AN INVESTIGATION TO DETERMINE THE EFFECTIVENESS OF
PICTORIAL EXPOSITION VERSUS SYMBOLIC EXPOSITION
OF TENTH-GRADE INCIDENCE GEOMETRY

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Abstract

The purpose of this investigation was to evaluate the effect of two modes of exposition of tenth-grade incidence geometry on logically evaluated problem solving ability. To achieve this purpose two classes of tenth-grade geometry students were chosen to be the experimental and control groups. The two treatments, which were of nine class hours duration per group, and were both taught by the investigator, involved the use of a set theoretic symbolic-nonrepresentational mode for the experimental group, and a pictorial-representational mode for the control group. The content of the treatments was Euclidean incidence geometry. At the termination of the treatment a criterion test was administered to both groups. The criterion test was composed of two types of problems- Type NR problems, which were believed to be most successfully solved by a symbolic-nonpictorial analysis, and Type R problems, which were believed to be most successfully solved by a pictorial analysis.

Two hypotheses, of null form, were considered: that the mean scores of both groups on Type NR problems would be equal and that the mean scores of both groups on Type R problems would be equal. Both hypotheses were tested by means of an appropriate t-statistic at the .05 level of significance. Analysis of the data indicated that both null hypotheses were not to be rejected. A difference in means on Test NR of the control over experimental group was observed at the .20 level of significance.
The implication of the analysis of the data and the restrictions imposed by the limitations of the study is that the pictorial-representational exposition was as effective as the experimental symbolic-nonrepresentational exposition for Type NR problems and for Type R problems. Since the pictorial-representational mode of exposition is generally considered standard practice in the teaching of tenth grade geometry it should be continued for the present.
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Chapter 1

INTRODUCTION

Background

Many mathematics educators accept, as does Albert Blank (7:14), the hypothesis that geometry is perhaps the most fertile part of mathematics for the development of both inductive and deductive thinking. Therefore these educators should be concerned to examine critically what is being taught in school geometry courses and how it is being taught. And if mathematics educators believe that geometry is an intellectual game which, to be played, draws on and develops both the player's spatial faculties and reasoning powers, then they should be concerned to determine whether students in school are actually taught to play this game or indeed are capable of playing it in any meaningful way, that is to say, that they have the capability of solving a wide range of problems which require the understanding of geometry concepts.

Geometry is concerned with a direct interplay between the world of physical-spatial experience and the world of abstract concepts. Albert Blank has called Euclidean geometry "applied mathematics" (7:15). However, in accord with Henkin (7:50), the writer believes Blank's interpretation is extreme, although it may not be an exaggerated view when pertaining to tenth-year geometry courses. Blank further states,

For geometry there is a setting in which the student can
experiment and formulate conjectures. Once he has begun to make his own conjectures, he is well motivated to test them logically against known propositions and other conjectures. In geometry, at least, we have not yet completely hidden the mode of thought which makes his subject so exciting to a research mathematician. Geometry has the advantage of being sufficiently close to common experience that long specialized training is not needed to manipulate the concepts. (7:15).

There are two aspects of geometry, one spatial-intuitive and the other abstract logical-deductive. That geometry should partake of both aspects, particularly in the secondary school, is sanctioned by many researchers and by proponents of curricular reform. Bruner (4:6ff.) suggests that mathematics learning should include ikonic or image manipulation in problem solving situations, and both Biggs (2:6) and Dienes (8:21) call for the need to foster concrete spatial-intuitive development of conceptualization. Employment of a formal or informal, but not wholly rigorous, axiomatic approach to mathematics may well begin in geometry since the student is capable of empirically deriving a "rich" collection of "interesting" theorems, many of whose proofs are within the scope of his deductive ability (Buck, 7:23). The student can investigate the significant underlying structures of geometry at a closer range than he can the structure of algebra (Buck, 7:23). That is to say, the student can comprehend and apply basic concepts of geometry by means of symmetries, or transformations, or vectors, for example, more easily than comparable basic concepts of algebra which, as Buck claims, "... must wait upon the development of the ring of polynomials over a field." (7:23) In geometry, the axiomatic structure can be a means to explore and make conjectures by using the restrictions imposed by the axioms to suggest consequences of the interaction of the geometric elements
of a problem. This is a use of the axiomatic structure as a component of a heuristic technique (Blank, 7:17). The role of axiomatics in geometry is stated by Buck as, "It is a valuable experience to learn to reason where intuition is not a guide." (6:470)

Any axiomatic development at this time should be "naive" or descriptive since the value of such an approach is to elucidate the structure (meaning) of geometry and not to treat abstract axiomatics for its own sake (Buck, 7:20). That the axiomatic method is only a part of mathematics is implicit in two statements, one by Hadamard, "Logical proofs merely sanction the conquests of intuition," (7:58) and one by Morris Kline, "Rigor will not refine intuition which is not free." (7:59) Suppes claims a need for the axiomatic method in heuristic approaches to problem solving, a use of axiomatics which he finds woefully lacking in the secondary school (7:70), and further he stresses that axioms are important tools in the process of discovery. (7:71). On the other hand a spatial-intuitive approach may be used to clarify issues concerning axioms. For example, in Euclid's Elements, and many modern textbooks which derive from it, there are logical deficiencies in dealing with order of points on a line. The remedy, at the secondary school level, is not the a priori introduction of suitable axioms, but the candid admission that the question is to be handled by inspection of various figures, drawn on a blackboard, which illustrate the meaning of the axioms (7:46).

The proposed import of the above discussion for the present study is that for tenth-grade geometry students, a distinction exists, relevant to learning concepts of axiomatic incidence geometry, between a pictorial representation of the consequences of axioms (a spatial-intuitive aspect of geometry) and a symbolic representation by set
notation of the consequences of the same axioms (an abstract logical-
deductive aspect of geometry). If the above distinction is shown to
have a significant influence on the ability of students to solve
incidence problems, then the teaching of axiomatic geometry should be
re-evaluated. The above issue is of particular importance with regard
to many incidence problems whose pictorial representations would,
at certain stages of the solution, involve situations not existing
in real world spatial experience. An example of such a situation is
the "picture" of a line crossing three sides of a triangle without
passing through a vertex. This situation arises in the proof that
such a line does not exist.

The Problem

The question arises as to whether the full range of a student's
potential geometry problem-solving ability can be developed in a
standard tenth year geometry course, one in which the exposition
stresses pictorial representation of the content as opposed to a more
indirect symbolic exposition. This study will consider two varie-
ties of tenth-year Euclidean incidence problems. Both of the vari-
eties involve problems which require logical-deductive reasoning,
even for informal solutions. One variety involves solutions that,
for tenth-year students, are, in the opinion of the writer, not directly
accessible by means of "visual manipulation" whereas the other variety
involves solutions that are directly accessible by this means.
The varieties of problems will be called Type NR and Type R respec-
tively. By a "standard course" is meant a one or two semester course
at grade-ten level, taught by a qualified instructor, whose mathe-
matics training is, in general, limited to undergraduate courses, and a course content which utilizes essentially the content of Geometry by Moise and Downs (17), but which typically excludes or does not stress the concepts of indirect proof, the more difficult aspects of line and plane incidence, or axiomatic systems. The Moise and Downs Geometry was chosen as the textbook for both groups because, in the opinion of the writer, it more clearly demonstrates the axiomatic nature of geometry and the properties of incidence than any of the other commonly used texts and because it is widely used in British Columbia.

Definition of Terms

Although it is doubtful that one can always categorically place such geometry problems as are likely to be considered in tenth-grade mathematics into classes that are defined by the specific means that students employ to solve them, the writer believes that the majority of problems can be characterized as to membership in one of two varieties according to the involvement of visual manipulation by the student in his solution. By a student's solution is meant his final considered analysis and reasoning composed to answer the question posed. Typically absent from solutions are abandoned approaches and indications of initial methods utilized to solve the problem.

The first variety (Type R) of problems is characterized by having the usual solution of the student developed by means of what will be called the "visual mode". "Visual mode" means the employment of visual-analytic techniques, that is, those techniques by which the
student modifies a figure, a drawing which represents the geometric elements associated with the problem, in order to obtain a pictorial representation of a situation which will suggest an analysis of the problem. The above modifications of the figure, collectively called the visual manipulations, are generally accomplished by either actually drawing or mentally visualizing ("picturing") auxiliary lines, standard constructions, extensions to more inclusive figures, redrawn transformations, or related supplementary figures. The analyses of, and relationships within the representation of the problem suggested by any if the above types of modification, are tested and verified by the student using visual observation. That is to say, he determines whether or not the figure behaves as it should according to previously learned behavior of chalk or ink lines, wire or bookshelves, steel girders or the many common notions of geometry, true or false, to which most children are exposed before they reach grade-ten. The visual observation may suggest to the student theorems and postulates which seem applicable to the problem. It is for Type R problems that figures can readily be constructed which obey the restrictions and conform to the limitations and requirements imposed by the problem statement, and, in general, unambiguously point to an analysis.

Examples involving the visual mode are the following.

1. Given \( \triangle ABC \), with \( B = M = C \), \( BM = MC \), and \( m \angle BAM = m \angle CAM \), prove that \( \triangle ABC \) is isosceles.

For this problem, the technique employed by five of the eight
persons who correctly solved the problem (twelve persons were questioned), was, after drawing the figure stated in the question, to draw auxiliary lines. Specifically lines $\overrightarrow{AM}$ and rays $\overrightarrow{AB}$, $\overrightarrow{AC}$ and line $p$, perpendicular to $\overrightarrow{AM}$ through $M$, were constructed. Then the solvers explored the resulting triangle congruences to find more information.

The combined use of construction and extension to an inclusive figure is illustrated by the proof of the following.

2. In $\triangle ABC$, if $\overrightarrow{AD}$ bisects $\angle A$ and $B-D-C$ then show $BD/CD = BA/CA$.

A common technique observed by the writer was to extend $\triangle ABC$ so that it was included in $\triangle BBC$ with $\overrightarrow{CE}$ parallel to $\overrightarrow{AD}$ and $B-A-E$. Type R problems are characterized as being those for which solutions are initiated and continued to completion by visual mode inspection and manipulation of the figure. Other examples of Type R problems and their solutions are to be found in Appendix II.

Type R problems are generally taught to grade-ten students by tactics which themselves employ techniques of the visual mode. This is a satisfactory approach for teaching the content related to these problems since they are generally analyzed and solved by students using visual mode techniques. However, the same tactics of teaching, utilizing techniques of the visual mode, which will be termed pictorial-representational mode, are frequently used in the teaching of geometry where the problems, for a student, may not be readily accessible to analysis using visual-analytic techniques. Such is often the case in the teaching of Type NR problems. The role of visual techniques in the pictorial-representational mode is elucidated by Moise who states,
It is customary, in elementary texts, for the reader to be assured that 'the proofs do not depend on the figure', but these promises are almost never kept (Whether such promises ought to be kept, in an elementary course, is another question, and the answer should probably be 'no'). (16:66)

The second variety of problems (Type NR) is characterized by having associated with the solution of the problem a figure which cannot be drawn with standard representations of segments, lines, rays, planes, etc. Such figures generally arise from the premise-and-not-conclusion statement of an indirect proof. And such figures as can be drawn do not readily provide information or relationships useful for an analysis of the problem when visual manipulations of the sort used for Type R are employed.

The student cannot readily make use of relationships learned from the physical environment to solve Type NR problems. For these problems, the writer contends, it is not true, as the statement from a teacher's manual claims, that

In geometry, the intuitive spatial-visual background is a part of common experience: we see, every day, objects which look like segments, rectangles, and circles; and mass production of consumer goods has made congruence a very familiar idea ... Formal geometry builds on these perceptions and extends them. (Moise and Downs, 18:13)

In the context of the present study, problem solving ability shall mean the ability to obtain a score, a number from zero to ten, when the solution attempts are logically evaluated by the techniques outlined in Appendix V. By logical evaluation of a problem solution is meant the analysis and scoring of the pattern of logical inference which composes the solution. That is to say, the syntax composed of phrases and diagrams which are stated, together with connectives,
to form implications or chains of implications is sought for in the solution protocol. These implications are then evaluated as to relevance of premise statement and validity of argument or parts of argument by the criteria outlined in Appendix V.

It is conjectured by the writer that use of techniques of the visual mode by the student, while appropriate for Type R problems, are inappropriate for the solution of Type NR problems and thus the pictorial-representational mode should not be used to teach Type NR problem solving.

The above conjecture was derived from observation by the writer of grade-ten students, the majority of whom devised pictorial models for Type NR problems. These models were such that the real world physical consequences of the visually perceived structure of the model did not logically lead the student in his reasoning to a valid solution of the problem. That is to say, for example, a student may have produced a pictorial model of a "line" crossing three sides of a triangle, avoiding the vertices, to prove the impossibility of the situation; but then the student deduced conclusions (correct or not) that were not valid consequences of the pictorial model used.

In the opinion of the writer, this inability to form mental "pictures" occurs because the subject's experience of perceiving his spatial surroundings supplies him with insufficient visual referents, from whose interaction he can draw logical consequences. For example, for most tenth-grade students, there is no visual referent on which to base the concept of several parallels to a given line through a given
point not on the line. On the other hand, there are many visual referents for the geometric concept of three planes whose intersection in space may produce 0, 1, 2, or 3 lines or a single point. Most tenth-grade geometry problems that fall into the category of Type NR are incidence theorems, many of which either because a direct proof is more difficult or is lacking require an indirect proof. The figures associated with indirect proof incidence problems are not amenable to easy visual manipulation because the figures must generally represent a state of affairs which is not only a geometric contradiction but is also contrary to ordinary physical possibility.

As a specific example of this lack of facility, consider the following problems.

1. Given convex quadrilateral ABCD with \( \overline{AC} \) bisecting \( \overline{BD} \), prove area \( \triangle ABC = \triangle ADC \).

2. Given that ray \( \overrightarrow{AB} \) intersects ray \( \overrightarrow{PQ} \) in segment \( \overline{AP} \), which of the following may be true: (a) Q-A-P-B, (b) A-B-Q-P, (c) A-Q-B-P, (d) A-P-Q-B, (e) P-Q-A-B?

3. Given that lines \( m \) and \( n \) are distinct, then prove that they cannot intersect in two points.

It was observed by the writer during a pilot study that approximately 60 percent of the students of two tenth-year classes which participated could, at the close of the 1969-1970 year's work, solve problems 1 and 2, whereas only two out of 50 could partially prove or informally justify a solution to problem 3, even though a discussion of the appropriate axioms had taken place.
The set of techniques to which Type NR problems are most susceptible will be called the nonvisual mode. Usually, the techniques of this mode employ various set theoretical interpretations of the problem and employ the logical rules of set theory and associated models, e.g. Venn and Euler diagrams and truth tables, to analyze the problem and to devise a proof. Examples of Type NR problems and their solutions are to be found in Appendix II.

Presentation of geometric content in a manner which utilizes techniques of the nonvisual mode, i.e. interpretation of geometric entities and relations in terms of set notation and set operations, will be called the symbolic-nonrepresentational mode of exposition.

Conjectures Relating to the Problem

After surveying appropriate literature and interpreting the results and after observing, while teaching, the actions of two geometry classes, it is the contention of the writer that:

I. The present program of high school geometry:

1. stresses the development of visual-analytic approaches (visual mode) to the exclusion of other techniques,

2. treats the concepts of logic and axiomatics as unrelated topics which are present in the curriculum at this point only because the ideas of geometry easily illustrate them,

3. gives little stress to heuristics, strategies, and techniques of problem solving and proof construction beyond implementing the obvious consequences of diagrams and pictures.

II. From past experience, the student possesses a repertoire of
heuristics and strategies that are almost exclusively of the visual-analytic form, and this:

1. impedes him in the solution of Type NR problems by producing a rigidity of thinking, or Einstellung, in his solution strategy, and

2. inhibits the acquisition of nonvisual techniques of problem solving.

Hypotheses (null form)

It is hypothesized that if one group of students at the grade-ten level in secondary school is exposed to a treatment (the experimental treatment) during which incidence geometry content is taught via a symbolic-nonrepresentational exposition, and another group of grade-ten students is exposed to a treatment (control group treatment) during which the same content is taught via a pictorial-representational exposition, then:

$H_1$: Students of the experimental group treatment will obtain scores on a criterion test which do not differ significantly from scores of control group treatment students on the same test when the content of the test is Type NR geometry problems.

$H_2$: Students of the experimental group treatment will obtain scores on a criterion test which do not differ significantly from scores of control group treatment students on the same test when the content of the test is Type R geometry problems.

Statistical Form of Hypotheses

$H_1$: The mean scores for the experimental group and control group will not differ significantly ($\alpha = .05$) on a test of Type NR problems.

$H_2$: The mean scores for the experimental group and the control group will not differ significantly ($\alpha = .05$) on a test of Type R problems.
Chapter 2

REVIEW OF RELATED RESEARCH AND LITERATURE

There is good reason to believe that the visual mode is a natural consequent of a child's development. For Piaget, this development is a psychological genesis of perceptions of the spatial-visual environment from topological (in which only gross characteristics are preserved by transformation) to projective (in which form, shape, outline, and other finer qualitative properties are preserved) and finally to Euclidean (in which quantitative and metric properties are preserved). Each stage involves experiential clarification of the restrictions and the conservation properties of the previous category (Furth, 10:23 ff.). This is to say that the young child learns progressively which qualities of his physical environment are preserved and which qualities are altered by his manipulation of the environment. The implication for the present study is that the grade ten students will confront a geometry problem as though it were a physical problem, analyzing it in terms of its physical counterpart. As Lovell claims, for the child, geometry is essentially a system of internalized physical operations, since mathematical concepts are derived from manipulation of real world objects (15:216). It is not the use of the visual mode of analyzing questions which the writer believes inhibits problem solving, for Polya and others state that it is a valid and valuable heuristic for the solution of geometry problems (19:59), but the sole reliance on visual intuition and on the
limited realm of one's experience of visual causality.

The relationship of visual experience to problem solving is expressed by VanDeGeer who states that if a subject has worked with Euclidean geometry, a visually oriented study, the content of the topic becomes his field, his reservoir of information, his repertoire of associated techniques, but if he attempts a problem in non-Euclidean geometry, for example, he is out of his field. "The familiar field brings a high transparency in the situations which belong to the field. But its visual self-evidence may become a fixation hampering the solution of problems in some cases." (21:143) Bruner and Kenney believe that young children are strongly guided by the perceptual nature of tasks and that they attempt them by analyzing one visual feature of them at a time. Although older children have greater problem solving ability than do younger ones, they are equally oriented toward use of visual aspects of problems, but they analyze several visual facets simultaneously (5:163). The implication for the present study is that the form of representation of a problem will be an important factor which determines successful solution by students trained in the "usual techniques" or by common experience. This conjecture is exemplified in a study by J. Sherrill in which it was found that in solving problems which could be solved with or without reference to a figure, a group of students performed significantly better when an accurate diagram was presented to accompany the problem than did a comparable group for whom no diagram was given with the problem. Sherrill found, too, that the latter group performed significantly better than a group which was given the problem with a
misleading, distorted, or ambiguous diagram. This study was based on earlier conjectures by Trimble and Brownell which are referred to in Sherrill's work (20:31 ff.).

Becker and MacLeod found that when abstract models were used, if the instructor emphasized critical reasoning rather than algorithmic techniques, transfer of problem solving ability to related tasks increased (1:103). Based on consideration of the above study it is inferred that the nonrepresentational mode treatment of the experimental group on Type NR problems will result in increased ability to solve Type R problems. Also of interest is a study by Rugg in which a descriptive geometry program induced increased ability in different aspects of geometry (12:15). Sandiford in his article, "Reciprocal Improvement in Learning", says, when quoting Judd,

"It is not far from the truth to assert that any subject taught with a view to training pupils in methods of generalization is highly useful as a source of mental training, and that any subject which emphasizes particular items of knowledge and does not stimulate generalization is educationally barren. (12:21)

Thus, it may be expected that the experimental group will score higher than the control group on both types of problems by virtue of the former's exposure to an abstract, logical, and generalizable mode of exposition.

Of concern at this point is a study by Hall, in which he examined the effects of the training of logical proof on the geometric ability of grade ten students (13:22). His experimental group was taught a program of logic followed by a program on geometric similarity. Both groups were then given an achievement test in geometry. He found no significant difference between the groups and concluded
that "such teaching does not seem to increase the ability to reason deductively in geometry as measured by performance on tests of geometric proof." (13:36) However, the unit in logic and deduction that Hall taught is alien to the context of geometry, in that very few of his examples are in a geometric framework and the unit appears to fall into what Shanks calls a "ritualized and memorized exercise with no understanding of meaning." (7:63) The present study will try to develop deductive ability in both treatment groups by incorporating the logical structure intrinsic to the representational mode and non-representational mode expositions.

It is not contended that geometric problem solving ability as construed for the present study is solely dependent on the mode of exposition; the role of a visual factor is still an open question (Werdelin, 23:38). However, the writer attempted to ensure that the present study was restricted to one class of problems so that the influence of mode of exposition would become more evident. The evidence concerning Einstellung is also ambiguous. Hudgins suggests that drill-oriented teaching induces Einstellung and the presentation of problems with multiple approaches of solutions lessens it (14:36–40). However in a study very similar to the classic one of Luchins, order of presentation of problem types had no significant effect on problem solving (9:138). The consequence for the present study was that drill was avoided in the treatment for both groups, and, except for question I which was chosen for its simplicity, the questions on the criterion test were in random order.

That visual representation of geometric concepts is relevant
to problem solving is stated by Brian as, "... much of ... problem solving behavior is associated with geometric and visual representation of problems," and chosen in his study of problem solving behavior was "a standard measure of spatial relations ability as the criterion instrument." (3:1202-A)
Chapter 3

DESIGN OF THE STUDY

Method

In Chapter 1 it was proposed that an experiment be conducted to compare the effectiveness of the representational mode of exposition with the nonrepresentational mode of exposition. To this purpose, two intact classes of 28 and 31 tenth-year geometry students in a public secondary school were chosen as the control and experimental groups respectively. The instruction for the experimental group (group E) consisted of nine class hours of symbolic-nonrepresentational mode exposition of incidence geometry concepts and the instruction for the control group (group C) consisted of nine class hours of the same geometry content presented via the pictorial-representational mode exposition. The instruction for both groups was initiated at the beginning of the semester for geometry. For a detailed outline of the treatment material for both modes of exposition, see Appendix III.

At the conclusion of the treatment period (approximately two weeks) a criterion test was administered to both groups. The intent of the test was to measure the ability of the students to solve problems whose content was based on material presented during the treatment period. The test was composed of four Type R and five Type NR problems, as indicated in Appendices I and II. In order to minimize subjective bias in the categorization of problems as to type, a panel of four doctoral candidates of the Department of Mathematics Education
at the University of British Columbia was chosen to categorize sample items. Based on criteria supplied by the writer, the panel unanimously categorized the problems selected for the test. Except for question I, which was chosen for its simplicity, all other questions were placed on the test in random order.

The reason for utilizing geometric problems involving incidence properties rather than metric properties was to avoid, at least in the initial stages of solution, involvement of algebraic and computational abilities. If the problems were accessible wholly from a metric approach then it could not be determined to what extent the student utilized spatial-visual mechanisms; neither could it be determined whether difficulties arose in translating elements of the geometric problems into algebraic form in the initial stages of the problem solution. In order to avoid the above difficulties the problems used in test and treatment were so restricted that they involved algebra of, at most, grade-nine level. Furthermore, problems considered were almost exclusively those for which the involvement of algebra, if present, occurred after the initial analysis had begun. The pool of problem items and the presentation form on the criterion test of the problems selected were determined by an analysis of two pilot studies on different groups of grade-ten students, both conducted by the experimenter. The results of the pilot studies, which consisted of two to four hours of treatment followed by a test (see Appendix VII for tests), were interpreted for determination of a feasible level of difficulty for questions and treatment material. Completed questions were analyzed to familiarize the writer with the style, vocabulary,
and syntax common to grade-ten student test responses.

The problem solutions from the criterion test were evaluated on a ten point scale per item. Two aspects of the solution of a problem were considered, the use of legitimate information, that is relevant rules (axioms), definitions, representations (pictures) or notation (set language), and, whether relevant or not, the use of logical argument on the information. Since all relevant information, rules, definitions and examples, was included in the test, and since the students had access to their notes, the presence of legitimate information in a response was considered less important than the logical argument employed. Credit was given to logical use of incorrect information to a maximum of four points, and the presence of correct information with no logical argument was given a maximum of two points. For examples of scoring technique, see Appendix V.

Subjects

Two classes of grade-ten mathematics students were selected from a public secondary school (in a rural, middle to low socio-economic community in the Greater Vancouver area). Because of absence, the registered class sizes of 33 and 30 varied by two or three each day. In neither group was any student absent for more than 2 days, with the exception of one individual who did attend class but did not write the criterion test. The class sizes for the criterion test were: group E, 31 and group C, 28. The students of both groups were heterogeneous with regard to mathematics background; see Appendix VIII for grades for the previous two years.
Procedure

Both groups were taught by the experimenter for a total of nine class hours of 60 minutes each, although the effective time per class was approximately 50 minutes. The first five class periods consisted of exposition, the sixth period consisted of a training test which was intended to familiarize the students with the form of the questions which they would receive on the criterion test, the seventh to ninth periods consisted of exposition and the tenth period contained the criterion test.

The experiment was begun one week after the start of the semester, and the instructor normally assigned to both classes reported that no geometry had been discussed within that semester. It may be assumed, therefore, that both groups had equal knowledge of geometry—specifically only the content of informal and common experience or the informal and peripheral content of previous mathematics courses.

The method of instruction for the control group was implementation of standard practice in which visual representation of problems were used by the instructor to explain the analysis of a problem and to justify the use of the statements in a proof for the problem. The actual drawings and discussion employed were derived from two sources, (i) observation of Type NR and R material comparable to the material included in the present study, when taught by instructors of grade ten, and (ii) from a reading of standard texts and the teachers' manuals which accompany them. For illustration of the different exposition techniques applied to specific problems, consult Appendix IV.
Statistical Procedures

The data for an individual consisted of two scores, one the sum of points credited for Type NR problems and one the sum of points credited for Type R. The above scores were denoted Test NR score and Test R score respectively. The mean Test NR score from group E was compared for significant difference from equality with the mean Test NR score from group C with a t-statistic. The mean Test R score from group E was compared for significant difference from equality with the mean Test R score from group C by use of a t-statistic. Both comparisons were made at the .05 level of significance. For each group of subjects, the Pearson product-moment correlation of IQ and test score was computed on both tests. This yielded four correlations. The significance of the difference of each of the correlations from zero was tested at the .10 level. The level of .10 was chosen since the concern of the writer was whether any positive correlation existed; thus an $\alpha$ as "coarse" as .10 seemed appropriate as it is the maximum suggested by Glass and Stanley in Statistical Methods in Education and Psychology (11:282). Comparisons of correlations using the Fisher $z$-transformation were tested at the .10 level.
Chapter 4

RESULTS OF THE STUDY

Introduction

The results of the statistical analysis of the data are presented to support acceptance or rejection of the two hypotheses and to aid in determination of implications of the study.

Tests of the Hypotheses

Hypothesis I:

The mean scores for group C and group E will not differ significantly ($\alpha = .05$) on Test NR.

Table 1

Statistics Concerning Hypothesis I

<table>
<thead>
<tr>
<th></th>
<th>N</th>
<th>Mean</th>
<th>S.D.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Group C</td>
<td>28</td>
<td>13.429</td>
<td>9.697</td>
</tr>
<tr>
<td>Group E</td>
<td>31</td>
<td>10.548</td>
<td>6.073</td>
</tr>
</tbody>
</table>
Table 2
Results of Statistics Concerning Hypothesis I

<table>
<thead>
<tr>
<th></th>
<th>Results for t-statistic</th>
<th>Results for t'-statistic</th>
</tr>
</thead>
<tbody>
<tr>
<td>s pooled</td>
<td>7.981</td>
<td>7.981</td>
</tr>
<tr>
<td>d.f.</td>
<td>57</td>
<td>43</td>
</tr>
<tr>
<td>t or t' ( \alpha/2 ), = .05</td>
<td>2.000(approx)</td>
<td>2.020</td>
</tr>
<tr>
<td>t or t' observed</td>
<td>1.384*</td>
<td>1.354*</td>
</tr>
<tr>
<td>level of significance of observed t or t'</td>
<td>.20</td>
<td>.20</td>
</tr>
</tbody>
</table>

* not significant at \( \alpha = .05 \)

Table 2 indicates that the observed value of t is 1.384. This value of t is such that \(-2.000 \leq t_{\text{obs}} \leq 2.000\) for \( t \alpha/2 = 2.000 \) at \( \alpha = .05 \). Therefore hypothesis I is accepted and it can be concluded that there is no significant difference between the mean scores of group C and group E on Test NR.

It should be noted that the standard deviation of scores for the two groups on Test NR was different, 6.0 versus 9.7. An F-test confirmed the significance of the difference at the .05 level. Therefore the data was subjected to the t'-test in accord with the recommendation of Introduction to Statistics, by Walpole (22:230). The result of the t'-test was that \( t' \) observed is 1.354. This value of \( t' \) is such that \(-2.020 \leq t'_{\text{obs}} \leq 2.020\) for \( t' \alpha/2 = 2.020 \) at \( \alpha = .05 \). Therefore hypothesis I is still accepted.
Hypothesis II:
The mean scores for group C and group E will not differ significantly (\( \alpha = .05 \)) on Test R.

Table 3
Statistics Concerning Hypothesis II

<table>
<thead>
<tr>
<th></th>
<th>N</th>
<th>Mean</th>
<th>S.D.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Group C</td>
<td>28</td>
<td>18.960</td>
<td>7.804</td>
</tr>
<tr>
<td>Group E</td>
<td>31</td>
<td>20.840</td>
<td>8.280</td>
</tr>
</tbody>
</table>

Table 4
Results of Statistics Concerning Hypothesis II

<table>
<thead>
<tr>
<th>Results for t-statistic</th>
</tr>
</thead>
<tbody>
<tr>
<td>s pooled</td>
</tr>
<tr>
<td>d.f.</td>
</tr>
<tr>
<td>t ( \alpha/2, = .05 )</td>
</tr>
<tr>
<td>t observed</td>
</tr>
<tr>
<td>level of significance of observed t</td>
</tr>
</tbody>
</table>

* not significant at \( \alpha = .05 \)

The observed value of t in Table 4 is 0.892. This value is such that \(-2.000 < t_{\text{obs}} < 2.000\) for \( t \alpha/2 = 2.000 \) at \( \alpha = .05 \).
Therefore hypothesis II is accepted. Thus it is concluded that there is no significant difference between the mean scores of group C and group E on Test R.

Correlations between test score and IQ were computed to determine if there existed a positive relationship. For each group of students, the Pearson product-moment correlation was computed between both Test R and IQ and Test NR and IQ. The IQ scores for each individual were the most recent entered on the individual's record, and consisted, in the main, of Otis (Form C) and Otis Higher (Form C) tests. The results of the correlational study appear in Table 5.

Table 5

Correlations Between Individual Test Scores and IQ

<table>
<thead>
<tr>
<th>group</th>
<th>correlation of:</th>
<th>correlation, r obs.</th>
<th>critical value of r for $\alpha=.10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>group E</td>
<td>A: Test R and IQ</td>
<td>.49*</td>
<td>.301</td>
</tr>
<tr>
<td>(mean IQ 110.2)</td>
<td>B: Test NR and IQ</td>
<td>.35*</td>
<td>.301</td>
</tr>
<tr>
<td>group C</td>
<td>C: Test R and IQ</td>
<td>.097</td>
<td>.317</td>
</tr>
<tr>
<td>(mean IQ 107.5)</td>
<td>D: Test NR and IQ</td>
<td>.36*</td>
<td>.317</td>
</tr>
</tbody>
</table>

* significant at $\alpha=.10$

Each of the observed correlation values, r, was tested for significant difference from the value of zero at the $\alpha=.10$ level,
using the t-statistic: \( t_{\text{obs}} = r / ((1 - r^2) / (n - 2))^{1/2} \) with d.f. = \( n - 2 \).

As is indicated in Table 5, correlations A, B, and D are significantly positive at \( \alpha = .10 \), and correlation C is not significantly different from zero.

It was noted that correlation C appeared considerably less than correlation A, the corresponding correlation between individual IQ of a group and Test R score. Therefore the hypothesis that correlation C did not differ from correlation A against the alternative hypothesis that correlation A exceeded correlation C was tested utilizing the Fischer z-transformation at the .10 level. It was found that the former hypothesis would be rejected at a significance level of \( \alpha = .076 \). Therefore it is concluded that correlation A was greater than correlation C.
Chapter 5

SUMMARY

Summary of study

The purpose of this study was to investigate the effects of the pictorial-representational mode of exposition versus the symbolic-nonrepresentational mode of exposition of grade-ten incidence geometry upon student ability in solving two varieties of geometry problems. To accomplish this purpose two grade-ten classes were selected, one as experimental group (E) and one as control group (C). At the termination of the treatment period of nine class hours which consisted of identical content presented via a symbolic-nonrepresentational mode to group E and a pictorial-representational mode to group C, a criterion test was presented to both groups. The test was composed of two varieties of incidence geometry problems. They were chosen from a pool of problems similar to and including those attempted or solved by students in pilot studies. Problems from the pool were submitted to a panel who classified them as to type according to criteria supplied by the writer (see Appendix II).

In the opinion of the writer, an opinion based on analysis of problem solutions by students in pilot studies, the first variety of problem, classified as Type R, would be more readily solvable by visual-analytic techniques than the other problems in the pool. The second variety of problem, classified as being Type NR, were chara-
terized as problems for which a visual-analytic technique of solution appeared inappropriate. For the latter type of problem, it was believed that symbolic-nonrepresentational techniques of solution would prove more successful than techniques of the visual-analytic approach. Data consisting of two scores per subject were obtained (a Test NR score and a Test R score, derived from Type NR and Type R problems respectively). The means on Test NR from group C and group E were compared by an appropriate t-statistic as were the means on Test R from groups C and group E. The above statistics were tested at the .05 level of significance.

Conclusions and Implications

Results of the statistical analysis confirmed both null hypotheses: that the mean on Test R for group C was not significantly different from the mean on Test R for group E and that the mean on Test NR for group C was not significantly different from the mean on Test NR for group E at the .05 level.

Correlation of subjects' IQ and test scores was computed for each group on Tests NR and R to ascertain whether IQ was a highly related factor in determining test score. As Table 5 (see page 26) indicates, IQ was significantly positively correlated with test score at $\alpha = .10$ except for the case of IQ for group C and Test R scores where no significant relationship was found.

It is inferred from the results of the present study that a pictorial-representational mode of exposition, as is generally standard practice in grade-ten, and a symbolic-nonrepresentational mode produced virtually equivalent problem solving ability as measured by Test R test items. The mean of group C did not significantly exceed
that of group E on Test R, although the mode of exposition used in
the treatment of group C employed visual mode techniques similar to
those which both groups used to solve Type R problems. It is possible
that group E learned and firmly established visual mode techniques
prior to their participation in this study, and that the treatment for
group C did not increase their ability to solve Type R problems beyond
that of group E.

It is probable that a similarity between the treatment material
presented to group C and Type R problems (see Appendix III), together with
past familiarity with visual geometric concepts overshadowed any
positive IQ-Test R score correlation since the correlation observed
was not significantly different from zero at $\alpha = .10$. It is probable
that for group E, the dissimilarity between the style of presentation
and Type R problems made IQ a more potent factor for Test R as indi­
cated by the significant correlation (at $\alpha = .10$) (see Table 5).

Even though both null hypotheses were accepted it should be
noted to what extent they fell short of being rejected. The signifi­
cance levels for which Hypotheses I and II would have been rejected
were .2 and .4, respectively. It was subjectively judged by the
experimenter that the structure of logical inference in attempted
solutions of Type NR problems was essentially the same for both groups.
By structure of logical inference is meant the pattern by which phrases
or diagrams are worded to state implications, or chains of implications,
or various combinations of statements with logical connectives.
Representational diagrams were used by both groups, although more fre­
quently by group C, and the notation accompanying them reflected that
used in the different expositions respectively. Thus it appears that the content was learned in the context presented and was utilized in that manner.

It must be emphasized here that Type NR problems were scored primarily for consistent use of a valid logical argument, not for the formal completeness of a symbolized statement. Thus an informal pictorial illustration of elimination-of-cases was given a higher score than a correct set theoretic statement which was not utilized logically. It is also to be noted that although group E received the geometric content via a set theoretic exposition, the use of "symbolic logic" was avoided; thus both groups had at their disposal the same mechanisms for logical manipulation of information. Since the content and logical form of material in the treatments was essentially the same, it appears that the relevant difference between the groups was the form, symbolic or pictorial, by which the geometric information of a problem was interpreted by the student. This implies that a visual-pictorial interpretation is equally susceptible to the logical techniques of a grade-ten student as is a symbolic-nonrepresentational interpretation. It was observed that group E students learned the geometric concepts and terminology taught, and that they did so by a symbolic-nonrepresentational interpretation as demonstrated by their ability to produce many correct or partially correct proofs involving content novel to them, but utilizing symbolism peculiar to group E exposition. Therefore the symbolic-nonrepresentational interpretation of Type NR items by group E subjects appears to have made these items as capable of solution as was measured by this study as the pictorial interpretation.
of group C.

In view of the results of the correlations of test score and IQ and the manner by which items were scored, it is probable that for either group there was a strong reliance on general intelligence (in the manner that IQ indicates this) for Type NR problems. The role of general intelligence and IQ in the scores of Type R problems is unclear. Type R items did not require inference as elaborate or abstract as for Type NR items and the significance of the difference in IQ-test score correlation of group E over group C, significant at a .10 level, presents no obvious relationship.

These conclusions must be viewed in light of the limitations of the study. Apart from the possession of a cursory set of common notions, both groups were naive with respect to incidence geometry and with respect to logically precise inference via implication and indirect argument. Both groups, although of heterogeneous academic background, had a knowledge of set notation and set operations. Therefore its use in the treatment of group E did not represent an unbalanced introduction of new concepts of notation. It is possible that the duration of treatment, nine class hours per group, although sufficient to convey the concepts and terminology of the geometry, was insufficient to convey the concept of logical proof - the vehicle by which much of the material was presented, and, as a result, may have masked any long term inferior or superior effectiveness of the symbolic mode of exposition.

Recommendations for Instructions

As a consequence of this study it is recommended that the
standard practice of teaching incidence geometry concepts to grade-ten students by the pictorial-representational exposition should be continued. Furthermore the inferential structure of the geometry content should also be developed in this manner. Until further research clarifies its role in problem solving the symbolic-nonrepresentational mode, as employed in the present study, should be minimally stressed. This should apply even to those situations arising from indirect argument where a pictorial representation does not provide, from its own structure, an adequate basis for deducing correct logical consequences. It is suggested that a symbolic-nonrepresentational mode of exposition involving geometric concepts be deferred until more sophistication concerning logical inference and proof is attained. Perhaps the symbolic-nonrepresentational mode should be introduced in a context which does not admit the strongly entrenched visual approaches—in courses such as abstract algebra or number theory.

Recommendations for Further Research

It is recommended that further research be conducted to clarify issues arising from this study. A full year or half year comparison of the treatments of this study might resolve the issue of whether subjects are more capable of utilizing geometric information symbolically interpreted after becoming acquainted with and proficient in the techniques of logical proof as are visually trained subjects. It is suggested that comparisons of the treatments be conducted with groups of a more homogeneous composition than those of the present study. This might clarify the observed difference in correlation of IQ with Test R scores for groups E and C, and might isolate other factors which influence test scores. It is also recommended that a systematic and quantitative scheme be constructed in order to classify and rank
the logical inference structure of problem responses with more precision than was possible in the present study.
BIBLIOGRAPHY


APPENDIX I

CRITERION TEST
QUIZ (2)

Name: __________________

Justify all answers with diagrams or set notation.

I. If ray $\overrightarrow{AB}$ intersect ray $\overrightarrow{PQ}$ is $\overrightarrow{AP}$, state the betweenness relationship of A, B, P, and Q.

II. Prove that an angle, $\angle ABC$, together with its interior is a convex set.

III. Prove that 2 distinct lines which cross are contained in exactly one plane.

Note: You can use all of the rules except the note in Definition II.

Hint: Prove this directly.

IV. If A, B, C, and D are 4 distinct points in a plane, what is the maximum number of lines in the plane I can use if each line is to contain exactly one pair of points? How are the points arranged?

V. Note: Three points in space which are not collinear determine exactly one plane, the unique plane which contains them.

1. Prove that if the least number of planes determined by the distinct points A, B, C, and D is 4, then A, B, and C cannot be collinear.

2. Can the least number of planes determined by A, B, C, and D be 3? Yes ( ), No ( ).

3. If the least number of planes determined by A, B, C, and D is
one, must any 3 of A, B, C, and D be collinear?

Yes ( ), No ( ).

VI. Assume that k, m, and n are 3 distinct lines in plane E such that the lines cross in 2 points. Show that the following statement is false: "No pair of the lines is parallel."

VII. A, B, C, and D are 4 distinct points on line m, with C-A-D and B-A-C.

1. Where must B be if \( \overrightarrow{AD} \) intersect \( \overrightarrow{BD} \) is \( \overrightarrow{BD} \)?

2. Where must B be if \( \overrightarrow{AD} \) intersect \( \overrightarrow{BD} \) is \( \overrightarrow{AB} \)?

3. State the betweenness relationship of A, B, C, and D if \( \overrightarrow{BA} \) intersect \( \overrightarrow{BD} \) is B.

VIII. If a, b, and c are 3 distinct lines in plane E with a parallel to c and b parallel to c, then prove a is parallel to b.

IX. Given triangle \( \triangle ABC \) and line m, both in the same plane such that m does not contain A, B, or C (a vertex of the triangle), prove that m cannot cross all 3 sides of the triangle.
RULES:

Rule I: If P and Q are any 2 points, then there is EXACTLY one line m which contains them.

Rule II:
1. Any plane contains at least 3 points which are not on a line.
2. SPACE has at least 4 points not on a plane.

Rule III: If P and Q are any 2 points in plane E, then the line m which contains P and Q lies completely in plane E.

Rule IV:
1. If P, Q, R are any 3 points (in SPACE) then there is at least one plane which contains P,Q,R.
2. If P, Q, R are any 3 points (in SPACE) which do not lie on some line, then there is EXACTLY one plane E which contains P,Q,R.

Rule V: If P is ANY point not on line m then there is EXACTLY one line t which contains P and is parallel to m.

DEFINITIONS:

I. Points are collinear if there is at least one line which contains all of them.

Note: 2 points are ALWAYS collinear.

3 points MAY be collinear.

II. Points are coplanar if there is at least one plane which contains them.
Note: 2 points are **ALWAYS** coplanar.
3 points are **ALWAYS** coplanar.
4 points **MAY** be coplanar.
A line and a point not on it are always coplanar.
2 lines which cross are always coplanar.

---

III. 2 lines m, n are parallel, m \parallel n, if
1. m and n are different lines,
2. m does not intersect (cross) n,
3. m and n lie in the same plane.

---

IV. A set of points is **CONVEX** if, for **ANY** choice of 2 points P, Q in the set, the segment PQ lies completely in the set.

Note: Any line is convex.
Any ray is convex.
Any segment is convex.
A circle is **not** convex.
A circle with its 'interior' is convex.

---

V. Note: Lines have an infinite number of points.

---

VI. Note: If line \overrightarrow{PQ} lies in plane E then so does segment \overline{PQ}.

---

VII. The interior of angle \angle BAC formed by rays \overrightarrow{AB} and \overrightarrow{AC} is the intersection of half planes formed by \overrightarrow{AB} on side C and \overrightarrow{AC} on side B.
VII. Note: P-Q-R-S means as a diagram:

\[ \begin{array}{cccc}
\text{P} & \text{Q} & \text{R} & \text{S} \\
\end{array} \]

or

\[ \begin{array}{cccc}
\text{S} & \text{R} & \text{Q} & \text{P} \\
\end{array} \]
APPENDIX II

EXAMPLES OF PROBLEM TYPES AND SELECTION

PROCESS AND CRITERIA
A pool of potential criterion test items was obtained from questions similar to those actually used in pilot study treatments and tests. The problems in the pool were either solved completely or substantially by at least 1/3 of the pilot study groups or the problems were similar to those generally discussed in grade-ten geometry. Therefore, it was believed that any of the problems which appeared on the criterion test, whether solved at the time or not, was within the range of difficulty of problems to which grade-ten students are normally exposed.

Sixteen problems of the pool were chosen such that, in the opinion of the writer, eight of the problems clearly exemplified the characteristics of Type R and eight the characteristics of Type NR. These sixteen problems were submitted in random order to a panel for independent categorization as to type according to the following criteria.

Categorize the item as Type R if you believe that it is likely that an average grade-ten student would utilize techniques of the visual-analytic mode to produce a solution.

Categorize the item as Type NR if you believe:
1. that an average grade-ten student would not generally employ visual mode techniques to solve the problem, or
2. that if the student did employ visual mode techniques his solution would probably not be a valid logical argument.

A description of the terms used above was given to each member of the panel as were several examples. Those items for which there was unanimous agreement among the panel and writer as to type were chosen to compose the criterion test.
Examples of problem types

Examples of both Type NR and Type R problems are to be found in the criterion test, training test and pilot tests in the following order.

Criterion test (Appendix I)
Type NR: Questions II, III, VI, VIII, IX
Type R: Questions I, IV, V, VII

Training test (Appendix VI)
Type NR: Questions III, IV, V, VI
Type R: Questions I, II, VII

Pilot test (A) (Appendix VII)
Type NR: Questions II, III, V, VI, VII
Type R: Questions I, IV, VIII

Pilot test (B) (Appendix VII)
Type NR: VI
Type R: I, II, III, IV, V, VII, VIII, IX, X

Further examples of problems of both types are the following which were used in the treatment exposition and pilot study training programs.

Examples of Type R problems:

1. Prove that in a plane the perpendicular bisector of a segment AB is the set of all points P such that PA = PB.
2. Prove that the line segment joining the midpoints of adjacent sides of a triangle is parallel to the third side and one-half its length.

3. Prove that if a point $P$ is not on a line $m$ then there exists at least one perpendicular from $P$ to $m$.

4. If two parallel lines are cut by a transversal then prove that the bisectors of any pair of corresponding angles are parallel.

5. In a plane if a line $m$ intersects a parallelogram dividing its interior into two regions of equal area, then prove that $m$ intersects the diagonals of the parallelogram at their point of intersection.

6. In a plane, if two circles $C_1$ and $C_2$ are externally tangent at $D$ and the circles are congruent, with $\overline{AD}$ and $\overline{BD}$ diameters of $C_1$ and $C_2$ respectively, and line $\overline{NC}$ tangent to $C_1$ at $C$ and intersecting $C_2$ at $E$, then prove $m(\overline{AC}) = m(\overline{DC}) + m(\overline{DE})$.

7. Given a line $\overrightarrow{BC}$ and a point $A$ not on $\overrightarrow{BC}$ such that triangle $ABC$ has $AB > AC$, and $P$ any point on $\overrightarrow{BC}$ such that $P-B-C$, prove $AP > AB$.

8. If $A$, $B$, $C$, $D$ are four different points in space, how many lines can pass through pairs of them if:
   (i) $A$, $B$, and $C$ are collinear?
   (ii) no three are collinear?
   (iii) the points are noncoplanar?
   (iv) $A$, $B$, $C$, and $D$ are coplanar?

9. Given triangle $\triangle ABC$ with $B-T-C$ and $\overrightarrow{AT}$ is the bisector of $\angle BAC$ and $\overrightarrow{AT}$ is the median, prove that triangle $\triangle ABC$ is isosceles.
10. Point D is in the interior of triangle \( \triangle ABC \). Prove that angle \( \angle ADB \) is greater than angle \( \angle ACB \), i.e., if \( m(\angle ADB) > m(\angle ACB) \), then \( \angle ADB > \angle ACB \).

Examples of Type NR problems:

1. Show that a half plane contains at least three noncollinear points.
2. Show that if \( m \) is parallel to \( n \) and \( m \) is parallel to \( p \), where \( m, n \) and \( p \) are three distinct lines in space, then \( n \) is parallel to \( p \), without considering perpendiculars.
3. Prove that if \( a, b, t, \) and \( m \) are different lines all in plane \( E \) such that \( t \) is parallel to \( a \), \( t \) is parallel to \( b \), and \( m \) intersects \( b \) and \( t \), then \( m \cap a \neq \emptyset \).
4. Given a triangle \( \triangle ABC \) and a line \( m \) in the same plane, prove that if \( m \) contains no vertex of the triangle, then \( m \) cannot intersect all three sides.
5. If \( A-M-C \) on line \( m \), then prove that \( A \) and \( M \) are on the same side of any line \( n \) different from \( m \) which contains point \( C \).
6. If \( B-M-C \) and \( A \), a point not on \( BC \), then prove that \( M \) is in the interior of angle \( \angle BAC \).
7. Show that a half plane \( H \) contains at least two distinct points.
8. Given triangle \( \triangle ABC \) and \( m \), a line, both in the same plane, then prove that if \( m \) intersects \( \overline{AB} \) between \( A \) and \( B \), then \( m \) must intersect either \( \overline{AC} \) or \( \overline{BC} \).
9. Given a plane \( E \) and a line \( m \) in \( E \) which forms two half-planes \( G \) and \( H \), prove that \( G \neq \emptyset \) and \( H \neq \emptyset \).
10. If $a$, $b$, and $t$ are different lines in a plane $E$, and $t$ is parallel to $a$, and $t$ is parallel to $b$, then prove that $a$ is parallel to $b$.

11. Prove that given a line $m$ in space and a point $P$ not on $m$, then there is exactly one plane containing both the point $P$ and the line $m$.

12. If $m$ and $n$ are two different lines in space which intersect, then prove that their intersection contains exactly one point.

13. Prove that every ray is convex.

14. If $m$ and $n$ are two different lines which intersect, prove that there is exactly one plane $E$ which contains them.
APPENDIX III

OUTLINE OF TREATMENT SCHEDULE
1. Discussion of what a proof accomplishes with regard to making a general statement.

2. Example, "6 < average of 6 and 8 < 8." Do you think that the average of two numbers is always between the smaller and larger value?

3. Class asked if above property of average is true for all positive numbers, one positive and one negative number, or for two negative numbers.

4. Discussion of whether testing some possibilities would guarantee that the statement about average is always true.

5. Statement was symbolized: \( A < (A+B)/2 < B \), when \( A, B \) are any two numbers with \( A < B \).

6. Proof of \( A < (A+B)/2 < B \), with \( A < B \) given.

   i.e. 1. \( A < B \), was assumed.
   2. \( A + A < A + B \), by a rule of algebra.
   3. \( 2A < A + B \) therefore \( A < (A+B)/2 \), by another rule of algebra.

7. It is not that the two rules of algebra used above made reference to numbers that was important, but that they made reference to a whole class of numbers that was important.

8. Discussion of idea that a proof will make a statement about a large set of possible situations.
9. Discussion and proof of other examples.

1. The sum of two odd numbers is always even.
2. The product of two odd numbers is always odd.

End of session

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Session 1, Group E

Same content as Session 1, Group C.

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Session 2, Group C

1. Geometry discussed as a game related to the real world in certain ways.
2. Discussion that the objects in geometry have many of the properties of common objects; for example, many objects have edges which one could stretch a string (line-like) along and many objects have surfaces like the floor or blackboard (plane-like).
3. Geometry has objects which resemble stretched string-- call them lines and things which resemble blackboard-- call them planes.
4. The class was asked what a line in geometry "looked" like.

5. The discussion concluded with the idea that lines could not be seen but only representations of them could be seen.

6. The class was asked to define a line.

7. The discussion concluded that any definition made use of "line" or "straight" or "curve" whose definitions need the idea of line to begin with.

8. It was concluded that a line could not be defined without involving other concepts which could not be defined like points.

9. It was suggested that while lines could not be seen or defined they could be described by talking about how they behaved, that is, if one had a list of rules that specified what line could "do" and not "do", one could get a good idea of what lines were.

10. It was suggested that the rules should, if possible, make lines and planes behave like the objects which suggested them.

11. It was suggested that, in order to talk about lines we should have a picture or caricature of a line, something which has some of the properties of a line.

12. Fig. 1, 2, 3 were drawn on the blackboard of two lines crossing.

Fig. 1

Fig. 2

Fig. 3
13. The issue of what picture would best represent the behavior of two lines crossing was discussed.

14. Opinions were obtained that where lines cross they should cross in one point.

15. It was decided that the picture in Fig. 1 would be a good way to represent the situation of two lines crossing.

16. The idea of how a "point" should be pictured was discussed and it was concluded that a chalk dot no larger in diameter than the width of a chalk mark for a line would be a good idea.

End of Session

Session 2, Group E

1. Geometry discussed as a game related to the real world in certain ways.

2. Discussion that objects in geometry have many of the properties of common objects; for example, many objects have edges along which one could stretch a string (line-like) and many objects have surfaces like floors or blackboards (plane-like).

3. Geometry has objects which resemble stretched strings—call them lines and things which resemble blackboards—
call them planes.

4. The class was asked what a line in geometry "looked" like.

5. The discussion concluded with the idea that lines could not be seen but only representations of them could be seen.

6. The class was asked to define a line.

7. The discussion concluded that any definition made use of "line" or "straight" or "curve" whose definitions needed the idea of line to begin with.

8. It was concluded that a line could not be defined without involving other concepts which could not be defined, like point.

9. It was suggested that, while lines could not be seen or defined, they could be described by talking about how they behaved, that is if one had a list of rules that specified what lines could "do" and not "do", one could get a good idea of what lines were.

10. It was suggested that these rules should, if possible, make lines and planes behave like the objects which suggested them.

11. It was suggested that, in order to talk about lines, we should have a means of describing some of their properties in a convenient fashion.

12. The idea of a line or a set of points was suggested, and it was claimed that set intersection was, for example, a convenient way to explore the situation of two lines crossing.

13. The notation to describe this was introduced as $p \cap q$, 
which was to be the set of points where p crossed q.

14. The issue of what \( p \cap q \) would be could be described by having \( p \cap q = \{ \frac{7}{3} \} \) or \( \{ A, B \} \).

15. It was decided that, for example, if \( p \cap q \) represented what two lines which crossed had in common at a place where they crossed then \( p \cap q = \{ A \} \), a set of one point.

End of Session

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Session 3, Group C

1. It was suggested that we should formally state some of the basic rules of geometry.

2. Geometry would be played with things called points, sets of points called lines and sets of points called planes, the totality of points being called space.

3. The first rule was stated:

   Rule I: If P and Q are two distinct points then there is exactly one line, \( m \), which contains P and Q.

4. The phrase "exactly one" was explained to mean that there was a line which contained P and Q and there was only one line.

5. To illustrate the idea of "exactly one" the solution of equations was considered: We say that "\( x^2 = a \)" has a solution, in fact it has two, but \( 3x + 5 = 23 \) has a solution and has only one solution.
6. To illustrate Rule I consider Figs. 1, 2.

Fig. 1

Fig. 2

7. In Fig. 1, it is shown that line $m$ contains P and Q (We can't have two points such that no line may contain the.). Fig. 2 shows "two lines" containing P and Q, but Rule I tells us that these two lines must be the same.

8. It was agreed that capital letters would name points and lower case letters would name lines and script letters would name planes.

9. It was agreed that the following conventions would be used to depict lines and planes, see Fig 3, 4.

Fig. 3

Fig. 4

10. The conventions were described as "caricatures" of points, lines and planes, not the objects themselves or even their "pictures" if such things existed.

11. A line could be named by one symbol $m$ or as $\overrightarrow{AB}$ from any two points on it.

12. Note that the caricatures "behave" in a fashion similar to what we expect for lines and planes.

End of Session
1. It was suggested that we should formally state some of the basic rules of geometry.

2. Geometry would be played with things called points, sets of points called lines and sets of points called planes, the totality of points is called space.

3. The first rule was stated:
   Rule I: If \( P \) and \( Q \) are two distinct points then there is exactly one line, \( m \), which contains \( P \) and \( Q \).

4. The phrase "exactly one" was explained to mean that there was a line which contained \( P \) and \( Q \) and only one such line.

5. To illustrate the idea of "exactly one" the solution of equations was considered: We say that \( x^2 = a \) has a solution, in fact it has two, but \( 3x + 5 = 23 \) has a solution and only one solution.

6. To symbolize Rule I consider:
   Rule I: If \( P \) and \( Q \) are distinct points, i.e. \( P \neq Q \), then there is exactly one line \( m \), such that \( \{ P, Q \} \subset m \), i.e. \( P \in m \), \( Q \in m \).

7. Thus Rule I states: If \( P, Q \) are distinct points then we always have a line \( m \), such that \( \{ P, Q \} \subset m \) and if \( \{ P, Q \} \subset m \), and \( \{ P, Q \} \subset n \), then \( m = n \).

8. It was agreed that the following convention would be used
to stand for points, lines and planes.

9. A line could be named by one symbol as "m" or by "AB" where A and B are any two points on the line.

10. Note that then we can express the idea of a point on a line by $P \in m$ or a line in a plane as $m \subseteq E$.

End of Session

Session 4, Group C.

1. The following definition of "collinear" was given: A set of points is collinear if there is some line, m, which contains them.

2. In Fig. 1, A,B,D are collinear, but A,B,C are not.

3. Note that 3 points may or may not be collinear, but 2 points always are collinear.

4. Discussion of what "straightness" of a line meant.

5. The idea that Rule I contributed to what straightness would mean was discussed.

6. It was noted that the representation for a line satisfied the restriction of Rule I.

7. A definition of "coplanar" was given as a set of points (points and lines or a set of lines) is coplanar if there is some plane which contains them.
8. In Figs. 2, 3, 4 note that A, B, C, D are not coplanar, but A, B, C, E are coplanar.

9. Note that in Fig. 3, A, B, M are coplanar and in Fig. 4, m and n are coplanar.

10. Note that 3 points seem to always be coplanar, but 4 or more points may or may not be coplanar.

11. A rule was proposed to make planes "bigger" than lines and space "bigger" than planes.

12. Discussion lead up to a statement of Rule II.

Rule II: 1. Every plane contains at least 3 points which are not collinear.

2. Space contains at least 4 points which are not coplanar.

13. A definition of parallel lines was given:

Two lines m, n are parallel, m \parallel n if
1. m and n are distinct, and
2. m and n lie in the same plane, and
3. m does not cross n.

14. This idea was illustrated in Figs. 5, 6, 7.
15. The possibilities of 2 lines m and n being parallel (Fig. 5), both in the same plane but intersecting (Fig. 6), and skew lines (Fig. 7) were discussed.

16. Fig. 7 was used to illustrate the need for condition 2 in the definition of parallel lines.

17. The idea of what would make a plane flat was discussed.

18. To express "flatness" Rule III was stated.

Rule III: If P and Q are 2 points in plane $E$ then the line which contains P and Q lies completely in plane $E$.

![Fig. 8](image)

![Fig. 9](image)

19. Fig. 8 was drawn to show how Rule III prevented the situation depicted.

20. Fig. 9 was drawn to show how Rule III made a plane "flat" in all directions unlike a cylinder which is "flat" in only one direction.

21. Rule IV was stated to express the idea that planes are "thin."

Rule IV: 1. Given any 3 points P, Q, R there is at least one plane which contains them.

2. Given any 3 points P, Q, R which are not collinear then there is exactly one plane which contains them.

22. Thus, as Fig. 10 shows, planes are "thin."
1. The following definition of "collinear" was given: A set of points is collinear if there is some line \( m \) which contains them.

2. Note we can say \( P, Q, R \) are collinear by \( \{P, Q, R\} \subset m \).

3. Two points are always collinear, but 3 points may or may not be collinear.

4. Discussion of what "straightness" of a line meant.

5. The idea that Rule I contributes to what straightness would mean was discussed.

6. It was noted that our notation \( \{P, Q\} \subset m \) for every \( P, Q \) and \( \{P, Q\} \subset m \) and \( \{P, Q\} \subset n \) implies that \( m = n \), obeyed the wording of Rule I.

7. A definition of "coplanar" was given: A set of points (points and lines or lines and other lines) is coplanar if there is some plane which contains them.

8. Our notation for this situation is: If \( P, Q, m \) are coplanar then we write \( P \in \mathcal{E}, Q \in \mathcal{E}, m \in \mathcal{E} \).

9. Note that 3 points seem to always be coplanar, but 4 or more points may or may not be coplanar.

10. A rule was proposed to make planes "bigger" than lines and space "bigger" than planes.
11. Rule II was stated.

Rule II: 1. Every plane contains at least three noncollinear points.
2. Space contains at least four noncoplanar points.

12. A definition of parallel lines was given:

Two lines m and n are parallel, written $m \parallel n$ if
1. m and n are distinct ($m \neq n$), and
2. m and n are in the same plane ($m \subseteq \mathcal{E}, n \subseteq \mathcal{E}$), and
3. $m \cap n = \emptyset$.

13. Consider the three possibilities for two lines.

1. $m \parallel n$
2. $m \cap n \neq \emptyset$, and $m \subseteq \mathcal{E}, n \subseteq \mathcal{E}$
3. $m \cap n = \emptyset$ and $m \cup n \subsetneq$ for any plane; the two lines in 3 are said to be skew.

14. The idea of what would make a plane flat was discussed.

15. To express this idea, Rule III was stated.

Rule III: If P and Q are two points in plane $\mathcal{E}$ then the line which contains P and Q, line m, lies completely in plane $\mathcal{E}$.

16. This rule was expressed as follows: If $P, Q \subseteq \mathcal{E}$ then $m \subseteq \mathcal{E}$, when $P, Q \subseteq m$.

17. It was noted that Rule III made planes flat in all directions, unlike a cylinder which was flat in only one direction.

18. Rule IV was stated to express the idea of "thin-ness" of
a plane.

Rule IV: 1. Given any 3 points P, Q, R there is at least one plane which contains them.

2. Given any 3 noncollinear points P, Q, R there is exactly one plane which contains them.

Thus, if P, Q, R are noncollinear then one cannot have $\mathcal{E}_{P, Q, R} \subset \mathcal{E}$, and $\mathcal{E}_{P, Q, R} \subset \mathcal{F}$ with $\mathcal{E} \neq \mathcal{F}$.

End of Session

Session 5, Group C.

1. The idea of betweenness was discussed and defined as:

A-B-C means A, B, C are collinear and B is between A and C, i.e. the distance of A to B plus the distance of B to C is equal to the distance of A to C.

2. Rays and segments were defined in terms of betweenness as:

AB means segment AB is defined as A, B, and all the points between A and B.

3. Fig. 1 illustrated the idea of betweenness.

\[
\begin{array}{c}
A \quad B \quad C
\end{array}
\]

Fig. 1

4. We write A-B-C or C-B-A to symbolize the arrangement of points in Fig. 1.

5. A ray was defined as follows:
In Fig. 2, the segment $\overline{AB}$ and all points $X$ such that $A-B-X$ make ray $\overrightarrow{AB}$; $A$ is called the vertex.

Fig. 2

6. The idea of convexity was defined as: A set of points is convex if whenever $P$ and $Q$ are in the set $PQ$ is in the set.

7. Examples were given in Fig. 3, 4, 5.

Fig. 3  
Fig. 4  
Fig. 5

8. Lines, rays and segments are convex as illustrated by Fig. 3, 4, 5.

9. The concept of indirect proof was introduced and discussed.

10. The technique of considering the implication as (premise) implies (statement) and then assuming (premise) and (opposite of statement) and deriving a contradiction was discussed.

End of Session

Session 5, Group E

1. The idea of betweenness was discussed and defined as: $A-B-C$ means $A, B, C$ are collinear and $B$ is between $A$ and $C$, i.e.
the distance of A to B plus the distance of B to C is equal to the distance of A to C.

2. Rays and segments were defined in terms of betweenness:
   $\overrightarrow{AB}$ means segment AB is defined as A, B, and all the points between A and B.

3. A ray was defined as follows: Ray $\overrightarrow{AB}$ is $\overrightarrow{AB} \cup X$, where A-B-X; A is called the vertex.

4. The idea of convexity was defined as: A set of points is convex if whenever P and Q are in the set, $\overrightarrow{PQ}$ is in the set.

5. This was stated as: T is convex if $\{P, Q\} \subseteq T$ implies $\overrightarrow{PQ} \subseteq T$.

6. It was noted that rays, lines, and segments were convex.

7. Examples of convex and nonconvex sets were mentioned.

8. The concept of indirect proof was introduced and discussed.

9. The technique of considering the implication as (premise) implies (statement) and then assuming (premise) and (opposite of statement) and deriving a contradiction was discussed.

End of Session
1. Problems from the training test were done in class.

2. Prove that a plane is a convex set. See Fig. 1.

![Fig. 1](image)

3. As in Fig. 1, to show plane $\mathcal{E}$ is a convex set we need to choose ANY two points $P, Q$ and ask if $\overline{PQ}$ is in $\mathcal{E}$.

4. By Rule III, $PQ$ is in $\mathcal{E}$, so $\overline{PQ}$ which is part of $PQ$ is in $\mathcal{E}$; thus $\mathcal{E}$ is convex.

5. Prove that two distinct lines cannot intersect in two points.

6. Fig. 2 shows the "opposite statement."

![Fig. 2](image)

7. As in Fig. 2, $m$ and $n$ intersect in $P$ and $Q$, two different points, but from Rule I we know exactly one line contains $P$ and $Q$, $\overrightarrow{PQ}$, so $m = n = \overrightarrow{PQ}$; this is a contradiction since we were given $m \neq n$.

8. Prove that if a line intersects a plane $\mathcal{E}$, and does not lie completely in $\mathcal{E}$, then the intersection of the line and the plane is only one point.

9. Fig. 3 shows the "opposite statement."

![Fig. 3](image)
10. As in Fig. 3, line \( m \) crosses plane \( \mathcal{E} \) in at least two points.

11. But by Rule III, the line which contains any two of these points must lie completely in \( \mathcal{E} \), this contradicts the assumption that \( m \) does not lie in \( \mathcal{E} \).

End of Session

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Session 7, Group E

1. Problems from the training test were proved.

2. Prove that a plane is a convex set.

3. To show plane \( \mathcal{E} \) convex, we choose any two points in \( \mathcal{E} \), \( P \) and \( Q \), and are asked to show \( \overline{PQ} \subseteq \mathcal{E} \).

4. We know by Rule III, that if \( \overrightarrow{PQ} \subseteq \mathcal{E} \) then \( \overrightarrow{PQ} \subseteq \mathcal{E} \) and \( \overline{PQ} \subseteq \mathcal{E} \), so \( \overline{PQ} \subseteq \mathcal{E} \), and \( \mathcal{E} \) is convex.

5. Prove that two distinct lines cannot intersect in two points.

6. The opposite statement is \( m \nmid n \), and \( m \cap n = \overrightarrow{P,Q} \), \( P \neq Q \).

7. By Rule I, we know that there is exactly one line, \( m \), which contains \( \overrightarrow{P,Q} \).

8. This contradicts \( m \nmid n \).
9. Prove that if a line $p$ intersects a plane $\pi$, and does not lie completely in $\pi$ then the intersection of $p$ and $\pi$ is only one point.

10. The opposite statement is $p \cap \pi = \{P_1, P_2, \ldots \}$.

11. We are given $p \cap \pi \neq \emptyset$, and $p \not\subset \pi$.

12. By Rule III, $\overrightarrow{P_1P_2} \subset \pi$, but this line $\overrightarrow{P_1P_2} = p \not\subset \pi$.

End of Session

Session 8, Group C

1. The plane separation postulate was stated as... See Fig. 1.

   If line $m$ lies in plane $\pi$ then:

   1. The plane is divided into 3 convex regions $m, H, K$, where $H, K$ are called half-planes, and

   2. If $P$ lies in $H$ and $Q$ lies in $K$ then $PQ$ crosses $m$.

   Fig. 1

2. The class was asked to guide the instructor in the proof of the following theorem:

3. If $m$ lies in $\pi$ and ray $\overrightarrow{AB}$ lies in $\pi$ and $A$ is on $m$, then ray $\overrightarrow{AB}$ (except for $A$) lies completely on one side of $m$.

4. The following figure was drawn, Fig. 2, which illustrated possibilities for the opposite of the statement.
5. It was noted that if any part of ray $\overrightarrow{AB}$ was in $K$, call that point $C$, then $\overrightarrow{CB}$ would cross $m$ at $A$, as is seen in the figure since a ray can cross a line in only one point.

6. Then we have Fig. 2(c) as our opposite statement.

7. Then, it was noted that we would have $C-A-B$, but this is impossible since $A$ is the vertex of the ray.

End of Session
7. Then \( \overline{CB} \cap m \neq \emptyset \), in fact \( \overline{CB} \subset \overline{AB} \) so \( \overline{CB} \cap m = A \) since \( \overline{AB} \cap m = A \).

8. But then \( C-A-B \), which is impossible since \( A \) is the vertex.

End of Session

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**Session 9, Group C**

1. The class was asked to guide the instructor in the proof of the following theorem:

2. Line \( m \), and \( \triangle ABC \) lie in plane \( \ell \). Show that if \( m \) crosses \( \overline{AB} \) between \( A \) and \( B \), that \( m \) must cross \( \overline{BC} \) or \( \overline{AC} \).

3. Fig. 1,2 show possibilities for the opposite statement and Fig. 3 shows the desired state of affairs.

![Fig. 1](image1)

![Fig. 2](image2)

![Fig. 3](image3)

4. The situation in Fig. 1 was shown to be impossible since \( m \) cannot cross \( \overline{AB} \) more than once.

5. Fig. 2 was shown to be impossible since, if it were true, then \( A \) and \( B \) would be on opposite sides of \( m \) by the plane separation postulate and since \( C \) is not on \( m \), it must be on the "A" side or "B" side of \( m \); if it is on the "A" side then \( B \) and \( C \) are on opposite sides of \( m \), thus \( BC \) crosses \( m \), a contradiction.

6. The same argument for \( C \) on the "B" side of \( m \).

End of Session
1. The class was asked to guide the instructor in the proof of the following theorem:

2. Line m, and \( \triangle ABC \) lie in plane \( \mathcal{E} \). Show that if m crosses \( \overline{AB} \) between A and B, that m must cross \( \overline{BC} \) or \( \overline{AC} \).

3. It was noted that if \( m \cap \overline{AB} \neq \emptyset \) then \( m \cap \overline{AB} = m \cap \overline{AB} = \mathcal{P} \).

4. Then, by the plane separation postulate, A and B are on opposite sides of m.

5. We know \( C \not\in m \), so \( C \in H \) or \( C \in K \), when H is the "A" half plane and K is the "B" half plane, i.e. \( A \in H, B \in K \).

6. If \( C \in H \), then \( \overline{CB} \cap m \neq \emptyset \) and if \( C \in K \) then \( \overline{CA} \cap m \neq \emptyset \).

7. Both of these situations are impossible.

End of Session
APPENDIX IV

EXAMPLES OF EXPOSITION TECHNIQUES
In order to illustrate the use of the pictorial-representational mode of exposition, consider the following treatment of incidence chosen from a geometry text (17). The student is informed early in the text that he already knows certain "facts about geometry", for example, that "Two straight lines cannot cross each other in more than one point" (17). The student is also informed that "Postulates describe fundamental properties of space" and ". . . the idea of point, line and plane are suggested by physical objects" and "When we use the term line, we shall always have in mind the idea of a straight line. A straight line extends infinitely far in both directions. Usually we shall indicate this in . . . illustration by putting arrowheads at the ends of the part of a line we draw" (17). Even though the student is warned that these statements are not definitions, the idea that geometry is a formalization of the physical world is implied. The standard teaching tactic (pictorial-representational) approaches incidence theorems, as the two that follow, in the manner outlined below.

Th. 1- If a line m intersects a plane E and does not lie in E
then prove that m intersects E in exactly one point.

Th. 2- Given triangle \( \triangle ABC \) and a line m in the same plane,
prove that if m contains a point between A and B
then m must intersect one of the other sides, \( \overline{AC} \) or \( \overline{BC} \).
The negation-of-conclusion-statement for an indirect proof of Theorem 1 is often depicted as in Fig. 4 or Fig. 5. Through inspection of these figures a contradiction of previous theorems or postulates is sought. Specifically, for the depiction in Fig. 4, the contradiction sought is that line $m$ containing the two points of intersection, $A$ and $B$, with plane $E$ does not lie in plane $E$ and yet is equal to line $\overrightarrow{AB}$. Note that in Fig. 5, which also is a depiction of the negation-of-the-conclusion of theorem 1, is less suitable for consideration of contradictions, since, as drawn, the presence of lines $AB$ and $BC$ may tend to confuse the direction of reasoning. Also note that Fig. 6, a depiction of the "desired state of affairs", does not clearly suggest a course of action for analysis of the problem, and thus would not be utilized in the visual mode analysis of the problem except as a preliminary statement of what is to be proved.

For theorem 2 a similar but more complicated situation exists. The depiction in Fig. 7 represents the negation-of-the-conclusion
of theorem 2 and is a possible diagram to be used in the visual mode program as follows: (Note that Fig. 8 represents the "desired state of affairs.")

A pictorial-representational mode exposition of the proof of theorem 2 could be outlined as follows:

1. As can be seen in the diagram in Fig. 7, if line m does not intersect \( \overline{AC} \) or \( \overline{BC} \), then

2. by the plane separation postulate, \( A, B, \) and \( C \) are all on the same side of line \( m \).

3. But \( A \) and \( B \) are on opposite sides of \( m \). Therefore we have a contradiction.

4. Thus we must reject the assumption that \( m \) does not cross \( \overline{AC} \) or \( \overline{BC} \).

The same problems, theorems 1 and 2, are taught via the symbolic-nonrepresentational mode of exposition by set theoretical ideas. It should be noted that the use of Venn diagrams to illustrate concepts
of set theory is included in the symbolic-nonrepresentational mode since a Venn diagram is not derived from visual manipulation of the geometric elements of the problem involved. Theorem 1 is stated in the same form as for the visual mode: if a line m intersects a plane E and does not lie in E then m intersects E in exactly one point, and is then translated into set language: if m\cap E \neq \emptyset and m \notin E then

1. assume m\cap E \neq \emptyset, i.e.
\[ m \cap E = \{P_1, P_2, \ldots \} \quad \text{or} \quad m \cap E = \emptyset. \]
2. We are given m\cap E \neq \emptyset .
3. We are given m \notin E .
4. Statement 2 rules out m\cap E = \emptyset .
5. P_1 and P_2 lie in E implies that the line that contains them, \overline{P_1P_2}, lies completely in E .(This postulate had been studied in class.)
6. Therefore, m = \overline{P_1P_2} \subseteq E.
7. Statement 3 rules out m \subseteq E,
8. therefore statement 1 must be rejected.
9. Thus we have m \cap E = \{P\} .

The symbolic-nonrepresentational mode for teaching the proof of theorem 2 employs a set theoretic statement of the plane separation postulate. This form of the postulate was taught together with the postulate statement to the experimental group. The control group was taught a pictorial form of the postulate statement. A symbolic-nonrepresentational proof of theorem 2 is outlined as follows.

1. If a line m and \triangle ABC lie in the same plane and m intersects \overline{AB} between A and B, then prove that m must intersect \overline{BC} or \overline{AC}.
2. \( m \) intersects \( \overline{AB} \) at \( T \), \( A-T-B \).

3. Assume \( m \) does not intersect \( \overline{BC} \) or \( \overline{AC} \). Then \( m \cap \overline{BC} = \emptyset \) and 
\( m \cap \overline{AC} = \emptyset \).

4. By the plane separation postulate, \( A \) and \( B \) are on opposite sides of \( m \), i.e. \( A(m)B \).

5. By the plane separation postulate, if \( m \cap \overline{AC} = \emptyset \) then \( A \) and \( C \) are on the same side of \( m \), i.e. \( A,C(m) \).

6. If \( A(m)B \) and \( A,C(m) \), then \( B(m)C \).

7. Also by the plane separation postulate, \( B(m)C \) implies that 
\( m \cap \overline{BC} \neq \emptyset \).

8. Thus we have a contradiction to the hypothesis in 3.

9. Conclusion: \( m \) intersects \( \overline{AC} \) or \( \overline{BC} \).
APPENDIX V

EXAMPLES OF SCORING TECHNIQUES
The responses from the subjects to the problems of the criterion test were scored by the writer in the following manner.

A response, which consists of sentences and phrases, diagrams, or set theoretic symbols and connections such as arrows, was read by the writer. The response was analyzed for logical structure. That is to say, the response was interpreted to consist of statements and implications between the statements and finally a chain of implications forming a logical argument. A response which was interpreted to consist of statements containing only those rules (axioms) and definitions relevant to the problem and having a valid logical argument leading to the desired conclusion was given full credit (10 points). The presence of irrelevant information caused loss of 1 point from the final score. Relevant information with no valid argument present was given a maximum score of 2 points and a logical argument using completely irrelevant information was given a maximum score of 4 points.

In order to illustrate the scoring for responses, the following are a range of solution possibilities. The solution possibilities are stated in the logical form against which the logically interpreted response of the students were compared. The problem chosen for the illustration is of question VIII on the criterion test.

If \(a, b,\) and \(c\) are 3 distinct lines in plane \(E\) with \(a\) parallel to \(c\) and \(b\) parallel to \(c\), then prove \(a\) is parallel to \(b\).

\[\text{Example 1.}\]

(diagram: A)
1. assume opposite of the statement, i.e. (a is not parallel to b.)
2. Then a and b cross, at P.
3. a is parallel to c, therefore P is not on c.
4. b is parallel to c, therefore a and b are parallel to c through P.
5. Rules state that there is exactly one parallel to c through P, therefore statement 1 is false.
6. Therefore a is parallel to b.

Score: 10 points.

Comment: logic is valid throughout, diagram used for reference only, only relevant information used

Example 2.
(diagram: A)

![Diagram A]

1. If a is not parallel to b, then a and b cross at P.
2. Rule V states that there can be only one line parallel to c through P.
3. Therefore (by 1 and 2) a does not cross b and is then parallel to b.

Score: (8-10) points.

Comment: no reason stated that P is not on c, only relevant information used, diagram used for reference only, indirect proof implicit, not stated

Example 3.
(no diagram used)
1. c || a, c || b is given
2. Assume $a \not\parallel b$. 
3. Then by 2 \[ a \cap b = \{ P \} \]
4. $a \parallel c$ implies $P \notin c$. 
5. Rule V states that there exists exactly one line parallel to $c$ through $P$. 
6. Therefore, statement 2 is false, and $a \parallel b$. 

Score: 10 points 
Comment: logic valid throughout, only relevant information used

Example 4. 
(no diagram used) 
1. If $a \not\parallel b$ then $a \cap b = \{ P \}$. 
2. Therefore both $a$ and $b$ are parallel to $c$ at $P$. 
3. But Rule V states that there exists exactly one line parallel to $c$ at $P$. 
4. Therefore $a \parallel b$. 

Score: (9-10) points 
Comment: argument does not account for $P \in c$. 

Example 5. 
(diagrams A,B,C,D) 

1. There are 2 possibilities for $a \parallel c$ and $b \parallel c$ (A or B).
2. If \((A)\) is false and \(a \nparallel b\) then we have \((C)\), which is impossible by the plane separation postulate.

3. If \((B)\) is false and \(a \nparallel b\) then we have \(D\) which is false by Rule V.

Score: (6-8)

Comment: diagram makes implicit use of plane separation postulate and convexity, but this is not stated

Example 6.
(no diagram used)

1. If \(a \nparallel b,\ then a \cap b \neq \emptyset\).

2. If \(a \parallel c,\ then b \nparallel c\) since Rule V states there exists exactly one parallel at a point \(P\).

3. Therefore \(a \parallel b\).

Score: (6-8)

Comment: argument from 1 to 2 not stated, assumption of 2 is not needed

Example 7.
(diagram: A)

1. If \(a \nparallel b\) and \(a \parallel c\), then \(b \nparallel c\), by Rule V.

2. Therefore \(a \parallel b\).

Score: (6-8)

Comment: argument from 1 to 2 not stated, diagram indirectly used.
Example 8.
(diagram: A)

1. If a crosses b, then (A).
2. But by Rule V there is only one parallel to c through P.
3. Therefore a\parallel b.

Score: (4-6)
Comment: argument from 1 and (A) to 3 not stated.

Example 2.
(diagram: A)

1. If a\parallel b then a crosses b.
2. Therefore a or b crosses c by (A).
3. But a\parallel c and b\parallel c.
4. Therefore a\parallel b.

Score: (2-4)
Comment: argument from 1 and (A) to 4 not stated.

Example 10.
(no diagram used)

1. If a\parallel b then a crosses b at P.
2. Therefore a, b are both parallel to c at P.
3. Rule V states that there is exactly one parallel to c at P (call it \(m\)).
4. Therefore there cannot be 3 parallels (a, b, m).

5. Therefore a \parallel b.

Score: (4-6)

Comment: argument from 1 to 2 not stated, argument 2 to 4 not stated.

Example 11.
(no diagram used)

1. If a \parallel b, then a \cap b = \{ P \}.
2. But a \parallel c.
3. Therefore a \parallel b.

Score: (2-4)

Comment: argument from 1, 2 to 3 not stated.

Example 12.
(diagram: A)

1. If a \parallel b then (A).
2. Therefore if b is extended in (A), b will cross c.
3. Therefore b \parallel c.
4. But b \perp c.
5. a \parallel b.

Score: (2-4)

Comment: argument from 1, (A) to 2 not stated.
Example 13.
(diagram:A)

1. There is no other way to draw A.
2. a \parallel b

Score: (0-2)
Comment: argument from (A) to 1 not stated.

The remainder of the Type NR problems, II, III, VI, IX, are evaluated in the same fashion as problem VIII. Question I is given 10 points for a correct statement of betweenness or lacking this, 0-8 points for a plausible diagram of the situation. Question IV is given 10 points if answer "6" is given; if not, then 0-8 points for a plausible diagram. Question V is given 10 points- 6 points for #1, 2 points for #2, and 2 points for #3. Question VII is given 10 points- 2 to #1, 2 to #2, and 6 to #3.
APPENDIX VI

TRAINING TEST
Quiz (1)

Name: ______________________

I. Given A, B, C, and D are 4 distinct points on a line, and given that B-C-D, and A-C-D

(1) must you conclude A-B-C?  yes  no  
(2) is B-A-D possible?    
(3) is B-A-C possible?    
(4) is A-C-B possible?    

II. Given A, B, C, D, and E are 5 distinct points on a line and given that B-C-E, D-B-C, and A-B-D

(1) must you conclude B-A-C?  yes  no  
(2) must you conclude D-B-E?    
(3) is A-B-C possible?    
(4) is B-C-A possible?    

III. Prove that a plane is a convex set.

IV. Prove that two distinct line p and q cannot intersect in two points.

   Hint: Prove that the opposite statement is false.

V. Prove that if a line p intersects a plane E and p does not completely lie in E, then the intersection of p and E is only one point.

   Hint: Prove that the opposite statement is false.

VI. Prove: If p, q, t are 3 distinct lines in a plane and p is parallel to q, and t intersects p, then t must intersect q.

   Hint: Prove that the opposite statement is false.
VII. A, B, and C are 3 distinct collinear points. C is on ray $\overrightarrow{AB}$.

If ray $\overrightarrow{AB}$ is intersected with ray $\overrightarrow{CB}$, which of the following (one or more) describe the possibilities for the intersection of $\overrightarrow{AB}$ and $\overrightarrow{CB}$?

<p>| | |</p>
<table>
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<tbody>
<tr>
<td>(1) $\overrightarrow{AB}$</td>
<td>(4) $\overrightarrow{CB}$</td>
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<tr>
<td>(2) $\overrightarrow{AC}$</td>
<td>(5) $\overrightarrow{AB}$</td>
</tr>
<tr>
<td>(3) $\overrightarrow{BC}$</td>
<td>(6) $\overrightarrow{BC}$</td>
</tr>
</tbody>
</table>
RULES:

Rule I: If P and Q are any 2 points then there is EXACTLY one line m which contains them.

Rule II:
1. Any plane contains at least 3 points which are not on a line.
2. SPACE has at least 4 points not on a plane.

Rule III: If P and Q are any 2 points in plane E, then the line m which contains P and Q lies completely in plane E.

Rule IV:
1. If P, Q, R are any 3 points (in SPACE) then there is at least one plane which contains P, Q, R.
2. If P, Q, R are any 3 points (in SPACE) which do not lie on some line, then there is EXACTLY one plane E which contains P, Q, R.

Rule V: If P is ANY point not on line m then there is EXACTLY one line m which contains P, and is parallel to m.

DEFINITIONS:

I. Points are collinear if there is at least one line which contains them.

   Note: 2 points are ALWAYS collinear.

   3 points MAY be collinear.

II. Points are coplanar if there is at least one plane which contains them.

   Note: 2 points are ALWAYS coplanar.

   3 points are ALWAYS coplanar.
4 points MAY be coplanar.
A line and a point not on it are always coplanar.
2 lines which cross are always coplanar.

III. 2 lines m, n are parallel, m||n, if
1. m and n are different lines,
2. m does not intersect (cross) n,
3. m and n lie in the same plane.

IV. A set of points is CONVEX if, for any choice of 2 points P,Q in
the set, the segment $\overline{PQ}$ lies completely in the set.

Note: Any line is convex.
Any segment is convex.
Any ray is convex.
A circle is not convex.
A circle with its 'interior' is convex.

V. Note: Lines have an infinite number of points.

VI. Note: If line $\overrightarrow{PQ}$ lies in plane E then so does segment $\overline{PQ}$. 
APPENDIX VII

PILOT STUDY TESTS
INSTRUCTIONS: Answer as many questions as you can on the answer sheet. Show all work. When doing a proof, you may informally give statement and reasons; you may use the numbers of the postulates on the attached sheet and need not write out their names in full.

I. Given A, B, C, and D as four distinct points on a line, and given that B-C-D and A-C-D,

(1) must you conclude A-B-C? \( \text{yes} \) \( \text{no} \)
(2) is B-A-D possible? \( \text{yes} \) \( \text{no} \)
(3) is B-A-C possible? \( \text{yes} \) \( \text{no} \)
(4) is A-C-B possible? \( \text{yes} \) \( \text{no} \)

II. Prove that a plane is a convex set. Write out the proof on the paper provided.

III. Given A, B, C, and D as four distinct points in space,

(1) Prove that the least number of planes needed to contain A, B, C, and D is either four or one; what will determine whether it is four or one?
(2) Prove that if the least number of planes needed to contain A, B, C, and D is four, then A, B, and C cannot be collinear.

IV. Given A, B, C, D, and E as five distinct points on a line and that B-C-E, A-B-D, and D-B-C,

(1) must you conclude B-A-C? \( \text{yes} \) \( \text{no} \)
(2) must you conclude D-B-E? \( \text{yes} \) \( \text{no} \)
V. Prove that a half-plane is a convex set.

VI. Prove that two distinct lines $p$ and $q$ cannot intersect in two or more points.

VII. Prove that if a line $p$ intersects a plane $E$ and $p$ does not lie in $E$, then the intersection of $p$ and $E$ is only one point.

VIII. $A$, $B$, and $C$ are three distinct collinear points and $C$ is on ray $\overrightarrow{AB}$. If you intersect $\overrightarrow{AB}$ with $\overrightarrow{CB}$, which of the following (one or more) describe the possibilities for the intersection of $\overrightarrow{AB}$ and $\overrightarrow{CB}$?

1. $\overrightarrow{AB}$ ( )
2. $\overrightarrow{BC}$ ( )
3. $\overrightarrow{AC}$ ( )
4. $\overrightarrow{CB}$ ( )
5. $\overrightarrow{AC}$ ( )
6. $\overrightarrow{AB}$ ( )
7. $\overrightarrow{BC}$ ( )
8. $\overrightarrow{BC}$ ( )
## Postulate List for Test A

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Point, Line, or Space Postulate</th>
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<tr>
<td>L1</td>
<td>For every two points in space, there is exactly one line which contains both points.</td>
</tr>
<tr>
<td>L2</td>
<td>Every plane contains at least three non-collinear points.</td>
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<tr>
<td>L3</td>
<td>Space contains at least four noncoplanar points.</td>
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<tr>
<td>L4</td>
<td>If two points lie in a plane then the line which contains them lies in that plane.</td>
</tr>
<tr>
<td>L5</td>
<td>Any three points lie in at least one plane.</td>
</tr>
<tr>
<td>L6</td>
<td>Any three noncollinear points lie in exactly one plane.</td>
</tr>
<tr>
<td>L7</td>
<td>If two planes intersect, then their intersection is a line.</td>
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### Separation Postulate

| S1           | Given a line and a plane containing it, the points of the plane that do not lie on the line form two sets such that (1) each of the sets is convex, and (2) if P is in one of the sets and Q is in the other, then the segment $PQ$ intersects the line. |

Recalling the definition of two lines being parallel as: two lines are parallel ($p \parallel q$) if (1) $p$ and $q$ are different lines, (2) $p$ and $q$ lie in the same plane, and (3) $p$ and $q$ do not cross, we have the parallel postulate.

### Parallel Postulate

| P1           | Through a given external point there is only one parallel to a given line. |
INSTRUCTIONS: For each of the problems 1-6, draw a line through the number of the correct answer.

I. If you are given ray $\overrightarrow{AB}$ which contains point C (different from A or B), then ray $\overrightarrow{AB}$ intersected with $\overrightarrow{CA}$ is

- (1) the line $\overrightarrow{AB}$
- (2) the point A
- (3) the point B
- (4) the segment $\overrightarrow{AC}$
- (5) the segment $\overrightarrow{CB}$

II. If you intersect ray $\overrightarrow{AB}$ with $\overrightarrow{CB}$, you get

- (1) either $\overrightarrow{AB}$ or $\overrightarrow{BC}$ (not both)
- (2) either $\overrightarrow{AC}$ or $\overrightarrow{CB}$
- (3) either $\overrightarrow{AC}$ or $\overrightarrow{CB}$
- (4) either $\overrightarrow{AB}$ or $\overrightarrow{CB}$
- (5) either $\overrightarrow{BC}$ or $\overrightarrow{CB}$

III. If the answer to $\overrightarrow{AB}$ intersected with $\overrightarrow{CB}$ is $\overrightarrow{CB}$, then

- (1) B is between A and C
- (2) C is between A and B
- (3) A is between B and C
- (4) cannot determine; not enough information

IV. Which of the following (one or more) is true?

- (1) Three points may lie on the same line but not lie in one plane.
- (2) Three points may not necessarily lie on one line but still can lie in one plane.
- (3) Three points must always lie on two or more lines.
- (4) You can always find three lines to contain four points.
V. Which of the following are true?

(1) It can happen that of four points, three can lie on a line and all four can lie in one plane.

(2) There may be no plane which holds all four, and yet any three of them lie in one plane.

(3) A circle may be found which passes through any three points if you cannot find a line to pass through them.

(4) Of any five points chosen in a plane, six lines in that plane are always sufficient to contain them and join every pair of points.

VI. Given a \( \triangle ABC \) and a line \( m \) which intersects \( AB \) (not A or B, however), then

(1) \( m \) must intersect \( C \)

(2) \( m \) must intersect \( AC \) (but not at \( C \))

(3) \( m \) can intersect \( AC \) and \( BC \) (not in \( C \))

(4) \( m \) must intersect \( BC \) if \( BC > AC \).

A, B, C, D, E all lie in one plane. (I shall write \( A-C-E \) to mean \( C \) is between \( A \) and \( E \).)

Answer true or false for each of the following by circling.

VII. \( A-C-E \) and \( B-C-D \) must yield \( A-C-D \) or \( A-C-B \).   T   F

VIII. \( A-C-B \) and \( C-B-D \) must yield \( A-B-D \).   T   F

IX. \( A-C-B \) and \( G-B-D \) and \( E-C-D \) must yield \( A-E-D \).   T   F

X. \( A-B-C \) and \( B-C-D \) must yield \( A-B-D \) and \( A-C-D \).   T   F
APPENDIX VIII

DATA ON CLASS COMPOSITION
Table 6
Past Math Achievement of Control Group

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* indicates subject had taken grade ten geometry previously

** indicates a deferred standing grade
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* indicates subject had taken grade ten geometry previously  
** indicates a deferred standing grade