INVESTMENT DECISIONS IN A DYNAMIC ENVIRONMENT

by

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ABSTRACT

The explicit consideration of certain types of uncertainty, in the analysis of investment opportunities, has become practical for the modern day decision maker. However, uncertainty analysis has generally been concerned with the probabilistic nature of future cash flows of investment opportunities. The uncertainty of cash flows, while extremely important in analysis, is by no means the only type of uncertainty which faces the decision maker. This thesis investigates another dimension of uncertainty by explicitly considering the possibility of better investment opportunities occurring in the future.

A model is developed which approximates the interarrival time of investment opportunities by a sequence of independent and identically distributed random variables. The interarrival times are assumed to have a negative exponential probability density function, which corresponds to an Erlang family of probability density functions for the n th - order interarrival time. The present value, at the time the investment opportunity occurs, is assumed to be an element of a sequence of independent and identically distributed random variables. The distribution from which these random variables are drawn is assumed to be known. Using these independent families of random variables, a general model is developed.

The above model is extended to take into account the effect of spending money in search of better investment opportunities. The amount spent on search is assumed to have an effect on the arrival rate of investment opportunities.

Using the model described above, a number of interesting problems are analyzed. For a large class of problems the analysis determines the following:

a) The expected value of continuing in search of better investment opportunities for a particular investment policy.

b) The optimal investment policy which maximizes the expected value of continuing to search.

c) The optimal level of search for a particular investment policy.

d) The expected time until an acceptable investment, as defined by the investment policy, occurs.

Numerical examples are formulated and numerical results for the above are determined.

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CHAPTER I

INTRODUCTION

Recently, a good deal of work has been done in connection with the optimal selection of investments under uncertainty. Byrne, Charnes, Cooper and Kortanek,[1] Näslund,[2] Weingartner,[3] and Ziemba[4] have developed models to analyze investment decisions under uncertainty.[5] However, in each of the above models, the uncertainty is with respect to future cash flows of given investment opportunities. Often, in practical situations, there exists uncertainty with respect to the possibility that better investments may come along shortly after the initial investment has been made. It is this aspect of uncertainty which this thesis is going to explore.

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[1]R. Byrne, A. Charnes, W. W. Cooper, and K. Kortanek, "A Chance-Constrained Approach to Capital Budgeting With Portfolio Type Payback and Liquidity Constraints and Horizon Posture Controls," Journal of Financial and Quantitative Analysis, Vol. II (1967), pp. 339-364.

[2]Näslund, "A Model of Capital Budgeting Under Risk," Journal of Business, Vol. XXXIX (1967), pp. 257-271.

[3]H.M. Weingartner, "Capital Budgeting of Interrelated Projects: Survey and Synthesis," Management Science, Vol. 12 (1966), pp. 485-516.

[4]W. Ziemba, "A Myopic Capital Budgeting Model" Journal of Financial and Quantitative Analysis, Sept. 1969, pp. 305-327.

[5]For a brief discussion of the first two models see R. H. Bernhard, "Mathematical Programming Models for Capital Budgeting--A Survey, Generalization, and Critique," Journal of Financial and Quantitative Analysis, Vol. IV (1969), pp. 111-158.

The underlying assumption which exists in the above models as well as models discussed in standard capital budgeting and finance texts is that the optimal investment policy can be obtained by choosing optimally among the given investment opportunities. Van Horne states, "If capital is to be allocated optimally, we must take into account possible differences in the future mobility of funds when evaluating investment proposals."[6] If this is true, then surely the possibility of not investing at the present, and maintaining a high mobility of funds must be considered. However, instead of analyzing the possibility of future investments in order to estimate the value of the mobility of funds, Van Horne only considers existing investment opportunities.

Weston and Brigham, in emphasizing the importance of capital budgeting, state, "The fact that the results continue over an extended period means that the decision maker loses some of his flexibility."[7] However, no attempt is made to consider how the loss of this flexibility effects the decision making procedures. In fact, Weston and Brigham state, "Aside from the actual generation of ideas, the first step in the capital budgeting process is to assemble the proposed new investments, together with the data necessary to appraise them."[8] It seems reasonable, in order to

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[6] J. C. Van Horne, <u>Financial Management and Policy</u>, Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1968, p. 103.

[7] J. F. Weston and E. F. Brigham, <u>Essentials of Managerial Finance</u>, Holt, Rinehart, and Winston, Inc., 1968, p. 125.

[8] <u>Ibid</u>., p. 126.

take into account this loss of flexibility, that possible future investment opportunities should be taken into consideration. However, as before, the investment decision is made, based only on existing investment opportunities.

Quirin defines the first three steps in the capital budgeting procedure as; project generation, project evaluation, and project selection.[9] Again, project selection is based only on existing investment opportunities.

Mao states in his chapter on investment decisions under uncertainty, that one section of the chapter will "..... illustrate with examples how this probabilistic information can help financial management optimize its accept-reject decisions involving individual risky investments."[10] However, the probabilistic information mentioned is in regards to the cash flows of given investments and no consideration is made of future possible investment opportunities.

Recent articles have emphasized uncertainty with respect to future cash flows of given investment opportunities as well as capital budgeting texts; thus, sufficient consideration has not been given to the possibility that better investments may come along shortly after the initial investment has been made. The basic

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[9] G. D. Quirin, The Capital Expenditure Decision, Richard D. Irwin, Inc., 1967.

[10] J. C. T. Mao, Quantitative Analysis of Financial Decisions, The Macmillan Co.: Collier-Macmillan Canada Ltd., 1969, p. 271.

problem which this thesis will analyze is determining the optimal policy between investing now, given a selection of present investment opportunities as well as some probabilistic information about future investment opportunities, or waiting in expectation of better opportunities arising.

This basic problem can be restated as: determine an optimal search stopping rule. That is, when would the decision maker stop and invest, and when should he continue searching for better investment opportunities. Chow and Robbins,[11] Radner,[12] Randolph,[13] and MacQueen and Miller[14] have analyzed the following general problem:

> Given a set of independent and identically distributed random variables which are sampled one at a time for a cost, what is the optimal policy at each stage; that is, should the decision maker stop or sample one more. If the decision is to stop the

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[11] Y. S. Chow and H. Robbins, "A Maringale System Theorem and Applications", Preceedings Fourth Berkeley Symposium on Mathematical Statistics and Probability, 1961, pp. 93-104.

[12] R. Radner, "Math. Specification of Goals for Decision Problems" in M. Shelly, Human Judgement and Optimality.

[13] P. Randolph, "Optimal Stopping Rules" Operations Research, (U. S.) 15 (1967), pp. 562-564.

[14] J. MacQueen, and R. G. Miller Jr., "Optimal Persistence Policies", Operations Research, (U. S.) Vol. 8 (1960), pp. 362-379.

decision maker receives the maximum of the random variables previously sampled, and if he goes on, the cost of searching for one more is a function of the current stage.

It is assumed that the probability distribution function of the random variables is known. An optimal result for the above problem can be determined which specifies the optimal policy at each stage.

The basic problem in applying the above model to the capital budgeting problem is that the following points, which are necessary in the capital budgeting model, are not included in the model that was analyzed:

1) The time value of money,
2) The cost of proceeding to the next stage is not known explicitly, and
3) The time to the next sampling becomes important.

However, the basic idea of the above problem is used in formulating the models presented later.

Further extensions to the above problem are considered by Randolph[15] and Yahav.[16] They consider the case where the probability density function is not known. By using past data, estimates of the optimal policy are determined. However, because of the

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[15] P. Randolph, op. cit.

[16] J. A. Yahav, "On Optimal Stopping," Annals of Mathematical Statistics, 37, pp.30-35.

complexity involved, this thesis will consider only the case where the probability density function is known.

Additional searching techniques are discussed by Marschak and Yahav[17] and MacQueen.[18] These two papers discuss models which optimize the division of effort between searching for new possibilities and further evaluation of existing possibilities. While these searching techniques appear to be applicable to the capital budgeting problem they will not be discussed in this thesis.

Summarizing, the basic purpose of this thesis is to make use of the ideas contained in optimal search stopping rules in order to determine the optimal policy between investing now and continuing waiting or searching for a better investment.

As in any study of this type, the decision maker must first estimate the profitability of the study, even if this is done subjectively. He must consider the expected return on the study and the expected cost of this study. The author believes, as is pointed out in a paper by B. Schwab,[19] that due to the dynamic

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[17]T. A. Marschak, and J. A. Yahav, "The Sequential Selection of Approaches to a Task" <u>Management Science</u>, Vol. 12, pp. 625-647.

[18]J. MacQueen, "Optimal Policies for a Class of Search and Evaluation Problems", <u>Management Science</u>, Vol. 10, pp.746-759.

[19]B. Schwab, "Investment Evaluation in a Dynamic Environment: Some notes on the Notion of Flexibility and the Role of Uncertain Future Opportunities," <u>Working Paper No. 14</u>, Faculty of Commerce and Business Administration, University of British Columbia.

nature of investment opportunities, there exist many circumstances where a study of possible future investments would be advantageous to the decision maker. It is under this assumption that this thesis is written.

CHAPTER II

DESCRIPTION OF THE PROBLEM AND THE MODEL

Generally, the decision maker would have a limited amount of funds which he is required to invest to the best of his ability. He would normally have some knowledge of existing investment opportunities, and, in addition, he might have some idea of future possible investment opportunities. Currently, if this probabilistic information about future investment opportunities is used at all by the decision maker, it is done so entirely subjectively. In most cases, the decision maker rarely considers the costs and benefits of searching or waiting for better investment opportunities.

Throughout this thesis waiting and searching will be defined as follows: "Waiting" is the process of holding funds in expectation of better investment opportunities. "Searching" is the process of holding funds and spending money searching for new investments in expectation of finding a better investment opportunity. In both of these processes the time value of money must be considered.

The following examples illustrate these concepts. An investor is waiting to invest his money if he expects a new issue of Canada Savings Bonds to be issued with returns that are better than existing Canada Savings Bonds. Note particularly that no money is being spent seeking this new investment. A firm is searching for new investment opportunities if it is carrying on an active program of product development. That is, it is carrying out research, at a cost, in order to develop investment opportunities. Another example of search would be an investor paying a stockbroker to locate good investment opportunities.

Throughout this thesis it is assumed that the decision maker can gain probabilistic information with regards to future investment opportunities at no cost. An interesting refinement would be to consider improving this probabilistic information at a cost. However, this refinement is considered outside the scope of this thesis.

For simplification it is also assumed throughout that abandonment is not possible; that is, once a firm is committed to an investment it cannot sell out. This would be the case where a firm has to make a large initial investment in a project and the abandonment value of the project is relatively small. If abandonment is possible, either at no loss or even at a relatively reasonable loss, the above problem becomes less important. This is due to the fact that the decision maker can choose optimally as to whether or not to invest in the new opportunity. This is done by comparing: a) the value of abandoning the initial investment and reinvesting in the new investment, and, b) the value of remaining with the previously accepted investment. It is interesting to note that when abandonment is possible, the policy of comparing a and b is not necessarily an optimal policy. This is due to the fact that although abandonment is possible it is often costly, and some of this cost could be eliminated by taking into account possibilities of future investment opportunities.

It is also assumed throughout that the decision maker can rank investment opportunities by using present value methods. As pointed out by many texts on investment decisions it is important to include risk when ranking investments.[20] However, it will be assumed that investments are ranked by standard methods without consideration of risk regarding the project's future cash flows.

In addition, only investment opportunities which are mutually exclusive will be considered. No attempt will be made to consider more general cases of interrelationships which may exist between investment opportunities.

Summarizing the above assumptions, and listing other minor ones, the assumptions made throughout are:

1) Probabilistic information with regards to future investment opportunities is given.
2) Abandonment is not possible.
3) Investment opportunities are ranked by present value methods with no consideration of risk.

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[20] See J. C. VanHorne, op.cit., G. D. Quirin, op. cit. J. T. C. Mao, op. cit. and F. S. Hillier, The Evaluation of Risky Interrelated Investments, North-Holland Publishing Company, Amsterdam, London, 1969.

4) Investment opportunities are all mutually exclusive.

5) The utility function of money gains and losses is linear.

and

6) The time value of money is given and is constant over time.

Given the problem as described above, the decision maker is faced with a number of alternative actions:

1) He could invest all available funds in existing investment opportunities. That is, he could choose optimally, using presently available optimization techniques, from investment opportunities presently available, or

2) He could hold all available funds and wait or search for better investment opportunities, or

3) He could use a combination of the above possible actions. For example, he could invest a portion of his funds now and hold the remaining funds in order to capitalize on possible attractive future investments.

The decision as to which course of action is optimal would, in general, have to be taken as each new investment occurred. Clearly, at each stage, (that is, at each time a new investment opportunity occurred) the decision maker would have to examine the following question: Is the expected gain in waiting or searching for new investments greater than the expected cost of waiting or searching for these new opportunities? The expected cost would include the time value of money until an acceptable investment occurred plus the cost of search, if search is carried out, plus, other penalties associated with waiting, such as deadline penalties. Clearly, if we are trying to maximize the expected monetary value of our investments, the optimal policy would be to wait or search for new investments if, and only if, the expected cost of waiting or searching for an acceptable investment is less than the expected gain associated with waiting for an acceptable investment. The expected gain in waiting would be the expected present value of an acceptable investment when it occurred discounted to the present, less the present value of the best presently available investment opportunity. Mathematically,

$$EG = E(PV_a) \cdot e^{-rt} - PV^+$$

where EG = the expected gain associated with waiting,

$E(PV_a)$ = the expected present value of an acceptable investment discounted to time t,

r = discount factor, and

PV^+ = the best presently available investment opportunity.

Next, consider the interarrival times between successive investment opportunities. These interarrival times can be represented by random variables which are independent and identically distributed. Hence, via the independence assumption, if we are given that investments occurred extremely close together last month, this gives no useful information in estimating the interarrivals this month; and, the identically distributed assumption means that if last month we estimated that the probability of interarrival times of less than one week was 90%, then we must have this same probability now and in the future of an interarrival time of less than one week.

Also, let the values which are used for ranking investments be represented by random variables which are also independent and identically distributed. In order to get a clear picture of the model which is being presented see Figure 1.

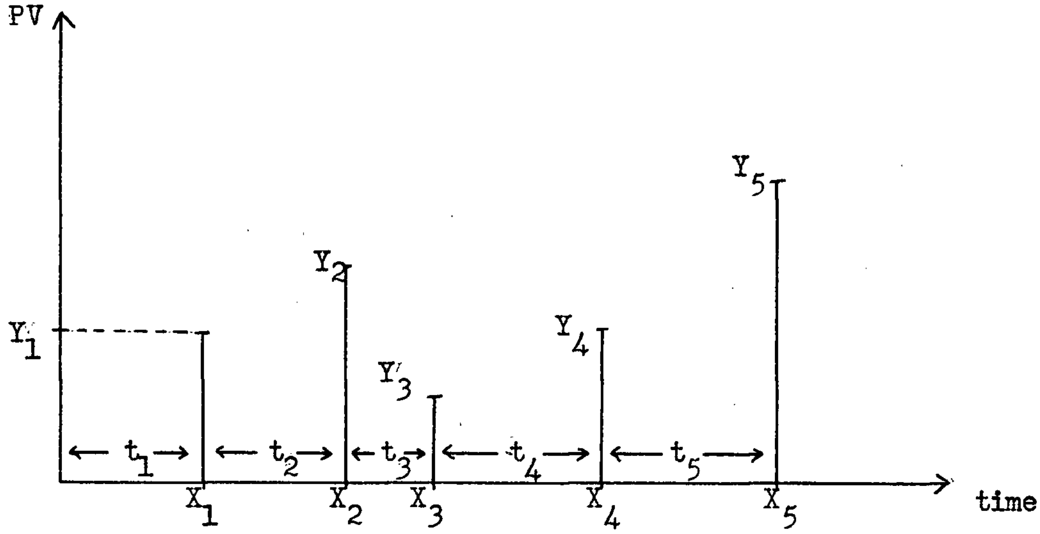

Figure 1. Illustration of the Model

In Figure 1, t_1, t_2, ... t_5 represent the independent and identically distributed random variable -- the time between successive investment occurrences. X_1, X_2 ....X_5 represent the time until the occurrence of the first, second, ... and fifth investment respectively. And Y_1, Y_2, ... Y_5 represent the independent and identically distributed random variable -- the present value of the first, second, ... fifth investment respectively, discounted to the time the investment occurs.

This model is analogous to the following inventory model:[20] The T's represent the interarrival times of customers arriving at a store and the Y's would represent the quantity that each customer purchases at the store. Again, the T's and the Y's are independent and identically distributed random variables.

This paper will be concerned with optimal decision rules of this general problem as well as with optimal decision rules to simpler problems which might direct us toward the optimal solution to the general problem.

Consider the following framework:

1) The arrival of investment opportunities is random, with known arrival rate λ, which is constant over time, which corresponds to a Poisson distribution of arrivals or to a negative exponential distribution of interarrival times. As mentioned in the paper by B. Schwab[21], this assumption seems intuitively reasonable; however, its validity will have to be tested empirically.

2) Money spent on search has the effect of decreasing the expected interarrival time of investment opportunities but has no effect on the distribution of the present

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[20] G. Hadley and T. M. Whitin, Analysis of Inventory Systems, Englewood Cliffs, N. J.: Prentice-Hall, Inc., 1963.

[21] B. Schwab, op. cit.

values of new investment opportunities.[22] Define λ_o as the constant arrival rate if no money is spent on search. Then the expected interarrival time if no money is spent on search is $1/\lambda_o$. Define S as the amount of money spent on search per unit time, and, define $1/\lambda_s$ as the expected interarrival time if S is spent on search. Therefore $1/\lambda_s$ is a function of S. As search dollars increase, one might expect that the law of diminishing returns would take effect. Thus, we assume that the expected interarrival time of investment occurrences decreases exponentially as search dollars are increased, as shown in Figure 2.

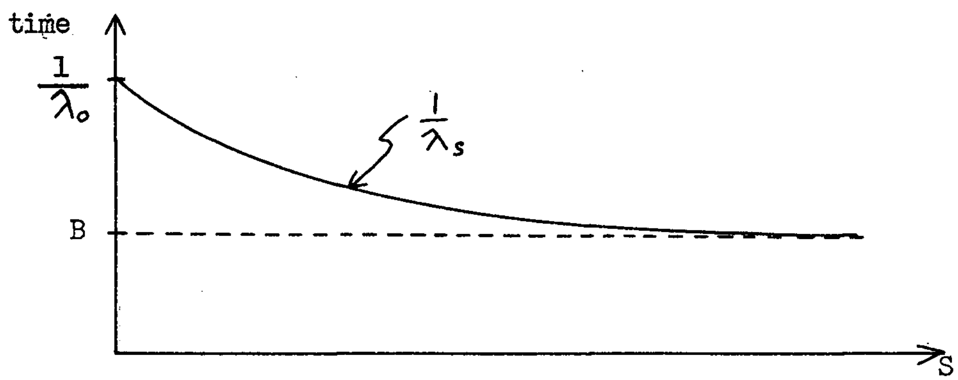

Figure 2. Interarrival time as a function of S.

[22]This assumption seems reasonable in many situations; however, some situations may be characterized by search dollars increasing the arrival rate of only the better investment opportunities. For example, product research could be directed to encourage product development in a known, profitable product line.

Mathematically, Figure 2 can be expressed as follows:

$$1/\lambda_s = B - (B - 1/\lambda_o)\, e^{-bS}$$

where $1/\lambda_o$, B and b are assumed to be known.

3) The distribution from which the present values (Y's) are obtained is known. That is,

$$P(Y \leqslant y) = F(y)$$

is known. From which the probability density function f(y), if it exists, can be calculated as follows:

$$f(y) = \frac{d}{dy} F(y)$$

Given the above information, one can determine, for certain types of investment problems, the optimal policy of investment. That is:

1) When the decision maker should invest and when he should wait or search for better investment opportunities.
2) The optimal amount of search that should be spent.
3) The expected time until an acceptable investment occurs.

CHAPTER III

DISCUSSION OF SUBPROBLEMS

A number of subproblems which seem relevant will now be discussed along with examples which will illustrate these subproblems.

An interesting, yet relatively simple problem is the special case where investments of only two types can occur. One type yields a present value equal to PV_1 and the other type yields a present value of PV_2. Each type of investment can occur many times and the distribution of interarrival times is negative exponential. If an investment yielding the lower of the two present values occurs first, should the firm invest in this investment or should it wait for the better type of investment to occur?

This type of problem could occur in a firm which has two product lines and a new product development group within each product line. If the firm has sufficient funds for only one new investment, and at present the only investment available is from the product line which yields the lower of the two present values, should the firm invest now or wait for an investment from the product line yielding the higher present value?

In the above problem, it is assumed that the cost of new product development is fixed and therefore the optimal policy is independent of this cost. The second problem which is of interest is the case where there are again only two possible types of investments; only now, the decision maker can increase the arrival rate of the investments by spending money on search. The decision maker is now faced with two decisions:

1) What is the optimal amount that should be spent on search?
2) Once this is determined what should the optimal investment policy be?

This problem could occur in circumstances similar to those in the first example with the exception that the product development group is now one independent division. It now becomes difficult to increase expenditures on the development of the product line yielding the higher present value and therefore additional expenditures increase developments in both product lines.

The above special case considered only two possible investment types. An extension to this is the case where there are more than two investment types. An optimal policy will be derived which defines a cut-off point PV*, such that, if an investment with present value PV_j has just occurred, it is acceptable if $PV_j \geq PV^*$ and unacceptable if $PV_j < PV^*$. An example of this subproblem is just an extension of the example with only two possible investment types.

The following problem may be of more practical importance. The present value can be any value from a known continuous distribution, and a constant aspiration level is subjectively assigned. That is, investments with present values (discounted to when they occur) below this aspiration level are not acceptable, while investments with present values above this aspiration level are acceptable. While the policy is determined in this particular case; that is, whether to accept an investment or not is determined, it is valuable to know the expected time until an acceptable investment occurs.

An example where this model becomes useful is in a firm that has a written policy which indicates when an investment is acceptable and when it is not. Another example is the individual who sets a minimum present value on money that he intends to invest.

A further refinement to the above problem is the problem where the possibility of search exists. The decision maker must now determine what is the optimal level of search. It is also useful to know the expected time until an acceptable investment occurs if the optimal level of search is known.

In many cases the aspiration level is not constant but varies with time. This is particularly true of the individual investor who may set out with a constant aspiration level, as in the example above. However, as time passes and no investment opportunities occur equal to or above his aspiration level, the individual often lowers his aspiration level. Simon states,

> The aspiration level, which defines a satisfactory alternative, may change from point to point in a sequence of choice situations. A vague principle would be that as the individual, in his exploration of alternatives, ...finds it difficult to discover satisfactory alternatives, his aspiration level falls.[23]

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[23] H. A. Simon, Models of Men, New York: Wiley, 1955, p. 253.

Cyert and March present the following explicit principle.[24] The current aspiration level equals the weighted sum of one's most recent aspiration level, the most recent experience, and an index of the recent performance of others. In view of the above discussion, the aspiration level will not be known as a function of time along. However, it will be interesting to formulate the problem assuming that it is only a function of time.

Another interesting problem occurs when a deadline exists. This problem can be divided into several problems according to whether investment opportunities disappear or not after they have been rated as unacceptable at the time when they occurred. In the previous problems, as well as in the general problem to be discussed later, there has been no mention of investments disappearing or not. This is due to the fact that once a project is rated unacceptable it will remain unacceptable except in the case of variable aspiration level and the case of a deadline. This will become clear in the analysis of the results obtained in the next two sections. If a deadline does exist and particularly if there exists a monetary penalty for not having found and accepted an investment prior to the deadline, the decision maker would like to know what the optimal policy is for the cases where investments disappear, where they do not disappear, and where they have a probability of disappearing. In addition, the decision maker would

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[24] R. M. Cyert and J. G. March, <u>A Behavioral Theory of the Firm</u>, Englewood Cliffs: Prentice-Hall, 1963.

also like to know the optimal level of search for each of these two problems.

A good example where a firm would find it advantageous to have the above information is the following. A firm may find it advantageous to make an investment before the budget period is up. For example, a growth firm may wish to acquire a new firm before year end so as to show continued growth in the year end financial statement and thus keep its stock in a favourable market position.

And finally, the general problem with no search will be analyzed. In this problem it is assumed that investment opportunities which occur over time become known to the investor at no cost. This would be the case if the investor was able to obtain free information about investment opportunities as they occur. In addition, the general problem where search is possible will be considered. For this problem the relevant results such as the optimal policy, the optimal amount of search and the expected time until an acceptable investment occurs will be developed.

CHAPTER IV

ANALYSIS OF SUBPROBLEMS

Results, such as the optimal policy, the optimal level of search, and the expected time until an acceptable investment opportunity occurs will be sought for the above subproblems. In certain explicit cases, analytical results are obtained. A general solution procedure is also developed for other cases where analytical results appear at least difficult.

i) Two possible investment types with no search.

In this case, we assume that investment opportunities occur with present value equal to either PV_1 or PV_2. Where PV_1 is greater than PV_2. Also, investment opportunities occur with a constant arrival rate λ .

Additional given information is the probability distribution of the present values. Given that an investment has occurred, the probability that it equals PV_1 is p. And, since only two possible investments can occur, the probability that it equals PV_2 is 1- p. These probabilities can be stated as follows:

$$P(Y = PV_1) = p$$

and $$P(Y = PV_2) = 1 - p.$$

The optimal policy for this problem can be stated as the following: If an investment with present value equal to PV_1 occurs first, it should be accepted. However, if an investment with present value equal to PV_2 occurs first it should only be accepted if the expected value of waiting, discounted to the present, is greater than PV_2. Let V be defined as the value of waiting discounted to the present. Then V is equal to the value of the first investment to occur, discounted to the present, if the first investment to occur has present value of PV_1; plus the value of the second investment to occur, discounted to the present, if this second investment has present value equal to PV_1 and if the first investment had present value equal to PV_2 (that is, the first investment was not accepted); plus the value of the third investment to occur, discounted to the present, if this third investment has present value equal to PV_1 and if all previous investments were not accepted; plus ...

Defining I as the indicator function,

$$I_{(Y_i = PV_j)} = \begin{cases} 1 & \text{if } Y_i = PV_j \\ 0 & \text{if } Y_i \neq PV_j \end{cases}$$

V can be expressed as follows:

$$\begin{aligned} V = {} & Y_1 \cdot e^{-rX_1} \cdot I_{(Y_1 = PV_1)} \\ & + Y_2 \cdot e^{-rX_2} \cdot I_{(Y_2 = PV_1)} \cdot I_{(Y_1 = PV_2)} \\ & + Y_3 \cdot e^{-rX_3} \cdot I_{(Y_3 = PV_1)} \cdot I_{(Y_2 = PV_2)} \cdot I_{(Y_1 = PV_2)} \\ & + \ldots \end{aligned}$$

The expected value of waiting (EV) can be calculated as follows:

$$
\begin{aligned}
EV = \ & E\left\{Y_1 \cdot e^{-rX_1} \cdot I_{(Y_1 = PV_1)}\right\} \\
& + E\left\{Y_2 \cdot e^{-rX_2} \cdot I_{(Y_2 = PV_1)} \cdot I_{(Y_1 = PV_2)}\right\} \\
& + E\left\{Y_3 \cdot e^{-rX_3} \cdot I_{(Y_3 = PV_1)} \cdot I_{(Y_2 = PV_2)} \cdot I_{(Y_1 = PV_2)}\right\} \\
& + \ldots \qquad (25)
\end{aligned}
$$

$$
\begin{aligned}
= \ & E\left\{Y_1 \cdot I_{(Y_1 = PV_1)}\right\} \cdot E\left\{e^{-rX_1}\right\} \\
& + E\left\{Y_2 \cdot I_{(Y_2 = PV_1)}\right\} \cdot E\left\{e^{-rX_2}\right\} \cdot E\left\{I_{(Y_1 = PV_2)}\right\} \\
& + E\left\{Y_3 \cdot I_{(Y_3 = PV_1)}\right\} \cdot E\left\{e^{-rX_3}\right\} \cdot E\left\{I_{(Y_2 = PV_2)}\right\} \\
& \quad \cdot E\left\{I_{(Y_1 = PV_2)}\right\} \\
& + \ldots \qquad (26)
\end{aligned}
$$

Taking the expected value of each term yields,

$$
\begin{aligned}
EV = \ & p \cdot PV_1 \cdot \frac{\lambda}{r+\lambda} + p \cdot (1-p) \cdot PV_1 \cdot \frac{\lambda}{r+\lambda} \\
& + p \cdot (1-p)^2 \cdot PV_1 \cdot \left(\frac{\lambda}{r+\lambda}\right)^2 + \ldots \qquad (27)
\end{aligned}
$$

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[25] Utilizing the fact that the expected value of a sum is equal to the sum of the expected values.

[26] Utilizing the fact that the expected value of a product of independent terms is equal to the product of the expected values.

[27] See Appendix I for proof of:

$$E\left\{e^{-rX_n}\right\} = \left(\frac{\lambda}{r+\lambda}\right)^n$$

$$EV = p \cdot PV_1 \cdot \frac{\lambda}{r + \lambda} \cdot \left[1 + \frac{(1-p)\lambda}{r+\lambda} + \left(\frac{(1-p)\lambda}{r+\lambda} \right)^2 + \cdots \right]$$

where $(1 - p)$, since it is a probability, is less than or equal to 1; and $\frac{\lambda}{r + \lambda}$, since λ and r are positive, is less than 1.

Therefore,

$$\frac{(1-p)\lambda}{r+\lambda} \leqslant 1$$

and the sum of the infinite series above is,

$$\frac{1}{1 - \frac{(1-p)\lambda}{r+\lambda}}$$

Therefore,

$$EV = p \cdot PV_1 \cdot \frac{\lambda}{r+\lambda} \cdot \frac{1}{1 - \frac{(1-p)\lambda}{r+\lambda}}$$

$$= \frac{p\lambda}{r + p\lambda} \cdot PV_1$$

The optimal investment policy becomes: Wait for an investment with present value equal to PV_1 if,

$$PV_2 < \frac{p\lambda}{r + p\lambda} \cdot PV_1$$

If the optimal policy is to wait for an investment with present value equal to PV_1, what is the expected time until such an investment occurs? In order to calculate this, let us first define T as the time until an investment is accepted with present value equal to PV_1, and ET as the expected time until such an investment occurs. Then T is equal to the time to the first investment if the first investment to occur is acceptable; plus the time to the second investment if the second investment is acceptable and if the first was not acceptable; plus the time to the third investment if the third investment is acceptable and if all previous investments were unacceptable; plus ...

Mathematically,

$$T = X_1 \cdot I_{(Y_1 = PV_1)} + X_2 \cdot I_{(Y_2 = PV_1)} \cdot I_{(Y_1 = PV_2)}$$
$$+ X_3 \cdot I_{(Y_3 = PV_1)} \cdot I_{(Y_2 = PV_2)} \cdot I_{(Y_1 = PV_2)} + \cdots$$

And,

$$\begin{aligned} ET = {} & E\left\{ X_1 \cdot I_{(Y_1 = PV_1)} \right\} \\ & + E\left\{ X_2 \cdot I_{(Y_2 = PV_1)} \cdot I_{(Y_1 = PV_2)} \right\} \\ & + E\left\{ X_3 \cdot I_{(Y_3 = PV_1)} \cdot I_{(Y_2 = PV_2)} \cdot I_{(Y_1 = PV_2)} \right\} \end{aligned}$$

$$\begin{aligned} = {} & E\left\{ X_1 \right\} \cdot E\left\{ I_{(Y_1 = PV_1)} \right\} \\ & + E\left\{ X_2 \right\} \cdot E\left\{ I_{(Y_2 = PV_1)} \right\} \cdot E\left\{ I_{(Y_1 = PV_2)} \right\} \\ & + E\left\{ X_3 \right\} \cdot E\left\{ I_{(Y_3 = PV_1)} \right\} \cdot E\left\{ I_{(Y_2 = PV_2)} \right\} \\ & \quad \cdot E\left\{ I_{(Y_1 = PV_2)} \right\} + \cdots \end{aligned}$$

$$ET = \frac{1}{\lambda} \cdot p + \frac{2}{\lambda} \cdot p \cdot (1 - p) + \frac{3}{\lambda} \cdot p \cdot (1 - p)^2 + \ldots \qquad (27)$$

$$= \frac{p}{\lambda} \cdot \left[1 + 2\,(1 - p) + 3\,(1 - p)^2 + \ldots \right]$$

The sum of the infinite series is:

$$\frac{1}{(1 - (1 - p))^2}$$

Therefore,

$$ET = \frac{p}{\lambda} \cdot \frac{1}{(1 - (1 - p))^2}$$

$$= \frac{1}{p\,\lambda}$$

A numerical example will help illustrate the above relationships. The following information is given:

$PV_1 = \$10,000$

$PV_2 = \$9,500$

$\lambda = 2$ per month

$r = .02$ per month

$P(Y = PV_1) = .4$

$P(Y = PV_2) = .6$

The optimal policy is continue waiting if

$$\$9,500 < \frac{.4 \times 2}{.02 + .4 \times 2} \times \$10,000 = \$9,756$$

---

[27]See Appendix II for proof of $E\left\{X_n\right\} = \dfrac{n}{\lambda}$

Hence it is optimal to wait for an investment to occur with present value equal to \$10,000. The expected waiting time until such an investment occurs is

$$ET = \frac{1}{.4 \times 2} = 1.25 \text{ months}$$

ii) Two possible investment types with Search.

This problem is similar to the previous problem except that the investor can now spend money on search and thereby increase the arrival rate of investment opportunities. If search dollars are assumed to be paid each time an unacceptable project occurs, to cover the cost of search until the next investment occurs, then the value of searching can be formulated as follows:

$$\begin{aligned}
V = \; & Y_1 \cdot I_{(Y_1 = PV_1)} \cdot e^{-rX_1} - S \cdot X_1 \cdot I_{(Y_1 = PV_1)} \\
& + Y_2 \cdot I_{(Y_2 = PV_1)} \cdot e^{-rX_2} \cdot I_{(Y_1 = PV_2)} \\
& \quad - \left(S \cdot X_1 + S\,(X_2 - X_1)\, e^{-rX_1} \right) \cdot I_{(Y_2 = PV_1)} \cdot I_{(Y_1 = PV_2)} \\
& + Y_3 \cdot I_{(Y_3 = PV_1)} \cdot e^{-rX_3} \cdot I_{(Y_2 = PV_2)} \cdot I_{(Y_1 = PV_2)} \\
& \quad - \left(S \cdot X_1 + S\,(X_2 - X_1)\, e^{-rX_1} + S\,(X_3 - X_2)\, e^{-rX_2} \right) \cdot \\
& \qquad I_{(Y_3 = PV_1)} \cdot I_{(Y_2 = PV_2)} \cdot I_{(Y_1 = PV_2)} \\
& + \ldots
\end{aligned}$$

An alternative formulation would be to assume that search costs are paid out continuously until an acceptable investment is found.

The expected value of searching becomes,

$$EV = \frac{p\lambda_s}{r + p\lambda_s} \cdot PV_1 \quad - \quad EC \qquad \text{(28)}$$

where EC is the expected cost of search.

$$EC = S \cdot E\left\{X_1\right\} \cdot E\left\{I_{(Y_1 = PV_1)}\right\}$$

$$+ \ S \cdot \left[E\left\{X_1\right\} + E\left\{(X_2 - X_1)\right\} \cdot E\left\{e^{-rX_1}\right\}\right] \cdot E\left\{I_{(Y_2 = PV_1)}\right\}$$

$$\cdot E\left\{I_{(Y_1 = PV_2)}\right\}$$

$$+ \ S \cdot \left[E\left\{X_1\right\} + E\left\{(X_2 - X_1)\right\} \cdot E\left\{e^{-rX_1}\right\} + E\left\{(X_3 - X_2)\right\} \cdot E\left\{e^{-rX_2}\right\}\right]$$

$$\cdot E\left\{I_{(Y_3 = PV_1)}\right\} \cdot E\left\{I_{(Y_2 = PV_2)}\right\} \cdot E\left\{I_{(Y_1 = PV_2)}\right\}$$

$$+ \ \ldots$$

$$= \frac{S\,p}{\lambda_s} + \left[\frac{1}{\lambda_s} + \frac{1}{\lambda_s}\left(\frac{\lambda_s}{r + \lambda_s}\right)\right] \cdot S \cdot p \cdot (1 - p)$$

$$+ \left[\frac{1}{\lambda_s} + \frac{1}{\lambda_s}\left(\frac{\lambda_s}{r + \lambda_s}\right) + \frac{1}{\lambda_s}\left(\frac{\lambda_s}{r + \lambda_s}\right)^2\right] \cdot S \cdot p \cdot (1 - p)^2 \ + \ \ldots$$

---

[28]The first term was obtained from EV on page 25.

$$EC = \frac{S\,p}{\lambda_s} \cdot \left\{ 1 + \left[1 + \frac{\lambda_s}{r + \lambda_s} \right] \cdot (1 - p) + \left[1 + \frac{\lambda_s}{r + \lambda_s} + \left(\frac{\lambda_s}{r + \lambda_s} \right)^2 \right] \cdot (1 - p)^2 + \dots \right\}$$

However,

$$1 + \frac{\lambda_s}{r + \lambda_s} + \left(\frac{\lambda_s}{r + \lambda_s} \right)^2 + \dots + \left(\frac{\lambda_s}{r + \lambda_s} \right)^n = \frac{1 - \left(\frac{\lambda_s}{r + \lambda_s} \right)^{n+1}}{1 - \frac{\lambda_s}{r + \lambda_s}}$$

therefore,

$$EC = \frac{S\,p}{\lambda_s} \cdot \sum_{n=0}^{\infty} \frac{1 - \left(\frac{\lambda_s}{r + \lambda_s} \right)^{n+1}}{1 - \frac{\lambda_s}{r + \lambda_s}} \cdot (1 - p)^n$$

$$= \frac{S\,p}{\lambda_s} \cdot \frac{1}{1 - \frac{\lambda_s}{r + \lambda_s}} \cdot \sum_{n=0}^{\infty} (1 - p)^n$$

$$- \frac{\lambda_s}{r + \lambda_s} \cdot \sum_{n=0}^{\infty} \left(\frac{\lambda_s}{r + \lambda_s} \right)^n \cdot (1 - p)^n$$

$$= \frac{S\,p}{\lambda_s \left(1 - \frac{\lambda_s}{r + \lambda_s} \right)} \cdot \left\{ \frac{1}{p} - \frac{\frac{\lambda_s}{r + \lambda_s}}{1 - \frac{\lambda_s}{r + \lambda_s} \cdot (1 - p)} \right\}$$

Which simplifies to

$$EC = \frac{S\ (r + \lambda_s)}{\lambda_s(r + p\ \lambda_s)}$$

Thus,

$$EV = \frac{p\ \lambda_s}{r + p\lambda_s} \cdot PV_1 - \frac{S\ (r + \lambda_s)}{\lambda_s(r + p\ \lambda_s)}$$

Since we wish to maximize the expected value of searching, we can find S*, the optimal amount of search, by setting the derivative of EV with respect to S equal to zero, and solving for S*. Except in extremely simple cases, solving this equation is difficult; however, unconstrained optimization techniques can easily be applied.[29]

Once S* has been found, the optimal policy can be stated as follows: Continue searching, spending S* on search, if

$$PV_2 < \frac{p\ \lambda_s^*}{r + p\ \lambda_s^*} \cdot PV_1 - \frac{S^*\ (r + \lambda_s^*)}{\lambda_s^*\ (r + p\ \lambda_s^*)}$$

where $\lambda_s^* = f(S^*)$.

The expected time until an investment opportunity occurs

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[29]W. I. Zangwill, <u>Nonlinear Programming: A Unified Approach</u>, Prentice-Hall, Inc., Englewood Cliffs, N. J., 1969.

with present value equal to PV_1 if S* is spent on search is

$$ET = \frac{1}{p\,\lambda_s^*} \qquad (30)$$

The following numerical example will help illustrate these results. In addition to the data in the last numerical example, the interarrival time as a function of S is given as,

$$\frac{1}{\lambda_s} = .25 + .25\,e^{-.05\,S}$$

Substituting the given information into the expression obtained for the expected value of continuing to search for an investment with present value equal to PV_1, the following is obtained:

$$EV = \frac{.4\,\lambda_s}{.02 + .4\,\lambda_s}\,10{,}000 - \frac{S\,(.02 + \lambda_s)}{(.02 + .4\,\lambda_s)}$$

where λ_s is defined above.

Calculating EV for several values of S, the graph in Figure 3 was easily obtained. Appendix III contains the computer program and the results from which this graph is obtained.

The optimal amount of search would be \$50, the EV of searching would be \$9,833, and the expected time to an acceptable investment would be .676 months.

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[30] ET was obtained from the previous subproblem with λ replaced by λ_s^*.

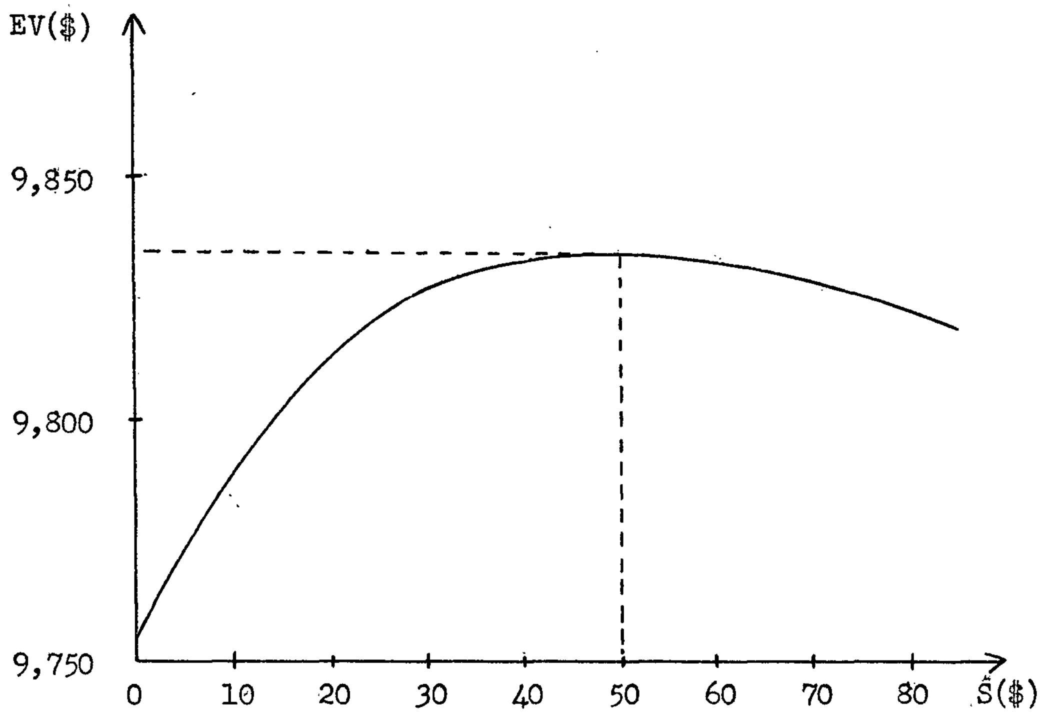

Figure 3. Expected value as a Function of Search.

iii) k possible investment types.

The probability that the present value of an investment which has just occurred is equal to PV_j, $j = 1, 2, \ldots, k$ is given as:

$$P(Y = PV_j) = p_j \qquad j = 1, 2, \ldots, k$$

where $\sum_{j=1}^{k} p_j = 1.$

Also $PV_1 > PV_2 > \ldots > PV_k$

Next let the optimal policy be stated as --invest only if an investment with present value greater than or equal to A occurs. The value of continuing can now be stated as follows:

$$V = Y_1 \cdot e^{-rX_1} \cdot I_{(Y_1 \geqslant A)}$$
$$+ Y_2 \cdot e^{-rX_2} \cdot I_{(Y_2 \geqslant A)} \cdot I_{(Y_1 < A)}$$
$$+ Y_3 \cdot e^{-rX_3} \cdot I_{(Y_3 \geqslant A)} \cdot I_{(Y_2 < A)} \cdot I_{(Y_1 < A)}$$
$$+ \cdots$$

The expected value of continuing is:

$$EV = E\left\{Y_1 \cdot I_{(Y_1 \geqslant A)}\right\} \cdot E\left\{e^{-rX_1}\right\}$$
$$+ E\left\{Y_2 \cdot I_{(Y_2 \geqslant A)}\right\} \cdot E\left\{e^{-rX_2}\right\} \cdot E\left\{I_{(Y_1 < A)}\right\}$$
$$+ E\left\{Y_3 \cdot I_{(Y_3 \geqslant A)}\right\} \cdot E\left\{e^{-rX_3}\right\} \cdot E\left\{I_{(Y_2 < A)}\right\} \cdot E\left\{I_{(Y_1 < A)}\right\}$$
$$+ \cdots$$

$$= \sum_{n=A}^{k} p_n \cdot PV_n \cdot \frac{\lambda}{r+\lambda}$$
$$+ \sum_{n=A}^{k} p_n \cdot PV_n \cdot \left(\frac{\lambda}{r+\lambda}\right)^2 \cdot \sum_{n=1}^{A-1} p_n$$
$$+ \sum_{n=A}^{k} p_n \cdot PV_n \cdot \left(\frac{\lambda}{r+\lambda}\right)^3 \cdot \left(\sum_{n=1}^{A-1} p_n\right)^2$$
$$+ \cdots$$

$$EV = \frac{\lambda}{r+\lambda} \cdot \sum_{n=A}^{k} p_n \cdot PV_n \cdot \left[1 + \frac{\lambda}{r+\lambda} \cdot \sum_{n=1}^{A-1} p_n + \left(\frac{\lambda}{r+\lambda} \cdot \sum_{n=1}^{A-1} p_n \right)^2 + \cdots \right]$$

$$= \frac{\lambda}{r+\lambda} \cdot \sum_{n=A}^{k} p_n \cdot PV_n \cdot \frac{1}{1 - \frac{\lambda}{r+\lambda} \cdot \sum_{n=1}^{A-1} p_n}$$

$$= \frac{\frac{\lambda}{r+\lambda} \cdot \sum_{n=A}^{k} p_n \cdot PV_n}{1 - \frac{\lambda}{r+\lambda} \cdot \sum_{n=1}^{A-1} p_n}$$

The value of A in the above expression which maximizes EV can be found by substituting $A = PV_j$, $j = 1, 2, \ldots k$, into this expression and choosing the optimal PV_j. The optimal investment policy has now been determined.

iv) Constant aspiration level with no search.

If the decision maker sets a minimum present value, PV_a, (the present value of the aspiration level at the time an investment occurs), then investments with present value greater or equal to PV_a are acceptable, while investments with present value less than PV_a are not acceptable. In this case the policy is given, therefore the only unknowns are the expected time until an acceptable investment occurs and the present value of the policy. In addition to the constant arrival rate λ, the constant aspiration level PV_a, the distribution of present values of investment opportunities, $F(y)$, must be known. Given this information, the time to an acceptable investment opportunity occurring is the following:

$$T = X_1 \cdot I_{(Y_1 \geqslant PV_a)} + X_2 \cdot I_{(Y_2 \geqslant PV_a)} \cdot I_{(Y_1 < PV_a)}$$

$$+ X_3 \cdot I_{(Y_3 \geqslant PV_a)} \cdot I_{(Y_2 < PV_a)} \cdot I_{(Y_1 < PV_a)} + \cdots$$

The expected time,

$$ET = E\left\{X_1\right\} \cdot E\left\{I_{(Y_1 \geqslant PV_a)}\right\}$$

$$+ E\left\{X_2\right\} \cdot E\left\{I_{(Y_2 \geqslant PV_a)}\right\} \cdot E\left\{I_{(Y_1 < PV_a)}\right\}$$

$$+ E\left\{X_3\right\} \cdot E\left\{I_{(Y_3 \geqslant PV_a)}\right\} \cdot E\left\{I_{(Y_2 < PV_a)}\right\} \cdot E\left\{I_{(Y_1 < PV_a)}\right\}$$

$$+ \cdots$$

$$ET = \frac{1}{\lambda} \cdot \int_{y=PV_a}^{\infty} f(y)\,dy + \frac{2}{\lambda} \cdot \int_{y=PV_a}^{\infty} f(y)\,dy \cdot \int_{y=0}^{PV_a} f(y)\,dy$$

$$+ \frac{3}{\lambda} \cdot \int_{y=PV_a}^{\infty} f(y)\,dy \left(\int_{y=0}^{PV_a} f(y)\,dy\right)^2 + \cdots$$

But,

$$\int_{y=PV_a}^{\infty} f(y)\ dy = 1 - F(PV_a)$$

and,

$$\int_{y=0}^{PV_a} f(y)\ dy = F(PV_a)$$

Therefore,

$$ET = \frac{1}{\lambda} \cdot (1 - F(PV_a)) \cdot \left[1 + 2\,F(PV_a) + 3 \cdot (F(PV_a))^2 + \dots \right]$$

where the infinite sum is equal to

$$\frac{1}{(1 - F(PV_a))^2}$$

Therefore,

$$ET = \frac{1}{\lambda(1 - F(PV_a))}$$

The value of the policy determined by the constant aspiration level, PV_a, is the following:

$$V = Y_1 \cdot I_{(Y_1 \geqslant PV_a)} \cdot e^{-rX_1}$$

$$+ Y_2 \cdot I_{(Y_2 \geqslant PV_a)} \cdot e^{-rX_2} \cdot I_{(Y_1 < PV_a)}$$

$$+ Y_3 \cdot I_{(Y_3 \geqslant PV_a)} \cdot e^{-rX_3} \cdot I_{(Y_2 < PV_a)} \cdot I_{(Y_1 < PV_a)}$$

$$+ \dots$$

The expected value of the policy is,

$$
\begin{aligned}
EV &= E\left\{Y_1 \cdot I_{(Y_1 \geqslant PV_a)}\right\} \cdot E\left\{e^{-rX_1}\right\} \\
&\quad + E\left\{Y_2 \cdot I_{(Y_2 \geqslant PV_a)}\right\} \cdot E\left\{e^{-rX_2}\right\} \cdot E\left\{I_{(Y_1 < PV_a)}\right\} \\
&\quad + E\left\{Y_3 \cdot I_{(Y_3 \geqslant PV_a)}\right\} \cdot E\left\{e^{-rX_3}\right\} \cdot E\left\{I_{(Y_2 < PV_a)}\right\} \cdot E\left\{I_{(Y_1 < PV_a)}\right\} \\
&\quad + \cdots \\
&= \int_{y=PV_a}^{\infty} y\ f(y)\ dy \cdot \frac{\lambda}{r+\lambda} \\
&\quad + \int_{y=PV_a}^{\infty} y\ f(y)\ dy \cdot \left(\frac{\lambda}{r+\lambda}\right)^2 \cdot \int_{y=0}^{PV_a} f(y)\ dy \\
&\quad + \int_{y=PV_a}^{\infty} y\ f(y)\ dy \cdot \left(\frac{\lambda}{r+\lambda}\right)^3 \cdot \left(\int_{y=0}^{PV_a} f(y)\ dy\right)^2 \quad + \cdots \\
&= \frac{\lambda}{r+\lambda} \cdot \int_{y=PV_a}^{\infty} y\ f(y)\ dy \cdot \left[1 + \frac{\lambda}{r+\lambda} \cdot F(PV_a) \right. \\
&\qquad \left. + \left(\frac{\lambda}{r+\lambda} \cdot F(PV_a)\right)^2 + \cdots\right]
\end{aligned}
$$

$$EV = \frac{\frac{\lambda}{r+\lambda} \cdot \int_{y=PV_a}^{\infty} y\, f(y)\, dy}{1 - \frac{\lambda}{r+\lambda} \cdot F(PV_a)}$$

A numerical example will now be given to illustrate the above results. If the probability density function of the present value of new investment opportunities is assumed to be normally distributed with mean \$10,000 and standard deviation \$1,000, the arrival rate is 2 per month and the aspiration level is set at \$11,500, then we can calculate the expected value of continuing and the expected time until an acceptable investment occurs as follows:

$$ET = \frac{1}{\lambda(1 - F(PV_a)}$$

$$F(11,500) = .9332$$

$$ET = \frac{1}{2\ (1 - .9332)}$$

$$= 7.5 \text{ months.}$$

And, $$\int_{y=11,500}^{\infty} y\, f(y)\, dy = 798$$

Therefore, $$EV = \frac{\frac{2}{.02 + 2} \times .798}{1 - \frac{2}{.02 + 2} \times .9332}$$

$EV = \$ 10{,}384$

v) Constant aspiration level with search.

This problem is similar to the last problem except that the decision maker can now increase the arrival rate by spending money on search. The value of searching, spending S dollars on search per unit time, is the following:

$$V = Y_1 \cdot I_{(Y_1 \geqslant PV_a)} \cdot e^{-rX_1} - S \cdot X_1 \cdot I_{(Y_1 \geqslant PV_a)}$$

$$+ Y_2 \cdot I_{(Y_2 \geqslant PV_a)} \cdot e^{-rX_2} \cdot I_{(Y_1 < PV_a)}$$

$$- \left[S \cdot X_1 + S\,(X_2 - X_1)\, e^{-rX_1} \right] \cdot I_{(Y_2 \geqslant PV_a)} \cdot I_{(Y_1 < PV_a)}$$

$$+ Y_3 \cdot I_{(Y_3 \geqslant PV_a)} \cdot e^{-rX_3} \cdot I_{(Y_2 < PV_a)} \cdot I_{(Y_1 < PV_a)}$$

$$- \left[S \cdot X_1 + S\,(X_2 - X_1)\, e^{-rX_1} + S\,(X_3 - X_2)\, e^{-rX_2} \right]$$

$$\cdot I_{(Y_3 \geqslant PV_a)} \cdot I_{(Y_2 < PV_a)} \cdot I_{(Y_1 < PV_a)}$$

$$+ \ldots$$

The expected value of searching is the following:

$$EV = \frac{\dfrac{\lambda_s}{r+\lambda_s} \cdot \displaystyle\int_{y=PV_a}^{\infty} y\ f(y)\ dy}{1 - \dfrac{\lambda_s}{r+\lambda_s} \cdot F(PV_a)} - EC \tag{31}$$

where EC is the expected cost of search, which equals the following:

$$\begin{aligned}
EC = {} & S \cdot E\left\{X_1\right\} \cdot E\left\{I_{(Y_1 \geqslant PV_a)}\right\} \\
& + S \cdot \left[E\left\{X_1\right\} + E\left\{(X_2 - X_1)\ e^{-rX_1}\right\}\right] \cdot E\left\{I_{(Y_2 \geqslant PV_a)}\right\} \\
& \qquad \cdot E\left\{I_{(Y_1 < PV_a)}\right\} \\
& + S \cdot \left[E\left\{X_1\right\} + E\left\{(X_2 - X_1)\ e^{-rX_1}\right\} + E\left\{(X_3 - X_2)\ e^{-rX_2}\right\}\right] \\
& \qquad \cdot E\left\{I_{(Y_3 \quad PV_a)}\right\} \cdot E\left\{I_{(Y_2 \quad PV_a)}\right\} \cdot E\left\{I_{(Y_1 \quad PV_a)}\right\} \\
& + \cdots \\
= {} & S \cdot \frac{1}{\lambda_s} \cdot (1 - F(PV_a) \\
& + S \cdot \left[\frac{1}{\lambda_s} + \frac{1}{\lambda_s} \quad \frac{\lambda_s}{r+\lambda_s}\right] \cdot (1 - F(PV_a)) \cdot F(PV_a) \\
& + S \cdot \left[\frac{1}{\lambda_s} + \frac{1}{\lambda_s} \quad \frac{\lambda_s}{r+\lambda_s} + \frac{1}{\lambda_s}\left(\frac{\lambda_s}{r+\lambda_s}\right)^2\right] \cdot (1 - F(PV_a)) \cdot (F(PV_a))^2 \\
& + \cdots
\end{aligned}$$

[31] The first term is identical to the EV in the previous subproblem with λ replaced by λ_s.

which simplifies to,

$$EC = \frac{S\ (r + \lambda_s)}{\lambda_s (r + \lambda_s (1 - F(PV_a)))} \qquad (32)$$

Therefore,

$$EV = \frac{\dfrac{\lambda_s}{r + \lambda_s} \cdot \displaystyle\int_{y=PV_a}^{\infty} y\ f(y)\ dy}{1 - \dfrac{\lambda_s}{r + \lambda_s} \cdot F(PV_a)} - \frac{S\ (r + \lambda_s)}{\lambda_s (r + \lambda_s (- F(PV_a)))}$$

The optimal value of S, S*, can be found for simple problems by setting the derivative of EV with respect to S equal to zero, and solving for S*. In more general problems unconstrained optimization techniques can easily be applied.[33] Once S* has been calculated the expected time until an acceptable investment occurs given that S* is spent on search can be calculated as follows:

$$ET = \frac{1}{\lambda_s^* \ (1 - F(PV_a))} \qquad (34)$$

where $\lambda_s^* = f(S^*)$.

A numerical example which illustrates the above results follows. In addition to the data given in subsection iv) the arrival rate as a function of S is required. Let this function be the following:

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[32] Simplification is similar to that on pages 29, 30, and 31.

[33] W. I. Zangwill, <u>op. cit.</u>

[34] Obtained from subsection iv by replacing λ with λ_s^*.

$$\lambda_s = \frac{1}{.25 + .25 \ e^{-.05 \times S}}$$

Calculating EV for several values of S, the graph in Figure 4 is easily obtained. Appendix IV contains the computer program and the results from which this graph is obtained.

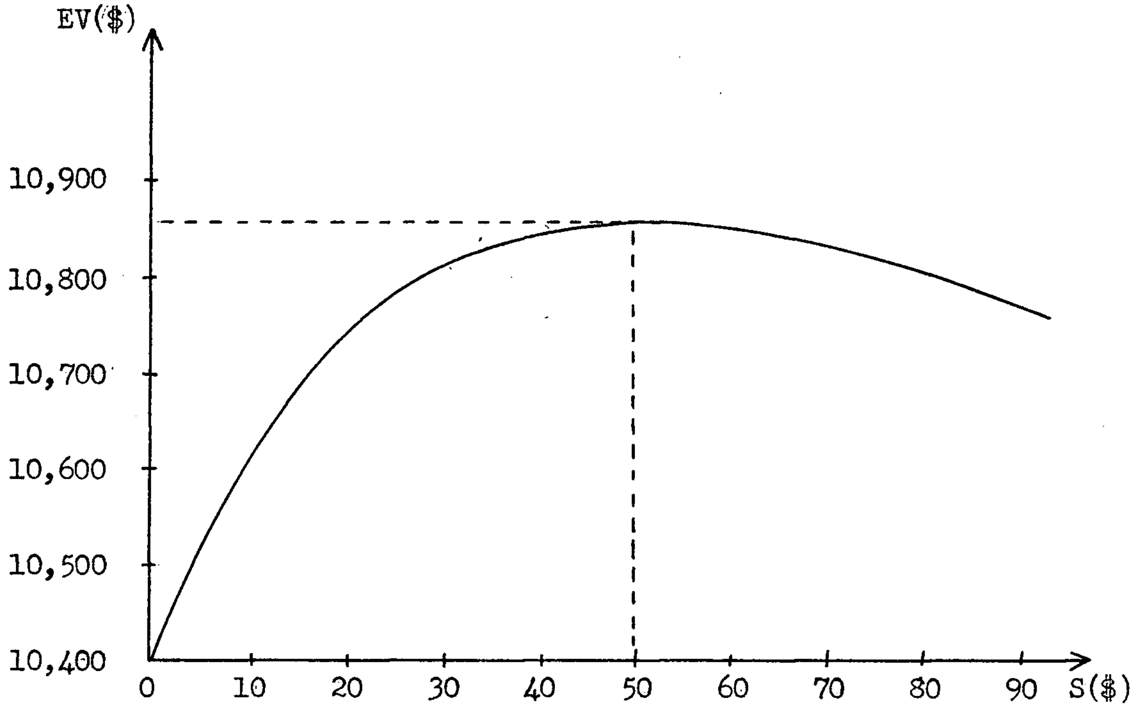

Figure 4. Expected value as a Function of Search.

From this graph the optimal level of search is observed to be $50, and the expected value of searching is $10,863. The expected time until an acceptable investment occurs can easily be calculated as follows:

$$ET = \frac{1}{\lambda_s^* (1 - F(PV_a))}$$

where $$\lambda_s^* = \frac{1}{.25 + .25 \quad e^{-.05 \times 50}}$$

$$= 3.7 \text{ per month}$$

Therefore, $$ET = \frac{1}{3.7\ (1 - .9332)}$$

$$= 4.05 \text{ months.}$$

vi) Variable aspiration level.

In the analysis of this problem it is important to realize the possibility of having an unacceptable investment opportunity become acceptable at a later time. Therefore, additional information with regard to the duration each investment opportunity exists is necessary.

Three cases will be of particular interest. The first, is the case of investment opportunities disappearing immediately after they have occurred. An example would be in the stock market, where the value of investment opportunities often vary rapidly. The second, is the case of investment opportunities remaining for a relatively long period of time. An example would be investments such as new product developments which a firm has developed and could shelve for a few years, with little chance of another firm developing the same idea. The third and possibly the closest to reality would be the case of investment opportunities remaining a available a probabilistic length of time, depending on the value of the investment.

The above cases are represented in Figure 5, where $PV_a(t)$ is the aspired present value at time t. As mentioned in the introduction the aspired present value at time t would, in general, be a function of the aspiration level in the past, the present values of past investment opportunities, the inter-arrival times of past investment opportunities and possibly the recent performance of others.

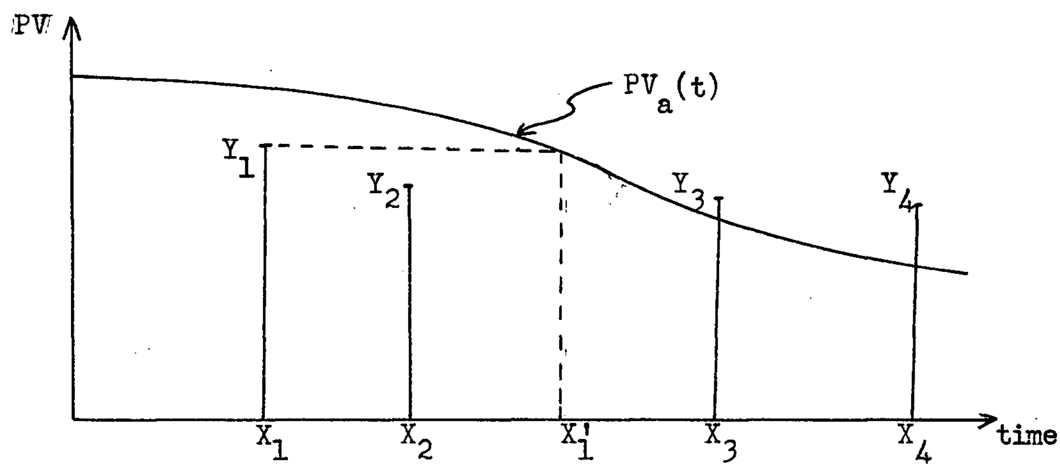

Figure 5. Variable aspiration level model.

To illustrate the difference between the two extreme cases, the time to an acceptable investment and the present value of that investment for the example given in Figure 5 are:

a) For investments which disappear,

$$T = X_3 \text{ and } PV_{accepted} = Y_3$$

b) For investments which do not disappear,

$$T = X_1' \text{ and } PV_{accepted} = Y_1.$$

For the case where investments disappear immediately after their occurrence, the time to an acceptable investment can be formulated as follows:

$$\begin{aligned} T = {} & X_1 \cdot I_{(Y_1 \geq PV_a(X_1))} \\ & + X_2 \cdot I_{(Y_2 \geq PV_a(X_2))} \cdot I_{(Y_1 < PV_a(X_1))} \\ & + X_3 \cdot I_{(Y_3 \geq PV_a(X_3))} \cdot I_{(Y_2 < PV_a(X_2))} \cdot I_{(Y_1 < PV_a(X_1))} \\ & + \cdots \end{aligned}$$

The expected time until an acceptable investment occurs is the following:

$$\begin{aligned} ET = {} & E\left\{X_1 \cdot I_{(Y_1 \geq PV_a(X_1))}\right\} \\ & + E\left\{X_2 \cdot I_{(Y_2 \geq PV_a(X_2))}\right\} \cdot E\left\{I_{(Y_1 < PV_a(X_1))}\right\} \\ & + E\left\{X_3 \cdot I_{(Y_3 \geq PV_a(X_3))}\right\} \cdot E\left\{I_{(Y_2 < PV_a(X_2))}\right\} \\ & \qquad \cdot E\left\{I_{(Y_1 < PV_a(X_1))}\right\} \\ & + \cdots \end{aligned}$$

which appears to have no easily obtainable solution for the general case.

The value of continuing can be formulated as the following:

$$
\begin{aligned}
V = {} & Y_1 \cdot e^{-rX_1} \cdot I_{(Y_1 \geqslant PV_a(X_1))} \\
& + Y_2 \cdot e^{-rX_2} \cdot I_{(Y_2 \geqslant PV_a(X_2)} \cdot I_{(Y_1 < PV_a(X_1))} \\
& + Y_3 \cdot e^{-rX_3} \cdot I_{(Y_3 \geqslant PV_a(X_3))} \cdot I_{(Y_2 < PV_a(X_2))} \\
& \qquad \cdot I_{(Y_1 < PV_a(X_1))} \\
& + \ldots
\end{aligned}
$$

It is obvious from observing V that the expected value of V cannot be determined easily.

However, a solution giving the distribution of T and V and their expected values could be readily obtained using simulation techniques.

For the second case we assume investments remain indefinitely after they have occurred. The present value, at the time the project is undertaken, of cash flows from the project is assumed to be constant no matter when the project is undertaken. Denote $Y_n^+ = \max(Y_1, Y_2, \ldots Y_n)$ and denote X_n' as the time when the aspiration level has decreased sufficiently in order that Y_n^+ becomes acceptable. X_n' is calculated by solving the following equation:

$$Y_n^+ = PV_a(X_n')$$

Figure 6 shows the relationship of Y_n^+ and X_n'.

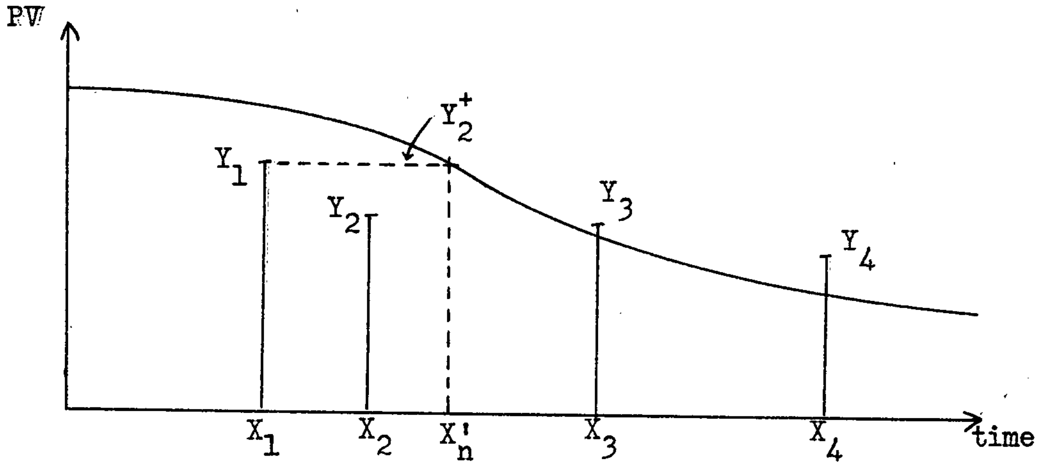

Figure 6. Intersection of Y_n^+ and $PV_a(t)$.

The value of continuing can now be stated as the following:

$$\begin{aligned} V = \; & Y_1 \cdot I_{(Y_1 \geqslant PV_a(X_1))} \cdot e^{-rX_1} \\ & + Y_1^+ \cdot I_{(Y_1^+ \geqslant PV_a(X_2))} \cdot e^{-rX_1'} \cdot I_{(Y_1^+ < PV_a(X_1))} \\ & + Y_2 \cdot I_{(Y_2 \geqslant PV_a(X_2))} \cdot e^{-rX_2} \cdot I_{(Y_1^+ < PV_a(X_2))} \\ & + Y_2^+ \cdot I_{(Y_2^+ \geqslant PV_a(X_3))} \cdot e^{-rX_2'} \cdot I_{(Y_2^+ < PV_a(X_2))} \\ & + \cdots \end{aligned}$$

The time until an acceptable investment occurs is the following:

$$
\begin{aligned}
T = {} & X_1 \cdot I_{(Y_1 \geqslant PV_a(X_1))} \\
& + X_1' \cdot I_{(Y_1^+ \geqslant PV_a(X_2))} \cdot I_{(Y_1^+ < PV_a(X_1))} \\
& + X_2 \cdot I_{(Y_2 \geqslant PV_a(X_2))} \cdot I_{(Y_1^+ < PV_a(X_2))} \\
& + X_2' \cdot I_{(Y_2^+ \geqslant PV_a(X_3))} \cdot I_{(Y_2^+ < PV_a(X_2))} \\
& + \dots
\end{aligned}
$$

The analytical solution of the above equations, along with the equations of their expected values, is again complicated, and once more simulation techniques could effectively be used to obtain approximate solutions. Using simulation techniques the decision maker could analyze various aspiration functions ($PV_a(t)$) and through the use of search techniques arrive at a close to optimal aspiration functions for specific situations.

The use of a step-function to approximate the variable aspiration level should be analyzed in order to see if approximate analytical solutions could be obtained for the variable aspiration level problem.

vii) The deadline problem.

A good example of a situation where an aspiration level which varies with time would be an optimal policy is the deadline problem. In general, the deadline has a monetary penalty associated with passing the deadline without having made an investment. The most interesting case is that of investments which disappear immediately after their occurence. For the case where investment opportunities remain indefinitely, the deadline penalty becomes irrelevant. For cases where deadlines are relevant the penalty associated with passing the deadline would be given. For example, assume the deadline penalty is $10,000. The optimal policy an instant before the deadline would obviously be to accept an investment if it yielded a present value greater than $PV_a(t)$. Where most certainly $PV_a(t)$ at the deadline would be not greater than $10,000. Graphically the optimal policy would be accept an an investment if its present value is greater or equal to $PV_a(t)$, where $PV_a(t)$ is some function which decreases to the penalty cost. Figure 7 shows several curves which have this property.

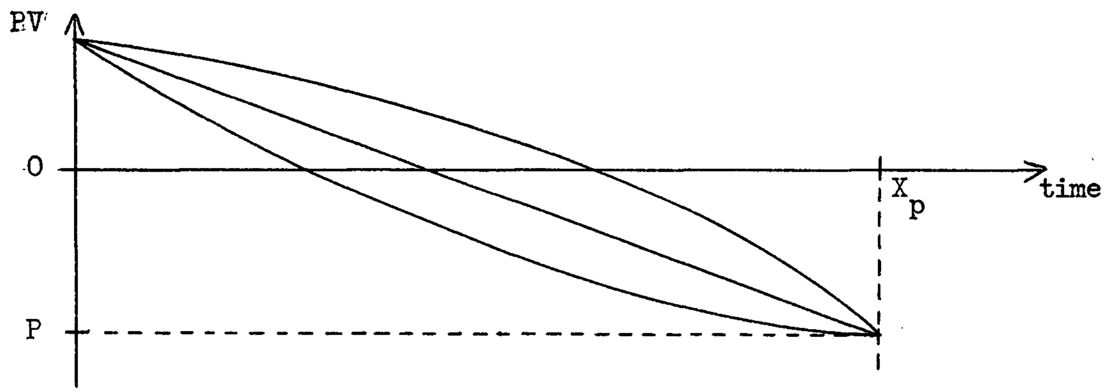

Figure 7. Aspiration levels for deadline penalty.

Let P be the penalty cost in dollars and X_p be the deadline time. If we are presently at time zero, and investments disappear if they are not accepted, then the value of continuing is the value of the first investment, discounted to the present, if it occurs before the deadline and if it is acceptable minus the penalty cost if the first investment occurs after the deadline; plus the value of the second investment, discounted to the present, if it occurs before the deadline and if it is acceptable and if the first investment was not acceptable minus the penalty cost if the second investment occurs after the deadline and if the first investment was not acceptable and if it occured before the deadline; plus ... Mathematically,

$$
\begin{aligned}
V = {} & Y_1 \cdot I_{(Y_1 \geqslant PV_a(X_1))} \cdot I_{(X_1 < X_p)} \cdot e^{-rX_1} \\
& - P \cdot I_{(X_1 > X_p)} \cdot e^{-rX_p} \\
& + Y_2 \cdot I_{(Y_2 \geqslant PV_a(X_2))} \cdot I_{(Y_1 < PV_a(X_1))} \cdot I_{(X_2 < X_p)} \cdot e^{-rX_2} \\
& - P \cdot I_{(Y_1 < PV_a(X_1))} \cdot I_{(X_2 > X_p)} \cdot e^{-rX_p} \\
& + \cdots
\end{aligned}
$$

Again the expected value of continuing is the sum of the expected values of the terms in V; however, these expected values are complex and in general no analytical solution exists to find the optimal aspiration level. Again in this situation, as in the previous situations where the aspiration levels were given as functions of time, simulation techniques can be used to yield the expected value of continuing for a given aspiration level as a function of time. Since there exists an infinite number of possible functions, only a near optimal solution could be obtained. However, if the model is reasonably accurate, the near optimal solution would be of great value in certain cases.

CHAPTER V

ANALYSIS OF THE GENERAL PROBLEM

The general problem for the model described is finding the optimal policy, the optimal amount of search, if search is carried out, and the expected time until an acceptable investment occurs for the case where the present value of future investment opportunities and the interarrival times are assumed to be independent and identically distributed random variables from known distributions. Also it is assumed for the general problem that no deadline exists.

i) The general problem with no search.

Each time a new investment opportunity occurs the decision maker must decide whether to accept that investment or to continue waiting for a better investment opportunity. The optimal policy will surely be to invest only if the expected value of continuing is greater than the value of the presently available investment opportunities. And clearly, because of the assumptions that the present values and interarrival times are identically distributed, the expected value of continuing will be the same at every point in time that a decision must be made. Therefore the optimal policy will be to invest if the expected value of continuing, which is a constant, is greater than the value of presently available investment opportunities. A rigorous proof that in fact a constant aspiration level is the optimal policy for a similar model to the one presented here is given by Chow and Robbins.[35] The problem of finding an optimal policy is now reduced to finding the optimal aspiration level.

Define A as the aspiration level, and A* as the optimal aspiration level. The value of continuing with an aspiration level A is the following:

$$
\begin{aligned}
V = {} & Y_1 \cdot I_{(Y_1 \geqslant A)} \cdot e^{-rX_1} \\
& + Y_2 \cdot I_{(Y_2 \geqslant A)} \cdot e^{-rX_2} \cdot I_{(Y_1 < A)} \\
& + Y_3 \cdot I_{(Y_3 \geqslant A)} \cdot e^{-rX_3} \cdot I_{(Y_2 < A)} \cdot I_{(Y_1 < A)} \\
& + \ldots
\end{aligned}
$$

The expected value of continuing is the following:

$$
EV = \frac{\dfrac{\lambda}{r+\lambda} \cdot \displaystyle\int_{y=A}^{\infty} y\, f(y)\, dy}{1 - \dfrac{\lambda}{r+\lambda} \cdot F(A)} \qquad (36)
$$

However, from the previous discussion we note that EV in fact, for the optimal policy, just equals A. Therefore,

---

[35] Y. S. Chow, and H. Robbins, op. cit.

[36] This expression is identical to the expected value of continuing derived on pages 38 and 39 except PV_a is replaced with A.

$$A^* = \frac{\dfrac{\lambda}{r+\lambda} \cdot \displaystyle\int_{y=A^*}^{\infty} y\, f(y)\, dy}{1 - \dfrac{\lambda}{r+\lambda} \cdot F(A^*)}$$

Since the only unknown in the above equation is A*, A* can be found, and the optimal policy determined. Two alternative methods of solving for A* are by setting the derivative of EV equal to zero and solving for A*, or by using unconstrained search techniques.

The data required for the following numerical example is given in subsection iv. The expected value of continuing under various aspiration levels can be calculated from the following:

$$EV = \frac{\dfrac{2}{.02+2} \cdot \displaystyle\int_{y=A}^{\infty} y\, f(y)\, dy}{1 - \dfrac{2}{.02+2} \cdot F(A)}$$

The results are shown graphically if Figure 8. Appendix V contains

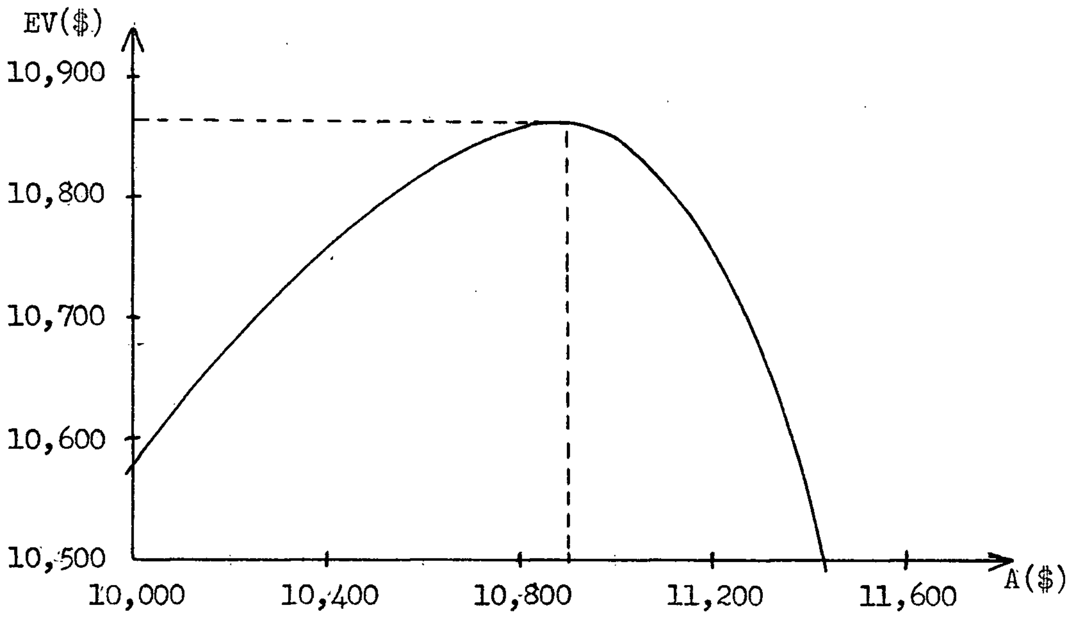

Figure 8. Expected Value as a function of Aspiration Level.

the computer program and results from which this graph is obtained.

The expected time until an acceptable investment opportunity occurs is given by the following:

$$ET = \frac{1}{\lambda(1 - F(A^*))} \quad (37)$$

$$= \frac{1}{2\ (1 - F(10{,}900)\ )}$$

$$= 2.7 \text{ months.}$$

Therefore the optimal policy is to wait for an investment with present value greater or equal to \$10,900. The expected value of this policy is \$10,856 and the expected time until an acceptable investment occurs is 2.7 months.

ii) The general problem with search.

This problem is identical to the previous problem except the decision maker can now increase the arrival rate of new investment opportunities by spending money on search. The expected value of continuing, spending S dollars on search per unit time, is the expected value of continuing where a constant aspiration level exists, only now, the constant aspiration level, PV_a, becomes an unknown, A. Therefore,

---

[37]Obtained form ET on page 37.

$$EV = \frac{\frac{\lambda_s}{r+\lambda_s} \cdot \int_{y=A}^{\infty} y\, f(y)\, dy}{1 - \frac{\lambda_s}{r+\lambda_s} \cdot F(A)} - \frac{S\,(r+\lambda_s)}{\lambda_s(r+\lambda_s(1-F(A)))} \tag{38}$$

The optimal amount of search, and the optimal aspiration level, S* and A*, in extremely simple cases, can be obtained by maximizing EV using classical optimization techniques. Set,

$$\frac{\partial EV}{\partial S} = 0$$

and $$\frac{\partial EV}{\partial A} = 0$$

and solve the two equations simultaneously to obtain S* and A*. In more practical situations where the solution of these two equations is difficult, unconstrained search techniques such as the cyclic-coordinate-ascent method could be used to maximize EV.[39]

The following numerical example will help illustrate the above problem. The data for this example is the same as for the previous examples, except that now the constant aspiration level and amount of search are both unknowns.

Defining x = (A,S), and using the cyclic-coordinate-ascent method, the following procedure and results were obtained at each iteration:

---

[38]This expression is obtained form EV on page 42.

[39]W. I. Zangwill, <u>op. cit.</u>, Chapter 5.

1. Set $x^0 = (10{,}000\ ,\ 0)$, and maximize in A direction.

   Appendix V, as well as Figure 8 in the previous subsection show the following results:

   maximum EV = 10,856 at $x^1 = (10{,}900\ ,\ 0)$

2. Set A at 10,900 and maximize in the S direction.

   Appendix VI shows the following results:

   maximum EV = 11,047 at $x^2 = (10{,}900\ ,\ 50)$

3. Set S at 50 and maximize in the A direction.

   Appendix VI shows the following results:

   maximum EV = 11,064 at $x^3 = (11{,}100\ ,\ 50)$

4. Set A at 11,100 and maximize in the S direction.

   Appendix VI shows the following results:

   maximum EV = 11,064 at $x^4 = (11{,}100\ ,\ 50)$

5. $x^* = (11{,}100\ ,\ 50)$ and EV at $x^* = 11{,}064$.

Therefore the optimal policy becomes: continue searching, spending $50 per month on search, until an investment with present value equal or greater than 11,100 occurs. The expected value in continuing under this policy is $11,064. The expected time until such an investment occurs is the following: [40]

$$ET = \frac{1}{\lambda^* (1 - F(A^*))}$$

$$= \frac{1}{3.7(1 - .8643)} = 2 \text{ months.}$$

---

[40] This expression is obtained from ET on page 42.

CHAPTER VI

CONCLUSION

Uncertainty with respect to future investment opportunities can be analyzed if probabilistic information about these future investment opportunities is available. In cases where the aspiration level is constant, whether it is given or determined as the optimal policy, the expected value of continuing, the optimal policy and the expected time until an acceptable investment opportunity occurs can be calculated analytically. In other cases, where deadlines exist or where the aspiration level varies for some other reason, analytical solutions are difficult; however, other techniques such as simulation could readily be applied to yield results similar to those above.

A summary and comparison of the results obtained from the last four numerical examples will now be given. The following information is assumed to be known:

1. The arrival rate of new investment opportunities as a function of search is given by the following expression:

$$\lambda_s = \frac{1}{.25 + .25\, e^{-.05 \times S}}$$

2. The present value of an investment when it occurs, is normally distributed with mean \$10,000 and standard deviation \$1,000.

The results corresponding to specific cases are shown in Table I.

	GIVEN INFORMATION		CALCULATED RESULTS			
	S	A	S*	A*	EV	ET
1	$0	$11,500	-	-	$10,384	7.5 months
2	-	$11,500	$50	-	$10,863	4.1 months
3	$0	-	-	$10,900	$10,856	2.7 months
4	-	-	$50	$11,100	$11,064	2.0 months

Table I. Summary of numerical results.

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APPENDIX I

Proof of: $E\left\{e^{-rX_n}\right\} = \left(\frac{\lambda}{r+\lambda}\right)^n$

The probability density function for the continuous random variable X_n (the time until the n th arrival) is the following: (41)

$$f_{X_n}(x) = \frac{\lambda^n \cdot x^{n-1} \cdot e^{-\lambda x}}{(n-1)!}$$

$$E\left\{e^{-rX_n}\right\} = \int_{x=0}^{\infty} e^{-rx} \cdot f_{X_n}(x)\, dx$$

$$= \frac{\lambda^n}{(n-1)!} \cdot \int_{x=0}^{\infty} x^{n-1} \cdot e^{-(r+\lambda)x}\, dx$$

$$= \frac{\lambda^n}{(n-1)!} \cdot \frac{n-1}{r+\lambda} \cdot \int_{x=0}^{\infty} x^{n-2} \cdot e^{-(r+\lambda)x} \cdot dx$$

Therefore,

$$E\left\{e^{-rX_{n+1}}\right\} = \frac{\lambda^{n+1}}{n!} \cdot \frac{n}{r+\lambda} \cdot \int_{x=0}^{\infty} x^{n-1} \cdot e^{-(r+\lambda)x}\, dx$$

But,

$$\int_{x=0}^{\infty} x^{n-1} \cdot e^{-(r+\lambda)x}\, dx = \frac{(n-1)!}{\lambda^n} \cdot E\left\{e^{-rX_n}\right\}$$

---

41 A. W. Drake, <u>Fundamentals of Applied Probability Theory</u>, New York: McGraw-Hill, Inc., 1967, p. 141.

Therefore,

$$E\left\{e^{-rX_{n+1}}\right\} = \frac{\lambda^{n+1}}{n!} \cdot \frac{n}{r+\lambda} \cdot \frac{(n-1)!}{\lambda^{n}} \cdot E\left\{e^{-rX_n}\right\}$$

$$= \frac{\lambda}{r+\lambda} \cdot E\left\{e^{-rX_n}\right\}$$

Since $X_0=0$, it is clear that

$$E\left\{e^{-rX_n}\right\} = \left(\frac{\lambda}{r+\lambda}\right)^n$$

APPENDIX II

Proof of: $E\left\{X_n\right\} = \dfrac{n}{\lambda}$

X_n is made up of n independent interarrival times each with an expected value of $\dfrac{1}{\lambda}$. Therefore, the expected value of the sum is just the sum of the expected values which yields $n \times \dfrac{1}{\lambda}$.

Thus,

$$E\left\{X_n\right\} = \frac{n}{\lambda}$$

TIME: 13:32:20 UNIVERSITY OF B. C. COMPUTING CENTRE DATE: 04-15-70

******************************** OTHERS ********************************

$COMPILE R. STEELE

****

```
          DO 8 J=1,4
    8     PRINT 9
    9     FORMAT ('0')
   10     S=0.0
   20     A=1.0/(0.25+0.25*EXP(-0.05*S))
   31     B=0.4*10000.0*A/(0.02+0.4*A)
   41     C=S*(0.02+A)/(A*(0.02+0.4*A))
   50     EV=B-C
   60     PRINT,S,A,B,C,EV
   70     S=10.0+S
   80     IF(S.GT.100.0) GO TO 91
   90     GO TO 20
   91     STOP
  170     END
```

$DATA

S	A	B	C	EV
0.0000000E 00	0.2000000E 01	0.9756090E 04	0.0000000E 00	0.9756090E 04
0.1000000E 02	0.2489838E 01	0.9803133E 04	0.9922212E 01	0.9793207E 04
0.2000000E 02	0.2924234E 01	0.9831895E 04	0.1692604E 02	0.9814965E 04
0.3000000E 02	0.3270298E 01	0.9849414E 04	0.2272649E 02	0.9826688E 04
0.4000000E 02	0.3523188E 01	0.9860074E 04	0.2814510E 02	0.9831926E 04
0.5000000E 02	0.3696568E 01	0.9866547E 04	0.3354440E 02	0.9833000E 04
0.6000000E 02	0.3810296E 01	0.9870480E 04	0.3906111E 02	0.9831418E 04
0.7000000E 02	0.3882751E 01	0.9872867E 04	0.4472733E 02	0.9828137E 04
0.8000000E 02	0.3928055E 01	0.9874313E 04	0.5053181E 02	0.9823777E 04
0.9000000E 02	0.3956052E 01	0.9875191E 04	0.5644897E 02	0.9818742E 04
0.1000000E 03	0.3973229E 01	0.9875723E 04	0.6245192E 02	0.9813270E 04

COMPILE TIME= 0.22 SEC,EXECUTION TIME= 0.10 SEC,OBJECT CODE= 752 BYTES,ARRAY AREA= 0 BYTES,UNUSED= 60688 BYTES

TOTAL CPU TIME USED: .38 SECONDS

TIME: 13:34:41 UNIVERSITY OF B. C. COMPUTING CENTRE DATE: 04-15-70

** OTHERS ************************************

$COMPILE R. STEELE

```
  1          DO 8 J=1,4
  2     8    PRINT 9
  3     9    FORMAT ('0')
  4    10    S=0.0
  5    20    A=1.0/(0.25+0.25*EXP(-0.05*S))
  6    30    B=798.0*A/(0.02+0.0668*A)
  7    40    C=S*(A+0.02)/(A*(0.02+0.0668*A))
  8    50    EV=B-C
  9    60    PRINT,S,A,B,C,EV
 10    70    S=10.0+S
 11    80    IF(S.GT.100.0) GO TO 91
 12    90    GO TO 20
 13   91     STOP
 14    170   END
```

$DATA

S	A	B	C	EV
0.0000000E 00	0.2000000E 01	0.1039063E 05	0.0000000E 00	0.1039063E 05
0.1000000E 02	0.2489838E 01	0.1066380E 05	0.5410187E 02	0.1060969E 05
0.2000000E 02	0.2924234E 01	0.1083659E 05	0.9351208E 02	0.1074307E 05
0.3000000E 02	0.3270298E 01	0.1094415E 05	0.1265788E 03	0.1081757E 05
0.4000000E 02	0.3523188E 01	0.1101044E 05	0.1575376E 03	0.1085290E 05
0.5000000E 02	0.3696568E 01	0.1105104E 05	0.1883280E 03	0.1086271E 05
0.6000000E 02	0.3810296E 01	0.1107580E 05	0.2197045E 03	0.1085610E 05
0.7000000E 02	0.3882751E 01	0.1109088E 05	0.2518564E 03	0.1083902E 05
0.8000000E 02	0.3928055E 01	0.1110005E 05	0.2847344E 03	0.1081531E 05
0.9000000E 02	0.3956052E 01	0.1110561E 05	0.3182075E 03	0.1078740E 05
0.1000000E 03	0.3973229E 01	0.1110899E 05	0.3521345E 03	0.1075686E 05

COMPILE TIME= 0.22 SEC,EXECUTION TIME= 0.10 SEC,OBJECT CODE= 744 BYTES,ARRAY AREA= 0 BYTES,UNUSED= 60696 BYTES

TOTAL CPU TIME USED: .42 SECONDS

```
******************************************** OTHERS ********************************************

$COMPILE    R. STEELE

**********************************************************************************************************

           DO 8 J=1,4
     8     PRINT 9
     9     FORMAT ('0')
    10     S=0.0
    18     AL=10000.
    19     R=(AL-10000.)/1000.
    20     A=1.0/(0.25+0.25*EXP(-0.05*S))
    21     BAA=(A/(.02+A))*1000.*EXP(-.5*R*R)/SQRT(2.*3.1429)
    22     BAB=(A/(.02+A))*10000.*.5*ERFC(R/SQRT(2.))
    23     BB=1.-(A/(.02+A))*(1.-.5*ERFC(R/SQRT(2.)))
    24     BA=BAA+BAB
    25     B=BA/BB
    26     C=S*(.02+A)/(A*(.02+.5*A*ERFC(R/SQRT(2.))))
    50     EV=B-C
    60     PRINT,AL,S,EV
    71     AL=AL+100.
    80     IF(AL.GT.12000) GO TO 91
    90     GO TO 19
   91      STOP
    170    END

$DATA
```

AL	S	EV
0.1000000E 05	0.0000000E 00	0.1058600E 05
0.1010000E 05	0.0000000E 00	0.1063140E 05
0.1020000E 05	0.0000000E 00	0.1067549E 05
0.1030000E 05	0.0000000E 00	0.1071746E 05
0.1040000E 05	0.0000000E 00	0.1075638E 05
0.1050000E 05	0.0000000E 00	0.1079109E 05
0.1060000E 05	0.0000000E 00	0.1082024E 05
0.1070000E 05	0.0000000E 00	0.1084215E 05
0.1080000E 05	0.0000000E 00	0.1085476E 05
0.1090000E 05	0.0000000E 00	0.1085557E 05
0.1100000E 05	0.0000000E 00	0.1084149E 05
0.1110000E 05	0.0000000E 00	0.1080876E 05
0.1120000E 05	0.0000000E 00	0.1075276E 05
0.1130000E 05	0.0000000E 00	0.1066793E 05
0.1140000E 05	0.0000000E 00	0.1054760E 05
0.1150000E 05	0.0000000E 00	0.1038398E 05
0.1160000E 05	0.0000000E 00	0.1016820E 05
0.1170000E 05	0.0000000E 00	0.9890617E 04
0.1180000E 05	0.0000000E 00	0.9541387E 04
0.1190000E 05	0.0000000E 00	0.9111602E 04
0.1200000E 05	0.0000000E 00	0.8594855E 04

APPENDIX VI


```
TIME: 13:50:24                    UNIVERSITY OF B. C. COMPUTING CENTRE                    DATE:  04-16-70

              ***********************************  OTHERS   ***********************************

              $COMPILE      R. STEELE

              ****************************************************************************************

          DO 8 J=1,4
     8    PRINT 9
     9    FORMAT ('0')
    10    S=0.0
    18    AL=10900.
    19    R=(AL-10000.)/1000.
    20    A=1.0/(0.25+0.25*EXP(-0.05*S))
    21    BAA=(A/(.02+A))*1000.*EXP(-.5*R*R)/SQRT(2.*3.1429)
    22    BAB=(A/(.02+A))*10000.*.5*ERFC(R/SQRT(2.))
    23    BB=1.-(A/(.02+A))*(1.-.5*ERFC(R/SQRT(2.)))
    24    BA=BAA+BAB
    25    B=BA/BB
    26    C=S*(.02+A)/(A*(.02+.5*A*ERFC(R/SQRT(2.))))
    50    EV=B-C
    60    PRINT,AL,S,EV
    71    S=S+5.
    80    IF(S.GT.100.0) GO TO 91
    90    GO TO 19
   91     STOP
    170   END

     $DATA
```

AL	S	EV
0.1090000E 05	0.0000000E 00	0.1085557E 05
0.1090000E 05	0.5000000E 01	0.1090616E 05
0.1090000E 05	0.1000000E 02	0.1094566E 05
0.1090000E 05	0.1500000E 02	0.1097611E 05
0.1090000E 05	0.2000000E 02	0.1099922E 05
0.1090000E 05	0.2500000E 02	0.1101638E 05
0.1090000E 05	0.3000000E 02	0.1102875E 05
0.1090000E 05	0.3500000E 02	0.1103728E 05
0.1090000E 05	0.4000000E 02	0.1104274E 05
0.1090000E 05	0.4500000E 02	0.1104572E 05
0.1090000E 05	0.5000000E 02	0.1104674E 05
0.1090000E 05	0.5500000E 02	0.1104618E 05
0.1090000E 05	0.6000000E 02	0.1104438E 05
0.1090000E 05	0.6500000E 02	0.1104158E 05
0.1090000E 05	0.7000000E 02	0.1103798E 05
0.1090000E 05	0.7500000E 02	0.1103376E 05
0.1090000E 05	0.8000000E 02	0.1102903E 05
0.1090000E 05	0.8500000E 02	0.1102391E 05
0.1090000E 05	0.9000000E 02	0.1101847E 05
0.1090000E 05	0.9500000E 02	0.1101279E 05
0.1090000E 05	0.1000000E 03	0.1100689E 05

TIME: 09:46:48 UNIVERSITY OF B. C. COMPUTING CENTRE DATE: 04-16-70

****************************** OTHERS ******************************

$COMPILE R. STEELE

```
          DO 8 J=1,4
    8     PRINT 9
    9     FORMAT ('0')
   10     S=50.
   18     AL=10000.
   19     R=(AL-10000.)/1000.
   20     A=1.0/(0.25+0.25*EXP(-0.05*S))
   21     BAA=(A/(.02+A))*1000.*EXP(-.5*R*R)/SQRT(2.*3.1429)
   22     BAB=(A/(.02+A))*10000.*.5*ERFC(R/SQRT(2.))
   23     BB=1.-(A/(.02+A))*(1.-.5*ERFC(R/SQRT(2.)))
   24     BA=BAA+BAB
   25     B=BA/BB
   26     C=S*(.02+A)/(A*(.02+.5*A*ERFC(R/SQRT(2.))))
   50     EV=B-C
   60     PRINT,AL,S,EV
   71     AL=AL+100.
   80     IF(AL.GT.12000) GO TO 91
   90     GO TO 19
   91     STOP
  170     END
```

$DATA

AL	S	EV
0.1000000E 05	0.5000000E 02	0.1065521E 05
0.1010000E 05	0.5000000E 02	0.1070698E 05
0.1020000E 05	0.5000000E 02	0.1075854E 05
0.1030000E 05	0.5000000E 02	0.1080929E 05
0.1040000E 05	0.5000000E 02	0.1085857E 05
0.1050000E 05	0.5000000E 02	0.1090552E 05
0.1060000E 05	0.5000000E 02	0.1094918E 05
0.1070000E 05	0.5000000E 02	0.1098832E 05
0.1080000E 05	0.5000000E 02	0.1102145E 05
0.1090000E 05	0.5000000E 02	0.1104674E 05
0.1100000E 05	0.5000000E 02	0.1106187E 05
0.1110000E 05	0.5000000E 02	0.1106399E 05
0.1120000E 05	0.5000000E 02	0.1104948E 05
0.1130000E 05	0.5000000E 02	0.1101388E 05
0.1140000E 05	0.5000000E 02	0.1095155E 05
0.1150000E 05	0.5000000E 02	0.1085557E 05
0.1160000E 05	0.5000000E 02	0.1071740E 05
0.1170000E 05	0.5000000E 02	0.1052678E 05
0.1180000E 05	0.5000000E 02	0.1027165E 05
0.1190000E 05	0.5000000E 02	0.9938430E 04
0.1200000E 05	0.5000000E 02	0.9512672E 04

```
************************************ OTHERS ************************************

$COMPILE     R. STEELE

************************************************************************************

          DO 8 J=1,4
    8     PRINT 9
    9     FORMAT ('0')
   10     S=0.0
   18     AL=11100.
   19     R=(AL-10000.)/1000.
   20     A=1.0/(0.25+0.25*EXP(-0.05*S))
   21     BAA=(A/(.02+A))*1000.*EXP(-.5*R*R)/SQRT(2.*3.1429)
   22     BAB=(A/(.02+A))*10000.*.5*ERFC(R/SQRT(2.))
   23     BB=1.-(A/(.02+A))*(1.-.5*ERFC(R/SQRT(2.)))
   24     BA=BAA+BAB
   25     B=BA/BB
   26     C=S*(.02+A)/(A*(.02+.5*A*ERFC(R/SQRT(2.))))
   50     EV=B-C
   60     PRINT,AL,S,EV
   71     S=S+5.
   80     IF(S.GT.100.0) GO TO 91
   90     GO TO 19
  91      STOP
   170    END

$DATA
```

AL	S	EV
0.1110000E 05	0.0000000E 00	0.1080876E 05
0.1110000E 05	0.5000000E 01	0.1087593E 05
0.1110000E 05	0.1000000E 02	0.1092856E 05
0.1110000E 05	0.1500000E 02	0.1096923E 05
0.1110000E 05	0.2000000E 02	0.1100016E 05
0.1110000E 05	0.2500000E 02	0.1102316E 05
0.1110000E 05	0.3000000E 02	0.1103977E 05
0.1110000E 05	0.3500000E 02	0.1105124E 05
0.1110000E 05	0.4000000E 02	0.1105859E 05
0.1110000E 05	0.4500000E 02	0.1106261E 05
0.1110000E 05	0.5000000E 02	0.1106399E 05
0.1110000E 05	0.5500000E 02	0.1106325E 05
0.1110000E 05	0.6000000E 02	0.1106084E 05
0.1110000E 05	0.6500000E 02	0.1105708E 05
0.1110000E 05	0.7000000E 02	0.1105225E 05
0.1110000E 05	0.7500000E 02	0.1104657E 05
0.1110000E 05	0.8000000E 02	0.1104023E 05
0.1110000E 05	0.8500000E 02	0.1103335E 05
0.1110000E 05	0.9000000E 02	0.1102604E 05
0.1110000E 05	0.9500000E 02	0.1101840E 05
0.1110000E 05	0.1000000E 03	0.1101048E 05