A STUDY OF VARIOUS COMPUTATIONAL METHODS FOR DETERMINING
TIME OPTIMAL CONTROL OF TIME DELAY SYSTEMS

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ABSTRACT

In this thesis some numerical techniques for obtaining the time optimal control of a class of time delay systems are studied and compared. The delays may be fixed or time varying. The delay systems considered, which need not be linear or time invariant, are those for which the time optimal control is bang-bang.

The optimal control is found by carrying out a search in switching interval space. The method of Rosenbrock (2, 3) is used to find the switching intervals which maximize a performance index of the final states and terminal time. Kelly's (21) method of gradients is shown to be applicable to systems with time varying time delays by using the costate equations of ref. [10]. The perturbations in the control are chosen in such a way that the descent in function space is changed to a steepest descent in switching interval space. In a third approach, a technique similar to that of Bryson and Denham (19) is used to account for the terminal conditions directly. All the methods are illustrated by examples.

The advantages of the direct search based on Rosenbrock's method are a) ease of programming and b) rapid convergence close to the optimum. However, initial convergence is slow when compared to that of either gradient method. Of the two gradient methods, that based on a penalty function approach was superior in ease of programming and convergence close to the optimum to that based on a descent to the final target set. Neither gradient scheme could match the rapid convergence of the Rosenbrock method close to the optimum.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABSTRACT</td>
<td>i</td>
</tr>
<tr>
<td>TABLE OF CONTENTS</td>
<td>ii</td>
</tr>
<tr>
<td>LIST OF ILLUSTRATIONS</td>
<td>iii</td>
</tr>
<tr>
<td>ACKNOWLEDGEMENT</td>
<td>iv</td>
</tr>
<tr>
<td>I. INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>II. GENERAL BACKGROUND</td>
<td>3</td>
</tr>
<tr>
<td>2.1 Maximum Principle</td>
<td>3</td>
</tr>
<tr>
<td>2.2 Proposed Techniques</td>
<td>6</td>
</tr>
<tr>
<td>III. THE EXTENSION OF ROSENBROCK'S METHOD TO FINDING TIME OPTIMAL</td>
<td>8</td>
</tr>
<tr>
<td>CONTROLS OF TIME DELAY SYSTEMS</td>
<td></td>
</tr>
<tr>
<td>3.1 Description of Rosenbrock's method</td>
<td>8</td>
</tr>
<tr>
<td>3.2 Time Optimal Control of Time Interval Adjustment</td>
<td>9</td>
</tr>
<tr>
<td>3.3 Computational Aspects</td>
<td>11</td>
</tr>
<tr>
<td>3.4 Examples and Results</td>
<td>12</td>
</tr>
<tr>
<td>3.5 Conclusions</td>
<td>17</td>
</tr>
<tr>
<td>IV. A GRADIENT METHOD</td>
<td>18</td>
</tr>
<tr>
<td>4.1 Development of the Perturbation Equations</td>
<td>19</td>
</tr>
<tr>
<td>4.2 Penalty Function Approach</td>
<td>24</td>
</tr>
<tr>
<td>4.3 Algorithm Description</td>
<td>29</td>
</tr>
<tr>
<td>4.4 Comments on Computational</td>
<td>30</td>
</tr>
<tr>
<td>4.5 Examples and Results</td>
<td>30</td>
</tr>
<tr>
<td>4.6 Conclusions</td>
<td>35</td>
</tr>
<tr>
<td>V. DESCENT TO FINAL TARGET SET</td>
<td>36</td>
</tr>
<tr>
<td>5.1 Perturbation Analysis</td>
<td>36</td>
</tr>
<tr>
<td>5.2 Algorithm Description</td>
<td>40</td>
</tr>
<tr>
<td>5.3 A Detailed Example</td>
<td>41</td>
</tr>
<tr>
<td>5.4 Conclusions</td>
<td>45</td>
</tr>
<tr>
<td>VI. CONCLUSIONS</td>
<td>47</td>
</tr>
<tr>
<td>APPENDIX I</td>
<td>48</td>
</tr>
<tr>
<td>BIBLIOGRAPHY</td>
<td>51</td>
</tr>
</tbody>
</table>
## LIST OF ILLUSTRATIONS

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1</td>
<td>Time Delays</td>
<td>15</td>
</tr>
<tr>
<td>3.2</td>
<td>Convergence of Rosenbrock's Method, 7th Order System</td>
<td>16</td>
</tr>
<tr>
<td>4.1</td>
<td>Nominal Control</td>
<td>23</td>
</tr>
<tr>
<td>4.2</td>
<td>New Control</td>
<td>23</td>
</tr>
<tr>
<td>4.3</td>
<td>Change in Control</td>
<td>23</td>
</tr>
<tr>
<td>4.4</td>
<td>New Control</td>
<td>23</td>
</tr>
<tr>
<td>4.5</td>
<td>Change in Control</td>
<td>23</td>
</tr>
<tr>
<td>4.6</td>
<td>Convergence of Gradient Method, 2nd Order System</td>
<td>32</td>
</tr>
<tr>
<td>4.7</td>
<td>Convergence of Gradient Method, 7th Order System</td>
<td>34</td>
</tr>
<tr>
<td>5.1</td>
<td>$</td>
<td>\text{Error}</td>
</tr>
</tbody>
</table>
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I. INTRODUCTION

A time delay system is one in which the state equations involve terms, either in the states or the control, which are delayed by some positive quantity, \( \Theta(t) \), called the time delay. If more than one delay exists in the system they are denoted by \( \Theta_1(t), \Theta_2(t), \ldots, \Theta_k(t) \).

Optimal control of time delay systems presents difficulties not found in problems which have no delays\(^{11,14}\), since such systems are theoretically of infinite order.\(^{11}\) Also the state and costate equations are nearly impossible to solve analytically even for the simplest systems\(^{11,14}\). Therefore numerical or iterative techniques of determining the optimal control offer, in most cases, the only practical methods of solution. Unfortunately some numerical methods which work well for systems with no delays, such as descent in costate space, cannot be applied for time delay systems; the costate equations by their nature being insoluble other than backwards in time.\(^{14}\)

If the control is restricted by magnitude constraints and, as is often the case for time optimal problems, lies on the boundary of the constraint set, then the derivative of the Hamiltonian with respect to the optimal control is not necessarily zero\(^{22}\). Thus methods such as the Newton-Raphson algorithm, and the min \( H \) strategy which make use of this fact cannot be used. A search in function space is also restricted since the allowable perturbations in the control are not free.

In this thesis three numerical methods designed to overcome these problems are presented:

a) Rosenbrock's\(^2\) method of maximizing a function is used to determine the time optimal control of time delay systems.

b) Kelly's method of gradients\(^{21}\) is shown to be applicable to systems with time varying time delays. The resulting descent in function space is modified by an unusual choice of perturbation in the control to
ensure that the new control is admissible. For such a choice of perturbations, the descent in function space reduces to steepest descent in switching interval space, with attendant reduction of storage and computation requirements.

This result is important because the gradient of any function with respect to the switching intervals is not, in general, available.

c) The theory of Banks, (10) the final target set approach of Bryson and Denham, (19) and the aforementioned perturbation in the control are all combined to yield a steepest descent in switching interval space which accounts for the terminal conditions directly.

Comparisons of these methods via examples are made, and the results discussed in the conclusions.
II. GENERAL BACKGROUND

A maximum principle, which was derived by Banks for systems with time varying time delays, may be used to show that the time optimal control is bang-bang. In such cases the control can be represented by a finite number of parameters, the switching instants. Ordinary descent in function space has difficulty in dealing with such controls; however, the descent in function space can be changed to a descent in parameter space.

2.1 The Maximum Principle

Consider a system defined by

\[ \dot{x}(t) = f(x(t), x(w(t)), u(t), t) \quad t_0 \leq t \leq t_f \]  

(2.1)

where the dot denotes the derivative with respect to time, \( t \), the independent variable and

- \( x(t) \) is an \( n \) dimensional column vector,
- \( x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T \),
- \( f \) is an \( n \) dimensional vector function,
- \( u \) is an \( m \) dimensional control vector,
- and \( w(t) \) is the delayed time defined by \( w(t) = t - \theta(t) \) where \( \theta(t) \) is the time varying time delay.

The time delay is usually restricted by physical considerations so that

\[ \theta(t) \geq 0 \quad \forall t \]  

(2.2a)

\[ \dot{\theta}(t) < 1 \quad \forall t \]  

(2.2b)

The reason for the first restriction is clear enough, for if the delay becomes negative then \( w(t) = t - \theta(t) \) is no longer a delayed time but an advanced time. The reasoning behind the second restriction is slightly more obscure. The time delay may decrease very rapidly, provided that it is always continuous and everywhere greater than zero. However, if the value of the time delay
is increasing faster than time itself, then the delayed time, $w(t)$, will not be monotonically increasing, and $r(t)$, the inverse function of $w(t)$, may not exist. Since the existence of $r(t)$ is a prerequisite to the formation of the costate equations\(^\text{(10)}\), $\dot{\theta}(t)$ must always be strictly less than one.

A system of the form 2.1 requires initial conditions

$$x(t) = g(t) \quad w(t_o) \leq t \leq t_o$$  \hspace{1cm} (2.3)

where $g(t)$ is an $n$-vector function of time.

It is desired to maximize some performance index

$$J = \phi(x_f, t_f) + \int_{t_o}^{t_f} f_o(x, u, t) dt$$  \hspace{1cm} (2.4)

subject to constraints 2.1 and terminal conditions

$$\psi(x_f, t_f) = 0$$

Here, $\phi$ is a scalar function, $\psi$ is a vector function, and the subscript "f" means that the corresponding quantities are evaluated at the final time.

When the problem consists of controlling the state $x(t_o) = x_o$ to the origin in minimum time the terminal conditions are

$$\psi(x_f, t_f) = x_f = 0$$

(the final time is free, the final point is the origin) and the performance function is $J = \int_{t_o}^{t_f} f \, dt$.

Banks in [10] has shown that the costate equations of system 2.1 are

$$p^T(t) = -p^T(t) \left[ \frac{\partial f}{\partial x}(x(t), x(w(t)), u(t), t) \right]$$

$$- p^T(r(t)) \left[ \frac{\partial f}{\partial x(w(t))}(x(r(t)), x(t), u(r(t)), r(t)) \right]$$

$$\frac{\partial f}{\partial x}(x(t), x(w(t)), u(t), t)$$

$$p^T(t_f) = -p^T(t) \left[ \frac{\partial f}{\partial x}(x(t), x(w(t)), u(t), t) \right]$$

$$w(t_f) \leq t \leq t_f$$  \hspace{1cm} (2.5)
r(t) is a scalar function defined as the inverse of w(t) such that \( r(w(t)) = t = w(r(t)) \), or in short notation \( r(t) = w^{-1}(t) \). The notation \( w^{-1}(t) \) refers, not to the reciprocal, but to the functional inverse, of \( w(t) \). \( p(t) \) is the n-dimensional costate vector

\[
p(t) = [p_1(t), p_2(t), \ldots, p_n(t)]^T.
\]

The term \( \frac{\partial f}{\partial x(w(t))} \) means the functional derivative of \( f(x(t), x(w(t)), u(t), t) \) with respect to its second argument. The Hamiltonian is then defined as

\[
H(x, p, u, t) = p^T(t) f(x(t), x(w(t)), u(t), t) - f_o(x(t), x(w(t)), u(t), t).
\]

In all subsequent work it is assumed that:

1) the optimal control exists
2) the optimal control is unique
3) the maximum principle as derived by Banks holds, and hence that all assumptions made by him about continuity, differentiability etc. of \( f(x(t), x(w(t)), u(t), t) \), \( x(t) \), and \( u(t) \) hold.

(See appendix I for more details.)

This last is necessary for if Bank's maximum principle does not hold then there is no theoretical justification for considering bang-bang optimal controls. All the examples considered in the thesis were chosen to satisfy these assumptions.

4) The vector function \( \hat{f}^T = (f_0^T, f)^T \) has the form \( \hat{f}(x(t), x(w(t)), u(t), t) = A(x(t), x(w(t)), t) + B(x(t), x(w(t)), t) u(t) \), where \( A(x(t), x(w(t)), t) \) is an \((n + 1) \times 1\) vector, \( B(x(t), x(w(t)), t) \) is an \((n + 1) \times m\) matrix and \( u(t) \) is an \(m \times 1\) vector.

5) There is no finite subinterval \((t_1, t_2)\) of the interval \([t_0, t_f]\) over which any element of the vector \( p^T(t) B(x(t), x(w(t)), t) \) is zero.
6) The control is restricted by $|u_i| < \beta_i^{l+1, \ldots, m}$ or in vector form $|u| < \beta$.

Since the numerical methods considered later optimize the performance index by a correct choice of switching intervals it is necessary to show that the optimal control is bang-bang.

**Lemma**: Under the above assumptions the optimal control is bang-bang.

**Proof**: By the maximum principle of Banks (10) the optimal control must maximize the Hamiltonian

$$H(x, p, u, t) = p^T(t)A(x(t), x(w(t)), t) + p^T(t)B(x(t), x(w(t)), u(t)),$$

By the Schwarz inequality the above is a maximum if

$$u(t) = -\text{SGN} [B^T(x(t), x(w(t)), t) p(t)] \quad (2.6)$$

for then the second term of $H$ is positive with the largest possible value of $u$.

The above lemma, which is a special case of Bank's maximum principle, determines the form of the time optimal control. The next section details a computing scheme which makes use of this knowledge.

### 2.2 Proposed Techniques

In order to synthesize the optimal control therefore three things must be known:

- a) the number of switching intervals
- b) their durations
- c) the sign of the control in the first interval.

A common method of determining the optimal control is to assume some initial conditions on the costates and to solve equations 2.1 and 2.5 forward in time while synthesizing the control at each step from 2.6. Such a method will not work for the time delay problem because of the form of equations 2.5.
The assumptions made about the delay ensures that \( w(t) \leq t \) and hence that 
\[ r(t) = w^{-1}(t) \] 
is greater than or equal to \( t \) [ref. 10]. Thus to solve equations 2.5 forward in time requires knowledge of the future. If the optimal control is bang-bang one cannot use schemes such as the min-H strategy which try to drive the gradient of the Hamiltonian with respect to the control to zero. If the perturbations in the control are chosen in the usual way for descent in function space the magnitude constraints on the control may be violated. It is shown in Chapter IV that the perturbations in the control can be chosen in such a way that the magnitude constraints on the control are not violated and that steepest descent in switching interval space results.

It is proposed that one method of solution would be to treat the performance index as a function of the switching intervals and to try to minimize it by correct choice of same. Any attempt to do so however must recognize that in practice the derivatives of the performance index with respect to the switching intervals are unavailable.

Fletcher\(^{(17)}\) has surveyed a number of methods of maximizing a function without evaluating derivatives. His results tentatively identify Rosenbrock's scheme as the best one to use for functions of a large (> 4) number of variables. Indeed, Davison and Monroe\(^{(3)}\) have applied Rosenbrock's procedure to nonlinear time varying systems and identified the time optimal control of some large order systems with a good degree of success. It has not been shown however that such a method will work for systems which contain constant or time varying time delays. The results presented in the next chapter show that their algorithm will work for such systems and that convergence is especially good around the optimum. However, the initial convergence of the method is slow when compared to those methods which use gradient information.\(^{(17)}\)

In this thesis this has also been shown to be the case for time delay systems.
III. THE EXTENSION OF ROSENBROCK'S METHOD TO FINDING
TIME OPTIMAL CONTROLS OF TIME DELAY SYSTEMS

In this chapter Rosenbrock's (2) method of maximizing a function is outlined. The application of this method to the time optimal problem is described and is illustrated by several examples.

3.1 Description of Rosenbrock's method

The Rosenbrock method is a searching algorithm which maximizes a function $J$ of several variables $T_i$, $i = 1$ to $n$ when the $T_i$'s are subject to constraints

$$g_i(T_1, T_2 \ldots T_n) \leq T_i \leq h_i(T_1, T_2 \ldots T_n) \quad i = 1, 2, \ldots n$$

$$g_{n+j}(T_1, T_2 \ldots T_n) \leq T_{n+j} \leq h_{n+j}(T_1, T_2 \ldots T_n) \quad j = 1, 2, \ldots p$$

The search is carried out by taking steps in the $T_i$'s along vectors $V_i$, so that $J$ is extremized, all the while ensuring that the constraint equations are not violated.

The size of the steps taken in each direction is determined automatically by the algorithm and the step directions are adjusted periodically. Whenever sufficient progress has been made, the direction vectors $V_i$ are rotated by the Schmidt procedure. Rosenbrock and Fletcher (17) argue that after several rotations the direction of search is that of steepest descent.

Because both Rosenbrock's (2) and Davison and Monroe's (3) papers contain misprints in the description of the Schmidt procedure it is rederived here. Suppose that $d_1$ is the algebraic sum of all successful steps $e_1$ in direction $V_1^k$ etc., where $V_1^k$ is the $k^{th}$ choice of $V_1$, then let

$$A_1 = d_1V_1^k + d_2V_2^k + \ldots + d_nV_n^k$$
\[ A_2 = d_2 v_2^k + \ldots + d_n v_n^k \]
\[ A_n = d_n v_n^k \]

Thus \( A_1 \) is the vector joining the initial and final points obtained by use of vectors \( v_1^k, v_2^k, \ldots, v_n^k \). \( A_2 \) is the vector sum of all advances made in direction other than first etc.

Orthogonal unit vectors, \( v_1^{k+1}, v_2^{k+1}, \ldots, v_n^{k+1} \) are now obtained by forming

\[ v_1^{k+1} = \frac{A_1}{||A_1||} \]

\[ B_2 = A_2 - (A_2 \cdot v_1^{k+1})v_1^{k+1} \]

\[ v_2^{k+1} = \frac{B_2}{||B_2||} \]

\[ \vdots \]

\[ B_n = A_n - \sum_{j=1}^{n-1} (A_n \cdot v_j^{k+1})v_j^{k+1} \]

\[ v_n^{k+1} = \frac{B_n}{||B_n||} \]

The maximization of the function, \( J \), is considered to be complete when the change in the variables \( T_1 \) is less than some specified value.

3.2 Time optimal control by switching interval adjustment

It is desired to bring the state vector whose dynamics are constrained by

\[ \dot{x}(t) = A(x(t), x(w(t)), t) + B(x(t), x(w(t)), t) u(t) \]

to the origin, \( x(t_f) = 0 \), from the initial state, \( x(t_0) = x_0 \), in minimum time under the assumptions made in chapter 2. As was shown in chapter 2 and appendix I the control which will do this is bang-bang. Therefore, the number
and lengths of the switching intervals are required along with the sign of the control in the first interval.

Since the switching intervals are to be adjusted, and since they are constrained by $0 \leq T_i$, and since the derivatives of the performance function with respect to the switching intervals are unobtainable, the hill climbing method of Rosenbrock seems ideally suited to the task.

Davison and Monroe (3) have reported an algorithm based on Rosenbrock's method which will find the time optimal control of time varying or non-linear systems. Because it would be useful to know if the time optimal controls of time delay systems can be found in this way, and because any results obtained in this fashion would form a basis for comparison with the algorithms derived later in chapters IV and V, their algorithm, and some examples of its application to time delay systems are presented here. The Davison and Monroe algorithm consists of two main parts: a) determining the sign of the control in the first switching interval and the correct number of switching intervals by maximizing a composite performance index

$$J_1 = -C t_f - x^T(t_f) x(t_f)$$

where $C$ is a weighting factor, often $=1$ and

b) determining the sizes of the switching intervals, after the correct number of intervals has been found in part a, by minimizing the distance squared

$$J_2 = x^T(t_f) x(t_f).$$

The algorithm is started by assuming a number (NSW) of switching intervals and the control (+$\delta$) in the first interval. These two choices are equivalent to choosing a nominal control. The state equations are then solved forward in time using the assumed nominal control. The performance index $J_1$ is evaluated at the assumed final time and the switching intervals are adjusted by the Rosenbrock hill climbing procedure. The state equations are solved and the switching intervals are adjusted until the performance index $J_1$ has
been maximized.

Once $J_1$ has been maximized the square of the distance to the origin, $J_2 = x^T(t_f) \cdot x(t_f)$ may or may not have approached zero. If $J_2$ has not approached zero this implies that the number of switching instants was too small or that the choice of $u = +\beta$ in the first interval was wrong. (The reason for this is the assumed uniqueness of the optimal control). In either case the number of switching intervals is increased by one and the maximization of $J_1$ by the hill climbing procedure is repeated.

If the function $J_2$ has approached zero some switching interval(s) $T_i$ may or may not have tended to disappear (by tending to zero). If some switching intervals have tended to disappear, the uniqueness of the optimal control implies that there are too many switching intervals and the correct number of intervals is then obtained by eliminating the vanishing ones, and counting the remainder. Note that if the initial choice of $u = +\beta$ was wrong then the first time interval will tend to disappear as hill climbing proceeds. When this time interval is eliminated the initial value of $u(t)$ will be correct.

At this stage the correct number of switching intervals is known, as well as their approximate values. More accurate values for the switching intervals are then found by maximizing the function $J_2(x_f) = -x_f^T \cdot x_f$.

3.3 Computational aspects

Because of the unavoidable errors in computation it is unlikely that $J_2$ can ever be made exactly zero. Hence an implicit assumption in the above treatment is that a control which reduces $J_2$ "close to" zero is, for practical purposes, the optimal control and is not too different from the theoretical optimum.

One shortcoming of this method is that there is no rule to help in deciding, at the end of the maximization of $J_1$, whether $J_2$ is unacceptably
large or not. One way of solving this problem is to increase the assumed 
number of switching intervals by one, and repeat the maximization of $J_1$ until 
one or more of the switching intervals tends to zero at the maximum.

In the maximizing of $J_1$ a correct choice of the weighting factor $C$, 
can do much to speed up convergence. However, experience or trial and error 
are the only guides to a correct choice of $C$.

3.4 Examples and results

1. The first system for which the above algorithm was implemented was 
the one studied theoretically by Wells and Kashewagi in [5].

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -x_1(t - \theta(t)) + u(t)$$

with initial data

$$x_1(s) = 2.0 + 2s \quad s < 0$$
$$x_2(s) = 2.0$$

In the Wells and Kashewagi system $\theta(t)$ was a constant equal to 
0.3. The time taken to drive the states to within $10^{-6}$ of the origin is 7.9 
seconds. From the state trajectory supplied by Wells and Kashewagi the ap­
proximate theoretical minimum time is 7.7 seconds. The number of switching 
intervals was correctly identified as four. The algorithm was also used to 
find the time optimal control of the above system when $\theta(t)$ was time varying.

The results for different delay functions are

$$\theta(t) = 0.3 - \frac{t}{100} \quad t_f = 6.636 \quad \text{dist} = 2.5 \times 10^{-5}$$
$$\theta(t) = 0.3 e^{-t} \quad t_f = 5.577 \quad \text{dist} = 1 \times 10^{-4}$$
$$\theta(t) = 0.2 + 0.1 \sin t \quad t_f = 6.2115 \quad \text{dist} = 1.5 \times 10^{-2}$$
2. A large order system

The next problem considered was a modified form of the seventh order system described in [3]

\[
\begin{align*}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= x_3(t - \varphi(t)) \\
\dot{x}_3(t) &= x_4(t - \varphi(t)) \\
\dot{x}_4(t) &= x_5(t) \\
\dot{x}_5(t) &= x_6(t) \\
\dot{x}_6(t) &= -x_7(t) \\
\dot{x}_7(t) &= -720 x_2(t) - 1764 x_3(t) - 1624 x_4(t) - 735 x_5(t) - 175 x_6(t) \\
&\quad - 21 x_7(t) + u(t)
\end{align*}
\]

with initial function of zero and initial conditions at \( t_0 = 0 \) of

\[
x^T(0) = (0, -10, 0, 0, 0, 0, 0).
\]

The control \( u \) is restricted by \( |u| < 1000 \).

Again it is desired to drive the state vector to the origin in minimum time. The results in the case of two different time delays are given below.

<table>
<thead>
<tr>
<th>Delay ( \Theta(t) )</th>
<th>number of intervals</th>
<th>distance from origin at final time</th>
<th>trials</th>
<th>minimum time</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Theta(t) = 0.2 )</td>
<td>11</td>
<td>1.7</td>
<td>79</td>
<td>36.86</td>
</tr>
<tr>
<td>( \Theta(t) = \frac{0.5}{t+1} )</td>
<td>11</td>
<td>1.4</td>
<td>92</td>
<td>36.1</td>
</tr>
</tbody>
</table>

These results are of the same order of magnitude as those reported in [3] for the non delay case.

Finally it was decided to try to determine what effect approximations in modelling the delay would have on system response. Accordingly the optimal control which drove the states of the following system to within \( 1.2 \times 10^{-4} \)
of the origin in 8.2 seconds was found.

\[ \dot{x}_1(t) = x_2(t) \]
\[ \dot{x}_2(t) = -x_1(t - \theta(t)) + u(t). \]

The exact time delay \( \theta(t) \) was chosen as

\[ \theta(t) = 0.2 + 0.1 \sin\left(\frac{3\pi}{8}t\right) \quad 0 < t < 4 \text{ sec} \]

\[ \theta(t) = 0.2 + 0.35 \left(1 - e^{-(t-4)}\right) \quad t > 4 \text{ sec} \]

(see fig. 3.1)

The same control was then applied to the above system with the exact delay approximated by

\[ \theta(t) = 0.36 \quad 0 < t \leq 2.6 \]
\[ \theta(t) = 0.25 \quad 2.6 < t \leq 4.6 \]
\[ \theta(t) = 0.4 \quad 4.6 < t \leq 5.4 \]
\[ \theta(t) = 0.525 \quad 5.4 < t \quad (\text{see fig. 3.1}) \]

The results were not encouraging as the miss distance was 0.55. This leads one to suspect that small changes in time delay will greatly affect the system response. Therefore when modelling time delay systems it is important to a) determine the sensitivity of the system to variation in delay and b) if this sensitivity is high, that is a small change in delay results in a large change in system response, ensure that the delay is modelled as exactly as possible.

As a non-linear example, Duffing's equation with delay in the control action was chosen. The state equations for this system are

\[ \dot{x}_1(t) = x_2(t) \]
\[ \dot{x}_2(t) = -x_1(t) - x_3(t) - x_2(t) + u(t - 0.3) \]

with \( |u| < 1 \)

and initial conditions \( x_1(0) = 1 \)
\[ x_2(0) = 1 \]
Fig. 3.1 Exact Time Delay and its Approximation
Fig. 3.2 Convergence of Rosenbrock's Method for a Seventh Order System with a Delay of 0.2
The hill climbing technique in 270 trials produced a control which drove
the system to the final state
\[ x_1(t) = 0.006 \]
\[ x_2(t) = -0.8 \times 10^{-6} \]
in 4.75 seconds.

3.5 Conclusions

The examples presented here show that the idea of a search in
switching interval space for an optimal control of a class of systems with
time varying time delays results in an algorithm which does not intercept any
local minima and has an acceptable convergence rate. The speculation by Wells
and Kashewagi in [5] that increases in the time delay would lead to increases
in the minimum time was also verified indirectly, even in the case of time
varying time delays. The Rosenbrock scheme showed some difficulty in getting
started. The reason for this is that the steepest descent direction is found
by trial and error. Therefore, many inefficient steps must be taken before
rapid convergence results. It was also shown that for at least one system,
even small errors in modelling the delay lead to quite large differences
between the model and system outputs.
IV. A GRADIENT METHOD

The previous chapter showed how the Rosenbrock hill climbing scheme can be adapted to find a time optimal control of a class of systems for which this optimal control is bang-bang. While the computing time increases only linearly with system order, storage requirements because of the Schmidt procedure and direction vectors increase as the square of the number of switching instants. Since an increase in system order nearly always brings about an increase in switching instants especially for non-linear or time delay systems the storage requirements increase at least as fast as the system order squared.

Further, nothing is known from an analytical viewpoint about convergence of the Rosenbrock procedure or about the efficiency of the searching method. Fletcher believes that with sufficient care an approach utilizing gradient information is always faster than non-gradient procedures. The gradient approach developed in this chapter did in fact converge more rapidly in the initial stages. However, it was slower near the optimum.

In order to implement a gradient search in function space the perturbations in the control had to be chosen to ensure that the performance function was minimized at each step and that the resulting controls were admissible. This last restriction not usually encountered in standard descent in function space was circumvented by choosing only those perturbations which kept the control inside the constraint set. The derived algorithm makes descent in function space possible for bang-bang controls and leads, if the control perturbations are chosen correctly, to a steepest descent in switching interval space.
4.1 Development of the perturbation equation

As before, it is desired to minimize a performance index $J(x_f, t_f)$ subject to the constraints

$$\dot{x}(t) = f(x(t), x(w(t)), u(t), t); \quad t_o \leq t \leq t_f \quad (4.1)$$

with initial data $x(s) = g(s) \quad w(t_o) < s < t_o$

and terminal constraints $\psi(x_f, t_f) = 0$.

All the terms used above have been defined in chapter 2.

Define $y = x(w(t))$

$$\delta y = \delta x(w(t)) = \left. \delta x(s) \right|_{s=t-\theta(t)}$$

$$\frac{\partial f}{\partial x} (x(\cdot), t) = \frac{\partial f}{\partial x} (x(t), x(w(t)), u(t), t).$$

If the control $\tilde{u}(t)$ is changed by an amount $\delta u(t)$ then the performance index $J$ will change by an amount

$$\delta J = \frac{\partial J}{\partial x_f} \delta x_f.$$

To relate $\delta x_f$ and $\delta u(t)$ consider

$$\delta \dot{x} = \frac{\partial f}{\partial x} \delta x + \frac{\partial f}{\partial y} \delta y + \frac{\partial f}{\partial u} \delta u \quad (4.1)$$

and

$$\frac{d}{dt} (p^T \delta x) = p^T \delta x + p^T \left( \frac{\partial f}{\partial x} \delta x + \frac{\partial f}{\partial y} \delta y + \frac{\partial f}{\partial u} \delta u \right).$$

Integrating both sides from $t_o$ to $t_f$ gives

$$p^T(t) \delta x(t) \bigg|_{t_o}^{t_f} = \int_{t_o}^{t_f} [p^T(t) + p^T(t) \frac{\partial f}{\partial x} \delta x(t) + \int_{t_o}^{t_f} p^T(t) \frac{\partial f}{\partial y} \delta y(t) + \int_{t_o}^{t_f} p^T(t) \frac{\partial f}{\partial u} \delta u(t) \quad (4.2)$$

Consider now the second integral on the right hand side of the above equation.

If $w = t-\theta(t)$ then let $t = r(w)$. If the integration with respect to $t$ is changed to integration with respect to $w$ the second term becomes
\[
\int_{t_0}^{t_f} -\theta(t_f) 
+ p^T(r(w)) \frac{\partial f}{\partial y} (r(w)) \delta x(w) \frac{dr}{dw} (w) \, dw.
\]

Because the initial data is fixed and therefore \(\delta x(w) = 0\) on the interval \(w(t_0) < s < t\), the above integral can be rewritten as

\[
\int_{t_0}^{t_f} -\theta(t_f) 
+ p^T(r(w)) \frac{\partial f}{\partial y} (r(w)) \delta x(w) \frac{dr}{dw} \, dw.
\]

The dummy variable of integration, \(w\), can be replaced by another dummy variable \(t\). Equation 4.2 then becomes

\[
\int_{t_0}^{t_f} -\theta(t_f) 
+ p^T(r(w)) \frac{\partial f}{\partial y} (r(w)) \delta x(t) \, dt
\]

\[
+ \int_{t_f}^{t_f-\theta(t_f)} [p^T(t) + p^T(r(t)) \frac{\partial f}{\partial x} (r(t))] \delta x(t) \, dt
\]

\[
+ \int_{t_0}^{t_f} p^T(t) \frac{\partial f}{\partial u} (t) \delta u(t) \, dt.
\]

(4.3)

In order to relate the change in performance function to the change in control directly, \(p(t)\) may be chosen arbitrarily as

\[
p(t_f) = \frac{\partial J}{\partial x_f}
\]

\[
\dot{p}(t) = -p^T(t) \frac{\partial f}{\partial x} (t) - p^T(r(t)) \frac{\partial f}{\partial y} (r(t)) \quad t_0 \leq t \leq w(t_f)
\]

\[
\dot{p}(t) = -p^T(t) \frac{\partial f}{\partial x} (t), \quad t_f-\theta(t_f) \leq t \leq t_f
\]

(4.4)

On closer examination it can be seen that the equations 4.4 are the costate equations derived by Banks in [10].
If the Hamiltonian is defined as

\[ H(x,y,p,u,t) = p^T(t)f(x,y,u,t) \]

and equations 4.4 used to eliminate the terms in \( \delta x(t) \), equation 4.2 becomes

\[ p(t_f)\delta x(t_f) = \int_{t_0}^{t_f} Hu(x,y,p,u,t)\delta u(t) \, dt. \quad (4.5) \]

MacAuley (6) has given a similar proof for systems with more than one constant time delay.

The change in performance index \( \Delta J \) that results from a perturbation \( \delta u \) in the control is given by (23)

\[ \Delta J = \delta J \bigg|_{t_f} + J \delta t_f \text{ where } \delta J = \frac{\partial J}{\partial x_f} \delta x_f \]

and \( J \) is the total derivative of \( J \) with respect to time

\[ \dot{J} = \left[ \frac{\partial J}{\partial t} + \frac{\partial J}{\partial x_f} \frac{\partial x_f}{\partial t} \right] \bigg|_{t_f}. \]

It has been shown earlier that the optimal control for the class of systems under discussion is bang-bang. For this reason the usual choice of \( \delta u = -kHu \) cannot be used because the perturbation \( \delta u \) must allow for the fact that the control is bounded. However, use may be made of the a priori knowledge about the form of the optimal control to choose \( \delta u \). In the following discussion refer to diagrams 4.1 through 4.5. The nominal control \( \bar{u} \) has been assumed bang-bang. Since the optimal control is known to be bang-bang the new control \( u \) is also assumed bang-bang. The perturbation \( \delta u \) is then defined by the equation \( u = \bar{u} + \delta u \). The diagrams 4.1 through 4.5 show that the perturbations in the control are essentially changes, positive or negative in the switching interval lengths and that \( \delta u(t) \) exists only around the switching times.
Lemma: The equation describing $\delta u$ when $u$ is restricted by $u_{\min} < u < u_{\max}$ is

$$
\delta u_{\hat{T}_i} = (u_{\max} - u_{\min}) \cdot \text{sgn} \left[ u(T_i^-) - u(T_i^+) \right] \text{sgn} \left[ \sum_{j=1}^{i} \epsilon_j \right]
$$

where:

a) $T_i = \sum_{k=1}^{i} T_k$ is the $i^{th}$ switching time,

b) $\epsilon_j$ is the amount of change in the switching interval $T_j$.

c) $u(T_i^-), u(T_i^+)$ are the control $u$ evaluated to the left and right of the switching instant $T_i$ respectively.

Proof: Consider first the case illustrated by diagrams 4.1 and 4.2. Here the switching interval $T_1$ is to be lengthened by the amount $\epsilon_1$. This means that switching time $T_1$ will be changed by an amount $\epsilon_1$. The control in switching interval $T_1$ is $u_{\max}$ and in switching interval $T_2, u_{\min}$. In the subinterval $[T_1, T_1 + \epsilon_1]$ the difference between the new control, $u_{\max}$, and the old one $u_{\min}$ is $(u_{\max} - u_{\min})$.

Had the interval $T_1$ been shortened by $\epsilon_1$ the new control in the subinterval $[T_1 - \epsilon_1, T_1]$ would have been $u_{\min}$ and the difference between the new and old controls $(u_{\min} - u_{\max})$ or $-(u_{\max} - u_{\min})$. Thus for the case just considered, that is switching time $T_1$ changed by an amount $\pm \epsilon_1$, $\delta u(T_1) = (u_{\max} - u_{\min}) \text{sgn} \epsilon_1$. A similar argument holds in the case where the control in the first interval is $u_{\min}$. The change $\delta u(T_1)$ is then given by $(u_{\min} - u_{\max}) \text{sgn} (\epsilon_1)$. These two results may be combined into one equation thus:

$$
\delta u(T_1) = (u_{\max} - u_{\min}) \text{sgn}[u(T_1^-) - u(T_1^+)] \text{sgn} (\epsilon_1).
$$

The diagrams 4.1, 4.2 and 4.3 also show that a change in the length of switching interval $T_1$ will affect all subsequent switching times $T_2, T_3$ etc. The argument just given for evaluating $\delta u(T_1)$ when $T_1$ has been shifted by an amount $\epsilon_1$ can
Fig. 4.1 Nominal Control

Fig. 4.2 New Control with Switching Interval $T_1$ Perturbed

Fig. 4.3 The Change in the Control
Fig. 4.4 New Control with all Switching Intervals Perturbed

Fig. 4.5 The Change in the Control of Fig. 4.1 to Obtain the Control of Fig. 4.4
be applied in these cases. Therefore

\[ \delta u(T_1) = (u_{\max} - u_{\min}) \text{sgn} [u(T_1^-) - u(T_1^+)] \text{sgn} (\varepsilon_1) . \]

A change of \( \varepsilon_2 \) in the length of switching interval \( T_2 \) will not affect switching time \( \hat{T}_1 \), but it will affect switching times \( \hat{T}_2, \hat{T}_3 \), etc. (see fig. 4.4 and 4.5). If these switching times have already been shifted by \( \varepsilon_1 \) they are now shifted by \( \varepsilon_1 + \varepsilon_2 \).

Since \( \varepsilon_2 \) need not be of the same sign or magnitude as \( \varepsilon_1 \) the sign of the total change, \( \varepsilon_1 + \varepsilon_2 \), must be considered. Otherwise this case, and the cases when the other switching intervals are adjusted, are the same as when only one switching interval is changed. Thus the change in the control \( \delta u \), at switching time \( \hat{T}_1 \) is given by

\[ \delta u(\hat{T}_1) = (u_{\max} - u_{\min}) \text{sgn} [u(\hat{T}_1^-) - u(\hat{T}_1^+)] \text{sgn} \left[ \frac{\varepsilon_1}{\varepsilon_2} \right] \text{sgn} \left[ \frac{\varepsilon_1}{\varepsilon_2} \right] \text{QED} \]

In most cases \( u_{\max} > 0 \) and \( u_{\min} < 0 \) and the above equation simplifies to

\[ \delta u(\hat{T}_1) = (u_{\max} - u_{\min}) \text{sgn} [u(\hat{T}_1^-)] \text{sgn} \left[ \frac{\varepsilon_1}{\varepsilon_2} \right] . \]

If \( u_{\min} = -1 \) and \( u_{\max} = +1 \)

\[ \delta u(\hat{T}_1) = (2) \text{sgn} [u(\hat{T}_1^-)] \text{sgn} \left[ \frac{\varepsilon_1}{\varepsilon_2} \right] . \]

4.2 The penalty function approach

When trying to drive the system 4.1 from the initial state \( x(t_o) = x_o \) to the final state \( x(t_f) = 0 \), it is often convenient to try to maximize a penalty function (21)

\[ J = -K \sum_{i=1}^{n} x_i^2 (t_f) \] where \( t_f = \bar{t}_f + \delta t_f \)

For this choice of performance index \( \Delta J \) becomes, using eq. 4.6,

\[ \Delta J = -2K \sum_{i=1}^{n} x_i(\bar{t}_f) \delta x_i(\bar{t}_f) + [-2Kx^T(\bar{t}_f) f(x(\cdot), \bar{t}_f) + 0] \delta t_f \quad (4.7) \]
In order to make use of equation 4.5 to relate $\delta u$ to $\delta x(t_f)$ it is necessary to evaluate the integral

$$\int_{t_0}^{t_f} H_u(x,y,u,t) \delta u(t) \, dt$$

Because of the special form of $\delta u$, a) $t_f$ is not in general the same as $\hat{t}_f$ and b) the integral from $t_0$ to $t_f$ can be broken up into the sum of several integrals around the switching instants. Let $\int_{T_1} \hat{t}_i \, Adt$ denote

$$\left\{ \begin{array}{ll}
\int_{T_1} \hat{t}_i + \varepsilon_j = 1 \varepsilon_j & \text{if } \sum_{j=1}^i \varepsilon_j \text{ is positive or zero} \\
\int_{T_1} \hat{t}_i - \varepsilon_j = 1 \varepsilon_j & \text{if } \sum_{j=1}^i \varepsilon_j \text{ is negative} \\
\end{array} \right.$$

Then

$$\int_{t_0}^{t_f} H_u(x,y,p,u,t) = \int_{T_1} H_u(x,y,p,u,t) \left[u_{\max} - u_{\min}\right] \text{sgn}[u(T_1)] \cdot \text{sgn}[\varepsilon_1] \, dt$$

$$+ \int_{T_2} H_u(x,y,p,u,t) \left[u_{\max} - u_{\min}\right] \text{sgn}[u(T_2)] \cdot \text{sgn}[\varepsilon_1 + \varepsilon_2] \, dt + \ldots$$

$$\ldots + \int_{T_{nsw-1}} H_u(x,y,p,u,t) \left[u_{\max} - u_{\min}\right] \text{sgn}[u(T_{nsw-1})] \cdot \text{sgn}[\sum_{j=1}^{nsw-1} \varepsilon_j] \, dt$$

where $nsw$ is the number of switching intervals.

Because $\hat{t}_f \neq t_f$,

$$\delta x(t_f) = \delta x(\hat{t}_f) + \dot{x}(\hat{t}_f) \delta t_f + O(\delta t_f^2) = \delta x(\hat{t}_f) + \dot{x}(\hat{t}_f) \delta t_f$$

To satisfy the linearization requirements let the $\varepsilon_i$'s be restricted by $\sum_{j=1}^{nsw} \varepsilon_j^2 \leq \Delta^2$ where $\Delta$ is chosen small enough so that terms of order $(\varepsilon_i^2)$ are negligible with respect to terms of order $(\varepsilon_i)$. To evaluate the integrals consider first the case when $\sum_{j=1}^i \varepsilon_j \geq 0$. Then by definition

$$\int_{T_1} \hat{t}_i \, Adt$$
\[ \int_{\hat{T}_1}^{\hat{T}_1 + \sum_{j=1}^{i} \epsilon_j} \text{Hu}(x,y,p,u,t) \, 2 \ \text{sgn} \left[ u(\hat{T}_1^-) \right] (\pm 1) \, dt \]

If \( \Delta \) is small and \( \sum_{j=1}^{n} \epsilon_j^2 \leq \Delta^2 \) then \( \sum_{j=1}^{i} \epsilon_j \) is probably small, for no \( \epsilon_j \) can exceed \( \Delta \) in magnitude and all the \( \epsilon_j \)'s need not be of the same sign, then the above integral can be approximated, using the mean value theorem, by

\[ \int_{\hat{T}_1}^{\hat{T}_1 + \sum_{j=1}^{i} \epsilon_j} \text{Hu}(x,y,p,u,t) \, 2 \ \text{sgn} \left[ u(\hat{T}_1^-) \right] \, dt \]

\[ = \text{Hu}(x,y,p,u,\hat{T}_1) \, 2 \ \text{sgn} \left[ u(\hat{T}_1^-) \right] \cdot \left[ \hat{T}_1 + \sum_{j=1}^{i} \epsilon_j - \hat{T}_1 \right] \]

\[ = \text{Hu}(x,y,p,u,\hat{T}_1) \, 2 \ \text{sgn} \left[ u(\hat{T}_1^-) \right] \cdot \left[ \sum_{j=1}^{i} \epsilon_j \right] \]

If on the other hand \( \sum_{j=1}^{i} \epsilon_j \) is negative the integral

\[ \int_{\hat{T}_1}^{\hat{T}_1 - \sum_{j=1}^{i} \epsilon_j} \text{Hu}(x,y,p,u,t) \, 2 \ \text{sgn} \left[ u(\hat{T}_1^-) \right] [-1] \, dt \]

Again if \( \sum_{j=1}^{i} \epsilon_j \) is small enough the integral can be approximated by

\[ \text{Hu}(x,y,p,u,\hat{T}_1) \, 2 \ \text{sgn} \left[ u(\hat{T}_1^-) \right] \left[ \hat{T}_1 - \hat{T}_1 + \sum_{j=1}^{i} \epsilon_j \right] \]

\[ = \text{Hu}(x,y,p,u,\hat{T}_1) \, 2 \ \text{sgn} \left[ u(\hat{T}_1^-) \right] \left[ - \sum_{j=1}^{i} \epsilon_j \right] \]

since \( \sum_{j=1}^{i} \epsilon_j < 0 \)

\[ = \frac{1}{\sum_{j=1}^{i} \epsilon_j} = \sum_{j=1}^{i} \epsilon_j. \] Therefore, in either case

\[ \int_{\hat{T}_1}^{\hat{T}_1} \text{Hu}(x,y,p,u,t) \, 2 \text{sgn}[u(\hat{T}_1^-)] \, \text{sgn} \left[ \sum_{j=1}^{i} \epsilon_j \right] \, dt \]

\[ = \text{Hu}(x,y,p,u,\hat{T}_1) \, 2 \text{sgn}[u(\hat{T}_1^-)] \left[ \sum_{j=1}^{i} \epsilon_j \right] \] (4.8)

Define now for convenience

\[ W(\hat{T}_1) = \text{Hu}(x,y,p,u,\hat{T}_1) \, 2 \text{sgn}[u(\hat{T}_1^-)] \] (4.9)
Then using equation 4.6, 4.8 and 4.9

\[ p^T(t_f) \delta x(t_f) = W(T_1) \varepsilon_1 + W(T_2) [\varepsilon_1 + \varepsilon_2] + \ldots + W(T_{nsw-1}) [\varepsilon_1 + \ldots + \varepsilon_{nsw-1}] \]

and if \( p^T(t_f) \) is chosen as 

\[ -2K x(t_f) = \frac{\partial J}{\partial x_f} \]

then

\[ \Delta J = W(T_1) \varepsilon_1 + W(T_2) [\varepsilon_1 + \varepsilon_2] + \ldots + W(T_{nsw-1}) [\varepsilon_1 + \ldots + \varepsilon_{nsw-1}] \]

\[ + \left[ \frac{\partial J}{\partial x_f} x(t_f) + \frac{\partial J}{\partial t_f} \right] [\varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_{nsw}] \quad (4.10) \]

Remark: Note that \( \varepsilon_1 = \delta T_1, \varepsilon_1 + \varepsilon_2 = \delta T_2 \) etc.

Equation 4.10 can then be rewritten as

\[ \Delta J = W(T_1) \delta T_1 + W(T_2) \delta T_2 + \ldots + W(T_{nsw-1}) \delta T_{nsw-1} \]

\[ + \left[ \frac{\partial J}{\partial x_f} x(t_f) + \frac{\partial J}{\partial t_f} \right] \delta t_f . \]

Setting \( \delta t_f = \delta T_j = 0 \quad \forall j \neq i \) yields

\[ \Delta J = W(T_i) \delta T_i . \]

Dividing both sides by \( \delta T_i \) and taking the limit as \( \delta T_i \) tends to zero yields

\[ \frac{\partial J}{\partial T_i} = W(T_i) = H(x,y,p,u,T_i) \cdot [u_{\max} - u_{\min}] \sgn [u(T_i^-) - u(T_i^+)] \]

and the term on the right hand side is the gradient of the performance index with respect to switching instant \( \hat{T}_i \).

By adjoining the constraint \( \varepsilon_1^2 + \varepsilon_2^2 + \ldots + \varepsilon_{nsw}^2 = \Delta^2 \) to eq. 4.10, one obtains an augmented function

\[ \Delta J_A = \varepsilon_1 (W(T_1) + W(T_2) + \ldots + W(T_{nsw-1}) + \frac{\partial J}{\partial x_f} x(t_f) + \frac{\partial J}{\partial t_f} ) + \eta \varepsilon_1^2 \]
In the above \( \eta \) is a scalar Lagrange multiplier. In order to choose the \( \epsilon_i \) which maximize the increase \( \Delta J \) in the performance index subject to the constraint \( \sum \epsilon_i^2 = \Delta^2 \) use is made of the following equations

\[
\frac{\partial \Delta J}{\partial \epsilon_{\text{nsw}}} = 0 \text{ implies that } \left[ \frac{\partial J}{\partial x_f} \dot{x}(t_f) + \frac{\partial J}{\partial t_f} \right] + 2 \eta \epsilon_{\text{nsw}} = 0 \quad (4.12)
\]

\[
\frac{\partial \Delta J}{\partial \epsilon_{\text{nsw}-1}} = 0 \text{ implies that } W(T_{\text{nsw}-1}) + \left[ \frac{\partial J}{\partial x_f} \dot{x}(t_f) + \frac{\partial J}{\partial t_f} \right] + 2 \eta \epsilon_{\text{nsw}-1} = 0
\]

or making use of 4.12

\[
W(T_{\text{nsw}-1}) - 2 \eta \epsilon_{\text{nsw}} + 2 \eta \epsilon_{\text{nsw}-1} = 0 \quad ,
\]

In general, setting the partial \( \frac{\partial \Delta J}{\partial \epsilon_i} = 0 \) yields

\[
\left[ \frac{\partial J}{\partial x_f} \dot{x}(t_f) + \frac{\partial J}{\partial t_f} \right] + 2 \eta \epsilon_{\text{nsw}} = 0 \quad i = \text{nsw}
\]

\[
W(T_i) - 2 \eta \epsilon_{i+1} + 2 \eta \epsilon_i = 0 \quad i = 1, 2, \text{ to nsw-1} \quad (4.14)
\]

The required \( \epsilon \)'s can be found from the following recursive relations

\[
\epsilon_{\text{nsw}} = \frac{1}{2 \eta} \left[ \frac{\partial J}{\partial x_f} \dot{x}(t_f) + \frac{\partial J}{\partial t_f} \right]
\]

\[
\epsilon_i = \epsilon_{i+1} - \frac{1}{2 \eta} W(T_i) \quad i=1,2, \text{ to nsw-1} \quad (4.15)
\]

In order to solve for the Lagrange multiplier, \( \eta \), observe that 4.15 gives the \( \epsilon \)'s in the form \( \epsilon_i = -\frac{1}{2 \eta} B_i \).
Thus
\[ \sum_{j=1}^{nsw} \epsilon_j^2 = \frac{1}{\Delta t^2} \left[ \sum_{i=1}^{\Delta t} B_i^2 + \ldots + B_{nsw}^2 \right] = \Delta^2 \]

Then
\[ \frac{1}{2\eta} = \sigma = \pm \sqrt{\frac{\Delta^2}{B_1^2 + \ldots + B_{nsw}^2}} \] (4.16)

Since \( J \) is always negative and is to be maximized the sign of \( \sigma \) must be chosen to make \( \Delta J_A \) positive.

\[ \Delta J_A = B_1 \epsilon_1 + 2\eta \epsilon_1^2 + B_2 \epsilon_2 + 2\eta \epsilon_2^2 + \ldots + B_{nsw} \epsilon_{nsw} + 2\eta \epsilon_{nsw}^2 \]

then by the Schwarz inequality to maximize \( \Delta J_A \) the sign of the \( \epsilon_i \) should be \(-\text{sgn}[B_i] \). Thus \( \sigma \) should be chosen with a plus sign in 4.16.

4.3 Algorithm description

A short description of how the above analysis can be applied as a numerical algorithm follows. Several examples are then given to illustrate its use. The algorithm consists of 9 steps:

1. Assume an initial point in the switching interval space.
2. Solve the state equations forward in time.
3. At the final time compute \( J, \frac{\partial J}{\partial x_f} \).
4. Solve the costate equations backwards in time with the correct terminal conditions.
5. While doing step 4 calculate and store \( W(T_i) \) at various points along the trajectory.
6. Solve the recursive relations 4.15 for the \( \epsilon \)'s and equation 4.16 for the Lagrange multiplier.
7. Update the switching intervals.
8. Check to see if the stopping condition is satisfied.
9. If it is not satisfied repeat steps 2 to 8. If it is satisfied print out the results and stop.

4.4 Comments on Computational aspects

Two suitable stopping conditions in step 8 are, either the penalty function is close to zero or the stepsize $\Delta^2$ has become small. While doing step 3 the newest value of the performance index is compared to the best one obtained to date. If it is greater than the previous best, the step size $\Delta^2$ is multiplied by two; if it is less than the older maximum, the trial is termed a failure, the newest $T_i$'s discarded in favour of the best ones to date and $\Delta^2$ is divided by 10. The adjustments of $\Delta^2$ ensure that the step size does not stay small when large steps could be taken and thus speeds up convergence.

In contrast to the usual descent in function space, for linear systems information need only be stored at the switching times, not all along the trajectory. For non-linear systems only the states required to solve the costate equations need be stored. Otherwise storage space requirements increase only linearly with system order. The computing time per iteration is largely determined by the time it takes to integrate the state and costate equations.

An area of difficulty is the function $r(t)$ in the costate equations. A generalized method of finding $r(t)$ in closed form does not exist. In extreme cases it may be necessary to find an expression for $r(t)$ by curve fitting or to store it as a series of points.

4.5 Examples and Results

Example I

The numerical procedure outlined above was first applied to the system described in chapter III. That is
\[
\begin{align*}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= -x_1(t - \Theta(t)) + u(t) \text{ with initial data} \\
&\quad \text{with initial data} \\
x_1(s) &= 2.0 + 2.0s \quad s < 0 \\
x_2(s) &= 2.0 
\end{align*}
\]

The results for several different time delays are summarized in the table below and in fig. 4.7.

<table>
<thead>
<tr>
<th>Delay</th>
<th>distance to origin at final time</th>
<th>Trials</th>
<th>minimum time</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\Theta(t) = 0.3)</td>
<td>(1.7 \times 10^{-5})</td>
<td>300</td>
<td>7.82</td>
</tr>
<tr>
<td></td>
<td>(2.9 \times 10^{-5})</td>
<td>275</td>
<td></td>
</tr>
<tr>
<td>(\Theta(t) = 0.3 - t/100)</td>
<td>(2.1 \times 10^{-3})</td>
<td>425</td>
<td>6.684</td>
</tr>
<tr>
<td></td>
<td>(1.7 \times 10^{-3})</td>
<td>400</td>
<td></td>
</tr>
</tbody>
</table>

These figures are in close agreement with the results of Rosenbrock's method in chapter III. They also illustrate the drawback of any technique based on small scale linearization namely that convergence may be limited by linearity requirements. From the very first, however, the switching intervals are adjusted in an optimum manner and initial convergence is better than that of the method of chapter III. For example the gradient method reduces the square of the miss distance to 0.05 from 1.7 in only 10 trials while the Rosenbrock scheme required 18.

A large order system

Next the same 7th order system considered in chapter III was used. In this case the state equations are given by

\[
\begin{align*}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= x_3(t - \Theta(t)) \\
\dot{x}_3(t) &= x_4(t - \Theta(t)) \\
\dot{x}_4(t) &= x_5(t) \\
\dot{x}_5(t) &= x_6(t)
\end{align*}
\]
Fig. 4.6 Convergence of Gradient Method for a Second Order System with a delay of 0.3
\[ \dot{x}_6(t) = x_7(t) \]
\[ \dot{x}_7(t) = -720 x_2(t) - 1764 x_3(t) - 1624 x_4(t) - 735 x_5(t) - 175 x_6(t) - 21 x_7(t) + u(t). \]

The initial functions were all zero and the initial condition vector \( x_0 \) was \( (0, -10, 0, 0, 0, 0, 0) \). The control \( u \) is restricted by \( |u| < 1000 \). The results for 2 different time delays are given below.

<table>
<thead>
<tr>
<th>Delay</th>
<th>Distance to origin at final time</th>
<th>Trials</th>
<th>Minimum time</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ \theta(t) = 0.2 ]</td>
<td>1.4</td>
<td>28</td>
<td>36.1</td>
</tr>
<tr>
<td>[ \theta(t) = \frac{0.5}{t+1} ]</td>
<td>1.76</td>
<td>35</td>
<td>35.4</td>
</tr>
</tbody>
</table>

As an example of the effort required to obtain the inverse function \( r(t) \) consider the case when

\[ \theta(t) = \frac{a}{t+b}, \quad a > 0, \quad b > 0 \]

w(t) is therefore \( t - \frac{a}{t+b} \).

Since \( w(r(t)) = t \) then

\[ r(t) - \frac{a}{r(t)+b} = t \text{ or} \]

\[ [r(t)]^2 + r(t) [b-t] - [a+bt] = 0 \]

and

\[ r(t) = -\left\{ \frac{b-t}{2} \right\} \pm \sqrt{\frac{[b-t]^2}{4} + [a+bt]} \]

(4.19)

Since \( r(t) \geq t \), \( t_0 \leq t \leq t_f \). Then if \( t_0 = 0 \)

\[ r(0) = -\frac{b}{2} + \sqrt{\frac{b^2}{4} + a} \text{ must be > 0.} \]

Since both \( a, b > 0 \) the plus sign should be chosen in 4.19.

If \( \theta(t) = ae^{-t} \) then \( r(t) \) is a solution to \( r(t) - ae^{-r(t)} = t \). This equation has no closed form solution for \( r(t) \).
Fig. 4.7 Convergence of Gradient Method for a Seventh Order System with a Delay of 0.2
4.6 Conclusions

A comparison between the results of this chapter and those of chapter III for the large order system shows the marked superiority of the gradient method in the initial stages. (Compare figs. 3.2 and 4.7). For the constant delay system the gradient method took only 29 trials to reduce the miss distance from 1760 to 1.4 while the hill climbing scheme took 70 when started from the same point. Close to the optimum the Rosenbrock scheme was slightly superior to the gradient method for large order systems and markedly superior for the low order cases. For the example I with constant delay to reduce the miss distance from $10^{-5}$ to $10^{-8}$ the Rosenbrock method required only 9 trials while the gradient approach took 25 to reduce the miss distance from $3 \times 10^{-5}$ to $1.7 \times 10^{-5}$. 
V. DESCENT TO FINAL TARGET SET

Bryson and Denham (19) have derived a gradient method of descent in function space in which the steps are taken to reach the final target set instead of minimizing a penalty function. In this chapter it is shown that the descent in switching interval space can also be made so that terminal conditions are accounted for directly.

Once again the state vector $x(t)$ must be brought to the origin, $x(t_f) = 0$, from the initial point $x(t_o) = x_o$ while minimizing the performance index

$$ J(t_f) = \int_{t_o}^{t_f} dt = t_f - t_o. $$

Since the assumptions of chapter II, sec. 2.1 still hold the optimal control will again be bang-bang. When the state vector must be driven to the origin a suitable choice of terminal conditions is

$$ \psi(x_f) = 0 $$

where

$$ \psi_1(x_f) \overset{\Delta}{=} x_1(t_f) = 0 $$

$$ \vdots $$

$$ \psi_n(x_f) \overset{\Delta}{=} x_n(t_f) = 0. $$

5.1 Perturbation analysis

If the nominal control $u$ is changed by $\delta u$ so that

$$ \hat{u} = u + \delta u $$

$$ \hat{t}_f = t_f + \delta t_f $$

$$ \hat{x}(t_f) = x(t_f) + \delta x(t_f) $$

$$ \hat{x}(t_f) = x(t_f) + \delta x(t_f) + \dot{x}(t_f) \delta t_f $$

$$ \hat{J}(\hat{x}, \hat{t}_f) = J(t_f) + \frac{\partial J}{\partial x} \delta x_f + \dot{J} \delta t_f $$

$$ \hat{\psi}(\hat{t}_f) = \delta \psi(t_f) + \dot{\psi}(t_f) \delta t_f $$
In general $\delta u(t)$ must be chosen so that a) $J$ is decreased and b) the terminal functions $\psi_i(x_f, t_f)$ change by some specified amount $\beta_i$. As before (chapter IV, sec. 4.1) the change $\delta x(t_f) = x(t_f) - x(t_f)$ is related to the change in control by the equation 4.5

$$p(t_f) \delta x(t_f) = \int_{t_0}^{t_f} H_u(x,y,p,u,t) \delta u(t) \, dt \quad (4.5)$$

Adopting the admissible $\delta u$ of chapter IV, sec. 4.1, equation 4.5 becomes

$$p(t_f)\delta x(t_f) = H_u(x,y,p,u,T_1) \cdot (u_{\max} - u_{\min}) \cdot \text{sgn}[u(T_1^-) - u(T_1^+)] \varepsilon_1 + \ldots + H_u(x,y,p,u,T_{nsw-1}) \cdot (u_{\max} - u_{\min}) \cdot \text{sgn}[u(T_{nsw-1}^-) - u(T_{nsw-1}^+)] \cdot [\varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_{nsw-1}]$$

(5.1)

where all the terms used above have been defined in chapter IV. In order to determine $\delta x_j(t_f), p(t_f)$ is chosen so that

$$p_i(t_f) = 0 \quad i \neq j$$

$$p_j(t_f) = 1$$

Let $p(t)$ denote the solution to 4.4 with the above initial conditions, then

$$\delta x_i(t_f) = \sum_{k=1}^{nsw-1} H_u(x,y,i,p,u,T_k) \cdot (u_{\max} - u_{\min}) \cdot \text{sgn}[u(T_k^-) - u(T_k^+)] \sum_{j=1}^{k} \varepsilon_j + \dot{x}_i(t_f) \cdot [\varepsilon_1 + \ldots + \varepsilon_{nsw}]$$

Define

$$H_u(x,y,i,p,u,T_k) \cdot (u_{\max} - u_{\min}) \cdot \text{sgn}[u(T_k^-) - u(T_k^+)] = C_{iK}$$

Then

$$\delta x_i = C_{i1} \varepsilon_1 + C_{i2} (\varepsilon_1 + \varepsilon_2) + \ldots + C_{insw-1} (\varepsilon_1 + \ldots + \varepsilon_{nsw-1}) + \dot{x}_i(t_f) \cdot [\varepsilon_1 + \ldots + \varepsilon_{nsw}]$$
Rearranging terms

\[ \delta x_i(t_f) = \varepsilon_1 [C_{i1} + C_{i2} + \ldots + C_{i\text{nsw-1}} + \dot{x}_i(t_f)] \]

\[ + \varepsilon_2 [C_{i2} + C_{i3} + \ldots + C_{i\text{nsw-1}} + \dot{x}_i(t_f)] \]

\[ \vdots \]

\[ + \varepsilon_{\text{nsw}} [\dot{x}_i(t_f)] \]

Let \( G_{\text{insw}} = \dot{x}_i(t_f) \) \( i = 1 \) to \( n \)

and \( G_{ij} = C_{ij+1} + C_{ij} \) \( i = 1 \) to \( n \)

\( j = 1, \text{nsw-1} \)

Then \( \delta x_i(t_f) = G_{i1} \varepsilon_1 + G_{i2} \varepsilon_2 + \ldots + G_{i\text{nsw}} \varepsilon_{\text{nsw}} \) \( (5.2) \)

or in matrix notation

\[ \delta x = G \cdot \varepsilon \] \( (5.3) \)

In order to satisfy the terminal constraints a certain change \( \beta_i \) is specified for each \( \delta x_i \). Equation \( 5.3 \) can then be written

\[ G \cdot \varepsilon = \beta \] \( (5.4) \)

The change in performance index \( \delta J \) is given by

\[ \delta J = \delta t_f = \sum_{i=1}^{\text{nsw}} \varepsilon_i \] \( (5.5) \)

where \( I_c \) is the unit column vector \([1 \ 1 \ 1 \ \ldots \ 1]^T \).

The \( \varepsilon \)'s are restricted by

\[ \varepsilon^T W \varepsilon = \Delta^2 \] \( (5.6) \)

in order for equation 4.1 to hold. To find the \( \varepsilon \)'s which maximize \( \delta J \) subject to 5.4 and 5.6 consider the augmented change in performance index

\[ \delta J_A = \varepsilon^T I_c + \nu^T [G \varepsilon - \beta] + \eta (\varepsilon^T W \varepsilon - \Delta^2) \]

\[ = \varepsilon^T I_c + \varepsilon^T G^T \nu - \nu^T \beta + \eta (\varepsilon^T W \varepsilon - \Delta^2) \]

where \( \eta \) is a scalar and \( \nu \) is an \( n \times 1 \) vector.
Setting the differential \( \frac{\partial \delta J}{\partial \varepsilon} \) equal to zero yields

\[
I_c + G^T\nu + 2\eta W \varepsilon = 0
\]

Solving for \( \varepsilon \) gives

\[
\varepsilon = -\frac{1}{2\eta} [I_c + G^T\nu] \tag{5.7}
\]

By substituting 5.7 into 5.4 one obtains

\[
\beta = -\frac{1}{2\eta} GW^{-1}[I_c + G^T\nu].
\]

Isolating terms in \( \nu \) on the left hand side of the equation gives

\[
\frac{1}{2\eta} (GW^{-1}G^T)\nu = -[\beta + \frac{1}{2\eta} GW^{-1} I_c]
\]

Define \( Z = GW^{-1}G^T \), the \( Z \) has dimension \( n \times n \) and therefore

\[
\nu = -Z^{-1}[2\eta \beta + GW^{-1}I_c]. \tag{5.8}
\]

Solving equation 5.7 for \( \varepsilon \) using 5.8 yields

\[
\varepsilon = -\frac{1}{2\eta} W^{-1}[I_c + G^T\{-Z^{-1}(2\eta \beta + GW^{-1}I_c)\}]
\]

\[
\varepsilon = -\frac{1}{2\eta} [I - W^{-1}GZ^{-1}G]W^{-1}I_c + W^{-1}GZ^{-1} \beta.
\]

For brevity let

\[
R_G = \{I - W^{-1}GZ^{-1}G\}W^{-1}I_c
\]

and

\[
R_B = W^{-1}GZ^{-1}
\]

Then

\[
\varepsilon = -\frac{1}{2\eta} R_G + R_B \beta.
\]

Note that if \( G \) is square then \( W^{-1}G^T Z^{-1}G = W^{-1}G^T \{(G^T)^{-1}WG^{-1}\}G = I \) and hence

\[
\varepsilon = R_B \beta \tag{5.9}
\]

Next eq. 5.9 is substituted back into 5.6 to solve for \( \eta \). \( \varepsilon^T W \varepsilon = \Delta^2 \) becomes

\[
(-\frac{1}{2\eta} R_G^T + \beta^TR_B^T) W (\frac{1}{2\eta} R_G + R_B \beta) = \Delta^2
\]
\[ \Delta^2 = \frac{1}{4\eta^2} R_G^T W R_G - \frac{1}{2\eta} R_G^T W R_B \beta \frac{-1}{2\eta} \beta^T R_B^T W R_G + \beta^T R_B^T W R_B \beta \] 

(5.10)

Let \( \frac{1}{2\eta} = \sigma \) and observe that

\[ R_G^T W R_B \beta = \beta^T R_B^T W R_G = b \quad \text{a scalar} \]
\[ R_G^T W R_G = a \quad \text{a scalar} \]
\[ \beta^T R_B^T R_B \beta - \Delta^2 = c \quad \text{a scalar} \]

Equation 5.10 then becomes

\[ a\sigma^2 - 2b\sigma + c = 0 \]

Therefore

\[ \sigma = \frac{b \pm \sqrt{b^2 - ac}}{a} \] 

(5.11)

In order to minimize \( \delta J_A \) with respect to \( \varepsilon \) the second derivative

\[ \left( \frac{\delta^2 J_A}{\delta \varepsilon^2} \right) = 2\eta W^T \] should be > 0.

Therefore in equation 5.11 the plus sign is chosen for minima, the minus sign for maxima. Note also that in order for the step size constraint \( \sigma \) to be real, \( ac < b^2 \). If the \( \beta \)'s are chosen small enough in relation to the \( \Delta^2 \) then

\[ c = \beta^T R_B^T R_B \beta - \Delta^2 \]
is less than zero and \( \sigma \) is always real. The \( W \) is a positive definite weighting matrix which may be adjusted, by trial and error if necessary, to improve the convergence rate of the algorithm.

5.2 Algorithm description

The computing algorithm may be described by the 9 following steps:

1. Assume a control and solve the state equations forward.
2. Solve the costate equations backwards \( n \) times storing the necessary information at the switching points.
3. Choose values for \( \beta_1 \) and \( \Delta^2 \).

4. Form the G matrix.

5. Find the inverse of \( Z = G W^{-1} G^T \).

6. Form the matrices \( R_G, R_B \).

7. Calculate the step size constraint \( \sigma \).

8. Calculate the changes in the switching intervals.

9. Check to see if the stopping condition is satisfied. If it is not, repeat steps one to eight.

**Comments**

The \( \beta \)'s are usually chosen to be some percentage of the final states. The step size constraint \( (\Delta^2) \) is initially chosen arbitrarily and then adjusted during computation in a manner similar to that outlined in chapter IV. The procedure may be stopped therefore when \( \Delta^2 \) becomes so small that errors in the simulation become the limiting factors.

5.3 A detailed example

Consider the system

\[
\begin{align*}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= -x_1(t) + u(t) \\
x_1(0) &= -0 \\
x_2(0) &= -2
\end{align*}
\]  

(5.12)

with \( u(t) = \)

\[
\begin{align*}
-1 & \quad 0 < t < T_1 \\
+1 & \quad T_1 < t < T_2 \\
-1 & \quad T_2 < t < T_3
\end{align*}
\]

The solution to 5.12 is given by

\[
\begin{align*}
x_1(t_f) &= -2\sin t_f - 2 \cos(t_f - T_1) + 2 \cos(t_f - T_2) - \cos(0) + \cos t_f \\
x_2(t_f) &= -2 \cos t_f + 2 \sin(t_f - T_1) - 2 \sin(t_f - T_2) - \sin t_f
\end{align*}
\]  

(5.13)

for \( t_f > T_2 \).
Let the times $T_1$, $T_2$, $T_3$ become $T_1 + \delta T_1$, $T_2 + \delta T_2$, $T_3 + \delta T_3$. Then the changes in the switching intervals become $\epsilon_1 = \delta T_1$, $\epsilon_2 = \delta T_2 - \delta T_1$, $\epsilon_3 = \delta T_3 - \delta T_2$. Then expanding $x_1(t_f)$, $x_2(t_f)$ by Taylor's series one obtains

$$\delta x_1(T_3) = -2 \cos(T_3) \delta T_3 + 2 \sin(T_3 - T_1) [\delta T_3 - \delta T_1] - \sin(T_3) \delta T_3$$

$$-2 \sin[T_3 - T_2] [\delta T_3 - \delta T_2] + O(\delta^2)$$

$$\delta x_2(T_3) = 2 \sin(T_3) \delta T_3 + 2 \cos(T_3 - T_1) [\delta T_3 - \delta T_1] - \cos T_3 \delta T_3$$

$$-2 \cos[T_3 - T_2] [\delta T_3 - \delta T_2] + O(\delta^2)$$

where $O(\delta^2)$ represents all the terms of order $\delta^2$ or higher.

For convenience let

$T_3 = \frac{3\pi}{2}$

$T_2 = \pi$

$T_1 = \frac{\pi}{2}$

$x_1(T_3) = +3$ \hspace{1cm} $x_2(T_3) = -1$

$x_1'(T_3) = -1$ \hspace{1cm} $x_2'(T_3) = -4$

The exact changes in the states at the final time are, therefore,

$$\delta x_1'(T_3) = -\delta T_3 + 2\delta T_2 + O(\delta^2)$$

$$\delta x_2'(T_3) = -4\delta T_3 + 2\delta T_1 + O(\delta^2)$$

The costate equations are given by

$$\dot{p}_1(t) = p_2(t)$$

$$\dot{p}_2(t) = -p_1(t)$$

so that

$$\begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix} = \begin{bmatrix} \cos(t-T_3) & \sin(t-T_3) \\ -\sin(t-T_3) & \cos(t-T_3) \end{bmatrix} \begin{bmatrix} p_1(T_3) \\ p_2(T_3) \end{bmatrix}$$

To find $\delta x_1(T_3)$ put $p_1(T_3) = 1$

$p_2(T_3) = 0$
\[ H_u(x, 1_p, u, ^\hat{T}_K) = \frac{1}{p_2(T_K)} \]

\[ \frac{1}{p_2(T_1)} = -\sin\left(\frac{\pi}{2} - \frac{3\pi}{2}\right) = 0 \]

\[ \frac{1}{p_2(T_2)} = -\sin(\pi - \frac{3\pi}{2}) = +1 \]

\[ C_{12} = H_u(x, 1_p, u, ^\hat{T}_2) \left( u_{\text{max}} - u_{\text{min}} \right) \text{ sgn}\left[ u(T_2^-) - u(T_2^+) \right] = 2\]

\[ C_{11} = 0 \]

\[ C_{13} = -1 \]

\[ C_{12} = +1 \]

\[ C_{11} = +1 \]

One can find similarly that \[ G_{23} = -4 \]

\[ G_{22} = -4 \]

\[ G_{21} = -2 \]

Evaluating \( \delta x_1 \) and \( \delta x_2 \) by means of equation 5.3

\[ \delta x_1 = C_{11} e_1 + C_{12} e_2 + C_{13} e_3 \quad (5.3) \]

Yields

\[ \delta x_1 = e_1 + e_2 - e_3 \]

\[ \delta x_2 = -2e_1 - 4e_2 - 4e_3 \quad (5.15) \]

The change in switching intervals in terms of the switching times are

\[ e_1 = \delta T_1 \]

\[ e_2 = \delta T_2 - e_1 = \delta T_2 - \delta T_1 \]

\[ e_3 = \delta T_3 - e_1 - e_2 = \delta T_3 - \delta T_2 \]

Therefore equations 5.15 become

\[ \delta x_1 = \delta T_1 + \delta T_2 - \delta T_1 - \delta T_2 + \delta T_2 = 2\delta T_2 - \delta T_3 \quad (5.16) \]

\[ \delta x_2 = 2\delta T_1 - 4\delta T_3 \]
If terms of order \((\delta T)^2\) may be neglected (this is true if \(\delta T\)'s are sufficiently small) equations 5.16 are the same as equations 5.14.

For simplicity let \(W = I\)

Then \(Z = GC^T = \begin{bmatrix} 1 & 1 & -1 \\ -2 & -4 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -2 \end{bmatrix}\)

and \(Z^{-1} = \begin{bmatrix} \frac{36}{104} & -\frac{2}{104} \\ -\frac{2}{104} & \frac{3}{104} \end{bmatrix}\)

\(R \beta \) becomes

\[
\begin{bmatrix}
8/26 & \beta_1 - 1/26 \\
7/26 & \beta_1 - 5/52 \\
-11/26 & \beta_1 - 7/52 \\
\end{bmatrix}
\]

\([I - G^T Z^{-1} G][W^{-1} I_c] = \begin{bmatrix} 4 & -48 & 16 \\ -48 & 36 & -12 \\ 1/104 & 16 & -12 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1/104 \end{bmatrix} = \begin{bmatrix} 4/13 \\ -3/13 \\ 1/13 \end{bmatrix}\)

Then

\(\delta T_1 = \frac{\varepsilon_1}{1} = 8/26 \beta_1 - 1/26 \beta_2\)

\(\delta T_2 = \varepsilon_1 + \varepsilon_2 = -\frac{1}{13} \sigma + \frac{15}{26} \beta_1 - \frac{7}{52} \beta_2\)

\(\delta T_3 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = -\frac{2}{13} \sigma + \frac{4}{26} \beta_1 - \frac{7}{26} \beta_2\)
Since

\[ \delta x_1 = -\delta T_3 + 2\delta T_2 \]

Then

\[ \delta x_1 = \frac{2}{13} \sigma - \frac{4}{26} \beta_1 + \frac{7}{26} \beta_2 - \frac{2}{13} \sigma + \frac{30}{26} \beta_1 - \frac{7}{26} \beta_2 = \beta_1 \]

and

\[ \delta x_2 = -4\delta T_3 + 2\delta T_1 \]

\[ = \frac{8}{13} \sigma - \frac{16}{26} \beta_1 + \frac{28}{26} \beta_2 - \frac{8}{13} \sigma + \frac{16}{26} \beta_1 + \frac{2}{26} \beta_2 = \beta_2. \]

The algorithm will, in theory, give the required change in terminal states provided the terms \(O(\delta^2)\) are small. The algorithm when tried with the above system reduced the magnitudes of the states at the final time on each iteration until the miss distance at the final time was reduced below 0.2.

Further improvement was not possible for around the optimum \(\Delta^2\) must be made so small for equation 4.1 to hold that the changes in the intervals were of less importance than the errors in the numerical integration scheme. (See fig. 5.1 for behaviour of \% relative error vs final state).

The same sort of behaviour was observed when control of the system described in ref. 5 and of a Vander Pol equation were tried. Because of this drawback the algorithm is not suitable for searching around the optimum.

5.4 Conclusions

A new numerical method for finding time optimal control of time delay systems, based on Bryson and Denham's approach was developed. The method showed poor final, but good initial convergence. Therefore, if it could be incorporated in a two stage algorithm and used to get close to the optimum, the method of chapter III could then be used to obtain the final results.
Fig. 5.1 The magnitude of the Percentage Error vs. the State $X_1$ at Final Time for a Second Order System with Time Delay of 0.3
VI. CONCLUSIONS

Three different methods of determining bang-bang optimal controls have been studied. All the methods are based on the assumption that a bang-bang optimal control can be described by the number and sizes of switching intervals. Various types of searches can then be carried out in this parameter space.

The Rosenbrock method of maximizing a function can be used to search the parameter space without evaluating gradients. This procedure shows good convergence around the optimum, and is easily used with a wide variety of time varying time delays. Its main drawback is slow initial convergence.

A gradient method which minimizes a penalty function can also be used. This method cannot be easily programmed with some time varying time delays because of the need for the inverse function

\[ r(t - \theta(t)) = t \]

which may have no closed form solution. Convergence around the optimum is poor by comparison with the Rosenbrock scheme. However, initial convergence is very good and the savings in computer time are substantial for large order systems.

A gradient approach which satisfies the target set directly also showed good initial convergence. However, final convergence was so poor and the programming so much more complex that there would be no advantage in using it.

In all cases tested no problems with multiple minima were encountered. Since the performance index may thus be assumed unimodal in the parameter space, other searching algorithms such as a Fibonacci mini-max search(25) could also be tried.
BIBLIOGRAPHY


APPENDIX I. THE MAXIMUM PRINCIPLE WHEN THE TIME DELAYS ARE TIME VARYING

The following appendix is not intended to be a rigorous mathematical proof, a task admirably done by H. Banks in reference [10]. Rather it is intended to outline the assumptions he has made and the results obtained without the complex intermediate steps.

The results are then applied to quickly show that under the further assumptions made in chapter 2, the optimal control for the given problem is bang-bang.

The problem considered by Banks.

Given:

1. The state equation

\[
\dot{x}(t) = f(x(t), x(w_1(t)), \ldots, x(w_v(t)), u, t) \quad t > t_0
\]
\[
x(t) = g(t) \quad \text{for} \quad w_i(t) < t \leq t_0
\]

where

\[
\theta_0(t) = 0
\]
\[
\theta_{i-1}(t) \leq \theta_i(t) \leq \theta_{i+1}(t) \quad \forall i = 1, 2, \ldots, v
\]
\[
\theta_i(t) < 1
\]

and

\[
w_i(t) = \Delta t - \theta_i(t)
\]

\[x(t)\] an n-dimensional state vector \([x_1(t), x_2(t), \ldots, x_n(t)]^T\) and \(f\) a vector function \(f = [f_1(x(\cdot), t), \ldots, f_n(x(\cdot), t)]^T\) where \(x(\cdot)\) is short notation for the state vector in all its delayed and undelayed forms.

And if

1. \(f(x(t), x(w_i(t)), i-1, v, u, t)\) is continuous in all its arguments.
2. \(\exists m(t) \in L_1(t_0, T_f)\) s.t. \(f(x(\cdot), u, t) \leq m(t)\).
3. \[|f_{x(w_i(t))}(x(t), x(w_1(t)), \ldots, x(w_v(t)), u, t)| \leq m(t) \quad i=0, \ldots, v\)

where \(f_{x(w_i(t))}\) denotes the functional derivative with respect to the \(i+1\)th argument.
II. The performance index

\[ J = \int_{t_0}^{t_f} f_o(x(\cdot), u, t) \, dt \]

where \( f_o \) also satisfies the above assumption 1 to 3.

III. That the initial and final constraint sets form a manifold \( T \) of dimension < 2n-1, and \( T \) is attainable.

IV. The control \( u \) where \( u \in \Omega \) where \( \Omega \) satisfies certain common assumptions about compactness and convexity.

Define the functions \( r_i(t) \) such that

\[ r_i(w_i(t)) = t \quad t_o \leq t \leq t_f \]

These functions are always absolutely continuous and one to one because of the nature of \( w_i(t) \).

Then the following maximum principle holds.

If \( u^* \) is the optimal control and \( x^* \) and \( f^* \) are the corresponding optimal trajectories then

1) there exist vector functions

\[ \lambda(t) = [\lambda_o(t), \lambda(t)] \text{ defined on } [t_o, t_f] \]

2) satisfying

1) \( \lambda_o(t) = \text{constant} \leq 0 \)

2) \( \dot{\lambda}(t) = -\lambda(t) \hat{f}_x(x(\cdot), t) \)

\[ -\sum_{i=1}^{v} \chi_i(t) \dot{r}_i(t) \hat{f}_x(w_i(t))(x(\cdot), r(t)) \]

here \( \hat{f} \) is the augmented vector

\[ \begin{bmatrix} f_o \\ -f \end{bmatrix} \]

and \( \chi_i(t) \) is the characteristic function of the interval \( [t_o, w_i(t^*)] \)
that is
\[ x^{(x)}(t) = 0 \quad \forall (t) \leq t < t^* \]
\[ x(t) = 1 \quad t_0 \leq t \leq \forall (t) \]

3) The 2n-1 dimensional vector
\[ (-\lambda(t^*_0), \lambda(t^*_f), -\lambda(t^*_f) \hat{f}(t^*_f)) \] is orthogonal to the tangent plane to T at \((x^*(t^*_0), x(t^*_f), t^*_f)\).

4) \[ \sum_{i=0}^{j-1} \lambda(r_{v-i}(t)) f_x(w_{v-i}(t)) r_{v-i}(t) = 0 \quad \text{a.e. on } [w_{v+1-j}(t^*_0), w_{v-j}(t^*_0)] \]

5) \[ \lambda(t) \hat{f}^*(x^*(\cdot), u^*, t) \geq \]
\[ \lambda(t) \hat{f}^*(x^*(t), x^*(w_{1}(t)), \ldots x^*(w_{v}(t)), u, t) \quad \forall u \in \Omega \]

This last condition is the maximum principle. In chapter II it was assumed that \( \hat{f}(x(\cdot), t) \) is of the form \( \hat{A}(x(\cdot), t) + \hat{B}(x(\cdot), t) u \) and \( |u| \leq \beta \)

Then in order to maximize
\[ \lambda(t)\hat{A}(x(\cdot), t) + \lambda(t)\hat{B}(x(\cdot), t) u \]

by Schwartz inequality u must be
\[ u = -|\beta| \operatorname{sgn} [\lambda(t) \hat{B}(x(\cdot), t)] \]

and hence \( u \) is bang-bang.