ON THE SELF-WEIGHT SAG OF PLATE-LIKE STRUCTURES
WITH APPLICATION TO MIRROR SUBSTRATE DESIGN

by

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ABSTRACT

The thesis investigates the self-weight sag of plate-like structures with application to astronomical mirrors.

The exact analytical solution for the self-weight deflection of a circular disc is obtained by superposition of various elementary solutions due to Love, and the validity of the existing approximate procedures is examined.

Guided by the concept of arch-like structures for optimum design, a finite element formulation for an axi-symmetrical solid is developed from first principles in terms of triangular ring-elements. Solutions are obtained for various structural configurations constructed within the envelope of the disc. The superiority of arch-type designs over solid discs with respect to deflection and weight is established and their attractive potential demonstrated.

The extensive experimental programme involving frozen stress photoelasticity in conjunction with immersion analogy of gravitational stress and fringe multiplication, clearly emphasizes some of the limitations of this approach. It establishes that a success of the method is dependent upon the availability of a sufficiently stress free araldite. Although not generally followed by stress analysts, a direct measurement of frozen body-force induced displacements is
attempted. The phenomenon of continuous polymerization of the model material, hitherto overlooked by photoelasticians, appears to play a decisive role in the measurement of minute self-weight induced deformations.

The direct use of silicone rubber models successfully determines the displacements in mirrors, in the form of solid disc and arched dome, and establishes the superiority of the latter.
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<td>D</td>
<td>diameter, (2a); modulus of rigidity</td>
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<tr>
<td>[D]</td>
<td>elasticity matrix for isotropic material</td>
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<tr>
<td>E</td>
<td>Young's modulus</td>
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<tr>
<td>{F}</td>
<td>external nodal force</td>
</tr>
<tr>
<td>{F}_p</td>
<td>nodal force due to body force</td>
</tr>
<tr>
<td>{F}_T</td>
<td>nodal force due to thermal effect</td>
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<tr>
<td>F_z</td>
<td>body force intensity in the (z) direction</td>
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<tr>
<td>H</td>
<td>thickness, (2h)</td>
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<tr>
<td>[K]</td>
<td>stiffness matrix</td>
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<td>S</td>
<td>support circle radius/radius of the plate</td>
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<td>T</td>
<td>average temperature rise in an element</td>
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<tr>
<td>{f}</td>
<td>displacement function</td>
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<tr>
<td>g</td>
<td>acceleration due to gravity</td>
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<tr>
<td>{p}</td>
<td>body force vector</td>
</tr>
<tr>
<td>q</td>
<td>intensity of a continuously distributed load</td>
</tr>
<tr>
<td>t</td>
<td>time</td>
</tr>
<tr>
<td>u,(v,)(w)</td>
<td>displacements in (x,(y,)z) directions</td>
</tr>
<tr>
<td>(w_o)</td>
<td>mid-plane displacement</td>
</tr>
<tr>
<td>(x,(y,)z)</td>
<td>rectangular coordinates</td>
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<tr>
<td>(r,(\theta,)z)</td>
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\( \alpha \) coefficient of thermal expansion

\( \gamma_{rz} \) shear strain component in cylindrical coordinates

\( \{\delta\}^e \) element nodal displacement

\( \Delta \) area of the triangle

\( \varepsilon_r', \varepsilon_\theta', \varepsilon_z \) normal strain components in cylindrical coordinates

\( \{\varepsilon\} \) strain vector

\( \{\varepsilon_T\} \) thermal strain vector

\( \lambda \) wave length of light

\( \nu \) Poisson's ratio

\( \rho \) density

\( \sigma_r', \sigma_\theta', \sigma_z \) normal stress components in cylindrical coordinates

\( \{\sigma\} \) stress vector

\( \tau_{rz} \) shear stress component in cylindrical coordinates

\( \chi_1 \) stress function
Dedicated to my beloved parents
1. INTRODUCTION

1.1 The Self-Weight Sag of Thick Plates

This investigation is primarily concerned with developing and applying methods for predicting the self-weight sag of thick plates. The investigation was prompted by several problems arising in large mirror substrate design and it is in this area that the methods have been used. Traditionally, an astronomical mirror, large or small, is designed as a right cylindrical disc having a diameter to thickness ratio of 8 to 1 (small aperture) or 6 to 1 (large aperture). These ratios are rules of thumb proposed by Ritchey [1] on the basis of experience. The smaller ratio, recommended for large mirrors, reflects the importance of self-weight deflection, which is proportional to \((\text{Radius})^4/(\text{Thickness})^2\) for a mirror uniformly supported at its periphery. It is apparent that, with such small diameter to thickness ratios, a theoretical investigation cannot be accomplished by using conventional thin plate theory. It is necessary to treat the mirror, as a thick plate, that is, an axisymmetric body governed by the three-dimensional field equations of elasticity.
In many instances, the mirror may be slightly curved for reasons of optical surface form (parabolic) and arched for weight economy. The point serves to indicate the necessity of analyzing axisymmetric thick plates of variable thickness, that is, shallow domes or axisymmetric shallow arched structures.

In thin plate theory, a common technique for examining body force effects is to replace it by an equivalent surface load. This simplifying technique cannot be used for thick plates and it is necessary to account for gravity deformation. Apart from the desirable feature of reliability of the procedure, the inclusion of body forces in the theory automatically supplies a solution for thermal effects. It is only necessary to interpret the body force stresses appropriately and by the method of strain suppression in thermoelasticity, the results apply to thermal effects. The effects of thermal stresses in astronomical mirror operation is of paramount interest and any information obtainable from the present investigation may be considered useful.

1.2 Optical, Mechanical and Thermal Problems in Large Telescopes

An astronomical telescope mirror is basically a specular, reflective surface whose geometrical form is maintained by an underlying supporting structure or
substrate. If the substrate serves by its mere existence as a "creator" of a surface or interface, its elements become masses which are distributed in space and thus are subjected to gravitational attraction.

The mirror surface would change its shape if the physical state of the substrate alters. This may arise when the orientation of the mirror or the gravitational field changes. A change in geometry of the elastic structure can also be induced by variations in absolute temperature; or by the existence of temperature gradients. Furthermore, problems of dimensional stability may arise from relaxation of residual stresses introduced during processing.

For satisfactory performance of the mirror, the substrate has to be so designed that the geometric form of the reflective surface is maintained within an acceptable limit determined by diffraction optics. The current general philosophy of mirror support is to equilibrate the weight of each element of the mirror substrate as directly as possible by various suitably designed mechanisms. Thus the elastic deformations of the substrate, which occur as the mirror is oriented in different positions, are minimized.
These complex mechanisms are quite elaborate. Instead of designing the substrate as a massive structure with many supports, alternative forms of composite and built-up substrates have been designed and constructed. Here the aim is to develop high stiffness-to-weight ratio structures with a capability of resisting changes in shape by an acceptable degree of flexure. These light-weight mirror designs can be obtained by changing the conventional geometrical forms of the substrate through redistribution of material. As an alternative to using many closely spaced support mechanisms, mentioned above, light-weight substrates may be supported directly at three points or at six internal points on a limited system of balanced levers pivoted at three primary points on the frame of reference.

1.2.1 The Optical Problem

The geometrical form of the mirror surface must be maintained so that the images formed are diffraction-limited. For astronomical purposes, a surface is considered diffraction-limited if it would maintain the geometrical form accurate to a small fraction of the wave length of light being received.
Rays coming from distant stars form images as shown in Figure 1.1. Due to diffraction effects the images, instead of being points, would consist of bright central Airy's discs surrounded by alternate dark and bright rings. A parabolic reflecting surface gives a theoretically ideal focussing but, even with a perfect geometry, there is a lower limit to the angle of separation between two objects, 
\[
\alpha = \frac{1.22\lambda}{D}
\]
where \( D \) = diameter of the aperture and \( \lambda \) = wave length of the light. This expression is determined from Rayleigh's criterion which states that two images may be resolved when the central maximum of one falls over the first minimum of the other. As indicated by the above formula this limit for \( \alpha \) depends on \( D \). In order to make \( \alpha \) small, \( D \) has to be large. Any imperfection of the reflecting surface degrades this limit which is really fixed by \( \lambda \) since we cannot control the wave length of light.

A parabolic mirror focuses at a nominal point, called the "focus," only for principal rays. Oblique rays focus on a cusp-like surface (Figure 1.2). Hence, real mirror surfaces are not always formed as true paraboloids. They are corrected to effect some geometrical control of focussing in a focal plane.
Figure 1.1 Formation of images from distant stars
Figure 1.2 Cusp-like surface formed by oblique rays
1.2.2 The Mechanical and Thermal Problems

To maintain a specified surface form, an underlying substrate, on which the surface is formed, is required.

During observations, a mirror may be rotated into different attitudes in the gravitational field. If the mirror surface is designed to be accurate when supported in a horizontal position, the surface shape may change as the mirror is moved into different attitudes. Hence a 'rigid' substrate must be stiff enough to ensure that attendant elastic distortions are less than acceptable amounts.

There are at least two possible ways of supporting a mirror substrate:

(a) Since the primary object in mirror design is to create a surface on the boundary of a substrate and its atmosphere, a thin plate supported at many closely spaced points (Figure 1.3a), located directly under the lines of action of the body forces, would serve the purpose. In this case the mirror can be made as thin as possible as it merely provides a workable surface. The material in the substrate is then supported directly, and internal transmission is confined to direct compression through the thickness of the plate.
Figure 1.3 Possible ways of supporting a mirror substrate

Figure 1.4 The 200-inch Hale telescope mirror support mechanism
(b) By using a stiff substrate with widely spaced supports (Figure 1.3b).

Figure 1.4 [2] shows a typical control mechanism used by the thirty-six supports of the 200-inch Hale Telescope at Mt. Palomar. These mechanisms are installed in cavities in the mirror blank. As the mirror is moved into different attitudes the weights W and the lever arms shown in the figure are so arranged that a force is exerted on band B which just balances the component of gravity normal to the direction of the optic axis, on the section of the mirror assigned to this support. Similarly, a force is exerted on the band S, which balances the component parallel to the optic axis, of the pull of the gravity on the same section of the mirror. The mirror is therefore floating on the support system as it is moved into different attitudes.

Thermal gradients in a mirror substrate can seriously deform the optical surface. Elaborate measures are taken to counteract this effect by providing large telescopes with enclosures, normally in the form of hemispherical domes. Inside and outside temperatures are equalized on an anticipated long-term basis to suit the time of observation at night. Furthermore, when materials are chosen for mirror substrates, two important physical properties are taken into account; the thermal coefficient of expansion and conductivity of the material.
Essentially a material with a low coefficient of expansion is sought since its 'optical figure' would be stable under normal ambient temperature fluctuations and it is less likely to be susceptible to thermal effects. On the other hand, high thermal conductivity is necessary, since this would enable the mirror to reach thermal equilibrium quickly, especially during grinding when it is necessary to dissipate the heat generated by friction. Beside selecting the material for mirror substrates with proper values of the thermal coefficient of expansion and thermal conductivity, its mounts must also be suitably designed so that the thermal distortion is minimized.

Although scarce, a reference should be made to the relevant investigations in the field. Deflection analysis of a mirror substrate was initiated by Couder (1931) [3]. He investigated flat cylindrical mirrors by taking account of bending stresses only, the shear stresses being neglected. It was concluded from observations and an approximate analysis that the flexure of vertically mounted mirrors would not be detectable up to a diameter of about 120 inches. Schwesinger (1954) [4] analyzed a vertically mounted mirror and took exception to the conclusion of Couder. More recently, Malvick and Pearson (1968) [5] conducted deflection analysis of Cassegrainian-type mirror substrates with a spherically dished front surface. They used the numerical method of 'dynamic relaxation' to study mirror deformations at different attitudes in co-ordinate
space. Kenny (1968) [6] undertook elementary experimental investigation for cylindrical, arched and build-up arched substrates and found the deflection of the arched model to be smaller than that of the equivalent right cylindrical blank.

1.3 Purpose and Scope of the Present Investigation

The long established use of right cylindrical disc as substrate for small aperture has led, by extension, to the use of such solid substrates even for large aperture elements. For instance the ground-based reflection Telescope of Kitt Peak is 156 inches in diameter and 26 inches thick. This is characterized by a traditional aperture to thickness ratio of 6 to 1. A disc such as this still has to be supported at many discrete points by carefully calculated and controlled forces, arranged to balance the gravitational forces which tend to deform the substrate as the mirror is manoeuvred to occupy desired positions.

Thus it is apparent that with ground based mirrors, as the size increases, the $a^4/H^2$ relation for constant stiffness leads to excessive substrate weight with attendant problems of the support design. With space-borne telescopes the reduction of substrate weight becomes even more important. Questions such as total mass to be accelerated in space and changes in the substrate shape as the telescope
is taken from the one-g field to zero-g field have to be considered. One promising solution to the two major problems mentioned above is to design substrates having the highest possible stiffness-to-weight ratio.

The investigation reported here considers the self-weight sag of plates in three fundamental ways. Firstly, a theoretical investigation has been conducted for thick plates of constant thickness. This method makes use of some results due to Love [7] for axisymmetric bodies. Using his results, an exact analytical solution has been formulated for the self-weight sag of a thick plate supported by uniform shear forces along the periphery. Although this exact solution is primarily of academic interest it provides an excellent check for the subsequent investigation. Next, the attention is focused on a numerical solution in conjunction with the method of finite elements. Governing equations have been formulated to analyze body force deflections in any axisymmetric three-dimensional structure. The procedure is ideally suited to examine the effects of arching and support.

Thirdly, some experimental methods of examining body force deflections have been considered. The frozen stress technique coupled with the immersion analogy for gravitational stress effects is used on epoxy-resin models. A well known difficulty in using models is the low stress
levels induced by gravity. This low sensitivity may be overcome by immersing the model in a high density liquid, in this case mercury. Fringe interpretation and identification is further improved by the technique of fringe multiplication. Unfortunately, the effect of mottle, introduced during casting, makes accurate interpretation of the stress patterns rather difficult. This led to the direct measurement of frozen displacements. The magnitude of the displacements are ideal for oblique interferometric measurement, an oblique incidence interferometer being available. Finally experiments have been conducted on moulded silicone rubber models. The relatively low elastic modulus and fairly high density, makes this an ideal material for self-weight distortion studies. Deflections have been measured using a Moiré fringe technique applied only recently to the non-specular surfaces.

In brief, the objective of the present project is to investigate geometrical forms of substrate which would yield low weight and high stiffness. The configurations studied are indicated in Figure 1.5.
<table>
<thead>
<tr>
<th>Objective 1</th>
<th>Objective 2</th>
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</thead>
<tbody>
<tr>
<td>Analytical solution of constant thickness plates for various D/H ratios to</td>
<td>Finite element solution of symmetrical arch type structures to show their</td>
<td>Experimental investigation of models in the forms of solid disc and arched</td>
</tr>
<tr>
<td>show importance of shear deflection as D/H is lowered.</td>
<td>superiority over constant thickness substrates.</td>
<td>done to show superiority of the latter.</td>
</tr>
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![Figure 1.5 Table of substrate forms](image-url)
2. **AN EXACT ANALYTICAL SOLUTION FOR THE SELF-WEIGHT DEFLECTION OF CIRCULAR PLATES OF CONSTANT THICKNESS**

2.1 Preliminary Remarks

There are several theoretical formulae which are in current use for predicting the self-weight deflection of a solid, right circular disc.

The plates shown in column 1 of Figure 1.5 may be analyzed approximately by Lagrangian Theory which makes assumptions similar to those made in the one dimensional Bernoulli-Euler Theory (plane sections remain plane and stresses in the direction normal to the mid-plane are negligible). The results are but slightly in error if the ratio $H/D$, where $H$ is the thickness and $D$ the diameter, is small ($< \frac{1}{20}$).

For relatively large ratios, the assumptions of the Lagrangian Theory break down. The fibre stress is no longer linearly distributed across the section. The departure from linearity is due to the combined effects of surface pressure and associated shear and leads to a significant difference between the Lagrangian and true solutions. The difference, frequently called "the deflection due to shear," is calculated by the semi-rational Rankine-Grashoff method and added to
the Langrangian result. This leads to an improvement in the accuracy of the latter and provides a convenient approximate treatment of deflections in relatively thick flexural members.

In reviewing, briefly, some existing solutions for the flexure of plates of constant thickness, the method of Williams [8] should be mentioned. Williams determined the deformations of a symmetrically loaded circular plate, of constant thickness, supported by equally spaced point reactions along a single concentric circle. The solution has no restriction on the diameter of the support circle, which could be as large as the outside diameter of the plate itself. The approach, based on the work of Bassali [9], is exact and rigorous and leads to tabulation of the series summation by a computer. The theory still excludes the estimation of shear deflections. These could only be determined by three dimensional analytical or numerical methods based on the field theory of elasticity or those akin to Rakine-Grashoff's approach.

A simple approximate method of determining deformations of a uniformly loaded, point supported circular plate has been developed by Vaughan [10]. In this method the classical solution of Michell [11], for a clamped plate under a single point load, is extended to any number of point loads regularly spaced around a circle concentric with the plate edge. The boundary moments and shears are released by
superposition of edge loading solutions based on the use of a single sine wave as an approximation to a series.

Either of the above analyses, if used to solve the self-weight deflection problem, would require the distributed body force to be approximated by the substitution of a uniformly distributed external-surface load.

2.2 Self-Weight Deflection of a Cylindrical Plate Supported by Uniformly Distributed Shear at the Edge

As noted earlier, the existing solutions for self-weight loaded plates supported at a system of discrete points are, in general, approximate. An accurate determination of the self-weight induced displacements may be obtained considering several axisymmetric systems using methods of Love [7].

Assuming isotropic elastic material with constants \( E \) and \( v \), small deformations \( u, v, w \) in the directions \( x, y, z \) and a body force \( pg \) per unit volume, an analytical solution has been obtained for self-weight deflection of a circular plate for equipollent boundary conditions.
This means that the stress resultant at the boundary is exactly equal to the externally applied tractions though the localized stresses may vary greatly.

The solution of the above mentioned problem as represented in Figure 2.1 must satisfy the following boundary conditions:

\[(\sigma_z)_z = h = 0, \quad (\tau_{rz})_z = h = 0 \quad (2.1)\]

\[
\left\{ 
\begin{align*}
\int_{-h}^{h} (\sigma_z) r dr &= a dz = 0 \\
\int_{-h}^{h} (\sigma_z) r &= a z dz = 0 \\
\int_{-h}^{h} (\tau_{rz}) r &= a 2\pi adz = \text{weight of the plate.}
\end{align*}
\right. \quad (2.2)
\]

All the conditions of (2.1) and (2.2) can be satisfied by superposition of several elementary solutions and the expressions for displacements can be obtained following Love's treatment of moderately thick plates [7].

First let us consider the plate held by a uniformly distributed tension, Figure 2.1a. If the solution for a plate (without body force) subjected to uniform compression \(-2\rho gh\) (Figure 2.1b) over the upper face is superimposed,
Figure 2.1 Superposition of several elementary solutions
the face \( z = h \) will be free of normal stress and the resulting displacements will be entirely due to the body force. But due to the nature of the second solution, there will be a moment and a radial force present at the edge of the plate. To relieve the edge of the moment and the radial force, two other solutions need to be added, as shown in Figure 2.1c,d. This leads to the desired solution of self-weight deflection of a plate supported by uniform shearing forces along the edges.

The notations for the plate, used in the present investigation are shown in Figure 2.2. Body force displacements for a plate held by uniformly distributed tension on its upper face are

\[
\begin{align*}
    u &= -\nu \rho g (z+h)x/E \\
    v &= -\nu \rho g (z+h)y/E \\
    w &= \frac{\rho g}{2E} (z^2 + \nu x^2 + \nu y^2 + 2hz)
\end{align*}
\]

and in absence of a body force its displacements when subjected to uniform compression \(-2\rho gh\) over its upper face, are

\[
\begin{align*}
    u &= -\frac{(1+\nu)(2\rho gh)x}{8Eh^3} [(2-\nu)z^3 - 3h^2z - 2h^3 - \frac{3}{4} (1-\nu)z(x^2+y^2)] \\
    v &= -\frac{(1+\nu)(2\rho gh)y}{8Eh^3} [(2-\nu)z^3 - 3h^2z - 2h^3 - \frac{3}{4} (1-\nu)z(x^2+y^2)]
\end{align*}
\]
Figure 2.2: Notations for cylindrical plate
\[ w = \frac{(1+v)(2\rho gh)}{16Eh^3} \left[ (1+v)z^4 - 6h^2z^2 - 8h^3z + 3(h^2-vz^2)(x^2+y^2) \right. \]

\[ \left. - \frac{3}{8} (1-v)(x^2+y^2)^2 \right] \] (2.4)

Superimposition of the above two systems gives,

\[ u = - \frac{(1+v)(2\rho gh)x}{8Eh^3} \left[ (2-v)z^3 - 3h^2z - 2h^3 - \frac{3}{4} (1-v) \right. \]

\[ \left. z(x^2+y^2) \right] - \frac{\nu \rho g(z+h)x}{E} \]

\[ v = - \frac{(1+v)(2\rho gh)y}{8Eh^3} \left[ (2-v)z^3 - 3h^2z - 2h^3 - \frac{3}{4} (1-v) z(x^2+y^2) \right. \]

\[ \left. - \frac{\nu \rho g(z+h)y}{E} \right] \]

\[ w = \frac{\rho g}{2E} \left[ z^2 + 2hz + \nu(x^2+y^2) \right] + \frac{(1+v)(2\rho gh)}{16Eh^3} \left[ (1+v)z^4 - 6h^2z^2 \right. \]

\[ \left. - 8h^3z + 3(h^2-vz^2)(x^2+y^2) - \frac{3}{8} (1-v)(x^2+y^2)^2 \right] \] (2.5)

with the resultant moments
\[ G_1 = -D(K_1 + \nu K_2) + \left[ \frac{24 + 23\nu + 3\nu^2}{30(1-\nu)} \rho g h^3 \right] \]

\[ G_2 = -D(K_1 + \nu K_2) + \left[ \frac{24 + 23\nu + 3\nu^2}{30(1-\nu)} \rho g h^3 \right] \quad (2.6) \]

\[ H_1 = D(1-\nu) \tau \]

where, \( D = \frac{2}{3} \frac{Eh^3}{(1-\nu^2)} = \) Modulus of rigidity of the plate and

\[ K_1 = \frac{\partial^2 w_0}{\partial x^2}, \quad K_2 = \frac{\partial^2 w_0}{\partial y^2}, \quad \tau = \frac{\partial^2 w_0}{\partial x \partial y} \]

and

\[ w_0 = \text{deflection of middle plane at } z = 0. \]

It should be noted here that the underlined parts of the expression (2.6) represent moments in the pure bending of plates. The square bracketed term is due to the more accurate analysis.

From (2.5), by putting \( z = 0, \)

\[ w_0 = \frac{\rho g v}{2E} (x^2 + y^2) - \frac{\rho gh}{32D} (x^2 + y^2) [x^2 + y^2 - \frac{8h^2}{1-\nu}] \quad (2.7) \]

Differentiation of (2.7) yields values of \( K_1, K_2 \) and \( \tau \) which
when substituted in (2.6) gives moments at \( r = a \):

\[
G_1 = G_2 = \frac{(3+v)(2\rho gh)}{16} a^2 + \frac{3-v}{20} (2\rho gh) h^2 \tag{2.8}
\]

\( H_1 = 0 \)

The resultant force at the edge due to the superposition of the two systems (2.3) and (2.4) is

\[
T = \frac{1}{2} (2\rho gh) \cdot h \tag{2.9}
\]

Now, in order that solution (2.5) be free of edge moment (2.8) and radial force (2.9), deflection due to a moment opposite to (2.8) and due to a radial force opposite to (2.9) must be superimposed on (2.5).

The expressions for deflection due to uniform tension along the edge are given by,

\[
u = \frac{1-v}{2E} \theta_0 x
\]

\[
w = -\frac{v\theta_0 z}{E}
\]

In the present case \( \theta_0 = -\frac{1}{2} (2\rho gh) \). Substituting this value of \( \theta_0 \) in (2.10) one obtains,
\[ u = - \frac{(1-\nu)}{2E} (\rho gh) \cdot x \]

\[ v = - \frac{(1-\nu)}{2E} (\rho gh) \cdot y \]  \hspace{1cm} (2.11)

\[ w = \frac{\nu}{E} (\rho gh) \cdot z. \]

For determining deflection due to a moment opposite to that of (2.8), we choose a stress function \( \chi_1 \) of the form

\[ \chi_1 = \frac{1}{4} \beta (x^2 + y^2) + \gamma, \]  \hspace{1cm} (2.12)

where \( \beta, \gamma = \) constants, giving

\[ G_1 = \frac{2}{3} h^3 \frac{\partial^2 \chi_1}{\partial y^2}, \quad G_2 = \frac{2}{3} h^3 \frac{\partial^2 \chi_1}{\partial x^2} \]

Substituting the values of the derivatives of \( \chi_1 \) from (2.12) in the above expression,

\[ G_1 = G_2 = \frac{1}{3} h^3 \beta \]  \hspace{1cm} (2.13)

This must be negative of the value obtained in (2.8). Hence,
\[ \beta = - \frac{3}{h^3} \left[ \frac{3+\nu}{16} (2\rho gh) \ a^2 + \frac{3-\nu}{20} (2\rho gh) h^2 \right] \]

with \[7\]

\[ u = \frac{1}{E} \beta x z - \frac{1+\nu}{E} z \ \frac{\partial \chi_1}{\partial x} \]

\[ v = \frac{1}{E} \beta y z - \frac{1+\nu}{E} z \ \frac{\partial \chi_1}{\partial y} \]

\[ w = - \frac{\beta}{2E} (x^2 y^2 + \nu z^2) + \frac{1+\nu}{E} \chi_1 \] (2.14)

Taking the reference for deflection values at 0 \((r, z, w = 0)\),

\[ (w)_{r=0} = - \frac{\beta}{2E} (0) + \frac{1+\nu}{E} \left( \frac{1}{4} \beta \cdot 0 + \gamma \right) \]

\[ z=0 \]

\[ \gamma = 0 \]

and \[ \chi_1 = - \frac{3}{4h^3} \left[ \frac{3+\nu}{16} (2\rho gh) \ a^2 + \frac{3-\nu}{20} (2\rho gh) h^2 \right] (x^2 + y^2) \] (2.15)

Now substituting the values of \(\chi_1\) and its derivatives in (2.14) the deflections due to a moment opposite to (2.8) are obtained as:
\[ u = \frac{3xz}{2Ef^3} \left[ \frac{3+\nu}{16} (2\rho gh) a^2 + \frac{3-\nu}{20} (2\rho gh) h^2 \right] (\nu-1) \]

\[ v = \frac{3yz}{2Ef^3} \left[ \frac{3+\nu}{16} (2\rho gh) a^2 + \frac{3-\nu}{20} (2\rho gh) h^2 \right] (\nu-1) \]

\[ w = \frac{1}{4Ef^3} \left[ \frac{3+\nu}{16} (2\rho gh) a^2 + \frac{3-\nu}{20} (2\rho gh) h^2 \right] \left( 3x^2 + y^2 + 6vz^2 - 3x^2\nu - 3y^2\nu \right) \]

The superimposition of (2.11) and (2.16) on (2.5) leads to the desired solution for deflection of a plate due to its own weight when supported by uniformly distributed shear along its edges. In cylindrical co-ordinates the components of deflection can be expressed as:

\[ w = \frac{\rho g}{2E} \left[ z^2 + 2hz + vr^2 \right] + \frac{1+\nu}{16Ef^3} (2\rho gh) \left[ (1+\nu)z^4 - 6h^2z^2 - 8h^2z \right. \\
+ 3(h^2-\nu^2) r^2 - \frac{3}{8} (1-\nu) r^4 \left] + \frac{\nu}{E} (\rho gh) z + \frac{3r^2 + 6\nu z^2 - 3r^2\nu}{4Ef^3} \right. \\
\left[ \frac{3+\nu}{16} (2\rho gh) a^2 + \frac{3-\nu}{20} (2\rho gh) h^2 \right] \]

\[ u = \nu = -\frac{1+\nu}{8Ef^3} (2\rho gh) \left[ (2-\nu)z^3 x - 3h^2zr - 2h^3 x - \frac{3}{4} (1-\nu) zr^3 \right] \]
It should be noted here that Duncan [12], in 1958, derived a similar expression for $w$ in a simply supported plate subjected to uniform pressure $q$ over its upper face. This was achieved by starting with the Legendre polynomial solution of the basic equations of elasticity for an axisymmetric case as adopted by Timoshenko [13].

At this stage it would be logical to verify if the above relations satisfy the field equations and the required boundary conditions.

The field equations for displacements without body forces were obtained by Allen [14]. Manipulating stress equilibrium and stress displacement relations, including body forces due to gravity, the modified field equations for axisymmetric systems can be shown to be,

\[
(1-v) \left\{ \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{u}{r^2} \right\} + \frac{1-2v}{2} \frac{\partial^2 u}{\partial z^2} + \frac{1}{2} \frac{\partial^2 w}{\partial r \partial z} = 0
\]

\[
(1-v) \frac{\partial^2 w}{\partial z^2} + \frac{1-2v}{2} \left\{ \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} \right\} + \frac{1}{2} \frac{\partial^2 u}{\partial r \partial z} + \frac{1}{2} \frac{1}{r} \frac{\partial u}{\partial z} - \rho g \]

\[
\frac{(1+v)(1-2v)}{E} = 0
\]

(2.18)
The values of \( u \) and \( w \) and their derivatives from (2.17) identically satisfy equations (2.18).

Boundary conditions to be satisfied are:

i) \( \int_{-h}^{h} (\sigma_r)_{r=a} \, dz = 0; \)

ii) \( \int_{-h}^{h} (\sigma_r)_{r=a} \, z \, dz = 0; \)

iii) \( \int_{-h}^{h} (\tau_{rz})_{r=a} \, dz \cdot 2\pi a = \text{weight of the plate}. \)

From (2.17) the strain components for an axisymmetric system are obtained as:

\[
\varepsilon_z = \frac{\partial w}{\partial z} = \frac{\rho g}{E} (z+h) + \frac{1+\nu}{16Eh} (2\rho gh)[4(1+\nu)z^3 - 12h^2z - 8h^3 - 6\nu z r^2] + \frac{\nu}{E} (\rho gh) + \frac{3\nu z}{Eh^3} \left[ \frac{3+\nu}{16} (2\rho gh) a^2 + \frac{3-\nu}{20} (2\rho gh) h^2 \right]
\]

\[
\varepsilon_r = \frac{\partial u}{\partial r} = -\frac{1+\nu}{8Eh^3} (2\rho gh) [(2-\nu)z^3 - 3h^2z - 2h^3 - \frac{9}{4} - (1-\nu)z r^2] - \frac{\nu \rho g (z+h)}{E} - \frac{1-\nu}{2E} (\rho gh) + \frac{3z}{2Eh^3} \left[ \frac{3+\nu}{16} \right]
\]

\[
(2\rho gh) a^2 + \frac{3-\nu}{20} (2\rho gh) h^2 \right] (\nu-1)
\]
\[ \varepsilon_0 = \frac{u}{r} = - \frac{1+\nu}{8Eh^3} (2\rho gh) \left[ (2-\nu)z^3 - 3h^2z - 2h^3 - \frac{3}{4} (1-\nu)zr^2 \right] \]

\[ - \frac{\nu\rho g(z+h)}{E} - \frac{1-\nu}{2} (\rho gh) + \frac{3z}{2Eh^3} \left[ \frac{3+\nu}{16} \cdot (2\rho gh) \cdot a^2 \right. \]

\[ + \frac{3-\nu}{20} \cdot (2\rho gh) \cdot h^2 \left. \right] (\nu-1) \]

\[ \gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} = \frac{3(1+\nu)(2\rho gh)}{4Eh^3} (h^2r - z^2r) \quad (2.19) \]

For axisymmetric systems, the stress-strain relations are:

\[ \frac{1+\nu}{E} \sigma_r = \varepsilon_r + \frac{\nu}{1-2\nu} (\varepsilon_r + \varepsilon_\theta + \varepsilon_z) \]

\[ \frac{1+\nu}{E} \sigma_\theta = \varepsilon_\theta + \frac{\nu}{1-2\nu} (\varepsilon_r + \varepsilon_\theta + \varepsilon_z) \quad (2.20) \]

\[ \frac{1+\nu}{E} \sigma_z = \varepsilon_z + \frac{\nu}{1-2\nu} (\varepsilon_r + \varepsilon_\theta + \varepsilon_z) \]
Substituting the values of the strain components from (2.19) in (2.20) and simplifying one obtains,

\[
\sigma_r = \frac{1+v}{(1+v)(1-2v)} \left[ \frac{1+v}{E} \right] \frac{1-v}{16h^3} (2\rho gh) \left\{ \frac{v}{1-v} \left[ 4z^3(1+v) - 6h^2z - 4h^3 - 6vzr^2 \right] - 2z^3(2-v) + \frac{3}{2}(1-v)zr^2 \right\} - 2z^3(2-v) + \frac{3}{2}(1-v)zr^2 \right\} \\
+ \frac{\nu^2 \rho gh}{(1-v)} - \nu \rho g(z+h) - \frac{\rho gh}{2} + \left\{ \frac{3+v}{16} \cdot (2\rho gh)a^2 + \frac{3-v}{20} (2\rho gh)h^2 \right\} \\
\left\{ \frac{3\nu^2 z}{h^3(1-v)} - \frac{3z}{h^3} \right\} \\
\sigma_\theta = \frac{1-v}{(1+v)(1-2v)} \left[ \frac{1+v}{E} \right] \frac{1-v}{16h^3} (2\rho gh) \left\{ \frac{v}{1-v} \left[ 4z^3(1+v) - 6h^2z - 4h^3 - 6vzr^2 \right] - 2z^3(2-v) + \frac{9}{2}(1-v)zr^2 \right\} - 2z^3(2-v) + \frac{9}{2}(1-v)zr^2 \right\} \\
+ \frac{\nu^2 \rho gh}{(1-v)} - \nu \rho g(z+h) - \frac{\rho gh}{2} + \left\{ \frac{3+v}{16} \cdot (2\rho gh)a^2 + \frac{3-v}{20} (2\rho gh)h^2 \right\}
\[
\sigma_z = \frac{1-v}{(1+v)(1-2v)} \left[ \frac{1+v}{8h} \left( 2 \rho gh \right) \left\{ \frac{v}{1-v} \left[ -2z^3 (2-v) + 6h^2 z + 4h^3 \right] + 2z^3 + 2vz^3 - 6h^2 z - 4h^3 \right\} + \rho g (z+h) - \frac{2v^2 \rho g (z+h)}{1-v} \right]
\]

\[
\tau_{rz} = \frac{3}{4} \rho gr \left\{ 1 - \frac{2}{h} \right\}^2 \tag{2.21}
\]

The routine algebra shows that \((\sigma_{r})_{r=a}\) and \((\tau_{rz})_{r=a}\) satisfy the boundary conditions mentioned before.

This analysis for the self-weight deflection of a cylindrical plate, supported by uniformly distributed shear at the edge, is valuable due to the fact that astronomical mirror substrates are still designed in the form of solid discs for medium size apertures.

In this context of self-weight deflection of circular plates, it is of interest to compare the mid-plane deflection as obtained by Emerson \[15\],

\[
w_0 = \frac{1}{(2h)^2} \frac{\rho g}{16E} 3(1-v^2)(a^2-r^2) \left( \frac{5+v}{1+v} \cdot a^2-r^2 \right) + \frac{\rho g}{4E} \left( 3+v \right) \left( a^2-r^2 \right)
\]

\[
(2.22)
\]
The above expression assumes that the total body force of the plate is equivalent to an equal uniformly distributed surface pressure $q = 2\rho gh$ (Figure 2.3).

2.3 Results and Discussion

For the plate supported by uniformly distributed shear, the distribution of radial and shear stress at the end of the plate is shown in Figure 2.4. The variation of radial stress at an intermediate cross-section is indicated in Figure 2.5. It is clear that the net radial force at any cross-section is zero but only the edge of the plate is free of moment. Furthermore, the distribution of shear is parabolic.

The variation of axial stress $\sigma_z$ as obtained from the present investigation, equation (2.21), is shown in Figure 2.6a with the distribution of $\sigma_z$ for the boundary and loading conditions of Figure 2.3 shown in Figure 2.6b [13]. It is interesting to note that the expression (2.22), being approximate (distributed body force replaced by a uniform loading), violates the boundary condition of zero normal stress over the top face of the plate (Figure 2.6b) whereas the distribution of $\sigma_z$ obtained from the present analysis satisfies the boundary conditions of zero normal stress over the top and bottom faces of the plate exactly (Figure 2.6a). A comparison of the middle plane deflections as obtained from (2.17) and (2.22) is shown in Figure 2.7. Here the numerical values of physical parameters correspond
Figure 2.3 Total body force considered equivalent to an equal uniformly distributed surface pressure, $q = 2 \rho gh$

Figure 2.4 Distribution of radial and shear stresses at the end of the plate
Figure 2.5  Radial stress variation at an intermediate cross-section
Figure 2.6 Axial stress distribution
Comparison between the middle surface deflections as obtained from the present theory (equation 2.17) and Emerson's analysis (equation 2.22):

\[ E = 10.5 \times 10^6; \quad \nu = 0.17; \quad \rho = 0.0795 \]
to that for fused silica, a conventional substrate material.

As evident from equation (2.21), the distribution of radial stress is non-linear. This may be attributed to the effect of shearing stresses and normal pressures on the planes parallel to the surface of the plate. However, the effect of non-linearity in the case considered appears to be rather negligible (Figure 2.5). The radial stresses at the edge are not zero (Figure 2.4a) but the resultant of these stresses and their moments indeed vanish. Hence, on the basis of Saint Venant's Principle, we can say that the removal of these stresses does not affect the stress distribution in the plate significantly at some distance away from the edge.

The expression for lateral deflection $w$, equation (2.17), shows it to be a function of $r$ and $z$ thus suggesting different shapes for top and bottom surfaces compared to the mid-plane. This difference increases as the diameter to thickness ratio decreases. Hence for thick cylindrical mirror substrates, prediction of top surface deflection from the mid-plane analysis (as is done in the approximate theories) is not fully justified. Thus for accurate determination of the top surface deflection the present analysis should prove useful. Figure 2.8 compares the deflections of the middle and outer surfaces.

It should be noted here that, as the outer surfaces and middle plane (Figure 2.8) assume different shapes when flexed, their radii of curvature would not differ merely by their normal distance apart; this interval will not be a constant function of $r$. Thus the present analysis accurately
Figure 2.8  Middle surface and outer surface deflection profiles:

\[ E = 10.5 \times 10^6; \quad \nu = 0.17; \quad \rho = 0.0795 \]
shows the finer details of elastic deformation patterns, which may be relevant in calculating the curvatures of outer faces. The curvature of the flexed disc follows readily from (2.17).

In order to investigate the influence of shear, the results based on present analysis are compared with the thin plate theory in Figure 2.9. The discrepancy in deflection values increases rapidly as the diameter/thickness ratio is decreased (3.4% for $D/H=10$, 21.7% for $D/H=4$). Furthermore, the deflection diminishes quite rapidly as the $D/H$ is decreased. That explains massive character of mirror substrates where rigidity is built up more rapidly than the associated body forces.

It should be noted that a continuous support by way of the parabolically distributed shear is not practicable. A logical simplification is to concentrate shear reactions at three equispaced boundary points or at six internal points, as mentioned earlier in Chapter 1. These would induce more strain compared to a continuous support because of additional internal transmission of body forces to reactive points. However, when a mirror substrate is supported on three equispaced pads, the problem becomes strictly three dimensional and one has to resort to numerical methods aided by a digital computer or experimental studies of models.
A comparison between the present analysis and the thin plate theory results for several D/H ratios:

\[ E = 10.5 \times 10^6; \quad \nu = 0.17; \quad \rho = 0.0795 \]
As there is a similarity in the nature of the body force and thermally induced effects in elastic solids, it is appropriate, in this context, to comment on the thermally induced displacements.

When the temperature in a small portion of the body is increased by $T$, dilatation proportional to $T$ can be produced without a corresponding change in pressure. This implies extension of all linear elements by $\alpha T$, where $\alpha$ is the coefficient of thermal expansion. In addition, if forces are applied to the body, the corresponding strain obtained from stress-strain relation would augment the thermal effect. Thus it follows that the displacement due to the thermal effect is identical to that produced by a body force expressed as the gradient of $- \frac{\alpha E}{1-2\nu} T$, and normal surface pressure $\frac{\alpha E}{1-2\nu} T [7, 13]$. These would be in addition to the actual body force and surface traction present in the system.

Thus, the analytical expressions derived for the body force case may be used to determine thermally induced displacements. On the other hand, thermally induced displacements can be determined quite readily using finite element procedures as explained in the following chapter.
3. NUMERICAL ANALYSIS OF THICK CIRCULAR PLATES OF VARIABLE THICKNESS AND ITS APPLICATION TO THE DESIGN OF MIRROR SUBSTRATES

An axisymmetric elastic system may be described by cylindrical coordinates as shown in Figure 3.1. Due to symmetry in geometry and loading about the vertical axis \( z \), the system deforms only in radial and vertical directions. Also stresses and strains do not vary in the tangential direction. Thus, from a mathematical point of view, the problem is only two dimensional and it is sufficient to analyze any axial \( z-r \) plane as shown in Figure 3.2. The structural forms shown in column 2 of Figure 1.5 represent axisymmetric systems. Several numerical techniques are available which may be used for solving problems of the present kind. It would be useful to comment on these procedures before treating the problem in hand.

3.1 A Brief Review of the Numerical Techniques

3.1.1 Finite Difference Method

This is probably the most widely used numerical technique for solving a variety of field problems, including those in elasticity. The majority of the problems in stress analysis, when tackled by a theoretical approach, reduce to the solution of one or more differential equations with specified boundary conditions on stress or displacement.
Figure 3.1  Axisymmetric solid  Figure 3.2  Axial z-r plane
In the finite difference method, the governing equations of elasticity and boundary conditions are replaced by a finite number of unknown values of the dependant variables at a number of discrete nodes, formed by a grid-work, within and over the boundary of the domain.

The problem can be formulated in terms of stress functions [16] or displacement components [14], the choice being largely governed by the specification of the boundary conditions in terms of stress, displacement or both.

In the former, the values of stress functions obtained for the nodes give only an intermediate solution to a pair of governing equations. A combination of the derivatives of the stress functions yield the required stress components. On the other hand, with the displacement formulation, the components of displacements are obtained directly. A suitable combination of the derivatives of displacements give the stress components.

Once the general formulation of the problem is established, the elastic region to be studied is systematically divided up by a system of coordinate planes intersecting along lines and at nodes. Now governing equations in terms of nodal stress function values have to be satisfied. These finite difference equations represent a set of linear algebraic relations to be solved for unknown stress functions, in general, by a digital computer.
A solution obtained by the finite difference method is always in error, however small. These errors may be reduced in two ways [17]:

i) by employing a very fine mesh size so that the errors due to neglected higher order terms in Taylor's series expansions are negligible;

ii) by including sufficiently high order differences in finite difference formulae based on a coarse mesh.

Both the procedures have disadvantages. The first alternative may appear to be theoretically attractive as errors decrease indefinitely with a mesh size. However, with a fine mesh, the number of equations becomes very large and the efforts involved in obtaining a solution gets prohibitive. Furthermore, round off errors also increase. The alternative would be to use more nodal points in the governing equations. This leads to extensive manipulations to eliminate the fictitious function values representing the stress or displacement functions at nodes outside the boundary of the problem. The fictitious quantities are eliminated by using the specified boundary conditions. Further, due to the inclusion of remote nodes, the local accuracy might be affected.

A comment concerning a relatively new numerical procedure, developed by Day [18] and Otter et al. [19]
would be appropriate here. Referred to as Dynamic Relaxation, it represents an iterative method for use with a computer, to solve the finite difference formulations of stress-strain and equilibrium equations of elasticity. However, instead of using the static relations of equilibrium, the system uses the dynamic equilibrium equations of elasticity. For axisymmetric bodies with body forces due to gravity, they can be expressed as:

\[
\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = \rho \ddot{u} + c \dot{u}
\]

\[
\frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} + F_z = \rho \ddot{w} + c \dot{w}
\]

where \( F_z \) = body force intensity in the z direction
\( \rho \) = mass density
\( c \) = damping factor
\( \dot{u}, \ddot{u}, \dot{w}, \ddot{w} \) = velocity and acceleration components along the r and z directions, respectively.

The system is analyzed considering:

i) the motion of an element due to internal stresses and imposed body forces;

ii) the elastic relation between the stresses and displacements during the course of the motion.
The object of the calculation is not, however, to study the motion but to determine static stresses and displacements of a structure. This is achieved by an iterative procedure. Thus the method is essentially a step-by-step integration of damped vibration, using viscous damping, to ensure attainment of steady state solution.

A typical iteration starts at \( t = 0 \), with arbitrarily chosen initial conditions except at the boundary where displacements and/or stresses are specified. Initially, displacements and velocities may be chosen to be zero except at the boundary. Substitution of chosen displacements in the stress displacement equations gives stresses. The stresses are then introduced in the equilibrium equations to obtain new velocities. The new displacements at time \( t_1 = t + \Delta t \) are obtained from the velocity-displacement relation, \( u_1 = u + \dot{u}\Delta t \).

The next iterative step begins with the calculation of stresses from the stress-displacement relation. The new stresses are then substituted in the equilibrium equations. The process continues until velocities and accelerations diminish and the right hand side of the equilibrium equation approximates to zero, thus satisfying the condition of static equilibrium.

Dynamic relaxation seems to be a very powerful numerical technique, however, the reference system used should be such that the coordinate surfaces conform to the
surfaces of the body. For curved boundaries curvilinear coordinates have to be used. Mixed boundary conditions involving stresses and displacements do not pose any problem.

The main disadvantage of the method is the derivation of the time interval and the damping factor. Suitable values have to be determined by trial and error.

As can be expected, the accuracy of the method depends on the element size used. The results obtained by dynamic relaxation compare fairly well with those due to other methods [18, 19].

3.1.2 Integral Method

Integral methods are fundamentally different from the finite difference and finite element approaches. Here a number of fictitious concentrated or locally distributed loads, of arbitrary values, are applied along the boundary of the model. Their total effect at an interior point is expressed as an analytical summation of the effects of the individual loads [20].

In plane problems, for instance, the forces applied at the boundary can be concentrated loads giving rise to a radial stress [21] \( \sigma_r = \frac{2P}{\pi} \frac{\cos \theta}{r} \) within the field. At the boundary, the interior stresses must equilibrate the exterior stress and from this condition the distribution of the concentrated fictitious loads all over the boundary are determined.
For axisymmetric solids Boussinesq's solution of a force applied normally to the boundary and giving rise to stresses $\sigma_z, \sigma_\theta, \sigma_r, \tau_{rz}$ may be used. The integrated effect of these components of stresses at interior points for every fictitious load applied on the boundary has to be obtained. At the boundary, the sum of interior stresses due to all surface loads has to be equated to the externally applied boundary stress.

Integral methods involve fewer unknowns than finite difference or finite elements techniques. They also result in system of equations with full matrices (meaning there are not many zero terms in the coefficient matrix) whereas difference methods involve sparse matrices of higher order.

Usually the integral procedures give more accurate results than the finite difference method, as the stresses are obtained directly from the analytical expression for the summation of the separate effects of the individual boundary loads. On the other hand, in the finite difference method the stress components depend on approximate derivatives of the discrete stress functions.
3.1.3 Finite Element Method

The finite element method is essentially a generalization of the theory of structural analysis. It makes possible the analysis of two- and three-dimensional elastic continua by the technique similar to that used in the analysis of ordinary framed structures.

The method was introduced originally by Turner et al. [22] for the solution of complex structural problems encountered in the aircraft industry.

The important concept introduced by the finite element method is the use of two- or three-dimensional structural elements to represent an elastic continuum. The real continuum is assumed to be divided into a finite number of elements interconnected at a finite number of nodes. An approximation which is employed in finite element techniques is of a physical nature; a modified structural system is substituted for the actual continuum. This distinguishes the finite element technique from finite difference method, where, as seen before, the exact equations of the actual physical system are solved by approximate mathematical procedures.

It should be recognized that the concept of modelling an elastic continuum by an assembly of structural elements is not new. In fact, the development of the finite element concept has stemmed from an effort to improve on the
Hrennikoff-McHenry [23,24] 'lattice analogy' for representing plane stress systems [25]. Important steps involved in the analysis may be summarized as below [26,27]:

i) Structural Idealization - Division of the elastic continuum, to be analyzed, into an appropriately shaped finite elements.

ii) Element Analysis - Evaluation of the stiffness matrix for every element by relating the forces developed at the element nodal points to the corresponding element displacements.

iii) Assembly Analysis - Evolution of the assembly stiffness matrix for the complete structure by the superposition of appropriate element stiffnesses and determination of unknown nodal displacements by the solution of the equilibrium equations.

For the present problem, the finite element approach was considered to be well suited as the curved boundaries of the arched geometries can be readily idealized by the use of triangular elements. The required deflection can be obtained directly and the mixed boundary conditions pose no problem. Furthermore, thermal effects can be treated using the normal procedure.
3.2 Light-Weight Mirror Substrate Design Philosophies

The self-weight problem associated with large mirrors has long been recognized. The solid disc is not an optimum structure since material near the middle plane, when its axis coincides with the direction of gravity, is not fully stressed. Hence, it is attractive to develop, for both ground and space applications, structural forms in which such understressed material has been redistributed or removed to more effective locations. Several procedures for achieving this are apparent. The most common would be the use of sandwiched and ribbed structures, web and flange construction, arch concept, etc. [6,28,29].

Light weight mirrors have been successfully manufactured for satellite and ground applications by the adoption of sandwich construction technique. A sandwiched mirror consists of two plates, separated and fixed in a rigid relation to one another by spacers, one of the external surfaces acting as mirror. The arrangement offers a number of advantages. Firstly, it allows for an internal circulation of air or some other fluid which could aid in bringing the entire structure to thermal equilibrium more rapidly. Secondly, a well designed sandwich can be made as rigid as a solid mirror and thirdly, mounting is facilitated. These ideas have been embodied in two patents, one by Ritchey in 1924 and the other by Parsons
and Rands in 1929. Figure 3.3 shows Parson's sandwich mirror and Ritchey's cellular construction [29].

The use of ribbed type structure was suggested by Lord Ross [29] as early as in the mid-19th century. The 200 inch mirror substrate of the Hale Telescope was made by pouring pyrex into a mould in which 114 firebrick cores had been bolted. The final pattern had the appearance of a honeycomb. The scheme resulted in a mirror whose weight was approximately half that of a solid blank of the same size when the cores were removed.

A study of the cellular forms of mirror substrates employed to date shows that very few designers have made use of the obvious and logical account of certain fundamental theorems touching optimum structural design as enunciated in 1869 by James Clark Maxwell and extended in 1904 by A.G.M. Michell. Maxwell's theorem, which is demonstrated in a condensed article by Barnett [30], shows that one class of optimum structure is that in which all the material employed is stressed, to an acceptable limit, in the same sense; either compression or tension. Membranes, arches and shells have this general characteristic. Michell's corollary, also outlined in the article by Barnett, recognizes that structural configurations cannot always be arranged to have all the members in compression or tension. However, Michell demonstrated that for a structure where members in both compression and tension are present, they must be orthogonal
Figure 3.3 Sandwich and cellular mirror substrates

a. Parson's Sandwich Mirror

b. Ritchey's Cellular Mirror
at all points of intersection and stressed to equal limiting values for the structure to be optimum. Though developed and proved in two dimensions, these theorems are valuable in presenting a guiding philosophy in the search for optimum three-dimensional structures.

The material employed in conventional beams and plates is not ideally distributed in accordance with the above theorems. One well-known method of redistribution is the adoption of Web and Flange (I-beam) concept. This can be further improved by castellation [31] of an I-beam Web (Figure 3.4). The Web is cut along a line resembling an Acme thread. The resulting halves are then relocated and welded together. The logical extension of this idea into three dimensions leads to the concepts of lattices and geodesic frame works.

The domain between the foundation and mirror substrate elements (Figure 3.5) can be spanned by a structure designed according to Michell's theorem. Michell's structure could be defined in the domain, where arbitrary reaction locations and directions are specified and the body forces exerted by the substrate elements are applied on or in the domain. In such a case the body forces of the substrate and the distributed mass of the Mitchell's structure itself are transmitted to the rigid foundation through direct tension or compression of the structural members. Figure 3.5 illustrates this notion. Some such forms have been discussed in detail by Johnson [32].
Figure 3.4 Castellated beam

Figure 3.5 Michell structure
The alternate approach to optimization follows from Maxwell's original theorem. By arranging the material in arch-like configurations, an optimum design is likely to result, in which the material tends to be stressed in one sense. Kenny [33] has demonstrated, by comparing deflections of a disc to those of spherical shells, which may be constructed by removing materials from the disc, that a shell or an arch type structure is much superior to a constant thickness plate.

In the present investigation the second line of thought is pursued to optimize circular substrates having superior stiffness-to-weight ratios. It was thought that arch-type substrates would be less intricate than ribbed sandwiches or spacial lattices which reflect the philosophy behind Michell's corollary to Maxwell's theorem.

Finally it must be stressed that, the optimum configuration changes with each change in the attitude of a mirror. Thus, there can be no unique optimum configuration for a substrate which changes attitude in a gravitational field. However, it is customary to design a substrate with the axis of the mirror vertical and to assume the demonstrable fact that deformation due to gravity in particular will be less severe and not critical when the axis is horizontal.

Hence, as with other investigations, design studies to follow are concerned with substrates having extreme dimensions in a plane normal to the vertical.
3.3 **Finite Element Solutions for Axisymmetric Systems**

Several axisymmetric bodies subjected to axisymmetric loading were analyzed using the finite element approach. Details of the mathematical treatment is given in Appendix 1. The element stiffness matrices were linked with the existing, two dimensional finite element computer programme. This was possible since the deformation problem for an axisymmetric system is one of two dimensional displacement and three dimensional stress.

In order to check the accuracy of the finite element procedure developed here, the gravity induced deflection in a circular plate, supported around its circumference by a parabolic shear, was computed and compared with the analytical solution (Figure 3.6). A close agreement between the two solutions is apparent.

With confidence established in the accuracy of the finite element procedure, deflections were evaluated for the constant thickness plate uniformly supported at various radial locations. The aim was to establish the optimum location for minimum deflection. The results showed (Figure 3.7) the minimum deflection to occur when the support circle radius is 0.66 of the radius of the plate.

This is in close agreement with Emerson's [15] study for three point supported optical flats. He concluded that
Figure 3.6 The middle surface deflection as predicted by the finite element procedure and the analytical solution:

\[ E = 10.5 \times 10^6; \quad \nu = 0.17; \quad \rho = 0.0795; \quad \frac{D}{H} = 6 \]
Figure 3.7 Effect of support circle radius on the middle surface deflection:

\[ E = 10.5 \times 10^{-6}; \quad \nu = 0.17; \quad \rho = 0.0795 \]
the three points should be located at 0.7a for the minimum bending deflection. Studies conducted by Kenny and Williams [8], and Vaughan [10] also led to the same conclusion. With this background it was decided to proceed with the arched substrate analysis.

3.3.1 Comparative Analysis of Arched Structures and Solid Discs

It is important to recognize that in designing a light weight mirror substrate, a relatively higher stiffness-to-weight ratio may be obtained by removing materials to form an arch from a deep solid disc (D/H = 3) rather than from a moderately thick disc (D/H = 6). But the space occupied by the former will be larger than the latter. This in turn will need a larger housing and supporting structure for the mirror. So, in the substrate design, the question of space limitation has to be borne in mind. The substrate forms analyzed in the present study are limited within the space envelope of the conventional solid disc having diameter to thickness ratio of 6. Furthermore, a slight curvature of the reflecting surface is neglected. As the objective is to compare deflection-to-weight ratios resulting from changes in the geometry of the mirror substrate, the error introduced by this approximation is likely to be insignificant.
Figure 3.8 shows the plate previously considered with certain material removed so that the substrate proper remains but an arch transmits its body weight to the outer circular boundary. The structures were analyzed by the finite element method, which clearly demonstrated the effect of attendant redistribution of substrate material on the flexural deflection patterns. The deflections of the upper surface of these substrates are shown in Figures 3.9 and 3.10. For comparison, the deflection of the related solid disc is also included. Ratios of the deflections of the arch models to those of the solid disc are plotted in Figures 3.11 and 3.12 together with the average deflection ratios.

In designing mirror substrate one is interested in achieving a low weight with high stiffness. Thus in determining the merit of arched substrates compared to solid discs, a parameter involving both weight and deflection is desired. For this purpose a function called 'figure of merit' has been defined as the ratio of the product of weight and deflection for the solid disc to that for the arched substrate. Figure of merit values, based on the maximum deflection of the upper surface, for different arch models are shown in Table 3.1.
Figure 3.8  Arched structures constructed within the envelope of the solid disc of D/H = 6
Figure 3.9 Comparison of the upper surface deflections for solid disc (1) with that of arch models (1) and (2): (a) lateral deflection

- Solid Disc (1) .... 1.000
- Arch Model (1) .... 0.641
- Arch Model (2) .... 0.398
- Arch Model (2) with Prop
- Prop

\[ E = 10.5 \times 10^6; \ \nu = 0.17; \ \rho = 0.0795 \]
Figure 3.9 Comparison of the upper surface deflections for solid disc (1) with that of arch models (1) and (2): (b) radial deflection

$E = 10.5 \times 10^6$; $\nu = 0.17$; $\rho = 0.0795$
Figure 3.10 Upper surface deflections for solid disc (2) and arch model (3):

\[ E = 10.5 \times 10^6; \quad \nu = 0.17; \quad \rho = 0.0795 \]
Figure 3.11  Comparison of the upper surface deflections of arch models (1) and (2) as ratios of the deflection of solid disc (1): (a) lateral deflection
\[ E = 10.5 \times 10^6; \quad v = 0.17; \quad \rho = 0.0795 \]
Figure 3.11 Comparison of the upper surface deflections of arch models (1) and (2) as ratios of the deflection of solid disc (1): (b) radial deflection

\[ E = 10.5 \times 10^6; \quad v = 0.17; \quad \rho = 0.0795 \]
Figure 3.12 Upper surface deflections of arch model (3) as ratios of the deflections of solid disc (2):

\[ E = 10.5 \times 10^6; \quad \nu = 0.17; \quad \rho = 0.0795 \]
A comparison of deflection patterns clearly establishes the great potential of arching in designing lightweight mirror substrates. For ring support reactions at $S = 1$, the maximum lateral deflection $w$ for the solid disc is about two times higher than that of arch models (1) and (2). On the other hand the weight of the solid disc is 35.9% and 60.2% higher than that of arch models (1) and (2), respectively. In terms of average deflection ratio, the arch models (1) and (2) offer 29.7% and 17.6% reduction in $w$ deflection and the figure of merit values are 2.84 and 4.68, respectively. For ring support reactions at $S = 0.58$, the maximum $w$ deflection for the solid disc is about 2.8 times higher than that of arch model (3) whereas the weight of the solid disc is 42.3% higher than that of arch model (3). Considering the average deflection ratio, arch model (3) offers 50.39% reduction in the $w$ deflection and the figure of merit value is 4.87.

From above it appears that among the different geometries considered, arch model (3) is the most advantageous.

It should be noted here that for astronomical purposes the changes in slope and curvature of the reflecting surface of the mirror are of the main interest. These parameters being related to deflection, the present analysis is quite representative of the true situation.

Figure 3.9 suggests that the overhanging type sub-
strate, arch model (2), gives rise to reverse curvature. Obviously this has to be avoided, possibly by providing a suitable "prop" at the edge.

TABLE 3.1
FIGURE OF MERIT VALUES

<table>
<thead>
<tr>
<th>(Maximum w x weight) Solid disc (2)</th>
<th>= 4.87</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Maximum w x weight) Arch model (3)</td>
<td></td>
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<tr>
<td>-----------------------------------------</td>
<td>--------</td>
</tr>
<tr>
<td>(Maximum w x weight) Solid disc (1)</td>
<td>= 4.68</td>
</tr>
<tr>
<td>(Maximum w x weight) Arch model (2)</td>
<td></td>
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<tr>
<td>-----------------------------------------</td>
<td>--------</td>
</tr>
<tr>
<td>(Maximum w x weight) Solid disc (1)</td>
<td>= 2.84</td>
</tr>
<tr>
<td>(Maximum w x weight) Arch model (1)</td>
<td></td>
</tr>
</tbody>
</table>

A finite element analysis of arch model (2) was done by prescribing displacements at nodes b and c with reference to the node a. This simulates the presence of a "prop" at the edge. The top surface w displacement for the arch model (2) with a prop at the edge is shown in Figure 3.9a. This is for prescribed displacements of 0.000088 and
0.000112 in. at the nodes b and c, respectively. The displacement of c relative to a is 0.000710 in. when there is no "prop" (arch model (2)), and 0.000011 in. when there is no "hole," (arch model (1)). Thus the analysis with specified displacement of 0.000112 in. represents a situation between arch models (1) and (2). The analysis demonstrates the principle that the reverse curvature of arch model (2) can be avoided by providing a "prop" at the edge. The exact nature of the displacements will be governed by the size of the "prop".

The geometries studied here were selected to demonstrate the principle of arch action rather than establish its optimum shape. The study was initiated with a simple semi-circular arch, so that during experimental investigation the model can be easily machined.

Arch model (3) is the best among the configurations studied. This is because the support reactions are provided close to the centre of gravity. Also, the flexure of the overhang is reduced by removing its material in an arch-like manner.

A comment concerning the Cassegrainian substrate, which was analyzed by the finite element method would be appropriate. Its top surface deflection for different support circle radii is shown in Figure 3.13. It is evident that, for the geometries studied, the minimum deflection is obtained for a value of S lying between 0.75 and 0.66.
Figure 3.13  Top surface deflection of a Cassegrainian substrate as a function of support circle radius:
$E = 10.5 \times 10^{-6}$; $\nu = 0.17$; $\rho = 0.0795$
3.3.2 Deflection Due to Thermal Gradient

The deformation of a mirror surface due to thermal effects (changes in ambient temperature or presence of thermal gradients) is of utmost importance as it may exceed the tolerance limits of diffraction limited optics.

It is possible to treat thermal effects by the same finite element programme as that developed for body force loaded axisymmetric solids. Thermal effects can be included in the calculations directly by accounting for corresponding strains in each element while writing the stress-strain relations involved. This leads to an additional term in the force-displacement relation. The additional term represents the thermal force which acts at every node and hence appears in the final load matrix of the whole structure. The evaluation of thermal displacements is briefly discussed in Appendix 2.

The finite element programme developed for the displacement analysis of body-force loaded axisymmetric solids was modified to account for any thermal effect. The modified programme calculates thermally induced nodal forces by using the relationship (A2.4).

Thermal deflections were calculated for a cylindrical plate subjected to linear temperature gradient $T(z) = Kz$, where $K$ is a constant. The deflection plots are presented in Figures 3.14 and 3.15. Note that the top and bottom
Figure 3.14 Effect of temperature gradient $T = 2z$ on axial deflection of a thick cylindrical plate; $D/H = 6$, $\alpha = 0.2777 \times 10^{-6}$ in/in°F
Figure 3.15  Radial deflection in a thick cylindrical plate due to a temperature gradient $T = 2z$; $\alpha = 0.2777 \times 10^{-6} \text{ in/in}^\circ F$
surfaces of the plate assume different shapes. This is due to the fact that, in the example, the top surface undergoes a temperature change of 0°F as against the bottom surface temperature change of 40°F. For a plane disc, an absolute temperature change will not induce any variation in the curvature, but if surfaces are curved, absolute temperature change may modify the curvature.

It should be noted here that the body-force-induced displacement in a substrate for a particular loading is constant, but that due to thermal effects will depend on the magnitude of the temperature change. A comparison of the lateral deflections (Figures 3.6, 3.14) shows that a thermal gradient of 2°F/inch can induce displacements more than four times those due to gravity. The importance of equalizing the temperature within a telescope dome-enclosure is thus quite apparent.
4. EXPERIMENTAL TECHNIQUES FOR THE ANALYSIS OF BODY
FORCE DEFLECTION

The present project is essentially theoretical in character. However, it was thought appropriate to undertake experimental studies to substantiate the theoretical predictions of reduction in the weight to stiffness ratio for mirror substrates by removing materials in arch-like manner.

Experimental investigation of body force induced displacements in a model presents considerable difficulty. Gravity induced displacements are, in general, small and diminish further in proportion to the linear scale of the model. Thus the experimental techniques used are required to have sufficient resolution to identify low order stresses or displacements.

4.1 Review of the Experimental Studies for Mirror Substrate

Several investigators have conducted experimental studies of self-weight deflection of mirror substrates. Emerson [15] determined true contours (i.e., contours of uniformly supported plates in absence of bending) and the bending deflection curves for 10 5/8 inch diameter optical
flats of fused quartz. Based on the relationship that the bending deflection varies inversely as the square of the thickness, the true contours were obtained by using three optical flats of like material, properties and diameter. The plates were tested in pairs, arranged parallel to each other and resting, at the vertices of an equilateral triangle, over two sets of supports located one over the other. The algebraic sum of contours of the surfaces was determined along a diametric line, parallel to two of the supports, using a Pulfrich viewing instrument [35]. The technique gives quite accurate results but is restricted to mirror substrates in the form of a solid disc.

In this line of studies the stereoscopic approach due to Gates [36] should be mentioned. The method relies on the use of two interferograms in which the angle between the interfering surfaces is of opposite sign, but the spacing and position of the fringes are, as far as the irregularities of the surfaces will allow, the same. The relative displacements of the bands at corresponding points of each of the pair of interferograms are then measured with a double microscope. The technique allows measurement of displacements with a sensitivity better than 0.002 wavelength. An impression of the relief of the surface may also be obtained by viewing the interferograms in a suitable stereoscope.

Dew [37, 38] described a precise method of determining gravitational sag and undeflected figure of optical flats
using a Fizeau interferometer. The undeflected configuration of a flat was determined by taking the mean of the 'face up figure' and 'face down figure.' The gravitational sag was obtained by measuring firstly the figure of the flat in the manner under investigation and secondly the undeflected figure by the method described above. The technique gives quite accurate results but is restricted to substrates in the form of solid discs only.

Unwin [39] has established a technique of measuring accentuated gravitational deformations in mirror substrate models by using Shadow Moiré technique. The accentuation of gravitational deformations was obtained by immersing polyurethane foam models in water. This resulted in a sufficient number of well-defined fringes to make analysis possible.

The investigation by Kenny [6] though exploratory in nature should also be mentioned. He conducted measurements of frozen displacements on photoelastic models of mirror substrates. For thin plates, the theoretical and experimental results showed good agreement. However, the thick plate results exhibited considerable discrepancy. Also, the tests data for two similar thick plates showed considerable variation. The discrepancies may be attributed to unreliable experimental techniques and, possibly, to different case histories of the models.
4.2 Present Experimental Investigations

4.2.1 Frozen stress photoelasticity coupled with the immersion analogy for gravitational stresses

The frozen stress technique of photoelasticity relies on the ability of epoxy resins to retain at room temperature a relatively large elastic strain system induced at a higher temperature when a suitable temperature cycle is used [40-42]. The technique of "locking" deformations due to an applied load can be explained by means of the bi-phase theory. The materials consist of long chain hydrocarbon molecules, some of which are well bonded together by primary bonds, whereas a large mass of them are less solidly held through shorter secondary bonds. The properties of the primary bonds do not change appreciably with temperature whereas the secondary bonds become viscous as the temperature increases. When a load is applied at a high temperature the viscous component carries a very small portion of the loading (corresponding to its low modulus of elasticity), the main portion of the load being carried by the primary bonds. If the temperature is lowered gradually to room temperature while the load is being maintained, the secondary bonds become rigid again and "locks" the deformation of the primary bonds. On removal of the load at room temperature the primary bonds relax to a very small degree, but the main portion of the deformation is not recovered. As the "locking"
takes place on a microscopic scale, the deformation and the accompanying birefringence is maintained in a slice obtained by careful sawing of the model, avoiding the generation of appreciable heat.

The slice thus obtained from a three dimensional model can then be examined under the polariscope like a two dimensional model.

As mentioned earlier, the main difficulty in the experimental study of body force problems is the low stress level induced in models much smaller in size than the prototypes. This low sensitivity can be overcome by immersing the model in a liquid of high density. The analogy was first shown by Biot [43] for two dimensional bodies and by Serafim and Da Costa [44] for the three dimensional case. It was developed with the assumption that the ratio of the density of the model to the density of the fluid was small enough so that the stresses due to the former are negligible [42]. But for accurate analysis the weight of the model should be taken into account. This was accomplished by Parks et al. [45]. The schematic proof of the immersion analogy is shown in Fig. 4.1. It is clear that the advantage of immersing the model in a heavy liquid is to increase the response of the model by \((K-1)\), where \(K\) is the ratio of the density of the liquid in which the model is immersed to the density of the model material.
<table>
<thead>
<tr>
<th>FREE BODY DIAGRAM</th>
<th>LOADING CONDITIONS</th>
<th>STRESS SYSTEM</th>
</tr>
</thead>
</table>
| ![Free Body Diagram A](imageA) | a. \( \sigma_n = k \gamma \gamma \)  
  \( \tau_n = 0 \)  
  \( R = \text{reactions} \)  
  b. \( Y = \gamma \) | \( \sigma_e \)  
  \( \tau_e \) |
| ![Free Body Diagram B](imageB) | a. \( \sigma_n = -k \gamma \gamma \)  
  \( \tau_n = 0 \)  
  \( R = \text{reactions} \)  
  b. \( Y = k \gamma \) | \( \sigma_{x,y} = -k \gamma \gamma \)  
  \( \text{all } \tau_{(x,y)} = 0 \) |
| ![Free Body Diagram C](imageC) | a. \( \sigma_n = 0 \)  
  \( \tau_n = 0 \)  
  \( R = \text{reactions} \)  
  b. \( Y = -(k-1) \gamma \) | \( \sigma_{y} = (\sigma_e + k \gamma \gamma) \)  
  \( \tau_{y} = \tau_e \) |
| ![Free Body Diagram D](imageD) | a. \( \sigma_n = 0 \)  
  \( \tau_n = 0 \)  
  \( \frac{1}{k-1} R = \text{reactions} \)  
  b. \( Y = -\gamma \) | \( \sigma_y = \frac{1}{k-1} (\sigma_e + k \gamma \gamma) \)  
  \( \tau_y = \frac{1}{k-1} (\tau_e) \) |

A. Body of density \( \gamma \) submerged in a liquid of greater density \( k \gamma \).
B. Body of density \( k \gamma \) submerged in a liquid of its own density.
C. Difference of above two, multiple gravity.
D. Gravity

Figure 4.1 Immersion analogy for gravitational stresses [45]
In order to obtain the maximum accentuation of gravitational stresses mercury was used as the immersion liquid. In the stress freezing procedure the model immersed in mercury has to be raised to a temperature of the order of 140°C. This would give rise to some mercury vapour, as the boiling point of mercury is 180°C. As mercury vapour is readily absorbed via respiratory tract and has adverse effects on the human body, arrangements were made in designing immersion tank to pipe out the vapour and condense it in a test tube held in liquid nitrogen (Fig. 4.2).

The three-point supported solid disc mirror substrate and the calibration disc were loaded in an oven. The temperature was raised at the rate of 1°C per hour to 140°C. After 48 hours at 140°C, the temperature was lowered at the rate of 1°C per pour to room temperature. At the end of the loading cycle the model was removed from the oven and the desired slices were cut for examination, using a diamond impregnated slitting wheel with copious amount of liquid coolant.

Though accentuation of gravity induced stresses was achieved by immersion of the model in mercury, in order to improve the accuracy and to identify low order fractional fringes it was necessary to use fringe multiplication. The novel fringe multiplication technique due to Post [46, 47] was used in the evaluation of photoelastic results.
Figure 4.2 Schematic diagram showing model in oven, mercury vapour condensation technique and the location of slices taken for examination.
Fringe multiplication is a whole field compensation technique in which the model is put between two partially reflecting mirrors. One of the mirrors is slightly inclined. Due to the presence of partial mirrors, the light rays travel back and forth as shown in Fig. 4.3. It is clear from the figure that each ray emerges from the mirror in a direction depending on the number of times the ray has traversed the model. The inclinations shown in Fig. 4.3 are greatly exaggerated. The ray does not pass through an unique point each time it traverses the model. The length of the line over which the photoelastic effect is averaged depends on the distance between the mirrors and the angle of inclination of the mirror. As different rays are inclined at different angles with respect to the axis of the polariscope, any one of the rays can be isolated and observed, as shown in Fig. 4.4, without interference from other rays. If the ray which has passed through the model three times is observed, three times multiplication is obtained.

The fringe multiplication photographs obtained for a radial slice through a support point and a sector slice containing two support points, of the three-point supported model, are shown in Figs. 4.5-4.6.*

An examination of the fringe photographs indicates presence of severe mottle in the material. Photoelastic

*These photographs were taken at Dr. D. Post's laboratory. Averill Park, N.Y.
Figure 4.3 Light reflection and transmission between two slightly inclined partial mirrors

Figure 4.4 Partial mirrors as employed in a polariscope for fringe multiplication
Figure 4.5 Multiplied fringe patterns for a radial slice through a support point
Figure 4.6 Multiplied fringe patterns for a sector slice through two support points.
mottle is a random birefringence caused by localized rapid polymerisation due to presence of "hot spots" when the material approaches gelation [48]. Since mottle is extraneous birefringence, it can be thought of as noise superposed on photoelastic signal [49]. Although the visibility of mottle increases with multiplication, the signal to noise ratio is the same for the multiplied pattern as it is for the ordinary isochromatic view.

The multiplied fringe patterns (Fig. 4.5) for the radial slice show clear fringes up to 17X. These fringes represent the magnitude of the difference of the secondary principal stresses,

\[ \sigma_1' - \sigma_2' = \sqrt{(\sigma_r - \sigma_z)^2 + 4\tau_{rz}^2} \]

In this case the dominant stress component is \( \sigma_r, \sigma_z \) and \( \tau_{rz} \) components being very small. So the magnitude of the imposed \( \sigma_1' - \sigma_2' \), due to body force loading, is quite high compared to the residual birefringence caused by mottle. The fringe patterns for the radial slice can be interpreted fairly accurately by disregarding the localized effects due to mottle and drawing a smooth curve through the isochromatic fringes.

On the other hand, the multiplied fringe patterns for the sector slice (Fig. 4.6) are substantially disturbed by the residual birefringence due to mottle. The difference
in the secondary principal stresses,

\[ \sigma_1' - \sigma_3' = \sqrt{(\sigma_r - \sigma_\theta)^2 + 4\tau_{r\theta}^2} \]

is expected to be quite small as magnitudes of the \( \sigma_r \) and \( \sigma_\theta \) stresses are approximately of the same order and \( \tau_{r\theta} \) is quite small. The noise due to residual birefringence thus becomes very prominent and outweighs photoelastic signal making it impossible to analyse the fringes. Similar phenomenon was observed in the fringe patterns for the surface sub-slices taken from the radial slice. However, for the integration of the stress-strain relations, to determine displacements, stress components have to be evaluated accurately, both from the radial slice and its sub-slices.

It should be mentioned here that Parks et al. [45] investigated bodyforce induced stresses in a thick-wall hollow cylinder using immersion analogy. However, they only determined the tangential stresses along the inside boundary of a radial slice.

Thus, although the accentuation of gravity stresses by immersion and identification of low order fractional fringes by fringe multiplication appears to be quite promising, the method would demand high quality casting with very little residual birefringence.

In making castings for the present investigation the procedures laid down by Leven [50] and Kenny [51] were
followed very carefully. As noted by several investigators in the field of photoelasticity, the amount of mottle present in castings is not predictable. Identical castings produced by identical methods but at different times and with different batches of resins have exhibited varying degrees of mottle. Hence, until a reliable technique for producing high quality castings is established, it appears that the method can not be used for the determination of gravity induced displacements.

4.2.2 Deflection study of photoelastic models by direct measurement of frozen displacement

The photoelastic method is usually applied to a situation where appreciably large strains are induced in a model permitting accurate determination of the stresses. Displacement measurement from photoelastic models is not very common.

As pointed out in section 4.1, Kenny [6] has conducted some preliminary measurements of frozen displacements. Though, he did not get justifiable results from frozen displacement measurements of the thick plates, it was decided to undertake experimental determination of frozen displacements on models in the form of solid disc and arched dome. This was based on the conviction that if experiments are performed with proper care the technique might produce results comparable with the theory.
A theoretical investigation of displacements at 'transition temperature' of a solid epoxy disc, 9 inch in diameter and 1.5 inch in thickness, and a ring support at periphery, showed displacements to be of the order of $10^{-4}$ inch. This order of displacements is ideal for oblique incidence interferometric measurement [52]. Hence it was decided to use the oblique incidence interferometer, readily available in the department, for the present investigation. The development and design of this apparatus is described by Bajaj [53].

The procedure followed during the measurement of the body force induced displacements in photoelastic models may be briefly summarized as follows:

i) The models were subjected to an annealing cycle to remove the machining stresses.

ii) The surface, on which the deflection measurements were made, was then ground with an adequate supply of a liquid coolant and the initial displacement pattern was recorded by using the interferometer.

iii) Now the model was put in the oven with proper support conditions and subjected to a suitable temperature cycle, thus freezing the body force induced displacements at the transition temperature.

iv) The final displacement pattern was then recorded by placing the model again in the oblique incidence interferometer.
v) The body force induced displacement was deduced by subtracting the initial displacement from the final displacement.

Following this procedure, the fringe photographs for the 'normal' and 'reverse' loadings were obtained for the disc with ring support at the periphery (Fig. 4.7). Corresponding deflections are plotted in Fig. 4.8. The interferograms represent an area in the form of an ellipse, one inch minor diameter and nine inch (diameter of the disc) major diameter, compressed optically into a circularly observed field. It is apparent from Fig. 4.8 that the deflection obtained from the normal loading show good agreement with the theory. On the other hand the non-symmetric deflection pattern obtained from the reverse loading shows considerable discrepancy. Theoretically, the normal and reverse loadings should produce the same deflection patterns. This raises a doubt as to the applicability of the frozen displacement technique in measuring self-weight deformations.

Hence, instead of pursuing further investigations of mirror models, it was decided to conduct test on a simple beam to gain more insight into this aspect of the problem. The fringe photographs for the simply supported beam are shown in Figure 4.9 and the displacements obtained are represented in Fig. 4.10. It is apparent that displacements associated with normal and reverse loadings are quite different from each other and show a great discrepancy
Figure 4.7 Fringe photograph for the ring supported solid disc: (a) normal loading

Figure 4.7 Fringe photograph for the ring supported solid disc: (b) reverse loading
Figure 4.8 Self-weight deflection of the ring-supported solid disc
Figure 4.9 Fringe photograph for the simply supported beam: (a) normal loading

Figure 4.9 Fringe photograph for the simply supported beam: (b) reverse loading
Figure 4.10  Self-weight deflection of the simply supported beam
with the theoretical displacement pattern. It seems every
time the model undergoes a temperature cycle, some changes
take place in the physical properties of the epoxy
materials.

The experimental information available so far, though
not directly useful in achieving the final objective, provides
useful insight into the limitations of epoxy models in the
frozen displacement technique. Some of the important
observations are listed below:

i) It seems that every time the model undergoes
a thermal cycle the material property of the
epoxy changes. This could be due to continuous
polymerization of the model material. It seems
the polymerization is never entirely complete
though photoelasticians believe that by the
curing process the model material is rendered
chemically inert.

ii) As the experiments were performed in the presence
of air, the oxygen can react with the polymer
groups and change the characteristics of the
model material near the surface. May be by
conducting experiments in an inert atmosphere
better results could be obtained.

iii) Frozen stress photoelasticity is a well estab-
lished technique. Oppel, a pioneer in the field
of photoelasticity, has made displacement
measurements on slices obtained from three-dimensional photoelastic models. These were for externally applied loads where the displacements produced by the loading were far in excess of disturbances due to the continuous polymerization. However, for self-weight loaded systems the minute displacements are easily out-weighed by these disturbances and/or reactions of the polymer groups with the oxygen of the surrounding atmosphere.

On the basis of Kenny's experiments with thick plates and associated discrepancy with the theory as mentioned before (experiments with thin plates substantiated the theory), it appears that polymerization also depends on surface to volume ratio.

4.2.3 Deflection analysis of a solid disc and an arched dome using cold cure silicone rubber models

Due to the difficulties presented by the frozen stress and displacement photo-elastic approaches, it was decided to conduct experiments with cold cure silicone rubber models. The object here was to study the gross effect of geometry on body force induced displacements.

The low Young's modulus of the cold cure silicone rubber enables large self-weight deflections in the
models. The measurement of the self-weight deflection requires a technique which does not involve a physical contact with the model. The Shadow Moiré technique is, thus, quite suited for the purpose.

The technique, originally proposed by Weller et al. [54] and developed by Theocaris [55] may be used to obtain contour maps of non-specular surfaces of significant arbitrary curvature. It has been successfully applied in studying the geometry of the biological surfaces [56].

Here a grid of alternate transparent and opaque lines is placed in a close proximity of the object to be tested. Collimated light incident at an angle \( i \) casts shadows of the opaque lines onto the model surface. When the grating and its shadow are viewed together at an angle, \( o \), fringes indicative of variable intensity in the plane of illumination and observation reveal points of known relative level as shown in Fig. 4.11. It should be pointed out that the permissible gap between the grating and the surface is directly proportional to the pitch. This is due to diffraction around the bars which degrade rectilinear projection of shadows at a sufficient distance from the grating. To minimize this effect, the grating with three levelling screws was mounted directly on the top of the model. While taking fringe photographs the levelling screws were slowly adjusted in order to obtain fringe patterns as symmetrical as possible.
Relations

In ABC, $h \tan i + h \tan o = e'$

$h = \frac{e'}{\tan i + \tan o}$

e = Normal pitch grating
e' = Effective pitch grating,
e = Cosec $\alpha$

Figure 4.11 Formation of Shadow Moiré fringes
In the relation for contour interval 'h' (Figure 4.11), a provision for altering the sensitivity can be seen. Hence, instead of observing the fringes normally, they were photographed at an angle $0 = 45^\circ$. The grating used had a pitch of 200 lines per inch and was arranged at its normal pitch grating.

The models of solid disc and arched mirror substrates were prepared by pouring a mixture of liquid silicone rubber and a catalyst in carefully prepared moulds, as suggested in the Dow Corning bulletin 08-417 [57].

The Young's modulus of silicone rubber was determined by applying a finite compression on a cylinder of silicone rubber [58]. It was found to be 160 psi.

The highly flexible character of the silicone rubber material presented some difficulty in realizing proper boundary conditions. However, by using the supporting technique as indicated in Figure 4.12, it was possible to approximate the boundary conditions used in the analysis (section 3.3.1).

In order to obtain a precisely flat surface of the model, a mould was carefully constructed with the accuracy of 0.001 inch. However, the curing process resulted in some shrinkage of the model thus leading to an initial curvature of the surface.

In order to eliminate the uncertainty concerning the extent of the initial curvature present, the deflections
Figure 4.12 Support frames for models
were measured with respect to that initial deformed state.

As shown by Duncan [59], the gravity-induced displacements in a solid can be eliminated partially by immersing it in a liquid of the same density as the solid and fully if the solid is incompressible \((v = 0.5)\). For silicone rubber \(v\) being 0.5, it appears to be an ideal material for this purpose. The liquid selected was glycerine whose specific gravity is 1.2 compared to 1.18 for the model material.

The models while in their support frames were put in a tank. Glycerine was then slowly poured, avoiding any trapped air bubbles, until its level came close to the top surface of the model.

Now the Shadow Moiré fringe pattern was photographed. Next, the glycerine was carefully drained and the fringe photograph was taken again. These photographs are shown, respectively, in Fig. 4.13 a,b for the solid disc and in Fig. 4.14 a,b for the arched substrate. As they were taken at an angle \(\theta = 45^\circ\) the surfaces appear as ellipses.

By subtracting the initial displacements of the models while in glycerine from the final displacements in air, the body-force induced displacements were obtained. They are shown in Fig. 4.15. Note that the maximum \(w\) for the solid disc is 1.34 times that for the arch model. On the other hand, the weight of the solid disc is 35.9% higher than that of the arch model. Hence, the figure of
Figure 4.13  Shadow Moiré fringe patterns for the solid disc
(a) Model in glycerine

(b) Model in air

Figure 4.14 Shadow Moiré fringe patterns for the arched model
Figure 4.15 Comparison of gravity-induced upper surface deflections for the solid disc with that of the arch model
merit, based on the maximum deflection, is 2.09, thus clearly showing the superiority of the arched substrate.

4.3 Applicability of the Model Results to the Prototype Mirror Substrate Design

In applying model results to the prototype, it is important to account for the differing values of Poisson's ratio. Silicone rubber has $\nu = 0.5$, whereas the prototype mirror substrate materials have much lower values of $\nu$. Some of the likely materials for mirror substrate, such as fused silica and beryllium, have Poisson's ratio of 0.17 and 0.025, respectively. This thus raises a question regarding the applicability of the model test results.

Clutterbuck [60] from experimental and analytical studies found the peak stresses for $\nu = 0.5$ to be 10% higher than those for $\nu = 0.3$. This was substantiated by Kenny's [61] analysis which showed experimental investigations with models of $\nu = 0.5$, to predict peak stresses 15% higher for the beryllium prototype.

In the present case the effect of $\nu$ on stress distribution and displacement is evaluated using the analytical expressions (2.17) and (2.21). The distribution of radial stress $\sigma_r$ is shown in Fig. 4.16. This analysis shows that with increasing value of $\nu$, the peak stresses also increase whereas the maximum displacement decreases. Furthermore, it shows that the maximum displacement predicted
Figure 4.16  Variation of top surface $\sigma_r$ with Poisson's ratio:

$$E = 10.5 \times 10^6; \quad \rho = 0.0795$$
by models of $v = 0.49$ would be 28.8% lower for the fused silica ($v = 0.17$) prototype. Fortunately, this does not affect the conclusion concerning the figure of merit based on geometry.
5. CONCLUSIONS AND RECOMMENDATIONS FOR FUTURE STUDY

5.1 Conclusions

Important features of the investigation and conclusions based on it may be summarized as follows:

i) The self-weight problem of the traditional circular disc mirror substrate has been investigated and an analytical solution, not previously noted in the literature, has been obtained.

ii) The finite element solution, in terms of triangular ring-elements, clearly demonstrates the superiority of arch-type structures over solid discs. This would make arch designs particularly suited in the construction of an orbiting telescope where weight would be one of the major considerations.

iii) The modified finite element programme, capable of analyzing deformations due to thermal effects, shows that the presence of a small temperature gradient can produce appreciable deformations in the mirror substrate.

iv) The frozen stress photoelasticity experiment clearly indicates its limitations in evaluating the body force
induced displacements. This is primarily due to the difficulty in obtaining a sufficiently stress free model material. Fringe multiplication up to 17X was successfully obtained during the present experiments.

v) The frozen displacement studies provide useful insight into the limitations of epoxy models in measuring body force induced displacements. This is due to the phenomenon of continuous polymerization of epoxy resins.

vi) The investigation on cold cure silicone models clearly establishes their usefulness in studying self-weight deformations. These experiments show the superiority of an arched design compared to the disc configuration.

vii) A difference in the Poisson's ratio would naturally lead to a discrepancy between the model and prototype stress distributions. However, it is important to recognize that this in no way precludes the use of models in establishing the superiority of a particular geometric configuration.

5.2 Recommendations for Future Study

The investigation presented here aims at demonstrating the favourable influence of the arch as applied to a mirror substrate design. A search for the optimum arch represents
a logical extension to the study.

For a three-point support system, the model can be further modified by removing material in the form of an arch in the tangential direction between support points (Fig. 5.1). This would encourage tangential transmission of body forces to reactive points through arching rather than flexure.

The significance of arched substrate, thus established, leads to numerous variations in designs. For example, a design shown in Fig. 5.2 appears to be particularly promising and needs to be explored further. Here each little block of material is supported by the arch below. The radial and circumferential cuts release transmission of bending in the respective directions. There is a possibility of degradation of image due to diffraction around the edges of the blocks. However, a mirror in the form of a plate placed at the top of the arched substrate may correct this (dotted lines in Fig. 5.2).

The arched forms of substrates may present some difficulty while polishing the reflective surface of the mirror. There is also a possibility of general 'quilting' of the aperture due to thermal straining of substrates with variable thickness. These problems need detailed investigations.

The present experimental investigations show the limitations of the photoelastic approach for the evaluation
Figure 5.1  Arched substrate form for three-point supports
Figure 5.2 Arched substrate form with radial and circumferential grooves
of self-weight deflections. Techniques to produce castings with negligible mottle may be explored. Frozen displacement measurements should be conducted in an inert atmosphere to eliminate reaction of the polymer groups with the atmospheric oxygen.

For a general three-dimensional substrate system a finite element technique may prove to be ideal. But computer storage and large input data preparation may present some difficulties. However, the use of isoparametric finite elements would help streamline the procedure. In any case this general problem needs further attention.
REFERENCES


APPENDIX 1

FINITE ELEMENT ANALYSIS OF AXISYMMETRIC BODY-FORCE LOADED SOLIDS

Analysis

For the purpose of analysis the continuum is divided into a finite number of elements by imaginary surfaces which intersect radial sections in a net of lines (Figure A1.1) [34]. These elements are assumed to be interconnected at discrete number of nodal points lying on their boundaries.

The stiffness characteristic of an element is determined by assuming suitable displacement functions. For a triangular axisymmetric element we define the six displacement components as (Figure A1.2),

\[
\{\delta\}^e = \begin{bmatrix}
w_i \\
u_i \\
w_j \\
u_j \\
w_m \\
u_m \\
\end{bmatrix}
\]  

(A1.1)
Figure A1.1 Finite element idealisation of axisymmetric solid (a) Actual continuum; (b) Triangular element approximation.
Figure A1.2  Nodal co-ordinates and displacement components
The functions should be so selected as to satisfy the compatibility conditions. This is achieved by assuming displacements that vary linearly in each direction. The edges of the elements would then displace as straight lines, and no gaps can develop between them so long as nodal continuity is maintained [33].

Let the displacement functions be represented by linear polynomials as,

\[\begin{align*}
\{A\} &= \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \\
\{T\} &= \begin{bmatrix}
\begin{bmatrix} w \\ u \end{bmatrix} &= \begin{bmatrix} 0 & 0 & 0 & 1 & r & z \\ 1 & r & z & 0 & 0 & 0 \end{bmatrix}
\end{bmatrix}
\end{align*}\]  

(Al.2)

The constants \(a\)'s are determined by inserting the nodal coordinates of the element (Figure Al.2),
\[
\begin{align*}
\{\delta\}_e & \quad [c] & \quad \{A\} \\
\begin{bmatrix}
  w_i \\
  u_i \\
  w_j \\
  u_j \\
  w_m \\
  u_m 
\end{bmatrix} & \quad \begin{bmatrix}
  0 & 0 & 0 & 1 & r_i z_i \\
  & & & & \\
  1 & r_i z_i & 0 & 0 & 0 \\
  & & & & \\
  0 & 0 & 0 & 1 & r_j z_j \\
  & & & & \\
  1 & r_j z_j & 0 & 0 & 0 \\
  & & & & \\
  0 & 0 & 0 & 1 & r_m z_m \\
  & & & & \\
  1 & r_m z_m & 0 & 0 & 0 
\end{bmatrix} & \quad \begin{bmatrix}
  a_1 \\
  & & & & \\
  a_2 \\
  & & & & \\
  a_3 \\
  & & & & \\
  a_4 \\
  & & & & \\
  a_5 \\
  & & & & \\
  a_6 
\end{bmatrix}
\end{align*}
\]

or, \(\{A\} = [c]^{-1} \{\delta\}_e\) \hspace{1cm} (A1.3)

Substituting for \(\{A\}\) in (A1.2),

\[
\{f\} = \{w\}_u = \{T\} [c]^{-1} \{\delta\}_e = [N] \{\delta\}_e 
\]

The strain vector for an axisymmetric system is given by [19],
\[
\{\varepsilon\} = \begin{bmatrix}
\varepsilon_z \\
\varepsilon_r \\
\varepsilon_\theta \\
\gamma_{rz}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial w}{\partial z} \\
\frac{\partial w}{\partial r} + \frac{\partial u}{\partial \theta} \\
\frac{\partial u}{\partial r} + \frac{\partial v}{\partial \theta} \\
\frac{\partial v}{\partial z}
\end{bmatrix}
\]

Differentiating (A1.2),

\[
\{Q\} = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 \\
\frac{1}{r} & 1 & \frac{z}{r} & 0 & 0 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1
\end{bmatrix}
\]

\[
\{A\} = \begin{bmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\alpha_4 \\
\alpha_5 \\
\alpha_6
\end{bmatrix}
\]

and substituting the value of \{A\} from (A1.3) gives,

\[
\{\varepsilon\} = \{Q\} [c]^{-1} \{\delta\}^e = [B] \{\delta\}^e \quad (A1.5)
\]

Thus stress expression becomes,
where \([D]\) is the elasticity matrix which for isotropic materials is given by,

\[
[D] = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix}
1 & \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 0 \\
\frac{\nu}{1-\nu} & 1 & \frac{\nu}{1-\nu} & 0 \\
\frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 1 & 0 \\
0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)}
\end{bmatrix}
\]

Consider now the nodal forces

\[
\{F\} = \begin{bmatrix} F_i \\ F_j \\ F_m \end{bmatrix}
\]

which are equivalent to the boundary stresses and distributed loads due to body forces \(\{p\}\) associated with the element.
From virtual work considerations [32],

\[
\{F\} = \int [B]^T \{\sigma\} \, dv - \int [N]^T \{p\} \, dv
\]

where integration extends over the volume of the element.

Substituting for \{\sigma\} and \{B\} leads to,

\[
\{F\} = ( \int [B]^T [D][B] \, dv ) \{\delta\}_e - \int [N]^T \{p\} \, dv \quad (A1.7)
\]

with the stiffness matrix of the element as,

\[
[K] = \int [B]^T [D][B] \, dv \quad (A1.8)
\]

and nodal forces given by,

\[
\{F\}_p = - \int [N]^T \{p\} \, dv \quad (A1.9)
\]

By superposing individual element properties, the assemblage \([K]\) and \({F\}_p\) for the whole system can be obtained.
The equilibrium equations can now be solved using the following relationship for the whole system:

\[
\{F\} = [K] \{\delta\}_e - \{F\}_p
\]  

(Eq. 10)

**Evaluation of Element Stiffness Matrix and Body Forces**

Substituting for constants \(a\) from (A1.3) into (A1.2) gives the displacement functions as:

\[
\begin{align*}
\mathbf{u} &= \frac{1}{2\Delta} \left[ u_i c_1 + u_j c_2 + u_m c_3 + (u_i c_4 + u_j c_5 + u_m c_6) r + (u_i c_7 \\
&+ u_j c_8 + u_m c_9) z \right] \\
\mathbf{w} &= \frac{1}{2\Delta} \left[ w_i c_1 + w_j c_2 + w_m c_3 + (w_i c_4 + w_j c_5 + w_m c_6) r + (w_i c_7 \\
&+ w_j c_8 + w_m c_9) z \right]
\end{align*}
\]

(Eq. 11)

where, \(\Delta = \text{area of the triangle}\)
The strain components are now readily obtained as:

\[
\begin{bmatrix}
\varepsilon_z \\
\varepsilon_r \\
\varepsilon_\theta \\
\gamma_{rz}
\end{bmatrix} = \frac{1}{2\Delta} \begin{bmatrix}
c_7 & 0 & c_8 & 0 & c_9 & 0 \\
0 & c_4 & 0 & c_5 & 0 & c_6 \\
0 & c_{10} & 0 & c_{11} & 0 & c_{12} \\
c_4 & c_7 & c_5 & c_8 & c_6 & c_9
\end{bmatrix} \begin{bmatrix}
w_i \\
u_i \\
w_j \\
u_j \\
w_m \\
u_m
\end{bmatrix}
\]

where,

\[
c_{10} = c_1/r + c_4 + c_7 z/r
\]

\[
c_{11} = c_2/r + c_5 + c_8 z/r
\]

\[
c_{12} = c_3/r + c_6 + c_9 z/r
\]

Note that the terms inside the square bracket in (A1.12) is the \([B]\) matrix. Now the stiffness matrix for an
axisymmetric element, computed from (A1.8) by taking the volume integral over the whole ring of material, is given by:

\[
[K] = 2\pi \int [B]^T [D] [B] \ r \ d \ r \ d \ z
\]

where,

\[
[B]^T [D] [B] = \frac{d_1}{4\Delta^2}
\]

and,

\[
k_{11} = c_7c_7 + c_4c_4d_3
\]

\[
k_{12} = c_7c_4d_3 + c_7c_4d_2 + c_7d_2c_{10}
\]

\[
k_{13} = c_7c_8 + c_4c_5d_3
\]

\[
k_{14} = c_5c_7d_2 + c_{11}c_7d_2 + c_4c_8d_3
\]

\[
k_{15} = c_9c_7 + c_6c_4d_3
\]

\[
k_{16} = c_6c_7d_2 + c_7c_{12}d_2 + c_4c_9d_3
\]

\[
k_{22} = c_4c_4 + 2c_4c_{10}d_2 + c_{10}c_{10} + c_7c_7d_3
\]
\[ k_{23} = c_8 c_4 d_2 + c_8 c_{10} d_2 + c_5 c_7 d_3 \]
\[ k_{24} = c_5 c_4 + c_5 c_{10} d_2 + c_{11} c_4 d_2 + c_{10} c_{11} + c_7 c_8 d_3 \]
\[ k_{25} = c_4 c_9 d_2 + c_9 c_{10} d_2 + c_6 c_7 d_3 \]
\[ k_{26} = c_4 c_6 + c_6 c_{10} d_2 + c_4 c_{12} d_2 + c_{10} c_{12} + c_7 c_9 d_3 \]
\[ k_{33} = c_8 c_8 + c_5 c_5 d_3 \]
\[ k_{34} = c_5 c_8 d_2 + c_8 c_{11} d_2 + c_5 c_8 d_3 \]
\[ k_{35} = c_9 c_8 + c_6 c_5 d_3 \]
\[ k_{36} = c_6 c_8 d_2 + c_8 c_{12} d_2 + c_9 c_5 d_3 \]
\[ k_{44} = c_5 c_5 + 2c_5 c_{11} d_2 + c_{11} c_{11} + c_8 c_8 d_3 \]
\[ k_{45} = c_5 c_9 d_2 + c_9 c_{11} d_2 + c_6 c_8 d_3 \]
\[ k_{46} = c_5 c_6 + c_6 c_{11} d_2 + c_5 c_{12} d_2 + c_{11} c_{12} + c_8 c_9 d_3 \]
\[ k_{55} = c_9 c_9 + c_6 c_6 d_3 \]
\[ k_{56} = c_6 c_9 d_2 + c_{12} c_9 d_2 + c_6 c_9 d_3 \]
\[ k_{66} = c_6 c_6 + 2c_6 c_{12} d_2 + c_{12} c_{12} + c_9 c_9 d_3 \]

and \[ d_1 = \frac{E(1-\nu)}{(1-\nu)(1-2\nu)} \], \[ d_2 = \frac{\nu}{1-\nu} \], \[ d_3 = \frac{1-2\nu}{2(1-\nu)} \].

In evaluating the element stiffness matrix, term by term exact integration was performed.
Coming to the body forces,

\[ \{F\}_p = - \int [N]^T \{p\} \, dv \]  \hspace{1cm} (A1.9)

For body forces due to gravity only, \( \{p\} = \{\rho g\} \)

If the body forces are constant it can be shown that \([32]\),

\[ \{F\}_i \{}_{p} = \{\bar{F}\}_j \{}_{p} = \{\bar{F}\}_m \{}_{p} = - 2\pi \{\rho g\} \bar{r} \Delta/3 \, , \quad \text{where} \]

\[ \bar{r} = \frac{(r_i + r_j + r_m)}{3} \]  \hspace{1cm} (A1.13)

Although the above relation is not exact, it is found to give quite accurate results.

**Computer Programme**

The subroutine for computing element stiffness matrix reads element properties and nodal coordinates as input data and generates the matrix for each element which is then stored in a tape by the main programme. The programme then calls the subroutine which forms the assemblage stiffness matrix by superposing individual element effects.
The element nodal forces due to a body force is then calculated for each element by using equation (A1.13). Superposition of results leads to the assemblage nodal forces due to the body force. The external nodal forces (reactions) are then added to nodal body forces to obtain the structural load matrix. The load displacements equations for the whole structure is then solved to give the nodal displacements by using the 'banded' property of the matrix.
APPENDIX 2

EVALUATION OF DEFORMATIONS DUE TO THERMAL EFFECTS

The strain vector for an isotropic material due to an average temperature change $T$ in the element is given by:

$$\{\varepsilon_{\text{m}}\} = \begin{bmatrix} \alpha T \
\alpha T \
\alpha T \
0 \end{bmatrix}$$  \hspace{1cm} (A2.1)

where $\alpha = \text{coefficient of thermal expansion}$.

For an elastic material, the stresses at any point within the element are expressed in terms of the corresponding strains including thermal effects by the elastic stress-strain relation,

$$\{\sigma\} = \begin{bmatrix} \sigma_z \
\sigma_r \
\sigma_\theta \
\tau_{rz} \end{bmatrix} = \mathbf{[D]} \ (\{\varepsilon\} - \{\varepsilon_{\text{T}}\})$$  \hspace{1cm} (A2.2)

where $\mathbf{[D]}$ is the elasticity matrix defined in Appendix - 1.
With thermal effects, the force displacement relation (A1.7) of Appendix 1 modifies to [32],

\[
\{F\} = \left( \int [B]^T [D][B] \, dv \right) \{\delta \}^e - \int [N]^T \{p\} \, dv
- \int [B]^T[D] \{\varepsilon_T\} \, dv
\]

(A2.3)

where, \(- \int [B]^T[D] \{\varepsilon_T\} \, dv\) represents the nodal force due to thermal effects. Integrating over the ring element,

\[
\{F\}_T = -2\pi \int [B]^T[D] \{\varepsilon_T\} \, r \, dr \, dz
\]

Noting that \(\{\varepsilon_T\}\) is a constant,

\[
\{F\}_T = -2\pi [\bar{B}]^T[D] \{\varepsilon_T\} \bar{r} \, \Delta
\]

Substituting for \([\bar{B}]^T[D]\) and multiplying by the vector \(\{\varepsilon_T\}\) gives,
\[ \{F\}_T = -d_1 \bar{\pi} \alpha T \]

where \( d_1 \ldots d_3, \) \( c_1 \ldots c_9, \) \( \bar{r}, \) \( \Delta \) are the same as defined in Appendix 1 and,

\[
\begin{align*}
\bar{c}_{10} &= c_1 / \bar{r} + c_7 \cdot \bar{z} / \bar{r} + c_4, \quad \bar{c}_{11} = c_2 / \bar{r} + c_8 \cdot \bar{z} / \bar{r} + c_5 \\
\bar{c}_{12} &= c_3 / \bar{r} + c_9 \cdot \bar{z} / \bar{r} + c_6
\end{align*}
\]

The components of the nodal force for each element can now be calculated. Once this is achieved, the element thermal nodal forces can be superposed to obtain the assemblage nodal forces for the whole structure.

The computer programme developed can now easily
be modified to incorporate these thermal nodal forces while forming the structural load matrix. The solution of the final load displacement equations to obtain the nodal displacements follows the same procedure as that used for the body force.