ON-LINE OPTIMAL AND ADAPTIVE CONTROL OF A QUEUING SYSTEM

by

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ABSTRACT

A number of on-line control methods have been studied for the operational control of a queuing system. Time-series models have been used, in contrast to the probability models usual in the traditional approach to such problems.

It is shown that most queuing processes can be formulated as multistage control problems to which modern control theory can be applied. The various control techniques applicable to a queuing system can be divided into two classes: decision and regulator control. In obtaining the control strategies, this thesis draws heavily from dynamic programming, least-squares estimation, the discrete maximum principle and gradient techniques.

The uncertainties encountered in the queuing system can be overcome with an adaptive control method. The open-loop-feedback-optimal control technique has been stressed here due to its simplicity. Applications of the methods to various fields have also been studied. Extension of the method to long interval control is immediate in all the cases.

Although the optimal control of a queuing system has been discussed, the methods are general enough to be applied to other areas.

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1. INTRODUCTION

Queuing occurs in many facets of our daily lives. To catch a bus, we join a queue. To place a telephone call, we join a queue. Indeed, it is impossible to survive today without involuntarily participating in some sort of a queuing process.

As society gets more complex, so do the queuing processes one encounters. Consequently, the study of queuing processes is an important feature of the way man interacts with his environment.

1.1 Queuing System, Queuing Models and the Classical Queuing Theory

A queuing system, or a stochastic service system - to include cases where the theory of queues has been applied to situations in which no physical queues actually exist, is a process of "customers" waiting for service from "servers". The terms "customers" and "servers" were inherited from earlier researchers whose problems were mostly in connection with the telephone business. The theory of queues was established about half a century ago to study the behaviour of the mathematical models developed for the queuing process.

All queuing theory models are based upon the three major characteristics of the queuing processes: the input-process, the service-mechanism and the queue-discipline. With the field of statistical analysis already firmly established, it is not surprising to find that all the classical queuing models are probability models, that is models in which emphasis is placed upon the probability distribution itself, - in contrast to time-series models where the actual values of some parameters are considered[9]. Subsequently, mathematicians have been able to apply queuing theory successfully to obtain elegant analytic results for a wide variety of queuing processes.

Nevertheless, as Saaty has said(6):

> "The subject of queuing is not directly concerned with optimization. Rather it attempts to explore, understand, and compare various queuing situations and thus indirectly achieve optimization approximately".

The traditional spirit of the prolific amount of research on queuing systems has been towards the performance analysis rather than the control aspect.

An inefficiently controlled queuing process can be very frustrating and exasperating. From the engineer's point of view, it is obviously desirable to obtain a compatible optimization technique for the queuing processes one encounters. Just as the classical queuing theory works to better control through understanding the behaviour of a queuing process, today's control engineer should aim at implementing better control through his knowledge from the well-developed theory of optimal control.

1.2 Outline of Thesis

This thesis is an attempt to apply modern control theory to a queuing process. Particular emphasis is placed upon two different types of on-line control strategies: decision and regulator control. Several applications are included to demonstrate the versatility of the approach. The possibility of extending the method to adaptive control systems is also investigated. The structure of the thesis is as follows.

The purpose of the first part of Chapter 2 is to demonstrate the real-time approach to the queue-modelling problem where the classical approach has been via probability modelling. A time-series model is developed for a simple queuing process. Operational control strategies, in particular the decision- and regulator-type, are also discussed qualitatively. An adaptive control system is also proposed.

Chapter 3 is devoted to a detailed discussion of decision control for queuing systems. The discrete Maximum (Minimum) Principle will be applied extensively to obtain the necessary conditions for optimality. A gradient technique is used to compute the solution for the resulting two-point boundary value problem. This control algorithm is then applied to two optimal ordering problems in queuing and service channelling problems. Several numerical examples, a traffic control problem and a computer queuing problem, are presented. Extension of the method to long-term applications is discussed, using an adaptive technique known as "open-loop-feedback-optimal" strategy.

In Chapter 4 the regulator-type of control is discussed. This class of control is mainly confined to server-rate control. An on-line adaptive control method is proposed: Dynamic Programming is employed for optimization while simultaneously the system is identified by least-square methods. A suboptimal, but feedback, control is derived. The control strategy is then applied to a trafic control problem and an operation-scheduling problem in hospitals.

The major conclusions and some suggestions for further research in this area are presented in Chapter 5.

2. A GENERALIZED QUEUING MODEL

2.1 Introduction

The main concern of this section is to develop a mathematical model for a general queuing system with provision for on-line control. Most of the classical queuing models have been probability models that lead to results in terms of probabilities and expected values.

In recent years, there has been a tendency towards time-series modelling of queuing systems[1]. However, the major attempts have been aimed at performance analysis rather than operational control of queuing systems. Although a number of researchers have applied control theory to the classical probability model[2,3,4], the results they obtained were more tailored for system designing: off-line rather than on-line control.

2.2 A Time-series Model for the Queuing System

It is impractical to develop a highly generalized mathematical model that includes all queuing situations. However, it is important to obtain a generalized queuing model which includes all the general characteristics of a queuing process, i.e. input-process, queue-discipline and service-mechanism.

The simplest queuing process consists of:

1. M parallel streams of arrival customers
2. N parallel queuing channels or waiting lines
3. P parallel service channels or servers

The queuing phenomenon for such a system is depicted in Fig. 2.1. We shall refer to it as an elementary system unit because it contains all the essential features of a queuing system. More sophisticated systems can then be represented as a cascade or network of such queuing units.

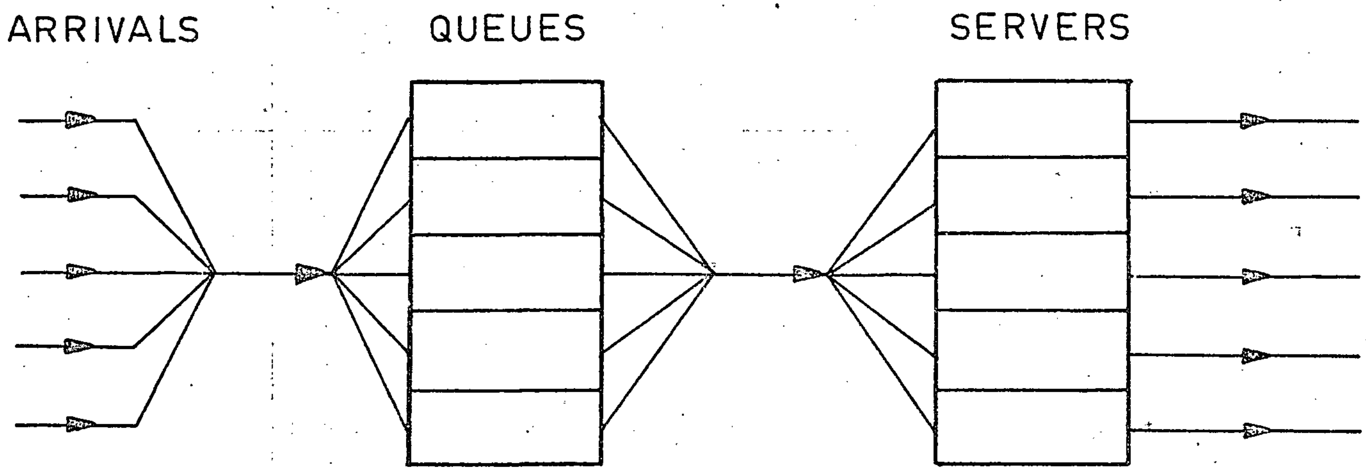


Fig. 2.1   A Fundamental Queuing System

A mathematical model for such a unit will be developed shortly. But first of all, the following remarks are appropriate:

1) In the classical queuing model, to obtain useful analytical results, certain probability distribution functions must be assumed on the input (poisson) and the service processes (negative-exponential). No such assumptions will be made here.

2) For on-line control purposes, it is suitable to have a mathematical model whose dynamics are depicted in discrete time unit.

3) Define the following quantities:

$\{....k-1, k, k+1......\}$: discrete sampling time instants, not necessarily equally spaced in time

$q_k \varepsilon R^N$: The number of customers waiting in the N queuing channels at the time k

$s_k \varepsilon R^P$: The number of customers each of the P servers is capable of servicing during $[k, k+1)$

$\lambda_k \varepsilon R^M$: The number of customers arriving from the M input streams during the interval $[k, k+1)$

Thus, the dynamics of the queuing process unit (Fig. 2.1) can be represented by the model

$$q_{k+1} = q_k + B_k \lambda_k - C_k s_k \tag{2-1}$$

where B_k is an (N x M) matrix, $[b_{ij}]_k$, in which $b_{ij} \triangleq$ fraction of the arriving customers from the j^{th} input channel who joined the i^{th} queuing channel during $[k, k+1)$, and C_k is an (N x P) matrix $[c_{il}]_k$ in which $c_{il} \triangleq$ the fraction of the customers served by the l^{th} servers during $[k, k+1)$, that had come from i^{th} queuing channel.

One can detect almost immediately the difference between this generalized queuing model and the classical probability model. Most classical models treat the evolution of the state in queuing process as a stochastic process. For instance, instead of using a queue size, q_k as the state, $P_r\{q_k \leq n\}$ would be used as the state of the model. For real-time control, a deterministic formulation of the stochastic queuing process would be more convenient. However, as the queuing process is basically a highly stochastic process, the deterministic model would be inadequate for describing the process over a long time period. Over a long span of time, in order to preserve a correct representation of the actual process, corrective adjustments would have to be taken on the model. Thus arises the need for an adaptive control model.

2.3 Operational Control of Queuing Systems

There are basically two classes of methods used in the operational control of queuing systems:

A) Decision Control - where the control is generally imbedded in the process of optimal selection. The optimal assignment of arriving customers to the available queuing channels is an example of this type of control.

B) Regulator Control - this type of control generally involves setting a control mechanism at certain levels so that overall optimal system performance is obtained. Examples are found in server-rate control and scheduling problems.

The objective of any type of control for queuing system is to prevent a congestion situation from building up. A certain performance criterion of the system is to be optimized. In most socio-economical problems, a cost would be associated with this performance criterion. For instance, there may be a cost associated with the queue size; in another situation we might want to maximize the use of a server though assigning a cost to each idle moment of the server. However, to define a cost at this stage would cause our generalized queuing model to lose much of its generality.

In the context of the generalized queuing model, the following remarks are appropriate:

1) The matrices B_k, C_k in fact represent the queuing and service discipline of the system.

   For example, $b_{ij} = 0 \Longrightarrow$ None of the customers from j^{th} input joins the i^{th} queue

   $c_{il} = 0 \Longrightarrow$ l^{th} server not serving customers from the i^{th} queue

   In a statistical sense, b_{ij} is the probability of an arrival customer from the j^{th} input joining the i^{th} queue; c_{il} is the probability of a customer in the i^{th} queue being served by the l^{th} server

2) From the control point of view, any type of decision control on the queuing or service discipline would directly affect the transition matrices B_k and C_k.

3) In our generalized model, $\{\lambda_k\}$, the arrival sequence is represented as a disturbance input whose dynamics are unknown to the controller. However, as far as the control of queues is concerned, such knowledge is not essential. Consequently, throughout the remainder of this thesis, it shall be assumed that information on the arrival sequence $\{\lambda_k\}$ (either predicted or scheduled) will be available to the controller.

4) In the case of totally unknown $\{\lambda_k\}$, the question of optimal

prediction can easily be studied through time-series analysis and a number of operations research techniques[9,10,11] which are beyond the scope of this thesis

2.4 Adaptive Control of Queuing Systems

The philosophy behind the generalized queuing model (2-1) aids in the implementation of different types of control to a queuing system. As mentioned earlier, the deterministic nature of the model is inadequate to represent the actual queuing process which is basically a stochastic system. Initial information on such a system is usually absent, or at best based on estimations. An effective control would be the adaptive control process, in which the state of knowledge about the system improves as it evolves. This updating of information is carried out by a learning mechanism. A simple adaptive control system (Fig. 2.2) would consist of a system identifier, which identifies the unknown system parameters in the system, and a feedback controller which optimizes the performance of the system. A parametrized control is usually obtained.

A schematic representation of an adaptive control system for the queuing model is shown in Fig. 2.3. The two classes of control, regulator and decision type, are applied simultaneously to achieve an overall on-line control for the queuing process.

There are many instants in the control of a queuing system where an adaptive scheme would be useful. Typical examples are found in cases where the actual arrival sequence may differ significantly from the predicted values due to some non-anticipated disturbances. In the case where the regulator type of control is carried out simultaneously but separately from decision control, knowledge of the effect of each control on the model parameters helps to establish an overall optimal

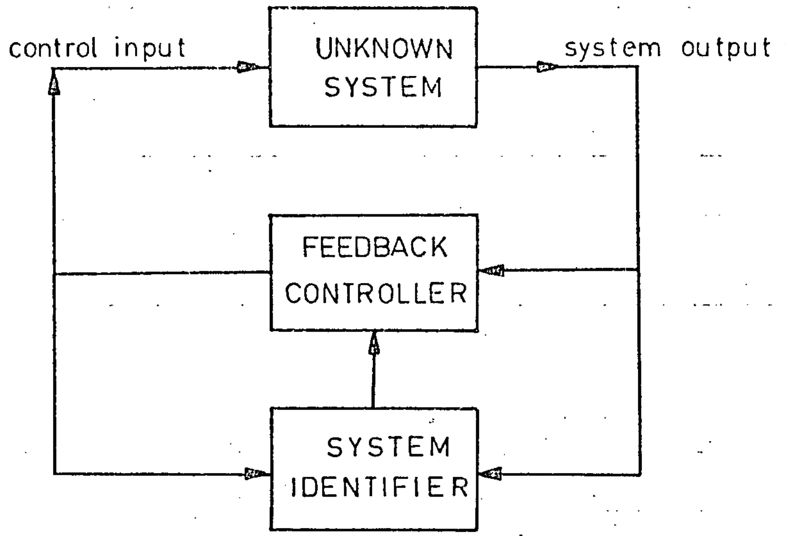


Fig. 2.2 A Simple Adaptive Control System

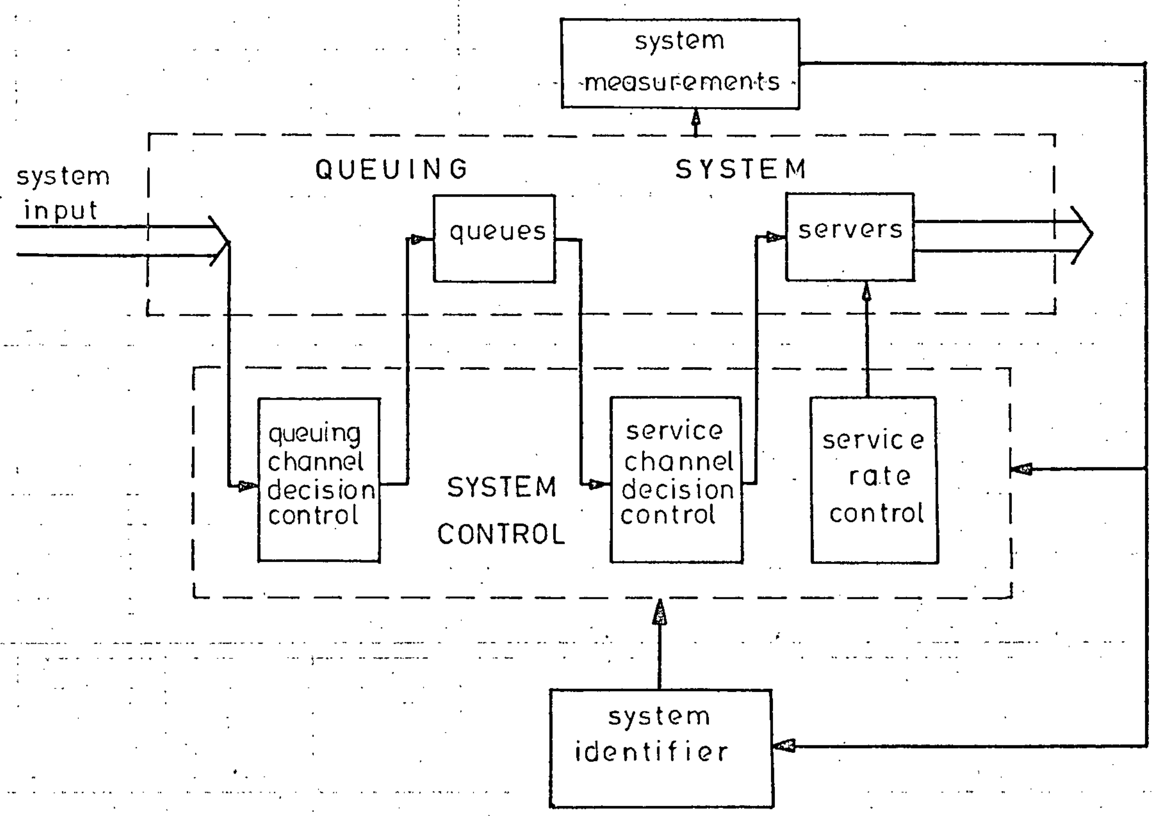


Fig. 2.3 Adaptive Control for a Simple Queuing System

control for the system.

2.5 Discussion

A generalized queuing model which includes all the basic features of a typical queuing process has been developed. Using such a model, it is possible to obtain various control strategies to the queuing system. The complexity of most queuing systems makes it necessary to separate the applicable control strategies into two major types: decision and regulator control. In the remainder of this thesis, these two types of control will be considered separately. Adaptive control techniques can be applied to both methods.

3. OPTIMAL DECISION CONTROL FOR QUEUING SYSTEMS

3.1 Introduction

Decision control in a queuing system arises whenever a selection process exists in the queuing order or service discipline. Customers entering into a queuing system are faced with a number of possible queues; customers emerging from queues are faced with the choice of a server. A good example is provided by a shopper checking out through a number of cashiers in a supermarket. Before joining any queue, each shopper makes a decision as to which queue would provide the fastest service. Very often a short queue does not imply fast service. This process of optimal selection can be formulated as a decision control problem.

Ireland et al.(4) and Esogbue(3) have handled problems of this nature through the application of control to the classical probability model. A time-series model with on-line control features will be described here.

3.2 Mathematical Formulation of the Optimal Decision Control Problem

Consider a decision control system which can be expressed as

$$x_{k+1} - x_k = f(x_k, u_k) \qquad k = 1,\ldots,K \tag{3-1}$$

where $x_k \in R^n$, is the system state and

$$u_k \in L_m \triangleq \left\{ \begin{pmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \cdot \\ \cdot \\ 0 \\ 0 \end{pmatrix}, \ldots\ldots, \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \\ 1 \end{pmatrix} \right\} \in R^m \text{ is the} \tag{3-2}$$

decision control.

As is usual, the performance measure for the optimal selection process will be expressed in the form

$$J = \sum_{j=1}^{K} I(x_j, u_j) + \Theta(x_{K+1}, K+1) \qquad (3\text{-}3)$$

Hence, we can formulate the decision control problem as a multistage decision problem:

Given: $x_{k+1} - x_k = f(x_k, u_k);\ c,\ x_k \ \varepsilon\ R^n,\ u_k \ \varepsilon\ R^m$ (3-1)

with $x_1 = c$

and $u_k \ \varepsilon\ L_m$ (3-2)

Find a control sequence $\{u_i\}$ $i = 1,\ldots K$ which would minimize the cost functional J. (3-3)

3.3 An Optimal Decision Control Algorithm

Looking at the problem stated, one tends to conclude that the solution naturally involves either dynamic programming(12) or the Discrete (Maximum) Minimum Principle (22). The high dimensionality of the problem practically eliminates the possibility of applying the exact Dynamic Programming technique (Appendix II). The peculiar form of control constraint (3-2) makes any dynamic programming technique(19), such as linear feedback control, formidably awkward to implement.

When applying the Discrete Minimum Principle, there are certain convexity conditions the system (3-1), (3-2), (3-3) has to satisfy(17,18). Because of the peculiar structure of the admissible control set L_m, such convexity conditions are not satisfied here. Nevertheless, the convexity condition can be relaxed in defining the Hamiltonian function and obtaining the necessary conditions for optimality(17). With this in mind, the Hamiltonian function is defined:

$$H(x_k, u_k, p_{k+1}, k) \triangleq I(x_k, u_k) + p'_{k+1}\, f(x_k, u_k) \qquad (3\text{-}4)$$

where p_k is the costate vector.

According to the Discrete Minimum Principle (Appendix I), if $\{u^*_k\}_{k=1,\ldots K}$ is the optimal control sequence and $\{x^*_k\}_{k=1,\ldots,K+1}$ is the optimal trajectory, the following relations hold:

(i) Canonical Equations:

$$x^*_{k+1} - x^*_k = \left.\frac{\partial H}{\partial p_{k+1}}\right|_* = f(x^*_k, u^*_k) \tag{3-5}$$

$$p^*_{k+1} - p^*_k = - \left.\frac{\partial H}{\partial x_k}\right|_* \tag{3-6}$$

where $f(x)|_* \triangleq f(x^*)$

(ii) Boundary Conditions:

$$x^*_1 = x_1 \tag{3-7}$$

$$p^*_{K+1} = \frac{\partial}{\partial x^*_{K+1}} \Theta(x^*_{K+1}, K+1) \tag{3-8}$$

(iii) Necessary Condition of Optimality

$$H(x^*_k, u^*_k, p^*_{k+1}, k) \leq H(x^*_k, u_k, p^*_{k+1}, k) \quad \forall\, u_k \in L_m \tag{3-9}$$
$$k = 1, 2, \ldots, K$$

An iterative solution to this non-linear Two-Point-Boundary-Value-Problem (TPBVP) can be obtained through gradient techniques[25]. The algorithm is initialized with a nominal control sequence $\{u^o_k\}_{k=1,\ldots K}$. This is used to generate $\{x^o_k\}_{k=1,\ldots, K+1}$ in the forward iteration of (3-5), (3-7), and $\{p^o_k\}_{k=1,\ldots, K+1}$ in the backward iteration of (3-6), (3-8). $\{H(x^o_k, u^o_k, p^o_{k+1}, k)\}_{k=1,\ldots K}$ can then be evaluated from (3-4). The new control sequence is generated by changing the components in the current control so as to decrease the Hamiltonian function. The number of control elements at which adjustment is made is determined by a predetermined step size factor. The complete algorithm takes the form:

1) Start with control function $\{u_k^i\}$ $k = 1,\ldots K$

2) Determine $\{x_k^i\}$ $k = 1,\ldots,(K + 1)$ from (3-5), (3-7)

3) Determine $\{p_k^i\}$ $k = 1,\ldots,(K + 1)$ from (3-6), (3-8)

4) Determine the Hamiltonian $\{H(x_k^i, u_k^i, p_{k+1}^i, k)\}$ $k=1,\ldots,K$ from (3-4)

5) Determine the "optimal" Hamiltonian values $\{\hat{H}_k^i\}$ $k=1, 2,\ldots,K$

where $\hat{H}_k^i \triangleq H(x_k^i, \eta_k^i, p_{k+1}^i, k)$

and $\eta_k^i \triangleq \arg \{\underset{\xi\varepsilon L_m}{\mathrm{Min}}\ H(x_k^i, \xi, p_{k+1}^i, k)\}$

where "arg" is used to denote the value of the argument at which minimization is achieved.

6) Determine the integer "step size" g

where $g \triangleq \arg \{\underset{\ell}{\mathrm{Max}} [\sum_{k=1}^{\ell} |\hat{H}_k^i - H(x_k^i, u_k^i, p_{k+1}^i, k)| \leq$

$$\delta h \sum_{k=1}^{K} |H(x_k^i, u_k^i, p_{k+1}^i, k)|]\}$$

δh: a predetermined step size factor

$|\cdot| \triangleq$ Absolute value of a vector

7) Update Control Function:

$$u_k^{i+1} = \eta_k^i \quad k = 1, \ldots, g$$

$$u_k^{i+1} = u_k^i \quad k = g + 1, \ldots, K$$

8) Return to Step (2)

Convergence occurs when

$$\left| \sum_{k=1}^{K} H\,(x_k^{i+1},\ u_k^{i+1},\ p_k^{i+1},\ k) - \sum_{k=1}^{K} H(x_k^i,\ u_k^i,\ p_{k+1}^i,\ k) \right| \leq \varepsilon$$

where ε is a small positive number.

It should be pointed out here that the integer step size g is in fact adjusted to limit the overall change in the Hamiltonian within a predetermined step size factor δh. Further refinements such as variable step size can also be included.

It is also interesting to note that the control constraint (3-2) is automatically satisfied in iteration step (5) of the algorithm. Hence the extremal solution for our relaxed problem will also be an extremal solution for the original problem.

3.4 An Illustrative Example

We shall demonstrate the technique by a simple numerical example for which the exact solution is easily obtainable by dynamic programming. This example is a simple decision control for queuing channels, to be discussed in a following section.

Consider the cost,

$$J = \sum_{i=1}^{5} u_i' \,[a_i\ (x_i + q) - \beta] + (x_6 + q)'\ F(x_6 + q)$$

where

$$x_{i+1} = x_i + u_i \quad ; \quad x_i,\ u_i\ \varepsilon\ \underset{\frown}{R}^2$$

$$x_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$u_i\ \varepsilon\ L_2 \triangleq \left\{ \begin{vmatrix} 0 \\ 1 \end{vmatrix}\ ,\ \begin{vmatrix} 1 \\ 0 \end{vmatrix} \right\}$$

$$a_i = 10.0 \qquad\qquad i = 1,2,\ldots,5$$

$$\beta = \begin{vmatrix} 10 \\ 20 \end{vmatrix} \quad ; \quad q = \begin{vmatrix} 0 \\ 3 \end{vmatrix} \quad ; \quad F = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

Table 3.1 Complete Dynamic Programming Solution for Queuing Channel Decision Control Example

K: 1	2	3	4	5	6
					$\begin{bmatrix}0\\5\end{bmatrix}$ 64
				$\begin{bmatrix}0\\4\end{bmatrix}^{*}$ 40** $\begin{bmatrix}1\\0\end{bmatrix}^{\#}$	$\begin{bmatrix}1\\4\end{bmatrix}$ 50
			$\begin{bmatrix}0\\3\end{bmatrix}$ 30 $\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}1\\3\end{bmatrix}$ 40 $\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}2\\3\end{bmatrix}$ 40
		$\begin{bmatrix}0\\2\end{bmatrix}$ 34 $\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}1\\2\end{bmatrix}$ 44 $\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}2\\2\end{bmatrix}$ 44 $\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}3\\2\end{bmatrix}$ 34
	$\begin{bmatrix}0\\1\end{bmatrix}$ 52 $\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}1\\1\end{bmatrix}$ 62 $\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}2\\1\end{bmatrix}$ 62 $\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}3\\1\end{bmatrix}$ 52 $\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}4\\1\end{bmatrix}$ 32
$\begin{bmatrix}0\\0\end{bmatrix}$ 62 $\begin{bmatrix}0\\1\end{bmatrix}$ $\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}1\\0\end{bmatrix}$ 72 $\begin{bmatrix}0\\1\end{bmatrix}$ $\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}2\\0\end{bmatrix}$ 72 $\begin{bmatrix}0\\1\end{bmatrix}$ $\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}3\\0\end{bmatrix}$ 62 $\begin{bmatrix}0\\1\end{bmatrix}$ $\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}4\\0\end{bmatrix}$ 42 $\begin{bmatrix}0\\1\end{bmatrix}$	$\begin{bmatrix}5\\0\end{bmatrix}$ 34

* : Admissible state

** : Minimum cost of reaching the final state at K = 6 for any given state

\# : Optimal control (or controls where both are optimal)

Table 3.2 Optimal Decision Control Solution for Example in 3.4

INITIAL TRAJECTORY			OPTIMAL TRAJECTORY*		
K	X(K)	U(K)	K	X(K)	U(K)
1	$\begin{bmatrix}0\\0\end{bmatrix}$	$\begin{bmatrix}0\\1\end{bmatrix}$	1	$\begin{bmatrix}0\\0\end{bmatrix}$	$\begin{bmatrix}1\\0\end{bmatrix}$
2	$\begin{bmatrix}0\\1\end{bmatrix}$	$\begin{bmatrix}1\\0\end{bmatrix}$	2	$\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}1\\0\end{bmatrix}$
3	$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}0\\1\end{bmatrix}$	3	$\begin{bmatrix}2\\0\end{bmatrix}$	$\begin{bmatrix}1\\0\end{bmatrix}$
4	$\begin{bmatrix}1\\2\end{bmatrix}$	$\begin{bmatrix}1\\0\end{bmatrix}$	4	$\begin{bmatrix}3\\0\end{bmatrix}$	$\begin{bmatrix}1\\0\end{bmatrix}$
5	$\begin{bmatrix}2\\2\end{bmatrix}$	$\begin{bmatrix}0\\1\end{bmatrix}$	5	$\begin{bmatrix}4\\0\end{bmatrix}$	$\begin{bmatrix}0\\1\end{bmatrix}$
6	$\begin{bmatrix}2\\3\end{bmatrix}$		6	$\begin{bmatrix}4\\1\end{bmatrix}$	
TOTAL COST = 90			TOTAL COST = 62		

* convergence after 3 iterations

The complete solution via Dynamic Programming (Appendix II) is shown in table 3-1. The optimal solution via Discrete Minimum Principle and gradient technique is shown in table 3-2.

3.5 Application I

3.5.1 An Optimal Decision Control Problem for Queuing Channels

The type of decision control required in a queuing problem is basically that of selecting an appropriate queue for an incoming customer. Let us assume that associated with each customer is a queuing cost α, such that the cost of joining a queue of size q will be $q\alpha$. We shall also assign a "reward", β to each queuing channel. For instance, a high reward will be assigned to a fast moving queue. In general, a customer has to decide how much "reward" he can obtain from joining a certain queue after paying for the queuing cost.

Consider the situation where K customers arrive at n parallel queuing channels. Let $x_k \varepsilon R^n$ denote the number of customers who have already joined the queues on the arrival of the k^{th} customer, then the queue selection process can be written as

$$x_{k+1} = x_k + u_k \tag{3-10}$$

where $u_k \varepsilon L_n$ is the decision vector; L_n is the set of n-dimensional unit vectors defined in (3-2).

The total cost of joining any one queue for the k^{th} customer would be $u_k' [\alpha_k (x_k + q) - \beta]$ where α_k is the queuing cost of the customer, $\beta \varepsilon R^n$ is the reward from the queuing channels, and q is the queue size before the arrival of the K customers.

The overall cost for assigning the K customers to the queuing channels would be

$$J = \sum_{i=1}^{K} u_i' [\alpha_i (x_i + q) - \beta] + (x_{K+1} + q)' F(x_{K+1} + q) \tag{3-11}$$

Applying the Discrete Minimum Principle, we obtain for the Hamiltonian:

$$H(x_k, u_k\ p_{k+1}, k) = u_k' [\alpha_k (x_k + q) - \beta] + p_{k+1}' u_k \qquad (3\text{-}12)$$

$$p_{k+1} = p_k - \alpha_k u_k \qquad (3\text{-}13)$$

$$p_{K+1} = 2F (x_{K+1} + q) \qquad (3\text{-}14)$$

$$H(x_k, u_k, p_{k+1}, k) \leq H(x_k, \eta_k, p_{k+1}, k) \ \forall \ \eta_k \in L_n \qquad (3\text{-}15)$$

where x_k, u_k, p_k are evaluated along the optimal trajectory. The algorithm from section 3.3 can then be applied.

3.5.2 Example - Optimal Queuing of Computer Batch Jobs

An interesting application of decision control is found in the optimal queuing of computer batch jobs. Programs with different priorities are submitted for service from a number of peripheral devices. Various devices have different execution speed. The priority of a job is a function of waiting cost. It is desirable to match the relative priorities of different jobs to the efficiency of the devices so as to achieve an overall optimal system performance.

Applying the model discussed in the last section, α_k would denote the priority of the k^{th} job submitted. β reflects the efficiency of a device. q is the number of jobs already lined up for service from the devices before the arrival of the batch of, say K jobs. F would measure the total cost of running the jobs.

A numerical example is shown below for optimal queuing of 10 jobs to 3 devices. The values of some of the parameters are:

$$\{\alpha_k\}_{k = 1,\ldots,10} = (1, 4, 10, 5, 2, 6, 2, 8, 4, 12)$$

$$\beta = \begin{vmatrix} 10 \\ 20 \\ 30 \end{vmatrix} \quad ; \quad u_k \in L_3 = \left\{ \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix}, \begin{vmatrix} 0 \\ 1 \\ 0 \end{vmatrix}, \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix} \right\}$$

$$q = \begin{vmatrix} 0 \\ 0 \\ 3 \end{vmatrix}$$

$$F = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix}$$

A number of runs with different initial control functions are made. The complete results are shown in Fig. 3-1 and Table 3-3. In all the runs, convergence is achieved in less than ten iterations. In addition, it is noted that the final optimal policies are different in all four cases. This is due to the ability of the gradient technique to track local optimalities.

3.5.3 Remarks

The control obtained above is obviously an open loop control. An assumption made is that $\{\alpha_k\}$ $k = 1,\ldots,K$ must be known. For on-line operation, one would prefer a feedback type of control. However, one can readily extend the above algorithm to an on-line application through an open-loop-feedback-optimal (OLFO) strategy[14,21]: At stage k, only the first element u_k of the optimal open-loop control sequence $\{u_i\}$ $i=k, k+1,\ldots K$ is transmitted. At the next stage $k + 1$, information on the system parameters, such as $\{\alpha_k\}$, β, or q, can be updated. A new open-loop control sequence $\{u_i\}$ $i = k + 1,\ldots,K$ is thus generated.

For long term optimizations, where $K \to \infty$, a "shifting interval" can be used. In this case it is assumed that at all times the controller can only see K intervals ahead, and an open-loop control is obtained for that interval. The procedure is repeated as the stage progresses. Of course, the overall decision control obtained will be sub-optimal, but from a computational point of view, such a scheme is desirable for on-line operations.

Table 3.3 Optimal Queuing Policies for Computer Batch Jobs

JOB	RUN 1		RUN 2		RUN 3		RUN 4	
	INITIAL POLICY	FINAL POLICY	INITIAL POLICY	FINAL POLICY	INITIAL POLICY	FINAL POLICY	INITIAL POLICY	FINAL POLICY
1	(1,0,0)*	(0,0,1)	(0,1,0)	(0,0,1)	(1,0,0)	(0,0,1)	(0,0,1)	(0,0,1)
2	(0,1,0)	(0,1,0)	(0,1,0)	(0,0,1)	(1,0,0)	(0,0,1)	(0,1,0)	(0,1,0)
3	(0,0,1)	(0,1,0)	(0,1,0)	(1,0,0)	(1,0,0)	(0,1,0)	(1,0,0)	(1,0,0)
4	(1,0,0)	(0,1,0)	(0,1,0)	(1,0,0)	(1,0,0)	(0,1,0)	(0,0,1)	(1,0,0)
5	(0,1,0)	(0,1,0)	(0,1,0)	(1,0,0)	(1,0,0)	(0,1,0)	(0,1,0)	(0,0,1)
6	(0,0,1)	(1,0,0)	(0,1,0)	(0,1,0)	(1,0,0)	(0,1,0)	(1,0,0)	(1,0,0)
7	(1,0,0)	(0,1,0)	(0,1,0)	(1,0,0)	(1,0,0)	(0,1,0)	(0,0,1)	(1,0,0)
8	(0,1,0)	(1,0,0)	(0,1,0)	(0,1,0)	(1,0,0)	(1,0,0)	(0,1,0)	(0,1,0)
9	(0,0,1)	(0,0,1)	(0,1,0)	(0,1,0)	(1,0,0)	(1,0,0)	(1,0,0)	(0,1,0)
10	(1,0,0)	(1,0,0)	(0,1,0)	(0,1,0)	(1,0,0)	(1,0,0)	(0,0,1)	(0,1,0)
No. of Itns.	10		7		6		8	
Initial Cost	89.0		312.0		312.0		101.0	
Final Cost	29.0		29.0		26.0		29.0	

* : The position of "1" denotes the device number to which the job is assigned.

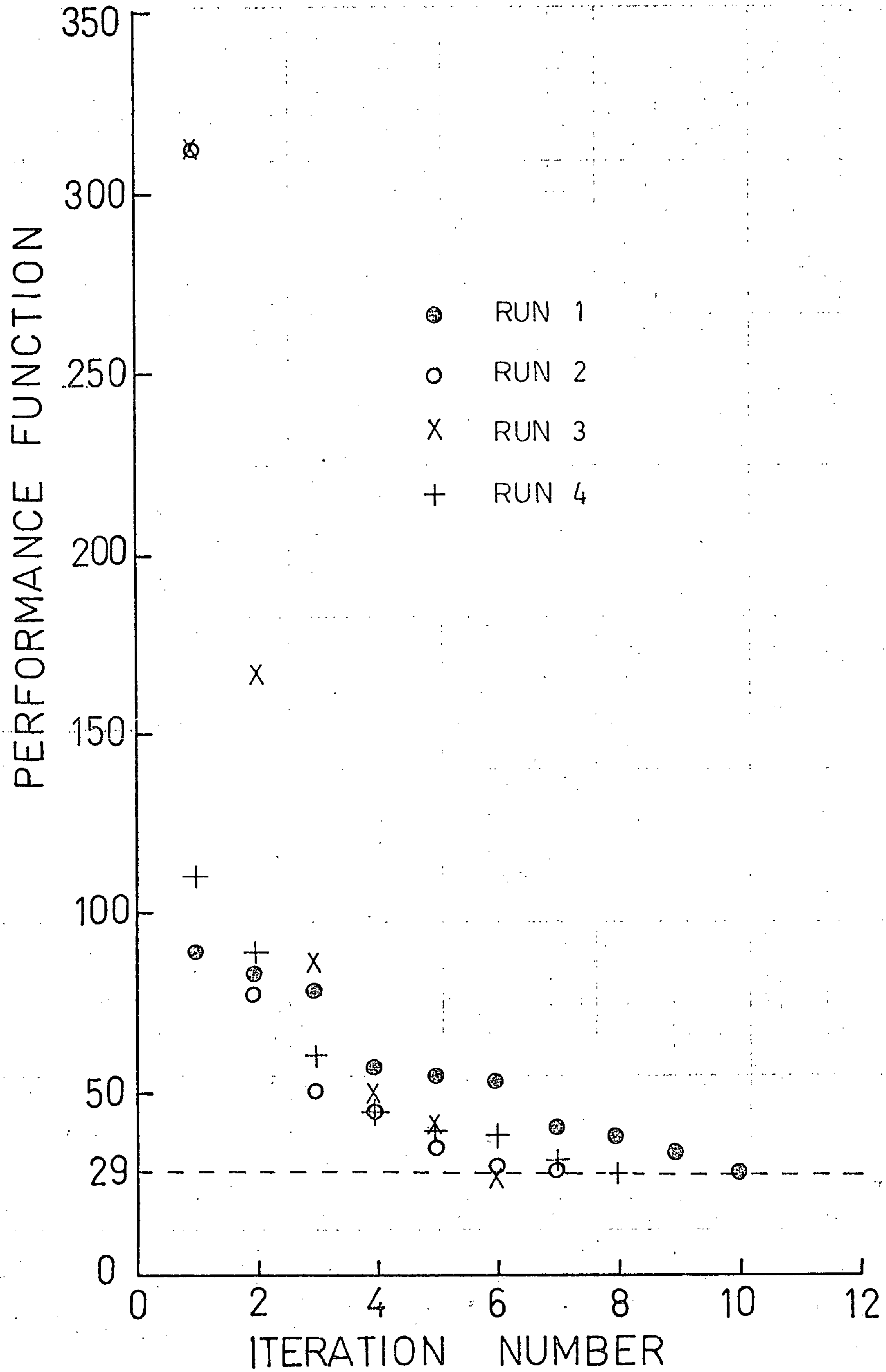


Fig. 3.1 Convergence of the Performance Function for the Computer-job Queuing Problem

3.6 Application II

3.6.1 An Optimal Decision Control Problem for Service Channels

A different kind of decision control problem occurs in the operation of service channels. Here the choice is usually between operating or shutting down a server. To keep a server active requires both operational and maintenance costs. On the other hand, to shut down a server at the expense of customers waiting in queues would incur another cost to the server.

Hence, the system performance function would consist of two terms: Total cost to the server system = (Idling cost) + (Operational cost) (3-16)

This cost structure will be applied to a practical example in the following section.

3.6.2 Example - Optimal Control of Traffic Congestion at an Intersection

A large amount of work on the traffic control of an intersection (5,7) has been done previously. The classical probability model is used in practically all of the cases. This example will adequately demonstrate the more realistic type of control one gets when the time-series model is employed.

Consider an intersection of two one-way traffic streams.

let: $x_k \in R^2$ denote the queue size at the intersection at time k

$\lambda_k \in R^2$ denote the number of vehicles arriving in the interval [k, k+1)

s denotes the number of vehicles allowed to pass the intersection in each green cycle of period equal to the interval [k, k+1).

If u_k is the decision control where $u_k \in L_2 = \left\{ \begin{vmatrix} 0 \\ 1 \end{vmatrix}, \begin{vmatrix} 1 \\ 0 \end{vmatrix} \right\}$, the dynamics of the process can thus be expressed as

$$x_{k+1} = x_k + \lambda_k - s\,u_k \tag{3-17}$$

Adopting the cost structure established in (3-16), the following cost functions are defined:

$c_k \,\varepsilon\, R^2$: The cost each traffic stream "pays" for a green cycle. This is the operational cost of the system.

Q_k, A(2 x 2) diagonal matrix whose elements are the queuing cost of each stream. This represents the cost each stream "pays" for a red cycle, which also reflects the idling cost of the intersection light. Hence, the cost functional can be written as:

$$J = \sum_{i=1}^{K-1} [x_i' \, Q_i \, x_i + u_i' \, c_i] + x_K' \, T \, x_K \tag{3-18}$$

Before applying the optimization algorithm of 3.3, it will be assumed that a predicted record of vehicle arrivals $\{\lambda_k\}$ $k = 1,\ldots,K-1$ is available K time instants ahead. A large number of current time-series analysis techniques[9,10,11] are available for such a prediction.

Application of the Discrete Minimum Principle yields the Hamiltonian:

$$H(x_k, u_k, p_{k+1}, k) = x_k' \, Q_k \, x_k + p_{k+1}' \, (\lambda_k - s\, u_k) + u_k' \, c_k \tag{3-19}$$

and costate equation:

$$p_k = p_{k+1} + 2\, Q_k \, x_k \tag{3-20}$$

with Boundary Conditions:

$$p_K = 2\, T\, x_K \tag{3-21}$$

$$x_1 = x \tag{3-22}$$

In the simulation below, the parameters assume the following values:

$$Q_k = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = T \; ; \quad c = \begin{vmatrix} 1 \\ 1 \end{vmatrix} \; ; \quad x = \begin{vmatrix} 10 \\ 30 \end{vmatrix}$$

$s = 10$

In addition, in order to simulate the effect of having inaccurate information on the arrivals $\{\lambda_k\}$, a noisy measurement on $\{\lambda_k\}$, is assumed:

$$\hat{\lambda}_k = \lambda_k + \gamma\xi_k$$

where $\gamma \triangleq$ noise index

ξ_k = a random sequence, with $|\xi_k| \leq 1$

Simulation results for 20 time units are shown in figures 3.2, 3.3. In Fig. 3.4, the effect of an open loop control based on inaccurate prediction on the arrivals is studied. It is apparent that for noise index $\gamma < 2$, the control resembles the optimal control where $\gamma = 0$.

It should also be noted that with zero noise index, the control obtained is optimal feedback control. This property is demonstrated in the complete result shown in Fig. 3.3.

3.6.3 Remarks

Remarks similar to those found in 3.5.3 can be applied to this section, that is a "short-term open-loop-feedback-optimal" control strategy can be employed in long-term on-line applications. Here, the information needed to update would be the vehicle arrival sequence $\{\lambda_k\}$. In addition, other parameters such as Q_k and s can also be updated as time progresses to accommodate the varying traffic demands of the day.

Finally, one can easily extend the above application to a traffic merging problem involving more than 2 streams of traffic.

3.7 Discussion

The decision control problem in queuing system has been formulated and the necessary condition of optimality have been derived by the application of the Discrete Minimum Principle. Solutions of the resulting TPBVP are obtained using gradient techniques. The simplicity of the

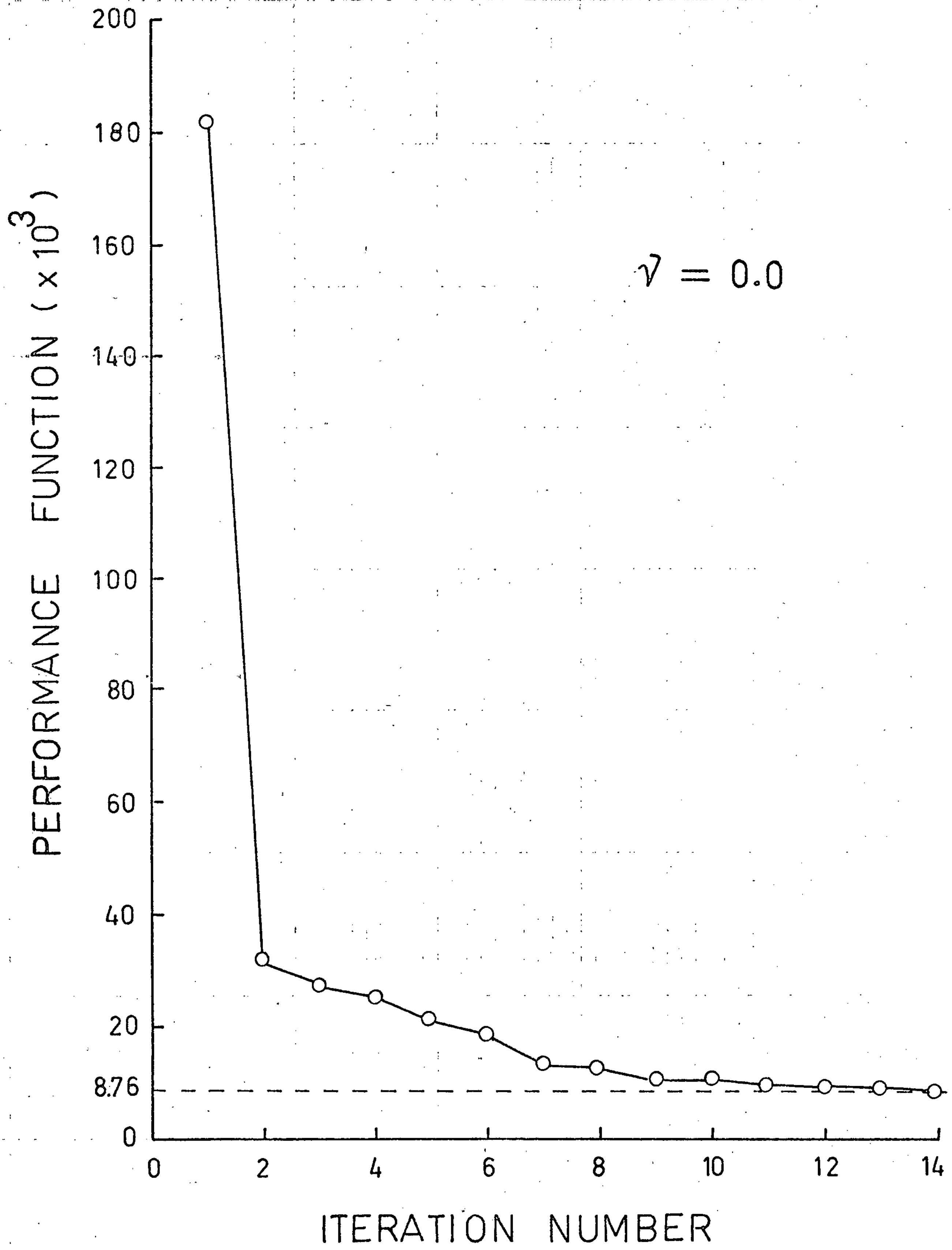


Fig. 3.2 Convergence of the Performance Function for the Traffic Intersection Control Problem

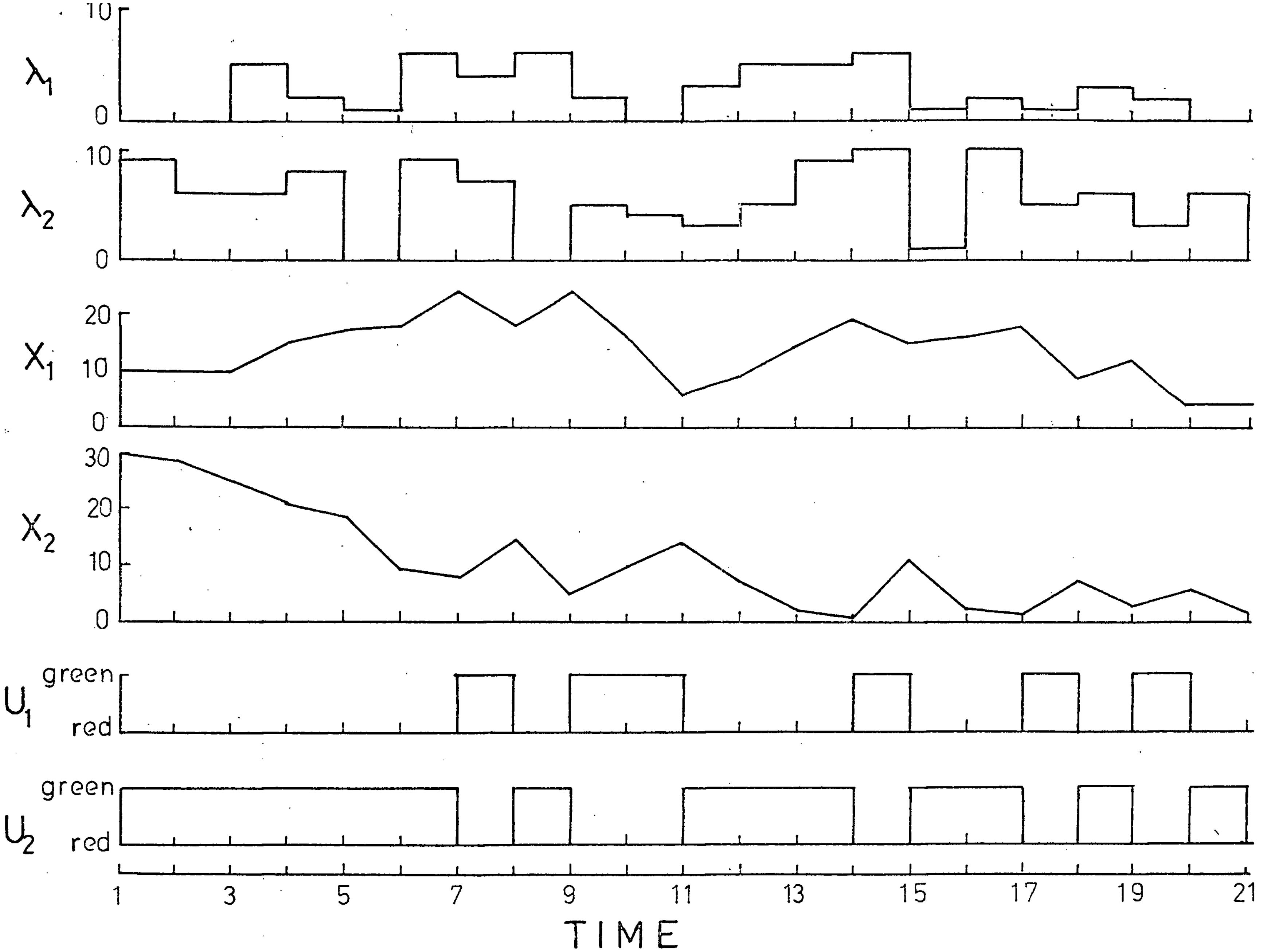


Fig. 3.3 Complete Result for the Traffic Intersection Control Problem

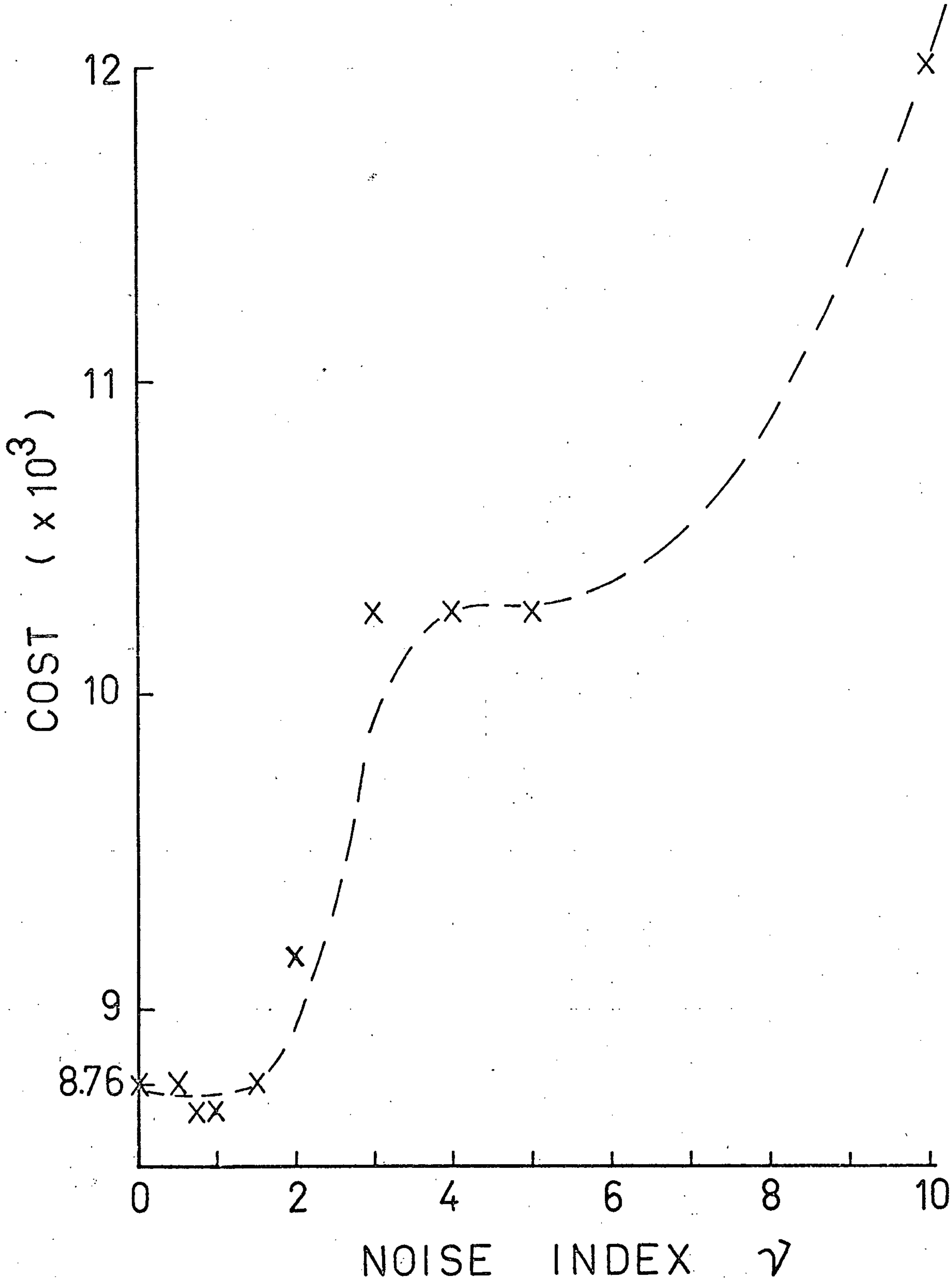


Fig. 3.4 Open-loop Control Cost versus Noise Index

suggested algorithm offers a feasible, if suboptimal, solution to a problem which otherwise would have been formidable using dynamic programming.

A fringe benefit of the algorithm is the possibility of extending it to on-line control applications. Using the "shifting interval" and open-loop-feedback-optimal technique, a short term adaptive scheme can be obtained.

Finally, one must observe the generality of the decision control model. The versatility of the model lies in the definition of the performance index (3-3). One can easily adapt the algorithm to accommodate different decision control situations by defining an appropriate cost model. This has been exemplified by the two applications in this chapter.

4. OPTIMAL AND ADAPTIVE REGULATOR CONTROL FOR QUEUING SYSTEMS

4.1 Introduction

The previous chapter treated the problem of decision control in queuing systems. As mentioned in Chapter 2, in queuing systems we are faced with another class of control, namely, regulator type of control. This type of control attempts to optimize the overall system performance by manipulation of a control mechanism, usually the service mechanism. The rest of this chapter will devote itself to such a problem. The generalized queuing model developed in Chapter 2 will be adopted. Both the optimal and adaptive control strategies will be discussed.

The generalized queuing model (2-1) is rewritten

$$q_{k+1} = q_k + B_k \lambda_k - C_k u_k$$

when $q_k \in R^n$, is the queue size at k

$\lambda_k \in R^m$, is the arrival in [k, k+1), and is assumed known or predictable

$u_k \in R^p$, is the server capacity control

B_k, C_k are (n x m) and (n x p) matrices respectively.

The results of decision control of the queuing or service channels will be registered in B_k and C_k. For instance, b_{ij} of B_k will denote the fraction of customers from the j^{th} arrival streams that selected the i^{th} queue; c_{il} of C_k will denote the fraction of customers from the i^{th} queuing channel that was served by the l^{th} server. As far as the server control is concerned, B_k and C_k must be known, or identified if unknown, before an optimal solution can be evaluated.

In the following sections, the question of identifying B_k and C_k will be treated separately from the optimal control problem. Combining the two schemes, an adaptive control algorithm is obtained.

4.2 An On-Line Identifier for Queuing Systems

Assume that B_k and C_k are either constant or quasi-stationary. Rewriting the queuing model

$$q_{k+1} = [I \,\vdots\, B \,\vdots\, C] \begin{bmatrix} q_k \\ \lambda_k \\ -u_k \end{bmatrix} = S^T \psi_{k+1} \tag{4-1}$$

Where $S^T \triangleq [I \,\vdots\, B \,\vdots\, C]$ is an $n \times (n + m + p)$ matrix and I is an $(n \times n)$ unit matrix.

$$\psi_{k+1} \triangleq \begin{bmatrix} q_k \\ \lambda_k \\ -u_k \end{bmatrix} , \quad \text{an } (n + m + p) \text{ - vector}$$

After making k observations of q_k, we have

$$\begin{bmatrix} q_1' \\ q_2' \\ \cdot \\ \cdot \\ \cdot \\ q_k' \end{bmatrix} = \begin{bmatrix} \psi_1' \\ \psi_2' \\ \cdot \\ \cdot \\ \cdot \\ \psi_k' \end{bmatrix} S \quad \text{or} \quad Q_k = \Theta_k S \tag{4-2}$$

where $(.)'$ denotes the transpose of a vector,

Q_k is a $(k \times n)$ matrix

Θ_k is a $(k \times (n + m + p))$ matrix

Note the similarity of (4-2) to the observation equation in state estimation problems[23]:

$$Z_k = H_k x \tag{4-3}$$

where Z_k = observation vector

H_k = observation matrix

x = unknown state to be estimated

Using the least squares state estimation[15] technique in (4-3) gives

$$x_k = (H_k^T H_k)^{-1} H_k^T Z_k$$

where $x_k \triangleq$ estimated value of x after k observations,

A matrix version of the above least squares estimation (MLSE) can similarly be derived (Appendix III) for (4-2).

We have
$$\hat{S}_k = (\Theta_k^T \Theta_k)^{-1} \Theta_k^T Q_k \tag{4-4}$$

which minimizes the estimation error functional

$$J(\hat{S}_k) \triangleq \sum_{i=1}^{k} (q_i' - \psi_i' \hat{S}_k)(q_i' - \psi_i' \hat{S}_k)'$$

$$= \mathrm{Tr}\,[(Q_k - \Theta_k \hat{S}_k)(Q_k - \Theta_k \hat{S}_k)^T] \tag{4-5}$$

where Tr (.) denotes the trace of a matrix. A sequential form for (4-4) is derived; this makes the identifier an on-line identifier for the system:

$$\hat{S}_{k+1} = \hat{S}_k + P_{k+1} \psi_{k+1} (q_{k+1}' - \psi_{k+1}' \hat{S}_k) \tag{4-6}$$

when

$$P_{k+1} = P_k - P_k \psi_{k+1} (I + \psi_{k+1}' P_k \psi_{k+1})^{-1} \psi'_{k+1} P_k \tag{4-7}$$

where $\hat{S}_k$: an n x (n + m + p) matrix

P_k: an (n + m + p) x (n + m + p) matrix

P_o is unknown. However, a number of simulation runs with $P_o = I$ yield fairly rapid convergence for the algorithm. Of course there are better ways of finding P_o[23], but such sophistication is beyond the scope of this thesis. A fringe benefit of the MLSE technique is that $(I + \psi'_{k+;} P_k \psi_{k+1})$ is now a scalar quantity. The rather arduous job of matrix inversion in vector LSE is eliminated here.

Finally, it should be noted that it has been assumed that the queue sizes, q_k, can be measured directly. This isnot too unreasonable an assumption in queuing systems. For low system noise level and small fluctuations of the model parameters, the MLSE can give fairly satisfactory results as will be demonstrated by an application below.

4.3 An On-Line Adaptive Controller for Queuing Systems

The stochastic service-rate control problem may be stated: Find a service-rate sequence: $\left\{ u_k \right\}_{k=o,\ldots,K-1}$ which minimizes the performance index

$$J = \underset{\{q_k\}_{k=o,\ldots,K}}{E} \left\{ \sum_{i=0}^{K-1} [q_i' Q_i q_i + u_i' R_i u_i] + q_K' Q_K q_K \right\} \qquad (4\text{-}8)$$

where $\underset{x}{E}\{f(x)\}$ denotes the expected values of the function f(.) of the random variable x.

Q_i is an (n x n) matrix, $\triangleq$ Cost of queuing in the system

R_i is an (n x p) matrix $\triangleq$ Cost of operation of server

$q_i \in R^n$; $u_i \in R^p$

Due to the unknown parameters in the systems, the predicted system model will be

$$q_{k+1}\big|_k = q_k + \hat{B}_k\big|_k \lambda_k - \hat{C}_k\big|_k u_k \qquad (4\text{-}9)$$

where $f_j\big|_i \triangleq$ expected value of the function f at time j, based on the information available up to and including time i.

$$q_{k+1}\big|_k \triangleq \underset{q_{k+1}}{E} (q_{k+1}\big| q_k, q_{k-1}, q_{k-2},\ldots,q_0; u_k, u_{k-1},\ldots,u_0,\lambda_k,\lambda_{k-1},\ldots,\lambda_0)$$

$$= \underset{q_{k+1}}{E} \left(q_{k+1} \middle| q_k, u_k, \lambda_k\right) \tag{4-10}$$

$\hat{B}_{k|k}$, $\hat{C}_{k|k}$ are the identified values of B_k and C_k matrices after k measurements. Equation (4-10) is a direct consequence of the markovian nature of the model (4-9). $\{\lambda_k\}_{k=o,1,..,(K-1)}$ will be assumed known or predictable.

Using dynamic programming (Appendix IV), a stochastic feedback control algorithm

$$u_{j|k} = A_{j|k}\, x_j + b_{j|k};\ j=k,\ k+1,\ldots,K-1 \tag{4-11}$$

can be obtained where the control gains $A_{j|k}$ and $b_{j|k}$ are a (pxn) matrix and a p-vector respectively. For j = k, k+1,...K-1,

$$A_{j|k} \triangleq \Lambda^{-1}_{j|k}\ \hat{C}^T_{k|k}\ L_{j+1|k} \tag{4-12}$$

$$b_{j|k} \triangleq \Lambda^{-1}_{j|k}\ [\hat{C}^T_{k|k}\ (m_{j+1|k} + L_{j+1|k}\ \hat{B}_{k|k}\ \lambda_j)] \tag{4-13}$$

with

$$\Lambda_{j|k} \triangleq R_j + \hat{C}^T_{k|k}\ L_{j+1|k}\ \hat{C}_{k|k} \tag{4-14}$$

$$\hat{B}_{k|k} \triangleq E(B_k | q_k, u_k, \lambda_k);\quad \hat{C}_{k|k} \triangleq E(C_k | q_k, u_k, \lambda_k) \tag{4-15}$$

$L_{j+k|k}$ and $m_{j+1|k}$ are obtained from the following recursive relations:

$$L_{j|k} = Q_j + [I - L^T_{j+1|k}\ \hat{C}_{k|k}\ \Lambda^{-1}_{j|k}\ \hat{C}^T_{k|k}]\ L_{j+1|k} \tag{4-16}$$

$$m_{j|k} = [I - L^T_{j+1|k}\ \hat{C}_{k|k}\ \Lambda^{-1}_{j|k}\ \hat{C}^T_{k|k}][m_{j+1|k} + L_{j+1|k}\ \hat{B}_{k|k}\ \lambda_j] \tag{4-17}$$

with the boundary conditions:

$$L_{K|k} = T > 0 \tag{4-18}$$

$$m_{K|k} = 0 \tag{4-19}$$

It is noted that the sequence $\{\lambda_k\}$, treated as a known disturbance input in (4-9), is assumed available at any stage, as shown in Equation (4-17). This assumption is essential if the control (4-11) is to be optimal; otherwise, a suboptimal control would be obtained.

An adaptive control is obtained when the feedback control (4-11) is applied simultaneously with the least-squares identifier of 4.2 to the system.

4.4 Application I

In section 3.6, we considered the traffic control problem using decision control. In this section, we shall investigate the application of regulator control to another traffic problem: the merging problem.

4.4.1 Optimal Traffic Control for a Highway Merging Problem

Consider a section of a highway consisting of n merging points located in series (Fig. 4.1), assuming that all the traffic is in the same direciton. Furthermore, the merge points are assumed equally spaced; and all the vehicles to move at the same constant speed, or the same constant average speed. The dynamics involved in vehicle acceleration will be ignored.

Let: $x_k \in R^n$ denote the total number of vehicles at each of the merging points at time k.

$\lambda_k \in R^n$ denote the total number of vehicles arriving at the merging points in the time interval [k, k+1).

$s_k \in R^n$ denotes the number of vehicles released from each merge point during [k, k+1).

$p > 0$, denotes the number of time units each vehicle takes to travel from one merge point to the next. In other words, the

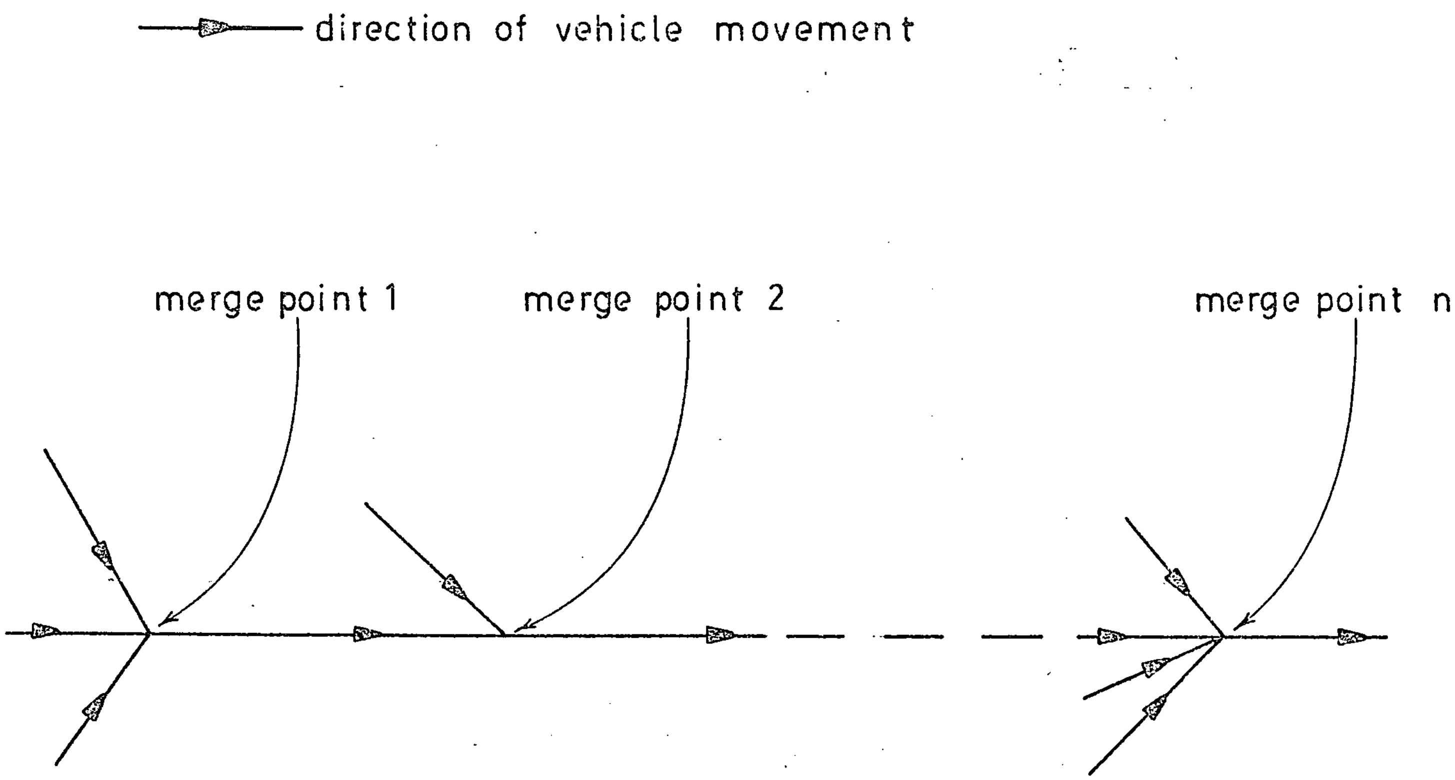


Fig. 4.1 Merging Points on a Highway

average speed of the vehicles is given by:

$$\text{Average vehicle speed} = \frac{\text{Distance between successive merge points}}{p} \tag{4-20}$$

Hence, at time k, the total input to the i^{th} merge point will be $\lambda_k^i + s_{k-p}^{i-1}$ where the superscript i denotes the i^{th} element of a vector.

The overall merging process can be represented by

$$x_{k+1} = x_k + \lambda_k + \bar{s}_{k-p} - s_k \tag{4-21}$$

where
$$\bar{s}_{k-p} \triangleq \begin{bmatrix} 0 \\ s_{k-p}^1 \\ s_{k-p}^2 \\ \vdots \\ s_{k-p}^{n-1} \end{bmatrix} \tag{4-22}$$

denotes the input into any junction from the preceding one.

To obtain an optimal overall control of the vehicle merge, we shall formulate the following control problem:

Given, the system dynamics represented by (4-21), (4-22), find a control sequence $\{s_k\}_{k=1,\ldots,K}$ which will minimize the quadratic cost function

$$J = \sum_{i=1}^{K} (x_i' Q_i x_i + s_i R_i s_i) + x_{K+1}' T x_{K+1} \tag{4-23}$$

where Q_i, R_i, which are (n x n) diagonal matrices, denote respectively the vehicle waiting cost and the cost of releasing vehicles at the junctions. T, a positive definite (n x n) matrix, is the cost of incomplete vehicle release at the end of the planning horizon [1, K].

Employing the control algorithm discussed in 4.3, we obtain the following feedback control strategy:

$$s_k = [R_k + P_{k+1}]^{-1} [m_{k+1} + P_{k+1} (x_k + \beta_k)] \tag{4-24}$$

where $\quad \beta_k \triangleq \lambda_k + \bar{s}_{k-p}$ (4-25)

P_k, m_k are computed from the recursive relations

$$P_k = Q_k + [I - P_{k+1}^T \Lambda_k^{-1}] P_{k+1} \tag{4-26}$$

$$m_k = [I - P_{k+1}^T \Lambda_k^{-1}] [m_{k+1} + P_{k+1} \beta_k] \tag{4-27}$$

where

$$\Lambda_k = R_k + P_{k+1} \tag{4-28}$$

and the boundary conditons

$$P_{K+1} = T > 0 \tag{4-29}$$

$$m_{K+1} = 0 \tag{4-30}$$

A system with 4 merging junctions and different arrival intensities at each junciton is simulated with the following parameter values:

$$Q_k = \begin{pmatrix} 10.0 & 0 & 0 & 0 \\ 0 & 10.0 & 0 & 0 \\ 0 & 0 & 10.0 & 0 \\ 0 & 0 & 0 & 10.0 \end{pmatrix} \; ; \; R_k = \begin{pmatrix} 20.0 & 0 & 0 & 0 \\ 0 & 20.0 & 0 & 0 \\ 0 & 0 & 20.0 & 0 \\ 0 & 0 & 0 & 20.0 \end{pmatrix}$$

$$T = \begin{pmatrix} 30.0 & 0 & 0 & 0 \\ 0 & 30.0 & 0 & 0 \\ 0 & 0 & 30.0 & 0 \\ 0 & 0 & 0 & 30.0 \end{pmatrix} \; ; \; K = 24$$

Simulation results for $p = 10$ and $p = 25$ are shown in Table 4-1 and Table 4-2 respectively. In Fig. 4.2, the relation between the junction delay time p and the total cost is studied. It appears that for all $p \geq K$, the total cost stays constant. In other words, the junctions are in effect decoupled as far as the overall control is concerned. However, for $p < K$, the control algorithm (4-24) to (4-30) is in fact repeated once every p time units in order to update $\bar{s}_k$. Consequently, a

MERGING POINT NO. 1

TIME	ARRIVAL	QUEUES	CONTROL
1	1.	50	30
2	2.	21	19
3	11.	4	18
4	27.	0	21
5	25.	6	19
6	12.	12	13
7	6.	11	9
8	3.	8	5
9	0.	6	3
10	6.	3	2
11	18.	7	18
12	18.	7	16
13	11.	9	12
14	4.	8	8
15	7.	4	7
16	2.	4	6
17	4.	0	6
18	10.	0	9
19	17.	1	9
20	8.	9	5
21	11.	12	17
22	14.	6	14
23	13.	6	12
24	10.	7	10
25		7	

MERGING POINT NO. 2

TIME	ARRIVAL	QUEUES	CONTROL
1	3.	50	46
2	19.	17	37
3	24.	9	34
4	9.	9	29
5	28.	0	31
6	17.	7	27
7	14.	7	24
8	16.	7	22
9	2.	11	16
10	26.	7	14
11	16.	29	53
12	12.	22	42
13	15.	11	36
14	9.	8	33
15	18.	5	30
16	8.	12	25
17	14.	8	22
18	12.	9	18
19	10.	8	15
20	32.	6	13
21	13.	27	40
22	13.	18	31
23	6.	16	23
24	13.	11	19
25		13	

MERGING POINT NO. 3

TIME	ARRIVAL	QUEUES	CONTROL
1	11.	50	52
2	21.	19	43
3	33.	7	41
4	18.	9	37
5	25.	0	37
6	41.	0	40
7	32.	11	35
8	7.	18	26
9	22.	9	22
10	10.	19	12
11	4.	27	75
12	44.	2	74
13	38.	9	70
14	36.	11	66
15	32.	10	61
16	37.	12	56
17	5.	20	47
18	34.	2	47
19	44.	11	41
20	38.	30	27
21	12.	55	93
22	35.	27	79
23	33.	25	67
24	31.	27	54
25		37	

MERGING POINT NO. 4

TIME	ARRIVAL	QUEUES	CONTROL
1	42.	50	73
2	35.	29	59
3	46.	15	53
4	2.	18	44
5	45.	0	59
6	84.	0	64
7	27.	30	49
8	2.	18	40
9	76.	0	49
10	48.	37	31
11	27.	64	122
12	69.	21	113
13	81.	20	103
14	5.	39	83
15	73.	0	86
16	35.	24	75
17	31.	24	63
18	25.	27	50
19	22.	28	37
20	16.	35	20
21	64.	43	130
22	0.	52	104
23	9.	22	94
24	78.	7	90
25		61	

TOTAL COST = 5.3067E 06

Table 4.1 Optimal Control for the Highway Merging Problem with $p = 10$

MERGING POINT NO. 1

TIME	ARRIVAL	QUEUES	CONTROL
1	1.	50	30
2	2.	21	19
3	11.	4	18
4	27.	0	21
5	25.	6	19
6	12.	12	14
7	6.	10	10
8	3.	6	7
9	0.	2	6
10	6.	0	10
11	18.	0	15
12	18.	3	14
13	11.	7	11
14	4.	7	8
15	7.	3	7
16	2.	3	6
17	4.	0	7
18	10.	0	11
19	17.	0	13
20	8.	4	11
21	11.	1	11
22	14.	1	12
23	13.	3	11
24	10.	5	8
25		7	

MERGING POINT NO. 2

TIME	ARRIVAL	QUEUES	CONTROL
1	3.	50	46
2	19.	17	37
3	24.	9	34
4	9.	9	29
5	28.	0	31
6	17.	7	28
7	14.	6	26
8	16.	4	25
9	2.	5	23
10	26.	0	30
11	16.	6	27
12	12.	5	25
13	15.	2	24
14	9.	3	23
15	18.	0	24
16	8.	4	22
17	14.	0	23
18	12.	1	24
19	10.	0	25
20	32.	0	31
21	13.	11	26
22	13.	8	23
23	6.	8	19
24	13.	5	16
25		12	

MERGING POINT NO. 3

TIME	ARRIVAL	QUEUES	CONTROL
1	11.	50	52
2	21.	19	44
3	33.	6	41
4	18.	8	37
5	25.	0	38
6	41.	0	41
7	32.	10	37
8	7.	15	30
9	22.	2	29
10	10.	5	28
11	4.	0	31
12	44.	0	49
13	38.	5	48
14	36.	5	45
15	32.	6	43
16	37.	5	41
17	5.	11	35
18	34.	0	45
19	44.	0	46
20	38.	8	43
21	12.	13	37
22	35.	0	39
23	33.	6	37
24	31.	12	31
25		22	

MERGING POINT NO. 4

TIME	ARRIVAL	QUEUES	CONTROL
1	42.	50	73
2	35.	29	60
3	46.	14	53
4	2.	17	44
5	45.	0	60
6	84.	0	66
7	27.	28	53
8	2.	12	47
9	76.	0	71
10	48.	15	63
11	27.	10	59
12	69.	0	71
13	81.	8	68
14	5.	31	54
15	73.	0	62
16	35.	21	51
17	31.	15	45
18	25.	11	40
19	22.	6	37
20	16.	1	37
21	64.	0	48
22	0.	26	35
23	9.	1	35
24	78.	0	52
25		36	

TOTAL COST = 2.9000E 06

Table 4.2 Optimal Control for the Highway Merging Problem with $p = 25$

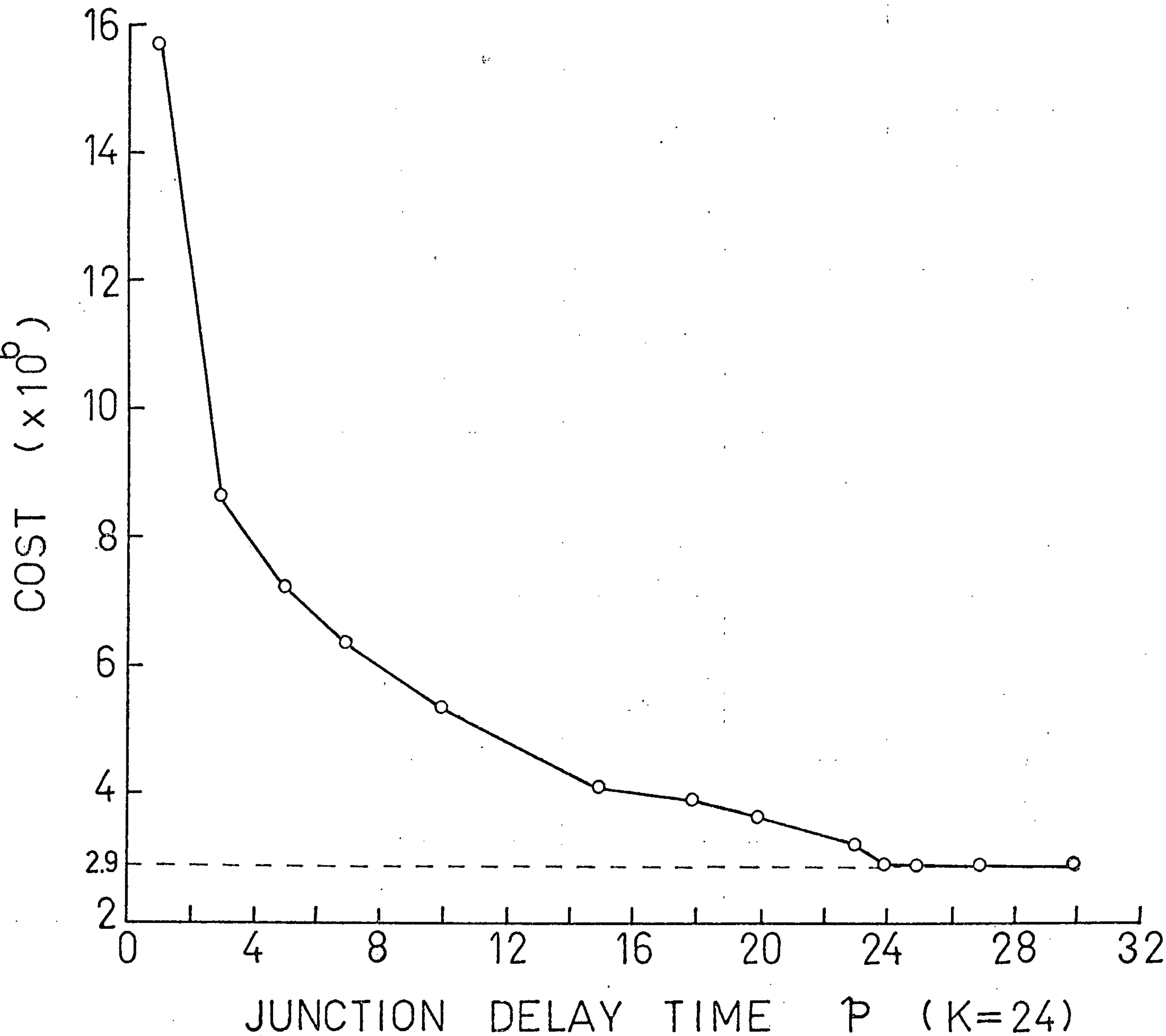


Fig. 4.2 The Relationship between the Total Cost and Junction Delay Time for the Highway Merging Problem

suboptimal control is obtained. In addition, the cost parameters Q_k, R_k, T can be made time-varying to reflect the fluctuating traffic demands of the day.

4.4.2 Remarks

Once again the familiar assumption of known arrival sequence $\{\lambda_k\}$ holds in this application. For long term operations ($K \to \infty$), $\{\lambda_k\}$ will have to be updated constantly in order to account for any prediction error. The "shifting interval" concept discussed in the last chapter can again be applied here. An on-line adaptive control will result through the use of the open-loop-feedback-optimal control technique combining with the prediction of $\{\lambda_k\}$.

4.5 Application II

In 4.3, we mentioned the possibility of an on-line adaptive control through simultaneous application of the system identifier of 4.2 and the controller of 4.3. In this section, an application ofthis technique to a scheduling problem will be discussed.

4.5.1 Optimal and Adaptive Scheduling of Operations in a Hospital

Consider a hospital with p operating rooms and enough surgical facilities for n types of operations. Patients demanding operations are first transferred from the hospital ward into the operating ward where they are screened for surgery or rejection. In the case of surgery, they are allowed to join a queue for the type of operation (one of the n-type) each requires. If patient's case is a "false alarm" or a relatively mild case compared to other emergency cases, he is rejected and returned to the hospital ward. The complete queuing phenomenon involved in the surgery process is shown in Fig. 4.3.

In view of the number of patients in the surgery queue and the

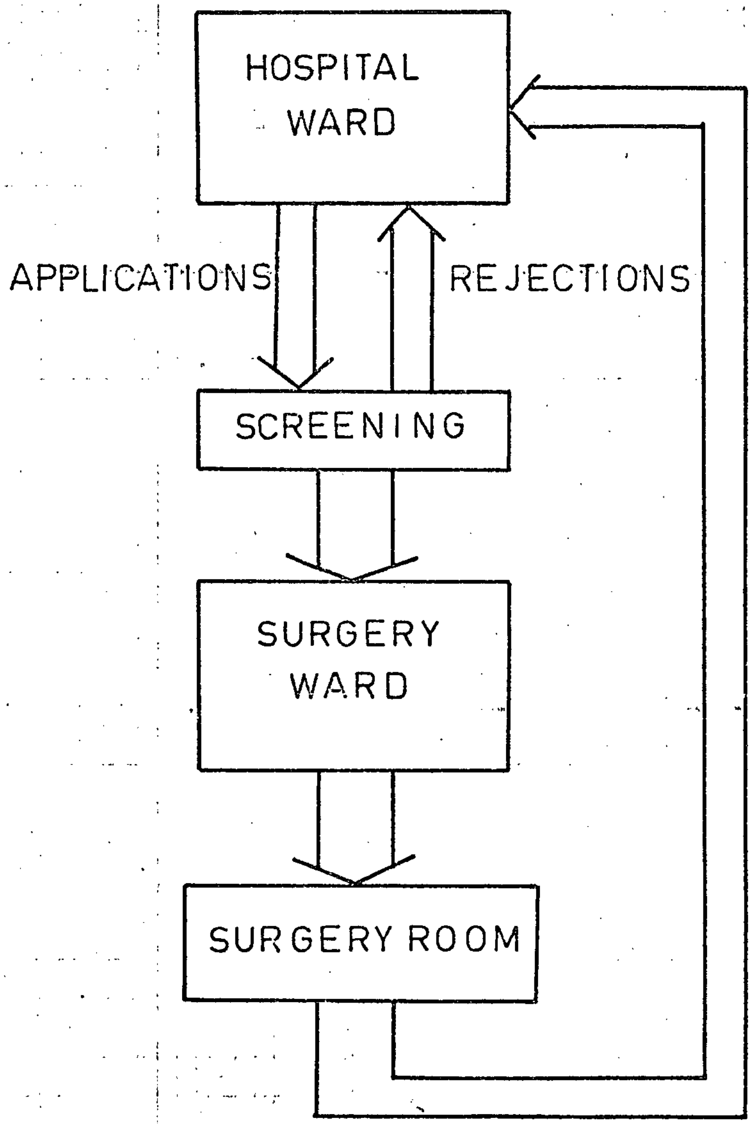


Fig. 4.3 The Queuing Phenomenon for a Surgery Process in a Hospital

cost of operations, the nurse has to decide on an optimal operation schedule at the beginning of each period (say a month). In addition, the type of operations performed in each operating room may vary from time to time (say every week) due to complications arising from the surgeon's schedule or the relocation of equipments. Hence the operation schedule decided on earlier will have to be revised constantly in order to cope with the varying situation. Under such circumstances, an on-line adaptive scheduling strategy would be most desirable.

The optimal scheduling problem can be formulated as a control problem as follows:

Let $x_k \in R^n$ denote the number of patients waiting for different types of operations in the operating ward at time k.

$\lambda_k \in R^n$ denote the number of new applications for the operations before screening process during the period [k, k+1).

$u_k \in R^p$ denote the number of operations performed in the operating rooms during the period [k, k+1).

Thus, the surgical queuing process can be expressed as:

$$x_{k+1} = x_k + \lambda_k - C_k\, u_k \tag{4-31}$$

where C_k, an (n x p) matrix, represents the fractional amount of the type of operations performed in different operating room. In effect, $c_{ij}\, u_j$ is the total number of the i^{th} -type operations performed in the j^{th} operating room in the period [k, k+1).

The mathematical statement of the scheduling problem is as follows:

Given the process dynamics (4-31), find a schedule $\{u_k\}_{k=0,\ldots,(K-1)}$ which will minimize the quadratic cost function

$$J = \sum_{i=0}^{K-1} [x_i' \; Q_i \; x_i + u_i' \; R_i \; u_i] \;+\; x_K' \; T \; x_K \tag{4-32}$$

where Q_k, an (n x n) matrix, denotes the cost of waiting for operation in the surgery ward ($x_k > 0$), or the cost of rejecting patients ($x_k < 0$).

R_k, a (p x p) matrix, denotes the cost of surgery and T, an (n x n) matrix, is the cost of incomplete operations at the end of the scheduling period.

In the case of adaptive scheduling, when C_k is unknown, the cost function (4.32) will be replaced by

$$J = \underset{\{x_k\} \; k=0,\ldots,K}{E} \left[\sum_{i=0}^{K-1} (x_i' \; Q_i \; x_i + u_i' \; R_i \; u_i) \;+\; x_K' \; Tx_K \right] \tag{4-33}$$

where x_k is now an random variable.

The on-line adaptive control algorithm (4.11) to (4.19) can then be applied with $\hat{B}_{k|k}$ replaced by I, a unit matrix. At any time k the optimal control sequence $\{u_i\}_{i=k,k+1,\ldots,K-1}$ can be computed from the following equations:

$$u_i = A_i \; x_i + b_i \tag{4-34}$$

where

$$A_i = \Lambda_i^{-1} \; \hat{C}_k^t \; P_{i+1} \tag{4-35}$$

$$b_i = \Lambda_i^{-1} \; (\hat{C}_k^t \; [m_{i+1} + P_{i+1} \; \lambda_i]) \tag{4-36}$$

$$\Lambda_i \triangleq R_i + \hat{C}_k^t \; P_{i+1} \; \hat{C}_k \tag{4-37}$$

$$P_i = Q_i + [I - P_{i+1}^t \; \hat{C}_k \; \Lambda_i^{-1} \; C_k^t] \; P_{i+1} \tag{4-38}$$

$$m_i = [I - P_{i+1}^t \; \hat{C}_k \; \Lambda_i^{-1} \; \hat{C}_k^t][m_{i+1} + P_{i+1} \; \lambda_i] \tag{4-39}$$

where $(.)^t$ denotes the transpose of a matrix, $\hat{C}_k$ is the identified value of C_k which is assumed constant during the interval [k,...,K-1]. The on-line identifier of 4.2 can be used for this purpose.

In the case of unknown C_k, only the first element of the sequence $\{u_i\}_{i=k,\ldots,K-1}$ is transmitted at time k. A new sequence is generated at k+1, and the same procedure continues. Hence, this is in fact an open-loop-feedback-optimal control strategy.

In the case where C_k is known, the complete sequence can be used, and the resulting control function will be optimal feedback.

In the simulation below, the following values have been used:

$$n = 2; \quad p = 3; \quad K = 30$$

$$C_k = \begin{pmatrix} 0.4 & 0.3 & 0.6 \\ 0.6 & 0.7 & 0.4 \end{pmatrix} \quad ; \quad Q_k = \begin{pmatrix} 10.0 & 0.0 \\ 0.0 & 10.0 \end{pmatrix} \quad ;$$

$$T = \begin{pmatrix} 50.0 & 0.0 \\ 0.0 & 50.0 \end{pmatrix}$$

$$R_k = \begin{pmatrix} 30.0 & 0.0 & 0.0 \\ 0.0 & 30.0 & 0.0 \\ 0.0 & 0.0 & 30.0 \end{pmatrix} \quad ; \quad x_0 = \begin{pmatrix} 50 \\ 50 \end{pmatrix}$$

The arrival sequence $\{\lambda_k\}_{k=1,\ldots,29}$ is assumed known (Fig. 4.4). Several runs have been made on the same system for both the optimal (C_k known) and the adaptive case (C_k unknown and estimated through a least-squares estimator). The identified values for C_k are shown in Fig. 4.5.

Fig. 4.6 gives a comparison between the optimal and adaptive state trajectories. Similar comparison can be made for the feedback control gain A_k and control bias b_k. Only one element from each is plotted in Fig. 4.7. The other elements behaved similarly. The complete control schedules are shown in Fig. 4.8 and Fig. 4.9.

Examination of the plots will demonstrate the nature of an

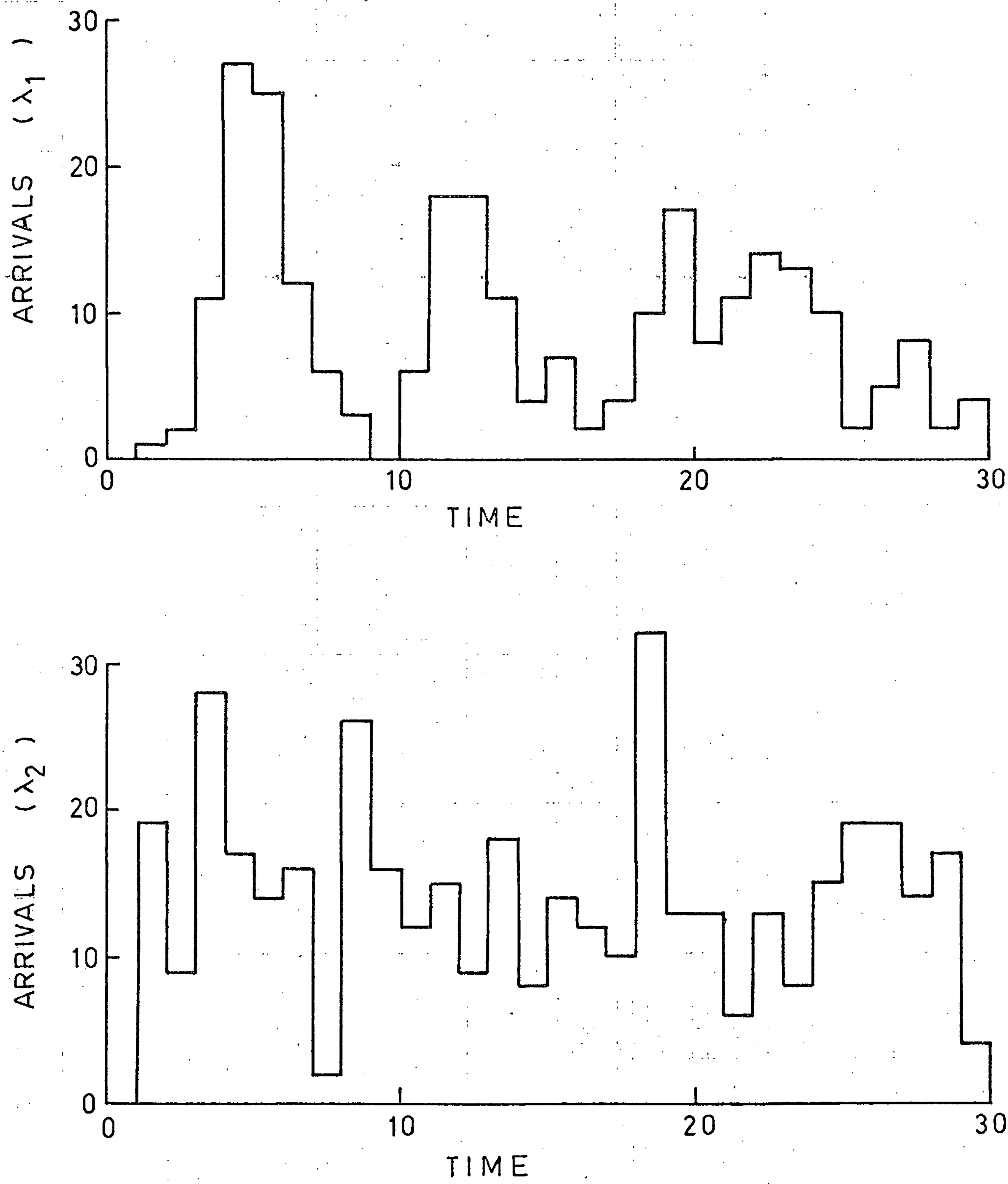
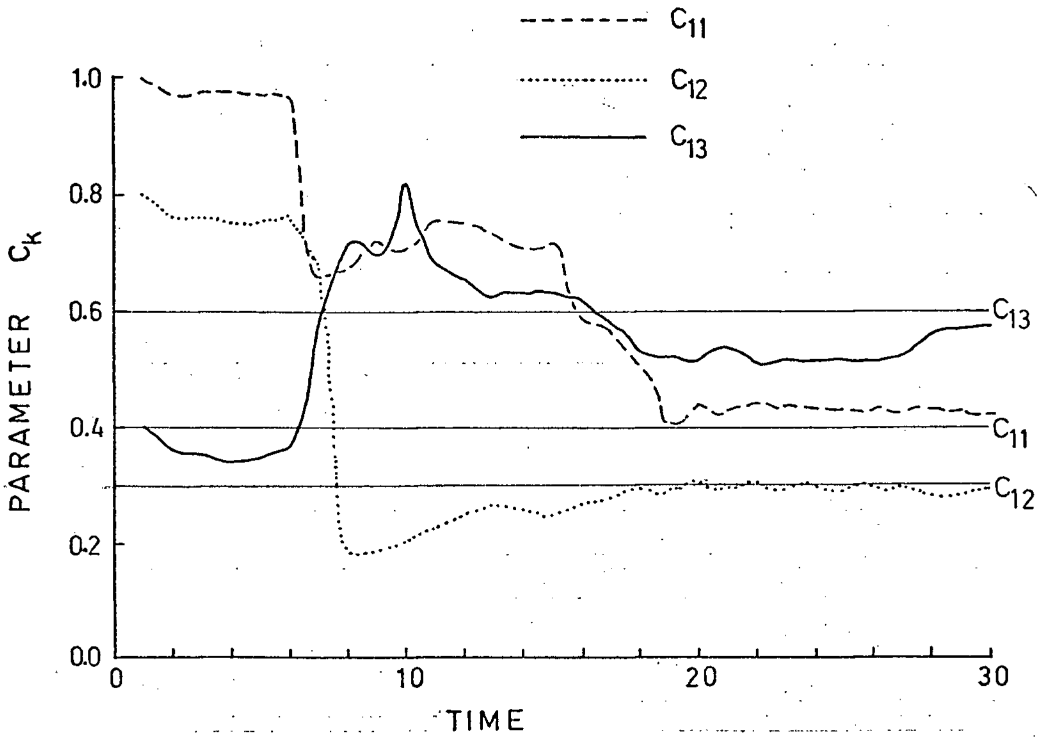


Fig. 4.4 The Arrival Sequences for the Optimal and Adaptive Operation-scheduling Problem



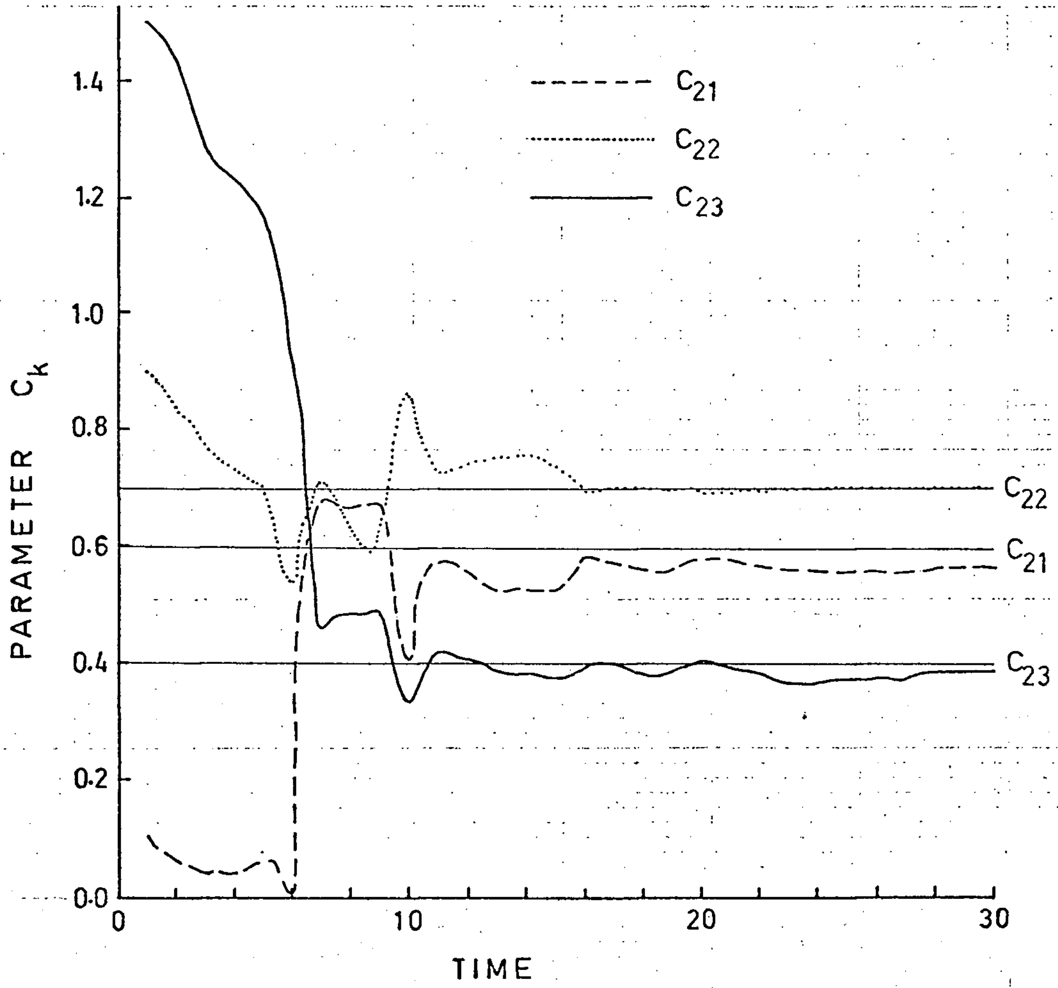


Fig. 4.5 On-line Identification of C_k

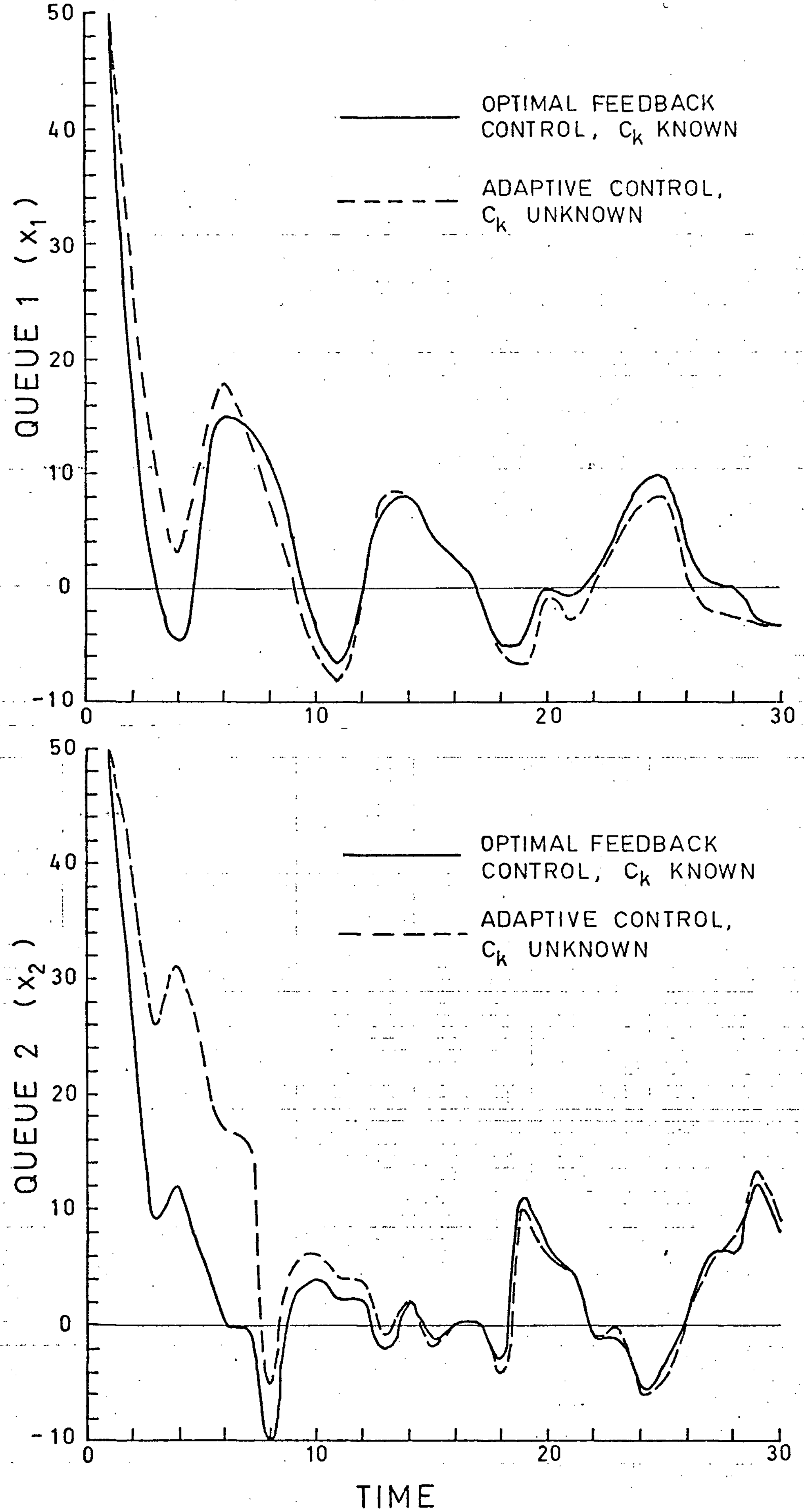


Fig. 4.6 Comparison Between the Optimal Feedback Control State Trajectory and the Adaptive Control State Trajectory

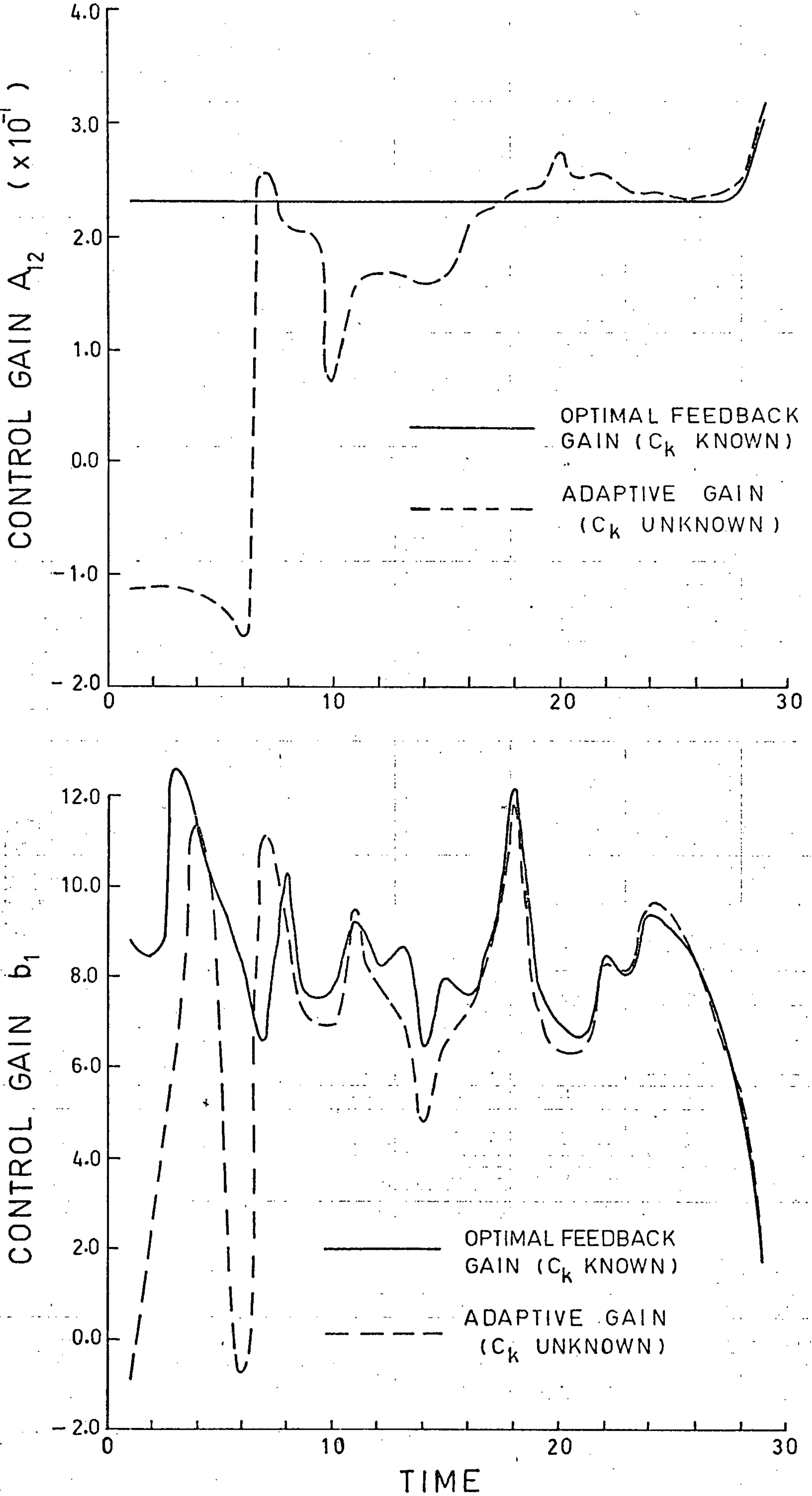


Fig. 4.7 Comparison Between the Optimal Feedback Control Gain and the Adaptive Control Gain

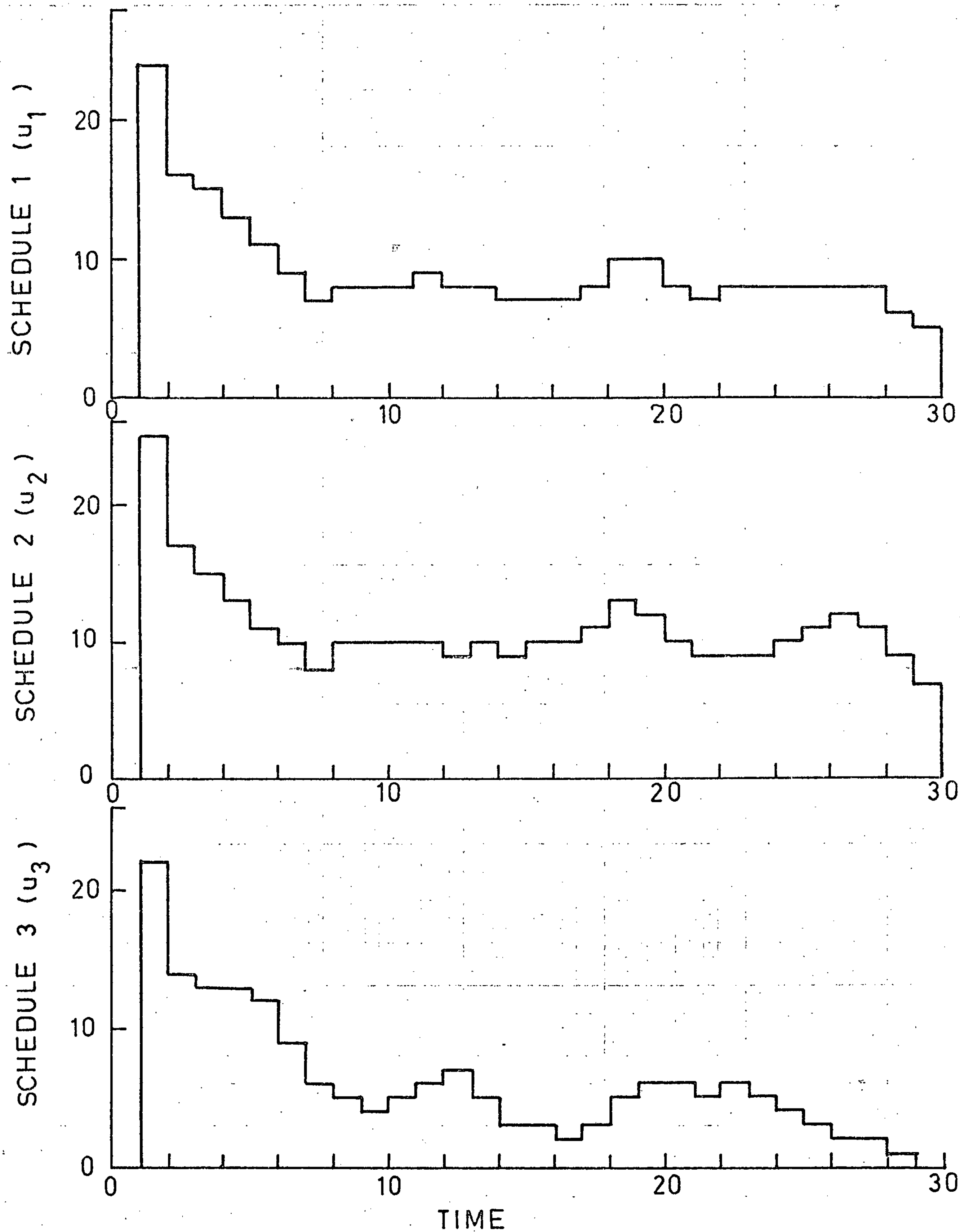


Fig. 4.8 Optimal Scheduling of Operations with C_k known

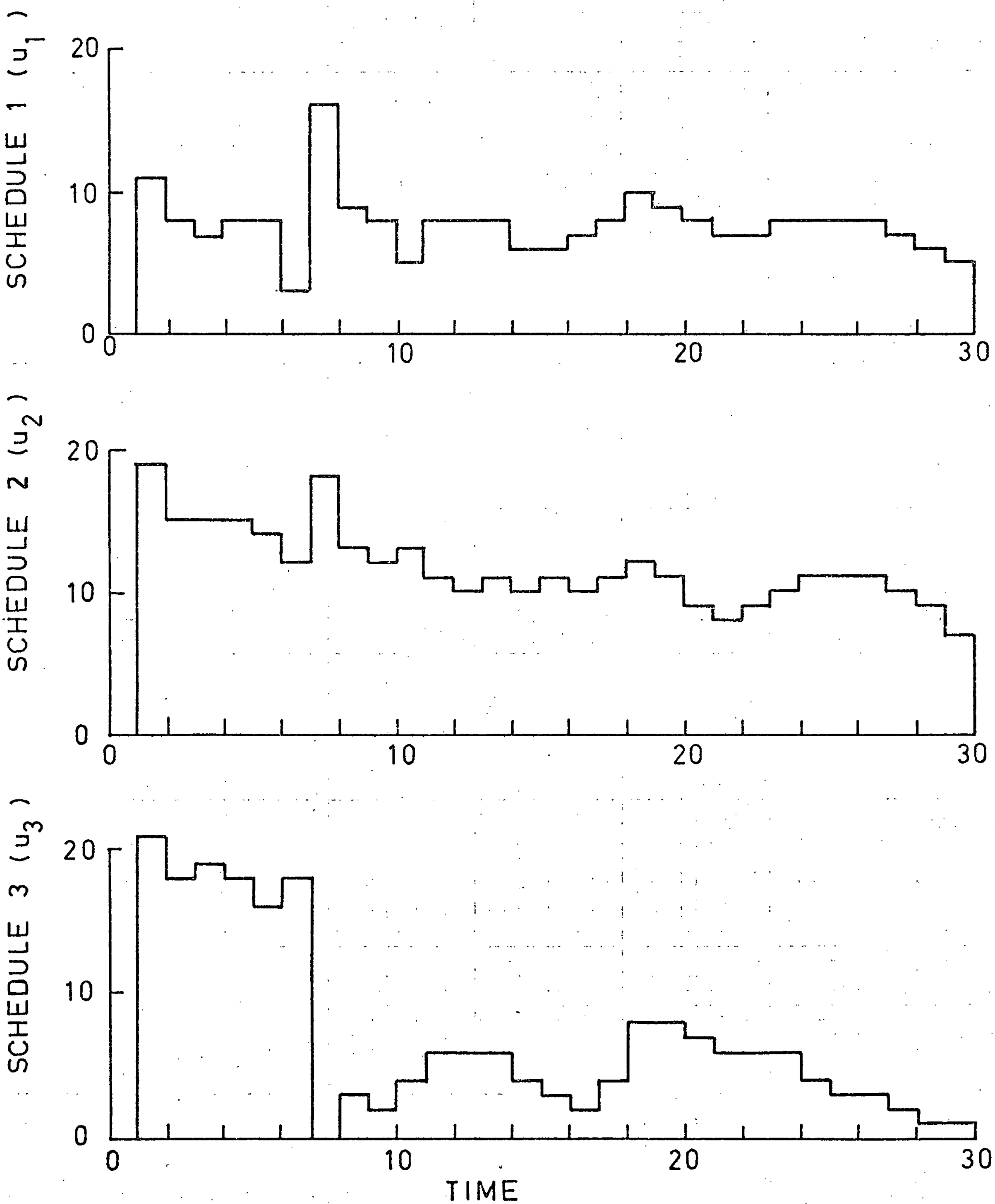


Fig. 4.9 Adaptive Scheduling of Operations with unknown C_k

adaptive system. During the initial stages ($p \leqslant k \leqslant 7$), the adaptive control behaves rather erratically. The uncertainty in the control arises from a lack of sufficient information on the system. The identifier gradually reacts properly as the number of measurements increases. The adaptive control collects information on the system through the identifier, and as time progresses, converges onto the optimal control. Hence, although the identifier is separated from the controller, there is an indirect interaction between the two.

Finally, it is noted that in both the optimal and the adaptive case above, the arrival sequence $\{\lambda_k\}$ has been assumed known. In the case of unknown $\{\lambda_k\}$, a suboptimal control can be obtained if the λ_j in (4.36) and (4.39) is replaced by λ_k for all $K-1 \geqslant j \geqslant k$. In other words, all future arrivals are assumed to be the same as the present arrivals. The results for this suboptimal strategy are shown in Fig. 4.10, Fig. 4.11 and Fig. 4.12.

4.5.2 Remarks

An on-line adaptive control method has been demonstrated for the operation-scheduling problem. This method is in fact of the open-loop-feedback-optimal (OLFO) type discussed in the last chapter. The same strategy can of course be extended to long-term scheduling problems ($K \to \infty$) if the infinite interval is approximated by a number of finite intervals. The resulting overall control would of course be suboptimal. Nevertheless, on-line control is achieved.

4.6 Discussion

Systems with time-varying parameters cannot be treated with the identification algorithm discussed in 4.2, unless the parameters vary slowly with time. However, there are always other more sophisticated

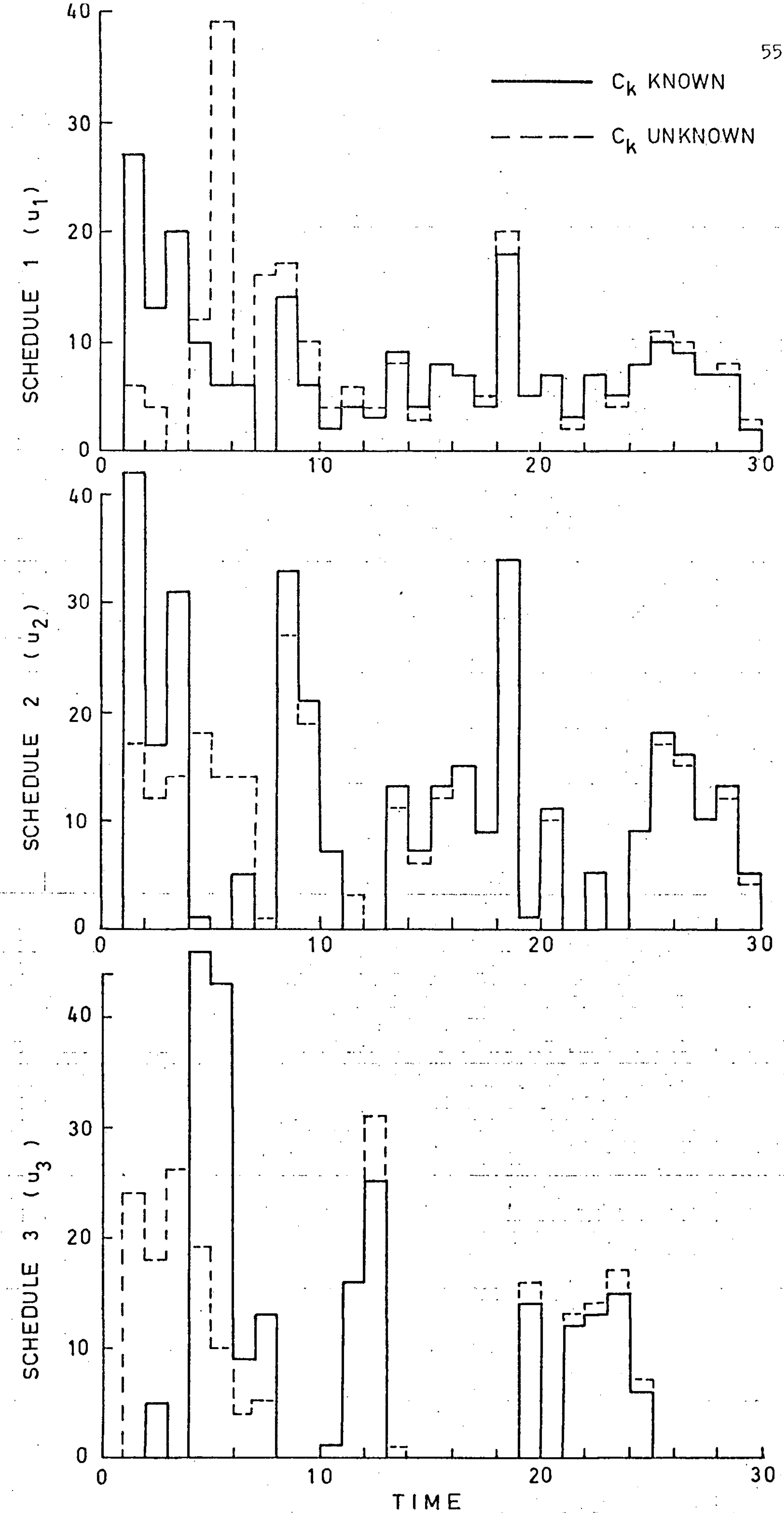


Fig. 4.10 Suboptimal Scheduling of Surgery with unknown arrivals $\{\lambda_k\}$

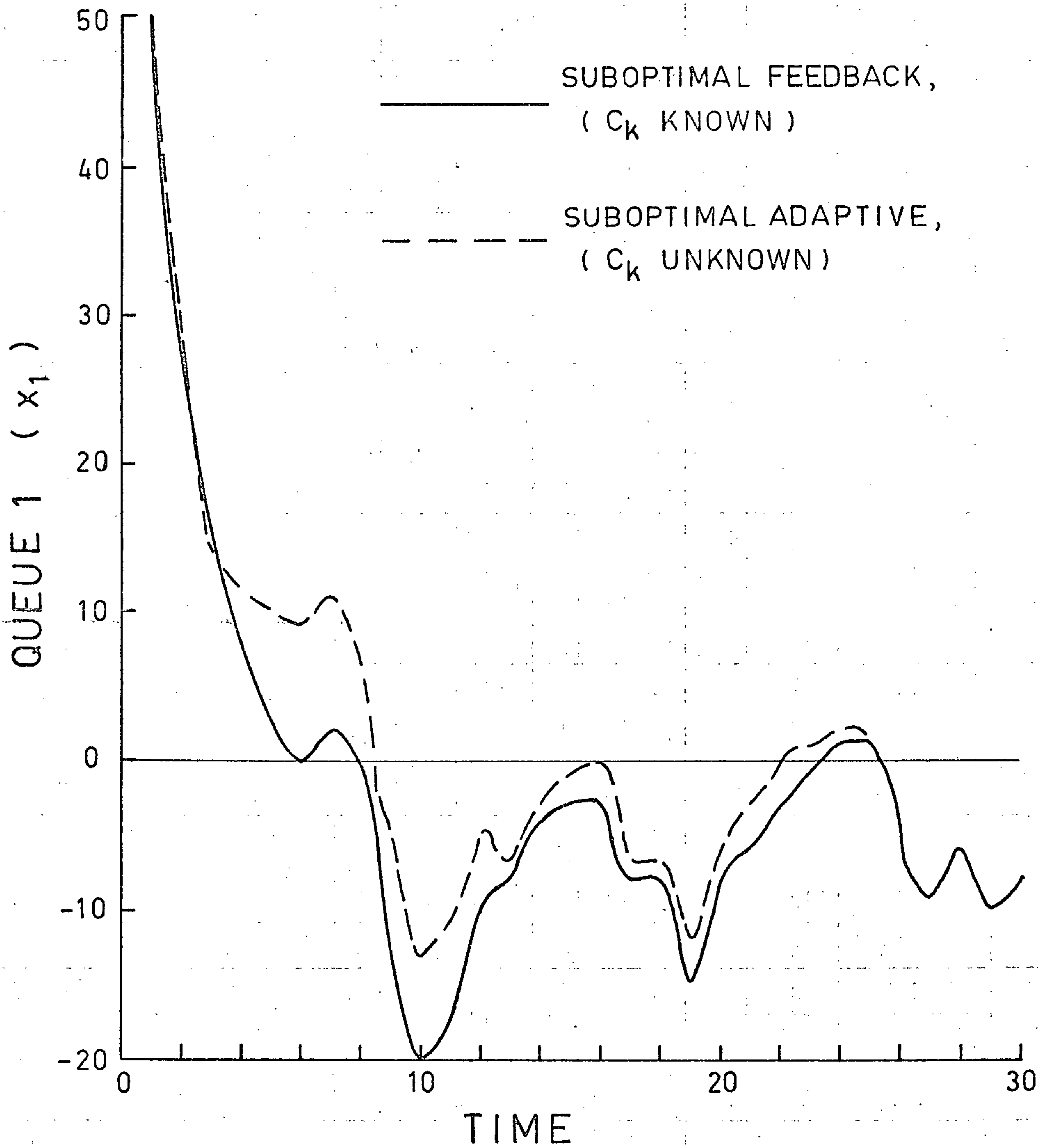


Fig. 4.11 Suboptimal Control State Trajectory with Unknown Arrivals $\{\lambda_k\}$. (x_2 behaves similarly)

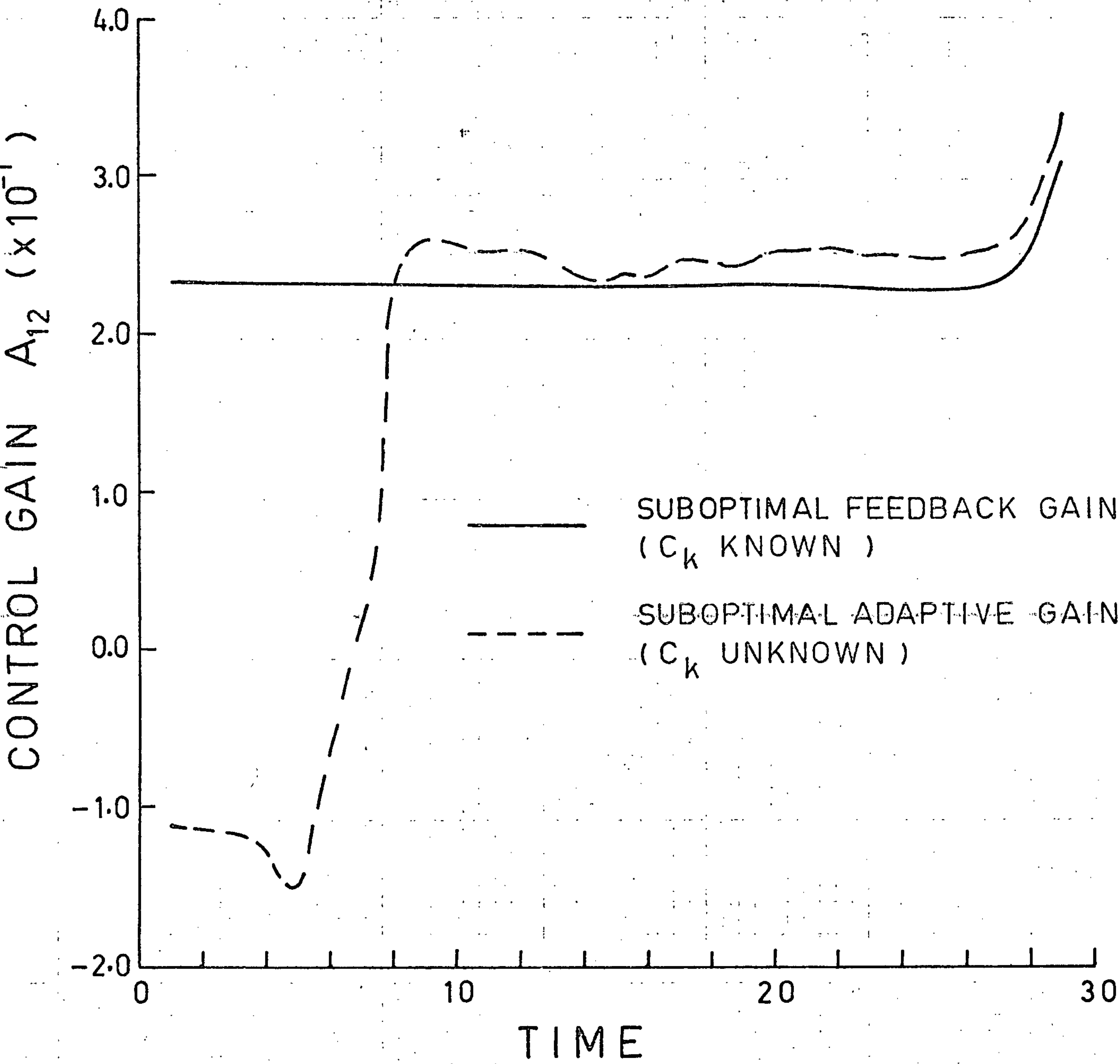


Fig. 4.12 Suboptimal Control Feedback Gain with Unknown Arrivals $\{\lambda_k\}$

identification techniques in the literature that can be applied.[13]

Finally, in view of the generalized queuing model discussed in Chapter 2, the system parameters C_k and B_k in fact represent the results from decision control in the system. Hence, simultaneous application of the regulator and decision control on the same system is possible. The coupling between the two controllers will be through the on-line identifier.

5. CONCLUSIONS AND SUGGESTIONS

5.1 Conclusions

Two different techniques have been demonstrated for the on-line control of a queuing system. Time-series models have been used. This is in direct contrast to the traditional approach where probability models are widely used.

It has been shown that decision control problems can be formulated as a control-constrained optimization problem. Application of the Discrete Maximum Principle and gradient techniques yields satisfactory results. Extension of the decision control strategy to the adaptive case is immediate through the open-loop-feedback optimal approach.

Regulator control is mainly applied to the control of a service mechanism. It has been shown that dynamic programming can be used to obtain an optimal feedback solution for such problems. An on-line adaptive control method has been demonstrated by applying the control and an identification scheme simultaneously but separately. A direct coupling of the regulator and decision control is thus possible through this identification process.

Least squares identification has been used because it does not require any statistical information on the system. More sophisticated techniques can of course be used to obtain better results. However, the philosophy behind the present work has been to demonstrate the feasibility of applying an on-line control strategy to a queuing system, rather than attempting an elaborate solution to any practical problem.

5.2 Suggestions

The techniques demonstrated here can theoretically be applied, without any major modification, to the control of large scale systems.

However, in any large scale system, there is always the problem of undesirable interactions among the variables. In addition, the optimality for the overall system is doubtful. Therefore, other large-scale system optimization techniques should be considered.

Finally, the successful application of on-line control to a queuing system can be the start of a trend for controlling other socio-economical systems. In view of the current awareness of the environment by scientists and engineers, it is apparent that large prospect exists in the field of environmental control. Successful application of modern control theory to such problems can be most rewarding.

APPENDIX I

The Discrete Maximum (Minimum) Principle

The Principle will be merely stated here for reference. A complete proof can be obtained from the literature[22,18].

Consider the system of discrete equations

$$x_{k+1} - x_k = f_k\ (x_k,\ u_k),\quad k = 0,\ 1,\ 2,\ \ldots,\ N-1 \tag{A1-1}$$

with $u_k \varepsilon L$, $x_k \varepsilon R^n$ and $u_k \varepsilon R^m$. Consider the scalar cost functional

$$J = \sum_{k=0}^{N-1} I_k\ (x_k,\ u_k) + \phi(x_N) \tag{A1-2}$$

Assuming that for every $k = 0, 1, \ldots, N-1$, and for every $x_k \varepsilon R^n$, the set $\{f_k\ (x_k,\ u_k)\ :\ u_k \varepsilon L\}$ is convex or at least directional convex[18]. (A1-3)

Define the Hamiltonian function

$$H\ (x_k,\ P_{k+1},\ u_k) \triangleq I_k\ (x_k,\ u_k) + p'_{k+1}\ f_k\ (x_k,\ u_k) \tag{A1-4}$$

where p_k is the costate vector.

The Minimum Principle states that at the optimal trajectory $\{x_k^*\}_{k=0,..,N}$ and optimal control $\{u_k^*\}_{k=0,\ldots,N-1}$, the following relations hold:

i) Canonical Equation

$$x_{k+1}^* - x_k^* = \left.\frac{\partial H}{\partial p_{k+1}}\right|_* \tag{A1-5}$$

$$p_{k+1}^* - p_k^* = -\left.\frac{\partial H}{\partial x_k}\right|_* \tag{A1-6}$$

ii) Boundary Conditions

$$x_o^* = x_o \tag{A1-7}$$

$$p_N^* = \frac{\partial \phi(x_N^*)}{\partial x_N^*} \tag{A1-8}$$

iii) Minimization of the Hamiltonian

$$H\,(x_k^*,\ p_{k+1}^*,\ u_k^*) \leq H\,(x_k^*,\ p_{k+1}^*,\ u_k) \tag{A1-9}$$

for every $u_k \varepsilon L$.

In the case of unconstrained control, (A1-9) yields

$$\left.\frac{\partial H}{\partial u_k}\right|_* = 0 \tag{A1-10}$$

APPENDIX II

Dynamic Programming Formulation of the Optimal Decision Control Problem

The illustrative example shown in 3.4 will be formulated below as a multistage decision process.

Given:

$$x_{k+1} = x_k + u_k \tag{A2-1}$$

$$x_k,\ u_k \ \varepsilon\ R^2$$

$$x_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$u_k \ \varepsilon\ E_2 \ \triangleq\ \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} \qquad k=1,\ldots,5$$

Find a control sequence $\{u_k \ \varepsilon\ E_2\}_{k+1,\ldots,5}$ which will minimize the cost

$$J = \sum_{i=1}^{5} L\,(x_i,\ u_i,\ i) + \phi(x_6,\ 6) \tag{A2-2}$$

where $\quad L\,(x_i,\ u_i,\ i) \triangleq u_i'\ [a_i\ (x_i + q) - \beta]$

$$\phi\,(x_6,\ 6) \triangleq (x_6 + q)' \quad F\,(x_6 + q)$$

Define an optimal value function

$$I(x_k,\ k) \triangleq \underset{u_k,\ldots,u_5 \varepsilon E_2}{\text{Min}} \left\{ \sum_{i=k}^{5} L(x_i,\ u_i,\ i) + \phi(x_6,\ 6) \right\} \tag{A2-3}$$

$$k = 1,2,\ldots,5$$

Because the cost at any stage is independent of any previous stage, we can write

$$I(x_k,\ k) = \underset{u_k \varepsilon E_2}{\text{Min}} \left[L\,(x_k,\ u_k,\ k) + \underset{u_{k+1},\ldots,u_5}{\text{Min}} \left\{ \sum_{i=k+1}^{5} L(x_i,u_i,i) + \phi(x_6,\ 6) \right\} \right]$$

By the definition of $I(x_k,\ k)$ in (A2-3), we have :

$$I\ (x_k,\ k) = \underset{u_k \varepsilon E_2}{\text{Min}}\ [L\ (x_k,\ u_k,\ k) + I\ (x_{k+1},\ k+1)]$$

$$= \underset{u_k \varepsilon E_2}{\text{Min}}\ [L\ (x_k,\ u_k,\ k) + I\ (x_k + u_k,\ k+1)] \qquad \text{(A2-4)}$$

Starting from the terminal condition

$$I\ (x_6,\ 6) = \phi(x_6,\ 6) = (x_6 + q)' \quad F\ (x_6 + q)$$

the functional equation (A2-4) can be solved completely. The optimal trajectory can be obtained for any initial point at any stage $1 \leq k \leq 5$.

Finally, it should be noted that (A2-4) is in fact the mathematical formulation for Bellman's Principle of Optimality.

APPENDIX III

Matrix Least Squares State Estimation

Consider the observation equation

$$x_k = S^T \psi_k \tag{A3-1}$$

or equivalently $$x_k' = \psi_k' S \tag{A3-2}$$

where $x_k \in R^n$, $\psi_k \in R^m$ and S is an (m x n) matrix.

$(.)^T$ denotes the transpose of a matrix, and

$(.)'$ denotes the transpose of a vector.

After k observations, we have

$$\begin{bmatrix} x_1' \\ x_2' \\ \cdot \\ \cdot \\ \cdot \\ x_k' \end{bmatrix} = \begin{bmatrix} \psi_1' \\ \psi_2' \\ \cdot \\ \cdot \\ \cdot \\ \psi_k' \end{bmatrix} S$$

or $$Z_k = H_k S \tag{A3-3}$$

where Z_k is now a (k x n) matrix and H_k is a (k x m) matrix. Consider finding a least squares estimate of S which would minimize the error function

$$J(\hat{S}_k) = \mathrm{Tr}[(Z_k - H_k \hat{S}_k)(Z_k - H_k \hat{S}_k)^T] \tag{A3-4}$$

where $\hat{S}_k$ denotes the estimate of S after k measurements and Tr(.) denotes the trace of a matrix.

Expanding (A3-4) we have

$$J(\hat{S}_k) = \mathrm{Tr}\{Z_k Z_k^T - Z_k \hat{S}_k^T H_k^T - H_k \hat{S}_k Z_k^T + H_k \hat{S}_k \hat{S}_k^T H_k^T\}$$

$$\frac{\partial J(\hat{S}_k)}{\partial \hat{S}_k} = 0 = -H_k^T Z_k - H_k^T Z_k + 2\, H_k^T H_k \hat{S}_k$$

Hence $$\hat{S}_k = (H_k^T H_k)^{-1} H_k^T Z_k \tag{A3-5}$$

Some of the gradient matrix identities used above are:

if A: n x m; B: m x n; C: p x n

then $$\frac{\partial \, Tr}{\partial A}(AB) = B^T = \frac{\partial \, Tr}{\partial A}(BA)$$

$$\frac{\partial \, Tr}{\partial A}(CAB) = C^T B^T$$

Now consider taking another measurement x_{k+1}:

$$Z_{k+1} = \begin{bmatrix} Z_k \\ ---- \\ x'_{k+1} \end{bmatrix} = \begin{bmatrix} H_k \\ ---- \\ \psi'_{k+1} \end{bmatrix} \hat{S}_{k+1} = H_{k+1} \hat{S}_{k+1}$$

where $$\hat{S}_{k+1} = (H_{k+1}^T H_{k+1})^{-1} H_{k+1}^T Z_{k+1} \tag{A3-6}$$

$$= p_{k+1} H_{k+1}^T Z_{k+1}$$

where $$p_{k+1}^{-1} \triangleq H_{k+1}^T H_{k+1} = H_k^T H_k + \psi_{k+1} \psi'_{k+1}$$

$$p_{k+1} = (H_k^T H_k)^{-1} - (H_k^T H_k)^{-1} \psi_{k+1} (I + \psi'_{k+1} (H_k^T H_k)^{-1} \psi_{k+1})^{-1} \psi'_{k+1} (H_k^T H_k)^{-1}$$

or $$p_{k+1} = p_k - p_k \psi_{k+1} (I + \psi'_{k+1} p_k \psi_{k+1})^{-1} \psi'_{k+1} P_k \tag{A3-7}$$

which is the consequence of a Matrix Inversion Lemma.

$(I + \psi'_{k+1} p_k \psi_{k+1})$ is a scalar quantity

Substituting (A3-7) into (A3-6) and expanding the terms, we have

$$\hat{S}_{k+1} = [I - p_k \psi_{k+1} (I + \psi'_{k+1} p_k \psi_{k+1})^{-1} \psi'_{k+1}] p_k H_k^T Z_k$$

$$+ p_k \psi_{k+1} [I - (I + \psi'_{k+1} p_k \psi_{k+1})^{-1} \psi'_{k+1} p_k \psi_{k+1}] x'_{k+1}$$

$$= [I - p_k \psi_{k+1} (I + \psi'_{k+1} p_k \psi_{k+1})^{-1} \psi'_{k+1}] \hat{S}_k$$

$$+ p_k \psi_{k+1} (I + \psi'_{k+1} p_k \psi_{k+1})^{-1} \{I + \psi'_{k+1} p_k \psi_{k+1} - \psi'_{k+1} p_k \psi_{k+1}\} x'_{k+1}$$

$$= \hat{S}_k + p_k \psi_{k+1} (I + \psi'_{k+1} p_k \psi_{k+1})^{-1} [x'_{k+1} - \psi'_{k+1} \hat{S}_k]$$

From the matrix Inversion Lemma

$$p_{k+1} \psi_{k+1} = p_k \psi_{k+1} [I + (I + \psi'_{k+1} p_k \psi_{k+1})^{-1} \psi'_{k+1} p_k \psi_{k+1}]$$

$$= p_k \psi_{k+1} [I + \psi'_{k+1} p_k \psi_{k+1}]^{-1}$$

Hence, we obtain the sequential form of (A3-5):

$$\hat{S}_{k+1} = p_{k+1} \psi_{k+1} (x'_{k+1} - \psi'_{k+1} \hat{S}_k) \qquad \text{(A3-8)}$$

where $$p_{k+1} = p_k - p_k \psi_{k+1} (I + \psi'_{k+1} p_k \psi_{k+1})^{-1} \psi'_{k+1} p_k \qquad \text{(A3-7)}$$

$\hat{S}_k$: m x n

p_k : m x m

x_k : n x 1

ψ_k : m x 1

$I + \psi'_{k+1} p_k \psi_{k+1}$: 1 x 1

APPENDIX IV

An Adaptive Feedback Control Algorithm via Dynamic Programming

Consider the optimization problem:

$$\text{Find: } \underset{\{u_k\}_{k=0,\ldots,N-1}}{\text{Min}} \left\{ \underset{\{x_k\}_{k=0,\ldots,N}}{E} \left[\sum_{j=0}^{N-1} (x_j' Q_j x_j + u_j' R_j u_j) + x_N' T x_N \right] \right\} \tag{A4-1}$$

subject to

$$x_{k+1} = x_k + B_k \lambda_k - C_k u_k \tag{A4-2}$$

where

$$x_k \varepsilon R^n, \quad \lambda_k \varepsilon R^m, \quad u_k \varepsilon R^p$$

If the parameters B_k and C_k are unknown, (A4-2) will be replaced by

$$x_{k+1}\big|_k = x_k + \hat{B}_k\big|_k \lambda_k - \hat{C}_k\big|_k u_k \tag{A4-3}$$

where $f_i\big|_j \triangleq$ estimate of the value of the function f at time i based on the total information available at time j.

$\hat{B}_k\big|_k$ and $\hat{C}_k\big|_k$ are the estimated values of B_k and C_k respectively.

$$x_{k+1}\big|_k \triangleq \underset{x_{k+1}}{E} (x_{k+1}\big|\Gamma_k) \text{ is the predicted value of } x_{k+1} \text{ at } k.$$

where $\underset{x}{E} (f(x)|g) \triangleq$ expected value of the random function f(x) over the random variable x, given information set g.

Γ_k is the information set available at time k. For the above problem, a sufficient set of information at k would be

$$\Gamma_k = \{x_k, \lambda_k, u_k, \hat{B}_k\big|_k, \hat{C}_k\big|_k\} \tag{A4-4}$$

Define the optimal value function

$$I(x_k,k) \triangleq \underset{u_k, u_{k+1}(.), \ldots, u_{N-1}(.)}{\text{Min}} \left\{ \underset{x_{k+1},\ldots x_N}{E} \left[\sum_{j=k}^{N-1} (x_j' Q_j x_j + u_j' R_j u_j) + x_N' T x_N \,\Big|\, \Gamma_k \right] \right\} \tag{A4-5}$$

Applying Bellman's Principle of Optimality (A2-4) to (A4-5), we obtain

$$I(x_k,k) = \underset{u_k}{\mathrm{Min}} \{x_k' Q_k x_k + u_k' R_k u_k + \underset{u_{k+1}(\cdot),\ldots,u_{N-1}(\cdot)}{\mathrm{Min}}$$

$$[\underset{x_{k+1},\ldots,x_N}{E} \left(\sum_{j=k+1}^{N-1} (x_j' Q_j x_j + u_j' R_j u_j) \qquad + x_N' \, T \, x_N \middle| \, \Gamma_k \right)]\}$$

$$= \underset{u_k}{\mathrm{Min}} \left\{ x_k' Q_k x_k + u_k' R_k u_k + \underset{x_{k+1}}{E} \, [\underset{u_{k+1},\ldots,u_{N-1}(\cdot)}{\mathrm{Min}} \right.$$

$$\left. \left\{ \underset{x_{k+2},\ldots,x_N}{E} \left(\sum_{j=k+1}^{N-1} (x_j' Q_j x_j + u_j' R_j u_j) + x_N' \, T \, x_N \, \middle| \Gamma_{k+1} \right) \right\} \middle| \, \Gamma_k \,] \right\}$$

Hence

$$I(x_k,k) = \underset{u_k}{\mathrm{Min}} \{x_k' Q_k x_k + u_k' R_k u_k + \underset{x_{k+1}}{E} \, (I(x_{k+1}, k+1) | \, \Gamma_k \,) \} \tag{A4-6}$$

Assuming a linear feedback structure of the optimal control, we obtain a linear quadratic optimal cost:

$$I(x_k,k) = x_k' \, P_k \, x_k + 2m_k' \, x_k + b_k \tag{A4-7}$$

where P_k : an (n x n) symmetric matrix

m_k : an n vector

b_k : a scalar quantity.

The reason for including m_k and b_k in (A4-7) is to account for the random input term λ_k in (A4-2).

Substituting (A4-7) and (A4-3) into (A4-6) and differentiating with respect to u_k, we obtain the optimal control

$$u_{k|k} = (R_k + \hat{C}^T_{k|k}\, p_{k+1}\, \hat{C}_{k|k})^{-1}\, \hat{C}^T_{k|k}\, [m_{k+1} + p_{k+1}(x_k + \hat{B}_{k|k}\lambda_k)] \tag{A4-8}$$

Replacing u_k in (A4-6) by the optimal function (A4-8), and replacing $I(x_k,k)$ by (A4-7), we obtain after equating the coefficient of the linear and quadratic terms of x_k:

$$p_k = Q_k + [I - p^T_{k+1}\, \hat{C}_{k|k}\, \Lambda^{-1}_{k|k}\, \hat{C}^T_{k|k}]\, p_{k+1} \tag{A4-9}$$

$$m_k = [I - p^T_{k+1}\, C_{k|k}\, \Lambda^{-1}_{k|k}\, C^T_{k|k}]\, [m_{k+1} + p_{k+1}\, \hat{B}_{k|k}\, \lambda_k] \tag{A4-10}$$

$$p_N = T \tag{A4-11}$$

$$m_N = 0 \tag{A4-12}$$

The complete control sequence is as follows:

$$u_{j|k} = \Phi_{j|k}\, x_j + \mu_{j|k} \qquad j = k,k+1,\ldots,N-1 \quad k = 0,1,2,\ldots,N-1 \tag{A4-13}$$

where

$$\Phi_{j|k} \triangleq \Lambda^{-1}_{j|k}\, \hat{C}^T_{k|k}\, p_{j+1|k} \tag{A4-14}$$

$$\mu_{j|k} \triangleq \Lambda^{-1}_{j|k}\, [\hat{C}^T_{k|k}\, (m_{j+1|k} + p_{j+1|k}\, \hat{B}_{k|k}\, \lambda_j)] \tag{A4-15}$$

$$\Lambda_{j|k} \triangleq R_j + C^T_{k|k}\, p_{j+1|k}\, \hat{C}_{k|k} \tag{A4-16}$$

$p_{j|k}$ and $m_{j|k}$ are obtained from the recursive equations:

$$p_{j|k} = Q_j + [I - p^T_{j+1|k}\, \hat{C}_{k|k}\, \Lambda^{-1}_{j|k}\, \hat{C}^T_{k|k}]\, p_{j+1|k} \tag{A4-17}$$

$$m_{j|k} = [I - p^T_{j+1|k}\, \hat{C}_{k|k}\, \Lambda^{-1}_{j|k}\, \hat{C}^T_{k|k}][m_{j+1|k} + p_{j+1|k}\, \hat{B}_{k|k}\, \lambda_j] \tag{A4-18}$$

$$p_{N|N-1} = T \;;\; m_{N|N-1} = 0 \tag{A4-19}$$

The above algorithm is in effect an approximation to the optimal control. At each stage k, the system parameters B_k and C_k are assumed equal to $\hat{B}_{k|k}$. $\hat{C}_{k|k}$ for all remaining stages [k, k+1,...,N-1]. This fact is illustrated in equations (A4-14) to (A4-18) where $\hat{B}_{k|k}$ and $\hat{C}_{k|k}$ are used instead of $\hat{B}_{j|k}$ and $\hat{C}_{j|k}$ for the optimal case. A further approximation can of course be made (when the random input $\{\lambda_k\}$ is unknown) by replacing λ_j with λ_k. In this case, all future random inputs are assumed to be the same as λ_k.

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