POWER SYSTEM STABILITY STUDY BY SZEGO'S METHOD AND A MAXIMIZED LIAPUNOV FUNCTION

by

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ABSTRACT

In this thesis Liapunov's direct method is applied to transient stability study of power systems. Szego's method is applied to a second order power system in chapter two and a quadratic Liapunov function applied to the same system in chapter three. The hypervolume enclosed by the quadratic V-function is maximized. Changes in the time derivative of the quadratic V function are then made to meet the conditions of Liapunov V and $\dot{V}$ functions. Finally a maximized modified Liapunov function is constructed from a tentative quadratic function for a three-machine system.

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NOMENCLATURE

$\underline{x}$	Vector of State Variables
$\dot{\underline{x}}$	Time derivative of $\underline{x}$
$\underline{f}(x)$	State variable equation vector
V	Liapunov function
V_m	Value of V defining stability region
$\dot{V}$	Time derivative of V
r	Series resistance of transmission system
x	Series reactance of transmission system
x_t	Transformer reactance
B	Transmission system shunt susceptance
G	Shunt conductance representing local load
P	Active power
P_e	Electrical power output of synchronous machine
P_i	Mechanical power input to synchronous machine
Q	Reactive power
V_t	Terminal voltage of synchronous machine
V_o	Infinite bus voltage
Z_{eq}	Equivalent impedence of local load and transmission system
r_e	Equivalent resistance of local load and transmission system
x_e	Equivalent reactance of local load and transmission system
V_o'	Equivalent infinite bus voltage

D	Damping coefficient
δ	Angle between quadrature axis of synchronous machine and infinite bus voltage or a reference frame rotating at synchronous spread in the case of multi-machine systems.
H	Inertia constant in KW. Sec./KVA
M	$H/(\pi f)$
f	System frequency = 60 c/s
t	Time
E_q'	Internal voltage of synchronous machine
B_1, B_2, B_3, B_4, β and γ	Constants in the expression for P_e
δ_o, δ_1^o, δ_2^o and δ_e^o	Steady state values for δ
δ^{us}	Value of δ at the unstable equilibrium position
p_i, q_i	Coeficients of expanded system equations
$\theta(x)$, $g(\xi(x))$	Scaler functions of $\underline{x}$
$\underline{x}'$	The prime on a vector or matrix indicates the transpose
$A(x)$	Square Symmatric matrix with variable elements
$a_{ij}(x)$	Elements of $A(x)$
$\alpha(x_1)$, $\xi(x_1)$	Polynomials in x_1
a_i, b_i	Coefficients of $\alpha(x_1)$ and $\xi(x_1)$ respectively
$A(x_1)$, $B(x_1)$, $C(x_1)$	Polynomials in x_1
x_d	Direct axis synchronous reactance
x_d'	Direct axis transient reactance

x_d''	Direct axis subtransient reactance
x_q	Quadrature axis synchronous reactance
x_q''	Quadrature axis subtransient reactance
τ_{do}'	Direct axis transient open-circuit time constant
τ_{do}''	Direct axis subtransient open-circuit time constant
τ_{qo}''	Quadrature axis subtransient open-circuit time constant
A	Square symmetric constant matrix
a_{11}, a_{12}, a_{22}	Elements of A
$\dot{V}_2$	Second degree terms in $\dot{V}$
$g_1, g_2, \ldots, g_6$	Constraint equations
λ_i	Eigenvalues of A
I	Hypervolume enclosed by $V=\underline{x}'A\underline{x}$
$\underline{Z}$	Augmented state variable vector
$\phi(z)$	Object function
$\phi_a(z)$	Augmented object function
$\underline{g}(z)$	Vector of constraint equations
$\underline{\nu}$	Vector of Lagrange multipliers
g_z	A matrix of elements $g_{zij} = \frac{\partial g_i}{\partial z_j}$
$\delta\underline{z}$, $\delta\underline{g}$, $\delta\phi_a$	Increments in Z, $\underline{g}$ and ϕ_a respectively
$\underline{\phi}_z$	A vector of components $\frac{\partial \phi}{\partial Z_i}$
P_g	Projection matrix
U	Unit matrix
$\delta \ell$	Step size
$V_{maximum}$	Maximum value of V describing a closed surface

V_T	Value of V tangent to $\dot{V}$=o
E_1, E_2, E_3	Internal voltages of respective machines
$\phi_1(x)$, ..., $\phi_4(x)$	Component functions of $\dot{v}$

1. INTRODUCTION

Oscillations in the power flow between synchronous machines have long been known to be present. Since no real power system is truly in the steady state and there are always disturbances, the system has to be continually adjusting to meet new operating conditions. In other words the power system has to have adequate transient stability margins.

The stability characteristics of a power system during transient disturbances are usually analyzed from a set of nonlinear differential equations known as the swing equations. The order of these equations depends on the detail of representation of the synchronous machines and associated controllers. The solution of these equations is usually obtained by step-by-step integration during and after the disturbance until the critical switching time is found.

The present trend towards interconnection of power systems in order to raise utility factors and to improve the load factors and so achieve more economical operation, increases the size and complexity of power systems making the step-by-step method for stability studies more tedious and costly. A need arises for a more economic and straightforward method for studying stability. For this, the direct method of Liapunov, [1], [2], is very useful. The method enables one to determine the stability of the equilibrium state without actual solution of the system's differential equations. With a suitably constructed Liapunov function the stability region of a power system can be established and the critical switching time can be obtained by carrying out only one forward integration of the swing equations.

The basic difficulty in the application is the absence of a unique method for constructing Liapunov functions although some formalized methods, [3] to [12], have been developed for certain classes of functions. Some of these methods have been applied also to power systems. Yu and Vongsuriya, [15], employed Zubov's method and a truncated power series of V-functions to study a one-machine-infinite bus system. Rao, [17], used Cartwright's procedure, [5], to study one-machine and three-machine systems. Applying Popov's theorem and Kalman's procedure, [4], Pai, et. al., [18], studied a one-machine system including governor action. The variable gradient method developed by Gibson and Schultz, [6], was applied by Rao and Desarkar, [20], to a third order model of a one-machine system. The generalized Popov criterion for multivariable feedback systems was used by Willems, [22], to develop a Liapunov function for multimachine systems. Others constructed Liapunov functions for one-machine, [14] [16], as well as multimachine systems, [13], [19], [21], based on energy integrals.

This thesis is an extension of the transient stability studies for power systems through the application of Liapunov's direct method. Szego's method, [7], is applied in chapter 2 to estimate the transient stability region of a power system. The Liapunov function obtained is in the form of a power series. A quadratic form Liapunov function is considered in chapter 3, and the hypervolume enclosed by this function is maximized subject to certain constraints on the Liapunov function, V, and its time derivative, $\dot{V}$. The results are further improved by eliminating the indefinite terms in $\dot{V}$ and by modifying the quadratic V-function. In chapter 4 a Liapunov function for a three-machine system is constructed. Starting with a quadratic V-function, the time derivative $\dot{V}$ is obtained and adjusted to be negative definite. The actual V-function is then formed and finally the volume enclosed by the quadratic portion of this new V-function is maximized.

2. A POWER SYSTEM STABILITY STUDY BY SZEGO's METHOD

Based on Zubov's work [23], [24], Szego [7] suggested a construction procedure to obtain Liapunov functions for systems with nonlinearities representable in polynomial form. The method is applied in this chapter to determine the stability region of a second order nonlinear power system.

The equations of a disturbed power system after final switching are written in state variable form, with the final equilibrium at the origin, as follows

$$\dot{\underline{x}} = \underline{f}(x) \quad , \quad f(0) = 0 \tag{2-1}$$

The stability region is expressed in the state space by its boundary surface as

$$V = V_m \tag{2-2}$$

where V is a Liapunov function and V_m is the maximum value of V that describes a closed surface tangent to $\dot{V} = 0$.

2.1. POWER SYSTEM EQUATIONS

A typical power system is shown in Fig. 2-1. It consists of a salient pole synchronous generator connected to an infinite bus through a high voltage transmission line. The transmission system is represented by a series resistance r and reactance x. The transformer is represented by a reactance x_t. The charging effect of the line and local reactive power are represented by a susceptance B and the local load is represented by a conductance G at the machine terminal. The total power output of the machine is P + jQ at a terminal voltage V_t. The infinite bus has a constant voltage V_o.

The following assumptions are made for the power system under study:

a - The internal induced voltage of the synchronous machine is constant.

b - The flux linkages in the rotor circuits of the synchronous machine are constant.

c - The mechanical input to the synchronous machine is constant.

d - The armature resistance is neglected.

The synchronous machine dynamics are represented by a second order differential equation with the voltage relations as shown in Fig. 2-2.

Applying Thevenin's theorem, the system shown in Fig. 2-1 can be reduced to the simpler form of Fig. 2-3 where

$$Z_{eq} = 1 \; / \; [G + jB + \frac{1}{r+j(x+x_t)}]$$

$$V_o' = V_o - \frac{V_o[r+j(x+x_t)]}{r+j(x+x_t)+1/(G+jB)}$$

Thus

$$r_e = \{G[r^2+(x+x_t)^2]+r\} \; /\Delta \tag{2-3}$$

$$x_e = \{(x+x_t)-B[r^2+(x+x_t)^2]\} \; /\Delta$$

$$V_o' = V_o/\sqrt{\Delta}$$

$$\Delta = 1 + 2[B(x+x_t)(1-Gr)+Gr]+(B^2+G^2)[r^2+(x+x_t)^2]$$

Although the damper winding circuits are not included in the machine equations, the damping effect is approximated [27], [15] by

$$D(\delta) = D_1 \cos^2\delta + D_2 \sin^2\delta$$

$$\left.\begin{aligned} D_1 &= V_o'^2 \; (x_q' - x_q'') \; \tau_{q_o}'' \; /(x_e+x_q')^2 \\ D_2 &= V_o'^2 \; (x_d' - x_d'') \; \tau_{d_o}'' \; /(x_e+x_d')^2 \end{aligned}\right\} \tag{2-4}$$

Including the energy conversion power output, $P_e(\delta)$, which is derived in appendix I, the swing equation of the machine has the form

$$M \frac{d^2\delta}{dt^2} + D(\delta)\frac{d\delta}{dt} + P_e(\delta) = P_i \tag{2-5}$$

where

$$P_e(\delta) = B_1 E_q'^2 + [B_2 \operatorname{Cos}(\delta+\beta) + B_3 \sin(\delta+\gamma)]E_q' + B_4 \sin(\delta+\gamma)\operatorname{Cos}(\delta+\beta) \tag{2-6}$$

Let $\delta = \delta_o$, $\frac{d\delta}{dt} = o$ and $\frac{d^2\delta}{dt^2} = o$ in the steady state, and let the state variables be chosen as

$$\begin{aligned} x_1 &= \delta - \delta_o \\ x_2 &= \frac{d\delta}{dt} \end{aligned} \tag{2-7}$$

The system equations in state variable form can be written as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{M}[P_i - P_e(x_1+\delta_o) - D(x_1+\delta_o)\, x_2] \end{aligned} \tag{2-8}$$

which can be expanded into a power series to give

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \sum_{i=1}^{N} p_i x_1^i + x_2 \sum_{i=1}^{N} q_i x_1^{i-1} \end{aligned} \tag{2-9}$$

For the stability study the series may be truncated [15] after N terms. The details of expansion are given in appendix II.

2.2. SZEGO'S METHOD

A brief summary of Szego's method is given as follows. A system represented by (2-1), if stable, will be either globally or locally stable. According to Szego, the sufficient condition for local stability is that the time derivative of the Liapunov function by

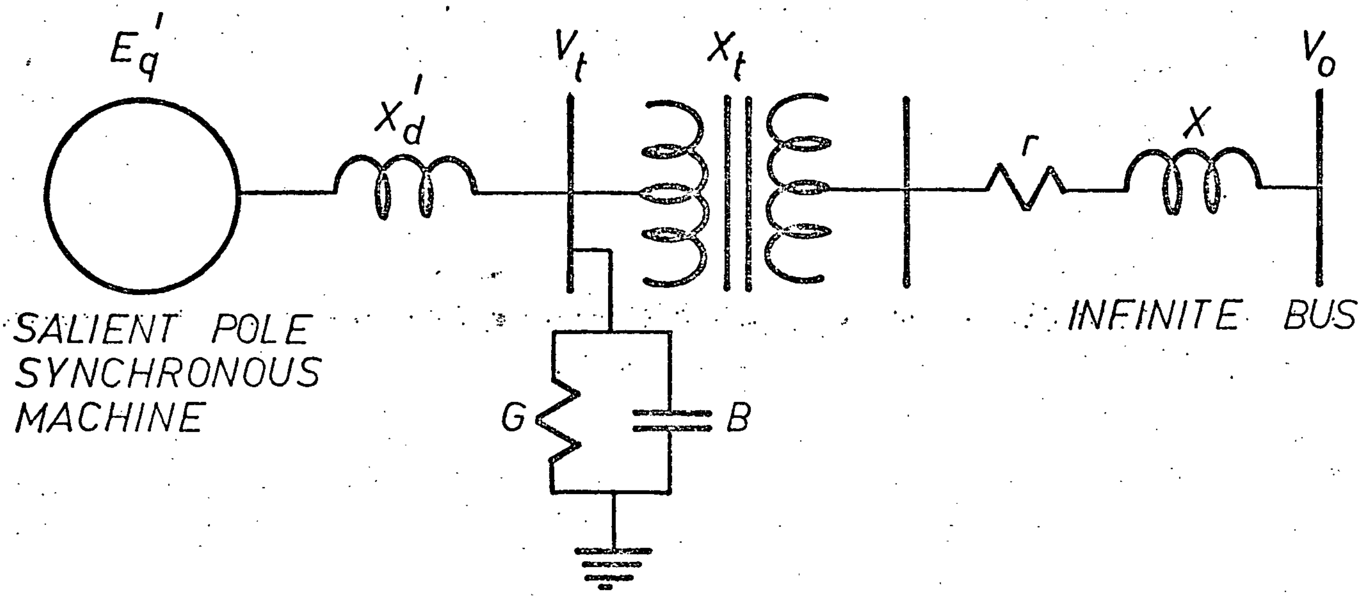


Fig. 2-1

A Typical Power System

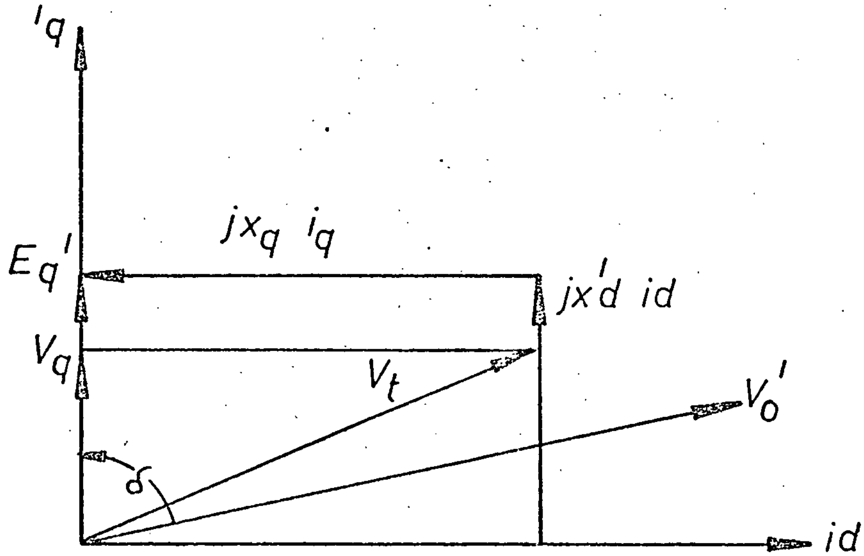


Fig. 2-2

Phasor Diagram of Salient Pole Synchronous Machine virtue of equation (2-1), has the form

$$\dot{V}(x) = \Theta(x) \cdot g(\xi(x)) \quad (2\text{-}10)$$

where $\Theta(x)$ is a semidefinite function not identically equal to zero on any nontrivial solution of the system (2-1), and $g(x)$ is indefinite on a closed surface, i.e. $\xi(x) = 0$ is a closed surface or family of surfaces and $g(u)$ is such that $g(0) = 0$ and $g(u)/u > 0$ for $u \neq 0$. Differing from LaSalle's and Zubov's methods, Szego required that $\dot{V}$ be indefinite on a closed surface as an approximate identification of the limit cycle.

A quadratic function with variable coefficients called the generating V-function is chosen for the Liapunov function

$$V(x) = \underline{x}' A(x)\, \underline{x} \quad (2\text{-}11)$$

where $A(x) = \{a_{ij}(x_i, x_j)\}$, $a_{ij}(x_i, x_j) = a_{ji}(x_i, x_j)$ and the elements $a_{ij}(x_i, x_j)$ do not contain x_n. The latter assumption is justified [7], by the fact that limit cycles of the most general nonlinear system in the phase space have at most two real intersections with each of the hyperplanes x_i = constant ($i=1,2,\ldots, n-1$).

Equation (2-11) is then differentiated, using (2-1), to give an expression for $\dot{V}$ which is then adjusted by changing the coefficients $a_{ij}(x_i,x_j)$ to get the desired form given by (2-10).

2.3. ALGORITHM [25]

Following Szego, the Liapunov function considered is

$$V = \alpha(x_1)\, x_1^2 + \xi(x_1)x_1x_2 + x_2^2 \quad (2\text{-}12)$$

By virtue of (2-9) the time derivative of V is

$$\dot{V} = \left[2\alpha(x_1)x_1 + x_1^2 \frac{d\alpha(x_1)}{dx_1} + \xi(x_1)x_2 + x_1x_2 \frac{d\,\xi(x_1)}{dx_1}\right] x_2$$

$$+ \left[\xi(x_1)x_1 + 2x_2\right]\left[\sum_{i=1}^{N} p_i x_1^i + x_2 \sum_{i=1}^{N} q_i\, x_1^{i-1}\right] \quad (2\text{-}13)$$

There are in general two steps in Szego's method. First, a suitable form for $\dot{V}$ is established and, secondly, from this form the actual V is constructed. However, a direct calculation of $\dot{V}$ is possible in our case. Let $\alpha(x_1)$ and $\xi(x_1)$ have the general form

$$\alpha(x_1) = \sum_{i=1}^{\infty} a_i x_1^{i-1}$$
$$\xi(x_1) = \sum_{i=1}^{\infty} b_i x_1^{i-1} \tag{2-14}$$

Substituting (2-14) into (2-13) gives

$$\dot{V}(x) = A(x_1)x_2^2 + B(x_1)x_2 + C(x_1) \tag{2-15}$$

where

$$A(x_1) = \sum_{i=1}^{\infty} b_i x_1^{i-1} + \sum_{i=2}^{\infty} (i-1)b_i x_1^{i-1} + 2\sum_{i=1}^{N} q_i x_1^{i-1}$$

$$B(x_1) = 2\sum_{i=1}^{\infty} a_i x_1^{i} + \sum_{i=2}^{\infty} (i-1)a_i x_1^{i} + 2\sum_{i=1}^{N} p_i x_1^{i} + \sum_{i=1}^{\infty}\sum_{j=1}^{N} b_i q_j x_1^{i+j-1}$$

$$C(x_1) = \sum_{i=1}^{\infty}\sum_{j=1}^{N} b_k p_j x_1^{i+j} \tag{2-16}$$

Equation (2-15) is of the second degree in x_2 and hence the equation $\dot{V} = 0$ will describe two curves in the state space. Now if $A(x_1)$, $B(x_1)$ and $C(x_1)$ are chosen such that

$$B^2(x_1) \equiv 4A(x_1)C(x_1) \tag{2-17}$$

then the two curves will coincide and $\dot{V}$ will not change sign along any line parallel to the x_2 axis. Condition (2-17) can be satisfied by setting both $A(x_1)$ and $B(x_1)$ identically equal to zero. $A(x_1) \equiv 0$ gives

constant term, $b_1 + 2q_1 = 0$

coeficient of x_1, $2b_2 + 2q_2 = 0$

coefficient of x_1^2, $3b_3 + 2q_3 = 0$

coefficient of x_1^{n-1}, $nb_n+2q_n = 0$

thus

$$b_i = \frac{-2}{i} q_i, \quad i = 1,2,\ldots, N$$

$$0 \quad , \qquad i>N \tag{2-18}$$

$B(x_1) \equiv 0$ gives

coefficient of x_1, $2a_1 + 2p_1 + b_1q_1 = 0$

coefficient of x_1^2, $3a_2 + 2p_2 + b_1q_2 + b_2q_1 = 0$

coefficient of x_1^3, $4a_3 + 2p_3 + b_1q_3 + b_2q_2 + b_3q_1 = 0$

thus

$$a_i = \frac{-1}{(i+1)} \left(2p_i + \sum_{j=1}^{i} b_i q_{i-j+1}\right) \quad , i=1,2,\ldots, 2N-1$$

$$0 \qquad , i \geq 2N \tag{2-19}$$

Equations (2-18) and (2-19) provide the algorithm for calculating the coefficients of the Liapunov function (2-12). The time derivative $\dot{V}$ now becomes

$$\dot{V} = C(x_1) \tag{2-20}$$

$$= \sum_{i=1}^{2N-1} \sum_{j=1}^{N} b_i p_j \, x_1^{i+j} \tag{2-21}$$

2.4. MAXIMUM VALUE OF THE LIAPUNOV FUNCTION

Using the Liapunov function derived in the previous section, the maximum value of V describing a closed curve tangent to $\dot{V} = 0$ is determined as follows. Consider equation (2-21), at the unstable equilibrium position $\delta = \delta^{us}$, $x_1 = \delta^{us} - \delta_o$ and

$$\frac{1}{M}[p_i - p_e(\delta^{us})] = \sum_{j=1}^{N} p_i x_1^j = 0 \tag{2-22}$$

Thus $\dot{V} = 0$ is a straight line parallel to the x_2 axis and passing through the point $x_1 = \delta^{us} - \delta_o$. Solving equation (2-12) for x_2 one gets

$$x_2 = \frac{-\xi(x_1)x_1 \pm \sqrt{\xi^2(x_1)x_1^2 - 4\alpha(x_1)x_1^2 + 4V}}{2} \tag{2-23}$$

For the curve $V = V_m$ to be tangent to $\dot{V} = 0$, the value of the square root must be equal to zero at $x_1 = \delta^{us} - \delta_o$, thus

$$V_m = [\alpha(\delta^{us} - \delta_o) - \frac{1}{4}\xi^2(\delta^{us} - \delta_o)](\delta^{us} - \delta_o)^2 \tag{2-24}$$

2.5. NUMERICAL EXAMPLE

Szego's method is now applied to study the stability of a particular power system. The synchronous machine under study has the following particulars:

$x_d' = 0.27$ p.u.	$\tau d_o' = 9$ sec.
$x_d = 1.0$ p.u.	$\tau_{d_o}'' = 0.04$ sec.
$x_q = 0.6$ p.u.	$\tau_{q_o}'' = 0.07$ sec.
$x_d'' = 0.22$ p.u.	H = 4 KW sec/KVA
$x_q'' = 0.29$ p.u.	

and is delivering a power of 0.753 + j 0.03 p.u. to the system at an initial terminal voltage of 1.05 p.u The transmission system particulars are shown on Fig. 2-4.

A sudden three-phase symmetrical short circuit to ground occurs at (x) on one of the transmission lines near the generator end causing bus A to ground. The faulty line is disconnected from the system at both ends after a fault duration of 8 cycles. The fault is then cleared and the line restored.

From the given initial terminal voltage V_t and power output P+jQ, the initial operating conditions determined, [26], are

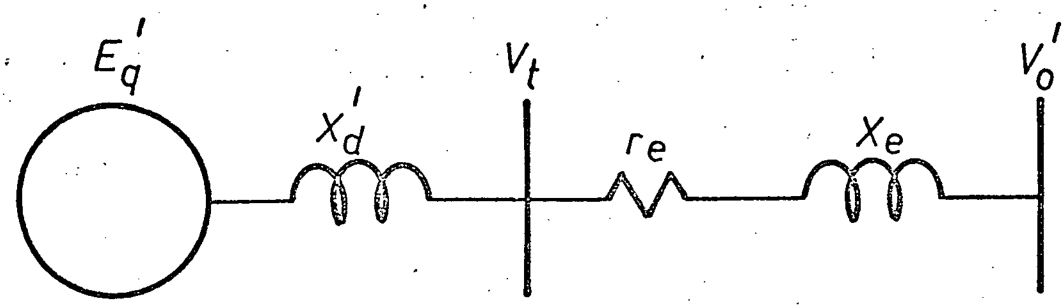


Fig. 2-3

Equivalent Power System

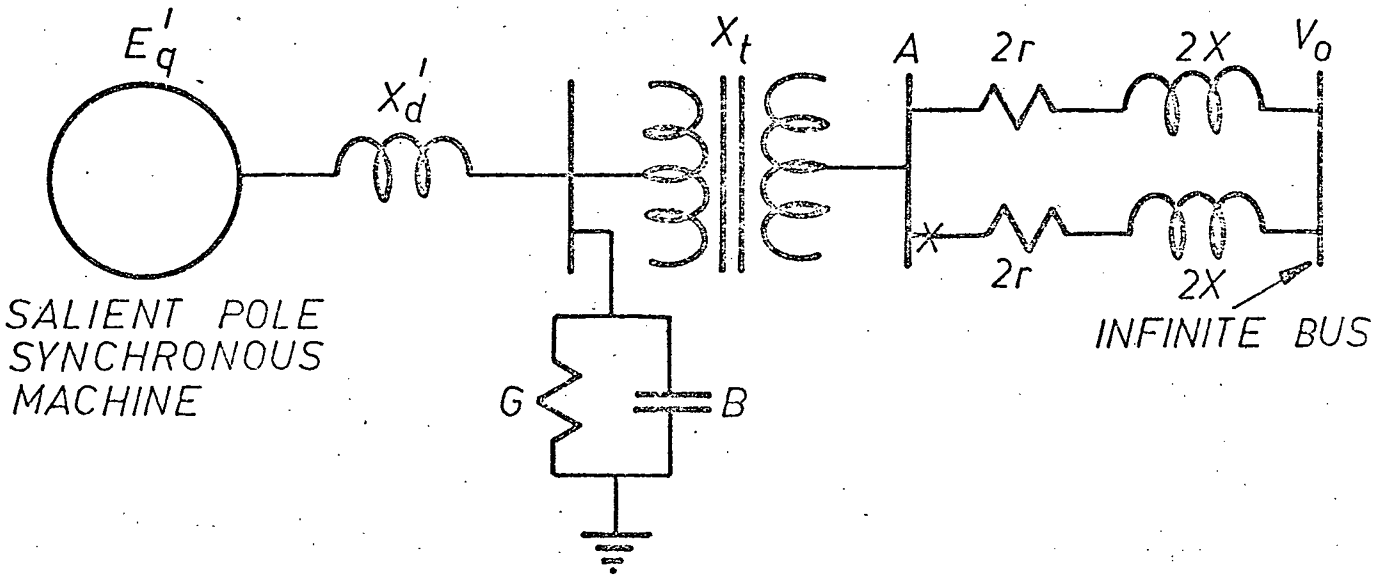


Fig. 2-4

Numerical Example

$X_t = 0.013$ p.u. $\quad V_t = 1.05$ p.u.

$X = 0.7488$ p.u. $\quad X_d' = 0.27$ p.u.

$r = 0.15$ p.u. $\quad P = 0.753$ p.u.

$B = 0.067$ p.u. $\quad Q = 0.03$ p.u.

$G = 0.18$ p.u.

$V_o' = 0.989$ p.u. $\qquad\qquad \delta_o' = 0.942$ radians

$E_q' = 1.053$ p.u. $\qquad\qquad = 53.9$ degrees

$$\delta^{us} = 3.04 \text{ radians}$$

$$= 174.28 \text{ degrees}$$

For the system considered the maximum value of V is found to be $V_m = 73$ and it gives a critical reclosing time of 23 cycles. The swing curve equations (2-8) are integrated forward using Runge-Kutta method. From the results it is found out that the critical reclosing time is 24 cycles. Fig. 2-5 shows the actual region of stability for the system considered along with the stability region defined by the Liapunov function and a system trajectory for a fault duration of 8 cycles and line reclosure after 23 cycles. A flow chart for the computer program used is shown in Fig. 2-6.

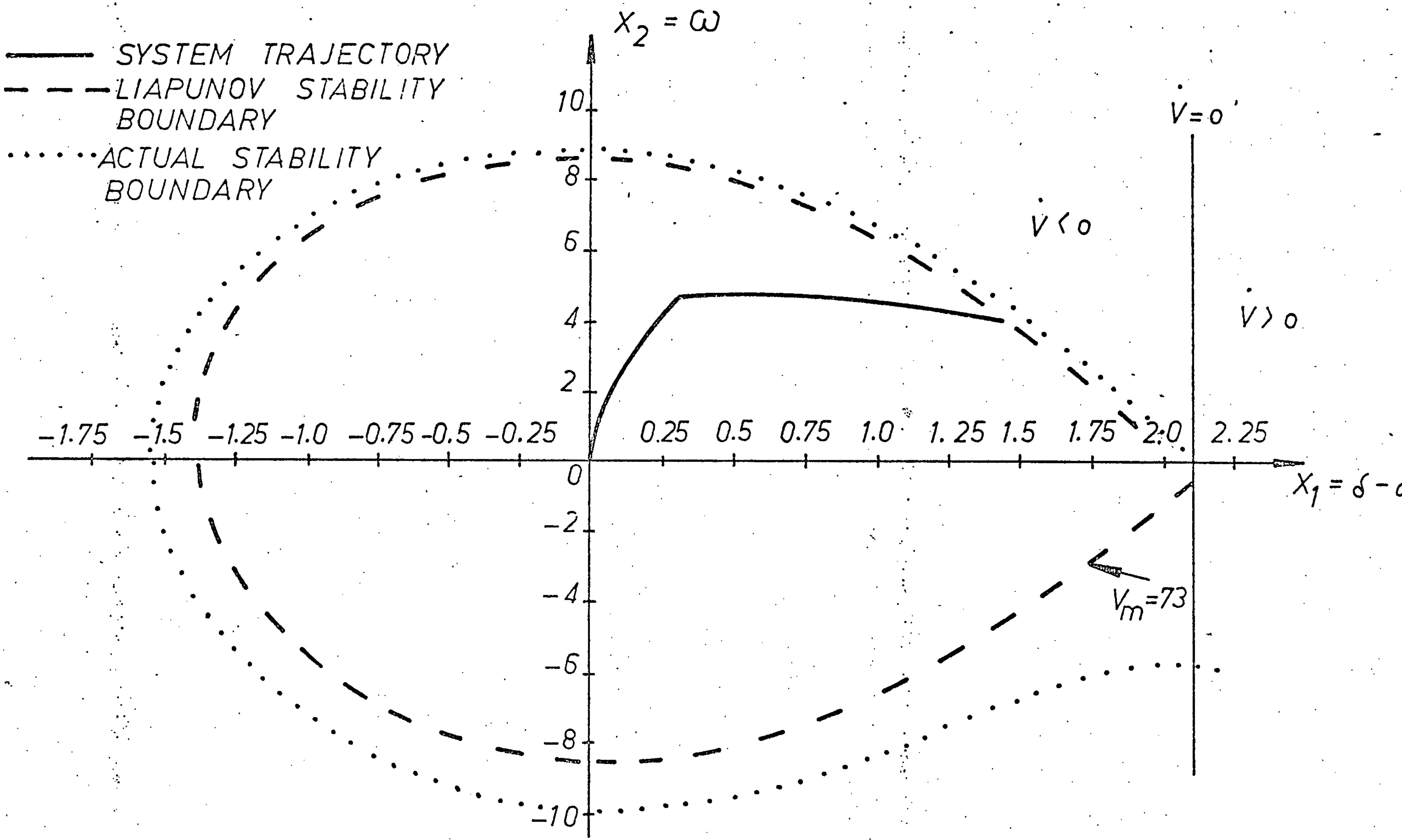


Fig. 2-5 STABILITY REGION BY SZEGO'S METHOD
___ SYSTEM TRAJECTORY
--- LIAPUNOV STABILITY BOUNDARY
... ACTUAL STABILITY BOUNDARY

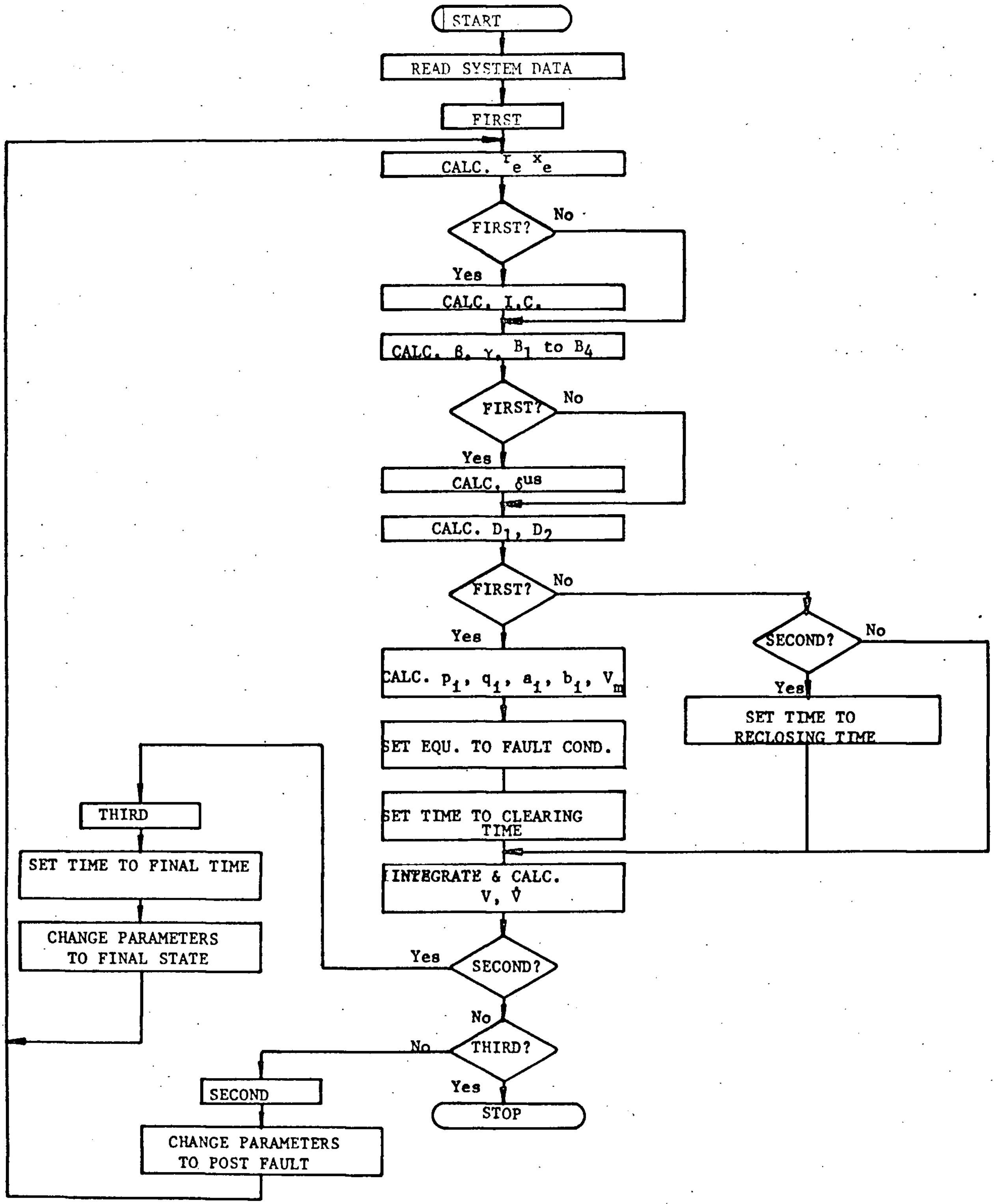


Fig. 2-6

FLOW CHART FOR SZEGO'S METHOD

3. MAXIMIZATION OF A QUADRATIC LIAPUNOV FUNCTION

In this chapter a quadratic Liapunov function of the form

$$V = \underline{x}' A \underline{x} \tag{3-1}$$

is considered, where A is a positive definite symmetric matrix. The hypervolume enclosed by (3-1) is sought and maximized subject to constraints arising from conditions on the Liapunov function and its time derivative. The stability region thus obtained for a power system is very restrictive. Since this does not serve the object of this study, a new Liapunov function is then sought by eliminating the indefinite terms in $\dot{V}$ and modifying V accordingly. The new V-function thus obtained gives a very good estimate of the stability region for a power system.

3.1. CONSTRAINTS ON A QUADRATIC V FOR A $2^{\underline{nd}}$ ORDER POWER SYSTEM

To establish asymptotic stability the Liapunov function must satisfy the following conditions

a - V is positive definite.

b - $\dot{V}$ is negative definite in an open region around the origin,

c - V_m tangent to $\dot{V} = 0$.

Consider the second order power system (2-8). Let A of (3-1) be

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \tag{3-2}$$

then one has

$$V = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 \tag{3-3}$$

The Sylvester conditions for V to be positive definite are

$$a_{11} > 0 \tag{a-1}$$

$$a_{11}a_{22} - a_{12}^2 > 0 \tag{a-2}$$

Next, using equation (2-8), the time derivative $\dot{V}$ is

$$\dot{V} = 2\{(a_{12} - a_{22}\frac{D(x_1)}{M})\, x_2^2 + [(a_{11} - a_{12}\frac{D(x_1)}{M})x_1 + a_{22}(\frac{p_i-p_e(x_1)}{M})]x_2 + a_{12}x_1(\frac{p_i-p_e(x_1)}{M})\} \tag{3-4}$$

To satisfy condition b, it is required that $\dot{V}$ be negative along the two axes x_1 and x_2, and also that $\dot{V}_2$, the second degree terms of $\dot{V}$, are negative definite. Along the x_1-axis, $\dot{V}$ is given by

$$\dot{V} = 2a_{12}x_1(\frac{p_i-p_e(x_1)}{M})$$

The term $x_1(\frac{p_i-p_e(x_1)}{M})$ is negative for

$$-(2\pi - \delta^{us} + \delta_o)<x_1<\delta^{us} - \delta_o$$

thus one must have

$$a_{12} > 0 \tag{b-1}$$

Along the x_2 axis, $\dot{V}$ is given by

$$\dot{V} = 2(a_{12} - \frac{a_{22}}{M}D(x_1))x_2^2$$

which is negative if

$$a_{12} - \frac{a_{22}}{M}D_{min}(x_1) > 0 \tag{b-2}$$

where D_{min} is the minimum value of the damping coefficient, given by

$$D_{min}(x_1)= \min_{x_1} \{\frac{D_1+D_2}{2} + \frac{D_1-D_2}{2}\operatorname{Cos}(2x_1+2\delta_o)\} = D_2$$

To find $\dot{V}_2$, equation (3-4) is expanded into a power series by the use of equation (2-9) to give

$$\dot{V} = 2\{(a_{11}x_1+a_{12}x_2)x_2+(a_{12}x_1+a_{22}x_2)(\sum_{i=1}^{N} p_i x_1^i + x_2\sum_{i=1}^{N} q_i x_1^{i-1})\}$$

The second degree terms may be written

$$\dot{V}_2 = 2(x_1 x_2)\begin{bmatrix} a_{12}p_1 & \frac{1}{2}(a_{11}+a_{12}q_1+a_{22}p_1) \\ \frac{1}{2}(a_{11}+a_{12}q_1+a_{22}p_1) & (a_{12}+a_{22}q_1) \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and the conditions for $\dot{V}_2$ to be negative definite are

$$a_{12}p_1 < 0 \qquad \text{(b-3)}$$

$$a_{12}p_1(a_{12}+a_{22}q_1) - \frac{1}{4}(a_{11}+a_{12}q_1+a_{22}p_1)^2 > 0 \qquad \text{(b-4)}$$

finally condition C implies

$$\dot{V} = 0 \qquad \text{(c-1)}$$

$$\frac{\partial V}{\partial x_1} \cdot \frac{\partial \dot{V}}{\partial x_2} - \frac{\partial V}{\partial x_2}\frac{\partial \dot{V}}{\partial x_1} = 0 \qquad \text{(c-2)}$$

Since p_1 is negative, conditions (b-1) and (b-3) are the same. Also condition (b-4) can be rewritten as

$$-p_1(a_{11}a_{22}-a_{12}^2) - \frac{1}{4}(a_{11}+a_{12}q_1-a_{22}p_1)^2 > 0$$

which automatically satisfies condition (a-2).

To summarize, the final set of constraints are:

$$g_1 = \dot{V} = 0$$

$$g_2 = \frac{\partial V}{\partial x_1} \cdot \frac{\partial \dot{V}}{\partial x_2} - \frac{\partial V}{\partial x_2} \cdot \frac{\partial \dot{V}}{\partial x_1} = 0$$

$$g_3 = a_{11} > 0$$

$$g_4 = a_{12} > 0$$

$$g_5 = \frac{a_{22}D_2}{M} - a_{12} > 0$$

$$g_6 = -p_1(a_{11}a_{22}-a_{12}^2) - \frac{1}{4}(a_{11}+a_{12}q_1-a_{22}p_1)^2 > 0 \qquad \text{(3-5)}$$

3.2. HYPERVOLUME BOUNDED BY $V = \underline{x}'A\underline{x}$

The hypervolume bounded by the surface $V = \underline{x}'A\underline{x}$, which is to be maximized, is found as follows. Since the matrix A of equation (3-1) is positive definite, its eigenvalues λ_i, $i=1,2,\ldots,n$ are all positive and the surface described by this equation is a closed one. The problem is then reduced to finding the volume bounded by

$$V = \sum_{i=1}^{n} \lambda_i \bar{x}_i^2 \qquad , \; i=1,2,\ldots,n \tag{3-6}$$

Let

$$C_i = \sqrt{V/\lambda_i} \qquad , \; i=1,2,\ldots,n \tag{3-7}$$

then

$$\sum_{i=1}^{n} \frac{\bar{x}_i^2}{c_i^2} = 1 \tag{3-8}$$

and the required volume is given by

$$I = 2\int_{-c_1}^{c_1} \int_{-c_2\sqrt{1-\bar{x}_1^2/c_1^2}}^{c_2\sqrt{1-\bar{x}_1^2/c_1^2}} \cdots \int_{-c_{n-1}\sqrt{1-\sum_{i=1}^{n-2}\bar{x}_i^2/c_i^2}}^{c_{n-1}\sqrt{1-\sum_{i=1}^{n-2}\bar{x}_i^2/c_i^2}} c_n\sqrt{1-\sum_{i=1}^{n-1}\bar{x}_i^2/c_i^2}\; dx_{n-1}dx_{n-2}\ldots dx_1 \tag{3-9}$$

resulting in

$$I = 2^n\left(\frac{\pi}{2}\right)^{\frac{n-1}{2}} \frac{1}{1.3.5\ldots n}\sqrt{V^n/|A|} \qquad n \text{ odd}$$

$$2^n\left(\frac{\pi}{2}\right)^{n/2} \frac{1}{2.4.6\ldots n}\sqrt{V^n/|A|} \qquad n \text{ even} \tag{3-10}$$

where $|A|$ denotes the determinant of matrix A. The details are given in appendix III.

3.3. OPTIMIZATION TECHNIQUE

Of the six constraints in equation (3-5), the last four are inequality constraints. These can be transformed into equality constraints [28], [29], by introducing some new free variables as follows. Let

$$g_3 = a_{11} - y_1^2 - \varepsilon = 0$$

$$g_4 = a_{12} - y_2^2 - \varepsilon = 0$$

$$g_5 = a_{22}\frac{D_2}{M} - a_{12} - y_3^2 - \varepsilon = 0$$

$$g_6 = -p_1(a_{11}a_{22} - a_{12}^2) - \frac{1}{4}(a_{11} + a_{12}q_1 - a_{22}p_1)^2 - y_4^2 - \varepsilon = 0 \qquad (3\text{-}11)$$

where y_1, y_2, y_3 and y_4 are free variables and ε is a small positive constant.

An augmented space, Z, consisting of the x-space, the a-space and the y-space is considered. The components of this new space are given by

$$Z = (Z_1 \; Z_2 \; \ldots \; Z_9)'$$

$$= (x_1 x_2 a_{11} a_{12} a_{22} y_1 \ldots y_4)' \qquad (3\text{-}12)$$

The problem is now defined as follows: minimize the cost function $\phi(Z) = -I(Z)$, subject to the constraints $g(Z) = 0$. The projected gradient method [30], [31], [32], as best explained in [32], is employed to solve this problem. The method is summarized as follows.

Consider an augmented cost function

$$\phi_a(Z) = \phi(Z) + \underline{g}'(Z)\underline{\nu} \qquad (3\text{-}13)$$

where $\underline{\nu}$ is a vector of Lagrange multipliers. Then

$$\delta\phi_a = (\underline{\phi_Z}+g_Z{}'\underline{\nu})'\,\delta\underline{Z} \tag{3-14}$$

where $\underline{\phi_Z}$, $\underline{\nu}$ and $\delta\underline{Z}$ are vectors and g_Z is a matrix (For notation see nomenclature). The steepest descent move is given by

$$\delta\underline{Z} = -k\,(\underline{\phi_Z}+g_Z{}'\underline{\nu}) \tag{3-15}$$

where k is the step size, k>0. The question now is how to choose $\underline{\nu}$. The increment $\delta\underline{Z}$ must be chosen so that the new point is in the constraint surface defined by $\underline{g}=0$. If a full correction is used for a nominal value of $\underline{g}$, $\delta\underline{Z}$ must be chosen so that

$$\delta\underline{g} = g_Z\delta\underline{Z} = -\underline{g} \tag{3-16}$$

or $\quad \delta\underline{g} + \underline{g} = 0$

Substituting (3-15) into (3-16) and solving for $\underline{\nu}$ yields

$$\underline{\nu} = (g_Z g_Z{}')^{-1}(\underline{g}/k - g_Z\underline{\phi_Z}) \tag{3-17}$$

substituting (3-17) into (3-15) gives

$$\delta\underline{Z} = -kP_g\underline{\phi_Z} - g_Z{}'(g_Z g_Z{}')^{-1}\underline{g} \tag{3-18}$$

where

$$P_g \triangleq U - g_Z{}'(g_Z g_Z{}')^{-1} g_Z \tag{3-19}$$

where U is the unit matrix. Let the desired step size be $\delta\ell$, thus

$$\delta\ell^2 = \delta\underline{Z}'\,\delta\underline{Z} \tag{3-20}$$

yielding

$$k = \sqrt{\frac{\delta\ell^2 - \underline{g}'(g_Z g_Z{}')^{-1}\underline{g}}{\underline{\phi_Z}'P_g\underline{\phi_Z}}} \tag{3-21}$$

3.4. NUMERICAL EXAMPLE

The same numerical example in chapter 2 is used here for the maximized quadratic Liapunov function stability study. The results obtained are:

$$A = \begin{bmatrix} 430 & 1.002 \\ 1.002 & 12.02 \end{bmatrix}$$

$V_m = 86$, tangent to $\dot{V} = 0$ at

$x_1 = 0.394$ and $x_2 = 1.238$

This V-function gives a critical reclosing time of 4 cycles after fault occurance which is a very restrictive result as shown in Fig. 3-1. A comparison between the stability region defined by this Liapunov function and the actual region obtained by integrating the system equations using Runge-Kutta method is given in the same figure. Fig. 3-2, shows a flow chart of the program used

3.5. A MODIFIED LIAPUNOV FUNCTION

From fig. 3-1, it is noticed that the reason for the poor estimation of stability region is due to the fact that the curve $\dot{V}=0$ is near to the origin. A better estimate will be obtained if this curve is shifted away.

Consider the expression for $\dot{V}$,

$$\dot{V}=2[[a_{12}-a_{22}\frac{D(x_1)}{M}]x_2^2 + [(a_{11}-a_{12}\frac{D(x_1)}{M})x_1 + a_{22}(\frac{p_i-p_e(x_1)}{M})]x_2$$
$$+ a_{12}x_1(\frac{p_i-p_e(x_1)}{M})] \qquad (3\text{-}4)$$

As pointed out in (b-2), $(a_{12}-a_{22}\frac{D(x_1)}{M})x_2^2$ is always negative. Also $a_{12}x_1(\frac{p_i-p_e(x_i)}{M})$ is negative for $-(2\pi-\delta^{us}+\delta_o)<x_1<\delta^{us}-\delta_o$. The

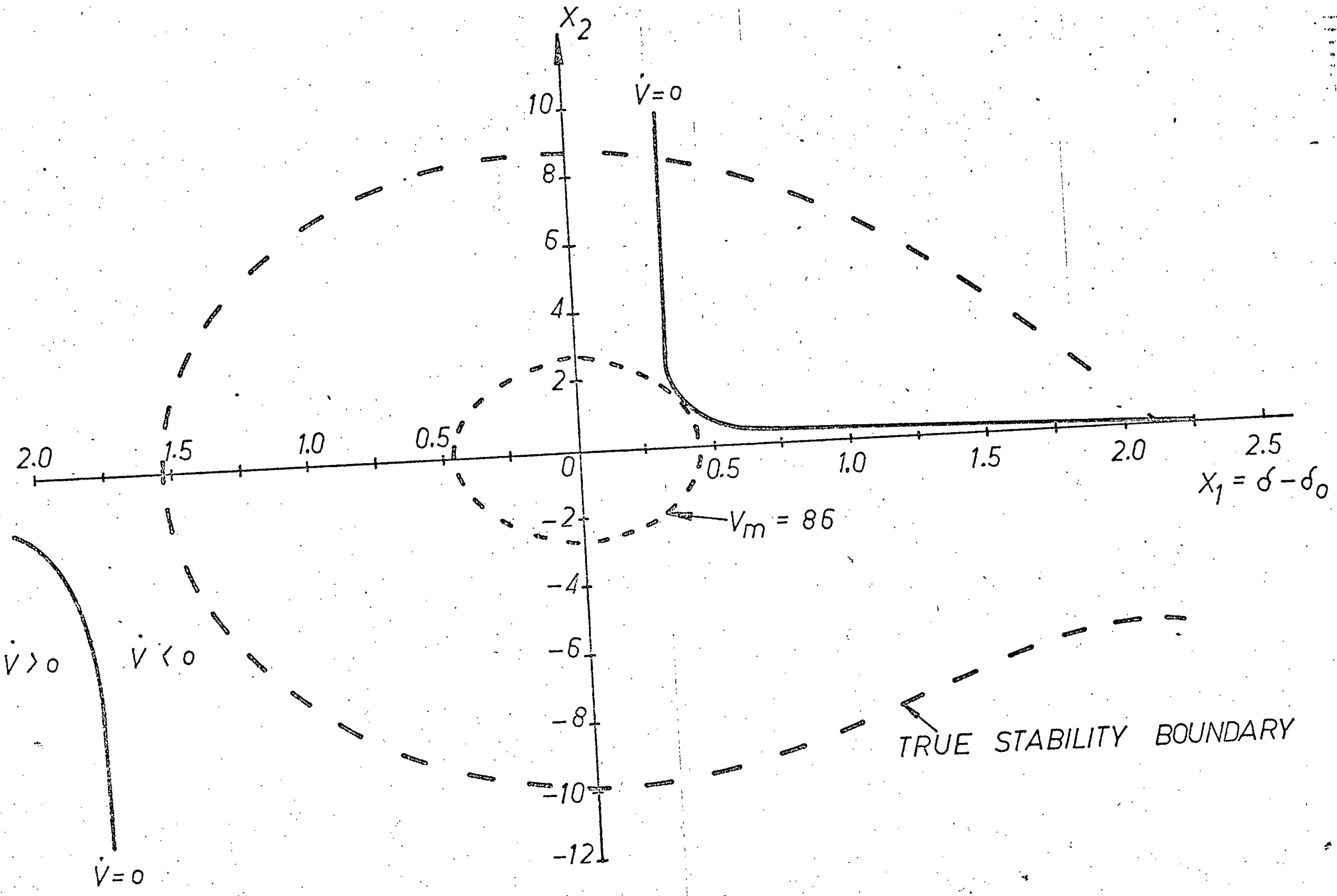


Fig. 3-1

STABILITY REGION BY A MAXIMIZED QUADRATIC

V-FUNCTION

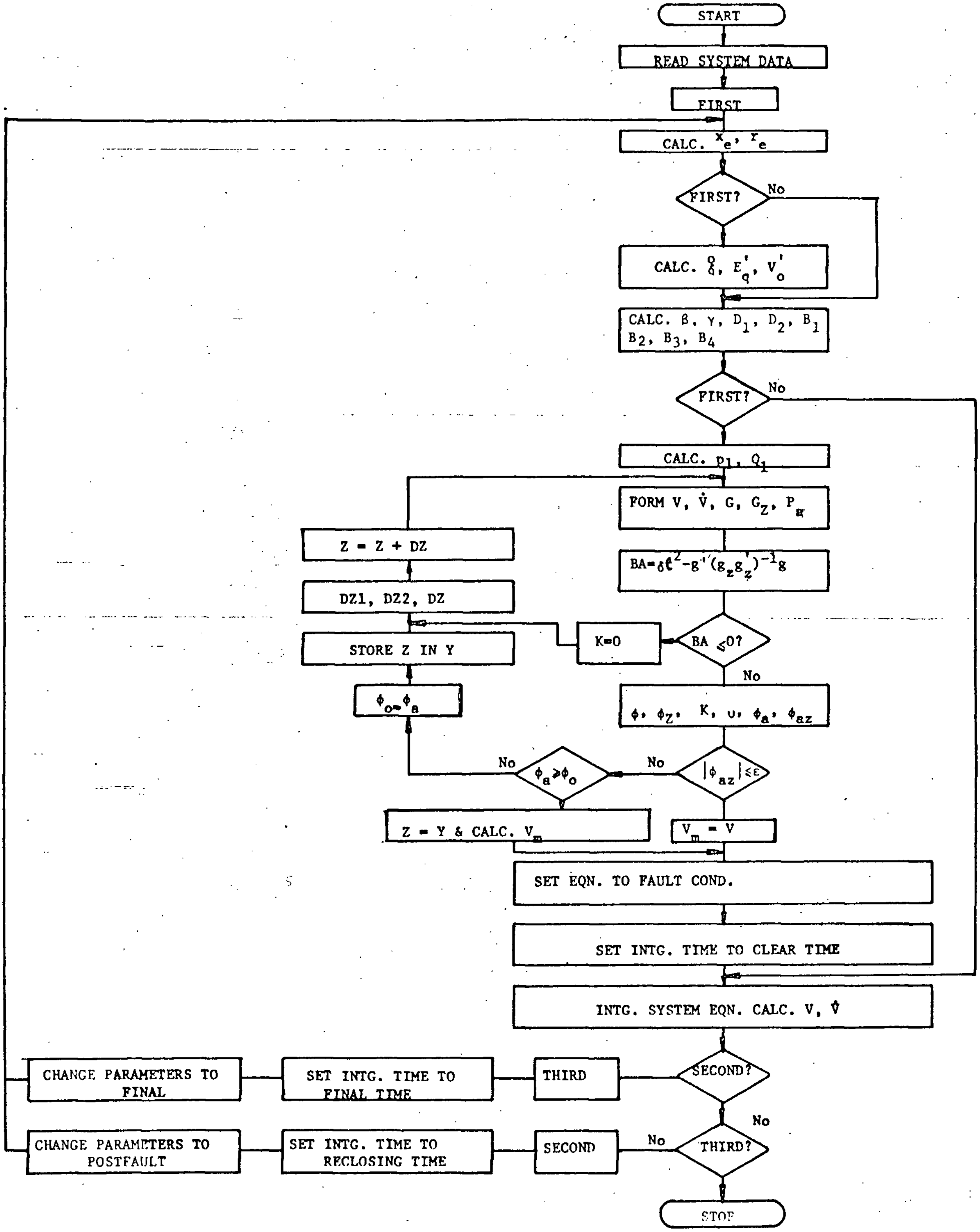


Fig. 3-2

FLOW CHART FOR MAXIMIZING A QUADRATIC V-FUNCTION remaining two terms, however, change sign according to values of x_1 and x_2. If these indefinite terms are eliminated from $\dot{V}$ by subtracting their integral, with respect to time, from the original V-function, then $\dot{V}$ will be negative definite. Since

$$\int_0^t 2x_2[(a_{11}-a_{12}\frac{D(x_1)}{M})x_1 + a_{22}(\frac{p_i-p_e(x_1)}{M})]dt$$

$$=2\int_0^{x_1} [(a_{11}-a_{12}\frac{D(x_1)}{M})x_1 + a_{22}(\frac{p_i-p_e(x_1)}{M})]dx_1$$

$$=(a_{11}-a_{12}\frac{D_1+D_2}{2M})x_1^2 - f(x_1) \tag{3-22}$$

where

$$f(x_1)=\frac{a_{12}(D_1-D_2)}{4M}[\text{Cos}(2x_1+2\delta_o)+2x_1\sin(2x_1+2\delta_o)-\text{Cos }2\delta_o]$$

$$+\frac{2a_{22}}{M}\{E_q'[B_2(\sin(x_1+\delta_o+\beta)-x_1\text{ Cos}(\delta_o+\beta)-\sin(\delta_o+\beta))$$

$$-B_3(\text{Cos}(x_1+\delta_o+\gamma)+x_1\sin(\delta_o+\gamma)-\text{Cos}(\delta_o+\gamma))]-\frac{1}{4}B_4$$

$$[\text{Cos}(2x_1+2\delta_o+\beta+\gamma)+2x_1\text{ Sin}(2\delta_o+\beta+\gamma)-\text{Cos}(2\delta_o+\beta+\gamma)]\} \tag{3-23}$$

let

$$V=\underline{x}'A\underline{x} - (a_{11}-a_{12}\frac{D_1+D_2}{2M})x_1^2 + f(x_1)$$

$$=a_{12}(\frac{D_1+D_2}{2M})x_1^2 + 2a_{12}x_1x_2+a_{22}x_2^2 + f(x_1) \tag{3-24}$$

Then

$$\dot{V}=2(a_{12}-\frac{a_{22}D(x_1)}{M})x_2^2 + 2a_{12}x_1(\frac{p_i-p_e(x_1)}{M}) \tag{3-25}$$

which is negative definite for $-(2\pi-\delta^{us}+\delta_o)<x_1<(\delta^{us}-\delta_o)$

From (3-24) one has

$$x_2 = \frac{-a_{12}x_1 \pm \sqrt{[a_{12}^2 - a_{12}a_{22}\frac{D_1+D_2}{2M}]x_1^2 - a_{22}f(x_1) + a_{22}V}}{a_{22}} \qquad (3\text{-}26)$$

For any curve V = constant to be a closed one, there must exist a value for x_1 such that the square root term equals zero resulting in

$$V = f(x_1) + x_1^2[\frac{a_{12}(D_1+D_2)}{2M} - \frac{a_{12}^2}{a_{22}}] \qquad (3\text{-}27)$$

Thus the maximum value of V describing a closed curve is obtained from (3-27) by differentiating the right hand side with respect to x_1 and equating to zero,

$$x_1 = \frac{a_{22}^2(p_i - p_e(x_1))}{a_{12}(a_{22}D(x_1) - a_{12}M)} \qquad (3\text{-}28)$$

The value of x_1 obtained from (3-28) is then substituted back in (3-27) to give $V_{maximum}$.

The value of V tangent to $\dot{V}=0$, V_T, is found by solving the two equations

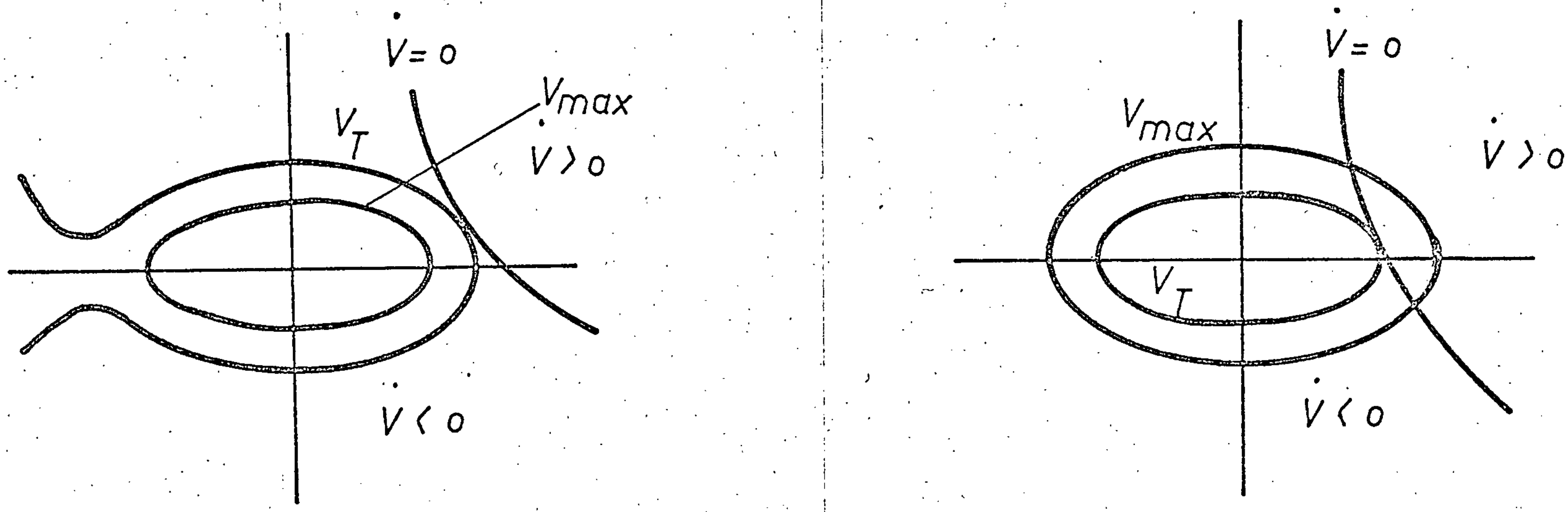
$$\dot{V}=0$$

$$\frac{\partial V}{\partial x_1} \cdot \frac{\partial \dot{V}}{\partial x_2} - \frac{\partial V}{\partial x_2} \cdot \frac{\partial \dot{V}}{\partial x_1} = 0 \qquad (3\text{-}29)$$

and the value of V to be used for stability is V_T or $V_{maximum}$ whichever is smaller, as shown in Fig. 3-3.

Applying the above procedure to the same example, (2-8), give $V_{maximum} > V_T$, $V_T = 878.4$ which is tangent to $\dot{V}=0$ at $x_1=2.1$ and $x_2=0$.

Fig. 3-4, shows the resulting stability region to be very close to the actual region obtained earlier.



(a)

$V_m = V_{max}$

(b)

$V_m = V_T$

Fig. 3-3

CHOOSING V_m .

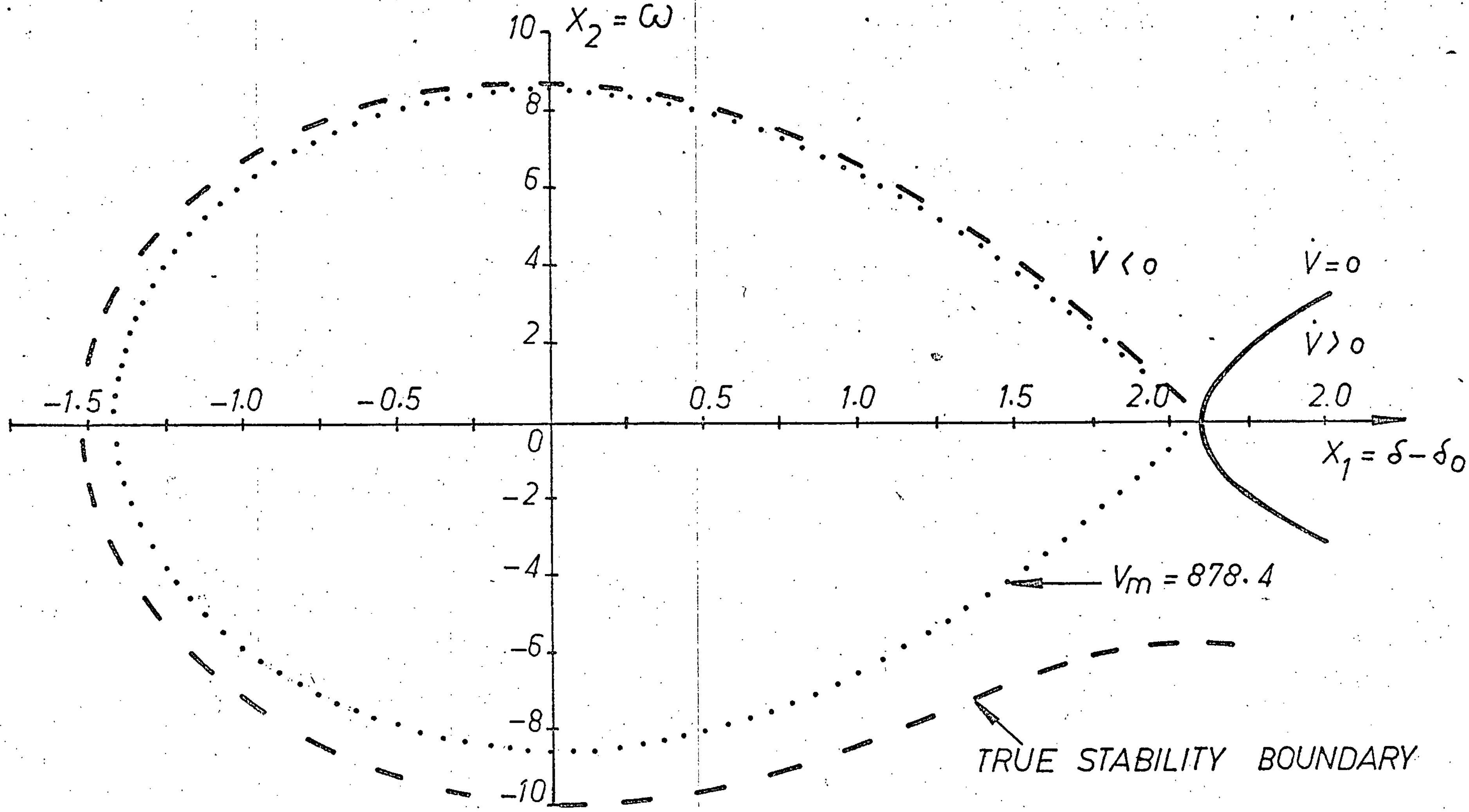


Fig. 3-4

STABILITY REGION BY MODIFIED V-FUNCTION

3.6. CONCLUDING REMARKS

Although a maximization technique for quadratic Liapunov functions has been reported recently by Davison and Kurak [33], the work of this chapter is independant. There are furthermore two major differences in our works

1. To ensure that $\dot{V}$ is negative everywhere inside and on the surface V, in Davison and Kurak's searching procedure they construct a grid in the n-dimensional space, calculate $\dot{V}$ at every point where the grid intersects the normalized surface $\underline{x}'A\underline{x}=V$, $0<V<1$ and then constrain the maximum value of $\dot{V}$ to be negative. In our case the following constraints are imposed for the same purpose a) The second degree terms of $\dot{V}$, $\dot{V}_2$, are to be negative definite and b) $\dot{V}=0$ is tangent to V.
2. Davison and Kurak state in their paper that the quadratic Liapunov function yields good estimates of stability regions for nonlinear systmes. We found the regions unsatisfactory in the case of a power system unless other terms were added to make sure that $\dot{V}$ is negative definite. The results then give a better estimate of the stability region.

4. A MAXIMIZED LIAPUNOV FUNCTION FOR A 3-MACHINE POWER SYSTEM

In the previous chapter it was found that a quadratic Liapunov function does not yield a good estimate of the stability region for a power system. It is also noticed that the expression of $\dot{V}$ has a great effect on the resulting stability region.

In the following a Liapunov function for a multimachine power system is constructed starting with a tentative quadratic Liapunov function. After the time derivative of this function is obtained, it is adjusted to be negative definite in a region around the origin. The actual Liapunov function is then formed, checked for positive definiteness and, as a final step, the quadratic portion is maximized.

4.1. EQUATIONS OF A 3-MACHINE SYSTEM

Consider a three-machine system as that of Fig. 4-1. In addition to the assumptions of chapter 2, the following assumptions are made.

a) The damping power is proportional to the slip frequency.

b) Resistance of transmission lines is neglected.

With these assumptions, the differential equations describing the motion of the system are

$$
\begin{aligned}
M_1 \frac{d^2 \delta_1}{dt^2} + D_1 \frac{d\delta_1}{dt} + P_{e1} &= P_{i1} \\
M_2 \frac{d^2 \delta_2}{dt^2} + D_2 \frac{d\delta_2}{dt} + P_{e2} &= P_{i2} \\
M_3 \frac{d^2 \delta_3}{dt^2} + D_3 \frac{d\delta_3}{dt} + P_{e3} &= P_{i3}
\end{aligned}
\qquad (4\text{-}1)
$$

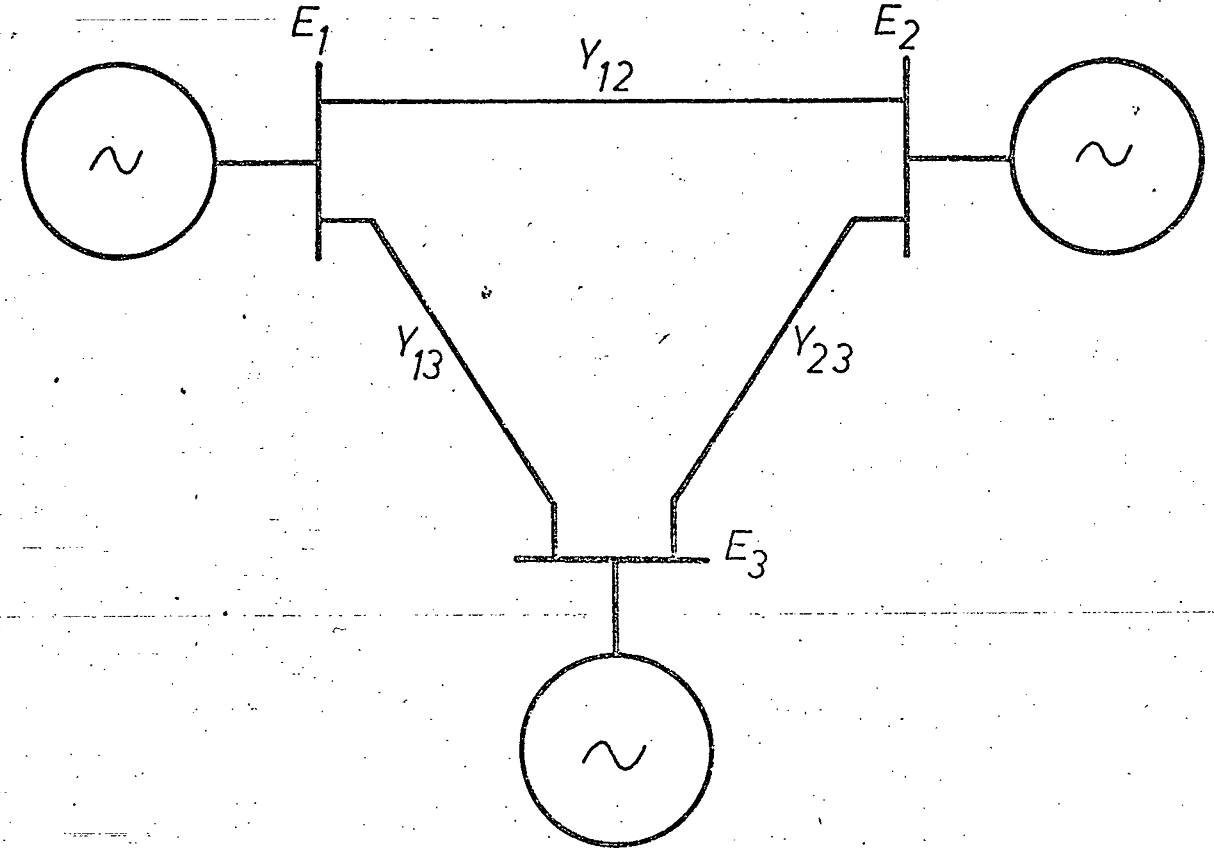


Fig. 4-1

THREE MACHINE POWER SYSTEM

where

$$P_{e1} = k_1 \sin(\delta_1-\delta_2) + k_3 \sin(\delta_1-\delta_3)$$

$$P_{e2} = k_1 \sin(\delta_2-\delta_1) + k_2 \sin(\delta_2-\delta_3)$$

$$P_{e3} = k_3 \sin(\delta_3-\delta_1) + k_2 \sin(\delta_3-\delta_2)$$

$$k_1 = E_1E_2 Y_{12}$$

$$k_2 = E_2E_3 Y_{23}$$

$$k_3 = E_1E_3 Y_{13} \tag{4-2}$$

At the stable equilibrium position we have

$$\delta_1 = \delta_1^o \quad , \quad \delta_2 = \delta_2^o \quad , \quad \delta_3 = \delta_3^o$$

$$\frac{d\delta_1}{dt} = 0 \quad , \quad \frac{d\delta_2}{dt} = 0 \quad , \quad \frac{d\delta_3}{dt} = 0$$

$$\frac{d^2\delta_1}{dt^2} = 0 \quad , \quad \frac{d^2\delta_2}{dt^2} = 0 \text{ and } \frac{d^2\delta_3}{dt^2} = 0 \tag{4-3}$$

Substituting (4-2) and (4-3) into (4-1) yields

$$P_{i1} = k_1 \sin(\delta_1^o - \delta_2^o) + k_3 \sin(\delta_1^o - \delta_3^o)$$

$$P_{i2} = k_1 \sin(\delta_2^o - \delta_1^o) + k_2 \sin(\delta_2^o - \delta_3^o)$$

$$P_{i3} = k_3 \sin(\delta_3^o - \delta_1^o) + k_2 \sin(\delta_3^o - \delta_2^o) \tag{4-4}$$

Let the state variables be

$$x_1 = \delta_1 - \delta_1^o$$

$$x_2 = \delta_2 - \delta_2^o$$

$$x_3 = \delta_3 - \delta_3^o$$

$$x_4 = \frac{d\delta_1}{dt}$$

$$x_5 = \frac{d\delta_2}{dt}$$

and $$x_6 = \frac{d\delta_3}{dt}$$

The system equations become

$$\dot{x}_1 = x_4$$

$$\dot{x}_2 = x_5$$

$$\dot{x}_3 = x_6$$

$$\dot{x}_4 = \frac{1}{M_1}\{k_1[\sin(\delta_1^o - \delta_2^o) - \sin(x_1 - x_2 + \delta_1^o - \delta_2^o)] + k_3[\sin(\delta_1^o - \delta_3^o) - \sin(x_1 - x_3 + \delta_1^o - \delta_3^o)] - D_1 x_4\}$$

$$\dot{x}_5 = \frac{1}{M_2}\{k_1[\sin(\delta_2^o - \delta_1^o) - \sin(x_2 - x_1 + \delta_2^o - \delta_1^o)] + k_2[\sin(\delta_2^o - \delta_3^o) - \sin(x_2 - x_3 + \delta_2^o - \delta_3^o)] - D_2 x_5\}$$

$$\dot{x}_6 = \frac{1}{M_3}\{k_3[\sin(\delta_3^o - \delta_1^o) - \sin(x_3 - x_1 + \delta_3^o - \delta_1^o)] + k_2[\sin(\delta_3^o - \delta_2^o) - \sin(x_3 - x_2 + \delta_3^o - \delta_2^o)] - D_3 x_6\} \qquad (4\text{-}5)$$

4.2. CONDITIONS TO ENSURE NEGATIVE DEFINITENESS OF $\dot{V}$

Consider a tentative quadratic Liapunov function of the form

$$V = \underline{x}'A\underline{x} \tag{4-6}$$

where A is a positive definite symmetric matrix. For the three machine case one has

$$A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_2 & a_7 & a_8 & a_9 & a_{10} & a_{11} \\ a_3 & a_8 & a_{12} & a_{13} & a_{14} & a_{15} \\ a_4 & a_9 & a_{13} & a_{16} & a_{17} & a_{18} \\ a_5 & a_{10} & a_{14} & a_{17} & a_{19} & a_{20} \\ a_6 & a_{11} & a_{15} & a_{18} & a_{20} & a_{21} \end{bmatrix} \tag{4-7}$$

Differentiating (4-6) with respect to time by virtue of (4-5) yields

$$\dot{V} = 2[\phi_1(x) + \phi_2(x) + \phi_3(x) + \phi_4(x)] \tag{4-8}$$

where

$$\phi_1(x) = (x_1\ x_2\ x_3) \begin{bmatrix} \frac{a_4}{M_1} & \frac{a_5}{M_2} & \frac{a_6}{M_3} \\ \frac{a_9}{M_1} & \frac{a_{10}}{M_2} & \frac{a_{11}}{M_3} \\ \frac{a_{13}}{M_1} & \frac{a_{14}}{M_2} & \frac{a_{15}}{M_3} \end{bmatrix} \begin{pmatrix} P_{i1}-P_{e1} \\ P_{i2}-P_{e2} \\ P_{i3}-P_{e3} \end{pmatrix} \tag{4-9}$$

$$\phi_2(x) = (x_4\ x_5\ x_6)\ T \begin{pmatrix} x_4 \\ x_5 \\ x_6 \end{pmatrix} \tag{4-10}$$

where

$$T = \begin{bmatrix} (a_4 - \frac{D_1}{M_1} a_{16}) & \frac{1}{2}(a_5+a_9- \frac{M_1D_2+M_2D_1}{M_1M_2} a_{17}) & \frac{1}{2}(a_6+a_{13}- \frac{M_1D_3+M_3D_1}{M_1M_3} a_{18}) \\ \frac{1}{2}(a_5 + a_9 - \frac{M_1D_2+M_2D_1}{M_1M_2} a_{17}) & (a_{10}- \frac{D_2}{M_2} a_{19}) & \frac{1}{2}(a_{11}+a_{14}- \frac{M_2D_3+ M_3D_2}{M_2M_3} a_{20}) \\ \frac{1}{2}(a_6+a_{13}- \frac{M_1D_3+M_3D_1}{M_1M_3} a_{18}) & \frac{1}{2}(a_{11}+a_{14}- \frac{M_2D_3+M_3D_2}{M_2M_3} a_{20}) & (a_{15}- \frac{D_3}{M_3} a_{21}) \end{bmatrix} \tag{4-10a}$$

$$\phi_3(x)=(x_1x_2x_3)\begin{bmatrix}(a_1-\frac{D_1}{M_1}a_4) & (a_2-\frac{D_2}{M_2}a_5) & (a_3-\frac{D_3}{M_3}a_6)\\ (a_2-\frac{D_1}{M_1}a_9) & (a_7-\frac{D_2}{M_2}a_{10}) & (a_8-\frac{D_3}{M_3}a_{11})\\ (a_3-\frac{D_1}{M_1}a_{13}) & (a_8-\frac{D_2}{M_2}a_{14}) & (a_{12}-\frac{D_3}{M_3}a_{15})\end{bmatrix}\begin{pmatrix}x_4\\ x_5\\ x_6\end{pmatrix} \qquad (4\text{-}11)$$

and

$$\phi_4(x)=(x_4x_5x_6)\begin{bmatrix}\frac{a_{16}}{M_1} & \frac{a_{17}}{M_2} & \frac{a_{18}}{M_3}\\ \frac{a_{17}}{M_1} & \frac{a_{19}}{M_2} & \frac{a_{20}}{M_3}\\ \frac{a_{18}}{M_1} & \frac{a_{20}}{M_2} & \frac{a_{21}}{M_3}\end{bmatrix}\begin{pmatrix}P_{i1}-P_{e1}\\ P_{i2}-P_{e2}\\ P_{i3}-P_{e3}\end{pmatrix} \qquad (4\text{-}12)$$

In order to ensure that $\dot{V}$ is negative definite in a region around the origin its component functions, ϕ's, must be either negative definite or semidefinite. Examining equations (4-9) to (4-12) we notice the following:

1. $\phi_1(x)$ is a function of x_1, x_2 and x_3 only and can be made negative in a region around the origin.
2. $\phi_2(x)$ is a function in x_4, x_5 and x_6 which is negative definite if the matrix T of equation (4-10a) is negative definite.
3. $\phi_3(x)$ and $\phi_4(x)$ are both indefinite. They can be eliminated from $\dot{V}$ either by setting each identically equal to zero or by integrating them with respect to time and then subtracting the result from the V-function.

In the following each component of $\dot{V}$ is examined to develop the necessary conditions for its elimination or to ensure the negative definiteness.

$\phi_1(x)$ can be made either identically equal to zero or negative definite. Knowing that the sum of the mechanical inputs to the system is equal to the sum of the electrical outputs, if we further set

$$\frac{a_4}{M_1} = \frac{a_5}{M_2} = \frac{a_6}{M_3} \quad ,$$

$$\frac{a_9}{M_1} = \frac{a_{10}}{M_2} = \frac{a_{11}}{M_3} \quad \text{and}$$

$$\frac{a_{13}}{M_1} = \frac{a_{14}}{M_2} = \frac{a_{15}}{M_3} \qquad (4\text{-}13)$$

in $\phi_1(x)$, this function will be identically equal to zero.

Substituting for P_e and P_i from (4-2) and (4-4) respectively into (4-9) one gets:

$$\phi_1(x) = k_1[\sin(\delta_1^o-\delta_2^o)-\sin(x_1-x_2+\delta_1^o-\delta_2^o)][x_1(\frac{a_4}{M_1} - \frac{a_5}{M_2} + x_2(\frac{a_9}{M_1} - \frac{a_{10}}{M_2})+x_3(\frac{a_{13}}{M_1} - \frac{a_{14}}{M_2})]$$

$$+ k_2[\sin(\delta_2^o-\delta_3^o)-\sin(x_2-x_3+\delta_2^o-\delta_3^o)][x_1(\frac{a_5}{M_2} - \frac{a_6}{M_3})+x_2(\frac{a_{10}}{M_2} - \frac{a_{11}}{M_3})+x_3(\frac{a_{14}}{M_2} - \frac{a_{15}}{M_3})]$$

$$+ k_3[\sin(\delta_1^o-\delta_3^o)-\sin(x_1-x_3+\delta_1^o-\delta_3^o)][x_1(\frac{a_4}{M_1} - \frac{a_6}{M_3})+x_2(\frac{a_9}{M_1} - \frac{a_{11}}{M_3})+x_3(\frac{a_{13}}{M_1} - \frac{a_{15}}{M_3})]$$

setting

$$(\frac{a_4}{M_1} - \frac{a_5}{M_2}) = -(\frac{a_9}{M_1} - \frac{a_{10}}{M_2}) = -(\frac{a_{14}}{M_2} - \frac{a_{15}}{M_3}) > 0 \quad ,$$

$$\frac{a_5}{M_2} = \frac{a_6}{M_3} \quad ,$$

$$\frac{a_9}{M_1} = \frac{a_{11}}{M_3} \quad \text{and}$$

$$\frac{a_{13}}{M_1} = \frac{a_{14}}{M_2} \qquad (4\text{-}14)$$

$\phi_1(x)$ becomes

$$\phi_1(x) = (\frac{a_4}{M_1} - \frac{a_5}{M_2})\{k_1(x_1-x_2)[\sin(\delta_1^o-\delta_2^o)-\sin(x_1-x_2+\delta_1^o-\delta_2^o)]$$

$$+k_2(x_2-x_3)[\sin(\delta_2^o-\delta_3^o)-\sin(x_2-x_3+\delta_2^o-\delta_3^o)]$$

$$+k_3(x_1-x_3)[\sin(\delta_1^o-\delta_3^o)-\sin(x_1-x_3+\delta_1^o-\delta_3^o)]\} \qquad (4\text{-}15)$$

which is negative definite for values of x_1, x_2 and x_3 satisfying

$$-\pi-2(\delta_1^o-\delta_2^o)<(x_1-x_2)<\pi-2(\delta_1^o-\delta_2^o)$$

$$-\pi-2(\delta_2^o-\delta_3^o)<(x_2-x_3)<\pi-2(\delta_2^o-\delta_3^o)$$

$$-\pi-2(\delta_1^o-\delta_3^o)<(x_1-x_3)<\pi-2(\delta_1^o-\delta_3^o) \qquad (4\text{-}16)$$

The expression $\phi_2(x)$ can be made identically equal to zero by setting

$$a_4 = \frac{D_1}{M_1} a_{16} \quad ,$$

$$a_{10} = \frac{D_2}{M_2} a_{19} \; ,$$

$$a_{15} = \frac{D_3}{M_3} a_{21} \; ,$$

$$a_5 + a_9 = (\frac{D_1}{M_1} + \frac{D_2}{M_2}) a_{17} \quad ,$$

$$a_6 + a_{13} = (\frac{D_1}{M_1} + \frac{D_3}{M_3}) a_{18} \quad \text{and}$$

$$a_{11} + a_{14} = (\frac{D_2}{M_2} + \frac{D_3}{M_3}) a_{20} \qquad (4\text{-}17)$$

On the other hand, for $\phi_2(x)$ to be negative definite, the conditions on matrix T are

$$a_4 - \frac{D_1}{M_1} a_{16} < 0$$

$$(a_4 - \frac{D_1}{M_1} a_{16})(a_{10} - \frac{D_2}{M_2} a_{19}) - \frac{1}{4}(a_5 + a_9 - \frac{M_1 D_2 + M_2 D_1}{M_1 M_2} a_{17})^2 > 0$$

$$(a_4 - \frac{D_1}{M_1} a_{16})[(a_{10} - \frac{D_2}{M_2} a_{19})(a_{15} - \frac{D_3}{M_3} a_{21}) - \frac{1}{4}(a_{11} + a_{14} - \frac{M_2 D_3 + M_3 D_2}{M_2 M_3} a_{20})^2]$$

$$-\frac{1}{2}(a_5+a_9-\frac{M_1D_2+M_2D_1}{M_1M_2}a_{17})[\frac{1}{2}(a_{15}-\frac{D_3}{M_3}a_{21})(a_5+a_9-\frac{M_1D_2+M_2D_1}{M_1M_2}a_{17})-$$

$$\frac{1}{4}(a_6+a_{13}-\frac{M_1D_3+M_3D_1}{M_1M_3}a_{18})(a_{11}+a_{14}-\frac{M_2D_3+M_3D_2}{M_2M_3}a_{20})]$$

$$+\frac{1}{2}(a_6+a_{13}-\frac{M_1D_3+M_3D_1}{M_1M_3}a_{18})[\frac{1}{4}(a_5+a_9-\frac{M_1D_2+M_2D_1}{M_1M_2}a_{17})(a_{11}+a_{14}-\frac{M_2D_3+M_3D_2}{M_2M_3}a_{20})$$

$$-\frac{1}{2}(a_{10}-\frac{D_2}{M_2}a_{19})(a_6+a_{13}-\frac{M_1D_3+M_3D_1}{M_1M_3}a_{18})]<0 \qquad (4\text{-}18)$$

$\phi_3(x)$ can be eliminated by setting

$$a_1 = \frac{D_1}{M_1}a_4 ,$$

$$a_2 = \frac{D_2}{M_2}a_5 = \frac{D_1}{M_1}a_9 ,$$

$$a_3 = \frac{D_3}{M_3}a_6 = \frac{D_1}{M_1}a_{13} ,$$

$$a_7 = \frac{D_2}{M_2}a_{10} ,$$

$$a_8 = \frac{D_3}{M_3}a_{11} = \frac{D_2}{M_2}a_{14} \quad \text{and}$$

$$a_{12} = \frac{D_3}{M_3}a_{15} \qquad (4\text{-}19)$$

On the other hand one may integrate $2\phi_3(x)$ with respect to time. It is noticed that

$$2(a_1-\frac{D_1}{M_1}a_4)\int x_1x_4dt = (a_1-\frac{D_1}{M_1}a_4)x_1^2$$

$$2(a_7-\frac{D_2}{M_2}a_{10})\int x_2x_5dt = (a_7-\frac{D_2}{M_2}a_{10})x_2^2$$

$$2(a_{12}-\frac{D_3}{M_3}a_{15})\int x_3x_6dt = (a_{12}-\frac{D_3}{M_3}a_{15})x_3^2$$

$$2(a_2-\frac{D_2}{M_2}a_5)\int x_1x_5dt + 2(a_2-\frac{D_1}{M_1}a_9)\int x_2x_4dt =$$

$$2(a_2-\frac{D_2}{M_2}a_5)x_1x_2+2(\frac{D_2}{M_2}a_5-\frac{D_1}{M_1}a_9)\int x_2x_4dt$$

$$2(a_3-\frac{D_3}{M_3}a_6)\int x_1x_6dt+2(a_3-\frac{D_1}{M_1}a_{13})\int x_3x_4dt =$$

$$2(a_3-\frac{D_3}{M_3}a_6)x_1x_3+2(\frac{D_3}{M_3}a_6-\frac{D_1}{M_1}a_{13})\int x_3x_4dt$$

$$2(a_8-\frac{D_3}{M_3}a_{11})\int x_2x_6dt+2(a_8-\frac{D_2}{M_2}a_{14})\int x_3x_5dt =$$

$$2(a_8-\frac{D_3}{M_3}a_{11})x_2x_3+2(\frac{D_3}{M_3}a_{11}-\frac{D_2}{M_2}a_{14})\int x_4x_5dt$$

if one sets $\frac{D_2}{M_2}a_5 = \frac{D_1}{M_1}a_9$,

$\frac{D_3}{M_3}a_6 = \frac{D_1}{M_1}a_{13}$ and

$$\frac{D_3}{M_3}a_{11} = \frac{D_2}{M_2}a_{14} \qquad (4-20)$$

one has

$$2\int\phi_3(x)dt=(x_1x_2x_3)\begin{bmatrix}(a_1-\frac{D_1}{M_1}a_4) & (a_2-\frac{D_2}{M_2}a_5) & (a_3-\frac{D_3}{M_3}a_6)\\ (a_2-\frac{D_2}{M_2}a_5) & (a_7-\frac{D_2}{M_2}a_{10}) & (a_8-\frac{D_3}{M_3}a_{11})\\ (a_3-\frac{D_3}{M_3}a_6) & (a_8-\frac{D_3}{M_3}a_{11}) & (a_{12}-\frac{D_3}{M_3}a_{15})\end{bmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} \qquad (4-21)$$

which will be subtracted from the tentative Liapunov function.

Finally $\phi_4(x)$ can be eliminated as follows. Knowing that the sum of mechanical inputs to the system is equal to the sum of electrical outputs, in order to reduce $\phi_4(x)$ to zero one may set

$$\begin{pmatrix} a_{17} \\ a_{18} \\ a_{19} \\ a_{20} \\ a_{21} \end{pmatrix} = \frac{a_{16}}{M_1^2} \begin{pmatrix} M_1M_2 \\ M_1M_3 \\ M_2^2 \\ M_2M_3 \\ M_3^2 \end{pmatrix} \tag{4-22}$$

On the other hand, similar to the case of $\phi_3(x)$, setting

$$a_{18} = \frac{M_3}{M_2} a_{17}$$

$$a_{19} = \frac{M_2 a_{16} + (M_2 - M_1) a_{17}}{M_1} ,$$

$$a_{20} = \frac{M_3}{M_1} a_{17} \quad \text{and}$$

$$a_{21} = \frac{M_3}{M_1} a_{16} + \frac{M_3(M_3 - M_1)}{M_1M_2} a_{17} \tag{4-23}$$

in $\phi_4(x)$ result in

$$\begin{aligned} 2\int\phi_4(x)dt = 2\left(\frac{a_{16}}{M_1} - \frac{a_{17}}{M_2}\right) \{ & k_1\int_o^{x_1-x_2}[\sin(\delta_1^o - \delta_2^o) - \sin(x_1-x_2+\delta_1^o-\delta_2^o)]d(x_1-x_2) \\ & + k_2 \int_o^{x_2-x_3}[\sin(\delta_2^o - \delta_3^o) - \sin(x_2-x_3+\delta_2^o - \delta_3^o)]d(x_2-x_3) \\ & + k_3 \int_o^{x_1-x_3}[\sin(\delta_1^o - \delta_3^o) - \sin(x_1-x_3+\delta_1^o - \delta_3^o)]d(x_1-x_3)\} \end{aligned} \tag{4-24}$$

which again will be subtracted from the tentative Liapunov function.

4.3. CONSTRUCTION OF LIAPUNOV FUNCTION AND MAXIMIZATION

There are sixteen different combinations of $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$ and $\phi_4(x)$ that result in a negative definite or semidefinite $\dot{V}$. These combinations are all investigated and a summary of the results obtained is given in table 4-1. Out of these sixteen combinations only one results in a positive definite V-function that has a negative definite time derivative. This is the case where $\phi_1(x)$ and $\phi_2(x)$ are made negative definite and $\phi_3(x)$ and $\phi_4(x)$ are integrated. Combining conditions (4-14), (4-18), (4-20) and (4-23) we end up with

$$V(x) = \underline{x}'\bar{A}\underline{x} - 2\int\phi_4(x)dt \tag{4-25}$$

where

$$\underline{x}'\bar{A}\underline{x} = \underline{x}'A\underline{x} - 2\int\phi_3(x)dt \tag{4-26}$$

and $-2\int\phi_4(x)\,dt$ is positive definite and is given by equation (4-24). The new matrix $\bar{A}$ is given by

$$\begin{bmatrix}
\frac{D_1}{M_1}a_4 & \frac{D_1D_2}{M_1M_2}a_{17} & \frac{D_1D_3}{M_1M_2}a_{17} & a_4 & \frac{D_1}{M_1}a_{17} & \frac{D_1M_3}{M_1M_2}a_{17} \\
\frac{D_1D_2}{M_1M_2}a_{17} & \frac{D_2}{M_1}\left(a_4+\frac{D_2-D_1}{M_2}a_{17}\right) & \frac{D_2D_3}{M_1M_2}a_{17} & \frac{D_2}{M_2}a_{17} & \frac{M_2}{M_1}a_4+\frac{D_2-D_1}{M_1}a_{17} & \frac{D_2M_3}{M_1M_2}a_{17} \\
\frac{D_1D_3}{M_1M_2}a_{17} & \frac{D_2D_3}{M_1M_2}a_{17} & \frac{D_3}{M_1}\left(a_4+\frac{D_3-D_1}{M_2}a_{17}\right) & \frac{D_3}{M_2}a_{17} & \frac{D_3}{M_1}a_{17} & \frac{M_3}{M_1}\left(a_4+\frac{D_3-D_1}{M_2}a_{17}\right) \\
a_4 & \frac{D_2}{M_2}a_{17} & \frac{D_3}{M_2}a_{17} & a_{16} & a_{17} & \frac{M_3}{M_2}a_{17} \\
\frac{D_1}{M_1}a_{17} & \frac{M_2}{M_1}a_4+\frac{D_2-D_1}{M_1}a_{17} & \frac{D_3}{M_1}a_{17} & a_{17} & \frac{M_2}{M_1}a_{16}+\frac{M_2-M_1}{M_1}a_{17} & \frac{M_3}{M_1}a_{17} \\
\frac{D_1M_3}{M_1M_2}a_{17} & \frac{D_2M_3}{M_1M_2}a_{17} & \frac{M_3}{M_1}\left(a_4+\frac{D_3-D_1}{M_2}a_{17}\right) & \frac{M_3}{M_2}a_{17} & \frac{M_3}{M_1}a_{17} & \frac{M_3}{M_1}\left(a_{16}+\frac{M_3-M_1}{M_2}a_{17}\right)
\end{bmatrix} \tag{4-27}$$

No.	ϕ_1	ϕ_2	ϕ_3	ϕ_4	$\dot{V}$	V	
1	0	0	0	0	0	p.s.d.	
2	0	0	0	$\int$	0	p.s.d.	
3	0	0	$\int$	0	0	p.s.d.	
4	0	n	0	0	n.s.d.	p.s.d.	
5	n	0	0	0	n.s.d.	p.s.d.	
6	0	0	$\int$	$\int$	0	———	conditions contradict
7	0	n	0	$\int$	n.s.d.	p.s.d.	
8	0	n	$\int$	0	n.s.d.	p.s.d.	
9	n	0	0	$\int$	n.s.d.	p.s.d.	
10	n	0	$\int$	0	n.s.d.	p.s.d.	
11	n	n	0	0	n.d.	p.s.d.	
12	0	n	$\int$	$\int$	n.s.d.	p.s.d.	
13	n	0	$\int$	$\int$	n.s.d.	p.s.d.	
14	n	n	0	$\int$	n.d.	p.d.	same as 16
15	n	n	$\int$	0	n.d.	p.s.d.	
16	n	n	$\int$	$\int$	n.d.	p.d.	same as 14

Table 4-1

n → negative

$\int$ → integrated

n.s.d. → negative semidefinite

n.d. → negative definite

p.s.d. → positive semidefinite

p.d. → positive definite

which is positive definite if

$$a_{17}>0$$

$$\frac{a_{16}}{M_1} - \frac{a_{17}}{M_2} >0$$

$$\frac{a_4}{D_1} - \frac{a_{17}}{M_2} >0 \tag{4-28}$$

Also

$$\dot{V} = 2[\phi_1(x) + \phi_2(x)] \tag{4-29}$$

where

$$\phi_1(x)=\left(\frac{a_4}{M_1} - \frac{a_{17}D_1}{M_1M_2}\right) \{k_1(x_1-x_2)[\sin(\delta_1^o -\delta_2^o)-\sin(x_1-x_2+\delta_1^o -\delta_2^o)]$$
$$+k_2(x_2-x_3)[\sin(\delta_2^o -\delta_3^o)-\sin(x_2-x_3+\delta_2^o -\delta_3^o)]$$
$$+k_3(x_1-x_3)[\sin(\delta_1^o -\delta_3^o)-\sin(x_1-x_3+\delta_1^o -\delta_3^o)]\} \tag{4-30}$$

and

$$\phi_2(x) = (x_4\ x_5\ x_6)T \begin{pmatrix} x_4 \\ x_5 \\ x_6 \end{pmatrix} \tag{4-31}$$

$$T = \begin{bmatrix} (a_4 - \frac{D_1}{M_1} a_{16}) & 0 & 0 \\ 0 & \left(\frac{M_2a_4-D_2a_{16}}{M_1} + \frac{M_1D_2-M_2D_1}{M_1M_2} a_{17}\right) & 0 \\ 0 & 0 & \left(\frac{M_3a_4-D_3a_{16}}{M_1} + \frac{M_1D_3 - M_3D_1}{M_1M_2} a_{17}\right) \end{bmatrix} \tag{4-32}$$

which is negative definite if

$$\frac{M_3}{D_3} \geq \frac{M_2}{D_2} \geq \frac{M_1}{D_1} \quad \text{and}$$

$$a_{16} > \frac{M_3}{D_3} a_4 \tag{4-33}$$

Thus we have a positive definite Liapunov function V(x) given by (4-25) and its time derivative $\dot{V}$ given by (4-29). Equations (4-28) and (4-33) define the relation between the three parameters a_4, a_{16} and a_{17}.

Finally the hypervolume enclosed by the quadratic part of V(x) shall be maximized subject to (4-28), (4-33) along with the tangent conditions

$$\dot{V} = 0$$

$$\frac{\partial V}{\partial x_1}\frac{\partial \dot{V}}{\partial x_2} - \frac{\partial V}{\partial x_2}\frac{\partial \dot{V}}{\partial x_1} = 0$$

$$\frac{\partial V}{\partial x_1}\frac{\partial \dot{V}}{\partial x_3} - \frac{\partial V}{\partial x_3}\frac{\partial \dot{V}}{\partial x_1} = 0$$

$$\frac{\partial V}{\partial x_1}\frac{\partial \dot{V}}{\partial x_4} - \frac{\partial V}{\partial x_4}\frac{\partial \dot{V}}{\partial x_1} = 0$$

$$\frac{\partial V}{\partial x_1}\frac{\partial \dot{V}}{\partial x_5} - \frac{\partial V}{\partial x_5}\frac{\partial \dot{V}}{\partial x_1} = 0$$

$$\frac{\partial V}{\partial x_1}\frac{\partial \dot{V}}{\partial x_6} - \frac{\partial V}{\partial x_6}\frac{\partial \dot{V}}{\partial x_1} = 0 \qquad (4\text{-}34)$$

The same algorithm of chapter 3 is used.

4.4. NUMERICAL EXAMPLE

The three-machine system considered has the following data:

$E_1=1.174 \angle 22.64^{\circ}$ p.u.	$P_1=0.8$ p.u.
$E_2=0.996 \angle 2.61^{\circ}$ p.u.	$P_2=0.3$ p.u.
$E_3=1.006 \angle -11.36^{\circ}$ p.u.	$P_3=-1.1$ p.u.
$H_1=3$ K.W.sec./K.V.A.	$\frac{D_1}{M_1} = 10$
$H_2=7$ K.W.sec./K.V.A.	$\frac{D_2}{M_2} = 7$
$H_3=8$ I.W.sec./K.V.A.	$\frac{D_3}{M_3} = 3$

A sudden 3-phase symmetrical short circuit to ground occurs on the transmission line connecting machines 2 and 3 of Fig. 4-1 close to bus 3. The critical clearing time obtained from the above V-function is 18 cycles. The actual critical clearing time obtained from the system's swing curves is 20 cycles.

The resulting Liapunov function, of the form (4-25), has the following particulars

$V_m = 33.65$ which is tangent to $\dot{V}=0$ at the point

$x = 2.195, -0.137, -0.005, 0.112 \times 10^{-3}, -0.278 \times 10^{-3}, -0.595 \times 10^{-3})$

$$\bar{A} = \begin{bmatrix} 3.26 & 5.32 & 2.28 & 0.326 & 0.76 & 0.868 \\ 5.32 & 8.69 & 3.72 & 0.532 & 1.24 & 1.42 \\ 2.28 & 3.72 & 1.6 & 0.228 & 0.532 & 0.608 \\ 0.326 & 0.532 & 0.228 & 0.129 & 0.076 & 0.087 \\ 0.76 & 1.24 & 0.532 & 0.076 & 0.402 & 0.203 \\ 0.868 & 1.42 & 0.608 & 0.087 & 0.203 & 0.489 \end{bmatrix}$$

Fig. 4-2 shows the function $V(x)=V_m$ plotted in the three dimensional space x_1, x_2 and x_3 with the other components x_4, x_5 and x_6 set to zero. This Fig. shows the maximum deviations in the three rotor angles, with respect to a reference frame rotating at synchronous speed, without losing synchronism with each other.

4.5. CONCLUDING REMARKS

It is interesting to notice that when setting

$$a_4 = 0$$

$$a_{17} = 0$$

$$\frac{a_{16}}{M_1} = 1 \qquad (4\text{-}35)$$

in the Liapunov function of equation (4-25), the resulting V-function is

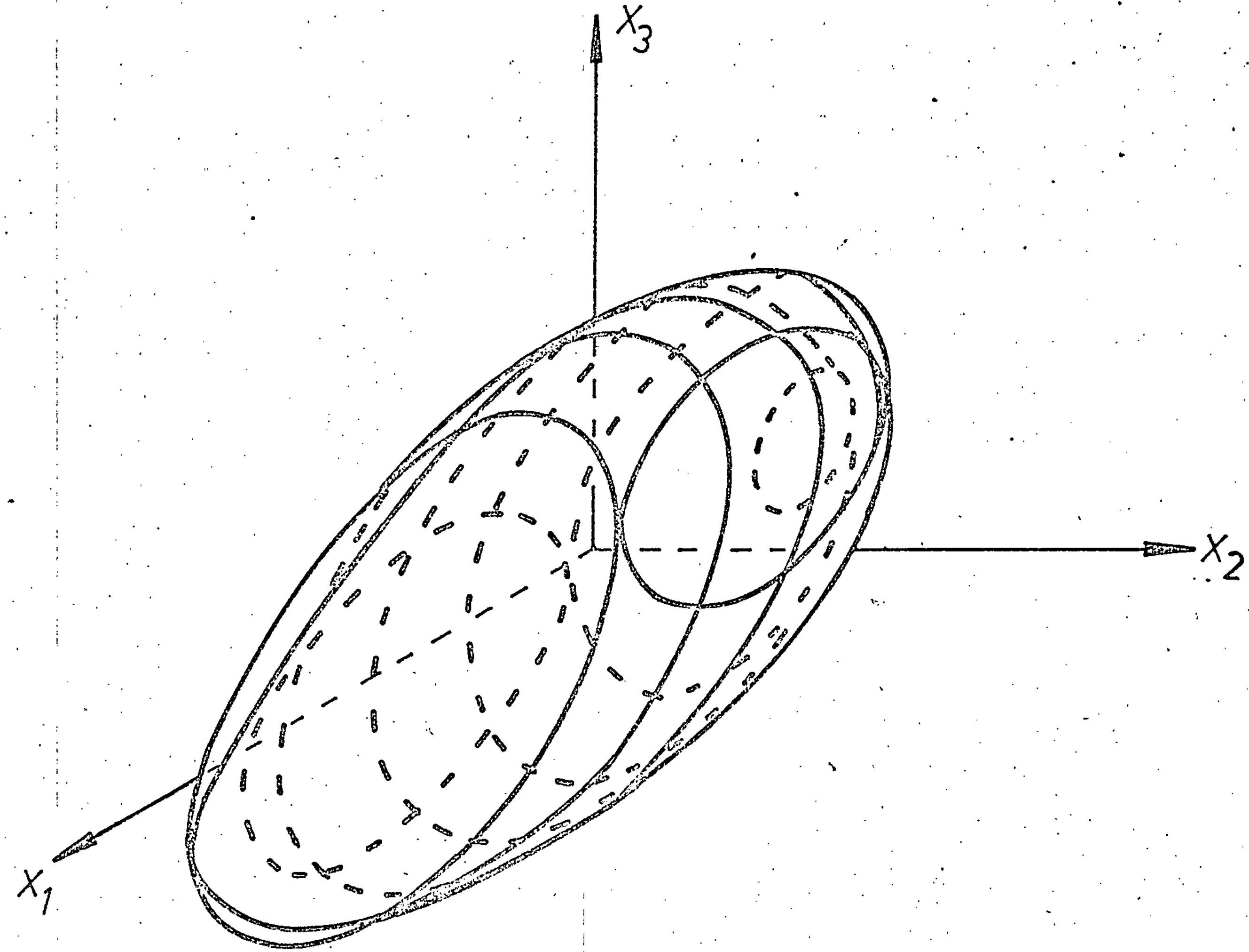


Fig. 4-2

V_m FOR THREE MACHINE SYSTEM WITH $x_4 = x_5 = x_6 = 0$

exactly the same as that of Willems [22] and is given by

$$V = (x_4\ x_5\ x_6) \begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{bmatrix} \begin{pmatrix} x_4 \\ x_5 \\ x_6 \end{pmatrix} +$$

$$2\{k_1 \int_o^{x_1-x_2} [\sin(x_1-x_2+\delta_1^o - \delta_2^o) - \sin(\delta_1^o - \delta_2^o)]\, d(x_1-x_2) +$$

$$k_2 \int_o^{x_2-x_3} [\sin(x_2-x_3+\delta_2^o - \delta_3^o) - \sin(\delta_2^o - \delta_3^o)]\, d(x_2-x_3) +$$

$$k_3 \int_o^{x_1-x_3} [\sin(x_1-x_3+\delta_1^o - \delta_3^o) - \sin(\delta_1^o - \delta_3^o)]\, d(x_1-x_3)\} \qquad (4\text{-}36)$$

When applying this V-function to the same numerical example considered in section 4.4., the resulting critical clearing time is 14 cycles compared to 18 cycles clearing time obtained from the V-function constructed in this chapter.

5. CONCLUSIONS

The direct method of Liapunov has been applied to the study of transient stability in power systems. The following conclusions are drawn:

1. Although Szego's procedure has been applied successfully to construct a Liapunov function for a second order single machine-infinite bus system, work remains to be done in developing algorithms for applying this method to higher order systems.

2. An expression for the hypervolume enclosed by a quadratic form function is developed and employed in maximizing the estimated stability region.

3. A construction procedure for optimized Liapunov functions for power systems has been developed. It starts with a quadratic form and is modified by the negative definite $\dot{V}$ constraints before maximization of the estimated stability region. The procedure has been applied successfully to a single machine-infinite bus system as well as a three machine system.

In general, it remains to develop procedures for the construction of Liapunov functions for multimachine power systems in which synchronous machines and controllers are represented in great detail.

APPENDIX I

Expression (2-6) for the electrical power output is obtained as follows. From the phasor diagram of Fig. 2-2, one has

$$V_t = \sqrt{V_d^{\,2} + V_q^{\,2}} \tag{I-1}$$

where

$$\begin{pmatrix} V_d \\ V_q \end{pmatrix} = V_o' \begin{pmatrix} \sin\delta \\ \cos\delta \end{pmatrix} + \begin{bmatrix} r_e & -x_e \\ x_e & r_e \end{bmatrix} \begin{pmatrix} i_d \\ i_q \end{pmatrix} \tag{I-2}$$

also

$$\begin{pmatrix} V_d \\ V_q \end{pmatrix} = \begin{pmatrix} 0 \\ E_q' \end{pmatrix} + \begin{bmatrix} 0 & x_q \\ -x_d' & 0 \end{bmatrix} \begin{pmatrix} i_d \\ i_q \end{pmatrix} \tag{I-3}$$

Solving (I-2) and (I-3) for V_d, V_q and i_d, i_q and gets

$$\begin{pmatrix} V_d \\ V_q \end{pmatrix} = \frac{V_o'}{\Delta} \begin{bmatrix} x_q(x_e+x_d') & -r_e x_q \\ r_e x_d' & x_d'(x_e+x_q) \end{bmatrix} \begin{pmatrix} \sin\delta \\ \cos\delta \end{pmatrix} + \frac{E_q'}{\Delta} \begin{pmatrix} r_e x_q \\ r_e^{\,2}+x_q(x_e+x_q) \end{pmatrix} \tag{I-4}$$

and

$$\begin{pmatrix} i_d \\ i_q \end{pmatrix} = \frac{V_o'}{\Delta} \begin{bmatrix} -r_e & -(x_e+x_q) \\ x_e+x_d' & -r_e \end{bmatrix} \begin{pmatrix} \sin\delta \\ \cos\delta \end{pmatrix} + \frac{E_q'}{\Delta} \begin{pmatrix} x_e+x_q \\ r_e \end{pmatrix} \tag{I-5}$$

where

$$\Delta = r_e^{\,2} + (x_e+x_q)(x_e+x_d') \tag{I-6}$$

Let

$$A_1 = x_q \sqrt{r_e^{\,2} + (x_e+x_d')^2} \,/\, \Delta$$

$$A_2 = r_e x_q / \Delta$$

$$A_3 = x_d' \sqrt{r_e^{\,2} + (x_e+x_q)^2} \,/\, \Delta$$

$$A_4 = [r_e^{\,2} + x_e(x_e+x_q)] \,/\, \Delta$$

$$A_5 = (x_e + x_q) / \Delta$$

$$\beta = \arctan\left(\frac{x_d' + x_e}{r_e}\right)$$

$$\gamma = \arctan\left(\frac{x_q + x_e}{r_e}\right) \tag{I-7}$$

Substituting (I-7) into (I-4) and (I-5) yields

$$\begin{pmatrix} V_d \\ V_q \end{pmatrix} = V_o' \begin{bmatrix} A_1 \sin\beta & -A_1 \cos\beta \\ A_3 \cos\gamma & A_3 \sin\gamma \end{bmatrix} \begin{pmatrix} \sin\delta \\ \cos\delta \end{pmatrix} + E_q' \begin{pmatrix} A_2 \\ A_4 \end{pmatrix}$$

and

$$\begin{pmatrix} i_d \\ i_q \end{pmatrix} = V_o' \begin{bmatrix} \frac{-A_3}{x_d'} \cos\gamma & \frac{-A_3}{x_d'} \sin\gamma \\ \frac{A_1}{x_q} \sin\beta & \frac{-A_1}{x_q} \cos\beta \end{bmatrix} \begin{pmatrix} \sin\delta \\ \cos\delta \end{pmatrix} + E_q' \begin{pmatrix} A_5 \\ \frac{A_2}{x_q} \end{pmatrix} \tag{I-9}$$

The electrical power output $P_e(\delta)$ is given by

$$P_e(\delta) = (V_d \ V_q) \begin{pmatrix} i_d \\ i_q \end{pmatrix} \tag{I-10}$$

Substituting (I-8) and (I-9) into (I-10) gives

$$P_e(\delta) = B_1 E_q'^2 + B_2[\cos(\delta+\beta) + B_3 \sin(\delta+\gamma)]E_q' + B_4 \sin(\delta+\gamma)\cos(\delta+\beta) \tag{I-11}$$

where

$$B_1 = A_2(A_5 + A_4/x_q)$$

$$B_2 = -V_o' A_1 (A_5 + A_4/x_q)$$

$$B_3 = V_o' A_2 A_3 \left(\frac{1}{x_q} - \frac{1}{x_d'}\right)$$

$$B_4 = -V_o'^2 A_1 A_3 \left(\frac{1}{x_q} - \frac{1}{x_d'}\right) \tag{I-12}$$

APPENDIX II

The expression (2-4) for the damping coefficient is expanded as follows

$$\frac{1}{M}D(\delta) = \frac{1}{M}(D_1 \cos^2\delta + D_2 \sin^2\delta)$$

$$= \frac{1}{M}(\frac{D_1+D_2}{2} + \frac{D_1-D_2}{2} \cos 2\delta)$$

Substituting for $\delta = x_1+\delta_o$ we get

$$\frac{1}{M} D(x_1) = \frac{1}{M}(\frac{D_1+D_2}{2} + \frac{D_1-D_2}{2} \cos(2x_1+2\delta_o))$$

$$= \frac{D_1+D_2}{2M} + \frac{D_1-D_2}{2M} [\cos 2\delta_o \cos 2x_1 - \sin 2\delta_o \sin 2x_1]$$

$$= \frac{D_1+D_2}{2M} + \frac{D_1-D_2}{2M} [\cos 2\delta_o (1- \frac{(2x_1)^2}{2!} + \frac{(2x_1)^4}{4!} -\ldots.)$$

$$-\sin 2\delta_o (2x_1 - \frac{(2x_1)^3}{3!} + \frac{(2x_1)^5}{5!} -\ldots.)$$

$$= -\sum_{i=1}^{\infty} q_i x_1^{i-1} \qquad \text{(II-1)}$$

where

$$q_1 = -\frac{D_1+D_2}{2M} - \frac{D_1-D_2}{2M} \cos 2\delta o$$

$$q_i = -\frac{2^{i-1}}{(i-1)!} (\frac{D_1-D_2}{2M}) \cos[2\delta_o + \frac{(i-1)\pi}{2}], i=2,3,\ldots \qquad \text{(II-2)}$$

At the equilibrium state:

$$P_i = P_e(\delta_o)$$

$$= B_1 E_q'^2 + [B_2 \cos(\delta_o+\beta) + B_3 \sin(\delta_o+\gamma)]E_q' + B_4 \sin(\delta_o+\gamma)\cos(\delta_o+\beta) \qquad \text{(II-3)}$$

Thus:

$$\frac{1}{M}(P_i-P_e(x_1)) = \frac{1}{M}\{B_2E_q'[\cos(\delta_o+\beta)-\cos(x_1+\delta_o+\beta)]+$$

$$B_3E_q'[\sin(\delta_o+\gamma)-\sin(x_1+\delta_o+\gamma)]+B_4[\sin(\delta_o+\gamma)\cos(\delta_o+\beta)$$

$$-\sin(x_1+\delta_o+\gamma)\cos(x_1+\delta_o+\beta)]\}$$

$$=\frac{1}{M}\{B_2E_q'[\cos(\delta_o+\beta)-\cos(x_1+\delta_o+\beta)]+$$

$$B_3E_q'[\sin(\delta_o+\gamma)-\sin(x_1+\delta_o+\gamma)]+$$

$$\frac{B_4}{2}[\sin(2\delta_o+\beta+\gamma)-\sin(2x_1+2\delta_o+\beta+\gamma)]\}$$

$$=\frac{1}{M}\{B_2E_q'[\cos(\delta_o+\beta)(1-\cos x_1)+\sin(\delta_o+\beta)\sin x_1]$$

$$+B_3E_q'[\sin(\delta_o+\gamma)(1-\cos x_1)-\cos(\delta_o+\gamma)\sin x_1]$$

$$+\frac{B_4}{2}[\sin(2\delta_o+\beta+\gamma)(1-\cos 2x_1)-\cos(2\delta_o+\beta+\gamma)\sin 2x_1]\}$$

$$=\frac{E_q'}{M}[B_2\cos(\delta_o+\beta)+B_3\sin(\delta_o+\gamma)][\frac{x_1^2}{2!}-\frac{x_1^4}{4!}+\ldots]$$

$$+\frac{E_q'}{M}[B_2\sin(\delta_o+\beta)-B_3\cos(\delta_o+\gamma)][x_1-\frac{x_1^3}{3!}+\ldots]$$

$$+\frac{B_4}{2M}[\sin(2\delta_o+\beta+\gamma)(\frac{(2x_1)^2}{2!}-\frac{(2x_1)^4}{4!}+\ldots)-\cos(2\delta_o+\beta+\gamma)$$

$$(2x_1-\frac{2x_1)^3}{3!}+\ldots)]$$

$$=\sum_{i=1}^{\infty} p_i x_1^i \qquad \text{(II-4)}$$

where

$$p_i = \frac{1}{i}\{\frac{E_q'}{M}[B_3\cos(\delta_o+\gamma+\frac{i-1}{2}\pi)-B_2\sin(\delta_o+\beta+\frac{i-1}{2}\pi)]$$

$$+\frac{2^{i-1}}{M}B_4\cos(2\delta_o+\beta+\gamma+\frac{i-1}{2}\pi) \qquad \text{(II-5)}$$

APPENDIX III

In equation (3-5) the hypervolume bounded by $V=\underline{x}'A\underline{x}$ is given by

$$I=2\int_{-c_1}^{c_1}\int_{-c_2\sqrt{1-\bar{x}_1^2/c_1^2}}^{c_2\sqrt{1-\bar{x}_1^2/c_1^2}} \cdots \int_{-c_{n-1}\sqrt{1-\sum_{i=1}^{n-2}\bar{x}_i^2/c_i^2}}^{c_{n-1}\sqrt{1-\sum_{i=1}^{n-2}\bar{x}_i^2/c_i^2}} \left[c_n\sqrt{1-\sum_{i=1}^{n-1}\bar{x}_i^2/c_i^2}\right]dx_{n-1}dx_{n-2}\ldots dx_1 \tag{3-9}$$

Consider the innermost integration

$$I_1 = \int_{-c_{n-1}\sqrt{1-\sum_{i=1}^{n-2}\frac{\bar{x}_i^2}{c_i^2}}}^{c_{n-1}\sqrt{1-\sum_{i=1}^{n-2}\bar{x}_i^2/c_i^2}} c_n\sqrt{1-\sum_{i=1}^{n-1}\bar{x}_i^2/c_i^2}\; dx_{n-1}$$

Let

$$\frac{\bar{x}_{n-1}}{c_{n-1}} = \sqrt{1-\sum_{i=1}^{n-2}\frac{\bar{x}_i^2}{c_i^2}}\;\sin\theta$$

hence

$$d\bar{x}_{n-1} = c_{n-1}\sqrt{1-\sum_{i=1}^{n-2}\frac{\bar{x}_i^2}{c_i^2}}\;\cos\theta d\theta$$

and

$$\sqrt{1-\sum_{i=1}^{n-1}\frac{\bar{x}_i^2}{c_i^2}} = \sqrt{1-\sum_{i=1}^{n-2}\frac{\bar{x}_i^2}{c_i^2}}\;\cos\theta$$

$$I_1 = c_n c_{n-1}\left(1-\sum_{i=1}^{n-2}\frac{\bar{x}_i^2}{c_i^2}\right)\int_{-\pi/2}^{\pi/2}\cos^2\theta\; d\theta$$

$$= \frac{\pi}{2}c_n c_{n-1}\left(1-\sum_{i=1}^{n-2}\frac{\bar{x}_i^2}{c_i^2}\right)$$

For the next step of integration use the substitution

$$\frac{\bar{x}_{n-2}}{c_{n-2}} = \sqrt{1-\sum_{i=1}^{n-3}\frac{\bar{x}_i^2}{c_i^2}}\ \sin\theta.$$

hence $d\bar{x}_{n-2} = c_{n-2}\sqrt{1-\sum_{i=1}^{n-3}\frac{\bar{x}_i^2}{c_i^2}}\ \cos\theta\ d\theta$

$$\left(1-\sum_{i=1}^{n-2}\frac{\bar{x}_i^2}{c_i^2}\right) = \left(1-\sum_{i=1}^{n-3}\frac{\bar{x}_i^2}{c_i^2}\right)\cos^2\theta$$

$$I_2 = \int_{-c_{n-2}\sqrt{1-\sum_{i=1}^{n-3}\frac{\bar{x}_i^2}{c_i^2}}}^{c_{n-2}\sqrt{1-\sum_{i=1}^{n-3}\frac{\bar{x}_i^2}{c_i^2}}} \frac{\pi}{2}\, c_n c_{n-1}\left(1-\sum_{i=1}^{n-2}\frac{\bar{x}_i^2}{c_i^2}\right) dx_{n-2}$$

$$= \frac{\pi}{2}\, c_n c_{n-1} c_{n-2}\left(1-\sum_{i=1}^{n-3}\frac{\bar{x}_i^2}{c_i^2}\right)^{3/2} \int_{-\pi/2}^{\pi/2}\cos^3\theta\ d\theta$$

$$= 2.\frac{2}{3}\cdot\frac{\pi}{2}\, c_n c_{n-1} c_{n-2}\left(1-\sum_{i=1}^{n-3}\frac{\bar{x}_i^2}{c_i^2}\right)^{3/2}$$

This procedure is repeadted (n-1) times to give the final answer. It is noticed that as a result of the $k^{\underline{th}}$ step, the term $\int_{-\pi/2}^{\pi/2}\cos^{k+1}\theta d\theta$ appeares, thus the last integral is $\int_{-\pi/2}^{\pi/2}\cos^{n}\theta d\theta$ which is equal to

$$2\int_0^{\pi/2}\cos^n\theta d\theta = 2.\ \frac{1.3.5...(n-1)}{2.4.6...n}\left(\frac{\pi}{2}\right) \qquad n \text{ even}$$

$$2.\ \frac{2.4.6...(n-1)}{1.3.5...n} \qquad n \text{ odd}$$

The volume required is thus given by

$$I = 2^n \left(\frac{\pi}{2}\right)^{\frac{n-1}{2}} \frac{1}{1.3.5\ldots n} \prod_{i=1}^{n} c_i \qquad n \text{ odd}$$

$$2^n \left(\frac{\pi}{2}\right)^{\frac{n}{2}} \frac{1}{2.4.6\ldots n} \prod_{i=1}^{n} c_i \qquad n \text{ even}$$

but $\prod_{i=1}^{n} c_i = \sqrt{V^n / \prod_{i=1}^{n} \lambda i}$

and the product of the eigenvalues of a matrix is equal to the matrix determinant. Thus

$$\prod_{i=1}^{n} c_i = \sqrt{V^n / |A|}$$

and $$I = 2^n \left(\frac{\pi}{2}\right)^{\frac{n-1}{2}} \frac{1}{1.3.5\ldots n} \sqrt{V^n / |A|} \qquad n \text{ odd}$$

$$2^n \left(\frac{\pi}{2}\right)^{\frac{n}{2}} \frac{1}{2.4.1\ldots n} \sqrt{V^n / |A|} \qquad n \text{ even} \qquad (3\text{-}10)$$

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