QUANTIZATION NOISE OF SIGNAL CORRELATORS

by

ROLF JERG KLINGLER

Diplom., E. T. H. Zurich, Switzerland

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Department of __**Electrical Engineering**__

The University of British Columbia
Vancouver 8, Canada

Date __**May 3rd 1972**__

ABSTRACT

In radio-astronomy, spectra of noisy signals are often computed using digital auto-correlation techniques. To simplify the design of the many high-speed multipliers and averagers, coarse quantization is employed, using only a few digital levels.

This thesis is a theoretical study of the penalty paid for such coarse quantization in the form of increased output noise. A degradation factor is defined and is calculated for a variety of logic schemes which have been used or proposed.

For each scheme, results are given as a function of sampling rate and it is demonstrated that there is often significant improvement in sampling at rates faster than the Nyquist rate.

A computer simulation technique was developed for verifying the computed results, and for extending the results to complicated schemes where analysis is very difficult.

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LIST OF ILLUSTRATIONS

LIST OF SYMBOLS

B	Bandwidth
D	Degradation Factor
K	Normalized sampling-rate $= f_s/B$
M	Number of simulated runs
MSD	Minimum detectable signal
N	Number of samples
P	Decision level of quantizer
Pr(1)	Probability of inputs to product merger lying in sector (1)
Pr(w=b)	Probability of satisfying equation in parentheses
R	$=R_{xy}(0)$ correlation factor of the signals x and y
$R_x(\tau)$	Autocorrelation function of x(t)
$R_{xy}(\tau)$	Cross correlation function of x(t) and y(t)
S	Spectral power density
Sa(x)	Sampling function $Sa(x) = \dfrac{\sin(x)}{x}$
S/N	Signal-to-noise ratio
T	Averaging Time of Averager
T_s	Sampling interval
W^2	Variance of w as a simulation result
a	Quantizer output level
	as a subscript: analog correlator or: asymmetrically sampled
d	Signal-to-(signal + noise) ratio
e	Subscript: correlator with least significant products eliminated
erf(x)	Error function $erf(x) = \dfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du$

erfc(x)	Complement error function $erfc(x) = 1 - erf(x)$
f	Frequency variable
f_s	Sampling frequency
$f(a_i, P_i)$	Quantizer function, defined on page 15, equation (III.1.2.13)
i	Subscript: input
m	Subscript: products merged
n,m	Subscript: number of quantized levels in the 2 sides of the correlator
n	Oversampling factor
n	Subscript "of noise signal"
n(t)	Noise waveform
o	Subscript: output
opt	Subscript: optimum
$p_x(x)$	Probability-density of a random variable x
$p_{x,y}(x,y)$	Joint probability-density function of 2 random variables x and y
q_x	quantized value of the signal x
r	Normalized correlation factor
s_o	Constant d-c signal
ss	Subscript: strong signal case
s(t)	Signal waveform
t	Time variable
v(t)	Delayed signal $y(t-\tau)$
w	Output of correlator = averaged value of q_z
x	Subscript: on the x-side of the correlator
x(t)	Unprocessed waveform on one (x-) side of the correlator
y	Subscript: on the y-side of the correlator
y(t)	Unprocessed waveform on one (y-) side of the correlator
z	Output of the multiplier for unprocessed signals x(t) and y(t)

ε Error of the simulated degradation factor

$\varepsilon_{nm}(K)$ Decomposition error

κ Relative step-size of quantizer

$\rho(\tau)$ Normalized autocorrelation function

σ Standard deviation

τ Time variable in correlation function, delay time

$\mathcal{F}\{h(t)\}$ Fourier transform of function h(t)

$\mathcal{F}^{-1}\{H(f)\}$ Inverse Fourier Transform of function H(f)

ACKNOWLEDGEMENT

I. INTRODUCTION

In radio-astronomy, as in other fields, it is often necessary to measure the amount of correlation between two signals. The signals usually are adequately modelled as white Gaussian noise, and the amount of correlation between the signals is typically very small. Hence, correlation coefficients must be measured by taking long-time averages using instruments free from drift. Such instruments tend to use analog to digital conversion followed by digital techniques for multiplication and averaging.

One particular application of correlators is in the determination of spectra of noisy signals. This can be done by first determining the autocorrelation function of the signals, multiplying $s(t)$ by $s(t-T_n)$ for many different values of T_n. A Fourier transform is then used to calculate a power spectrum. Similar techniques can be used to find the spectrum of a correlation signal coming from two antennas.

In a correlation spectrometer, two A/D converters feed a large number (over 100) of multipliers and averagers, all processing samples at high rates. It thus becomes important to simplify the design of these many repetitive units. Early instruments did this by using 2-level (1-bit) A/D converters[11]. Of late, more complicated logic schemes have been used and some thought has been given to reducing the penalty paid for coarse quantization. This penalty can be expressed as the degradation of the output S/N ratio, when compared with analog instrumention.

I.1 Literature Survey

Polarity-coincidence correlators have been studied by F. Bowers[5], Burns and Yao[6], Cheng[7], Ekre[9], and Yerbury[13]. Burns and Yao mention the

fact that the output signal-to-noise ratio does not change when the output of an analog correlator is oversampled beyond the Nyquist-rate. They claim that the actual shape of the input-filter (which is usually assumed to be rectangular) is important. Cheng found the Degradation factor for a polarity-coincidence correlator at an infinite sampling-rate for arbitrary signals and signal power. Ekre found the output signal-to-noise ratio vs. sampling-rate of a polarity-coincidence correlator. Yerbury investigated the effect of amplitude limiting the analog correlator to increase its stability. His special case of infinite limiting (infinite stability) is identical to the polarity-coincidence correlator. Watts[10] gives a mathematical description of a quantizer with an infinite number of levels. Cooper treats the 2 bit correlator sampled at Nyquist-rate and investigates "incomplete multiplication" where the least significant products are neglected.

I.2 Contribution of This Thesis

This work is a continuation of studies by F. Bowers[5]. He found the Degradation factor of different combinations of quantizers for Nyquist-rate-sampling. Here the Degradation factor of a correlator can be calculated as the product of two single channel Degradation-factors. He mentions that a Gaussian signal can be replaced by a DC-signal for small input signal-to-noise ratios and that the decision levels and stepwidths can be optimized.

The present work investigates the dependence of S/N ratio of quantized multi-level correlators on the sampling-rate. General formulae are presented. Numerical evaluations give the actual values of the

degradation factor for five different correlators. Procedures are
presented which optimize decision levels for four different correlators
and variable sampling-rates. Two special problems are also discussed:

(a) The degradation factor for a 3*5 level correlator where the
     two channels are sampled at a different rate (one channel at
     Nyquist-rate, the other channel oversampled) and;

(b) The effect of "overquantization". Here the product of two
     signals is limited to a small number of levels, but the
     signal before multiplication can have many more levels.

Finally, a simulation program to determine the degradation
factor of quantized correlators was developed. This is useful for
higher level correlators where theoretical analysis becomes too difficult.
The simulation results confirm the findings from theoretical calculations
for the five different correlators considered.

II. THEORETICAL MODEL OF A CORRELATION RECEIVER

Figure II.1 shows the model of a correlation receiver whose properties are to be investigated.
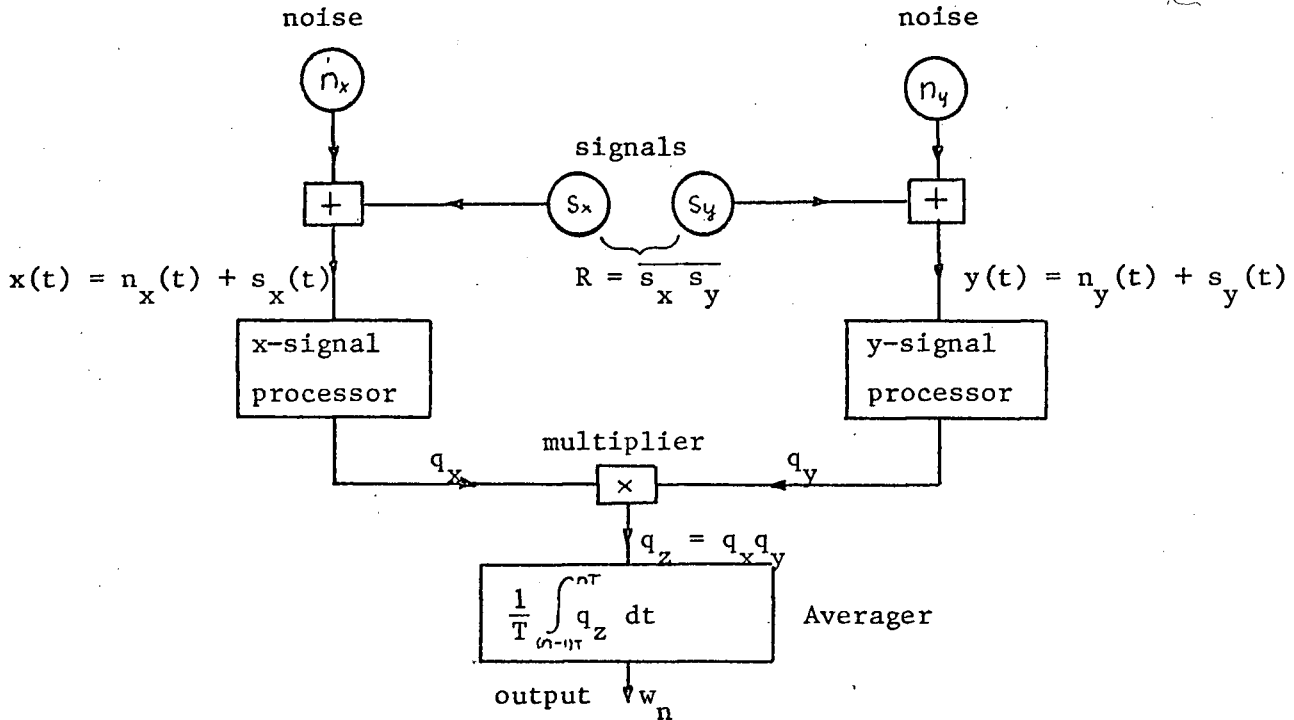
Fig. II.1 Model of a correlation receiver

Each of the waveforms, $x(t)$ and $y(t)$ is made up of two components: a relatively large amount of "noise" and a relatively small amount of "signal". The two noise sources are completely uncorrelated, while the signal sources may have a finite cross-correlation factor, $R = \overline{s_x \cdot s_y}$. The task of the instrument is to determine R as accurately as possible.

The two signal processors can take a variety of forms:

(a) There could be no processing at all, in which case we have an "analog correlator";

(b) the waveforms could be sampled (usually at rates higher than twice the bandwidth);

(c) the waveforms could be quantized into several discrete levels, with subsequent digital handling of the multiplication and averaging, or;

(d)   the waveforms could be both sampled and quantized. This is the common situation and is also the most general case. All other treatments can be regarded as limiting cases of this one.

Often the two signal processors are alike, but this is not necessary, and several instances of unequal processing will be investigated. The processors will normally have a transfer function symmetrical around zero volts, and this is assumed for simplicity in the calculations.

The analog correlator (a) above is normally difficult to instrument due to problems of drift. Theoretically it has the best output signal-to-noise ratio, and its performance is the standard by which all other instruments will be judged.

II.1 General Assumptions

In calculations using the model shown in Figure II.1, the following assumptions are made:

(1) The signals, s_x and s_y, and the noise sources, n_x and n_y, are all limited to the frequency band (0 to B).

(In practice, observations are usually made at higher frequencies. The waveforms x(t) and y(t) are then band-limited at that frequency and are translated down in frequency by heterodyning.)

(2) The noise sources, n_x and n_y, are both white Gaussian signals with zero mean values.

(Usually such noise is a mixture of "antenna noise" from sky background and of "receiver noise". The assumption of uniform spectral power density is true only because in practice the bandwidth observed is small compared with its centre frequency. The assumption of Gaussian statistics is an idealization, but is valid for Johnson noise and for many other noise sources).

(3) The two noise sources have the same average power, σ_n^2, and the

two signals have the same average power, σ_s^2.

(This is true for most examples of interest and simplifies the calculations, but the results can easily be generalized to the case of unequal powers.)

(4) The noise sources are statistically independent of each other.

(This is obviously true when the noise is generated in two different receivers. For some other sources of noise there may be some correlation, but if this is so, then the correlated components are separated out and are treated as part of the signal to be measured.)

(5) The signal powers are small compared with the average noise powers.

(This assumption simplifies the calculations considerably. It is not valid for all applications of correlators; but the greatest interest in optimizing signal-to-noise ratios arises when the signals are small.)

(6) The signals, s_x and s_y, are ergodic and they are not correlated with the noise sources, n_x and n_y.

(It is not necessary to make any other assumptions about the spectra or the statistics of the signals. It is shown in Appendix A2 that the degradation in signal-to-noise factor is independent of the character of the signals, and can be computed using a very special d-c case where $s_x(t) = s_y(t) = s_o = $ constant.)

II.2 Definitions

Let σ_n^2 be the average power (the variance) of the noise sources n_x and n_y, and R be the cross-correlation factor of the signals s_x and s_y.

We then define the input signal-to-noise ratio of a correlator as

$$(S/N)_i = R/\sigma_n^2 \qquad\qquad (II.2.1)$$

In the d-c case, where $R = s_o^2$, this becomes

$$(S/N)_i = s_o^2/\sigma_n^2 \qquad\qquad (II.2.2)$$

If, further, $\bar{w}$ is the expected value of the output, w, and σ_w^2 is the variance of w, then we define the output signal-to-noise ratio as

$$(S/N)_o = \bar{w}/\sigma_w \qquad\qquad (II.2.3)$$

In general we will find that the average correlator output, $\bar{w}$, will be some monotonically increasing function of R, as shown in Figure II.2.1.
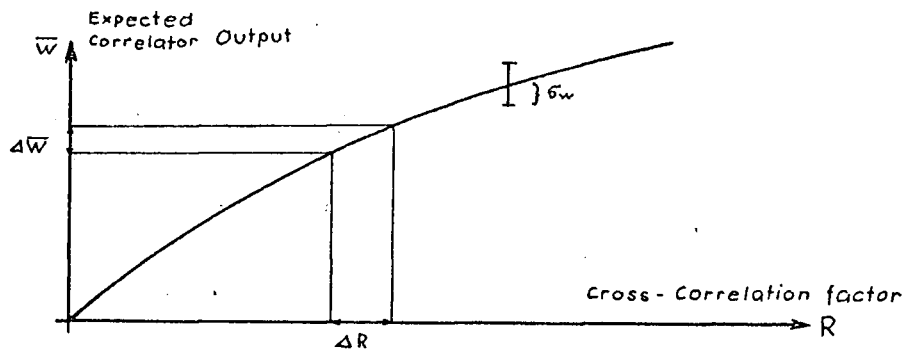
Fig. II.2.1 Typical variation of expected correlator output
with cross-correlation factor

A particular value of w, obtained by averaging over one time-interval, will show deviations from this expected value, $\bar{w}$, with standard deviation σ_w as shown.

If now R changes by ΔR, this will result in a change in the expected output by an amount $\Delta\bar{w}$. Whether such a change is detectable by a single observation of w depends on the relative size of σ_w and $\Delta\bar{w}$.

Quite arbitrarily, we define the change as "detectable" if $\Delta\bar{w}$ exceeds σ_w, and not detectable otherwise. This leads to the concept of

the <u>minimum detectable signal</u>, MDS. It is defined as that change in

R which will result in an expected output change exactly equal to the

standard deviation.

Hence

$$\text{MDS} = \sigma_w / (\frac{d\bar{w}}{dR}) \qquad (II.2.4)$$

The MDS will depend on the noise power, the bandwidth, the

integration time, and (for large signals) on the cross-correlation factor.

When all these are held constant, one can compare minimum detectable

signals for various correlators.

The best instrument, with least MDS, will be the analog

correlator. All other instruments pay some penalty in terms of higher

values of MDS. This finally leads to the definition of the <u>degradation</u>

<u>factor</u>, D, of a particular correlator as

$$D = \text{MDS}/(\text{MDS})_{\text{analog}} \qquad (II.2.5)$$

The calculation of this degradation factor for a variety of

correlators is the subject of this thesis.

II.3 <u>Degradation Factor for Small Signals</u>

When the signals are small compared with the noise (as is

assumed in most of the calculations), the degradation factor can also

be expressed in terms of the output signal-to-noise ratios.

Using Equation (II.2.4), each MDS can be expressed in terms of

σ_w and $(\frac{d\bar{w}}{dR})$. However when the signals are small R will be small, and

Figure II.2.1 can be linearized near the origin.

We then have

$$\text{MDS} = \sigma_w / (\frac{\bar{w}}{R}) = R / (\frac{\bar{w}}{\sigma_w}) = R / (\frac{S}{N}_0). \qquad (II.3.1)$$

so that D can be expressed as

$$D = \frac{(S/N)_o \text{ for analog correlator}}{(S/N)_o \text{ for system}}$$

(II.3.2)

The degradation factor is therefore a measure of the deterioration in output signal-to-noise ratio as a result of the insertion of the signal processors into the correlator.

III. RESULTS OF PRACTICAL INTEREST

III.1 Degradation Factor vs. Sampling Rate

In the following we assume the signal power σ_s^2 to be far below the noise power σ_n^2. Therefore equation (II.3.2) is applicable and we have to find the output signal-to-noise ratio of a correlator in order to compute the correlator degradation factor. We defined the output signal-to-noise ratio in equation (II.2.3).

For small signals it can be shown that σ_w does not depend on σ_s and can be calculated assuming $\sigma_s = 0$. The two inputs are then white, independent Gaussian noise sources[9]. In this case $\bar{w} = 0$ and

$$\sigma_w^2 = \overline{w^2} \tag{III.1.1}$$

The output signal-to-noise ratio then becomes:

$$\left(\frac{S}{N}\right)_0 \qquad \frac{[\bar{w}]\sigma_s}{[\sqrt{\overline{w^2}}]_{s=0}} \tag{III.1.2}$$

III.1.1 Replacement of Small Gaussian Signals by d-c

For small signals ($\sigma_s \ll \sigma_n$) it is shown in Appendix A2, that for the purpose of calculating the degradation factors, the Gaussian zero-mean signals can be replaced by a d-c signal

$$s_x(t) = s_y(t) = s_0 = \text{const.} \tag{III.1.1.1}$$

Then both input signals

$$x(t) = s_0 + n_1(t) \tag{III.1.1.2}$$

and

$$y(t) = s_0 + n_2(t) \tag{III.1.1.3}$$

are independent, Gaussian signals with mean value $\bar{x} = \bar{y} = s_0$ and with variance

$$\sigma_x^2 = \sigma_y^2 = \sigma_n^2 \tag{III.1.1.4}$$

The signals x and y are independent Gaussian random variables. Replacing the signals by a d-c signal, the probability-density functions of x and y become those of the noise signals $n_1(t)$ and $n_2(t)$, displaced by an offset equal to the d-c value, s_o:

$$p_x(x) = \frac{1}{\sqrt{2\pi}\sigma_n} e^{-\frac{1}{2\sigma_n^2}(x-s_o)^2} \qquad (III.1.1.5)$$

$$p_y(y) = \frac{1}{\sqrt{2\pi}\sigma_n} e^{-\frac{1}{2\sigma_n^2}(y-s_o)^2} \qquad (III.1.1.6)$$

Since both channels are statistically independent, the joint probability density, $p_{xy}(x,y)$ is given by the product of the probability-density functions of the two signals x and y:

$$p_{xy}(x,y) = p_x(x) * p_y(y). \qquad (III.1.1.7)$$

It can be shown (see Chapter III.1.2 and III.1.6) that the degradation factor D depends only on the input signal-to-noise ratio defined in II.2.2. Therefore the variance of the signals x and y, σ_n^2, can be arbitrarily set equal to 1 in all further calculations.

III.1.2 Output S/N-Ratio for a Quantized, Sampled Correlator

The block diagram of a quantized, sampled correlator is shown in Fig. III.1.2.1. Assuming f_s as the sampling frequency,

$$T_s = \frac{1}{f_s} \quad \text{denotes the sampling-period} \qquad (III.1.2.1)$$

and

$$K = \frac{f_s}{B} \quad \text{the normalized sampling-rate} \qquad (III.1.2.2)$$
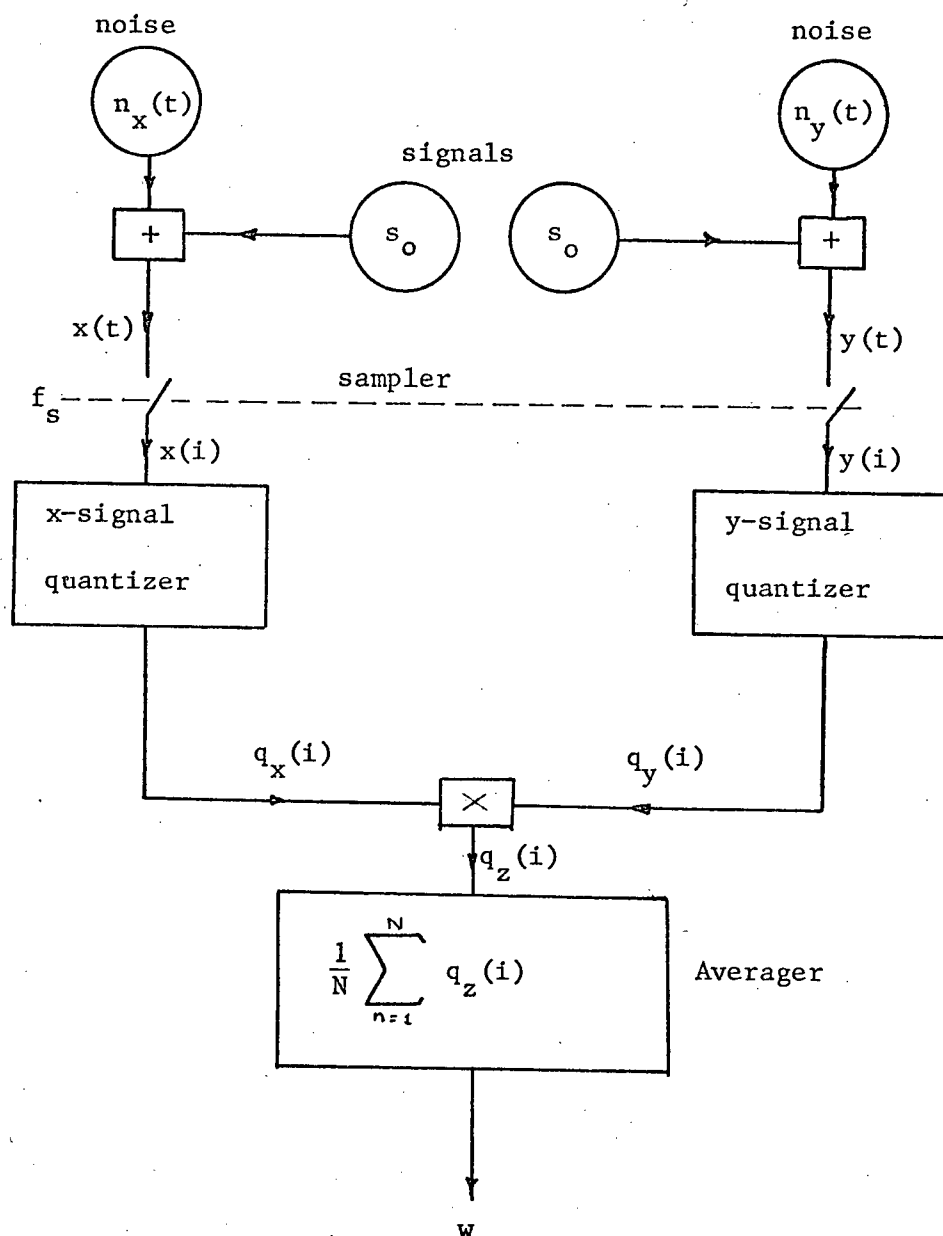
Figure III.1.2.1   Sampled quantized correlator

According to the sampling-theorem, the lowest sampling frequency that allows recovery of a signal from its samples is $f_s = 2B$. In this case the normalized sampling-rate K becomes equal to 2. This lowest possible sampling-rate is called the Nyquist-rate. The value of the time function x(t) in Fig. III.1.2.1 taken at $t = iT_s$ is denoted by x(i). The same is valid for the signals y, q_x, q_y and q_z.

The output signal-to-noise ratio of a correlator with small input signals is defined in (III.1.2.) Therefore we have to find (a) the expected value of the correlator output w when the signal s_o is present and (b) the standard deviation of w when we omit the signal s_o.

(a) Since the two signals x and y are considered to be statistically independent, the expected value of w is equal to the product of the expected value of q_x and q_y, i.e.

$$\bar{w} = \bar{q}_x \cdot \bar{q}_y$$

(III.1.2.3)

Fig. III.1.2.2 shows a symmetric n-level quantizer. $Pr(P_i \leqslant x < P_{i+1})$ denotes the probability of x being between the two decision levels P_i and P_{i+1}.
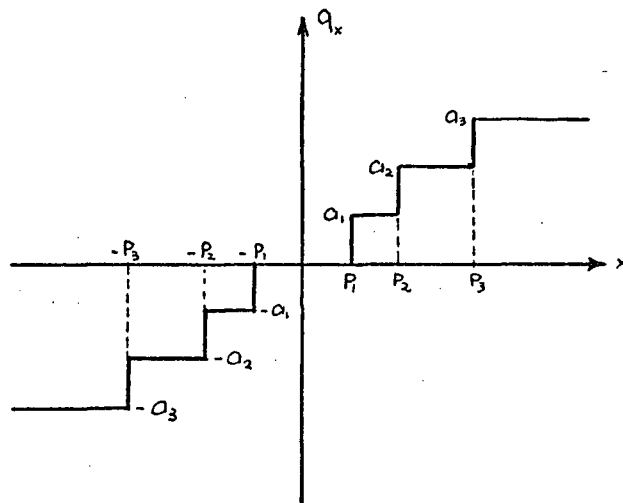
Fig. III.1.2.2   n-level quantizer

The expected value of x becomes

$$\bar{q}_x = \sum_{i=1}^{\frac{n-1}{2}} a_i Pr(P_i \leq x < P_{i+1}) - \sum_{i=1}^{\frac{n-1}{2}} a_i Pr(-P_{i+1} \leq x < -P_i)$$

(III.1.2.4)

where

$$Pr(P_i \leq x < P_{i+1}) = \int_{P_i}^{P_{i+1}} p_x(x)\,dx = \frac{1}{\sqrt{2\pi}} \int_{P_i}^{P_{i+1}} e^{-\frac{1}{2}(x-s_o)^2}\,dx$$

$$= erfc(P_i - s_o) - erfc(P_{i+1} - s_o)$$

(III.1.2.5)

and

$$Pr(-P_{i+1} \leq x < -P_i) = \int_{-P_{i+1}}^{-P_i} p_x(x)\,dx$$

$$= erfc(P_i + s_o) - erfc(P_{i+1} + s_o)$$

(III.1.2.6)

The complement error-function erfc (x) is defined as

$$erfc(x) = \int_x^\infty e^{-\frac{1}{2}x^2}\,dx.$$

(III.1.2.7)

Since $s_o \ll 1$, we can expand each of the terms in eq. (III.1.2.5) and (III.1.2.6) around 0 using a Taylor series.

Therefore

$$erfc(P_i - s_o) \approx erfc(P_i) + \frac{s_o}{\sqrt{2\pi}} e^{-\frac{P_i^2}{2}}$$

(III.1.2.8)

and

$$erfc(P_i + s_o) \approx erfc(P_i) - \frac{s_o}{\sqrt{2\pi}} e^{-\frac{P_i^2}{2}}.$$

(III.1.2.9)

Using (III.1.2.8) in (III.1.2.5) and (III.1.2.9) in (III.1.2.6) we obtain

$$Pr(P_i \leq x < P_{i+1}) \approx erfc(P_i) - erfc(P_i + 1) + \frac{1}{\sqrt{2\pi}} s_o (e^{-\frac{P_i^2}{2}} - e^{-\frac{P_{i+1}^2}{2}})$$

and

(III.1.2.10)

$$Pr(-P_{i+1} \leq x < -P_i) \approx erfc(P_{i+1}) - \frac{1}{\sqrt{2\pi}} s_o (e^{-\frac{P_i^2}{2}} - e^{-\frac{P_{i+1}^2}{2}})$$

(III.1.2.11)

Using III.1.2.4 we find finally

$$\bar{q}_x = \sqrt{\frac{2}{\pi}} \sum_{i=1}^{\frac{n-1}{2}} a_i (e^{-\frac{P_i^2}{2}} - e^{-\frac{P_{i+1}^2}{2}}) . s_o \qquad (III.1.2.12)$$

We define

$$f_x(a_i, P_i) = \sum_{i=1}^{\frac{n-1}{2}} a_i (e^{-\frac{P_i^2}{2}} - e^{-\frac{P_{i+1}^2}{2}}) . \qquad (III.1.2.13)$$

Since $\overline{q_y}$ is of the same form as $\overline{q_x}$, the general form of $\bar{w}$ is therefore

$$\bar{w} = \frac{2}{\pi} s_o^2 f_x(a_i, P_i) \cdot f_y(a_i, P_i) \qquad (III.1.2.14)$$

where $f_x(a_i, P_i)$ and $f_y(a_i, P_i)$ describe the actual quantizers.

(b)   To compute the variance σ_w^2 we can neglect the signal s and as

an approximation consider only two independent noise sources as inputs

to the correlator, i.e.

$$x(t) \approx n_x(t) \qquad (III.1.2.15)$$

and

$$y(t) \approx n_y(t) \qquad (III.1.2.16)$$

Therefore

$$w^2 \approx (\frac{1}{N} \sum_{i=1}^{N} q_z(i))^2 = \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} q_z(i) q_z(j) \qquad (III.1.2.17)$$

and

$$\overline{w^2} \approx \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \overline{q_z(i) q_z(j)} = \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} R_{q_z}(i-j) \qquad (III.1.2.18)$$

Using the relation

$$\sum_{n=1}^{N} \sum_{m=1}^{N} f(k + n-m) = \sum_{n=1}^{N-1} (N-n)(f(k+n) + f(k-n)) + Nf(k)$$

$$\qquad (III.1.2.19)$$

and recognizing that

$$R_{q_z}(k) = R_{q_z}(-k), \qquad (III.1.2.20)$$

it follows that

$$\overline{w^2} = \frac{1}{N} \left(2 \sum_{i=1}^{N-1} \left(1 - \frac{i}{N} \right) R_{q_z}(i) + R_{q_z}(0) \right) \tag{III.1.2.21}$$

and

$$\sigma_w = \sqrt{\overline{w^2}} = \frac{1}{\sqrt{N}} \left(2 \sum_{i=1}^{N-1} \left(1 - \frac{i}{N} \right) R_{q_z}(i) + R_{q_z}(0) \right)^{\frac{1}{2}} \tag{III.1.2.22}$$

The autocorrelation function of $q_z(t)$ is

$$R_{q_z}(\tau) = \overline{q_z(t) \, q_z(t + \tau)} = \overline{q_x(t) \, q_x(t + \tau)} \; \overline{q_y(t) \, q_y(t + \tau)} \tag{III.1.2.23}$$

$$= R_{q_x}(\tau) \, R_{q_y}(\tau) \tag{III.1.2.24}$$

since $q_x(t)$ and $q_y(t)$ are statistically independent random processes.

Note that

$$R_{q_z}(iT_s) \doteq R_{q_z}(i) = R_{q_x}(i) \, R_{q_y}(i). \tag{III.1.2.25}$$

The autocorrelation function of $q_x(t)$ is given by

$$R_{q_x}(\tau) = \overline{q_x(t) \, q_x(t + \tau)}. \tag{III.1.2.26}$$

Let the normalized autocorrelation function of $x(t)$ be denoted by

$$\rho_x(\tau) = \rho(\tau) = \frac{R_x(\tau)}{R_x(0)} \approx \frac{R_n(\tau)}{R_n(0)} = \frac{\mathcal{F}^{-1}\{S_n(f)\}}{\sigma_n^2} = Sa(2\pi B\tau) \tag{III.1.2.27}$$

where

$$Sa(\cdot) = \frac{\sin(\cdot)}{(\cdot)} \tag{III.1.2.28}$$

Let

$$v(t) = x(t + \tau). \tag{III.1.2.29}$$

Then x and v are jointly normal random variables with the joint probability-density function

$$P_{xv}(x,v) = \frac{1}{2\pi\sqrt{1-\rho^2(\tau)}} \, e^{- \frac{1}{2(1-\rho^2(\tau))} (x^2 - 2\rho(\tau)xv + v^2)} \tag{III.1.2.30}$$

Considering again the symmetric n-level correlator (Fig. III.1.2.1)

we see that

$q_x q_v = a_i a_j$ with probability $\Pr(P_i \leq x < P_{i+1},\ P_j \leq v < P_{j+1}) + \Pr(-P_{i+1} \leq x < -P_i,$

$$-P_{j+1} \leq v < -P_j) \qquad \text{(III.1.2.31)}$$

and

$q_x q_v = -a_i a_j$ with probability $\Pr(-P_{i+1} \leq x < -P_i,\ P_j \leq v < P_{j+1}) + \Pr(P_i \leq x < P_{i+1},$

$$-P_{j+1} \leq v < -P_j), \qquad \text{(III.1.2.32)}$$

where

$$i,j = 1,\ 2,\ \ldots\ \frac{n-1}{2}\ .$$

Therefore,

$$\overline{q_x q_v} = \sum_{i=1}^{\frac{n-1}{2}} \sum_{j=1}^{\frac{n-1}{2}} a_i a_j [\Pr(P_i \leq x \leq P_{i+1}, P_j \leq v < P_{j+1}) + \Pr(-P_{i+1} \leq x < -P_i,\ -P_{j+1} \leq v < -P_j]$$

$$\text{(III.1.2.33)}$$

$$-\ \sum_{i=1}^{\frac{n-1}{2}} \sum_{j=1}^{\frac{n-1}{2}} a_i a_j\ \Pr(-P_{i+1} \leq x < -P_i,\ P_j \leq v < P_{j+1}) + \Pr(P_i \leq x < P_{i+1},\ -P_{j+1} \leq v < -P_j)$$

Since $p_{xv}(x,v)$ is symmetric in x and v and $p_{xv}(x,v) = p_{xv}(-x,-v)$,

$\Pr(P_i \leq x < P_{i+1},\ P_j \leq v < P_{j+1}) = \Pr(-P_{i+1} \leq x < -P_i,\ -P_{j+1} \leq v < -P_j)$

$$= \int_{P_i}^{P_{i+1}} \int_{P_j}^{P_{j+1}} p_{xv}(x,v) dx dv = \frac{1}{\sqrt{2\pi}} \int_{P_i}^{P_{i+1}} [\text{erfc}(\frac{P_j - \rho(\tau)x}{\sqrt{1-\rho^2(\tau)}}) - \text{erfc}(\frac{P_{j+1} - \rho(\tau)x}{\sqrt{1-\rho^2(\tau)}})] e^{-\frac{1}{2}x^2} dx$$

$$\text{(III.1.2.34)}$$

and

$\Pr(P_i \leq x < P_{i+1},\ -P_{j+1} \leq v < -P_j) = \Pr(-P_{i+1} \leq x < -P_j,\ P_j \leq v < P_{j+1})$

$$= \int_{P_i}^{P_{i+1}} \int_{-P_{j+1}}^{-P_j} p_{xv}(x,v) dx dv = \frac{1}{\sqrt{2\pi}} \int_{P_i}^{P_{i+1}} e^{-\frac{1}{2}x^2} (\text{erfc}(\frac{P_j + \rho(\tau)x}{\sqrt{1-\rho^2(\tau)}}) - \text{erfc}(\frac{P_{j+1} + \rho(\tau)x}{\sqrt{1-\rho^2(\tau)}}) dx$$

$$\text{(III.1.2.35)}$$

and finally

$$R_{q_x}(\tau) = 2 \sum_{i=1}^{\frac{n-1}{2}} \sum_{j=1}^{\frac{n-1}{2}} a_i a_j [\Pr(P_i \leq x < P_{i+1}, P_j \leq v < P_{j+1}) - \Pr(+P_i \leq x < P_{i+1}, -P_{j+1} \leq v < -P_j)].$$

(III.1.2.36)

$$R_{q_x}(0) = \overline{q_x^2} = 2 \sum_{i=1}^{(\frac{n-1}{2})} a_i^2 \Pr(P_i \leq x < P_{i+1}).$$

(III.1.2.37)

The same calculations hold for the y-channel if x is replaced by y in the above formulas.

The output (S/N) - ratio is therefore

$$\left(\frac{S}{N}\right)_0 = \frac{\overline{w}}{\sigma_w} = \frac{\frac{2}{\pi}\sqrt{N}\, s_o^2 f_x(a_i, P_i)\, f_y(a_i, P_i)}{\left(2 \sum_{i=1}^{N-1}(1-\frac{i}{N})R_{q_x}(i) \cdot R_{q_y}(i) + R_{q_x}(0) \cdot R_{q_y}(0)\right)^{\frac{1}{2}}}$$

(III.1.2.38)

where $f_x(a_i, P_i)$ and $f_y(a_i, P_i)$ are given by equation (III.1.2.13)

$R_{q_x}(i), R_{q_y}(i)$ (III.1.2.36) for $i = iT_s$,

and $R_{q_x}(0), R_{q_y}(0)$ (III.1.2.37).

The total number of samples during the observation time T is

$$N = \frac{T}{T_s} = KBT$$

(III.1.2.39)

III.1.3 $\left(\frac{S}{N}\right)_0$ for Sampling at Nyquist-Rate

For the calculations in this chapter, the Nyquist sampling rate $f_s = 2B$ is assumed. i.e. K = 2.

The autocorrelation function of the bandlimited noise is:

$$\rho_n(\tau) = Sa(2\pi B\tau).$$

(III.1.3.1)

Therefore,

$$\rho_n(iT_s) = Sa(\pi i) = \begin{cases} 1, & \text{if } i = 0 \\ 0, & \text{if } i \neq 0 \end{cases},$$

(III.1.3.2)

i.e. the noise-samples taken $T_s = \frac{1}{2B}$ apart are uncorrelated, and since they are Gaussian, they are also statistically independent. It is easy to see that

$$R_{q_x}(iT_s) = R_{q_y}(iT_s) = 0 \quad \text{for} \quad K = 2, \qquad (\text{III.1.3.3})$$

since the quantizers are memoryless devices.

Replacing N by 2BT in (III.1.2.38) yields the output signal-to-noise ratio

$$\left(\frac{S}{N}\right)_o = \frac{2}{\pi} s_o^2 \sqrt{2BT} \; \frac{f_x(a_i, P_i) \cdot f_y(a_i, P_i)}{\sqrt{R_{q_x}(0) \cdot R_{q_y}(0)}} \qquad (\text{III.1.3.4})$$

It is remarkable that $\left(\frac{S}{N}\right)_o$ can be expressed as a product of two functions

$$\frac{f_x(a_i, P_i)}{\sqrt{R_{q_x}(0)}} \qquad \text{and} \qquad \frac{f_y(a_i, P_i)}{\sqrt{R_{q_y}(0)}}$$

which depend only on one channel of the correlator. Therefore, for sampling at Nyquist-rate, the dual channel correlator can be decomposed into single channel correlators. This result is due to F. Bowers[5] and is treated in more detail in Chapter V.2. It can be seen (III.1.2.38) that we improve $\left(\frac{S}{N}\right)_o$ by sampling faster than at Nyquist-rate and that sampling at rate infinity gives us the asymptotic value or the maximum $\left(\frac{S}{N}\right)_o$ of a quantized correlator.

III.1.4. $\left(\frac{S}{N}\right)_o$ for an Infinite Sampling-Rate

It can be seen that the limiting case, when K goes to infinity, corresponds to an unsampled but quantized correlator. Thus signal-to-noise ratio is maximized for this limiting case.

From III.1.2.13 and III.1.2.14 we see that $\bar{w}$ does depend on K.

Substituting $N = KBT$ in $(III.1.2.22)$ we get

$$\overline{w^2} = \frac{1}{KBT} \left(2 \sum_{i=1}^{KBT-1} (1 - \frac{i}{KBT}) R_{q_z}(i) + R_{q_z}(0)\right) \qquad (III.1.4.1)$$

Taking the limit we obtain

$$\lim_{K \to \infty} \overline{w^2} = \frac{1}{BT} \lim_{K \to \infty} \frac{1}{K} \left(2 \sum_{i=1}^{KBT-1} (1 - \frac{i}{KBT}) R_{q_z}(i) + R_{q_z}(0)\right]$$

$$= \frac{1}{T} 2 \int_0^T (1 - \frac{t}{T}) R_{q_z}(t) dt. \qquad (III.1.4.2)$$

This result can also be found by computing $\overline{w^2}$ for an unsampled correlator, where

$$\frac{1}{N} \sum_{i=1}^{N} q_z(i) \text{ is replaced by } \frac{1}{T} \int_0^T q_z(t) dt.$$

Finally

$$\lim_{K \to \infty} \left(\frac{S}{N}\right)_o = \frac{2}{\pi} \sigma_s^2 \sqrt{T} \frac{f_x(a_i, P_i) f_y(a_i, P_i)}{(2 \int_0^T (1 - \frac{t}{T}) R_{q_z}(t) dt)^{\frac{1}{2}}} \qquad (III.1.4.3)$$

It was found in II.3.6 that the degradation factor is the ratio of $\left(\frac{S}{N}\right)_o$ of the quantized correlator to $\left(\frac{S}{N}\right)_o$ of the analog correlator. The value of $\left(\frac{S}{N}\right)_o$ for the analog correlator is calculated in the following chapter.

III.1.5 $\left(\frac{S}{N}\right)_o$ of an Analog Correlator

We can omit the quantizers in the x- and y-channel or, equivalently, set

$$q_x = x \qquad (III.1.5.1)$$

and

$$q_y = y. \qquad (III.1.5.2)$$

Then

$$\overline{w} = \overline{x} \cdot \overline{y} \qquad (III.1.5.3)$$

and

$$\bar{x} = \bar{y} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} xe^{-\frac{1}{2}(x - s_o)^2} dx = s_o \tag{III.1.5.4}$$

Therefore

$$\bar{w} = s_o^2. \tag{III.1.5.5}$$

The variance of w is given by (III.1.2.21) as

$$\overline{w^2} = \frac{1}{KBT} (2 \sum_{i=1}^{KBT-1} (1 - \frac{i}{KBT}) R_x^2(i) + R_x^2(0)), \tag{III.1.5.6}$$

where

$$R_x(i) = \rho_n(i) = Sa(\frac{2\pi i}{K}). \tag{III.1.5.7}$$

For long integration time

$$\overline{w^2} = \frac{1}{BT} \frac{1}{K} (2 \sum_{i=1}^{\infty} Sa^2(\frac{2\pi i}{K}) + 1) = \frac{1}{2BT} . \tag{III.1.5.8}$$

$$\text{for any } K \geq 2$$

The relationship

$$\frac{1}{K} \sum_{i=-\infty}^{\infty} Sa^2(\frac{2\pi i}{K}) = \frac{1}{2} \tag{III.1.5.9}$$

$$\text{for any } K \geq 2$$

is proved in Appendix A1. Therefore,

$$(\frac{S}{N})_o = \frac{\bar{w}}{\sqrt{\overline{w^2}}} = s_o^2 \sqrt{2BT} \tag{III.1.5.10}$$

and this is independent of the sampling-rate K as long as $K \geq 2$

III.1.6 General Formula for the Degradation Factor

Let n be the number of quantizer-levels in the x-channel and let m be the number of quantizer-levels in the y-channel. Then $D_{n \times m}$ the degradation factor of an n×m level correlator sampled at rate K×B. $D_{n \times m}$ will be a function of K.

$$D_{n \times m}(K) = \frac{(\frac{S}{N})_{o,analog}}{(\frac{S}{N})_{o,n \times m}} = \frac{s_o^2 \sqrt{2BT}}{(\frac{S}{N})_{o,n \times m}} \qquad \text{(III.1.6.1)}$$

Substituting III.1.2.10 in III.1.6.1 and letting N = KBT gives:

$$D_{n \times m}(K) = \frac{\pi}{2} \sqrt{\frac{2}{K}} \; \frac{\left\{ 2 \sum\limits_{i=1}^{KBT-1} (1 - \frac{i}{KBT}) R_{q_x}(i) R_{q_y}(i) + R_{q_x}(0) R_{q_y}(0) \right\}^{\frac{1}{2}}}{f_x(a_i, P_i) \cdot f_y(a_i, P_i)} \qquad \text{(III.1.6.2)}$$

At the Nyquist-rate, since $R_{q_x}(i)$ and $R_{q_y}(i) = 0$ for $i \neq 0$, we have

$$D_{n \times m}(2) = \frac{\pi}{2} \frac{\sqrt{R_{q_x}(0) \cdot R_{q_y}(0)}}{f_x(a_i, P_i) f_y(a_i, P_i)} \qquad \text{(III.1.6.3)}$$

For K→∞, (III.1.4.3) substituted in (III.1.6.1) yields

$$D_{n \times m}(\infty) = \pi \; \frac{\left\{ B \int_o^T (1 - \frac{\tau}{T}) R_{q_x}(\tau) R_{q_y}(\tau) d\tau \right\}^{\frac{1}{2}}}{f_x(a_i, P_i) f_y(a_i, P_i)} \qquad \text{(III.1.6.4)}$$

For long integration time

$$\lim_{T \to \infty} D_{n \times m}(\infty) = \pi \; \frac{(B \int_o^\infty R_{q_x}(\tau) R_{q_y}(\tau) d\tau)^{\frac{1}{2}}}{f_x(a_i, P_i) \; f_y(a_i, P_i)} \qquad \text{(III.1.6.5)}$$

where $R_{q_x}(\tau)$ and $R_{q_y}(\tau)$ are functions of only $\rho(\tau) = Sa(2\pi B\tau)$. Substituting

$x = B\tau$, $D_{n \times m}(\infty)$ becomes independent of the bandwidth B. The degradation

factor D is independent of the input signals, the bandwidth, and the

integration time and is only a function of the quantizers, the multi-

plication scheme, and the sampling-rate. The definition of D allows

us to compare different correlators. In the following chapter the

degradation-factor D is calculated for four different quantized correlators.

III.2 Degradation Factor of the 2×2, 3×3, 2×3, 3×5 and 4×4 Level Correlators

III.2.1 2×2 Level Correlator

Both the x and the y-channel have quantizers shown in figure

(III.2.1.1)
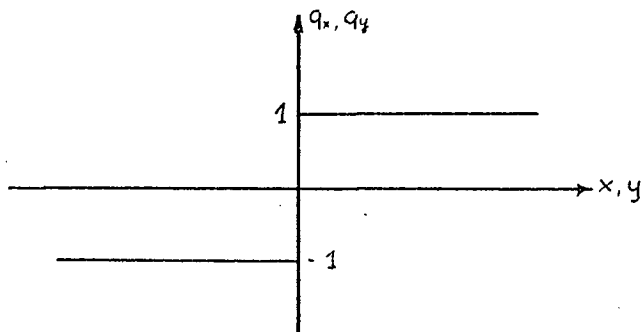
Fig. III.2.1.1 2-level quantizer

The quantized signals $q_x(t)$ and $q_y(t)$ are given by

$$q_x(t) = \text{sign}(x(t)) \tag{III.2.1.1}$$

and

$$q_y(t) = \text{sign}(y(t)). \tag{III.2.1.2}$$

Since x(t) and y(t) are statistically independent signals, the expected value of w is given by the product of the expected values of $q_x(t)$ and $q_y(t)$. Therefore

$$\bar{q}_x = \bar{q}_y = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{1}{2}(x-s_o)^2} dx - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 e^{-\frac{1}{2}(x-s_o)^2} dx = 1 - 2\text{erfc}(s_o) \tag{III.2.1.3}$$

Taking the first two terms of a Taylor series expansion of the complement error-function around $s_o = 0$ we obtain

$$\bar{q}_x = \bar{q}_y = \sqrt{\frac{2}{\pi}}\, s_o \tag{III.2.1.4}$$

therefore

$$\bar{w} = \frac{2}{\pi} s_o^2 \tag{III.2.1.5}$$

and the functions defined in (III.1.2.13) become then

$$f_x(a_i, P_i) = 1 \tag{III.2.1.6}$$

and
$$f_y(a_i, P_i) = 1 \qquad \text{(III.2.1.7)}$$

The autocorrelation functions $R_{q_x}(\tau)$ and $R_{q_y}(\tau)$ can easily be found with the van Vleck relation (or arcsin law, see[1] pg. 483) to be

$$R_{q_x}(\tau) = R_{q_y}(\tau) = \frac{2}{\pi} \arcsin(\rho_n(\tau)), \qquad \text{(III.2.1.8)}$$

where
$$\rho_n(\tau) = Sa(2\pi B\tau) \qquad \text{(III.2.1.9)}$$

and
$$R_{q_x}(0) = 1. \qquad \text{(III.2.1.10)}$$

Equations (III.2.1.6) to (III.2.1.10) used in (III.1.6.2) yield

$$D_{2\times2}(K) = \frac{\pi}{2}\sqrt{\frac{2}{K}}\, (\frac{8}{\pi^2} \sum_{i=1}^{KBT-1} (1-\frac{i}{KBT})(\arcsin(Sa(\frac{2\pi i}{K})))^2 + 1)^{\frac{1}{2}} \qquad \text{(III.2.1.11)}$$

At the Nyquist-rate we have

$$D_{2\times2}(2) = \frac{\pi}{2} \qquad \text{(III.2.1.12)}$$

For $K\to\infty$ using (III.1.6.5) it follows that

$$D_{2\times2}(\infty) = \sqrt{\frac{2}{\pi}}\, (\int_0^\infty (\arcsin(Sa(\tau)))^2 d\tau)^{\frac{1}{2}} \qquad \text{(III.2.1.13)}$$

The integral $\int_0^\infty (\arcsin(Sa(\tau)))^2\, d\tau$ cannot be solved analytically. However, an upper bound can be found to be (see Figure III.2.1.2):
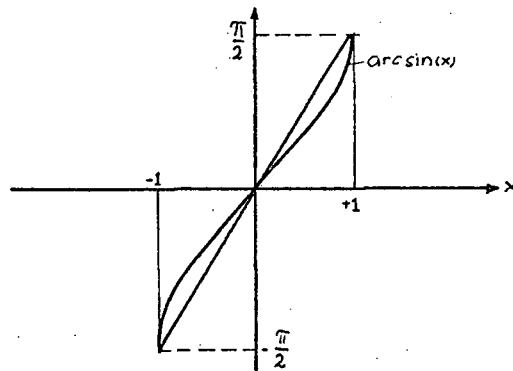
Fig. III.2.1.2 arcsin(x) vs. x

$$|\arcsin(x)| \leq \frac{\pi}{2} |x| \qquad \text{(III.2.1.14)}$$

$$|Sa(\tau)| = \left|\frac{\sin\omega}{\tau}\right| \leq \left|\frac{1}{\tau}\right| \qquad \text{(III.2.1.15)}$$

Letting $x = Sa(\tau)$, we obtain

$$|\arcsin(Sa(\tau))| \leq \frac{\pi}{2} |Sa(\tau)| \leq \frac{\pi}{2} \left|\frac{1}{|\tau|}\right| \qquad \text{(III.2.1.16)}$$

Therefore,

$$(\arcsin(Sa(\tau)))^2 \leq \frac{\pi^2}{4\tau^2} \qquad \text{(III.2.1.17)}$$

There exists an R such that

$$\int_R^\infty (\arcsin(Sa(\tau)))^2 d\tau \leq \frac{\pi^2}{4} \int_R^\infty \frac{1}{\tau^2} d\tau = \frac{\pi^2}{4R} \qquad \text{(III.2.1.18)}$$

The integral

$$\int_0^R (\arcsin(Sa(\tau)))^2 d\tau \text{ can be evaluated numerically.}$$

The error of the remainder can be bounded with any desired accuracy

using (III.2.1.18).

Evaluating $\int_0^R (\arcsin(Sa(\tau))^2 d\tau$ numerically for R = 1000 yields

1.2515 ± 0.00013.

Let

$$I = \int_0^{1000} (\arcsin(Sa(\tau))^2 d\tau \qquad \text{(III.2.1.19)}$$

and

$$\epsilon = \frac{\pi^2}{4 \cdot R \cdot I} \qquad \text{(III.2.1.20)}$$

Then

$$D_{2\times2}(\infty) \leq \sqrt{\frac{2I}{\pi}} \sqrt{1 + \epsilon} \leq \sqrt{\frac{2I}{\pi}} \cdot (1 + \frac{1}{2}\epsilon) \qquad \text{(III.2.1.21)}$$

A lower bound is given by the accuracy of the numerical integration of I.

Therefore,

$$1.2515 - 0.00013 \leq D_{2\times2}(\infty) \leq \sqrt{\frac{2I}{\pi}} (1 + \frac{1}{2}\epsilon) \qquad \text{(III.2.1.22)}$$

and

$$1.25137 \leq D_{2\times2}(\infty) \leq 1.2528 \qquad \text{(III.2.1.23)}$$

Yerbury[13] found $D_{2 \times 2}(\infty)$ by an approximation as $\frac{\pi}{\sqrt{6}} = 1.28$. He states that his value is 2-3% too high, which agrees with our result. $D_{2 \times 2}(K)$ is plotted in Fig. III.2.1. on page 34.

It is remarkable that we can achieve up to a 20% lower degradation factor for the 2×2 level correlator by sampling faster than at Nyquist-rate. At 4 times the Nyquist-rate the degradation factor is 18% lower and at twice the Nyquist-rate it is 14% lower.

III.2.2 3×3 Level Correlator

Both the x- and the y-channel have quantizers of the form illustrated in Fig. III.2.2.1:
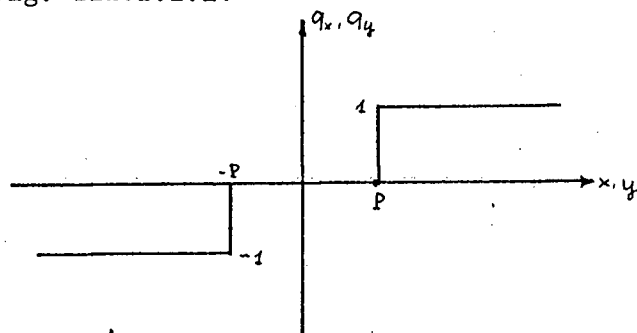
Fig. III.2.2.1 3-level quantizer

P is the decision level and should be optimized to yield a minimum degradation factor. The expected value of w is found by letting $P_2 \to \infty$ in (III.1.2.13). Then we get

$$f_x(a_i, P_i) = f_y(a_i, P_i) = e^{-\frac{P^2}{2}} \qquad (III.2.2.1)$$

and

$$\bar{w} = \frac{2}{\pi} s_o^2 e^{-P^2} \qquad (III.2.2.2)$$

To calculate the standard deviation σ_w, the autocorrelation functions $R_{q_x}(\tau)$ and $R_{q_y}(\tau)$ can be obtained from (III.1.2.36) and (III.1.2.37) as

$$R_{q_x}(\tau) = \sqrt{\frac{2}{\pi}} \left(\int_P^\infty e^{-\frac{x^2}{2}} \left(\text{erfc}\left(\frac{P-\rho(\tau)x}{\sqrt{1-\rho^2(\tau)}}\right) - \text{erfc}\left(\frac{P+\rho(\tau)x}{\sqrt{1-\rho^2(\tau)}}\right) \right) dx \right) \qquad \text{(III.2.2.3)}$$

and

$$R_{q_x}(0) = 2 \text{ erfc}(P) \qquad \text{(III.2.2.4)}$$

Above results in (III.1.6.2) substituted yield

$$D_{3\times3}(K) = \frac{\pi}{2}\sqrt{\frac{2}{K}} \left\{ 2 \sum_{i=1}^{KBT-1} \left(1 - \frac{i}{KBT}\right) R_{q_x}^2(i) + 4 \text{ erfc}^2(P) \right\}^{\frac{1}{2}} e^{P^2} \qquad \text{(III.2.2.5)}$$

For sampling at Nyquist-rate, $D_{2\times2}(2)$ becomes

$$D_{3\times3}(2) = \pi e^{P^2} \text{erfc}(P) \qquad \text{(III.2.2.6)}$$

and for $K \to \infty$ the degradation factor takes on the limit

$$D_{3\times3}(\infty) = \pi e^{P^2} \left(B \int_0^\infty R_{q_x}^2(\tau) d\tau \right)^{\frac{1}{2}} \qquad \text{(III.2.2.7)}$$

The decision level P can be optimized as shown in Chapter V.1.1. The optimum value of P, which depends on the sampling-rate, is 0.612 at Nyquist-rate and about 16% higher at infinite sampling rate.

III.2.3 2×3 Level Correlator

In this case the x-channel has a 2-level quantizer (Fig. III.2.2.1). The functions $f_x(a_i, P_i)$ have already been found in (III.2.2.1.6) and (III.2.2.1) respectively for the 2×2 and the 2×3 level correlators. Therefore,

$$\bar{w} = \frac{2}{\pi} s_o^2 e^{-\frac{P^2}{2}} \qquad \text{(III.2.3.1)}$$

The autocorrelation functions $R_{q_x}(\tau)$ and $R_{q_y}(\tau)$ have been found in (III.2.1.8) and (III.2.1.10) for the 2×2 and in (III.2.2.3) and (III.2.2.4) for the 3×3 level correlator.

Using (III.1.6.2) we obtain therefore,

$$D_{2\times3}(K) = \frac{\pi}{2}\sqrt{\frac{2}{K}}\ e^{\frac{P^2}{2}}\left\{2\sum_{i=1}^{KBT-1}(1-\frac{i}{KBT})R_{q_x}(i)R_{q_y}(i) + 2erfc(P)\right\}^{\frac{1}{2}} \qquad \text{(III.2.3.2)}$$

Sampled at Nyquist-rate, $D_{2\times3}(2)$ becomes

$$D_{2\times3}(2) = \frac{\pi}{2}\sqrt{2erfc(P)}\ e^{\frac{P^2}{2}} \qquad \text{(III.2.3.3)}$$

and as $K\to\infty$ $D_{2\times3}(\infty)$ takes on the limit

$$D_{2\times3}(\infty) = \pi e^{\frac{P^2}{2}}(B\int_0^\infty R_{q_x}(\tau)R_{q_y}(\tau)d\tau)^{\frac{1}{2}} \qquad \text{(III.2.3.4)}$$

Again P can be optimized as shown in Chapter V.1.2 and is 0.612 at Nyquist-rate and slightly higher than P for the 3×3 level correlator at higher sampling-rates.

III.2.4   3×5 Level Correlator

The x-channel has 3 levels; its quantizer is shown in Fig. III.2.2.1. The y-channel has a 5-level quantizer as shown in Fig. III.2.4.1.
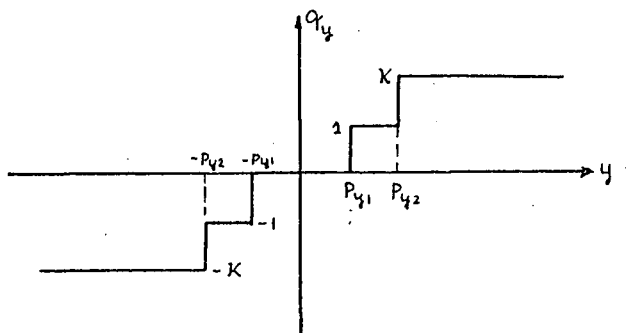
Fig. III.2.4.1   5-level quantizer

The functions $f_x(a_i, P_i)$ and $f_y(a_i, P_i)$ are given by (III.1.2.13). The function $f_x(a_i, P_i)$ was calculated in (III.2.2.1) and

$$f_y(a_i, P_i) = e^{-\frac{P_{y1}^2}{2}} + (\kappa - 1)e^{-\frac{P_{y2}^2}{2}} \qquad \text{(III.2.4.1)}$$

Therefore,

$$\bar{w} = \frac{2}{\pi} s_o^2 \, e^{-\frac{P^2}{2}} (e^{-\frac{P_{y1}^2}{2}} + (\kappa - 1)e^{-\frac{P_{y2}^2}{2}}) \qquad \text{(III.2.4.2)}$$

The autocorrelation function $R_{q_x}(\tau)$ is given by (III.2.2.3) and III.2.2.4)

and

$$R_{q_y}(\tau) = \sqrt{\frac{2}{\pi}} [\int_{P_{y1}}^{P_{y2}} e^{-\frac{x^2}{2}} (\text{erfc}(\frac{P_{y1} - \rho(\tau)x}{\sqrt{1-\rho^2(\tau)}}) - \text{erfc}(\frac{P_{y1} + \rho(\tau)x}{\sqrt{1-\rho^2(\tau)}})$$

$$+ \text{erfc}(\frac{P_{y2} + \rho(\tau)x}{\sqrt{1-\rho^2(\tau)}}) - \text{erfc}(\frac{P_{y2} - \rho(\tau)x}{\sqrt{1-\rho^2(\tau)}}))dx$$

$$+ 2\kappa \int_{P_{y1}}^{P_{y2}} e^{-\frac{x^2}{2}} (\text{erfc}(\frac{P_{y2} - \rho(\tau)x}{\sqrt{1-\rho^2(\tau)}}) - \text{erfc}(\frac{P_{y2} + \rho(\tau)x}{\sqrt{1-\rho^2(\tau)}}))dx$$

$$+ \kappa^2 \int_{P_{y2}}^{\infty} e^{-\frac{x^2}{2}} (\text{erfc}(\frac{P_{y2} - \rho(\tau)x}{\sqrt{1-\rho^2(\tau)}}) - \text{erfc}(\frac{P_{y2} + \rho(\tau)x}{\sqrt{1-\rho^2(\tau)}}))dx]$$

and $\qquad$ (III.2.4.3)

$$R_{q_y}(0) = 2(\text{erfc}(P_{y1}) + (\kappa^2 - 1)\text{erfc}(P_{y2})) \qquad \text{(III.2.4.4)}$$

Substituting the above results into equation (III.1.6.2), we get

$$D_{3 \times 5}(2) = \frac{\pi}{2} \frac{\sqrt{\frac{2}{K}} \{2 \sum_{i=1}^{KBT-1} [(1-\frac{i}{KBT})R_{q_x}(i)R_{q_y}(i)] + 4\text{erfc}(P) \cdot (\text{erfc}(P_{y1}) + (\kappa^2-1)\text{erfc}(P_{y2}))\}^{\frac{1}{2}}}{e^{-\frac{P^2}{2}}(e^{-\frac{P_{y1}^2}{2}} + (\kappa - 1)e^{-\frac{P_{y2}^2}{2}})}$$

$$\text{(III.2.4.5)}$$

Sampled at Nyquist-rate, $D_{3\times5}(2)$ becomes

$$D_{3\times5}(2) = \widetilde{\pi} \frac{(\text{erfc}(P)(\text{erfc}(P_{y1}) + (\kappa^2-1)\text{erfc}(P_{y2}))^{\frac{1}{2}}}{e^{-\frac{P^2}{2}}(e^{-\frac{P_{y1}^2}{2}} + (\kappa-1)e^{-\frac{P_{y2}^2}{2}})} \qquad (III.2.4.6)$$

and as K goes to infinity $D_{3\times5}(K)$ takes on the limit

$$D_{3\times5}(\infty) = \pi \frac{\sqrt{B \int_0^\infty R_{qx}(\tau)R_{qy}(\tau)d\tau}}{e^{-\frac{P^2}{2}}(e^{-\frac{P_{y1}^2}{2}} + (\kappa-1)e^{-\frac{P_{y2}^2}{2}})} \qquad (III.2.4.7)$$

P, P_{y1} and P_{y2} can be optimized as shown in Chapter V.1.4. At Nyquist-rate $P_{opt} = 0.612$, $P_{y1,opt} = 0.422$ and $P_{y2,opt} = 1.266$.

III.2.5 <u>4×4 Level Correlator</u>

Fig. III.2.5.1 shows the quantizer used in the x- and in the y-channel.
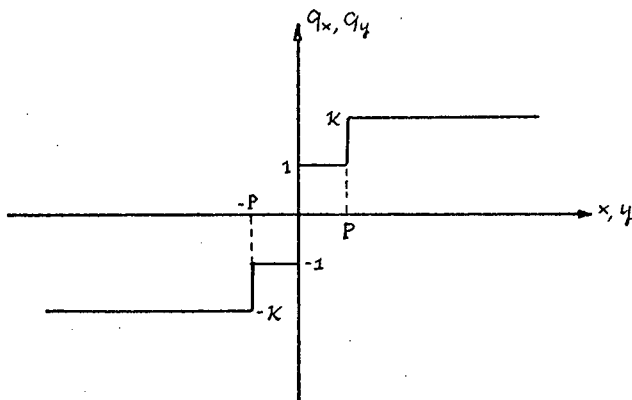
Fig. III.2.5.1 4-level quantizer

The formulae (III.1.2.13), (III.1.2.36) and (III.1.2.37) are easily applied to quantizers with an even number of levels by letting $P_1 = 0$, since a n-level quantizer (n even) is equal to a n+1 level

quantizer with the first decision level P_1 set to zero.

For a 4 level quantizer we set n=5 and take the a_i's and P_i's as shown below:

$$n=5$$

$$P_1=0 \qquad\qquad a_1 = 1$$

$$P_2=P \qquad\qquad a_2 = \kappa$$

$$P_3=\infty$$

Using III.1.2.12 we obtain

$$f_x(a_i, P_i) = f_y(a_i, P_i) = 1 + (\kappa - 1)e^{-\frac{P^2}{2}} \qquad (III.2.5.1)$$

and therefore

$$\bar{w} = \frac{2}{\pi} s_o^2 (1 + (\kappa - 1)e^{-\frac{P^2}{2}})^2 \qquad (III.2.5.2)$$

The autocorrelation functions $R_{q_x}(\tau)$ and $R_{q_y}(\tau)$ are the same and given by (III.1.2.36) and (III.1.2.37):

$$R_{q_x}(\tau) = R_{q_y}(\tau) = \sqrt{\frac{2}{\pi}}[\int_o^P e^{-\frac{x^2}{2}} (erfc(\frac{\rho(\tau)x}{\sqrt{1-\rho^2(\tau)}}) - erfc(\frac{\rho(\tau)x}{\sqrt{1-\rho^2(\tau)}})$$

$$+ erfc(\frac{P+\rho(\tau)x}{\sqrt{1-\rho^2(\tau)}}) - erfc(\frac{P-\rho(\tau)x}{\sqrt{1-\rho^2(\tau)}}))dx$$

$$+ 2\kappa \int_o^P e^{-\frac{x^2}{2}} (erfc(\frac{P-\rho(\tau)x}{\sqrt{1-\rho^2(\tau)}}) - erfc(\frac{P+\rho(\tau)x}{\sqrt{1-\rho^2(\tau)}}))dx$$

$$+ \kappa^2 \int_P^\infty e^{-\frac{x^2}{2}} (erfc(\frac{P-\rho(\tau)x}{\sqrt{1-\rho^2(\tau)}})-erfc(\frac{P+\rho(\tau)x}{\sqrt{1-\rho^2(\tau)}}))dx]$$

$$(III.2.5.3)$$

Note that $\quad$ erf$(\cdot)$ = 1-erfc$(\cdot)$ $\hspace{3cm}$ (III.2.5.4)

and

$$R_{q_x}(0) = R_{q_y}(0) = 1 + 2(\kappa^2-1)\text{erfc}(P) \hspace{2cm} \text{(III.2.5.5)}$$

above results used in (III.1.6.2) yield

$$D_{4\times4}(K) = \frac{\pi}{2}\sqrt{\frac{2}{K}}\,\frac{\left(2\sum\limits_{i=1}^{KBT-1}(1-\frac{i}{KBT})R_{q_x}^2(i)+(1+2(\kappa^2-1)\text{erfc}(P))^2\right)^{\frac{1}{2}}}{(1+(\kappa-1)e^{-\frac{P^2}{2}})^2} \hspace{1cm} \text{(III.2.5.6)}$$

Sampled at Nyquist-rate, $D_{4\times4}$ then becomes

$$D_{4\times4}(2) = \frac{\pi}{2}\,\frac{1+2(\kappa^2-1)\text{erfc}(P)}{(1+(\kappa-1)e^{-\frac{P^2}{2}})^2} \hspace{2cm} \text{(III.2.5.7)}$$

and as K goes to infinity, $D_{4\times4}$ takes on the limit

$$D_{4\times4}(\infty) = \pi\,\frac{\left(B\int_0^\infty R_{q_x}^2(\tau)d\tau\right)^{\frac{1}{2}}}{(1+(\kappa-1)e^{-\frac{P^2}{2}})^2} \hspace{2cm} \text{(III.2.5.8)}$$

The optimized values of P vs. sampling-rate are calculated in Chapter V.1.3. At Nyquist-rate, P_{opt} = 0.995 and increases about 17.5% at an infinite sampling-rate.

III.2.6 Conclusions

For long integration times, D is a function of the quantizers and the sampling-rate only. Two functions characterize a quantizer,

$$f_x(a_i,P_i) \quad \text{and} \quad R_{q_x}(\tau).$$

Referring to (III.1.2.12), the normalized, averaged output of one quantizer is

$$f_x(a_i, P_i) = \frac{\overline{q}_x}{\overline{x}}\sqrt{\frac{\pi}{2}} \qquad\qquad (III.2.6.1)$$

The autocorrelation function of the quantizer output, $R_{q_x}(\tau)$, was obtained in (III.1.2.36) and (III.1.2.37). These two functions are valid for any symmetric quantizer. Knowing $f_x(a_i, P_i)$ ($f_y(a_i, P_i)$) and $R_{q_x}(\tau)$ ($R_{q_y}(\tau)$) for the x- and the y-channels, we are able to compute $D_{n\times m}(K)$ for any combination of two quantizers. For sampling at the Nyquist-rate, the dual-channel degradation factor is simply the product of two single-channel degradation factors. It will be shown in Chapter V.2. that the single-channel degradation factor is the D obtained for a correlator with only one quantizer in one channel, the other channel left unquantized, i.e.

$$D_{n\times m}(2) = D_{n\times\infty}(2) \; D_{m\times\infty}(2) \qquad\qquad (III.2.6.2)$$

$D_{n\times m}(\infty)$ is the limiting value for D as K→∞, and is the minimum achievable degradation factor for a correlator receiver with n×m level quantization.

For an unquantized correlator receiver, sampling faster than at the Nyquist-rate does not change the degradation factor. For a quantized correlator, however, we obtain a lower degradation by "over-sampling" (K > 2).

Figure III.2.1, is a graph of the degradation factor vs. sampling rate K for the five combinations of quantizers considered in this chapter. As an example, it can be seen from that figure that a 4×4 level correlator sampled at Nyquist-rate has approximately the same degradation factor as a 3×3 level correlator at twice the Nyquist-rate.

Decision levels as well as stepwidths, a_i, can be optimized to minimize the degradation factor. Choosing the optimum quantization levels for minimization of the degradation results in impractical logic complications[12]. Choosing the quantization levels as integral multiples
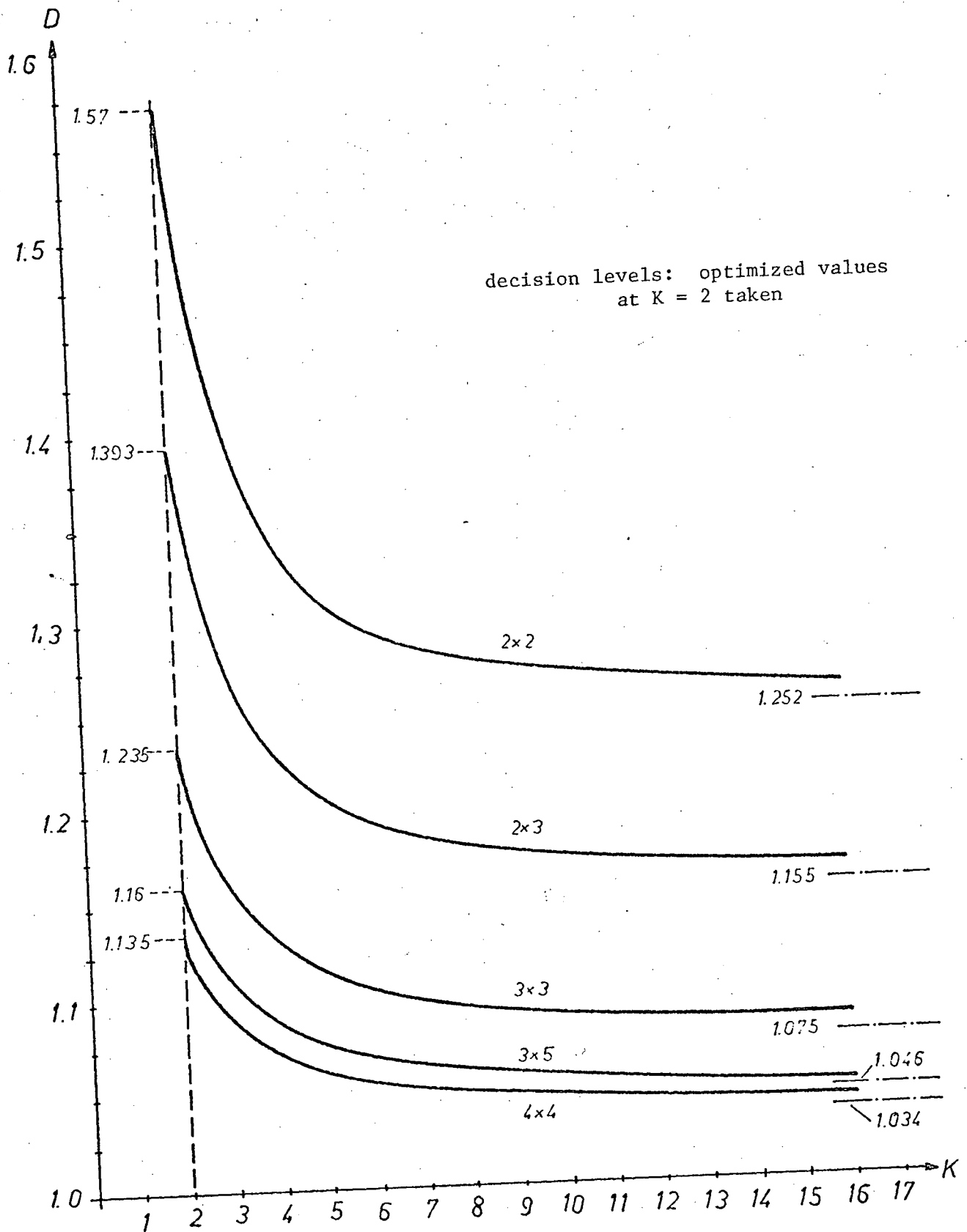
Figure III.2.1   Degradation factor, D, versus sampling rate K

of one another, preferably powers of two, yields a near optimum performance in terms of the degradation factor. As the decision levels can be set continuously on the analog-digital converters, non-integral size of P_i creates no additional difficulties. A method to optimize the P_i's is given in Chapter V.1.

In all previous calculations the x- and the y-channel were sampled at the same rate. The next section investigates the degradation in one case where the two channels are sampled at different rates.

III.3 3×5 Level Correlator at Unequal Sampling-Rate

A hardware construction of a 3×5 level correlator for Nyquist-sampling[12] has demonstrated that under certain circumstances the 5-level channel can be sampled at a faster rate with little increase in complexity. This chapter investigates whether there is anything to be gained by "oversampling" the 5-level channel.

A general model of the scheme under consideration is given in Fig. III.3.1.

Assume that every sample of the 3-level x-channel is multiplied with n samples of the 5-level y-channel. If the sampling rate of the x-channel is the Nyquist rate, 2B, then that of the y-channel will be

$$f_{sy} = 2n \cdot B \qquad\qquad (III.3.1)$$

If the averaging is done over N x-samples, it will include Nn products.

The multiplication of a given x-samples with y-samples at a variety of time-intervals will attenuate any high-frequency components in the correlated signal. Hence, for the purposes of this calculation, it is no longer legitimate to calculate degradation factors by replacing
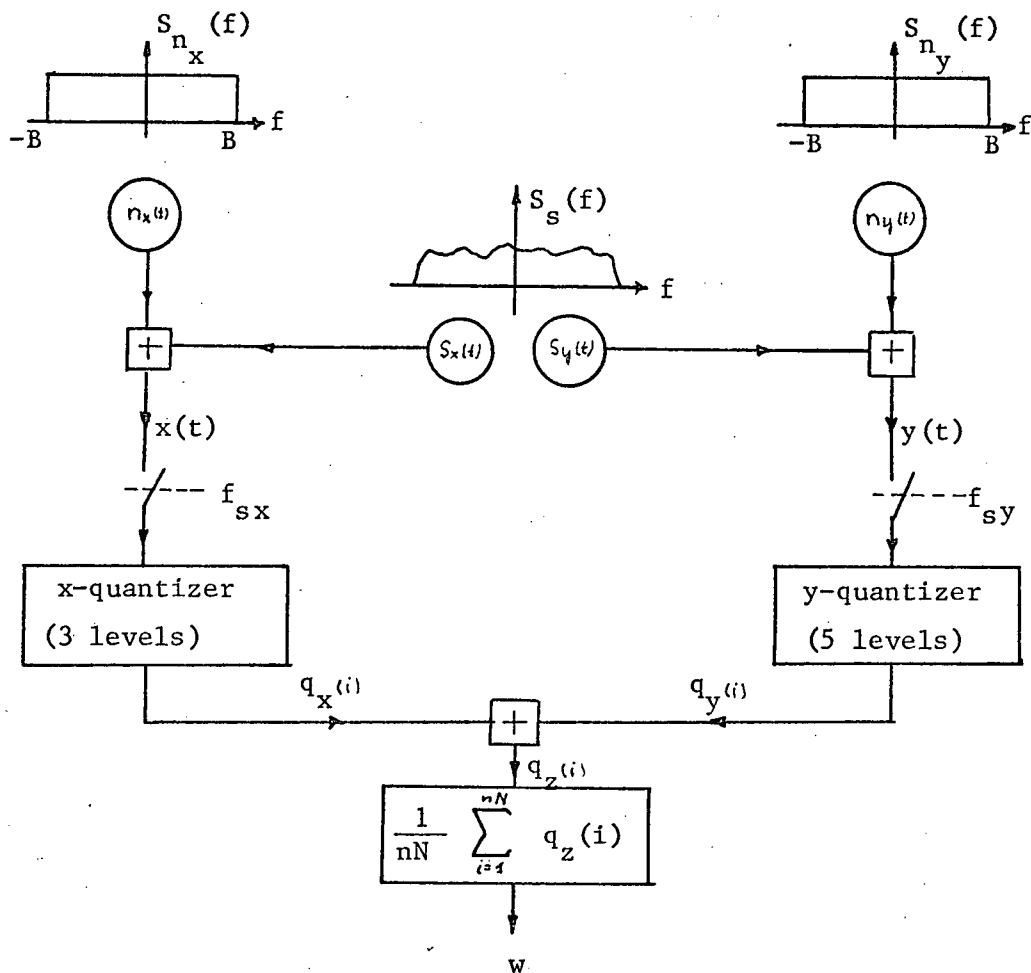
Fig. III.3.1 Model of correlator with unequal sampling rates

both signals with a d-c value. We can still simplify the calculations
by assuming that

$$s_x(t) = s_y(t) = s(t) \tag{III.3.2}$$

but we need to make some assumption about the spectral characteristics
of s(t), by specifying its power density spectrum $S_s(f)$.

Signals of interest in radio-astronomy will, in general, not
have a flat spectrum, but may contain spectral lines within the bandwidth B.
If after translation to baseband, such spectral lines occur near the
origin, the attenuation of the signal due to the time-displacement of the
samples will be negligible, and we would find a rather small amount of
degradation. If, on the other hand, there is much spectral power near

the upper end of the band, the degradation will be severe.

To obtain typical and realistic values of the degradation factor, we assume that the power density spectrum is not concentrated at either end. In fact, the calculations are carried out assuming white Gaussian noise for s(t). It must be remembered that the results so obtained are merely a representative value of the degradation factor, and that in a practical case the degradation could be better or worse, depending on the nature of the signal to be correlated.

III.3.1 Asymmetric Sampling

Each x-channel sample is taken to be synchronized with the first of a group of y-channel samples, as shown in Figure III.3.3.1.1.
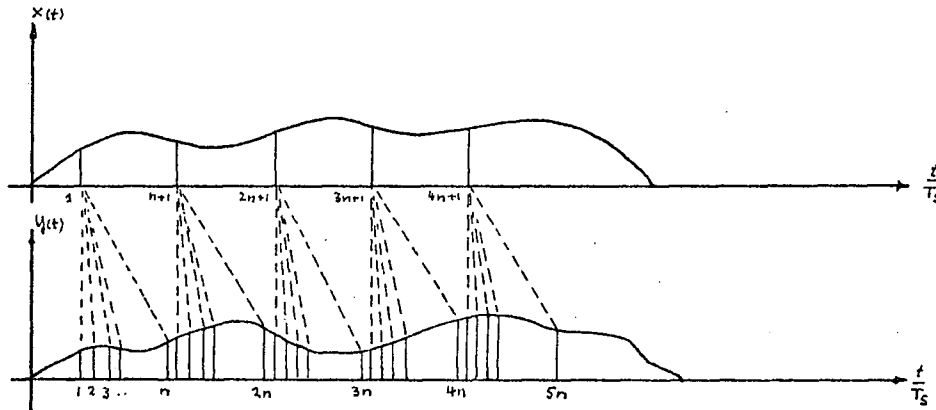
Fig. III.3.1.1   Signals x(t) and y(t), asymmetrically sampled

The output w of the correlator is then found to be

$$w = \frac{1}{nN} \sum_{i=1}^{n} \sum_{k=1}^{N} q_x((k-1)n+1)q_y((k-1)n + i) \qquad \text{(III.3.1.1)}$$

Therefore the expected value of w is given by equation (III.3.1.2).

$$\bar{w} = \frac{1}{nN} \sum_{i=1}^{n} \sum_{k=1}^{N} \overline{q_x((k-1)n+1)q_y((k-1)n+i)}$$

$$= \frac{1}{nN} \sum_{i=1}^{n} \sum_{k=1}^{N} R_{q_x q_x}(i-1) = \frac{1}{n} \sum_{i=o}^{n-1} R_{q_x q_y}(i) \qquad \text{(III.3.1.2)}$$

where $R_{q_x q_y}(i)$ is defined as the expected value of the product of the two ergodic random processes $q_x(t)$ and $q_y(t + iT_s)$

$$R_{q_x q_y}(i) = \overline{q_x(t) q_y(t + iT_s)} \qquad \text{(III.3.1.3)}$$

where T_s is the sampling interval

$$T_s = \frac{1}{2nB} \qquad \text{(III.3.1.4)}$$

Since the joint probability-density function of the signals x and y is given by

$$p_{xy}(x,y) = \frac{1}{2\pi\sqrt{1-r^2(\tau)}} e^{-\frac{1}{2} \frac{1}{1-r^2(\tau)} (x^2 - 2r(\tau)xy + y^2)} \qquad \text{(III.3.1.5)}$$

and

$$r(\tau) = \rho_s(\tau) \cdot \sigma_s^2 \ll 1 \qquad \text{(III.3.1.6)}$$

where

$$\rho_s(\tau) = \frac{R_s(\tau)}{R_s(0)} \qquad \text{(III.3.1.7}$$

we find $R_{xy}(\tau)$ as the expected value of the signals $x(t)$ and $y(t + \tau)$ or, equivalently as the cross-correlation-function of $x(t)$ and $y(t)$. Therefore,

$$R_{xy}(\tau) = \overline{x(t)\ y(t + \tau)}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy\ p_{xy}(x,y)\ dxdy$$

$$= r(\tau)$$

$$= \rho_s(\tau)\sigma_s^2 \qquad \text{(III.3.1.8)}$$

$R_{q_x q_y}(\tau)$ is a linear function of $R_{xy}(\tau)$ if $\sigma_s \ll 1$ and depends only on the quantizers and $R_{xy}(\tau)$.

In equation III.1.2.13 we used a d-c signal with the normalized autocorrelation function

$$\rho_s(\tau) = 1 = const. \qquad (III.3.1.9)$$

and therefore

$$R_{q_x q_y}(\tau) = const. = R_{q_x q_y}(0) = \overline{q_x q_y}. \qquad (III.3.1.10)$$

Since $R_{q_x q_y}(\tau)$ is proportional to $\rho_s(\tau)$ it follows that

$$R_{q_x q_y}(\tau) = \frac{2}{\pi} f_x(a_i, P_i) f_y(a_i, P_i) \rho_s(\tau) \sigma_s^2 \qquad (III.3.1.11)$$

and for the 3×5 level correlator it follows from III.2.4.2 that

$$R_{q_x q_y}(\tau) = \frac{2}{\pi} \sigma_s^2 e^{-\frac{P^2}{2}} (e^{-\frac{P_{y1}^2}{2}} + (\kappa-1)e^{-\frac{P_{y2}^2}{2}}) \rho_s(\tau) \qquad (III.3.1.12)$$

Letting $\tau = iT_s$ and denoting iT_s shortly by "i" we obtain, after substituting (III.3.1.12) into (III.3.1.2),

$$\overline{w} = \frac{1}{n} \frac{2}{\pi} \sigma_s^2 e^{-\frac{P^2}{2}} (e^{-\frac{P_{y1}^2}{2}} + (\kappa-1)e^{-\frac{P_{y2}^2}{2}}) \sum_{i=0}^{n-1} \rho_s(i) \qquad (III.3.1.13)$$

Since D is proportional to $\frac{1}{\overline{w}}$ and $\overline{w^2}$ is independent of s for small signals, it can be seen that the lowest degradation factor is obtained for a d-c signal where $\rho_s(i) = 1$. The more high-frequency components s(t) contains, the higher the expected degradation factor. If $s(t) = \cos 2\pi Bt$, then $\rho_s(i) = \cos \frac{i\pi}{n}$ and

$$\overline{w} \text{ is proportional to } \frac{1}{n} \sum_{i=0}^{n-1} \cos(\frac{i\pi}{n})$$

In the limiting case as $n \to \infty$, $\bar{w}$ is proportional to $\frac{1}{\pi} \int_0^\pi \cos(x)dx = 0$ and therefore the degradation factor goes to infinity. A low degradation factor can be expected for an unequal sampling-rate if the signal $s(t)$ has most of its spectral power at low frequencies. To compute the variance σ_w^2 we assume $x(t) \approx n_x(t)$ and $y(t) \approx n_y(t)$, i.e.

$$\overline{w^2} = \frac{1}{n^2 N^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{N} \sum_{\lambda=1}^{N} q_x((k-1)n+1)q_x((\lambda-1)n+1) \ q_y((k-1)n+i)q_y((\lambda-1)n+j)$$

$$= \frac{1}{n^2 N^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{N} \sum_{\lambda=1}^{N} R_{q_x}((k-\lambda)n) \ R_{q_y}((k-\lambda)n+i-j) \qquad \text{(III.3.1.14)}$$

Using the identity

$$\sum_{i=1}^{N-1} (N-i)(f(k+i) + f(k-i)) + Nf(k) = \sum_{i=1}^{N} \sum_{j=1}^{N} f(k+i-j) \qquad \text{(III.3.1.15)}$$

we obtain

$$\overline{w^2} = \frac{1}{nN^2} \sum_{k=1}^{N} \sum_{\lambda=1}^{N} R_{q_x}((k-\lambda)n)\{ \sum_{i=1}^{n-1} (1-\frac{i}{N})(R_{q_y}((k-\lambda)n+i) + R_{q_y}((k-\lambda)n-i))+R_{q_y}((k-1)n)\}$$

$$\text{(III.3.1.16)}$$

After using equation (III.3.1.15) again, we find

$$\overline{w^2} = \frac{2}{nN} \{ \sum_{k=1}^{N-1} (1-\frac{k}{n})R_{q_x}(nk)R_{q_y}(nk)+ \sum_{i=1}^{n-1} (1-\frac{i}{n})R_{q_x}(0)R_{q_y}(i)$$

$$+ \sum_{i=1}^{n-1} \sum_{k=1}^{N-1} (1-\frac{i}{n})(1-\frac{k}{N})R_{q_x}(nk)(R_{q_y}(nk+i)+R_{q_y}(nk-i))+ \frac{1}{2} R_{q_x}(0)R_{q_y}(0)\} \qquad \text{(III.3.1.17)}$$

We assume in what follows that the x-channel is sampled at Nyquist-rate. Since $R_{q_x}(nk)=R_{q_y}(nk) = 0$, (III.3.1.16) becomes

$$\overline{w^2} = \frac{1}{nN} \{2 \sum_{i=1}^{n-1} (1-\frac{1}{n})R_{q_y}(i) + R_{q_y}(0)\} R_{q_x}(0) \qquad \text{(III.3.1.18)}$$

and as for $u \to \infty$ (equivalent to x-channel sampled at Nyquist-rate, y-channel unsampled),

$$\overline{w^2} = \frac{2}{N n T_s} R_{q_x}(0) \int_0^{T_s} (1 - \frac{\tau}{T_s}) R_{q_y}(\tau) d\tau \qquad (III.3.1.19)$$

where $T_s = \frac{1}{2Bn}$ is the sampling-interval $R_{q_y}(\tau)$ is given by (III.2.4.4) and $R_{q_x}(0)$ by III.2.2.4.

The asymmetric degradation factor Da 3×5 (w) is therefore found

$$Da_{3\times5}(n) = \frac{\pi}{2} \sqrt{n} \frac{(R_{q_x}(0)[2 \sum_{i=1}^{n-1} (1-\frac{i}{n}) R_{q_y}(i) + R_{q_y}(0)])^{\frac{1}{2}}}{e^{-\frac{1}{2}P^2} (e^{-\frac{1}{2}P_{y1}^2} + (\kappa-1)e^{-\frac{1}{2}P_{y2}^2}) \sum_{i=0}^{n-1} \rho_s(i)} \qquad (III.3.1.20)$$

For $n \to \infty$

$$Da_{3\times5}(\infty) = \frac{\pi}{2} \frac{(R_{q_x}(0) B \int_0^{\frac{1}{2B}} (1-2B\tau) R_{q_y}(\tau) d\tau)^{\frac{1}{2}}}{e^{-\frac{1}{2}P^2} (e^{-\frac{1}{2}P_{y1}^2} + (\kappa-1)e^{-\frac{1}{2}P_{y2}^2}) B \int_0^{\frac{1}{2B}} \rho_s(\tau) d\tau} \qquad (III.3.1.21)$$

Since $R_{q_y}(\tau)$ and $\rho_s(\tau)$ are functions of $2\pi B\tau$, $Da_{3\times5}(\infty)$ becomes independent of B. The stepwidths P, P_{y1} and P_{y2} can be optimized. Since the x-channel is sampled at Nyquist-rate, $Da_{3\times5}(K)$ can be expressed as the product of an x and a y-part. P_{opt} for the x-channel (3-level-side) is equal to P_{opt} for a 3-level quantizer at Nyquist-rate. (=0.612). The values of $P_{y1,opt}$ and $P_{y2,opt}$ are slightly higher than $P_{y1,opt}$ and $P_{y2,opt}$ at Nyquist-rate. In Figure III.3.3.1 and III.3.3.2 however the Nyquist-rate values $P_{y1 opt} = 0.422$ and $P_{y2 opt} = 1.266$ are used, (see Chapter V.1.4) since the error of D is small enough to be neglected.

III.3.2 Symmetric Sampling

The x-samples are taken to be synchronized with the y-channel samples. Further we consider the center of a group of y-channel samples to be coincident with an x-channel sample (see Fig. III.3.2.1).
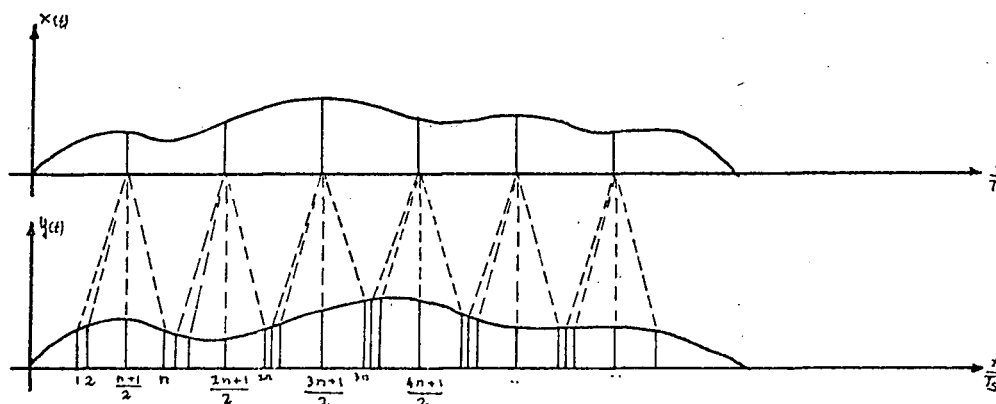
Fig. III.3.2.1 Signals x(t) and y(t), symmetrically sampled

The output w of the correlator is found to be

$$w = \frac{1}{nN} \sum_{i=1}^{n} \sum_{k=1}^{N} q_x\left(\frac{(2k-1)n+1}{2}\right) q_y((k-1)n+i) \qquad \text{(III.3.2.1)}$$

Therefore the expected value of w is given by equation (III.3.2.2)

$$\bar{w} = \frac{1}{nN} \sum_{i=1}^{n} \sum_{k=1}^{N} \overline{q_x\left(\frac{(2k-1)n+1}{2}\right)q_y((k-1)n+i)}$$

$$= \frac{1}{n} \sum_{i=1}^{n} R_{q_x q_y}\left(i - \frac{n+1}{2}\right) \qquad \text{(III.3.2.2)}$$

The cross-correlation function of the signals $q_x(t)$ and $q_y(t)$, $R_{q_x q_y}(\tau)$, was obtained in (III.3.1.11)

Therefore,

$$\bar{w} = \frac{1}{n} \frac{2}{\pi} \sigma_s^2 \, e^{-\frac{P^2}{2}} \left(e^{-\frac{P_{y1}^2}{2}} + (K-1)e^{-\frac{P_{y2}^2}{2}}\right) \sum_{i=1}^{n} \rho_s\left(i - \frac{n+1}{2}\right) \qquad \text{(III.3.2.3)}$$

The variance σ_w^2 is again calculated for noise-inputs only.

Therefore,

$$\overline{w^2} = \sigma_w^2 \qquad (III.3.2.4)$$

$$\overline{w^2} = \frac{1}{n^2 N^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{N} \sum_{\lambda=1}^{N} \overline{q_x(\frac{((2k-1)n+1)}{2}) q_x(\frac{((2\lambda-1)n+1)}{2})}$$

$$\overline{q_y((k-1)n+i) q_y((\lambda-1)n+j)}$$

$$= \frac{1}{n^2 N^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{N} \sum_{\lambda=1}^{N} R_{q_x}((k-\lambda)n) \cdot R_{q_y}((k-\lambda)n + i - j) \qquad (III.3.2.5)$$

The expression (III.3.2.5) is the same as (III.2.1.13) for asymmetric

sampling. Therefore $\overline{w^2}$ is given by (III.3.1.17) and the symmetric degrada-

tion factor $Ds_{3\times5}$ (n) becomes then

$$Ds_{3\times5}(n) = \frac{\pi}{2} \sqrt{n} \frac{\left\{ R_{q_x}(0) \cdot [2 \sum_{i=1}^{n-1} (1-\frac{i}{n}) R_{q_y}(i) + R_{q_y}(0)] \right\}^{\frac{1}{2}}}{e^{-\frac{P^2}{2}} [e^{-\frac{1}{2} P_{y1}^2} + (\kappa-1) e^{-\frac{1}{2} P_{y2}^2}] \sum_{i=1}^{n} \rho_s(i - \frac{n+1}{2})} \qquad (III.3.2.6)$$

Using (III.3.1.18) and taking the limit of (III.3.2.2) as $n \to \infty$ we find for

an infinite sampling-rate on the y-channel

$$Ds_{3\times5}(\infty) = \frac{\pi}{2} \frac{\left\{ R_{q_x}(0) \cdot B \int_0^{\frac{1}{2B}} (1-2B\tau) R_{q_y}(\tau) d\tau \right\}^{\frac{1}{2}}}{e^{-\frac{1}{2} P^2} [e^{-\frac{1}{2} P_{y1}^2} + (\kappa-1) e^{-\frac{1}{2} P_{y2}^2}] B \int_{-\frac{1}{4B}}^{\frac{1}{4B}} \rho_s(\tau) d\tau} \qquad (III.3.2.7)$$

Again $R_{q_y}(\tau)$ and $\rho_s(\tau)$ are functions of $2\pi B\tau$, i.e. $Ds_{3\times5}(\infty)$ is independent

of B. A high degradation factor can be expected where the signal s has

most of its spectral power near the bandlimit B. For symmetric sampling $Ds_{3\times5}(n)$ is proportional to

$$[\frac{1}{n}\sum_{i=1}^{n}\rho_s(i - \frac{n+1}{2})]^{-1}$$

and, since $\rho_s(\tau) = \rho_s(-\tau)$ is a monotonically decreasing function between 0 and $\frac{1}{4B}$ for any power density spectrum $S_s(f)$, it can be seen that symmetric sampling always results in a lower degradation than asymmetric sampling. The largest degradation factor is obtained where $S_s(f)$ has only one spectral line at $f = B$, i.e. $\rho_s(\tau) = \cos 2\pi B\tau$.

As $n\to\infty$ $Ds_{3\times5}(\infty)$ becomes proportional to

$$[\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \, dx]^{-1} = \frac{\pi}{2}$$

and does not go to infinity as in the asymmetric case.

It is of interest to note, however, that the optimum values of P, P_{y1} and P_{y2} are the same for the asymmetric and the symmetric case. Comparison of asymmetric/symmetric sampling for a white signal s(t): Under the assumption of s(t) having a flat power density spectrum over the bandwidth B, its autocorrelation function $\rho_s(\tau)$ becomes

$$\rho_s(\tau) = Sa(2\pi B\tau) \qquad\qquad (III.3.2.8)$$

For asymmetric sampling $\int_{0}^{\frac{1}{2B}} \rho_s(\tau)d\tau$ is the integral over the sampling-function from the origin to the first zero-crossing. (see Figure (III.3.2.2).
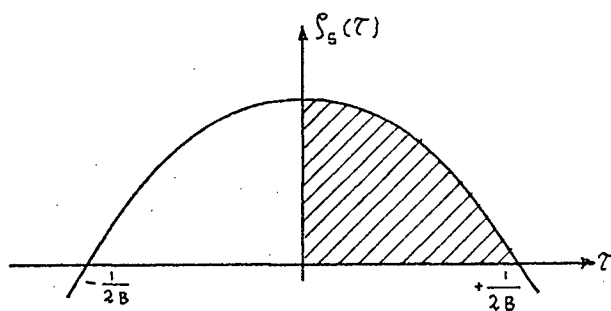
Fig. III.3.2.2 Autocorrelation function of s(t)

For symmetric sampling the integral is the shaded area illustrated in

Figure (III.3.2.3).
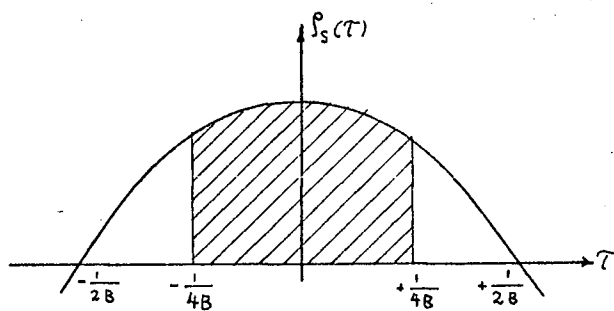
Fig. III.3.2.3 Autocorrelation function of s(t)

It is easy to see that

$$\int_{-\frac{1}{4B}}^{\frac{1}{4B}} \rho_s(\tau)\,d\tau \; > \int_{0}^{\frac{1}{2B}} \rho_s(\tau)\,d\tau \qquad\qquad\text{(III.3.2.9)}$$

From (III.3.1.20) and (III.3.2.7) it follows that the symmetric case will

result in a lower degradation factor than in the asymmetric case.

III.3.3 Conclusions

$Da_{3\times5}(n)$ and $Ds_{3\times5}(n)$ have been computed and plotted in Figures

III.3.3.1 and III.3.3.2 respectively, for a flat and for a triangular spectrum which we arbitrarily assumed. As expected, asymmetric sampling results in a higher degradation factor than symmetric sampling for both shapes of $S_s(f)$. For the two shapes considered, asymmetric sampling is not a good method to use in determining the autocorrelation function of $s(t)$. Considering Fig. III.3.3.2 , we see that for the triangular shape the degradation factor D is smaller than the D at Nyquist-rate at only one particular rate, namely twice the Nyquist-rate (n=2).

Symmetric sampling results in about a 2% smaller degradation factor beyond n=4 for the flat spectrum, and about an 8.5% reduction for the triangular spectrum.

Generally speaking, unequal, symmetric sampling is advantageous only if $S_s(f)$ has most of its spectral power at lower frequencies. But it may be preferable, in this case, to neglect the high frequency components and operate with a smaller bandwidth, considering only the relevant spectral lines of $S_s(f)$.

III.4 Degradation for Overquantized Correlators

We have seen in Chapter III.2 that, at a given sampling-rate, the degradation factor D becomes smaller as the number of levels in the quantizers increases. The lower bound (D=1) corresponds to an infinitely fine quantization or no quantization at all. However, more levels means a greater variety of products, q_z, to be handled by the averager, and this results in a greater complexity of the radiometer. In practice, therefore, there is a limit to the number of different products one is willing to handle. Given this limit, the question arises whether it might be advantageous to quantize each signal to many more levels, but to
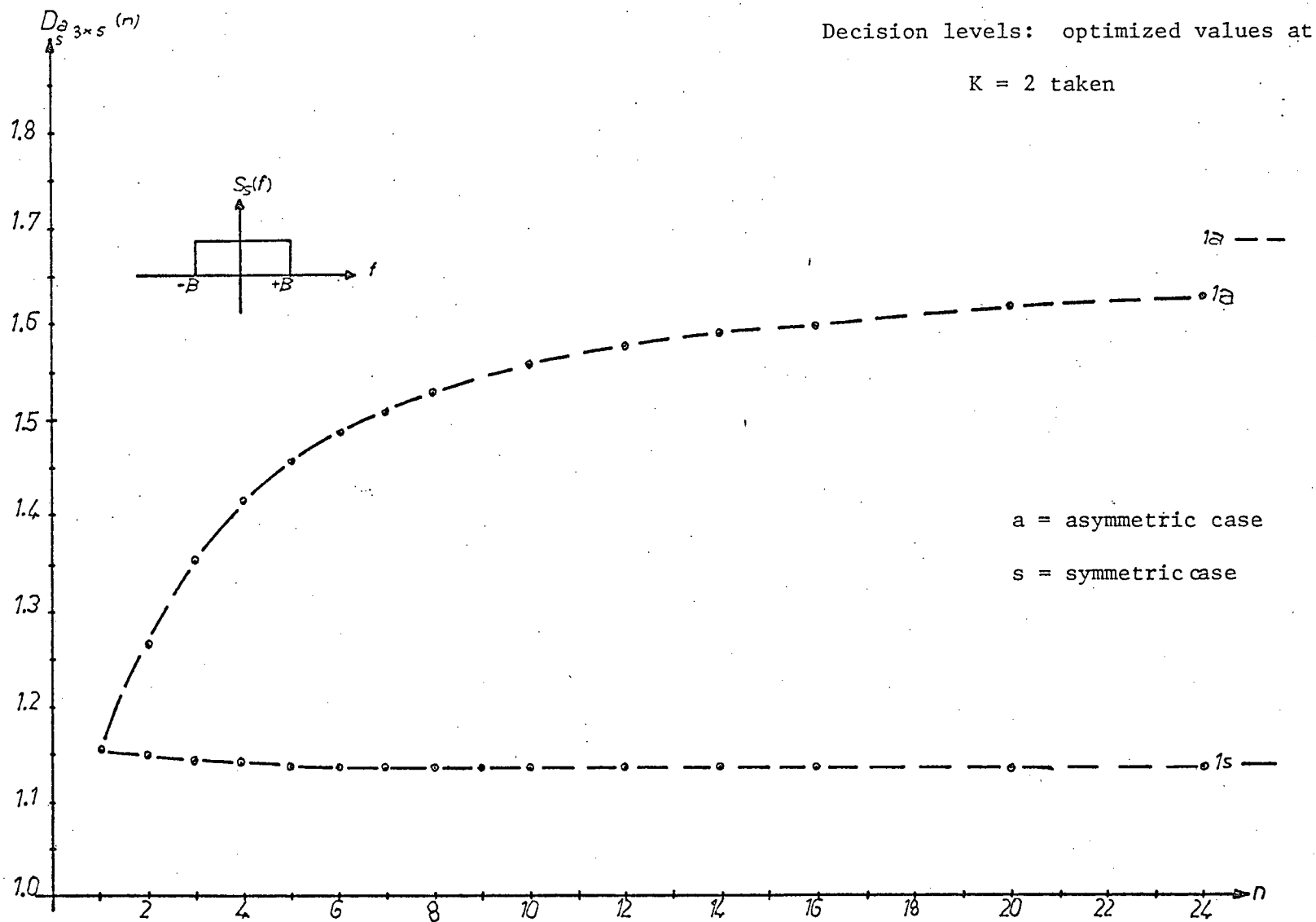
Figure III.3.1 Degradation factor D versus oversampling factor n for a
flat spectrum
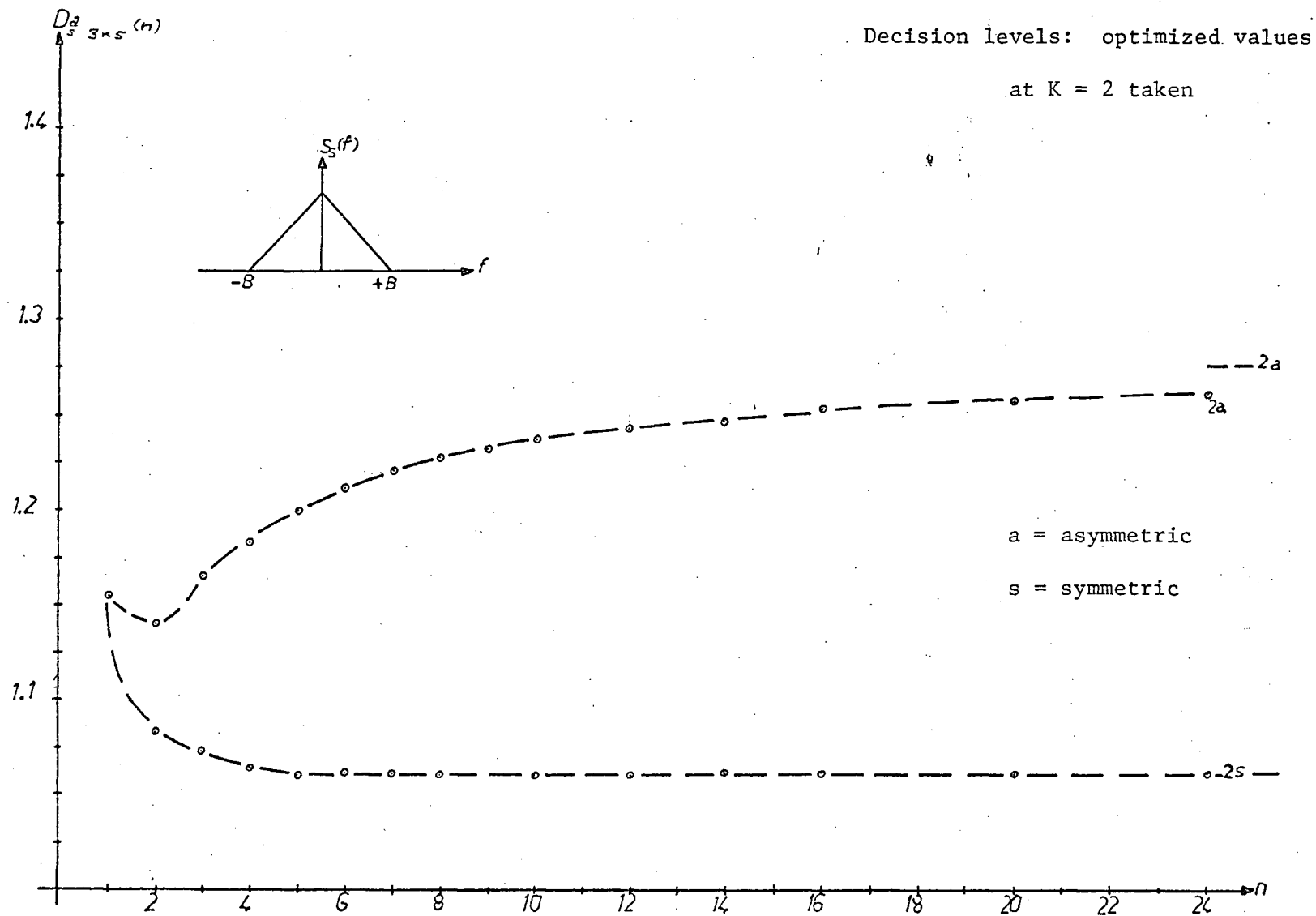
Figure III.3.2   Degradation factor D versus oversampling factor n for a
triangular spectrum

merge some of the resulting "products" to obtain only the desired number of different outputs.

One case of "incomplete multiplication" was investigated by Cooper[8], for a 4×4 level correlator where the least significant products were neglected. The present chapter is based on Cooper's technique and extends this technique to the 5×5 level correlator. Later, allowing only three different products, q_z, we consider, in section III.4.2, a 5×5 level correlator with the number of products reduced to 3 and, in section III.4.3, an analog correlator, where the quantization to three levels is done after the multiplication. This last case is interesting, as it represents a limiting case and tells us what we could gain for a given number of products by "overquantization".

III.4.1 Multiplication Using Four Possible "Products"

(a) Cooper's Scheme[8], 4×4 levels

The function $f(a_i, P_i)$ for a 4-level quantizer has been found in III.2.5.1:

$$f(a_i, P_i) = 1 + (\kappa-1) e^{-\frac{P^2}{2}}$$

Therefore $\bar{w}$ for a 4×4 level quantizer is given by

$$\bar{w} = \frac{2}{\pi} s_o^2 (1 + (\kappa-1)e^{-\frac{P^2}{2}})^2 \qquad (III.4.1.1)$$

or, multiplied out,

$$\bar{w} = \frac{2}{\pi} s_o^2 \left[(1-e^{-\frac{P^2}{2}})^2 + 2\kappa e^{-\frac{P^2}{2}} (1-e^{-\frac{P^2}{2}}) \right.$$
$$\left. + \kappa^2 e^{-P^2} \right] \qquad (III.4.1.2)$$

The following 6 different products have to be handled by the averager:

$$\pm 1, \pm\kappa \quad \text{and} \quad \pm\kappa^2.$$

Therefore the expression (III.4.1.2) has terms in 1, κ and κ^2 where:

only the products $\pm\kappa^2$ contribute to the terms in κ^2,

only the products $\pm\kappa$ contribute to the terms in κ and

only the products ± 1 contribute to the terms in 1.

Deleting a product pair cancels the corresponding term in equation (III.4.1.2).

Omitting the terms in ± 1, which are the least significant terms,

(III.4.1.2) becomes

$$\overline{w}_e = \frac{2}{\pi} s_o^2 \kappa e^{-\frac{P^2}{2}} (2(1- e^{-\frac{P^2}{2}}) + \kappa e^{-\frac{P^2}{2}}) \qquad (III.4.1.3)$$

where the subscript e refers to a correlator with the least significant

products eliminated. The variance of w for the least significant products

deleted is denoted as $(\sigma_w^2)e$ and calculated for sampling at Nyquist-rate.

Using equation (III.1.2.21), letting N = 2BT and recognizing that $R_{q_z}(i) = 0$

for $i \neq 0$ we find

$$(\sigma_w^2)_e = \frac{1}{2BT} R_{q_z}(0) \qquad (III.4.1.4)$$

The autocorrelation function $R_{q_z}(0)$ is given by

$$R_{q_z}(0) = [R_{q_x}^2(0)]_e \qquad (III.4.1.5)$$

where $[R_{q_x}^2(0)]e$ is found multiplying out equation (III.2.5.5)

$$[R_{q_x}^2(0)]_e = (1-2\text{erfc}(P))^2 + 4\kappa^2(1-2\text{erfc}(P))\text{erfc}(P) + 4\kappa^2 \text{erfc}^2(P)$$

$$(III.4.1.6)$$

and deleting the least significant term yields

$$[R_{q_x}^2(0)]_e = 4\kappa^2[(1-2\text{erfc}(P))\text{erfc}(P) + \kappa^2\text{erfc}^2(P)]. \qquad (III.4.1.7)$$

Equation (III.4.1.8) is obtained after substitution of (III.4.1.7) into

(III.4.1.4) as

$$(\sigma_w^2)_e = \frac{1}{2BT} \ 4\kappa^2 \{ erfc(P)(1-2erfc(P)) + \kappa^2 \ erfc^2(P) \} \qquad (III.4.1.8)$$

The output signal-to-noise ratio, $(\frac{S}{N})_{o,e}$ is then found using (II.2.3), as

$$(\frac{S}{N})_{o.e} = \frac{\overline{w}_e}{\sigma_{w,e}} = \frac{\sqrt{2BT}}{\pi} \ \frac{e^{-\frac{P^2}{2}} (2(1-e^{-\frac{P^2}{2}})+\kappa e^{-\frac{P^2}{2}})}{\sqrt{erfc(P)(1-2 \ erfc \ (P))+\kappa^2 erfc^2(P)}} \ s_o^2 \qquad (III.4.1.9)$$

$(\frac{S}{N})_o$ for the analog correlator is $s_o^2 \ \sqrt{2BT}$. The degradation factor, with the lowest order term deleted, becomes then:

$$[D_{4\times4}(2)]_e = \pi \ \frac{\{erfc(P)(1-2erfc(P))+\kappa^2 erfc^2(P)\}^{\frac{1}{2}}}{e^{-\frac{P^2}{2}} (2(1-e^{-\frac{P^2}{2}})+\kappa e^{-\frac{P^2}{2}})} \qquad (III.4.1.10)$$

Deleting the least significant products leaves only those in $\pm\kappa$ and $\pm\kappa^2$, i.e. the averager has to handle only four different products instead of 6 for the usual 4×4 -level correlator.

Figure (III .4.1.1) shows $[D_{4\times4}(2)]_e$ versus the decision level, P, for different values of the stepwidth, κ, as parameter. For comparison, $D_{4\times4}(2)$ for the regular 4×4 level correlator is plotted in the same figure. $[D_{4\times4}(2)]_e$ at the optimum decision level is about 0.97% higher than $D_{4\times4}(2)$, which is a small price to pay for the advantage of having four instead of 6 different products of q_z to be entered into the averager.

(b) 5×5 level correlator yielding 4 "products"

The function $f(a_i, P_i)$ for a 5-level quantizer was found in equation (III.2.4.1) as

$$f(a_i, P_i) = e^{-\frac{P_1^2}{2}} + (\kappa - 1)e^{-\frac{P_2^2}{2}} \qquad (III.4.1.11)$$
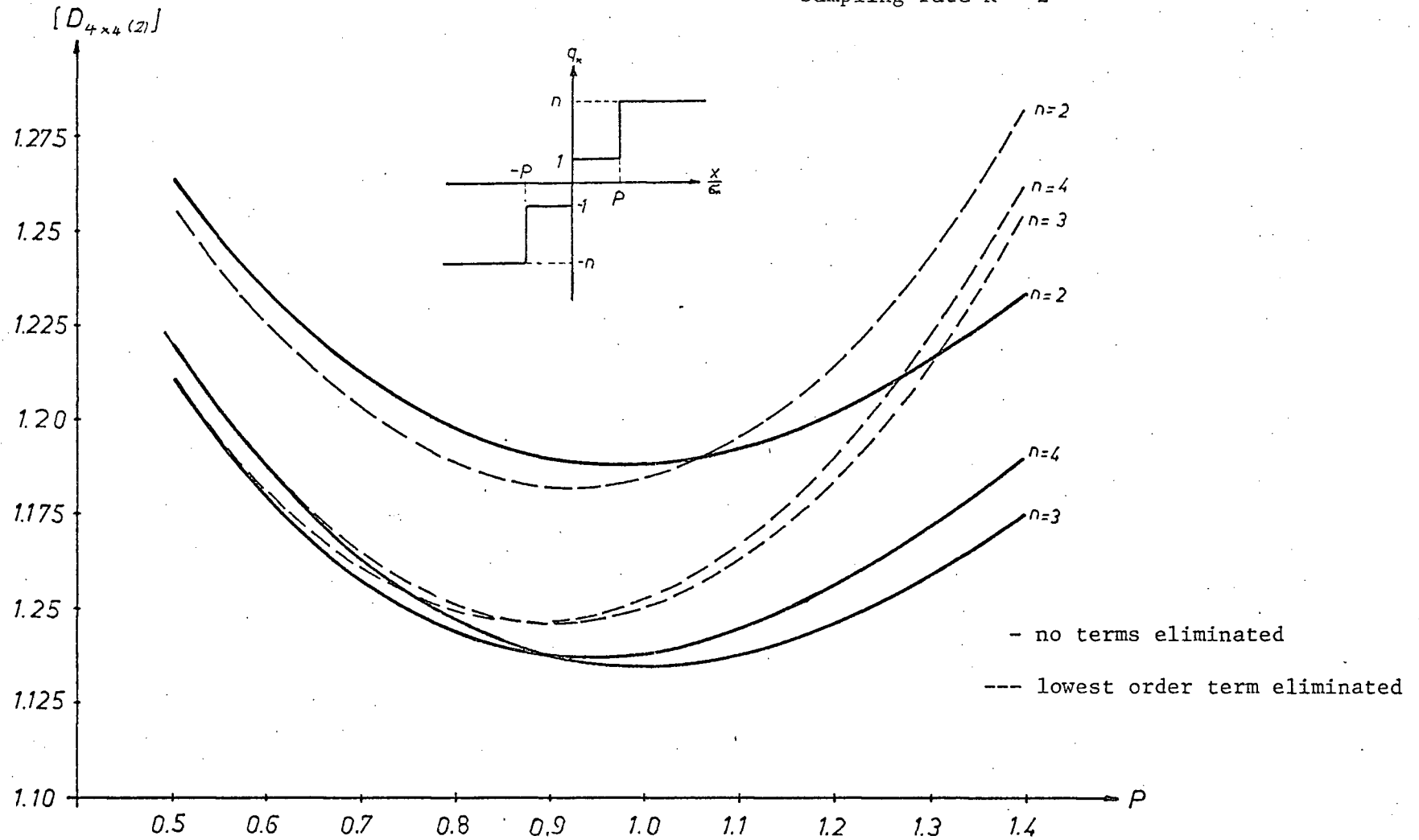
Figure III.4.1.1 Degradation factor versus decision level P for 4 × 4 levels, lowest order term eliminated

and the expected value of w is given by (III.1.2.14)

$$\bar{w} = \frac{2}{\pi} s_o^2 \left((e^{-\frac{P_1^2}{2}} - e^{-\frac{P_2^2}{2}})^2 + 2\kappa e^{-\frac{P_2^2}{2}} (e^{-\frac{P_1^2}{2}} - e^{-\frac{P_2^2}{2}}) + \kappa^2 e^{-P_2^2} \right)$$

(III.4.1.12)

after deleting terms in ±1 we obtain

$$\bar{w_e} = \frac{2}{\pi} s_o^2 \kappa e^{-\frac{P_2^2}{2}} \left[2(e^{-\frac{P_1^2}{2}} - e^{-\frac{P_2^2}{2}}) + \kappa e^{-\frac{P_2^2}{2}} \right]$$

(III.4.1.13)

Assuming sampling at Nyquist-rate, the variance $(\sigma_w^2)_e$ for the least

significant terms deleted can be obtained using (III.4.1.4) and (III.4.1.5)

where $[R_{q_x}(0)]_e$ is found by multiplying out equation (III.2.2.4) as:

$$R_{q_x}^2(0) = 4[\kappa^4 \ \mathrm{erfc}^2(P_2) + 2\kappa^2 \mathrm{erfc}(P_2)(\mathrm{erfc}(P_1) - \mathrm{erfc}(P_2)) + (\mathrm{erfc}(P_1)$$
$$- \mathrm{erfc}(P_2))^2]$$

(III.4.1.14)

and deleting the least significant term.

The variance, $\sigma_{w,e}^2$, is then obtained as

$$\sigma_{w,e}^2 = \frac{4\kappa^2}{\sqrt{2BT}} (\kappa^2 \mathrm{erfc}^2(P_2) + 2 \ \mathrm{erfc}(P_2)(\mathrm{erfc}(P_1) - \mathrm{erfc}(P_2)))$$

(III.4.1.15)

Therefore, the degradation factor $[D_{5\times5}(2)]_e$ is given using (II.2.3) and

(III.1.6.1)

$$[D_{5\times5}(2)]_e = \pi \frac{\{\kappa^2 \mathrm{erfc}^2(P_2) + 2 \ \mathrm{erfc}(P_2)(\mathrm{erfc}(P_1) - \mathrm{erfc}(P_2))\}^{\frac{1}{2}}}{e^{-\frac{P_2^2}{2}} [2(e^{-\frac{P_1^2}{2}} - e^{-\frac{P_2^2}{2}}) + \kappa e^{-\frac{P_2^2}{2}}]}$$

(III.4.1.16)

The 5×5 level correlator with the least significant term deleted has also

4 different products (±κ and ±κ^2) to handle.

From Figure (III.4.1.2) we see that $D_{5\times5}(2)_e$ is 2.76% higher than $D_{5\times5}(2)$ at optimized decision levels, but, compared with the 4×4 level correlator, still 2.46% lower than $[D_{4\times4}(2)]_e$, which also has 4 different values of the quantizer-products, q_z. The prices we pay for this lower degradation are an increase in complexity of both the quantizers and the "multiplier".

III.4.2. Overquantized 3-Product-Correlator

Under the general assumption stated in Chapter II we consider a sampled correlator with two 5-level quantizers as shown in Figure III.2.4.1. Seven products $q_x q_y$ are excited from the multiplier:

$$-\kappa^2, \quad -\kappa, \quad -1, \quad 0, \quad +1, \quad +\kappa, \quad +\kappa^2,$$
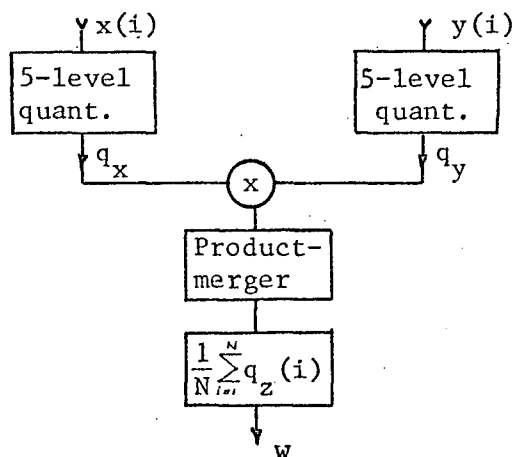
Fig. III.4.2.1 Correlator with product-merger

using the product-merger shown in Figure III.4.2.1.
The logic scheme of this product merger is shown in Fig. III.4.2.2 and probability chart as in Figure III.4.2.3, the signal $q_z(i)$ retains only the three products -1, 0 and +1.
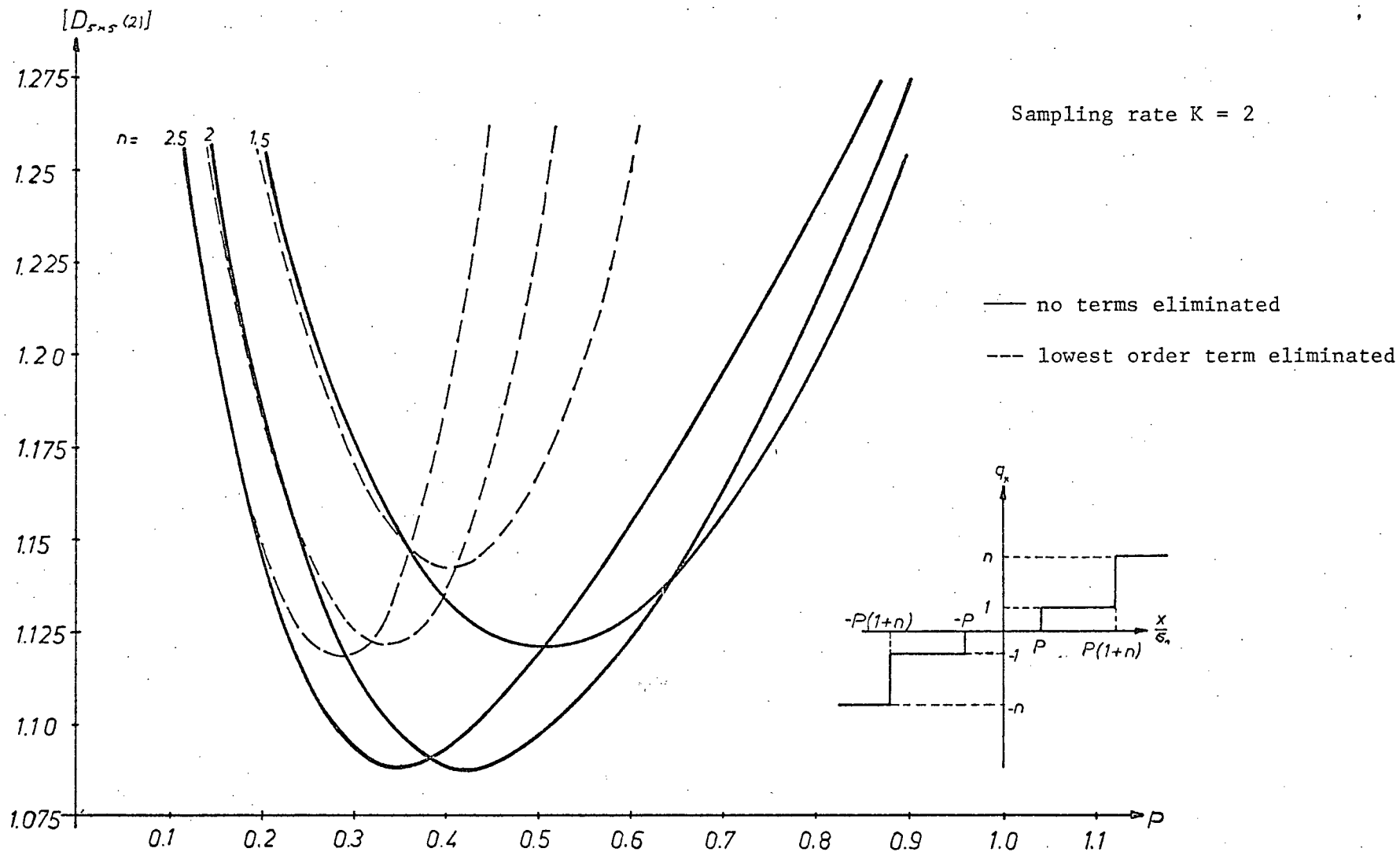
Figure III.4.1.2   Degradation factor versus decision level P for 5 × 5 levels, lowest order term eliminated

q_y \ q_x	$\mathcal{K}$	1	0	-1	$-\mathcal{K}$
$\mathcal{K}$	1	1	0	-1	-1
1	1	0	0	0	-1
0	0	0	0	0	0
-1	-1	0	0	0	1
$-\mathcal{K}$	-1	-1	0	1	1

Fig. III.4.2.2 Logic Scheme of Product merger
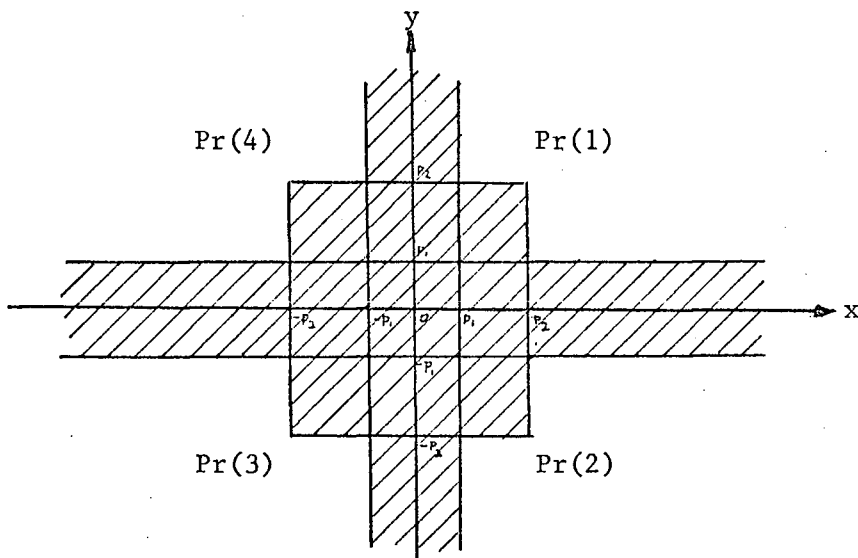
Fig. III.4.2.3 Probability chart of product merger

The signal q_z equals 0 if the signal-pair (x,y) is in the shaded area, equals +1 if (x,y) are in the area (1) or (3) and equals -1 if (x,y) are in (2) or (4).

Therefore the expected value of the output, w, is

$$\bar{w} = \overline{q_z} = \text{Pr}(1) + \text{Pr}(3) - \text{Pr}(2) - \text{Pr}(4) \qquad \text{(III.4.2.1)}$$

According to the assumptions made in Chapter III.1.1, the Gaussian input-signal s(t) can be replaced by a d-c signal $s(t) = s_o$ and since the

signals x and y are statistically independent, their probability density function is

$$p_{xy}(x,y) = p_x(x) \cdot p_y(y) \qquad \text{(III.4.2.2)}$$

where

$$p_x(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x - s_o)^2} \qquad \text{(III.4.2.3)}$$

and similarly for $p_y(y)$ using the probability chart of Figure III.4.2.3, the probabilities of finding (x,y) in area (1) is

$$Pr(1) = \int_{x=P_1}^{\infty} \int_{y=P_2}^{\infty} p_x(x) \, p_y(y) \, dx \, dy - \int_{x=P_1}^{P_2} \int_{y=P_1}^{P_2} p_x(x) \, p_y(y) \, dx \, dy$$

$$= \left[\int_{P_1}^{\infty} p_x(x) \, dx \right]^2 - \left[\int_{P_1}^{P_2} p(x) \, dx \right]^2$$

$$= [erfc(P_1 - s_o)]^2 - [erfc(P_1 - s_o) - erfc(P_2 - s_o)]^2$$

$$= erfc(P_2 - s_o) \, [2erfc(P_1 - s_o) - erfc(P_2 - s_o)] \qquad \text{(III.4.2.4)}$$

Similarly

$$Pr(3) = erfc(P_2 + s_o)[2 \, erfc(P_1 + s_o) - erfc(P_1 + s_o)]$$

$$\text{(III.4.2.5)}$$

For area (2) we have

$$Pr(2) = \int_{P_1}^{\infty} p_x(x) \, dx \int_{-\infty}^{P_1} p_y(y) \, dy - \int_{P_1}^{P_2} p_x(x) \, dx \int_{-P_2}^{P_1} p_y(y) \, dy$$

$$= erfc(P_1 - s_o)erfc(P_2 + s_o) + erfc(P_2 - s_o)erfc(P_1 + s_o) - erfc(P_2 + s_o)$$

$$erfc(P_2 - s_o) \quad \text{(III.4.2.6)}$$

and exactly the same result is obtained for $Pr(4)$

$$Pr(4) = Pr(2) \qquad \text{(III.4.2.7)}$$

Since $s_o \ll 1$, the error-functions can be linearized using the first two terms of their Taylor-series around $s_o = 0$

$$Pr(1) + Pr(3) = 2 \, erfc(P_2)(2 \, erfc(P_1) - erfc(P_2)) + \frac{s_o^2}{\pi} e^{-\frac{P_2^2}{2}} (2e^{-\frac{P_1^2}{2}} - e^{-\frac{P_2^2}{2}})$$

$$(III.4.2.8)$$

$$Pr(2) + Pr(4) = 2 \, erfc(P_2)(2 \, erfc(P_1) - erfc(P_2)) + \frac{s_o^2}{\pi} e^{-\frac{P_2^2}{2}} (2e^{-\frac{P_1^2}{2}} - e^{\frac{P_2^2}{2}})$$

$$(III.4.2.9)$$

Therefore,

$$\bar{w} = \frac{2}{\pi} s_o^2 \, e^{-\frac{P_2^2}{2}} (2 \, e^{-\frac{P_1^2}{2}} - e^{-\frac{P_2^2}{2}}) \qquad (III.4.2.10)$$

The variance σ_w^2 is obtained from the general equation (III.1.2.1). Considering sampling at Nyquist-rate, the autocorrelation function $R_{q_z}(i)$ differs from zero only at $(i=0)$, and, using $N = 2BT$, σ_w^2 is found as

$$\sigma_w^2 = \frac{R_{q_z}(0)}{2BT} \qquad (III.4.2.11)$$

Since q_z^2 has only the values 0 or 1,

$$R_{q_z}(0) = \overline{q_z^2}$$

$$= Pr(q_z = 1) + Pr(q_z = -1) \qquad (III.4.2.12)$$

Since we consider noise inputs only for the evaluation of σ_w^2, and since the signals x and y are statistically independent. $Pr(q_z = 1)$ and $Pr(q_z = -1)$ therefore can be obtained by putting $(s_o = 0)$ in the equations (III.4.2.4) to (III.4.2.7).

Therefore,

$$Pr(q_z = 1) = [Pr(1) + Pr(3)]_{(s_o=0)}, \qquad (III.4.2.13)$$

$$Pr(q_z = -1) = [Pr(2) + Pr(4)]_{(s_o=0)} \qquad (III.4.2.14)$$

and

$$\Pr(q_z^2 = 1) = \overline{q_z^2} = [P_1 + P_2 + P_3 + P_4]_{(s_o=0)}$$

$$+ 4 \, \text{erfc}(P_2)(2 \, \text{erfc}(P_1) - \text{erfc}(P_2)) \quad \text{(III.4.2.15)}$$

Using equation (III.4.2.11), the variance σ_w^2 is then found as

$$\sigma_w^2 = \frac{2}{BT} \, \text{erfc}(P_2)(2 \, \text{erfc}(P_1) - \text{erfc}(P_2)) \quad \text{(III.4.2.16)}$$

We denote by $[D_{5\times5}(2)]_3$ the degradation factor for this correlator, where q_z values have been merged to reduce them to three different values.

$$[D_{5\times5}(2)]_3 = \frac{\sqrt{2BT} \, s_o^2}{\left(\dfrac{\bar{w}}{\sigma_w}\right)} = \pi \frac{\left(\text{erfc}(P_2)(2 \, \text{erfc}(P_1) - \text{erfc}(P_2))\right)^{\frac{1}{2}}}{e^{-\frac{P_2^2}{2}}\left(2 \, e^{-\frac{P_1^2}{2}} - e^{-\frac{P_2^2}{2}}\right)} \quad \text{(III.4.2.17)}$$

Figure III.4.2.4 is a plot of this degradation factor versus P_1 with the decision level P_2 optimized.

III.4.3 3-Level Quantization After Analog Multiplication

We now assume that in the correlator described in Chapter II the signals x(t) and y(t) are sampled but not otherwise processed before multiplication, but that the resulting products z(i) are then quantized into three levels as shown in Figure III.4.3.1
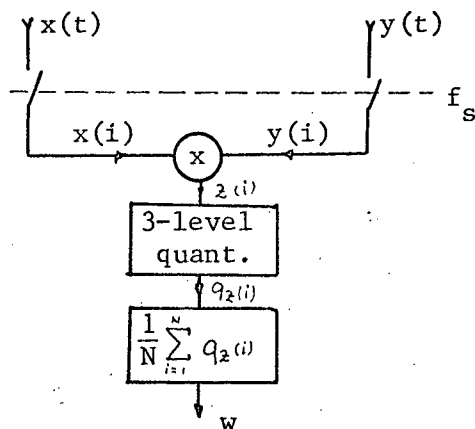
Fig. III.4.3.1 Quantization after multiplication

$[D_{S=S}(2)]_3$

1.54

1.50

1.46

1.42

1.38

1.34

1.30

1.26

1.22

1.18

x
quant.
q_x

y
quant.
q_y

⊗

logic
q_z

$\frac{1}{N}\sum_{i=1}^{N} q_z(i)$

w

q_q, q_y

2

$-P_2\ -P_1$

$-1\ P_1\ P_2$

x, y

-2

P₂ optimized

q_x \ q_y	-2	-1	0	1	2
-2	1	1	0	-1	-1
-1	1	0	0	0	-1
0	0	0	0	0	0
1	-1	0	0	0	1
2	-1	-1	0	1	1

$q_z =$

P_2 opt. =   1.07   1.06   1.05   1.04   1.05   1.06   1.07   1.09   1.12   1.15   no P_2 opt

.1 0.1   0.2   0.3   0.4   0.5   0.6   0.7   0.8   0.9   1.0   1.1   1.2 P_1
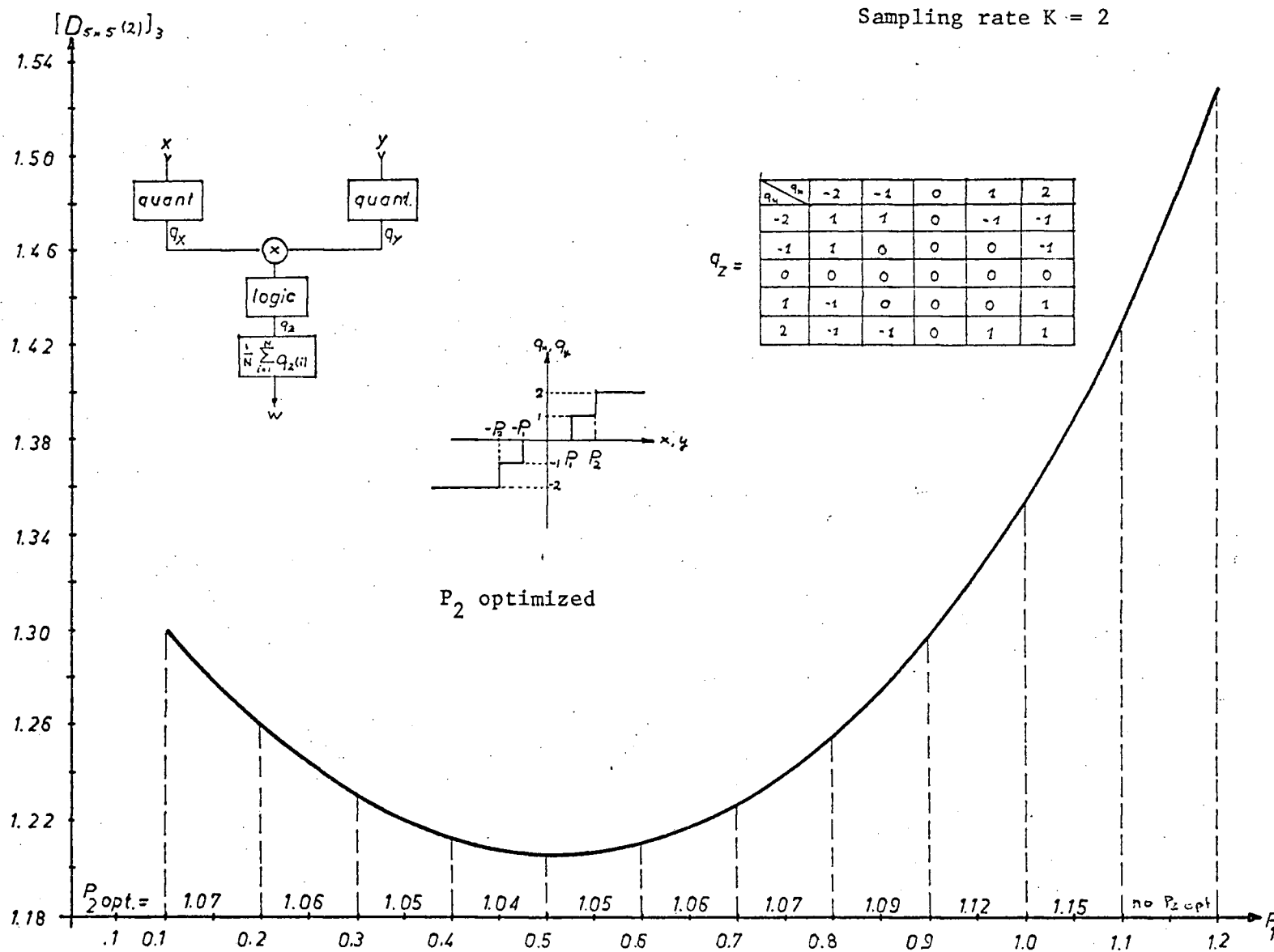
Figure III.4.2.4   Degradation factor versus decision level P_1 for merged product correlator

This arrangement is not very practical, but is considered as a limiting case of the process described in proceeding subsections, where we have "overquantization" with subsequent "merging" of products.

The expected value of w is easily found as

$$\bar{w} = \overline{q_z} = \Pr(z \geqslant P) - \Pr(z \leqslant -P) \qquad \text{(III.4.3.1)}$$

Signals x and y have normal distribution, variance equal to unity, and normalized correlation coefficient

$$r = s_o^2 \qquad \text{(III.4.3.2)}$$

Their joint probability-density is given by

$$p_{xy}(x,y) = \frac{1}{2\pi\sqrt{1-s_o^4}} \; e^{-\dfrac{1}{2(1-s_o^4)}(x^2 - 2 s_o^2 xy + y^2)} \qquad \text{(III.4.3.3)}$$

Then

$$\Pr(z \geqslant P) = \int_0^\infty \int_{\frac{P}{x}}^\infty p_{xy}(x,y)\; dx\; dy + \int_{-\infty}^o \int_{-\infty}^{\frac{P}{x}} p_{xy}(x,y)\; dx\; dy$$

$$= 2 \int_0^\infty \int_{\frac{P}{x}}^\infty p_{xy}(x,y)\; dx\; dy$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-\frac{x^2}{2}} \; \text{erfc}\left(\frac{\frac{P}{x} - s_o^2 x}{\sqrt{1 - s_o^4}}\right)\; dx \qquad \text{(III.4.3.4)}$$

Using the first two terms of a Taylor-series around $(s_o = 0)$, we obtain

$$\Pr(z \geqslant p) \approx \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-\frac{x^2}{2}} \; \text{erfc}\left(\frac{P}{x}\right) dx + \frac{s_o^2}{\pi} \int_0^\infty x e^{-\frac{1}{2}(x^2 + (\frac{P}{x})^2)}\; dx \quad \text{(III.4.3.5)}$$

and

$$\Pr(z \leqslant -P) = \int_{-\infty}^o \int_{-\frac{P}{x}}^\infty p_{xy}(x,y) dx\; dy + \int_0^\infty \int_{-\infty}^{-\frac{P}{x}} p_{xy}(x,y) dx\; dy$$

$$= 2 \int_{-\infty}^{0} \int_{-\frac{P}{x}}^{\infty} p_{xy}(x,y) \, dx \, dy$$

$$= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} e^{-\frac{x^2}{2}} \, \mathrm{erfc}\left(\frac{\frac{P}{x} + s_o^2 \, x}{\sqrt{1-s_o^4}}\right) \, dx \qquad \text{(III.4.3.6)}$$

$$\approx \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} e^{-\frac{x^2}{2}} \, \mathrm{erfc}\left(\frac{P}{x}\right) dx - \frac{s_o^2}{\pi} \int_{0}^{\infty} x e^{-\frac{1}{2}\left(x^2 + \left(\frac{P}{x}\right)^2\right)} dx$$

$$\text{(III.4.3.7)}$$

Therefore,

$$\bar{w} = \frac{2 s_o^2}{\pi} \int_{0}^{\infty} x e^{-\frac{1}{2}\left(x^2 + \left(\frac{P}{x}\right)^2\right)} \, dx \qquad \text{(III.4.3.8)}$$

The variance σ_w^2 is found using (III.1.2.24)

$$\sigma_w^2 = \frac{1}{KBT} \left[2 \sum_{i=1}^{KBT-1} \left(1 - \frac{i}{KBT}\right) R_{q_z}(i) + R_{q_z}(0) \right]$$

The autocorrelation function

$$R_{q_z}(\tau) = \overline{q_z(t) q_z(t+\tau)} \qquad \text{(III.4.3.9)}$$

can be found as follows:

Let $u(t) = z(t+\tau)$ (III.4.3.10)

Then

$$q_z(t) \, q_z(t+\tau) = 1 \quad \text{if} \quad uz \geqslant P$$
$$= -1 \quad \text{if} \quad uz \leqslant -P$$
$$= 0 \quad \text{if} \quad -P < uz < P \qquad \text{(III.4.3.11)}$$

Therefore,

$$R_{q_z}(\tau) = \mathrm{Pr}(uz \geqslant P) - \mathrm{Pr}(uz \leqslant -P) \qquad \text{(III.4.3.12)}$$

Considering sampling at Nyquist-rate, we find that

$$R_{y_z}(i) \neq 0 \text{ if } i \neq 0 \text{ and } R_{q_z}(0) = \overline{q_z^2} \qquad \text{(III.4.3.13)}$$

The squared signal, q_z^2, has the values

$$+ 1 \quad \text{with Prob } Pr(|xy| \geq P)$$

$$0 \quad \text{with Prob } Pr(|xy| < P),$$

Therefore,

$$Pr(|xy| \geq P) = Pr(xy \geq P) + Pr(xy \leq -P) \qquad (III.4.3.14)$$

The probabilities $Pr(xy \geq P)$ and $Pr(xy \leq -P)$ can be found by putting $s_o = 0$

in (III.4.3.4) and (III.4.3.6) as

$$Pr(xy \geq P) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-\frac{x^2}{2}} \text{erfc}(\frac{P}{x}) \, dx \qquad (III.4.3.15)$$

and

$$Pr(xy \leq -P) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-\frac{x^2}{2}} \text{erfc}(\frac{P}{x}) \, dx \qquad (III.4.3.16)$$

Therefore, from (III.4.3.13),

$$\overline{q_z^2} = R_{q_z}(0) = 2\sqrt{\frac{2}{\pi}} \int_0^\infty e^{-\frac{x^2}{2}} \text{erfc}(\frac{P}{x}) \, dx \qquad (III.4.3.17)$$

and using (III.1.2.22) for $N = 2BT$

$$\sigma_w = \frac{1}{2BT} \left\{ 2\sqrt{\frac{2}{\pi}} \int_0^\infty e^{-\frac{x^2}{2}} \text{erfc}(\frac{P}{x}) \, dx \right\}^{\frac{1}{2}} \qquad (III.4.3.18)$$

Let $[Da(2)]_3$ be the degradation factor of the analog correlator with three

level quantization after multiplication of the signals x and y. Then, for

sampling at Nyquist rate,

$$[Da(2)]_3 = \sqrt{2}(\frac{\pi}{2})^{\frac{3}{4}} \frac{\left(\int_0^\infty e^{-\frac{x^2}{2}} \text{erfc}(\frac{P}{x}) \, dx \right)^{\frac{1}{2}}}{\int x e^{-\frac{1}{2}(x^2+(\frac{P}{x})^2)} \, dx} \qquad (III.4.3.19)$$

$[Da(2)]_3$ is plotted in Figure (III.4.3.2) as a function of the decision level

P.
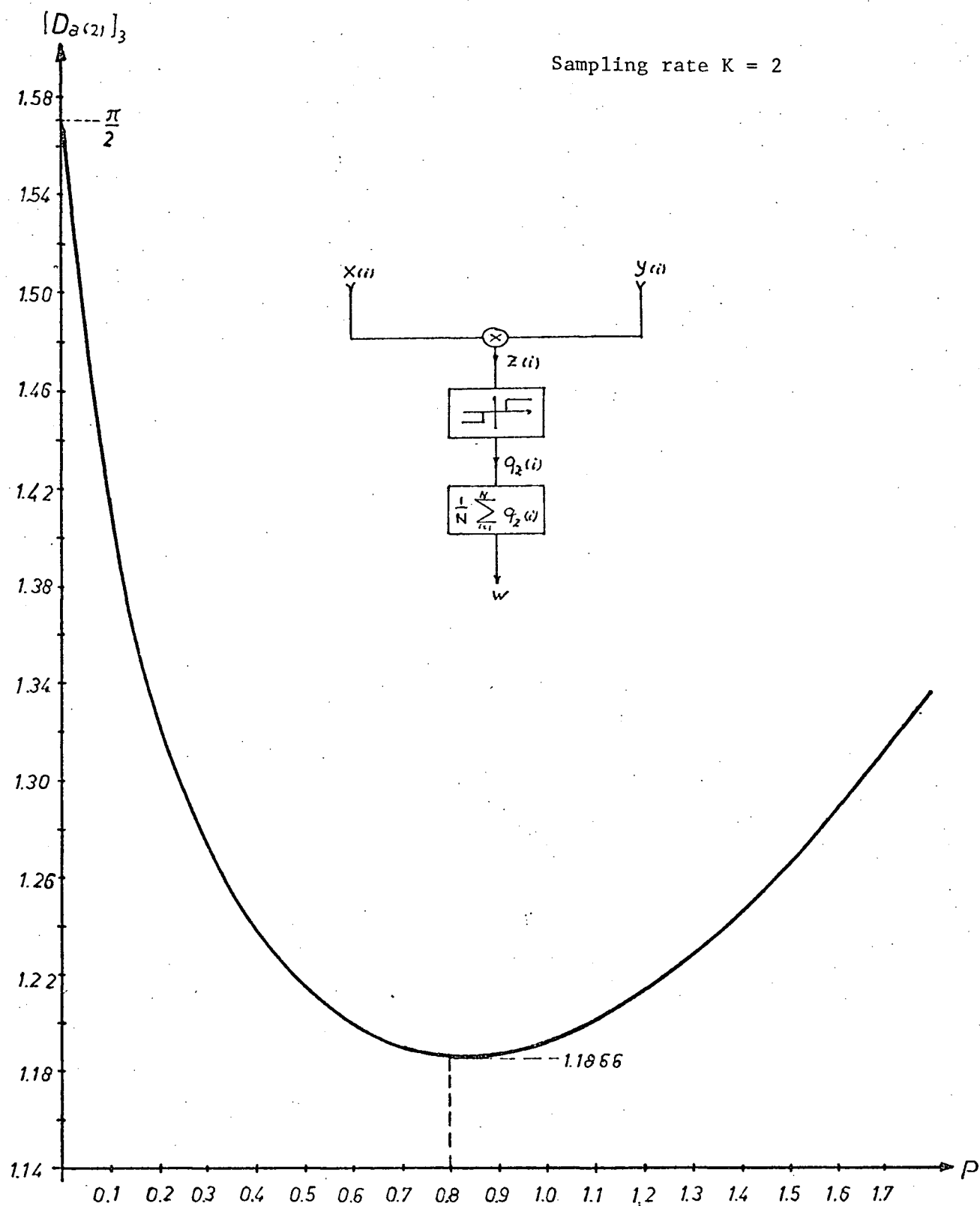
Figure III.4.3.2 Degradation factor versus decision level P
3-level quantizer after "multiplier"

III. 4.4 <u>Conclusions</u>

Figure (III.4.4.1) combines the results on correlators using three

values of the products, q_z which are to be averaged. It shows degradation

factors, D, plotted against decision level, P, in three cases

(1)　for the regular 3×3 level correlator, employing no overquantizing" of the signals.

(2)　for the 5×5 level correlator where products are merged to three values as described in section (III.4.2),

and

(3)　for the correlator studied in section (III.4.3), which uses an infinite number of levels before quantization and reduces the possible number of products to three afterwards.

It is seen that some improvement of degradation factor can result

from "overquantization", but that this is limited to about 4%. At

optimum decision levels, the degradation for case (3) is 4% less than that of

the regular 3×3 level correlator, (1).
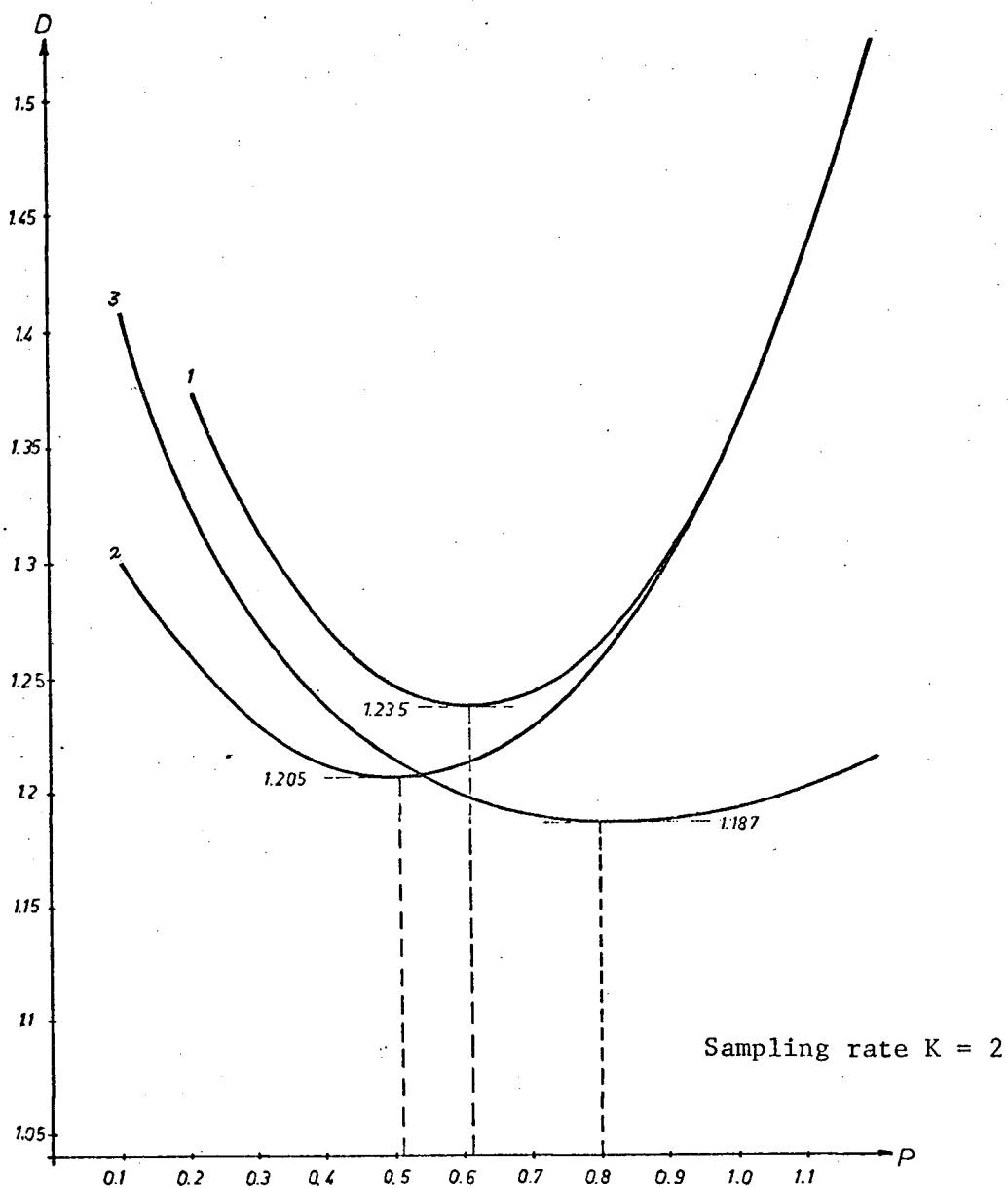
1.   regular 3 × 3 level correlator
2.   5 × 5 level correlator, products merged
3.   analog correlator with   3-level quantizer after the multiplier

Figure III.4.4.1 Degradation factor for   3-product correlators
               versus decision level P

IV. SIMULATION WITH RANDOM NUMBERS

The simulation is a software model of the actual correlator oper-
ating under the assumptions stated in II.1. and has two main purposes:

(a) To verify the theoretical results found in III.2.

(b) To evaluate the degradation factor for complicated quantizers
    where a theoretical computation of D would be too difficult.

The assumption $s_o \ll 1$ would make it necessary, as for a real radiometer,
to correlate over extremely long intervals of time. A direct simulation of
the whole system is therefore impractical. Since the expected value of w
is relatively easy to compute and does not depend on the sampling-rate, we
restrict ourselves to a simulation to determine the variance, σ_w^2. Accor-
ding to our original theoretical assumptions we neglect again the signal
s(t) and consider only the two independent Gaussian white noise sources as
inputs.

Our noise-samples are generated by a subroutine which produces
random numbers with normal distribution, which is available on the IBM-360.
It was found that 2^{14} = (16384) samples are needed in order to determine
σ_w with sufficient accuracy.

IV. 1. Creation of Correlated Samples

The random numbers generated by the computer program can repre-
sent samples of band-limited Gaussian noise taken at the Nyquist rate 2B.

For some calculations, we need to simulate samples taken at a
higher rate, KB, where K > 2. Such numbers will show some auto-correlation.
They were generated as illustrated in Figure IV.1.1.

Part (a) of that figure shows the situation to be simulated; a
noise source of bandwidth B is to be sampled at rate KB, where K > 2.
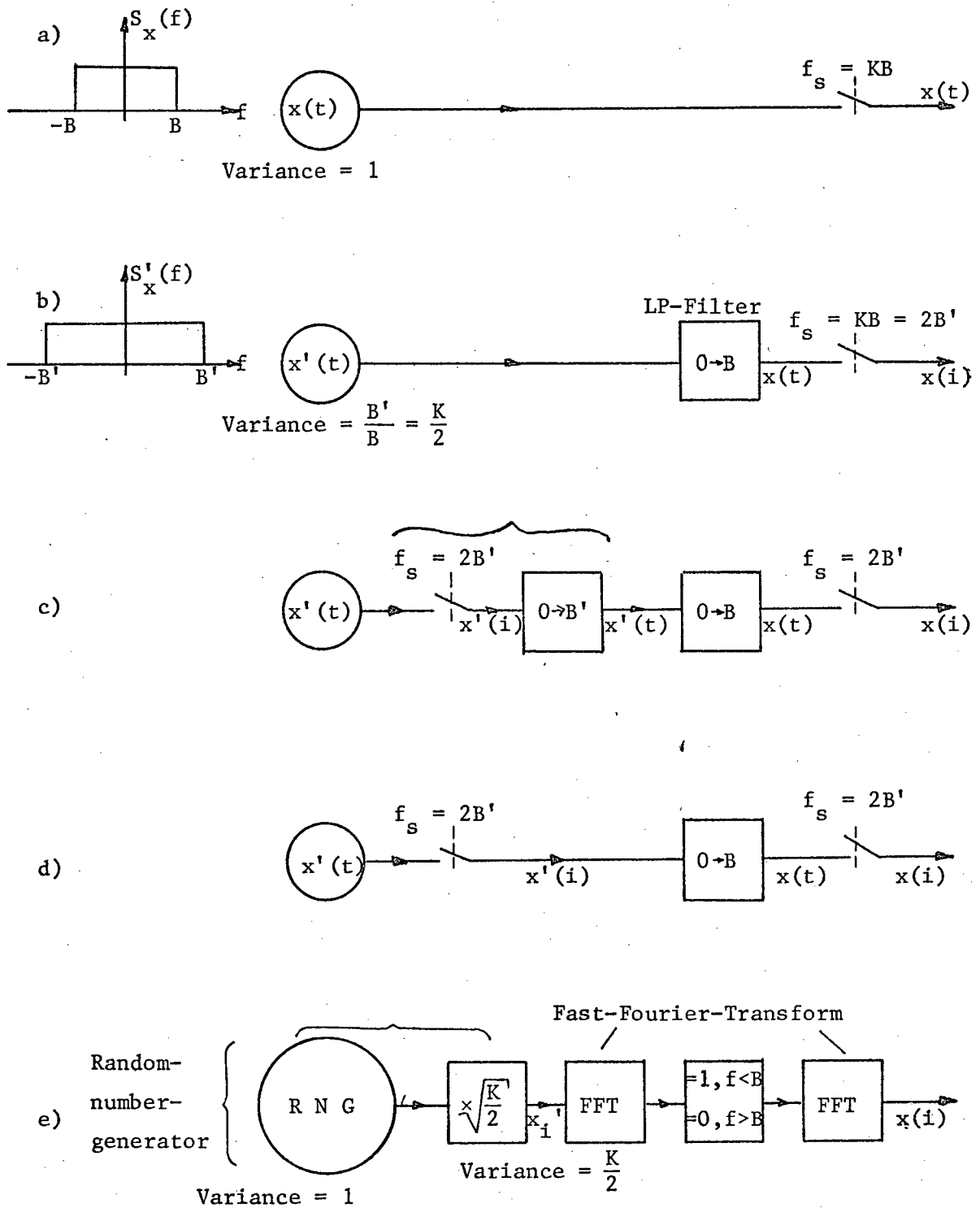
Figure IV.1.1   Generation of correlated noise samples

Part (b) shows an equivalent situation, where the noise source is thought of as having initially a wider bandwidth $B' = \frac{K}{2} B$, and the signal is then reduced to bandwidth B by an ideal low-pass filter. Note that the average power of the hypothetical source x'(t), should be (B'/B) times that of the real source, x(t).

Part (c) is equivalent to (b). Here a sampling switch at rate 2B' is inserted, with the samples passed through an ideal filter of bandwidth B', which allows the signal to be recovered completely.

Of course, a filter of bandwidth B' preceding one of lesser bandwidth, B, is redundant and can be removed, as is shown in part (d).

Finally, part (e) shows how the situation shown in (d) is simulated. Random numbers with Gaussian distribution are generated to represent samples of the hypothetical signal, x'(t) at intervals $(\frac{1}{2B'})$. They are multiplied by $\sqrt{K/2}$ to give them the required variance. Band-limiting is achieved by taking a Fast Fourier transform of a sequence of N = KBT such samples, rejecting components above frequency B, and performing an inverse Fourier transform on the remainder to recover the required time-samples.

IV. 2. Simulation of the Variance

Two independent sets of noise samples n(i) are used as inputs on the x-and y-channel, as shown in Figure IV.2.1. The samples in one set are correlated to represent samples of a bandlimited, white noise as discussed in IV.1. The samples for the x-channel are called x(1) .....x(NM) and for the y-channel y(1) ....y(NM).
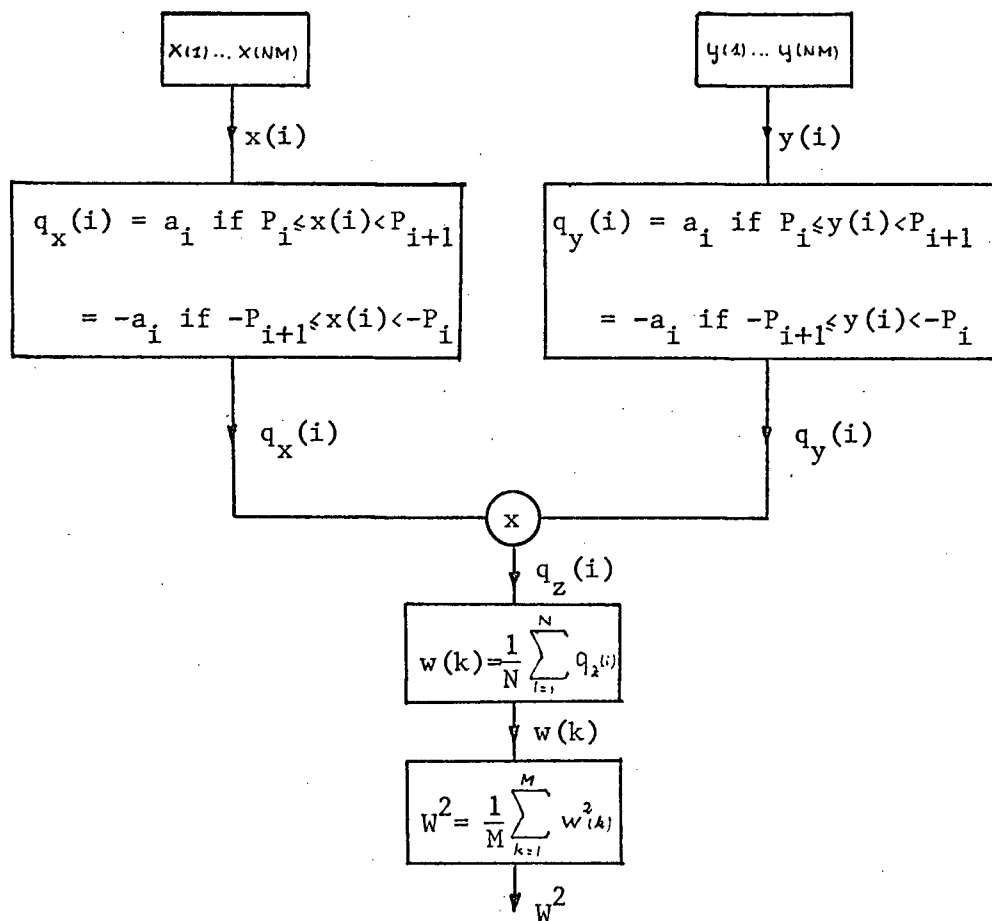
Figure IV.2.1 Simulation model of a quantized, sampled correlator

The expected value of w is zero since x and y are statistically independent, zero-mean samples and the quantizers are symmetric. Therefore the variance is equal to the expected value of w^2 and, denoting W^2 as the average over M values of w^2, the expected value of W^2, is found as

$$\overline{W^2} = \overline{w^2} = \sigma_w^2 \qquad (IV.2.1)$$

We are interested in $\sigma_{(W^2)}$, the standard deviation of W^2,

$$\sigma_{(W^2)} = \overline{W^4} - (\overline{W^2})^2 \qquad (IV.2.2)$$

in order to estimate the accuracy of σ_w^2 as found by the simulation method. The products, $q_z(i)$, are zero-mean random numbers with a non-Gaussian probability-density distribution. In fact, $p_{q_z}(z)$ is a set of Dirac delta

functions symmetrical about $q_z = 0$. However, the central limit theorem

states that the probability-density function of the sum of a large number

of random variables with arbitrary probability-density functions tends to

become Gaussian in the central region. Applying this to our case, we see

that $w(k)$ has a normal probability-density function in the central region,

assuming that N is large enough.

The expected value of W^4 is given by

$$\overline{W^4} = \frac{1}{M^2} \sum_{i=1}^{M} \sum_{j=1}^{M} \overline{w^2(i)\ w^2(j)}. \qquad (IV.2.3)$$

For $i = j$ we get

$$\overline{w^2(i)\ w^2(j)} = \overline{w^4} = 3\sigma_w^4 \quad (\text{see } [1], \text{ pg. } 148) \quad (IV.2.4)$$

and for $i \neq j$ we get

$$\overline{w^2(i)\ w^2(j)} = \overline{w^2(i)}\ \overline{w^2(j)} = (\overline{w^2})^2 \qquad (IV.2.5)$$

where the values of W^2 are assumed to be independent.

Therefore,

$$\overline{W^4} = \frac{1}{M^2} \sum_{i=1}^{M} \overline{w^4} + \frac{1}{M^2} 2 \sum_{i=1}^{M} \sum_{j=i+1}^{M} (\overline{w^2})^2$$

$$= \frac{1}{M} \overline{w^4} + \frac{M-1}{M} (\overline{w^2})^2 \qquad (IV.2.6)$$

$$= \overline{q^2} - (\overline{q})^2 = \frac{1}{M} 3\sigma_w^4 + \frac{M-1}{M} \sigma_w^4 - \sigma_w^4$$

$$= \frac{2}{M} \sigma_w^4 . \qquad (IV.2.7)$$

Therefore the standard deviation of W^2 is obtained as

$$\sigma_{(W^2)} = \sqrt{\frac{2}{M}} \ \sigma_w^2 \qquad\qquad \text{(IV.2.8)}$$

IV.3 Results of Simulation Runs

In chapter III it was seen that the degradation factor, D, is proportional to the standard deviation, σ_w, of the output of the quantized correlator.

Specifically it follows from equations III.1.2, III.1.2.14 and III.1.6.1 that

$$D = \frac{\pi}{2} \ \frac{\sqrt{2BT}}{f_x(a_i,P_i) \cdot f_y(a_i,P_i)} \quad \times \quad \sigma_w \qquad \text{(IV.3.1)}$$

A series of M simulation runs, each using N = KBT samples in each channel, gives a result W^2 whose expected value, $\overline{W^2}$, equals σ_w^2. Hence we can re-write equation (IV.3.1) in terms of N and $\overline{W^2}$ as

$$D^2 = \frac{\pi^2 N}{2K} \quad [f_x(a_i,P_i) \cdot f_y(a_i,P_i)]^{-2} \ \overline{W^2} \qquad \text{(IV.3.2)}$$

For any given correlator, all terms on the right hand side except $\overline{W^2}$ can be easily calculated. The quantity $\overline{W^2}$ is found from the M simulation runs with an uncertainty given by equation (IV.2.8) as

$$\frac{\sigma_{(W^2)}}{\overline{W^2}} \quad = \sqrt{\frac{2}{M}} \qquad\qquad \text{(IV.3.3)}$$

Hence the simulation runs can give us a value of D^2 with the same relative uncertainty

$$\frac{\sigma_{(D^2)}}{D^2} \quad = \quad \frac{\sigma_{(W^2)}}{\overline{W^2}} \quad = \sqrt{\frac{2}{M}} \qquad\qquad \text{(IV.3.4)}$$

or a value of D with uncertainty

$$\varepsilon = \frac{\sigma_D}{D} = \frac{1}{2} \frac{\sigma_{(D^2)}}{D^2} = \frac{1}{\sqrt{2M}} \qquad \text{(IV.3.5)}$$

To get an accurate value of D then requiries a very large number, M, of simulation runs. Each run also requires a large number, N, of samples if it is to represent a practical application where BT >> 1. It was found that N = 180 is a reasonable number for this purpose. The combination requires NM samples in each channel and this can soon produce exorbitant computing times. For the actual computer runs, 2^{14} = 16,384 samples were used in each channel. If these are "band-limited" samples, it requires about 32 sec of CPU time in the IBM 360 computer to generate the 2×2^{14} samples and a further 8 seconds to execute the simulation program, for a total of 40 seconds.

If the samples are then regarded as batches of N = 180 samples each, the number of batches will be

$$M = \frac{16,384}{180} = 91 \qquad \text{(IV.3.6)}$$

resulting in an uncertainty in the value of D given by equation (IV.3.5) as

$$\varepsilon = \frac{1}{\sqrt{2 \times 91}} = 7.4\% \qquad \text{(IV.3.7)}$$

This is too large an error for practical purposes. Reducing it by a factor of four requires an increase by a factor of 16 in the number of samples and hence in the computing time, which would increase to about 649 seconds.

Another more efficient way, which has proved very useful, is described here:

The samples x_i and x_j are correlated; with normalized auto correlation coefficient, $\rho(i-j)$, given by $\overline{x_1 x_j} = \overline{x^2} \rho(i-j) = \overline{x^2} s_a(\frac{2\pi(i-j)}{K})$ (IV.3.8)

Therefore $\quad |\rho(i-j)| \leq \dfrac{K}{2\tau(i-j)}$. $\hspace{4cm}$ (IV.3.9)

Therefore, taking two samples x_i and x_j far enough apart, their correlation coefficient is so small that the samples can be considered uncorrelated (and independent, since they are Gaussian). We now rotate the samples on one channel k steps and repeat the process, obtaining again 91 independent results of w^2. We can repeat this rotation 16384:k times and get therefore 91(16384:k) results, w_j^2 which can be considered to be independent, assuming that k is large enough.

$\qquad$ For our program, k = 1000 was used which results in a correlation coefficient for $\overline{x_i x_j}$

$$|\rho(i-j)| \leq \frac{K}{2000\pi} \hspace{3cm} \text{(IV.3.10)}$$

The highest sampling rate used was $K = 16$, and even in this case, the correlation coefficient is less than 2.5×10^{-3}, which is small enough to be neglected. With this method we obtain M = 1456 independent values of w^2 and from equation (IV.3.5),

$$\varepsilon = \frac{1}{\sqrt{2 \times 1456}} = 1.84\% \hspace{3cm} \text{(IV.3.11)}$$

Therefore the error of D is 1.84%, which is more tolerable.

$\qquad$ This second way results in the same error ε as that taking 16 times more samples, but only the execution part of the program runs longer:

Production of $2*2^{14}$ bandlimited samples:	32 sec
Execution of the simulation program:	<u>120 sec</u>
Total:	152 sec CPU-time

The total CPU-time needed is 152 sec, amounting to about \$23, at a charge of \$560/hr. We see therefore that this second scheme is roughly four times

less expensive than taking 16 times more samples.

Figure (IV.3.1) shows a comparison of calculated and simulated results for the degradation factor, D, for various normalized sampling rates, K.

The solid lines are the calculated results for five different quantized correlators.

The shaded areas show the domain of D within 1.84% of the calculated values for the 2×2 and 2×3 and 3×3 level correlators. The error domains for the other two correlators are not drawn to avoid confusion. The dots are the results obtained from simulation.

It is seen that the simulation runs verify the calculated results to the anticipated accuracy. The same simulation technique could therefore be relied on to provide approximate degradation factors for other schemes where calculations are impractical.
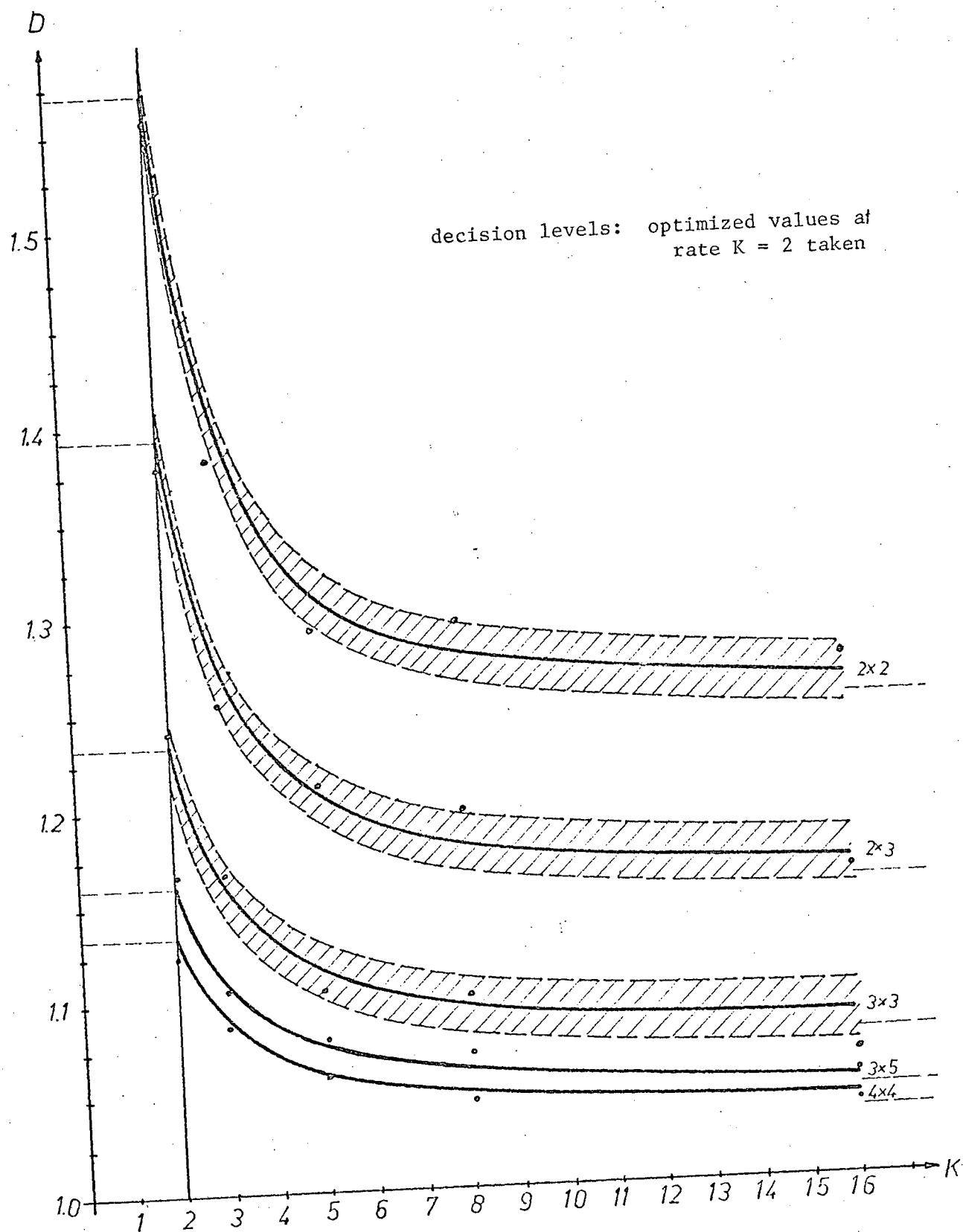
Figure IV.3.1 Simulated degradation factor versus sampling rate K

V.   RESULTS OF MORE THEORETICAL INTEREST

V.1   Optimization of Decision-Levels

In chapter III.1 we have stated that the decision levels P_i of a quantized correlator can be optimized.   If in a 3×3 level correlator P is allowed to go to zero, we get a 2×2 level correlator with a higher degradation factor than for 3×3 levels for any K as shown in Figure (III.2.1). As P goes to infinity, q_x and q_y tend to zero for all values of x and y and the correlator gives no information about the signal s(t).   Therefore D goes to infinity.   It follows that P_{opt} must exist.   Plotting D versus P as in Figure V.1.1   , we see that in every case D(P) has only one P_{opt}.

The 2×3, 3×3 and 4×4 level correlator have only one parameter, P, to optimize.   For higher level correlators several parameters P_i have to be simultaneously optimized, which leads to a nonlinear optimization problem of multiple parameters.   One method of solving this problem was found for the 3×5 level correlator and is discussed in section V.1.4.

V.1.1   Optimum Decision. Level for the 3×3 Level Correlator

The degradation factor $D_{3\times3}(P)$ is a concave function for P in $(0 \leqslant P < \infty)$ or equivalently,

$$\frac{d^2D}{dP^2} > 0 \qquad\qquad (V.1.1.1)$$

Therefore a minimum degradation factor for an optimum value of P_i, denoted by P_{opt} occurs for

$$\frac{dD}{dP} \Big/ P_{opt} = 0. \qquad\qquad (V.1.1.2)$$
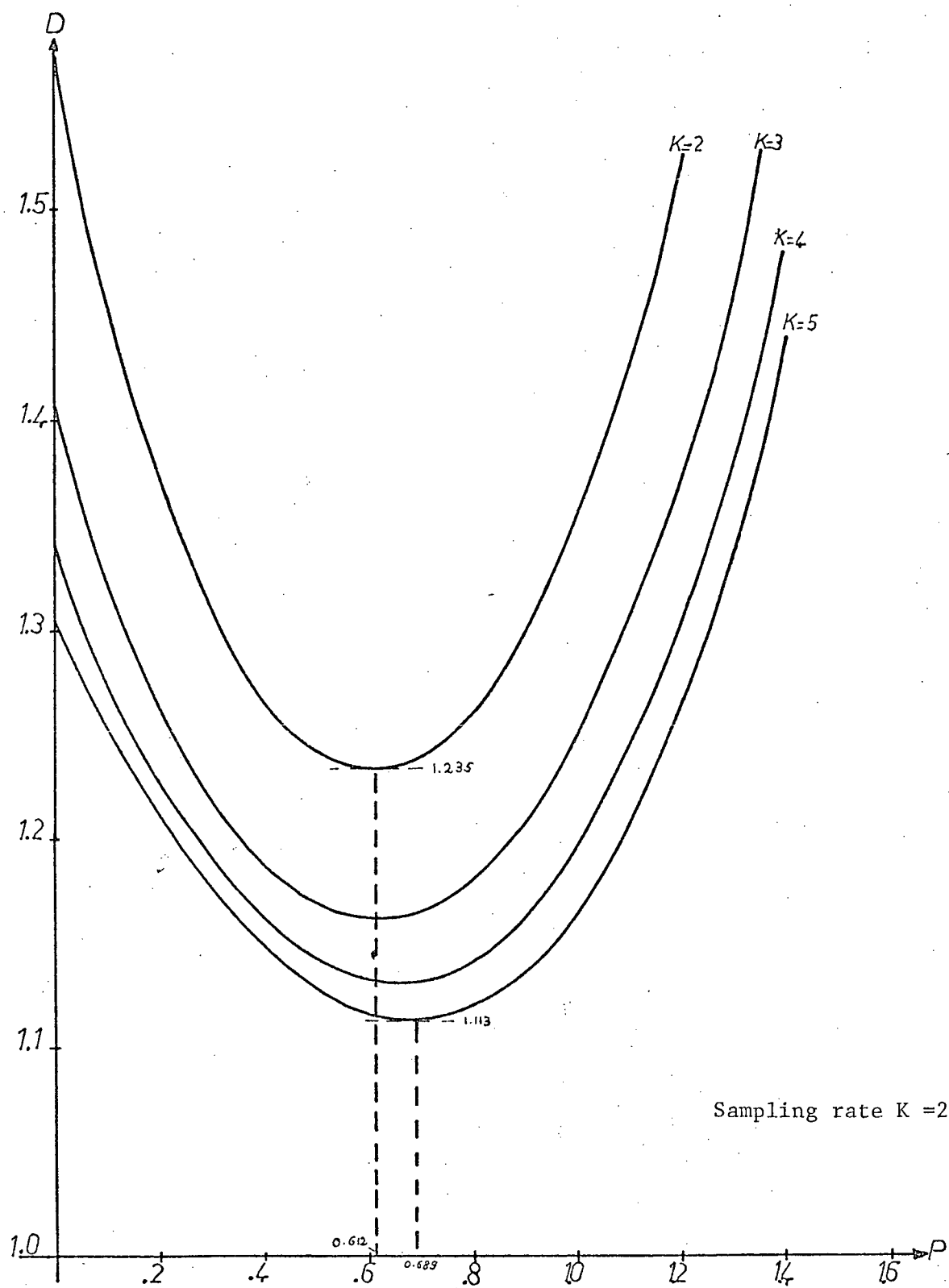
Figure V. 1.1 Degradation factor versus decision level P for a 3×3 level
correlator with sampling rate as a parameter

Using (III.1.2) in (III.1.6.1) we find

$$D = \sqrt{2BT} \; s_o \; 2\sqrt{\frac{\overline{w^2}}{\overline{w}}} \qquad\qquad (V.1.1.3)$$

where only $\overline{w^2}$ and $\overline{w}$ are functions of P.

We define the characteristic function f(P) as

$$f(P) = \frac{d}{dP}\left(\frac{\sqrt{\overline{w^2}}}{\overline{w}}\right) \qquad\qquad (V.1.1.4)$$

where

$$f(P)\Big|_{P_{opt}} = 0 \qquad\qquad (V.1.1.5)$$

or, equivalently,

$$\frac{\frac{d\overline{w}}{dP}}{\overline{w}}\Bigg|_{P_{opt}} = \frac{1}{2}\frac{\frac{d\overline{w^2}}{dP}}{\overline{w^2}}\Bigg|_{P_{opt}} \qquad\qquad (V.1.1.6)$$

The expected value of w of a 3×3-level correlator was found in (III.2.2.2). Therefore,

$$\frac{\frac{d\overline{w}}{dP}}{\overline{w}} = -2P \qquad\qquad (V.1.1.7)$$

The variance σ_w^2 was found in (III.1.2.21), where the autocorrelation function

$$R_{q_z}(i) = R_{q_x}^{\;2}(i), \qquad\qquad (V.1.1.8)$$

since both quantizers are equal. $R_{q_x}(i)$ was found in (III.2.2.3) and (III.2.2.4). Taking the derivative of $\overline{w^2}$,

$$\frac{d\overline{w^2}}{dP} = \frac{2}{N}\left\{2\sum_{i=1}^{N-1}(1-\frac{i}{N})\,R_{q_x}(i)\,\frac{dR_{q_x}(i)}{dP} + R_{q_x}(o)\,\frac{dR_{q_x}(o)}{dP}\right\} \quad (V.1.1.9)$$

and using Leibniz' rule

$$\frac{d}{dP} \int_P^\infty f(P,x) \, dx = \int_P^\infty \frac{df(P,x)}{dP} \, dx - f(P,P) \qquad (V.1.1.10)$$

to calculate the derivative of $R_{q_x}(i)$,

$$\frac{dR_{q_x}(i)}{dP} = -2\sqrt{\frac{2}{\pi}} \left[e^{-\frac{P^2}{2}} \left\{ \text{erfc} \left(P\sqrt{\frac{1-\varsigma_{(i)}}{1+\varsigma_{(i)}}}\right) - \text{erfc} \left(P\sqrt{\frac{1+\varsigma_{(i)}}{1-\varsigma_{(i)}}}\right) \right\} \right] \quad (V.1.1.11)$$

and

$$\frac{dR_{q_x}(o)}{dP} = -\sqrt{\frac{2}{\pi}} \, e^{-\frac{P^2}{2}} \qquad (V.1.1.12)$$

The characteristic function $f(P)$, defined in (V.1.1.4), is then found as

$$f(P) = \frac{\overline{\frac{dw^2}{dP}}}{\overline{w^2}} + 4P. \qquad (V.1.1.13)$$

Equations (III.1.2.21), (V.1.1.3) and (V.1.1.9) substituted into (V.1.1.13) yields

$$f(P) = \frac{2\left\{ 2 \sum_{i=1}^{N-1} (1-\tfrac{i}{N}) \, R_{q_x}(i) \, \frac{dR_{q_x}(i)}{dP} + R_{q_x}(o) \, \frac{dR_{q_x}(o)}{dP} \right\}}{2 \sum_{i=1}^{N-1} (1-\tfrac{i}{N}) \, R_{q_x}^2(i) + R_{q_x}^2(o)} + 4P \qquad (V.1.1.14)$$

This nonlinear equation was solved numerically using the subroutine RTWI is *SSP (IBM 360).

Figure (V.1.1.1) shows P_{opt} versus the sampling-rate K for the 3×3 level correlator. P_{opt} increases about 16% as K grows from 2 to 12. The fastest increase in P_{opt} occurs between K = 3 and K = 5. The variation of optimum decision level with sampling rate is an interesting and unexpected phenomenon, and there seems to be no obvious qualitative explanation for it.
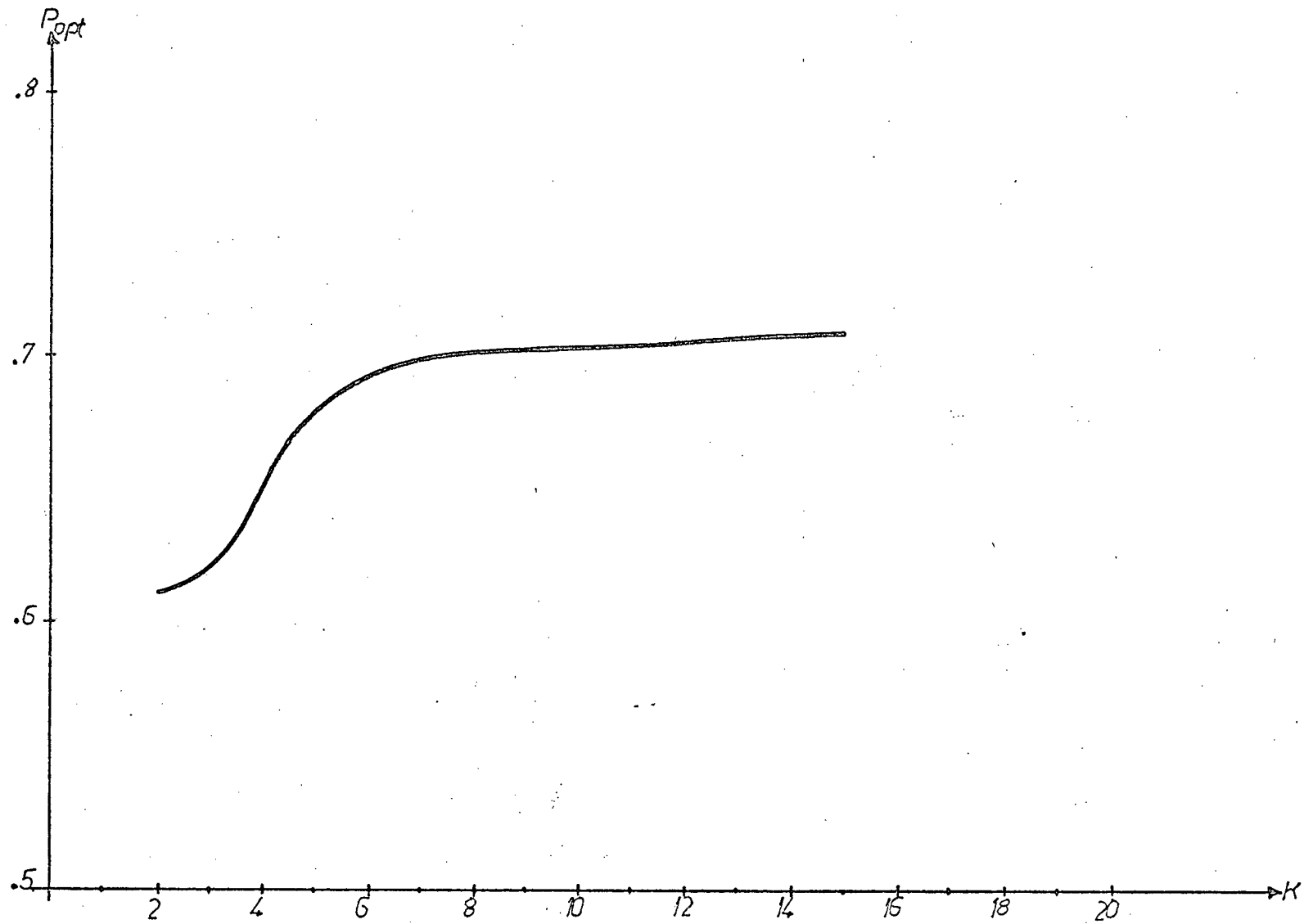
Figure V.1.1.1 Optimized decision level P_{opt} versus sampling rate K for a 3×3 level correlator

V.1.2 Optimum Decision Level for the 2×3 Level Correlator

The expected value of w was found in (III.2.3.1).

Therefore,

$$\frac{\dfrac{d\overline{w}}{dP}}{\overline{w}} = -P \tag{V.1.2.1}$$

The variance σ_w^2 was found in (III.1.2.21) where

$$R_{q_z}(i) = R_{q_x}(i) \cdot R_{q_y}(i) \tag{V.1.2.2}$$

$R_{q_x}(i)$ is the autocorrelation function of the 3-level quantized signal and given in (III.2.2.3) and (III.2.2.4) for $\tau = iT_s$, where the derivative of $R_{q_x}(i)$ with respect to P was found in (V.1.1.11) and (V.1.1.12).

The autocorrelation function for the 2 – level quantized signal, $R_{q_y}(i)$, was found in (III.2.1.8) to (III.2.1.10).

Since only $R_{q_x}(i)$ is a function of P, the derivative of $\overline{w^2}$ with respect to P becomes

$$\frac{\overline{dw^2}}{dP} = \frac{1}{N}\left\{ 2\sum_{i=1}^{N-1}\left(1-\frac{i}{N}\right) R_{q_y}(i)\frac{dR_{q_x}(i)}{dP} + R_{q_y}(o)\frac{dR_{q_x}(o)}{dP}\right\} \tag{V.1.2.3}$$

The function f(P) was defined in (V.1.1.4).

Therefore,

$$f(P) = \frac{\dfrac{\overline{dw^2}}{dP}}{\overline{w^2}} + 2P \tag{V.1.2.4}$$

and using (III.1.2.21), (V.1.2.2) and (V.1.2.3) in (V.1.2.4) we obtain

$$f(P) = \frac{2\sum\limits_{i=1}^{N-1}\left(1-\frac{i}{N}\right) R_{q_y}(i)\dfrac{dR_{q_x}(i)}{dP} + R_{q_y}(o)\dfrac{dR_{q_x}(o)}{dP}}{2\sum\limits_{i=1}^{N-1}\left(1-\frac{i}{N}\right) R_{q_x}(i) R_{q_y}(i) + R_{q_x}(o) R_{q_y}(o)} + 2P \tag{V.1.2.5}$$

Again the solution f(P) = 0 was found for different values of the sampling

rate K and plotted in Figure V.1.2.1.

V.1.3 4×4 Level Correlator

The expected value of w was found in (III.2.5.2). Therefore,

$$\frac{\frac{d\bar{w}}{dP}}{\bar{w}} = -2P \quad \frac{K-1}{e^{\frac{P^2}{2}} + (K-1)} \tag{V.1.3.1}$$

Using (III.1.2.21) with $R_{q_z}(i) = R_{q_x}(i)^2$ yields

$$\frac{\overline{dw^2}}{dP} = \frac{2}{N} \left[2 \sum_{i=1}^{N-1} (1-\frac{i}{N}) R_{q_x}(i) \frac{dR_{q_x}(i)}{dP} + R_{q_x}(o) \frac{dR_{q_x}(o)}{dP} \right] \tag{V.1.3.2}$$

where the autocorrelation function $R_{q_x}(i)$ is given by (III.2.5.3) and

(III.2.5.5) for $\tau = i\, T_s$.

Taking derivatives, we obtain

$$\frac{dR_{q_x}(i)}{dP} = -2\sqrt{\frac{2}{\pi}}\; e^{-\frac{P^2}{2}}\; (K-1)\left\{ (K-1)\left[\text{erfc}\,(P\sqrt{\frac{1-\rho(i)}{1+\rho(i)}}) - \text{erfc}(P\sqrt{\frac{1+\rho(i)}{1-\rho(i)}}) \right. \right.$$

$$\left. \left. + 1 - 2\,\text{erfc}\,(\frac{P}{1-\rho^2(i)}) \right\} \right. \tag{V.1.3.3}$$

and

$$\frac{dR_{q_x}(o)}{dP} = -(K^2-1)\sqrt{\frac{2}{\pi}}\; e^{-\frac{P^2}{2}} \tag{V.1.3.4}$$

The function f(P), defined in (V.1.4.1), becomes then

$$f(P) = 2\; \frac{\sum_{i=1}^{N-1} (1-\frac{i}{N}) R_{q_x}(i) \frac{dR_{q_x}(i)}{dP} + R_{q_x}(o) \frac{dR_{q_x}(o)}{dP}}{2 \sum_{i=1}^{N-1} (1-\frac{i}{N}) R_{q_x}^2(i) + R_{q_x}^2(o)} + 2P\; \frac{K-1}{K-1 + e^{-\frac{P^2}{2}}}$$

The equation $f(P)/_{P_{opt}} = 0$ was solved and plotted versus the sampling-rate
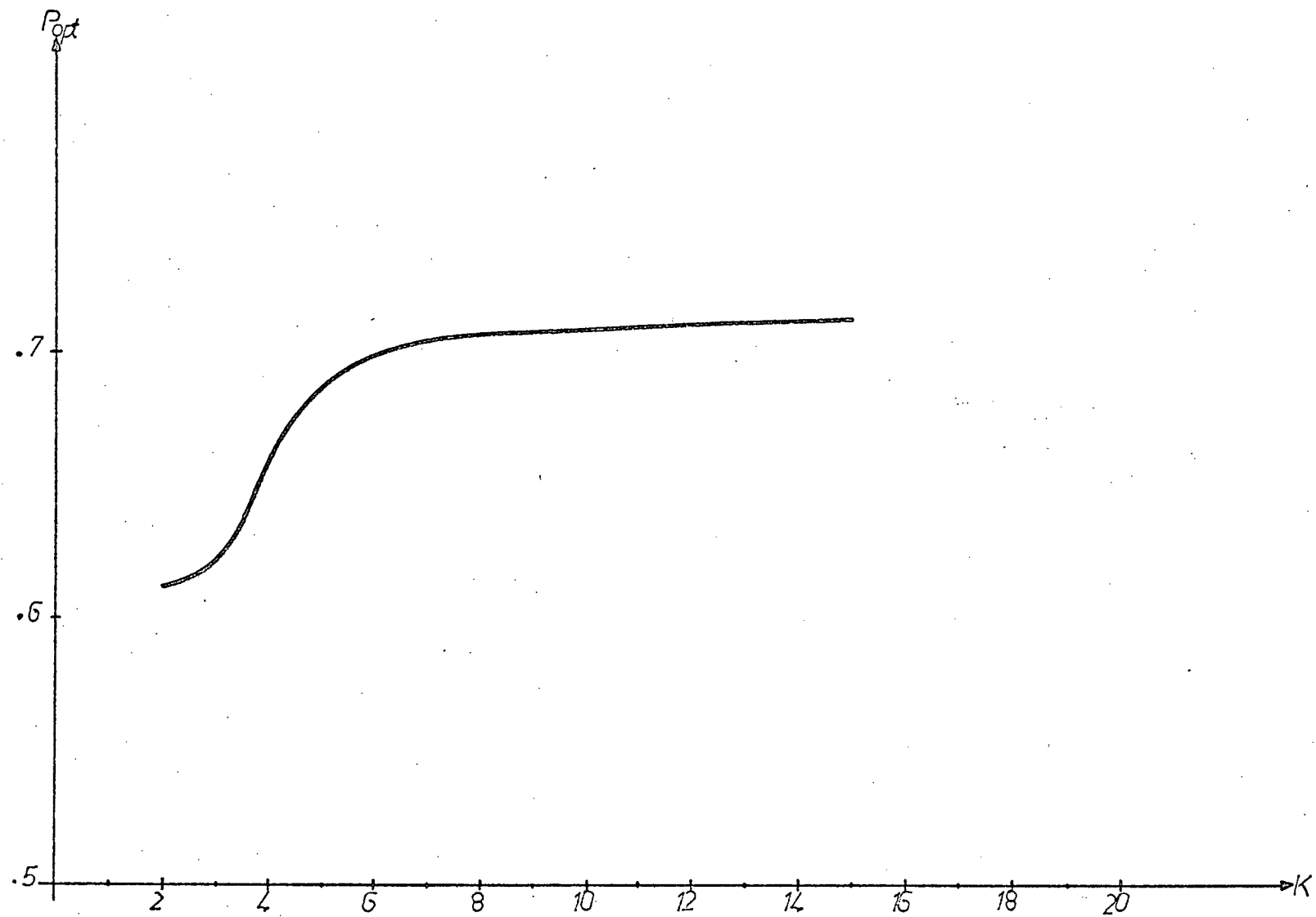
Figure V.1.2.1 Optimized decision level P_{opt} versus sampling rate K for a 2×3 level correlator

K in Figure (V.1.3.1).

V.1.4 3×5 Level Correlator

The quantizers for the x- and y-channel are shown in the figures (V.1.4.1) and (V.1.4.2):
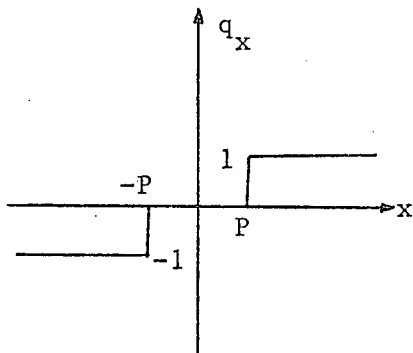
Figure V.1.4.1   3-level quantizer   Figure V.1.4.2   5-level quantizer
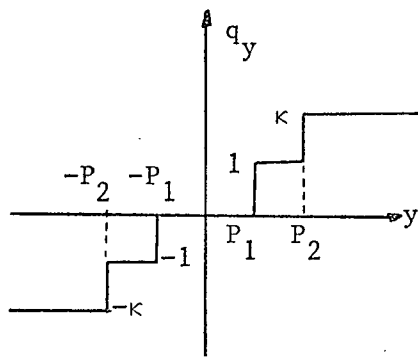
In this case, the decision levels P, P_1 and P_2 have to be optimized simultanously. It can be shown that $D(P, P_1, P_2)$ is analytic for ($0 \leqslant P < \infty$; $0 \leqslant P_1 < \infty$; $P_1 \leqslant P_2 < \infty$) and that

$$\frac{d^2 D}{dP^2} > 0 \quad , \quad \frac{d^2 D}{dP_1^2} > 0, \quad \frac{d^2 D}{dP_2^2} > 0 \qquad (V.1.4.1)$$

Defining $\vec{P}$ as an array of the variables P, P_1 and P_2, it can be shown that

$$\vec{P}_{opt} = (P_{opt}, P_{1\ opt}, P_{2\ opt}) \qquad (V.1.4.2)$$

exists such that

$$\frac{dD}{dP}\bigg|_{P_{opt}} = 0, \quad \frac{dD}{dP_1}\bigg|_{P_{opt}} = 0, \quad \frac{dD}{dP_2}\bigg|_{P_{opt}} = 0 \qquad (V.1.4.3)$$

From V.1.4.1 it follows that there is only one min. at $\vec{P}_{opt}$. We define
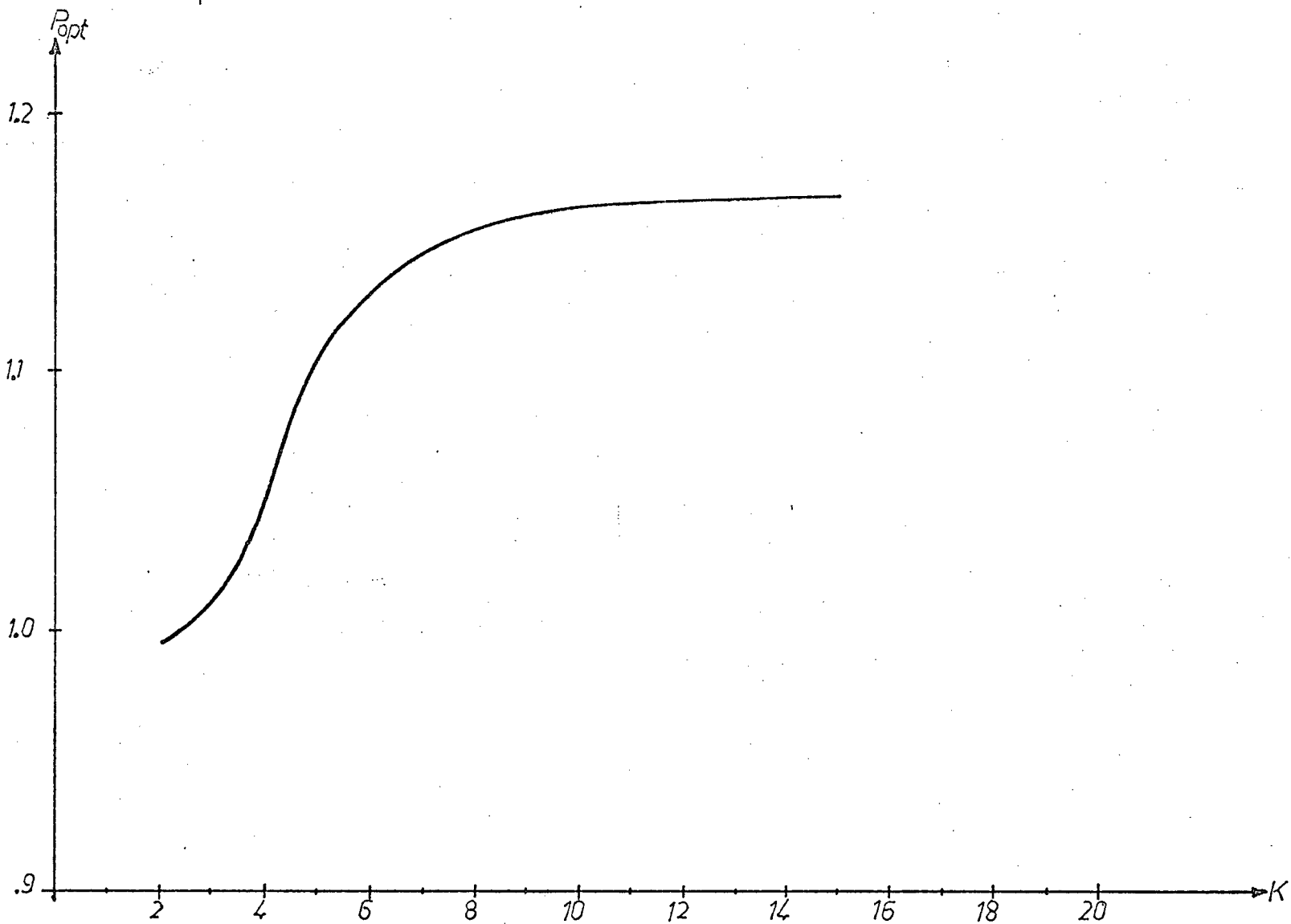
Figure V.13.1 Optimized decision level P_{opt} versus sampling rate K for a 4×4 level correlator

the partial derivatives of D with respect to P, P_1 and P_2 as

$$f_1(\vec{P}) = f_1(P, P_1, P_2) = \frac{dD}{dP} \qquad (V.1.4.4)$$

$$f_2(\vec{P}) = f_2(P, P_1, P_2) = \frac{dD}{dP_1} \qquad (V.1.4.5)$$

$$f_3(\vec{P}) = f_3(P, P_1, P_2) = \frac{dD}{dP_2} \qquad (V.1.4.6)$$

where $\vec{P}_{opt}$ has to be found such that

$$f_1(\vec{P}_{opt}) = f_2(\vec{P}_{opt}) = f_3(\vec{P}_{opt}) = 0. \qquad (V.1.4.7)$$

We consider the special case first, where we sample at Nyquist-rate. This case can easily be calculated, since $\vec{P}_{opt}$ does not change much for higher sampling-rates. The solution $\vec{P}_{opt}$ at Nyquist-rate can be used as the initial vector for an iterative method, which is discussed later on in this chapter. The degradation factor at Nyquist-rate (equation III.1.6.3) can be expressed as a product of 2 functions which depends only on the x- and y-channel respectively:

$$D_{3\times5}(2) = \frac{\pi}{2} \frac{\sqrt{R_{q_x}(o)}}{f_x(a_i, P_i)} \cdot \frac{\sqrt{R_{q_y}(o)}}{f_y(a_i, P_i)} \qquad (V.1.4.8)$$

As $R_{q_x}(o)$ is not a function of P_1 and P_2 and $R_{q_y}(o)$ does not depend on P, we see that

$$f_1 = f_1(P) , \qquad (V.1.4.9)$$

$$f_2 = f_2(P_1, P_2) \qquad (V.1.4.10)$$

and $$f_3 = f_3(P_1, P_2) . \qquad (V.1.4.11)$$

and therefore that both channels can be optimized separately. The optimization with respect to P has already been done for the 3×3-level correlator in chapter V.1.1 and P_{opt} was found to be

$$P_{opt} = 0.612.$$

Evaluating (V.1.4.5) and (V.1.4.6) we obtain

$$f_2(P_1, P_2) = P_1 + \frac{e^{-\frac{P_1^2}{2}} + (\kappa-1)\, e^{-\frac{P_2^2}{2}}}{2 e^{-\frac{P_1^2}{2}}} \; \frac{\dfrac{dR_{q_y}(o)}{dP_1}}{R_{q_y}(0)} \qquad (V.1.4.12)$$

and

$$f_3(P_1, P_2) = P_2 + \frac{e^{-\frac{P_1^2}{2}} + (\kappa-1)\, e^{-\frac{P_2^2}{2}}}{2(\kappa-1) e^{-\frac{P_1^2}{2}}} \; \frac{\dfrac{dR_{q_y}(o)}{dP_2}}{R_{q_y}(0)} \, . \qquad (V.1.4.13)$$

From (III.2.4.4) we obtain

$$R_{q_y}(o) = 2\,(\operatorname{erfc}(P_1) + (\kappa^2 - 1)\, \operatorname{erfc}(P_2)). \qquad (V.1.4.14)$$

Therefore,

$$\frac{dR_{q_y}(o)}{dP_1} = -\sqrt{\frac{2}{\pi}}\; e^{-\frac{P_1^2}{2}} \qquad (V.1.4.15)$$

and

$$\frac{dR_{q_y}(o)}{dP_2} = -\sqrt{\frac{2}{\pi}}\; (\kappa^2-1)\, e^{-\frac{1}{2}P_2^2} \qquad (V.1.4.16)$$

A necessary condition that $f_2(P_1, P_2)$ and $f_3(P_1, P_2)$ hold simultaneously is found dividing (V.1.4.12) by (V.1.4.13) as

$$\frac{P_1}{P_2} = (\kappa-1)\, e^{-\frac{1}{2}(P_2^2 - P_1^2)} \; \frac{\dfrac{dR_{q_y}(o)}{dP_1}}{\dfrac{dR_{q_y}(o)}{dP_2}} \qquad (V.1.4.17)$$

Using (V.1.4.15) and (V.1.4.16) in (V.1.4.17), we obtain

$$\frac{P_1}{P_2} = \frac{1}{1 + \kappa} \qquad (V.1.4.18)$$

This simple relationship allows us to reduce the joint equations $f_2 = 0$ and $f_3 = 0$ to one equation of one variable only. Solving the remaining equation, we finally obtain

$$P_{1\,opt} = 0.422 \qquad \text{and} \qquad P_{2\,opt} = 1.266.$$

In the general case, where $K > 2$, f_1, f_2 and f_3 are functions of all three variables P, P_1 and P_2. All three equations $f_1 = 0$, $f_2 = 0$ and $f_3 = 0$ have to be solved simultaneously. However, it has been found that $\vec{P}_{opt}$ does not change much (only a few %) for $K > 2$. Therefore we can use $\vec{P}_{opt}$ at Nyquist-rate as a first approximation for an iterative method to solve $f_1 = f_2 = f_3 = 0$.

A gradient method has proved to be useful to find $\vec{P}_{opt}(K)$: Given the degradation factor as a function of P, P_1 and P_2, we have to find

$$\vec{P}_{opt} = (P_{opt},\ P_{1\,opt},\ P_{2\,opt}),$$

such that the partial derivatives defined in (V.1.4.4) to (V.1.4.6) disappear for $\vec{P} = \vec{P}_{opt}$.

Defining $\vec{P}_o$ as the optimum decision vector at Nyquist-rate, the degradation factor in the neighborhood of $\vec{P}_o$ is then

$$D(\vec{P}_o - \Delta\vec{P}) \simeq D(P_o) - f_1\,\Delta P - f_2\,\Delta P_1 - f_3\,\Delta P_2 \qquad (V.1.4.19)$$

We choose $\Delta P = \Delta P_1 = \Delta P_2 = h$, the stepwidth we want to move in the direction of the gradient. The best value of h has to be found by trial and error. If h is too large, the sequence $P_{(o)}, P_{(1)}, P_{(2)}, \ldots\ldots P_{(i)}$ does not converge to $\vec{P}_{opt}$. If h is too small, the convergence is slow.

We normalize

$$h = \frac{s}{\sqrt{P^2 + P_1^2 + P_2^2}} \qquad (V.1.4.20)$$

and choose the desired accuracy

$$\mathcal{E} = \sqrt{f_1^2 + f_2^2 + f_3^2} \qquad \leq \qquad 10^{-5} \qquad\qquad \text{(V.1.4.21)}$$

A suitable stepwidth s was found as 0.65. For s = 1 no convergence occured and for s = 0.5 the convergence was too slow.

The iteration produces the sequence

$$\vec{P}_{(1)}, \quad \vec{P}_{(2)}, \quad \vec{P}_{(3)} \quad \cdots \cdots \vec{P}_{(n)}$$

where $\vec{P}_{(n)}$ can be close to $\vec{P}_{opt}$ with any desired accuracy.

Starting from the initial array $\vec{P}_o$, above sequence can be computed using

$$P_{(i)} = P_{(i-1)} - f_1(P_{(i-1)}, P_{1(i-1)}, P_{2(i-1)}) \cdot h \qquad\qquad \text{(V.1.4.22)}$$

$$P_{1(i)} = P_{1(i-1)} - f_2(P_{(i-1)}, P_{1(i-1)}, P_{2(i-1)}) \cdot h \qquad \text{(V.1.4.23)}$$

$$P_{2(i)} = P_{2(i-1)} - f_3(P_{(i-1)} {}^m P_{1(i-1)}, P_{2(i-1)}) \cdot h \qquad \text{(V.1.4.24)}$$

The results P_{opt}, P_{opt} and $P_{2\,opt}$ were computed and are plotted in Fig. V.1.4.3 vs. the sampling-rate K. All three curves $P_{opt\,(K)}$, $P_{1\,opt(K)}$ and $P_{2\,opt\,(K)}$ show the same characteristics, i.e. the P_{opt} increases by 1.65% to 1.8% between Nyquist-rate and rate K = 6. It can be seen that P_o found at Nyquist rate is a good approximation for any K >2.

V.2 Decomposition into Single Channel Correlators

Under certain circumstances D can be expressed as the product of two single-channel factors, D_x and D_y:

$$D = D_x D_y \qquad\qquad \text{(V.2.1)}$$

D_x and D_y are the degradation factors of a correlator which has only one quantizer in the x- and y-channel respectively, the other channel carrying
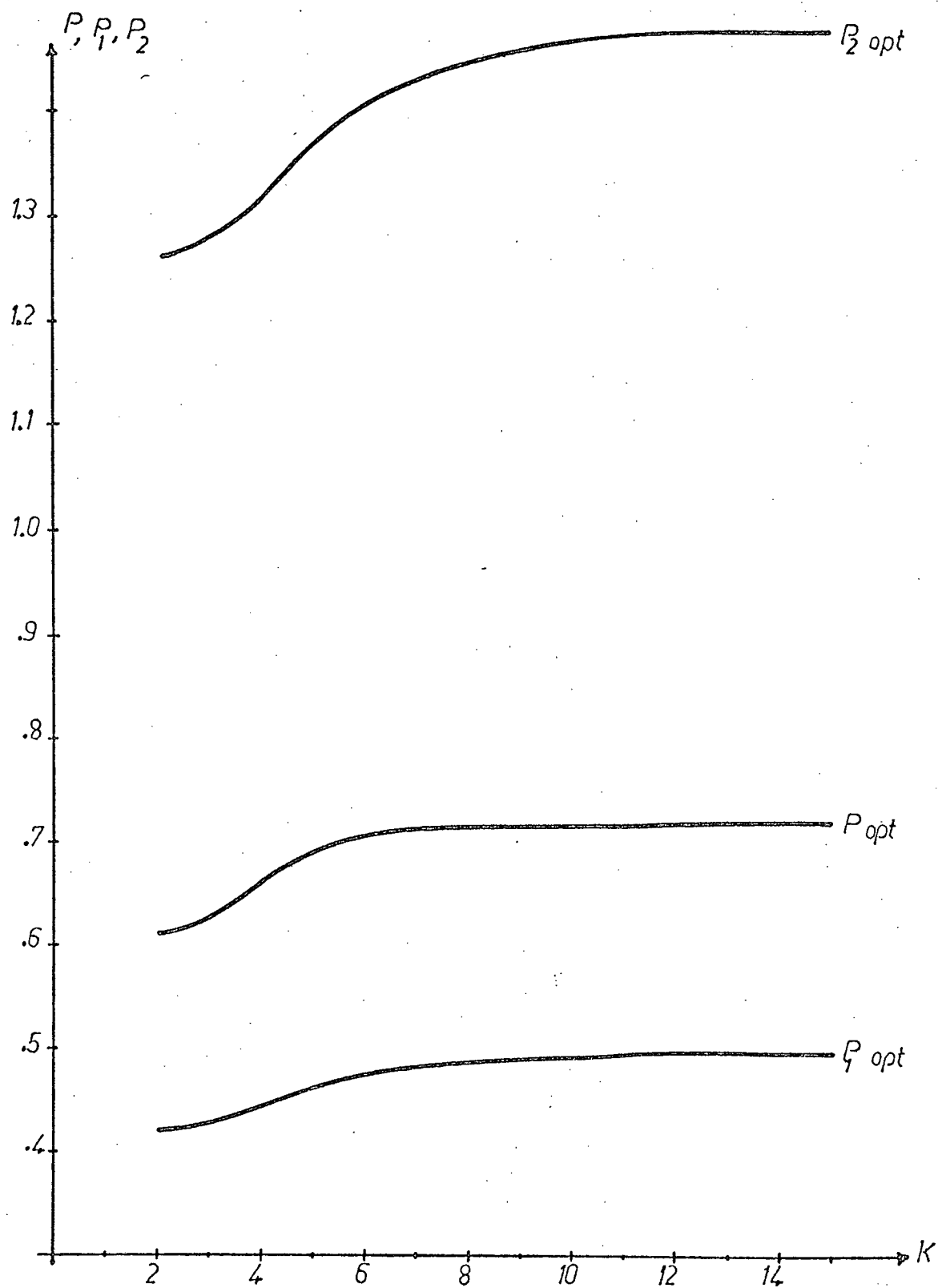
Figure V.1.4.3 Optimized decision levels P, P_1 and P_2 versus
sampling rate K for a 3×5 level correlator

analog signals.

We investigate in this section the circumstances under which such decomposition is valid, and the errors that occur if the decomposition is assumed to be always true.

It will be recalled that the degradation factor of a quantized correlator depends on its output signal to noise ratio:

$$D \quad \propto \quad \frac{\sigma_w}{\bar{w}} \qquad (V.2.2)$$

Now it is shown in Appendix A2 that for small signals, $\bar{w}$ can always be decomposed into an x-component and a y-component.

$$\bar{w} = \bar{q}_z = \overline{q_x q_y} = \bar{q}_x \; \bar{q}_y \qquad (V.2.3)$$

It remains to see whether it is possible to write σ_w in the same way. For small signals and symmetrical quantizers, $\sigma_w^2 = \overline{w^2}$ and, from equations (III.1.2.24) and (III.1.2.21) , we can express this variance as

$$\overline{w^2} = \frac{1}{N} \sum_{i=-(N-1)}^{N-1} (1-i/N) \; R_{q_x}(i) \cdot R_{q_y}(i) \qquad (V.2.4)$$

and since $\lim_{i\to\infty} R_{q_x}(i) = \lim_{i\to\infty} R_{q_y}(i) = 0$ \qquad (V.2.4)

$$\overline{w^2} \approx \frac{1}{N} \sum_{-(N-1)}^{N-1} R_{q_x}(i) \; R_{q_y}(i) \qquad (V.2.5)$$

for large N can be expressed as a product of a function of $R_{q_x}(i)$ and a function $R_{q_y}(i)$, only if

$$\sum_{i=-(N-1)}^{N-1} \sum_{j=-(N-1)}^{N-1} R_{q_x}(i) \; R_{q_y}(j) = 0 \qquad (V.2.6)$$

i.e. if $R_{q_x}(i)$ and $R_{q_y}(i)$ are orthogonal.

It can be seen that for any symmetric quantizer

$$\text{sign } (R_{q_x}(\tau)) = \text{sign}(\rho(\tau)) \qquad (V.2.7)$$

where τ is the normalized autocorrelation function of the unprocessed signal $x(t)$. So $R_{q_x}(\tau)$ has the same zero-crossings as $\rho(\tau)$. Since $x(t)$ and $y(t)$ have the same autocorrelation function,

$$\sum_{i=1}^{N-1} (1-\frac{i}{N}) R_{q_x}(i) R_{q_y}(i) \geqslant 0 ; \qquad (V.2.8)$$

equality holds only if $\rho(i) = 0$ for all $i \neq 0$, i.e. if the samples are uncorrelated. For the bandlimited signal this is only the case if we sample of Nyquist-rate where

$$\overline{w^2} = \frac{1}{N} R_{q_x}(o) R_{q_y}(o) \qquad (V.2.9)$$

Therefore D can be expressed as the product of 2 single channel degradation factors only if we sample at Nyquist-rate.

V.2.1 Single Channel Correlation Factors and Decomposition Error

Define $D_{n \ast \infty}(K)$ to be the "single-channel degradation factor", being the degradation factor to a correlator with an n-level quantizer in the x-channel and no quantizer in the y-channel. The expected value of the correlator output, w, is given by equations (III.1.2.3) and (III.1.5.4) as

$$\overline{w} = s_o \overline{q_x(i)} \qquad (V.2.1.1)$$

and the expected value of w^2 is obtained using equations (III.1.2.21), (III.1.2.36), (III.1.2.37) and (III.1.5.7).

Using these equations in (III.1.6.1) and (II.2.3) we find

$$D_{n \times \infty}(K) = \sqrt{\frac{\pi}{K}} \; \frac{\left\{2 \sum_{i=1}^{N-1} (1 - \frac{i}{N}) \, R_{q_x}(i) \cdot s_a(\frac{2\pi i}{K}) + R_{q_x}(o)\right\}^{\frac{1}{2}}}{f_x(a_i, P_i)} \qquad (V.2.1.2)$$

For Nyquist-rate sampling we define

$$D_x = D_{n \times \infty}(2). \qquad (V.2.1.3)$$

Using (V.2.1.2) for K = 2 we obtain

$$D_x = \sqrt{\frac{\pi}{2}} \; \frac{\sqrt{R_{q_x}(o)}}{f_x(a_i, P_i)} \; . \qquad (V.2.1.4)$$

Similarly, the single-channel degradation factor for the y-channel containing an m-level correlator, is

$$D_y = D_{m \times \infty}(2) = \sqrt{\frac{\pi}{2}} \; \frac{\sqrt{R_{q_y}(o)}}{f_y(a_i, P_i)} \qquad (V.2.1.5)$$

An n·m-level correlator, sampled at Nyquist-rate then has the degradation factor

$$D_{n \times m}(2) = D_x D_y = \frac{\pi}{2} \; \frac{\sqrt{R_{q_x}(o) \, R_{q_y}(o)}}{f_x(a_i, P_i) \, f_y(a_i, P_i)} \qquad (V.2.1.6)$$

Equation (V.2.1.6) agrees with (III.1.6.3) for sampling at Nyquist-rate.

We define a "decomposition error", $\mathcal{E}_{nm}(K)$, as

$$\mathcal{E}_{nm}(K) = \frac{D_{n \times m}(K) - D_x D_y}{D_{n \times m}(K)} \qquad (V.2.1.7)$$

This value measures the relative error made by decomposing the degradation factor $D_{n \times m}(K)$ into a product of two single channel degradation factors, $D_{n \times \infty}(K) \cdot D_{m \times \infty}(K)$. This "decomposition" error is plotted in Figure V.2.1.1 against the normalized sampling frequency, K.

There is no error at K=2. For higher values, the error soon settles down to a constant value of ≈13%. Decomposing the degradation factor under these circumstances leads to a small error, but is still
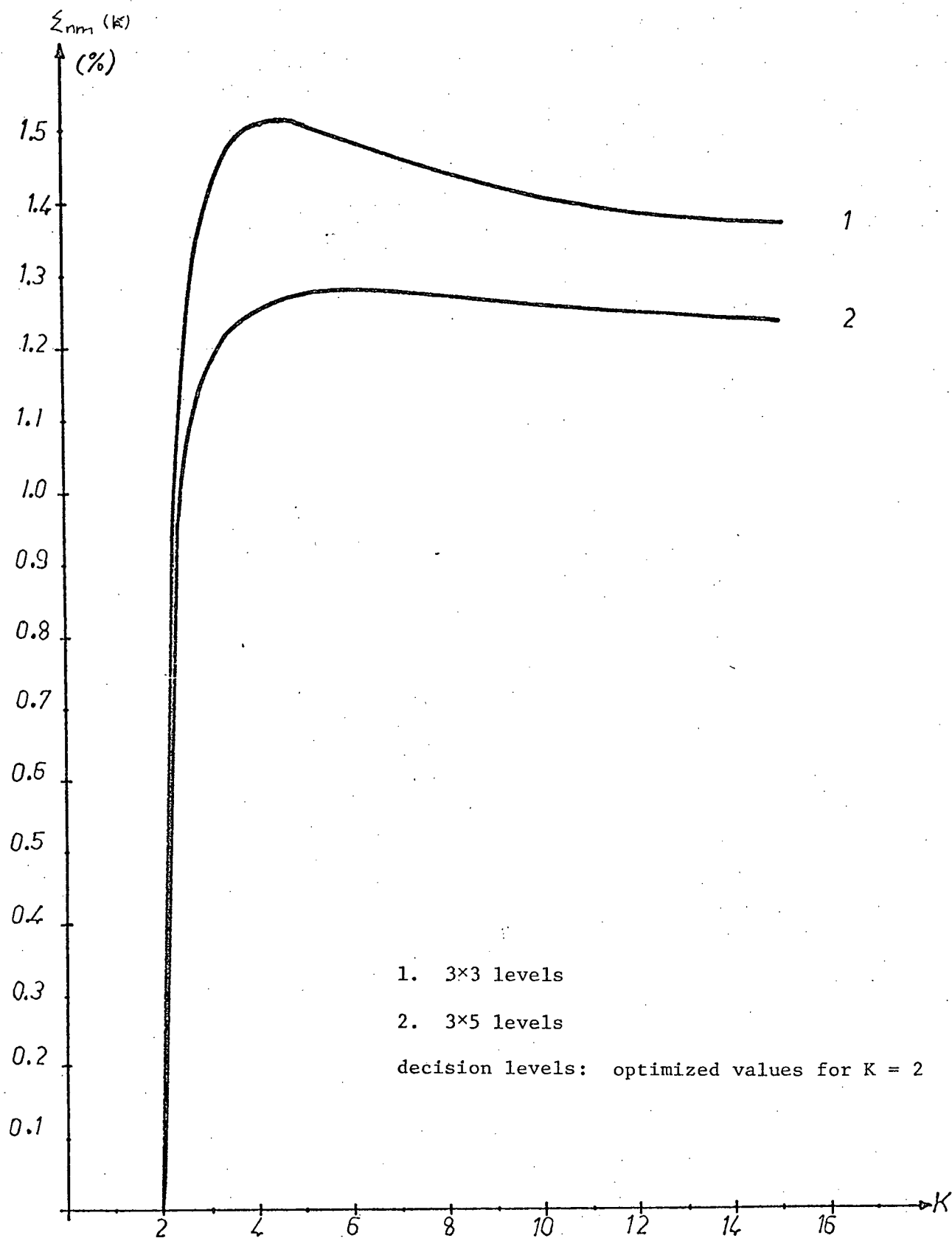
Figure V.2.1.1 Decomposition error ε_{nm} versus sampling rate K for 3×3 and 3×5 level correlator

1. 3×3 levels

2. 3×5 levels

decision levels: optimized values for K = 2

useful as a first approximation.

V.3 Degradation for Strong Signals

In all previous calculations the signal power was assumed much less than the noise power. This is true for most cases of practical interest. However, it seemed advisable to consider in at least one case how the degradation factor changes when the signal power is no longer small. This is done for a 2 × 2 level correlator at infinite sampling rate, i.e. a polarity coincidence detector.

The following two assumptions, made for small signals, do <u>not</u> hold for strong signals:

(a) The Gaussian signal s(t) can be replaced by a d-c signal

$$s_o = \sigma_s.$$

(b) The variance σ_w can be computed in absence of the signal, s(t).

For arbitrary signals, s(t), the general definition of D, given in equation (III.2.5) must be used.

The degradation of a 2 × 2 level correlator for strong signals has been treated by Cheng[7], with the difference that he used on RC-network for the integration. (His "degradation factor, r", is the square root of D defined in this work).

Fig. V.3.1 shows the block diagram of a polarity-coincidence correlator as used for strong signals.
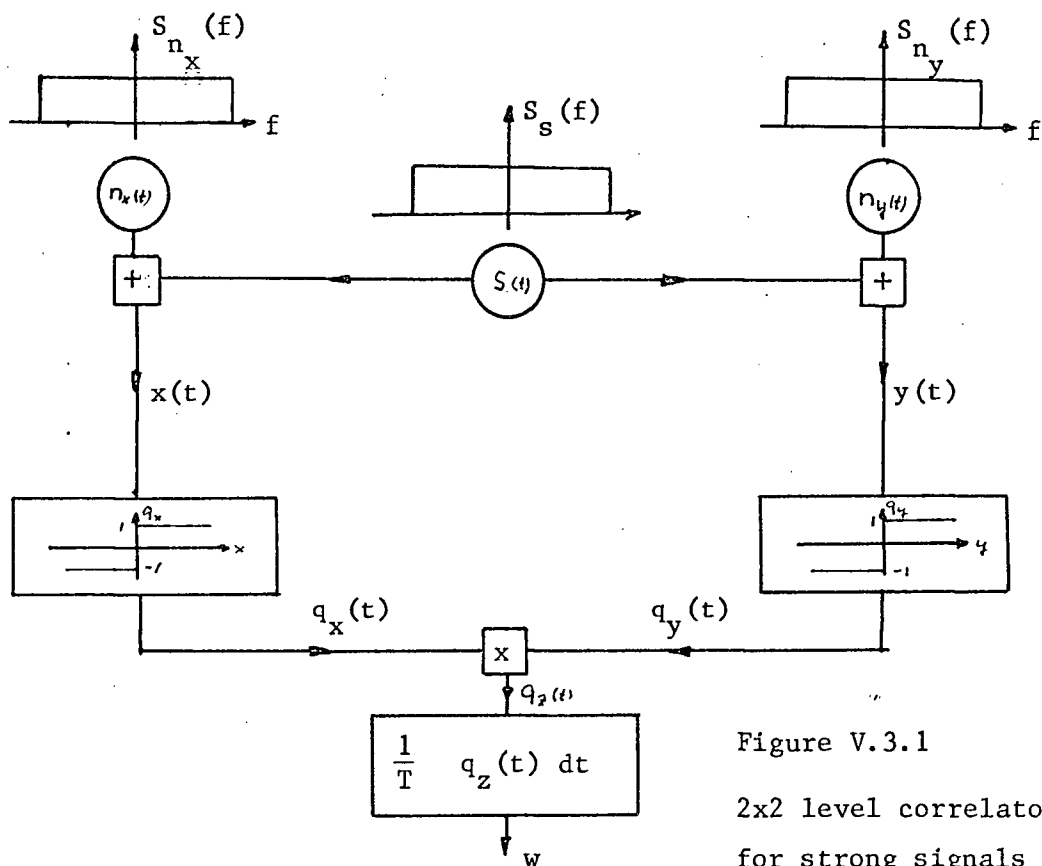
Figure V.3.1

2x2 level correlator

for strong signals

We assume the signals $s(t)$, $n_x(t)$ and $n_y(t)$ are bandlimited, statiscally independent, Gaussian, zero-mean signals, and have a flat power-density spectrum within B.

Therefore, the normalized autocorrelation function $\rho(\tau)$ is given by

$$\rho(\tau) = \frac{R_{n_x}(\tau)}{\sigma_n^2} = \frac{R_{n_y}(\tau)}{\sigma_n^2} = \frac{R_s(\tau)}{\sigma_s^2} \qquad (V.3.1)$$

V.3.1 Unquantized Correlator for Strong Signals

The expected value of w is found as

$$\bar{w} = \frac{1}{T} \int_0^T \overline{x(t)\ y(t)}\ dt = \overline{xy}$$

$$= \overline{n_x n_y} + \overline{n_x s} + \overline{n_y s} + \overline{s^2} \qquad (V.3.1.1)$$

Since $n_x(t)$ and $n_y(t)$ are assumed to be zero-mean,

$$\bar{w} = \overline{s^2} = \sigma_s^{\,2} \qquad\qquad (V.3.1.2)$$

The signals $s(t)$, $n_x(t)$ and $n_y(t)$ are assumed to be ergodic. Therefore the expected value of w^2 can be found as the time-average

$$\overline{w^2} \approx \frac{1}{T^2} \int_0^T \int_0^T \overline{z(\alpha)\,z(\beta)}\; d\alpha\; d\beta$$

$$= \frac{2}{T} \int_0^T (1-\frac{\tau}{T})\; R_z(\tau)\; d\tau \qquad\qquad (V.3.1.3)$$

where the autocorrelation function of $z(t)$,

$$R_z(\tau) = \overline{z(t)\,z(t+\tau)}$$

$$= R_{n_x}(\tau)\,R_{n_y}(\tau) + R_{n_x}(\tau)\,R_x(\tau) + R_{n_y}(\tau)\,R_s(\tau)$$

$$+ \overline{s^2(t)\,s^2(t+\tau)} - \sigma_s^{\,4} \qquad\qquad (V.3.1.4)$$

The expected value of $s^2(t)\,s^2(t+\tau)$ was found in [1] as

$$\overline{s^2(t)\,s^2(t+\tau)} = \sigma_s^{\,4}(1+2\rho^2(\tau)). \qquad\qquad (V.3.1.5)$$

Therefore,

$$R_z(\tau) = \rho^2(\tau)\;\;(\sigma_s^{\,4} + (\sigma_n^{\,2} + \sigma_s^{\,2})^2) \qquad\qquad (V.3.1.6)$$

$$= \sigma_s^{\,4}\;\;\rho^2(\tau)\;\;(\frac{1+d^2}{d^2}) \qquad\qquad (V.3.1.7)$$

where

$$d = \frac{\sigma_s^{\,2}}{\sigma_s^{\,2} + \sigma_n^{\,2}} \qquad\qquad (V.3.1.8)$$

Using equation (V.3.1.7) in (V.3.1.3) we finally obtain

$$\overline{w^2} \approx \frac{2}{T}\;\sigma_s^{\,4}\;(\frac{1+d^2}{d^2})\;\cdot\;\int_0^T (1-\frac{\tau}{T})\;\rho^2(\tau)\;d\tau + \sigma_s^{\,4} \qquad (V.3.1.9)$$

assuming the power density spectra of $s(1)$, $n_x(t)$ and $n_y(t)$ as which bandlimited to B, we obtain

$$\rho_{(\tau)} = Sa\;(2\pi B\tau) \qquad\qquad (V.3.1.10)$$

The standard deviation of w is defined as

$$\sigma_w = \sqrt{\overline{w^2} - (\overline{w})^2} \qquad (V.3.1.11)$$

Therefore,

$$\sigma_w = \sqrt{\frac{2}{T}} \ \sigma_s^2 \sqrt{\frac{1+d^2}{d^2}} \ [\int_0^T (1-\frac{\tau}{T}) \ \rho^2(\tau) \ d\tau]^{\frac{1}{2}} \qquad (V.3.1.12)$$

and using the definition (II.2.4)

$$MDS = \sigma_w \ (\frac{\sigma_s^2}{\overline{w}}) = \sigma_w = \sqrt{\frac{2}{T}} \ \sigma_s^2 \sqrt{\frac{1+d^2}{d}} \ [\int_0^T (1- \frac{\tau}{T}) \ \rho^2(\tau) \ d\tau]^{\frac{1}{2}}$$
$$(V.3.1.13)$$

assuming a large integration time T, we obtain, using (V.3.1.10)

$$\lim_{T \to \infty} \int_0^T (1-\frac{\tau}{T}) \ \rho^2(\tau) \ d\tau = \int_0^\infty Sa^2 \ (2\pi B\tau) \ d\tau = \frac{1}{4B} \qquad (V.3.1.14)$$

Therefore,

$$(MDS)_{analog} = \sigma_s^2 \sqrt{\frac{1+d^2}{d}} \frac{1}{\sqrt{2BT}} \qquad (V.3.1.15)$$

For large input signal-to-noise ratio, as $(\frac{S}{N})_i$ goes to infinity

$$(MDS)_{analog} = \sigma_s^2 \ \frac{1}{\sqrt{2BT}} \qquad (V.3.1.16)$$

is independent of d.

V.3.2 Application to a 2 $\times$ 2 Level Correlator

The two signals x(t) and y(t) (see Figure V.3.1) have the joint probability density

$$P_{xy}(x,y) = \frac{1}{2\pi\sqrt{1-r^2} \ \sigma_n^2} \ e^{- \frac{1}{2\sigma_n^2 \ (1-r^2)} (x^2-2rxy+y^2)} \qquad (V.3.2.1)$$

where

$$r = d = \frac{\sigma_s^2}{\sigma_s^2 + \sigma_n^2} \qquad (V.3.2.2)$$

The expected value of w is given by

$$\bar{w} = \overline{q_x q_y} = \int_0^\infty \int_0^\infty p_{xy}(x,y)\ dxdy + \int_{-\infty}^0 \int_{-\infty}^0 P_{xy}(x,y)\ dxdy$$

$$- 2 \int_0^\infty \int_{-\infty}^0 P_{xy}(x,y)\ dxdy \qquad (V.3.2.3)$$

Evaluating these integrals, we find

$$\int_0^\infty \int_0^\infty p_{xy}(x,y)\ dxdy = \frac{1}{2} - \frac{1}{2\pi}\ \text{arctg}\ \frac{\sqrt{1-d^2}}{d} \qquad (V.3.2.4)$$

and

$$\int_0^\infty \int_{-\infty}^0 p_{xy}(x,y)\ dxdy = \frac{1}{2\pi}\ \text{arctg}\ \frac{\sqrt{1-d^2}}{d} \qquad (V.3.2.5)$$

Therefore,

$$\bar{w} = 1 - \frac{2}{\pi}\ \text{arctg}\ \frac{\sqrt{1-d^2}}{d} = \frac{2}{\pi}\ \text{arcsin}\ d \qquad (V.3.2.6)$$

The expected value of w^2 is given by the time average

$$\overline{w^2} \approx \left(\frac{1}{T}\int_0^T q_z(t)dt\right)^2 = \frac{2}{T}\int_0^T (1-\frac{\tau}{T})\ R_{q_z}(\tau)\ d\tau \qquad (V.3.2.7)$$

and

$R_{q_z}(\tau)$ can be found as[7]

$$R_{q_z}(\tau) = \frac{4}{\rho 2}\ [(\text{arcsin}\ \rho(\tau))^2 - (\text{arcsin}\ \rho(\tau)d)^2] \qquad (V.3.2.8)$$

Therefore,

$$\sigma_w = \sqrt{\overline{w^2} - \overline{(w^2)}} = \frac{2}{\pi}\left(\frac{2}{T}\int_0^T (1-\frac{\tau}{T})\ \{(\text{arcsin}\rho(\tau))^2 - (\text{arcsin}(\rho(\tau)d))^2\}d\tau\right)^{\frac{1}{2}}$$

$$(V.3.2.9)$$

The derivative of $\bar{w}$ with respect to σ_s^2 is given by

$$\frac{d}{d(\sigma_s^2)}\ \bar{w} = \frac{2}{\pi}\ \frac{d}{\sigma_s^2\sqrt{1 + 2\ (\frac{\sigma_s}{\sigma_n})^2}} \qquad (V.3.2.10)$$

Using equation (II.2.4) with $R = \sigma_s^2$ we find

$$MDS = \frac{\sigma_s^2}{d} \sqrt{1 + 2(\frac{\sigma_s}{\sigma_n})^2} \; (\frac{2}{T} \int_0^T (1 - \frac{\tau}{T}) \{ (\text{arc } \sin\rho(\tau))^2 -$$

$$(\text{arc } \sin (\rho(\tau) \cdot d))^2 \} \; d\tau)^{\frac{1}{2}} \qquad\qquad (V.3.2.11)$$

Using the definition of the degradation factor stated in equation (II.2.5), the degradation factor of the strong signal correlator, denoted by $D_{ss_{2\times2}}(\infty)$ is obtained as

$$D_{ss_{2\times2}}(\infty) = \frac{MDS}{(MDS)_{analog}}$$

$$= \left\{ \frac{1 + d}{(1-d)(1+d^2)} \quad \frac{\int_0^T (1-\frac{\tau}{T}) \; [(\arcsin\rho(\tau) - (\arcsin\rho(\tau) \cdot d)^2] d\tau}{\int_0^T (1 - \frac{\tau}{T}) \; \rho^2(\tau) \; d\tau} \right\}^{\frac{1}{2}}$$

$$(V.3.2.12)$$

For large T, using (V.3.1.5) and (V.3.2.11) in (II.2.5) we find

$$D_{ss_{2\times2}}(\infty) = 2 \left\{ \frac{1 + d}{(1-d)(1+d^2)} \quad B \int_0^\infty [(\arcsin\rho(\tau))^2 - \right.$$

$$\left. (\arcsin \rho(\tau)d)] d\tau \right\}^{\frac{1}{2}} \qquad\qquad (V.3.2.13)$$

Since $\rho(\tau) = Sa \; (2\pi B\tau)$ (equation V.3.1.10), $D_{ss_{2\times2}}(\infty)$ is independent of B.

Bounds for $D_{ss_{2\times2}}(\infty)$ were found by Cheng[7]:

$$\sqrt{\frac{1+d}{1+d^2}} \leqslant D_{ss_{2\times2}}(\infty) \leqslant \left\{ \frac{1 + d}{(1-d)(1+d^2)} \; (\frac{\pi^2}{4} - (\arcsin d)^2) \right\}^{\frac{1}{2}} \quad (V.3.2.14)$$

The lower bound shows $D \geqslant 1$ for any d in $[0,\infty)$, which is a very loose bound.

Using equation (V.3.2.13) for small input signal-to-noise ratios (d << 1), we obtain

$$D_{ss_{2\times2}}(\infty) \simeq 2 \sqrt{B} \; [\int_0^\infty (\arcsin\rho(\tau)^2 d\tau]^{\frac{1}{2}}$$

$$=\sqrt{\frac{2}{\pi}} \ [\ \int_0^\infty (\arcsin Sa(x))^2 \ dx]^{\frac{1}{2}} \qquad (V.3.2.15)$$

Equation (V.3.2.15) agrees with the expression for $D_{2\times2}(\infty)$ found in (III.2.1.13) which was evaluated to be

$$D_{2\times2}(\infty) = 1.253.$$

In Figure V.3.2.1 $D_{ss_{2\times2}}(\infty)$ is plotted versus the input signal-to-noise ratio $(\frac{\sigma_s}{\sigma_n})^2$.

We notice that D goes to infinity for high input signal-to-noise ratios, which can be explained as follows:

In the limit, where $(\frac{\sigma_s}{\sigma_n})^2$ goes to infinity, we find

$$x(t) = s(t)$$

and $$y(t) = s(t)$$

so that x(t) and y(t) always have the same sign. Therefore $q_z \equiv 1$, so that q_z never changes and the output w has zero variance. In this limiting case, the correlator gives no information about the signal s(t) and obviously $D_{ss_{2\times2}}(\infty)$ goes to infinity, as the input signal-to-noise ratio goes to infinity.

This is not true, however, if more than 2 quantizer levels are taken, because even in the absence of noise the multiplier-output is a function of the signal amplitude. The strong signal degradation factor for higher level correlators has not been investigated, because the power of signals investigated in radio astronomy is always far below the noise power.

D stays virtually constant up to $(\frac{s}{n})^2$ 0.25, i.e. that our assumption $\sigma_s \ll \sigma_n$ is valid for $\sigma_s^2 < 1/4 \ \sigma_n^2$. This result was also found by Yerbury[12].
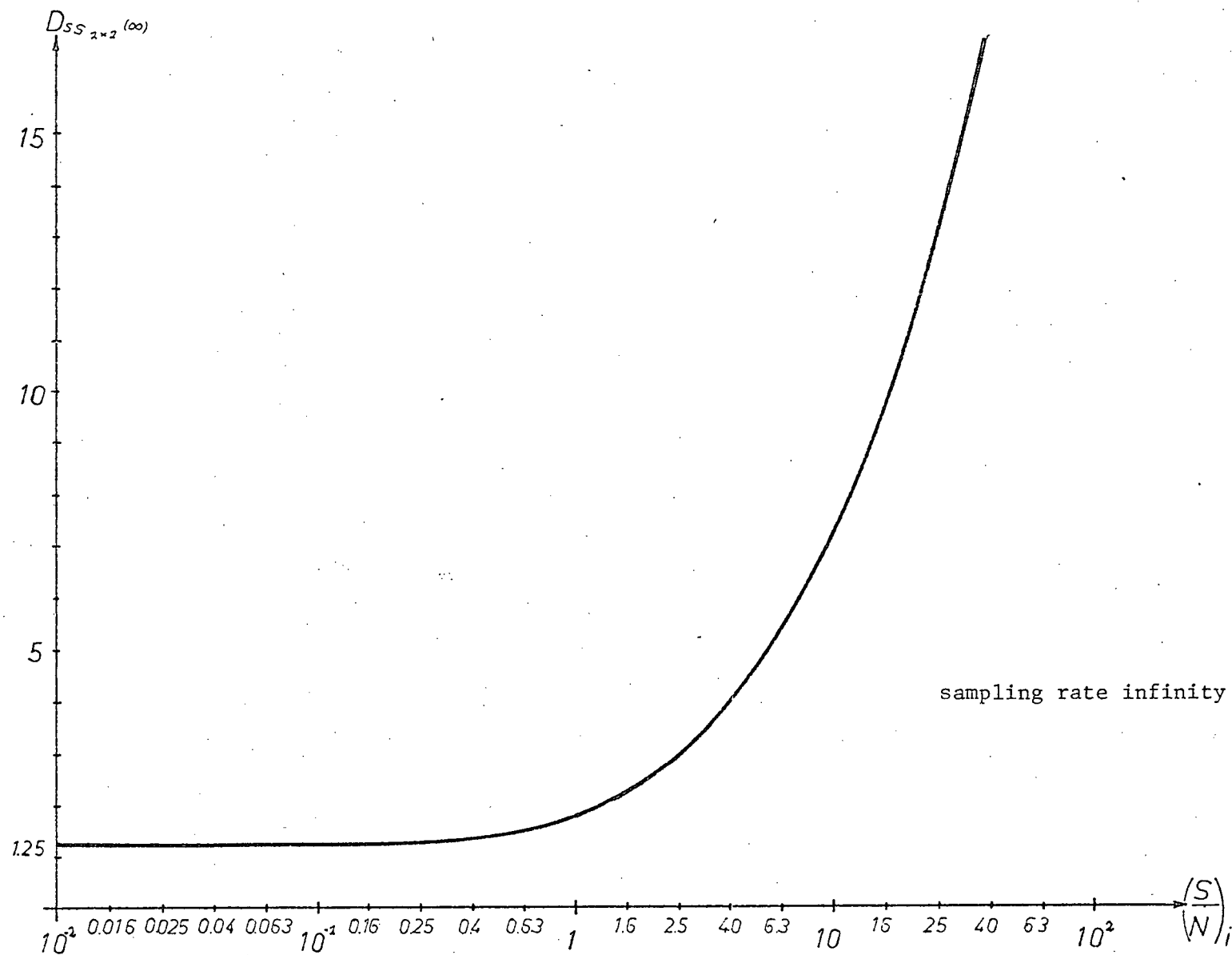
Figure V.3.2.1 Strong signal degradation factor versus input signal-to-noise ratio for a 2×2 level correlator

V1. OVERALL CONCLUSIONS

In designing a digital correlation spectrometer, various possible logic schemes can be considered. Those employing fine quantization with many digital levels degrade the measurements very little, but are costly and complicated to instrument. Simple schemes employing coarse quantization degrade the signal-to-noise ratio appreciably. A balance must be struck between excessive complexity and excessive degradation.

The results in this thesis help the designer make such a choice by giving him the degradation factors for a variety of logic schemes, not only for the Nyquist sampling rate but also for higher sampling rates.

In the course of the calculations, some interesting theoretical results were found, particularly concerning the variation of the optimum decision levels with sampling rate, and the possibility of decomposing the degradation factor into components.

The numerical results are of practical interest. For example, it is shown that a 3-level x 3-level logic scheme sampling at 4 times the bandwidth has a slightly lower degradation than the 3-level x 5-level scheme sampling at the Nyquist rate which is used by Whyte[12]. Such a 3×3 level scheme is probably also easier to build.

The numerical results are expected to fill all foreseen needs. As integrated circuit technology advances, it becomes progressively more practical to use many-level multiplication and averaging, and the degradation due to sampling then becomes insignificant.

APPENDIX A1

Proof that D = 1 for all K ≥ 2 for the Unquantized Correlator

Claim - An unquantized dual channel correlator which is sampled at Nyquist-rate or faster has a constant degradation factor D = 1 for any sampling-rate K. In particular we claim that

$$\frac{1}{K} \sum_{i=-\infty}^{\infty} Sa^2(\frac{2\pi i}{K}) = \frac{1}{2} \qquad (A1.1)$$

Proof - We pass white Gaussain noise $n(t)$ through 2 ideal low-pass filters. The first filter has a bandwidth $\frac{K}{2}$ B, the second filter has a bandwidth B. The output of the second filter n'(t), is sampled at a rate, KB, as shown in Figure A1.1
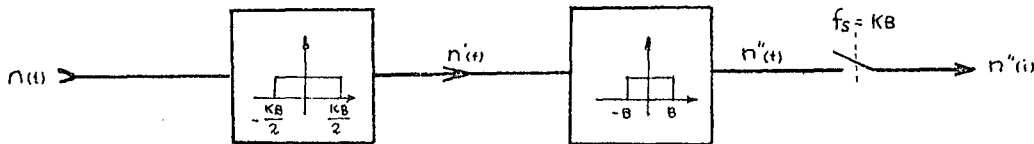
Figure A1.1 White noise bandlimited in two lowpass filters

The function n'(t) can be completely reconstructed from its samples taken 1/KB apart using the relation

$$n'(t) = \sum_{i=-\infty}^{\infty} a_i' \ Sa(\pi(KBt-i)) \qquad (A1.2)$$

$$a_i' = n'(i\tau) \qquad (A1.3)$$

$$\tau = \frac{1}{KB} \qquad (A1.4)$$

(for reference see 2 pg. 49).

We assume the average power of the signal n'(t) to be 1, i.e.

$$\int_{-\frac{KB}{2}}^{\frac{KB}{2}} S_n(f)\,df = 1 \qquad (A1.5)$$

or

$$S_n(f) = \frac{1}{KB} \qquad (A1.6)$$

for

$$|f| \leq \frac{KB}{2}$$

It follows that

$$\overline{n'(t)^2} = R_n(0) = 1 \qquad\qquad (A1.7)$$

where the autocorrelation function of n'(t) is given by the Fourier transform

of $S'_n(f)$

$$R'_n(\tau) = \mathcal{F}^{-1} S'_n(f) = Sa(KB\pi\tau) \qquad\qquad (A1.8)$$

Since we sample at a rate KB the Nyquist-rate for the bandwidth KB/2,

the a_i's are independent.

Now we process the signal n'(t) found in equation (A1.2) through the second

lowpass-filter. The average signal power of n''(t) is proportional to the

bandwidth of the second filter, since n'(t) is white.

Therefore,

$$\overline{n''(t)^2} = \int_{-B}^{B} \frac{1}{KB}\, df = \frac{2}{K} \qquad\qquad (A1.9)$$

The signal n''(t) can be reconstructed from its samples taken 1/KB apart,

because n''(t) is bandlimited to B and sampled at a rate K > 2.

Therefore,

$$n''(t) = \sum_{i=-\infty}^{\infty} a'_i \frac{2}{K} Sa\left(2\pi B\left(t - i\frac{1}{KB}\right)\right)$$

$$= \sum_{i=-\infty}^{\infty} \frac{2}{K} a_i' \, Sa\pi\left(2Bt - \frac{2}{K}i\right) \qquad\qquad (A1.10)$$

and

$$n''(0) = \frac{2}{K} \sum_{i=-\infty}^{\infty} a'_i \, Sa(2\pi i/K) \qquad\qquad (A1.11)$$

where

$$\overline{n''^2(0)} = \frac{4}{K^2} \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \overline{a'_i \, a'_j} \, Sa\left(\frac{2\pi i}{K}\right) Sa\left(\frac{2\pi j}{K}\right) \qquad (A1.12)$$

$$(A1.12)$$

The samples a'_i, and a'_j are taken at Nyquist-rate for the first filter output

and therefore are independent, i.e.

$$\overline{a'_i \, a'_j} = \quad 0, \quad \text{if } i \neq j$$

$$1, \quad \text{if } i = j, \qquad\qquad (A1.13)$$

since

$$\overline{a_i'^2} = R_n'(0) = 1 \tag{A1.14}$$

Therefore, the expected value of $n''^2(0)$ is given by

$$\overline{n''^2}(0) = \frac{4}{K^2} \sum_{i=-\infty}^{\infty} Sa^2(\frac{2\pi i}{K}) \tag{A1.15}$$

and since $n(t)$ is an ergodic process,

$$\overline{n''^2}(t) = \overline{n''^2}(0) = \overline{n''^2} \tag{A1.16}$$

Using the equation (A1.9) in (A1.15) we finally get

$$\frac{2}{K} = \frac{4}{K^2} \sum_{i=-\infty}^{\infty} Sa^2(\frac{2\pi i}{K}) \tag{A1.17}$$

or

$$\frac{1}{K} \sum_{i=-\infty}^{\infty} Sa^2(\frac{2\pi i}{K}) = \frac{1}{2} \tag{A1.18}$$

(A1.18) is equal to our claim (A1.1), which completes the proof. The above relation, used in (III.1.5.9), shows that the output signal-to-noise ratio of an analog correlator becomes independent of K, and therefore, the degradation factor becomes equal to unity for any $K \geqslant 2$.

APPENDIX A2

Replacement of General Signals by d-c in the Calculation of Degradation factors

The general model of a correlator shown in Figure II.2.1 shows

two signals $s_x(t)$ and $s_y(t)$, with a cross-correlation factor, $R = \overline{s_x \cdot s_y}$.

No assumptions are made about the spectral or statistical nature

of the signals, except that they are

    - ergodic

    -small compared with the noise

    -independent of the noise sources

    -limited in frequency to the range 0 to B

Calculations of degradation factors using such general signals

are clumsy, and it is shown in this appendix that the degradation factors

so found are identical with those obtained in a much simpler "d-c case",

where it is assumed that

$$s_x(t) = s_y(t) = s_o = \text{constant} \qquad (A2.1)$$

so that

$$R = \overline{s_x \cdot s_y} = s_o{}^2 \qquad (A2.2)$$

It will be recalled (Equations II.2.5 and II.2.4) that the

degradation factor, D, is defined as the ratio of two minimum detectable

signals

$$D = \frac{MDS}{(MDS)_{\text{analog}}}$$

and that the MDS for any given system can be written as

$$MDS = \sigma_w / (\frac{d\overline{w}}{dR})$$

Now, as long as the signal is small, the standard deviation of

the output, σ_w, is determined entirely by the noise sources and is independent

of the nature of the signals. It is usually evaluated assuming $s_x = s_y = 0$.

It remains then to show that for every correlator, the quantity $(\frac{d\bar{w}}{dR})$ is the same under these two assumptions

(1) $s_x(t) = s_y(t) = s_0$ = constant, - in the d-c case; and

(2) $s_x(t)$ and $s_y(t)$ have arbitrary spectra and statistics, subject

to the limitations set out above, - the general case.

A2.1 Evaluation of $\frac{d\bar{w}}{dR}$ for the d-c Case

In the absence of the signal (when $s_0 = 0$), the x-processor has at its input white Gaussian band-limited noise, with a mean value $\bar{x} = 0$. We assume, for simplicity, that the transfer function of the processor is symmetrical about zero volts. In that case its output, q_x, will also have mean value, $\overline{q_x} = 0$.

If now a small d-c signal, s_0, is present, the variance of $x(t)$ will not change but its mean value will no longer be zero,

$$\overline{x(t)} = s_0 \tag{A2.1.1}$$

Hence the output of the x-processor will no longer average exactly zero, but will have a mean value

$$\overline{q_x} = s_0 (\frac{\overline{dq_x}}{d\bar{x}}) \tag{A2.1.2}$$

where the derivative is a function of the processor used and of the amount of noise at the input. Since the signals are small, it can be evaluated at $\bar{x} = 0$.

Similarly

$$\overline{q_y} = s_0 (\frac{\overline{dq_y}}{d\bar{y}}) \tag{A2.1.3}$$

Now the correlator output will have a mean value given by

$$\bar{w} = \overline{q_z} = \overline{q_x \cdot q_y} \qquad\qquad (A2.1.4)$$

But q_x and q_y are statistically independent, since their variations are due to statistically independent noise sources. Therefore

$$\bar{w} = \overline{q_x \cdot q_y} = s_o^2 \cdot (\frac{\overline{dq_x}}{\overline{dx}}) \cdot (\frac{\overline{dq_y}}{\overline{dy}}) \qquad\qquad (A2.1.5)$$

and, since $R = s_o^2$, we have

$$\frac{d\bar{w}}{dR} = (\frac{\overline{dq_x}}{\overline{dx}}) \cdot (\frac{\overline{dq_y}}{\overline{dy}}) \qquad\qquad (A2.1.6)$$

A2.2 Evaluation of $\frac{d\bar{w}}{dR}$ for the General Case

In the general case we consider first the situation when the signal s_x happens to lie within interval ds_x of value s_1, and simultaneously s_y lies within interval ds_y of value s_2. The joint probability of this happening we can write as

$$\text{joint probability} = p_{s_x s_y}(s_1, s_2) \cdot ds_x \cdot ds_y \qquad\qquad (A2.2.1)$$

Given these assigned values for s_x and for s_y, what is now the expected value of w? We denote this as $(\bar{w})_{1,2}$. It is given by

$$(\bar{w})_{1,2} = (\overline{q_z})_{1,2} = (\overline{q_x \cdot q_y})_{1,2} \qquad\qquad (A2.2.2)$$

again, for fixed values of s_x and s_y, the fluctuations in q_x and q_y are independent, so that we can decompose the last expression into

$$(\bar{w})_{1,2} = (\overline{q_x})_1 \cdot (\overline{q_y})_2$$
$$= s_1(\frac{\overline{dq_x}}{\overline{dx}}) \cdot s_2(\frac{\overline{dq_y}}{\overline{dy}}) \qquad\qquad (A2.2.3)$$

To find now the overall average value of w for all possible combinations of s_x and s_y, we multiply each expression like that above by

the probability of its occurrence, and integrate over all values of s_x, s_y. Thus

$$\bar{w} = \int_{s_x} \int_{s_y} (\bar{w})_{x,y} \cdot P_{s_x s_y}(s_x, s_y) \cdot ds_x \cdot ds_y$$

$$= \iint_{S_x \, S_y} s_x \left(\frac{\overline{dq_x}}{d\bar{x}}\right) \cdot s_y \cdot \left(\frac{\overline{dq_y}}{d\bar{y}}\right) \cdot P_{s_x s_y}(s_x, s_y) \cdot ds_x \cdot ds_y$$

$$= \left(\frac{\overline{dq_x}}{d\bar{x}}\right) \left(\frac{\overline{dq_y}}{d\bar{y}}\right) \iint_{S_x \, S_y} s_x \cdot s_y \, P_{s_x s_y}(s_x, s_y) \, ds_x \cdot ds_y$$

$$= \left(\frac{\overline{dq_x}}{d\bar{x}}\right) \cdot \left(\frac{\overline{dq_y}}{d\bar{y}}\right) \overline{s_x s_y} \qquad\qquad (A2.2.4)$$

But $R = \overline{s_x s_y}$, so that

$$\frac{d\bar{w}}{dR} = \left(\frac{\overline{dq_x}}{d\bar{x}}\right) \cdot \left(\frac{\overline{dq_y}}{d\bar{y}}\right) \qquad\qquad (A2.2.5)$$

This is the identical equation to (A2.1.6) above.

This then proves that the degradation factor of a given system with arbitrary signals s_x and s_y can be evaluated using the simpler d-c case where both signals are made equal to a small constant.

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