THE RELATIONSHIP BETWEEN FIELD-INDEPENDENCE
AND INSTRUCTIONAL STRATEGY ON PERFORMANCE
ON ELEMENTARY MATHEMATICS ALGORITHMS

by

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ABSTRACT

THE RELATIONSHIP BETWEEN FIELD-INDEPENDENCE AND INSTRUCTIONAL STRATEGY ON PERFORMANCE ON ELEMENTARY MATHEMATICS ALGORITHMS

A study was conducted to determine the interaction effect, if any, between the field-independence construct and two instructional strategies, a pattern strategy which used diagrams extensively and an algebraic strategy which used algebraic field properties familiar to the child and was devoid of diagrams. Two algorithms classified as simple and two algorithms classified as complex formed the content of the instructional materials.

One half the children in each of twelve grade five classes, which were participating in a study conducted by a doctoral student, were randomly selected to form the sample of the study. The Children's Embedded Figures Test was individually administered to the sample.

Three null hypotheses were tested each at $\alpha = .05$. These were: (1) There is no significant difference in mean post-test scores between students taught by a pattern instructional strategy and students taught by an algebraic instructional strategy; (2) There is no significant difference in mean post-test scores between groups of students differing in degree of field independence; (3) There is no significant interaction between students' degree of field independence and instructional strategy.
Multiple linear regression techniques were used to analyse the data.

The results of the study indicated that extreme field independent children did respond differently to the two instructional strategies, although for the sample as a whole the two strategies did not produce significantly different results. For extreme field independent students, the algebraic strategy was superior to the pattern strategy.
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CHAPTER I

THE PROBLEM

Introduction

Today, in North American education, there is a revival of interest in individualized instruction. In mathematics programs such as IPI (Individually Prescribed Instruction) and SAMI (Systematic Approach to Mathematical Instruction) stress an individualized, sequential approach through the extensive use of diagnostic testing, work-week units, multi-media learning centres, teacher-student contracts and self pacing. Yet, as Gage and Unruh have noted, "... the fact is, that despite several decades of concern with individualization, few, if any, striking results have been reported."¹

Coop and Sigel have noted that "... few, if any, of these individualized programs have examined carefully the inter-individual variability of the learners, who will be exposed to their educational stimuli."² Yet, Bloom, Cronbach, Gagne, Glaser and Jensen, have suggested that there is no one instructional method which provided optimal learning for all students.³ Cronbach and Snow have also stated


that "... the search for generally superior methods must be supplemented by a search for ways of adapting instruction to the individual."\(^4\)

Bracht\(^5\) and Mitchell\(^6\) have strongly advocated investigations to discover significant interactions between personological variables of the learner and alternative instructional routes to a desired educational outcome. (They label this Aptitude-Treatment Interaction research.) But which personological variables are relevant and as Becker\(^7\) notes which methods combined with these personological variables are relevant to which desired educational outcomes? As yet, these questions remain for the most part, unanswered in this area of research. Cronbach has suggested that "... we will find these aptitude variables to be quite unlike our present aptitude measures ..."\(^8\)

The challenge to researchers, then, is to search for relevant personological variables. The challenge to curriculum developers is to design viable alternative instructional strategies.

\(^4\)Lee J. Cronbach and Richard E. Snow, "Individual Differences in Learning Ability as a Function of Instructional Variables" Final Report Stanford University, California School of Education, ED 029 001


Background of the Problem

Educators and psychologists have recently suggested that the psychological construct of cognitive style may be relevant to the problems of education. It has been suggested that research into the interaction between cognitive style and instructional processes may provide a theoretical and empirical basis for the optimal assignment of learners to alternative instructional processes. Spitler has further suggested that Witkin's construct of cognitive style (field-dependence-independence) may have important implications for mathematics education in terms of the types of curricular materials used.

A considerable portion of elementary school mathematics is concerned with algorithm instruction. Weinstein reviewed relevant literature and concluded that the meaningfulness of the algorithm for the child determines the child's ability to retain and appropriately apply the algorithm. She also concluded that the procedure used to justify an algorithm is one of the major factors influencing the meaningfulness of the algorithm for the child.

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Weinstein designed a study to investigate the relative effectiveness of two instructional strategies, which she labels pattern and algebraic, in the teaching of simple and complex algorithms. She classified algorithms commonly taught in the elementary grades as simple or complex on the basis of the number of prerequisites required for their acquisition. She then designed two alternative instructional strategies for each of two examples of simple algorithms and two examples of complex algorithms. Diagrams are used to justify the algorithm in the pattern instructional strategy, while appeal to definitions and algebraic field postulates are used in the algebraic instructional strategy. For each algorithm, a computation post-test and a generalization post-test were developed to measure achievement on the algorithm.

The Problem

The problem which this researcher sought to investigate was: Do children differing in their degree of field independence respond differently to the two instructional strategies developed by Weinstein? In other words, is there an interaction effect between field independence and instructional strategy? Two questions relevant to the problem were also investigated: Does one of the instructional strategies, on the average result in superior pupil performance? Is there differential achievement on the algorithms among students differing in their degree of field independence?

In order to answer these questions, the investigator randomly selected a sample of the students participating in the Weinstein study and measured these students on the field independence construct. These students participated fully in the Weinstein study, following either a
pattern or algebraic instructional strategy on a simple algorithm and a complex algorithm and taking the computation and generalization post-tests on these algorithms.

Witkin's Construct of Cognitive Style: Field-Dependence-Independence

The term cognitive style is used in psychological literature to denote individual consistencies in modes of functioning over a wide range of behavioral situations. However, since the term is a psychological construct, there is some disagreement among psychologists as to which specific observable behaviors are representative of the term. The term is therefore, necessarily investigator specific.

Witkin and his associates have developed a theory of individual self-consistency along divergent psychological growth patterns. "These patterns suggest consistency in psychological functioning which pervades the individual's perceptual, intellectual, emotional, motivational, defensive and social operations."¹²

The perceptual and intellectual components of Witkin's theory of "differentiation" are combined to form the cognitive dimension, the extremes of which are represented by an "analytical field approach" and a "global field approach" or "articulated" versus "global". Field-dependence-independence is an index of the perceptual component. Field independence represents the ability to overcome an embedding context and perceive an item as distinct from its background. The field-depen-

dence-independence dimension reflects the individual's ability to experience stimuli analytically. Witkin and his associates have demonstrated through research that a tendency to experience analytically, that is a tendency toward field independence, is strongly associated with a tendency to structure experience.\(^{13}\) The ability to both analyse and structure experience is referred to as an "articulated" way of experiencing as opposed to a "global" way of experiencing. At the global extreme... when the field is structured, there is a tendency for its organization, as given, to dictate the manner in which both the field as a whole and its parts are experienced; when the field lacks structure, experience tends to be global and diffuse."\(^{14}\) At the "articulated" extreme "... there is a tendency for experience to be delineated and structured, even when the material lacks inherent organization; parts of a field are experienced as discrete and the field as a whole as organised."\(^{15}\)

Witkin's longitudinal studies indicate that in children from the ages of 5 - 17 there is a progression toward greater field independence, but that there is also a high degree of relative stability. That is, children tend to maintain their position along the field-dependence-independence dimension in relation to their peers. There is a levelling-off about the age of 17 and then a gradual "dedifferentiation" toward

\(^{13}\) Ibid., pp. 81 - 114.


\(^{15}\) Ibid.
field dependence after the mid thirties.\textsuperscript{16}

Research also indicates that the construct of field-dependence-independence is independent of I. Q. level. Witkin et al. found significant correlation between the construct and a group of subtests of WISC, Block design, Object assembly and Picture completion\textsuperscript{17}; however, there was non-significant correlation between the construct and verbal comprehension and arithmetic subtest scores of WISC. Thus "... intelligence test scores cannot be interpreted to mean that field-independent children are of generally superior intelligence."\textsuperscript{18}

Hypotheses

Conjectured Outcomes. The investigator conjectured that students will prefer the instructional strategy which is most closely matched to their cognitive style, and that, consequently, this preference will be demonstrated by significant differences between mean scores on the computation and generalization post-tests between students with similar cognitive styles taught by different instructional strategies.

First, since the pattern instructional strategy utilizes an overriding physical analogy for justification of the algorithm and since for the field dependent child "... the organization of a field as a whole dictates the way in which its parts are experienced,"\textsuperscript{19}, it is expected that


\textsuperscript{17}Herman A. Witkin et al., \textit{Psychological Differentiation}(New York: Wiley Inc., 1962), 223.

\textsuperscript{18}Ibid., p.70.

\textsuperscript{19}Herman A. Witkin et al., \textit{A Manual For The Embedded Figures Tests} (Palo Alto, California: Consulting Psychologists Press, 1971), 7.
field dependent children will achieve higher group post-test scores on the pattern approach than on the algebraic approach.

Secondly, since the algebraic instructional strategy is basically an analytic procedure in which individual steps must be drawn together to form the whole and since for the field independent child "... parts of a field are experienced as discrete and the field as a whole as structured,"\(^\text{20}\) it is expected that field independent children will achieve higher group post-test scores on the algebraic approach than on the pattern approach.

**Null Hypotheses.** The results of the computation and generalization tests were analyzed separately for each of the two simple and two complex algorithms. The following null hypotheses were tested at \(\alpha = .05\) in each of the analyses.

\(H_1:\) There is no significant difference in mean post-test scores between students taught by a pattern instructional strategy and students taught by an algebraic instructional strategy.

\(H_2:\) There is no significant difference in mean post-test scores between groups of students differing in degree of field independence.

\(H_3:\) There is no significant interaction between students' degree of field independence and instructional strategy.

\(^{20}\)Ibid.
Definition of Terms

Algebraic Instructional Strategy—"... those explanations which consist purely of appeals to definitions, to rules of logic and to the algebraic field postulates or to combinations thereof."\(^{21}\)

Pattern Instructional Strategy—"... those explanations which use physical analogs for mathematical operations."\(^{22}\)

Simple Algorithms—"... those algorithms with a relatively small number of prerequisites for their acquisition..."\(^{23}\)

Complex Algorithms—"... those algorithms which require a substantially greater number of prerequisites for their acquisition."\(^{24}\)

Field Independent Child—a child whose score is above the sample mean on the Children's Embedded Figures Test

Field Dependent Child—a child whose score is below the sample mean on the Children's Embedded Figures Test

Degree of Field Independence—relative position of the student's score in the distribution of the sample scores on the Children's Embedded Figures Test. A high score represents a high degree of field independence.

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\(^{22}\) Ibid.

\(^{23}\) Ibid., p.5.

\(^{24}\) Ibid.
CHAPTER II

REVIEW OF THE LITERATURE

Introduction

This section is divided into four parts. The first two, Individual Differences in Learning and Individualized Instruction, are a review of learned opinions regarding individual differences and present attempts to adapt instruction to individual differences. The last two sections consist of a review of studies indirectly related to this study. The third section, Aptitude-Instruction Interaction in Mathematics, reviews studies in which the treatment variable is instruction in mathematics. The fourth section, Field-Dependence-Independence Studies, reviews studies in which the field independence construct is used as an aptitude variable in an educational setting.

A thorough review of the literature failed to produce any studies which investigated the relationship between the field independence construct and instructional strategy in mathematics.

Individual Differences in Learning

Gagné has noted that "... the question of how people differ in the rate, extent, style and quality of their learning is one which has concerned psychologists for a great many years. Yet ... we do not know much more about individual differences in learning than we did thirty years ago."¹

¹Robert M. Gagne, "Learning and Individual Differences: Introduction to the Conference", ed. R. M. Gagne, Learning and Individual Differences (Columbus, Ohio: Charles E. Merrill Books), 1967, XI.
Cronbach also notes the lack of a coherent theory of individual differences in learning. He calls for an experimental strategy of attempting to isolate aptitude variables relevant to individual differences in learning and designing alternative treatments to interact with those variables. He states: "I presume that an individual has greater aptitude for learning, say, to multiply from one method of teaching than from another method that is equally good on the average." He broadly defines aptitude as "... whatever promotes the pupil's survival in a particular educational environment, and it may have as much to do with styles of thought and personality variables as with the abilities covered in general tests." He further argues that instructional strategy, as opposed to learning rate, is the key to successful individualization because a person's learning rate is dependent upon the nature of instruction.

Jensen has also noted the lack of knowledge concerning individual differences in learning. In an article in which he attempted to delineate individual differences in learning he concluded: "... in this attempt to offer some description of the domain of ID's in learning at our present state of knowledge, or at least my own state of knowledge, I feel very much like one of the legendary blind men who tried to describe an elephant. At this stage more than one approach is obviously warranted."2


Individualized Instruction

Tyler, at the Abington Conference in 1967, reviewed the current situation in individualized instruction and identified six major concepts prevalent in individualized programs at that time. He identified "... the concept that the individual should be able to work at his own rate" as "... the concept most commonly followed in current efforts to individualize instruction." The other points he mentions are: (2) Pupil should be able to work at times convenient to him. (3) Slow learner should not be embarrassed by feeling that he is much slower than others. (4) Learning progress is linear. (5) A few factors, easily identifiable, interfere with progress. (6) With the wealth of media of communication, audio, visual, various kinds of forms, static and moving, the learner can select the one or more which is or are effective for him.

Tyler raises many of the nagging questions concerning individualization, and further suggests that we may be placing the proverbial cart before the horse. He asks: "Have we really devised the strategy for learning before we've tried to develop the individualization of it? If we don't do that, then maybe we are just assuming that everybody learns following the same strategy, and we merely provide more of the materials that we now have rather than materials required for an effective strategy."

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5 Ibid. pp. 4-5.

6 Ibid. p.7.
IPI (Individually Prescribed Instruction) is perhaps the most widely known of present attempts to individualize instructional systems. IPI provides a sequenced individualized program for grades K-6 in four subject areas; mathematics, reading, science and spelling. The three major components in the IPI program are detailed sequential "behaviorally-stated" instructional objectives, diagnostic testing and the individual prescriptions. The individual prescription is the most important of the three. The prescription is prepared by the teacher based on the diagnostic tests and cumulative personal record of the child. Once the objective is decided, the teacher chooses from among a number of instructional settings, one appropriate for the child. These instructional options include tutoring by the teacher, small-group instruction, work sheets, listening to tapes or discs, and viewing filmstrips. Glaser has noted: "At the present state of our knowledge, the decision rules for going from measures of student performance to instructional prescriptions may not be very complex, but little is known about the amount of complexity required. ... Sustained analysis of such information about individual difference-learning environment relationships should result in the ability to supply the teacher with the kind of data reduction and information that will enable him to manage the task of adapting to individual differences." 

Pieronek recently conducted a survey of individualized reading and mathematics programs. She personally visited schools across North America.
in which SAMI (Systematic Approach to Mathematical Instruction) and IPI were being used. She identifies self-pacing as the predominant feature of IPI and notes that IPI has an excellent detailed hierarchical structure in mathematics. She found SAMI very similar to the mathematics program of IPI. However, SAMI also features self-selection of learning materials. For any given unit, the student is free to choose his instructional materials from pre-packaged alternatives.\textsuperscript{10}

The literature reviewed in the above two sections indicates that very little progress has been made in the past thirty years toward a theory of individual differences in learning and that the lack of such a theory is severely hampering the success of current attempts to individualize instruction. It has been suggested by many educators, including Cronbach, Gagné and Tyler, that a two-pronged experimental strategy of isolating aptitude variables relevant to individual differences in learning and designing alternative instructional strategies to be tested for interaction with these variables is needed for the development of an effective systematic approach to the adaptation of educational stimuli to learners.

\textbf{Aptitude-Instruction Interaction in Mathematics}

Becker has commented that "... few studies have been designed to investigate the interaction between aptitude and instruction," and in particular, that "... only a small number of such studies deal directly with mathematics learning."\textsuperscript{11} He further states that there is as yet no evidence to sup-

\textsuperscript{10} Florence T. Pieronek, "A Survey of Individualized Reading and Mathematics Programs", Calgary Catholic School Board, Calgary, Alberta, 1971, ED 047894.

port the generalizability of research in other areas to mathematics.

Aiken\textsuperscript{12} also notes the paucity of studies in mathematics which resulted in significant interactions between aptitude and instructional strategy. He does, however, report three unpublished doctoral studies, two of which will be described below. These studies are also discussed in Cronbach's report.\textsuperscript{13} Cronbach's discussion of one of them is quoted below.

Becker\textsuperscript{14} used two programmed instructional strategies to teach high school algebra students to sum series. One approach was expository, the students were given the formulas and explanations relating these to the series in both verbal and symbolic form. The other was discovery oriented, the student was given examples of the relationships sought and asked to discover the formula. Two aptitude measures were used (verbal and numerical). However, no significant interactions between aptitudes and instructional methods were found.

Carry (1967) conducted a dissertation comparing geometric-graphical vs. algebraic-analytical presentations using programmed instructional materials in the mathematics of quadratic inequalities. Criterion measures representing both immediate recall and transfer to new problems were obtained for 181 high-school geometry students. Carry hypothesized that spatial visualization would be called for in the graphical treatment and so would predict success in it, more than in the algebraic treatment. He hypothesized also that general reasoning would relate more highly to learning from algebraic than from graphical instruction. The


\textsuperscript{13} Lee J. Cronbach and Richard E. Snow, "Individual Differences in Learning Ability as a Function of Instructional Variables" Final Report. Stanford University, California School of Education ED 029 001.

data did not confirm these hypotheses. No interactions were obtained with the recall criterion for either aptitude variable. Significant interaction was detected for the transfer measure, but the low internal consistency of this measure made overall findings suspect. Analyses at the item level showed two of the eight transfer items involved in interactions with aptitude. For both items, the reasoning measure was found predictive of responses in the graphical treatment but not in the analytic treatment. For one item, spatial aptitude also predicted graphic but not analytic achievement. Without confirmation, results such as these are uninterpretable.15

King, Roberts and Kropp16 administered a battery of aptitude tests to 426 fifth and sixth grade students (four fifth grade classes and four sixth grade classes). The students were then classified as verbal or figural and within these groups were classified as deductive or inductive. Four instructional strategies, verbal-deductive, verbal-inductive, figural-deductive and figural-inductive, were developed for a programmed two day unit on elementary set concepts. Classes were assigned intact to one of the four strategies. The investigators hypothesized that verbal ability measures would correlate significantly higher than figural ability measures with achievement on materials presented verbally. Similar effects were hypothesized between figural ability measures and materials presented figurally, between inductive measures and materials presented inductively, and between deductive measures and materials presented deductively.

The dependent measure was a criterion test of 24 items, 12 presented

15Lee J. Cronbach and Richard E. Snow, "Individual Differences in Learning Ability as a Function of Instructional Variables" Final Report Stanford University, California School of Education, ED 029 001, 125.

16F. J. King, Dennis Roberts and Russell P. Kropp, "Relationship Between Ability Measures and Achievement under Four Methods of Teaching Elementary Set Concepts", Journal of Educational Psychology, LX, 1969, 244-47.
verbally and 12 presented figurally. The data was analysed using regression analysis and t-tests were used to detect significant differences between pairs of regression coefficients. A summary of the results follows.

None of the t ratios for the verbal-figural comparisons was significant, so there was no support for ATI effects in the verbal or figural groups. However, the deductive-inductive contrasts support the hypothesis because two of the t ratios were significant at the .05 level and the differences in both cases were in the hypothesized direction. Thus the Inference Test(deduction) was a better predictor for the deductive materials than for the inductive materials. For the Word Grouping Test(induction) the converse was true. 17

A review of the literature confirmed Becker and Aiken's comments regarding the lack of aptitude-treatment interaction research in mathematics education. Only three studies were found and the diversity of the studies made it impossible to draw any conclusions with regard to aptitude-treatment interactions in mathematics education.

Field-Dependence-Independence Studies

Davis and Klausmeier 18 conducted two separate studies involving cognitive style and a concept identification task. The first varied cognitive style and level of complexity of the task. The second varied cognitive style and the training procedure. In both studies, the Hidden Figures Test (which correlates r = .62 with Witkin's Embedded Figures Test) was used to categorize students on the cognitive style factor. Students were identified as high, middle or low analytic depending upon their position in the distribution of scores on the test. High analytic represented the ability


to identify the hidden figures.

In experiment one, the three levels of cognitive style were used and complexity was defined as the number (1, 3 or 5) of bits of irrelevant information in the problem. Ninety senior high school males formed the study's sample. The data indicated a main effect due to cognitive style. The high analytics identified the concepts with greater ease than the low analytics. However, there was no interaction between cognitive style and complexity.

In the second experiment, the two extreme levels of the cognitive style factor, high analytic and low analytic, and four training conditions were used. The training conditions were prompt, verbal only, verbal-prompt, and control, the standard procedure for the task. Eighty senior high school males (40 high analytic and 40 low analytic) formed the sample. As in the first experiment, a significant main effect due to cognitive style was found. However, no significant interaction effect was found between cognitive style and training condition. Davis and Klausmeier concluded that "... these training procedures do not differentially influence concept identification for individuals manifesting different cognitive styles."¹⁹

Davis²⁰ investigated the relationship between cognitive style and two different concept identification tasks. The Hidden Figures Test was administered to 600 senior high school females. Thirty-six students who were classified as analytic (+1 standard deviation above the mean) and

¹⁹Ibid. p. 429.

thirty-six who were classified as global (-1 standard deviation below the mean) were chosen to participate in the study. The two concept identification tasks were labelled nonsign-differentiated (NSD) and sign-differentiated (SD). A problem whose solution "... is not based upon a particular sign or cue but rather upon the conditional relationship between figures"\(^21\) was labelled as NSD. An SD problem was one in which the corresponding NSD problem was changed to allow for its solution by isolating a relevant cue. That is, the SD problem could be solved using the conditional rules (as was required for the corresponding NSD) or the relevant cue. Davis hypothesized that global students would solve the NSD problem sooner than analytic students because the solution requires a global strategy to discover the relationship between the two figures. He further hypothesized that, since the SD problem could be solved by an analytic strategy, analytic students would solve the SD problem sooner than global students.

Two dependent measures were used to analyse the results: (1) the number of trials to criterion and (2) the number of errors to criterion. From the analysis of variance performed on the data, the following conclusions were drawn:

The results of this study only partially support the hypotheses. The overall performance of the analytic Ss was superior to the overall performance of the global Ss. The significant interaction of cognitive style and problem type, however, demonstrated that this superiority was restricted only to the SD problem. There was no difference between cognitive style levels on the NSD problem.

It was hypothesized that the analytic Ss would perform best on the SD problem and that the global Ss would perform best on the NSD problem. It was found that the analytic Ss performed significantly better on the SD problem than they did on the NSD problem, but that

\(^{21}\)Ibid. p.3.
the global Ss did not perform better on the NSD problem. In fact, their performance on both problems was virtually the same and in general was quite poor."

A study conducted by Davis and Grieve explored the relationship between cognitive style and instructional strategy in the teaching of geography. Two levels of instructional strategy, discovery and expository, and two levels of cognitive style, analytic and global, were factorially combined to give a 2 x 2 design. The Hidden Figures Test was administered to 117 grade nine students and a median split was used to classify students as analytic or global. The two instructional strategies each required eleven hours to complete and differed only in the placement of the verbalization of the generalization to be drawn from the instructional unit. In the discovery method the verbalization was delayed until the end. Whereas, in the expository method, the verbalization was the initial step in the sequence. Two dependent measures were used: (1) a multiple-choice test measuring a knowledge of the geography studied and (2) a multiple-choice test measuring the ability to use geographic materials in new situations.

The data was analysed twice. The first time, all the subjects were used. The second analysis utilized only those subjects who had been identified as extreme analytic or extreme global. (74 of the 117 students were classified as extreme) The major conclusions of the study were:


1. Neither cognitive style nor method of instruction had an overall effect on the acquisition of knowledge. With respect to the extreme analytic and extreme global Ss, however, it was found that extreme global males receiving the expository instruction experienced significant difficulty in acquiring knowledge of Japan's geography. This finding suggests that the expository method of instruction should be avoided when teaching extremely global males unless sufficient time is devoted to establishing those discriminations which are basic to the generalizations that are to be learned.

2. An individual's cognitive style was found to differentially influence his higher learning scores. Analytic Ss were better able to apply knowledge of geography to new situations than were global Ss. Neither of the methods of instruction were found to have an overall effect on higher learning performance, but the analysis of the extreme Ss indicated that global males receiving the expository instruction experienced significant difficulty in applying knowledge to new situations.\(^{24}\)

Hester and Tagatz\(^{25}\) study investigated the interaction effect of cognitive style and instructional strategy on concept attainment. Ten similar concept identification tasks were presented to 72 graduate education students by one of two different instructional strategies, commonality or conservative. These students had been identified as analytic or global according to their relative position to the median score of the group on the TIPT (Tagatz Information Processing Test). TIPT has been found to correlate significantly with the Hidden Figures Test at the .01 level. "Those receiving the commonality instruction were directed to look at the "focus" instance and the exemplars, and to determine the attributes common to these cards. The conservative strategy directed Ss to compare the "yes" and "no" cards with the "focus" instance. A "yes" card differed by one irrelevant attribute and a "no" card differed by

\(^{24}\)Ibid. p.141.

one relevant attribute. The dependent variable was the subjects time-to-criterion score on each task. The data supported the following conclusions:

(a) Ss displaying the analytic cognitive style can efficiently utilize either the conservative or commonality instructional strategy. The analytic cognitive style seems to be an inherent organismic characteristic that enables Ss to achieve the differentiation required by the more rigorous conservative instructional strategy.

(b) Ss displaying the global cognitive style are able to utilize efficiently the commonality instructional strategy, which does not require fine discriminations within the stimulus field and is therefore related to their cognitive style. Ss displaying the global cognitive style are unable to utilize and are inhibited by the more rigorous conservative instructional strategy.

Saarni investigated differences in problem solving as a function of the cognitive developmental level of the subject and the subject's cognitive style. It was hypothesized that Piaget's theory of the development of logical thinking would provide an over-all framework for understanding complex problem-solving performances and that Witkin's construct of field independence would prove fruitful in understanding individual differences within each Piagetian developmental level.

Sixty-four students (eight male and eight female students randomly selected from each of grades six, seven, eight and nine) participated in the study. Two Piagetian tasks were used to identify the students' present cognitive developmental level (formal operational, transitional or concrete operational). The portable rod and frame test was used to iden-

26 Ibid. p.232.
27 Ibid. p.236.
tify the student's level of field independence. Three levels of field independence were used: low, medium and high. The range of these levels was determined by ranking the rod and frame test scores and dividing the resulting distribution into thirds.

Two detective stories constituted the problem-solving tasks, and four dependent measures were used to measure performance on each of the stories. The data was analysed using multivariate analysis of variance. Among the conclusions of the study was the following:

The construct field independence appears to have doubtful implications for complex problem solving performance. The analyses indicate that field independence within each Piagetian level does not affect complex, multi-step problem solving performance as manifested in the Productive Thinking problems. This does not invalidate the role field independence might have in determining performance on problems which are more perceptually bound and/or relatively non-verbal. The results obtained here, however, cast doubt on the generality of the field independence construct as a "cognitive style" or as a consistent characteristic of the individual in his intellectual functioning.²⁹

Of the five studies reported above which used field-dependence as an aptitude variable, only one, the study by Davis and Grieve, utilized an instructional setting as the treatment variable. In this study, it was hypothesized that students would perform best when taught by a method consistent with their cognitive style. The results, however, confirmed this hypothesis only in the case of extreme global students.

Three of the studies used a concept identification task as the treatment variable and the fourth, the study by Saarni, used a problem-solving task as the treatment variable. Of the three studies which used

²⁹Ibid. pp.19-20
a concept identification task, the studies by Davis and Hester and Tagatz reported partial interactions between field independence and treatment. The Davis study hypothesized that students would perform best on the concept identification task for which the solution required a strategy most closely matched to the student's cognitive style. The Hester and Tagatz study hypothesized that students would perform best on the concept identification task when the training procedure matched the student's cognitive style. In the Davis study the hypothesized outcome was true only for the analytic student and in the Hester and Tagatz study the reverse outcome was observed.

Discussion of the Literature

The student in an individualized program in mathematics proceeds at his own pace, using a variety of audio-visual devices and pre-packaged instructional materials, through a detailed hierarchical model of mathematics content. Instructional strategies are not adapted to individual learning differences of the child, because as Gagne, Cronbach and Jensen have pointed out, very little is known at present about individual differences in learning relevant to an educational setting. Tyler has suggested, that until a coherent theory of individual differences in learning is developed, successes in the area of individualized instruction will be less striking than anticipated.

Only a few isolated studies of interaction between aptitude and instructional strategy in mathematics were found. Two of these reported significant interactions. In both of these studies, the aptitude variable was reasoning measure and the instructional strategies were presented in programmed booklets. However, the dearth of studies in this area,
makes it impossible to draw any conclusions concerning relevant aptitude variables to mathematics instruction.

Five studies using field independence as an aptitude variable were presented. Three of these studies, Davis, Hester and Tagatz, and Davis and Grieve, provide partial support for the hypothesized outcomes of this study: specifically that a student's performance will be best on the instructional strategy which is most closely matched to his cognitive style.

In conclusion, the literature indicates the need for extensive research both in the area of individual differences in learning and in the area of adaptation of instructional strategies to these individual differences. The review of the literature in mathematics education provided no guidance as to an appropriate aptitude variable to be considered in instruction in mathematics. The literature did indicate that Witkin's field-independence construct is a potential relevant aptitude variable in an educational setting. However, of the five studies which utilized field independence as an aptitude variable in an educational setting, three of the studies used a concept identification task as the treatment variable and a fourth used a problem-solving task as the treatment variable. Only one, the study by Davis and Grieve, used an instructional setting as the treatment variable. Thus, while the need for research into the interaction between aptitude variables and instructional strategies has been recognized for the past decade, very few studies of this nature have been conducted in a classroom instructional setting.
CHAPTER III

DESIGN AND PROCEDURE

INTRODUCTION

The study was carried out concurrently with one conducted by Marian Weinstein, a doctoral candidate at the University of British Columbia. The Weinstein study investigated the effects of four instructional strategies in the teaching of simple and complex algorithms at the grade five level. The instructional strategies were: (1) an algebraic justification approach, (2) a pattern justification approach, (3) a mixed approach, pattern followed by algebraic, and (4) a mixed approach, algebraic followed by pattern. The present study investigated the interaction effect between two of these approaches, pattern and algebraic, and the child's degree of field independence. All instructional materials, tests and covariates are those of the Weinstein study.

Population

The population consisted of twelve grade five classes in six schools in a lower mainland school district in British Columbia which were also participating in the Weinstein study. These classes had been assigned at random to one of four possible treatments; (1) a pattern approach on simple and complex algorithms 1, (2) an algebraic approach on simple and complex algorithms 1, (3) a pattern approach on simple and complex algorithms 2, and (4) an algebraic approach on simple and complex algorithms 2. There were three classes in each of the four treatments.
The grade five level was chosen because the students are unfamiliar with the algorithms used, yet possess the necessary prerequisites for the learning of the algorithms.

Sample

Using a table of random numbers, a random sample of one-half of the students who completed the Weinstein study in each of the twelve classes was selected to participate in the study. These students were then tested on the cognitive style factor.

INSTRUCTIONAL MATERIALS

Four algorithms, two classified as simple and two classified as complex, formed the basis of the instructional units. For each algorithm, two different instructional strategies were used; one called a pattern instructional strategy and the other an algebraic instructional strategy.

The pattern instructional strategy used diagrams extensively, whereas the algebraic approach relied heavily on renaming and the associative, distributive and commutative field properties. Diagrams were never used in the algebraic approach and conversely, renaming or the field properties were never used in the pattern approach.

The instructional units were divided into stages with practise for the student at each stage. The objectives for each stage were stated for the teacher as well as the suggested completion time. The simple algorithms required five instructional periods, while the complex algorithms required nine instructional periods. Worksheets were also provided with specific instructions as to when they were to be used. A
A description of each of the units for the four algorithms follows.

**S1: Product of a Mixed Number and a Fraction**

- **Stage 1:**
  - (a) whole number and a unit fraction \((\frac{1}{n})\)
  - (b) whole number and a proper fraction
  - (c) whole number and a mixed number

- **Stage 2:**
  - (a) unit fraction and a unit fraction
  - (b) proper fraction and a proper fraction
  - (c) proper fraction and a mixed number

**Pattern Approach.** At all stages, finding the areas of rectangles is the physical analogy used to justify the algorithm. Stage 1(c) used the principle of conservation of area by partitioning rectangles. See Appendix A for the development of stages 2(a) and 2(b). Stage 2(c) is developed by partitioning rectangles.

**Algebraic Approach.** Stages 1(a) and 1(b) made use of the repeated addition model for multiplication \((5 \times \frac{1}{2} = \frac{1}{2} + \ldots + \frac{1}{2})\). Stage 1(c) utilized renaming a mixed number as a whole number plus a fraction and then used the distributive principle. See Appendix A for the development of stages 2(a) and 2(b). Stage 2(c) was developed through renaming and the distributive principle.

**S2: Comparison of Fractions Using the Cross-Product**

- **Stage 1:**
  - (a) comparing a fraction with 1
  - (b) comparing a fraction with whole numbers

- **Stage 2:**
  - (a) generating equivalent fractions with the common denominator the product of the two denominators
  - (b) cross-product rule: \(\frac{a}{b} > \frac{c}{d}\) if \(a \times d > b \times c\)
**Pattern Approach.** Stages 1(a) and 1(b) were developed by renaming the whole numbers via diagrams. Stage 2(a) was developed through diagrams by cutting all pieces in the original diagram in the same manner. The student was taught that if he wanted to compare for example thirds and fourths, he would cut each of the thirds into fourths and each of the fourths in thirds giving twelfths in each diagram. See Appendix A for the development of stage 2(b).

**Algebraic Approach.** Stage 1(a) was developed by choosing an appropriate equivalent name for 1 and then relying on a multiplication argument. Stage 1(b) involved renaming the whole number as the whole number x 1 and then choosing an appropriate name for 1. Stage 2(a) was developed by writing the two fractions to be compared as multiplication statements and then multiplying by an appropriate equivalent form of 1. See Appendix A for the development of stage 2(b).

**Cl: Changing a Fraction to a Decimal**

**Stage 1: Prerequisites**

(a) decimal system
(b) division of decimals by whole numbers
(c) interpretation of a fraction as division \(a/b = a \div b\)

**Stage 2: The Algorithm**

(a) terminating decimals
(b) non-terminating decimals

The two approaches differed in their development of stage 1 yet were identical in the development of stage 2.
**Pattern Approach.** Stage 1(a) is developed through the use of rectangles divided into tenths and hundredths. See Appendix A for the development of stages 1(b) and 1(c).

**Algebraic Approach.** Stage 1(a) was developed by the renaming of ones, tenths, and hundredths ($1 = 10 \times 1/10; 1/10 = 10 \times 1/100$). See Appendix A for the development of stages 1(b) and 1(c).

**C2: Finding the Square Root of a Fraction**

**Stage 1:** Prerequisites

(a) multiplication of fractions  
(b) concept of the square root of a whole number  
(c) square root of a fraction as $\sqrt{\text{numerator}}/\sqrt{\text{denominator}}$

**Stage 2:** The Algorithm

(a) division technique for the square root of wholes  
(b) division technique for the square root of fractions  
(c) approximating square roots

For each of the approaches, stage 1(a) was developed in the same manner as stages 2(a) and 2(b) of the corresponding approach in algorithm SI. See Appendix A. Stage 2(b) is a drawing together of stages 1(c) and 2(a).

**Pattern Approach.** Stage 1(b) was presented as the factors of a number which will produce a square with area that number. Stage 1(c) was developed by dividing a $1 \times 1$ square into an equal number of parts, each part being $1/\text{denominator}$ of the area. The number of these parts
needed to make up the numerator were selected and a square was made. Stage 2(a) was justified by using an area of rectangles argument. A process of squeezing between width and height of the rectangles was demonstrated. See Appendix A for the development of stage 2(c).

**Algebraic Approach.** In stage 1(b) the square root was immediately defined as that number which when multiplied by itself gives the whole number in question. The square root is found by listing all the facts associated with that whole number. Stage 1(c) was developed through a renaming of both wholes and fractions. Stage 2(a) used a closing-in argument. The student was taught that as one factor gets larger, its corresponding factor gets smaller. See Appendix A for the development of stage 2(c).

**MEASURING INSTRUMENTS**

**Criterion Pretests**

"Each of the criterion pretests for the four algorithms consisted of free response items designed to test knowledge of that algorithm's prerequisites as determined by a panel of mathematics education judges. The pretests were given in order to adjust criterion scores for differences among classes in 'readiness' for the instructional material."¹ A breakdown of items of the four pretests as well as the KR-20 reliability coefficients is given in Table 1 below.

<table>
<thead>
<tr>
<th>Type of Item</th>
<th>S1</th>
<th>S2</th>
<th>C1</th>
<th>C2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Multiplication facts</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>Diagrammatic representation of fractions</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>Reciprocals</td>
<td>3</td>
<td>3</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>Fractions as multiplication statements</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>1 as the multiplicative identity</td>
<td>4</td>
<td></td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Commutativity and associativity</td>
<td>3</td>
<td></td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>Distributive law</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Area formula</td>
<td>5</td>
<td></td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>Long division algorithm</td>
<td></td>
<td></td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>Rewriting a mixed number as a sum</td>
<td>4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Conservation of area</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Comparing fractions using diagrams</td>
<td></td>
<td></td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>Fractional names for 1</td>
<td>3</td>
<td></td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>Division as sharing</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Reversibility of multiplication and division</td>
<td></td>
<td></td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>Preservation of inequalities when multiplied by a positive</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>42</td>
<td>32</td>
<td>30</td>
<td>35</td>
</tr>
</tbody>
</table>

KR-20 Reliability Coefficients

.90 .84 .90 .90
Criterion Computation Tests

"Each of the computation tests for the four algorithms consisted of free response items designed to test the student's ability to perform that algorithm." Descriptions of the items of the four tests as well as the KR-20 reliability coefficients are given below in Tables 2, 3, 4 and 5.

Table 2
Types of Items of S1, Product of a Fraction and a Mixed Number, Computation Test and KR-20 Reliability Coefficient

<table>
<thead>
<tr>
<th>Item Type</th>
<th>Number of Items</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fraction x whole number</td>
<td>6</td>
</tr>
<tr>
<td>Mixed x whole number</td>
<td>6</td>
</tr>
<tr>
<td>Fraction x fraction</td>
<td>6</td>
</tr>
<tr>
<td>Mixed x fraction</td>
<td>6</td>
</tr>
<tr>
<td>Total</td>
<td>24</td>
</tr>
<tr>
<td>KR-20</td>
<td>.94</td>
</tr>
</tbody>
</table>

Ibid p.24
Table 4
Types of Items of C1, Changing a Fraction to a Decimal, Computation Test and KR-20 Reliability Coefficient

<table>
<thead>
<tr>
<th>Item Type</th>
<th>Number of Items</th>
</tr>
</thead>
<tbody>
<tr>
<td>Converting fractions to terminating decimals</td>
<td>8</td>
</tr>
<tr>
<td>Approximating the decimal equivalent of fractions</td>
<td>10</td>
</tr>
<tr>
<td>Total</td>
<td>18</td>
</tr>
</tbody>
</table>

| KR-20                                                 | .90             |

Table 5
Types of Items of C2, Finding the Square Root of a Fraction, Computation Test and KR-20 Reliability Coefficient

<table>
<thead>
<tr>
<th>Item Type</th>
<th>Number of Items</th>
</tr>
</thead>
<tbody>
<tr>
<td>Square root of a perfect square fraction</td>
<td>8</td>
</tr>
<tr>
<td>Square root of a non-perfect square fraction</td>
<td>7</td>
</tr>
<tr>
<td>Total</td>
<td>15</td>
</tr>
</tbody>
</table>

| KR-20                                                 | .90             |
Criterion Generalization Tests

"Each of the generalization tests for the four algorithms consisted of 30 free response items. The tests were based on five types of items—those designed to measure: the ability to shortcut the algorithm because of the numbers involved (Type A), the ability to define the entire problem when given the solution and a part of the problem (Type B), the ability to extend the algorithm to more than two operands (Type C), the ability to use the algorithm with numbers other than the type studied (Type D), and the ability to explain an alternate approach to the algorithm (Type E)."  

The types of items of the four tests as well as the KR-20 reliability coefficients are reported in Tables 6, 7, 8 and 9 below.

Table 6

Types of Items of S1, Product of a Fraction and a Mixed Number, Generalization Test and KR-20 Reliability Coefficient

<table>
<thead>
<tr>
<th>Item Type</th>
<th>Number of Items</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>8</td>
</tr>
<tr>
<td>B</td>
<td>6</td>
</tr>
<tr>
<td>C</td>
<td>5</td>
</tr>
<tr>
<td>D</td>
<td>4</td>
</tr>
<tr>
<td>B-C</td>
<td>3</td>
</tr>
<tr>
<td>E</td>
<td>4</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>30</strong></td>
</tr>
</tbody>
</table>

| KR-20 | .83 |

3Ibid. p.25.
Table 7
Types of Items of S2, Comparison of Fractions, Generalization Test and KR-20 Reliability Coefficient

<table>
<thead>
<tr>
<th>Item Type</th>
<th>Number of Items</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>8</td>
</tr>
<tr>
<td>B</td>
<td>6</td>
</tr>
<tr>
<td>C</td>
<td>8</td>
</tr>
<tr>
<td>D</td>
<td>4</td>
</tr>
<tr>
<td>E</td>
<td>4</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>30</strong></td>
</tr>
</tbody>
</table>

**KR-20** .82

Table 8
Types of Items of CL, Changing a Fraction to a Decimal, Generalization Test and KR-20 Reliability Coefficient

<table>
<thead>
<tr>
<th>Item Type</th>
<th>Number of Items</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>8</td>
</tr>
<tr>
<td>B</td>
<td>9</td>
</tr>
<tr>
<td>D</td>
<td>9</td>
</tr>
<tr>
<td>E</td>
<td>4</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>30</strong></td>
</tr>
</tbody>
</table>

**KR-20** .80
Table 9

Types of Items of C2, Finding the Square Root of a Fraction, Generalization Test and KR-20 Reliability Coefficient

<table>
<thead>
<tr>
<th>Item Type</th>
<th>Number of Items</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>8</td>
</tr>
<tr>
<td>B</td>
<td>9</td>
</tr>
<tr>
<td>D</td>
<td>9</td>
</tr>
<tr>
<td>E</td>
<td>4</td>
</tr>
<tr>
<td>Total</td>
<td>30</td>
</tr>
</tbody>
</table>

KR-20 .88

Testing on Cognitive Style Factor

The Children’s Embedded Figures Test is a modified version of Witkin’s Embedded Figures Test. The twenty-five item test consists of eleven pictures of complex figures in which a triangular shape is embedded and fourteen pictures in which a house-shaped form is embedded.

The test is a modification by Karp and Konstadt of the Children’s Embedded Figures Test developed by Goodenough and Eagle in 1963. Goodenough and Eagle's test, although it has high reliability was considered too bulky and costly for use in this study.

The Embedded Figures Test was found to be too difficult and frustrating for children aged ten and under and to require modifications in...
administration for children in the ten to eleven age bracket.5 The Children's Embedded Figures Test reduced the difficulty and frustration of the Embedded Figures Test through the use of simple and complex forms more familiar to the child and through the elimination of the pressure of a time limit. The child is also given more than one opportunity to locate the figure, although only first tries are actually used for scoring purposes.

The test was standardized using one hundred and sixty children, ranging in age from five to twelve years, randomly selected from elementary schools in Brooklyn. "Because of the small N's involved, these normative data can be considered only tentative."6 Reliability estimates for children in the age groups 9-10 and 11-12 are recorded in Table 10 below. The children in the present study were in the 10-11 age bracket.

Table 10
CEFT Reliability Estimates and Validity Coefficients

<table>
<thead>
<tr>
<th>Age</th>
<th>N</th>
<th>Internal Consistency r</th>
<th>r CEFT, EFT</th>
</tr>
</thead>
<tbody>
<tr>
<td>9-10</td>
<td>40</td>
<td>.88</td>
<td>.71</td>
</tr>
<tr>
<td>11-12</td>
<td>40</td>
<td>.87</td>
<td>.85</td>
</tr>
</tbody>
</table>

Sources:
Internal Consistency r: A Manual For The Embedded Figures Tests, 1971, Table 4, p.25.


5Ibid. p.17.

6Ibid. p.24.
PROCEDURE

Prior to the actual beginning of the study in the classrooms, a meeting was held between the researchers (Weinstein and O'Brien) and the participating teachers. At this time, the two basic approaches were outlined and the necessity of following the materials carefully was emphasized. A General Information Sheet was given to the teachers. This sheet included reminders on do's and don't's for using the materials (already discussed at the meeting) and the home phone numbers of the researchers.

An individual class followed either an algebraic or pattern approach throughout S1 followed by C1 or S2 followed by C2. There were no mixed sequences of algorithms. A flow chart of the general procedure is contained in Figure 1.

**Figure 1**

*Flow Chart of the Procedure*

- Pretest on simple algorithm → instruction on simple algorithm → computation and generalization tests on simple algorithm → pretest on complex algorithm → instruction on complex algorithm → computation and generalization tests on complex algorithm → administration of the Children's Embedded Figures Test

The correcting of all tests, pretests, computation and generalization tests, was done by the two researchers.

Following the completion of the instructional and testing sequence on the algorithms, testing on the cognitive style factor was con-
ducted by this investigator. The test was administered according to the procedure outlined in the test manual.\footnote{Ibid. pp.26-28.}

**CONTROLS**

There were three controls on teacher variation. These were: (1) detailed daily instructional guides and worksheets were provided; (2) the researchers visited each of the twelve teachers every second day to discuss the progress of the study and (3) all tests were corrected by the two researchers.

Hawthorne effect was controlled by: (1) all teaching and testing on the instructional materials was conducted by the classroom teacher and (2) the individual testing on the cognitive style factor was carried out only after the completion of instruction in and testing on the two algorithms.

A possible "differences in rater" problem in testing on the cognitive style factor was overcome by the researcher solely doing the testing.

Two statistical controls were also used. These were: (1) pretest scores were used as covariates to adjust for initial differences among students and (2) three classes were assigned to each of the four possible treatments to help in the control of teacher variation.
Each of the data sets of the eight post-tests was analyzed separately using multiple linear regression techniques. The computing facilities at the University of British Columbia were used and a multiple linear regression program contained in the personal file of Dr. Seong Soo Lee, Faculty of Education of the University of British Columbia, was used.
CHAPTER IV

ANALYSIS OF THE DATA

The reader will recall that two post-tests, a computation test which measured ability to perform the algorithm and a generalization test which measured ability to shortcut, extend and explain the algorithm, were administered upon completion of instruction on each of the four algorithms. The two simple algorithms were S1, the product of a fraction and a mixed number, and S2, the comparison of fractions. C1, changing a fraction to a decimal, and C2, finding the square root of a fraction were classified as complex algorithms. For the purposes of analysis, each of these eight post-tests was treated as an independent unit of data. Multiple linear regression techniques were used to test the three null hypotheses for each data set. Where significant interactions were found, the interactions were graphed. The following discussion of the statistical technique applies to each of the eight separate analyses of the data.

A linear relationship between the dependent variable, score on the post-test, and independent variables was assumed. The independent variables were method of instruction, field independence and interaction between method of instruction and field independence. This relationship may be expressed as:

\[ Y_1 = \beta_0 + \beta_1 V_{11} + \beta_2 V_{21} + \beta_3 V_{31} + \beta_4 V_{41} + \epsilon_i \]

where \( Y_1 \) = observation of \( i \)th subject on the post-test
\[ V_{1i} = \text{observation of } ith \text{ subject on the covariate} \]
\[ V_{2i} = \text{observation of } ith \text{ subject on the method of instruction} \]
\[ V_{3i} = \text{observation of } ith \text{ subject on the field independence measure} \]
\[ V_{4i} = \text{observation of } ith \text{ subject on the interaction between method of instruction and field independence} \]

\[ \beta_1, \ldots, \beta_4 \text{ are the population regression coefficients} \]
\[ u_i \text{ is the departure of } Y_i \text{ from the linear model} \]

Smillie states:

We may use the analysis of variance to test the hypothesis that the last \( p-k \) independent variables, \( X_{k+1}, X_{k+2}, \ldots, X_p \), for some \( k \leq p \) do not make a significant contribution to the regression sum of squares, \( SSR \) computed for all \( p \) independent variables. Suppose that the regression sum of squares computed for the regression model with only the first \( k \) independent variables included is \( SSR' \). Then it may be shown that the difference \( SSR - SSR' \) is distributed as \( \chi^2 \) with \( p-k \) degrees of freedom. Thus the ratio

\[
P = \frac{(SSR - SSR')}{(p-k)} \frac{(SSR - SSR'')}{SSSE / (n-p-1)}
\]

has an \( F \)-distribution with \( p-k \) and \( n-p-1 \) degrees of freedom, and may be used to test the hypothesis that \( \beta_k = \beta_{k+1} = \beta_{k+2} \ldots = \beta_p = 0 \).\(^1\)

Restricted regression models were thus defined in order to test the following null hypotheses.

\( H_0 \): There is no significant difference in mean post-test scores between students taught by a pattern instructional strategy and students taught by an algebraic instructional strategy.

\( H_1 \) was tested by computing the sum of squares of the regression coefficients for the regression model:

(2) \( Y_1 = \beta_0 + \beta_1 V_{11} + \beta_2 V_{21} + \beta_3 V_{31} + \beta_4 V_{41} + \epsilon_i \) (omitting method, \( V_{21} \)) and applying formula I.

\( H_2 \): There is no significant difference in mean post-test scores between groups of students differing in degree of field independence. \( H_2 \) was tested by computing the sum of squares of the regression coefficients for the regression model:

(3) \( Y_1 = \beta_0 + \beta_1 V_{11} + \beta_2 V_{21} + \beta_3 V_{31} + \beta_4 V_{41} + \epsilon_i \) (omitting field independence, \( V_{31} \)) and applying formula I.

\( H_3 \): There is no significant interaction between students' degree of field independence and instructional strategy. \( H_3 \) was tested by computing the sum of squares of the regression coefficients for the regression model:

(4) \( Y_1 = \beta_0 + \beta_1 V_{11} + \beta_2 V_{21} + \beta_3 V_{31} + \beta_4 V_{41} + \epsilon_i \) (omitting interaction, \( V_{41} \)) and applying formula I.

In addition, for each data set, the significance of the contribution of the covariate to the regression sum of squares of the linear model chosen was tested by computing the sum of squares of the regression coefficients for the regression model:

(5) \( Y_1 = \beta_0 + \beta_2 V_{21} + \beta_3 V_{31} + \beta_4 V_{41} + \epsilon_i \) (omitting covariate, \( V_{11} \)) and applying formula I.

The eight separate analyses of the data are reported in Tables 11-18 below. The distribution of scores on the field independence mea-
sure are contained in Appendix C. The raw data are also contained in Appendix C.

In each of the tables below, SSR and SSR refer to the sum of squares of the regression coefficients of the full model, (1), and the sum of squares of the regression coefficients of the restricted models, (2), (3), (4), or (5), respectively. The numbers in each of these columns correspond to the regression models defined above. The computer program uses n-p as the degrees of freedom for the denominator as opposed to n-p-1. Sixty-four subjects took the S1 and C1 post-tests. Thus, in the analyses of S1 and C1 computation and generalization scores, the computed F-values have one degree of freedom for the numerator and sixty degrees of freedom for the denominator. Seventy-seven subjects took the S2 and C2 post-tests. Thus, in the analyses of S2 and C2 computation and generalization scores, the computed F-values have one degree of freedom for the numerator and seventy-three degrees of freedom for the denominator.

<p>| Table 11 |</p>
<table>
<thead>
<tr>
<th>Analysis of S1 Computation Scores</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
<tr>
<td><strong>Significance of covariate</strong></td>
</tr>
<tr>
<td>(1) .08228</td>
</tr>
<tr>
<td>(5) .05974</td>
</tr>
<tr>
<td>1.4737</td>
</tr>
<tr>
<td>.22949</td>
</tr>
<tr>
<td><strong>Significance of method</strong></td>
</tr>
<tr>
<td>(1) .08228</td>
</tr>
<tr>
<td>(2) .08217</td>
</tr>
<tr>
<td>.0071</td>
</tr>
<tr>
<td>.93306</td>
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<tr>
<td><strong>Significance of field indep.</strong></td>
</tr>
<tr>
<td>(1) .08228</td>
</tr>
<tr>
<td>(3) .06194</td>
</tr>
<tr>
<td>1.3295</td>
</tr>
<tr>
<td>.25346</td>
</tr>
<tr>
<td><strong>Significance of interaction</strong></td>
</tr>
<tr>
<td>(1) .08228</td>
</tr>
<tr>
<td>(4) .08228</td>
</tr>
<tr>
<td>0.0</td>
</tr>
<tr>
<td>1.00000</td>
</tr>
</tbody>
</table>

*With large N's, such as in the sample of this study, there is no significant difference between using n-p and n-p-1.*
In the analysis of S1 Computation Scores presented in Table 11, the three null hypotheses of no significant main effect due to instructional strategy, no significant main effect due to degree of field independence and no significant interaction effect between method and degree of field independence were not rejected at \( \alpha = .05 \). The F-value for the covariate was significant at \( p = .22949 \).

**Table 12**  
Analysis of S2 Computation Scores

<table>
<thead>
<tr>
<th></th>
<th>SSR</th>
<th>SSR'</th>
<th>F-Value</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Significance of covariate</td>
<td>(1) .19443</td>
<td>(5) .16895</td>
<td>2.3092</td>
<td>.13292</td>
</tr>
<tr>
<td>Significance of method</td>
<td>(1) .19443</td>
<td>(2) .17842</td>
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<td>.23224</td>
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<tr>
<td>Significance of field indep.</td>
<td>(1) .19443</td>
<td>(3) .19426</td>
<td>.0151</td>
<td>.90253</td>
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<td>Significance of interaction</td>
<td>(1) .19443</td>
<td>(4) .19111</td>
<td>.3008</td>
<td>.58509</td>
</tr>
</tbody>
</table>

From Table 12 it can be seen that in the analysis of S2 Computation Scores, the three null hypotheses of no significant main effect due to instructional strategy, no significant main effect due to degree of field independence and no significant interaction effect between method and degree of field independence were not rejected at \( \alpha = .05 \). The F-value for the covariate was significant at \( p = .13292 \).

In the analysis of S1 Generalization Scores presented in Table 13, the three null hypotheses of no significant main effect due to instruction-
al strategy, no significant main effect due to degree of field independence and no significant interaction effect between method and degree of field independence were not rejected at $\alpha = .05$. The F-value for the covariate was significant at $p = .00808$.

Table 13
Analysis of S1 Generalization Scores

<table>
<thead>
<tr>
<th></th>
<th>SSR</th>
<th>SSR'</th>
<th>F-Value</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Significance of covariate</td>
<td>(1) .33100</td>
<td>(5) .24728</td>
<td>7.5087</td>
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<tr>
<td>Significance of method</td>
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<td>(2) .33053</td>
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<tr>
<td>Significance of field indep.</td>
<td>(1) .33100</td>
<td>(3) .31496</td>
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<td>.23501</td>
</tr>
<tr>
<td>Significance of interaction</td>
<td>(1) .33100</td>
<td>(4) .32617</td>
<td>.4332</td>
<td>.51294</td>
</tr>
</tbody>
</table>

Table 14
Analysis of S2 Generalization Scores

<table>
<thead>
<tr>
<th></th>
<th>SSR</th>
<th>SSR'</th>
<th>F-Value</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Significance of covariate</td>
<td>(1) .18560</td>
<td>(5) .09223</td>
<td>8.3700</td>
<td>.00502</td>
</tr>
<tr>
<td>Significance of method</td>
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<td>(2) .16575</td>
<td>1.7793</td>
<td>.18639</td>
</tr>
<tr>
<td>Significance of field indep.</td>
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<td>(3) .16591</td>
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<td>.18809</td>
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<tr>
<td>Significance of interaction</td>
<td>(1) .18560</td>
<td>(4) .16221</td>
<td>2.0969</td>
<td>.15187</td>
</tr>
</tbody>
</table>
In the analysis of S2 Generalization Scores presented in Table 14, the three null hypotheses of no significant main effect due to instructional strategy, no significant main effect due to degree of field independence and no significant interaction effect between method and degree of field independence were not rejected at $\alpha = .05$. The F-value for the covariate was significant at $p = .00502$.

Table 15
Analysis of C1 Computation Scores

<table>
<thead>
<tr>
<th></th>
<th>SSR</th>
<th>SSR'</th>
<th>F-Value</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Significance of covariate</td>
<td>(1) .34542</td>
<td>(5) .15500</td>
<td>17.4544</td>
<td>.00010</td>
</tr>
<tr>
<td>Significance of method</td>
<td>(1) .34542</td>
<td>(2) .32787</td>
<td>1.6087</td>
<td>.20956</td>
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<tr>
<td>Significance of field indep.</td>
<td>(1) .34542</td>
<td>(3) .32587</td>
<td>1.7918</td>
<td>.18576</td>
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<tr>
<td>Significance of interaction</td>
<td>(1) .34542</td>
<td>(4) .32518</td>
<td>1.8551</td>
<td>.17826</td>
</tr>
</tbody>
</table>

Table 16
Analysis of C2 Computation Scores

<table>
<thead>
<tr>
<th></th>
<th>SSR</th>
<th>SSR'</th>
<th>F-Value</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Significance of covariate</td>
<td>(1) .29443</td>
<td>(5) .14279</td>
<td>15.681</td>
<td>.00017</td>
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<tr>
<td>Significance of method</td>
<td>(1) .29443</td>
<td>(2) .28271</td>
<td>1.2128</td>
<td>.27440</td>
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<tr>
<td>Significance of field indep.</td>
<td>(1) .29443</td>
<td>(3) .25451</td>
<td>4.1302</td>
<td>.04576</td>
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<tr>
<td>Significance of interaction</td>
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<td>(4) .28788</td>
<td>.6773</td>
<td>.41321</td>
</tr>
</tbody>
</table>
From Table 15 it can be seen that in the analysis of C1 Computation Scores, the three null hypotheses of no significant main effect due to instructional strategy, no significant main effect due to degree of field independence and no significant interaction effect between method and degree of field independence were not rejected at $\alpha = .05$. The F-value for the covariate was significant at $p = .00010$.

In the analysis of C2 Computation Scores presented in Table 16, the two null hypotheses of no significant main effect due to instructional strategy and no significant interaction effect between method and degree of field independence were not rejected at $\alpha = .05$. The null hypothesis of no significant main effect due to degree of field independence was rejected at $\alpha = .05$. The F-value for the covariate was significant at $p = .00017$.

Table 17

<table>
<thead>
<tr>
<th></th>
<th>SSR</th>
<th>SSR'</th>
<th>F-Value</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Significance of covariate</td>
<td>.51876</td>
<td>.34471</td>
<td>21.6996</td>
<td>.00002</td>
</tr>
<tr>
<td>Significance of method</td>
<td>.51876</td>
<td>.48083</td>
<td>4.7291</td>
<td>.03361</td>
</tr>
<tr>
<td>Significance of field indep.</td>
<td>.51876</td>
<td>.48124</td>
<td>4.6782</td>
<td>.03454</td>
</tr>
<tr>
<td>Significance of interaction</td>
<td>.51876</td>
<td>.46123</td>
<td>7.1725</td>
<td>.00954</td>
</tr>
</tbody>
</table>
In the analysis of C1 Generalization Scores presented in Table 17, the three null hypotheses of no significant main effect due to instructional strategy, no significant main effect due to degree of field independence and no significant interaction effect between method and degree of field independence were rejected at $\alpha = .05$. The F-value for the covariate was significant at $p = .00002$.

Table 18
Analysis of C2 Generalization Scores

<table>
<thead>
<tr>
<th></th>
<th>SSR</th>
<th>SSR'</th>
<th>F-Value</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Significance of covariate</td>
<td>(1) .63736</td>
<td>(5) .21713</td>
<td>84.5927</td>
<td>.00000</td>
</tr>
<tr>
<td>Significance of method</td>
<td>(1) .63736</td>
<td>(2) .62134</td>
<td>3.2247</td>
<td>.07668</td>
</tr>
<tr>
<td>Significance of field indep.</td>
<td>(1) .63736</td>
<td>(3) .62623</td>
<td>2.2401</td>
<td>.13880</td>
</tr>
<tr>
<td>Significance of interaction</td>
<td>(1) .63736</td>
<td>(4) .60991</td>
<td>5.5250</td>
<td>.02145</td>
</tr>
</tbody>
</table>

In the analysis of C2 Generalization Scores presented in Table 18, the null hypotheses of no significant main effect due to instructional strategy and no significant main effect due to degree of field independence were not rejected at $\alpha = .05$. The null hypothesis of no significant interaction between method and degree of field independence was rejected at $\alpha = .05$. The F-value for the covariate was significant at $p = .00000$. 
Graphing of Significant Results

Since the null hypothesis of no significant main effect due to degree of field independence was rejected in the analysis of C2 computation scores and since the null hypothesis of no significant interaction effect between method and field independence was rejected in the analysis of C1 and C2 generalization scores, the results of the C2 computation test and the C1 and C2 generalization tests were graphed to aid in the interpretation of the results.

The UBC computer program CGROUP\(^3\) was used to identify natural range groupings on the field independence measure. CGROUP optimally clustered subjects by matching subjects on field independence scores and dependent variable scores in such a way as to minimize variation within the created groups. The scores on the field independence measure ranged from 9-25. Five optimal groupings were identified by CGROUP. These were: (1) 9-15, (2) 16-18, (3) 19-20, (4) 21-22 and (5) 23-25.

The C2 computation scores were adjusted using the C2 pretest as a covariate, and the C1 and C2 generalization scores were adjusted using the C1 and C2 pretests as covariates. The means of the resulting residual scores were calculated for each of the five range groups on both the pattern and algebraic approaches. The graphs of these means follow. The means and the number of observations for each group are contained in Appendix C.

\(^3\)This program is on file at the University of British Columbia Computing Centre.
Discussion of the Figures

The reader should bear in mind that the graphs of the results do not provide statistical support on which to base conclusions, but rather serve only as an aid in interpreting significant results indicated by the statistical analysis of the data. The reader should further keep in mind that the group sizes were unequal.

The analysis of C2 computation scores indicated a significant main effect due to degree of field independence, but no significant main effects due to method or interaction between method and degree of field independence. Figure 2 indicates that on both the pattern and algebraic approaches, except for the extreme field independent group on the pattern approach, with an increase in degree of field independence there is a corresponding increase in level of achievement.

The analysis of C1 generalization scores indicated a significant main effect due to method, a significant main effect due to degree of field independence and a significant interaction effect. Figure 3 indicates this interaction effect is due to the performances of the extreme ranges. For extreme field dependent students, 9-15 range, the pattern instructional strategy resulted in a higher level of achievement than the algebraic instructional strategy; while for extreme field independent students, 23-25 range, the algebraic instructional strategy resulted in a higher level of achievement than the pattern instructional strategy. The figure also indicates that the significant main effect of method is due to the high level of achievement of the field independent group on the algebraic instructional strategy. This high level of achievement of the field independent group
Figure 2 Mean Residual Scores on C2 Computation Scores

Pattern

Algebraic

Mean Residual Scores

Field Independence Ranges
Figure 3 Mean Residual Scores on Cl Generalization Scores

Pattern

Algebraic

Field Independence Ranges
on the algebraic instructional strategy also accounts for the significant
main effect due to degree of field independence. Due to this high level
of achievement of the extreme field independent group taught by the al­
gebraic approach, the overall mean of the extreme field independent group
is greater than the mean of all other groups.

The analysis of C2 generalization scores indicated a significant
interaction effect. Figure 4 indicates that students in the extreme field
independent group, 23-25 range, achieved higher on the average than all
other students when taught by the algebraic approach and achieved lower on
the average than all other students when taught by the pattern approach.
There is a similar, but less extreme, effect in the 21-22 range group.
The analysis of the data also indicated a trend toward a main effect due
to method, but the null hypothesis was not rejected. Figure 4 indicates
that the trend is toward the algebraic approach, but again the performance
of the extreme field independent group inflates the overall effect of the
algebraic approach and deflates the overall effect of the pattern approach.

Figures 3 and 4 indicate that, except for the two extreme field
independent ranges, 21-22 and 23-25, the interaction results are inconclu­
sive. In the 16-18 range group and the 19-20 range group there is no inter­
action with method. For the extreme field dependent group, 9-15 range,
there is some evidence of interactions. However, the interactions on C1
and C2 for this group are in the opposite directions and hence no conclu­
sions can be drawn for this group. The inconsistent results for this
group may be due to the small number of subjects in this range, six on C1
and eight on C2. This range contained the smallest number of subjects of
all the ranges of the field independence dimension.
Figure 4 Mean Residual Scores on C2 Generalization Scores

Pattern

Algebraic

Mean Residual Scores

Field Independence Ranges
However, the results are consistent with respect to the two extreme field independent ranges, 21-22 and 23-25. For the 21-22 range, with regard to performance on the generalization tests on the complex algorithms, there is a trend toward superiority of the algebraic instructional strategy. For the 23-25 range, with regard to performance on the generalization tests on the complex algorithms, the algebraic instructional strategy was strongly superior to the pattern instructional strategy.

Discussion of the Results

The high total sums of squares of the full models, SSR in each of the separate analyses indicate that the model chosen was appropriate to the data. The statistical analysis of the data also indicates that the covariates were well chosen. In each of the data analyses, the covariate accounted for the largest portion of the sum of squares of the regression coefficients and in each of the data analyses, except for S1 and S2 computation scores, the contribution of the covariate to the total sum of squares of the regression coefficients was significant at $\alpha = .01$. For S1 the contribution of the covariate was significant at $\alpha = .23$ and for S2 at $\alpha = .14$.

The results of the analysis of the data indicate that for the two examples of simple algorithms chosen, on both the computation and generalization post-tests, there was no statistically significant main effect due to instructional strategy, no statistically significant main effect due to degree of field independence and no statistically significant interaction effect between instructional strategy and field independence.
Thus, for the simple algorithms, the two instructional strategies did not produce significantly different results and there was no differential achievement effect due to degree of field independence. There was also no evidence of differential achievement on the two instructional strategies within groups of students similar in their degree of field independence.

The statistical analysis of the data indicates that for performance on the computation tests on the complex algorithms, there is no significant main effect due to instructional strategy and no significant interaction effect between instructional strategy and degree of field independence. Thus, with regard to producing computational ability with the complex algorithms, the two instructional strategies are not significantly different overall, and also do not produce significantly different results within groups similar in their degree of field independence. However, the analysis of the data, does indicate that there is differential achievement on the C2 computation test among students differing in degree of field independence. The graphing of the results of the C2 computation test indicate that with increasing degree of field independence, the mean level of achievement increases with a slight tapering off at the extreme field independent level.

The statistical analysis of C1 generalization scores indicates a significant main effect due to degree of field independence. However, the analysis of the C2 generalization scores did not support this finding.

The statistical analysis of C1 generalization scores indicates a significant main effect due to instructional strategy and the analysis of C2 generalization scores indicates a strong trend, F-value has p = .077,
toward a main effect due to instructional strategy. The graphs of the results of the C1 and C2 generalization tests indicate that this significant main effect is due to the high level of performance of students at the two highest levels of the field independence dimension on the algebraic instructional strategy.

The statistical analyses of the C1 and C2 generalization scores indicate a significant interaction effect between instructional strategy and degree of field independence. The graphs of these results, however, provide conclusive information only for the two extreme levels of field independent students, 21-22 and 23-25 ranges. For these students the algebraic approach is superior to the pattern approach and is dramatically superior for the students in the 23-25 range.
CHAPTER V

CONCLUSIONS AND IMPLICATIONS

SUMMARY

A study was conducted to determine the interaction effect, if any, between the field independence construct and two instructional strategies, a pattern strategy which used diagrams extensively and an algebraic strategy which used algebraic field properties familiar to the child and was devoid of diagrams. A review of the relevant literature indicated that cognitive style may hold the key to understanding individual differences in learning and at the same time that very few studies have investigated the interaction effect between cognitive style and instructional strategy. No studies were found which investigated the interaction effect between field independence and instructional strategy in mathematics.

The study was designed to answer three major questions: (1) Do children differing in their degree of field independence respond differently to the two instructional strategies?; (2) Is one of the instructional strategies, on the average, superior to the other?; (3) Is there differential achievement among students differing in their degree of field independence?

Twelve grade five classes formed the population of the study. These classes were part of the sample of a study being conducted by Marian Weinstein, a doctoral candidate at the University of British Columbia. Ms. Weinstein developed the two instructional strategies and the pretests and post-tests used in the study. Ms. Weinstein classified algorithms as sim-
ple or complex on the basis of the number of prerequisites required for their acquisition and two examples of each type were selected for use in the study. The two simple algorithms chosen were the product of a fraction and a mixed number, S1 and the comparison of fractions, S2. The two complex algorithms chosen were changing a fraction to a decimal, C1 and finding the square root of a fraction, C2. Each class went through the following instructional and testing sequence, going through either S1 and C1 or S2 and C2: Pretest on simple algorithm → instruction on simple algorithms(5 sessions) → computation and generalization tests on the simple algorithm → pretest on complex algorithm → instruction on complex algorithm(9 sessions) → computation and generalization tests on complex algorithm.

In each of the twelve classes, one half of the students who completed the Weinstein study were randomly selected to form the sample of this study. The Children's Embedded Figures Test was then individually administered to these students.

Multiple linear regression techniques were used to analyse the data. Eight separate analyses of the data were performed: the results of the computation and generalization tests were analysed separately for each of the two simple and two complex algorithms. Three null hypotheses were each tested at $\alpha = .05$: (1) There is no significant difference in mean post-test scores between students taught by a pattern instructional strategy and students taught by an algebraic instructional strategy; (2) There is no significant difference in mean post-test scores between groups of students differing in degree of field independence; (3) There is no sig-
significant interaction between students' degree of field independence and instructional strategy.

The analysis of the data resulted in acceptance of the three null hypotheses for the simple algorithms and for the C1 computation test. On the C2 computation test, the data resulted in rejection of the null hypothesis of no significant main effect due to degree of field independence. The analysis of the complex algorithm generalization tests resulted in rejection of the null hypothesis of no significant interaction between instructional strategy and degree of field independence.

LIMITATIONS

The research agreement with the school district stipulated the use of existing intact grade five classes. Thus, only classes and not subjects were randomly assigned to treatments.

Teacher preferences for mode of mathematical instruction was an uncontrolled factor in the study. No assessment was made of the teacher's instructional strategy preferences nor of the teachers attitude toward the instructional strategy randomly assigned to them. However, since the teachers participated on a voluntary basis, it is assumed that their attitude towards the experimental materials was positive.

The subjects' previous instructional experiences in mathematics is an uncontrolled variable in the study. No assessment was made of the types of instructional strategies to which the subject had been exposed in his schooling prior to the study, nor of the instructional strategy predominantly used in his mathematics experiences.
The use of the familiar classroom setting and time-schedule minimized the Hawthorne Effect. However, the amount and length of the testing done in the study was a departure from the norm for mathematics and can be considered to have contributed to a Hawthorne Effect. However, the testing schedule was uniform for all subjects.

There is an uncontrolled cumulative learning effect. It is impossible to separate the effects of instruction on the simple algorithm from the effects of instruction on the complex algorithm, because instruction on the simple algorithm preceded instruction on the complex algorithm.

DISCUSSION OF THE RESULTS

The analyses of the computation and generalization scores for the two simple algorithms, product of a fraction and a mixed number and comparison of fractions, resulted in not rejecting the null hypotheses of no significant difference between the two instructional strategies and of no significant interaction between degree of field independence and instructional strategy. There appears to be two equally plausible explanations for these findings. The first is that for simple algorithms, the algebraic and pattern instructional strategies are equally effective both for the sample as a whole and for groups of the sample differing in their degree of field independence. The second is that the period of exposure to the two approaches, five sessions, was insufficient to allow for differential effects to emerge.

The analysis of the C2 computation test scores indicated that, with the exception of the extreme field independent students, as the degree of field independence increased, the level of achievement increased. The data
for the CI computation test, however, did not support this finding. Closer examination of the two complex algorithms may provide an insight into these findings. Although the two complex algorithms, changing a fraction to a decimal and finding the square root of a fraction, are similar in their degree of complexity, defined by the number of prerequisites required for their acquisition, the investigator claims that the algorithms differ in their degree of abstractness. Changing a fraction to a decimal involves an extension of the place value system to the right and the learning of division in this extended system. Finding the square root of a fraction cannot rely on such an extension of previously learned mathematical concepts, nor does it naturally arise out of the real world experiences of the child, thus making it more abstract for the child than changing a fraction to a decimal. The investigator tentatively suggests that at the grade five level, there may be a positive correlation between the child's degree of field independence and his ability to cope with abstract concepts.

The significant interaction effects indicated by the analyses of the complex algorithm generalization tests partially supported the investigator's hypothesized outcomes. The investigator had hypothesized that children differing in their degree of field independence would respond differently to the two instructional strategies. That is, children tending toward field dependency would perform better on the pattern approach than on the algebraic approach, and children tending toward field independency would perform better on the algebraic approach than on the pattern approach. No conclusive results could be drawn for field dependent children. However,
for children tending toward field independency, with regard to performance on the complex algorithm generalization tests, the algebraic approach was superior to the pattern approach, as had been hypothesized. In fact, the performance of the extreme field independent students on the complex algorithm generalization tests, when taught by the algebraic instructional strategy, was superior to the performance of all other students. However, the performance of these students on the complex algorithms generalization tests, when taught by the pattern instructional strategy, was dramatically inferior to their performance when taught by the algebraic instructional strategy and on the C2 generalization test, was dramatically inferior to the performance of all other students. Thus, while on the average the pattern instructional strategy and the algebraic instructional strategy produce equivalent results, they do not produce equivalent results with respect to field independent children. For these children, the algebraic instructional strategy appears to be superior to the pattern instructional strategy.

CONCLUSIONS

The following conclusions were drawn from the results of the study:

1. The field independence construct is a profitable aptitude variable to be considered in the adaptation of mathematical instructional strategies to individual learners.

2. For students at the grade five level, with the exception of extreme field independent students, an instructional strategy which uses diagrams extensively is equally effective as an instructional strategy
which is devoid of diagrams and which uses algebraic properties.

3. Students at the grade five level, with perhaps the exception of extreme field dependent students, can cope adequately with instructional explanations which are based solely on algebraic properties familiar to the students and which are devoid of concrete or pictorial aids.

4. For extreme field independent students in this study, with regard to instruction in complex algorithms, as measured by pupil performance, an instructional strategy in mathematics which relies extensively on the field properties of the number system and which is devoid of diagrams is superior to an instructional strategy which relies extensively on diagrams.

IMPLICATIONS OF THE STUDY

The review of the literature in mathematics education revealed the lack of a systematic approach to adapting mathematics instructional strategies to individual learners. Research studies in mathematics have been primarily concerned with the search for the superior instructional strategy for mathematics at specific grade levels. Yet this study suggests that although two instructional strategies may produce equivalent results on the average, they do not necessarily produce equivalent results for individual groups of students.

There is a need for a concentrated, coordinated research effort to establish a theoretical basis for the assignment of learners to instructional strategies in mathematics. This will require drawing on the
resources of both learning theorists and developers of mathematics curriculum. The learning theorists can aid in the isolation and characterization of relevant individual differences in learners. The curriculum developers can aid in the designing of alternative instructional strategies. Together, through research studies similar to the one reported in this paper, both the learning theorist and the curriculum developer can refine their contributions to this area of research. Out of this refined body of knowledge new research studies should arise and the refining process repeated. This study suggests that the area of cognitive style, and in particular the construct of field independence, may prove to be a fruitful starting point for such a research effort.

The study also has implications for mathematics education with respect to instruction of an algebraic nature, in particular with respect to algebra "readiness" and to the introduction of algebraic instructional strategies in grades six to nine. This study suggests that field independent children may be better able to cope with instructional material of an algebraic nature than their field dependent counterparts, and that knowledge of the child's degree of field independence may provide profitable insight into the child's readiness for algebraic instructional materials.


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I have 9 (3 x 3) rectangles, each of area \( \frac{1}{3} \), so I have a total area of 6 \( \times \frac{1}{3} = \frac{6}{3} \), as I said I should.

Let's find the product: \( \frac{3}{5} \times 4 \).

I want to find the area of a rectangle with dimensions \( \frac{3}{5} \times 4 \); I draw:

\[ \frac{3}{5} \]

\[ \frac{1}{1} \]

\[ \frac{1}{1} \]

\[ \frac{1}{1} \]

Notice that I have 4 columns each of area \( \frac{3}{5} \times 1 = \frac{3}{5} \), so I have a total area of \( 4 \times \frac{3}{5} = \frac{12}{5} \).

Therefore, \( \frac{3}{5} \times 4 = \frac{12}{5} \)

What would be the product: \( \frac{3}{4} \times 6 \)?

Anticipate the answer \( \frac{18}{4} \). Then draw the diagram to verify:

\[ \frac{3}{4} \]

\[ \frac{1}{1} \]

\[ \frac{1}{1} \]

\[ \frac{1}{1} \]

\[ \frac{1}{1} \]

\[ \frac{1}{1} \]

I have 6 columns each of area \( \frac{3}{4} \), so I have a total area of \( \frac{18}{4} \), as suggested.

Ask the students to find these products:

\[ \frac{3}{7} \times 2 \]

\[ \frac{4}{8} \times 3 \]

\[ \frac{1}{5} \times 4 \]

Explain that they may always refer back to a diagram to make certain.

Have three students come to the board to diagrammatically explain their answers.

** Here the students will learn to find the products of unit fractions by finding the area of appropriate rectangles; he will do problems such as \( \frac{1}{2} \times \frac{1}{3} \) or \( \frac{1}{5} \times \frac{1}{6} \).

Draw a large unit square on the board.
Label the sides as indicated.

What happens if I cut the square like this? What are the dimensions of the rectangles I have shaded? Notice that the original square had area 1.

Expect the answer 1 by \( \frac{1}{3} \).

What is the area of each of these rectangles?

Expect the answer \( \frac{1}{3} \).

What happens if I also cut the square this way?

What are the dimensions of the new smaller rectangles formed?

Expect the answer \( \frac{1}{2} \) by \( \frac{1}{3} \).

Now the area of each of these smaller rectangles can be represented by a multiplication statement, namely \( \frac{1}{2} \times \frac{1}{3} \).

What is the area of each of these smaller rectangles? Notice that I know the area of the entire square was 1 square unit, so the area of each of these rectangles must be a fraction of that and the fraction is determined by the number of rectangles there are in the figure.

How many rectangles are there of dimensions \( \frac{1}{2} \) by \( \frac{1}{3} \)?

Expect the answer 6.

Therefore, each of the rectangles has what area?

Expect the area \( \frac{1}{6} \).
Since we already said this area could be represented in a multiplication way as $\frac{1}{2} \times \frac{4}{3}$, that means

$$\frac{1}{2} \times \frac{4}{3} = \frac{1}{6}$$

How would we find $\frac{1}{4} \times \frac{1}{7}$?

We draw a rectangle with dimensions 1 by 1, like so: 1

1

And cut it: 1

How many small rectangles are there in the unit square? What's the area of each?

Expect the answers 23 and $\frac{1}{28}$ respectively.

Since each area can also be represented as $\frac{1}{4} \times \frac{1}{7}$, that means that

$$\frac{1}{4} \times \frac{1}{7} = \frac{1}{28}$$

What do you think $\frac{1}{3} \times \frac{1}{6}$ will be?

Expect the answer $\frac{1}{18}$.

Let's check with a diagram. I can draw:

1

And each rectangle, which is of dimensions $\frac{1}{3}$ by $\frac{1}{6}$, does have area $\frac{1}{18}$ since there are 18 of them in the unit square.

What about $\frac{1}{9} \times \frac{1}{4}$?

Expect the answer $\frac{1}{36}$.

Let's check. I can draw:
AND I SEE THAT THERE ARE 36 SMALL RECTANGLES, SO EACH HAS AREA $\frac{1}{36}$.

But since the dimensions of each are $\frac{1}{4}$ by $\frac{1}{4}$, then

$$\frac{1}{4} \times \frac{1}{4} = \frac{1}{36}.$$

Do you notice that in each of these problems, if I multiply $\frac{1}{4}$ by $\frac{1}{4}$, I get a figure with $\square \times \square$ small rectangles in it, each of area, therefore, $\frac{1}{4} \times \frac{1}{4}$? Check back to our diagram for $\frac{1}{3} \times \frac{1}{3}$.

$$\frac{1}{3} \times \frac{1}{3} \text{ and } \frac{1}{4} \times \frac{1}{4}.$$

** Here the student will learn how to find products of any two proper fractions by finding the areas of appropriate rectangles; he will do problems such as

$$\frac{4}{5} \times \frac{2}{3} \text{ or } \frac{1}{5} \times \frac{3}{7}.$$

Suppose we want to multiply $\frac{2}{3}$ by $\frac{3}{4}$. We know that we can find the area of a rectangle with these dimensions to solve our problem.

So let's start out with a square again.

IF WE WANT ONE OF THE DIMENSIONS TO BE \( \frac{2}{3} \), WE CUT IT SO:

IF WE WANT THE OTHER DIMENSION TO BE \( \frac{3}{4} \), WE CUT IT SO:

THEN THE AREA OF THE SHADED REGION IS THE AREA I WANT.

How many of the small rectangles are shaded in?

Expect the answer 6.
TANGLES SHADED IN.
WHAT IS THE AREA OF EACH OF THESE RECTANGLES?

Expect the answer $\frac{4}{12}$.

WE ALREADY LEARNED THAT SINCE THERE ARE $3 \times 4$ OF THESE RECTANGLES ALTOGETHER, EACH HAS AREA $\frac{4}{12}$.

THEREFORE, WE HAVE $2 \times 3$ PIECES EACH OF AREA $\frac{4}{12}$, SO WE HAVE A TOTAL AREA OF $\frac{6}{12}$.

NOTICE THAT THE NUMERATOR OF MY ANSWER TELLS ME THE NUMBER OF SHARED PIECES AND THE DENOMINATOR TELLS ME THE TOTAL NUMBER OF PIECES IN THE FIGURE.

LET'S FIND $\frac{3}{5} \times \frac{2}{3}$ USING THIS APPROACH:

I DRAW:

```
\begin{center}
\begin{tikzpicture}
\draw[very thick] (0,0) grid (2,2);
\end{tikzpicture}
\end{center}
```

THEN THE NUMBER OF SHARED SQUARES IS $3 \times 2$. EACH OF THESE SQUARES HAS AREA $\frac{1}{15}$ SINCE THERE ARE $5 \times 3$ OF THEM IN THE ENTIRE UNIT SQUARE, SO I HAVE A TOTAL AREA OF $\frac{6}{15}$.

THEREFORE, $\frac{3}{5} \times \frac{2}{3} = \frac{6}{15}$.

WHAT WOULD $\frac{4}{6} \times \frac{2}{5}$ BE?

Expect the answer: $\frac{8}{30}$

LET'S CHECK WITH A DIAGRAM:

```
\begin{center}
\begin{tikzpicture}
\draw[very thick] (0,0) grid (2,2);
\end{tikzpicture}
\end{center}
```

WE DO HAVE $4 \times 2$ SQUARES EACH OF AREA $\frac{1}{30}$ SHARED IN, SO A TOTAL AREA OF $\frac{8}{30}$

NOTICE THAT THE NUMERATOR OF MY PRODUCT IS THE PRODUCT OF MY NUMERATORS
(8 = 4 \times 2) \text{ just as the denominator of my product was the product of my denominators } (39 = 6 \times 5).

Ask the students to find the following products:

\[
\frac{3}{4} \times \frac{1}{5} \\
\frac{2}{5} \times \frac{4}{3} \\
\frac{4}{7} \times \frac{2}{5}
\]

Have three students come to the board to diagramatically explain their answers.

** Here the student will learn to find products of fractions and mixed numbers by finding the areas of appropriate rectangles with dimensions like \( \frac{1}{2} \) by \( \frac{3}{4} \) or \( \frac{2}{3} \) by \( 4 \frac{1}{6} \). **

Now that we know how to find areas of rectangles like \( \frac{3}{4} \) by 2 and \( \frac{3}{4} \) by \( \frac{1}{5} \), we are ready to find areas of rectangles like \( \frac{3}{4} \) by \( 2 \frac{1}{3} \) in order to find products like \( \frac{3}{4} \times 2 \frac{1}{3} \).

Suppose I want to find the solution to: \( \frac{3}{4} \times 2 \frac{1}{3} \).

I can draw a diagram like the following:

\[
\begin{array}{c}
\frac{3}{4} \\
\hline
\frac{1}{5} \\
\hline
2
\end{array}
\]

Recalling the review we did earlier about splitting up rectangles without changing area, I realize that I can split this rectangle into two parts so as to make the computation simpler, one rectangle with dimensions \( \frac{3}{4} \) by 2 and another with dimensions \( \frac{3}{4} \) by \( \frac{1}{3} \) and the area remains the same so long as I add these two numbers.

I can easily find the areas of each of these smaller rectangles.
Explain that they may always go back to a renaming statement to verify their answers.

** Here the students will learn to find the products of unit fractionssuch as
\[
\frac{1}{2} \times \frac{1}{3} \quad \text{or} \quad \frac{1}{5} \times \frac{1}{6} \quad . **
\]

SUPPOSE I WANT TO FIND THE ANSWER TO A MULTIPLICATION QUESTION WHERE
BOTH OF THE NUMBERS TO BE MULTIPLIED ARE FRACTIONS. FOR EXAMPLE,
WE MIGHT TRY \( \frac{1}{2} \times \frac{1}{3} \).
WHAT DO YOU SUGGEST THE ANSWER MIGHT BE?

Wait for a response of \( \frac{1}{6} \) before proceeding.

** LET'S SEE IF THIS IS REASONABLE. WHAT DO WE KNOW ABOUT \( \frac{1}{6} \)? ONE THING
WE KNOW IS THAT \( \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{5}{6} = 6 \times \frac{1}{6} = 1 \). SO,
WE KNOW THAT \( 6 \times \frac{1}{6} = 1 \). THEREFORE, \( \frac{1}{6} \) IS A NUMBER I CAN MULTIPLY BY 6
TO GET ONE.

WILL THERE BE ANY OTHER NUMBERS I CAN MULTIPLY BY 6 TO GET ONE?
IF THERE WERE, THEY WOULD HAVE TO EQUAL \( \frac{1}{6} \) SINCE THAT IS EXACTLY WHAT WE
MEAN BY \( \frac{1}{6} - (6 \times \frac{1}{6} = 1) \).

THEN, IF \( 6 \times (\frac{1}{6} \times \frac{1}{3}) = 1 \), THAT WOULD MEAN THAT \( \frac{1}{2} \times \frac{1}{3} = \frac{1}{6} \) SINCE
\( \frac{1}{6} \) WAS THE ONLY NUMBER I COULD MULTIPLY BY 6 TO GET ONE. BUT, \( 6 \times (\frac{1}{6} \times \frac{1}{3}) \)
\( = 3 \times 2 \times (\frac{1}{2} \times \frac{1}{3}) \)
\( = (3 \times \frac{1}{3}) \times (2 \times \frac{1}{2}) \)
\( = 1 \times 1 \)
\( = 1 \)

NOTICE THAT ALL I DID WAS TO DECIDE WHETHER \( \frac{1}{2} \times \frac{1}{3} = \frac{1}{6} \)
WAS TO SEE IF \( 6 \times (\frac{1}{2} \times \frac{1}{3}) = 1 \).
HOW WOULD WE FIND $\frac{3}{4} \times \frac{1}{7}$?

WE CHECK TO SEE IF $\frac{3}{4} \times \frac{1}{7} = \frac{3}{28}$ BY SEEING IF $28 \times (\frac{3}{4} \times \frac{1}{7}) = 1$.

BUT,

$$28 \times (\frac{3}{4} \times \frac{1}{7})$$

$$= 7 \times 4 \times (\frac{3}{4} \times \frac{1}{7})$$

$$= (7 \times \frac{1}{7}) \times (4 \times \frac{1}{4})$$

$$= 1 \times 1$$

$$= 1$$

SO, $28 \times (\frac{3}{4} \times \frac{1}{7}) = 1$.

SO, $$\frac{3}{4} \times \frac{1}{7} = \frac{1}{28}$$

WHAT DO YOU THINK $\frac{4}{3} \times \frac{2}{5}$ WILL BE?

Expect the answer $\frac{4}{28}$.

LET US CHECK THIS:

IF THIS IS SO, THEN $18 \times (\frac{4}{3} \times \frac{2}{5}) = 1$.

BUT,

$$18 \times (\frac{4}{3} \times \frac{2}{5})$$

$$= 6 \times 3 \times (\frac{4}{3} \times \frac{2}{5})$$

$$= (6 \times \frac{1}{3}) \times (3 \times \frac{4}{5})$$

$$= 1 \times 1$$

$$= 1$$

SO, $18 \times (\frac{4}{3} \times \frac{2}{5}) = 1$, so $\frac{4}{3} \times \frac{2}{5} = \frac{1}{28}$.

DO YOU NOTICE THAT IN EACH OF THESE PROBLEMS, IF I MULTIPLY $\frac{A}{B}$ BY $\frac{1}{\Delta}$, I GET AS AN ANSWER A FRACTION $\frac{A}{\Delta \times \Delta}$. THIS SEEMS REASONABLE SINCE IN EACH CASE IF I MULTIPLIED $\Delta \times \Delta$ BY $\frac{1}{\Delta}$, I GOT $(\frac{\Delta}{\Delta} \times \frac{A}{B}) \times (\Delta \times \frac{1}{\Delta}) = 1 \times 1 = 1$.

Here the student will learn how to find products of any two proper fractions.
Suppose we want to multiply \( \frac{2}{3} \) by \( \frac{3}{4} \). We know that we can do some rearranging to find this answer.

\( \frac{2}{3} \) can be rewritten as \( 2 \times \frac{1}{3} \) and \( \frac{3}{4} \) as \( 3 \times \frac{1}{4} \).

Therefore,
\[
\frac{2}{3} \times \frac{3}{4} = (2 \times \frac{1}{3}) \times (3 \times \frac{1}{4})
\]

Since I can multiply in any order, this is the same as
\[
(2 \times 3) \times (\frac{1}{3} \times \frac{1}{4})
\]

We already know that \( \frac{1}{3} \times \frac{1}{4} = \frac{1}{12} \).

We also know that \( (2 \times 3) \times \frac{1}{12} = \frac{6}{12} \).

Therefore,
\[
\frac{2}{3} \times \frac{3}{4} = \frac{6}{12}
\]

Let's find \( \frac{3}{5} \times \frac{2}{3} \) using this approach:

I write:
\[
\frac{3}{5} \times \frac{2}{3} = (3 \times \frac{4}{5}) \times (2 \times \frac{1}{3})
\]
\[
= (3 \times 2) \times (\frac{4}{5} \times \frac{1}{3})
\]
\[
= (3 \times 2) \times \frac{4}{15}
\]
\[
= \frac{6}{15}
\]
Therefore,
\[
\frac{3}{5} \times \frac{2}{3} = \frac{6}{15}
\]

What would \( \frac{4}{6} \times \frac{2}{5} \) be?

Expect the answer: \( \frac{8}{30} \)

Let's check with a rearranging:
\[
\frac{4}{6} \times \frac{2}{3} = (4 \times \frac{1}{6}) \times (2 \times \frac{1}{3})
\]
\[
= (4 \times 2) \times (\frac{1}{6} \times \frac{1}{3})
\]
\[
= (4 \times 2) \times \frac{1}{18}
\]
Ask the students to find the following products:

\[
\begin{align*}
\frac{3}{4} \times \frac{1}{6} \\
\frac{2}{5} \times \frac{1}{3} \\
\frac{4}{7} \times \frac{3}{5}
\end{align*}
\]

Have three students come to the board to show the renaming that leads to the answer.

Point out to the students that the numerator of their answer comes from the numerator of the two fractions they are multiplying since they are collecting numerators in their renaming tasks just as they are collecting denominators in terms of the fractions \( \frac{4}{11} \) and \( \frac{4}{8} \).

** Here the student will learn to find products of fractions and mixed numbers such as \( \frac{1}{2} \times 3\frac{4}{3} \) or \( \frac{2}{3} \times 4\frac{4}{5} \). **

** Now that we know how to find products like \( \frac{3}{4} \times 2 \) and \( \frac{3}{4} \times \frac{4}{5} \), we are ready to find products like \( \frac{3}{4} \times 2\frac{4}{3} \).**

Suppose I want to find the answer to: \( \frac{3}{4} \times 2\frac{4}{3} \).

I can write this as:

\[
= \frac{3}{4} \times \left( 2 + \frac{4}{3} \right)
\]

\[
= \frac{3}{4} \times 2 + \frac{3}{4} \times \frac{4}{3}
\]

Notice that I am splitting the mixed number here just as I did when I was finding the products of whole numbers and mixed numbers.

Then I can find the products of the individual parts and total them.

The first product is \( \frac{3}{4} \times 2 = \frac{6}{4} \) which I can verify by renaming:

\[
\frac{3}{4} \times 2
\]

\[
= (3 \times \frac{3}{4}) \times 2
\]

\[
= \left( \frac{3 \times 2}{} \right) \times \frac{4}{4}
\]

\[
= \frac{6}{4}
\]
Have three students show their work on the board. If any students use denominators other than the products of the given ones, explain that they are correct (if they are), but ask them to use the product as a guarantee that the procedure will work. For example, for \( \frac{1}{2} \) and \( \frac{3}{4} \), we might use the denominator 4, but we could certainly not do this for \( \frac{1}{5} \) and \( \frac{3}{7} \); point out that we can always use the product.

Be sure that the students realize that the product of the two denominators can always be used since we are insuring that each of the small pieces into which each of the diagrams is divided can be written as an integral number of the new subdivisions, and therefore the total fractions we start with can also be written as integral multiples.

**Here the student will learn that \( \frac{a}{6} > \frac{c}{8} \) only when \( a \times d > b \times c \), for example, \( \frac{3}{8} > \frac{2}{5} \) only because \( 3 \times 5 > 2 \times 6 \).**

Now that we already realize that an easier way to compare two fractions to decide which is larger is to rewrite them as fractions with a common denominator, we have practiced rewriting equivalent fractions.

Suppose I want to compare \( \frac{1}{2} \) with \( \frac{4}{6} \) to find which is greater.

I can rewrite these as equivalent fractions with the same denominator.

What fractions would I use?

Expect the answers: \( \frac{6}{12} \) and \( \frac{8}{12} \).

I can check this with a diagram.
THEN, I NOTICE THAT I AM COMPARING \( \frac{6}{12} \) WITH \( \frac{8}{12} \).

Which is greater?

Expect the answer: \( \frac{8}{12} \)

But where did the 6 in \( \frac{6}{12} \) come from? It came from \( 1 \times 6 \),

the one piece out of the two that I started with was cut into 6 pieces.

Where did the 8 come from? It came from \( 4 \times 2 \), the four pieces out of

the 6 that I started with were cut into 2 pieces each.

Why were the original halves cut into 6 pieces?

Because the other denominator was 6.

Why were the original sixths cut into 2 pieces each?

Expect the answer: Because the other denominator was 2.

So, in comparing \( \frac{1}{2} \) and \( \frac{4}{6} \), I am really comparing 6 and 3,

or \( 1 \times 6 \) and \( 4 \times 2 \).

To compare \( \frac{1}{2} \) and \( \frac{4}{6} \), I say \( 1 \times 6 < 4 \times 2 \), so \( \frac{1}{2} < \frac{4}{6} \).

Suppose I want to compare \( \frac{1}{2} \) with \( \frac{3}{5} \) to find which is greater.

I can rewrite these as equivalent fractions with the same denominator.

What fractions would I use?

Expect the answers: \( \frac{5}{10} \) and \( \frac{6}{10} \).

I can check with a diagram:
Thus, I notice that I am comparing \( \frac{5}{10} \) with \( \frac{6}{10} \).

Which is greater?

Expect the answer: \( \frac{6}{10} \)

But in comparing \( \frac{5}{10} \) and \( \frac{6}{10} \), I am really comparing 5 and 6, or \( 1 \times 5 \) and \( 3 \times 2 \). Numbers that \( 1 \times 5 \) tells me I originally had 1 piece which was subdivided into 5, and the \( 3 \times 2 \) that I originally had 3 pieces, each subdivided into 2. The subdivisions are now all equal so I only need count the numbers of each to tell which fraction is larger.

Since \( 1 \times 5 < 3 \times 2 \), then \( \frac{1}{2} < \frac{3}{5} \).

Which of these two fractions is greater: \( \frac{3}{4} \) or \( \frac{5}{6} \)? How can we tell?

Expect the answer: change both into twentieths and then see if the numerator of the first is greater than the one of the second. Or, expect: multiply \( 3 \times 6 \) and compare it to \( 4 \times 5 \).

Now we have two alternatives. We could convert both fractions to twentieths and compare like so:

\[
\frac{3}{4} = \frac{18}{24} \\
\frac{5}{6} = \frac{20}{24}
\]

Or else we might simply notice that this is exactly the same as comparing \( 3 \times 6 \) with \( 4 \times 5 \), since \( 3 \times 6 \) says I originally had 3 pieces, each of which was cut into sixths and so now I have \( 3 \times 6 \), and \( 4 \times 5 \), that I originally had 4 pieces, each of which was cut into fifths, and so I now have \( 4 \times 5 \) pieces, all of these of the same size. Notice this is the same as comparing the numerators in the above method.
Can you see why this is called the cross-product rule for comparing fractions? I take the product across the fractions and compare these:

For example, in $\frac{3}{4}$ and $\frac{5}{6}$, I compared $3 \times 6$ with $4 \times 5$ which comes from $\frac{3 \times 5}{4 \times 6}$.

Ask the students to find the larger in each of these pairs by using the cross product rule. Have a student show each of these on the board diagrammatically explaining why the rule works.

<table>
<thead>
<tr>
<th></th>
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<tbody>
<tr>
<td>$\frac{3}{5}$</td>
<td>$\frac{2}{4}$</td>
<td></td>
</tr>
<tr>
<td>$\frac{3}{8}$</td>
<td>$\frac{4}{5}$</td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{5}$</td>
<td>$\frac{2}{6}$</td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{3}$</td>
<td>$\frac{3}{5}$</td>
<td></td>
</tr>
</tbody>
</table>
Take sure the students realize that we examine the denominator of the other fraction to get the name for 1 to be used in renaming a fraction.

** Here the student will learn that \( \frac{a}{b} > \frac{c}{d} \) only when \( a \times d > b \times c \), for example, \( \frac{3}{4} > \frac{2}{5} \) only because \( 3 \times 5 > 2 \times 6 \). **

** Now that we already realize that an easier way to compare two fractions to decide which is larger is to rename them as multiplication expressions and with a common part. We have practiced renaming fractions, we can go on to comparing them. **

** Suppose I want to compare \( \frac{4}{3} \) with \( \frac{4}{5} \) to find which is greater. I can rewrite these as multiplication expressions with a common part. **

** What will these expressions be? **

Expect the answers: \( 1 \times 6 \times (\frac{4}{3} \times \frac{1}{5}) \) and \( 4 \times 2 \times (\frac{4}{5} \times \frac{1}{2}) \).

** I can check this:**

\[
\frac{4}{3} = 1 \times \frac{4}{3} \\
= 1 \times \frac{4}{3} \times (6 \times \frac{1}{5}) \\
= 1 \times 6 \times \frac{4}{3} \times \frac{1}{5} \\
= 6 \times \frac{4}{3} \times \frac{1}{5} \\
= 6 \times \frac{4}{15} \\
= \frac{24}{15} \\
= \frac{8}{5}
\]

** Then, I notice that I am comparing 6 with 8. **

** Which is greater? **

Expect the answer: 8

** But where did the 6 in \( 6 \times \left( \frac{4}{3} \times \frac{1}{5} \right) \) come from? It came from \( 1 \times 6 \), the 1 from the original expression for \( \frac{1}{2} \) and the 6 from multiplying that by \( 6 \times \frac{1}{5} \). Where did the 8 come from? It came from \( 4 \times 2 \), the \( 4 \) from the original expression for \( \frac{4}{5} \) and the 2 from multiplying... **
NOTICE THAT THE FRACTION WITH 2 AS DENOMINATOR WAS MULTIPLIED BY 6 x \( \frac{1}{6} \), SINCE THE OTHER DENOMINATOR WAS 6.

WHY WAS THE FRACTION WITH 6 AS DENOMINATOR MULTIPLIED BY 2?

Expect the answer: because the other denominator was 6.

SO, IN COMPARING \( \frac{1}{2} \) AND \( \frac{4}{6} \), I AM REALLY COMPARING 6 AND 3, OR 1x6 AND 4x2.

TO COMPARE \( \frac{1}{2} \) AND \( \frac{4}{6} \), I SAY 1x6 < 4x2, SO \( \frac{1}{2} < \frac{4}{6} \).

SUPPOSE I WANT TO COMPARE \( \frac{1}{2} \) WITH \( \frac{3}{5} \) TO FIND WHICH IS GREATER.

WHAT EXPRESSIONS WOULD I USE?

Expect the answers: \( 1x5 \times (\frac{1}{2} \times \frac{4}{5}) \) and \( 3x2 \times (\frac{4}{5} \times \frac{1}{2}) \).

I CAN CHECK:

\[
\begin{align*}
\frac{1}{2} &= 1 \times \frac{1}{2} \\
&= 1 \times \frac{1}{2} \times (5 \times \frac{1}{5}) \\
&= 1 \times 5 \times (\frac{1}{2} \times \frac{1}{5}) \\
\frac{3}{5} &= 3 \times \frac{1}{5} \\
&= 3 \times \frac{1}{5} \times (2 \times \frac{1}{2}) \\
&= 3 \times 2 \times (\frac{1}{5} \times \frac{1}{2})
\end{align*}
\]

THEN, I NOTICE THAT I AM COMPARING 1x5 WITH 4x2.

WHICH IS GREATER?

Expect the answer: 4x2.

REMEMBER, THAT 1x5 TELLS US THAT I ORIGINALLY HAD A MULTIPLE OF 1 OF SOME FRACTION AND THEN MULTIPLIED THAT BY \( 5 \times \frac{1}{5} \) AND THE 3x2 TELLS ME THAT I ORIGINALLY HAD 3 TIMES SOME FRACTION AND THEN MULTIPLIED BY \( 2 \times \frac{1}{2} \). THEN BOTH EXPRESSIONS NOW HAVE A PART \( \frac{1}{2} \times \frac{4}{5} \).

SO, I NEED ONLY COUNT HOW MANY OF THESE I HAVE FOR EACH.

SINCE 1x5 < 3x2, THEN \( \frac{1}{2} < \frac{3}{5} \).
WHICH OF THESE TWO FRACTIONS IS GREATER: $\frac{3}{4}$ OR $\frac{5}{6}$

Expect the answer: multiply the first by $6 \times \frac{1}{6}$ and the second by $4 \times \frac{1}{4}$.

or else compare $3 \times 6$ with $4 \times 5$.

NOW WE HAVE TWO ALTERNATIVES. WE COULD CONVERT BOTH FRACTIONS TO EQUIVALENT EXPRESSIONS AND COMPARE THEM LIKE SO:

\[
\frac{3}{4} = 3 \times \frac{1}{4} = 5 \times \frac{1}{6} = 5 \times \frac{1}{6} \times (6 \times \frac{1}{6}) = 5 \times 4 \times (\frac{1}{6} \times \frac{1}{6})
\]

or else we might simply notice that this is exactly the same as comparing $3 \times 6$ with $4 \times 5$, since $3 \times 6$ tells me I originally had $3 \times$ a fraction and then multiplied it by $6 \times \frac{1}{6}$, and the $5 \times 4$ that I originally had $5 \times$ another fraction and then multiplied it by $4 \times \frac{1}{4}$.

Now each expression has some multiple of $(\frac{1}{4} \times \frac{1}{5})$.

Can you see why this is called the cross-product rule for comparing fractions. I take the product across the fractions and compare these.

For example, in $\frac{3}{4}$ and $\frac{5}{6}$,

I compared $3 \times 6$ with $5 \times 4$ which comes from $\frac{3}{4} \times \frac{5}{6}$.

Ask the students to find the larger in each of these pairs by using the cross product rule. Have a student show each of these on the board diagrammatically explaining why the rule works.
Write each of the following as a decimal:

\[
\begin{align*}
5 \frac{2}{100} \\
6 \frac{13}{100} \\
28 \frac{14}{100}
\end{align*}
\]

Have the students who read out their answers explain in terms of either a diagram or an explanation of their reasoning how they got the correct answer.

Similarly, I might read and write decimals into the thousands.

How would I read 34,529?

Expect the answers: 34 ones and 5 tenths and 2 hundredths and 9 thousandths, or 34 ones and 529 thousandths, or 34,529 thousandths. If not all of these answers came up, mention the ones that did not. Point out the analogy to the hundredths situation.

In particular, I can rewrite a whole number, say 4, as tenths or hundredths or thousands. That is, 4 = 4.0 = 4.00 = 4.000, so 4 can be read as 4, 40 tenths, 400 hundredths, or 4000 thousandths.

Hand students worksheet 1 to complete.

"Here the students will be introduced to the idea of dividing decimals by wholes."

Suppose we would like to find the answer to \(1.5 \div 3\)?

We know that \(15 \div 3\) means to separate 15 into 3 groups of the same size and then find the size of each of these groups.

This diagram would show what we mean:

\[
\begin{array}{ccc}
\bigcirc & \bigcirc & \bigcirc \\
\bigcirc & \bigcirc & \bigcirc \\
\bigcirc & \bigcirc & \bigcirc
\end{array}
\]

Well, we should mean a similar thing by \(1.5 \div 3\) if we should separate 1.5 into 3 groups of the same size, and the answer is that each group should be 0.5.
We already know that 1.5 can be read as $\frac{15}{10}$ or 15 tenths.

So to divide 15 tenths by 3, I merely put the 15 tenths into 3 groups and find the size of each group. I can draw:

![Diagram of 15 tenths divided into 3 groups]

Notice that this was the same as putting 15 objects of any sort into 3 groups and then remembering the type of object being dealt with. That is, the answer was $(15 \div 3)$ tenths.

Let's find $1.6 \div 4$. What do you expect the answer to be?

Expect the answer: 4 tenths or .4.

Let's draw a picture. $1.6 = \frac{16}{10} = 16$ tenths. Then, to put 16 tenths into 4 groups, I draw:

![Diagram of 16 tenths divided into 4 groups]

And we can see that indeed I have 4 tenths = .4 in each of the groups.

Have the students predict the answers to:

$4.5 \div 9$

$5.6 \div 8$

$4.5 \div 6$

Ask students to read out their answers. If there is any difficulty with any of these, draw a diagram as was done for $1.6 \div 4$ to illustrate.

What would we do if we were trying to find $1.35 \div 5$? That means that...
I want to divide 125 hundredths into 5 groups and find the sum of each group. I can begin a drawing like so:

![Drawing of 5 groups of hundredths]

Notice that what I am really doing is dividing 125 by 5 and then extending that I am dealing with hundredths.

Ask the students to predict the answers to:

\[
\begin{array}{c}
0.25 \div 5 \\
1.25 \div 25 \\
2.00 \div 4 \\
\end{array}
\]

Have three students explain their answers. If there is any difficulty, draw diagrams as indicated above.

Now we can write this scheme in a way that we usually use to write division questions.

For example, to find \(4.2 \div 6\), I write:

\[
\begin{array}{c|c}
6 & 4.2 \\
\hline
\end{array}
\]

\[
\begin{array}{c|c}
6 & \text{42 tenths} \\
\hline
\end{array}
\]

And to find the answer: \(6 \big/ 42\) tenths

I write:

\[
\begin{array}{c|c|c}
6 & \text{42 tenths} \\
\hline
\text{42 tenths} & 7 \text{ tenths} \\
\hline
\text{0 tenths} & 7 \text{ tenths} = .7 \\
\end{array}
\]

How could I write out this problem: \(4.5 \div 9\)? I might write:

\[
\begin{array}{c|c}
9 & 4.5 \\
\hline
\end{array}
\]

\[
\begin{array}{c|c|c}
9 & \text{45 tenths} \\
\hline
\text{45 tenths} & 5 \text{ tenths} \\
\hline
\text{0 tenths} & 5 \text{ tenths} = .5 \\
\end{array}
\]

How would we write \(.36 \div 6\)? I write:
Hand students worksheet 2 to complete.

**Here the student will now leave decimals for a while to discover that another interpretation for a fraction is division. That is, \( \frac{4}{3} = 4 \div 3 \). This will tie in to decimals when he discovers that \( \frac{1}{2} = 1 \div 2 \), and so to get the decimal for \( \frac{1}{2} \), he divides 2 into 1. **

**Now as we said before, the whole idea of introducing decimals was to eventually show you a way to write fractions as decimals. But we have forgotten all about fractions. Let us return to this subject temporarily. Suppose I have the fraction \( \frac{12}{4} \). Let us see what whole number this is another name for. Do you know already?**

Expect the answer: 3. In any case, go on with the following discussion.

\( \frac{12}{4} \) tells me that I have twelve pieces each of size \( \frac{1}{4} \). One way to find the number of wholes there are in \( \frac{12}{4} \) is to collect fourths into wholes until I have used all 12 pieces. I could draw:

![Diagram of whole numbers](image)

And so I see that \( \frac{12}{4} = 3 \).

Let's do the same with \( \frac{42}{7} \). What should I draw and what whole number do I get as equal to \( \frac{42}{7} \)?

Expect the students to draw:
Notice that \( \frac{42}{7} = 6 \).

Now we can do the same for \( \frac{18}{6} \). What is the whole number for \( \frac{18}{6} \)?

Expect the answer: 3.

Draw:

My drawing indicates that 3 is the correct answer.

Notice that each time I was finding how many groups of the denominator there were in the numerator (groups of 4 in 12, groups of 7 in 42, and groups of 6 in 18).

We already know that finding the number of groups of one number in another is the same as dividing that second number by the first. But finding the number of groups on one number there are in another (like the number of fours in twelve) is the same as finding the size of each of that number of groups in the larger number (like the size of each of four groups of twelve in total).

For example, to show \( 12 \div 4 \), I might draw:

\[
\begin{array}{ccc}
1 & 2 & 1 \ 2 \\
3 & 4 & 3 \ 4
\end{array}
\]

Or I might draw:
IN EITHER CASE, I GET THE SAME ANSWER, NAMELY 3.

SO SINCE \( \frac{12}{4} \) WAS FOUND BY FINDING THE NUMBER OF GROUPS OF 4 IN 12,
OR DIVIDING 12 BY 4, I COULD ALSO FIND \( \frac{12}{4} \) BY FINDING THE SIZE OF
EACH OF 4 GROUPS OF EQUAL SIZE TO SHARE ALL 12 OBJECTS.

LET'S CHECK TO SEE IF FINDING \( \frac{42}{7} \) BY DIVIDING 42 BY 7 TO FIND THE
NUMBER OF SEvens IN 42 IS THE SAME AS FINDING THE SIZE OF EACH OF 7
GROUPS TO EQUALLY SHARE 42 OBJECTS. I CAN DRAW (TO SHOW HOW MANY 7's
IN 42):

I CAN DRAW THE DIAGRAM HERE TO SHOW HOW MANY OBJECTS IN EACH OF THE 7
GROUPS FOR SHARING THE 42 OBJECTS, AND I GET THE SAME ANSWER.

So, \( \frac{42}{7} = 42 \div 7 \).

SIMILARLY, \( \frac{18}{6} = 3 = 18 \div 6 \) SINCE I WAS FINDING THE NUMBER OF 6's IN 18,
WHICH IS THE SAME AS FINDING THE SIZE OF EACH OF 6 GROUPS TO SPLIT UP 18 OBJECTS.

FOR EACH OF THESE FRACTIONS, GIVE A DIVISION SENTENCE THAT WOULD GET
A WHOLE NUMBER NAME FOR THE FRACTION:

\( \frac{16}{8} \), \( \frac{15}{3} \), \( \frac{14}{7} \)

Ask 3 students to read out their answers and then diagrammatically explain them.
fewer by dividing the numerator by the denominator, even when the division is not even."

Now we have seen that for some fractions, another way of finding a name for the fraction is to divide the denominator into the numerator. We would like to find out if this is always true.

Suppose we start with \( \frac{11}{5} \). If our rule works, \( \frac{11}{5} \) should be the same as \( 11 \div 5 \): that is, if I take 11 objects and put them in 5 groups, the size of each group should be \( 11 \div 5 \), or \( \frac{11}{5} \).

Suppose we take 11 objects, say, 11 circles. I want to put them in 5 groups. We can set up 5 piles.

Then we can keep putting circles on the piles evenly until all the circles are used up.

To begin, let us put one circle on each pile.

\[
\begin{array}{cccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

Then I still have 6 circles left.

So I can put another circle on each pile and I have used up \( 5 \times 5 = 10 \) circles, and have only 1 left.

\[
\begin{array}{cccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

Since I only have one more circle to put onto all 5 piles equally, I must split it up into 5 equal parts. That means, each part becomes \( \frac{1}{5} \).

So, on each pile, I can put \( 2 \frac{1}{5} \) circles.
$3 \times 2 \frac{1}{3} = 2 \frac{4}{5}$, but $2 \frac{1}{6}$ is another name for $\frac{14}{5}$ since I can split each and whole circle into 5 parts before I count up the total.

Sells of each pile, and I see that each pile has $\frac{5}{5} + \frac{5}{5} + \frac{1}{5} = \frac{11}{5}$ or 11.

Let's try the same idea with $3 \frac{1}{3}$. If our rule works, we should expect that if we put 3 circles onto 3 equal piles, each pile will be of size $\frac{8}{3}$.

I can start with putting 2 objects on each pile, and I have used up 6 and have 2 left.

Then, I can cut each of the remaining two circles into thirds and separately split each of them up and put them evenly on the piles, like so:

And then:

Then I count up and see that on each pile, I have $3 \frac{1}{3} = 2 \frac{2}{3}$ circles.

But I can then split each whole into 3 pieces, and rename my groups as

Sells of size $\frac{3}{3} + \frac{3}{3} + \frac{2}{3} = \frac{8}{3}$. 
Can anyone show me why these things: \[ \frac{5}{4} = 5 \div 4? \]

Expect someone to draw 4 piles and try to split 5 objects evenly among them, like so:

If no one does this, you draw the diagram.

Notice each group has \[ \frac{1}{4} + \frac{1}{4} = \frac{2}{4} \] as its size.

Can anyone tell me why these things: \[ \frac{3}{4} = 3 \div 4? \]

I can draw 4 piles and try to put my circles evenly on them.

But since I only have 3 circles, I can't even put a whole one on each pile. What I can do is split each of the three wholes into fourths and put one fourth of each on each of the 4 piles, like so:

Clearly, the size of each pile is \[ \frac{1}{4} \] day.

Hand students worksheet 3 to complete.

**Here the student will finally tie together his work on decimals and fractions to find the decimal equivalent to a fraction.**

Now that we know how to rewrite a fraction as a division question, we can proceed to tie together some of our decimal ideas with this one.

Let's look at \[ \frac{4}{10} \]. We know \[ \frac{4}{10} = 4 \div 10 \] by drawing a diagram like this:
ON THE OTHER HAND, WE ALSO KNOW THAT $\frac{4}{10}$ CAN BE WRITTEN AS A DECIMAL AS $0.4$.

IT WOULD BE INTERESTING TO BE ABLE TO SHOW DIRECTLY THAT $4 \div 10 = 0.4$.

WELL, $4 \div 10$ CAN BE WRITTEN $10 \div 4$. BUT IF I TRY TO DO THIS DIVISION, I ONLY GET AN ANSWER OF 0 WITH REMAINDER 4, WHICH DOES NOT HELP AS MUCH.

SO I MIGHT TRY TO REPRESENT $4 = \frac{40}{10} = 40$ TENTHS.

THEN $4 \div 10$ CAN BE WRITTEN:

\[
\begin{align*}
10 \div 4 & \rightarrow 10 \div 4.0 \rightarrow 10 \div 40 \\
& \rightarrow 10 \div 40 \text{ TENTHS} \rightarrow 4 \text{ TENTHS} \\
& \rightarrow 4 \text{ TENTHS} = 0.4
\end{align*}
\]

WE HAVE DIRECTLY CONVERTED, THEN, $\frac{4}{10}$ INTO A DECIMAL BY USING DIVISION.

LET US TRY CONVERTING $\frac{2}{10}$ TO A DECIMAL. WE KNOW THAT OUR RESULT WOULD BE $0.2$.

BUT WE CAN TRY TO FIND THIS THROUGH DIVISION:

\[
\begin{align*}
10 \div 2 & \rightarrow 10 \div 2.0 \rightarrow 10 \div 20 \\
& \rightarrow 10 \div 20 \text{ TENTHS} \rightarrow 2 \text{ TENTHS} \\
& \rightarrow 2 \text{ TENTHS} = 0.2
\end{align*}
\]

WE SHOULD NOT TRY TO FIND DECIMALS WE MIGHT NOT ALREADY KNOW. SUPPOSE WE WANT THE DECIMAL FOR $\frac{1}{2}$.

$\frac{1}{2} = 1 \div 2$, SO WE WRITE:

\[
\begin{align*}
2 \div 1 & \rightarrow 2 \div 1.0 \rightarrow 2 \div 10 \\
& \rightarrow 2 \div 10 \text{ TENTHS} \rightarrow 5 \text{ TENTHS} \\
& \rightarrow 5 \text{ TENTHS} = 0.5
\end{align*}
\]

SO THE DECIMAL FOR $\frac{1}{2} = 0.5$.

THIS MAKES SENSE SENSE $0.5 = \frac{5}{10} = \frac{1}{2}$.

NOTICE THAT I CONVERTED THE 2 INTO 1.0 WITH DECIMALS I WOULD NOT.
GET ANOTHER BY DIVIDING 1 \( \div \) 2, IN THAT FORM. I WOULD ONLY HAVE PUT 1
6, REMAINDER 1.

NOW LET US TRY ANOTHER FRACTION. SUPPOSE WE START WITH \( \frac{3}{4} \) AND WE
WANT THE DECIMAL FOR THIS.

WE WRITE \( \frac{3}{4} = 3 \div 4 \), SO WE WRITE:

\[
\begin{array}{c|c|c|c|c}
4 & 3 & \rightarrow & 4 \div 3.0 & \rightarrow 4 \rightarrow 30 \text{ TENTHS} & 7 \text{ TENTHS} \\
25 & 20 & & 0 & 7 \text{ TENTHS} \\
2 & 0 & & 0 & 7 \text{ TENTHS}
\end{array}
\]

SO I GET A REMAINDER WHICH I AM NOT SURE HOW TO HANDLE. ONE POSSIBILITY
IS TO CHANGE 3 INTO 3.00 INSTEAD. THIS SEEMS REASONABLE: I CHANGED IT
FROM 3 TO 3.0 ONLY TO AVOID THE PROBLEMS OF A REMAINDER, SO I MIGHT TRY
TO GO ONE STEP FURTHER.

\[
\begin{array}{c|c|c|c|c|c}
4 & 3.00 & \rightarrow & 4 \rightarrow 300 \text{ HUNDREDTHS} & 70 \text{ HUNDREDTHS} \\
220 & 200 & \rightarrow 20 & 7 \text{ HUNDREDTHS} \\
20 & 20 & \rightarrow 5 \text{ HUNDREDTHS} \\
75 & 75 & \rightarrow 75 \text{ HUNDREDTHS} = .75
\end{array}
\]

SO THE DECIMAL EQUIVALENT FOR \( \frac{3}{4} = .75 \).

WE WILL NOT ALWAYS GET RID OF THE REMAINDER BY STOPPING AT THE TENTHS PLACE,
OR EVEN THE HUNDREDTHS OR THOUSANDTHS, BUT WE CAN KEEP TRYING PLACES UNTIL
WE EITHER STOP OR ARE CONVINCED WE CANNOT STOP GETTING REMAINDERS.

LET'S TRY ONE MORE PROBLEM WITH HUNDREDTHS TOGETHER. WE CAN GET THE
DECIMAL FOR \( \frac{2}{25} \).

\[
2 \div 25 = 25 \div 2 \rightarrow 25 \div 2.00 \rightarrow 25 \div 2.00 \rightarrow 25 \div 200 \text{ HUNDREDTHS} = .08
\]

\[
\frac{2}{25} = .08
\]

Suggested

Hand students worksheet 4 to complete.

Day 5
Expect the answer: 13.51.

Write each of the following as a decimal:

- \[5 \frac{2}{100}\]
- \[6 \frac{13}{100}\]
- \[28 \frac{44}{100}\]

Have the students who read out their answers explain in terms of renaming how they got the correct answer.

Similarly, I might read and write decimals into the thousandths. How would I read 34.529?

Expect the answers: 34 ones and 5 tenths and 2 hundredths and 9 thousandths, or 34 ones and 529 thousandths, or 34,529 thousandths. If not all of these answers come up, mention the ones that did not. Point out the analogy to the hundredths situation.

In particular, I can rewrite a whole number, say 4, as tenths or hundredths or thousandths. That is, 4 = 4.0 = 4.00 = 4.000, so 4 can be read as 4, 40 tenths, 400 hundredths, or 4000 thousandths.

Hand students worksheet 1 to complete.

Here the students will be introduced to the idea of dividing decimals by whole numbers.

Suppose we would like to find the answer to \(1.5 \div 3\).

We know that if \(1.5 \div 3 = \square\), then \(\square \times 3 = 1.5\).

We have never done any multiplication with decimals, so we might simplify it...
OUR PROBLEM BY ADJUSTING $1.5 = \frac{15}{10} = 15 \times \frac{1}{10}$.

We know that $1.5 = \frac{15}{10}$ from our previous discussion and we know that \( \frac{15}{10} = 15 \times \frac{1}{10} \). Since \( \frac{15}{10} = \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} \),

Then I am looking for a number to multiply by 3 to get 15 tenths.

Notice that if I write this as:

\[
\square \times 3 = 15 \times \frac{1}{10},
\]

I may have to replace \( \square \) with a product rather than just a single number.

For example, to solve \( 3 \times \square = 3 \times 5 \times 4 \), I replace \( \square \) by \( 5 \times 4 \).

We can do the same thing here.

If \( \square \times 3 = 15 \times \frac{1}{10} \), we see that we can replace \( \square \) by \( 5 \times \frac{1}{10} \).

And then \( \square \times 3 = (5 \times \frac{1}{10}) \times 3 = 5 \times 3 \times \frac{1}{10} = 15 \times \frac{1}{10} = 1.5 \).

So \( (5 \times \frac{1}{10}) \times 3 = 5 \text{ tenths} \times 3 = \frac{5}{10} \times 3 = 1.5 \).

Notice that this was the same as finding what I had to multiply 3 by to get 15 and then remembering to multiply that answer by \( \frac{1}{10} \). So the answer was \( (15 \div 3) \) tenths.

Let's find \( 1.6 \div 4 \). What do you expect the answer to be?

Expect the answer: 4 tenths or 0.4.

Let's write a multiplication sentence:

\[
\square \times 4 = 1.6 = 16 \text{ tenths},
\]

\[
4 \times 4 = 16, \text{ so } 4 \text{ tenths} \times 4 = 16 \text{ tenths}, \text{ so } \square = 4 \text{ tenths} = 0.4.
\]

Have the students predict the answers to:

\[
\begin{align*}
2.5 & \div 5 \\
3.5 & \div 3 \\
4.3 & \div 4
\end{align*}
\]
Ask students to read out their answers. If there is any difficulty with any of these, rewrite as multiplication sentences to find the solutions.

WHAT WOULD WE DO IF WE WERE TRYING TO FIND 1.25 ÷ 5? THAT MEANS THAT I WANT TO FIND WHAT NUMBER TO MULTIPLY 5 BY TO GET 125 HUNDREDTHS.

SO WHAT I AM REALLY DOING IS SOLVING:

\[ \square \times 5 = 125 \text{ HUNDREDTHS} \]

NOTICE THAT WHAT I AM REALLY DOING IS DIVIDING 125 BY 5 AND THEN REMEMBERING THAT I AM DEALING WITH HUNDREDTHS.

Ask the students to predict the answers to:

\[
\begin{align*}
.25 \div 5 \\
1.25 \div 25 \\
2.00 \div 4
\end{align*}
\]

Have three students explain their answers. If there is any difficulty, rename as multiplication sentences as illustrated above.

NOW WE CAN WRITE THIS SUCHE IN A WAY THAT WE USUALLY USE TO WRITE DIVISION QUESTIONS.

FOR EXAMPLE, TO FIND \(4.2 \div 6\), I WRITE:

\[
\begin{array}{c|c}
6 & 4.2 \\
\hline
6 & 42 \text{TENTHS} \\
\hline
0 & 7 \text{TENTHS} = .7
\end{array}
\]

AND TO FIND THE ANSWER:

I WRITE:

\[
\begin{array}{c|c}
6 & 42 \text{TENTHS} \\
\hline
6 & 42 \text{TENTHS} \\
\hline
0 & 7 \text{TENTHS} = .7
\end{array}
\]

HOW COULD I WRITE OUT THIS PROBLEM: \(4.5 \div 9\)? I MIGHT WRITE:

\[
\begin{array}{c|c}
9 & 4.5 \\
\hline
9 & 45 \text{TENTHS} \\
\hline
0 & 5 \text{TENTHS}
\end{array}
\]
Here the student will now leave decimals for a while to discover that another interpretation for a fraction is division. That is, \( \frac{4}{3} = 4 \div 3 \). This will tie in to decimals when he discovers that \( \frac{1}{2} = 1 \div 2 \), and so to get the decimal for \( \frac{1}{2} \), he divides 2 into 1. **

** As we said before, the whole idea of introducing decimals was to eventually show you a way to write fractions as decimals. But we have forgotten all about fractions. Let us return to this subject temporarily.

Suppose I have the fraction \( \frac{12}{4} \). Let us see what whole number this is another name for. Do you know already?

Expect the answer: 3. In any case, go on with the following discussion.

\[
\frac{12}{4} \text{ tells me that I have } 12 \times \frac{1}{4} \text{. I can rewrite this,}
\]

\[
12 \times \frac{1}{4} = (3 \times 4) \times \frac{1}{4}
\]

\[
= 3 \times (4 \times \frac{1}{4}) = [4 \times \frac{1}{4} = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{4}{4} = 1]
\]

\[
= 3 \times 1
\]

\[
= 3
\]

And so I see that \( \frac{12}{4} = 3 \).

Let's do the same with \( \frac{42}{7} \). What should I write for my renaming and what whole number do I get equal to \( \frac{42}{7} \)?

Expect the students to suggest:
\[
\begin{align*}
&\frac{42}{7} \times \frac{1}{1} \\
&= (6 \times 7) \times \frac{1}{7} \\
&= 6 \times (7 \times \frac{1}{7}) \\
&= 6 \times 1 \\
&= 6\end{align*}
\]

If they do not suggest this, you rename for them.

\[
\text{NOTICE THAT } \frac{42}{7} = 6.
\]

\[
\text{NOW WE CAN DO THE SAME FOR } \frac{45}{9} \text{. WHAT IS THE WHOLE NUMBER FOR } \frac{45}{9} ?
\]

Expect the answer: 3.

Rename:
\[
\frac{45}{9} = 13 \times \frac{4}{3}
\]
\[
= (3 \times 13) \times \frac{4}{3}
\]
\[
= 3 \times (6 \times \frac{4}{3})
\]
\[
= 3 \times 1 \\
= 3
\]

My renaming indicates that 3 is the correct answer.

\[
\text{NOTICE THAT EACH TIME I WAS FINDING A WAY OF WRITING THE NUMERATOR AS A PRODUCT WHERE ONE OF THE MULTIPLIERS WAS THE DENOMINATOR( 4 \times \square = 12, 7 \times \square = 42, \text{ AND } 6 \times \square = 15).}
\]

We already know that finding what number to multiply one by to get another is the same as dividing that second number by the first.

So, in finding the whole number for \( \frac{42}{7} \), for example, I was finding out how to write 42 = \( \square \) \( \times \) 7, or 42 = 7 \( \times \) \( \square \). And that answer was the whole number for \( \frac{42}{7} \). But that answer is also the answer to 42 \( \div \) 7 = \( \square \) or 42 \( \div \) \( \square \) = 7.
For each of these fractions, give a division sentence that would get a whole number name (another name) for the fraction:

<table>
<thead>
<tr>
<th>Fraction</th>
<th>Division Sentence</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{16}{8} )</td>
<td>2</td>
</tr>
<tr>
<td>( \frac{15}{3} )</td>
<td>5</td>
</tr>
<tr>
<td>( \frac{14}{2} )</td>
<td>7</td>
</tr>
</tbody>
</table>

Ask 3 students to read out their answers and then explain them by renaming in a multiplication sentence.

** Here the student will learn that another name for a fraction can always be found by dividing the numerator by the denominator, even when the division is not even. **

Now we have seen that for some fractions, another way of finding a name for the fraction is to divide the denominator into the numerator. We would like to find out if this is always true.

Suppose we start with \( \frac{14}{5} \). If our rule works, \( \frac{14}{5} \) should be the same as \( 11 \div 5 \): that is, if I find the solution to \( \square \times 5 = 11 \), that number \( \square \) should be \( \frac{14}{5} \).

Let's do this.

Suppose we write: \( \square \times 5 = 11 \). If \( \square = \frac{14}{5} \), then it should be true that \( \frac{14}{5} \times 5 = 11 \).

But \( \frac{14}{5} \times 5 \)

\[ = (11 \times \frac{4}{5}) \times 5 \]
\[ = 11 \times (5 \times \frac{4}{5}) \]
\[ = 11 \times 1 \]
\[ = 11 \]

So, \( \frac{14}{5} \times 5 = 11 \), and therefore, \( 11 \div 5 = \frac{14}{5} \).

Let's try this same idea with \( \frac{2}{3} \). If our rule works, we should expect...
THAT THE SOLUTION TO $\Box \times 3 = 3$ SHOULD BE $\frac{8}{3}$.

Well, $\frac{8}{3} \times 3 = (3 \times \frac{1}{3}) \times 3 = 3 \times (3 \times \frac{1}{3}) = 3 \times 1 = 3$

So, $\Box \times 3 = 3$ AND $\Box \div 3 = \frac{8}{3}$.

Can anyone tell me why they think $\frac{5}{4} = 5 \div 4$?

Expect someone to suggest: If $\Box \times 4 = 5$, then $\Box = \frac{5}{4}$ since

$$\frac{5}{4} \times 4 = (5 \times \frac{1}{4}) \times 4 = 5 \times (4 \times \frac{1}{4}) = 5 \times 1 = 5$$

If no one does this, you write out this renaming scheme.

Can anyone tell me why they think $\frac{3}{4} = 3 \div 4$?

I can rewrite $3 \div 4 = \Box$ into $\Box \times 4 = 3$.

Then, $\Box \times 4 = \frac{3}{4} \times 4 = (3 \times \frac{1}{4}) \times 4 = 3 \times (4 \times \frac{1}{4}) = 3 \times 1 = 3$

So, $\Box = 3 \div 4 = \frac{3}{4}$.

Hand students worksheet 3 to complete.
**Here the student will finally tie together his work on decimals and fractions to find the decimal equivalent to a fraction.**

Now that we know how to rewrite a fraction as a division question, we can proceed to tie together some of our decimal ideas with this one.

Let's look at $\frac{4}{10}$. We know $\frac{4}{10} = 4 \div 10$ because

If $4 \div 10 = \Box$, then $\Box \times 10 = 4$, and

$$\frac{4}{10} \times 10 = \left(\frac{4}{10}\right) \times 10 = 4 \times \left(10 \times \frac{1}{10}\right) = 4 \times 1 = 4$$

So, $\frac{4}{10} \times 10$ does equal 4, so $\frac{4}{10} = 4 \div 10$.

On the other hand, we also know that $\frac{4}{10}$ can be written as a decimal as $0.4$.

It would be interesting to be able to show directly that $4 \div 10 = 0.4$.

Well, $4 \div 10$ can be written as $10 \sqrt{4}$. But if I try to do this division, I only get an answer of 0 with remainder 4, which does not help us much.

So I might try to rewrite $4 = 4.0 = \frac{40}{10} = 40$ tenths.

(Since $4.0 = 4$ ones $= 4 \times 1 = 4 \times \left(10 \times \frac{1}{10}\right) = (4 \times 10) \times \frac{1}{10} = 40$ tenths.)

Then, $4 \div 10$ can be written:

$$10 \sqrt{4} \rightarrow 10 \sqrt{4.0} \rightarrow 10 \sqrt{40 \text{ tenths}}$$

We have directly converted, then, $\frac{4}{10}$ into a decimal by using division.

Let us try converting $\frac{6}{10}$ to a decimal. We know that our result should be $0.6$.

But we can try to find this through division:

$$10 \sqrt{6} \rightarrow 10 \sqrt{6.0} \rightarrow 10 \sqrt{60 \text{ tenths}}$$

We should now try E. Find decimals we might not already have. Suppose 10
Want the decimal for \( \frac{1}{2} \).

\[
\frac{1}{2} = 1 \div 2, \text{ so we write:}
\]

\[
2 \lceil \frac{1}{2} \rceil \rightarrow 2 \rceil 1.0 \rightarrow 2 \rceil \frac{10 \text{ TENTHS}}{0 \text{ TENTHS}} \frac{5 \text{ TENTHS}}{5 \text{ TENTHS}} = .5
\]

So the decimal for \( \frac{1}{2} = .5 \)

This makes sense since \( .5 = \frac{5}{10} = \frac{1}{2} \).

Notice that I converted the 1 into \( .0 \) only because I would not get anywhere by dividing 1 by 2, in that form. I would only have gotten 0, remainder 1.

Now let us try another fraction. Suppose we start with \( \frac{3}{4} \) and we want the decimal for this.

We write \( \frac{3}{4} = 3 \div 4, \text{ so we write:} \)

\[
4 \lceil \frac{3}{4} \rceil \rightarrow 4 \lceil 3.0 \rceil \rightarrow 4 \rceil \frac{30 \text{ TENTHS}}{26 \text{ TENTHS}} \frac{7 \text{ TENTHS}}{7 \text{ TENTHS}} \frac{0 \text{ TENTHS}}{0 \text{ TENTHS}} \frac{5 \text{ TENTHS}}{5 \text{ TENTHS}} = .75
\]

So I get a remainder which I am not sure how to handle. One possibility is to change 3 into 3.00 instead. This seems reasonable. I changed it from 3 to 3.0 only to avoid the problems of a remainder, so I might try to go one step further.

\[
4 \lceil 3.00 \rceil \rightarrow 4 \lceil 300 \text{ HUNDREDTHS} \rceil \frac{70 \text{ HUNDREDTHS}}{70 \text{ HUNDREDTHS}} \frac{0 \text{ HUNDREDTHS}}{0 \text{ HUNDREDTHS}} \frac{75 \text{ HUNDREDTHS}}{75 \text{ HUNDREDTHS}} = .75
\]

So the decimal equivalent for \( \frac{3}{4} = .75 \).

We will not always get rid of the remainder by stopping at the tenths place, or even the hundredths or thousandths, but we can keep trying places.
UNTIL WE EITHER STOP OR ARE CONVINCED WE CANNOT STOP IMPENDING REVENGE.

LET'S TRY OUR FIRST PROBLEM WITH HUNDREDTHS TOGETHER. WE CAN GET THE
DECIMAL FOR \( \frac{2}{25} \):

\[
\frac{2}{25} = 25 \sqrt{2} \rightarrow 25 \sqrt{2.0} \rightarrow 25 \sqrt{2.00} \rightarrow 25 \sqrt{200 \text{ HUNDREDTHS}} = .08
\]

So, \( \frac{2}{25} = .08 \). THIS SEEMS CLEAR SINCE

\[.08 = \frac{8}{100} = \frac{2}{25} \]

Hand students worksheet 4 to complete.

Suggested

Day: 5
Hand students worksheet 3 to complete.

Day 5: Here the students will learn about approximating square roots of wholes. **

Suppose we now want to find \( \sqrt{5} \). We know that 2 is too small, since \( 2 \times 2 = 4 \) and 3 is too large since \( 3 \times 3 = 9 \). So we have a problem we had not previously faced.

One technique we might use is to convert 5 to a fraction and try to find a fraction near it. For example, we know that \( 5 = \frac{125}{25} \), and \( \frac{125}{25} \) is close to \( \frac{121}{25} \), which has an "even" square root, namely \( \frac{11}{5} \).

But this might prove very long and tedious if we don't find the right fraction right away.

Let's try to develop a method.

Suppose we try to draw a square with area of 5. We might draw something like:

```
   2
  __
  |  |
  |  |
  |  |
  |  |
  |__|
```

Where sections 1, 2 and 3 have a total area of 1 unit. However, we do not know the size of the square which extends from 2.

What is the area of section 1?

Expect the answer: \( 2 \times \square \).

What is the area of section 2?

Expect the answer: \( 2 \times \square \).

What is the area of section 3?

Expect the answer: \( \square \times \square \).

Now in order of size, sections 1 and 2 come before 3, so let us pretend temporarily that they make up most of the 1 unit that sections 1, 2, and 3
MAKE UP ALTOGETHER.

SO, THE TOTAL AREA OF SECTIONS 1 AND 2 IS \((2 \times \square) + (2 \times \square)\) IS ABOUT 1.

BUT, \((2 \times \square) + (2 \times \square)\) IS ANOTHER WAY OF WRITING \(4 \times \square\) SINCE I CAN DRAW:

\[
\begin{array}{c}
\square \\
4 \left( \begin{array}{c}
2 \times \square \\
2 \times \square
\end{array} \right)
\end{array}
\]

NOW IF \(4 \times \square\) IS ABOUT 1, WHAT IS \(\square\)?

Expect the answer: about \(\frac{1}{4}\).

NOW SUPPOSE WE PRETEND THAT \(\square\) IS \(\frac{1}{4}\). WHAT IS THE AREA OF SECTION 3, IN OTHER WORDS, WHAT IS \(\square \times \square\)?

Expect the answer: \(\frac{1}{16}\).

SINCE THIS NUMBER IS LESS THAN \(\frac{1}{2}\), WE CAN SAY THAT WE HAVE USED UP MOST OF THE EXTRA 1 UNIT OF AREA THAT IS NOT IN THE 2 BY 2 SECTION OF THIS SQUARE:

\[
\begin{array}{c}
\square \\
2 \times \square
\end{array}
\]

AND SO WE CAN SAY, THAT \(\sqrt{5}\) IS ABOUT \(2 \frac{1}{2}\). WE WRITE THIS \(\sqrt{5} \approx 2 \frac{1}{2}\).

WE MIGHT HAVE USED THE DIVISION PROCEDURE AGAIN, AS WELL.

SO OUR FIRST GUESS MIGHT HAVE BEEN 2. THEN WE WOULD DIVIDE 5 BY 2, AND GET \(2 \frac{5}{4} = 2 \frac{1}{4}\).

UNFORTUNATELY, THIS GETS US INTO DIVISION WITH FRACTIONS SINCE OUR NEXT CHOICE WOULD BE A NUMBER BETWEEN 2 AND \(2 \frac{1}{2}\). AND SO,

WE WILL AVOID THE DIVISION TECHNIQUE FOR THE TIME BEING.

SUPPOSE WE WANT TO FIND \(\sqrt{13}\).
We know that the answer must be between 3 and 4, since $3 \times 3 = 9$ and $4 \times 4 = 16$, and 13 is between 9 and 16.

So, we again draw a square like so:

![Diagram of a square with sections labeled 1, 2, and 3.]

We have used up 9 units of the 13 in the 3 by 3 square, leaving us with 4 more units distributed in sections 1, 2, and 3.

Now what is the area of section 1?

Expect the answer: $3 \times \Box$.

What is the area of section 2?

Expect the answer: $3 \times \Box$.

What is the area of section 3?

Expect the answer: $\Box \times \Box$.

Again, we will go from largest to smallest. If we temporarily ignore section 3, we have used up most of our 4 units in sections 1 and 2 with combined area: $3 \times \Box + 3 \times \Box = 6 \times \Box$.

If $6 \times \Box = 4$, what is $\Box$?

We know that $6 \times \frac{1}{6} = 1$ by drawing:

![Diagram of a unit square.]

So that $6 \times \frac{4}{6} = 4$.
AND CERTAINLY WE NEED TO MULTIPLY BY SOMETHING 4 TIMES AS BIG AS \( \sqrt{\frac{1}{36}} \) TO GET 4.

Then, if we guess \( \sqrt{\frac{1}{36}} = \frac{4}{6} \), we find that the neglected area is \( \frac{4}{6} \times \frac{4}{6} = \frac{16}{36} \), which is less than \( \frac{1}{2} \) (which is \( \frac{18}{36} \)) and so we say that

\[
\sqrt{\frac{13}{36}} \approx 3 \frac{4}{6}.
\]

Suppose we want to find \( \sqrt{37} \). If we did not know in advance to try a number near 6, we might use our division procedure to close in on a number between 6 and 7.

For example, we might do:

Guess the length of the square with area 37 to be 4. Then the width = \( 37 \div 4 \)

\[
\begin{array}{c|c|c}
4 & \sqrt{37} & 9 \\
\hline
26 & 37 & 9 \\
\hline
1 & 9 \\
\end{array}
\]

So we try a number between 4 and 9 for the length, say 7.

So the width = \( 37 \div 7 \)

\[
\begin{array}{c|c|c}
7 & \sqrt{37} & 5 \\
\hline
25 & 37 & 5 \\
\hline
2 & 6 \\
\end{array}
\]

So we still did not get a square. We now try a number between 5 and 7, 6.

If the length = 6, the width = \( 37 \div 6 \),

\[
\begin{array}{c|c|c}
6 & \sqrt{37} & 6 \\
\hline
26 & 37 & 6 \\
\hline
1 & 6 \\
\end{array}
\]

And we are back to the point where our division process cannot help us.

However, we do know for sure now that \( \sqrt{37} \) is between 6 and 7.
So let us draw:

\[
\begin{array}{c}
\underline{6} \\
\underline{6} \\
\underline{6} \\
\underline{6}
\end{array}
\]

What are the areas of each of the sections 1, 2 and 3?

Expect the answers: \(6 \times \square\), \(6 \times \square\), and \(\square \times \square\).

Since section 3 is smallest, we might ignore it temporarily and see that sections 1 and 2 have a total area of \((6 \times \square) + (6 \times \square) = 12 \times \square\).

Since there is only one unit of the 37 not in the 6 by 6 square, we have to assume that \(12 \times \square\) is about 1; what value is \(\square\)?

Expect the answer: about \(\frac{1}{12}\).

Then what is the area of section 3?

Expect the answer: \(\square \times \square = \frac{1}{144}\).

This is certainly smaller than \(\frac{1}{2}\), so we can assume that

\[\sqrt{37} \approx 6 \frac{1}{12}\.

Notice that if I had been trying to find \(\sqrt{38}\), I would have done the same thing until I said that \(12 \times \square\) is about 1. Then I would have said that \(12 \times \square\) is about 2, and so \(\square\) is about \(2 \times \frac{1}{12} \times \frac{2}{12}\).

I still would have checked \(\square \times \square\) to see that it was smaller than \(\frac{1}{2}\), and since \(\frac{2}{12} \times \frac{2}{12}\) is smaller than \(\frac{1}{2}\), I would have accepted this approximation.
Ask the students to work out the approximate square roots for each of the following and then ask students to come to the board to show their work.

8
18
50
67

** Here the students will learn how to approximate the square roots of fractions whose denominators are perfect squares, but whose numerators are not. **

NOW SUPPOSE I WANT TO FIND THE APPROXIMATE SQUARE ROOT FOR \( \sqrt{\frac{5}{4}} \).
I HAVE ALREADY LEARNED THAT I CAN SEPARATELY ATTACK THE NUMERATORS AND DENOMINATORS AND PUT TOGETHER THESE SQUARE ROOTS TO FORM THE SQUARE ROOT FRACTION.

SUPPOSE WE DO THAT HERE.
WE FOUND THAT \( \sqrt{5} \approx 2.4 \) FROM OUR PREVIOUS WORK AND WE KNOW THAT \( \sqrt{4} = 2 \).

SO THE APPROXIMATE SQUARE ROOT OF \( \frac{5}{4} \) IS \( \frac{2.4}{2} \).

SUPPOSE WE ARE TRYING TO FIND \( \sqrt{\frac{13}{4}} \).
WE KNOW THAT \( \sqrt{13} \approx 3.6 \) AND \( \sqrt{4} = 2 \), SO \( \sqrt{\frac{13}{4}} \approx \frac{3.6}{2} \).

LET'S DO THE SAME THING FOR \( \sqrt{\frac{37}{4}} \).
WE CAN WRITE THAT THE APPROXIMATE SQUARE ROOT OF \( \frac{37}{4} \) IS \( \frac{6.1}{2} \).

Have the students find approximate square roots for each of the following; have a few students show their work on the board.

\( \sqrt{\frac{32}{9}} \), \( \sqrt{\frac{47}{25}} \), \( \sqrt{\frac{29}{49}} \)
AND \[ 27 \times 27 = 729, \text{ so } \sqrt{729} = 27. \]

Suggest hand students worksheet 3 to complete.

**Here the students will learn about approximating square roots of wholes.**

Suppose we now want to find \( \sqrt{5} \). We know that 2 is too small, since \( 2 \times 2 = 4 \) and 3 is too large since \( 3 \times 3 = 9 \). So we have a problem we had not previously faced.

One technique we might use is to convert 5 to a fraction and try to find a fraction near it with an "even" square root. For example, we know that \( 5 = 5 \times 1 = 5 \times 25 \times \frac{1}{25} = 125 \times \frac{1}{25} = 121 \times \frac{1}{25} \) which has a square root of \( 11 \times \frac{1}{5} = \frac{11}{5} \). But this might prove very long and tedious if we don't find the right fraction right away.

Let's try to develop a method.

Suppose we try to find \( \sqrt{5} \).

We know that the solution is something between 2 and 3. Let us call it \( 2 + \Box \), where \( \Box \) is smaller than 1.

Then, since \( (2 + \Box) \) is the square root of 5, we have

\[ (2 + \Box) \times (2 + \Box) = 5. \]

How can we multiply these numbers out?

Well, just as \( 4 \times (5 + 6) = 4 \times 5 + 4 \times 6 \), we can consider:

\[ (2+\Box) \times (2 + \Box) = (2+\Box) \times 2 + (2+\Box) \times \Box. \]

Then we can rewrite each of the expressions on the right:

\[ (2+\Box) \times 2 = 2 \times 2 + \Box \times 2 \quad \text{and} \quad (2+\Box) \times \Box = 2 \times \Box + \Box \times \Box. \]

Notice that the use of the distributive law here is very involved and it may be necessary to go back to consideration of simpler examples, like \( 2 \times (4 + 5) = 2 \times 4 + 2 \times 5 \), or \( (2+3) \times (4 + 5) = (2+3) \times 4 + (2+3) \times 5 \) and then distribute.
again. Try to use underlining to indicate the common term and make sure that the entire first expression is viewed as the common term the first time, but as the usual addends the second time through. That is, the first time \((2 + \square)\) on the left is treated as a united quantity which travels together in order to break up the right hand expression, but afterwards it too is broken up.

**Then, we have that**  
\[(2 \times 2) + (2 \times \square) + (\square \times 2) + (\square \times \square) = 5.\]

Since \(2 \times 2 = 4\), that means that \((2 \times \square) + (\square \times 2) + (\square \times \square) = 5 - 4 = 1\).

**Suppose we pretend temporarily that we can ignore the \(\square \times \square\) part.**

Then we have \((2 \times \square) + (\square \times 2) = 1.\)

But \(2 \times \square = \square \times 2\), so we have \((2 \times \square) + (2 \times \square) = 1.\)

But \((2 \times \square) + (2 \times \square) \neq 4 \times \square\) using the distributive principle.

So, \(4 \times \square = 1\) so \(\square\) is \(\frac{1}{4}\).

If we assure \(\square\) is \(\frac{1}{4}\), then we notice that \(\square \times \square\) is only what number?

Expect the answer: \(\frac{1}{16}\).

Since this number is less than \(\frac{1}{2}\), we can say that we have used up most of the 1 in the other expressions besides the \(\square \times \square (2 \times \square + \square \times 2)\). And so we say that \(\sqrt{5}\) is about \(2 \frac{1}{4}\). We write this:

\[\sqrt{5} \approx 2 \frac{1}{4}.\]

We might have used the division procedure again, as well.

So our first guess would have been 2. Then we would divide 5 by 2, and get \(2) \frac{5}{4} 2 \frac{1}{2}\) (we have used up with fractions since our next choice would be a number between 2 and 2 \(\frac{5}{2}\) and so we will avoid the division technique for the time being).

Suppose we want to find \(\sqrt{13}\).
WE KNOW THAT THE ANSWER MUST BE BETWEEN 3 AND 4, SINCE 3 x 3 = 9 AND 4 x 4 = 16, AND 13 IS BETWEEN 9 AND 16.

SO AGAIN, WE ASSUME THAT THE SQUARE ROOT IS A LITTLE OVER 3, SAY, 3 + \( \Box \).

THEN, BECAUSE IT IS A SQUARE ROOT,

\[
(3 + \Box) \times (3 + \Box) = 13.
\]

WE CAN REWRITE:

\[
(3 + \Box) \times (3 + \Box) = (3 + \Box) \times 3 + (3 + \Box) \times \Box
\]

\[
= (3 \times 3) + (\Box \times 3) + (3 \times \Box) + (\Box \times \Box)
\]

\[
= 9 + (3 \times \Box) + (3 \times \Box) + (\Box \times \Box).
\]

THEN, IF \( 9 + (3 \times \Box) + (3 \times \Box) + (\Box \times \Box) = 13 \),

\[
(3 \times \Box) + (3 \times \Box) + (\Box \times \Box) = 4 \cdot (13 - 9).
\]

AGAIN, LET US IGNORE THE \( \Box \times \Box \) TEMPORARILY. IF WE DO, WE GET

\[
(3 \times \Box) + (3 \times \Box) = 4.
\]

BUT \( (3 \times \Box) + (3 \times \Box) = 6 \times \Box \)

AND IF \( 6 \times \Box = 4 \), THEN \( \Box = 4 \times \frac{1}{6} \) SINCE

\[
6 \times (4 \times \frac{1}{6}) = 4 \times (6 \times \frac{1}{6}) = 4 \times 1 = 4
\]

SO BY IGNORING \( \Box \times \Box \), WE GET \( \Box = \frac{4}{6} \).

THEN, IF WE INCLUDE \( \Box \times \Box \), WE GET \( \Box \times \Box = \frac{4}{6} \times \frac{4}{6} = \frac{16}{36} \).

BUT SINCE THIS IS LESS THAN \( \frac{1}{2} \), WE DO NOT WORRY ABOUT IT, AND SAY THAT WE ARE CLOSE ENOUGH TO BE SATISFIED.

WE WRITE:

\[
13 \approx 3 \frac{4}{6}
\]

SUPPOSE WE WANT TO FIND \( \sqrt{37} \). IF WE DID NOT KNOW IN ADVANCE TO TRY

A NUMBER NEAR 6, WE MIGHT USE OUR DIVISION PROCEDURE TO CLOSE IN ON A NUMBER BETWEEN 6 AND 7.
FOR EXAMPLE, WE MIGHT DO:

\[ 4 \times □ = 37 \]  TO SEE IF □ = 4.

WE DIVIDE:

\[
\begin{array}{c}
4 \sqrt{37} \\
36 \\
\hline
9
\end{array}
\]

AND WE FIND THAT □ IS ABOUT 9. SO NOW WE TRY A NUMBER FOR THE SQUARE ROOT BETWEEN 4 AND 9, LIKE 7.

SO WE SOLVE:  \[ 7 \times □ = 37 \]  TO SEE IF □ = 7.

\[
\begin{array}{c}
7 \sqrt{37} \\
35 \\
\hline
5
\end{array}
\]

AND WE FIND THAT □ IS ABOUT 5. SO NOW WE TRY A NUMBER BETWEEN 7 AND 5, SAY, 6.

\[
\begin{array}{c}
6 \sqrt{37} \\
36 \\
\hline
5
\end{array}
\]

AND WE ARE BACK TO THE POINT WHERE OUR DIVISION PROCESS CANNOT HELP US.

HOWEVER, WE DO KNOW FOR SURE NOW THAT □ SITS BETWEEN 6 AND 7.

SO WE GO BACK TO OUR PROCEDURE OF BEFORE.

WE KNOW THAT \[ \sqrt{37} = 6 + □ \] .

SO \[ (6+ □) \times (6 + □ ) = 37 \] .

BUT, \[ (6+ □) \times (6 + □ ) \]

\[ = (6+ □) \times 6 + (6+ □) \times □ \]

\[ = (6 \times 6) + (□ \times 6) + (6 \times □) + (□ \times □) \]

\[ = 36 + (6 \times □) + (6 \times □) + (□ \times □) \]

THE, \[ 36 + (6 \times □) + (6 \times □) + (□ \times □) = 37, \]  SO

\[ (6 \times □) + (6 \times □) + (□ \times □) = 1(= 37 - 36) \]

TEMPORARILY IGNORING THE □ \times □, WE FIND THAT \[ (6 \times □) + (6 \times □) = 12 \times □ \]

IS ABOUT 1, SO □ IS ABOUT \[ \frac{1}{12} \]

IF WE THEN CALCULATE □ \times □, WE GET \[ \frac{1}{144} \]  WHICH IS MUCH SMALLER THAN \[ \frac{1}{2} \] .
SO WE ARE NOT TOO FAR OFF, AND WE CAN SAY THAT \( \sqrt{37} \approx 6 \frac{1}{12} \)

NOTICE THAT IF I HAD BEEN TRYING TO FIND \( \sqrt{38} \), I WOULD HAVE DONE THE SAME THING UNTIL I SAID THAT \( 12 \times \square \) IS ABOUT 1. THEN I WOULD HAVE SAID THAT \( 12 \times \square \) IS ABOUT 2, AND SO \( \square \) IS ABOUT \( 2 \times \frac{1}{12} = \frac{1}{6} \). I STILL WOULD HAVE CHECKED \( \square \times \square \) TO SEE THAT IT WAS SMALLER THAN \( \frac{1}{2} \), AND SINCE \( \frac{1}{12} \times \frac{1}{12} \) IS SMALLER THAN \( \frac{1}{2} \), I WOULD HAVE ACCEPTED THIS APPROXIMATION.

Ask the students to work out the approximate square roots for each of the following and then ask students to come to the board to show their work.

8
18
50
67

**Here the students will learn how to approximate the square roots of fractions whose denominators are perfect squares, but whose numerators are not.**

NOW SUPPOSE I WANT TO FIND THE APPROXIMATE SQUARE ROOT OF \( \sqrt{\frac{5}{4}} \).

I HAVE ALREADY LEARNED THAT I CAN SEPARATELY ATTACK THE NUMERATORS AND DENOMINATORS AND PUT TOGETHER THESE SQUARE ROOTS TO FORM THE SQUARE ROOT FRACTION.

SUPPOSE WE DO THAT HERE.

WE FOUND THAT \( \sqrt{5} \approx 2 \frac{1}{4} \) FROM OUR PREVIOUS WORK AND WE KNOW THAT \( \sqrt{4} = 2 \).

SO THE APPROXIMATE SQUARE ROOT OF \( \sqrt{\frac{5}{4}} \) IS \( 2 \frac{1}{4} \).
Suppose I am trying to find \( \sqrt{\frac{13}{4}} \).

We know that \( \sqrt{13} \approx 3.6 \) and \( \sqrt{4} = 2 \), so \( \sqrt{\frac{13}{4}} \approx \frac{3.6}{2} \).

Let's do the same for \( \sqrt{\frac{37}{4}} \).

We can write that the approximate \( \sqrt{\frac{37}{4}} \) is \( \frac{6.12}{2} \).

Have the students find approximate square roots for each of the following; have a few students show their work on the board.

\( \sqrt{\frac{31}{9}} \), \( \sqrt{\frac{47}{25}} \), \( \sqrt{\frac{28}{49}} \).
APPENDIX B

THE MEASURING INSTRUMENTS
Items of Product of a Mixed Number and a Fraction Pretest

1. 6 x 3 = □
2. 2 x 9 = □
3. 8 x 4 = □
4. 6 x 8 = □
5. 7 x 5 = □
6. 9 x 7 = □
7. 6 x 7 = □
8. 8 x 9 = □
9. 6 x 6 = □
10. 8 x 8 = □
11. 3 1/2 = 3 + □/Δ
12. 6 1/4 = □ + 1/4
13. 2 9 5/9 = □ + 5/9
14. □ 6/8 = 5 + 6/8
15. Give a fraction name to the shaded portion below:

16. Shade in 1/3 of the diagram on the right.

17. Give a fraction name to the shaded portion below:

18. Draw a diagram with 2/3 shaded in on the line at the right.

19. Draw a diagram with 3/4 shaded in on the line at the right.

20. □ x 1/5 = 1
21. □ x 1/8 = 1
22. 6 x □/Δ = 1
23. 3/2 = 3 x □/Δ
24. $\frac{8}{9} = \square \times \frac{1}{9}$
25. $\frac{16}{5} = \square \times \frac{1}{5}$
26. What is the length of a rectangle with a width of 2 feet and an area of $6 \times 2 \text{ sq. ft.}$?
27. What is the width of a rectangle with a length of 8 feet and an area of $8 \times 3 \text{ sq. ft.}$?
28. What is the area of a rectangle with width 3 ft. and length 5 ft.?
29. What operation would you use to find the area of a rectangle with length 89 feet and width 38 feet? (Would you add, subtract, multiply or divide?)
30. Using the idea that to find area, you find the number of one-unit squares in a figure, show how you would find the area of the figure below:

```
       2
      /
     /
    /
   /
```
31. $1 \times 6 = \square$
32. $\square \times 8 = 8$
33. $\times 239 = 239$
34. $1 \times \frac{1}{4} = \square / \Delta$
35. $33 \times 24 \times 51 = 24 \times \square \times 33$
36. $56 \times 19 \times \square = 19 \times \Delta \times 48$
37. $27 \times 56 \times 65 \times 41 = 27 \times \square \times 56 \times 41$
38. $55 \times (\square + 38) = (55 \times 42) + (55 \times 38)$
39. $\square \times (23 + 872) = 8950$
40. $6 \times (84 + \square) = (6 \times 84) + (6 \times 156)$
41. What is the area of the figure below marked with the question mark?
42. What is the area of the figure below marked with the question mark?
### Items of Product of a Mixed Number and a Fraction Computation Test

1. $2 \times \frac{6}{7}$
2. $4 \times \frac{3}{8}$
3. $6 \times \frac{8}{9}$
4. $5 \times \frac{5}{7}$
5. $9 \times \frac{8}{11}$
6. $7 \times \frac{8}{9}$
7. $2 \times 2 \frac{3}{8}$
8. $4 \times 2 \frac{2}{13}$
9. $8 \times 4 \frac{3}{25}$
10. $6 \times 5 \frac{4}{29}$
11. $9 \times 6 \frac{5}{70}$
12. $7 \times 7 \frac{6}{50}$
13. $\frac{2}{7} \times \frac{1}{2}$
14. $\frac{2}{3} \times \frac{8}{9}$
15. $\frac{4}{6} \times \frac{5}{6}$
16. $\frac{4}{5} \times \frac{7}{8}$
17. $\frac{8}{9} \times \frac{8}{9}$
18. $\frac{5}{6} \times \frac{9}{10}$
19. $\frac{1}{2} \times 8 \frac{6}{9}$
20. $\frac{2}{3} \times 7 \frac{4}{5}$
21. $\frac{4}{5} \times 7 \frac{4}{6}$
22. $\frac{3}{4} \times 8 \frac{7}{9}$
23. $\frac{8}{9} \times 8 \frac{6}{7}$
24. $\frac{7}{9} \times 7 \frac{8}{9}$
1. If you know that \( \frac{1}{81} \times \frac{1}{63} = \frac{1}{5103} \), you could also tell that
\[ \frac{2}{81} \times \frac{1}{63} = □/△. \]

2. If you know that \( \frac{53}{69} \times \frac{48}{73} = \frac{2544}{5037} \), find values for □ and △ so that \( \frac{48}{69} \times \frac{53}{△} = \frac{□}{5037} \)

3. If you know that \( \frac{13}{64} \times \frac{12}{15} = \frac{156}{960} \), you could also tell that
\[ \frac{13}{64} \times \frac{12}{10} \times 15 = □/△. \]

4. If you know that \( \frac{13}{5} \times \frac{4}{27} = \frac{52}{135} \), you could also tell that
\[ \frac{13}{10} \times \frac{5}{27} = \△/□. \]

5. Write the following as a multiplication statement.
\[ 2 \frac{3}{5} + 2 \frac{3}{5} + 2 \frac{3}{5} + 2 \frac{3}{5} \]

6. Write the following as strictly a multiplication expression (not involving addition). Do not compute the answer.
\[ \frac{5}{6} \times 2 \frac{4}{9} + \frac{5}{6} \times \frac{5}{9} \]

7. Write the following as strictly a multiplication expression (not involving addition). Do not compute the answer.
\[ \frac{3}{4} \times 2 \frac{1}{4} + \frac{3}{4} \times 2 \frac{3}{4} \]

8. Find the answers to a, b, and c. Write the answer to (d) on the line to the right. The first three answers are only meant to help you with the answer to (d). (□ is a whole number)
(a) \( \frac{3}{4} \times \frac{4}{3} = □ \)
(b) \( \frac{8}{9} \times \frac{9}{8} = □ \)
(c) \( \frac{5}{6} \times \frac{6}{5} = □ \)
(d) \( \frac{879}{432} \times \frac{432}{879} = △ \)
9. Find the answers to a, b, and c. Write the answer to (d) on the line to the right. (□ is a whole number)

(a) \(3 \times 2 \frac{1}{3} = \square\)
(b) \(4 \times 2 \frac{1}{4} = \square\)
(c) \(9 \times 2 \frac{1}{9} = \square\)
(d) \(\Delta \times 2 \frac{1}{\Delta} = 17\)

10. If \(3 \times 4 \frac{\Delta}{\Delta} = 12 \frac{6}{9}\), what is \(\square\)

11. If \(5 \times \square \frac{\Delta}{\Delta} = 20 \frac{5}{8}\), what are \(\square\) and \(\Delta\)?

12. If \(\square \times 3 \frac{5}{16} = 9 \frac{15}{16}\), what is \(\square\)?

13. If \(7/2 \times \square \frac{1}{4} = 35/2 + 7/8\), what is \(\square\)

14. If \(2/3 \times 5 \frac{\square}{\Delta} = 10/3 + 8/15\), what is \(\square/\Delta\)

15. How would you use the rule for multiplying a fraction by a mixed number, for example, to multiply: \(2 \frac{1}{4} \times 3 \frac{1}{5}\). Show all of your steps in the space below and write the answer on the line to the right.

16. Find \(1/2 \div 2 \times 24 \frac{2}{5}\)

17. Find \(2/3 \div 4 \times 16 \frac{9}{11}\)

18. Find the answers to a, b, and c. Write the answer to (d) on the line to the right. The first three answers are only to help you with (d).

(a) \(2 \frac{1}{3}\)
\[
\times 3 \frac{1}{4}
\]
\[
6 = 3 \times 2
\]
\[
1 = 3 \times 1/3
\]
\[
2/4 = 2 \times 1/4
\]
\[
+ 1/12 = 1/3 \times 1/4
\]
\[
\square \frac{\Delta}{\square}
\]

(b) \(2 \frac{1}{3}\)
\[
\times 4 \frac{1}{5}
\]
\[
8 = 4 \times 2
\]
\[
4/3 = 4 \times 1/3
\]
\[
2/5 = 2 \times 1/5
\]
\[
+ 1/15 = 1/3 \times 1/5
\]
\[
\square \frac{\Delta}{\square}
\]
(c) 3 1/5  
\[ \times 1 \frac{4}{6} \]
\[ 3 = 1 \times 3 \]
\[ 1/5 = 1 \times 1/5 \]
\[ 12/6 = 3 \times 4/6 \]
\[ + \frac{4}{6} = 1/5 \times 4/6 \]
\[ \Delta \triangleleft \frac{\Delta}{\square} \]

19. What is \((6 \times 2/9) \times 5 \frac{1}{3}\)

20. What is \((1/3 \times 1/2) \times 2 \frac{5}{8}\)

21. What is \(3/4 \times (2/3 \times 3)\)

22. What is \((4 \times 1/3 \times 1/5) \times 2 \frac{1}{6}\)

23. If \((3 \times 1/2) \times 3 \frac{1}{4} = \frac{\Delta}{\Delta} \times 9 \frac{3}{4}\), what is \(\frac{\square}{\Delta}\)

24. If \((6 \times \frac{\Delta}{\Delta}) \times 5 \frac{1}{3} = 30/3 + 6/9\), what is \(\frac{\square}{\Delta}\)

25. If \((4 \times 2/3) \times \frac{\Delta}{\Omega} = 16/3 + 8/15\), what is \(\frac{\square}{\Delta}\)

26. Find values for \(\Delta\) and \(\Delta\) so that
\[ 4 \times \frac{\square}{\Delta} \times 2 \frac{5}{\Delta} = 40/3 + 25/27 \]

27. Susie has suggested another rule for multiplying fractions. To multiple, for example, \(2/3 \times 4/5\), she doubles the first fraction's numerator and doubles the second fraction's denominator and then multiplies. \(4/3 \times 4/10 = 16/30\), so \(2/3 \times 4/5 = 16/30\). She claims that her answer is equivalent to the one which the teacher gets by the method taught in class. \((2 \times 4/3 \times 5 = 8/15\) Will she always get equivalent answers to yours? Why do you think she should? Use \(3/4 \times 2/9\) as an example.
28. Johnny has suggested another rule for multiplying fractions. To multiply, for example, 2/3 x 4/5, he finds the answer to 4/3 x 2/5. He switches the numerators of the fractions and then multiplies. He claims that his answer is equivalent to the one which the teacher gets by the method taught in class. Will he always get equivalent answer to yours? Why do you think he should? Use 3/4 x 2/9 as an example.

29. Sam has suggested a rule for multiplying two fractions, too. To find the answer to, for example, 2/3 x 4/5 Sam finds the answer to 3/2 x 5/4 and then turns the fraction answer upside down. In other words, he turns both fractions upside down, multiplies, and then turns his answer upside down. He claims that his answer is the same as the one the teacher gets by the method taught in class. Will he always get the same answer as yours? Why do you think he should? Use 3/4 x 2/9 as an example.

30. Judy has suggested a way to check multiplication of fraction answers. She says that one can be sure that a/b x c/d = P/_) if P/c = a and P/d = b. For example, it is true that 3/5 x 2/9 = 6/45 since 6 ÷ 2 = 3 and 45 ÷ 9 = 5. Can you be sure if your answer is right by checking with Judy's method? Why do you think so?
Items of Comparing Fractions Pretest

1. $6 \times 3 =$
2. $2 \times 9 =$
3. $8 \times 4 =$
4. $6 \times 8 =$
5. $7 \times 5 =$
6. $9 \times 7 =$
7. $6 \times 7 =$
8. $8 \times 9 =$
9. $6 \times 6 =$
10. $8 \times 8 =$

11. Give a fraction name to the shaded portion below:

![Fraction 1](image1)

12. Shade in $\frac{1}{3}$ of the diagram on the right.

![Diagram 2](image2)

13. Give a fraction name to the shaded portion below:

![Diagram 3](image3)

14. Draw a diagram with $\frac{2}{3}$ shaded in on the line at the right.

15. Draw a diagram with $\frac{3}{4}$ shaded in on the line at the right.

16. Is the fraction representing area (a) more than, less than, or the same as that representing area (b)

(a) ![Diagram 4](image4)  (b) ![Diagram 5](image5)
17. Is the fraction representing area (a) more than, less than, or the same as that representing area (b)?

18. Is the fraction representing area (a) more than, less than, or the same as that representing area (b)?

19. Is the fraction representing area (a) more than, less than, or the same as that representing area (b)?

20. Is the fraction representing area (a) more than, less than, or the same as that representing area (b)?

21. What statement does the diagram below suggest?

For example: suggests \( \frac{1}{2} = \frac{2}{4} \)

22. What statement does the diagram below suggest:
23. What statement does the diagram below suggest?

24. $33 \times 24 \times 31 = 24 \times \square \times 33$

25. $56 \times 19 \times 48 = 19 \times \square \times 48$

26. $27 \times 56 \times 65 \times 41 = 27 \times \square \times 56 \times 41$

27. $\square \times 1/5 = 1$

28. $\square \times 1/8 = 1$

29. $6 \times \square/\Delta = 1$

30. $3/2 = 3 \times \square/\Delta$

31. $8/9 = \square \times 1/9$

32. $16/5 = \square \times 1/5$

33. Which is larger, $44 \times 13$ or $52 \times 13$

34. Which is larger, $69 \times 158$ or $32 \times 158$

35. Which is greater, $15 \times 1/4$ or $16 \times 1/4$
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</tr>
<tr>
<td>14. 4/6 or 6/7</td>
<td></td>
</tr>
</tbody>
</table>
Items of Comparison of Fractions Generalization Test

1. Which is greater: \( \frac{\square}{37} \) or \( \frac{\square}{38} \)

2. Which is greater: \( \frac{87}{\square} \) or \( \frac{88}{\square} \)

3. For each, find which is greater. Write the answer to (d) on the line to the right. The first three answers are only meant to help you with the answer to (d).
   (a) \( \frac{2}{3} \) or \( \frac{1}{2} \)
   (b) \( \frac{3}{4} \) or \( \frac{4}{5} \)
   (c) \( \frac{7}{8} \) or \( \frac{6}{7} \)
   (d) \( \frac{\square}{\square} + 1 \) or \( \frac{\square}{\square} - 1 \)

4. For each, find which is greater. Write the answer to (d) on the line to the right. The first three answers are only meant to help you with the answer to (d).
   (a) \( \frac{2}{3} \) or \( \frac{2}{5} \)
   (b) \( \frac{3}{17} \) or \( \frac{3}{28} \)
   (c) \( \frac{13}{53} \) or \( \frac{13}{78} \)
   (d) \( \frac{\square}{834} \) or \( \frac{\square}{729} \)

5. For each, find which is greater. Write the answer to (d) on the line to the right. The first three answers are only meant to help you with the answer to (d).
   (a) \( \frac{1}{14} \) or \( \frac{2}{14} \)
   (b) \( \frac{21}{4} \) or \( \frac{33}{4} \)
   (c) \( \frac{40}{58} \) or \( \frac{33}{58} \)
   (d) \( \frac{6}{\square} \) or \( \frac{3}{\square} \)
6. For each, find which is greater. Write the answer to (d) on the line to the right.

(a) $33 - 1/33$ or $33/33 + 1$

(b) $33/33 + 9$ or $33 - 9/33$

(c) $33 - 18/33$ or $33/33 + 18$

(d) $33/33 + □$ or $33 - □/33$

7. If $384/529 \geq 384/□$, what can you say about □?

8. If $483 \times 25 \geq 365 \times 18$, which is greater:
   
   $2 \times 483/365$ or $2 \times 18/25$

9. If $6 \times 18 \geq 13 \times 8$, which is greater:
   
   $6/13$ or $8/18$

10. If $17 \times 11 \leq 14 \times 15$, which is greater:
    
    $14/17$ or $11/15$

11. If $82 \times 15 \geq 63 \times □$, which is greater:
    
    $82/□$ or $63/15$

12. If $8 \times (5 + □) \geq 6 \times 3$, which is greater:
    
    $5 + □/6$ or $3/8$

13. If $425 \times 211 \leq 310 \times 316$, which is greater:
    
    $425-310/310$ or $316-211/211$

14. If $6 \times (3 - □) \geq 2 \times 5$, which is greater:
    
    $6/5$ or $2/3 - □$

15. Use the rule for comparing fractions to compare two whole numbers, 5 and 7. Show all of your steps in the space below and write the greater of the two numbers on the line to the right. (Hint: Write them first as fractions.)
16. Use the rule for comparing fractions to compare the two mixed numbers, 2 1/3 and 2 2/5. Show all of your work in the space below and write the greater of the two numbers on the line to the right.

17. Use the rule for comparing fractions to compare 2 1/5 and 20/9. Write the larger on the line to the right.

18. Use the rule for comparing fractions to compare 3 2/3 and 31/9. Write the larger on the line to the right.

19. If 22 x Δ = 83 x Δ, is 22/83 more than, less than, or equal to Δ/Δ.

20. You know that 22/Δ = 83/Δ if 22 x Δ = ___.

21. You know that Δ/Δ = Δ/Δ if ___ = ____.

22. Arrange these three fractions from largest to smallest:
   3/5  19/30  7/12

23. Arrange these three fractions from largest to smallest:
   4/9  2/3  6/11

24. Arrange these three fractions from largest to smallest:
   8/3  25/10  25/9

25. Find three fractions less than 2/5.

26. Arrange these fractions from largest to smallest;
   3/13  6/25  14/52

27. Susie has suggested another rule for comparing fractions. To decide, for example, if 2/3 > 4/9, she compares 2/4 to 3/9. If 2/4 > 3/9, then 2/3 > 4/9. She switches the first denominator with the second numerator and then compares these to decide if the original first fraction is larger than the original second fraction. She claims that her answer is always the same as the one which the teacher gets by her
method. Will Susie always get the right answer? Why do you think she should? Show how Susie would compare $\frac{7}{11}$ and $\frac{5}{8}$; explain her method and why it works.

28. Johnny has suggested a method for comparing fractions when the numerator of the second goes into the numerator of the first exactly and the denominator of the second goes into the denominator of the first exactly. For example, his method works for comparing $\frac{15}{16}$ and $\frac{3}{4}$ since 3 goes into 15 exactly and so does 4 into 16. Johnny decided if the first numerator divided by the second is greater than the first denominator divided by the second, then the first fraction is larger than the second. For example, $15 \div 3 = 5$ and $16 \div 4 = 4$ and $5 > 4$, so $\frac{15}{16} > \frac{3}{4}$. Does Johnny's method always work for these kinds of problems? Explain why, using the problem of comparing $\frac{24}{25}$ and $\frac{4}{5}$ as an example.

29. Sam says that $\frac{a}{b} > \frac{c}{d}$ whenever $\frac{1}{a} \times d < \frac{1}{b} \times c$ and only then, so to compare, for example, $\frac{2}{3}$ and $\frac{4}{9}$, he discovers that $\frac{1}{2} \times 9 < \frac{1}{3} \times 4$, so he concluded that the first fraction is greater than the second. Does Sam's method always work for comparing fractions? Explain what Sam would do and why he is correct by comparing $\frac{7}{11}$ and $\frac{5}{8}$ by his method.

30. Judy says that she can tell if $\frac{a}{b} > \frac{c}{d}$ right away, if $a > (b \times c) \div d$, then the first fraction is larger. For example, to compare $\frac{2}{3}$ and $\frac{4}{9}$ she says $2 > (4 \times 3) \div 9 = 1 \frac{1}{3}$. Therefore, the first fraction is greater. Does Judy's method always work for comparing fractions? Explain what Judy would do and why she is correct by comparing $\frac{7}{11}$ and $\frac{5}{8}$ by her method.
Items of Changing Fractions to Decimals Pretest

1. Give a fraction name to the shaded portion below:

2. Give a fraction name to the shaded portion below:

3. Shade in 1/5 of the diagram on the right.

4. Draw a diagram with 3/4 shaded in on the line to the right.

5. Draw a diagram with 2/6 shaded in on the line to the right.

6. What equation would you write to describe the action in sharing 27 marbles among 3 people?

7. What equation would you write to describe the action in sharing 18 cupcakes among 6 children?

8. What problem would you write to describe the action in sharing 24 pencils among 8 students?

9. Draw a diagram below showing why 8 ÷ 4 = 2.

10. Draw a diagram showing why 12 ÷ □ = 4. Use the space below for the drawing and find the value of □.

11. Divide

   \[ \frac{425}{11} \]
12. Divide
   \[ 20 \div 5834 \]

13. Divide
   \[ 7 \div 2859 \]

14. Divide
   \[ 25 \div 4632 \]

15. Divide
   \[ 6 \div 384 \]

16. Write the following multiplication statement as a division statement:
   \[ 532 \times 18 = 9576 \]

17. Write the following division statement as a multiplication statement:
   \[ 492 \div 123 = 4 \]

18. Write the following division statement as a multiplication statement:
   \[ \Box \div 6 = \Delta \]

19. Write the following multiplication statement as a division statement:
   \[ \Box \times 6 = \Delta \]

20. Write the following multiplication statement as a division statement:
   \[ \Box \times \Delta = 0 \]

21. \[ \Box \times \frac{1}{9} = 1 \]

22. \[ 8 \times \frac{\Box}{\Delta} = 1 \]

23. \[ \frac{14}{9} = 14 \times \frac{\Box}{\Delta} \]

24. \[ \frac{3}{7} = \frac{\Box}{1/7} \]

25. \[ \frac{38}{6} = \frac{\Box}{1/6} \]

26. \[ 74 \times 83 \times 77 = \Box \times 74 \times 77 \]

27. \[ 312 \times 25 \times \Box \times 87 = 43 \times \Delta \times 25 \times 312 \]
28. $45 \times 203 \times 87 = \square \times 203 \times 87$

29. $1 \times 576 = \square$

30. $\square \times (374 + 596) = 374 + 596$
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<thead>
<tr>
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<td>18.</td>
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Items of Changing Fractions to Decimals Generalization Test

1. If you know that the decimal for $\frac{4}{3} \approx 1.333$, you can say that the decimal for $2 \times \frac{4}{3} \approx$

2. For each part, find the decimal equivalent required. Write the answer to (d) on the line to the right. The first three answers are only meant to help you with the answer to (d).

(a) If you know that the decimal for $\frac{4}{6} \approx 0.666$, you can also tell that the decimal for $\frac{4}{10} \times 6 \approx$

(b) If you know that the decimal for $\frac{3}{5} = 0.600$, you can also tell that the decimal for $\frac{3}{10} \times 5 =$

(c) If you know that the decimal for $\frac{6}{7} \approx 0.857$, you can also tell that the decimal for $\frac{6}{10} \times 7 \approx$

(d) If you know that the decimal for $\frac{6}{\Box} \approx 0.295$, you can also tell that the decimal for $\frac{6}{10} \times \Box \approx$

3. For each part, find the decimal equivalent required. Write the answer to (d) on the line to the right. Do all other work below.

(a) If you know that the decimal for $\frac{4}{6} \approx 0.666$, you can also tell that the decimal for $10 \times \frac{4}{6} \approx$

(b) If you know that the decimal for $\frac{3}{5} = 0.600$, you can also tell that the decimal for $10 \times \frac{3}{5} =$

(c) If you know that the decimal for $\frac{6}{7} \approx 0.857$, you can also tell that the decimal for $10 \times \frac{6}{7} \approx$

(d) If you know that the decimal for $\frac{6}{\Box} \approx 0.295$, you can also tell that the decimal for $10 \times \frac{6}{\Box} \approx$
4. If you know that the decimal for $7/25 = .28$, you can also tell that the decimal for $7/25 \times 2 = 7/50 =$ __________

5. If you know that the decimal for $3/50 = .06$, and the decimal for $9/50 = .18$, then you can also tell that the decimal for $3 + 9/50 = 12/50 =$ __________

6. If you know that the decimal for $9/15 = .600$ and the decimal for $7/15 \approx .467$, then you can also tell that the decimal for $9-7/15 = 2/15 \approx$

7. If you know that the decimal for $17/16 = 1.0625$, then you can also tell that the decimal for $1/16 =$ __________

8. If you know that the decimal for $8/14 \approx .571$ and that the decimal for $7/14 = .5$, then you can also tell that the decimal for $8-7/14 = 1/14 \approx$

9. For each part, find the decimal equivalent required. Write the answer to (d) on the line to the right. Show all other work below.

(a) If you know that the decimals for $4/9 \approx .444$ and $2/9 \approx .222$, then you can also tell that $.666$ is the approximate decimal for what fraction $\square / \triangle$

(b) If you know that the decimal for $3/16 \approx .187$ and $8/16 = .500$, then you can also tell that $.687$ is the approximate decimal for what fraction $\square / \triangle$

(c) If you know that the decimals for $15/20 = .750$ and $4/20 = .200$, then you can also tell that $.950$ is the approximate decimal for what fraction $\square / \triangle$

(d) If you know that the decimals for $37/\square = .074$ and $12/\square = .024$, then $.098$ is the approximate decimal for what fraction $\triangle / \square$. Do not find the value of $\square$.
10. If you know that the decimal for $2/3 \approx .666$, then you could also tell that $3 \times \Box \approx 2$, where $\Box$ is a decimal.

11. If you know that the decimal for $2/5 = .400$, then you could also tell that $5 \times \Box = 2$, where $\Box$ is a decimal.

12. If the decimal for $\Box/11 \approx .45$, what whole number is $\Box$?

13. If the decimal for $9/\Box \approx .81$, what whole number is $\Box$?

14. If the decimal for $8/\Box \approx .88$ and the decimal for $5/\Box \approx .55$, then $.33$ is the approximate decimal for what fraction $\Delta/\Box$ Do not find the value of $\Box$.

15. If the decimal for $2/\Box = .04$, what whole number is $\Box$?

16. Suppose the decimals for $\Box/50 = .56$ and for $\Delta/50 = .48$.

   If the decimal $(.56 - .48) = .08$ is the decimal equivalent for the fraction $\sigma/50$, what can you say about the relationship between $\Box$, $\Delta$, and $\sigma$.

17. If you know that the decimal for $\Box/\Delta = .89$, then what decimal can I multiply $\Delta$ by to get $\Box$?

18. How would you use the rule for changing a fraction to a decimal to find the decimal for a mixed number, like $2 \ 1/2$ in two different ways. Show all of your work in the space below and write the actual decimal equivalent on the line to the right.

19. Find the decimal equivalent for $2/.3$. Notice that the denominator is $0.3$ (or $3$ tenths)

20. Find the decimal equivalent for $8/0.2$

21. Find the decimal equivalent for $0.7/2$

22. Find the decimal equivalent for $1/2 /4$

23. Find the decimal equivalent for $1/3 /2$
24. Find the decimal equivalent for \(\frac{4}{9} / 2\)

25. Find the decimal equivalent for \(\frac{3}{5} / \frac{8}{10}\)

26. Find the decimal equivalent for \(\frac{4}{6} / \frac{5}{3}\)

27. Susie has suggested another rule for finding the decimal equivalent of a fraction \(\frac{a}{b}\). She multiplies each of the numerator and denominator by ten, and then she divided the denominator into the numerator. For example, to find the decimal for \(\frac{3}{5}\), she divided

\[
50 \div 30 = 50 \div 30.0 = 50 \div 300 \text{ tenths} = 6 \text{ tenths}
\]

\[
300 \text{ tenths}
\]

\[
0 \text{ tenths} 6 \text{ tenths} = .6
\]

She claims that her answer is always correct, and the same as the one the teacher gets by using the method taught in class. Will Susie always get the right answer? Why do you think so? Show how Susie would find the decimal for \(\frac{5}{8}\) and explain why her method seems to work.

28. Johnny has suggested another method for finding the decimal equivalent for a fraction when the numerator is larger than the denominator. He subtracts the denominator from the numerator and then divided this number by the denominator. For example, to find the decimal for \(\frac{6}{4}\) he writes:

\[
6 - 4 = 2, \text{ so I divide 2 by 4}
\]

\[
4 \div 2 = 4 \div 2.0 = 4 \div 20 \text{ tenths} = 5 \text{ tenths}
\]

\[
20 \text{ tenths}
\]

\[
0 \text{ tenths} 5 \text{ tenths} = .5
\]

He then adds one to the answer, to get \(.5 + 1 = 1.5\). He claims that his method always works for problems where the numerator is larger than the denominator. Is he correct? Show how Johnny would find the decimal for \(\frac{8}{5}\) and explain why his method seems to work.
29. Sam's rule for finding the decimal equivalent to a given fraction is very much like Susie's, but instead of multiplying numerator and denominator by 10, he multiplies each by 100 and then divides. For example, to find the decimal for $3/5$, he divides

$$
\begin{array}{c}
500 \div 300 = 500 \div 300.0 = 500 \div 3000 \text{ tenths} \\
3000 \text{ tenths} \\
0 \text{ tenths} \\
6 \text{ tenths} = 0.6
\end{array}
$$

He claims that his answer is always correct. Will Sam always get the correct answer? Why do you think so? Show how Sam would find the decimal for $5/8$ and explain why his method seems to work.

30. Judy's method for finding the decimal for a fraction is very much like Johnny's, only hers works for any fraction. She adds the denominator to the numerator and then divides this number by the numerator. For example, to find the decimal for $6/4$, she writes:

$$
6 + 4 = 10, \text{ so I divide } 10 \text{ by } 4
$$

$$
4 \div 10 = 4 \div 10.0 = 4 \div 100 \text{ tenths} \\
100 \text{ tenths} \\
0 \text{ tenths} \\
25 \text{ tenths}
$$

Then she subtracts one from this answer, to get $2.5 - 1 = 1.5$.

She claims that her method always works for problems of this sort. Show how Judy would find the decimal for $5/8$ and explain why her method seems to work.
Items of Finding the Square Root of a Fraction Pretest

1. 8 × 7 = □
2. 4 × 9 = □
3. 6 × 5 = □
4. 9 × 2 = □
5. 7 × 5 = □
6. 6 × 8 = □
7. 4 × 5 = □
8. 9 × 8 = □
9. 6 × 7 = □
10. 7 × 7 = □

11. Give a fraction name to the shaded portion below:

12. Give a fraction name to the shaded portion below:

13. Shade in 1/5 of the diagram on the right.

14. Draw a diagram with 3/4 shaded in on the line to the right.

15. Draw a diagram with 2/6 shaded in on the line to the right.

16. Write the following multiplication statement as a division statement:
    532 × 18 = 9576

17. Write the following division statement as a multiplication statement:
    492 ÷ 123 = 4
18. Write the following division statement as a multiplication statement:
\[ \square \div 6 = \triangle \]

19. Write the following multiplication statement as a division statement:
\[ \square \times 6 = \triangle \]

20. Write the following multiplication statement as a division statement:
\[ \square \times \triangle = \sigma \]

21. \( \square \times 1/9 = 1 \)

22. \( 8 \times \square / \triangle = 1 \)

23. \( 14/9 = 14 \times \square / \triangle \)

24. \( 3/7 = \square \times 1/7 \)

25. \( 38/6 = \square \times 1/6 \)

26. \( 74 \times 83 \times 77 = \square \times 74 \times 77 \)

27. \( 312 \times 25 \times \square \times 87 = 43 \times \triangle \times 25 \times 312 \)

28. \( 45 \times 203 \times 87 = \square \times 203 \times 87 \)

29. \( 1 \times 576 = \square \)

30. \( \square \times (374 + 596) = 374 + 596 \)

31. What is the length of a rectangle with a width of 2 feet and an area of 6 x 2 sq. ft.

32. What is the width of a rectangle with a length of 8 feet and an area of 8 x 3 sq. ft.

33. What is the area of a rectangle with width 3 feet and length 5 feet?

34. What operation would you use to find the area of a rectangle with length 89 feet and width 38 feet? (Would you add, subtract, multiply or divide?)
35. Using the idea that to find area, you find the number of one-unit squares in a figure, show how you would find the area of the figure below:

36. Divide

\[ 11 \overline{\div 425} \]

37. Divide

\[ 20 \overline{\div 5834} \]

38. Divide

\[ 7 \overline{\div 2859} \]

39. Divide

\[ 25 \overline{\div 4632} \]

40. Divide

\[ 6 \overline{\div 384} \]
# Items of Finding the Square Root of a Fraction Computation Test

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<thead>
<tr>
<th>Item</th>
<th>Expression</th>
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<tbody>
<tr>
<td>1.</td>
<td>( \sqrt{1089/4} )</td>
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<td>( \sqrt{2764/9} )</td>
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<td>( \sqrt{961/16} )</td>
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<td>4.</td>
<td>( \sqrt{1849/25} )</td>
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<td>( \sqrt{3025/16} )</td>
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<td>15.</td>
<td>( \sqrt{93/25} )</td>
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Items of Finding the Square Root of a Fraction Generalization Test

1. If you know that $\sqrt{169/225} = 13/15$, then you can also tell that $\sqrt{4 \times 169/225} = \square/\triangle$.

2. If you know that $\sqrt{256/25} = 16/5$, then you can also tell that $\sqrt{256/25 \times 9} = \square/\triangle$.

3. If you know that $\sqrt{9/16} = 3/4$, and $\sqrt{25/49} = 5/7$, you can also tell that $\sqrt{9 \times 25/16 \times 49} = \square/\triangle$.

4. For each part, find the square root required. Write the answer to (d) on the line to the right. The first three questions are only meant to help you with the answer to (d).
   (a) If you know that $\sqrt{9/16} = 3/4$, you can also tell that $\sqrt{16/9} = \square/\triangle$
   (b) If you know that $\sqrt{16/25} = 4/5$, you can also tell that $\sqrt{25/16} = \square/\triangle$
   (c) If you know that $\sqrt{81/100} = 9/10$, you can also tell that $\sqrt{100/81} = \square/\triangle$
   (d) If you know that $\sqrt{x/y} = 8/5$, you can also tell that $\sqrt{y/x} = \square/\triangle$

5. If you know that $\sqrt{8/9} \approx 2 \, 3/4 / 3$, you can also tell that $\sqrt{4 \times 8/9} \approx \square/\triangle$.

6. If you know that $\sqrt{25/15} \approx 3 \, 7/8 / 5$, you can also tell that $\sqrt{15/25 \times 100} \approx \square/\triangle$

7. If you know that $\sqrt{8/9} \approx 2 \, 3/4 / 3$, you can also tell that $\sqrt{9 \times 8/9} \approx \square/\triangle$.

8. If you know that $\sqrt{16/25} = 4/5$ and $\sqrt{49/64} = 7/8$, you can also tell that $\sqrt{16 \times 64/25 \times 49} = \square/\triangle$.
9. For each part, find the square root required. Write the answer to (d) only on the line to the right.

(a) \( \sqrt{1/9} = 1/3 \times \sqrt{9} \)

(b) \( \sqrt{1/25} = 1/5 \times \sqrt{25} \)

(c) \( \sqrt{1/100} = 1/10 \times \sqrt{100} \)

(d) \( \sqrt{1/1000} = 1/1000 \times \sqrt{1000} \)

10. \( \sqrt{\Box /81} = 5/9 \)

11. \( \sqrt{49/\Box} = 7/8 \)

12. \( \sqrt{\Box /4 \times 2/9} = 2/6 \)

13. How do a and b compare if \( \sqrt{a/b} \) is larger than 1?

14. If you know that \( \sqrt{8/9} \approx 2 3/4 /3 \) and \( \sqrt{11/25} \approx \Box /\Box /5 \) then you can also tell that \((2 3/4 /5 \times \Box /\Box /5) \times (2 3/4 /5 \times \Box /\Box /5) = \Box /\Box /5 \)

15. If you know that \( \sqrt{\Box \times 6/9} = 24/3 \), what is \( \Box \)

16. Find values for \( \Box \) and \( \Delta \) so that \( \sqrt{8 \times \Box /\Delta} = 12/5 \)

17. If you know that \( \sqrt{\Delta /49} = 161/7 \) and \( \sqrt{81/\Box} = 9/87 \), then you can also tell that \( 161/7 \times 9/87 \times 161/7 \times 9/87 = \Box /\Box /5 \). Do not find values for \( \Box \) and \( \Delta \). Express \( \Box /\Box /5 \) in terms of \( \Box \) and \( \Delta \).

18. Find a fraction \( \approx \sqrt{4/5} \)

19. Find a fraction \( \approx \sqrt{5/8} \)

20. Using \( \sqrt{18/1} \) as an example, show how you use the rule for finding the square roots of fractions to find square roots of wholes?

21 How would you use the rule for finding the square roots of fractions to find the square root of a mixed number, like 1 9/16. Show all of your steps and write the actual square root on the line to the right.
22. Find \( \sqrt{16/25} \) /49
23. Find \( \sqrt{4/25} \) /64
24. Find \( \sqrt{8/16} \) /2
25. Find \( \sqrt{81/25} \) /49
26. Find \( \sqrt{16/100} \) /25

27. Susie has suggested another method for finding square roots of fractions. She says that if she wants to find \( \sqrt{a/b} \), she finds \( \sqrt{4 \times a/b} \) and then multiplies the answer by 1/2. For example, to find \( \sqrt{16/25} \), she says that \( 4 \times 16 = 64 \); then, \( \sqrt{64/25} = 8/5 \) and \( 1/2 \times 8/5 = 8/10 \). She claims that her answer is always correct, and equivalent to the one the teacher gets by using the method taught in class. Will Susie always get an answer equivalent to the teacher's if she follows her directions correctly? Show how Susie would find \( \sqrt{36/100} \) using her method and explain why it seems to work.

28. Johnny says that another way to find \( \sqrt{a/b} \) is to find \( \sqrt{a \times b} \) and then find \( a/\sqrt{a \times b} \). For example, to find \( \sqrt{16/25} \), he says \( 16 \times 25 = 400 \), \( \sqrt{400} = 20 \). Therefore, \( 16/20 = \sqrt{16/25} \). He claims that he always gets an answer equivalent to the teacher's using the method taught in class. Is he correct? Show how Johnny would find \( \sqrt{36/100} \) and explain why his method seems to work.
29. Sam says that he has still another way to find $\sqrt{a/b}$. He says $\sqrt{a/b} = \sqrt{a \times b/b}$. For example, to find $\sqrt{16/25}$, he says $16 \times 25 = 400$, and then $\sqrt{16/25} = \sqrt{400/25} = 20/25$. He claims that his method is correct. Show how Sam would find $\sqrt{36/100}$ and explain why his method seems to work.

30. To find $\sqrt{a/b}$, Judy finds $\sqrt{a/4 \times b}$ and then multiplies her answer by 2. For example, to find $\sqrt{16/25}$, she says $25 \times 4 = 100$ and then $\sqrt{16/100} = 4/10$ and $2 \times 4/10 = 8/10$. So $\sqrt{16/25} = 8/10$. She claims that her method is correct. Show how Judy would find $\sqrt{36/100}$ and explain why her method seems to work.
### Table 19

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$\text{Mean} = \frac{\sum f_k x_k}{N} = \frac{2776}{141} = 19.7$