when a comparison between various combinations such as friction-velocity or friction-time traces, was required during one cycle of vibration.

4.3.3 One Cycle Sequence Triggering System

The one cycle sequence triggering system used the displacement signal produced by the strain gauges in the
friction system. It was important that switching load effects from the triggering system would in no way affect the displacement signals being received and recorded. A vacuum tube d.c. amplifier with a 10 megohms input impedance before the triggering unit eliminated switching load effect. The output impedance of this amplifier had a low value and was well suited to the input impedance of the triggering unit.

The essential parts of the sequence circuit consisted of three relays which were activated by the displacement signal. The negative electrical displacement signal was chosen as the start point. To obtain a one cycle display also meant that the negative displacement signal must also serve as an end point signal. It was necessary therefore to interlock this switching circuit so that the start and end point relays could neither operate simultaneously nor operate closure before start. Fig. 4.3.3a shows a diagram of the signal. The use of a third relay to the system provided the means of sequencing events to the desired operation. This additional relay was energised by the positive electrical displacement signal and again had to be sequenced so that a negative start signal had to occur before this relay would energise. A fourth relay provided the means for synchronous triggering of the recording instruments.
The amplitude of the electrical displacement signal was not constant and was dependent on the degree of displacement of the metal beam. It was necessary therefore to provide variable controls for adjusting the electrical signal in the triggering unit so as to encompass the varying signal levels thus ensuring relay closure. Electrical noise had to be eliminated so as to prevent accidental triggering of the unit.

4.3.4 Speed Counting System

The speed of the rotating disc was measured by a counting disc and LDR system. The counting system required no driving mechanism thus the counter unit could be placed in a remote area away from the friction apparatus. This procedure minimised the possibility of mechanical noise being picked up by the recording instruments. The counting disc had two rows of \( \frac{1}{6} \) in. dia. holes. The outer row with 60 equally spaced holes was used for low speed counting and the inner row with 12 equally spaced holes was used for high speed counting. A stroboscope was not suitable for the low speed range of 100 rpm.

The counting device consisted of two electrical pulse counters. One counter registered the number of holes scanned by the LDR and the other registered the elapsed time in minutes. The counting procedure was actuated by a multiple interval timer driven by a one rpm synchronous
motor. In addition, a 600 rpm synchronous motor together with a LDR circuit was used as a built-in calibration unit. Light sources for the LDR circuits were supplied from a built-in variable d.c. power supply. Provision was also made to transform the on-off signals received from a LDR circuit into near sine wave output which could be displayed on the oscilloscope screen. This method was particularly useful for providing an instant close estimate of the speed.

4.4 Specimens

Preliminary investigation indicated that several material and lubricant combinations gave the desired quasi-harmonic vibration characteristic. The following combinations were studied.

(i) Steel slider running on steel disc: Many methods for preparing the surfaces were tried. It was found that the most satisfactory surfaces were obtained by grinding and then lapping with fine lapping compound. Initially a steel slider specimen prepared from Keewatin steel (Appendix V) hardened to $R_c 55$ and a disc of Atlas Nutherm steel (Appendix V) hardened and annealed to $R_c 53.5$ were used. Inconsistency in friction results was observed, and uniform quasi-harmonic oscillation could not be achieved with this combination of friction materials. Later, the Keewatin steel slider was replaced by a mild steel slider and more
satisfactory results were obtained although some inconsistency still existed. It is likely that the nonuniformity of the disc surface was felt with less influence from a slider of softer material resulting in more uniformly deformed contacting areas. In general, stick-slip oscillation was observed for this combination of friction materials after a run-in period, often with the formation of a fine black deposit on the disc track. The behaviour was probably due to the fine lapping compound being embedded on the top layer of the surface structure during the lapping process.

The mild steel slider on hardened steel disc combination was also tested in the presence of petrolatum (U.S.P.) as the lubricant. The results were found to be more consistent along the disc track.

(ii) Blotting paper slider running on the same disc as (i) with automatic transmission fluid as lubricant: This particular combination simulates the behaviour of some automatic transmissions. The ordinary commercial facing for automatic transmission clutch plates is usually some kind of paper material which is resilient, spongy and readily absorbs lubricant. The material usually consists of cellulosic fibres which are coated with phenolic resin. It was found that blotting paper against steel gave friction results similar to those obtained from clutch facing materials (Appendix V). The combination with certain types
of automatic transmission fluid or their neutral fluids gave quasi-harmonic oscillation. It was decided that all tests would be carried out using blotting paper on the slider surface against a steel disc with automatic transmission fluids as the lubricants. This decision was governed by the fact that uniform vibration results were obtained.

The lubricants were supplied by Cities Service Oil Corp. of New Jersey. These included some neutral oils and some transmission fluids of various viscosities (Appendix V).

The steel disc used in both cases was made from Atlas Nutherm steel hardened and annealed to $R_c$ 53.5. The disc is 4 in. dia. and 1 in. thick. The normal running track had a mean diameter of $3\frac{1}{2}$ in. The slider was $\frac{3}{8}$ in. dia.

(iii) Polymer slider running on the same disc as (i): The slider were prepared by the Centre for Material Science, U.B.C. The first slider was prepared from polyester resin material. The slider surface measured $\frac{1}{4}$ in. dia. A second slider of carbon fibres composite of the resin had the same dimension as the resin slider. Carbon filaments of approximately 6 μm diameter were grouped in bundles of approximately 0.25 mm diameter. These bundles of carbon fibres were arranged perpendicular to the sliding surface as shown in Fig. 5.1.11a. With its light weight and high strength, carbon fibre composites have received wide attention in recent years, particularly in the air-
craft industry where turbine blades have been made from carbon fibre material.

(iv) Rubber slider running on same disc as (i): The slider was prepared by cementing a thin piece of Neoprene, \( \frac{1}{4} \text{ in. dia. x } \frac{1}{16} \text{ in. thick} \) onto a steel slider. The hemispherical shaped slider holder and the retaining cup were replaced by a cylindrical shaped slider holder. This alternation prevented the slider holder from rolling over due to the high friction force.

4.5 Experimental Method

4.5.1 Preliminary Investigation of System

Tests were conducted to study the effect of various parameters; these included disc speed, load, lubricants, external forcing amplitude and external forcing frequency. The influence of the stiffness of supporting system was investigated in the early stages of the research. Three cantilever supporting beams were tested, they were \( \frac{1}{8} \text{ in.}, \frac{3}{16} \text{ in.}, \text{ and } \frac{1}{4} \text{ in. thick} \) respectively. The beams were made from Atlas Nutherm steel and were annealed, hardened and tempered to obtain the best combination for toughness and hardness. It was found that a \( \frac{1}{4} \text{ in. thick x 1 in. deep} \) beam provided sufficiently high friction force with a displacement still low enough not to have significant curvature effect.
The standard friction system, using the $\frac{1}{4}$ in. thick beam had an equivalent vibration weight of 1.2 lb., the equivalent beam stiffness at the slider was 60 lb/in. and the viscous damping coefficient was in the region of 0.01 lb/in/sec. (Appendix III). The damping coefficient was obtained by performing a damped free vibration test with the slider clear of the surface. A displacement-time curve was obtained from such a test and the coefficient was calculated by the logarithmic decrement method. The system damping could be also obtained from the $(m\ddot{x} + kx)$ vs $x$ trace in which the coefficient was obtained directly from the slope of the trace (Appendix III). As it could be observed from the oscilloscope trace in Fig. A2 that the trace was almost linear therefore the value could be considered as viscous damping coefficient. The forgoing values gave a damped natural frequency of 138 rad/sec. for the slider supporting system. Owing to the smallness of the system damping the natural frequency was virtually the same as the damped case. The beam stiffness and the system natural frequency could be varied by varying the length of the supporting cantilever beam. The calibration of the system is described in Appendix IV.

4.5.2 Specimen Preparation

The steel surfaces were prepared by first grinding and then lapping on a cast iron lapping plate with
petroladum (U.S.P.) and alum powder as a cutting agent. The resultant finish was 25-30 microinch AA. Prior to each test the surfaces were relapped and thoroughly cleaned with hexane.

4.5.3 Test Procedure

The standard procedure for the investigation of the friction-induced vibration consisted of running the disc at its lowest speed and increasing the speed gradually to its maximum or until vibration decay occurred. During each speed increment a set of oscilloscope traces and chart records were taken, these usually included the friction-velocity and displacement-velocity traces from the storage oscilloscope, and velocity-time and acceleration-time or displacement-time traces from the chart recorder. All the results were taken at the same point on the disc. Prior to the completion of each test, several random disc speeds were re-run and records were again taken at these speeds. These results were compared with the results obtained in the first time. The foregoing procedure permitted the observation of any significant changes due to environmental factors or surface wear during a test. For each set of tests only one parameter was varied. Prior to each test, the slider was set at its neutral position and all instruments were checked for reference level.
Similar tests were carried out for different combinations of friction materials and for different lubricants. However, tests for the investigation of the effect of load were carried out only in two cases, \#9 oil and \#4 oil as the lubricant. It was felt that these tests would give sufficient information to show the effect of the load on the friction and on the friction-induced vibration.

As it has been described earlier the external excitation was applied by means of unbalanced rotating mass. The force thus obtained following the relationship $m \omega^2 r/g$; thus varying the frequency also varied the external force amplitude. Therefore it would seem impossible to vary either the external frequency or the external force amplitude alone. However, by adjusting the two parameters $m$ and $r$ in the relationship it was possible to obtain a reasonable range of values for either the frequency or the amplitude, although this type of variation during the conduct of a test was usually very cumbersome, and was not always satisfactory. A force amplitude ranged from $0.00011 N^2$ lb. using the smallest mass and the smallest radius, to $0.00065 N^2$ lb. using the largest mass and the largest radius, could be obtained from the present arrangement, where $N$ is the external excitation frequency in rev./sec.

Tests were performed by varying the frequency while keeping the disc velocity constant. In some cases attempts were made to keep the external force magnitude constant
by adjusting the parameters $m$ and $r$. Photographs of the displacement-velocity phase plane trace on the storage oscilloscope and chart records of displacement-time and velocity-time were taken for each frequency setting. The complete process was repeated for four different types of lubricant, namely; #5, #1, #9 and #7. These four lubricants displayed three distinctly different friction characteristics.
Experimental friction velocity traces obtained from the oscilloscope were fitted by exponential expressions and by $n^{th}$ order polynomials. These mathematical expressions were used in the theoretical verification of the experimental results. Owing to the complicated algebraic manipulation involved, even with a low order polynomial in the theoretical analysis of the non-autonomous cases, the polynomials expressions did not exceed $9^{th}$ order. This somewhat limited the accuracy of the polynomial fitted curves, however, the analysis of the non-autonomous cases was supplemented by applying the more accurate exponential expression to the equation of motion and solving the equation by numerical methods.

5.1 **Autonomous Case**

5.1.1 **Blotting Paper on Steel**

As it had been indicated earlier the blotting paper on steel combination together with automatic transmission fluid as lubricant simulated the behaviour of some automatic transmissions. Several automatic transmission fluids and their neutral fluids (Appendix V) were used in order to obtain various forms of friction-velocity characteristic. Three general forms of friction-velocity curves as indicated in Fig. 3.2.1 and Fig. 3.2.2 of the theoretical section were investigated. In the discussion throughout this chapter,
the friction-velocity curves having the general forms resembling those of Fig. 3.2.2, Fig. 3.2.1b and Fig. 3.2.1a will be referred to as Type A, B and C respectively. Type A, the humped form of Fig. 3.2.2 is discussed first, followed by Type B, the curve with initial negative slope, and finally Type C, the almost linear curve with positive slope is considered.

(a) Friction-Velocity Curve of Type A

Fig. 5.1.1 illustrates a phase plane oscilloscope trace of displacement \( x \) versus vibration velocity \( \dot{x} \) with #7 oil as the lubricant, at a load of 5.4 lb. and a disc speed of 1.05 in/sec. A plot of friction force versus velocity is displayed in the same diagram. The foregoing results were obtained during one cycle of the vibration. Similar traces were obtained for a sequence of disc velocities and the results of Fig. 5.1.2 were obtained by plotting vibration amplitude versus disc velocity. A plot of vibration frequency as a function of disc velocity is also shown in Fig. 5.1.2.

The exponential and polynomial functions of eq. (3.2.1) and eq. (3.2.2) were computer fitted to the experimental friction-velocity curve of Fig. 5.1.2 which was reproduced from the oscilloscope trace of Fig. 5.1.1 allowing for a small correction for the system damping. The exponential equation was applied to the theory developed earlier eq. (3.3.13), to give the theoretical amplitude-velocity curve displayed in Fig. 5.1.2. Theoretical amplitude values
derived from eq. (3.3.13) were inserted into eq. (3.3.14) in order to check the stability. A similar procedure was followed for the polynomial approximation employing eq. (3.3.19) and eq. (3.3.20). In Fig. 5.1.2 the amplitude velocity curve for the polynomial theory is shown for the region A to B only for the reason stated earlier in Chapter III, namely that the polynomial expression was applicable only within the velocity region for which the curve was originally fitted.

Fig. 5.1.3 illustrates four oscilloscope traces of the friction-velocity characteristic curve including system damping recorded at disc velocities of 0.84, 1.06, 1.30 and 1.64 in/sec respectively. A comparison of the curves is also shown in the same diagram. The humped shape appears to be very consistent.

(b) Friction-Velocity Curve of Type A2

The curves of Fig. 5.1.4 were obtained using #9 oil as the lubricant. The results were obtained following the same procedure as outlined earlier. The friction-velocity characteristic curve obtained from this combination has a similar form as the Type A1 except that the hump is less pronounced.

The amplitude-velocity curves of Fig. 5.1.2 and Fig. 5.1.4 illustrate that the experimental results and the predictions by the exponential and polynomial theories are in reasonable agreement. It should be noted that
the experimental points to the left of A in Fig. 5.1.4 represent stick-slip amplitude values. In theory and by experiment the quasi-harmonic oscillation commences at a discrete velocity A and the amplitude of vibration increases in a near-linear fashion with increasing velocity until sudden decay occurs at an upper critical velocity, B. Theory predicts that the vibration is stable between the velocity limits A and B. To the right of B instability is possible, and in fact two vibration amplitudes are predicted at each velocity. The vibration of larger amplitude is stable whereas the lower amplitude curve represents an unstable condition. If the system is perturbed to an amplitude between the two amplitude curves, vibration will grow until its amplitude reaches the stable amplitude curve. Alternatively, if the perturbation displacement is below the unstable amplitude curve, vibration will die out. Vibration in the region A to B corresponds to soft self-excitation whereby the system departs from an unstable singularity and arrives at a state of steady state vibration with a stable amplitude [40]. To the right of B the singularity is stable and hard self-excitation prevails. Under these conditions the barrier presented by an unstable limit cycle must be crossed before a stable vibration can exist (Ref. Fig. 3.1.1). Experimentally, the vibration tended to exist further to the right of point B largely because inconsistencies in friction along the disc track
generated small perturbations which carried the slider over the barrier formed by the unstable amplitude curve. However, at still higher disc velocities, small perturbations are not sufficient to carry the slider over the barrier, and vibration would not occur.

(c) Distinction Between Stick-Slip and Quasi-Harmonic Oscillations

The scales of Figures 5.1.1, 5.1.2 and 5.1.4 are dimensionless. The non-dimensionalized parameters are respectively: \( x = Xh \), \( \dot{x} = X\omega h \) and \( t = \tau/\omega \), where \( \omega h \) is equal to unity. In the amplitude of vibration versus velocity curve of Fig. 5.1.2 and Fig. 5.1.4 the dimensionless scales have the advantage of distinguishing the difference between the stick-slip type vibration and the quasi-harmonic form, since the 45° line in the amplitude-velocity diagram is a close approximation to the boundary between the stick-slip and the quasi-harmonic forms of vibration. The distinction between the two types of vibration can be easily visualized from the \( x-\dot{x} \) phase plane diagrams of Fig. 3.2.3. In the case of quasi-harmonic oscillation, the plot does not reach the zero sliding velocity axis, thus the amplitude of vibration is less than the disc velocity in terms of dimensionless scales, provided the phase plane is very nearly circular. Thus, if the amplitude of vibration appeared above the 45° line in the amplitude-velocity diagram the vibration would be of the stick-slip type.
(d) **Correlation Between the Amplitude of Vibration and the $f_{\mu_k}$-V Curve**

An examination of the friction-velocity curve and the amplitude-velocity curve of Fig. 5.1.2 and Fig. 5.1.4 reveals that the upper critical velocity occurred near a point where the slope of the friction-velocity curve began to change from negative to positive. It was also noted that the velocity $A$ where quasi-harmonic oscillation commenced, appeared near the peak of the hump. No quasi-harmonic oscillation was observed in the region to the left of the peak. However, stick-slip type oscillation could occur in this region depending on the slope of the friction-velocity curve over this region and the static friction characteristics of the friction combination. If the friction material combination has small rise in static friction or the slope of the friction-velocity curve over this region is very steep thus giving heavy surface damping, then under such conditions stick-slip oscillation may not occur. In fact, stick-slip oscillation was not observed with the Type A1 friction-velocity curve.

(e) **Effect of External Damping with Type A2 Friction-Velocity Curve**

The constants $C_1$ to $C_5$ for the exponential function of the friction-velocity curve of Fig. 5.1.4 were substituted into eq. (3.3.28) in order to obtain an estimate of the damping coefficient required for complete extinction of
vibration over the entire range of sliding velocities. The damping coefficient obtained from this calculation was 0.055. A test was performed using a permanent magnet as damper which had a damping coefficient of approximately 0.08. With this damper only stick-slip oscillation was observed in the low velocity region and quasi-harmonic oscillation did not occur. The amplitude-velocity curve obtained from this test is also shown in Fig. 5.1.4. Hence it is possible to introduce controlled damping into the system in order to prohibit quasi-harmonic oscillations. This finding could be of utility in the design of practical systems where vibration is undesired.

(f) Effect of Varying the Normal Load

Fig. 5.1.5 shows three experimental amplitude-velocity curves under different normal loads, 5.4 lb, 3.85 lb and 2.4 lb respectively. The friction-velocity curve has the form of Type A2. The results were obtained by varying the disc velocity and during each velocity setting the normal load was varied. It should be noted that the amplitude of vibration increased almost linearly as the disc velocity was increased in all three cases. Very little change in the amplitude of vibration was observed when the normal load was varied. However, there was a distinct difference in relation to the upper critical velocity, the smaller the normal load the earlier the decay commenced. It will be noted later in Fig. 5.1.7
that a similar phenomenon was also observed in the case with the Type B2 friction-velocity curve. A study of the one cycle friction-velocity characteristic traces revealed that there was a slight change in the slope of the friction-velocity curve as the normal load was varied. It was noted that the negative slope became less steep as the normal load was reduced to 2.4 lb. The foregoing observation provides an explanation of the behaviour. It is apparent that when the viscous damping of the system was added to surface damping provided by the friction-velocity curve, the point where the slope changed from negative to positive would occur earlier if the initial negative slope was smaller. This also explains the slight change in amplitude of vibration which is shown in Fig. 5.1.5. However, further study would be required in order to explain the slight variation in the friction-velocity curve due to normal load.

(g) Friction-Velocity Curve of Type B1

The experimental curves of Fig. 5.1.6 were obtained using #1 oil as the lubricant. The form of the friction-velocity curve indicates that the hump is almost insignificant, it is very close to the zero sliding velocity axis and that the negative slope of the curve is steeper than the two previous cases. Analytically, this form of friction-velocity curve suggests that over a large velocity
range the amplitude of vibration would fall on the left side of the 45° line of a dimensionless amplitude-velocity plot and only in the higher velocity region under the combined effect of the hump and of the system viscous damping does the amplitude of vibration gradually fall back to the right of the 45° line. Experimentally, those amplitudes on the left side of the line would appear in the form of stick-slip vibration. Under such circumstances, the quasi-harmonic theory would not provide accurate predictions, since the theory does not take into account the static friction characteristic of the friction couple. Nevertheless, the amplitude-velocity curves of Fig. 5.1.6 illustrate that the experimental results and the predictions by the quasi-harmonic theory are still in reasonable agreement. The larger discrepancy in the high velocity region is likely due to the fact that when the amplitude of vibration becomes large, the system damping may become nonlinear. The damping may be proportional to some power of the velocity instead of being in direct proportion to the velocity.

(h) **Friction-Velocity Curve of Type B2**

The experimental friction-velocity curve obtained from using #4 oil as the lubricant is shown in Fig. 5.1.7. The curve was reproduced from a one cycle oscilloscope trace at a disc velocity of 0.8 in/sec and with a normal
load of 3.85 lb. The curve appears to be almost linear with slight negative slope in the low velocity region and levels off rapidly. Due to the absence of the hump in the friction-velocity curve, quasi-harmonic oscillation was not predicted by the theory. In fact, the amplitude-velocity curves of Fig. 5.1.7 show that all the oscillations are of the stick-slip type and that the upper critical velocity where decay commences is reached as soon as the friction-velocity curve levels off.

The amplitude-velocity curves of Fig. 5.1.7 were obtained with normal loads of 5.4 lb, 3.85 lb, 2.4 lb and 1.0 lb respectively. The curves show that the normal load not only affected the upper critical velocity at which decay commences, but also the amplitude of the stick-slip oscillation near the low velocity end where the stick-slip oscillation is mainly under the influence of the static friction characteristic. Under such circumstances, it is quite apparent that the lower the normal load the lower the maximum static friction force and therefore the smaller the amplitude of stick-slip oscillation.

(i) Frequency of the Friction-Induced Vibration

The frequency of the friction-induced vibration was plotted as a ratio versus the disc velocity in Figures 5.1.2, 5.1.4 and 5.1.6. It is to be noted that the frequency of stick-slip oscillation was generally lower than
the damped natural frequency of the elastic system, and the frequency of oscillation approached the damped natural frequency as the oscillation transformed to the quasi-harmonic type. In Fig. 5.1.2 where stick-slip type oscillation was not recorded, the frequency-velocity plot appears to be constant at a frequency very nearly equal to the damped natural frequency of the system.

(j) **Friction-Velocity Curve of Type C**

In Fig. 5.1.8 is shown two friction-velocity curves quite different from those described earlier. The friction-velocity curve obtained by using #5 oil as the lubricant shows a horizontal section in the very low velocity region which is followed by a steep positive slope at higher velocities; the positive slope section is almost linear. The friction-velocity curve using #6 oil as the lubricant shows a curve with a steep positive slope in the very low velocity region with the slope gradually diminishing at higher velocities; this type of friction-velocity curve provides good performance in some automatic transmission clutches [53]. Quasi-harmonic oscillation was not observed for these cases, which is predictable because of the form of the friction-velocity curves which show no negative slope region. However, some stick-slip oscillation was observed in the very low velocity region in the case of the #5 oil, which is no doubt due to the presence of a small horizontal section in the friction-velocity curve. It should be
noted that the friction-velocity curves of Fig. 5.1.8 were obtained by plotting the friction force values from a sequence of disc velocities at a normal load of 5.4 lb. A one cycle trace was not obtained since there was no quasi-harmonic oscillation. However, as it has been indicated earlier that in the absence of friction-induced vibration the friction-velocity curve obtained by measuring displacement alone gives accurate results.

5.1.2 Steel on Steel

The nature of the tests required repeated runs over the same track at various speeds, this usually resulted in early damage of the test track with the dry steel-on-steel combination. In order to minimise the damage, petrolatum was used as the lubricant for the steel-on-steel combination. The combination had the propensity to execute stick-slip oscillation. However, near-circular phase plane diagrams were obtained on the oscilloscope at higher velocities.

(a) Stick-Slip Oscillation

Fig. 5.1.9a illustrates a typical $x-\dot{x}$ phase plane oscilloscope trace of a stick-slip type oscillation obtained for steel-on-steel surfaces with petrolatum (U.S.P.) as the lubricant. The normal load was 3.85 lb and the disc speed was 0.08 in/sec. A slight amount of ripple during the early portion of the stick portion of the cycle was attributed to the inertia force associated with the resi-
dual acceleration of the vibrating mass which appears in the early portion of the stick period. The frequency of this secondary oscillation is considered to be some combination of the frequency of the driving system and of the 2nd mode of the elastic supporting system. A computer simulation was made using the observed frequency and a roughly estimated mass for the driving system. The results of this simulation study appeared to agree with the experimental observations. Fig. 5.1.9b illustrates a near-circular x-x phase plane oscilloscope trace obtained at a disc speed of 2.45 in/sec, a slight trace of stick still exists. The small loop appears near the zero sliding velocity axis is due to the small amount of residual backlash of the driving system. For practical reasons, the backlash takeup described in the apparatus section was used for tests at low disc velocities only.

The one cycle trace of the friction-velocity curve of Fig. 5.1.9b was reproduced in Fig. 5.1.10. It would be noted that the curve has the form as that of the Type B except it is almost linear in this case; the average slope is equivalent to a negative damping of 0.045 lb/in/sec. In the low velocity region the equivalent negative damping is approximately 0.075 lb/in/sec and the curve levels off at velocities above 6 in/sec. The quasi-harmonic theory did not give an accurate prediction for this type of friction-velocity curve since some stick-portion existed.
in almost all vibrations throughout the entire velocity range. In the higher velocity region the stick portion was small so that the phase plane was almost circular. The experimental amplitude-velocity curve is shown in Fig. 5.1.10 and it can be seen that all the amplitudes are on the left side of the 45° line which is an indication of the stick-slip type oscillation.

(b) **Effect of External Damping**

The effect of heavy external damping was studied using a powerful permanent magnet which was applied to the vibratory system during friction-induced vibration. The system damping was increased from approximately 0.009 to 0.08 lb/in/sec. The experimental amplitude-velocity results obtained with the heavy external damping are also shown in Fig. 5.1.10. It should be noted that some stick-slip type oscillation still exists in the low velocity region, however, an upper critical velocity also appears. Vibration was not observed as the disc velocity was increased beyond 2.5 in/sec. Furthermore the amplitude of stick-slip oscillation in the very low velocity region with the presence of external damper are substantially lower than those obtained without external damping; in fact the amplitude-velocity curve appears to be almost linear throughout the entire length.

One cycle oscilloscope traces of the friction-velocity curves obtained with the heavy external damping
showed that there was significant modification on the curve due to the presence of the heavy damping. The negative slope became less steep in the higher velocity region and the curve leveled off at a disc velocity of about 3 in/sec. It should be noted that the one cycle method includes the damping term and that the external damping is viscous in nature therefore it has less effect in the low velocity region. Due to the high damping coefficient, phase relationships between the acceleration, velocity and displacement signals became significant, therefore the curve so obtained was not considered as an accurate representation of the friction-velocity characteristic with heavy damping.

5.1.3 Polymer Material on Steel

Microphotographs of one of the bundles of carbon fibres are shown in Figures 5.1.11a, b, c and d, the magnifications are 100, 200, 400 and 800 respectively. The pictures were taken from the slider surface after the test was completed.

It was observed that the friction results were critically dependent on the presence of resin material on the disc surface. Therefore great care was exercised during each test and friction results were obtained at increasing disc velocities as well as decreasing disc velocities. Several similar tests were performed. Owing to the absence of friction-induced vibration the two curves of Fig. 5.1.12
were obtained by plotting the friction force against the disc velocity. The normal loads were 5.4 lb and 3.85 lb respectively. It was found that while the friction-velocity characteristic curve showed a consistent positive slope in every test, variation of as much as 15% in static displacement was observed from test to test. This variation was believed to be due to the presence of resin material on the disc surface.

The curves illustrate that the friction force increases as the disc velocity is increased and levels off at disc velocity around 2.0 in/sec. This type of friction-velocity curve suggests heavy surface damping and no quasi-harmonic oscillation would be expected. In fact, frictional oscillation of any form was not observed in the experimental tests. It would be of interest to note that in their studies of static coefficient of friction of polymer material with relative to time, Weiter and Schmidt [79] showed that while PTFE (Teflon) is highly time dependent whereas teflon-graphite composite is not time dependent.

A second slider prepared from resin material without the carbon fibres was used in a test to study the behaviour of resin-steel combination. The coefficient of friction of this combination was found to be much higher than that of the carbon-resin composition. Again, the results were found to be greatly influenced by the presence of resin deposit on the disc surface. Results were inconsistent
even within the confines of one test. The broken line curve of Fig. 5.1.12 was obtained by assessing the results of several similar tests. In general, the friction-velocity characteristic shows a slight positive slope at the low velocity end and levels off at disc velocities around 1.0 in/sec. Quasi-harmonic oscillation together with high frequency oscillation was observed at disc velocities above 1 in/sec. This behaviour was attributed to the presence of resin deposit on the disc surface. Since a large part of the friction-velocity characteristic is almost linear and level and is only preceded by a small amount of positive slope, the system is in some kind of neutral state at disc velocities above 1 in/sec. Accordingly any outside disturbance such as when the resin slider rode over spots of resin deposit on the disc surface would change the system into an unstable state and resulted in quasi-harmonic self-induced vibration.

The carbon fibres and steel combination with its substantial positive slope in the friction-velocity characteristic has a definite advantage in many engineering applications where vibration is to be avoided. For instant, the carbon fibres would provide greater resistance to wear and low friction when used as bearing materials.

5.1.4 Rubber on Steel

The viscoelastic properties of rubber are subject to variation with temperature [47], [80], therefore reliable
friction-velocity characteristic curves can only be obtained at low disc velocities. Fig. 5.1.13 shows a friction-velocity curve obtained at a disc velocity of 0.39 in/sec. The curve illustrates a distinctly humped form with the maximum occurring at disc velocities around 0.4 in/sec. In general, frictional oscillation was not observed at disc velocities below 0.3 in/sec. Some oscillation was observed in some tests at disc velocities between 0.3 in/sec. to 0.4 in/sec. The frictional oscillation became consistent at disc velocities above 0.4 in/sec although the fundamental oscillation was usually accompanied by high frequency oscillations and under these circumstances a clear one cycle trace could not be obtained. It is believed that the high frequency oscillation is related to the temperature variation of the rubber slider caused by the high sliding velocity. It was also observed that the friction force was generally higher as the system started from a stationary state and gradually decreased to a lower friction value as sliding continued at a constant disc velocity. The high initial friction value applied to both static and kinetic friction forces. This finding suggests that the temperature effect as well as the velocity should be investigated in the studies of the frictional characteristics of the rubber on steel combination.
5.1.5 Summary

By using various types of friction couples and lubricants, it was possible to show a variety of friction-velocity characteristics. However, the fundamental reasons for the shape of the various friction-velocity characteristic curve were not investigated. Another full investigation could be devoted to this topic. Nevertheless, the accuracy of the measuring technique in the present investigation has made it possible to categorise the various forms of dynamic friction-velocity characteristic curves in relation to friction-vibration behaviour.

The investigation revealed that the dynamical behaviour of a sliding couple is critically dependent on the shape of the friction-velocity characteristic curve. The curve forms employed were designated as type A, B and C for the humped, decreasing and increasing characteristics respectively (Ref. Figures 3.2.2, 3.2.1b and 3.2.1a). Generally, a friction-velocity characteristic curve of Type A is the major condition for the existence of quasi-harmonic type friction-induced vibration, although stick-slip type oscillation may exist in the low velocity region for this type of curve depending on the static friction characteristic. When quasi-harmonic oscillation occurs, the amplitude of vibration increases as the driving velocity is increased. However, the system becomes unstable when a critical velocity is reached; the vibration may decay or
grow depending on whether the slider displacement and velocity coordinates are inside or outside the unstable limit cycle on the phase plane. The blotting paper-steel combination with automatic transmission fluid as the lubricant as well as polymer-steel and rubber-steel combinations give this type of friction-velocity characteristic. Good agreement was obtained between the experimental and theoretical results.

Pure sliding was observed in friction couples exhibiting increasing friction-velocity characteristic (Type C). This type of characteristic was observed in the blotting paper-steel combination with certain type of automatic transmission fluid as the lubricant.

Friction couples such as the steel-steel combination usually exhibit friction-velocity characteristics of Type B. In general, the friction-velocity characteristic with a continuous negative slope has the propensity to execute stick-slip oscillation. It should be noted that the present investigation does not take into consideration the static friction characteristic which is generally the governing factor for stick-slip type friction-induced vibration. However, when stick-slip oscillation does occur, the form of the dynamic friction characteristic does influence the amplitude of the oscillation and the transition to the quasi-harmonic oscillation. Thus the amplitude variation with driving velocity of the stick-slip type
oscillation depends on the shape of the friction-velocity characteristic as well as the static friction characteristic. Generally, in the higher velocity region the vibration becomes near sinusoidal although there still exists a small portion of stick period.

The application of the external viscous damping has the effect of extinguishing the friction-induced vibration, except in the very low velocity region when the oscillation is mainly under the influence of the static friction characteristic. Under these circumstances the external damping reduces the amplitude of the stick-slip type oscillation.

The frequency of the quasi-harmonic type friction-induced vibration is close to the damped natural frequency of the vibratory system, whereas the frequency of the stick-slip oscillation depends on the stick period and is generally lower than the damped natural frequency of the system.

5.2 Non-Autonomous Cases

The blotting paper on steel combination was used for all frictional tests with transverse external excitation. Three types of transmission fluid which gave three distinct forms of friction-velocity characteristics, namely Type A, B, and C, were used. The results show the transition from a linear characteristic to a non-linear characteristic. Initially, forced vibration with the slider clear of the lower surface was studied. The positive slope friction-
velocity characteristic, Type C, was investigated next which could be considered as equivalent to a linear system with heavy damping. This study was followed by an investigation of the frictional characteristic with the almost linear curve having negative slope, and with the hump close to the zero sliding velocity axis. Finally, detailed investigations which included the non-resonance, fundamental resonance and subharmonic entrainment cases, were conducted with the humped type friction-velocity characteristic, Type A.

5.2.1 **Forced Vibration** (linear case)

Fig. 5.2.1 shows a plot of the magnification factor, $x/x_0$ versus frequency ratio $\nu/\omega$. The results were obtained with the slider clear of the disc surface and by varying the rotational speed of the unbalanced weight. The amplitudes of vibration $x$ were recorded from the oscilloscope, and the static displacements $x_0$ were calculated from the unbalanced forces. A theoretical curve using the linear vibration theory with a damping ratio $r/r_c$ of 0.01, where $r_c$ is the critical damping coefficient, is also shown in Fig. 5.2.1. In general, the experimental and theoretical results are in reasonably good agreement. These results indicate that the system parameters, particularly the system damping estimated from the free vibration test were of the correct order.
5.2.2 Friction-Velocity Characteristic Curve of Type C

The experimental results of Fig. 5.2.2 were obtained using the #5 oil as the lubricant, its friction-velocity characteristic has the form as shown in Fig. 5.1.8. The friction curve has a continuous positive slope and is almost linear in the region shown which suggests that the curve could be considered simply as viscous damping having an equivalent damping coefficient of 0.20 lb/in/sec including the system damping. The theoretical curve of Fig. 5.2.2 was obtained by applying the equivalent damping coefficient of 0.2 to eq. (3.4.1) of the linear vibration theory. Experimental results were taken at constant disc velocities of 1.085 in/sec and 1.325 in/sec and at a normal load of 5.4 lb. The results are in good agreement with the theory. Friction-induced oscillation does not exist in the system due to the presence of the continuous positive slope of the friction-velocity curve; the oscillation is due solely to the external excitation. The good agreement between the experimental and theoretical results again illustrates the accuracy and the self-consistency of the apparatus and the measuring instrumentation.

5.2.3 General Discussion of a Non-Autonomous Friction-Induced Vibration System

A nonlinear system subject to forced vibration was obtained when the friction-velocity characteristic has the
form of a Type B1 curve as illustrated in Fig. 5.1.6. In this case, the linear vibration theory could no longer be applied; in fact, as it would be noted later even the nonlinear theory could not give accurate predictions for certain types of friction-velocity curves, particularly when the frequency of the external excitation is close to the fundamental resonance. Around this region the amplitude of oscillation tends to exceed the zero sliding velocity axis in the $x-\dot{x}$ phase plane plot. The position of the zero sliding velocity axis in the $x-\dot{x}$ phase plane is determined by the disc velocity. In theory, when the trajectory goes beyond this axis, its path will be under the influence of a friction-velocity characteristic with reversed sign. In the autonomous case, when the trajectory reaches the zero sliding velocity axis stick may occur and the oscillation transforms into some form of stick-slip type oscillation. However, in the case where the magnitude of the external excitation predominates, the trajectory may be forced to the other side of the axis and result in some distorted form of oscillation. In Fig. 5.2.3 two oscil­loscope traces are shown, one of which illustrates the distorted form of oscillation.

A plot of the magnification factor versus frequency ratio is shown in Fig. 5.2.4. The results were obtained at disc velocities of 1.85, 1.6 and 1.06 in/sec, the normal load was 5.4 lb. No attempt was made to investigate this
case in detail. Nevertheless, the curve shows that sub-harmonic entrainment exists in the system due to the non-linearity of the friction-velocity characteristic. Sub-harmonic entrainment occurs when the frequency ratio \( \alpha \) is very close or equal to 2, 3, etc., the system vibrates at a frequency which is a sub-multiple of the external excitation frequency and is close to the frequency of the autoperiodic oscillation.

Fig. 5.2.5 shows five curves obtained at various disc velocities and with different external excitation forces. The results were obtained with \#9 oil as the lubricant and the friction-velocity characteristic has a Type A2 curve as illustrated in Fig. 5.1.4. The normal load was 5.4 lb. The curves of Fig. 5.2.5 illustrate a very similar pattern although the occurrence of the peak varies over a large range. A comparison of curves (1) and (2), and curves (3) and (5) shows that for the same disc velocity, the higher the external excitation force the lower the peak and also the earlier the peak occurs. It should be noted that the curves were plotted as magnification factor versus frequency ratio. In the linear case, the curve would be independent of the external excitation force. The variation with the external excitation force suggests that the friction force is nonlinear. The variation with the external excitation force would be due to the reason indicated earlier whereby the disc velocity,
which determines the location of the zero sliding velocity axis in the $x-\dot{x}$ phase plane, exerts a certain restriction on the amplitude of oscillation whenever its trajectory tends to grow beyond this axis in the $x-\dot{x}$ phase plane. Any increase in amplitude exceeding the limit would be partially restricted and the resulting vibration would be of the form shown in Fig. 5.2.3b which is somewhat similar to stick-slip type oscillations. In fact, a comparison of curves 2, 4, and 5 further verifies this point. The curves were obtained at three different disc velocities with the same external force having a magnitude large enough such that the amplitude of vibration in the region near the fundamental resonance frequency tended to exceed the limit. The results illustrate that increasing the disc velocity also increases the peak of the curve and delays its occurrence. It is apparent that increasing the disc velocity extends the limit where the restriction would apply. Under these circumstances the amplitude of oscillation would be able to grow further before it reaches the barrier.

Fig. 5.2.6 shows four diagrams illustrating the effect of the external excitation on the average displacement at zero absolute velocity. In each diagram both the amplitude of vibration and average displacement were plotted against the frequency ratio. The approximate limit of the quasi-harmonic oscillation is also shown in the diagrams. A star shown near the vertical axis indi-
cates the average displacement when the external excitation was absent. It is immediately clear that whenever the amplitude of vibration exceeds the limit for quasi-harmonic oscillation the corresponding average displacement begins to drop and that the higher the amplitude value the lower the average displacement becomes. The average displacement returns to its normal value once the amplitude of oscillation is within the limit. In Fig. 5.2.6d where the amplitude of oscillation nowhere exceeds the limit, the average displacement remains constant over the full frequency range. The results of (a) and (b) were obtained having a Type Bl friction-velocity curve while (c) and (d) were obtained having Type A2 and Type C friction-velocity curve respectively, in all four cases the normal load was 5.4 lb.

Further investigation of the  \( x - \dot{x} \) phase plane diagrams obtained from the oscilloscope revealed that when the quasi-harmonic oscillation limit was exceeded the  \( x - \dot{x} \) phase plane resembled stick-slip type oscillation, although the stick portion was not well defined due to the presence of the external excitation. The positive displacement or the upper half of the phase plane seemed to be under the influence of the stick portion which in turn was due to the external excitation, whereas the negative displacement or the lower half of the phase plane was free from this kind of restriction. Thus it would appear that the increase
in amplitude was mainly accomplished by the increase in the lower portion of the phase plane plot thus giving an apparent decrease in the average displacement.

The above results illustrate the importance of choosing a proper range of external excitation force and of disc velocity so that meaningful results can be obtained from a full scale investigation of a non-autonomous system with friction-induced vibration.

5.2.4 Investigation of a Non-Autonomous Friction-Induced Vibration System Having a Type Al Friction-Velocity Characteristic Curve

The friction-velocity characteristic curve of Type Al as illustrated in Fig. 5.1.2 was chosen for the detailed studies of a non-autonomous system, for its well defined humped form which is a necessary condition for the existence of quasi-harmonic oscillation. The Type Al curve also provides a friction-velocity characteristic from which the amplitude of the self-excited vibration was well within the limit for quasi-harmonic oscillation.

In the theoretical analysis the friction force was expressed as a seventh order polynomial because the algebraic manipulation was comparatively easy whereas the exponential form of expression would inevitably involved Bessel functions and would become extremely complicated in the non-autonomous case. However, the exponential
expression was used for obtaining more accurate solutions when a numerical method was used.

(a) Non-Resonance Case

The curves of Fig. 5.2.7 were prepared by applying eq. (3.4.13) and eq. (3.4.14) of the non-resonance case. By varying the external parameters \( L = F_o \alpha/(1 - \alpha^2) \) the steady state amplitude of the autoperiodic oscillation was varied, and as a critical value of \( L \) was reached, the amplitude polynomial had no positive real root which indicated that the autoperiodic oscillation was no longer present. If the absolute value of \( L \) was further increased, only heteroperiodic oscillation at the frequency of the external excitation existed in the system. When \( L = 0 \), the steady state amplitude of the vibration is same as that obtained from the autonomous theory. Thus the curves of Fig. 5.2.7 indicate whether purely heteroperiodic oscillation with external frequency or combined autoperiodic and heteroperiodic oscillation with beat frequency existed in the system. It may be observed that the curves of Fig. 5.2.7 were expressed in a generalised form where \( L \) contains both the external excitation force magnitude and frequency. In fact, a whole series of curves can be re-plotted from any single curve of Fig. 5.2.7.

In Fig. 5.2.8 the curve was plotted for a constant frequency of 1.79 and the corresponding values of \( F_o \) were
derived from the expression of L. Curve (1) of Fig. 5.2.8 was replotted from curve (3) of Fig. 5.2.7 replacing the L-axis by the $F_q$-axis. Curve (3) of Fig. 5.2.8 was prepared from eq. (3.4.17) when the autoperiodic oscillation was absent and it represents the heteroperiodic oscillation in the system. When the autoperiodic oscillation is present, the effect of the external force is felt only in the second approximation which is normally small when $F_q$ is small. However, when $F_q$ becomes sufficiently large, the effect of the external force is significant; the dotted line of curve (3) in Fig. 5.2.8 was obtained from eq. (3.4.17) by considering the autoperiodic oscillation to be absent. In the non-resonance analysis the original d.e. in terms of variable X was transformed by introducing a new variable as shown in eq. (3.4.4), therefore in addition to the autoperiodic and heteroperiodic oscillations as shown by curves (1) and (3), the amplitude of the actual oscillation in terms of variable X should be obtained from eq. (3.4.4) with the additional forcing term which was introduced in the original d.e. The magnitude of this forcing term was plotted as curve (2). Thus in considering these curves, it should be noted that close to the left hand side axis the autoperiodic oscillation predominates; 'beat' started to show as the external force $F_q$ was increased; around $F_q = 0.75$ where the amplitude of the autoperiodic oscillation equals the amplitude of the external excitation.
the beat would become most significant. As $F_o$ was further increased only heteroperiodic oscillation at the external frequency existed.

Fig. 5.2.9 shows six sets of computer plotted displacement-time and velocity-time traces. The traces were obtained by applying the exponential expression for the friction-velocity characteristic of Fig. 5.1.2 (Type A1) in eq. (3.4.2) and solving the equation by a numerical method. It should be noted that for $F_o = 0.2$ the oscillation was almost autoperiodic with a frequency equal to that of the autonomous system. At $F_o = 0.4$ and 0.6 the oscillations were of the combined autoperiodic and heteroperiodic, as $F_o$ reached 0.8 the oscillation turned into a well defined 'beat' showing that the amplitudes of the autoperiodic and of the heteroperiodic cases were about equal. The last two sets of traces at $F_o = 1.05$ and 2.5 showed purely heteroperiodic oscillation at a frequency equal to that of the external excitation. Comparing the traces of Fig. 5.2.9 and the curves of Fig. 5.2.8, some slight discrepancy is apparent which is probably associated with the difference between the two expressions for the friction force function.

(b) **Fundamental Resonance Case**

The analysis in the foregoing section is applicable only when the external frequency is not too close to the resonance frequency.
The solid curve of Fig. 5.2.10 was obtained from eq. (3.4.24) of the fundamental resonance theory. It was noted that the lower sections of the curve, at frequency ratios below 0.9 and above 1.1, were merely part of the curves of the non-resonance case (Ref. Curve (2), Fig. 5.2.8). Thus Fig. 5.2.10 illustrates the transition from the non-resonance case to fundamental resonance and back to the non-resonance case again. The two dashed line curves are the amplitudes of autoperiodic oscillation obtained from the non-resonance theory (Ref. Curve (1), Fig. 5.2.8). The solutions obtained by the numerical method using the exponential expression are also shown in the same diagram. Six sets of these computer plotted solutions as well as six sets of corresponding experimental traces are shown in Fig. 5.2.11 and Fig. 5.2.12 respectively. It should be noted that while in theory the external excitation force can be kept constant for various frequencies, whereas in the experimental investigation this was not always possible. The variable parameters \( p \) and \( e \), namely the out-of-balance mass and the eccentricity, are in step sizes rather than continuous variation, hence some discrepancy between the theoretical and experimental results is to be expected.

It would be noted from Fig. 5.2.10 that with an external excitation force \( F_0 = 0.2 \), almost periodic oscillation or beat oscillation would occur at frequen-
cies below 0.87 and above 1.17. Within the region of frequencies of 0.87 and 1.17, harmonic oscillations could exist in the system. The numerical solutions of Fig. 5.2.11 substantiates the findings of the analytical methods. Again, a slight discrepancy exists due to the two different expressions which were used in the two methods.

Fig. 5.2.13 shows curves plotted with magnification factor against frequency ratio. The experimental results were obtained at a disc velocity of 1.05 in/sec and a normal load of 5.4 lb. It should be noted that good agreement exists between the experimental curve and the curve obtained by the numerical method using the exponential expression. Reasonably good agreement also existed between the experimental curve and the curve obtained from the fundamental resonance theory using the polynomial expression. The greater amount of discrepancy is likely related to the inaccuracy of the polynomial curve fitting. In fact, Fig. 5.2.13 demonstrates the importance of an accurate friction-velocity characteristic.

Both theoretical curves of Fig. 5.2.13 show higher magnification values than the experimental curve, particularly near a frequency ratio of 1. The experimental vibration amplitude slightly exceeded the limit for quasi-harmonic oscillation around the resonance frequency region, resulting with slight restriction on the amplitude of vibration as its trajectory tended to grow beyond the
zero sliding velocity axis in the $x-\dot{x}$ phase plane. On the other hand, in the theoretical analysis a restriction of this kind was not incorporated. Nevertheless, in both theoretical and experimental investigations the external excitation parameters were chosen such that the vibration amplitude would be within or at least near the quasi-harmonic limit.

(c) **Sub-harmonic Entrainments**

When the frequency of the external excitation force was near or equal to a multiple of the frequency of the autonomous system, subharmonic entrainment was observed in the experiments, during which the system oscillated at a frequency equal or close to the frequency of the autonomous system. The amplitude of vibration was higher than the amplitude of vibration without the external force. In the non-resonance case the amplitude of vibration is normally lower than that of the self-excited vibration alone.

Figures 5.2.14 to 5.2.17 illustrate the wave forms and $x-\dot{x}$ phase planes of the $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ harmonics. On the left hand side of each figure the experimental traces from the oscilloscope and oscillograph are shown. The computer plotted traces on the right hand side were obtained by applying the exponential equation of the friction-velocity curve to eq. (3.4.2) and solving the equation numerically. All the experimental results were
obtained at a disc velocity of 1.05 in/sec and a normal load of 5.4 lb. The scales of the computer plotted traces are dimensionless. The dimensionless parameter $\omega h$ is equal to unity and $\omega$ is 138 rad./sec. The scales of the experimental traces are shown with units of 0.001 in, in/sec, and second for displacement, velocity and time respectively.

The $x-x$ phase plane of the $\frac{1}{2}$ harmonic resembles a cardiac pattern was shown in Fig. 5.2.14 and Fig. 5.2.15. The external excitation forces were 0.245 and 0.895 (dimensionless) respectively. Extremely good simulations of the experimental traces were obtained from the numerical method. Increase in amplitude was noted when the external excitation force was increased. The frequency of the oscillation was equal to the frequency of the autonomous system. Figures 5.2.16 and 5.2.17 illustrate two different $x-x$ phase plane patterns for the $\frac{1}{3}$ and $\frac{1}{4}$ harmonic cases. The dimensionless external force magnitude was 1.35 for the $\frac{1}{3}$ harmonic case and 1.00 for the $\frac{1}{4}$ harmonic case. Again good agreement between the experimental traces and the computer plotted traces was obtained.

Fig. 5.2.18 shows a summary of the subharmonic cases and the non-resonance cases. The curves were prepared using the harmonic balance method. The seventh order polynomial expression representing the friction-velocity curve of Type Al was applied to the equations derived
from the harmonic balance method (Appendix II). The curve for the \( \frac{1}{2} \) harmonic case shows a rise in autoperiodic vibration amplitude as the magnitude of the external excitation force was increased. The amplitude of the autoperiodic oscillation started to decline as the external force reached a certain value, in fact, two real roots were obtained for each external force magnitude near the end of the curve, this suggested that the autoperiodic oscillation became unstable just prior to its being completely quenched by the increasing external force. Further increase of the external force magnitude resulted in harmonic oscillation at the frequency of the external excitation. The amplitude curve of the heteroperiodic term is shown as dotted lines in the same diagram. A comparison of the results from Figures 5.2.14 and 5.2.15 with the theoretical curve of Fig. 5.2.18 shows they are in good agreement.

The curves for the \( \frac{1}{3} \) and \( \frac{1}{4} \) harmonic are also shown in Fig. 5.2.18. The amplitudes of autoperiodic oscillation vary slightly with the external excitation force and drops off rapidly as the external force exceeds a certain limit. Again, the results from Figures 5.2.16 and 5.2.17 are in reasonably good agreement with the \( \frac{1}{3} \) and \( \frac{1}{4} \) harmonic curves of Fig. 5.2.18. It should be noted from Fig. 5.2.18 that the amplitude of oscillation \( X \) comprises two terms of different frequency, therefore a direct comparison between the experimental and theoretical results is difficult.
A comparison of the non-resonance curves of Fig. 5.2.18 with those of the Fig. 5.2.8 shows they are in very good agreement. Similarly, eq. (3.4.34) of the harmonic balance method is almost the same as that of eq. (3.4.23) of the fundamental resonance case. Thus in the first approximation the K and B method for the non-resonance case and fundamental resonance case give very nearly the same result as the harmonic balance method.

The curve of Fig. 5.2.19 was obtained by plotting the amplitude of the external excitation force magnitude at which $a_e$ of eq. (3.4.37) became zero, against the frequency ratio $\alpha$. Oscillation occurring in the region above the solid line would be of a harmonic oscillation at the frequency of the external excitation. Below the solid line, oscillation would have either a combined frequency or entrained frequency. Thus the curve illustrates the limits of the external excitation force which separates the oscillation from that of the resonance or of the combined frequency type to that of the purely harmonic type. Stability investigation was not carried out in this case, therefore it is not possible to define exactly the subharmonic regions. Six sets of computer plotted displacement-time and velocity-time traces for $\alpha = 4$ using the numerical method are shown in Fig. 5.2.20. The traces illustrate the gradual transformation of the subharmonic oscillation to the harmonic oscillation as the external
excitation force magnitude is increased. Thus, it can be concluded from Fig. 5.2.19 that subharmonic entrainment occurs within a narrow range of the external frequency whereas the harmonic entrainment occurs at any external force frequency provided the external force magnitude $F_0$ is sufficiently large.

No attempt was made to perform similar sequence of tests for various disc velocities. It is believed that for disc velocities within the range of quasi-harmonic oscillation, the results would follow a similar pattern except the amplitude of oscillation at and around frequencies of resonance or subharmonic resonance would be larger for higher disc velocities. Fig. 5.2.21 shows two sets of experimental results plotted and computer plotted $x-x$ phase planes obtained at a disc velocity of 1.35 in/sec for frequency ratios of 3 and 4. A comparison of Fig. 5.2.21 and Fig. 5.2.17 illustrates that the patterns are the same but the amplitudes are larger for the case at a disc velocity equal to 1.35 in/sec.

(d) Subharmonic Resonance Case Using a Linearized Friction-Velocity Curve

Fig. 5.2.22 illustrates two sets of computer plotted displacement-time and velocity-time traces using a linearised friction-velocity curve. It shows that the oscillation would either decay into a harmonic oscillation at the frequency of the external excitation or grow in ampli-
tude until it exceeds the limit for quasi-harmonic oscillation depending on whether the linearised friction-velocity curve has a slightly positive or slightly negative slope. In both cases the oscillation never reached a steady state subharmonic entrainment such as shown by Fig. 5.2.18. The illustration further proves that the humped type friction-velocity curve of Fig. 5.1.2 is a necessary condition for the existence of quasi-harmonic oscillation.

5.2.5 Summary

When under the influence of external transverse excitation, frictional systems having an increasing friction-velocity characteristic (Type C) behaved in a manner similar to a linear vibratory system with heavy damping; whereas subharmonic entrainment as well as resonance and non-resonance oscillations were observed in systems having friction-velocity characteristic curves of Type A or Type B. In the case of Type B friction-velocity curve, owing to the continuous decreasing characteristic, there was a tendency for the trajectory of the x-x phase plane to overshoot the zero sliding velocity axis when the frequency ratio associated with the external excitation was about unity. Under these circumstances the form of the oscillation was distorted.

The effect of external transverse excitation on a frictional system subject to quasi-harmonic type oscillation
was demonstrated by using the #7 oil as the lubricant on a blotting paper-steel combination. The extinguishing of the autoperiodic oscillation by the external excitation force and the occurrence of the harmonic and subharmonic entrainment as predicted by the theoretical analysis were observed. The theoretical and experimental results are in reasonably good agreement.

5.2.6 Dynamically Loaded System

Two types of investigation have been conducted. The first employing the steel-on-steel combination as friction materials executed stick-slip type friction-induced vibration when the external excitation was absent. The second investigation was performed in the presence of quasi-harmonic type friction-induced vibration.

(a) Stick-Slip Type Oscillation

The results of Fig. 5.2.23 were obtained from a steel slider on steel disc combination which has a type C friction-velocity characteristic curve and has the propensity of executing stick-slip type oscillation. The disc velocity was 0.15 in/sec and the normal load (static) was 6.4 lb. The curve of Fig. 5.2.23 was plotted with the amplitude of the stick-slip oscillation versus $\beta$, the ratio of the external excitation force magnitude (oscillating load) to the static load. The frequency of the corresponding external excitation force is also shown in
the same diagram. The values of \( \beta \) were taken in random order, the number inside the points indicates the sequence the result was taken. This has the significance of showing the change in amplitude was not due to the effect of the number of traverses of the disc track. The results show that the amplitude of the stick-slip oscillation decreases as the external excitation force magnitude is increased. As a critical value of \( \beta \) is reached, pure sliding exists in the system. Fig. 5.2.24 shows four oscillograph traces illustrating first the stick-slip oscillation in the absence of external excitation, next the diminishing stick-slip oscillation when the external excitation is present and finally the pure sliding with the traces showing high frequency oscillation due to the external excitation. It appears that the frequency of the external excitation (within the range 20-60 cps) has no significant effect on the amplitude of the stick-slip oscillation, rather, the magnitude of the external excitation is the major factor. It is of interest to note that the critical value of \( \beta \) for the complete quenching of the stick-slip oscillation is only 0.15, when the normal load (static) is 6.4 lb. It is believed that the critical \( \beta \) value varies with the static normal load, at higher static normal loads, the critical \( \beta \) value may become smaller. However, a more detailed study of the friction mechanism is required in order to reveal the relationship between the normal load and the critical \( \beta \) value.
A curve of maximum static friction force versus $\beta$ is also shown in Fig. 5.2.23. It shows that the maximum static friction force decreases as $\beta$ is increased until a pure sliding friction value is reached. It is believed that the decrease in maximum static friction is associated with the breakdown of junctions between the contacting surfaces.

(b) Quasi-Harmonic Type Oscillation

The test was carried out using the #7 oil as the lubricant and the friction material combination was blotting paper on steel which has a friction-velocity characteristic curve of Type Al in the autonomous case. It was found that the results are very similar to those of the transverse external excitation case, beat frequencies as well as subharmonic entrainment were observed, although the theoretical results do not give as good agreement to the experimental results as in the transverse vibration case. The theoretical results were obtained by applying the external excitation parameters, namely $\beta$ and $\alpha$, to eq. (3.4.41) and solving the equation by a numerical method. In fact, eq. (3.4.41) is very similar to eq. (3.4.2) of the transverse external excitation case, except the non-linear function $yG(\dot{X})$ has extra periodic coefficient terms. It is apparent that when $\beta$ is small, that is when the effect of the periodic coefficient terms of the nonlinear function is not significant, eq. (3.4.41) would behave as
in the non-autonomous case with transverse excitation. However, when $\beta$ becomes large, the effect of the periodic coefficient terms becomes significant, the system would not follow a similar pattern as the transverse case. Fig. 5.2.25 shows six sets of experimental results of the displacement-time and velocity-time traces illustrating the similar behaviour as in the transverse vibration case. It shows the combined autoperiodic and heteroperiodic oscillation with beat frequency when $\alpha$ is not close to unity or multiples of one; the resonance when $\alpha = 1$ and the subharmonic entrainment when $\alpha = 2, 3$ etc. The last set of traces (Fig. 5.2.25f) illustrates the pure harmonic oscillation at the frequency of the external excitation when $\beta$ is sufficiently large. The results were obtained at a disc velocity of 1.05 in/sec and with a normal load of 6.4 lb. Fig. 5.2.26 shows a $x-x$ phase plane diagram and the displacement-time and velocity-time traces for $\alpha = 3$ and $\beta = 0.23$. The diagrams on the left are experimental oscilloscope and oscillograph traces and those on the right are computer plotted traces obtained by applying the external excitation parameters to eq. (3.4.41). It is believed that with high $\beta$ values, the external excitation may have caused variation in the contact conditions between the sliding surfaces; under these conditions, the friction-velocity characteristic may be quite different from that obtained in the autonomous case and eq. (3.4.41) will not give correct predictions.
(c) **Summary**

The results obtained from a system having normal external excitation show that with a sufficiently high external force magnitude, both the stick-slip type and the quasi-harmonic type friction-induced vibration may be extinguished. The normal external excitation reduces the maximum static friction in the stick-slip type vibration case. However, reduction in sliding friction or average displacement during quasi-harmonic oscillation was not observed.
VI  CONCLUSION

In overall summary of the research, the following conclusions may be listed:

1. A reliable apparatus free from unwanted external vibration and noise was developed for the investigation of quasi-harmonic type friction-induced vibration. The apparatus together with the instrumentation techniques developed permitted experimental results of reasonable accuracy to be obtained.

2. The advantage of the one cycle method (Acceleration-velocity-displacement) in determining the friction-velocity characteristic curve is clearly demonstrated by the fact that a single cycle of oscillation across the surface completely defines the entire dynamic characteristic curve. Furthermore, the presence of the hump in the low velocity region could not be verified until the one cycle method was used.

3. The results from the investigation of the autonomous case demonstrated that a humped shape friction-velocity curve is a condition necessary for the existence of quasi-harmonic oscillation providing that the negative sloped section of the curve is not too steep. The investigation also clearly defines the distinction between the quasi-harmonic and the stick-slip forms of friction-induced vibration.
4. In the humped friction-velocity characteristic, stick-slip oscillation may occur in the low velocity region. In general, quasi-harmonic oscillation would start at a disc velocity near the peak of the hump. The amplitude of the quasi-harmonic oscillation increases as the driving velocity is increased. At a driving velocity near the point of inflection from negative slope to positive slope of the friction-velocity curve the vibration becomes unstable and self-excited vibration cannot start from rest.

5. Other types of friction-velocity characteristic give rise to stick-slip type friction-induced vibration or stable displacement (pure sliding), depending on whether the curve has a decreasing characteristic or an increasing characteristic. In the low velocity region the amplitude of the stick-slip oscillation is mainly governed by the static friction characteristic. However, in the higher velocity region, the shape of the friction-velocity characteristic curve becomes important.

6. The application of the external viscous damping diminishes or extinguishes the friction-induced vibration. In the low velocity region when the oscillation is mainly under the influence of the static friction characteristic, the effect of the external damping is less significant. The amount of external viscous damping required for extinguishing the quasi-harmonic type friction-induced vibration is related
to the coefficients of the friction-velocity characteristic equation and can be determined from the theory.

7. The dynamic friction characteristic curves were represented by $n$th order polynomials and by exponential expressions. These expressions were applied to the first approximation method of Krylov and Bogoliuboff. The theoretical predictions were in good agreement with the experimental results.

8. Transverse vibrations obtained by applying rotating out-of-balance masses were applied to the frictional system in the direction of the friction force. The harmonic balance method as well as the first approximation methods of Krylov and Bogoliuboff were used in the theoretical analysis. The experimental results and the theoretical prediction for a pair of friction materials having a type A friction-velocity characteristic curve indicated that subharmonic entrainment could occur, and when the external excitation magnitude was sufficiently high, harmonic entrainment at the external excitation frequency existed in the system. This finding suggests that high frequency external excitation with a small vibration amplitude, may be used as a means of extinguishing the unwanted quasi-harmonic type friction-induced vibration.

9. By applying the external excitation vertically at the loading end, the effect of dynamic loading was simulated.
The experimental results showed that stick-slip type oscillation may be extinguished with a dynamic to static load ratio of 0.15. The normal external excitation has the effect of reducing the maximum static friction of the stick-slip type vibration. In the quasi-harmonic case, the nonlinear function has variable coefficients. Reduction in sliding friction or average displacement during quasi-harmonic oscillation was not observed. However, quasi-harmonic oscillation may still be extinguished with a sufficiently high external force magnitude. The simulated theoretical results did not give accurate predictions in this case suggesting that the friction mechanisms between the sliding surfaces may have been changed during dynamic loading.
REFERENCES


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54. Ibid., 45, pp. 178-184.


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58. Ibid., 56, p. 11.


60. Ibid., 40, p. 329.


63. Ibid., 41, p. 293.

64. Ibid., 41, p. 289.

65. Ibid., 40, p. 360.

66. Ibid., 40, p. 329.

67. Ibid., 41, p. 292.

68. Ibid., 40, p. 368.

69. Ibid., 41, pp. 313-319.

70. Ibid., 40, p. 377.


73. Ibid., 72, p. 153.

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78. Ibid., 76, p. 253.


APPENDIX I

DERIVATION OF STEADY STATE AMPLITUDE AND PHASE EQUATIONS FROM THE EXPONENTIAL EXPRESSION

Substitution of eq. (3.3.10) into eq. (3.3.7) and eq. (3.3.8) yields

\[
\begin{align*}
\dot{a} &= -\frac{D_1}{2E\pi} \int_0^{2\pi} \cos^2 \psi \, d\psi - D_2 \int_0^{2\pi} e^{C_2 \cos \psi} \cos \psi \, d\psi + D_3 \int_0^{2\pi} \alpha e^{C_2 \cos \psi} \cos^2 \psi \, d\psi \\
\dot{\psi} &= 1 + \frac{D_1}{aE^2\pi} \int_0^{2\pi} \frac{a}{2} \sin^2 \psi \, d\psi - D_2 \int_0^{2\pi} e^{C_2 \cos \psi} \sin \psi \, d\psi \\
&\quad + D_3 \int_0^{2\pi} \alpha e^{C_2 \cos \psi} \cos \psi \sin \psi \, d\psi
\end{align*}
\]

(A.1)

(A.2)

From the Table of Integrals, Series and Products [81], we have

\[
\int_0^{2\pi} e^{(p\cos x + q\sin x)} \sin(\alpha \cos x + b \sin x - mx) \, dx
\]

\[
= i \pi [(b-p)^2 + (a+q)^2]^{-\frac{m}{2}} [\frac{A+iB}{2} I_m(C+iD)^\frac{3}{2} - (A-iB)^2 I_m(C+iD)^\frac{1}{2}]
\]

\[
\int_0^{2\pi} e^{(p\cos x + q\sin x)} \cos(\alpha \cos x + b \sin x - mx) \, dx
\]

\[
= \pi [(b-p)^2 + (a+q)^2]^{-\frac{m}{2}} [\frac{A+iB}{2} I_m(C+iD)^\frac{3}{2} + (A-iB)^2 I_m(C+iD)^\frac{1}{2}]
\]

where \((b-p)^2 + (a+q)^2 > 0\), \(m = 0, 1, 2, \ldots\)
and \[ A = p^2 - q^2 + a^2 - b^2 \], \[ B = 2(pq+ab) \], \[ C = p^2 + q^2 - a^2 - b^2 \], \[ D = -2(ap+bq) \].

Application of the integrals to eq.(A.1) and noting that

\[ I_2(C_3a) = I_0(C_3a) - 2I_1(C_3a)/C_3a \]

yields

\[ \dot{a} = -\frac{D_1}{2E} [a + 2aD_2I_0(C_3a) - 2(D_2 + \frac{D_2}{C_3^2})I_1(C_3a)] \] \hspace{1cm} (A.3)

Similarly, we have from eq.(A.2)

\[ \psi = 1 \] \hspace{1cm} (A.4)

Differentiating eq.(A.3) with respect to \( a \) gives

\[ \frac{d\dot{a}}{da} = -\frac{D_1}{2E} [1 + 2D_2C_3aI'(C_3a) + 2D_2I_0(C_3a) - 2C_3(D_2 + \frac{D_2}{C_3^2})I'(C_3a)] \]

Noting \[ I_1'(C_3a) = I_0(C_3a) - \frac{I_1(C_3a)}{C_3a} \] and \[ I_0'(C_3a) = I_1(C_3a) \]

we finally have

\[ \phi_a(a) = -\frac{D_1}{2E} [1 - 2C_3D_2I_0(C_3a) + \frac{2}{C_3a} D_2(C_3a)^2 + D_2C_3 + D_3 I_1(C_3a)] \] \hspace{1cm} (A.5)
APPENDIX II
HARMONIC BALANCE METHOD

A solution of eq. (3.4.38) can be sought in the form

\[ X = a \sin \tau + a_e \sin \alpha \tau \]

where \( a \) and \( a_e \) are considered to be constants.

For a seventh order polynomial

\[ \gamma G(\dot{X}) = \frac{D_1}{E}(\dot{X} - D_2 \dot{X}^2 + \ldots + D_7 \dot{X}^7) \]

Expansion of the \( \dot{X}, \dot{X}^2 \) etc, and writing \( A = a_e \alpha \) yields

\[ \dot{X} = a \cos \tau + A \cos \alpha \tau \]

\[ \dot{X}^2 = \frac{1}{2}(a^2 + A^2) + \frac{1}{2}a^2 \cos 2 \tau + \frac{1}{2}A^2 \cos 2 \alpha \tau + aA [\cos(1+\alpha) \tau + \cos(1-\alpha) \tau] \]

\[ \dot{X}^3 = \frac{3}{4}(a^3 + 2aA^2) \cos \tau + \frac{3}{4}a^3 \cos 3 \tau + \frac{1}{4}A^3 \cos 3 \alpha \tau + \frac{3}{4}(2a^2A + A^3) \cos \alpha \tau \]

\[ + \frac{3}{4}a^2A \left[ \cos(2+\alpha) \tau + \cos(2-\alpha) \tau \right] + \frac{3}{4}A^2 \left[ \cos(1+2\alpha) \tau + \cos(1-2\alpha) \tau \right] \]

\[ \dot{X}^4 = \frac{3}{8}(a^4 + 4aA^2 + A^4) + \frac{1}{2}(a^4 + 3a^2A^2) \cos 2 \tau + \frac{1}{2}(3a^2A^2 + A^4) \cos 2 \alpha \tau \]

\[ + \frac{1}{8}a^4 \cos 4 \tau + \frac{1}{8}A^4 \cos \alpha \tau + \frac{3}{2}(a^3A + aA^3) \left[ \cos(1+\alpha) \tau + \cos(1-\alpha) \tau \right] \]

\[ + \frac{1}{2}a^3A \left[ \cos(3+\alpha) \tau + \cos(3-\alpha) \tau \right] + \frac{1}{2}aA^3 \left[ \cos(1+3\alpha) \tau + \cos(1-3\alpha) \tau \right] \]

\[ + \frac{3}{4}a^2A^2 \left[ \cos 2(1+\alpha) \tau + \cos 2(1-\alpha) \tau \right] \]
\[ x^5 = \frac{5}{8} (a^5 + 6a^3 A^2 + 3a A^4) \cos \tau + \frac{5}{16} (a^5 + 4a^3 A^2) \cos 3 \tau + \frac{1}{16} a^5 \cos 5 \tau \\
+ \frac{5}{6} (a^5 + 6a^3 A^2 + 3a A^4) \cos \alpha \tau + \frac{5}{16} (a^5 + 4a^3 A^2) \cos 3 \alpha \tau + \frac{1}{16} a^5 \cos 5 \alpha \tau \\
+ \frac{5}{8} (2a^4 + 3a^2 A^3) \left[ \cos(2\alpha) \tau \cos(2\alpha) \tau + \cos(4\alpha) \tau \tau + \cos(4\alpha) \tau \right] \\
+ \frac{5}{16} a^4 \cos(4\alpha + 1 \tau) \tau + \cos(4\alpha - 1 \tau) \tau + \frac{5}{8} (3a^3 A^2 + 2a^4) \left[ \cos(1 + 2 \alpha) \tau + \cos(1 - 2 \alpha) \tau \right] + \frac{5}{8} a^2 A^3 \\
\left[ \cos(2 + 3 \alpha) \tau + \cos(2 - 3 \alpha) \tau \right] \\
\] 

\[ x^6 = \frac{45}{16} (a^6 + a^4 A^2 + a^2 A^4 + 9a^6) + \frac{15}{32} (a^6 + 8a^4 A^2 + 6a^2 A^4) \cos 2 \tau + \frac{1}{32} a^6 \cos 6 \tau \\
+ \frac{1}{16} (a^6 + 5a^4 A^2) \cos 4 \tau + \frac{15}{32} (a^6 + 8a^4 A^2 + 6a^2 A^4) \cos 2 \tau + \frac{3}{16} (a^6 + 5a^4 A^2) \cos 4 \tau \\
+ \frac{15}{32} (a^6 + 5a^4 A^2) \cos 4 \tau + \frac{15}{32} (a^6 + 5a^4 A^2) \cos 4 \tau + 3 \left[ \cos(1 + \alpha) \tau + \cos(1 - \alpha) \tau \right] \\
+ \frac{15}{16} (a^6 + 2a^3 A^3) \left[ \cos(3 + \alpha) \tau + \cos(3 - \alpha) \tau \right] + \frac{3}{16} a^5 \left[ \cos(5 + \alpha) \tau + \cos(5 - \alpha) \tau \right] \\
+ \frac{3}{16} a^5 \left[ \cos(1 + 5 \alpha) \tau + \cos(1 - 5 \alpha) \tau \right] + \frac{15}{16} (a^5 + 2a^3 A^3) \left[ \cos(1 + 3 \alpha) \tau + \cos(1 - 3 \alpha) \tau \right] + \frac{15}{32} a^4 A^2 \\
\left[ \cos(2 + \alpha) \tau + \cos(2 - \alpha) \tau \right] + \frac{15}{32} a^2 A^4 \left[ \cos(2 + 2 \alpha) \tau + \cos(2 - 2 \alpha) \tau \right] + \frac{5}{8} a^3 A^3 \left[ \cos(1 + \alpha) \tau + \cos(1 - \alpha) \tau \right] \\
\] 

\[ x^7 = \frac{35}{64} (a^7 + 12a^5 A^2 + 18a^3 A^4 + 4a A^6) \cos \tau + \frac{21}{64} (a^7 + 10a^5 A^2 + 10a^3 A^4) \cos 3 \tau \\
+ \frac{7}{64} (a^7 + 6a^5 A^2) \cos 5 \tau + \frac{1}{64} a^7 \cos 7 \tau + \frac{1}{64} a^7 \cos 7 \tau + \frac{7}{64} (a^7 + 6a^5 A^2) \cos 5 \tau \\
+ \frac{21}{64} (a^7 + 10a^5 A^2 + 10a^3 A^4) \cos 3 \tau + \frac{35}{64} (a^7 + 12a^5 A^2 + 18a^3 A^4 + 4a A^6) \cos \tau \\
\]
\[ + \frac{105}{64} (a_A^6 + 4a_A^4 A^3 + 2a_A^2 A^5) [\cos(2a_1 - \alpha_1) \tau + \cos(2a_1 - \alpha_2) \tau] + \frac{105}{64} (a_A^6 + 4a_A^4 A^3 + 2a_A^2 A^5) [\cos(2a_1 + \alpha_1) \tau + \cos(2a_1 + \alpha_2) \tau] + \frac{21}{32} a_A^6 [\cos(4a_1 - \alpha_1) \tau + \cos(4a_1 - \alpha_2) \tau] + \frac{7}{64} a_A^6 [\cos(6a_1 - \alpha_1) \tau + \cos(6a_1 - \alpha_2) \tau] + \frac{7}{64} a_A^6 [\cos(1+6a_1) \tau + \cos(1-6a_1 - \alpha_2) \tau] + \frac{35}{64} (3a_A^2 A^2 + 4a_A^4 A^3) [\cos(3+2a_1) \tau + \cos(3-2a_1 - \alpha_2) \tau] + \frac{35}{64} (3a_A^2 A^2 + 4a_A^4 A^3) [\cos(2+3a_1) \tau + \cos(2-3a_1 - \alpha_2) \tau] + \frac{21}{64} a_A^2 A^2 [\cos(5+2a_1) \tau + \cos(5-2a_1 - \alpha_2) \tau] + \frac{21}{64} a_A^2 A^2 [\cos(5+2a_1) \tau + \cos(5-2a_1 - \alpha_2) \tau] + \frac{35}{64} a_A^4 A^3 [\cos(4+3a_1) \tau + \cos(4-3a_1 - \alpha_2) \tau] + \frac{35}{64} a_A^4 A^3 [\cos(4+3a_1) \tau + \cos(4-3a_1 - \alpha_2) \tau] + \frac{105}{64} a_A^4 A^3 [\cos(4+3a_1) \tau + \cos(4-3a_1 - \alpha_2) \tau]

For \( \alpha = 1 \) but \( \neq 1 \), we have

\[ H_1(a, a_e) = \frac{D_1}{E} [a + \frac{3}{4} D_3 (a^3 + 2a_A^2) + \frac{5}{8} D_5 (a^5 + 6a_A^2 A^2 + 3a_A^4)] + \frac{35}{64} D_7 (a_A^7 + 12a_A^5 A^2 + 18a_A^3 A^4 + 4a_A^6)
\]

\[ H_2(a, a_e) = \frac{D_1}{E} [a + \frac{3}{4} D_3 (A^3 + 2Aa^2) + \frac{5}{8} D_5 (A^5 + 6A^3 a^2 + 3Aa^4)] + \frac{35}{64} D_7 (A_A^7 + 12A_A^5 a^2 + 18A_A^3 a^4 + 4A_A^6)
\]
(1) **System Parameters**

For the cantilever beam as shown in Fig. A.3, the deflection and slope at \( \text{a} \) due to a force \( \text{P} \) acting at \( \text{b} \) are

\[
\delta_\text{a} = \frac{\text{Pl}_1^2}{3\text{EI}} + \frac{\text{Pl}_2\text{l}_1^2}{2\text{EI}} = \delta_\text{a} \left(1 + \frac{3\text{l}_2}{2\text{l}_1}\right)
\]

\[
\theta_\text{a} = \frac{\text{Pl}_1^2}{2\text{EI}} + \frac{\text{Pl}_2\text{l}_1^2}{\text{EI}} = \theta_\text{a} \left(\frac{3}{2\text{l}_1} + \frac{3}{\text{l}_1}\right)
\]

\[\therefore\] static deflection at \( \text{b} \) due to \( \text{P} \) acting at \( \text{b} \) is

\[
\delta_\text{b} = \delta_\text{a} + \theta_\text{a} \text{l}_2 = \delta_\text{a} \left[1 + \frac{3}{\text{l}_1} + \frac{3}{\text{l}_1}\left(\frac{\text{l}_2}{\text{l}_1}\right)^2\right]
\]

The estimated equivalent stiffness of the elastic system \( S_\text{b} = \text{P}/\delta_\text{b} = 62.3 \text{ lb/in} \)

The natural frequency of the system can be estimated from the equation given in [82]

\[
\omega_\text{n} \approx \left(\frac{S_\text{b}}{m}\right)^{\frac{1}{2}}\left(1 - \frac{33 \text{ mb}}{280 \text{ m}}\right) = 134 \text{ rad./sec}
\]

The experimentally recorded stiffness and damped natural frequency of the system are respectively

\[S_\text{b} = 60 \text{ lb/in} \quad \omega = 138 \text{ rad./sec}\]
(2) **Determination of System Damping Coefficient**

a) **Logarithmic decrement method**

In order to determine system damping, the elastic beam was given an initial displacement and then released in free vibration with the slider clear of the lower surface. Oscillograph chart records of the free vibrations were obtained for five similar tests. The logarithm of the vibration amplitudes were plotted against the cycle numbers. The curve was found to be almost linear thus suggesting that the damping coefficient of the system was proportional to the velocity of the vibration. From each test the amplitudes of vibration of every tenth cycle were recorded and the ratio between consecutive amplitudes was calculated. The average amplitude ratio and the frequency from the five tests were used for determining the damping coefficient. Fig. A.2a shows an oscillograph trace of the free vibration wave form.

The equation of motion is

\[ m\ddot{x} + r\dot{x} + kx = 0 \]

the solution of the d.e. is

\[ x = e^{-\Delta t} [d_1 \cos \omega t + d_2 \sin \omega t] \]
where $\Delta = \frac{r}{2m}$ and $\omega = \sqrt{\frac{k}{m} - \frac{r^2}{4m^2}}$.

For any two maxima in the decaying sine wave the amplitude of the vibration diminishes from $e^{-\Delta t}$ to $e^{-\Delta(t + \frac{2\pi}{\omega})}$.

Therefore for ten intervals, we have

$$\frac{x_n}{x_{n+10}} = \frac{e^{-\Delta t}}{e^{-\Delta(t + 20\pi/\omega)}} = e^{20\pi\Delta/\omega}$$

or $\Delta = \frac{\omega}{20\pi} \log e^{\frac{x_n}{x_{n+10}}}$

and $\frac{\omega_n^2}{\omega_n^2} = \omega^2 + \Delta^2$

From experimental results we have $\frac{x_n}{x_{n+10}} = 1.925$ and $\omega = 22.1$ cps

$\therefore \Delta = 1.45$ rad./sec, and $\omega_n = 138.5$ rad./sec

The equivalent mass of the system $m = \frac{k}{\omega_n^2} = 1.205$ lb,

The damping coefficient of the system $r = 2m\Delta = 0.009$ lb/in/sec.

The coefficient for critical damping $r_c = 2m\omega_n = 0.86$ lb/in/sec.
b) **Direct measurement from oscilloscope**

Externally applied viscous damping as well as the system damping can be calibrated directly from the oscilloscope. By applying scaled accelerometer and displacement signals to the differential amplifier of an oscilloscope, and the velocity signal to the horizontal display amplifier, a straight line was obtained. The slope of the straight line is \( \frac{m\ddot{x} + kx}{-\dot{x}} = r \). Fig. A.2b shows an example of this trace obtained with the heavy permanent magnet as external damper.
APPENDIX IV
CALIBRATION AND SCALING OF DISPLACEMENT, VELOCITY
AND ACCELERATION SIGNALS

1) **Displacement**

   A depth micrometer was rigidly mounted with its spindle lying horizontally and imposed perpendicular to the side of the slider mount. The beam was then gradually deflected by the micrometer. The variation in output signal from the oscilloscope was recorded. A calibration curve was obtained by plotting the beam deflection in 1/1000 inch against output signal in millvolt. For deflections within 50/1000 inches the curve is linear.

2) **Velocity**

   For a damped free vibration system the amplitude of vibration is

   \[ x = Ae^{-\Delta t} \cos(qt + \phi) \]

   \[ \dot{x} = Ae^{-\Delta t} [-\Delta \cos(qt + \phi) - q\sin(qt + \phi)] \]

   \[ \ddot{x} = Ae^{-\Delta t} [(\Delta^2 - q^2) \cos(qt + \phi) + 2q\Delta \sin(qt + \phi)] \]

   where \( \Delta = \frac{r}{2m} \) and \( q = \sqrt{\frac{\omega^2 - \Delta^2}{}} \)

   From Appendix III \( q >> \Delta \)
Therefore for calibration purpose, it is sufficient to consider \( \dot{x} = q \dot{x} \) and \( \ddot{x} = q^2 \ddot{x} \), where \( x, \dot{x} \) and \( \ddot{x} \) are the peak values of \( x, \dot{x} \) and \( \ddot{x} \). The calibration of the velocity signal was fulfilled by performing a free oscillation test of the vibratory system. A \( x-x \) phase plane plot was obtained from the oscilloscope. The d.c. output signals of \( X \) and \( \dot{X} \) were recorded. The actual displacement in terms of inches was obtained from the calibration curve of \( l \). Knowing \( q \) from Appendix III, the actual velocity in terms of in/sec was obtained from \( \ddot{x} = q \dot{x} \). Comparing the velocity in in/sec with its corresponding d.c. output signal, a relationship in terms of d.c. output per in/sec could be obtained.

3) **Scaling of Displacement and Acceleration Signals**

The accelerometer is a commercial unit and has a pre-calibrated output of 0.1 volt per g. Calibration of the accelerometer is therefore not required. However, scaling of the acceleration force term and the spring force term was required so that when \( \ddot{x} = 0 \), \( m \ddot{x} + kx = 0 \). The scaling procedure was carried out by applying the displacement and accelerometer signals to the differential amplifier of the oscilloscope and the velocity signal to the horizontal display amplifier during free vibration test and adjusting the gain control of the bridge amplifier meter until the combined displacement and accelerometer signal
formed a straight line. Alternatively, the time based displacement and accelerometer signals could be fed to separate amplifiers of the oscilloscope and the gain control of the bridge amplifier meter was adjusted until the displacement signal equaled the accelerometer signal.
APPENDIX V

SPECIMEN COMPOSITION

A. Atlas Nutherm Steel
   C  0.7 %
   Mn 2.0 %
   Si 0.3 %
   P  0.014 %
   S  0.010 %
   Cr 1.0 %
   Mo 1.35 %

B. Atlas Keewatin Steel
   C  0.9 %
   Mn 1.2 %
   Si 0.3 %
   Cr 0.5 %
   V  0.2 %
   W  0.5 %

C. Automatic Transmission Fluid
   The following oil samples and information were received from the Cities Service Oil Company, Cranbury, New Jersey.
   (1) Cities Service 100 Neutral Oil
   (2) Cities Service 200 Neutral Oil
(3) Cities Service 350 Neutral Oil
(4) Cities Service 650 Neutral Oil
(5) Cities Service 150 Bright Stock
(6) Cities Service 100 Neutral Oil + 1% Victablube 5810
(7) CITGO Automatic Transmission Fluid Type "A"
   Suffix "A"
(8) CITGO Automatic Transmission Fluid Type F
(9) Cities Service 100 Neutral Oil + 1% Sulfurized Sperm Oil

The first five are base stocks with no additives. Their viscosities are 100, 200, 350, 670 and 2650 Saybolt Seconds at 100°F. in the order of the sample numbers and the viscosities of the Automatic Transmission Fluids are 190 Seconds for Type A and 180 Seconds for the Type F.

The Kodak Blotter gives friction data similar to even as vastly different a material as the R-3681-22 clutch facing material against steel.

Oil Sample 8 shows the hump in the friction curve and behaves in a chattery and squawky manner despite the fact that static friction is lower than kinetic friction in most cases.
Figure 1.1.1 Displacement-Time Waveforms of Friction-Induced Vibration

(a) STICK-SLIP

(b) QUASI-HARMONIC
Figure 1.1.2 Schematic Diagrams of Three Frictional Systems
Figure 3.1.1 Topological Diagrams Illustrating Soft and Hard Self-Excitations
Figure 3.2.1 Possible Forms of Friction-Velocity Characteristic Curves
Figure 3.2.2  Humped Form of a Friction-Velocity Characteristic Curve
Figure 3.2.3  $x\cdot \dot{x}$ Phase Plane Diagrams
Figure 3.3.1  Humped Form of a Friction-Velocity Characteristic Curve Represented by a Simplified Exponential Expression
\[ C_3V + \frac{C_1C_3}{C_2} = 1 \]

\[ V = \frac{C_2 - C_1C_3}{C_2C_3} \] (REF. FIG.3.3.1)

Figure 3.3.2 Theoretical Amplitude of Vibration versus Velocity Curve From the Humped Friction-Velocity Curve of Fig. 3.3.1
Figure 3.3.3  $y_1$, $y_2$ vs Velocity Plot Showing Regions of stability and instability
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Figure 4.2.3  Isometric Diagram of Apparatus
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Figure 5.1.1
One Cycle Oscilloscope Trace of a Friction-Velocity Curve and a x-x Phase Plane Diagram.
Figure 5.1.3  One Cycle Oscilloscope Traces at Various Disc Velocities
- Type Al

**DISC VELOCITY, v**
- □ 0.84 in/sec
- × 1.06
- ○ 1.30
- △ 1.64

**w = 5.4 lb**
Figure 5.1.4
Graph of Experimental and Theoretical Curves - Type A2
Figure 5.1.5  Graph of Experimental Amplitude of Vibration versus Velocity Curves at Various Normal Loads - Type A2
Figure 5.1.7  Graph of Experimental Amplitude of Vibration versus Velocity Curves at Various Normal Loads - Type B2
Figure 5.1.9  x-\dot{x} Phase Plane Diagrams of the Stick-Slip Type

(a)  \( v=0.08 \text{ in/sec}, \ w=3.85 \text{ lb} \)

(b)  \( v=2.45 \text{ in/sec}, \ w=5.4 \text{ lb} \)
Figure 5.1.10 Graph of Experimental Curves Showing Effect of External Damping - Steel on Steel
Figure 5.1.11 Microphotographs of a Carbon Fibre-Resin Slider
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Friction-Velocity Characteristic Curves - Carbon Fibre-Resin
Figure 5.1.13 Friction-Velocity Characteristic Curve Reproduced From A One Cycle Oscilloscope Trace – Rubber on Steel
Figure 5.2.1 Graph of Magnification Factor vs Frequency Ratio - Linear System
Figure 5.2.2 Graph of Magnification Factor vs Frequency Ratio - Type C

Figure 5.2.2 Graph of Magnification Factor vs Frequency Ratio - Type C
Figure 5.2.3  \( x - \dot{x} \) Phase Plane Diagram During Forced Vibration - Type Bl

(a)  \( \alpha = 0.83 \)
    \( F = 0.29 \)

(b)  \( \alpha = 0.88 \)
    \( F = 0.325 \)

\( V = 0.98 \)
Figure 5.2.4 Graph of Magnification Factor vs Frequency Ratio - Type B1
Figure 5.2.5 Graph of Magnification Factor vs Frequency Ratio - Type A2
Figure 5.2.6
Grabs Showing Effect of Amplitude of Vibration on the Average Displacement of the Slider During Forced Vibration.
Figure 5.2.7  Graph of Amplitude of Vibration vs External Force Parameters at Various Disc Velocities - Non-Resonance
Figure 5.2.8 Graph of Amplitude of Vibration vs External Force Magnitude - Non-Resonance
Figure 5.2.9  Displacement-Time and Velocity-Time Traces - Non-Resonance
Figure 5.2.10  Graph of Amplitude of Vibration vs Frequency Ratio - Fundamental Resonance

F_0 = 0.2  V = 1.05

- FUNDAMENTAL RESONANCE THEORY
- NON-RESONANCE THEORY
- NUMERICAL (EXPONENTIAL)
Figure 5.2.11 Displacement-Time and Velocity-Time Traces at Various External Excitation Frequencies
Figure 5.2.12 Experimental Oscillograph Traces at Various External Excitation Frequencies
Figure 5.2.13 Graph of Magnification Factor vs Frequency Ratio - Type Al
Traces - 1 Harmonic

Figure 5.2.14: Displacement-Time, Velocity-Time and Displacement-Velocity

Phase Plane

Velocity

Displacement

Time

Velocity in/sec

Displacement in

EXT. F/R = 2630 RPM; FORCE = 0.107 lbf

VEL = 1.050, FRE = 2.000
Figure 5.2.15 HARMONIC DISPLACEMENT-TIME, VELOCITY-TIME, AND DISPLACEMENT VELOCITY

Phase Plane

Velocity

Displacement

Time

Velocity

Displacement
Figure 5.2.16
Displacement-Time, Velocity-Time and Displacement-Time Trace, 1 Harmonic
Figure 5.2.17: Displacement-Time, Velocity-Time and Displacement-Time Harmonic Traces - 1/4 Harmonic

Phase Plane

Displacement x 0.001 in.
Velocity x 0.001 in.

Time [sec]

Velocity [in/sec]

Displacement [in]

EXT FR = 5300 rpm, FORCE = 0.435 lb

VEL = 1.050, FRI = 0.010
Figure 5.2.18 Graph of Amplitude of Vibration vs External Force Magnitude
Figure 5.2.19 Graph of External Excitation Magnitude for the Extinction of Autoperiodic Oscillation vs External Excitation Frequency
Figure 5.2.20 Displacement-Time and Velocity-Time Traces at Various External Excitation Magnitudes
Figure 5.2.21 Displacement-Velocity Phase Plane Diagrams
Figure 5.2.22 Displacement-Time and Velocity-Time Traces - Linearised Friction-Velocity Curve
NORMAL LOAD = 6.4 lb

FREQ. RATIO $\alpha$

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Figure 5.2.23 Graph of Amplitude of Stick-Slip Vibration and Maximum Static Friction Force vs Load Ratio
Figure 5.2.24 Experimental Oscillograph Traces Illustrating the Extinction of Stick-Slip Vibration Due To Normal Excitation
Figure 5.2.25 Experimental Oscillograph Traces at Various Normal Excitation - Type A1
Figure 5.2.26 x-t, x-t and x-x Traces at $\frac{1}{3}$ Harmonic - Normal Excitation
Figure A.1  Calibration Curves of Elastic Beam

STEEL BEAM
1 in x 1/4 in
Figure A.2 Logarithmic Decrement Oscillograph Trace and One Cycle Oscilloscope Trace For The Determination of System Damping Coefficient