AN INVESTIGATION OF THE HEURISTICS USED BY SELECTED GRADE ELEVEN ACADEMIC ALGEBRA STUDENTS IN THE SOLUTION OF MATHEMATICAL PROBLEMS

by

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ABSTRACT

This normative survey investigated the question, "What general heuristics are used by selected grade eleven academic algebra students in the solution of mathematical problems?" The investigator was interested in determining if students who had either A or B mathematics eleven grades used any heuristics.

Forty-two students, who were enrolled in nine schools, were interviewed. Each student was given two mathematical problems to solve. These problems could be solved using two of nine general heuristics namely, cases, deduction, inverse deduction, invariation, analogy, symmetry, preservation of rules, variation, and extension.

The researcher requested the students to think aloud. The student was encouraged to attempt the problems any way he chose. They were asked to be more concerned with revealing as much of their thought processes as possible, as with the accuracy of their solution. All the interviews were taped.

The investigator found evidence that eight of the nine heuristics were used. The heuristics were cases, deduction, inverse deduction, invariation, analogy,

preservation of rules, variation, and extension. Thirty-eight of the forty-two students interviewed showed evidence of using one or more of the heuristics. Eighteen of the students used cases, seven used deduction, three used invariation, two used inverse deduction, seven used analogy, two used preservation of rules, three used variation, and seven used extension. The investigator also found evidence that a heuristic which was not mentioned previously was used by eleven of the students. For the purpose of this investigation the heuristic was called "successive variation." When the heuristic of successive variation is used a possible solution to the given problem is chosen at random. If the answer is not correct, the student determines what changes must be made. Then the possible solution is varied successively until the correct answer is found. The students' command of the heuristics was not developed and therefore they could not use these techniques efficiently and effectively to solve the problems they were given.

TABLE OF CONTENTS

LIST OF TABLES

CHAPTER I

THE PROBLEM AND DEFINITION OF TERMS

I. THE PROBLEM

Background of the problem. Most students encounter difficulty when solving problems in mathematics. Washburne and Osborne stated that, "training children to solve arithmetic problems is one of the hardest and most discouraging tasks of the teacher."[1]

Techniques have been devised to help students learn to solve mathematical problems. Some of these procedures are formal analysis, selecting correct procedures, and solving a wide variety of problems. These adult-devised techniques generally have little effect in helping students to become better problem solvers. Allowing the student to choose his own method when solving a problem (student-generated sequences) is sometimes better than a specific adult-devised method.[2] Kaplan suggested that the classroom teacher should spend time investigating student-generated sequences.[3]

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[1]Carleton W. Washburne and Raymond Osborne, "Solving Arithmetic Problems," The Elementary School Journal, XXVII (November, 1926), 219.

[2]Paul R. Hanna, "Methods of Arithmetic Problem Solving," The Mathematics Teacher, XXIII (November, 1930), 442-450.

[3]Jerome D. Kaplan, "An Example of Student-Generated Sequences in Mathematics Instruction," The Mathematics Teacher, LVII (May, 1964), 298-302.

Grossnickle stated that:

> Before giving specific helps in problem solving, it is necessary to consider different levels of maturity which characterize problem by employing different patterns of mathematical thinking.[4]

Grossnickle further stated that, "it should not be inferred that a pupil should always follow one particular pattern in solving a problem."[5] In his doctoral dissertation, Kilpatrick said that:

> To make our knowledge of problem-solving behavior relevant to education, we must eventually study how students solve problems of the sort they meet in the classroom. If at present there are too many uncontrollable or even unknown sources of variation for careful experimental studies, we should at least attempt analyses of behavior. Such analyses . . . are necessary for directing future experimentation.[6]

The investigator hypothesizes that the adult-devised methods referred to earlier are not general enough to be used to attack all types of mathematics problems. According to Luchins' research, drill in one specific method of problem solving may have a tendency to blind the student to other approaches to solving problems. The pupil may develop a set and be unable to solve the problems that do not fit the

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[4]Foster E. Grossnickle, "Verbal Problem Solving," The Arithmetic Teacher, XI (January, 1964), 12.

[5]Ibid., p. 14.

[6]Jeremy Kilpatrick, "Analyzing the Solution of Word Problems in Mathematics: An Exploratory Study," (unpublished doctoral dissertation, Stanford University, Stanford, 1967), p. 2.

type encountered previously.[7] Hence it would appear useful to teach methods of problem solving that are so general that they can be used to attempt to solve many different types of problems. The use of certain general heuristics appears to produce this wide applicability.

Several writers have discussed heuristics and a few studies have been done on some aspect of the topic. Polya urged teachers to make use of specific heuristic questions such as, "What is the unknown?" and "What are the data?"[8] Ashton[9] tried to apply Polya's suggestions while Kilpatrick[10] attempted to discover if students use heuristic questions when they solve mathematical problems. Wills,[11] Larson,[12]

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[7]Abraham Luchins, "Mechanization in Problem Solving," Psychological Monographs, LIV (whole no. 248, 1942).

[8]George Polya, How to Solve It (second edition; New York: Doubleday and Company, Inc., 1957), p. 7.

[9]Sister Madeleine Rose Ashton, "Heuristic Methods in Problem Solving in Ninth Grade Algebra," (unpublished doctoral dissertation, Stanford University, Stanford, 1962).

[10]Jeremy Kilpatrick, op. cit.

[11]Herbert Wills III, "Transfer of Problem Solving Ability Gained Through Learning by Discovery," Dissertation Abstracts, XXVIII (1967), 1319-A.

[12]Charles McLoud Larsen, "The Heuristic Standpoint in the teaching of Elementary Calculus," Dissertation Abstracts, XXI (1961), 2632-2633.

Wilson,[13] Washburne and Osborne,[14] and Schaaf[15] attempted to teach various heuristics to students by having the teacher use these strategies. Henderson,[16] Jones[17] and Bruner[18] urged teachers to use stratagems in the hope that students would learn how to use these methods.

None of these studies investigated the question, "Do students use any general heuristics, such as analogy or inverse deduction, when they solve problems using any method they choose?" The purpose of this investigation is to see if students use any of the nine heuristics proposed by MacPherson,[19] when they attempt to solve

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[13]James William Wilson, "Generality of Heuristics as an Instructional Variable," Dissertation Abstracts, XXVIII (January, 1968), 2575-A.

[14]Carleton W. Washburne and Raymond Osborne, op. cit., p. 219-226, 296-304.

[15]Oscar Schaaf, "Student Discovery of Algebraic Principles as a Means of Developing Ability to Generalize," The Mathematics Teacher, XLVIII (May, 1955), 324-327.

[16]Kenneth B. Henderson, "Strategies for Teaching by the Discovery Method," Updating Mathematics (high school teachers edition; New London, Connecticut: Croft Educational Services, 1964), p. 57-64.

[17]Phillip S. Jones, "Discovery Teaching From Socrates to Modernity," The Mathematics Teacher, LXIII (October, 1970), 501-508.

[18]Jerome S. Bruner, The Process of Education (New York: Vintage Books, 1960).

[19]Eric D. MacPherson, (unpublished paper on heuristics, The University of British Columbia, Vancouver, 1969). (mimeographed).

mathematical problems that can be solved by using at least one of these heuristics. The type of investigation is a normative survey.

Statement of the problem. What general heuristics are used by selected grade eleven academic algebra students in the solution of mathematical problems?

Importance of the study. MacPherson's[20] list of general heuristics is an adult model of heuristics that can be used in mathematical problem solving. Research has shown, however, that students who are taught adult-devised methods seldom become better problem solvers.[21] If this normative survey finds evidence that students use certain of the heuristics in attempting to solve mathematical problems, then the investigator proposes that an effort be made to teach students these general heuristic procedures in the hope that they will become better problem solvers.

Since there are always new problems to be solved, it is advantageous for the students to know several general techniques that can be used to attempt to find solutions to novel problems.

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[20]Eric D. MacPherson, loc. cit.

[21]Carleton W. Washburne and Raymond Osborne, op. cit., p. 219-226; Latney Conrad Welker, "A Study of Interrelationships in Arithmetical Problem Solving," (unpublished doctoral dissertation, The University of Southern Mississippi, Hattiesburg, 1962).

The investigator will also observe if any other general heuristics, which could be useful in solving a wide variety of problems, are used by the students.

II. DEFINITION OF TERMS

Heuristic. The term heuristic has been variously defined in the literature. Bruner defined a heuristic as an approach one follows to solve problems.[22] Henderson referred to a heuristic as a stratagem, which he defined as a plan of action.[23] Polya called heuristic, "The study of means and methods of problem solving."[24] Ashton's definition, based on Polya's, is:

> Heuristic methods are based on the mental operations involved in the art of invention. They do not consist of a series of formal steps, since these might well hinder rather than aid thought. They are intended rather to lead the problem solver to recognize and then consciously apply himself to certain modes of thinking about problems.[25]

Newell *et al.* used the term heuristic "to denote any principle or device that contributes to the reduction in the average search to solution." [26] MacPherson defined

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[22] Jerome S. Bruner, *op. cit.*, p. 27.

[23] Kenneth B. Henderson, *op. cit.*, p. 57.

[24] George Polya, *Mathematical Discovery* (New York: John Wiley and Sons, Inc., 1962), I, p. vi.

[25] Sister Madeleine Rose Ashton, *op. cit.*, p. 1.

[26] Allen Newell, J.C. Shaw, and Herbert A. Simon, "The Process of Creative Thinking," *Contemporary Approaches to Creative Thinking* (New York: Atherton Press, 1964), p. 78.

heuristic as, "an alogical algorithm which is used for the purpose of discovering some order of mathematical generalization in a novel situation."[27] Basically all these definitions described a heuristic procedure as a general or specific plan of action which is used to attempt to solve a problem. In this paper heuristic is defined as a general plan of attack or process of reasoning that can be employed in the search for a solution to a particular problem.

The nine general heuristics proposed by MacPherson,[28] namely, cases, deduction, inverse deduction, invariation, analogy, symmetry, preservation of rules, variation, and extension, will subsequently be defined. A problem and its solution also will be given to illustrate the use of each heuristic. Some of these procedures have been discussed in the literature as techniques that the teacher can use in discovery teaching. Henderson stated that the stratagems he lists, "can be used to plan the teaching of a mathematical concept or principle so that the student discovers the knowledge rather than listens while the teacher tells it to him."[29] None of the heuristics

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[27]Eric D. MacPherson, op. cit.

[28]Ibid.

[29]Kenneth B. Henderson, op. cit., p. 61.

appeared to have been fully defined in the literature from the point of view of the use of the heuristic in the solution of a problem.

Heuristic I: Cases. Leask defined simple enumeration as:

> The presentation of many instances of the generalization to be discovered. The students form hypotheses based on the example and test those to determine which is correct. One counter-example is sufficient to warrant rejection of a hypothesis.[30]

This definition applies to the heuristic as it could be used by the teacher in presenting his lesson. The heuristic, cases, will now be defined as it applies to the student's use of the technique. The heuristic of cases is a method for finding the solution to a problem by considering all or some of the possible solutions, and testing them until the correct solution is discovered. Either a random or a systematic sampling of some of the solutions can be considered. If a systematic examination of cases is tried, it can be a critical search which involves looking for the exception to the rule, or a sequential search through the possible solutions. The

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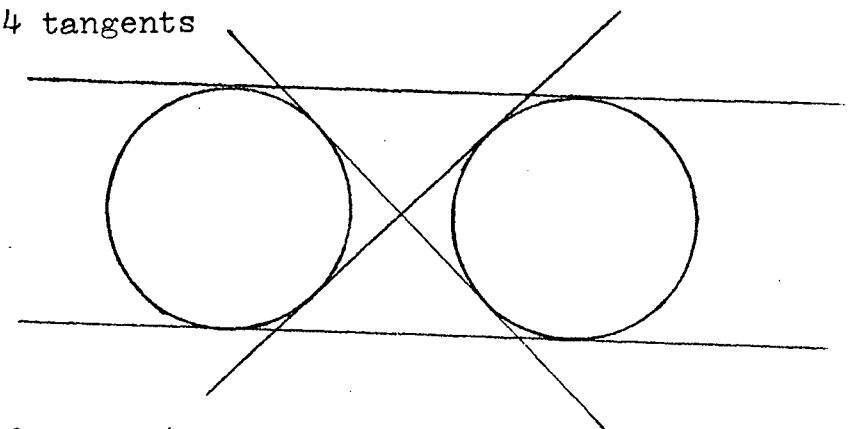
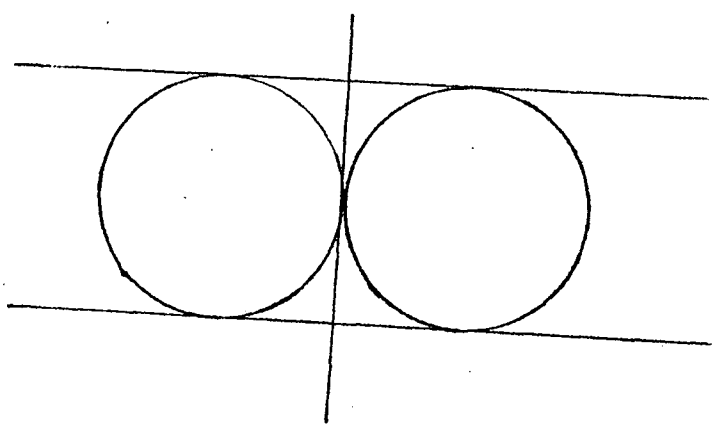
[30]Isabel Campbell Leask, "The Effectiveness of Simple Enumeration as a Strategy for Discovery," (unpublished Master's thesis, The University of British Columbia, Vancouver, 1968), p. 8.

following problem illustrates how the heuristic of cases could be used in its solution: How many tangents common to both circles can two equal circles in the same plane have?[31] Consider two disjoint circles moving together to become a single circle. The following three cases result.

Case 1 4 tangents

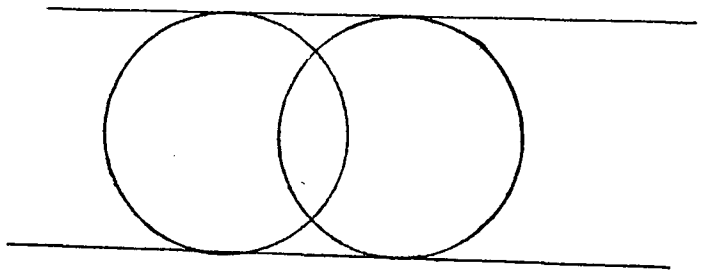


Case 2 3 tangents



[31]Charles T. Salkind, The Contest Problem Book (New York: Random House, Inc., 1961), p. 18.

Case 3 2 tangents[32]



Heuristic II: Deduction. The heuristic of deduction can be sub-classed into the two heuristics, direct deduction and hypothetical deduction. Direct deduction is the procedure of discovering a consequence from a set of premises, by logical reasoning. In hypothetical deduction it is assumed that a sub-problem of the original problem can be solved. Then acting on this assumption, the consequences that follow are determined. If these consequences are valuable in that they lead to a solution of the original problem, then an attempt is made to solve the sub-problem, as a first step in finding the solution to the original problem. Deduction could be used in the solution of the following problem: Triangle ABC is isosceles with side AC equal to 12 inches and side AB equal to 6 inches. What is the length of the radius CD of the circle drawn through the vertices of triangle ABC?[33]

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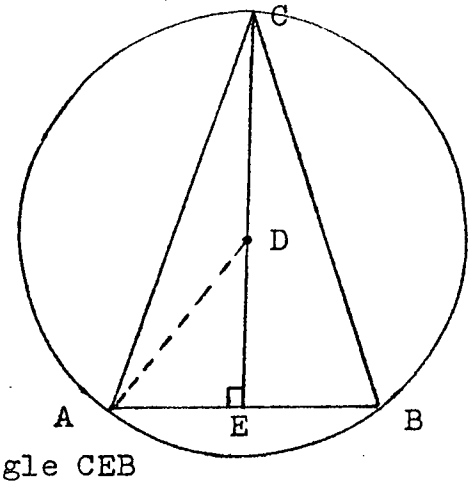
[32] Ibid., p. 93.

[33] Ibid., p. 28.

If the length of CE was known, the right triangle AED could be solved for AD. Since AD = CD, first consider the sub-problem; find the length of CE.



Consider triangle CEB

1. angle CEB = 90°	1. by construction, angle AEB = angle AEC + angle CEB angle AEB = 180° angle AEC = 90°
2. CB = AC = 12 inches	2. triangle ABC is isosceles AC = 12 inches
3. AE = EB = 3 inches	3. AB = 6 inches triangle ABC is isosceles
4. $(CE)^2 = (CB)^2 - (EB)^2$ $CE = \sqrt{(CB)^2 - (EB)^2}$ $= \sqrt{(12)^2 - (3)^2}$ $= \sqrt{144 - 9}$ $= \sqrt{135}$	4. Pythagorean theorem

Consider triangle ADE

5. $(AD)^2 = (AE)^2 + (ED)^2$	5. Pythagorean theorem
$AD = \sqrt{(3)^2 + (\sqrt{135} - CD)^2}$	AD = CD therefore
$= \sqrt{9 + 135 - 2\sqrt{135}\,CD + (CD)^2}$	$(AD)^2 = (CD)^2$
$= \sqrt{144 - 2\sqrt{135}\,AD + (AD)^2}$	
$2\sqrt{135}\,AD = 144$	
$AD = 144/2\sqrt{135} = 72/\sqrt{135}$ [34]	

Heuristic III: Inverse Deduction. Polya referred to this heuristic as working backwards. He stated, "We concentrate upon the desired end, we visualize the final position in which we would like to be. From what foregoing position could we get there?"[35] Newell *et. al.* discussed the strategy of working backwards and stated its usefulness in a problem situation that has "many possible starting points versus a single end."[36] Inverse deduction is defined in this paper as a method of working from a known or an assumed conclusion backwards to the original statement of the problem. The following problem illustrates the use of the heuristic of inverse deduction: How could you bring up from a river exactly 6 quarts of water using only two containers, a

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[34] *Ibid.*, p. 103.

[35] George Polya, *How to Solve It* (second edition; New York: Doubleday and Company, Inc., 1957), p. 230.

[36] Newell, *op. cit.*, p. 80.

4 quart pail and a 9 quart pail.[37] The aim is to end with 6 quarts of water in the 9 quart pail. If there were 1 quart of water in the 4 quart pail, it would be possible to fill up the 9 quart pail and pour out 3 quarts of water into the 4 quart pail and then be left with 6 quarts of water in the 9 quart pail. Suppose it is possible to get 1 quart of water in the 4 quart pail. How could this be done? Fill the 9 quart pail and pour from it 4 quarts into the smaller pail, dump it out, and fill it again. Then there will be 1 quart of water left in the 9 quart pail. The 1 quart can be poured into the 4 quart pail and the solution is accomplished.[38]

Heuristic IV: Invariation. When this heuristic is employed in attempting to solve a problem, either some variable in the problem is fixed or else some of the data is ignored. Then an attempt is made to see if the problem can be solved under these new conditions. The solution of the simplified problem may provide insight into the solution of the original problem. The following problem will illustrate the heuristic: Construct a triangle ABC given the length of the

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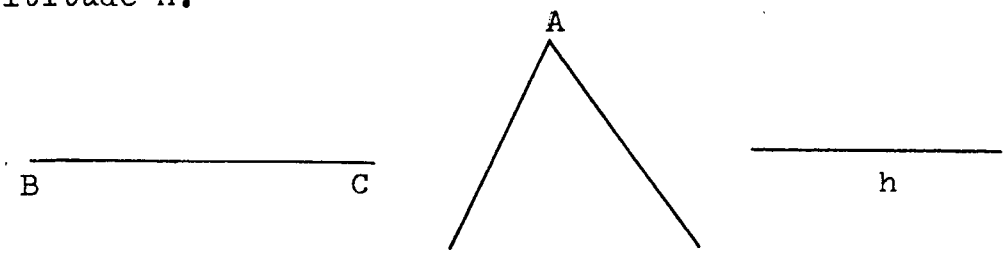
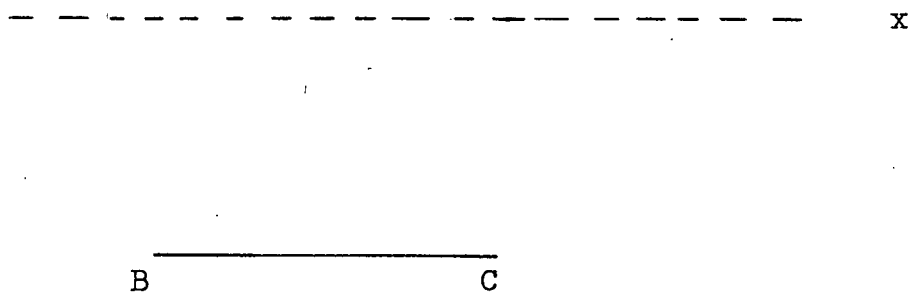
[37]George Polya, op. cit., p. 226.

[38]Ibid., p. 227-230.

base BC, the measure of angle A, and the length of the altitude h.[39]



In order to solve the problem, consider the two conditions separately. First ignore the measure of the angle, and consider constructing a triangle given the length of the base BC and the length of the altitude h. The vertex A of the triangle would lie along a line x which is parallel to side BC and at a distance h from it.



Second consider constructing a triangle using the length of the base BC and the measure of angle A. Several triangles can be constructed with base BC and angle A as the angle opposite this base. Since the sum of the measures of the angles of a triangle is 180°, extend BC through C to D.

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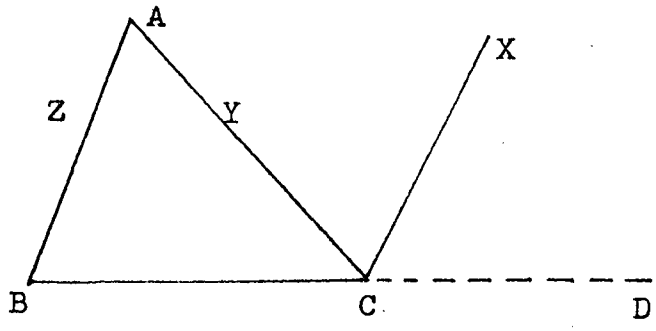
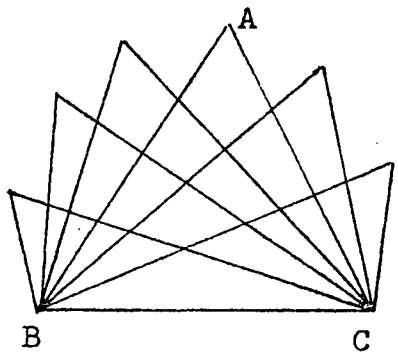
[39] *Ibid.*, p. 80.

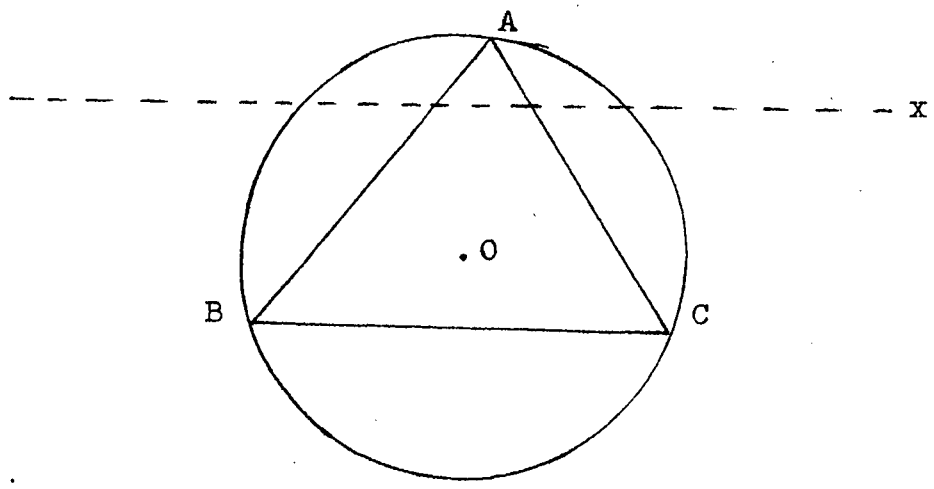
Construct angle XCD = angle A. Divide angle BCX into any two angles BCY and YCX. Construct angle CBZ = angle YCX. Rays BZ and CY will intersect at point A. Now triangle ABC has base BC and angle A as the angle opposite BC.



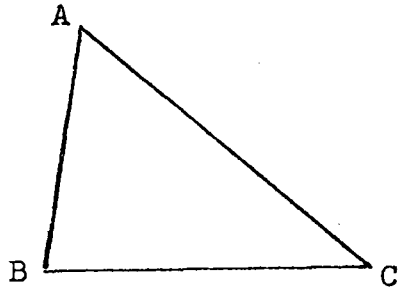
By dividing angle BCX into different parts, the following six triangles can be drawn.

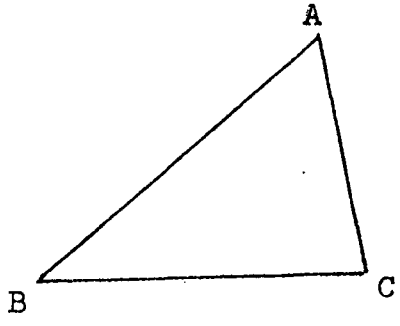


The locus of the vertex A of the triangle ABC is a circle with chord BC. The centre of the circle can be constructed using the three points B, C, and A and the fact that the perpendicular bisector of a chord of a circle passes through the centre of the circle. The centre of the circle is point O.



Now the intersection of the circle with the line x shows the two solutions to the problem.[40] The following diagrams illustrate these two answers to the problem:





[40] Ibid., p. 81-82.

Heuristic V: Analogy. Henderson described the stratagem of analogy as a teaching technique which is useful in discovery learning. He planned the lesson so that the new problem situation was very similar to a problem that the student already knew how to solve.[41] Jones illustrated analogy as the perception of a new problem as a variation of another problem that one already could solve. He implied that the problem previously solved acts as a mental model in suggesting a solution for the newer problem.[42] Polya on the other hand, attempted a definition of the heuristic of analogy as "a sort of similarity." [43] Analogy is also defined as a sort of isomorphism between structures. MacPherson stated that this heuristic enables one to know what questions to ask in a new situation because there is a similarity between this new question and a problem that was previously studied.[44] Analogy will be defined in this study as the perception of a similarity between the problem to be solved and one that has already been solved. The same methods of attack may be tried on the new problem as were used previously. The

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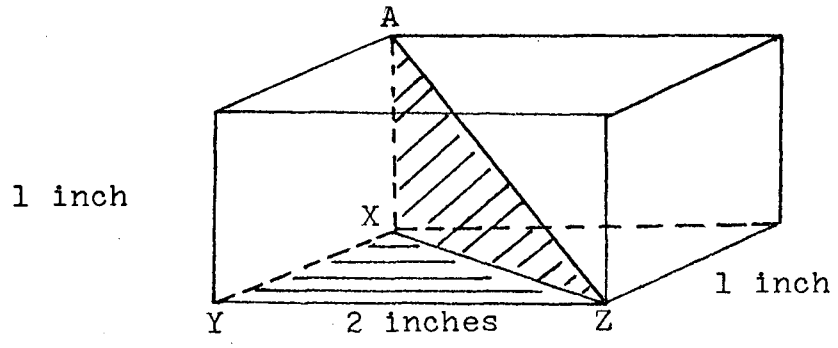
[41]Kenneth B. Henderson, op. cit., p. 57-58.

[42]Phillip S. Jones, op. cit., p. 507-508.

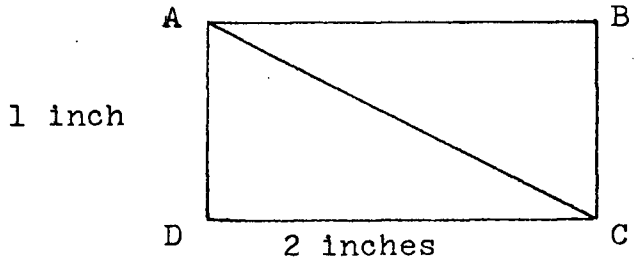
[43]George Polya, op. cit., p. 37.

[44]Statement by Eric D. MacPherson, personal interview.

following problem illustrates how the heuristic of analogy can be used in the solution of the problem: Find the length of the diagonal of the rectangular box (the length of the line joining points A and Z).[45]



Consider the two-dimensional case of a rectangle. To find the length of the diagonal (the length of the line joining points A and C) use the Pythagorean theorem.



$AC = \sqrt{(AD)^2 + (CD)^2} = \sqrt{1 + 4} = \sqrt{5}$

$AC = \sqrt{5}$ inches

In the three-dimensional case of a rectangular box, there are two triangles, triangle XYZ and triangle AZX. Consider triangle XYZ. Since angle Y = 90°, $XZ = \sqrt{(XY)^2 + (YZ)^2}$

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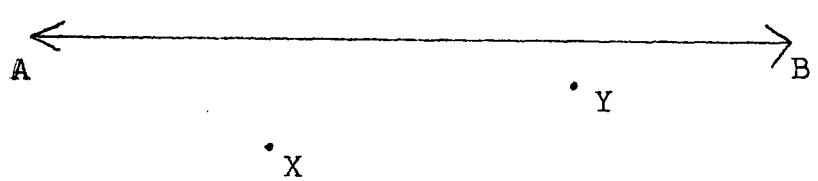
[45]George Polya, op. cit., p. 7.

$XZ = \sqrt{1 + 4} = \sqrt{5}$. Consider triangle AZX. Since angle X = 90°, $AZ = \sqrt{(XZ)^2 + (AX)^2}$. $AZ = \sqrt{5 + 1} = \sqrt{6}$. Therefore AZ = $\sqrt{6}$ inches.

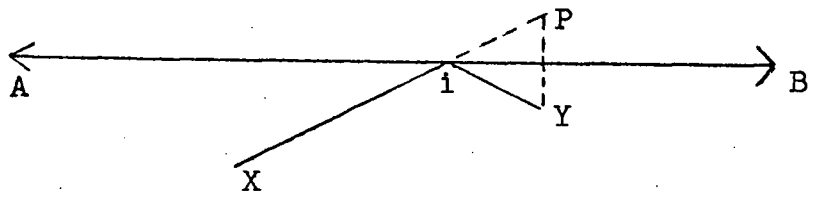
Heuristic VI: Symmetry. The use of the heuristic of symmetry involves an attempt to find some inherent symmetry in the problem itself. The perception of symmetry may lead to an addition of data that will now help to solve the new problem. It is hoped that by omitting the added information, a solution to the original problem will clearly be seen. Polya said that a solution to the problem may depend on noticing that some parts of the data are interchangeable. He said that interchangeable parts should be treated in the same manner.[46] The following problem illustrates the use of symmetry: Find the shortest path between points X and Y that passes throught point X, touches line AB, and then passes through point Y.



Suppose there is a point P located the same distance from line AB as point Y, but in the other half plane. The shortest distance between points X and P would be the

[46] George Polya, op. cit., p. 199.

straight line joining them which passes through line AB at point i. Triangle YiP is isosceles with Yi = iP, therefore the solution to the original problem is the line joining points X, i, and Y.



Heuristic VII: Preservation of Rules. The heuristic preservation of rules would be used when one system is extended to create a new system. The use of the heuristic would involve asking a question such as, how many of the rules of the previous system apply to the new system? If the rules of the previous system are preserved, then it may be possible to solve the problem in the new system by an analogy to the method of solution in the previous system. In other words the heuristic, preservation of rules, provides an initial starting point in attempting to solve the problem. The problem illustrated below could be used to test the use of the heuristic of preservation of rules: Consider a pseudo-ternary system with the three numberals -1, 0, and +1. Now replace -1 by -, +1 by + and leave 0 unchanged.

Consequently, -, 0, and + are the only permissible digits in pseudo-ternary arithmetic. Given the fact that 8_{ten} = $(+0-)_{p.t.}$, change 54_{ten} to its equivalent representation in pseudo-ternary arithmetic.[47] Suppose that the rules follow for the representation of numbers in pseudo-ternary arithmetic as they would in other base systems. In other words the representation of 8_{ten} should be $(+1 \times 3^2) + (0 \times 3^1) + (-1 \times 3^0)$, which does in fact equal 8_{ten}. Therefore following the method used for converting base ten numbers to their equivalents in other systems,

$$
\begin{array}{r|l l}
3 & 54 & \\
\hline
 & 18 & 0 \\
\hline
 & 6 & 0 \\
\hline
 & 2 & -1 \\
\hline
 & +1 &
\end{array}
$$

And therefore $54_{ten} = +-000_{p.t.}$[48]

<u>Heuristic VIII</u>: <u>Variation</u>. In using variation some fact in the problem is changed. The attempt is made to find out what other facts are now either variant or invariant. This heuristic could extend the knowledge of a particular topic since its use involves asking the question, What would happen if . . . ? Any answers either to the new problems suggested by the use of variation, or to the original problem

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[47] Winifred Asprey, "Pseudo-Ternary Arithmetic," <u>The Mathematics Teacher</u>, LIX (March, 1966), 241.

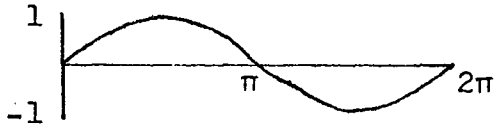
[48] <u>Ibid</u>., p. 241.

would be generated by other heuristics. This heuristic provides an initial starting point in the search for a solution to the problem. Polya said that if the solution to a problem can not be readily seen, the data should be varied.[49] The following problem illustrates how variation could be used. Investigate the graph of $y = a \sin bx$. The graph of the equation $y = a \sin bx$ could be investigated as follows:

Suppose $b = 1$, then $y = a \sin x$. What happens as a takes on different values?

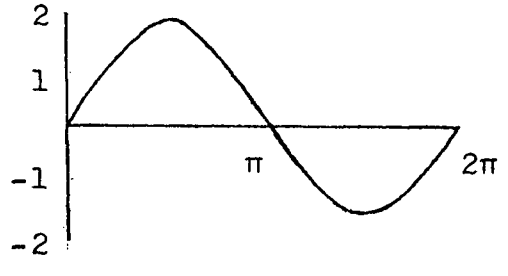
$a = 1 \quad y = \sin x$

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
y	0	1	0	-1	0



$a = 2 \quad y = 2 \sin x$

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
y	0	2	0	-2	0



As a changes the amplitude of the wave varies. Suppose

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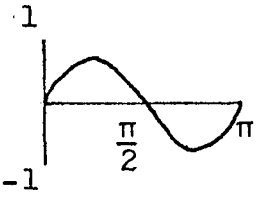
[49]George Polya, op. cit., p. 210-211.

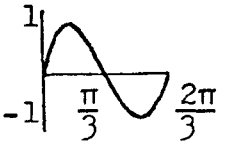
a = 1, then y = sin bx. What happens as b takes on different values? b = 2 y = sin 2x

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{2}$	π
y	0	1	0	-1	0



b = 3 y = sin 3x

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
y	0	1	0	-1	0



As b changes the wavelength varies.

<u>Heuristic IX</u>: <u>Extension</u>. Polya's heuristic of generalization, which he defined as a "passing from the consideration of a given set of objects to that of a larger set, containing the given one," could be classed here.[50] This heuristic is used in the generation of new problems. Some fact is added to or subtracted from the present system and then the question, What can now be discovered? is asked. Other heuristics would then be used to solve the new problem that had been suggested by the use of the heuristic of extension. The following problem illustrates the use of the heuristic of extension: Solve $x^2 + 1 = 0$. The solution to the equation $x^2 + 1 = 0$ remained unsolved until the real number system was extended to include the complex number i, which was defined as $\sqrt{-1}$.

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[50] <u>Ibid</u>., p. 12.

The solution to the problem is $x = i$, since $i^2 = -1$.

The heuristics, preservation of rules, variation, and extension are used when initially attacking a new problem or formulating a new problem. The problem will usually be solved using one of the other heuristics, most probably, cases, deduction, inverse deduction, or analogy. In other words the three heuristics, preservation of rules, variation, and extension suggest an initial question to be asked when faced with the problem, but the actual solution will usually involve the use of the other heuristics.

CHAPTER II

REVIEW OF THE LITERATURE

The purpose of this study is to investigate the use of general heuristics by high school students when they solve mathematical problems. Some authors have urged teachers to use various heuristics when they demonstrate the solution to problems or have urged them to teach these procedures to their students. Several studies were reviewed in which certain heuristic procedures were taught to students in order to improve their problem-solving skills. One study was concerned with the heuristics students use when asked to solve a problem using any method they chose. This chapter reviews some of the more important studies and opinions related to the topic of this investigation. For organizational purposes the literature is classified according to studies having general relevance to the problem, studies having specific relevance to the problem, opinions having general relevance to the problem, and opinions having specific relevance to the problem.

Studies having general relevance to the problem. The investigator reviewed several studies in which attempts were made to improve the student's problem-solving ability by requiring the instructor to use a heuristic method when teaching solutions to problems.

Ashton's research tested the effectiveness of teaching problem solving in ninth grade algebra by a heuristic method as opposed to a conventional method where the solution to a particular type of problem is demonstrated and then similar problems are given for the students to solve. The heuristic method used was based on Polya's writings and involved frequent questioning techniques such as: What is the unknown?, What are the data?, and What are the conditions?[1] A total of 134 students were taught using the heuristic method and 110 were taught by the conventional method. Each teacher taught two classes, one with the heuristic method and the other using the conventional method. After the ten-week teaching period, the results showed that all classes gained in their scores from pre-test to post-test. However, the gains of the heuristic group were greater, and were significant at the 0.01 level for classes in four of the five participating schools.[2]

Wills tried to determine if the student's problem-solving ability was improved if a particular set of discovery stratagems were used by the teacher. The procedure

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[1]George Polya, How to Solve It (second edition; New York: Doubleday and Company, Inc., 1957).

[2]Sister Madeleine Rose Ashton, "Heuristic Methods in Problem Solving in Ninth Grade Algebra," (unpublished doctoral dissertation, Stanford University, Stanford, 1962).

followed was:

(a) Students were presented with a difficult problem which could be solved easily by anyone aware of a certain generalization.
(b) Simpler problems similar to the initial task were presented developmentally.
(c) The results of these simpler tasks were organized in the form of a table which served to reveal a pattern to the students.
(d) The generalization suggested by the pattern was applied to the initial task.
(e) The results were checked.[3]

The pre-test and post-test contained sixty problems that involved a wider range of mathematical topics than the current instructional unit. The experimental group generally doubled their pre-test results, while the control group made only a slight score gain. Wills found no difference between the experimental group that was taught by the teacher who merely used the stratagems, as opposed to the group who took an active role in discussing the stratagems and their relevance to problem solving.

Larsen wished to determine if general heuristic suggestions, which would facilitate the student's solution of other mathematical problems, could be given independently of the detail of a particular calculus problem. The results suggested that students can learn to use heuristic methods when taught heuristics, but that this learning may be at the

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[3] Herbert Wills III, "Transfer of Problem Solving Ability Gained Through Learning by Discovery," Dissertation Abstracts, XXVIII (1967), 1319-A.

expense of normal course content.[4]

Wilson investigated the problem-solving performance of 144 high school students on learning and transfer tasks, after instruction in which the level of the generality of the problem-solving heuristics was varied. Three levels of generality were used; means-ends heuristics, task specific heuristics, and planning. Results showed that for the learning tasks the performance tended to be independent of the level of generality of heuristics used. However, for the transfer tasks the subjects seemed to benefit from having a wide range of heuristics available.[5]

Schaaf's investigation was made to determine if students could learn to be better generalizers if the teaching procedure was a discovery method which emphasized such procedures of generalizing as simple enumeration, analogy, continuity of form, statistical procedures, deduction, variation, formal analogy, and inverse deduction. Results showed that most of the students made an improvement in their ability to generalize and that this learning was

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[4]Charles McLoud Larsen, "The Heuristic Standpoint in the Teaching of Elementary Calculus," Dissertation Abstracts, XXI (1961), 2632-2633.

[5]James William Wilson, "Generality of Heuristics as an Instructional Variable," Dissertation Abstracts, XXVIII (January, 1968), 2575-A.

not at the expense of the mastery of algebraic concepts.[6]

Washburne and Osborne attempted to train students in grade six and seven to see the analogy between a difficult problem and a simple problem of the same type. The pupils were required to write a simple problem that was analogous to the more difficult problem they were required to solve. The students then observed how they solved the simple problem and applied this process to the original problem. As contrasted with the other two methods tested (solving many problems and formal analysis) this method resulted in approximately a 2% gain in the final test scores.[7]

Russell discussed a study by Doty who interviewed 151 grade four students and thirty-one grade six students to determine what methods they used to solve verbal arithmetic problems. Among the procedures which led to correct answers was recalling analogous problems.[8]

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[6]Oscar Schaaf, "Student Discovery of Algebraic Principles as a Means of Developing Ability to Generalize," The Mathematics Teacher, XLVIII (May, 1955), 324-327.

[7]Carleton W. Washburne and Raymond Osborne, "Solving Arithmetic Problems," The Elementary School Journal, XXVII (November, 1926), 219-226; (December, 1926), 296-304.

[8]David H. Russell, Children's Thinking (Toronto: Blaisdell Publishing Company, 1956), p. 275.

Crutchfield attempted to teach creative problem solving by using a series of auto-instructional materials. "Each lesson is a complete problem-solving episode, containing all of the principle steps and processes inherent in creative problem-solving."[9]

> The lessons are constructed not only to give the reader repeated experiences in the solution of interesting problems, but also directly to instruct him in helpful strategies or heuristic procedures for creative problem solving, by showing him how he can use them in the concrete problems. The procedures pertain to the formulation of the problem, the asking of relevant questions, the laying out of a plan of attack, the generation of many ideas, the search for uncommon ideas, the transformation of the problem in new ways, the evaluation of hypotheses, the sensitivity to odd and discrepant facts, and the openness to metaphorical and analogical hints leading to solution.[10]

Crutchfield carried out two major studies using the materials in fifth and sixth grade classes. The designs of the two studies were essentially the same, except that the training and testing materials were revised and lengthened for the second study. A total of 267 children were given the training materials and another 214 formed the control group. The first experiment took three weeks and the second lasted four weeks.

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[9]Richard S. Crutchfield, "Creative Thinking in Children: It's Teaching and Testing," Intelligence: Perspective 1965 (New York: Harcourt, Brace, and World, Inc., 1966), p. 43.

[10]Ibid., p. 44.

Each student worked individually at his own pace. The post-test results showed that the problem-solving ability of the trained children surpassed that of the control group. According to Crutchfield:

> The trained children asked a greater number of relevent questions while working on the problems, were more sensitive to significant clues and factual discrepancies, generated more ideas and better ideas, and were better able to utilize clues and hints as help in getting to solutions.[11]

Studies having specific relevance to the problem. Only one study attempted to discover which heuristics the students used to solve problems.

Kilpatrick's main aim was to develop a system to analyze the heuristics students used in solving word problems. Fifty-six grade nine students were individually interviewed and asked to think aloud as they solved mathematical problems. The heuristic processes identified were based on Polya's writings.[12] Kilpatrick designed four different systems for coding the student's response. The first system attempted to record instances of the pupil asking himself any of thirty-six questions. Some of the questions were; what is the unknown?, what is the condition?, and "Have I seen the problem before?" Kilpatrick concluded that this list was not useful for characterizing the student's

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[11] Ibid., p. 51.

[12] George Polya, loc. cit.

problem-solving behaviour, for two reasons; (1) there was not enough time to complete the form and still maintain rapport with the student and (2) the catagories were not precisely defined. The second system was a list of eleven heuristics that were put as actions the student might take, rather than as questions that he might ask himself. Four of these heuristics were; drawing a figure, rephrasing the problem, using successive approximations, and checking that the result is reasonable. Although this list was more manageable, the results were also unsatisfactory. According to the study, two of the main reasons why the second list was not successful were; (1) many actions of the student were not being recorded and (2) no indication of the sequence of the heuristics used by the student was indicated. The third system was an attempt to identify sixty-four processes and the sequence used during the student's solution to a problem. This system proved to be unmanageable. Kilpatrick then developed his fourth and final system. He tried to combine the ideas of system two and system three into a manageable list. The processes used were reading and trying to understand the problem, deduction from the condition, setting up an equation, trial and error, and checking the solution. The subject's ability was rated as (1) incomplete (2) impasse (3) intermediate result (4) incorrect result (5) correct result. The rest of the

coding system dealt with thirty items of supplementary information such as; draws a figure, uses a related problem in the solution, and admits confusion. The results indicated that the students used fifteen of the items but only eight were frequently used. Kilpatrick concludes, "The check list yields few variables that can be measured reliably."[13] From the process-sequence list Kilpatrick observed that deduction was likely to result in an incorrect solution whereas trial and error was likely to result in a correct solution. The main conclusion Kilpatrick arrived at seemed to be that mathematics teachers should devote more time to trial and error as a legitimate problem-solving method.

Kilpatrick was mainly interested in analyzing the problem-solving process. The present study is not concerned with the process or sequence taken in the solution of the problem, but only interested in determining if certain general heuristics are used by students when they are asked to attempt to solve a problem for which they do not know an algorithm.

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[13]Jeremy Kilpatrick, "Analyzing the Solution of Word Problems in Mathematics: An Exploratory Study," (unpublished doctoral dissertation, Stanford University, Stanford, 1967), p. 63.

Opinions having general relevance to the problem. Several writers have urged that teachers employ a heuristic method when teaching.

Polya stressed a heuristic teaching procedure. He described such heuristics as synthesis, analysis, analogy, cases, and working backwards and showed how they could be used when teaching mathematical problem solving.[14]

Henderson discussed seven stratagems which he felt can be employed by the teacher so that students will be helped to discover mathematical truths. The stratagems are; (1) analogy (2) simple enumeration (3) agreement (4) difference (5) difference and agreement (6) concomitant variation and (7) independent action. Although he urged the use of the stratagems (1) to (6) by the teacher, it would seem as if Henderson believed that the stratagems would be picked up and used by the students from observing the teacher. His last stratagem, independent action, allows the student to attempt to find an answer to a problem on his own. Henderson stated, "Quite possibly this stratagem would not be used except with very bright students, or

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[14]George Polya, loc. cit.

students who have had considerable experience with other strategies."[15]

Jones illustrates the use of heuristic techniques that could be used by teachers in planning discovery-type lessons. The techniques are induction, permanence of form, structure and deduction, models, logical form of statements, generalization and specialization, and analogy and mental models.[16]

Bruner stated:

> It might be wise to assess what attitudes or heuristic devices are most pervasive and useful, and then an effort should be made to teach children a rudimentary version of them that might be further refined as they progress through school.[17]

Bruner felt that intuitive thinking which he defines as "the training of hunches" should be taught.[18] He said that heuristic rules such as the use of analogy, the appeal to symmetry, the examination of limiting conditions, and the

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[15]Kenneth B. Henderson, "Strategies for Teaching by the Discovery Method," Updating Mathematics (high school teachers edition; New London, Connecticut: Croft Educational Services, 1964), p. 64.

[16]Philip S. Jones, "Discovery Teaching From Socrates to Modernity," The Mathematics Teacher, LXIII (October, 1970), 501-508.

[17]Jerome S. Bruner, The Process of Education (New York: Vintage Books, 1960), p. 27.

[18]Ibid., p. 13.

visualization of the solution, when used frequently by the teacher should promote intuitive thinking in students.[19]

Opinions having specific relevance to the problem. Kilpatrick saw the necessity for studying how students solve problems and for determining what heuristics were used.

> We have so little knowledge of how problem solving is and could be taught that we must do more, however, than simply exhort teachers to pay greater attention to it. We should encourage them to study the problem-solving behavior of their students.[20]

Summary. Ashton, Wills, Larsen, Wilson, and Schaaf[21] all urge teachers to use heuristic methods when teaching mathematics. In all cases the researchers found that if heuristic teaching methods were used, the problem-solving ability of the students was improved. Washburne and Osborne[22] attempted to train students to use analogy. They found that the students could solve problems better after they had been taught to use the heuristic of analogy. Several writers have urged that teachers employ a heuristic method when teaching.[23] Kilpatrick, however, felt that the

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[19]Ibid., p. 64.

[20]Kilpatrick, op. cit., p. 105.

[21]Ashton, loc. cit.; Wills, loc. cit.; Larsen, loc. cit.; Wilson, loc. cit.; Schaaf, loc. cit.

[22]Washburne and Osborne, loc. cit.

[23]Polya, loc. cit.; Henderson, loc. cit.; Jones, loc. cit.; Bruner, loc. cit.

investigator should attempt to determine what heuristics the students use and then an attempt should be made by the teacher to improve the pupil's problem-solving ability.[24]

Limitation of previous studies. The main limitation of previous research is that no attempt was made to investigate whether or not the students use any general type heuristics such as, analogy or inverse deduction, to solve an unfamiliar problem. Since heuristics are useful and worthwhile tools to use when attempting to solve a variety of problems, students should be taught these methods. However if the students use some heuristics and not others, the students should first be taught to use the heuristics that they understand.

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[24]Kilpatrick, loc. cit.

CHAPTER III

THE DESIGN AND STATISTICAL TREATMENT

I. THE DESIGN

Each of the forty-two grade eleven students interviewed was given two mathematical problems to solve. These problems can be solved using two of the nine general heuristics discussed earlier. Although the students were allowed to use pencil and paper, the researcher requested the students to think aloud.

Various methods are used to obtain information on the method of solution of problems. They range from a study of the written work of a student, retrospection, introspection, and thinking aloud. In retrospection studies, the subject is asked to give an account of his thinking after he has completed the solution to the problem. Two disadvantages to the use of this technique appear to be, (1) the subject may edit his report and omit errors and false leads, (2) the subject may not remember all the steps and the order in which they occurred.

Introspection attempts to analyze thought processes by obtaining reports from the student as he is in the process of solving a problem. Since this technique requires the student to observe and analyze his thinking, the student would have to be aware of his use of general heuristics.

The investigator feels that few students, if any, are aware of their use of any general heuristics. Duncker stated that the instruction "think aloud,"

> . . . Is not identical with the instruction to introspect. . . . While the introspector makes himself as thinking the object of his attention, the subject who is thinking aloud remains immediately directed to the problem, so to speak allowing his activity to become verbal.[1]

When the student thinks aloud he does not have to be aware of the methods he is using to solve the problem. It is up to the researcher to analyze the response to determine if the student shows that he used certain heuristics. Another objection to the thinking aloud technique is that the student may change his method of attack on the problem because he is asked to vocalize his thinking. However, Kilpatrick stated, "we can ignore questions about the changes in thinking produced by vocalization . . . , if we restrict our use of the method to producing hypotheses rather than to validating them."[2] The researcher was only attempting to discover if the pupil knew any general heuristics and not whether he would always attack the given problems in this way. Taylor stated that, "thinking aloud" has repeatedly

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[1]Karl Duncker, "On Problem Solving," trans. Lynne S. Lees, Psychological Monographs, LVIII (whole no. 270, 1945), p. 2.

[2]Kilpatrick, op. cit., p. 8.

proved fruitful in the analysis of process."[3]

A sample of the form which the student was required to fill out is shown in Appendix A. The two problems were typed on separate sheets of paper. After the student had attempted the first problem, he was given the second one. The standard interview format that was used with every student is shown in Appendix B. The student was encouraged to attempt the problems any way he chose. They were asked to be more concerned with revealing as much of their thought processes as possible, as with the accuracy of their solution. All interviews were taped. When the student requested assistance, he was aided only if the investigator felt that the required information would not help the student to plan his method of solution. The student was given no indication of the correctness of any of the steps in his solution. All subjects were individually interviewed by the writer, and worked in an area where distractions were minimized.

The Problems Used. The following is a list of the nine problems used in this study. Each one is classified as to the method by which it could be solved. Each individual student was presented with two of these nine problems. The total number of thirty-six different

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[3]Donald W. Taylor, "Discussion of Papers by Adrian de Groot and by Jeffery M. Paige and Herbert A. Simon," Problem Solving: Research, Method, and Theory (New York: John Wiley and Sons, Inc., 1966), p. 123.

combinations of the nine problems were distributed among the forty-two subjects. A complete solution which shows how a given heuristic technique could be used to solve the problem is shown in Appendix C.

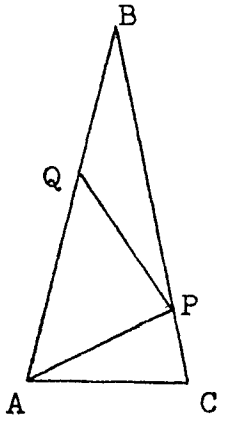
1. Cases.

Find all real numbers x which satisfy the following equation[4]

$$|x + 1| - |x| + 3|x - 1| - 2|x - 2| = x + 2$$

2. Deduction.

Triangle ABC is isosceles (the two sides AB and BC have the same length) with base AC. Point P is in CB and point Q is in AB such that AC = AP = PQ = QB. What is the number of degrees in angle B?[5]



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[4] Steven J. Bryant, George E. Graham, and Kenneth G. Wiley, Nonroutine Problems in Algebra Geometry and Trigonometry (New York: McGraw-Hill Book Company, 1965), p. 5.

[5] Charles T. Salkind, The MAA Problem Book II (New York: Random House, Inc., 1966), p. 13.

3. Inverse Deduction.

Bill, Ron, and Ted each have some money of their own but they do not all have the same amount of money. Now, Bill decides to give some of his money to Ron and some of his money to Ted. He gives Ron as much money as Ron already has and also he gives Ted as much money as Ted already has. After this Ron decides to give some of his money to Bill and to Ted. He gives each of them as much money as each of them now has. Then Ted decides that he will give Bill and Ron as much money as each of them has. Now Bill has 16¢ and Ron has 16¢ and Ted has 16¢. How much money did Bill originally have before he gave any of his money away?[6]

The problem was originally presented as:

A gives B as many cents as B has and C as many cents as C has. Similarly, B then gives A and C as many cents as each of them has. C, similarly, then gives A and B as many cents as each of them has. If each finally has 16 cents, with how many cents does A start? Student 1, student 2, and student 3 answered the original problem. It was observed that these three students took too much time to comprehend the problem. Since most of the time allotted each student should be spent on the solution, it was necessary to minimize the comprehension time. The revised form reduced the time the students spent

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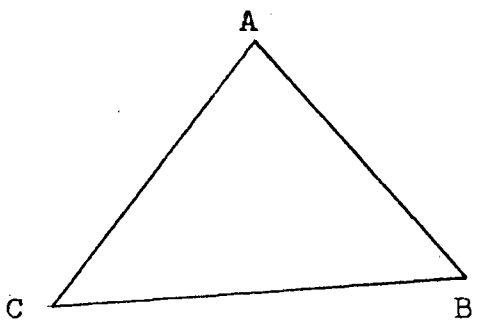
[6] Ibid., p. 26.

in comprehending the problem.

4. Invariation.

Using a compass and a straightedge construct a square inside the given triangle ABC such that two of the vertices of the square are on the base CB of the triangle and the other two vertices of the square are on the other two sides AC and AB of the triangle, one vertex on each side.[7]



5. Analogy.

Find the values of the five unknowns x, y, z, u, and t that satisfy the following set of five simultaneous linear equations.[8]

$$\begin{aligned} 4y - 3x + 2u &= 5 \\ 9x - 2z + u &= 41 \\ 7y - 5z - t &= 12 \\ 3y - 4u + 3t &= 7 \\ 7z - 5u &= 11 \end{aligned}$$

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[7]George Polya, How to Solve It (second edition; New York: Doubleday and Company, Inc., 1957), p. 23.

[8]Reverend Alfred Wrigley, A Collection of Examples and Problems in Pure and Mixed Mathematics with Answers and Occasional Hints (London: Longmans, Green, and Company, 1875), p. 38.

6. Symmetry.

Using a compass and a straightedge construct an isosceles triangle (two of the sides have the same length) given the altitude (the perpendicular distance from the vertex to the side opposite) equal to $1\frac{1}{2}$ inches and the perimeter equal to 5 inches.[9]

7. Preservation of Rules.

Using the following mathematical system containing six dabas d_1, d_2, d_3, d_4, d_5, and d_6 and the definition "an aba is a set containing three dabas" (i.e. $\{d_1, d_2, d_3\}$ and $\{d_2, d_4, d_6\}$ are abas),

(i) List all the abas

(ii) Decide if the following statements are true or false

Statement 1 If p and q are two dabas, then there exists one and only one aba containing both p and q.

Statement 2 If L is an aba, there exists a daba not in L.

Statement 3 If L is an aba, and p is a daba not in L, then there exists one and only one aba containing p and not containing any daba that is in L.[10]

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[9]Edmund C. Plant, Geometrical Drawing (London: MacMillan and Company, Limited, 1899), p. 103.

[10]Betty Plunkett, "Aba Daba Daba," The Mathematics Teacher, LIX (March, 1966), 236-237, citing Howard Eves, and C.V. Newsom. An Introduction to the Foundations and Fundamental Concepts of Mathematics (New York: Holt, Rinehart and Winston, 1964), p. 80.

8. Variation.

Construct a trapezoid (a closed plane figure with four sides, two of which are parallel) being given the lengths a, b, c, and d of its four sides.[11]

a	b	c	d

9. Extension.

$1 + 8 + 27 + 64 = 100$

$1^3 + 8 + 27 + 64 = 100$ What can you discover?[12]

The problem was originally presented as:

$1 + 8 + 27 + 64 = 100$ What can you discover? Student 13, student 14, and student 15 answered the original problem. It was observed that the students did not notice that the left-hand side was $1^3 + 2^3 + 3^3 + 4^3$ or that the right-hand side was 10^2. Since the problem was not to discover the fact that $1^3 + 2^3 + 3^3 + 4^3 = 10^2$, but to see if the student would extend the problem, to see if, for example, $1^3 + 2^3 + 3^3 + 4^3 + 5^3 = (x)^2$ where x is a positive integer, the problem was revised.

The Group Used. This investigation was a normative survey of forty-two subjects enrolled in nine academic grade

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[11] Polya, op. cit., p. 211.

[12] Ibid., p. 108.

eleven algebra classes in nine schools. Based on their Christmas and Easter mathematics eleven grades, the students' mathematical ability was A or B. The investigator was interested in determining if the student who had a high mathematical ability used any heuristics. It is felt that this type of study was justified since as Kilpatrick stated:

> The researcher . . . who chooses to investigate problem solving in mathematics is probably best advised to undertake clinical studies of individual subjects . . . because our ignorance in this area demands clinical studies as precursors to larger efforts.[13]

II. STATISTICAL TREATMENT

This normative survey was concerned with the identification of any heuristic which the students used in attempting to solve the mathematical problems used in this investigation. The survey was also concerned with the frequency of the use of the heuristics. A tabulation of the heuristics used by the students and a count of the number of heuristics each student used is presented. Each heuristic and its use by the students is discussed.

Limitations of the Study. This study attempted to find out what heuristics were used by grade eleven academic mathematics students. The pupil's use of heuristics may differ at various grade levels. Consequently all heuristics

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[13]Jeremy Kilpatrick, "Problem Solving in Mathematics," Review of Educational Research, XXXIX (October, 1969), 523-534.

may not be able to be taught effectively at all levels, but may have to be simplified or even ignored. It may be that it is only practical to teach certain heuristics at one level and others in higher grades. Only a limited sample of students was used and therefore it may be that other heuristics would be uncovered if the sample was larger. This investigation says nothing about which heuristics the student does not know or if he knows a heuristic but can not use it, but only which heuristics he happened to use when solving the problems used in this study. Nine problems were used in this study. If different problems were presented to the students, it may be that other heuristics would have been discovered or some of the heuristics may have been used more frequently. Suppose a student attempted to solve a problem using a heuristic which was inappropriate. He then could have been given another problem where the use of this heuristic was appropriate. It could then be determined if the student usually used this heuristic when attempting problems and whether he could use it correctly.

CHAPTER IV

RESULTS

Identification and frequency of student usage of heuristics. The investigation found evidence that eight of the nine heuristics were used by the forty-two students interviewed. The students used cases, deduction, inverse deduction, invariation, analogy, preservation of rules, variation, and extension. No student used the heuristic of symmetry. Three students noticed symmetry in the problem they solved, but they did not use symmetry to help them in their solution of the problem. Eighteen of the students tried to use cases to solve the problems they were presented. Table 1, page 68, shows the number of students who used heuristics when attempting to solve a problem and the number of students who were presented with each of the problems. Appendix D shows sample interviews and analyses. Appendix E gives information on the students' use of the general heuristics, Appendix F gives information on the students included in this survey, and Appendix G gives information on the schools sampled in this survey. The students' use of each of the nine heuristics will now be discussed.

Cases. Eighteen students used the heuristic of cases. Most students considered a random sample of the possible

solutions to the problem. Usually the students felt that this method was too time consuming and did not feel it was an effective approach to the problem. Problem 1 which could be solved efficiently by the use of cases was not solved correctly by any of the six pupils who tried to solve the problem by using this heuristic. They appeared to have no method for efficiently selecting the cases to be tried, and for narrowing down their search through the possible solutions. Three of the students called this method trial and error. The way each of the students used cases in the problems they attempted will now be discussed.

Student 2 Problem 1

A random sample of positive and negative integers was substituted for x. See Appendix D.

Student 3 Problem 1

Two cases were considered--x a positive value and x a negative value.

Student 5 Problem 4

A figure was drawn inside the triangle but since it appeared to be a rectangle, the use of the heuristic of cases was abandoned.

Student 7 Problem 5

The student suggested that numbers be substituted for the different variables until the equations were satisfied.

Student 9 Problem 8

The student randomly varied the placement of the sides and the measure of the angles in order to draw a trapezoid. The student referred to this method as trial and error.

Student 13 Problem 5

Two different sets of values for y, x, and u that satisfied the first equation were chosen at random. The third equation was then used to find values for z and t. These values were checked in equation 2. Since the problem was not solved correctly, another approach was chosen. Later the student returned to using cases. The student called this method trial and error.

Student 15 Problem 6

Various figures were drawn at random in an attempt to solve the problem.

Student 16 Problem 5

The student suggested that the problem could be solved by substituting numbers for the variables, but that this procedure would be time consuming.

Student 17 Problem 1

Two values for x were considered--x negative and x positive. See Appendix D.

Student 21 Problem 8

Several figures were drawn. Each time the placement of the sides was randomly varied. The student called the procedure guessing and stated that there was no method to the procedure chosen.

Student 23 Problem 8

The sides of the trapezoid were placed in different positions at random in an attempt to construct the correct figure.

Student 26 Problem 1

The student first changed some of the plus signs to minus and the minus signs to plus, and then all of the signs were changed to the opposite sign.

Student 29 Problem 8

Various positions for the sides and the angles were tried in the attempt to draw the trapezoid. The student felt that the various arrangements for the sides and angles would have to be tried until a solution was found.

Student 30 Problem 1

Various values were substituted into part of the equation, $|x + 1| + |x - 1|$, to see if there was any relationship between the values substituted and the answers obtained. Values were then substituted into the left-hand side of an equation that the student had incorrectly derived, $x - |x - 2| = (x + 2)/6$.

Student 31 Problem 8

Several figures were drawn at random to attempt to solve the problem. The student referred to this method as trial and error.

Student 32 Problem 4

The student randomly tried various lengths to see if a square could be constructed.

Student 35 Problem 1

Two cases were considered--x a positive value and x a negative value.

Student 37 Problem 8

Various figures were drawn. The positions of the sides and the measures of the angles were randomly varied in order to get the required trapezoid.

Deduction. Seven students used direct deduction to attempt to solve problem 2. Three of the seven students, who thought they found the solution, correctly derived false conclusions from false premises. All the students seemed reasonably confident of their method of attack. Two students attempted to use hypothetical deduction in the solution of problem 2 and problem 3(b). Problem 2 which could be solved using the technique of deduction was not solved correctly by any of the students. The analysis of the solutions in which deduction was used will now be discussed.

Student 1 Problem 2

From the following three facts, (i) AC = AP = PQ = QB (ii) triangle ABC, triangle APC and triangle PQB are isosceles and (iii) the base angles of an isosceles triangle are equal, the student incorrectly concluded that angle A = angle C = angle APC = angle QPB = angle B. Since angle A + angle C + angle B = 180°, angle B = 60°. See Appendix D.

Student 4 Problem 2

From the following three incorrect statements (i) angle QPB + angle APQ = 90°, (ii) triangle QPA is equilateral, therefore angle QPA = 60° and (iii) angle QPB = angle B, the student concluded that angle B = 30°.

Student 11 Problem 3(b)

The student felt that the amount of money Bill had after he gave some of his money to Ron and Ted, could be found. Then the student said it would be possible to work back to the original amount Bill had. The investigator would classify the student's technique as hypothetical deduction.

Student 18 Problem 2

Using the fact that two angles were supplementary or that the sum of the angles in a triangle was equal to 180°, an attempt was made to deduce the measure of angle B.

Student 24 Problem 2

The student felt that the answer could be obtained by adding or subtracting angles. The student noticed that angle QAP + angle PAC = angle QAC, angle QPB + angle APQ = angle APB, and angle PAC = angle BAC - angle QAP.

Student 31 Problem 2

The student stated that he would work from angle APQ if it was 90°. The student said that if angle A or angle C could be found, then angle B could be obtained since the sum of the angles in a triangle is 180°. The student called this procedure working backwards. The investigator classified the method as hypothetical deduction.

Student 33 Problem 2

An attempt to find angle B was made by combining angles that were supplementary or using the fact that the sum of the angles in a triangle equals 180°. The student felt that angle B could be expressed in terms of other angles in various ways and that these statements could be combined by addition or subtraction, to get the value for angle B. See Appendix D.

Student 42 Problem 2

The student expressed angle B incorrectly in terms of other angles. The student thought that the sum of the angles in a triangle was 360°. The student also said that one of the measures of the angles would have to be given

before the number of degrees in angle B could be determined.

Inverse Deduction. Two students suggested using the heuristic of inverse deduction to solve problem 3. Problem 3 could be solved by this heuristic, but neither student was able to use the technique. The students both felt that the problem could be solved by working backwards, but they did not know how to proceed.

Student 2 Problem 3(a)

The student suggested that the problem could probably be solved by working backwards.

Student 11 Problem 3(b)

The student attempted to find a number closer to the original amount of money that Bill had. The student incorrectly found the amount of money Bill had after he had given money to Ron and Ted and then wanted to work back to the amount Bill had originally.

Appendix D contains the interviews for the two students who used inverse deduction.

Invariation. Three students used the heuristic of invariation. In all cases problem 1 was attempted by ignoring some of the data. Problem 4 which could have been solved by using invariation was not given to these three students. The investigator felt that this heuristic was used by students 1 and 30 because they were very uncertain how to solve absolute value questions.

Student 1 Problem 1

The student decided to ignore the absolute value signs, rewrite the problem, and then try to solve it.

Student 2 Problem 1

The student decided to rewrite the equation leaving out the absolute value signs.

Student 30 Problem 1

The student attempted to ignore the absolute values and work the resulting equation.

Analogy. Seven students used the heuristic of analogy. Six of the students used the heuristic in attempting to solve problem 5. Problem 5 could be solved by the method used to solve three simultaneous linear equations in three variables. Four of the students suggested elimination of variables and two students suggested solving for one of the variables and substituting it into the next equation where the same variable appeared.

Student 10 Problem 7

The student could not remember how many sets of three dabas could be constructed from a total of six dabas, so the student attempted to decide what the formula was by considering three numbers grouped in sets of two and then four numbers grouped in sets of two.

Student 10 Problem 8

The student noticed that the trapezoid formed two triangles. The student said that to construct a triangle one line and two angles would be necessary. The student then tried to relate the method for constructing the triangle to the given problem.

Student 13 Problem 5

The student thought that if two of the equations had the same unknowns they could be added or subtracted so that one of the variables could be cancelled.

Student 14 Problem 5

Equations one and two were solved for x and then these two equations were combined in order to solve for y. The student then wished to solve equations four and five for u and continue until one of the variables was known.

Student 16 Problem 5

The student described how to solve two simultaneous linear equations in two variables by solving for one variable and substituting it in the second equation. The student said that she had never seen the three variable case before, but she would try the problem the same way as the two variable case. See Appendix D.

Student 25 Problem 5

After first looking for two equations in two unknowns, the student thought that it didn't matter how many variables

there were as long as there were like terms in two equations. Then the equations could be subtracted or added to eliminate one of the unknowns.

Student 38 Problem 5

The student thought that there were two equations with two variables the same in each of the two equations, and therefore the equations could be subtracted to cancel out one of the unknowns. The student felt that the same procedure would solve the given problem because it was the method used to solve two simultaneous linear equations in two unknowns. See Appendix D.

Student 39 Problem 5

The student thought that if two equations were combined, one variable could be eliminated.

Symmetry. None of the forty-two students interviewed used the heuristic of symmetry. Only three students appeared to notice symmetry in a problem, but they did not use this symmetry to help find a solution. Problem 6 which could be solved using symmetry, was usually tried by cases or by the heuristic of variation. Student 25, student 27 and student 34, who were given problem 6, all noted that the altitude of an isosceles triangle bisects the base of the triangle.

Preservation of Rules. At the high school level, the investigator felt that the heuristic could not be tested in the sense of giving the students a problem in another system

and determining if they would see if various rules in the familiar systems they had studied held, before they proceeded to try the problem in the new system by a method analogous to the method used for similar problems in the familiar system. However it was felt that the heuristic could be tested to see if students can remain working in a new system. A small system was defined and then the student was supposed to determine if statements about the system were true or false. They had to preserve the definition given and not switch out of the system to more familiar systems. Two of the eight students who were given problem 7 showed that they really could preserve the rules of the system, in order to correctly answer the questions. A discussion of the solutions to problem 7 by all the students who were given the problem, will follow.

Student 8 Problem 7

The student showed no preservation of rules. The student felt that statement 1 was false because there were only two dabas listed and there must be three in an aba. The student apparently thought that the statement meant "If p and q are two dabas, then there exists one and only one aba containing only p and q." The student felt that statement 2 was false because there are three dabas in an aba so there would be none left over.

Student 10 Problem 7

The student showed no preservation of rules. The student thought there were more than six dabas.

Student 19 Problem 7

The student showed no preservation of rules. The student was not certain of the difference between a daba and an aba and thought that there could be only one set containing p and q, since there were only two numbers p and q for the set.

Student 20 Problem 7

The student showed no preservation of rules. The student thought there could be only one set containing both p and q, since there were only two numbers for the set. For statement 2, the student felt that there were only two sets, L and the empty set.

Student 22 Problem 7

The student showed no preservation of rules. The student said statement 1 was false because there couldn't be an aba since there were only two dabas, and an aba required three dabas. The student apparently thought the statement was, "If p and q are two dabas, then there exists one and only one aba containing only p and q."

Student 26 Problem 7

The student showed preservation of rules. See Appendix D.

Student 28 Problem 7

The student showed preservation of rules. See Appendix D.

Student 39 Problem 7

The student showed preservation of rules. The student thought there was an infinite number of dabas in the system.

Variation. Three students used the technique of variation. Two of the students varied the problem that was to be solved, while the other student varied the data that was given. Two of the students used variation as they attempted problem 8. Problem 8 could be solved by variation, but it was usually attempted by cases. Student 25 and student 31 apparently felt that variation was not a legitimate technique of problem solving, but a way of cheating in order to arrive at an answer. Actually the way in which these students employed variation, they did not carry the procedure far enough. They wanted to vary the data in order to get a solution instead of varying the data in order to help in the search for the solution.

Student 23 and Student 29 Problem 8

The students both suggested that the problem could be solved by making one of the given lengths longer. The students were thinking of adding to the length of one of the given sides in order to draw a closed figure, and not

to determine that if the side was longer a parallelogram could be drawn that would help in the solution of the problem, using the original lengths of the sides. See Appendix D.

Student 29 Problem 4

The student suggested drawing a circle inside the triangle. The student then felt that it would be possible to contruct a square using the circle, because there were lines indicating equal distances from the middle of the triangle.

Student 32 Problem 6

The student thought that if a circle with a 3/4 inch radius was drawn it would help in the construction of the required triangle.

Extension. The investigator felt that at the high school level it was not possible to determine if the student could add or delete certain facts from a given system in order to see what discoveries could now be made. However, it was felt that the heuristic of extension is employed when a student attempts to generalize a given statement. Seven students used the heuristic of extension. Three of the students solved problem 9(b) which was given to see if any of the students would use extension. All three of these students had been exposed to computer programming. The three students who used this heuristic for problem 4 and

the one student who attempted problem 2, were not given problem 9(b).

Student 1 Problem 2

Line AP was extended past line BC. The student looked at the problem to see if the solution was now evident. The student concluded that the line would just add more confusion.

Student 5 Problem 4

The student decided to draw the three altitudes of the triangle. The student then attempted to see if a square could be drawn.

Student 25 Problem 4

The student thought that if the base line CB was extended it might help to solve the problem. It became evident later in the interview that the student could not visualize a square inside the triangle.

Student 32 Problem 4

The student drew the bisectors of the angles in order to find the centre of the triangle. The student wanted to construct a square with centre the point in the triangle which is equidistant from the three vertices, since there would be equal space on all sides.

Student 33 Problem 9(b)

The student noticed that $1^3 + 2^3 + 3^3 + 4^3 = (1 + 2 + 3 + 4)^2$ and then extended the series to determine

what happened with $1^3 + 2^3 + 3^3 + 4^3 + 5^3$. The student added on 6^3, 7^3, and then 8^3 and finally concluded $Fn = 1^3 + 2^3 + 3^3 + \ldots + n^3 = (1 + 2 + 3 + \ldots + n)^2$. See Appendix D.

Student 37 Problem 9(b)

The student incorrectly concluded that $1^3 + 2^3 + 3^3 + 4^3 = \sqrt{100}$. The student then hypothesized that the sum of any cubes equals the square root of some integer. For example, the student suggested $5^3 + 6^3 + 7^3 + 8^3 = x$ where x is a positive integer.

Student 41 Problem 9(b)

The student noticed that $1^3 + 2^3 + 3^3 + 4^3 = (1 + 2 + 3 + 4)^2$. The student then extended the sum by adding 5^3 and 6^3 and concluded that $1^3 + 2^3 + 3^3 + \ldots + n^3 = (1 + 2 + 3 + \ldots + n)^2$. He then investigated $2^3 + 3^3$, $2^3 + 3^3 + 4^3$, $3^3 + 4^3$, $4^3 + 5^3$ and concluded that $n^3 + (n + 1)^3 + \ldots + (n + x)^3 = (n + (n + 1) + \ldots + (n + x)) \times (n + (n + 1) + \ldots + (n + x) + (n^2 - n))$. The student next looked at $\sqrt[2]{1} + \sqrt[3]{2} + \sqrt[3]{3}$ to determine if it was equal to $\sqrt{6}$. See Appendix D.

The students' use of other heuristics. The investigator found evidence that another heuristic was used by eleven of the forty-two students interviewed. The investigator calls the method successive variation. A definition of the heuristic successive variation follows.

A problem and its solution also is given to illustrate the use of the heuristic.

Successive Variation. A probable solution to the given problem is chosen at random. If the answer is not correct, the student determines what changes must be made. Then the probable solution that is chosen is varied successively until the student reaches the correct answer. Higgins refers to a problem-solving technique that involves "guessing an answer, working out its consequences, and--by comparing these with the original conditions of the problem--improving the original guess."[1] The investigator feels that this heuristic has most value if the problem requires only an approximate answer. The following problem illustrates how this heuristic could be effectively used: Find an approximation to three decimal places for $\sqrt{27}$. The answer can be successively bracketed as follows: To one decimal place $5 < \sqrt{27} < 6$. Since $5^2 = 25$ and $6^2 = 36$, the answer is closer to 5 than to 6. To two decimal places try $(5.1)^2$. The answer to two decimal places is $5.1 < \sqrt{27} < 5.2$. Since $(5.1)^2 = 26.01$ and $(5.2)^2 = 27.04$, the answer is closer to 5.2 than to 5.1. Hence to three decimal places try $(5.19)^2$. The answer to three decimals is $5.9 < \sqrt{27} < 5.20$. The way each of the students used

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[1]Jon L. Higgins, "A New Look at Heuristic Teaching," The Mathematics Teacher, LXIV (October, 1971), 489.

successive variation in the problems they attempted will now be discussed.

Student 3 Problem 4

An estimated guess determined the position of the initial figure drawn inside the triangle. Then the placement of the figure was varied until a square was obtained. The student referred to this method as trial and error.

Student 6 Problem 4

A figure was drawn and then the placement of the lines was varied to attempt to form a square.

Student 7 Problem 8

One figure was drawn and then the sides and angles were varied until the figure was a trapezoid.

Student 10 Problem 8

A figure was drawn and the positions of the sides and the measures of the angles were varied until a solution was obtained.

Student 11 Problem 3(b)

An estimated guess of the amount of money Bill, Ron, and Ted originally had was made. The student then suggested that these values be adjusted until the correct answer was obtained.

Student 12 Problem 4

A figure was drawn and then the positions of the sides were moved until a square was constructed.

Student 17 Problem 6

The compass was set at different positions and two equal sides were drawn from the altitude to the base line. The student drew the triangle and then measured the perimeter. If the perimeter was less than 5, the student lengthened the sides of the triangle for the next trial. The student referred to the method as trial and error and said it would take forever.

Student 22 Problem 4

An attempt was made to draw a square in the triangle by varying the placement of the sides of the square.

Student 24 Problem 6

The student suggested that some triangles be drawn and then measured to see if they were correct. The student referred to this technique as trial and error.

Student 34 Problem 6

Different lengths for the sides of the triangle were drawn and measurements were taken to determine if the perimeter was 5 inches.

Student 36 Problem 4

An approximate figure was drawn. Then the positions of the sides were varied until a perfect square was obtained. The student called this method trial and error.

TABLE I

THE HEURISTICS USED FOR EACH PROBLEM

Problem	Number of Students Presented the Problem	Number of Students Using the Heuristics								
		Cases	Deduction	Inverse Deduction	Invariation	Analogy	Preservation of Rules	Variation	Extension	Successive Variation
1	9	5			3					
2	9		7						1	
3(a)	3			1						
3(b)	9		1	1						1
4	10	3						1	3	5
5	9	3				6				
6	8	1						1		3
7	8					1	2			
8	8	6				1		2		2
9(a)	3									
9(b)	8								3	

CHAPTER V

SUMMARY AND CONCLUSIONS

I. SUMMARY

The purpose of this normative survey was to determine if students used any of the nine heuristics namely, cases, variation, preservation of rules, deduction, inverse deduction, invariation, extension, analogy, and symmetry, when they attempted to solve mathematical problems that could be solved by using at least one of these heuristics. Each of the forty-two grade eleven students that was interviewed was given two mathematical problems to solve. Although the students were allowed to use pencil and paper, the researcher requested the students to think aloud.

The investigation found evidence that eight of the nine heuristics namely, cases, deduction, inverse deduction, invariation, analogy, preservation of rules, extension, and variation, were used. Eighteen students used cases, seven used deduction, three used invariation, two used inverse deduction, seven used analogy, two used preservation of rules, three used variation, and seven used extension. The investigator also found evidence that a heuristic which was not mentioned previously in the study was used by eleven of the students. For the purposes of this investigation the heuristic was called "successive variation." When the heuristic of successive variation is used a probable solution to the

given problem is chosen at random. If the answer is not correct, the student determines what changes must be made. Then the probable solution is varied successively until the correct answer is found. The students' command of the heuristics was not developed and therefore they could not use these techniques efficiently and effectively to solve the problems they were given. The investigator felt that in most cases the students used the heuristics in a very elementary manner.

Need for Further Study. The following problems could be investigated,

1. Do students at different grade levels show evidence of using heuristics when solving mathematical problems?
2. Do students who achieve C or lower letter grades in mathematics use heuristics?
3. Can students be taught to use heuristics?
4. Are any other heuristics used by students?
5. Does computer programming aid the student to efficiently use heuristics?
6. Does flow charting help the student to use heuristic methods more effectively?

II. CONCLUSIONS

This investigation found evidence that most of the grade eleven algebra students interviewed, used heuristics when solving mathematical problems. Nine heuristics were

identified namely, cases, deduction, inverse deduction, analogy, preservation of rules, variation, extension, invariation, and successive variation. The students' use of these heuristics was generally very poorly developed. The knowledge of a particular heuristic usually was of little help to the student. However, the knowledge of a heuristic enabled the student to make an effective attempt to solve the problem even though the student usually could not easily and correctly find the solution to the problem. The student who did not show the use of any heuristics usually was unable to make a constructive attempt to find the solution to the problem.

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APPENDICES

APPENDIX A

STUDENT INFORMATION FORM

Date _____

Name _____

School _____

Grade _____

Birthdate _____ _____ _____
Month Day Year

Mathematics Letter Grade _____ _____
Christmas Easter

Mathematics Teacher _____

Do you plan to take Math 12? Yes ___ No ___

Do you plan to take mathematics either in grade 13 or in first year at a junior college or university?

Yes ___ No ___

Try your best to solve the following two problems, using any method you wish. You may use pencil and paper, but please think aloud as you work. Be concerned with telling all your thoughts. Don't worry about the accuracy of your answer. If the method that you first choose does not solve the problem, you are encouraged to use as many other methods as you can to attempt to solve the problem.

APPENDIX B

INTERVIEW FORMAT

My name is Laurie Dinsmore. I'm attending U.B.C. where I'm completing my M.A. in Mathematics Education. I already have a B.Sc. in Mathematics and Chemistry. I've been interviewing some grade 11 math students, those who make good marks like yourself, to find out how students solve particular math problems that I give them. I'm interested in how you solve mathematical problems. I'm interested in your thoughts as you attempt to solve a problem and not in the final answer. I want you to think out loud as you work. You may use pencil and paper but always remember to tell me your thoughts. Just as a practice in thinking out loud tell me how you would attempt to solve this problem. Try to tell me all your thoughts. I may ask you some questions while you work. I'm interested in how you try to solve the problem and not in the accuracy of your solution.

The student was given one of the following three problems to solve.

(a) 215 tickets were sold for a concert. Adult tickets were 75¢ each and children's tickets were 35¢ each. The sale of both kinds of tickets amounted to $135.25. How

many adult tickets were sold?[1]

(b) How long is a rectangular lot if its length is 10 feet longer than its width and its perimeter is 52 feet?[2]

(c) The sum of three consecutive odd integers is 273. What are the integers?[3]

The student was given encouragement and training in thinking aloud, according to the individual students, in order to develop his ability for telling all his thoughts. The student was then given the form shown in Appendix A. The student was given the two problems, one at a time.

No assistance was given unless the investigator felt that the requested information would not help the student to plan his solution. Questions such as; What are you thinking now?, Any reason for that choice?, Finding anything?, and What did you do there? were frequently asked. An attempt was made to get all the student's thoughts, including his reasons for certain steps or methods of solution. It was considered important to make certain the student understood the problem without giving him any clues as to how he might solve it. The students were not told if a particular step was correct,

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[1]Eugene D. Nichols, Wagner G. Collins, and Eric D. MacPherson, Modern Elementary Algebra (Toronto: Holt, Rinehart and Winston of Canada, Limited, 1961), p. 194.

[2]Ibid., p. 197.

[3]Ibid., p. 201.

but were encouraged to keep going until they either were satisfied that they had a solution or until they gave up.

APPENDIX C

SOLUTIONS TO THE PROBLEMS USED IN THE INVESTIGATION

APPENDIX C

SOLUTIONS TO THE PROBLEMS USED IN THE INVESTIGATION

1. Cases.

Find all real numbers x which satisfy the following equation[1]

$$|x + 1| - |x| + 3|x - 1| - 2|x - 2| = x + 2$$

Suppose that x is a real number satisfying the equation. Consider the five cases:

Case 1 Let $x \geq 2$

then

$$|x + 1| - |x| + 3|x - 1| - 2|x - 2| =$$
$$x + 1 - x + 3(x - 1) - 2(x - 2) =$$
$$x + 2$$

therefore $x + 2 = x + 2$ and the equation holds for all $x \geq 2$

Cases 2 Let $1 \leq x < 2$

then

$$|x + 1| - |x| + 3|x - 1| - 2|x - 2| =$$
$$x + 1 - x + 3(x - 1) - 2(2 - x) =$$
$$5x - 6$$

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[1]Steven J. Bryant, George E. Graham, and Kenneth G. Wiley, Nonroutine Problems in Algebra Geometry and Trigonometry (New York: McGraw-Hill Book Company, 1965), p. 5.

therefore $5x - 6 = x + 2$

$4x = 8$

$x = 2$ so there is <u>no</u> solution

Case 3 Let $0 \leqslant x < 1$

then

$$|x + 1| - |x| + 3|x - 1| - 2|x - 2| =$$

$$x + 1 - x + 3(1 - x) - 2(2 - x) =$$

$$-x$$

therefore $-x = x + 2$

$x = -1$ so there is <u>no</u> solution

Case 4 Let $-1 \leqslant x < 0$

then

$$|x + 1| - |x| + 3|x - 1| - 2|x - 2| =$$

$$x + 1 + x + 3(1 - x) - 2(2 - x) =$$

$$x$$

therefore $x = x + 2$ so there is <u>no</u> solution

Case 5 Let $x < -1$

then

$$|x + 1| - |x| + 3|x - 1| - 2|x - 2| =$$

$$-x - 1 + x + 3(1 - x) - 2(2 - x) =$$

$$-x - 2$$

therefore $-x - 2 = x + 2$

$-2x = 4$

$x = -2$ is the only solution[2]

Thus the solutions are $x \geqslant 2$ or $x = -2$

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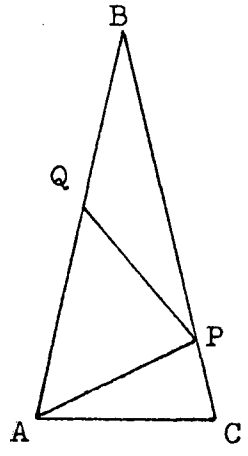
[2] <u>Ibid</u>., p. 33.

2. Deduction.

Triangle ABC is isosceles (the two sides AB and BC have the same length) with base AC. Point P is in CB and point Q is in AB such that AC = AP = PQ = QB. What is the number of degrees in angle B?[3]



Statement	Reason
Let m_B be the magnitude of angle	
1. angle QPB = m	1. triangle QBP is isosceles with side QP equal to side QB
2. angle BQP = 180° - 2m	2. sum of the angles in a triangle is 180° 180° = angle B + angle QPB + angle BQP = m + m + angle BQP
3. angle AQP = 2m	3. angle AQB = angle AQP + angle PQB = 180° angle AQB = angle AQP + 180° - 2m = 180°

[3]Charles T. Salkind, The MAA Problem Book II (New York: Random House, Inc., 1966), p. 13.

Statement	Reason
4. angle QAP = 2m	4. triangle QAP is isosceles with side AP equal to side QP
5. angle QPA = 180° - 4m	5. sum of the angles in triangle QAP = 180 = angle AQP + angle QAP + angle QPA = 2m + 2m + angle QPA
6. angle APC = 3m	6. angle BPC = 180° = angle QPB + angle QPA + angle APC = m + (180° - 4m) + angle APC
7. angle C = 3m	7. triangle APC is isosceles with side AP = side AC
8. angle A = 3m	8. triangle ABC is isosceles with side AB = side BC
9. angle B = $25\ 5/7^\circ$	9. sum of the angles of triangle ABC = 180° = angle A + angle B + angle C = 3m + 3m + m

The heuristic of hypothetical deduction is used. If the subproblem of finding the base angles of the isosceles triangle ABC can be solved, then angle B can be found from the relation, the sum of the angles of the triangle is 180°. The solution of the problem involves finding the magnitude of successive angles, in terms of the unknown angle B, until the magnitude of the base angle is found.[4]

3. <u>Inverse Deduction</u>.

Bill, Ron, and Ted each have some money of their own but they do not all have the same amount of money. Now,

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[4]<u>Ibid</u>., p. 53.

Bill decides to give some of his money to Ron and some of his money to Ted. He gives Ron as much money as Ron already has and also he gives Ted as much money as Ted already has. After this Ron decides to give some of his money to Bill and to Ted. He gives each of them as much money as each of them now has. Then Ted decides that he will give Bill and Ron as much money as each of them has. Now Bill has 16¢ and Ron has 16¢ and Ted has 16¢. How much money did Bill originally have before he gave any of his money away?[5]

Working backwards from the last condition to the first, the following table can be derived.[6]

amounts in cents

	step 4	step 3	step 2	step 1
Bill	16	8	4	26
Ron	16	8	28	14
Ted	16	32	16	8

4. Invariation.

Using a compass and a straightedge construct a square inside the given triangle ABC such that two of the vertices of the square are on the base CB of the triangle and the other two vertices of the square are on the other two sides

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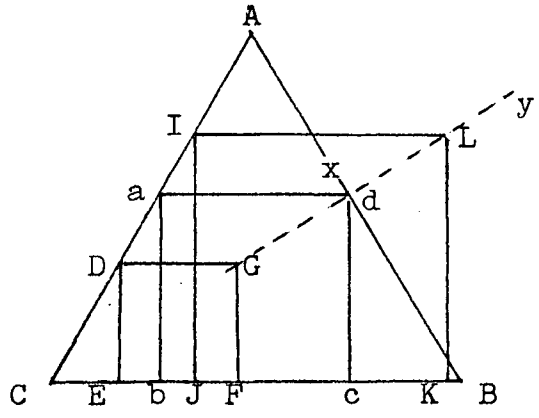
[5]Ibid., p. 26.

[6]Ibid., p. 75.

AC and AB of the triangle, one vertex on each side.[7]



Suppose only a part of the condition is retained. Draw a square DEFG with two vertices on the base of the given triangle and one of the other two vertices on one of the two sides of the triangle. Then draw square IJKL of different size with the same requirements. The locus of the fourth vertex of the square is a straight line. Draw the straight line y. Using the point of intersection x of this line with the side of the triangle, construct the square abcd with side the perpendicular distance from this point to the base of the triangle.[8]

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[7]George Polya, How to Solve It (second edition; New York: Doubleday and Company, Inc., 1957), p. 23.

[8]Ibid., p. 23-25.

5. Analogy.

Find the values of the five unknowns x, y, z, u, and t that satisfy the following set of five simultaneous linear equations.[9]

$$4y - 3x + 2u = 5$$
$$9x - 2z + u = 41$$
$$7y - 5z - t = 12$$
$$3y - 4u + 3t = 7$$
$$7z - 5u = 11$$

Consider the set of three simultaneous linear equations

$$2x + 4y + 5z = 49 \quad (1)$$
$$3x + 5y + 6z = 64 \quad (2)$$
$$4x + 3y + 4z = 55 \quad (3)$$

According to the algorithm, (a) eliminate the variable x using (i) equations (1) and (2) to obtain equation (4)

$$3 \times [2x + 4y + 5z = 49] \qquad 6x + 12y + 15z = 147$$
$$-2 \times [3x + 5y + 6z = 64] \qquad -6x - 10y - 12z = 128$$
$$2y + 3z = 19 \quad (4)$$

(ii) equations (1) and (3) to obtain equation (5)

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[9]Reverend Alfred Wrigley, A Collection of Examples and Problems in Pure and Mixed Mathematics with Answers and Occasional Hints (London: Longmans, Green, and Company, 1875), p. 38.

$4 \times [2x + 4y + 5z = 49]$ $\quad 8x + 16y + 20z = 196$

$-2 \times [4x + 3y + 4z = 55]$ $\quad -8x - 6y - 8z = -110$

$10y + 12z = 86 \quad (5)$

(b) eliminate the variable y using equations (4) and (5)

$-5 \times [2y + 3z = 19]$ $\quad -10y - 15z = -95 \quad -3z = -9 \quad z = 3$

$10y + 12z = 86 \qquad 10y + 12z = 86$

(c) solve for y using equation (4)

$2y + 3(3) = 19 \quad 2y + 9 = 19 \quad 2y = 10 \quad y = 5$

(d) solve for x using equation (1)

$2x + 4(5) + 5(3) = 49 \quad 2x + 20 + 15 = 49 \quad 2x = 14 \quad x = 7$

Now to consider the problem of five simultaneous linear equations

$9x - 2z + u = 41 \quad (1)$

$7y - 5z - t = 12 \quad (2)$

$4y - 3x + 2u = 5 \quad (3)$

$3y - 4u + 3t = 7 \quad (4)$

$7z - 5u = 11 \quad (5)$

Using the algorithm for three simultaneous linear equations, the variables x, y, z, and t could be eliminated in the five simultaneous equation case. Applying the algorithm analogously

(a) eliminate the variable x using equations (1) and (3) to obtain equation (6)

$9x - 2z + u = 41$ $\qquad$ $9x - 2z + u = 41$

$-3 \times [4y - 3x + 2u = 5]$ $\qquad$ $-9x + 12y + 6u = 15$

$$-2z + 12y + 7u = 56 \quad (6)$$

(b) eliminate the variable y using

(i) equations (6) and (2) to obtain equation (7)

$7 \times [-2z + 12y + 7u = 56]$ $\qquad$ $84y - 14z + 49u = 392$

$-12 \times [7y - 5z - t = 12]$ $\qquad$ $-84y + 60z + 12t = -144$

$$46z + 49u + 12t = 248 \quad (7)$$

(ii) equations (6) and (4) to obtain equation (8)

$-2z + 12y + 7u = 56$ $\qquad$ $-2z + 12y + 7u = 56$

$-4 \times [3y - 4u + 3t = 7]$ $\qquad$ $-12t - 12y + 16u = -28$

$$-2z - 12t + 23u = 28 \quad (8)$$

(c) eliminate the variable z using

(i) equations (7) and (8) to obtain equation (9)

$46z + 49u + 12t = 248$ $\qquad$ $46z + 49u + 12t = 248$

$23 \times [-2z - 12t + 23u = 28]$ $\qquad$ $-46z + 529u - 276t = 644$

$$578u - 264t = 892 \quad (9)$$

(ii) equations (7) and (5) to obtain equation (10)

$7 \times [46z + 49u + 12t = 248]$ $\qquad$ $322z + 343u + 84t = 1736$

$-46 \times [7z - 5u = 11]$ $\qquad$ $-322z + 230u = -506$

$$573u + 84t = 1230 \quad (10)$$

(d) eliminate the variable t using equations (9) and (10)

$7 \times [578u - 264t = 892]$ $\qquad$ $4046u - 1848t = 6244$

$22 \times [573u + 84t = 1230]$ $\qquad$ $12606u + 1848t = 1230$

$$16652u = 33304 \qquad u = 2$$

(e) solve for t using equation (9)

$578(2) - 264t = 892 \quad 1156 - 264t = 892 \quad -264t = -264 \quad t = 1$

(f) solve for z using equation (7)

$46z + 49(2) + 12(1) = 248 \quad 46z + 98 + 12 = 248 \quad 46z = 1138$

$z = 3$

(g) solve for y using equation (6)

$-2(3) + 12y + 7(2) = 56 \quad -6 + 12y + 14 = 56 \quad 12y = 48$

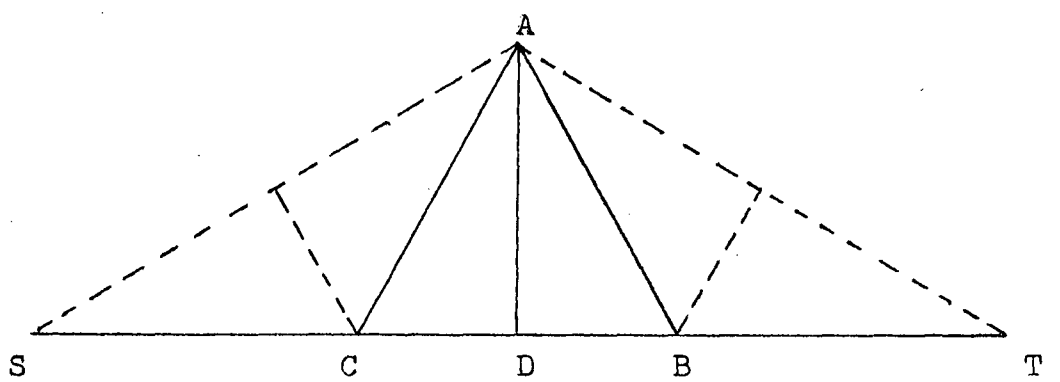
$y = 4$

(h) solve for x using equation (3)

$4(4) - 3x + 2(2) = 5 \quad 16 - 3x + 4 = 5 \quad -3x = -15 \quad x = 5$

6. Symmetry.

Using a compass and a straightedge construct an isosceles triangle (two of the sides have the same length) given the altitude (the perpendicular distance from the vertex to the side opposite) equal to $1\frac{1}{2}$ inches and the perimeter equal to 5 inches.[10]



[10]Edmund C. Plant, Geometrical Drawing (London: MacMillan and Company, Limited, 1899), p. 103.

Draw any isosceles triangle ABC with altitude AD equal to $1\frac{1}{2}$ inches. Since AC = AB and angle C = angle B, the triangle is symmetric with respect to the altitude AD. Therefore triangle ACD is congruent to triangle ADB and DC = DB. Suppose that this is the correct triangle. In order to preserve the symmetry, the perimeter (length ST) could be introduced into the picture along CB so that SC = AC and BT = AB. Noticing that triangle SAC and triangle ABT are isosceles, the altitudes from C and B would bisect SA and AT respectively. Therefore the construction of triangle ABC could now proceed as follows.

Draw ST equal to 5 inches. Bisect ST by AD. Draw AD perpendicular to ST and equal to $1\frac{1}{2}$ inches. Join AS and AT. Bisect SA and AT by lines perpendicular to SA and AT which meet ST in C and B respectively. Join A and C and A and B. ABC is the triangle required.[11]

7. <u>Preservation of Rules</u>.

Using the following mathematical system containing six dabas d_1, d_2, d_3, d_4, d_5, and d_6 and the definition "an aba is a set containing three dabas" (i.e. $\{d_1, d_2, d_3\}$ and $\{d_2, d_4, d_6\}$ are abas),

(i) List all the abas

(ii) Decide if the following statements are true or false

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[11]Edmund C. Plant, <u>loc</u>. <u>cit</u>.

Statement 1 If p and q are two dabas, then there exists one and only one aba containing both p and q.

Statement 2 If L is an aba, there exists a daba not in L.

Statement 3 If L is an aba, and p is a daba not in L, then there exists one and only one aba containing p and not containing any daba that is in L.[12]

Preserving the rules using the definition and the elements (dabas) in this system results in the following solution.

abas

$a_1 = \{d_1, d_2, d_3\}$	$a_2 = \{d_1, d_3, d_4\}$	$a_3 = \{d_1, d_4, d_5\}$
$a_4 = \{d_1, d_5, d_6\}$	$a_5 = \{d_1, d_2, d_4\}$	$a_6 = \{d_1, d_2, d_5\}$
$a_7 = \{d_1, d_2, d_6\}$	$a_8 = \{d_1, d_3, d_5\}$	$a_9 = \{d_1, d_3, d_6\}$
$a_{10} = \{d_1, d_4, d_6\}$	$a_{11} = \{d_2, d_3, d_4\}$	$a_{12} = \{d_2, d_3, d_5\}$
$a_{13} = \{d_2, d_3, d_6\}$	$a_{14} = \{d_2, d_4, d_5\}$	$a_{15} = \{d_2, d_4, d_6\}$
$a_{16} = \{d_2, d_5, d_6\}$	$a_{17} = \{d_3, d_4, d_5\}$	$a_{18} = \{d_3, d_4, d_6\}$
$a_{19} = \{d_3, d_5, d_6\}$	$a_{20} = \{d_4, d_5, d_6\}$	

Statement 1 False

If p and q are two dabas, then there exists more than one aba containing both p and q.

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[12]Betty Plunkett, "Aba Daba Daba," The Mathematics Teacher, LIX (March, 1966), 236-237, citing Howard Eves, and C.V. Newsom. An Introduction to the Foundations and Fundamental Concepts of Mathematics (New York: Holt, Rinehart and Winston, 1964), p. 80.

Let $p = d_1$ and $q = d_2$

then $a_1 = \{d_1, d_2, d_3\}$

$a_5 = \{d_1, d_2, d_4\}$

$a_6 = \{d_1, d_2, d_5\}$

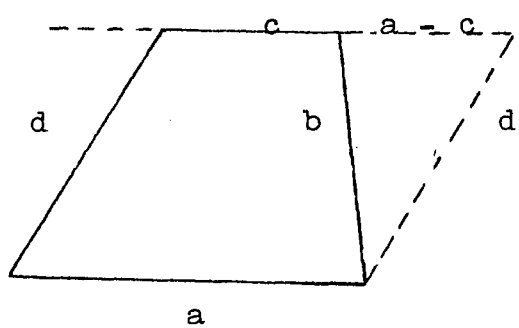
$a_7 = \{d_1, d_2, d_6\}$

Statement 2 True

Statement 3 True[13]

8. Variation.

Construct a trapezoid (a closed plane figure with four sides, two of which are parallel) being given the lengths a, b, c, and d of its four sides.[14]



Let a be the base and c the top with a and c parallel but unequal. The other two sides b and d are not parallel. Suppose we vary c. What happens when c increases until it becomes equal to a? The trapezoid becomes a parallelogram. The triangle that has now been added has sides b, d, and

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[13] Ibid., p. 237.

[14] Polya, op. cit., p. 211.

a - c. Since we have the lengths a, b, c, and d we can construct the triangle and then by completing the parallelogram the original problem can be solved.[15]

9. Extension.

$$1 + 8 + 27 + 64 = 100$$

$$1^3 + 8 + 27 + 64 = 100$$ What can you discover?[16]

The first observation is,

$$1^3 + 2^3 + 3^3 + 4^3 = 10^2 = (1 + 2 + 3 + 4)^2$$

Now the question can be asked, "Does it happen that the sum of other consecutive cubes is a square?" The sum of the consecutive numbers can be extended to,

$$1^3 + 2^3 + 3^3 + 4^3 + 5^3 = 1 + 8 + 27 + 64 + 125$$
$$= 225 = 15^2 + (1 + 2 + 3 + 4 + 5)^2$$

Then this can be continued to suggest that,[17]

$$1^3 + 2^3 + 3^3 + \ldots + n^3 + (1 + 2 + 3 + \ldots + n)^2$$

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[15] Ibid., p. 211-212.

[16] Ibid., p. 108.

[17] Ibid., p. 108.

APPENDIX D

SAMPLE INTERVIEWS AND ANALYSES

Heuristic	Interview	Problem	Student	Page Interview	Page Analysis
Cases	1	1	2	96	99
	2	1	17	100	102
Deduction	3	2	1	103	105
	4	2	33	106	108
Inverse Deduction	5	3	2	109	113
	6	3	11	114	118
Analogy	7	5	16	119	122
	8	5	38	123	124
Preservation of Rules	9	7	26	125	128
	10	7	28	129	130
Variation	11	8	23	131	134
	12	8	29	135	138
Extension	13	9	33	139	141
	14	9	41	142	144
Successive Variation	15	4	3	145	146
	16	4	22	147	148

Interview 1
Problem 1
Student 2
Heuristic:Cases

Student Well first I'm trying to just get the overall picture. I think I'll take them like out of the absolute value things and just try and work it out like that.

Interviewer Any reason for that choice?

Student Well you can't really work with it when it's in the absolute value things, so I'll write it down. Well now I think I'll just try and solve for x and see what happens. Boy I've got a problem; have 0 = -2 so.

Interviewer So what are you thinking about now? What did you think when you saw that?

Student Well that can't be. You can't have two numbers equal. So it could mean that there's no solution or else I figured it out wrong.

Interviewer Well you have a choice now. Do you think there's no solution or do you think you, do you think you figured it out wrong?

Student I don't know I probably figured it out wrong. You know there's got to be probably a solution.

Interviewer Why would there have to be a solution?

Student I don't know.

Interviewer What are you thinking about now? What are you looking at now?

Student Oh what I'm just trying to figure it out if there's any way, how the absolute value would affect it. You know to see if there's any way we could get around it.

Interviewer What about this absolute value, what, what's it doing or what are you thinking when you said you said you were trying to get around the absolute value?

Student Well if the absolute value wasn't there, well then there'd be no, x would not equal, anything you can have it unless it's imaginary number or something. But with the absolute value there it could be either two different numbers, so don't know how to do that.

Interviewer Is there any other, do you think you could try it any other way? Do you have any other any other ideas?

Student Oh I don't know. I suppose you could try substituting a number for x, you know if you want to just guess or something.

Interviewer Well you can try anything you want. That's what I'd like to, you know, see what you want to try.

Student I think I'll just try substituting in just to see if anything will work. I think it's probably the null set or something.

Interviewer You can, you know, whenever you figure you're finished, but if you want to keep on at that that's fine. So you tried putting zero in and that didn't work for you?

Student No, it didn't.

Interviewer You just, what are you doing now? Putting other numbers in, or or what?

Student No, I'm just kind of thinking if I could somehow get this equal, the equation, to see if it will correspond to x + 2.

Interviewer What do you mean by making the equation equal?

Student Oh I don't know exactly. Just like say this was x like here, just saying let's say -x was and just trying that out.

Interviewer So you put a -x in there for the first one. What are you going to do when you come to that that next absolute value one? You're putting, you're leaving that unchanged now?

Interview 1 (continued)

Student Cause if it was negative then the opposite of it would be still positive. It works out that way you get $x + 2 = x + 2$.

Interviewer And what does that say to you?

Student That if x was positive it wouldn't work out. If it was negative maybe it would. So I think I'll try a negative number and see if it works out.

Interviewer Okay.

Student Yes it works out that way. So I I think I could assume that all negative numbers would probably work.

Analysis of Interview 1
Problem 1
Student 2

Cases

"you could try substituting a number for x . . . I'll just try substituting in just to see if anything will work . . . I'll try a negative number and see if it works out."

The student tried 0, -x, and -2.

Interview 2
Problem 1
Student 17
Heuristic: Cases

Student First I'd, can I write this down?

Interviewer Yes.

Student Okay first I'd multiply out the brackets, what's in front of them.

Interviewer Those are absolute values, okay?

Student Oh oh start over. Well then first of all I'd add add the x's together. x - x no that doesn't look right.

Interviewer Any reason why you thought that wouldn't work?

Student Let's see. I have to make it equal. There's two values for the absolute value.

Interviewer Okay. What do you mean by two values for absolute value?

Student Well one x could equal negative and this absolute value could, makes it positive and then x could be positive and still might get the same answer. And let's see, let's see remove the absolute value and find x. Let's see now, I've, add the x's up 1 + 3 is 4 - 3 is 1 + 5 = x + 2 and now I'd bring the x's over to this side and then I get zero equals -3. Oh that didn't work.

Interviewer Well. What do you think now? Is there any reason why you just left out the absolute values there and solved for x?

Student No. I thought I was going to make, find one way, find x first whatever it came out to be. Like maybe if x was, equalled -3, then I could substitute it in there and see if it was true, and then just try the positive value of it.

Interview 2 (continued)

Interviewer Oh I see.

Student But it didn't work.

Interviewer Any other ideas?

Student Oh I don't know.

Interviewer Think you could do anything else with it?

Student Oh I can't even remember what I did with, used to do, cause I haven't done these for so long.

Interviewer You used to do things like that?

Student Yes, about three grades ago.

Interviewer Is that what you're trying to do now, remember how you did them?

Student Yes. I'm not sure. Would you multiply? If I'd multiply a negative, the things in front of these absolute values, if you'd multiply three times x or just leave it.

Interviewer Well you can try whatever you'd, you know, whatever you'd like to try you can try it and see what happens. I'm just interested in how many ideas you have. You can, you know, when you're ready, when you run out of ideas of course you can stop, but.

Student Let's see.

Interviewer Thinking of anything now?

Student No I don't think so. I can't think of what I'd do.

Interviewer Do you want to try another problem or do you think you might have another idea?

Student Yes. I don't think I have another one.

Analysis of Interview 2
Problem 1
Student 17

Cases

"there's two values for the absolute value . . . x could equal negative . . . and then x could be positive . . . I thought I was going to make, find one way, find x first whatever it came out to be. Like maybe if x was, equalled -3, then I could substitute it in there and see if it was true, and then just try the positive value of it."

Interview 3
Problem 2
Student 1
Heuristic:Deduction

Student Well these, this is an isosceles according to this and so is this. So all isosceles triangles. This is in geometry in grade 10. So AC = AP. Do you mind if I mark this?

Interviewer No I don't mind.

Student So I can find out what this is like. So now if I can figure out which angles are equal. Oh these if it's isosceles the angles should be equal, is that right?

Interviewer What angles?

Student Well the ones at the bottom. So that should equal to that and this should be equal to that and this should be equal to that and this should be equal to that and this should be equal to that. So that should be 180 divided by three. Sixty degrees?

Interviewer That's your, that's good. That's your solution?

Student I think so, yes. Let me check that over again for mistakes. Angle B. So these two will have to be equal. Now that has to be equal to that. No these two won't necessarily have to be equal. I don't, just trying to figure out which one that can be equal to.

Interviewer What are you thinking now?

Student I'm trying to figure out which angle will relate to this somehow. It won't be that, it's not parallel or anything. And that's definitely out. If I put a line there.

Interviewer So what are you saying now?

Student It's not 60 degrees like I thought. These two are equal and that's equal. I'll just confuse it some more if I bring the line out across P. It'll just confuse it some more I think so.

Interview 3 (continued)

Interviewer Tell me all the things that you're thinking about doing. Like you just told me now you, you know, you were thinking maybe you'd bring that line out but that you didn't didn't feel that was going to work. Tell me all these little things that you're trying.

Student Well I'm just generally looking it over and trying to relate everything together.

Interviewer Anything you've been looking for? You mentioned something awhile ago.

Student Oh, here's another, here's another isosceles triangle QPA, that I didn't see before. So this angle here should be equal to angle Q, if that's of any help. It's probably just more confusion in the end. It's probably just confusion. Let's see, still trying to relate everything together. This will be, what's that 180? Yes. This angle BQP will be P, will be 180 minus the other angle, if that's of any help. It's crazy. Sometimes at this stage of the game I take a wild guess, so. But I don't think I want to now.

Interviewer Well you can try as many different approaches as you, as you feel you can. And then if you want to quit, well.

Student Let's see. If that equal to that. A is equal to C and C will be equal to APC and that's fine and dandy to there. And QAP will be equal to AQP and QPB will be equal to QBP. Everything is equal, but it doesn't come out. I think I'm just wasting time.

Interviewer You've got no more ideas?

Student No, I'm afraid not.

Analysis of Interview 3
Problem 2
Student 1

Deduction

"if it's isosceles the angles should be equal . . . the ones at the bottom. So that should equal to that and this should be equal to that and this should be equal to that and this should be equal to that. So that should be 180 divided by three. Sixty degrees? . . . here's another isosceles triangle QPA . . . so this angle here should be equal to angle Q . . . angle BQP will be P, will be 180 minus the other angle."

Interview 4
Problem 2
Student 33
Heuristic: Deduction

Student Okay. Well first of all, by all those sides equal, these are all isosceles triangles. And so I haven't been given any numbers, so I'll just do it almost algebraically. Angle C. Let's see now. Yes angle C. Sorry, angle B = angle P, angle C = angle BAC and I can, I see that I can use some possibly supplementary angles. I know the angle sum of a triangle is 180 degrees and I could use supplementary angles to say that if B is equal to B. Two times angle B equals angle PQA because two angle B is going to be supplementary to that and that's supplementary to that, you know.

Interviewer Supplementary to the Q there, yes I see.

Student Okay angle PQA is equal to PAQ so. And angle, so those two angles. It will be angle C, 180 degrees minus angle C and angle CAQ. Or, sorry it'll be 180 minus two times angle CAQ.

Interviewer Well, could I ask you why for that?

Student Why? Cause I'm going to try and work, dividing CAQ into that angle plus that angle which will I hope isolate angle PAQ. And I've already got a relationship with angle B and PAQ and a I'm just starting to get one now, if I can just find it. Angle B = 180° - 2PAC - 2PAQ. That, so two times angle B = angle PAQ and you could say that angle B, 180° - 2PAC - 4angle B. Add 4angle B to both sides. 5B that equals 180°. That's 2PAC. Now I get that angle. I don't really think I, that's going to get me anywhere.

Interviewer Could I ask you what your reason was for the 2PAC there?

Student 2PAC was there from the time before. PAC plus PAQ is equal to the QAC and.

Interview 4 (continued)

Interviewer Okay. What, where did that 2PAC come from in the first step?

Student Oh, angle B was 180° - angle 2 - 2angle CAQ. Now if you break down angle 2CAQ you get two of these plus two of these, okay?

Interviewer I get it. Okay.

Student Now if I could get, try to work from PAC. Now 2PAC is going to equal PAC = 180° - 2C and 2PAC = 360° - 4C. Let's see if I've got angle C up here. Anyways, yes angle PAC. So 5B = 4C. I can substitute -4C for that. Now I get -180° - 4C or 5B + 180° = 4C, sorry, negative 4C which means that and also 180 degrees, 180 degrees minus 2C = B. So B = 180° - 2C. So I should be able to get it from this and I, I'm just going to add both equations to both sides. 6B + 180° = 180° negative 4C plus that. So it's negative 4C plus. So it's 180° - 6C, 6B, negative 6C which means that I've done something wrong with.

Interviewer What's your overall plan for solving it? Could you just explain you know, your?

Student My overall plan's right. You can get values for B by, in terms of other different angles, and you can by working it, by using equations, you can get a value for B in terms of other angles, you combine that and make it so that it could, and could be, it could be 30 degrees or it could be some other value. You get it in this. All these isosceles triangles make it quite easy to work it around so you can get it in terms of say you get angle B in terms of C in two different ways, like I did there, except I've done something wrong. So I think I was on the right track there.

Analysis of Interview 4
Problem 2
Student 33

Deduction

"I'll just do it almost algebraically . . . angle B = angle P, angle C = angle BAC . . . I can use some possibly supplementary angles. I know the angle sum of a triangle is 180 degrees and I could use supplementary angles to say that if B is equal to B. Two times angle B = angle PQA because two angle B is going to be supplementary to that and that's supplementary to that . . . angle PQA is equal to angle PAQ so and angle. So those two angles. It will be angle C, 180 degrees minus angle C and angle CAQ . . . you can get values for B by, in terms of other different angles, and you can by working it, by using equations, you can get a value for B in terms of other angles, and you could combine that and make it so that it could, and could be, it could be 30 degrees or it could be some other value. You get in this. All these isosceles triangles make it quite easy to work it around so you can get it in terms of, say you get angle B in terms of C in two different ways."

Interview 5
Problem 3
Student 2
Heuristic: Inverse Deduction

Student Well first of all I just read through the problem, just to get a general idea. Now I am going to try, you know, figure it out. Okay I'll say B has, let's say x cents. So so B has a certain amount of cents, call it x and A will give him x more cents so he will have 2x. Does this mean like? Okay. Let's say B doesn't have anything yet, but A gives him how much and that's how much he has, or does it mean he already has something?

Interviewer It means A has something and he gives something to B and to C. Then when that is done B gives a little bit to A and to C. So this is done one after the other. First of all A has something and he gives to B and C.

Student Do B and C have anything to begin with?

Interviewer Well I don't.

Student I guess I have to figure out. Okay.

Interviewer Finding anything?

Student No. I'm still trying to, you know, just think it out.

Interviewer Seeing anything in the problem? When you are trying to figure out, are you saying anything to yourself?

Student I, I'm just kind of reading the question over and just kind of concentrating on it, you know. I'm not doing anything. Well I think that seeing that A is giving each of them something I think he is giving B a certain amount and C a certain amount so he's just giving back what he got, sort of. So now I'm just going to try and figure out, you know, working with the 16, sort of, and just trying.

Interview 5 (continued)

Interviewer Working with the 16?

Student Yes. So to begin with A's going to have a certain amount that B has, plus a certain amount that C has. So we'll say that's C + B. That's the certain amount he has.

Interviewer What's going through that head now Theresa?

Student I don't know. I'm just trying to figure it out, get things together. It's kind of confused, you know.

Interviewer You're thinking about A now are you?

Student Yes.

Interviewer That B + C is, you're saying that, what's happening, he's getting some from B plus C?

Student Well if A's got B + C a certain amount of cents and he gives some to B and he gives some to C well I'm trying to figure out if he has anything left or does he still have some.

Interviewer What do you think? Does he have anything left at all?

Student Does this mean that each, both A, B, and C, all of them have the 16¢?

Interviewer Yes. See, you see it says if each finally has 16¢. So what are you thinking now?

Student I'm just trying to figure out, somehow make an equation and get.

Interviewer Got any new ideas, or are you just still trying to make an equation?

Student I'm still just trying to figure it out.

Interviewer Make an equation?

Interview 5 (continued)

Student Yes.

Interviewer See, you know, is there anything like, are you thinking of anything when you're trying to figure it out? Like, you know, is there anything going through your head? Like, you know, are you trying different things, or are you?

Student Yes. I'm just thinking of different things to do.

Interviewer Well tell me some of these different things you're thinking of.

Student Oh, okay.

Interviewer Cause I'd like to know.

Student Well first of all I'm thinking that maybe I, I can't just assume that A has nothing to start with, or that B and C don't. So I was thinking that A gives B, B will have B + x, he'll give him B and he will always, already have x and C will have C + y or something and.

Interviewer Well how, how is that coming? Are you getting, how are you coming along with that?

Student I'm thinking that if I did make an equation anyway. I'm trying, see I know, cause there's three different variables like A, B, and C. So can't just have one equation and have three variables. I've got something. Okay. So first of all A has a certain amount, A has x, and he gives B. A just has a certain amount then he gives B, B plus what he already has, say it's B + x, and he gives C, C + x. Then B, which has B + x, gives A, B + x, so A now will have the original amount plus what B gave him back. And then C is going to give A, which will be C + y, and so A will have the original amount plus B + x + C + y.

Interviewer And now what happens?

Interview 5 (continued)

Student Well since I can't work with three variables, I'm trying to figure out if I can get, maybe put one in terms of the others, something like that. So I think I, I'll just write down, you know, rather than thinking of something, write it down. Okay. Well say A has a certain amount plus what B gave him which would be B + x plus what C gave him, which is C + y. Just trying to work backwards. And so you end up with 16. Okay. And some is going to give to C, and some of that to B, and then he'll end up with the original amount.

Analysis of Interview 5
Problem 3
Student 2

Inverse Deduction

"just trying to work backwards. And so you end up with 16. Okay. And some is going to give to C, and some of that to B, and then he'll end up with the original amount."

Interview 6
Problem 3
Student 11
Heuristic: Inverse Deduction

Student Lookee, it's kind of long. Alright. Well I think I'll start at the beginning and work down. They do not have the same amount of money. Well, there's three amounts of money, I know that. I think I'll probably give them variables, just for the heck of it.

Interviewer So you have x, y, and z.

Student Yes. He gives Ron as much money as Ron already has. So that would mean Ron is going to have twice as much as he has. So that's be 2y. And this would be x - y. And be the same for Ted. 2x, 2z, and y - z. There still seems funny.

Interviewer Now what are you thinking at the moment?

Student Well he has this much. Ron has 2y and he gives him twice this and him twice this and then it works out the same thing over again. He gives twice this and twice this again for, and he would have subtracted from here 2z and. This is going to make a lot of nothing once I get there, but x - y - z, a lot of variables. And this would be -2y - 2x - 5z. There I'm done.

Interviewer Now, what are you going to do next?

Student Now finally Bill has 16, Ron has 16, and Ted has 16. Okay. I'd like to change it into one variable, if I could. Oh, they all have 16. Okay. Okay, 16. Before he gave away any of his money, did Bill originally have.

Interviewer What are you thinking now?

Student Well I don't know. I just, I can't work three variables, I know that. I can work two variables, but a, but I'm just trying to figure out how to get rid of some of them.

Interviewer Have you got any ideas there?

Student There must be some way to connect the, the amounts they have by a an addition or something. I don't know what. So he has 16 now.

Interviewer Now, what's that idea coming?

Student Well, Bill has 16 now, right at the end, and a of this 16, part of it, a well you can find out how much he had after he gave to Ron and Ted because he has it doubled over twice. Like a Ron and, or a, yes Ron gave him twice as much as he had, or the same amount over, and Ted gave him the same amount over.

Interviewer Now, where is this you're getting that from?

Student From the second thing here.

Interviewer The second sentence?

Student Or this, or Ron decides to give some of his money to Bill and to Ted. He gives them as much money as each of them now has. So Bill has x and he is given twice that amount. And then Ted gives him twice that which would be 4x and that should be equal to 16. What he has now and that should work out. I hope, I don't think it will though. I think I did something wrong. No. It should be 2x times 2 which would be 4x = 16 which would be 4.

Interviewer Could you explain that, a 2 times the 2x there?

Student Well he had, he has x amount of money after he gives the money to Ron and Ted. And he has this x doubled by Ron, which is 2x = 16, the number we are looking, working with. And then he has it doubled again. So it would be 2 times the 2x = 16 he now has. Which works out to 4. So he had 4 after he gave the money to Ron and Ted. And that should work out for the other two, I think. One kind of equation like that.

Interviewer Well, what do you mean by working out for the other two?

Student Taking their, the number they now have, and working with that.

Interview 6 (continued)

Interviewer Oh yes. And then you would know then what each of them had at the beginning? Is that what you're doing or? I'm not sure.

Student If, eventually I should be able to find out.

Interviewer Oh. Well what is that 4 then?

Student The 4 is the number that Bill had after he gave his money away and so.

Interviewer Well where in the problem would that be?

Student Money to Ted.

Interviewer So it's right there in that sentence, now Bill decides to give some of his money.

Student He gives this to Ron and he gives this to Ted.

Interviewer So after he gives it to Ron and to Ted he has?

Student 4 cents.

Interviewer 4 cents. Now where would you go from there Marie?

Student Possibly if I could find out how much Ron and Ted had after a, oh, or a. Like, well 16 does me no good, that Ted and Ron have right now, because it's been doubled over and double over. And, and what I want to do is find a number closer to the original number they had. So I could find out how much Bill gave Ron and Ted, and add it back to what Bill had.

Interviewer Well how might you find a closer number to there?

Student Well that's where I'm just looking at. So Ron ends up with 16. They all end up with the same amount. Everybody has their a money four times over I think, like.

Interviewer So what could you do with that? What would be, you be thinking to do with that?

Interview 6 (continued)

Student Well Bill would have to have more money than anyone to start with. And there would have to be some kind of a, I don't know how to say this, a divisible [sic] of 4, more than likely, or they all are, as it ends out. And a I was just wondering if it, a if I could work it out without an equation. I mean not really without an equation, but a that it might work out just by a guess, you know. I mean like an estimated guess of how much more Bill would have in the beginning, to have to give that much to Ron and Ted.

Interviewer And then what would you do with your estimated guess?

Student I'd see if it works. Like take it through the equation, the whole problem, and see if it worked. And I would be tempted to give Bill 32 and Ron 8 and Ted 8 for the first, you know. Which would mean, wouldn't work, I know now.

Interviewer Okay. Suppose it didn't work, what would you do?

Student I know that Bill would have to have less than 32 right now, because the way it works now, if he gave 8 to Ted and 8 to Ron he would have 16 left and he's supposed only, will have 4 left. I know that. So I can bring it down to a possibly a divisible [sic] of 12, which is a good round number, maybe 24.

Analysis of Interview 6
Problem 3
Student 11

Inverse Deduction

"what I want to do is find a number closer to the original number they had. So I could find out how much Bill gave Ron and Ted, and add it back to what Bill had."

Interview 7
Problem 5
Student 16
Heuristic:Analogy

Student Well, this is like five separate questions, is it?

Interviewer No. Five equations all together. That's why we call them simultaneous.

Student Yes. So find these five values that will satisfy all these equations?

Interviewer Right.

Student Oh very well. I'll solve each one separately and then search for common values. I guess we used to do this also. We'll take 4y - 3x, the first one. We'll try to solve with, let me see now. Get both variables equal to 1, such as 4y a 4y = 5 + 3x - 2u. Take them over to this side and I got a value for 4y. And also 2u would equal 5 + 3x - 4y. Oh I'm still stuck with a y. This isn't going to work. That's not good enough. So maybe it will, equals 5 - 3x - 2u. I think I'm going to get myself mixed up here. The 2u is equal to this, which will all cancel now. There's some way, there must be some way to get rid of all these variables and be left with the one and solve for that. And then work it back to find a value for each of these.

Interviewer I see. Well just let me make you go on for just a second. Suppose you did that, with that one, then what would you do?

Student Okay. I wanted to solve here for 4y and find the value for 4y, substitute it in so that 3x has some value as 4y. There's some way that it equals 4y, some fraction perhaps. And 2u is also equal in some way to 4y which is equal to 5. It's all, all added, subtracted together to give you 5.

Interviewer I see. So you, you'd have one variable up there then?

Interview 7 (continued)

Student Yes. And then I already know what 4y is somehow related to 3x and relate that back to find the value for x. And do the same thing with u. And I've got there, I've got my one, two, and three values for this equation. Go on and do the same for each equation. And then given, then I'll, I'll just go through and pick the common values from each one. There will be several values from each one, there must be, I can see how there will be.

Interviewer Okay. So that's your, you're sure that would work and that's what you're going to do?

Student Yes I think I'd take a lot of time but it must work that way. I can solve for this one it's shorter. I'm not too sure how we'd go about three variables.

Interviewer Try that last one for me.

Student Okay, well there's got to be a better way here. Trust me to multiply here.

Interviewer You took the last one and solved for 7z, okay, for 7z and then what was that next step there?

Student Oh subtract 5. No. That would get rid of the whole variable. Oh, oh dear heavens. Well I'm solving for 7z. 7z will equal 11 plus 5u. That, that's just by, I forget what we call it now, taking, subtracting more or less 5u, or adding 5u on each side. That will cancel that off. $7z = 11 + 5u$, substitute this back into this equation. But then again I need another equation. You need two equations. That, that's what I'm, that's where I'm thinking of. You need two equations to solve this and I've only got the one.

Interviewer Well how did you come up with the idea that you need two equations to solve it?

Interview 7 (continued)

Student Well given say x + y equals oh, z, some number. You're given two of these values. No, you're given the one value, this z perhaps. Given that x - z equals another value. Again say, well you'd find a value for x and you can substitute it into here, into this second equation. And solve it and you get a say that's what you're looking for. And you'd get a, by finding a value for this, for all these variables in this first equation. This one is going to have an algebraic, like value, is going to have some letter in it, some variable. Substitute it into here. So like in here it'll equal z - y. Substitute z - y into this here equation. So you get z - y equals a. I think that's how I go about it.

Interviewer Those are two x's and those are two z's are they, or is that y and z there?

Student No. I take this equation here and I get this one from it by subtracting y.

Interviewer Oh yes, your z is over there. Fine, yes.

Student Yes, and then I take this here, this value for x, and take this and put in.

Interviewer Put it in there? Yes.

Student Solve. This is z - z, they cancel. And then -y = a. And you've got y, that's a constant.

Interviewer You've done these type before?

Student Yes.

Interviewer When you had two of them?

Student Yes. Well I've never seen this three variables before, but I think that's how I'd go about it. Yes, I think so. But that's how I would go about it. I think I'd try to find a value for this variable and substitute it in, in each, and then find that value for the.

The interviewer failed to notice that the tape had run out as the student made her last comment. The student suggested using cases but said it would "take all day."

Analysis of Interview 7
Problem 5
Student 16

Analogy

"you need two equations to solve this . . . well given say x + y equals oh, z, some number. You're given two of these values. No, you're given the one value, this z perhaps. Given that x - z equals another value. Again say, well you'd find a value for x and you can substitute it into here, into this second equation. And solve it and you get a say that's what you're looking for. And you'd get a, by finding a value for this, for all these variables in this first equation. This one is going to have an algebraic, like value, is going to have some letter in it, some variable. Substitute it into here. So like in here it'll equal z - y substitute z - y into this here equation. So you get z - y = a. I think that's how I go about it . . . Then I take this here, this value for x, and take this and put in . . . and solve. This is z - z, they cancel. And then -y = a. And you've got y, that's a constant . . . well I've never seen this three variables before, but I think that's how I'd go about it . . . I'd try to find a value for this variable and substitute it in, in each, and then find that value."

Interview 8
Problem 5
Student 38
Heuristic: Analogy

Student I'm thinking, like maybe if you make them, so that they're equal, and subtract them, so that you can cancel one of the, one of the unknowns out.

Interviewer Okay. Could you explain to me what you mean when you say getting them equal?

Student Well multiplying the equation so that they, like maybe multiply this one by 2, so that that would be 2y. And then you subtract them so that that would cancel out.

Interviewer Any particular reason why you decided that that might work?

Student It's the only thing I can think of that looks like this, that we've done.

Interviewer What kind of things did you do before?

Student What?

Interviewer Well you said it reminded you of something you'd done before. What kind of things had you done before?

Student Oh yes. Well working with two variables. I think I'd probably work with the bottom one cause it's just got the two. So take one of the other ones and cancel, so that you can cancel them out.

Interviewer So that's going to be your method of attack is it?

Student I'd try that first, yes.

Analysis of Interview 8
Problem 5
Student 38

Analogy

"I'm thinking, like maybe if you make them, so that they're equal, and subtract them so that you can cancel one of the, one of the unknowns out . . . it's the only thing I can think of that looks like this, that we've done . . . working with two variables. I think I'd probably work with the bottom one cause it's just got the two. So take one of the other ones and cancel, so that you can cancel them out."

Interview 9
Problem 7
Student 26
Heuristic: Preservation of Rules

Student List all the abas. Oh, for example this, do I have to list this and this?

Interviewer No.

Student d_2d_3 and d_4, $d_3d_4d_5$.

Interviewer Okay, I'll let you do it a short way. You can just write down the 2, 3, 4's, okay? So we can save time. Well I mean just so we can save time. The next one you write down, like just write down the numbers, okay? So we don't have to bother with the brackets. It just saves a little time.

Student Okay. 345, 456, 561, and 612, and 134, 154, 156, 234. I have that already 245, 246, 356, and. Oh can I go back again? I think so. 261, 263, 264, oh.

Interviewer Now what are you thinking there?

Student I was wondering if I can go back, you know. I was just a, at first I was just listing when I read it. But I'm, but then I was thinking if I can list it like 56 and 1. That means, that's a, order counts. Does it mean if I put it in another order, it's another aba or whether it?

Interviewer What do you think about sets? That's a set.

Student I think the order matters.

Interviewer Okay, you list them.

Student 3 let me see 356. Did I have 346? 346, 361, 362. Can I use the, like can I use 363, you know, the same number? No?

Interviewer Oh, not the same number twice. No.

Interview 9 (continued)

Student Not the same number. 364.

Interviewer So what you're doing now is, you're explain it to me, what you're doing now.

Student I'm just going over the number. Like, I have a system. Like, that I go 123 and the 124 and 125 and then like that and so I.

Interviewer Sure, okay. Now what are you doing down here? What's your system for getting these, these ones here, you're going back you said now?

Student Yes. I'm, if I start with 3, like I wrote, I wrote 345 already, so I skip this number and 346 and 341, 342. And then after I finished with the 4, I use 356 and 351, 352, and like that. And then afterwards, I use the 636 a 1, 362, 364.

Interviewer Have you done all the, yet, or?

Student I think so.

Interviewer Okay take a look at statement, the second part of the question there.

Student Dabas are these number. Only, one and only one aba. P and q are two dabas. No, I think it's false. Because, like for 1, 2, these are two dabas and if I want to put it into an aba I only have to add one more. And I can use 123, 124, 125.

Interviewer Good. What about statement two?

Student If L is an aba there exists a daba not in L. If L is an aba there exists a daba not in L. I don't quite get the question. There exists a daba not in L. Yes.

Interviewer Okay. I'd like you to.

Student Because if, if L is an aba that means there's only three in this case and there's six dabas here. And if there's only three, well, well an aba, you know, any of the others, not, could not be in the L.

Interview 9 (continued)

Interviewer Good reason. Fine. Take a look at three.

Student If L is an aba and p is a daba not in L, then there exists one and only one aba, aba containing p and not containing any daba that is in L. If L is an aba and p is a daba not in L, then there exists one and only one aba. Yes. I think it's possible. If, if L is only one an aba, because, and there's six numbers here. And for instance, if 123 is in L and 456 isn't, and well, for example, 6 is is p, then it can use 4 and 5 just to have another aba which is not consisted [contained] in L.

Analysis of Interview 9
Problem 7
Student 26

Preservation of Rules

Statement 1

"I think it's false because . . . if I want to put it into an aba I only have to add one more. And I can use 123, 124, 125."

Statement 2

"yes because if, if L is an aba that means there's only three in this case and there's six dabas here. And if there's only three, well, well an aba, you know, any any of the others, not, could not be in the L."

Statement 3

"I think it's possible. If, if L is only one an aba, because, and there's six numbers here. And for instance, if 123 is in L and 456 isn't, and well, for example, 6 is in p, then it can use 4 and 5 just to have another aba which is not consisted [contained] in L."

Interview 10
Problem 7
Student 28
Heuristic:Preservation of Rules

Student This may be a little primitive method, but I think I'd just start by using the first daba and then combining it with a the two others. And just keep going on like that. So it'd be $d_1d_2d_3$ and $d_1d_2d_4$, $d_1d_2d_5$ . . . 18. If p and q are two dabas I'd say the, oh, two. These things are dabas. Well the first one's false.

Interviewer Okay. You give me a reason why.

Student Well, just a, well just sort of the way I did it. Like a you have the d_1 and the d_2 in the first and a d_1 and d_2 in the second so that obviously comes out there being only one set that a contains the two of them.

Interviewer Good reason. Take a look at statement two for me.

Student A daba. Yes, because there can only be three sets of or three dabas in an aba and there are six dabas. So obviously one of them may not be in a set with the three of them.

Analysis of Interview 10
Problem 7
Student 28

Preservation of Rules

Statement 1

"well the first one's false . . . like you have the d_1 and the d_2 in the first and a d_1 and d_2 in the second so that obviously comes out there being only one set that contains the two of them."

Statement 2

"there can only be . . . three dabas in an aba and there are six dabas. So obviously one of them may not be in a set with the three of them."

Interview 11
Problem 8
Student 23
Heuristic: Variation

Student Construct a trapezoid? Just, just draw one? Is that the idea?

Interviewer Yes, and those are the lengths of the sides.

Student Those are the lengths?

Interviewer Do you know what a trapezoid looks like?

Student Yes. Well, so these are the sides you want me to use to?

Interviewer Yes

Student To use, oh I see.

Interviewer I want you to use those sides and make me a trapezoid.

Student First of all I'd find out which two were the same length, if any of them. What happens if none of them are the same length?

Interviewer Okay, then what happens?

Student You wouldn't have a trapezoid. Oh, just a minute, parallel to it, parallel. Oh, I'd take the two longest ones to a, to a make the two parallel sides. And make sure they were parallel by using my eyes. And then a, it would be hard to make this parallel.

Interviewer The two sides are parallel?

Student The two sides are parallel.

Interviewer Then you have the two sides that are parallel now?

Interview 11 (continued)

Student Yes. But if I have to use these exact lengths, it's not going to connect. So maybe have to use different, different a two to do it with. Because if I use those two, then I use these two for my ends. This one would be too short and it wouldn't, either one go and not the other. You want me to connect?

Interviewer Yes. The trapezoid has the two sides parallel and then the other two sides. It's closed, that's what I mean by closed.

Student Yes, closed. Well, it wouldn't be closed if I'd used the sides that I've used. I don't think it would be. No, it wouldn't be closed if I'd used the sides that I've used. I don't think it would be. No, it wouldn't.

Interviewer So, any ideas?

Student Well I could do it the easy way and add on.

Interviewer What's the easy way?

Student Well you could always cheat and make this one longer. But no, you've got to do it that way.

Interviewer I see, you could make that one longer.

Student Make that one longer or shorten one or put these farther apart. But that wouldn't solve it. Oh, I know what I'm going to do.

Interviewer You mean you just wanted to change what I gave you?

Student No. No, I know what I'm going to do, I think. I don't know if this will work.

Interviewer No, but the idea. You had an idea there of changing what I gave you?

Student Yes. But that wouldn't, you couldn't do that. Easy way to answer for mathematics. Okay.

Interviewer Now you're doing what at the moment?

Interview 11 (continued)

Student No. I don't think, that wouldn't do. The trapezoids nave to be like this or can they be like this?

Interviewer Well they have two sides parallel and the other two sides can be anywhere you want.

Student Anywhere you want?

Interviewer As long as it's closed.

Student Okay, well then I'll put these two sides parallel and see what happens.

Interviewer Okay, so you're going to put b parallel to?

Student Yes. That will work.

Interviewer b parallel to what?

Student b parallel to, oh just a minute, b parallel to a. No, yes b parallel to a. And I think just have to fiddle around with the different sides until you get the a, you know. Let's see, what I'd do is I'd just fiddle around with the different sides until I, I got them in such a position where I could maybe one end was closer for d is shorter than a, so that if, if, I don't know if this would work or not.

Interviewer Okay. So that's what your plan of attack's going to be then?

Student Yes. I'd just sort of fiddle around with it until it works. I don't think an equation would get you anywhere. I don't think so. What I'd do, I'd just have to experiment, you know fiddle around with each of the different sides until I was able to a fit them in so that they worked. By putting, you know, maybe this one farther this way and that one farther that way, yet still being parallel.

Analysis of Interview 11
Problem 8
Student 23

Variation

"well I could do it the easy way and add on . . . Well you could always cheat and make this one longer . . . make that one longer or shorten one or put these farther apart."

Interview 12
Problem 8
Student 29
Heuristic:Variation

Student You want to know how I'd go about? Okay. How I'd go about constructing the trapezoid, if I was given the lengths of a, b, c, and d. Four sides.

Interviewer Yes. These are the lengths of the sides. So I gave you the lengths of the sides.

Student Oh. You mean you want me to take, take these there and , and just kind of put them together?

Interviewer And make a trapezoid for me.

Student Well.

Interviewer Do you know what a trapezoid looks like?

Student Yes. It looks, c and b like that. So, I don't know. I think I'd probably measure them first, maybe. Okay, just, just to see. I just want to see how well. I don't know. I've, look I, I can see the lengths, you know, not exact lengths, but a. I'd probably take b and, b because c is the smallest. And I'd probably put the c, c on the top. And a is the longest, so I'd probably put that one on the bottom. And then b and d, b and d on the sides. You want to know why? I don't have any reason, except.

Interviewer No. I'd like you to try it.

Student Try to put them there?

Interviewer Yes.

Student You mean kind of measure it out and put them?

Interviewer Yes. I'd like you to try it and tell me what you'd do now. I want you to try it.

Student Oh, okay. Okay b and d, I think.

Interview 12 (continued)

Interviewer So you have drawn a for. Now you're going to take b and d?

Student Well it might be better. Oh, change, I'm sorry. I'm thinking maybe I should put b and d first. Okay. So I'll do that.

Interviewer You've got b?

Student Well I don't know which one's going to work right, so I just do them.

Interviewer That's b?

Student Oh, should be. It should be more slanted, okay.

Interviewer Why, why did you want to slant b more?

Student Because I don't think the angles would come out right. It'd be more like a right angle there, okay. And then I'd put a down and then I'd put. I just had b and this is a, so then I'd put d. I think. Okay. Not going to come out, I don't think. It's not going to, won't come out right. Oh, well. And then I'd take c and it doesn't fit. Go like that, and it's wrong.

Interviewer Okay, what would you do now that it doesn't fit?

Student I don't know. I don't know. I think I'd take, let's see. I used. This is a and this is c, that's d. I think, well you actually, I don't know. I'd think that you'd need a longer, a longer base except you can't do that. So I don't know, I think maybe. I guess I don't know.

Interviewer What do you mean by making it much more wider?

Student Well like, like you can't just make, it'd be a rectangle then. I mean the angles here, like you know, you can have it like that or except I couldn't really take it much, much wider cause it'd, it'd turn into a right angle, so a I'd a.

Interview 12 (continued)

Interviewer What was your idea? You said something just before that idea. You said something about if you had a longer, but you couldn't do that. What were you thinking?

Student A longer?

Interviewer Yes, what were you thinking of? What if a was longer, would that make any difference?

Student Well these two would be farther away and therefore that one would fit.

Interviewer Oh, I see. Your b and d would be farther away.

Student But, but maybe if you make c longer. If you had c. If instead of using c here you used a and d and then put b and c in the middle. You want me to try that, or? I was saying like if you had a and d or, and then let the b and d or, and then let the b and c over here, the shorter ones. But I don't really know.

Interviewer But that's the way you'd go about solving it?

Student I think so. A unless, unless a. Yes I think so. But then you'd have to go through all the different combinations. But I don't see another way. I guess you could measure them exactly and then, I don't know. Maybe you're supposed to know some kind of a, something about trapezoids, that I don't know.

Analysis of Interview 12
Problem 8
Student 29

Variation

"I'd think that you'd need a longer, a longer base except you can't do that . . . but maybe if you make c longer."

Interview 13
Problem 9
Student 33
Heuristic: Extension

Student Oh well these are the cube roots. It's a 1 cubed plus 2 cubed plus 3 cubed plus 4 cubed equals 100. And so I'd say that a faulty conclusion you might come to is, if you added the terms together and made 10 cubed equals 100. Which is wrong cause you can't combine those. I don't think, let's see.

Interviewer I'd like to know all the ideas that occur to you. You know, anything you think of doing or trying, or you know. Don't let anything slip by. Give me an idea what you're looking for or trying.

Student Oh, I see. Okay. I'll break this down into prime factors, I suppose. Equals, let's see 100 is 2, 2, 5, 5.

Interviewer Now did you do that for any reason, or when you got it did you look for anything?

Student No, not really. A oh well I was sort of thinking of combining to begin with, but a well, let's see. I'm just trying to develop a pattern. Just continue it, 25, 6 cubed, 7 cubed is.

Interviewer So you've written down there a 53, yes 73. And what are you looking for now?

Student To see if a any pattern exists in it because well first four.

Interviewer The first four what?

Student Well the first four seem to add up to 100. And see if I can a, well I'll just take a, well you can a divide these like. A 1, well square everything there. No that'd be sort of useless. Let's see a I, I can't seem to a.

Interview 13 (continued)

Interviewer Now what are you thinking right now?

Student I'm wondering, unless a you just x + y + z all those cubed equals a number. But if you add them all together then square it, it equals the same number.

Interviewer Oh yes, I see.

Student So a. But there's no way it would work if you continued cause a those numbers, if you added it might work. I don't know, let's see.

Interviewer Show me what you mean there.

Student Yes. Well I'll just use 5, 6, 5, 6, 7, and 8 and see if it works for the next four numbers. So 8 cubed going to be about 512, I think. So I'll add up these. That equals.

Interviewer Oh, you're adding up the cubes, yes.

Student Yes, 196. 5, 26 squared, 26 times 26. No. So it doesn't seem to be, it doesn't work that much. Now I guess I'll try it with any four numbers that add up to 10 and see if that works. Let's see.

Interviewer Why did you pick that particular combination, like any four that add up to 10?

Student Well these were four numbers that added up to 10, and I'm just trying. Well I'll try for three numbers that add up to 10. 5, 3, and 2. I know it's not going to work because 5 cubed is over 100. A you see that it works here it 1 + 8 = 9, 1 + 2 squared equals 9 and a it works for the next 3, 1 + 2 + 3 = 6 squared is 36, 27, 35 and 36. So I think any pattern. Well just add 5 to that and you get 225 and 5 + 10 is 15 squared is 225. So one if you just progress by integers 1 + 2 + 3. A you could get a general term for this. Meaning that Fn equals for any, let's see, equals 1 + 2 cubed + 3 cubed etc. plus n cubed and a equals 1 + 2 + 3 etc. + n all squared.

Analysis of Interview 13
Problem 9
Student 33

Extension

"I'm just trying to develop the pattern. Just continue it 25, 6 cubed, 7 cubed is . . . I'll just use 5, 6, 5, 6, 7, and 8 and see if it works for the next four numbers."

Interview 14
Problem 9
Student 41
Heuristic: Extension

Student One cubed. Oh looks like that's 2 cubed, 3, 3, that's 3 cubed and 4 cubed equals 100. A 1, 2 equals 1 and and 4 all squared. And then just discover from this, or can I jot down some sort of formula?

Interviewer I want to know what you can do with it? If there's anything else you can discover.

Student I just throw in 5, 25, 125.

Interviewer So now you threw in a what for me?

Student I'm throwing in a 5 cubed, 3 cubed plus, plus 5 cubed equals. What does that equal? 25, 225 and that's 15 a 75, 150. Yes that's, that's 15 squared. And so it looks like sort of a general. Plus 2 cubed. I'll try 6 too. 216 and that'd give.

Interviewer And that is what?

Student 6, 6 cubed and I put that in with this now. 441. And then a, I'll try adding on a, a 6 to the end of the sequence. And a it'd give 15 and 20, 21 and I'll try squaring that. Yes, that gives the same. So you could probably, that's about all you could do I suppose. Let's see if it would work for, if you had to start with one. So I'd just do these. 35. But that's not a square. A divide by 2 + 3, 5 and you'd get something like 7. A 7. Oh I might as well try another. 99. Take another 10, divide by.

Interviewer Where did the 99 comes from? I'd like to ask you.

Student A 8 + 27 + 64. I'm getting rid of the 1 now. And I get a 2 + 3 + 4 is 9. I get 11. Can I just try one more?

Interview 14 (continued)

Interviewer Sure. You can do anything you want. I want to know what you can discover.

Student A 99 + 125. You get 99 and 125, 214, yes, 214 and a 9 plus the last was 5. You'd come up with a boob. It doesn't work, no. 5, 5, and 14. Sorry, that's once. So 7, 4, 624. No, that doesn't work. Wait, a oh, 99, one twenty. It's 84 divided by 14. A 6 so might be some relationship between 5 and 7, 9, 11, and 14 and 16. Well what's the very first one. Just take 8 alone, divide by your first one, two. You'd come 4. Seems to be 2 difference. A the first time it came out as exact square but a a I guess I'll just start with this one now. A 27 dividing by 3 for the first one, come up with 9. 6 difference now these two. A 91 come up with 3 + 4 is 7. No that's prime, isn't it. 1, 13, oh, there's 6. Oh, for starting with one you'd come up with zero. You come up with a squares. Starting with 2 you'd come up with 2 difference between them, the two things. And if you starting with 3, you'd get 6 difference. And let's do 4 quickly. 64. The first would be 4, 16, yes, 12 difference. If it were a 189 divided by 4 + 5, 9, you'd come up with 2, 21. That's 12 difference again. Yes. So you'd have a, what can I make out of that. 2 between there 4, 6. A I'd have to find some sequence or. Just a, this squared minus the original. Yes, see you'd have a, starting off with the first n cubed plus n + 1 cubed plus n plus oh, however far you want to go plus x or something cubed and that will equal the n + n + 1 onto n + x. And if you started with one it'll be squared but if, if you don't it will have to be a the n + the n + 1, n + x plus that's n squared. Difference 12, n, 6 squared, 2, and if 1 - 1 yes plus 0. So it would be squaring it plus n squared -n. These are all big brackets and you'd come up with an equation sort of like that. Yes, that's your equation then, and then you could sort of try seeing whether or not it would work with, instead of 2 + cube root of 3. Does it equal the square root of 6. Maybe I should try that. That's 1, that's 1 a little over one, that's a 1.3. See you'd come up with about 3.4 over the square root of 6 equals 1. Square root of 6 is about a 2.6 or something. No. Which is too small cause that's right here.

Analysis of Interview 14
Problem 9
Student 41

Extension

"I'll try adding on a, a 6 to the end of the sequence . . . let's see if it would work for, if you had to start with one . . . then you could sort of try seeing whether or not it would work with, instead of square, square roots, a cube root of 1 + cube root of 2 + cube root of 3."

Interview 15
Problem 4
Student 3
Heuristic: Successive Variation

Student First measure the sides. This CB is 8 centimeters. 5 and 7/8. And that's roughly 8 again.

Interviewer And why did you do that?

Student Well, I have to get two a vertices on here and one on each side. That way you know, I don't know, I just thought I'd do that.

Interviewer Okay.

Student Now I'm supposed to construct a square. I guess it's by trial and error.

Interviewer So you're trying to do what now?

Student Well, to see like what number. You know, which way one side would be like. Mostly think it would be between 3 and 4, or something like that. Now I think I got it. Like the way I did it, I thought it would be between like 3 and 4. Then I tried out 3. And I figured and I, you know, made a mark here to show where it is. And I went parallel to the base line. And it showed that it was about $3\frac{1}{2}$. So I went up a bit more. So I make it about $3\frac{1}{4}$. And that came out $3\frac{1}{4}$. So it came out right.

Interviewer You think that's right?

Student Yes.

Analysis of Interview 15
Problem 4
Student 3

Successive Variation

"which way one side would be like. Mostly I think it would be between 3 and 4 . . . I thought it would be between like 3 and 4. Then I tried out 3. And I figured and I, you know, made a mark here to show where it is. And I went parallel to the base line. And it showed that it was about $3\frac{1}{2}$. So I went up a bit more. So I make it about $3\frac{1}{4}$. And that came out $3\frac{1}{4}$."

Interview 16
Problem 4
Student 22
Heuristic: Successive Variation

Student I don't like geometry. I can't do it. Oh, let me see. Well, I really wouldn't know where to start. I guess a like first of all I'd just, I don't know why, but I'd just measure to some extent. And like make it more clear. And see if, well, wait a second. I'd better do that over again. And then measure down here to see if that, like make it perpendicular. Like a I gather I have to make this one, I don't know why, I'd make this, maybe a little bit bigger, to have the, no. That still wouldn't work.

Interviewer Okay, first of all why did you want to make it bigger?

Student To make this smaller.

Interviewer To make the one going up and down smaller?

Student Yes. Because I needed it smaller. Cause like that's up there and that's down there and I need to make it smaller. But then that would make this bigger, and it wouldn't work.

Interviewer Oh, I see. Yes.

Student So a I guess I'd just keep on doing that until I got them both right. Either that or I'd just draw an imaginary one in my head, and just do that.

Interviewer What do you mean by that?

Student Well like I'd just, I'd just look at it and construct a square in my head. And then do it from that. That's why I happened to choose that point. Cause like I was looking at a square in my head, and I saw it there.

Interviewer I see what you mean, yes. So you'd try to do that would you?

Student Yes.

Analysis of Interview 16
Problem 4
Student 22

Successive Variation

"I guess like first of all I'd just . . . measure to some extent . . . And then measure down here to see if that, like make it perpendicular . . . I'd a make this, maybe a little bit bigger, . . . to make it smaller . . . just keep on doing that until I got them both right"

APPENDIX E

INFORMATION ON THE STUDENTS INTERVIEWED

Grade 11 Student	Problems Tried		Sex Male	Sex Female	School	Age Years	Age Months	Math 11 Grades Easter	Math 11 Grades Christmas	Plans to take Math 12 Yes	Plans to take Math 12 No	Plans to take Math 12 Undecided	Plans to take math after high school Yes	Plans to take math after high school No	Plans to take math after high school Undecided
1	1	2	x		1	16	8	A	B	x			x		
2	1	3		x	1	16	6	A	A	x			x		
3	1	4	x		1	16	7	A	B	x			x		
4	2	3		x	1	16	5	A	B	x			x		
5	2	4	x		1	16	6	B	B	x			x		
6	3	4		x	1	16	11	B	B	x			x		
7	5	8	x		2	17	6	A	A	x			x		
8	6	7	x		2	16	7	B	B			x		x	
9	6	8	x		2	16	8	A	A	x			x		
10	7	8		x	3	16	0	A	A	x			x		
11	3	9		x	3	15	8	A	A	x			x		
12	4	9		x	3	16	7	A	A	x			x		

APPENDIX E (continued)

Grade 11 Student	Problems Tried		Sex Male	Sex Female	School	Age Years	Age Months	Math 11 Grades Easter	Math 11 Grades Christmas	Plans to take Math 12 Yes	Plans to take Math 12 No	Plans to take Math 12 Undecided	Plans to take math after high school Yes	Plans to take math after high school No	Plans to take math after high school Undecided
13	5	9		x	3	17	1	A	A	x			x		
14	3	5		x	3	18	3	A	A	x			x		
15	3	6		x	3	16	7	B	B	x					x
16	1	5		x	4	16	11	A	A	x			x		
17	1	6		x	4	16	6	A	A	x			x		
18	2	5	x		4	16	2	A	A	x			x		
19	2	7	x		4	17	0	A	A	x					x
20	3	7		x	4	16	9	B	A	x			x		
21	3	8		x	4	17	0	B	A	x				x	
22	4	7		x	6	16	1	A	A	x					x
23	1	8		x	6	17	4	A	A	x			x		
24	2	6		x	5	16	8	A	A	x					x
25	4	5		x	5	16	7	A	A			x			x
26	1	7		x	6	16	4	A	A	x			x		
27	5	6		x	6	15	9	A	A			x		x	

APPENDIX E (continued)

Grade 11 Student	Problems Tried		Sex Male	Sex Female	School	Age Years	Age Months	Math 11 Grades Easter	Math 11 Grades Christmas	Plans to take Math 12 Yes	Plans to take Math 12 No	Plans to take Math 12 Undecided	Plans to take math after high school Yes	Plans to take math after high school No	Plans to take math after high school Undecided
28	7	9		x	6	16	1	A	A	x					x
29	4	8		x	6	16	6	A	A	x					x
30	1	9		x	7	16	6	A	B	x					x
31	2	8	x		7	17	6	A	A	x			x		
32	4	6	x		7	16	6	A	A	x			x		
33	2	9	x		8	16	10	B	B	x			x		
34	6	9	x		8	16	8	A	A	x			x		
35	1	3	x		8	k7	2	A	A	x			x		
36	3	4	x		8	16	11	A	A	x			x		
37	8	9	x		9	16	5	A	A	x			x		
38	5	9		x	9	16	2	B	A	x			x		
39	5	7	x		9	16	8	B	B	x				x	
40	4	9	x		9	17	3	B	B	x					x
41	3	9	x		8	17	4	B	B	x					x
42	2	3	x		8	17	0	A	A	x			x		

APPENDIX F

THE STUDENTS' USE OF HEURISTICS

Student	Problem	Heuristics Used								
		Cases	Deduction	Inverse Deduction	Invariation	Analogy	Preservation of Rules	Variation	Extension	Successive Variation
1	1				x					
	2		x						x	
2	1	x			x					
	3(a)			x						
3	1									
	4									x
4	2		x							
	3(a)									
5	2									
	4	x							x	
6	3(a)									
	4									x

APPENDIX F (continued)

Student	Problem	Heuristics Used								
		Cases	Deduction	Inverse Deduction	Invariation	Analogy	Preservation of Rules	Variation	Extension	Successive Variation
7	5	x								
	8									x
8	6									
	7									
9	6									
	8	x								
10	7					x				
	8					x				x
11	3(b)		x	x						x
	9(a)									
12	4									x
	9(a)									
13	5	x				x				
	9(a)									

APPENDIX F (continued)

Student	Problem	Heuristics Used								
		Cases	Deduction	Inverse Deduction	Invariation	Analogy	Preservation of Rules	Variation	Extension	Successive Variation
14	3(b)									
	5					x				
15	3(b)									
	6	x								
16	1									
	5	x				x				
17	1	x								
	6									x
18	2		x							
	5									
19	2									
	7									
20	3(b)									
	7									

APPENDIX F (continued)

Student	Problem	Heuristics Used								
		Cases	Deduction	Inverse Deduction	Invariation	Analogy	Preservation of Rules	Variation	Extension	Successive Variation
21	3(b)									
	8	x								
22	4									x
	7									
23	1									
	8	x						x		
24	2		x							
	6									x
25	4								x	
	5					x				
26	1	x								
	7						x			
27	5									
	6									

APPENDIX F (continued)

Student	Problem	Heuristics Used								
		Cases	Deduction	Inverse Deduction	Invariation	Analogy	Preservation of Rules	Variation	Extension	Successive Variation
28	7						x			
	9(b)									
29	4							x		
	8	x						x		
30	1	x			x					
	9(b)									
31	2		x							
	8	x								
32	4	x							x	
	6							x		
33	2		x							
	9(b)								x	
34	6									x
	9(b)									

APPENDIX F (continued)

Student	Problem	Heuristics Used								
		Cases	Deduction	Inverse Deduction	Invariation	Analogy	Preservation of Rules	Variation	Extension	Successive Variation
35	1	x								
	3(b)									
36	3(b)									
	4									x
37	8	x								
	9(b)								x	
38	5									
	9(b)					x				
39	5					x				
	7									
40	4	x								
	9(b)									
41	3(b)									
	9(b)								x	
42	2		x							
	3(b)									

APPENDIX G

INFORMATION ON THE SCHOOLS SAMPLED

School	Grades Enrolled	Type	School Enrollment	Math 11[a] Enrollment	Sample Size	Social Class[b]	Mathematics Teacher Male	Mathematics Teacher Female
1	9-12	private coeducational	450	43	6	3		x
2	8-12	private boys	350	12 16	4 1	3	x	
3	8-12	private girls	205	34	6	2		x
4	8-12	private coeducational	165	29	6	4		x
5	8-12	private girls	250	22	2	1	x	
6	1-12	private girls	305	30	6	2		x
7	8-12	public	1050	25	3	3	x	
8	8-12	public	1050	33	6	1	x	
9	8-12	public	920	31	4	1	x	

[a]This figure represents the enrollment of the class where the students that were interviewed were enrolled.

[b]Joseph A. Kahl, The American Class Structure (New York: Holt, Rinehart, and Winston, 1957), p. 215-216.