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TWO DIFFERENT INSTRUCTIONAL PROCEDURES FOR A MULTIPLICATION ALGORITHM AND THEIR TRANSFER EFFECTS TO A HIGHER-ORDER ALGORITHM.

by

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B.Sc., University of British Columbia, 1965

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF ARTS

in the Department

of

Mathematics Education

We accept this thesis as conforming to the

required standard

THE UNIVERSITY OF BRITISH COLUMBIA

May, 1972.

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Abstract

This was a study to determine the effects of two instructional procedures for a multiplication algorithm on the ability of elementary school children to extend this algorithm to the solving of computational tasks involving the use of a higher-order algorithm.

Each of two groups was given preliminary instruction in solving multiplication problems via the application of the distributive law. After this readiness phase was completed, students were randomly assigned to either a T1 or T2 treatment group. The T1 subjects were taught a rote-type standard multiplication algorithm for determining the solution of 2 x 1 and 3 x 1 products. No explicit instruction was given to indicate the relationships between the two learning tasks, viz. the acquisition of the distributive law and the standard multiplication algorithm. Unlike the T1 instructional sequence, the T2 instructional sequence was designed to promote the learning of the relationships between the series of learning tasks. That is, the T2 subjects were taught a standard multiplication algorithm that required the explicit use of the distributive law and other acquired algebraic skills. It was hypothesised that this continual integration of learning tasks would enable the T2 subjects to exhibit superiority over the T1 subjects in extending their standard multiplication algorithm to computational tasks requiring the use of an untaught higher-order algorithm. A total of 238 subjects and 8 teachers were used in all phases of the experiment.

A mixed model of analysis of variance was used to validate the performance hypothesis. It was found that the T1 subjects were significantly better than the T2 subjects in the performance of the standard multiplication algorithm. An analysis of covariance was performed to determine the validity of the transfer hypothesis. A subject's score on the performance test was used as a covariate in order to equate the disparate computational abilities of the T1 and T2 subjects. Although the mean score of the T2 subjects was higher than that of the T1 subjects on the transfer test, this difference was not statistically significant.

TABLE OF CONTENTS

LIST OF TABLES

CHAPTER I

OUTLINE OF THE PROBLEM

INTRODUCTION

Most modern arithmetic programs are in agreement that the field postulates for the system of arithmetic should form an integral part of arithmetic content. Both mathematicians and psychologists have advised that the understanding of many of these postulates be included as elementary school objectives. Participating mathematicians at the Cambridge Conference on School Mathematics stressed that students be familiar with part of the "global structure" of mathematics.[1] They felt that a very solid mathematical superstructure can be erected which will help pupils in more advanced mathematical fields. Although the idea of "global structure" was never clearly defined there is little doubt, after examining their recommendations for curriculum content, that the field postulates formed part of it.[3]

Jerome Bruner, again avoiding the knotty problem of definition, stated "there are at least four general claims that can be made for teaching the fundamental structure of a subject, claims in need of

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[1] Goals for School Mathematics, (New York: Houghton Mifflin, 1963), p. 8.

[2] Ibid., p. 8.

[3] Ibid., p. 36.

detailed study". He listed the following as supportive claims:

1. Understanding fundamentals makes a subject more comprehensible.

2. Unless detail is placed into a structured pattern it is rapidly forgotten.

3. Understanding of fundamental principles and ideas leads to transfer of training.

4. By constantly reexamining material taught in elementary and secondary schools for its fundamental character, one is able to narrow the gap between "advanced" knowledge and "elementary" knowledge.[4]

David Ausubel claims that "precise and integrated understandings are, presumably, more likely to develop if the central, unifying ideas of discipline are learned before more peripheral concepts and information are introduced".[5] In his opinion, "the most significant advances that have occurred in recent years in the teaching of such subjects as mathematics, chemistry, physics and biology are predicated on the assumption that efficient learning and functional retention of ideas and information are largely dependent upon the adequacy of cognitive structure, i.e. upon the adequacy of an individual's existing organization, stability and clarity of knowledge in a particular subject-matter field".[6]

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[4]Jerome S. Bruner, The Process of Education, (New York: Vintage Books, 1963), p. 23-26.

[5]David P. Ausubel, The Psychology of Meaningful Verbal Learning, (New York: Grune and Stratton, 1963), p. 21.

[6]Ibid., p. 26.

In this writer's opinion Ausubel supports the early understanding of the field postulates when he claims that: "the acquisition of adequate cognitive structure, in turn, has been shown to depend upon both substantive and programmatic factors using for organizational and integrative purposes those substantive concepts and principles in a given discipline that have the widest explanatory power, inclusiveness, generalizability, and relatability to the subject-matter content of that discipline".[7]

Although much has been hypothesised about the pedagogical benefits of subject-matter structure, little validation has been attempted. Moreover, those studies that have been concerned with such issues have rarely attempted to offer suitable psychological explanations of the role of subject-matter structure in arithmetic understanding. Assuming that the field postulates form part of mathematical structure, the intent of this study is to provide both plausible psychological explanations and empirical data related to the role of the understanding of these field postulates in promoting arithmetic understandings.

GENERAL STATEMENT OF THE PROBLEM

Since computational algorithms are commonly given logical justification by using the field postulates, it is hypothesised that the learning of the field postulates will facilitate understanding, and through understanding, the learning of such algorithms. More specifically, this study will attempt to determine under what instructional conditions

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[7] Ibid., p. 26.

the understanding of the field postulates promotes ease of extension to untaught computational algorithms. Moreover, an attempt will be made to provide a psychological rationale for the inclusion of these postulates in a contemporary arithmetic program.

DEFINITION OF TERMS

In order to avoid an ambiguous and lengthy statement of hypotheses it was felt necessary to define the following terms:

Algebraic principles. These are also referred to as field axioms, field principles, and field postulates. In this study the subset of field postulates with which we are concerned is the set of postulates that apply to the whole numbers.

Algorithm. Any rule or ordered set of procedures that can be used to produce a correct solution to a computational task independent of the individual using that algorithm; for example, the usual column addition algorithm.

Internal algorithm. Any algorithm whose primary function is that it is used in the generation of other algorithms. It is internal in the sense that it is considered a means to an end rather than an end in itself. That is, its prime instructional purpose is to serve as an algebraic prerequisite for more complex computational algorithms. The writer will use the term for mainly referential purposes and will not attribute any special psychological properties to internal algorithms. The internal algorithm used in this study is the annexation algorithm; the reader should examine the T2 Instructional Sequence on page 9 for an explanation of this algorithm. Appendix A describes another internal algorithm.

Standard multiplication algorithm. For the purposes of this study the standard multiplication algorithm will refer to those procedures used to compute products such as axb where either a or b has a one digit numeral and the other has a two or three digit numeral. For example:

$$\begin{array}{r} 12 \\ \underline{\times\ 9} \end{array} \qquad \begin{array}{r} 132 \\ \underline{\times\ 9} \end{array} \qquad \begin{array}{r} 7 \\ \underline{\times\ 18} \end{array} \qquad \begin{array}{r} 6 \\ \underline{\times\ 132} \end{array}$$

Hereafter such products will be referred to as 2x1 and 3x1 products.

Higher-order algorithm. For the purposes of this study a "higher-order algorithm" will refer to an algorithm used to compute products such as axb where neither a nor b has a one digit numeral and where either a or b may have more than two digits in the numeral. For example:

$$\begin{array}{r} 1001 \\ \underline{\times\ 77} \end{array} \qquad \begin{array}{r} 132 \\ \underline{\times\ 111} \end{array} \qquad \begin{array}{r} 12 \\ \underline{\times\ 1002} \end{array}$$

These algorithms are "higher" in the sense that the standard multiplication algorithm must be conceptually modified in order to compute novel products. Further elaboration is given later in the chapter.

Performance tasks. This refers to those tasks requiring the application of the standard multiplication algorithm. Level of performance was measured by a written test described in Chapter III.

Transfer tasks. A solution of a transfer task required the successful extension of the previously taught standard multiplication algorithm. Level of transfer was measured by a written test described in Chapter III.

T1 group. Those students who completed the T1 Instructional Sequence. The reader is referred to page 9 for details of this sequence.

T2 group. Those students who completed the T2 Instructional Sequence. The reader is referred to page 9 for details of the sequence.

DISCUSSION AND SIGNIFICANCE OF THE PROBLEM

The role of algorithms in arithmetic programs has changed considerably over the past twenty years. Previously, considerable instructional time was devoted to increasing a student's proficiency with an algorithm rather than his understanding of that algorithm. Arithmetic content was treated as if it were a series of logically unrelated algorithmic tasks rather than an integrated set of relationships between relatively simple concepts. With advances in technology less stress has been placed on mere performance of computational algorithms, although computational algorithms still form the main substance of most modern arithmetic programs. Thus, the modern curriculum developer has been primarily concerned that children understand the rationale of an algorithm; i.e. concerned about the ability of children to explain the relationships between the algorithm and other previously acquired algebraic principles.

Since computational algorithms are logically related to the properties of place value systems and the field principles, it has frequently been claimed by some mathematics educators that these logical relationships enhance the understanding of computational algorithms. Eric MacPherson expresses this view when he states, "the child who understands arithmetic is the child who sees how each algorithm follows from these principles".[8] It would be erroneous to conclude from such statements that children who understand the field principles are able to derive spontaneously the usual standardized computational algorithms. Rather such views imply that when a child understands the role of the principle in an algorithm, (e.g. recognizes an instance of the principle in an algorithm, demonstrates that a 'step' in an algorithm is another application of some previously learned principle, etc.) he is more likely to understand the rationale of other related algorithms. However, what seems to be lacking in the arguments of "structure advocates" is a reasonable psychological interpretation of the role of subject-matter structure in effecting understanding. More specifically, in what sense does understanding of the role of the field postulates in specific algorithms promote ease of extension to untaught related algorithms? For the purposes of this study, it would seem that of the many learning psychologists, David P. Ausubel and Robert M. Gagné are two whose views seem particularly relevant.

In order to demonstrate the relevance of these psychological views to this study, it is necessary to refer constantly to specific instructional sequences used in this study. Hence it seems appropriate, first, to explain the nature of these instructional sequences. The

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[8] Eric D. MacPherson, "The Foundations of Elementary School Mathematics", Modern Instructor, Volume 33 (October 1964), p. 70.

reader is referred to Figure 1. on page 9 for a diagrammatic explanation of these sequences.

A. The T1 Instructional Sequence

This sequence is typical of many that occur in modern textbooks. The first skill taught in this sequence is use of the distributive law. A child is assumed to understand the distributive law when he can:

a) use the distributive law to solve such algebraic expressions as

$$9 \times 5 = (9 \times 3) + (9 \times 1) + (9 \times \square)$$

$$8 \times 6 = (2 \times 6) + (2 \times 6) + (\triangle \times 6)$$

$$9 \times 7 = (6 + 3) \times (5 + 2) = (6 \times 5) + (6 \times \triangle) + (3 \times \square) + (3 \times 2)$$

$$(2 \times 6) + (2 \times 6) + (\square \times 6) = 8 \times 6$$

$$(8 \times 3) + (8 \times 5) + (8 \times 1) = 8 \times \square$$

b) compute products such as 9 x 8 by application of the distributive law:

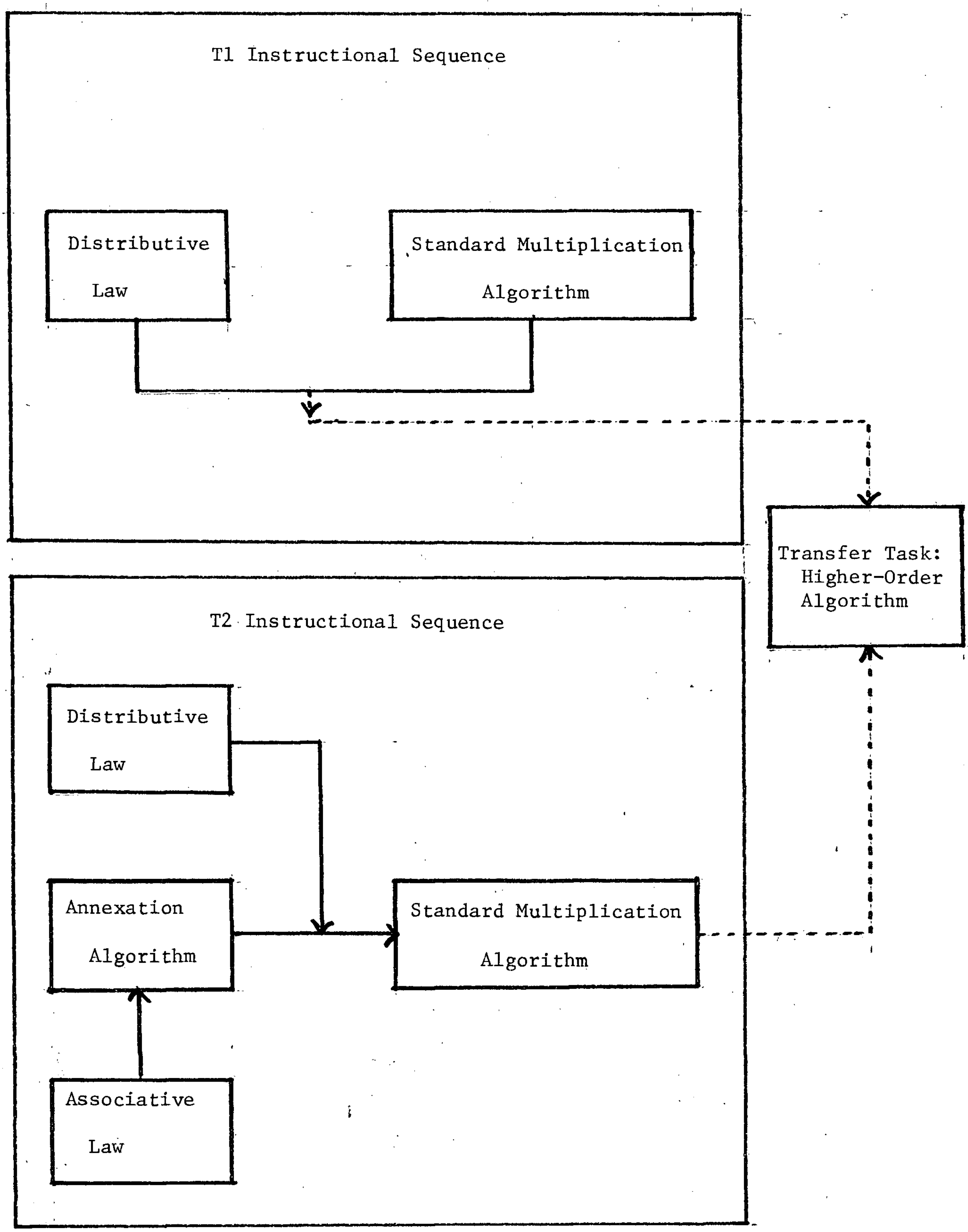
$$9 \times 8 = 9 \times (2 + 6) = 18 + 54 = 72$$

$$9 \times 8 = 9 \times (2 + 5 + 1) = 18 + 45 + 9 = 72$$

$$9 \times 8 = (4 + 5) \times (2 + 6) = 8 + 24 + 10 + 30 = 72$$

The next objective in the sequence is the acquisition of a rote-type standard multiplication algorithm, the algorithm is being considered to be rote-type in the sense that no attempt is made explicitly to indicate the relationships between the previously mastered skill and this algorithm.

FIGURE 1. THE INSTRUCTIONAL SEQUENCES



B. <u>The T2 Instructional Sequence</u>

As with the T1 sequence, the T2 instructional sequence incorporates the understanding of the distributive law as an initial learning objective. However, additional algebraic skills are also considered necessary. These skills involve the use of the associative law and an internal algorithm, in this case the annexation algorithm. The child is taught to compute products in which 10 is a factor by "annexing the zeros". For example the product of 7 x 200 is initially computed by using the associative law in the following manner:

$$7 \times 200 = 7 \times (2 \times 100) = (7 \times 2) \times 100 = 14 \times 100 = 1400$$

or 7×2 hundreds $= (7 \times 2)$ hundreds $= 14$ hundreds $= 1400$

Later computation simply involves direct annexation. For example,

$$\underline{7} \times \underline{2}\,00 = \underline{14}\,00$$

The standard multiplication algorithm utilized in this sequence validates procedural "steps" by explicitly pointing out instances of the prior learned skills.

This writer is primarily interested in the effects of each instructional sequences on the amount of transfer to computational tasks that involve an untaught higher-order algorithm. As mentioned earlier in this chapter, the views of Gagné and Ausubel would seem to provide possible explanations of these transfer differences.

Gagné has developed what he considers a hierarchy-of-learning model.[9] Before a specified learning task can be mastered, Gagné would

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[9] Robert M. Gagné, <u>The Conditions of Learning</u>, (New York: Holt, Rinehart and Winston, 1970).

claim that a number of subordinate concepts must also be mastered. These concepts in turn depend upon other subordinate concepts so that it can be argued that Gagné's model ultimately resembles that of S→R learning. As Gagné explains,

> when such an analysis (selecting appropriate prerequisite tasks) is continued progressively to the point of delineating an entire set of capabilities having an order relation to each other (in the sense that in each case prerequisite capabilities are represented as subordinate in position, indicating they need to be previously learned), one has a learning hierarchy. The analytic process may be carried out if desired, until the simplest kinds of learnings (Ss→R's, chains, discriminations) are reached and identified.[10]

Thus once the terminal task is clearly specified, the problem is to select hypothesised prerequisites and arrange these in a hierarchical manner. Although initially these prerequisites are selected logically on an a priori basis, a hypothesised prerequisite is concluded to be pedagogically necessary only after empirical investigation. As Gagné explains: "a subordinate skill is determined to be pedagogically necessary if it facilitates the learning of the higher-level skill to which it is related. In contrast, if the subordinate skill has not been previously mastered, there will be no facilitation of the higher-level skill. This latter condition does not mean that the higher-level skill cannot be learned -- only that, on the average, in the group of students for whom a topic sequence has been designed, learning will not be accomplished readily".[11] Thus if transfer differences between the T1 and T2 groups were observed, Gagné, rather than trying

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[10] Gagne, op. cit., p. 238.

[11] Ibid., p. 239-240.

to explain the differences in terms of any particular learning theory, would probably attribute these differences to the selections and arrangement of prerequisites, since he seems to be more concerned with the development of empirically validated hierarchies than the validation of contemporary psychological theories. Hence this study could prove to be valuable for the curriculum designer if it produced a more effective instructional sequence for teaching initial multiplication skills.

Ausubel would view the potential efficiency of each instructional sequence for promoting transfer in quite a different sense than would Gagné. For Ausubel, the amount of transfer brought to a learning task depends on an individual's cognitive structure, where "cognitive structure" means an individual's organization, stability, and clarity of knowledge in a particular subject-matter field at any given time.[12] That is existing cognitive structure is regarded as the major factor influencing the learning and retention of potentially meaningful new material in the same field. According to Ausubel, a major criterion determining whether learning material is potentially meaningful is its relatability to the particular cognitive structure of a particular learner. As Ausubel states:

> for meaningful learning to occur in fact, it is not sufficient that the new material simply be relatable to relevant ideas in the abstract sense of the term. The cognitive structure of the particular learner must include a requisite intellectual capacities, ideational content and experiential background.[13]

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[12] Ausubel, op. cit., p. 26.

[13] Ibid., p. 23.

The key concern of this study is the effect of these instructional sequences on cognitive structure. That is, which of the T1 and T2 sequences might be best integrated by the learner and in what sense this act of integration promotes greater transfer to tasks requiring the use of an untaught higher-order algorithm.

According to Ausubel, new learning is sometimes incorporated into cognitive structure by correlative subsumption.[14] This psychological phenomenon occurs when a learner somehow determines that new learning material is related to relevant cognitive subsumers via some general principle. Thus new learning material may be best incorporated into an individual's cognitive structure if those principles which require the least extension act as subsumers. In Ausubel's terms one might suppose that the learning of the algebraic principles of arithmetic may affect the learning of logically related computational algorithms in the same sense as 'advance organizers'. Thus it is hypothesised that the T2 instructional tasks might form relatively stronger subsumers than the T1 tasks, for future transfer tasks requiring the use of a higher-order algorithm.

For example, consider the possible differing complexity of extension from the standard multiplication algorithm to the higher-order algorithm that each treatment group must make for successful solution of such a transfer task as 107 x 11.

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[14] Ausubel, op. cit., p. 77.

A typical solution that might be exhibited by the T2 group could be as follows:

$$\begin{array}{r} 11 \\ \underline{\times\ 107} \end{array} \longrightarrow \begin{array}{r} 10 + 1 \\ \underline{100 + 7} \end{array} \quad \left\{ \begin{array}{l} \text{partitioning both factors into} \\ \text{binary sums involving powers of ten} \end{array} \right.$$

$$\longrightarrow \begin{array}{r} 10 + 1 \\ \underline{100 + 7} \\ 7 \\ 70 \\ 100 \\ \underline{1000} \end{array} \quad \left\{ \begin{array}{l} \text{application of distributive} \\ \text{law and annexation algorithm} \end{array} \right.$$

It is assumed that no 'new' concept or skill is required for successful extension from the standard multiplication algorithm to the higher-order algorithm. (The skill of partitioning both factors, rather than just one factor, before application of the distributive principle was included in both instructional sequences.)

The extension of the rote-type standard multiplication algorithm to the standardized higher order algorithm by the T1 procedure seems a very remote possibility:

$$\begin{array}{r} 11 \\ \underline{\times\ 107} \end{array} \longrightarrow \begin{array}{r} 11 \\ \underline{\times\ 107} \\ 107 \\ \underline{107\ \ } \end{array} \quad \left\{ \begin{array}{l} \text{"move over one space to the left} \\ \text{when multiplying by a factor of ten"} \end{array} \right.$$

Suppose a T1 group member attempts to compute such products as 107 x 11 by considering the 11 as 'one digit' and proceeds as with the standard multiplication algorithm:

$$\begin{array}{r} {}^{10}11 \\ \underline{\times\ 107} \\ 7 \end{array} \qquad 107 \times 1 = 107$$

as with standard multiplication algorithm, place 'seven' and 'carry ten'

$$\begin{array}{r} {}^{10}11 \\ \underline{\times\ 107} \\ 1177 \end{array} \qquad (107 \times 1) + 10 = 117$$

place 'seven' and 'carry 11'

Although such partial products as 7 x 11 could be computed by using the T1 standard multiplication algorithm, this extended procedure seems much more difficult than the hypothesised T2 procedure. Transfer to tasks involving 3 x 3 and 4 x 3 products would seem even more unlikely considering the complexity of extending the T1 standard algorithm.

HYPOTHESES

Most textbooks and practitioners are being urged by curriculum specialists to promote the understanding of algebraic principles. The arguments for the inclusion of such principles are based on the belief that much of arithmetic, and especially computational algorithms, may be better understood through the learning of algebraic principles. Hence from both a practical and a theoretical point of view, it seems worthwhile to investigate the validity of the following hypotheses:

<u>Hypothesis One</u> -- The T1 group will score significantly higher than the T2 group in the performance of the standard multiplication algorithm, as measured by the performance test.

<u>Hypothesis Two</u> -- The T2 group will score significantly higher than the T1 group on the test of transfer from the standard multiplication algorithm to a higher-order algorithm.

CHAPTER II

SURVEY OF THE LITERATURE

In reviewing the literature, one soon realizes that very few studies have been concerned with children's acquisition or use of the field of postulates to generate algorithms.

Children's understanding of the field postulates without formal instruction was studied by Crawford in 1964.[15] Using a multiple choice test of 45 items, he tested each of the eleven field axioms once at each level of Bloom's taxonomy. He found that the mean scores increased significantly, from one even numbered grade to the next, in a linear manner. Students exposed to 'modern mathematics' content in grades 9 and 10 had scores significantly superior to those of students in all other programs at the same level. This study seems important in that it provides data on developmental processes which were occuring without explicit teaching.

A study by Hall attempted to determine whether the rote learning of certain multiplication combinations could be accomplished more effectively through teaching procedures emphasising the commutative and ordered pair approach in conjunction with practice on related combinations.[16] This procedure was compared to teaching procedures

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[15] Douglas Crawford, "An Investigation of Age-Grade Trends in Understanding the Field Axioms," Dissertation Abstracts, Syracuse University, 1964.

[16] Kenneth D. Hall, "An Experimental Study of Two Methods of Instruction for Mastering Multiplication Facts at the Third-Grade Level," Doctoral Dissertation, Duke University, 1967.

employing the traditional approach with practice on commuted combinations. He found no significant difference between the groups on both arithmetic computation and achievement in multiplication. This result lends support to the notion that there is no advantage in the mere acquisition of a field postulate.

Gray, in 1964, tried to determine how a method of teaching introductory multiplication which stressed development of an understanding of the distributive law would relate to pupil development as measured in terms of achievement, transfer, retention and progress toward maturity of understanding of multiplication.[17] He used two treatment groups. One group, T1 was taught according to what was judged to be the best of current methods. The other group, T2, was provided with introductory multiplication using an understanding of the distributive principle. Pre-experimental achievement and I.Q. were covaried. He constructed written pre-test, post-test, retention, and transfer tests. Individual interviews of 110 random subjects measured maturity of understanding. His results warranted the following conclusions:

1. A program of arithmetic instruction which introduces multiplication by a method which stressed understanding of the distributive property produced results superior to those of current methods.

2. Understanding of the distributive property enables children to proceed independently to the finding of products of

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[17]Roland F. Gray, "An Experimental Study of Introductory Multiplication," Doctoral Dissertation, University of California, Berkeley, 1964.

novel multiplication combinations, to a greater extent than those children not introduced to the distributive principle.

3. These children appeared not to develop an understanding of the distributive property unless it was specifically taught.

There have been relatively few studies which have been concerned with the relationship between understanding of the field postulates and learning of computational algorithms. In most studies the algorithms were illustrated using physical devices. However, Schrankler tried to evaluate the effectiveness of two pre-algorithm treatments in combination with two algorithms for teaching the multiplication of whole numbers at three intelligence levels.[18] Effectiveness was evaluated in terms of computational skills, speed in computation, understanding of the multiplication process, problem solving and retention of the four previous criteria. The readiness phase placed emphasis on the 100 multiplication facts for group B_1. Emphasis was placed on the commutative, associative and distributive properties for group B_2. Following this period, these groups were subdivided into algorithmic groups. Group A_1 subjects were taught the indent unit-skills algorithm:

$$\begin{array}{r} 57 \\ \underline{\times\ 28} \\ 456 \\ \underline{114} \\ \underline{1596} \end{array}$$

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[18] William Schrankler, "A Study of the Effectiveness of Four Methods of Teaching Multiplication of Whole Numbers in Grade Four," Dissertation Abstracts, University of Minnesota, 1966.

Group A_2 subjects were taught the partial products algorithm:

$$\begin{array}{r} 57 \\ \underline{\times\ 28} \\ 56 \\ 400 \\ 140 \\ \underline{1000} \\ \underline{1596} \end{array}$$

No mention was made of the use of the annexation algorithm in the partial products algorithm. Students in each of the treatment groups, A_1B_1, A_2B_1, A_1B_2, A_2B_2, were identified at one of three levels of intelligence. Schrankler found that the A_2B_2 group tested higher on the test of understanding than the other groups. This same group also tested higher on the retention test of understanding. The fact that the A_2B_2 group was found to be superior to the A_1B_2 group on the test of understanding of the multiplication algorithm is of particular interest. This result suggests that the understanding of computational algorithms is best promoted by the _explicit_ application of previously acquired algebraic principles. Studies such as Schrankler's have been restricted to examining the use of algebraic principles in promoting understanding of already _acquired_ computational algorithms. No studies were found which examined the use of algebraic principles in promoting transfer to untaught higher-order algorithms.

CHAPTER III

DESIGN OF THE EXPERIMENT

THE SAMPLE

The experimenter decided to use grade three students as subjects in the study since they had had some experience with multiplication but had not as yet been taught the standard multiplication algorithm. Eight grade three classes were selected from six British Columbia schools. All eight of the teachers involved in the study were volunteers.

After the readiness phase, which will be described in the next section, students in each classroom were randomly assigned to either the T1 or T2 group. A student's test scores were omitted from the study if more than one treatment lesson was missed. A total of 238 subjects were used to obtain the final set of data; 44 subjects were used for test analysis, and the remaining 194 subjects for testing the hypotheses.

THE INSTRUCTIONAL SEQUENCES

A. The Readiness Phase.

During this phase, all the subjects were taught the skills which were considered to be prerequisites for the treatment phase.

A set of lesson plans was provided for each teacher involved in the study. Briefly, these lessons stressed:

- the relationship between multiplication and arrays.

  For example,

  3 x 4 means a "3 by 4" array

- the distributive law; both the right hand and the left hand. This was to be accomplished by breaking arrays into the "sums" of smaller arrays. For example:

```
        4 x 5          4 x  (2 + 3)     (4 x 2)  +  (4 x 3)

     x x x x x         x x   x x x        x x        x x x
     x x x x x         x x   x x x        x x        x x x
     x x x x x         x x   x x x        x x        x x x
     x x x x x         x x   x x x        x x        x x x
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- the application of the distributive law to multiplication problems.

Only the techniques of breaking a product into the sums of smaller products was stressed and no attempt was made to have children provide a final numerical answer. For example,

```
     28        20 +  8        20 +  8
   x 19           x 19           x 19
   ----        -------        -------
                               19 x  8
                             + 19 x 20
                             ---------
```

For a full description of these lessons, the reader is referred to Appendix B.

In order to parallel typical teaching practices and thus increase the generalizability of this study, the writer did not demand a fixed criterion of mastery of the distributive law. Rather, all teachers were instructed to terminate this phase when, in <u>their</u> judgement, the students indicated a mastery of the distributive law.

The teachers reported that this phase generally took about five hours of classroom instruction.

B. The Treatment Phase.

Every teacher was provided with a set of written lesson plans suitable for each treatment lesson. The lessons contained the general dialogue, examples and seatwork to be used. The teachers met with the writer twice during this phase to ensure that they understood the lesson materials. To minimize the effect of teacher differences each teacher taught both groups within her class. To minimize pupil interaction, it was arranged to have the groups separated during a treatment lesson. All pupils were told by their teacher they they were involved in an experiment. To minimize outside influences, teachers were instructed to give no homework during this phase. Both the T1 and T2 groups had approximately four hours of treatment time. A brief description of both treatments is provided in the following section but the reader is referred to Appendix C for the lesson plans used.

The T1 Instructional Sequence

The T1 group was taught the rote-type algorithm described in Chapter I. The algorithm was restricted to 2 x 1 and 3 x 1 products. To convince the students of the legitimacy of this algorithm, all answers were initially checked using the distributive law. For example, the check might be made as follows:

$$\begin{array}{r} 12 \\ \underline{\times\ 9} \end{array} \longrightarrow \begin{array}{r} 2 + 2 + 2 + 3 + 3 \\ \underline{\times\ 9} \end{array} \longrightarrow 18 + 18 + 18 + 27 + 27 = 108$$

In contrast to the T2 Instructional Sequence, no explicit application of the distributive principle was stressed. Once the students were convinced that this rote-type algorithm yielded correct products, the objective of the succeeding lessons was merely to provide further practice.

The T2 Instruction Sequence

The T2 subjects were first taught the annexation algorithm. All computation of 2 x 1 and 3 x 1 products were accomplished by using the distributive principle in conjunction with the annexation algorithm. The teachers were instructed to use the same examples and seatwork with both groups.

THE MEASURING INSTRUMENTS

Both the performance and transfer tests were written tests constructed by the experimenter. The teachers knew the general nature of each test prior to the treatment phase but did not see the actual test items until the test administration date. Teachers were instructed to give students ample time to complete both tests. Any solution by repeated addition was disregarded for both tests. The reader is referred to Appendix D for the actual tests used.

A. Performance Test.

This test consisted of twenty items that required the use of the standard multiplication algorithm. The total number of correct responses was considered a measure of an individual's performance. In order to delete items that were either excessively difficult or easy, a point biserial correlation was calculated for every item. It was decided to reject an item if the point biserial r was less than .20 in magnitude.[19] As a result of this analysis, all items of the original test were retained. Since this test was designed to measure a very specific trait, (viz. the ability to use the standard multiplication algorithm), it was felt that a measure of item homogeneity should be determined. Thus a KR20 was calculated for the twenty item test and was found to be .93. This value indicated that the performance test was high in item homogeneity. The results of the items analysis can be found in Table I.

B. Transfer Test.

This test consisted of fourteen items which were intended to measure the ability to compute novel products requiring the use of a higher-order algorithm. Neither the T1 group nor the T2 group had been previously exposed to any of these items. The total number of correct responses was considered a measure of the ability to extend the standard multiplication algorithm. As with the performance test,

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[19] Nunnally, J.C., Psychometric Theory, (New York: McGraw-Hill Book Company, 1967), p. 242.

a KR20 was calculated to evaluate item homogeneity. The KR20 of the final fourteen item test was found to be .78. It is possible that the KR20 might have been increased in magnitude by including additional test items. However, this lengthening procedure was felt to be inappropriate since a very lengthy test might have had the undesirable effect of increasing test anxiety of such young and 'test immature' students. The results of the item analysis can be found in Table II.

TABLE I

ANALYSIS OF THE PERFORMANCE TEST

Item Number	Point Biserial	Item Number	Point Biserial
1	.45	11	.53
2	.61	12	.73
3	.50	13	.74
4	.77	14	.72
5	.64	15	.49
6	.75	16	.67
7	.74	17	.72
8	.70	18	.64
9	.54	19	.74
10	.64	20	.67

TABLE II

ANALYSIS OF THE TRANSFER TEST

Item Number	Point Biserial	Item Number	Point Biserial
1	.50	13	0.0 *
2	.59	14	0.0 *
3	.43	15	0.0 *
4	.59	16	0.0 *
5	.65	17	0.0 *
6	.65	18	0.0 *
7	0.0 *	19	0.0 *
8	.56	20	.24
9	.46	21	.47
10	.43	22	0.0 *
11	.65	23	.69
12	.51		

*Deleted items.

CHAPTER IV

ANALYSIS OF THE DATA

EXPERIMENTAL RUN

A. The Performance Hypothesis.

The statistical hypotheses to be tested were:

H_0: There will be no significant differences between the means of the T1 and T2 groups as measured by the performance test.

That is: $\mu_{T_1} = \mu_{T_2}$

H_1: The mean of the T1 group will be significantly greater than the mean of the T_2 group as measured by the performance test.

That is: $\mu_{T_1} > \mu_{T_2}$

Each classroom teacher taught both the T1 and T2 groups in her classroom. Thus a subject in a classroom was given either the T1 or T2 instruction by his or her usual classroom teacher. The experimenter considered the differences in teacher performance to be a random effect, while difference in treatment were considered to be a fixed effect. In other words, a mixed analysis of variance model was felt to be the most appropriate statistical model to test the hypothesis. The linear model chosen was:

$Y_{ijk} = \mu + \alpha_i + \tau_j + (\alpha\tau)_{ij} + \varepsilon_{ijk}$; where

α_i indicates the ith teacher (random effect) and

τ_j represents the jth treatment level (fixed effect).

The experimenter made the usual assumptions underlying an ANOVA but did not test for these as the F test is reasonably robust to violations of these assumptions.[20] The assumptions made were:

a) the teachers, used in the experiment, were randomly selected from a normal population. i.e.

α_i are NID $(0, \sigma_\alpha^2)$ and

$(\alpha\tau)_{ij}$ are NID $(0, \sigma_{\alpha\tau}^2)$,

b) the ε_{ijk} are normally distributed, i.e.

ε_{ijk} are NID $(0, \sigma_\varepsilon^2)$,

c) the treatment variances are homogeneous, i.e.

$\sigma^2_{T1} = \sigma^2_{T2}$

The reader is reminded that the denominator in the test for treatment (fixed) effects in a mixed model is the interaction term and not the usual error term.[21] The null hypothesis was considered to be rejected if the probability of obtaining an F value, under the null hypothesis, was less than or equal to $\alpha = .05$. All calculations were done at the University of British Columbia Computer Centre using the BMD-X64 program. This program allows for differing numbers of subjects in a cell by using the least squares estimate technique. The results of this analysis are summarized in Table III.

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[20] Lindquist, E.F., Design and Analysis of Experiments in Psychology and Education (Boston: Houghton, Mifflin Company, 1953), pp. 78-90.

[21] William L. Hays, Statistics (New York: Holt, Rinehart and Winston, 1963), pp. 433-444.

TABLE III

ANALYSIS OF VARIANCE: PERFORMANCE HYPOTHESIS

Source of Variation	df	Sum of Squares	Mean Squares	F	P
Teacher	7	812.197	116.028	5.258	
Treatment	1	1775.630	1775.630	215.491	.0000083
Interaction	7	57.679	8.240	0.373	
Error	178	3927.752	22.066		

Mean for T1 group was 14.418
Mean for T2 group was 8.330

Since the probability of obtaining an F-value of 215.491 was calculated to be far less than .05, the null hypothesis H_0 was rejected and the alternate hypothesis H_1 was accepted.

B. <u>The Transfer Hypothesis</u>.

The statistical hypotheses to be tested were:

H_0: There will be no significant differences between the means of the T1 and T2 group as measured by the transfer test.

That is: $\mu_{T1} = \mu_{T2}$

H_1: The mean of the T2 group will be greater than the mean of the T1 group as measured by the transfer test.

That is: $\mu_{T2} > \mu_{T1}$

Originally the experimenter had hoped to terminate the treatment phase only when both groups had reached a specified performance criterion. That is, until there were no significant differences between the two groups on the performance of the standard multiplication algorithm. Thus, if any degree of correlation existed between the performance and transfer tasks, this preliminary equating would minimize any differences between the groups on the transfer test that might be a result of differences between the means on the performance tasks. However, to bring about the equality of the groups on the performance test, performance scores were covaried with transfer scores. Thus the linear model used to test the transfer hypothesis was:

$$Y_{ijk} = \mu + \alpha_i + \tau_j + (\alpha\tau)_{ij} + \beta'_w (\bar{X}... - X_{ijk}) + \varepsilon_{ijk}$$

where

α_i, τ_j, $(\alpha\tau)_{ij}$ and ε_{ijk} were previously defined;

β'_w is an estimate of the common population regression coefficient;

X_{ijk} is a subject's performance score and

$\bar{X}...$ is the grand mean of the total sample on the performance test.

In addition to the necessary assumptions underlying an ANOVA that were discussed in the previous section, the use of this model necessitates the following additional assumptions:

a) the population within-cell regression coefficients are homogeneous, i.e.

$$\beta'_w = \beta'_{w_{ij}}, \text{ for all ij.}$$

Because little is known about the F test with respect to violation of the foregoing assumption, it was decided to test this assumption at a level of significance equal to .10. Using the BMD-X82 computer program, which adjusts for differing numbers of subjects in a cell, an F of 1.06, with a numerator and denominator of 15 and 162 degrees of freedom respectively, was obtained. Since the probability of obtaining such an F, under the null hypothesis is .398, homogeneity of the regression coefficients was assumed.

b) the pooled estimate β'_w is not zero.

In testing this assumption at the .05 level of significance, an F of 99.14, with a numerator and denominator of 1 and 177 degrees of freedom respectively, was obtained. Since the probability of obtaining such an F, under the null hypothesis, is less than 10^{-8} the hypothesis of zero slope was easily rejected.

The results of the statistical analysis of the transfer hypothesis are summarized in Table IV.

TABLE IV

ANALYSIS OF COVARIANCE: TRANSFER HYPOTHESIS

Source of Variation	df	Adjusted Sum Square	Mean Square	F	P
Teacher	7	179.984	25.712	3.961	
Treatment	1	24.363	24.363	2.725	.141
Interaction	7	62.570	8.939	1.377	
Error	177	1149.077	6.492		

Adjusted mean for T1 group was 3.776
Adjusted mean for T2 group was 4.544

Since the probability of obtaining an F of 2.725 is .141, the null hypothesis was accepted. That is, the mean of the T2 group was higher, but not significantly higher, than the mean of the T1 group.

CHAPTER V

CONCLUSIONS AND IMPLICATIONS FOR FURTHER STUDY

DISCUSSION OF CONCLUSIONS

A. Performance Hypothesis.

With respect to the performance hypothesis, it was found that subjects taught a rote-type algorithm did significantly better on tasks requiring the use of a standard multiplication algorithm than did the subjects taught a standard multiplication algorithm using previously learned algebraic principles. In fact, the performance level of the T2 group was so inferior to that of the T1 group that this researcher suspected that one of the teachers had not followed the recommended treatment procedures. It was quite possible that, since most teachers had never used an instructional sequence like the T2 sequence, they may have had an experimental bias towards the rote T1 sequence. Perhaps more frequent observations of teacher performance would have eliminated such a bias towards treatment.

B. Transfer Hypothesis.

With respect to the transfer hypothesis, it was found that subjects taught a standard multiplication algorithm using algebraic principles appeared to exhibit superior positive transfer to tasks requiring the use of a higher-order algorithm. However, this difference in the amount of transfer was not statistically significant at the α = .05 level of significance.

Because of the nature of the treatments, a T2 subject needed more time fo format correctly a computational problem than did a T1 subject. Thus, teachers were instructed to give students at least one hour to attempt all fourteen items of the transfer test. However, after a brief discussion with the teachers, it was noted that some had allowed students about thirty minutes to complete this test. In fact, one teacher who obviously misunderstood the intent of the transfer test, stated that she gave children about fifteen minutes on this test because "the students weren't taught to compute such large products". This situation could not be remedied by another test administration because school holidays immediately followed the test administration date.

PROBLEMS FOR FURTHER STUDY

Since the results of this study must remain inconclusive because of important uncontrolled factors, a replicate study employing controls to minimize teacher misunderstandings should be conducted.

This writer also suggests that a study be conducted to examine the effect of instructional sequences that use algebraic principles to teach computational algorithms on a student's attitude toward arithmetic. It is postulated that instructional sequences that maximize the use of previously learned algebraic principles may enable a student to view arithmetic as a series of integrated tasks. This integrated view of arithmetic, might, in turn, have a positive effect on a student's attitude towards arithmetic.

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APPENDIX A

THE IDENTIFICATION OF ANOTHER INTERNAL ALGORITHM

THE DIVISION OF FRACTIONS ALGORITHM

AND THE EQUAL FACTORS ALGORITHM

In addition to the mere rote performance of an algorithm, most modern programs attempt to provide some rationale of that algorithm. Perhaps the most difficult algorithm to explain reasonably to the average elementary school child is the division of fractions algorithm. In an attempt to provide this rationale, a typical approach is as follows:[22]

Step 1	$5\frac{1}{2} \div \frac{3}{4} = W$	The work below shows how to divide $5\frac{1}{2}$ by $\frac{3}{4}$. Use $\frac{11}{2}$ as another name for $5\frac{1}{2}$.
Step 2	$\dfrac{\frac{11}{2}}{\frac{3}{4}}$	Express the division in this way.
Step 3	$\dfrac{\frac{11}{2}}{\frac{3}{4} \times \frac{4}{3}}$	First you need to get 1 for the divisor, you multiply $\frac{3}{4} \times \frac{4}{3}$ to get 1.
Step 4	$\dfrac{\frac{11}{2} \times \frac{4}{3}}{\frac{3}{4} \times \frac{4}{3}}$	$\frac{3}{4}$ has been multiplied by $\frac{4}{3}$. So you must also multiply $\frac{11}{2} \times \frac{4}{3}$.
	$\dfrac{\frac{11}{2} \times \frac{4}{3}}{1}$	You do not need to write the divisor when it is 1.
	$\frac{11}{2} \times \frac{4}{3}$	So now you can write the computation in this way
Step 5	$\frac{11}{2} \times \frac{4}{3} = \frac{44}{6} = 7\frac{1}{3}$	You found $7\frac{1}{3}$ by multiplying $\frac{4}{3}$ by $\frac{11}{2}$
	$5\frac{1}{2} \div \frac{3}{4} = 7\frac{1}{3}$	

[22]Maurice L. Hartung, et al. Seeing Through Arithmetic 6, Scott, Foresman and Co., Chicago, p. 198.

One apparent assumption that has been made is that the procedures taken in Steps 3 and 4 can be followed by the elementary school child. However, the validity of these two steps must be blindly accepted by the child since no preliminary work has been done that could be used to justify these steps. One wonders what advantages this modern treatment has over the rote "invert and multiply" algorithm because apparently we have merely substituted a long <u>rote</u> algorithm for a short <u>rote</u> algorithm.

What is needed to validate steps 3 and 4 is an internal algorithm; the equal factors algorithm. This algorithm states that if the divisor and dividend are multiplied or divided by any non-zero rational number, the quotient remains unchanged. For example:

$$(8 \div 4) = (2 \times 8) \div (2 \times 4)$$

If this internal algorithm is mastered, the division of fractions becomes much more reasonable to the elementary school child.

Step 1	$\frac{11}{2} \div \frac{3}{4} = (\frac{11}{2} \times \frac{4}{3}) \div (\frac{3}{4} \times \frac{4}{3})$	equal factors algorithm
Step 2	$(\frac{11}{2} \times \frac{4}{3}) \div (\frac{3}{4} \times \frac{4}{3}) = (\frac{11}{2} \times \frac{4}{3}) \div 1$	multiplication of reciprocals
Step 3	$(\frac{11}{2} \times \frac{4}{3}) \div 1 = \frac{11}{2} \times \frac{4}{3}$	property of one
Step 4	$\frac{11}{2} \div \frac{3}{3} = \frac{11}{2} \times \frac{4}{3}.$	

APPENDIX B

READINESS PHASE LESSON PLANS

THE READINESS PHASE

These three lessons should enable most students to acquire the necessary prerequisite skills before the actual experimental treatment begins. The teacher will find that all lesson plans are quite detailed including examples to use, questions to ask, answers one can expect, and seatwork problems to be used after each lesson. In order to minimize any misunderstanding that may result, will the teachers please observe closely the following instructions:

1. Carefully read the lesson plans at least a day before the presentation. If you have any questions or suggestions, please don't hesitate to contact me. The phone number is 736-0595.
2. Try to give the answers to all seatwork questions before the students leave school for that day. Give NO HOMEWORK as outside influences must be discouraged.
3. Record any absenteeisms on the list provided.
4. If more examples are needed to illustrate any concept before the seatwork is attempted, please feel free to do more.
5. If you feel that another period may be necessary, then extend this phase for another period.

LESSON 1: MULTIPLICATION AND ARRAYS

The basic objectives of the lesson are:

A. To introduce the concepts of an array and its relationship to multiplication.

B. To illustrate the commutative principle for multiplication; a x b = b x a (in this case, an a x b array, though drawn differently, has the same number of elements as a b x a array).

1. Introduction of an Array

"Today we will see how we can multiply using an array."

(Write the word array on the board).

"Here is an example of an array."

x x x
x x x

"This array is called a 2 x 3 array since it has 2 rows of 3 crosses."

2 x 3	x x x	row 1
	x x x	row 2

"We usually write the words '2 by 3' as '2 x 3'."

Draw a 4 x 3 array on the board; ask children to give reasons for their responses.

x x x
x x x
x x x
x x x

"This is a 4 x 3 array because it has 4 rows of 3."

	x x x	row 1
4 x 3	x x x	row 2
array	x x x	row 3
	x x x	row 4

Draw the following examples on the board (one at a time) and ask the children to name each. Ask children to give reasons for their responses.

x x x		
x x x		
x x x		
x x x	x x x x x x	
x x x	x x x x x x	x x x x x x
5 x 3 array	2 x 6 array	1 x 6 array
(5 rows of 3)	(2 rows of 6)	(1 row of 6)

"Here is the name of an array." (Put 3 x 6 on the board).

"This time, try to draw what this array would look like."

(Give children a few moments and then check individual pupil's work).

Answer:

	x x x x x x	row 1
3 x 6	x x x x x x	row 2
array	x x x x x x	row 3

Ask the children to draw the following arrays:

1 x 4

2 x 5

3 x 7

Check pupils' work and ask reasons for their responses.

2. <u>The Commutative Principle for Multiplication; a x b = b x a</u>

"Can anyone come up to the board and draw a 2 x 4 array?"

(Have a pupil come to the board and draw the array; ask the child how many crosses are in this array).

"Can anyone come up to the board and draw us a 4 x 2 array?"

Draw attention to the fact that a 2 x 4 array and a 4 x 2 array have the same number of elements but are drawn differently.

Repeat the same procedure using the 5 x 4 array and a 4 x 5 array.

Draw an 8 x 4 array on the board.

```
x x x x
x x x x
x x x x
x x x x
x x x x
x x x x
x x x x
x x x x
```

Ask the children if they can find another array which would have the same number of crosses, but would be drawn differently.

Note: several answers are possible, but draw attention to the fact that if we rotate the array we end up with a 4 x 8 array.

```
x x x x
x x x x
x x x x
x x x x
x x x x          x x x x x x x x
x x x x          x x x x x x x x
x x x x          x x x x x x x x
x x x x          x x x x x x x x
8 x 4 array          4 x 8 array
```

"Here is a very large array (13 x 8). Can anyone tell me another array that would be drawn differently but would have the same number of crosses?" (Answer: 8 x 13).

If needed, do other examples to emphasize the point that an a x b array has the same number of elements as a b x a array.

3. <u>Seatwork</u>

These series of questions are to provide additional practice with the concepts covered in Lesson 1. Please allow enough time for marking the seatwork as this will enable you to determine if most of your class will be ready for Lesson 2.

A. Name the following arrays.

1.

```
x x x
x x x
x x x
x x x
x x x          (Answer: 5 x 3)
```

2.

```
x x x x x x x   (Answer: 1 x 7)
```

3.

```
x x x x x
x x x x x
x x x x x      (Answer: 3 x 5)
```

4.

```
x
x
x
x
x
x
x              (Answer: 7 x 1)
```

B. Draw the following arrays.

1. 2 x 6
2. 8 x 2
3. 10 x 4
4. 1 x 11
5 11 x 1

C. Name or draw another array which would have the same number of x's but would look different.

1. 2 x 6

2.

x x x

x x x

x x x

x x x

x x x

LESSON 2: THE DISTRIBUTIVE LAW

The basic objective of this lesson is:

To introduce both the left hand and the right hand distributive law. The left hand law states that a x (b + c) = (a x b) + (a x c).

For example: 4 x 7 = 4 x (4 + 3) = (4 x 4) + (4 x 3).

The right hand law states that (b + c) x a = (b x a) ± (c x a).

For example: 8 x 6 = (3 + 5) x 6 = (3 x 6) + (5 x 6).

The teaching of both principles will be accomplished by dividing an array into smaller arrays.

Please do not use the <u>terms</u> right hand and left hand distributive laws with the children as this only leads to confusion.

1. <u>Review</u>

   a) Draw a 6 x 7 array on the board and ask the children the name of this array. Children should give reasons for their answers.

   Example: there are six rows of seven x's.

   b) Have a child come to the board and draw a 4 x 2 array.

2. "Let us look at the following array."

x x x x x x x
x x x x x x x
x x x x x x x
x x x x x x x
x x x x x x x

"What is the name of this array?" (5 x 7)

"How could we find out how many crosses there are in that array?"

(Children will probably offer suggestions such as counting the individual elements, adding 5 seven's etc.).

"All of these methods are fine, but here is another interesting way. Let's break up the 5 x 7 array into smaller arrays like this."

Step 1	Step 2	Step 3
5 x 7	5 x (4 + 3)	(5 x 4) + (5 x 3)
x x x x x x x	x x x x x x x	x x x x x x x
x x x x x x x	x x x x x x x	x x x x x x x
x x x x x x x	x x x x x x x	x x x x x x x
x x x x x x x	x x x x x x x	x x x x x x x
x x x x x x x	x x x x x x x	x x x x x x x

"Notice that the 5 x 7 array equals a 5 x 4 array plus a 5 x 3 array. Can any of you think of other ways of breaking up this array?"

(Let children suggest other possibilities).

For example:

Step 1	Step 2	Step 3
5 x 7	5 x (2 + 2 + 3)	(5 x 2) + (5 x 2) + (5 x 3)
x x x x x x x	x x x x x x x	x x x x x x x
x x x x x x x	x x x x x x x	x x x x x x x
x x x x x x x	x x x x x x x	x x x x x x x
x x x x x x x	x x x x x x x	x x x x x x x
x x x x x x x	x x x x x x x	x x x x x x x

Allow children to break up a 7 x 8 array. Try to emphasize Step 1, 2 and 3.

For example: 7 x 8 = 7 x (3 + 5) = (7 x 3) + (7 x 5).

3. <u>The Right Hand Distributive Law</u>

"Here is another array."

x x x x	
x x x x	
x x x x	5 x 4
x x x x	
x x x x	

"We have been breaking these arrays up by renaming the <u>second</u> number."

For example: 5 x 4 = 5 x (5 x 2) + (5 x 2)

"We can also break an array into smaller arrays by renaming the <u>first</u> number."

For example:

Step 1		Step 2		Step 3
x x x x	3	x x x x x x x x x x x x	3 x 4	x x x x x x x x x x x x
x x x x x x x x x x x x x x x x	2	x x x x x x x x	2 x 4	x x x x x x x x
5 x 4		(3 + 2) x 4		(3 x 4) + (2 x 4)

Ask children for further ways of breaking up this array by renaming the first number.

For example:

5 x 4 = (1 + 1 + 2 + 1) x 4 = (1 x 4) + (1 x 4) + (2 x 4) + (1 x 4).

"Now we should be able to break up any array into smaller arrays be renaming the <u>second</u> number or renaming the <u>first</u> number."

Note: Several more examples will probably be needed at this stage. The teacher should emphasize the techniques or renaming both the first number and the second number.

4. Seatwork

A. Break up each of these arrays by renaming the second number.

1) 6 x 7 =

2) 3 x 8 =

B. Break up each of the following arrays by renaming the first number.

1) 6 x 7 =

2) 8 x 4 =

C. Provide the numeral which makes the sentence true.

1) 6 x 7 = 6 x (4 + 3) = (6 x ?) + (6 x 3)

2) 4 x 8 = (4 x 2) + (? x 6)

3) 3 x 8 = (3 x 2) + (3 x 2) + (3 x ?)

4) 7 x 5 = (4 + ?) x 5 = (4 x 5) + (? x 5)

5) 7 x 9 = (7 x 8) + (7 x ?)

APPENDIX C

TREATMENT PHASE LESSON PLANS

TREATMENT ONE

LESSON 1: MULTIPLICATION OF TWO BY ONE PRODUCTS

The objective of this lesson is to teach the rote multiplication algorithm for 2 by 1 products; both with and without carrying.

A. <u>Without Carrying</u>

"Let us look at the following multiplication problem"

$$\begin{array}{r} 11 \\ \underline{\times\ 6} \end{array}$$

"Can anyone suggest a way of solving this problem by renaming the top number?"

(One possible answer might be):

$$\begin{array}{r} 11 \\ \underline{\times\ 6} \end{array} \qquad \begin{array}{r} 5+5+1 \\ \underline{\times\ 6} \end{array} \qquad \begin{array}{rr} & 5+5+1 \\ & \underline{\times\ 6} \\ & 6 \times 1 = 6 \\ + & 6 \times 5 = 30 \\ + & \underline{6 \times 5} = \underline{30} \\ & 66 \end{array}$$

The teacher should leave the work for 11 x 6 on the board and write down 11 x 6 somewhere else.

"Today we will learn another way that is probably faster than breaking up a multiplication problem. We merely have to work in the following way."

$$\begin{array}{r} 11 \\ \underline{\times\ 6} \end{array}$$

"We first ask ourselves what is 6 x 1? Then we place the 6 ones in the ones position."

$$\begin{array}{r} 11 \\ \underline{\times\ 6} \\ \underline{6} \end{array}$$

"Then we ask ourselves again what is 6 x 1? This time we have 6 tens and must place the 6 tens in the tens position."

$$\begin{array}{r} 11 \\ \underline{\times\ \ 6} \\ 66 \end{array}$$

To confirm the answer, the teacher should refer to the problem 11 x 6 done by the distributive principle (first example).

At this point the teacher should ask one part of the T1 group to try the problem,

$$\begin{array}{r} 11 \\ \underline{\times\ \ 7} \end{array}$$

by renaming the top number. The other half should try the new algorithm. When both groups have finished, the answers should be compared. If needed, try the problem of 11 x 9 in the same suggested manner.

B. <u>With Carrying</u>

Write the problem $\begin{array}{r} 12 \\ \underline{\times\ \ 6} \end{array}$ on the board.

Ask for suggestions as to how to solve this problem by renaming the <u>top</u> number. One suggestion might be:

$$\begin{array}{r} 12 \\ \underline{\times\ \ 6} \end{array} \qquad \begin{array}{r} 6 + 6 \\ \underline{\times\ 6} \end{array} \qquad \begin{array}{rcr} 6 + 6 & & \\ \underline{\times\ 6} & & \\ 6 \times 6 & = & 36 \\ +\ 6 \times 6 & = & \underline{+36} \\ & & 72 \end{array}$$

"We can solve this problem using our new way."

$$\begin{array}{r} 12 \\ \underline{\times\ \ 6} \end{array}$$

"What is 6 x 2? This time we have 12 <u>ones</u>. Let's break this up into 1 <u>ten</u> and 2 <u>ones</u>. Now we can place the 2 in the one's place as before."

$$\begin{array}{r} 12 \\ \times\ \ 6 \\ \hline 2 \end{array}$$

"We should place the 1 ten in the ten's place."

$$\begin{array}{r} 1\ \ \\ 12 \\ \times\ \ 6 \\ \hline 2 \end{array}$$

"Now we ask ourselves--what is 6 x 1? This time we get 6 tens. But since we have another group of ten underneath, we must add it to the 6. Then we place the 7 tens in the ten's place."

$$\begin{array}{r} 1\ \ \\ 12 \\ \times\ \ 6 \\ \hline 72 \end{array}$$

The teacher should then try a problem like 6 x 22 which involves the placement of a 1 in the hundreds place. Use the same steps as before.

Using the new algorithm, the pupils should attempt the following:

$$\begin{array}{r} 21 \\ \times\ \ 6 \\ \hline \end{array} \qquad \begin{array}{r} 11 \\ \times\ \ 9 \\ \hline \end{array} \qquad \begin{array}{r} 14 \\ \times\ \ 5 \\ \hline \end{array}$$

During this time help can be given to individuals as needed.

C. One by Two Products

This involves the handling of a problem such as:

$$\begin{array}{r} 6 \\ \times\ 23 \\ \hline \end{array}$$

Since in the readiness phase the commutative law for multiplication was taught, it should be easy to convince the child that with this type of problem we merely "turn it upside down."

$$\begin{array}{r} 6 \\ \times\ 23 \\ \hline \end{array} \qquad\qquad \begin{array}{r} 23 \\ \times\ 6 \\ \hline \end{array}$$

Now the child should be able to solve this type of problem.

D. Seatwork

It must be emphasized again that only these listed problems should be attempted. It is also important that the answers be given to the children before they leave school for that day.

Multiply:

1. $\begin{array}{r} 65 \\ \times\ 3 \\ \hline \end{array}$ 2. $\begin{array}{r} 7 \\ \times\ 13 \\ \hline \end{array}$ 3. $\begin{array}{r} 89 \\ \times\ 5 \\ \hline \end{array}$ 4. $\begin{array}{r} 15 \\ \times\ 8 \\ \hline \end{array}$ 5. $\begin{array}{r} 49 \\ \times\ 4 \\ \hline \end{array}$ 6. $\begin{array}{r} 99 \\ \times\ 2 \\ \hline \end{array}$

7. $\begin{array}{r} 6 \\ \times\ 41 \\ \hline \end{array}$

LESSON 2: MULTIPLICATION OF THREE BY ONE PRODUCTS

The objective of this lesson is to teach the rote multiplication algorithm for 3 x 1 products; both with and without carrying. Since the procedures for 3 x 1 products are very simple extensions of those for 2 by 1 products, a detailed lesson would be redundant. However, the teacher is urged to restrict all computation to only the examples given.

A. Review

Examples to use:

$$\begin{array}{r} 3 \\ \underline{\times\ 37} \end{array} \qquad \begin{array}{r} 11 \\ \underline{\times\ \ 8} \end{array} \qquad \begin{array}{r} 78 \\ \underline{\times\ \ 3} \end{array}$$

Emphasize the steps taken to get the final answer.

B. Without Carrying

Ask for suggestions to solve the problem

$$\begin{array}{r} 132 \\ \underline{\times\ 3} \end{array}$$

Most children will probably suggest extending the procedures used to solve 2 x 1 products.

A typical explanation of the procedures to use might go as follows:

$$\begin{array}{r} 132 \\ \underline{\times\ 3} \end{array}$$

"Multiply the 3 x 2; we get 6 ones and have to place this 6 in the one's position."

$$\begin{array}{r} 132 \\ \underline{\times\ 3} \\ \underline{6} \end{array}$$

"Multiply the 3 x 3; we get 9 <u>tens</u> and place this 9 in the ten's position."

$$\begin{array}{r} 132 \\ \underline{\times\ 3} \\ 96 \end{array}$$

"Finally, multiply the 3 x 1; we get 3 <u>hundreds</u> and place the 3 in the hundred's position."

$$\begin{array}{r} 132 \\ \underline{\times\ 3} \\ 396 \end{array}$$

Children should attempt: $\begin{array}{r} 102 \\ \underline{\times\ 3} \end{array} \quad \begin{array}{r} 412 \\ \underline{\times\ 2} \end{array} \quad \begin{array}{r} 210 \\ \underline{\times\ 4} \end{array}$.

After sufficient time, ask the children to explain the procedure in addition to the final answer.

C. <u>With Carrying</u>

Example to use: $\begin{array}{r} 213 \\ \underline{\times\ 7} \end{array}$

Again children will probably extend procedures for 2 x 1 products. Go through steps as in problem without carrying, but stress breaking up 7 x 3 = 21 <u>ones</u> = 2 <u>tens</u> + 1 <u>one</u>.

Children should attempt: $\begin{array}{r} 120 \\ \underline{\times\ 7} \end{array} \quad \begin{array}{r} 108 \\ \underline{\times\ 5} \end{array} \quad \begin{array}{r} 223 \\ \underline{\times\ 6} \end{array}$

D. <u>Solution of</u> $\begin{array}{r} 6 \\ \underline{\times\ 142} \end{array}$

Again, as in 2 x 1 products, children should be urged to turn problem "upside down" and then solve.

$$\begin{array}{r} 6 \\ \underline{\times\ 142} \end{array} \qquad \begin{array}{r} 142 \\ \underline{\times\ 6} \end{array}$$

E. Seatwork

1. $\begin{array}{r} 222 \\ \underline{\times\ 7} \end{array}$ 2. $\begin{array}{r} 107 \\ \underline{\times\ 7} \end{array}$ 3. $\begin{array}{r} 24 \\ \underline{\times\ 101} \end{array}$ 4. $\begin{array}{r} 27 \\ \underline{\times\ 550} \end{array}$

5. $\begin{array}{r} 636 \\ \underline{\times\ 5} \end{array}$ 6. $\begin{array}{r} 101 \\ \underline{\times\ 4} \end{array}$ 7. $\begin{array}{r} 2 \\ \underline{\times\ 191} \end{array}$

LESSON 3: A REVIEW OF THE ROTE TYPE ALGORITHM FOR 3 x 1 AND 2 x 1 PRODUCTS

This lesson is needed to allow the T2 group to finish their treatment. Since most of the students in the T1 group will have mastered the rote-type algorithm, this lesson is probably unnecessary for this group. However, it is essential for the purposes of this study and can be used as merely a practice lesson. The teacher should use only the problems given in this lesson. Please do not give extra problems to those who finish early. The teacher should have ample time to give individual help during this period. In addition to giving answers to the problems, the teacher should explain the procedures used to get the final answer in 3 or 4 problems.

Multiply:

1.	62 x 6	2.	1 x 49	3.	23 x 6	4.	40 x 2	5.	425 x 5
6.	8 x 280	7.	3 x 949	8.	108 x 2	9.	39 x 3	10.	4 x 49
11.	57 x 7	12.	523 x 4	13.	730 x 5	14.	9 x 424	15.	61 x 5
16.	99 x 4	17.	45 x 9	18.	9 x 30	19.	253 x 4	20.	208 x 8

TREATMENT TWO

LESSON 1: THE BEGINNINGS OF THE ANNEXATION ALGORITHM

The objective of this lesson is to teach children a technique for multiplying any number by 10, 100, or 1000.

1. Ten as a Factor

The teacher should first quickly review multiplication as repeated addition.

e.g. $3 \times 8 = 8 + 8 + 8 = 24$

List the following series of questions somewhere on the board.

$2 \times 10 = ?$	$6 \times 10 = ?$
$3 \times 10 = ?$	$7 \times 10 = ?$
$4 \times 10 = ?$	$8 \times 10 = ?$
$5 \times 10 = ?$	$9 \times 10 = ?$

Starting with $2 \times 10 = ?$ ask children how to solve by adding (10 + 10). Solve each problem by adding.

It should be pointed out to the students that in each problem the one digit number has changed places.

For example $\underline{2} \times 10 = \underline{2}0$

"The 2 was originally in the one's place but after multiplication by ten it shifted to the ten's place and a zero was placed to the right."

The children should quickly realize that to multiply by 10 we merely place a zero to the right of the other multiplier.

The following series of questions should then be placed on the board.

10 x 11 = ?	10 x 21 = ?
12 x 10 = ?	18 x 10 = ?
13 x 10 = ?	

Solve at least 2 or 3 problems by adding. Again have the students note that when multiplying by 10 the digits of the other multiplier all shift to the left and a zero is placed to the right.

2. One Hundred as a Factor

Again the teacher should list a series of questions such as:

2 x 100 = ?	4 x 100 = ?
3 x 100 = ?	5 x 100 = ?

Solve each by adding. This time it should be noted that students should recognize that the digits have shifted two places (from the one's place to the hundred's place) and two zero's are then placed to the right.

The following series of questions should then be placed on the board.

11 x 100 = ?	13 x 100 = ?
12 x 100 = ?	26 x 100 = ?

After solving the first problem or so by adding, the children should be able to quickly generalize that 26 x 100 = ? can be solved by "placing two zeros to the right of the 26."

(26 x 100 = 2600)

It is probably advisable to show the pupils how the 2 digit and 6 digit of 26 have shifted two places to the left.

3. Thousand as a Factor

By now the students should be able to generalize to problems such as:

6 x 1000 = ? --- 6000

12 x 1000 = ? --- 12000

To convince some pupils of the legitimacy of this technique it may be necessary to solve a problem or two by adding.

Again the pupils should realize that the digits have shifted three places to the left and three zeros have been placed to the right.

4. Seatwork

1. 8 x 10 = ?
2. 10 x 12 = ?
3. 15 x 100 = ?
4. 12 x 100 = ?
5. 10 x 1000 = ?
6. 100 x 9 = ?
7. 1000 x 8
8. 12 x 100
9. 10 x 12
10. 100 x 9
11. 72 x 10 = ?
12. 98 x 100 = ?
13. 100 x 13
14. 1000 x 3
15. 28 x 10

When marking the teacher should have students explain how they determined their final answers.

LESSON 2: THE ANNEXATION ALGORITHM

The objectives of this lesson are:

1) to complete the annexation algorithm $\underline{3} \times \underline{80} = \underline{240}$

2) to begin applying the annexation algorithm and distributive principle to solve 2 x 1 products.

1. Review

a) ask children to multiply the following:

$3 \times 10 = ?$ $\qquad 100 \times 11 = ?$

$18 \times 100 = ?$ $\qquad 1000 \times 19 = ?$

Explain the procedures used to get final answer.

Example: (multiply by 100; we place two zeros to the right of the other factor; ie: etc.).

b) review breaking up a product into the sum of smaller products by renaming the top or bottom number. Final answer not important.

$$\begin{array}{r} 12 \\ \underline{\times\ 7} \end{array} \qquad \begin{array}{r} 10 + 2 \\ \underline{\times\ 7} \\ 7 \times 2 \\ \underline{+\ 7 \times 10} \end{array} \qquad \text{and} \qquad \begin{array}{r} 7 \\ \underline{\times\ 13} \end{array} \qquad \begin{array}{r} 7 \\ \underline{\times\ 10 + 3} \\ 3 \times 7 \\ \underline{+\ 10 \times 7} \end{array}$$

2. Put the following series of questions on the board.

$3 \times 20 = ?$ $\qquad 110 \times 3 = ?$ $\qquad 300 \times 4 = ?$

$70 \times 5 = ?$ $\qquad 20 \times 3 = ?$ $\qquad 110 \times 2 = ?$

$3 \times 200 = ?$

Show the children how to solve any of the above in the following manner.

For example: $3 \times 20 = ?$

a) "How many tens are there in 20? (Ans. 2)

"we can rewrite 20 as 2 x 10

$$3 \times 20 = 3 \times \underline{2 \times 10}$$

b) Now the order in which we multiply in a question does not matter, so; $3 \times 20 = 3 \times 2 \times 10 = \underline{6} \times 10$.

c) We have already learned how to multiply a problem such as this" (place a zero to the right).

$$3 \times 20 = 3 \times 2 \times 10 = 6 \times 10 = 60$$

d) To have the children see the emerging pattern for the series of questions, the teacher should underline the following:

$$\underline{3} \times \underline{2}0 = 60$$

If the teacher does a few more examples in the above manner it is hoped that the child will see how to multiply $3 \times 200 = ?$ (Simply multiply 3×2 and place 2 zeros to the right $\underline{3} \times \underline{2}00 = \underline{6}00$).

Note: In solving $70 \times 5 = ?$ the teacher should rewrite 70 as 10×7 rather than 7×10 since:

$70 \times 5 = 7 \times 10 \times 5$ (have to commute 7 and 10 to solve)

$\underline{7}0 \times \underline{5} = 10 \times \underline{7} \times \underline{5} = 10 \times 35 = \underline{35}0$

The next series of questions should be assigned to the pupils. This will enable the teacher to quickly determine whether or not the class is ready to continue. If not, more example should be used to increase the competency with the annexation algorithm.

$30 \times 6 = ?$	$200 \times 8 = ?$
$9 \times 20 = ?$	$100 \times 10 = ?$
$110 \times 3 = ?$	

3. <u>Multiplication of 2 by 1 products</u>

"Now we are ready to do some difficult multiplication problems

like $\begin{array}{r} 12 \\ \underline{\times\ 6} \end{array}$

a) rename "top" number as

$$\begin{array}{r} 12 \\ \underline{\times\ 6} \end{array} \qquad\qquad \begin{array}{r} 10 + 2 \\ \underline{\times\ 6} \end{array}$$

b) we know how to multiply this type;

$$\begin{array}{r} 12 \\ \times\ 6 \end{array} \qquad\qquad \begin{array}{r} 10 + 2 \\ \underline{\times\ 6} \\ 6 \times 2 \\ \underline{+\ 6 \times 10} \end{array}$$

c) Now it becomes easy since 6 x 2 = 12 and we know that

$\underline{6} \times \underline{1}0 = \underline{6}0$

$$\begin{array}{r} 12 \\ \underline{\times\ 6} \\ 72 \end{array} \qquad\qquad \begin{array}{rcr} 10 + 2 & & \\ \underline{\times\ 6} & & \\ 6 \times 2 & = & 12 \\ \underline{+\ 6 \times 10} & = & \underline{+60} \\ & & \underline{72} \end{array}$$

The teacher should demonstrate:

$$\begin{array}{r} 21 \\ \underline{\times\ 6} \end{array} \qquad\qquad \begin{array}{rcr} 20 + 1 & & \\ \underline{\times\ 6} & & \\ 6 \times 1 & = & 6 \\ +\ 6 \times 20 & = & \underline{120} \\ & & \underline{126} \end{array}$$

Allow children to try $\begin{array}{r} 13 \\ \underline{\times\ 4} \end{array} \qquad \begin{array}{r} 28 \\ \underline{\times\ 3} \end{array}$

If class appears to be acquiring some mastery and if time still permits, continue to the next section.

4. <u>Renaming bottom numbers</u>

Problems such as $\begin{array}{r} 6 \\ \underline{\times\ 23} \end{array}$ should be attacked in the following manner.

$$\begin{array}{r} 6 \\ \underline{\times\ 23} \end{array} \qquad \begin{array}{rcr} & 6 & \\ & \underline{20 + 3} & \\ 3 \times 6 & = & 18 \\ +\ 20 \times 6 & = & \underline{120} \\ & & \underline{138} \end{array}$$

Other examples to use might be:

$$\begin{array}{r} 4 \\ \underline{\times\ 31} \end{array} \qquad \begin{array}{r} 8 \\ \underline{\times\ 51} \end{array}$$

5. Seatwork

Multiply the following:

1. $2 \times 120 = ?$
2. $3 \times 70 = ?$
3. $2 \times 600 = ?$
4. $100 \times 100 = ?$
5. $11 \times 300 = ?$
6. $\begin{array}{r} 65 \\ \underline{\times\ \ 3} \end{array}$
7. $\begin{array}{r} 7 \\ \underline{\times\ 13} \end{array}$
8. $\begin{array}{r} 89 \\ \underline{\times\ \ 5} \end{array}$
9. $\begin{array}{r} 15 \\ \underline{\times\ 8} \end{array}$
10. $\begin{array}{r} 6 \\ \underline{\times\ 41} \end{array}$

The next series of questions should be assigned if some students appear to have mastered the 2 x 1 products.

11. $\begin{array}{r} 6 \\ \underline{\times\ 102} \end{array}$
12. $\begin{array}{r} 100 + 10 + 1 \\ \underline{\times\ 7} \end{array}$
13. $\begin{array}{r} 140 \\ \underline{\times\ 2} \end{array}$

LESSON 3: SOLUTION OF 3 BY 1 PRODUCTS

Hopefully, this should be the last period of treatment for the T2 group. The objective of this lesson is to teach the techniques for solving several types of 3 by 1 products.

1. Review

Pupils should be assigned the following:

$$8 \times 20 = ? \qquad \begin{array}{r} 34 \\ \underline{\times\ \ 6} \end{array} \qquad \begin{array}{r} 9 \\ \underline{\times\ 23} \end{array}$$

In addition to final answers, the procedures used to solve each should be re reviewed.

For example: $\underline{8} \times \underline{20} = \underline{160}$

$$\begin{array}{r} 34 \\ \underline{\times\ \ 6} \end{array} \qquad \begin{array}{rcr} 30 + 4 & & \\ \underline{\times\ 6} & & \\ 6 \times 4 & = & 24 \\ \underline{+\ 6 \times 30} & = & \underline{+180} \end{array}$$

$$\begin{array}{r} 9 \\ \underline{\times\ 23} \end{array} \qquad \begin{array}{rcr} 9 & & \\ \underline{20 + 3} & & \\ 3 \times 9 & = & 27 \\ \underline{+\ 20 \times 9} & = & \underline{+180} \end{array}$$

2. Renaming the Top Number

Put the problem $\begin{array}{r} 102 \\ \underline{\times\ 3} \end{array}$ on the board. Ask for suggestions for possible solution. Rename the top number in the following way. The rationale for each step should be explained in detail.

$$\begin{array}{r} 102 \\ \underline{\times\ 3} \end{array} \qquad \begin{array}{r} 100 + 2 \\ \underline{\times\ 3} \end{array} \qquad \begin{array}{rcrl} 100 + 2 & & & \\ \underline{\times\ 3} & & & \\ 3 \times 2 & = & 6 & \text{multiplication fact} \\ +\ 3 \times 100 & = & \underline{+300} & \text{multiplication by 100} \\ & & \underline{306} & \end{array}$$

Other examples that should be demonstrated by the teacher are:

$$\text{a)}\quad \begin{array}{r} 134 \\ \underline{\times\ 3} \end{array} \qquad \begin{array}{r} 100 + 20 + 4 \\ \underline{\times\ 3} \end{array} \qquad \begin{array}{rcl} 100 + 30 + 4 & & \\ \underline{\times\ \ 3} & & \\ 3 \times 4 & = & 12 \\ + \underline{3} \times \underline{30} & = & +190 \\ \underline{+ \underline{3} \times 100} & = & \underline{+300} \\ & & \underline{402} \end{array}$$

$$\text{b)}\quad \begin{array}{r} 240 \\ \underline{\times\ 4} \end{array} \qquad \begin{array}{rcl} 240 + 40 & & \\ \underline{\times\ \ 4} & & \\ \underline{4} \times \underline{40} & = & 160 \\ \underline{+ \underline{4} \times 200} & = & \underline{+800} \\ & & \underline{960} \end{array}$$

3. <u>Renaming the Bottom Number</u>

The pupils should realize that sometimes it is advantageous to rename the <u>bottom</u> number. These examples should illustrate the techniques to be used.

$$\text{a)}\quad \begin{array}{r} 8 \\ \underline{\times\ 101} \end{array} \qquad \begin{array}{rcl} 8 & & \\ \underline{\times\ 100 + 1} & & \\ 1 \times 8 & = & 8 \\ \underline{+ 100 \times 8} & = & \underline{800} \\ & & \underline{808} \end{array}$$

$$\text{b)}\quad \begin{array}{r} 3 \\ \underline{\times\ 246} \end{array} \qquad \begin{array}{r} 3 \\ \underline{\times\ 200 + 40 \times 6} \end{array} \qquad \begin{array}{rcl} 3 & & \\ \underline{\times\ 200 + 40 + 6} & & \\ 6 \times 3 & = & 18 \\ + \ 40 \times 3 & = & +120 \\ \underline{+ 200 \times \underline{3}} & = & \underline{+600} \\ & & \underline{738} \end{array}$$

$$\text{c)}\quad \begin{array}{r} 7 \\ \underline{\times\ 120} \end{array} \qquad \begin{array}{r} 7 \\ \underline{100 + 20} \end{array} \qquad \begin{array}{rcl} 7 & & \\ \underline{100 + 20} & & \\ 20 \times 7 & = & 140 \\ +100 \times 7 & = & +700 \\ \hline & & \underline{840} \end{array}$$

The preceding examples should be enough to enable most students to acquire some proficiency for solving 3 x 1 products. The seatwork to be assigned will be good practice for all students and is lengthy enough to allow the teacher to give individual help.

4. Seatwork

Multiply:

1.	3 x 280	2.	730 x 5	3.	201 x 6	4.	485 x 5	5.	9 x 424
6.	61 x 5	7.	208 x 5	8.	9 x 14	9.	45 x 3	10.	8 x 101

APPENDIX D

THE MEASURING INSTRUMENTS

Performance Test

Name __________________________________ School _______________________
First Last

Part A - Multiply the following

<u>Please show all work</u>

1. 87 x 5	2. 53 x 6	3. 60 x 2	4. 93 x 2	5. 22 x 4
6. 3 x 41	7. 4 x 15	8. 7 x 23	9. 4 x 68	10. 7 x 30
11. 732 x 6	12. 623 x 2	13. 201 x 4	14. 840 x 2	15. 604 x 8
16. 4 x 606	17. 4 x 433	18. 6 x 218	19. 5 x 330	20. 7 x 111

Transfer Test

Name ______________________________ School ______________________
First Last

Part B - Multiply the following

Please show all work

1. $\begin{array}{r} 1001 \\ \times\ 6 \\ \hline \end{array}$	2. $\begin{array}{r} 3 \\ \times\ 1234 \\ \hline \end{array}$	3. $\begin{array}{r} 6 \\ \times\ 1100 \\ \hline \end{array}$
4. $\begin{array}{r} 3461 \\ \times\ 3 \\ \hline \end{array}$	5. $\begin{array}{r} 12 \\ \times\ 11 \\ \hline \end{array}$	6. $\begin{array}{r} 11 \\ \times\ 26 \\ \hline \end{array}$
7. $\begin{array}{r} 13 \\ \times\ 64 \\ \hline \end{array}$	8. $\begin{array}{r} 25 \\ \times\ 12 \\ \hline \end{array}$	9. $\begin{array}{r} 1001 \\ \times\ 11 \\ \hline \end{array}$
10. $\begin{array}{r} 12 \\ \times\ 2010 \\ \hline \end{array}$	11. $\begin{array}{r} 1111 \\ \times\ 15 \\ \hline \end{array}$	12. $\begin{array}{r} 16 \\ \times\ 1100 \\ \hline \end{array}$

13. $\begin{array}{r} 111 \\ \underline{\times\ 101} \end{array}$	14. $\begin{array}{r} 203 \\ \underline{\times\ 122} \end{array}$	15. $\begin{array}{r} 120 \\ \underline{\times\ 102} \end{array}$
16. $\begin{array}{r} 113 \\ \underline{\times\ 201} \end{array}$	17. $\begin{array}{r} 1001 \\ \underline{\times\ 101} \end{array}$	18. $\begin{array}{r} 1200 \\ \underline{\times\ 110} \end{array}$
19. $\begin{array}{r} 101 \\ \underline{\times\ 1111} \end{array}$	20. $\begin{array}{r} 122 \\ \underline{\times\ 12} \end{array}$	21. $\begin{array}{r} 101 \\ \underline{\times\ 18} \end{array}$
22. $\begin{array}{r} 16 \\ \underline{\times\ 211} \end{array}$	23. $\begin{array}{r} 11 \\ \underline{\times\ 103} \end{array}$	

APPENDIX E

THE EXPERIMENTAL DATA

TABLE V

EXPERIMENTAL DATA

TREATMENT

	T1			T2		
	Sub-ject	Perfor-mance	Trans-fer	Sub-ject	Perfor-mance	Trans-fer
	1	15	4	1	8	2
	2	16	3	2	8	5
	3	18	2	3	9	0
	4	11	1	4	14	8
	5	11	8	5	8	0
	6	16	4	6	14	0
TEACHER	7	16	4	7	6	0
1	8	16	4	7	1	0
	9	15	3	9	4	0
	10	11	2	10	9	2
	11	5	2	11	1	0
	12	6	1	12	0	0
	13	12	3	13	7	3
	14	19	4	14	0	0

TABLE V

EXPERIMENTAL DATA

TREATMENT

	T1			T2		
	Sub-ject	Perfor-mance	Trans-fer	Sub-ject	Perfor-mance	Trans-fer
	1	20	11	1	5	0
	2	10	1	2	14	0
	3	18	8	3	14	0
	4	10	4	4	10	3
TEACHER	5	3	2	5	14	9
2	6	13	3	6	0	0
	7	3	1	7	0	0
	8	13	3	8	7	0
	9	9	2	9	0	1
	10	5	1	10	0	0

TABLE V

EXPERIMENTAL DATA

TREATMENT

	T1			T2		
	Subject	Performance	Transfer	Subject	Performance	Transfer
TEACHER 3	1	18	4	1	17	4
	2	15	9	2	15	7
	3	14	12	3	7	5
	4	11	0	4	8	5
	5	18	4	5	2	2
	6	15	6	6	0	1
	7	12	5	7	0	0
	8	13	2	8	7	0

TABLE V

EXPERIMENTAL DATA

TREATMENT

	T1			T2		
	Sub-ject	Perfor-mance	Trans-fer	Sub-ject	Perfor-mance	Trans-fer
	1	19	5	1	17	8
	2	17	7	2	17	8
	3	19	10	3	15	9
	4	20	4	4	16	3
	5	12	5	5	14	6
TEACHER	6	16	3	6	11	12
4	7	17	4	7	6	3
	8	19	12	8	9	1
	9	19	4	9	6	3
	10	15	4	10	10	3
	11	14	3	11	2	3
	12	14	5	12	8	4

TABLE V

EXPERIMENTAL DATA

TREATMENT

	Subject	Performance	Transfer	Subject	Performance	Transfer
	1	20	12	1	19	12
	2	19	14	2	12	11
	3	18	8	3	19	10
	4	18	7	4	17	12
	5	17	12	5	14	8
	6	16	9	6	6	5
	7	19	12	7	13	7
	8	19	10	8	15	6
TEACHER 5	9	19	9	9	14	7
	10	18	12	10	13	5
	11	19	8	11	20	6
	12	16	11	12	10	4
	13	12	6	13	8	2
	14	17	11	14	6	1
	15	16	4	15	4	1
	16	12	1	16	4	0
	17	16	1	17	1	0

TABLE V

EXPERIMENTAL DATA

TREATMENT

	T1			T2		
	Subject	Performance	Transfer	Subject	Performance	Transfer
TEACHER 6	1	15	6	1	8	4
	2	19	14	2	12	7
	3	19	9	3	15	9
	4	11	4	4	7	2
	5	18	17	5	9	3
	6	14	4	6	9	1
	7	4	3	7	11	5
	8	16	7	8	6	0
	9	16	3	9	13	4
	10	9	4	10	3	0
	11	16	3	11	0	0
	12	17	3	12	3	0
	13	9	2	13	0	0
	14	0	0	14	0	0

TABLE V

EXPERIMENTAL DATA

TREATMENT

	T1			T2		
	Sub-ject	Perfor-mance	Trans-fer	Sub-ject	Perfor-mance	Trans-fer
TEACHER 7	1	18	5	1	5	2
	2	19	8	2	5	6
	3	12	3	3	11	5
	4	17	2	4	14	7
	5	18	4	5	0	0
	6	12	4	6	0	0
	7	8	2	7	9	2
	8	6	0	8	0	1

TABLE V

EXPERIMENTAL DATA

TREATMENT

	T1			T2		
	Subject	Performance	Transfer	Subject	Performance	Transfer
	1	18	13	1	15	10
	2	15	4	2	12	7
	3	19	4	3	16	8
	4	13	4	4	3	0
	5	17	4	5	7	7
TEACHER 8	6	12	3	6	7	2
	7	15	1	7	11	0
	8	17	3	8	15	1
	9	19	4	9	9	3
	10	18	2	10	9	7
	11	12	3	11	11	0
	12	12	3	12	7	0
	13	14	4	13	4	0
	14	11	2	14	7	0