AN INVESTIGATION OF ALGORITHM JUSTIFICATION IN ELEMENTARY SCHOOL MATHEMATICS

by

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Abstract

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It was the purpose of this study to determine by experimental procedures whether there are any significant differences either in computational skill with an algorithm or in ability to extend that algorithm among elementary school pupils taught a mathematical algorithm by different methods of justification. The four types of justification methods were: pattern, algebraic, pattern followed by algebraic, and algebraic followed by pattern. A pattern justification is one based on an analog to two-dimensional phyiscal actions, whereas an algebraic justification is one based on the algebraic principles for rational numbers, as well as the rules of logic. Differences in performance among treatment groups were examined for four algorithms varying in both mathematical operation accomplished by the algorithm and in complexity of the algorithm. The latter is determined by the number of steps and processes required for its execution. The two simple algorithms were multiplication of a fraction and a mixed number and comparison of fractions using the cross-product rule. The two complex algorithms were conversion of a fraction to a decimal and calculation of the square root of a fraction.

Three classes were given a strictly pattern justification, three a strictly algebraic justification, one a pattern followed by algebraic justification, and one an algebraic followed by pattern justification for one simple and one complex algorithm. The same assignment was made for the other simple and complex algorithm pair. In total, 16 grade five classes participated in the experiment.

The results of the multiple analysis of covariance indicated no

significant differences in the case of all four algorithms tested among the comparisons between students taught by a strictly pattern approach and students taught by a strictly algebraic approach. However, there is evidence to indicate that students taught by an algebraic approach, as a group, tend to do better on extension tests than their pattern taught counterparts and that students taught by a pattern method, as a group, tend to do better on simple algorithm computation tests than their algebraically taught counterparts. No trend is evident in terms of performance of groups on the complex algorithm computation tests. Furthermore, the existence of a significant between-groups-within-treatments effect indicates the strong possibility of a teacher by treatment interaction which might be further investigated.

Although some significant differences were found among the algebraic followed by pattern and pattern followed by algebraic comparisons in favor of the algebraic followed by pattern groups, these results must be considered in light of the possibility that a group difference is what is being indicated confounded with a treatment difference. There did appear a trend, although nonsignificant, indicating that the algebraic followed by pattern taught students performed better, in general, on the extension tests.

Finally, the data indicated the plausibility of a model for research into algorithm learning in elementary mathematics which incorporates two dimensions--type of justification provided for the algorithm and complexity of the algorithm-- as useful determinants of student performance on computation and extension tests based on that algorithm.

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Chapter 1

THE PROBLEM

BACKGROUND

By examining the chapter titles in virtually any series of elementary school mathematics textbooks, one can discern that a considerable part of the elementary mathematics program is concerned with instruction in the use of standard computational procedures. The most common such computational procedures in grades one through seven are concerned with addition, subtraction, multiplication, and division of whole numbers, fractions, and decimals. These procedures are generally called algorithms; they satisfy the conditions of being routine, well-defined, and of guaranteeing a correct result if properly applied.

Many researchers in mathematics education have categorized elementary school algorithms in terms of the mathematical operations accomplished. For example, multiplication algorithms would be considered as one class of procedures for study. Within such categories, these researchers have looked at methods for teaching different algorithms in terms of their resulting effects either on the computational skill of the learners or on the students' ability to apply the procedures in appropriate situations. For example, Scott(40) conducted a study at the grade three level comparing the learning of two division algorithms to the learning of one division algorithm. The criterion examined was the students' ability to use a correct algorithm given a word problem whose solution required division. Schrankler (39) compared the effectiveness of four methods of teaching multiplication on computational skill

and speed, and on understanding of the multiplication process. Similarly, Dilly (10) compared the results of two methods of teaching division on computational speed and accuracy. Other examples of this type of research comparing various algorithms for a given operation are abundant, for example, (Gray [15] , Hall [17] , MacSchell [27]).

For at least the past fifteen years, there has been an increased emphasis on student understanding and the meaningfulness of learning in any learning situation, including the learning of mathematics algorithms. Bruner, in his book, The Process of Education (7), suggested that it was both possible and important for a student to understand the particular learning process in which he was engaged. Ausubel (1), too, reported on meaningful instruction. He defined the intent in meaningful teaching as being the imparting to the student of a mental set to relate "... substantive aspects of the new information to relevant components of existing cognitive structure" (1:22). This is to be contrasted with the verbatim internalization by the student of material as an end in itself. Ausubel (1) views meaningful teaching as desirable for two reasons:

(1) it is often difficult for a child to learn by rote all of the occasions upon which a given rule is applicable, and

(2) much classroom evidence exists that meaningful material is learned more quickly and retained longer (1:45).

Thus, it appears that because algorithms in arithmetic are usually applicable to a wide variety of problems, and because it is impossible to teach specifically for every kind of problem, meaningfulness is especially important with respect to instruction in arithmetic algorithms.

It has been suggested by Phenix (31:312) that the selction of a particular justification procedure for the rationale of an algorithm

is likely to be the single most important factor in determining its meaningfulness. He writes that the "... important point is that meaning is defined by the method of validation. If knowledge is claimed but no way of testing it can be indicated, it is meaningless" (31:312). "Validation here refers to the evaluation of meaning. In school situations, the method of validation available to students is usually derived from the explanation, or justification, for the procedure provided by the teacher. Thus, it would seem productive to conduct research about meaningful algorithm learning in mathematics so as to include a comparison of the possible types of justifications for the algorithms given to the students.

The writer proposes a scheme for organizing an analysis of algorithm learning in the elementary school mathematics curriculum in which justification is a component. Components other than justification are:

1) the mathematical operation accomplished by the algorithm
2) the number of steps and processes required for the execution of the algorithm (simplicity/complexity), and
3) the number form to which the algorithm applies.

The mathematical operation accomplished by an elementary school mathematics algorithm is generally addition, subtraction, multiplication, or division, but other possibilities exist. For example, it is possible to consider the operation of taking the square root. The number form to which the algorithm applies consists of some subset of the real numbers--for example, the whole numbers only, or the fraction numbers only.

This scheme derives substantially from B.O. Smith's (42) model of rule ventures and procedure ventures in classroom learning. It is this

writer's intention in proposing the analysis and using it for research to provide a framework for examination of an important area in mathematics education. It is only by making use of such a framework that a model for algorithm learning can be built. For the purposes of this particular study, the only two components to be experimentally examined were justification and number of steps and processes required for execution of the algorithm.

In attempting to determine a reasonable categorization of the component of justification in mathematics teaching, the writer consulted several sources. Brumfiel (6) notes that three procedures are possible. According to Brumfiel, some studies which examine justification include no justification as one possible procedure. The rote procedure, however, will not be considered here because this study is solely concerned with situations where some type of justification is attempted. Another justification technique is one in which the rationale is based on the description of a particular set of physical objects; an example of this procedure, henceforth to be known as a pattern justification, is found in the following description: To compute the sum $\frac{3}{5} + \frac{1}{5}$, one could use the physical analogy that if one cuts a pie into five equal parts, takes three of them, and then takes another of the parts, it is equivalent to having originally taken four of the five equal parts. Hence the sum is $\frac{4}{5}$. The third procedure is one in which the rationale is based on a set of precise rules for dealing with symbols to create entities with new symbolic names. An example of this procedure, henceforth to be known as algebraic justification, is found in the following argument: To compute the product 6×13, one could employ the distributive principle of multiplication over addition and rewrite the indicated product

as $6 \times 10 + 6 \times 3$. Hence the product is $60 + 18 = 78$. Frequently in instructional practice, the two justification procedures, pattern and algebraic, together with nonjustification as a deliberate procedure in its own right, are interwoven in various sequences.

What has traditionally been known as rigorous proof can be based only on logical rules applied to a set of axioms; that is, rigorous justification is algebraic. Yet it has been increasingly emphasized, as by Phenix (31:344), that a physical world rationale may often lead to a student's acceptance of a procedure or theorem. Thus, justification based upon a physical analogy has attained a position of acceptability by mathematics educators, particularly for the elementary grades.

Hopefully, research can help teachers and text writers to decide whether there are optimal sequences of algorithm justification for algorithm learning as examined along two dimensions -- the ability of students to compute with the algorithm and the ability of students to exhibit higher-level understandings of the algorithm. These higher-level understandings include <u>extensions</u> of the material learned, specifically, the student's ability to :

1) shortcut the algorithm in particular cases,
2) solve equations with missing operands requiring use of the algorithm,
3) use the algorithm with number forms other than the type studied,
4) extend the algorithm to more than two operands, and
5) explain an alternate version of the algorithm.

The purpose of this study was to begin to study the question of optimal algorithm justification in terms of the computation dimension and the extension aspect of the higher-level understanding dimension.

THE PROBLEM

In this study, the writer attempted to determine by experimental procedures whether there were any significant differences, either in computational skill with an algorithm or in ability to extend that algorithm, among pupils taught by one of four types of justification for a given algorithm involving fractions. These four types of justification were: pattern, algebraic, pattern followed by algebraic, and algebraic followed by pattern. For the mixed justifications, the first justification type was given slightly longer emphasis. Examination of algorithms requiring few steps and processes for execution and also algorithms requiring a larger number of steps and processes for execution was undertaken.

DEFINITION OF TERMS

Certain terms occur throughout the discussion of the study and are, therefore, defined here.

The term <u>algebraic justification</u> refers to those explanations which consist purely of appeals to definitions, to rules of logic, and to the algebraic field postulates, or to combinations thereof. These postulates are the laws which govern the operations for a given number system. A list of these would include the following:

(i) the commutative laws of addition and multiplication

(i.e. for any real numbers 'a' and 'b', $a+b = b+a$ and $a \cdot b = b \cdot a$)

(ii) the associative laws of addition and multiplication

(i.e. for any real numbers 'a', 'b', and 'c', $(a+b)+c = a+(b+c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$).

A complete list of these postulates can be found in Spoonder and Mentzer(43).

An example of an algebraic justification, then, might be the following: To determine the product of 3 and 7, one could compute $3 \cdot 5 + 2 \cdot 3$ since

(i) by the distributive principle, $3 \cdot (5+2) = 3 \cdot 5 + 3 \cdot 2$,

(ii) by the commutative principle of multiplication, $3 \cdot 2 = 2 \cdot 3$, and

(iii) by the logical rule of substitution, $3 \cdot (5+2) = 3 \cdot 5 + 2 \cdot 3$.

The term <u>pattern justification</u> refers to those explanations which make use of physical analogs for mathematical operations. These physical analogs may be either two or three-dimensional, but for the purposes of this study, only two-dimensional analogs were used. An example of such a justification, then, might be the following: To determine the product of 3 and 7, one could compute $3 \cdot 5 + 2 \cdot 3$ since

(i) $3 \cdot (5+2)$ can be represented as

```
x x x x x x x
x x x x x x x
x x x x x x x
```

(ii) The diagram can be rearranged to represent $3 \cdot 5 + 3 \cdot 2$ as

```
x x x x x   x x
x x x x x   x x
x x x x x   x x
```

(iii) If rotated through 90°, the array representing $3 \cdot 2$ can represent $2 \cdot 3$

```
x x
x x        x   x   x
           x   x   x
x x
```

(iv) Therefore, $3 \cdot (5+2) = 3 \cdot 5 + 2 \cdot 3$

```
x x x x x
                x   x   x
x x x x x       x   x   x

x x x x x
```

The term simple algorithm refers to those algorithms requiring few steps and processes for execution. This implies that simple algorithms have a small number of immediate mathematical prerequisites for their acquisition. The term complex algorithm refers to those algorithms which require many steps and processes for their execution. Thus, complex algorithms have a greater number of immediate mathematical prerequisites for their acquistion than do simple algorithms. The writer is not implying a dichotomy, but rather a complexity continuum, and, for the purposes of this study, the distinction between the simple and complex algorithms was based on decisions of a panel of judges. An example of a simple algorithm might be multiplication using common fractional numbers, where knowledge of the multiplication facts is almost the sole prerequisite for learning the procedure. A complex algorithm, however, might be "long division" where decisions about estimates must be made and multiplications and subtractions performed as well as the use of division facts.

The terms restricted and extended algorithms refer to the scope of the algorithm under consideration. Most algorithms can be considered extensions of more restricted algorithms; the restricted algorithm, therefore, becomes a special case of the extended algorithm. For example, an algorithm dealing with the addition of integers can be considered to be an extended version of the restricted algorithm dealing with the addition of whole numbers. This distinction between restricted and extended algorithms is dependent on context; an algorithm may be considered as restricted in one context, yet extended in another context. For the purposes of this study, the writer chose to use lower case letters to indicate a restricted usage of an algorithm and upper case letters to indicate an extended usage.

THE USE OF MODELS IN RESEARCH

In embarking upon the study of a particular learning area, one chooses to examine certain factors and to ignore others. This is unavoidable, since for any such area the number of factors is potentially enormous. The choice of the factors to be examinedboth determines the usefulness of the results and limits the value of the examination.

Since some choice must be made, it is best for the researcher to be cognizant of the model he is bringing to bear on the issue, in order to minimize important oversights. Lachman (24) describes a model as a separate system which "brings to bear an external organization of ideas, laws, or relationships upon the hypothetical propositions of a theory or the phenomena it encompasses" (24:114). Some organization is vital,even though it may have limitations. Willer (46) emphasizes that "models are never exhaustive, never descriptive of all aspects of the phenomena" (46:16), and Lachman (24) explains that "more than one model generally functions for a theory" (24:114).

Models have important uses. Kaplan (22) notes that models make clear to others what one has in mind and so aid communication, that models often distinguish between definitions and empirical propositions, and that models provide a means of data organization. Willer (46:19) sees the purpose of the model as relationship generating. It can be argued, therefore, that a model should be judged not by its exhaustiveness, but by its applicability, its precision of communication, its parsimony, and the number of new hypotheses suggested by it.

Mathematics education is a relatively new field, and researchers, such as Becker (3), have pointed out that mathematics educators must now

attempt to focus their attention on asking the right questions through the use of models. Becker maintains that mathematics educators can contribute towards a theory of mathematical learning by employing the right kinds of models. To date, few such models have been formulated, and it is the need for the development of such models for mathematics education that stimulated the present study.

A Proposed Model for Algorithm Learning

The model described in this study is an outgrowth of the work of B.O. Smith and several of his colleagues (42) at the University of Illinois, who have assembled and categorized hundreds of classroom transcripts in terms of the types of "ventures" exhibited. Ventures are defined as segments of discourse dealing with one single topic and relevant to some cognitive objective and are composed of teacher actions called "moves" (42:46). The Smith study lists eight types of identifiable classroom ventures classified according to their objectives: causal, conceptual, evaluative, particular, interpretative, procedural, reason, and rule (42:23). To illustrate a conceptual venture, one might suggest a sequence of moves to indicate criteria for determining class membership,for example, to indicate criteria for classifying an act as a crime. The sequence of moves might include, in particular, a move Smith calls an "instance enumeration move", listing, in this case, felonies and misdemeanors as types of crimes. Two of the venture types described by Smith, rule ventures and procedure ventures, can be considered to constitute a basis for an analysis of algorithm instruction in mathematics.

"Rules" refer to conventional ways of doing things or to analytic relationships used to guide action (42:196). Rules include such statements as mathematical equations resulting from definitional and axiomatic

considerations; the statement that $\frac{a}{b} = a \times \frac{1}{b}$ can be considered a rule. A procedure refers to the "abstraction that may be described independently of a particular sequence of actions by which an end may be achieved" (42:221). For example, the sequence of steps in performing an addition computation can be considered the components of a procedure. Each of these types of action, rule or procedure, can be seen to be involved in algorithm instruction.

Smith included as rule venture moves those moves centering on "rule formulation", those centering on "rule justification", and those centering on "rule application" (42:198). A rule formulation move for the rule $\frac{a}{b} = a \times \frac{1}{b}$ might suggest that the equation holds for positive numbers replacing 'a' and 'b'. A rule justification move withing a justification venture might suggest that since $\frac{a}{b} = \frac{1}{b} + \frac{1}{b} + \ldots + \frac{1}{b}$ where the series has 'a' terms, then $\frac{a}{b}$ can be rewritten as $a \times \frac{1}{b}$ where multiplication is a way of expressing repeated addition. A rule application move might suggest that if $a = 3$ and $b = 2$, then $\frac{3}{2} = 3 \times \frac{1}{2}$.

Procedural venture moves include "problem-centered", "performance-centered," and "procedure-centered" moves (42:228). A problem-centered move in an addition procedure venture might serve only to focus attention on the existence of a situation calling for addition. A performance-centered move might involve discussing the placement of the digits in the sum. A procedure-centered move might involve discussing the generalization of a two-column addition procedure to a three-column one.

It should be clear that rule venture moves and procedure venture moves are involved in algorithm instruction. The rule venture moves contribute primarily to a discussion of the scope of an algorithm and its justification, while the procedure venture moves pertain primarily to a

discussion of the actual steps to be taken in performing the algorithm.

Based on Smith's analysis, the writer proposes the following model for research about algorithm instruction in elementary school mathematics.

The Model. The basic assumption is that differences in overall computation and extension performance in elementary school mathematics algorithms can be explained in terms of:

1) the mathematical operation accomplished by the algorithm,
2) the number of steps and processes required for the execution of the algorithm (simplicity/complexity),
3) the number form to which the algorithm applies, and
4) the type of justification given in teaching the algorithm.

This model does not attempt to examine individual differences among pupils on computation or extension measures, but seeks to predict general performance.

The first and third components of the model reflect Smith's rule formulation and rule application moves; they indicate the scope of application of the rule. The first and third components also account for moves involving the teaching of the type of problems to which the rule applies, another aspect of scope of application. The second component reflects the aspect of complexity which Smith might expect to arise in performance-centered procedural moves. The fourth component, justification, reflects Smith's rule justification moves category. Most of the above analysis, as indicated, derives from Smith's model; however, the author's model has the added advantage of being particularly applicable to mathematics education research concerning algorithms. This research is often concerned with number systems and mathematical operations.

Discussion of the Model. It should be noted, as with any other model, many factors influencing the learning of mathematics-- the form of presentation, the amount of student discovery, or the attitudes which may be acquired, for example-- have been ignored. But the model does take into account the frequently considered factors of mathematical operation and complexity of algorithm and allows for a study of meaningful algorithm learning because of the inclusion of the justification component.

Another important aspect of the model is that it is hypothesis generating. It allows for an examination of any combination of particular instances of the four components and can be used to describe much of the presently conducted research. For example, research comparing two types of multiplication algorithms fits completely within component one, the mathematical operation accomplished by the algorithm. Research comparing performance on division and multiplication questions can, for example, fit within component two, the complexity of the algorithm, if division is considered a more complex procedure. Other research suggested by the model might investigate whether the kinds of justifications for algorithms should depend on the kind of number system involved.

Research derived from the model might also lead to a simplification of the components that need to be considered in future research into algorithm learning. It is the intention of this study to use the model for this purpose. Specifically, it was decided to examine components two and four, simplicity/complexity and justification type, to determine if information about these two components might be sufficient to prescribe optimal algorithm instruction for a variety of instances of the other two components, mathematical operation accomplished by the algorithm and number form to which the algorithm applies.

The decision to study justification types was based on the alleged importance of meaningfulness and the alleged role of justification in determining meaningfulness, as previously discussed. The decision to examine simplicity/complexity as a dimension of this study as opposed to mathematical operation accomplished, component 1, or number form to which the algorithm applies, component 3, was based on two further assumptions:

1) Most algorithms in the elementary grades can be classified as either simple or complex, whereas algorithms to accomplish only one operation or which apply to only one number form are fewer in number. Thus, a conclusion involving the simplicity/complexity component would be a relevant one for a larger proportion of the mathematics curriculum.
2) Furthermore, the simplicity/complexity dimension may interact with the justification component where a particular type of justification procedure may be more important for explaining long, involved procedures, but not as important for simpler ones.

Thus, this study attempted to determine the most effective method for justifying simple and complex algorithms as evidenced by resultant computational ability and ability to extend the algorithm.

Use of the Model in this Study

Because the use of this model with only one algorithm would not yield information about the two components of justification and simplicity/complexity independently of the other components, operation and number form, it was decided to apply the same research design to several algorithms involving several different operations. Four algorithms were chosen, two simple ones and two complex ones. All four, however, were restricted to the same type of number form, common fractional numerals for rational numbers. Since the number of classes available to the writer prohibited

variation of both number form and kind of operation, the writer decided that variation in number form could be sacrificed, particularly because all operations on fractions inevitably involve whole number operations.

In considering the choice of algorithms to be taught, all four algorithms were required to meet two criteria. First, all of the choices were required to lend themselves easily to both pattern and algebraic justifications. Second, the algorithms were required to be unfamiliar to most grade five students in British Columbia and useful to them in their later work. The decision as to whether an algorithm met both of the latter requirements was checked against curriculum guides and the advice of teachers.

Then to decide which particular algorithms to choose for the simple algorithms, S1 and S2, and the complex algorithms, C1 and C2, the experimenter selected S1 and S2 to be as disparate as possible. C1 and C2 were also chosen to be as disparate as possible. This was done so that any parallel in terms of the differences in performance among treatments could be attributed to the treatment in conjunction with the simplicity/complexity factor, and not to the kind of operation or kind of number form. On the other hand, it was reasonable for algorithms 1 and 2 in each category to differ in some previously defined way so that hypotheses might be readily tested to account for any differences in optimal justification sequence within pairs. The analysis of the components of algorithm performance previously discussed was of help in the selection process.

On the basis of the above considerations, the two simple algorithms chosen were : (1) multiplication of a mixed number and a fraction, and (2) comparison of two fractions using the cross-product rule. The two complex algorithms chosen were: (1) conversion of a common

fraction to its decimal equivalent, and (2) computing the square root of a common fraction. Thus, for both pairs, S1-S2 and C1-C2, the number form to which the algorithm applies was held constant, while the operations varied. It should be noted here that the writer is exercising license in not distinguishing between such things as numbers and numerals or fractions and rational numbers, but it seems that to rely on common usage of the terms will minimize confusion for the reader. A summary of the algorithm choices can be found in Figure 1 below.

Figure 1

Algorithms Used in the Study

S1- Finding the Product of a Mixed Number and a Fraction

S2- Comparing Fractions Using the Cross-Product Rule

C1- Finding the Decimal Equivalent to a Fraction

C2- Finding the Square Root of a Fraction

SIGNIFICANCE OF THE STUDY

The results of this study may have effects in several areas, but its most important contribution may be that the writer has added to the current store of theoretical models available to mathematics education researchers. By using a large overall model to examine the effects on algorithm learning of different kinds and sequences of justifications, many questions can be settled in fewer studies. For example, a decision that an optimal justification sequence for simple algorithm instruction involving fractions is pattern followed by algebraic could bear on the teaching of multiplication of fractions, division of fractions, transla-

tions of mixed numbers to improper fractions, and possibly multiplication of whole numbers by tens; in other words, it could apply to many other simple algorithms. The use of such encompassing, but workable, models may greatly simplify the work of future researchers.

By drawing attention to a delineation of the components of algorithm instruction, it is hoped that the teacher, the textbook writer, and the researcher may be better able to devise more effective instructional sequences for the teaching of algorithms. In particular, because it is justification of algorithms that is being studied, it is likely that any results, if made known to teachers, could lead to a better awareness on the teacher's part of different kinds of teaching strategies that are available to them. Teachers frequently do not consider the possibility that they often have alternatives for explaining why an algorithm 'works'.

Finally, this study may have some bearing on the short-term value of mathematical rigor, especially of rationales based upon algebraic principles, in the elementary grades. It may also effect future research about the place of physical models in the elementary curriculum, in particular, research concerning rationales based upon physical analogy.

It should be noted also that the time allowed for instruction in this study was comparable to the instructional time typical of common practice in many classrooms and the results will not, thereby, exhibit the lack of generalizability that occurs with much short-term research.

RESEARCH HYPOTHESES

Four major hypotheses were tested in this study:

H_{01}: On tests measuring the ability to <u>compute</u> using algorithm S1 and to <u>extend</u> it, the algebraic groups and the algebraic-followed-by-pattern

groups will perform better than the pattern and pattern-followed-by-algebraic groups, respectively.

H_{02}: On tests measuring the ability to compute using algorithm S2 and to extend it, the algebraic groups and the algebraic-followed-by-pattern groups will perform better than the pattern and pattern-followed-by-algebraic groups, respectively.

H_{03}: On tests measuring the ability to compute using algorithm C1 and to extend it, the algebraic groups and the algebraic-followed-by-pattern groups will perform better than the pattern and pattern-followed-by-algebraic groups, respectively.

H_{04}: On tests measuring the ability to compute using algorithm C2 and to extend it, the algebraic groups and the algebraic-followed-by-pattern groups will perform better than the pattern and pattern-followed-by-algebraic groups, respectively.

ANALYSIS OF POSSIBLE RESULTS

An analysis of the theoretically possible results of the study for comparing justifications is summarized in Figure 2. The term used in the figure, "identical statistical decisions are made for S1, S2, C1, and C2" implies that a particular treatment, for example, algebraic justification, achieved the highest scores for each of the algorithms S1, S2, C1, C2.

Figure 2

Evaluation of Possible Results

Case A: Identical statistical decisions are made for S1,S2, C1, and C2.

Case B: Identical statistical decisions are made for S1 and S2, identical statistical decisions are made for C1 and C2, but these two pairs of decisions differ.

Case C: Identical statistical decisions are made for S1 and S2, but the statistical decisions for C1 and C2 differ.

Case D: Identical statistical decisions are made for C1 and C2, but the statistical decisions for S1 and S2 differ.

Case E: The statistical decisions for S1 and S2 differ, as do the statistical decisions for C1 and C2.

In the instance of Case A in Figure 2, if all null hypotheses were rejected in the same direction, one could conclude that, for groups, if not for individuals, there may be an optimal justification sequence for all algorithm instruction for fractions. If all null hypotheses were accepted, one could conclude that there would probably be no difference among pattern and algebraic treatment groups for any algorithm instruction involving fractions.

If Case B occurred and the null hypotheses were rejected for S1 and S2 in the same direction, but accepted for both C1 and C2, one could conclude that there may be an optimal justification sequence for simple algorithm instruction but no optimal sequence for complex algorithm instruction. If Case B occurred and the null hypotheses were rejected for S1 and S2 in one direction and for C1 and C2 in another direction, one could conclude that there may be one type of justification to be preferred

for simple algorithm instruction and another for complex algorithm instruction. If Case B occurred and the null hypotheses were accepted for S1 and S2, but were rejected for C1 and C2 in the same direction, one could conclude that there may be an optimal justification sequence for complex algorithm instruction, but none for simple algorithm instruction.

If Case C occurred and the null hypotheses were rejected for S1 and S2 in the same direction, but the decisions for C1 and C2 differed, one could conclude that there may be an optimal justification sequence for simple algorithm instruction. Additionally one could conclude that it is possible that the mathematical operation may interact with complexity in determining an optimal justification sequence for complex algorithms. If the null hypotheses were accepted for both S1 and S2, but the decisions for C1 and C2 differed, one could conclude that there is probably no optimal justification sequence for simple algorithms and hypothesize that the mathematical operation may play a part in determining the best justification procedure for complex algorithms.

If Case D occurred, all of the discussion about Case C would pertain if the words 'simple' and 'complex' were interchanged.

If Case E occurred, one could conclude that mathematical operation accomplished by an algorithm is of importance in determining the optimal justification sequence for both simple and complex algorithms. Therefore, simplifying the model for algorithm research to just the two components of justification and simplicity/complexity could have been insufficient for examination of algorithm instruction in general.

Chapter 2

A REVIEW OF THE RELATED LITERATURE

INTRODUCTION

To date, no research appears yet to have been conducted which directly tests the model upon which this study is based. However, there is some research in areas that did affect the design of the present study. This literature is divided into three sections here. Most important is the work of a number of researchers as discussed in section one below on techniques to analyze rule, principle, and algorithm learning. Additionally, the research on justification techniques and representational modes in section two and the literature on the four particular algorithms taught in this study described in section three contributed to the development of the design and materials.

LITERATURE ON RULE, PRINCIPLE, AND ALGORITHM LEARNING

B.O. Smith's work (42) contributed most directly to the model described in this study. Smith compiled transcripts of classroom exposition and assembled and categorized these. He named the unit of categorization a venture, meaning a segment of discourse dealing with one single topic and "having a single overarching content objective"(42:6). He labled the sub-units which comprise ventures as venture moves.

In his study of classroom behaviors, two of the commonly found venture types which Smith has identified are rule ventures and procedural ventures (42:23). "Rule", as used here, refers to "conventional ways of doing things or to analytic relationships which may be used to guide

actions... They may be called principles." (42:196). An example of a rule in geometry that Smith cites concerns the formula for areas of triangles, expressed as $A = \frac{1}{2} b h$ (42:205). "Procedure", as used here, refers to a "purposeful sequence of actions by which an end may be achieved,"(42:220) and, in particular, to the "abstraction that may be described independently of a particular sequence of actions utilized in achieving a desired outcome in a certain situation,..." (42:221). Smith explains that one type of mathematical procedure might be an established way of attacking a given problem in geometry, for example, starting with the given information. Because Smith did not discuss algorithms per se, and, because, to this writer, algorithm learning seems to contain elements of both rule and procedural ventures, further analysis of Smith's categories was undertaken in order to find a place for algorithmic ventures in his scheme.

Smith identified three types of emphasis in rule venture moves: those centering on "rule formulation," those centering on "rule justification," and those centering on "rule application" (42:198). Formulation moves concern themselves with presenting, clarifying, or formulating a rule, or indicating the scope, range, or purpose of the rule's application. A statement that "we are now going to learn a way to get square roots of whole numbers" would be a rule formulation move. Another example of a rule formulation move in the materials for the present study is the statement,"Now that we have learned how to multiply a fraction and a whole number and a fraction and a fraction, we can now learn to multiply fractions and mixed numbers."

Justification moves are further subdivided into rule verification and rule derivation moves (42:198). A rule verification move

implies a specific situation where the rule is applied and the result is tested against "some explicit or implicit standard appropriate to such an action in these situations" (42:202). These moves arise often in pattern justifications. For example, to find that $\sqrt{16} = 4$, the student might construct exemplars for all rectangles with whole number side lengths and area 16. When he discovers that the 4 by 4 rectangle is a square, he can accept that $\sqrt{16} = 4$. In other words, "the function of the move is to determine whether or not the result of applying the rule to a particular case does, in fact, lead to a correct or satisfactory outcome" (42:202). A rule derivation move is usually concerned with the development or justification of one rule by deriving it from another rule which is taken to be true. These moves arise often in algebraic justifications. For example, the student might learn that $\frac{1}{a} \times \frac{1}{b} = \frac{1}{a \times b}$ by using the properties of reciprocals and the commutative and associative properties of rational numbers. When he multiplies $(a \times b) \times (\frac{1}{a} \times \frac{1}{b})$, by using the properties, he gets $(a \times \frac{1}{a}) \times (b \times \frac{1}{b}) = 1 \times 1 = 1$, so he sees that $\frac{1}{a} \times \frac{1}{b}$ is the reciprocal of $(a \times b)$, namely, $\frac{1}{a \times b}$. Although rule verification moves can arise in algebraic justifications and rule derivation moves can arise in pattern justifications, there seems to be a closer parallel between verification and pattern and derivation and algebraic than is true of the reverse analogy.

The above description of these two types of rule moves comprises all aspects of algorithm learning as discussed in the author's model except the teaching of the actual procedure to be followed. That is, rule ventures would encompass explanations of the purpose of the algorithm, of reasons which would justify the algorithm, and of descriptions of the types of problems to which the algorithm could be legitimately applied.

Smith notes that a large percentage of the total ventures in mathematics classes observed, in fact, 1 out of each 1.8 ventures observed, were rule ventures (42:208). In particular, it was found that the most common ventures in high school geometry were those made up of formulation and application, but relatively few involved justification (42:211).

Similarly, Smith distinguishes among three types of emphasis in procedural ventures: problem-centered, performance-centered, and procedure-centered moves (42:228). Problem-centered moves are used to analyze the problem or situation, as, for example, identifying the problem or explaining the information given. One might, for example, identify that the problem under consideration is $\frac{1}{3} \times \frac{4}{5}$. Performance-centered moves describe the necessary performances involved in the procedure. Here the teacher might suggest that $\frac{1}{3} \times \frac{4}{5}$ can be found by multiplying 1 x 4 for the numerator and by multiplying 3 x 5 for the denominator. Procedure-centered moves are those moves aimed at helping the student to abstract the performances illustrated for a particular problem to a more general idea of the required procedure in all such problems. For example, the teacher, having completed discussing that $\frac{1}{4} \times \frac{1}{5} = \frac{1}{20}$, $\frac{1}{6} \times \frac{1}{3} = \frac{1}{18}$, and $\frac{1}{6} \times \frac{1}{3} = \frac{1}{18}$, might suggest that, in general, $\frac{1}{a} \times \frac{1}{b} = \frac{1}{a \times b}$. This categorization of procedural ventures (Smith [42]:228) seems to describe the teaching of algorithmic procedures. It should be noted that the problem-centered moves, although similar to rule formulation moves, are much more specific. For example, a teacher might say, "Let's try to do question 6 on page 15," and this would be a problem-centered procedural move. The procedure-centered moves, although perhaps pointing out the relevant component concepts in the procedure in their attempts to generalize from a particular problem, do not go as far as rule justification moves do in

explaining the connections between the components.

It was found that procedural ventures were most prevalent in the science classes Smith observed, but that they occurred in about 1 out of every 4.4 ventures in the high school mathematics classes examined (42:238). Most procedural ventures were of the form in which the problem was identified and a performance given without any attempt to extract a procedure from the performance (42:240). Smith found that procedure-centered moves rarely occurred (42:239).

It is the disparity between the observed paucity of rule justification moves and procedure-centered moves and their theoretical importance that motivated the writer to direct attention to their study. Smith's analysis of rule and procedural ventures, therefore, influenced to a large extent both the model for algorithm learning devised by the author and the examination by the writer of the particular component of justification in algorithm learning.

Whereas Smith has attempted to categorize current classroom practice, Scandura's aim (35) has been to create a research model to learn more about the area of principles and procedures. In noting that most investigators today accept the premise that meaningful learning involves the ability to perform successfully whole classes of tasks rather than simply one specific task, and that S-R language does not directly deal with this type of learning, Scandura has proposed a new language, SFL (Set Function Language), whose basic unit is a "rule", rather than an association. For Scandura, a rule is considered to be the analog of a mathematical function and consists of three elements: 1) a set of stimulus properties (D), 2) a set of response properties (R), and 3) an operation (O) between the stimulus and response elements such that each speci-

fic stimulus is associated with exactly one response (Scandura [35]). An example of a rule would be the rule for adding two numbers, where D consists of a set of pairs of numbers, R consists of a set of single numbers, and O consists of the counting of the union of two disjoint sets to find a sum. In this scheme, an association would be considered a rule with one-element sets for both D and R. For example, the relationship $(3,2) \rightarrow 5$ is strictly an association. A concept would be considered a rule with a one-element set for R. For example, D might consist of pairs (1,4),(2,3), (3,2), and (4,1) and R of the number 5 where the concept may be a sequence of associations for the addition facts for 5. It should be noted that one component of algorithm instruction of the proposed model, the number form to which the algorithm applies, is inherent in the choice of D.

The components of algorithm instruction dealing with kind of mathematical operation accomplished by the algorithm and number of steps and processes required for execution of the algorithm (simplicity/complexity) are reflected in the achievement of the understanding of the rule as a whole, but particularly in the achievement of understanding of the operation,O. Scandura has reserved the word "principle" for situations where the student has knowledge both of a rule and its applicability and here the algorithm component of justification might be important. Using the SFL model, Scandura generated and then investigated several research problems concerning rule learning,which are relevant to the study of algorithm instruction.

In one study of rule generality and response consistency, Scandura, Woodward, and Lee (38) sought information on how abstract the presentation of a rule should be. Generality was measured by the "scope" of the

statement of the rule, where "scope" was defined in terms of the (D,O,R) denotation of rules. Rule A was considered more general than rule B if D_B was a subset of D_A, O_B of O_A, and R_B of R_A (36). In the experiment (Scandura, Woodward and Lee [38]), 85 college subjects were assigned to three treatment and two control groups differing in terms of the generality of the rules provided them for playing the number game NIM. Results indicated that the most specific rule was easiest to learn (36); however, subjects who had learned the most specific rule were generally unable to extend that rule to cope with problems outside the scope of that specific rule but within the scope of its more general extensions (36). Scandura conducted another study with J. Durnin (37) also to investigate rule generality but with high school subjects, and, again, the most specific rule group showed no extra-scope transfer, It is for the cultivation of the ability to generalize or extend a situation, therefore, that algebraic justification may become important, since an algebraic argument is usually of a more general form than is a pattern argument. This generality of algebraic arguments is probably due to the small number of rules of logic and field postulates as opposed to the greater variety of physical analogs for mathematical situations.

Scandura also did some research on rule learning that was not directly tied to his SFL model. In one study, he examined the role of symbolism in learning mathematical rules (34). He was attempting to find whether verbally or symbolically stated rules are more easily memorized and whether knowledge of correct use of the constituent symbols is necessary and/or sufficient to apply a memorized rule correctly. He found that those subjects given symbolic rules were successful only if they had

learned the symbolism, that symbolic statements were always learned more quickly than verbal ones, and that mastery of the constituent symbolism seemed to be almost sufficient for correct application of the rule (34:361). It may be that in the algorithmic learning of elementary school mathematics, pictorial-algebraic justification is an analog of the verbal-symbolic dimension of Scandura's study. Therefore, although Scandura's findings were based on subjects of college age, they did motivate in the present study the consideration of algebraic justification as well as pattern justification.

Gagné (14) has also studied rule learning. In his work, he has defined a rule as "an inferred capability that enables the individual to respond to a class of stimulus situations with a class of performances, the latter being predictably related to the former by a class of relations" (14:191). This definition is much like Scandura's. Gagné notes that a rule is typically composed of several concepts and that learning a rule is actually the same as learning the correct sequence for the involved concepts. Much of his research has been an attempt to show that the teaching of rules is facilitated by prior instruction or recall of the subconcepts inherent in them (14:193). He has also suggested a sequence of instruction for rule learning; the four-step procedure consists of the following:

i) a statement of the general nature of the performance expected at the conclusion of instruction,

ii) verbal instructions to invoke recall of component concepts,

iii) verbal cues for the rule as a whole, and

iv) questions to ask the student to demonstrate the rule.

He adds that there should be reinforcement and repetition throughout (14:202). It was this procedure for rule instruction that served as a basis for the presentation in the instructional materials used in this study.

LITERATURE ON JUSTIFICATION TECHNIQUES AND REPRESENTATIONAL MODES

The literature on justification techniques falls into two categories: literature discussing possible justification techniques in mathematics in general, and literature discussing pattern and algebraic type justifications in particular. First, the general literature will be discussed.

In suggesting methods available to teachers for structuring mathematical knowledge in the classroom, Cooney and Henderson (9) list two types of explaining that teachers can utilize in helping students to understand a principle in mathematics. The first type is called justifying and "involves challenging an assertion whereupon the person challenged is expected to provide facts and generalizations from which the assertion follows: (9:427). The teacher might ask, for example, "How do we know $\frac{a}{b} = a \times \frac{1}{b}$?" The second type of explaining is called implicating and involves providing students certain conditions and "asking them to formulate the conclusion following from these conditions" (9:428). Here, the teacher might ask, "Suppose $4 \times 5 \times (\frac{1}{4} \times \frac{1}{5}) = 1$. What can we conclude about the product of $\frac{1}{4}$ and $\frac{1}{5}$?" It is the first type of justification by a teacher which predominates in the notion of justification examined in the present study. That is, the teachers are expected to provide a basis of facts from which the techniques employed

in the algorithm follow, and the teachers are expected to explain the connection between these facts and the algorithmic procedure. It is mostly in the conclusion sections, after the use of the justifying by the teacher, that the student is called on to make implications. It might also be noted that implicating is a substantial part of the criterion measure of student ability to extend an algorithm.

A further categorization of justification techniques can be found in the work of Richard Wolfe, who conducted a study similar to B.O. Smith's (42) to investigate the "verbal activity of justification as it is carried out in the classroom by teachers of secondary school mathematics" (47:334). Primarily two types of justification were considered-- justification of an assertion (validation) and justification of an action (vindication). For example, the assertion $\frac{a}{b} = a \times \frac{1}{b}$ might be validated by renaming $\frac{a}{b} = \frac{1}{b} + \frac{1}{b} + \frac{1}{b} + \frac{1}{b} + \ldots + \frac{1}{b} = a \times \frac{1}{b}$. But the action of putting the sum of the units digits in the addition question: 45 + 56 + 168 in the units place of the answer might be vindicated by discussing what confusion would result if it were placed in, perhaps, the tens place.

The most frequently employed strategy, or pattern of moves, of validation was found by Wolfe to be one in which a statement is justified by means of a subsuming generalization, a second statement which takes the first as a special case; this strategy occurred in 33% of the observed ventures (47:334). For example, the fact that $\frac{3}{4} = 3 \times \frac{1}{4}$ would be validated by reference to the more general assertion that $\frac{a}{b} = a \times \frac{1}{b}$ for any whole numbers 'a' and 'b'.

The most frequently employed strategy of vindication was found to be one in which an action is justified by an argument primarily based

on practical grounds; this was noted in 12% of the observed ventures (47:335). For example, the fact that in dividing 312 by 6, one should not take away one group of 6 at a time to find the quotient may be vindicated by referring to the subsequent length of the question if this practice is pursued.

These three particular types of justification, accompanied by three more-- a supporting instance, the search for a counter-instance, and mathematical notation for a verbal argument-- were listed as the only six justification techniques found in secondary mathematics classrooms by Wolfe (47:334). It can be argued that validation moves are prevalent in both the pattern and algebraic justifications employed in the present study, particularly in the use of subsuming generalizations. This was manifested both in the use of restricted versions of each of the four algorithms prior to their extended versions and in the use of subsuming generalization techniques employed within the instructional sub-units. Vindication moves were less prevalent because procedural moves, of which vindication might be taken as an example, occurred less frequently than rule moves, which are more related to validation. However, the notion of vindication, if extended to encompass any argument based primarily on practical grounds, might form the basis of most pattern justification. Furthermore, validation moves, if restricted to rule justification rather than rule formulation or rule application moves, might form the basis for the ideas behind most algebraic arguments. Deductive proof is much akin to what, in this study, is called algebraic justification; the only difference between the two is the rigor of argument required. For elementary school purposes, deductive proof may be less valuable than a simple algebraic rationale.

Three of the justification types Wolfe listed were not included

in the present study. The use of mathematical notation for a verbal argument was not considered as a justification type primarily because it must be found within any other mathematical justification which is not totally symbolic, and in the elmentary grades a totally symbolic argument would be a rarity. The search for a counter-instance was not included because it does not apply to the generation of an algorithmic procedure which, by its nature, operates on all members of a class of questions and so has no counter-example. Supporting instances, too, were not considered as rationalizations of algorithms which, by their definition, apply to an entire class of questions rather than a particular one.

Thus, Wolfe's list of six possible justification types, for the purposes of a study of elementary school mathematics algorithm justification, can be reduced to the three: validation, vindication, and deductive proof, which are bases for both pattern and algebraic rationales.

Phenix (31:342) points out that, in general, there are only three major positions to take on how mathematics may be justified-- the empirical, the intuitionist, and the formalist positions. The empiricist depends on the properties of the world perceived by the senses; Phenix argues that the primary contribution of the empiricist's point of view for mathematics may be its logic of discovery (31:344). The intuitionist concerns himself exclusively with the philosophical issue of the existence of abstract number, and so cannot contribute directly to the creation of instructional material in mathematics at an elementary level. The formalist can and would argue for a strongly algebraic approach. The empirical and formalist positions can be construed as the philosophical progenitors of the pattern and algebraic justifications that are examined in this study. It should be noted that although Wolfe discovered six justifi-

cation procedures in his observations, these six can be considered as sub-categories of Phenix's empiricist and formalist positions; that is, vindication, supporting instances, and the search for a counter-instance tend toward a pragmatic empirical outlook, while validation, the use of mathematical notation for a verbal argument, and deductive proof tend toward the formalist position.

At one time, the pattern or physical rationale in mathematics was used almost only in the lower grades of the elementary school, but increasingly more instances of this type of justification are being found at the secondary level. For example, Bidwell (4) wrote an article describing a physical model for teaching factoring of quadratic polynomials. He claimed that the particular advantage of that method was that it led the learner from the concrete stage to the abstract stage, and so probably led to more meaningful learning (Bidwell [4]). Similarly, Edmonds (2) considered a "more intuitive approach to the teaching of algebra" in teaching square numbers and the factoring of squares in terms of diagrammatic materials. Many other examples, too, might be found of physical models in the secondary school (Vance [45]). Thus, both pattern and algebraic justifications have been considered useful for students at levels other than that of the primary grades.

Among the opinions on the relative merits of pattern and algebraic justification, Bates (2) suggests that most mathematics lessons require constant shifting "between what we generally regard as the 'real' world of physical objects and the 'non-real' world of mathematical abstractions" (2: 354). Using the example of teaching division of fractions, he showed several pattern and algebraic arguments which could be used to justify the division of fractions algorithm. Although he claims that

either rationale can be used to teach with understanding, Bates(2) believes that it is in the melding of these two points of view that most arithmetic becomes best understood by children. Bates is in basic agreement with Hartung (18) who feels that "mathematics is a conceptual subject, and the physical activity that is used to promote learning is mainly a means to this end." (18: 280). Fennema(12) conducted a study comparing the relative effectiveness of a symbolic and a concrete model in learning the principle of multiplication as the cardinality of the union of disjoint sets. She found that among the children tested, all of ages seven to eight, those children who learned using a symbolic model performed at a higher level than those who learned using the physical method employing Cuisenaire rods, on tests requiring extension of the principle (12:237). Although a manipulative mode of presentation cannot be considered as precisely analogous to the pattern mode of this study, it is a reasonable parallel because of its dependence on a description of a physical set of objects for its rationale.

It can be seen that most of the little existing discussion of the justification techniques available to mathematics teachers has supported any method that leads to understanding and has advocated, in particular, both the pattern and the algebraic methods. Although most writers have not compared the two types of justification discussed in this study, there seems to be a greater emphasis on the pattern approach in their work. It is hoped that the results of the current study might better determine the actual merits of the two approaches.

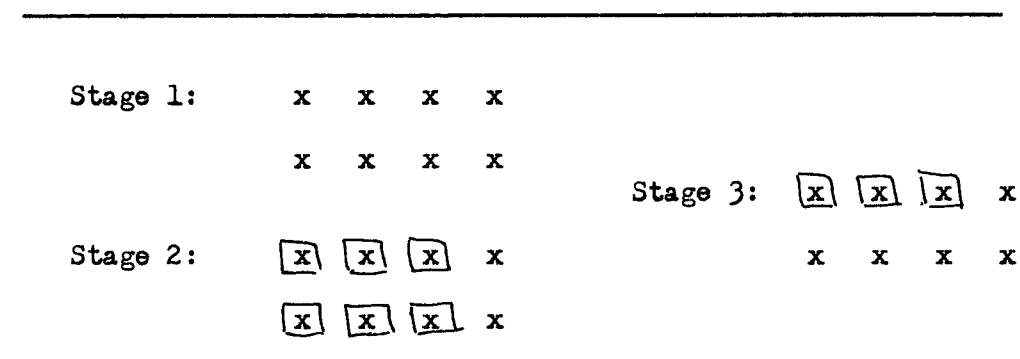
LITERATURE ON TECHNIQUES TO TEACH THE FOUR ALGORITHMS USED IN THE STUDY

In order to complete a discussion about the related research, it is necessary to include some mention of the work that has been done dealing with the four algorithms employed in the present study. This will indicate how such literature affected the specific teaching methods used by the writer.

Mathematics educators have suggested techniques and have conducted experimental studies to find optimal ways for teaching each of the four algorithms considered in the present study. Among those who suggested possible techniques were Nelson and Nelson(30), who demonstrated the use of pegboards for multiplication of fractions. They used pegs to make rectangular outlines with dimensions equal to the denominators of the two fractions being multiplied. For example, for solving the question $\frac{1}{2} \times \frac{3}{4} = \frac{\square}{\triangle}$, they would set up a 2 by 4 rectangle. They then indicated the maneuvers to be performed to demonstrate the multiplication process. The demonstration is as in Figure 3 below.

Figure 3

Pegboard Multiplication

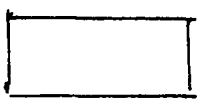


The answer would be read as the fractional part of the total number of pegs covered by straws in the final indicated stage. This technique is related to the partitioning scheme used in the pattern justification for algorithm S1 in the present study. Kolesnik (23) more directly suggested the partitioning idea to represent multiplication of fractions. For example, to represent $\frac{1}{2} \times \frac{3}{4}$, he would use three stages--first the drawing of an arbitrary unit rectangle, then the partitioning of that rectangle into fourths along its length and the shading in of three of those fourths, and finally, the cross-partitioning into halves along its width and the further shading in of one half of those three fourths already shaded. A diagram of this scheme can be found in Figure 4.

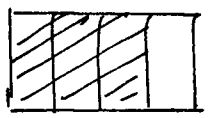
Figure 4

Multiplication of Fractions by Partitioning

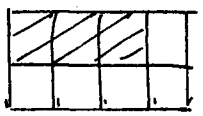
Stage 1:



Stage 2:



Stage 3:



In terms of research into algorithm S1, Romberg (35) found that in the NLSMA study, the National Longitudinal Study in Mathematics Achieve-

ment, a substantial proportion of the incorrect answers on multiplication of fractions tests resulted from the students' not reducing to lowest terms what were actually correct answers. For this reason, the writer did not require students to reduce their answers on the S1 criterion tests.

Brueckner and Elwell (5) found that in a test for individual diagnosis of ability to multiply fractions, at least three example of each type of sub-skill should be included to get a high degree of consistency. Thus, the S1 computation test was constructed in terms of four sub-tests, each with six items of a type, to test each of the sub-skills taught.

Collier (8) discovered that in learning to multiply fractions, children find it easier first to learn to find the product of a whole number and a fraction in that order, such as $4 \times \frac{1}{3}$, and later the product of a fraction and a whole number in that order, such as $\frac{1}{3} \times 4$. This was the sequence followed in the S1 instructional materials. It may be noted that this finding may be peculiar to countries where a x b is interpreted as b+b+...+b, that is, 'a''b's', rather than as a+a+...+a, 'b' 'a's'.

Also of relevance to the S1 materials was Green's finding(16) that taking an area approach to multiplication of fractions was more successful than taking an approach depending on finding fractional parts of sets of objects. For example, it proved more successful to find $\frac{1}{2} \times 4$ by using the diagram approach as in Figure 3 above than by taking $\frac{1}{2}$ of a set of 4 objects. It was the area approach that was employed in the pattern justification for algorithm S1.

Very little has been written about methods for teaching compari-

son of fractions using the cross-product rule, but Matthews(28) suggested that one should begin by asking students to name pairs of unlike fractions such that the first was greater, then pairs where the second was greater, and finally equal pairs. It was only then that he recommended changing the fractions to equivalent ones within pairs and then explaining the cross-product rule. This suggestion was adopted in the S2 instructional materials; the notion of "greater than" for fractions was introduced prior to the actual determination of which of two fractions was greater. Determination of which fraction was greater was first achieved by the use of equivalent fractions and later by the cross-product rule. The other major contribution to the techniques employed in S2 was Hinkelman's (20) finding that, relative to other principles dealing with fractions, the principle that if two fractions have the same denominator, the one with the greater numerator is greater, is relatively well-understood by grade five and six students. This principle was depended upon heavily for both algebraic and pattern approaches.

In a study comparing long division and short division for one digit divisors, John (21) found that, although the short division group showed greater speed and accuracy initially, the long division group soon moved ahead on both. For this reason and because of the familiarity of grade five British Columbia students with long division, the long division algorithm was used in teaching the conversion of fractions to their decimal equivalents in algorithm C1 of this study.

The last major suggestion offered by the literature for the materials used in this study was the method of estimating square roots of non-perfect squares by Frederiksen (13). He suggested that in order

to estimate the square root of 'b', one might use the partitioned binomial square with area 'b' and dimensions 'a+ ☐ ' where 'a' was the closest whole number less than the square root of 'b'.

In the opinion of this writer, the lack of research in and original examination of techniques of instruction for the four algorithms employed in this study is difficult to justify. Most other articles the writer encountered concerning the four algorithms of this study seemed either to deal with a single curiosity about one of the algorithms or else they reiterated the standard techniques which were identified by Kolesnik (23), Matthews (28), John (21), and Frederiksen (13). It is hoped that the model proposed in this study will suggest new techniques for future research concerning these four particular algorithms.

SUMMARY OF THE RELATED LITERATURE

In conclusion, the literature from three major areas has influenced the design of the present study. The work on rule and principle learning led to the model being used in this study, an adaptation of B.O. Smith's model of rule and procedural ventures. It also led to the consideration of the possible effects of algebraic justification on the ability of students to extend an algorithm and to adoption of the particular format of the instructional materials.

The work on justification procedures and representational modes reinforced consideration of both pattern and algebraic justification schemes and also supported the use of the technique of sub-suming generalization prevalent in the instructional materials.

Finally, the literature on the four algorithms themselves suggested instructional techniques, namely, the partitioning technique

used in S1, the multiplication of whole numbers and fractions in that order before the multiplication of fractions and whole numbers in that order in S1, the use of the principle that if two fractions have the same denominator, the one with the greater numerator is greater in S2, the long division algorithm in C1, and the binomial square in C2. It is hoped that the use of the writer's model will unify these three disparate areas into a more usable body of knowledge.

Chapter 3

DESIGN AND PROCEDURE

DESIGN

This study was designed to examine two independent variables connected with algorithm instruction in arithmetic: the first variable was the type of justification for the algorithm and the second, the complexity of the algorithm. Thus, four experimental conditions varying in type of justification were employed for each of two simple and two complex algorithms. Treatment 1, or pP, subjects were instructed using materials based on pictorial justification, henceforth called pattern justification, for both restricted and extended versions of an algorithm, while treatment 2, or aA, subjects were instructed using materials based on algebraic justification for both the restricted and extended versions of that algorithm. The subjects in treatments 3 and 4, labelled pA and aP treatment conditions, were taught using materials employing a combination of pictorial and algebraic justifications; the former group received pattern justification for the restricted version of the algorithm followed by algebraic justification for the extended version, while the latter group received the reverse treatment.

Immediate mathematical prerequisites for each of the two simple algorithms were determined, based on decisions by a panel of judges who were specialists in the area of mathematics education. For all pupils, scores on a criterion pretest constructed by the writer and measuring knowledge of these prerequisites were first gathered. These scores were used as covariates to adjust for initial differences among

treatment groups with respect to this knowledge.

All groups were then instructed by their usual teacher in one of the two simple algorithms-- either finding the product of a mixed number and a proper fraction (algorithm S1) or comparing fractions using the cross-product rule (algorithm S2). The instructional materials written by this author and used by the teachers reflected either the pattern, algebraic, or combination justification, depending on the treatment condition to which the group had been assigned--pP, aA, pA, or aP. Each class was taught as a unit. The wording, examples, and exercises for the sets of teaching material for a given algorithm were parallel in presentation for all treatment conditions. This parallelism can be observed by examining pages 155 and 172 in Appendix A.

For three forty-five minute sessions, the students were taught a restricted version of either S1 or S2 using either a pictorial(p) or algebraic (a) approach. Algorithm S1, when confined to multiplication of a mixed number by a whole number rather than a mixed number by any fraction, was considered the restricted version of S1. Algorithm S2, when confined to comparison of a fraction and a whole number, rather than comparison of any two fractions, was considered the restricted version of S2. A restricted version of each algorithm was taught first in order that those students on a purely algebraic or purely pattern approach might also learn something new during the time in which those students in the mixed justification groups were learning something new by being exposed to the second type of approach. The students completed worksheets covering the material taught during the three sessions, and these worksheets were returned to the writer by the teachers. The sheets were not graded, but were provided by the writer as exercise sheets for

the students so that no teacher was required to produce supplementary work.

Each of the groups was then taught the extended version of the algorithm it was studying, using either the pattern (P) or algebraic (A) approach, depending on the treatment condition to which the group had been assigned. This instruction continued for two sessions.

At the conclusion of the five sessions of instruction on the simple algorithms, two criterion tests were administered to each class. The first test measured computational ability with the algorithm studied. The second test measured the ability to extend mathematically the material studied. The extension test included five types of items designed to measure the ability to :

1) shortcut the algorithm in particular cases,
2) solve equations with missing operands requiring use of the algorithm,
3) use the algorithm with number forms other than the type studied,
4) extend the algorithm to more than two operands, and
5) explain an alternate version of the algorithm.

Next, a second set of pretests, instruction, and criterion tests were presented to each class on one of the two complex algorithms. The S1 students were taught an algorithm for converting fractions to their decimal equivalents (algorithm C1), while the S2 students were taught an algorithm for finding the square root of a fraction using long division (algorithm C2). The same treatment assignment as had been employed in the simple algorithm instruction was repeated with each group for the complex algorithm. For example, groups assigned to treatment pA for the simple algorithms were also assigned to pA for the complex algorithms.

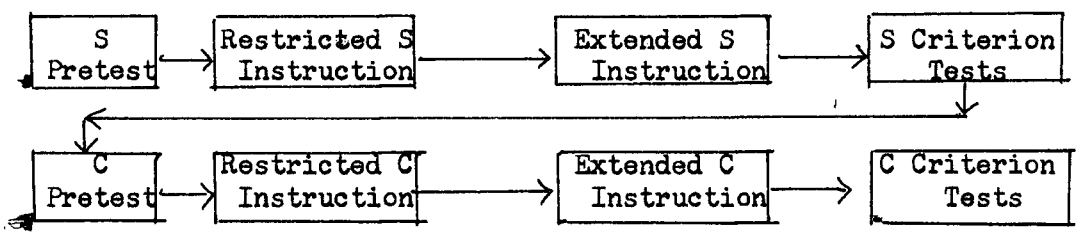
After administration of pretests based on immediate mathematical prerequisites for the complex algorithms-- the prerequisites again determined by a panel of judges-- the students were given five sessions of instruction

on the restricted version of either C1 or C2. For each of the two complex algorithms, the restricted versions dealt only with cases where the answer could be found exactly, while the extended versions dealt with approximation techniques. For example, an exercise such as computing $\sqrt{16}$ might arise in the restricted version of C2, whereas an exercise such as computing $\sqrt{17}$ might arise in the extended version of C2. The instruction on the extended portions of the algorithms continued for three more sessions. During the course of instruction, all classes completed a number of worksheets provided for the convenience of the teachers which were returned to the experimenter. At the conclusion of all instruction, a computation test and an extension test based on the complex algorithm studied were administered.

The overall procedure was as indicated in Figure 5 below.

Figure 5

Sequence of Instruction



SUBJECTS

Sixteen classes of grade five children in nine schools in the Lower Mainland area of British Columbia, Canada, were made available to the experimenter. In order to control as much as possible for teacher-method interaction, three classes with three different teachers

were to be assigned to each of the treatment conditions. The experimenter, therefore, randomly assigned twelve classes, three to each of the pP and aA conditions for algorithm set S1-C1 and three to each of the pP and aA conditions for algorithm set S2-C2. The remaining four of the sixteen classes were randomly assigned-- one to the pA and one to the aP conditions for each of the S1-C1 and S2-C2 sets. A diagram of the assignment can be found in Figure 6 below.

Figure 6

Assignment of Subjects to Treatments

	Treatment Condition			
	pP	aA	aP	pA
S1-C1	Class 1 Class 2 Class 3	Class 4 Class 5 Class 6	Class 13	Class 14
S2-C2	Class 7 Class 8 Class 9	Class 10 Class 11 Class 12	Class 15	Class 16

It was impossible to replicate these mixed conditions pA and aP in other classes due to the difficulty in obtaining the larger number of classes that would be required. The choice of assigning three classes to each of the pure algebraic and pattern conditions, rather than the mixed conditions, was motivated by the writer's judgment that information based upon the former conditions would be more immediately useful to classroom practice. The writer believes that if a classroom teacher did use both pattern and algebraic modes, he would be likely to mix them more

intimately than in this study, perhaps as papapaPA, rather than as simply pA or aP. On the other hand, the writer believes that many teacher would often use, and do often use, a purely pattern mode, while others often use a purely algebraic mode. Due to the difficulty in deciding what degree of mixture is most commonly employed by teachers, the mixed approach was not further examined.

The fifth grade level was selected because it was decided that such students would be sufficiently unfamiliar with the content to be learned that treatment effects could be studied, but sufficiently familiar with prerequisite knowledge that the content would be teachable. This was confirmed by the teachers of all experimental classes. All teachers involved also agreed that the material would be useful in reviewing prior learning of their classes and would at the same time provide their students with new information.

The individual class, each with its own teacher, rather than the individual student, was regarded as the experimental unit for this study. This situation resulted from the virtual impossibility in educational research of obtaining anything other than complete classes to work with, as well as the impossibility of assigning one teacher to teach in each of 16 classrooms at the same time of year, in several schools, over a three week to four week period. The fact that each class was taught by one teacher as a unit, and so the treatment was not independent for individuals, necessitated allocation of classes as experimental units. Thus, the need for at least two classes, or experimental units, in each treatment became obvious. Furthermore, it was decided that there should, if possible, be more than two classes per treatment, not only to control further for teacher and class differences, but also to provide sufficient

degrees of freedom to allow for significant statistical results. It is for this reason that the maximum available number of classes was assigned to each treatment; unfortunately, no more than three could be arranged for each treatment.

This type of assignment of classes, rather than individual subjects, to treatments was recognized by the experimenter as a calculated risk since it decreases the minimal control for teacher and class differences and it reduces degrees of freedom for statistical analysis. However, this risk was deemed a reasonable one for several reasons. First, if results indicated that the between-groups-within-treatments variance was not significant, significant statistical results would be possible. Second, if results indicated significant between-groups-within-treatments variance, information would be available regarding the extent to which individual class means differed, indicating tentative evidence for treatment by teacher interactions which might be further investigated. Third, even if no clear-cut results could be found because of substantial between-groups-within-treatments variance, the experimenter would have provided a useful design for future research teams with sufficient funds and administrative power to find the large number of classes needed to research the problem. A pilot study would have been desirable to permit some estimate of between-classes-within-treatment variance, but, practically, only a certain number of classes could be obtained for experimental use and if any of these were used for the pilot, no guarantee could be made that other classes would be available to replace them in the main study.

CONTROLS

An attempt was made to control for several possible contaminating variables-- differences among classes on prerequisite knowledge, different use of materials by the teachers involved, differences in frequency of feedback to the teachers from the writer, and novelty effects.

The criterion pretest scores were used in an analysis of covariance on the criterion posttest scores. This was done to adjust for initial differences among groups in prerequisite knowledge for the algorithms that were studied. In this way, groups which were better prepared for learning the particular algorithms assigned to them were not awarded spuriously high final scores on the criterion tests following covariate adjustment.

In order to control for diversity in the use of materials by the teachers and differences in frequency of feedback to them from the writer, several measures were taken. First, the writer distributed the materials to all teachers five days prior to the start of the experimental period. Teachers were encouraged to ask questions of the writer at any time so that the material would be clear to them. The teachers were visited regularly by the writer to provide them with an opportunity to ask any questions they had concerning the project. All teachers agreed to follow the material as written. The teachers were cautioned particularly to use the assigned justification approach exclusively in response to student questions or remarks. All teachers received a copy of their pupils' grades on each of the pretests and criterion tests no later than two teaching days after collection of the tests by the writer, who graded all tests. Thus, all teachers had the opportunity to be equally aware

of the progress of each of their students. No teacher was informed of the standing of his class in comparison with other classes.

Because all classes received instruction from their regular teachers and because all classes received instruction clearly different from usual, differences among groups with respect to novelty effects were judged to be sufficiently minimized.

INSTRUCTIONAL MATERIALS

General Construction of Materials

Materials for each teacher included a written teaching exposition, a prescribed set of practice exercises to be used by the students, and a written explanation of each new learning objective for the teacher. Indications were also included as to the points in the instructional sequence at which to give practice questions and the points at which to distribute experimenter-prepared worksheets.

To provide for uniformity, all materials were written to conform with Gagné's (14: 202) scheme for the teaching of rules. Each new learning objective was stated to the students in order to prepare them for the material which followed. The immediate mathematical prerequisites for the material were reviewed next and only then was the new content taught. At the conclusion of each section of new material, the students were allowed to practice the learned algorithm on a set of exercise questions. Gagné's scheme, as described earlier, was chosen because it is a common one in arithmetic textbooks and would not be particularly difficult for the students to follow. To achieve even further uniformity among classes, teachers were given instructions in the use of the materials prior to the experimental period. Suggested

amounts of material to be covered each session were also indicated to the teachers.

The writer ensured that, so far as possible, pattern materials and algebraic materials were parallel in terms of wording, statement of objectives,and sample exercises. Differences in wording occurred only where the type of justification necessitated them.

In constructing the materials, the writer first prepared outlines of the four algorithms for each of the justification approaches. After a panel of judges, composed of members of the Mathematics Education Department at the University of British Columbia, had made suggestions for changes in the manner of justification, a further elaboration of the materials was constructed. Several practicing teachers who were not participating in the study read these elaborations and made suggestions for improvement based on their ability to follow the material and based on their usual classroom practices. The final draft of the materials was also read by several practising teachers. A few further corrections were made and following this, the materials were distributed to the teachers involved in the study.

S1 Materials--Multiplication of a Mixed Number and a Proper Fraction

Pattern Approach for the Restricted Version of S1. The presentation of the pattern approach for the restricted version of this algorithm, the multiplication of a mixed number and a whole number, consisted of four steps.

1. The students were introduced to the relationship between calculating the area of a rectangle and multiplication, namely, that computing the product of two numbers can be interpreted as finding the area of a

rectangle for which these two numbers represent measures of the sides. For example, to compute the product of 3 and 2, one could find the area of a 3 by 2 rectangle.

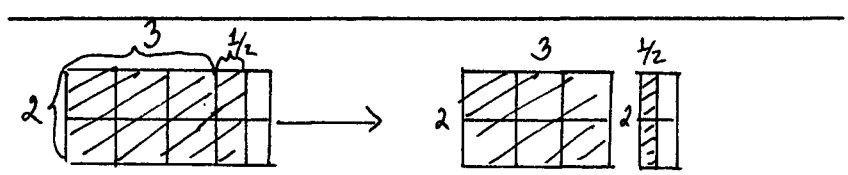
2. Using this area idea, the students were shown how to compute the products of whole numbers and fractions with numerators of 1, such as $3 \times \frac{1}{4}$. This was accomplished by drawing rectangles with the proper dimensions and calculating the areas of these rectangles.

3. The students were taught to compute products of whole numbers and proper fractions with numerators other than 1, for example, $3 \times \frac{2}{4}$. Again, the area argument was used.

4. The students were instructed in computing products of mixed numbers and whole numbers by drawing rectangles of the proper dimensions and calculating their areas. The students were told to deal separately with the areas of the whole number by whole number section of the rectangle and the whole number by fraction section of the rectangle. For example, it was suggested that in computing $2 \times 3\frac{1}{2}$, they should separately find the areas of the 2 by 3 rectangle and the 2 by $\frac{1}{2}$ rectangle as indicated in Figure 7 below, and then write the total product as the sum of these two partial products. Simplification of answers in this and all succeeding sections was not required or encouraged.

Figure 7

Area Approach to Multiplication of
a Fraction and a Whole Number

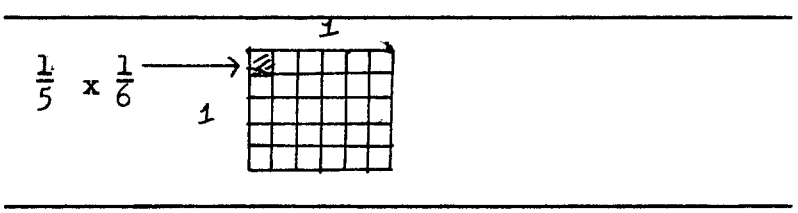


<u>Pattern Approach for the Extended Version of S1</u>. The presentation of the pattern approach for the extended version of this algorithm also consisted of four steps.

1. The students were reminded of the relationship between the area of a rectangle and the multiplication of the two numbers which were dimensions of that rectangle. This was followed by a review of computing the products of fractions and whole numbers, as indicated in the restricted version sequence above.
2. The students were shown how to compute the product of two fractions, both with numerators of 1, for example, $\frac{1}{5} \times \frac{1}{6}$. This was accomplished by first drawing a 1 by 1 square, then marking off its sides into fifths and sixths respectively, and using these marks to partition the unit square, as in Figure 8 below.

Figure 8

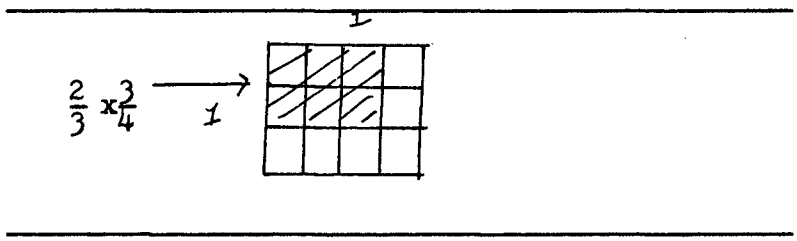
Multiplication of Unit Fractions



3. The teacher explained how to compute the products of any two proper fractions, regardless of the magnitude of their numerators, using, again, portions of a unit square, where the numbers to be multiplied were marked in as dimensions on the sides as in the example of $\frac{2}{3} \times \frac{3}{4}$ in Figure 9.

Figure 9

Multiplication of Two Fractions



4. The students were taught to compute the products of mixed numbers and proper fractions, using the total of the areas of two rectangles as in stage 4 of the instructional sequence for the restricted version of S1.

<u>Algebraic Approach for the Restricted Version of S1</u>. The presentation of the algebraic approach for the restricted version of S1 was presented in four steps.

1. The students were introduced to the idea that another way of naming a fraction is by a multiplication expression, for example, $\frac{4}{5}$ can be renamed as $4 \times \frac{1}{5}$.
2. Using this idea, they were taught to compute the products of whole numbers and fractions with numerators of 1, An example such as $3 \times \frac{1}{4}$ was considered to be an instruction to rename the product as the fraction $\frac{3}{4}$.
3. The teacher demonstrated how to compute products of whole numbers and fractions with numerators other than 1. This was justified using the associative principle of multiplication; for example, $5 \times \frac{3}{4}$ was rewritten as $5 \times (3 \times \frac{1}{4})$, which was itself rewritten as $(5 \times 3) \times \frac{1}{4}$, then as $15 \times \frac{1}{4}$, and then as $\frac{15}{4}$.

4. The students were shown how to compute products of mixed numbers and whole numbers by using the distributive principle of multiplication over addition, rewriting the mixed number as the sum of a whole number and a fractional number. For example, in computing the product $2 \times 3\frac{1}{4}$, they were taught to use the following procedure: $2 \times 3\frac{1}{4} = 2 \times (3 + \frac{1}{4}) = 2 \times 3 + 2 \times \frac{1}{4}$, and then to add, therefore, $6 + \frac{2}{4}$ to calculate the product desired.

<u>Algebraic Approach for the Extended Version of S1</u>. The presentation of the algebraic approach for the extended version of S1 also consisted of four stages.

1. The students were taught to compute the products of fractions and wholes, using the renaming of a fraction as a product. For example, $\frac{4}{5} \times 3$ was treated as $(4 \times \frac{1}{5}) \times 3 = (4 \times 3) \times \frac{1}{5} = 12 \times \frac{1}{5} = \frac{12}{5}$.
2. They were shown how to multiply any two fractions, each with numerator of 1. This was accomplished by first establishing the idea that there is only one number that can be multiplied by a given number to get a product of 1, namely, the reciprocal of that number. Thus, to show that, for example, $\frac{1}{4} \times \frac{1}{5} = \frac{1}{20}$, the teacher would show that $20 \times (\frac{1}{4} \times \frac{1}{5})$ is $(5 \times 4) \times (\frac{1}{4} \times \frac{1}{5}) = (5 \times \frac{1}{5}) \times (4 \times \frac{1}{4}) = 1 \times 1 = 1$. Therefore, since $20 \times (\frac{1}{4} \times \frac{1}{5}) = 1$, $\frac{1}{4} \times \frac{1}{5} = \frac{1}{20}$.
3. The students were instructed in computing the products of any two proper fractions, regardless of the magnitude of their numerators. This instruction utilized the associative and commutative principles of multiplication. For example, $\frac{2}{3} \times \frac{3}{4}$ would be renamed as $2 \times \frac{1}{3} \times 3 \times \frac{1}{4}$ which would then be renamed as $(2 \times 3) \times (\frac{1}{3} \times \frac{1}{4}) = 6 \times \frac{1}{12} = \frac{6}{12}$.
4. The teacher demonstrated how to compute the products of mixed numbers

and fractions, first writing the mixed number as the sum of a whole number and a fractional number and then applying the distributive principle of multiplication over addition to this sum, as in stage 4 for the restricted algebraic version described above.

S2 Materials-- Comparison of Fractions Using the Cross-Product Pattern Approach for the Restricted Version of S2. The presentation of the pattern approach for the restricted version of S2, comparison of a fraction and a whole number, consisted of four steps.

1. The students were taught that in order to compare two fractions, they could determine which of two shaded regions of two congruent unit diagrams had the greater area. For example, they were shown that the fractions associated with diagram (a) in Figure 10 below must be greater than the fraction associated with diagram (b) in Figure 10.

Figure 10

Comparison of Fractions by Area

2. Because the students were to be dealing with whole numbers in a fractional form, they were shown how to find other fractional names for 1, for example, as $\frac{4}{4}$ or $\frac{5}{5}$. This was done by indicating the partitioning of a whole into pieces, as shown in Figure 11.
3. The students were taught to use the knowledge that they had gained about equivalent names for 1 in order to find fractional names for other whole numbers. Again, they were taught to use diagrams and

sum the number of 1's needed to arrive at the whole number indicated. For example, they would be expected to find another name for 4 in terms of sixths as $\frac{6}{6} + \frac{6}{6} + \frac{6}{6} + \frac{6}{6} = \frac{24}{6}$. The diagram for this might be similar to the one in Figure 12.

Figure 11

Fractional Names of 1

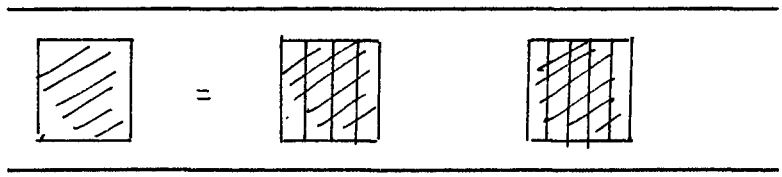
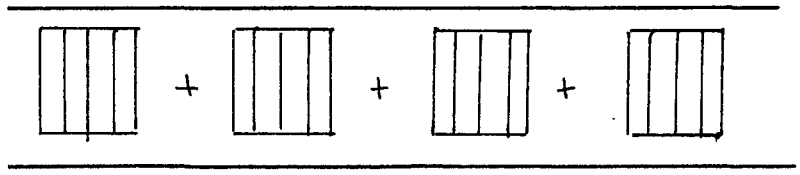


Figure 12

Fractional Names for Whole Numbers



4. They were instructed in determining which of a whole number or a fraction was greater by renaming the whole number as a fraction with the same denominator as the given fraction and then comparing the two numerators. For example, to compare 3 and $\frac{14}{4}$, they would be expected to rename 3 as $\frac{14}{4}$, and compare $\frac{12}{4}$ to $\frac{14}{4}$ by comparing 12 to 14.

<u>Pattern Approach for the Extended Version of S2</u>. The presentation of the pattern approach for the extended version of S2 also involved four stages.

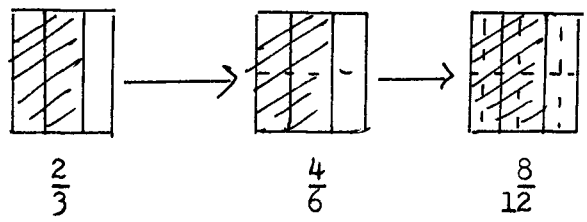
1. The teacher led a discussion about finding equivalent names for a fraction by looking at pairs of diagrams where equivalent areas were

shaded.

2. This led to the idea of partitioning a region to find equivalent names for the fraction represented by the region. For example, to find equivalents to $\frac{2}{3}$, diagrams such as those in Figure 13 would be shown.

Figure 13

Equivalent Fractions by Partitioning



3. Given two fractions with different denominators, the students were shown that they could always find equivalents for each of these fractions so that the two equivalents had the same denominator, namely, the product of the original two denominators. This was expressed diagrammatically using the partitioning technique discussed earlier in stage 2. The students were shown that having done this renaming, it would be easier to determine which of the two fractions was greater.
4. The teacher directed the students to decide which of a pair of fractions was greater by comparing the numerators of their equivalents. Here, the students reviewed the procedure in stage 3 and were shown that the numerator of one of these equivalents resulted from the multiplication of the original numerator by the original denominator of the other fraction which determined the partitioning. That is, in comparing $\frac{2}{5}$ and $\frac{3}{4}$, the equivalents to $\frac{2}{5}$ and $\frac{3}{4}$ would be

$\frac{8}{20}$ and $\frac{15}{20}$, respectively. 8 resulted from the multiplication of the original numerator of 2 in $\frac{2}{5}$ by the denominator of the other fraction, 4 in $\frac{3}{4}$, and 15 from the multiplication of the original numerator of 3 in $\frac{3}{4}$ by the denominator of the other fraction, 5, in $\frac{2}{5}$. Thus, the comparison of $\frac{2}{5}$ and $\frac{3}{4}$ was equivalent to the comparison of $\frac{8}{20}$ and $\frac{15}{20}$ or 8 and 15, 2x4 and 5x3, the cross products of the two given fractions.

Algebraic Approach for the Restricted Version of S2. The presentation of the algebraic approach for the restricted version of S2 also comprised four steps.

1. The teacher led a discussion about comparing fractions with the same denominator by comparing their numerators.
2. Because the students were to deal with whole numbers, they were shown a method to find fractional names for 1, for example, $\frac{4}{4}$ or $\frac{5}{5}$. This was based on the renaming of, say, $\frac{4}{4}$ as $4 \times \frac{1}{4}$, defined to be 1.
3. They were taught to use this knowledge to find fractional names for other whole numbers. This development made use of the properties of multiplication by 1 and associativity of multiplication. For example, to find another name for 2, they would rewrite 2 as follows:

$$\begin{aligned} 2 &= 2 \times 1 \\ &= 2 \times (3 \times \tfrac{1}{3}) \\ &= (2 \times 3) \times \tfrac{1}{3} \\ &= 6 \times \tfrac{1}{3} \\ &= \tfrac{6}{3}. \end{aligned}$$

4. The students were shown that to compare whole numbers to fractions they could rename the whole number as a fraction with the same denominator as the given fraction and then compare numerators. For example, to compare 4 and $\frac{26}{6}$, they would rename 4 as $\frac{24}{6}$.

<u>Algebraic Approach for the Extended Version of S2</u>. The presentation of the algebraic approach for the extended version of S2 also involved four stages.

1. The students were taught to find equivalent expressions for a given fraction by multiplication by some form of 1.
2. In particular, the teacher explained they might multiply using some reciprocal product form of 1 without changing the value of an expression. For example, they might rename $\frac{2}{5}$ as follows:

$$\begin{aligned}\frac{2}{5} &= \frac{2}{5} \times 1 \\ &= \frac{2}{5} \times (3 \times \frac{1}{3}) \\ &= (2 \times \frac{1}{5}) \times (3 \times \frac{1}{3}) \\ &= (2 \times 3) \times (\frac{1}{5} \times \frac{1}{3}) \\ &= 6 \times (\frac{1}{5} \times \frac{1}{3})\end{aligned}$$

3. Given any two fractions, the teacher instructed them to multiply each by some form of 1 so that the unit fractional parts of the renamed quantities were the same. For example, for $\frac{2}{5}$ and $\frac{3}{4}$, they would rename $\frac{2}{5}$ as $(2 \times \frac{1}{5}) \times (4 \times \frac{1}{4})$ and $\frac{3}{4}$ as $(3 \times \frac{1}{4}) \times (5 \times \frac{1}{5})$. This was done to make it easier to compare the two fractions $\frac{2}{5}$ and $\frac{3}{4}$ since now both were written as multiples of a given quantity, namely, $\frac{1}{4} \times \frac{1}{5}$. $\frac{2}{5}$ is (2×4) multiplied by this quantity and $\frac{3}{4}$ is (3×5) multiplied by this quantity.
4. The students were asked to compare the expressions to decide which was the greater. They would now be comparing the two multiples of $\frac{1}{4} \times \frac{1}{5}$, $(2 \times 4) \times (\frac{1}{4} \times \frac{1}{5})$ and $(3 \times 5) \times (\frac{1}{4} \times \frac{1}{5})$. It was shown that these multiples could be compared by simply comparing 2×4 with 3×5, and further, that these products could be considered as the crossproducts obtained from the original two fractions.

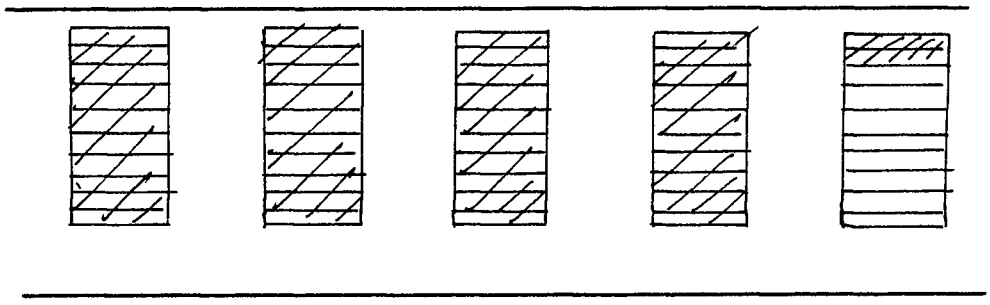
C1 Materials-- Converting a Fraction to a Decimal

Pattern Approach for the Restricted Version of C1. The presentation of the pattern approach for the restricted version of this algorithm, conversion of specific fractions to terminating decimal equivalents, consisted of six stages.

1. The students were introduced to decimal notation for tenths and then for hundredths and thousandths by analogy to the place value positions to the left of the decimal point. That is, just as 1 is $\frac{1}{10}$ of 10, since 10 x 1 = 10, to name the decimal place to the right of 1, one looks at $\frac{1}{10}$ of 1, or $\frac{1}{10}$. Diagrams were used to indicate the relationships between place holder names.
2. The concept of the equivalence of mixed number, decimal, and improper fraction names for the same number was developed. For example, in the case of the equivalence of 4.2, $4\frac{2}{10}$, and $\frac{42}{10}$, a diagram was drawn representing four wholes and two tenths in which the wholes were later partitioned into tenths and then the tenths totalled, as in Figure 14. Similarly, the students were shown the equivalence of 5.28, $5\frac{28}{100}$, and $\frac{528}{100}$, again using diagrams.

Figure 14

Equivalence of Mixed Numbers and Improper Fractions



The students were also shown that they might write, say, $\frac{42}{10}$ as '42 'tenths' for later convenience in dividing. In particular, the students were told that a whole number such as 8 could be rewritten as 8.0, 80 tenths, 8.00, 800 hundredths, 8.000, or 8000 thousandths.

2. The students were introduced to division of a decimal by a whole number, for example, $1.5 \div 3$. The partitioning explanation of the division process took the form of first indicating that if 1.5, or 15 tenths, were to be equally divided into three groups, each group would contain 5 tenths and therefore could be represented by .5 . Diagrams such as the one shown in Figure 15 below were first used and later a more standard division algorithm was employed.

Figure 15

Division of a Decimal by a Whole

$1.5 \div 3 = 15 \text{ tenths} \div 3$

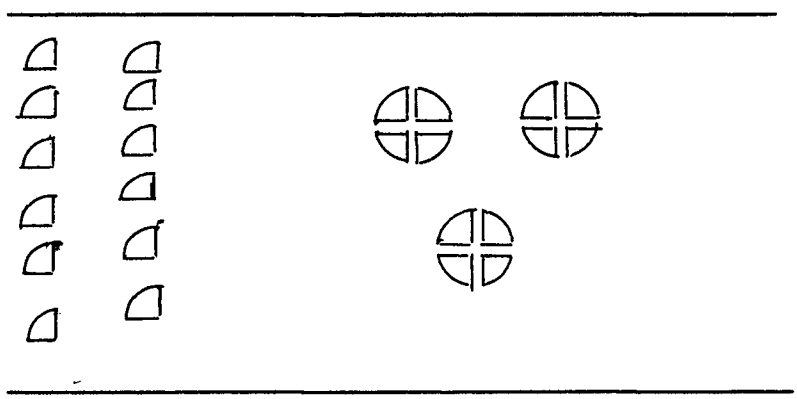
tenth	tenth	tenth
tenth	tenth	tenth
tenth	tenth	tenth
tenth	tenth	tenth
tenth	tenth	tenth
5 tenths	5 tenths	5 tenths

4. The interpretation of the fraction bar as a division sign was developed by the teacher. First, the equivalence of $\frac{c}{d}$ and $c \div d$ was shown for fractions whose numerators were multiples of their denomina-

tors. In these cases, the students were taught that finding a whole number name for the fraction involved computing the number of "groups of the denominator in the numerator"; it was indicated that the whole number name could, therefore, be calculated by dividing the numerator by the denominator. For example, to show that $\frac{12}{4}$ was interpretable as $12 \div 4$, a diagram such as the one in Figure 16 was used, where the twelve fouths were grouped into units and the total number of units counted.

Figure 16

Equivalence of $\frac{n \times a}{a}$ to $n \times a \div a = n$

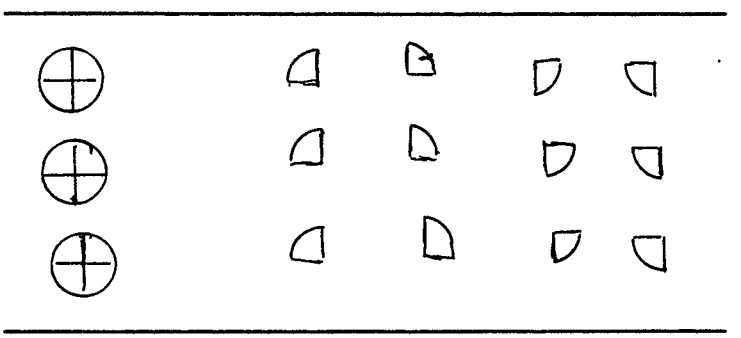


5. Once the students were made aware of the interpretation of the fraction bar as a division sign, the teacher discussed how this could be extended to any fraction, First, improper fractions were used with the partitioning interpretation of division. The students were reminded that one meaning of $c \div d$ is the numerical value of 'c' objects shared equally among 'd' groups. Thus, $11 \div 5$ is the size of each share when 11 objects are shared equally by 5 groups. The students were shown that one could put two objects in each of the 5 groups for the $11 \div 5$ question, but that one object would be left. Then the remaining

object could be cut into fifths and one fifth placed on each group, so that $11 \div 5$ would be $2 + \frac{1}{5}$, or $2\frac{1}{5}$. Each whole would then be partitioned into fifths and $11 \div 5$ was found to be $\frac{5}{5} + \frac{5}{5} + \frac{1}{5} = \frac{11}{5}$ The interpretation of the fraction bar as a division sign was then extended to any proper or improper fraction. For example, $3 \div 4 = \frac{3}{4}$ would be illustrated using the diagram shown in Figure 17.

Figure 17

Equivalence of $\frac{3}{4}$ and $3 \quad 4$



6. The students were asked to find the decimal equivalents for fractions chosen to produce terminating decimals upon conversion. The students had earlier been shown that then could set up the computation as: $4 \overline{)3}$ since they knew that another name for a fraction such as $\frac{3}{4}$ is $3 \div 4$. Based on their earlier experience with decimals, the students were allowed to rename 3 as 3.00 or 3.000, convert these expressions to the names '300 hundredths' or '3000 thousandths', respectively, and then divide.

<u>Pattern Approach for the Extended Version of C1</u>. The presentation of the pattern approach for the extended version of C1 consisted of two stages.

1. The interpretation of the fraction bar as a division sign was retaught

as indicated in stage 5 above using the diagram approach there indicated.

2. The students were then given fractions whose decimal equivalents did not terminate and were taught to approximate these equivalents to three decimal places using the pattern approach indicated in stage 6 of the restricted version, pattern approach. For example, to convert $\frac{5}{7}$ to a decimal, they would rename '5' as 5.000 or 5000 thousandths and divide 500 thousandths by 7 .

Algebraic Approach for the Restricted Version of C1. The presentation of the algebraic approach for the restricted version of C1 consisted of the same six subtasks as those for the pattern approach for the restricted version.

1. The students learning by the algebraic method, too, were introduced to decimal notation for tenths and then for hundredths and thousandths. An analogy was made to the place value positions to the left of the decimal point. That is, just as 1 is $\frac{1}{10}$ of 10, since $10 \times 1 = 10$, to name the next decimal place to the right of 1, one solves the equation : $10 \times \square = 1$. Since $10 \times \frac{1}{10} = 1$, the place to the right of the ones place must be the tenths place.
2. The teacher led a discussion about the equivalence of mixed number, decimal, and improper fraction names for fractions. For example, they were taught to rename 4.2 as $4\frac{2}{10}$, and then $\frac{42}{10}$. This was accomplished by using the property that multiplication by 1 does not change the value of an expression and by knowledge that the product of any number and its reciprocal is 1. Therefore, to find an improper fraction name for $4\frac{2}{10}$, they would rename 4 as follows:

$$
\begin{aligned}
4 &= 4 \times 1 \\
&= 4 \times (10 \times \tfrac{1}{10}) \\
&= (4 \times 10) \times \tfrac{1}{10} \\
&= 40 \times \tfrac{1}{10} \\
&= \tfrac{40}{10}
\end{aligned}
$$

and then $4.2 = 4\frac{2}{10} = \frac{40}{10} + \frac{2}{10} = \frac{42}{10}$. Similarly, the students had an opportunity to convert decimals of the form 'a.bc' into hundredths. These students were also told to rename a fraction with, say, denominator 10, for example $\frac{42}{10}$, and to write this as '42 tenths', for later convenience in division.

3. The students taught by the algebraic method were introduced to division of a decimal by a whole number in terms of missing factors multiplication. For example, to calculate $1.5 \div 3$, they were told they should try to find the value of $\square$ so that $3 \times \square = 1.5 = 15$ tenths; thus, they would see that $\square = (15 \text{ tenths}) \div 3 = 5 \text{ tenths} = .5$. Later the pupils were allowed to use a standard division algorithm to find answers to such questions.

4. In showing in an algebraic manner that a fraction bar is equivalent to a division sign, fractions equivalent to whole numbers were used first. Since the students had been taught that, say, $\frac{12}{4} = 12 \times \frac{1}{4}$, it was now suggested they might rewrite 12 as a product so that one of its factors would be 4, so they now would be expected to say:

$$
\begin{aligned}
\tfrac{12}{4} &= 12 \times \tfrac{1}{4} \\
&= (3 \times 4) \times \tfrac{1}{4} \\
&= 3 \times (4 \times \tfrac{1}{4}) \\
&= 3 \times 1 \\
&= 3.
\end{aligned}
$$

Thus, they were taught to solve for $\square$ in $12 = \square \times 4$, or in its equivalent statement, $\square = 12 \div 4$.

5. This same procedure was extended to other improper fractions and later proper fractions. The students were reminded that if $b \times \square = a$, then $\square = a \div b$; so, to see if, for example, $\frac{11}{5} = 11 \div 5$, they would be expected to multiply $\frac{11}{5}$ by 5 to see if they got 11. They might write:

$$\begin{aligned} \frac{11}{5} \times 5 &= (11 \times \tfrac{1}{5}) \times 5 \\ &= 11 \times (5 \times \tfrac{1}{5}) \\ &= 11 \times 1 \\ &= 11. \end{aligned}$$

Then, since $\frac{11}{5} \times 5 = 11$, $11 \div 5 = \frac{11}{5}$.

6. These students were asked to calculate the decimal equivalents for those fractions which were chosen to produce a terminating decimal upon conversion. They would rename the numerator as a fraction with denominator 10, or 100 or 1000 before proceeding with the division required for the conversion.

<u>Algebraic Approach for the Extended Version of C1</u>. The presentation of the algebraic approach for the extended version of this algorithm consisted of two stages.

1. First a review of the interpretation of the fraction bar as a division sign using algebraic methods as indicated in stages 4 and 5 above was conducted.
2. These students were taught to convert fractions with nonterminating decimal equivalents to their three place approximations. For example, to convert $\frac{5}{7}$ to a decimal, they might recall that $\frac{5}{7}$ is another name for $5 \div 7$, which is another name for 5000 thousandths $\div 7$; to find the decimal, they would use the latter form of $\frac{5}{7}$.

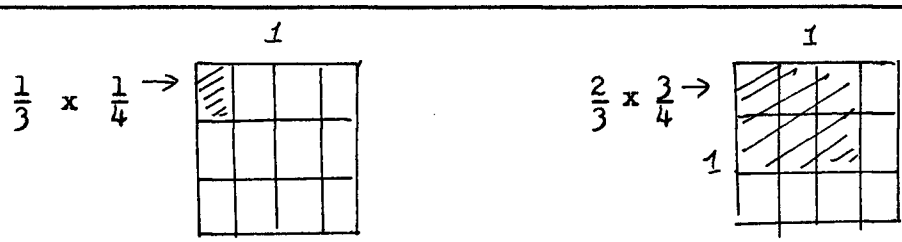
C2 Materials-- Finding the Square Root of Fractions

Pattern Approach for the Restricted Version of C2. The presentation of the pattern approach for the restricted version of algorithm C2, finding the square roots of such fractions as have perfect squares for both numerator and denominator, consisted of six stages.

1. The students were introduced to the relationship between calculating the area of a rectangle and multiplication of the two numbers which are measures of the rectangle's length and width.
2. They were shown how to apply this relationship to computing first the products of unit fractions like $\frac{1}{3} \times \frac{1}{4}$ and then the products of any two proper fractions. Diagrams such as those indicated in Figure 18 were used.

Figure 18

Diagrams to Show Multiplication of Two Fractions

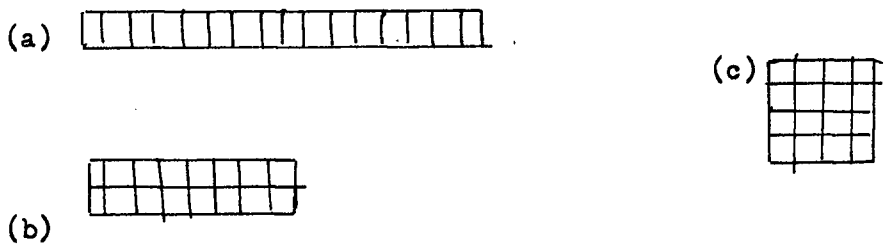


3. Next, they were taught the meaning of the term 'square root'. Using whole numbers only, they were shown a method for drawing all of the rectangles with a given area; they were told to take that rectangle which also happened to be a square and call its side length the square root of the area. For example, for computing the square root of 16, they would be expected to draw the rectangles indicated in

Figure 19. They would then be expected to realize that diagram (c) was a square and thus, that $\sqrt{16} = 4$, and that $4 = \sqrt{16}$ is equivalent to $4 \times 4 = 16$.

Figure 19

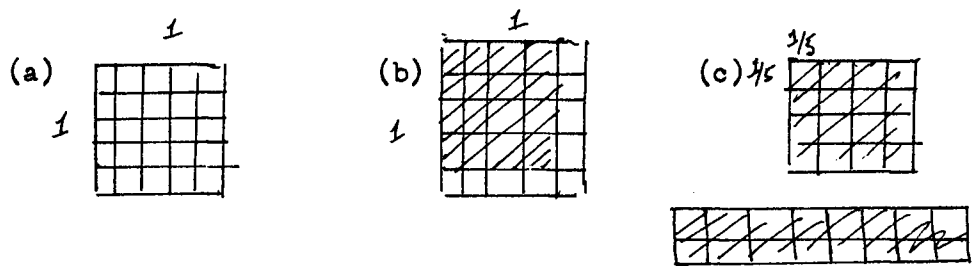
Using Rectangles with a Given Area to Find a Square Root



4. Fractions with perfect squares for numerator and denominator were developed next. For example, for finding the square root of $\frac{16}{25}$, the students were told first to partition a unit square into 25 congruent pieces to form the 'units' of the rectangle to be formed with area $\frac{16}{25}$. The diagram they might begin with could look like the one in Figure 20(a) below. It was then suggested they draw rectangles with the shaded area, as in Figure 20(b); they could then compute the square root, in this case, $\frac{4}{5}$, as indicated in Figure 20(c).

Figure 20

Diagrams to Compute $\sqrt{\frac{16}{25}}$



The teacher would then discuss that since $\frac{4}{5} \times \frac{4}{5} = \frac{16}{25}$, the students could just as easily compute $\sqrt{16}$ and $\sqrt{25}$ separately, as was essentially the case by separating the step shown in Figure 20(a) from that shown in Figure 20(c), and then form the fraction $\sqrt{\frac{16}{25}}$. Thus, computing the square root of a fraction was seen to involve simply finding the square roots of the two whole numbers named by the numerator and the denominator.

5. The issue of what to do to find the square roots of large whole numbers where the drawing of rectangles became inconvenient was introduced; the students were shown how they might use an iterative division procedure to compute this number. The approach was as follows: After making an initial guess for the square root, say, a guess of 11 for $\sqrt{441}$, they might try to sketch a rectangle of length 11 and area 441 and see if the width of this rectangle were also 11. If so, then 11 would be $\sqrt{441}$. Alternatively, they might divide 441 by 11 to find the width. They might observe that the width was not 11, so they would have a rectangle of length 11, but width much greater than 11, so that it is not a square. Because this rectangle was too wide for the length to be a square, it became necessary to try a larger number

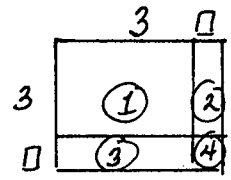
for the length. The students were taught to choose a number between 11 and 40, say, 25. They would repeat the procedure until the resultant width equalled the length chosen, in which case, the square root would be computed. After some experience with this procedure, there was discussion about the choice of initial guesses in terms of comparison of the square with 10^2, 20^2, 30^2, and 40^2. The students were then taught to apply this same technique to numerators of fractions which were perfect squares, and whose denominators were perfect squares less than 100, and so were within the realm of the multiplication facts of the students. In this way, the transition from the whole number procedure was direct.

Pattern Approach for the Extended Version of C2. The presentation of the pattern approach for the extended version of C2 consisted of three steps.

1. The students were led in a discussion reviewing the iterative procedure for finding the square roots of whole numbers as indicated in stage 5 above.
2. The students were then taught to find approximate square roots of whole numbers which are not perfect squares. For example, to compute $\sqrt{10}$, they would be expected first to see that the answer must be between 3 and 4, say, $3+\square$, where $\square$ is less than 1. So a square would be drawn as in Figure 21 and the teacher would list the areas of the numbered sections of the square in order of size and remind the students that the total of these areas would be 10.

Figure 21

The Binomial Square for Approximating Square Roots



Therefore, the students would compute the area of section ① to be 9, of sections ② and ③ to be 3x □ each, and of section ④ to be □x □ . They would then write: 9 + (3 x □) + (3 x □) + □ x □ = 10. It was explained that since □ < 1, □ x □ < 1. Therefore, the rest of the area would be close to 10, or 9 + (3 x □) + (3 x □) is about 10. They were taught to rename (3 x □) + (3 x □) as 6 x □ . They would then solve the equation and find that □ was about $\frac{1}{6}$. The allowable margin of error on all problems was □x□ < $\frac{1}{2}$. Since the area of section ④ , □ x □ = $\frac{1}{36}$ was less than $\frac{1}{2}$, the students agreed to be satisfied with their approximation of $\sqrt{10} \approx 3\frac{1}{6}$.

3. The students were taught to extend this approximation technique to fractions with denominators which were perfect squares, but numerators which were not perfect squares. The denominators were small to ensure that no question required too great a time for solution.

<u>Algebraic Approach for the Restricted Version of C2</u>. The presentation of the algebraic approach for the restricted version of C2 consisted of six stages.

1. The teacher demonstrated a procedure for computing the product of two unit fractions and then any two proper fractions. To multiply

two unit fractions, it was suggested the students might first guess an answer, for example, that $\frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$. The teacher then discussed the idea that this could be true only if $8 \times (\frac{1}{2} \times \frac{1}{4}) = 1$ because only one number could be multiplied by 8 to get 1, namely, $\frac{1}{8}$. The students were then instructed to rewrite 8 as 4 x 2 and show that it follows that:

$$\begin{aligned} 8 \times (\tfrac{1}{2} \times \tfrac{1}{4}) &= (4 \times 2) \times (\tfrac{1}{2} \times \tfrac{1}{4}) \\ &= (4 \times \tfrac{1}{4}) \times (2 \times \tfrac{1}{2}) \\ &= 1 \times 1 \\ &= 1. \end{aligned}$$

From this, the pupils agreed that $\frac{1}{2} \times \frac{1}{4}$ must indeed be $\frac{1}{8}$.

2. To multiply any two fractions, say, $\frac{5}{6} \times \frac{3}{4}$, they were taught to use the associative and commutative principles of multiplication and write $\frac{5}{6} \times \frac{3}{4}$ as follows:

$$\begin{aligned} \tfrac{5}{6} \times \tfrac{3}{4} &= (5 \times \tfrac{1}{6}) \times (3 \times \tfrac{1}{4}) \\ &= (5 \times 3) \times (\tfrac{1}{6} \times \tfrac{1}{4}) \\ &= 15 \times \tfrac{1}{24} \\ &= \tfrac{15}{24}. \end{aligned}$$

3. The students were introduced to the meaning of the term 'square root'. Using a whole number, the teacher listed all pairs of whole number factors whose product was that number. The class was asked to look on the list for a pair where the two factors were identical. If such a pair was found, the factor was called the square root of the product.

4. The students were led to discussing fractions with perfect squares for numerators and denominators. For example, to compute $\sqrt{\frac{16}{25}}$, they were taught to rewrite $\frac{16}{25} = 16 \times \frac{1}{25}$. The students were then to look for all the facts they could for this product, for example, $(16 \times 2\frac{1}{5}) \times 1$,

$(8 \times \frac{1}{5}) \times (2 \times \frac{1}{5})$, looking for a pair with two identical factors. They were then shown that they could separately compute $\sqrt{16}$ and $\sqrt{25}$ and then regard the fraction $\sqrt{\frac{16}{25}}$ as if it were $\frac{\sqrt{16}}{\sqrt{25}}$.

5. In order to compute the square roots of large whole numbers where it became impractical to list all of the factor pairs for this number, these students were shown the iterative procedure, using division, to compute square roots. They were told to make an initial guess for the square root, say, 11 for $\sqrt{441}$. They were then told to determine whether replacing $\square$ by 11 in the equation $11 \times \square = 441$ would make it true. To compute the value of $\square$, they were taught to divide 441 by 11. In this case, the quotient would be considerably greater than 11. They were next to compare the two factors, the divisor and the quotient to see if they were equal; if not, it was explained that they might raise the value of their first guess or lower it by choosing a number between that first guess and the quotient obtained and using this number as a next guess. They were shown that repetition of this procedure would close in on the appropriate square root.
6. This same technique was then applied to fractions which had perfect squares for both numerator and denominator. In all cases, the numerators of these fractions were large enough to require the procedure discussed in step 5, but the denominators were small enough so that the square root would be immediately known to the students.

<u>Algebraic Approach for the Extended Version of C2</u>. The presentation of the algebraic approach for the extended version of this algorithm consisted of three stages.

1. The students were led in a review of the iteration procedure for computing square roots of whole numbers as described in stage 5 above.

2. The teacher then explained how to find approximate square roots of whole numbers which were not perfect squares. For example, to find $\sqrt{10}$, the students were shown that they might first decide that the answer was between 3 and 4, say, $3 + \square$, where $\square$ is less than 1. Then, since $(3+\square) \times (3+\square) = 10$, the students were taught how to multiply out the left hand side of the equation, resulting in $9 + (3 \times \square) + (3 \times \square) + (\square \times \square) = 10$. It was shown that since $\square < 1$, $\square \times \square < 1$, and the students' attention was drawn to the fact that $9 + (3 \times \square) + (3 \times \square)$ is almost 10. Because $(3 \times \square) + (3 \times \square)$ could be renamed as $6 \times \square$, the students would be expected to see that $\square$ would be about $\frac{1}{6}$. Then, since $\square \times \square = \frac{1}{36}$ which is less than $\frac{1}{2}$, the allowable margin or error, the students would agree that $3\frac{1}{6}$ would be an acceptable approximation for $\sqrt{10}$.
3. The students were led to extend the approximation technique to fractions with denominators which were perfect squares but with numerators which were not perfect squares.

MEASURING INSTRUMENTS

Criterion Pretests

Each of the criterion pretests for the four algorithms consisted of free response items designed to test knowledge of that algorithm's immediate mathematical prerequisites, as determined by a panel of judges. The pretests were given in order to adjust criterion scores for differences among classes in 'readiness' for the instructional material. KR-20 reliabilities were calculated for each pretest and each criterion test to provide evidence about the extent to which each test could be considered a consistent measure of a single ability. These

measures were also used to indicate the probable stability of the tests over short intervals of time. An alternate forms reliability coefficient with a delay equal to the duration of the experiment would be expected to be little lower than the immediate alternate forms coefficient, of which the KR-20 can be taken to be an estimate. This is so particularly because cognitive abilities are likely to be fairly stable, and these are the abilities being measured. It was decided that any reliability less than .8 would be considered as unacceptable, for, in such cases, over 20% of the variance in test scores could be attributed to factors extraneous to those being measured, including error of measurement.

Figures 22-25 indicate the content and number of items for each of the pretests for S1, S2,C1, and C2, respectively. Also indicated are the KR-20 reliability coefficients for each of these four tests. It should be noted that all of these reliabilities exceeded .80 and were, therefore, considered acceptable for the experimental purposes.

Figure 22

Pretest for Algorithm S1

Prerequisites for S1	Number of Items
Multiplication Facts	10
Rewriting a Mixed Number as a Sum	4
Use of a Diagram to Represent a Fraction	5
Property of Reciprocals	3
Renaming of a Fraction as a Multiplication Expression	3
Calculating the Area of a Rectangle	5
1 as a Multiplicative Identity	4
Commutativity and Associativity of Multiplication	3
Distributive Property of Multiplication over Addition	3
Conservation of Area of a Rectangle	2
	42

KR-20 reliability coefficient = .90
(based on 201 subjects)

Figure 23

Pretest for Algorithm S2

Prerequisites for S2	Number of Items
Multiplication Facts	10
Use of a Diagram to Represent a Fraction	5
Use of Diagrams to Find the Greater of Two Fraction Representations	5
Use of Diagrams to Find Fractional Names for 1	3
Property of Reciprocals	3
Commutativity and Associativity of Multiplication	3
Renaming of a Fraction as a Multiplicative Expression	3
Preservation of an Inequality after Multiplication	3
	35

KR-20 reliability coefficient = .84
(based on 236 subjects)

Figure 24

Pretest for Algorithm C1

Prerequisites for C1	Number of Items
Use of a Diagram to Represent a Fraction	5
Division to Solve a Sharing Problem	5
Long Division Skills	5
Inverse Relation of Multiplication and Division	5
Property of Reciprocals	2
Renaming of a Fraction as a Multiplicative Expression	3
Commutativity and Associativity of Multiplication	3
One as a Multiplicative Identity	2
	30

KR-20 reliability coefficient = .90
(based on 204 subjects)

Figure 25

Pretest for Algorithm C2

Prerequisites for C2	Number of Items
Multiplication Facts	10
Inverse Relationship of Multiplication and Division	5
Use of a Diagram to Represent a Fraction	5
Property of Reciprocals	2
Renaming of a Fraction as a Multiplicative Expression	3
1 as a Multiplicative Identity	2
Commutativity and Associativity of Multiplication	3
Computing the Area of a Rectangle	5
Lond Division Skills	5
	40

KR-20 reliability coefficient = .90
(based on 235 subjects)

Criterion Computation Tests

The computation tests for each of the four algorithms consisted of free response items designed to test the students' ability to perform that algorithm. Figures 26-29 indicate the content and number of items for each of the computation tests for S1, S2, C1, and C2, respectively. Also indicated are the KR-20 reliability coefficients.

Figure 26

S1 Computation Test

Content	Number of Items
Products of Fractions and Whole Numbers	6
Products of Mixed Numbers and Whole Numbers	6
Products of Two Fractions	6
Products of Mixed Numbers and Fractions	6
	24

KR-20 reliability coefficient = .94
(based on 201 subjects)

Figure 27

S2 Computation Test

Content	Number of Items
Comparison of a Fraction and a Whole Number	9
Comparison of Two Proper Fractions	9
Comparison of Two Improper Fractions	9
	27

KR-20 reliability coefficient = .95
(based on 232 subjects)

Figure 28

C1 Computation Test

Content	Number of Items
Conversion of a Fraction to a Terminating Decimal	8
Conversion of a Fraction to a Nonterminating Decimal	10
	18

KR-20 reliability coefficient = .93
(based on 193 subjects)

Figure 29

C2 Computation Test

Content	Number of Items
Finding the Square Root of a Perfect-square Fraction	8
Finding the Square Root of a Nonperfect-square Fraction	7
	15

KR-20 reliability coefficient = .92
(based on 225 subjects)

Criterion Extension Tests

Each of the extension tests for the four algorithms consisted of 30 free response items. The tests were based on five types of items designed to measure the abilities to:

1) shortcut the algorithm in particular cases (Type A),
2) solve equations with a missing operand requring use of the algorithm (Type B),
3) extend the algorithm to more than two operands (Type C),
4) use the algorithm with number forms other than the type studied (Type D), and
5) explain an alternate version of the algorithm (Type E).

These tests were administered in order to see whether justification treatments affected the ability of students to extend, as measured by these tests, as well as the ability to compute. Types of extensions A,C, and D included in the tests are frequently used in arithmetic classes to extend already learned material. Types B and E were included as measures of the student's knowledge of the processes involved in the algorithm studied. An example of a Type A question might be, "If you know that $\frac{1}{81} \times \frac{1}{63} = \frac{1}{5103}$, you could also tell that $\frac{2}{81} \times \frac{1}{63} = \frac{\square}{\triangle}$. What is $\frac{\square}{\triangle}$?" An example of a Type B question might be," If $6 \times 18 > 13 \times 8$, which is greater: $\frac{6}{13}$ or $\frac{8}{18}$?" An example of a Type C question for algorithm S1 might be," What is $(6 \times \frac{2}{9}) \times 5\frac{1}{3}$?" An example of a Type D question for this algorithm might be, "Find $\frac{\frac{1}{2}}{2} \times 24\frac{2}{5}$." An example of a Type E question might be, " Susie has suggested another rule for finding the decimal equivalent for a fraction $\frac{a}{b}$. She multiplies each of the numerator and by ten, and then she divides the denominator into the numerator. For

example, to find the decimal for $\frac{3}{5}$, she divides 50 into 30 , or 50 into 300 tenths to get an answer of 6 tenths. She claims that her answer is always correct and the same as the one the teacher gets by using the method taught in class. Will Susie always get the right answer? Why do you think so? Show how Susied would find the decimal for $\frac{5}{8}$ and explain why her method seems to work."

The S1 extension test consisted of eight Type A, six Type B, five Type C, four Type D, three Type B-C, and four Type E questions. The KR-20 reliability coefficient, based on 200 subjects, was found to be .83. The S2 extension test consisted of eight Type A, six Type B, eight Type C, four Type D, and four Type E questions. The KR-20 reliability coefficient, based on 198 subjects, was found to be .82.

The C1 and C2 extension tests each consisted of eight Type A, nine Type B, nine Type D, and four Type E items. The KR-20 reliability coefficients for the C1 test, based on 184 subjects, was found to be .80 and for the C2 test, based on 224 subjects, was found to be .88.

It should be noted that all of the extension tests had reliabilities of at least .80 and were judged, therefore, to be adequate for the purposes of this study.

STATISTICAL PROCEDURES

The statistical analyses described below were performed in parallel fashion for all four algorithms. Because intact groups were the only ones available to the writer, it was decided to use a covariate measure, the score on the criterion pretest, to adjust for differences among the groups in entering mathematical skills. However, as Lindquist (25:9) has pointed out, one must also consider Type G

error, where there may be factors with one effect on the members of a particular treatment group, and a different effect or no effect on the members of another treatment group, particularly when using intact classes. Such a factor might be teacher abilitiy differences or group rate of learning differences. In order to minimize the possibility of a Type G error, a groups-within-treatments design was selected for the comparison of the pP-aA treatment groups. Wright(48:3) discussed two types of groups-within-treatments analyses, one with the subject as the unit of analysis and one with a pooled subject and group analysis.

As was discussed earlier, a decision about which of the two designs to use might be based on the amount of variance contributed by the groups-within-treatments factor. The legitimate unit of analysis must be class groups because it was groups that were randomly assigned to treatments and because only such an analysis can adequately cope with Type G error. However, as can be seen from the analysis of covariance model below, a simpler model could be achieved if the term $\tau_{j(i)}$ could be deleted, that is, if statistical and logical considerations strongly suggested that differences among groups within treatments were zero in the population. This deletion would have been justifiable had the groups been constituted randomly. The legitimate deleting of $\tau_{j(i)}$ from the model would permit the use of a subjects-within-groups-within-treatments error term, with greatly increased statistical power. Despite the possibility that the classes may have not been considered equivalent in all possible relevant measures, for example, IQ measures, it was decided to test the hypothesis that the $\tau_{j(i)}$ were zero. This was done at the .25 level of significance to reduce the likelihood of falsely accepting the hypothesis. The purpose of making this test was to see

whether , but for the nonrandom assignment of subjects to groups, a simpler and more powerful model would be justified. If so, then an alternative analysis would be available-- an analysis which, though requiring careful interpretation because of possible nonrandomness of assignment of subjects to groups-- might provide additional useful information.

Although the groups-within-treatments analysis was used to analyze the pP-aA treatment comparisons, a non-nested design was necessary for the pA-aP comparisons because only one class unit was obtainable for each treatment. Thus, for the pA-aP analyses, subject and group factors were necessarily confounded, making findings indicative rather than dependable.

Because of the likelihood of a substantial positive correlation between scores on the computation and extension tests for each algorithm, analyses were first performed using a multiple analysis of covariance model to obatin an overall test of significance. Later, the univariate tests for the two dependent variables for each algorithm were performed.

The multiple analysis of covariance, or MANOCOVA, model for the pP-aA test is:

$$Y_{ijkh} = \mu_h + \alpha_{ih} + \tau_{j(i)h} + \beta_h X_{ijk} + e_{ijkh}$$

where:

μ_h = general level parameter of the hth response

α_{ih} = effect of the justification treatment i on h^{th} response

$\tau_{j(i)h}$ = effect of the group condition j nested in treatment i on the hth response

$\beta_{h\,ijk}X$ = effect of the concomitant variable of pretest score X for person k in group j within treatment i on the hth response

e_{ijkh} = usual multinormal random variable error term for h^{th} response for person k in group j within treatment i

It should be noted that in the notation above, h = 1 or 2; h = 1 for the criterion computation test, and h = 2 for the criterion extension test. In this model, i = 1 or 2, depending on assignment to the pP or aA groups, and j = 1, 2, or 3, depending on assignment to a particular class within a treatment.

The MANOCOVA model for the pA-aP test is the same with the exception of the missing $\tau_{j(i)h}$ term for groups-within-treatments effects.

Thus, for the pP-aA tests, the F-ratio computed was based on a measure of the between-groups matrix of sums of squares and sums of products and the between-justification-treatments matrix of sums of squares and sums of products. For the pA-aP tests, the F-ratio computed was based on a measure of the between-justification-treatments matrix of sums of squares and sums of products and the error matrix of sums of squares and sums of products. The F-ratio used in these multivariate tests is that due to Rao (Tatsuoka [44] :200). It is discussed in more detail in the following chapter.

Chapter 4

RESULTS OF THE STUDY

RESULTS

The means and standard deviations for the scores on the pretests, computation tests, and extension tests for algorithms S1, G1, S2, and C2 are shown in Tables 1, 2, 3, and 4, respectively. In all tables, scores are supplied for these justification treatments: pattern only (pP), algebraic only (aA), pattern followed by algebraic (pA), and algebraic followed by pattern (aP), as well as for individual classes involved in each treatment. Classes are numbered in these tables as on page 45.

Table 1

Means and Standard Deviations for S1 Pretest, Computation Test, and Extension Test

	Pretest		Computation		Extension	
	Mean	S.D.	Mean	S.D.	Mean	S.D.
Treatment pP	26.2713	(5.9561)	15.0247	(6.9946)	4.0370	(3.1283)
Class 1	24.2903	(6.0619)	16.1613	(7.2300)	2.4194	(2.2624)
Class 2	25.6667	(6.5080)	11.0566	(5.6305)	3.2222	(2.4628)
Class 3	28.5312	(4.8326)	16.1562	(6.8208)	6.0625	(3.1206)
Treatment aA	31.9844	(6.4359)	13.4000	(7.5763)	7.1538	(4.3020)
Class 4	33.8928	(6.5283)	13.0714	(7.2210)	5.6429	(4.1563)
Class 5	38.5000	(2.1381)	23.1250	(0.6409)	11.8750	(3.8336)
Class 6	28.3448	(4.7905)	11.0345	(6.9204)	7.3103	(3.6460)
Treatment pA	24.3333	(6.0265)	11.2500	(6.8414)	3.9167	(3.2958)
Treatment aP	32.4666	(5.6553)	22.8000	(1.9714)	9.6667	(4.2873)
Maximums	42		24		30	

Table 2

Means and Standard Deviations for C1 Pretest, Computation Test, and Extension Test

	Pretest		Computation		Extension	
	Mean	S.D.	Mean	S.D.	Mean	S.D.
Treatment pP	16.4742	(5.6997)	7.2564	(5.0874)	1.2308	(1.5536)
Class 1	14.2308	(4.9259)	6.2692	(4.2854)	1.2308	(1.3056)
Class 2	13.6875	(5.2373)	5.6875	(4.6003)	0.5000	(1.0954)
Class 3	19.3333	(5.1659)	8.6667	(5.5549)	1.5556	(1.7959)
Treatment aA	21.0665	(5.8682)	9.4500	(5.1599)	3.9167	(3.6233)
Class 4	18.3750	(6.5196)	7.8750	(5.2859)	2.0833	(2.1853)
Class 5	26.3750	(2.1998)	16.2500	(1.5811)	10.8750	(2.9001)
Class 6	21.8571	(4.7276)	8.8571	(4.1786)	3.5000	(2.2194)
Treatment pA	16.3667	(6.9107)	7.2000	(5.6715)	2.6000	(2.3134)
Treatment aP	24.5000	(4.4004)	17.8333	(0.3895)	4.0833	(2.8110)
Maximums	30		18		30	

Table 23

Means and Standard Deviations for S2 Pretest, Computation Test, and Extension Test

	Pretest		Computation		Extension	
	Mean	S.D.	Mean	S.D.	Mean	S.D.
Treatment pP	26.3330	(5.0805)	23.6126	(4.5684)	9.4086	(4.7873)
Class 7	25.7273	(4.6856)	23.5757	(5.5511)	6.6667	(3.4157)
Class 8	28.4193	(4.6100)	23.5806	(4.2172)	12.0323	(3.9029)
Class 9	24.7931	(5.4076)	23.6897	(3.7807)	9.7241	(5.3911)
Treatment aA	28.2561	(4.4207)	19.0254	(8.2791)	11.1410	(4.7392)
Class 10	33.5882	(1.9705)	24.9412	(3.0919)	16.1765	(4.2461)
Class 11	27.6552	(2.5394)	12.7931	(7.6410)	8.1379	(3.3988)
Class 12	25.9688	(4.4030)	21.5312	(7.1119)	11.1875	(3.7021)
Treatment pA	26.4737	(4.4643)	19.1053	(7.7810)	7.8421	(6.0485)
Treatment aP	27.3448	(4.5378)	19.2069	(8.5956)	9.7241	(4.6051)
Maximums	35		27		30	

Table 4

Means and Standard Deviations for C2 Pretest, Computation Test, and Extension Test

	Pretest		Computation		Extension	
	Mean	S.D.	Mean	S.D.	Mean	S.D.
Treatment pP	26.0975	(7.0944)	8.3369	(4.6392)	6.0543	(4.5918)
Class 7	23.0000	(7.1694)	4.6977	(3.9029)	2.1290	(1.9957)
Class 8	30.6000	(5.7871)	10.0000	(4.3865)	9.0000	(4.4952)
Class 9	24.8387	(6.0888)	10.0968	(3.7090)	7.1290	(3.8275)
Treatment aA	28.6873	(6.5556)	8.7375	(5.2476)	9.2375	(5.4290)
Class 10	35.5000	(4.0988)	13.2500	(2.5690)	16.6250	(4.0146)
Class 11	26.1000	(5.7317)	3.5333	(3.2455)	5.5000	(2.9798)
Class 12	27.7647	(6.0756)	11.2059	(3.5911)	9.0588	(4.0297)
Treatment pA	25.5625	(7.1830)	8.0000	(4.7889)	5.5625	(5.7268)
Treatment aP	28.0000	(6.9282)	8.4762	(5.3816)	7.3810	(5.2199)
Maximums	40		15		30	

Examination of the preceeding means and standard deviations suggests the likelihood of a groups-within-treatments effect in terms of both the great differences among group means within treatments and also the great differences among groups variances within treatments. It is particularly those groups with exceptionally high means on the criterion tests which show unusually small within-group variance. This may be noted by examining the means and standard deviations for class 5 and group aP in Tables 1 and 2, and class 10 in Table 3. This situation might have resulted from a Type G error arising from differences in teacher ability or possibly from a teacher-treatment interaction leading to a ceiling effect which prevented many subjects from scoring much higher than the group's mean score. Analysis of the data, therefore, demanded particular attention to the usual assumption of homogeneity of variance for groups for an analysis of variance test. However, based on Norton's (Lindquist [25]:83) study of the effect of marked heterogeneity of variance on the form of the F-distribution, the writer decided to proceed with the analysis of variance design originally proposed. Norton suggested that although the F-test is robust with respect to violation of the homogeneity of variance assumption, one should consider that a 5% level test of significance might, in fact, be a 7% or 8% level test, given violation of the variance assumption (Lindquist [25]: 83). It was decided, as well, that any significant results among the pP-aA comparisons would be reexamined using nonparametric statistical methods where the assumption of homogeneity of variance is not required.

The pretests for the four algorithms were administered to obtain covariates to adjust for differences in immediately prerequisite knowledge for study of the experimental material. However, the correlation

coefficients indicating the degree of linear relationship between each of the pretests and the corresponding measures were examined prior to the use of analysis of covariance to ensure that such an analysis was warranted. These coefficients of correlation for treatments and for groups-within-treatments for the computation and extension tests are shown in Tables 5 and 6 below. Also shown are the regression coefficients for the lines used to adjust criterion scores based on pretest scores, in Tables 8 and 9. The regression slopes were examined to check for violations of the homogeneity of regression assumption for analysis of covariance.

Table 5

Correlations of Pretest Scores with Computation Scores for S1, C1, S2, and C2

	S1	C1		S2	C2
Overall pP-aA	.2756	.4400	Overall pP-aA	.1879	.4924
Treatment pP	.2573	.4441	Treatment pP	.2566	.5235
Class 1	.2187	.5559	Class 7	.2273	.6457
Class 2	.5287	.1423	Class 8	.4809	.3464
Class 3	.1814	.3724	Class 9	.1452	.4302
Treatment aA	.4433	.6130	Treatment aA	.2329	.5701
Class 4	.2225	.5048	Class 10	.0676	.0063
Class 5	-.4692	-.4826	Class 11	.0201	.4846
Class 6	.3939	.6176	Class 12	.1829	.6509
Overall pA-aP	.4212	.7404	Overall pA-aP	.3945	.6331
Treatment pA	.4651	.7969	Treatment pA	.3008	.5485
Treatment aP	.4255	-.0531	Treatment aP	.4486	.6947

Table 6

Correlations of Pretest Scores with Extension Scores for S1, C1, S2, and C2

	S1	C1		S2	C2
Overall pP-aA	.4927	.5154	Overall pP-aA	.3626	.5352
Treatment pP	.5046	.5331	Treatment pP	.4122	.5736
Class 1	.4089	.3770	Class 7	.3905	.3821
Class 2	.3352	.6914	Class 8	.4735	.5448
Class 3	.5090	.5461	Class 9	.3140	.5616
Treatment aA	.5153	.6204	Treatment aA	.5096	.7387
Class 4	.6674	.6569	Class 10	.5323	.5914
Class 5	.0261	.1651	Class 11	-.0026	.6673
Class 6	.5601	.5330	Class 12	.5723	.7457
Overall pA-aP	.6979	.6140	Overall pA-aP	.5323	.5914
Treatment pA	.7173	.6134	Treatment pA	.3733	.5947
Treatment aP	.6933	.7166	Treatment aP	.6747	.5890

The correlations of pretest scores with extension scores are generally quite high both for treatments and for individual groups. However, many of the group correlations were below the .4 generally considered minimal for use of an analysis of covariance design, rather than an analysis of variance design (Myers [29]:324). The correlations which are low can be found mostly where expected, among the correlations for those classes with particularly small within-group variance on one or both of the pretest and extension test, as, for example class 5 for the S1 tests. Low correlations do not occur for all such groups, however, In the case of class 5, for example, it can be argued that because the

variance of grades within the class was so small, it is quite possible for those students who performed above the mean on the pretest to perform below the mean on the criterion test, and vice versa, since only a few marks, easily within the error of measurement, might affect a person's position above or below the mean. Thus, the correlation between the two tests might just as easily be zero as positive or negative. These few exceptions did not, however, in the opinion of the writer, necessitate abandonment of the covariance procedure.

Although many of the pretest-computation correlations were not greater than .4, particularly for the simple algorithms, because it was suspected that scores on the two dependent measures, the computation test and the extension test, might be sufficiently correlated to warrant multivariate statistical procedures, the multiple analysis was undertaken using the covariate, pretest score, on both dependent measures. The actual correlations obtained between computation and extension scores are displayed in Table 7.

Regression coefficients were also computed for groups and for treatments for predicting computation and extension scores from pretest scores. This was done to determine whether the homogeneity of regression assumption for use of an analysis of covariance design had been violated. These coefficients are displayed in Tables 8 and 9.

Table 7

Correlations of Computation and Extension Scores for S1, C1, S2, and C2

	S1	C1		S2	C2
Overall pP-aA	.3851	.3802	Overall pP-aA	.2532	.5615
Treatment pP	.3233	.3046	Treatment pP	.1897	.6754
Class 1	.2301	.3388	Class 7	.0813	.1503
Class 2	.7711	.2844	Class 8	.4221	.7013
Class 3	.2981	.2339	Class 9	.2147	.6049
Treatment aA	.5096	.6865	Treatment aA	.5378	.7349
Class 4	.4143	.5731	Class 10	.4388	.3329
Class 5	-.5160	.3193	Class 11	.1483	.5099
Class 6	.5176	.5591	Class 12	.4555	.7488
Overall pA-aP	.4971	.5373	Overall pA-aP	.5229	.6451
Treatment pA	.6122	.6634	Treatment pA	.6142	.5761
Treatment aP	.4987	.2631	Treatment aP	.4689	.7016

Table 8

Regression Coefficients for Pretest Scores on Computation Scores for S1, C1, S2, and C2

	S1	C1		S2	C2
Overall pP-aA	.3309	.4028	Overall pP-aA	.2491	.3008
Treatment pP	.3022	.3964	Treatment pP	.2307	.3423
Class 1	.2608	.4836	Class 7	.2693	.3515
Class 2	.4574	.1250	Class 8	.4399	.2626
Class 3	.2560	.4004	Class 9	.1015	.2621
Treatment aA	.5218	.5390	Treatment aA	.4362	.4564
Class 4	.2461	.4093	Class 10	.1061	.0040
Class 5	-.1406	-.3468	Class 11	.0605	.2744
Class 6	.5690	.5488	Class 12	.2954	.3847
Overall pA-aP	.3955	.5662	Overall pA-aP	.7249	.4620
Treatment pA	.5280	.6540	Treatment pA	.5243	.3657
Treatment aP	.1483	-.0047	Treatment aP	.8498	.5396

Table 9

Regression Coefficients for Pretest Scores on Extension Scores for S1, C1, S2, and C2

	S1	C1		S2	C2
Overall pP-aA	.2886	.1881	Overall pP-aA	.3426	.3214
Treatment pP	.2650	.1453	Treatment pP	.3884	.3713
Class 1	.1526	.0999	Class 7	.2847	.1064
Class 2	.1268	.1446	Class 8	.4009	.4232
Class 3	.3287	.1898	Class 9	.3130	.3530
Treatment aA	.3444	.3831	Treatment aA	.5463	.6118
Class 4	.4249	.2202	Class 10	.4544	.1706
Class 5	.0468	.2177	Class 11	-.0035	.3469
Class 6	.4263	.2502	Class 12	.4812	.4946
Overall pA-aP	.4388	.2390	Overall pA-aP	.6160	.4573
Treatment pA	.3923	.2053	Treatment pA	.5058	.4741
Treatment aP	.5256	.4578	Treatment aP	.6847	.4438

The regression coefficients for the prediction of extension scores from pretest scores, as well as those for the prediction of computation scores from pretest scores, showed considerable lack of homogeneity. In particular, class 5 and class 10 showed marked difference from the overall regression slope in their within-group slopes, as was the case with the correlation coefficients. Because covariance assumptions cannot be violated with impunity, caution must be taken in interpreting results of this study, particularly for those few groups whose regression slopes differed substantially from the overall slopes. However, a sufficient number of within-group regression coefficients seemed to

be similar enough to make use of a covariance design a reasonable procedure.

For all algorithms, the pP-aA comparisons were made independently of the pA-aP comparisons because the former were based on a nested design, with groups nested in treatment, while the latter were simple one-way analyses of covariance. Although multivariate procedures were used, univariate analyses for the two dependent measures were also examined in order to discuss the two sets of results independently.

The four null hypotheses for the research hypotheses stated in Chapter 1 can each be stated in vector form:

$$H_o: \begin{bmatrix} \alpha_{11} \\ \alpha_{12} \end{bmatrix} = \begin{bmatrix} \alpha_{21} \\ \alpha_{22} \end{bmatrix}.$$

To test the pP-aA comparisons for significance, the overall F-ratio due to Rao (Tatsuoka [44]:200) was computed. This F-ratio can be expressed as follows:

$$\frac{1 - \Lambda^{1/s}}{\Lambda^{1/s}} \cdot \frac{s(2f_e - 2c - p + f_h - 1) - pf_h + 2}{2pf_h}$$

where: degrees of freedom are pf_h for the numerator and

$\frac{1}{2}(s\ (2f_e - 2c - p + f_h - 1) - pf_h + 2)$ for the denominator

and:

p = number of dependent variables

c = number of covariates

f_h = univariate degrees of freedom for the effect under test

f_e = univariate degrees of freedom for the "error"

$s = \sqrt{\frac{p^2 f_h^2 - 4}{p^2 + f_h^2 - 5}}$ or 2 if indeterminate

Because classes, not pupils, were randomly assigned to treatments, the nested design was used. The question of what error term to use is dependent on the model. Strictly, the error for testing the H_o's in this nested design should be $\tau_{j(i)h}$. Recalling the earlier discussion in Chapter 3, however, it can be noted that if $\tau_{j(i)h}$ can be conceived to be zero in the population, it can legitimately be deleted from the model, and then one has an alternative analysis. Although it was expected that there might be differences among group means, it was decided that if this F-ratio indicated non-significant groups-within-treatments differences at the .25 level, the null hypothesis for comparison of the treatment means would be tested using the within cells variance, rather than the groups-within-treatments variance. For each of the four algorithms, the overall F-ratios for the groups-within-treatments test, along with the critical F-values for a .25 significance level are listed in Table 10. A .25 level of significance was chosen in order to reduce the probability of accepting the null hypothesis of equality of mean vectors if, in fact, it ought to be rejected.

Table 10

Overall F-Ratios for the Multivariate Tests of Equality of Adjusted Mean Vectors for Groups-within-Treatments

	Obtained F	Degrees of Freedom	Critical F at .25 Level
Algorithm S1	7.21655*	8 ; 276	1.29
Algorithm S2	11.29581*	8 ; 326	1.28
Algorithm C1	12.73003*	8 ; 260	1.29
Algorithm C2	20.21288*	8 ; 328	1.28

*Significant at the .25 level

Because the F-ratios for the groups-within-treatment tests were all significant at the .25 level, and the term $\tau_{j(i)h}$ could not be deleted from the model, the tests of the treatment hypotheses were performed using the groups-within-treatments factor as an error term. The overall F-ratios for the multivariate tests of equality of mean vectors for the pP-aA comparisons are listed in Table 11 below. It can be observed that none of the overall F-ratios proved significant at the .05 level.

The univariate analyses of covariance for each of the computation and extension tests, along with group and treatment means adjusted for the covariate, for algorithms S1, C1, S2, and C2 can be found in Tables 12, 13, 14, and 15, respectively.

Table 11

Overall F-ratios for the Multivariate Tests of Equality of Adjusted Mean Vectors for Treatments pP-aA

	Obtained F	Degrees of Freedom	Critical F at .05 Level
Algorithm S1	1.78395	2 ; 4	6.59
Algorithm S2	1.77292	2 ; 4	6.59
Algorithm C1	1.63609	2 ; 4	6.59
Algorithm C2	1.49990	2 ; 4	6.59

Table 12

Univariate Analyses of Covariance for the Computation and Extension Tests for the pP-aA Comparison on S1

Computation ANOCOVA

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square	F	Prob.
Justification (Treatment)	1	351.0721	351.0721	1.6514	.2682
Groups-within-Treatment	4	850.3697	212.5924	5.0753*	.0008
Error	139	5822.364	41.8876		

Adjusted Means

Treatment pP	15.866	Treatment aA	12.351
Class 1	17.659	Class 4	11.391
Class 2	12.097	Class 5	19.920
Class 3	16.250	Class 6	11.190

Extension ANOCOVA

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square	F	Prob.
Justification (Treatment)	1	61.2116	61.2116	.7505	.4382
Groups-within-Treatment	4	326.2639	81.5660	9.9818*	.0000
Error	139	1135.831	8.1714		

Adjusted Means

Treatment pP	4.771	Treatment aA	6.239
Class 1	3.725	Class 4	4.177
Class 2	4.131	Class 5	9.080
Class 3	6.144	Class 6	7.446

*Significant at the .05 level

Table 13

Univariate Analyses of Covariance for the Computation and Extension Tests for the pP-aA Comparison on C1

Computation ANOCOVA

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square	F	Prob.
Justification(Treatment)	1	3.3430	3.3430	.0690	.7913
Groups-within-Treatment	4	193.6710	48.4178	2.6324*	.0367
Error	131	2409.496	18.393		

Adjusted Means

Treatment pP	8.061	Treatment aA	8.405
Class 1	7.977	Class 4	7.914
Class 2	7.614	Class 5	13.067
Class 3	8.319	Class 6	7.493

Extension ANOCOVA

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square	F	Prob.
Justification (Treatment)	1	93.7698	94.7698	1.1581	.3439
Groups-within-Treatment	4	323.8798	80.9699	30.3962*	.0000
Error	131	348.9602	2.664		

Adjusted Means

Treatment pP	1.606	Treatment aA	3.428
Class 1	2.029	Class 4	2.101
Class 2	1.400	Class 5	9.288
Class 3	1.393	Class 6	2.863

* Significant at the .05 level

Table 14

Univariate Analyses of Covariance for the Computation and Extension Tests for the pP-aA Comparison on S2

Computation ANOCOVA

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square	F	Prob.
Justification (Treatment)	1	1034.745	1034.745	2.3858	.1965
Groups-within-Treatment	4	1734.827	433.7068	13.9762*	.0000
Error	164	5089.227	31.0319		

Adjusted Means

Treatment pP	23.831	Treatment aA	18.765
Class 7	23.945	Class 10	23.353
Class 8	23.280	Class 11	12.682
Class 9	24.292	Class 12	21.841

Extension ANOCOVA

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square	F	Prob.
Justification (Treatment)	1	46.4612	46.4612	.2563	.6404
Groups-within-Treatment	4	725.0459	181.2615	12.7670*	.0000
Error	164	2328.421	14.197		

Adjusted Means

Treatment pP	9.709	Treatment aA	10.783
Class 7	7.175	Class 10	13.991
Class 8	11.618	Class 11	7.986
Class 9	10.552	Class 12	11.613

* Significant at the .05 level

Table 15

Univariate Analyses of Covariance for the Computation and Extension Tests for the pP-aA Comparison on C2

Computation ANOCOVA

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square	F	Prob.
Justification (Treatment)	1	5.8535	5.8535	.0194	.8646
Groups-within-Treatment	4	1205.185	301.296	29.0670*	.0000
Error	165	1710.319	10.366		

Adjusted Means

Treatment pP	8.699	Treatment aA	8.321
Class 7	6.262	Class 10	10.784
Class 8	9.008	Class 11	3.895
Class 9	10.838	Class 12	11.067

Extension ANOCOVA

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square	F	Prob.
Justification (Treatment)	1	225.7897	225.7897	1.007	.3758
Groups-within-Treatment	4	902.5420	225.6355	23.9341*	.0000
Error	165				

Adjusted Means

Treatment pP	6.441	Treatment aA	8.792
Class 7	3.512	Class 10	13.991
Class 8	7.940	Class 11	5.886
Class 9	7.921	Class 12	8.910

* Significant at the .05 level

It should be noted that although no differences among pP-aA computation or extension performances for treatment effects are significant at the .05 level, the adjusted means for the aA treatments are greater than those for the pP treatments on all four extension tests. It might also be noted that the adjusted means for the pP treatments are higher than those for the aA treatments on the two simple algorithms computation tests, but that there is almost no absolute difference between the treatment adjusted means on the complex algorithm computation tests. In no case, however, are all of the groups-within-treatments means for a given treatment greater than those for the other treatment groups, with the exception of the extension test for C1, where all algebraic groups outperformed all pattern groups. Furthermore, even using a χ^2 test for combined probabilities from the four separate tests in the case of the extension tests and the two simple algorithm computation tests, there is no assurance that the overall H_o should not be rejected at the .05 level.

The multivariate analysis of covariance model for all pA-aP comparisons was the same as that used before except that the $\tau_{j(i)h}$ term was deleted because only one group was assigned to each treatment. The four null hypotheses to correspond to the research hypotheses in Chapter 1 are of the identical form as for the pP-aA comparisons.

The overall F-ratios for the multivariate tests of equality of mean vectors for the pA-aP comparisons are listed in Table 16. It can be observed that the F-ratios for the S1 and C1 tests are significant, while those for the S2 and C2 tests are not.

Table 16

Overall F-Ratios for the Multivariate Tests of Equality of Adjusted Mean Vectors for Treatments pA-aP

	Obtained F	Degrees of Freedom	Critical F at .05 Level
Algorithm S1	8.4718*	2 ; 35	3.28
Algorithm S2	.7442	2 ; 44	.051
Algorithm C1	11.2000*	2 ; 38	3.25
Algorithm C2	.3844	2 ; 33	.051

The univariate analyses of covariance for each of the computation and extension tests, along with treatment means adjusted for the covariate, for algorithms S1, C1, S2, and C2 can be found in Tables 17,18,19, and 20, respectively.

It can be noted in these tables that where significant differences are found for the mixed justification comparisons for the four algorithms, aP subjects outperformed pA subjects in terms of adjusted treatment means.

Table 17

Univariate Analyses of Covariance for the Computation and Extension Tests for the pA-aP Comparison on S1

Computation ANOCOVA

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square	F	Prob.
Justification	1	434.3534	434.3534	16.8092*	.0003
Error	36	930.2442	25.8400		

Adjusted Means

Treatment pA	12.487	Treatment aP	20.821

Extension ANOCOVA

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square	F	Prob.
Justification	1	29.7544	29.7544	4.1178*	.0474
Error	36	260.1265	7.2257		

Adjusted Means

Treatment pA	5.289	Treatment aP	7.470

*Significant at the .05 level

Table 18

Univariate Analyses of Covariance for the Computation and Extension Tests for the pA-aP Comparison on C1

Computation ANOCOVA

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square	F	Prob.
Justification	1	229.8931	229.8931	21.2384*	.0001
Error	39	422.1512	10.8244		

Adjusted Means

Treatment pA 8.516 Treatment aP 14.544

Extension ANOCOVA

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square	F	Prob.
Justification	1	1.3414	1.3414	.3468	.5663
Error	39	150.8459	3.8678		

Adjusted Means

Treatment pA 3.155 Treatment aP 2.695

* Significant at the .05 level

Table 19

Univariate Analyses of Covariance for the Computation and Extension Tests for the pA-aP Comparison on S2

Computation ANOCOVA

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square	F	Prob.
Justification	1	3.1930	3.1930	.0539	.8027
Error	45	2667.083	59.268		

Adjusted Means

Treatment pA	19.487	Treatment aP	18.957

Extension ANOCOVA

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square	F	Prob.
Justification	1	20.5858	20.5858	1.0323	.3163
Error	45	897.3669	19.9408		

Adjusted Means

Treatment pA	8.166	Treatment aP	9.512

* Significant at the .05 level

Table 20

Univariate Analyses of Covariance for the Computation and Extension Tests for the pA-aP Comparisons on C2

Computation ANOCOVA

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square	F	Prob.
Justification	1	3.7190	3.7190	.2286	.6404
Error	34	553.2126	16.2710		

Adjusted Means

Treatment pA	8.639	Treatment aP	7.987

Extension ANOCOVA

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square	F	Prob.
Justification	1	4.3621	4.3621	.2200	.6465
Error	34	674.2760	19.8316		

Adjusted Means

Treatment pA	6.195	Treatment aP	6.899

DISCUSSION OF RESULTS

The data presented in the present research indicated no significant differences in performance among groups taught by a purely algebraic approach and those taught by a purely pattern approach on computation using several distinct algorithms which are different both with respect to the operation accomplished by the algorithm and with respect to complexity of the algorithm. Nor did significant differences occur in performance on extensions of the learned algorithms. However, consistently, students taught by an algebraic approach performed better, if not significantly better, on extension tests, than students taught by the pattern approach. There also appeared to be a trend for students taught by a pattern approach to outperform their algebraic counterparts on the simple algorithm computation tests; again, however, the performance was not significantly better.

The data further indicated that students taught by an algebraic followed by pattern approach did significantly better than students taught by a pattern followed by algebraic approach on one of the simple algorithm computation tests, one of the simple algorithm extension tests, and one of the complex algorithm computation tests. No other significant differences were found. However, the significant differences that did result must be considered in light of the confounding of teacher with treatment in this segment of the study.

Statistical hypothesis H_{01}, which predicted no significant differences in computation and extension performance for S1 for groups taught by different justification procedures, was not rejected for the pP-aA comparison with a .05 level of significance. However, an examination of

adjusted means reveals that the pP treatment mean was greater than the aA treatment mean on the computation test, while the aA treatment mean was greater than the pP treatment mean on the extension test. Statistical hypothesis H_{01} was rejected at the .05 level of significance for the pA-aP comparison. Nevertheless, in that a teacher-class factor was operating along with the treatment condition, it may well be that the significant difference cannot be attributed to the treatment.

Statistical hypothesis H_{02}, which predicted no significant differences in computation and extension performance on S2 tests for groups taught by different justification procedures, was not rejected for either the pP-aA comparison or the pA-aP comparison with a .05 level of significance. Again, an examination of adjusted means reveals that the pP treatment mean was greater than the aA treatment mean on the computation test, while the reverse was true on the extension test.

Statistical hypothesis H_{03}, which predicted no significant differences in computation and extension performance on C1 tests for groups taught by different justification procedures, was not rejected for the pP-aA comparison with a .05 level of significance, although it was rejected for the pA-aP comparison. An examination of adjusted means again indicates that the aA treatment mean was greater than the pP treatment mean on the extension test; the difference on the computation test between the pP-aA treatment means was extremely small. In terms of the pA-aP comparisons, the aP group performed significantly better than the pA group on the C1 computation test, although not on the extension test. Because the same teacher-class factor was operating along with the treatment condition here as was operating in the S1 situation, it is possible that the significant differences in both cases are due to the extraneous teacher-class factor.

Statistical hypothesis H_{04}, which predicted no significant differences in computation and extension performance on C2 tests for groups taught by different justification procedures, was not rejected for either the pP-aA comparison or the pA-aP comparison with a .05 level of significance. An examination of the adjusted means for this algorithm also shows that the aA treatment mean was greater than the pP treatment mean on the extension test, while the difference for the two groups on the computation test was extremely small.

No firm conclusions can be drawn about the relative effects of pA and aP justification procedures because of the operation of extraneous factors in the design. With one exception, in the case of C1, there was, however, a tendency for the aP groups to do better than the pA groups on the extension test, but no clear-cut trend was observable on the computation tests. In talking to the teachers using the mixed justification procedures, the writer discovered that all of the classes involved expressed a preference for learning by the algebraic approach, although none of the teachers had used exclusively one mode of justification prior to the experimental period. Furthermore, it became evident that the classes first exposed to the algebraic approach were more reticent to switch to the pattern approach than the classes started on the pattern approach were to switch to an algebraic approach. Because it was impossible to interpret the significant results from the pA-aP comparisons, it is clear that an investigation exactly along the lines of this study, but involving more groups-within-treatments, would be needed to make any reasonably certain statements about which justification procedure is optimal.

Again, no firm conclusions can be drawn about the relative effects

of pP and aA justification procedures because of the lack of power in the statistical testing of the four null hypotheses. But several things should be noted.

First, it might be recalled that the writer had originally proposed a model for research into algorithm learning in elementary school mathematics which consisted of four components: the mathematical operation accomplished by the algorithm, the complexity of the algorithm, the number form to which the algorithm applied, and the justification used in teaching the algorithm. It was the purpose of this study to determine whether the mathematical operation accomplished by an algorithm has some bearing on a preferred justification sequence. If it was found that the same justification sequence seemed to be preferred for algorithms for many different mathematical operations, the proposed simplification of the model to eliminate the component of operation accomplished by an algorithm would seem to be a reasonable alteration. In that no significant differences were found for S1, C1, S2, and C2 among the pP=aA comparisons and that, therefore, the four statistical decisions were identical, the simplification of the model for research into algorithm learning in elementary mathematics was not invalidated. Furthermore, because the trends on all four extension tests were in the same direction, namely that the algebraically taught treatment mean was greater than the pattern taught treatment mean, there seems to be an even stronger case for continuing research with the proposed model or some alteration of it. The results on the computation test also support retention of the model; pP-taught students tended to perform better on the simple algorithms, while no trend was evident for the complex algorithms, thus distinguishing appropriate justification type for simple algorithm and complex algorithm computation instruction.

Unfortunately, between-groups-within-treatments variablility was so large for each of the four algorithms among the pP-aA comparisons that statistical significance could not be attained with the small number of groups available. With one exception, that of the C1 extension test, there was no case in which all groups in one treatment outperformed all groups in the other treatment, strongly suggesting that a treatment-by-teacher interaction might be further investigated. In fact, even when covariance adjustments were made for initial performances, classes differed markedly in their post-treatment performance on all measures used. This seems still to be the case, even when allowance is made for the fact that several of the adjusted means might have been slightly misleading due to the lack of homogeneity of regression slopes within treatment groups. Since classes and teacherss are confounded, this may well mean that the success of a justification treatment may depend more on the teacher than the student. In particular, it may be that the approach teachers learn in their mathematics methods training courses, or the basic approach of the students' texts, or the teacher's personal attitudes toward teaching mathematics has an effect on the treatment with which their students learn best. Thus, the next research task may be to identify which combination of justifications are best for which teachers; that is, the next task may be to add to the model upon which this research is based-- addition of a component involving teacher preference.

If the trends indicated in this study were to persist for a larger number of groups, they would indicate that algebraically taught students generally perform better than pattern taught students, as a group, on extension tests. Although the results from the data analysis were not significant, two other considerations make this possibility all the more

likely. First, there was some adjusting of the class means because of the covariates which may not have been as appropriate for all classes. Thus, the algebraic means, after adjustment, might have been even greater than the analysis indicated, making significant differences more plausible. Second, the homogeneity of variance assumption was violated, but it was the smaller groups which showed the unusually low variance; therefore, the actual significance level of the test might have been smaller than the nominal .05 level. This result, that algebraic justification of algorithms leads to greater ability to extend them, is in accord with Scandura's (34:361) results showing that symbolically taught students generalize better than verbally taught ones. The trends also indicate that pattern-taught students perform better on simple algorithm computation, where the thread of steps is probably short enough to allow a consistent pattern to aid the student; on the complex algorithm computation tests, no consistent trend developed, seeming to indicate that the consistent pattern was of no more use in the long thread of steps necessary to reach the conclusion than was the generality of the algebraic principles. However, investigation into treatment-by-teacher interactions, student ability-by-treatment interactions, or other cognitive variable-by-treatment interactions may suggest that significant interactions, as well as significant main effects, exist. In such a case, the results of this study would not apply to any randomly selected class, but may only suggest an optimal treatment plan over the long run. In the case of a particular class, these interactions could become important.

Additionally, it would be valuable to repeat this study in a situation allowing for simultaneous comparisons of pP, aA, pA, and aP justification procedures using a groups-within-treatments design.

Chapter 5

SUMMARY AND CONCLUSIONS

THE PROBLEM

It was the purpose of this study to determine by experimental procedures whether there are any significant differences either in computational skill with an algorithm or in ability to extend that algorithm among pupils taught an algorithm by different justification procedures. The ability to extend an algorithm was defined, for the purposes of this study, to involve five skills; the abilities to:

1) shortcut the algorithm in particular cases,
2) solve equations with missing operands requiring use of the algorithm,
3) use the algorithm with number forms other than the type studied,
4) extend the algorithm to more than two operands, and
5) explain an alternate version of the algorithm.

The four types of justification procedures were : pattern, algebraic, pattern followed by algebraic, and algebraic followed by pattern. A pattern justification is one based on an analog using two-dimensional physical actions, whereas an algebraic justification is one based on the algebraic principles for rational numbers, as well as the rules of logic. Differences in performance on the criterion variables were examined for four algorithms differing in both mathematical operation accomplished by the algorithm and in complexity of the algorithm. Complexity was determined by the number of steps and processes required for the algorithm's execution.

THE FINDINGS

The results of the data analysis indicate no significant differences in the case of all four algorithms tested in the comparisons between students taught by a strictly pattern approach and students taught by a strictly algebraic approach on computation or extension ability. This lack of significance is probably due in large measure to the existence of a large between-groups-within-treatments effect for each analysis,always a problem in field research in which each class has its own teacher, and particularly a problem where classes are intact and not composed of randomly selected pupils. The result is an F-test with class variance rather than pupil variance as an error term, resulting in a statistical test with few degrees of freedom in the denominator, and hence relatively low power.

However, there is evidence to indicate that students taught by an algebraic approach, as a group, tendttoddobbetter on extension tests than their pattern taught counterparts and that students taught by a pattern method, as a group, tend to do better on simple algorithm computation tests than their algebraically taught counterparts. No trend is evident in terms of performance of groups on the complex algorithm computation tests. This evidence is based on the consistency of the relative sizes of the pattern and algebraic adjusted treatment means for all four algorithms. Furthermore, the existence of a significant between-groups-within-treatments effect indicates the strong possibility of a teacher-by-treatment interaction which might be further investigated. This is borne out by inspection of the class means within treatments; these means consistently differ quite widely.

Although significant differences were found for the algorithm S1 computation test, the algorithm C1 computation test, and the algorithm S1 extension test in favor of the algebraic followed by pattern group over the pattern followed by algebraic group, these results must be considered in light of the possibility that a Type G error is being made--that a group difference confounded with a treatment difference is what is being measured. No other significant differences were found for the mixed justification comparisons. There did appear, however, to be a trend, although a nonsignificant one, indicating that, in general, the algebraic followed by pattern students perform better than the pattern followed by algebraic taught students on the extension tests. This result, along with the trends previously indicated, and having regard to the limitations of the study, may indicate that an algebraic introduction to an algorithm leads to better generalizing ability with it on the part of the students; however, lack of significance in the results makes this suggestion highly tentative. Because algebraic versus algebraic followed by pattern comparisons were not possible in this study, there is no way to determine from the experimental data the effectiveness of pattern reinforcement of the algebraic introduction.

Finally, the data indicate the plausibility of a model for research into algorithm learning in elementary mathematics which incorporates these two dimensions: type of justification provided for the algorithm and complexity of the algorithm, as useful determinants of student performance on computation and extension tests based on that algorithm. That is, student performance for a given justification type was consistent for the two simple algorithms examined in the study, as well as for the two complex algorithms examined. Thus, there seems reason to believe that,

for a given complexity of algorithm, there may be main effects in the study of children's algorithm performance due to justification approach provided for any algorithm, and, further, that this may be true for any algorithm, regradless of the mathematical operation accomplished by the algorithm or the number form to which it applies. This is not to say that interactions of justification type with other variables do not exist.

IMPLICATIONS

As a consequence of the findings of this study, several inferences can be drawn.

It is clear that, in the intermediate grades, the algebraic justification of at least some algorithms is feasible both for teachers and students. Further , in that all teachers using the algebraic approach expressed to the writer a favorable attitude toward it and since the teachers using the mixed justification approach preferred the algebraic section of the work, there is some reason to consider teaching of the use of algebraic justification in preparing prospective arithmetic teachers and in preparing instructional materials for students.

Evidence is presented to support the likelihood that an algebraic approach, or at least an algebraic followed by pattern approach, rather than a pattern only approach, leads to greater development of students' generalizing skill with mathematical algorithms. It was suggested earlier that this result might have been expected in light of Scandura's (37) results that more general rules, if learned, can be used to apply to a more extensive set of questions than more specific rules; algebraic principles might be considered a more general method of justification than a set of patterns based on many physical analogies. This is not to say,

however, that algebraic justification is inherently superior to pattern justification; rather, one might merely suspect that it could be. It is equally feasible that other teacher or class factors, such as teacher preference for a particular justification type, had a bearing on this result. However, the existence or possible existence of a main effect based on type of justification provided, for whatever reason, cannot be overlooked.

Therefore, because the skill of extending algorithms is a valuable one both in developing sophisticated ideas about the algorithm and in giving it a sense of coherence, as well as in preparing students for future work in mathematics, an algebraic justification may be most useful even for those teachers whose primary objective is the development of computational competence. This result potentially affects all mathematical algorithm learning in the elementary school, including the learning of the four basic algorithms for addition, subtraction, multiplication, and division.

Evidence is also presented to support the likelihood that a strictly pattern approach in the teaching of simple algorithms, rather than a strictly algebraic approach, leads to greater immediate development of computational skill. From the experimental data, no conclusions can be drawn about the relative merits of algebraic and pattern justifications in the performance of complex algorithm computation. Thus, it would seem that, until research is conducted to refute it, one might consider the use of an algebraic followed by pattern justification approach as a preferred one for simple algorithm instruction, and a strictly algebraic approach as a preferred one for complex algorithm instruction if the principal aim is to maximize performance in ability to compute with the

algorithm and ability to extend it. Again, the same caution must be taken in explaining the reason for this result as must be taken in explaining the extension result.

The existence of significant between-groups-within-treatments effects for all four algorithms examined suggests the existence of teacher-by-treatment interactions. The most likely contributors to this interaction are the teacher's previous preference in justification approach and his mathematical knowledge. In particular, examination of each of five variables and their interactions with treatment may lead to research conclusions with immediate classroom instruction applicability. These variables are: the teacher's stated preference for algebraic or pattern justification, the basic approach the teacher had learned in his teacher training mathematics methodology course, the basic approach taken by the students' text from which the teacher currently teaches, the number of mathematics courses previously taken by the teacher, and the teacher's attitude toward teaching arithmetic. Similarly, the possible existence of interactions between treatment and other variables, such as student's general ability, or student's particular mathematical ability, or student's field dependence or field independence, could be investigated.

Some of the more important implications of this study can also be asserted from considering the model behind it. The results of the study seem to indicate that differences in ability of students to extend algorithms may be accounted for as completely by the type of justification provided in instruction, as by this factor in conjunction with information about the degree of complexity of the algorithm and the operation accomplished by it. If this can be further verified with replicate studies, it seems

that the proposed model for this study might be reduced to even fewer components. The model originally incorporated four components-- the type of mathematical operation accomplished by the algorithm, the complexity of the algorithm, the number form to which the algorithm applies, and the type of justification provided in instruction for the algorithm. Because, however, a difference was found between simple and complex algorithm instruction in terms of resulting computational ability for students learning by the same justification approach, it seems unlikely that the simplicity/complexity component can be eliminated from the model. Thus, it is perhaps the mathematical operation accomplished by an algorithm or the number form to which it applies, if not both, which may have the least effect on determining the most useful type of justification to be employed in instruction.

It may also be true that a model such as the one proposed, or a simplified version of it, may apply to research into areas of mathematics other than algorithm instruction, as, for example, concept development. Certainly it is possible to define algebraic and pattern rationales that may aid in concept development. For example, an algebraic concept rationale might lead to the acquisition of the concept of $\frac{a}{b}$ as $a \times \frac{1}{b}$, where the corresponding pattern rationale might lead to the acquisition of the concept of $\frac{a}{b}$ as 'a' regions of a whole initially sectioned into 'b' equivalent sections. It is also possible to define simple and complex concepts. For example, a simple concept might be a definition, such as the concept of a factor as being a number which divides another whole number, while a complex concept might be one, such as that of addition, which requires the continual extension of an initial idea about the concept to encompass more and more situations of greater and greater variety. For

the concept of addition, this extension involves inclusion at first of only whole numbers, but later, rational numbers, and then all real numbers, and the extension also involves the growing variety of verbal problems requiring the use of addition for solution by the students. The work of Henderson (19), Rector (32), Shumway (41), and Smith (42) in the area of concept development provide starting points in this area. It might be particularly noted that if it were found that many algebraic concepts can be acquired by elementary school students, it could have implications for the mathematics of grades 7 and 8; at present, much of the mathematics of these grades is concerned with the "algebrization" of previously learned algorithms and previously developed concepts.

The results of this study also imply that computational ability should probably be treated independently of extension ability, and so a teacher must make judgments about the relative weights to put on these two objectives, particularly with simple algorithms, before choosing a justification approach. Probably some kind of taxonomy of abilities is implied by the differences between this study's outcomes for computational and extension criterion scores. That is, computational ability may be considered as a low-level cognitive ability, whereas ability to extend may be considered a relatively high level cognitive ability. Refinement of the taxonomy of abilities of which computational ability and ability to extend are members might shed even further light on the variables operating within the model for research into algorithm learning.

LIMITATIONS OF THIS STUDY

The limitations of the study result primarily from the fact that a vast problem is being examined in one individual study. Because arith-

metic algorithms in general, rather than only one particular algorithm, are under consideration, it was necessary to sample from among a considerable set of possible algorithms. However, due to physical limitations, this sampling was minimal; only two algorithms in each of the categories under consideration were studied. Thus, it would be useful to apply this same design to other algorithms, varying both in the mathematical operation accomplished by them and the number form to which they apply, to see if similar results can be established. If so, there is every reason to believe that the trends indicated in this study for main effects are valid. It would also be useful to apply the same design using younger subjects with algorithms appropriate to these subjects. It is often claimed that such young subjects are not ready for an abstract, algebraic approach.

It might be noted that because of the need to remove as much as possible the effects of earlier algorithm learning so as to determine the actual treatment effects, it was necessary in this study to look at four somewhat unusual algorithms, rather than the standard addition, subtraction, multiplication, and division algorithms for whole numbers. However, the writer believes that the results apply to these standard algorithms as well, while recognizing that it would take extended study of children in mathematics classes throughout several grade levels to decide whether the hypothesis of generality is reasonable.

Because intact groups are the ones generally available to experimenters in applied research in education and because such groups were, in fact, used in this study, it become important to replicate the design of the study with other classes. Certainly, for the comparisons between the pattern followed by algebraic and algebraic followed by pattern groups, where only one class was used in each treatment condition and treatment

effects were confounded with group effects, it is necessary to replicate the conditions with other classes. But even with the purely pattern and purely algebraic comparisons, where an attempt was made to control for extraneous factors which might have one effect on one class but a different effect on others, it would be useful to determine whether the trends indicated here are observed with other classes in another setting. In any case, a study which allows simultaneous examination of all four justification types would be desirable.

Another constraint on interpretation of the results of the study is the lack of precision inherent in the definition of simplicity and complexity. Although it is relatively easy to categorize some algorithms as simple or complex, it becomes quite difficult to order algorithms that seem to be somewhere in the middle of the complexity continuum. For this reason, it is difficult to extend the results of the study to all algorithms in the elementary mathematics program.

Similarly, the preciseness of the definition of the ability to extend an algorithm also puts a constraint on interpretation of the results. The definition of extension, for the purposes of this study, encompassed five precisely specified abilities. Frequently, interest in higher level understandings that a student might have extends beyond these five abilities or includes only some of them. For this reason, caution must be observed in discussing the role of algebraic justification in students' ability to generalize an algorithm, unless the meaning of generalization is as indicated in this study.

SUGGESTIONS FOR FUTURE RESEARCH

The suggestions for further research fall into several main categories: extensions of the study to examine aptitude-treatment

interactions, extension of the study over time, extension of the study over algorithms sampled, extension of the study over age levels, extension of the study over criterion variables examined, and extension or modification of the underlying model, including allowance for teacher-by-treatment interaction studies.

The purpose of this study was to examine main effects of justification type provided for an algorithm as it related to students' ensuing ability to compute with and extend that algorithm. However, the inferences drawn here from the results of the data analysis of teacher by treatment interactions suggest the importance of an examination of treatment interactions with other variables. Some of these variables would stem from an examination of aspects of the teacher's knowledge of and attitude toward mathematics teaching. Specifically, the teacher's stated preference for a particular approach, his experience with a particular justification approach in the past, and his attitude toward teaching and learning mathematics may interact with justification treatment in determining student performance on computation and extension tests for that algorithm.

In addition, certain student variables might interact with treatment in prescribing a preferred approach for algorithm instruction. It may be that students with particularly high mathematics ability, as determined by previous achievement in arithmetic, may perform better with an algebraic approach, while students with less mathematical ability may perform better if taught by a pattern approach. This estimate of mathematical ability may, however, to a large extent, reflect the student's previous exposure to algebraic concepts. Thus, a measure of this previous ability must be carefully considered.

Because of the importance of the physical analogy to the pattern justification approach, the possibility that the cognitive style variable of field dependence might interact with the type of justification in leading to the student's best performance is also reasonable. The criterion is the student's ability to differentiate an object from a surrounding field. It might be that field dependent students do not do as well with an algebraic approach as with a pattern approach, whereas the field independent student might do as well with either approach.

Other cognitive or affective variables might also be examined for interactions with the treatment variables. To get at these interactions, examination of these variables for high proficiency pattern students and low proficiency algebraic students to see if any common factor exists might be a starting point. Similar analyses for low proficiency pattern and high proficiency algebraic, low proficiency pattern followed by algebraic and high proficiency algebraic followed by pattern, and high proficiency pattern followed by algebraic and low proficiency algebraic followed by pattern students might be undertaken.

Although there are only slight indications that one or another justification approach may be superior in promoting the immediate ability to compute with an algorithm or the immediate ability to extend it, it may be that one or the other approach of justification is considerably more valuable over long term instruction. For example, it may be that algebraic justification is more valuable in terms of the ability of students to compute with an algorithm or extend it, but only after a reasonably long exposure to this justification approach. The only way to determine this would be to compare groups taught for at least a year, perhaps more, exlusively with one justification technique. That is, every

algorithm taught would be taught using the same approach, be it a pattern approach, an algebraic approach, a pattern followed by algebraic approach, or an algebraic followed by pattern approach. Results would be gathered for ability to compute with each algorithm learned and to extend each algorithm learned. Improvement over time could then also be considered. The results of such a study might have many implication for the role of juniro secondary mathematics in grades seven and eight. A trend study might be desirable here.

Another important question still unanswered by this study is whether the results are applicable at younger age levels than the eleven year old level of most grade five students. For example, it is possible that primary students perform better both in the areas of computation and extension if taught by the pattern approach, whereas older elementary students do not. This might be discovered by replication of a study such as this one at other elementary school grade levels.

It is also desirable that the design of the study be replicated with other algorithms, both simple ones and complex ones. The algorithms studied should include both the standard computational algorithms for addition, subtraction, multiplication, and division, and other algorithms which arise in the curriculum. This would lead to information about the place of the component, number form to which the algorithm applies, in the proposed model for algorithm learning research.

Certainly the criterion variable "ability to extend the algorithm" employed in this study can be further categorized, perhaps even into the five separate components of the writer's definition of extension, and the effects of justification type can be examined for these categorized criterion variables. For example, it may be that algebraic justification

leads to a greater ability to shortcut the algorithm in special cases, but that a pattern justification leads to a greater ability to solve equations with missing operands requiring use of the algorithm. Similarly, a new definition of "extending" might lead to examination of transfer effects of different methods of justification on the learning of new algorithms.

Reexamination of the proposed model for algorithm instruction leads to an even greater number of suggestions for further research. First, a further examination of the component of justification is possible, particularly in extending the definition of pattern justification to include use of three-dimensional physical analogy. The effects of this newly defined justification type might then be examined using the design of the current study. Similarly, other mixtures of pattern and algebraic justifications might be researched.

Second, other combinations of components of the model can be examined; for example, the interaction of justification type with number form to which the algorithm applies can be examined by looking at several algorithms accomplishing the same operations for different number forms; for example, addition for whole numbers, addition for fraction, and addition for decimals.

Finally, there remains much research to be done in an attempt to extend the proposed model. As was suggested previously, the model lends itself to examination of justification of mathematical concepts, as well as mathematical algorithms. Pattern and algebraic justifications could be employed with simple and complex concepts to investigate the effects of justification type on the students' ability to solve word problems based on the concept or perhaps the effects on the students' ability to explain that concept. The model might also incorporate a

teacher preference component.

Only by pursuing these kinds of studies will it be clear whether the model for research into algorithm learning, or modifications of it, proposed in this study is a useful one.

REFERENCES

1. Ausubel, David P. The Psychology of Meaningful Verbal Learning. New York: Grune and Stratton, 1963.

2. Bates, Thomas. "The Road to Inverse and Multiply," The Arithmetic Teacher, XV (April, 1968), 347-354.

3. Becker, Jerry P. "Research in Mathematics Education: The Role of Theory and of Aptitude-Treatment-Interactions," Journal for Research in Mathematics Education, I (January, 1970), 19-28.

4. Bidwell, James K. "A Physical Model for Factoring Quadratic Polynomials," The Mathematics Teacher, LXV (March, 1972), 201-206.

5. Brueckner, Leo J., and Mary Elwell. "Reliability of Diagnosis of Error in Multiplication of Fractions," Journal of Educational Research, XXVI (November, 1932), 175-185.

6. Brumfiel, Charles. "Mathematical Systems and their Relationships to the Real World," The Arithmetic Teacher, XVII (November, 1970), 563-573.

7. Bruner, Jerome S. The Process of Education. New York: Vintage Books, 1963.

8. Collier, Myrtle. "Learning to Multiply Fractions," School Science, and Mathematics, XXII (April, 1922), 324-329.

9. Cooney, Thomas J., and Kenneth B. Henderson. "Structuring Knowledge in Mathematics and Science," School, Science, and Mathematics, LXXII (May, 1972), 425-430.

10. Dilly, Clyde Alan. "A Comparison of Two Methods of Teaching Long Division," Dissertation Abstracts, XXXI: 2248-A, 1970.

11. Edmonds, George F. "An Intuitive Approach to Square Numbers," The Mathematics Teacher, LXIII (February, 1970), 113-120.

12. Fennema, Elizabeth H. "The Relative Effectiveness of a Symbolic and a Concrete Model in Learning a Selected Mathematical Principle," Journal for Research in Mathematics Education, III(Novemebr,1972), 233-238.

13. Frederiksen, John G. "Square Root +," The Arithmetic Teacher, XVI (November, 1969), 549-555.

14. Gagne, Robert M. The Conditions of Learning, 2nd edition. New York: Holt, Rinehart, and Winston, 1970.

15. Gray, Roland F. "An Experimental Study of Introductory Multiplication," Dissertation Abstracts, XXV: 6432, 1964.

16. Green, Gerladine A. "A Comparison of Two Approaches, Area and Finding a Part of, and Two Instructional Materials, Diagrams and Manipulative Aids, on Multiplication of Fractional Numbers in Grade 5," Dissertation Abstracts, XXXI: 676-A, 1970.

17. Hall, Kenneth Dwight. "An Experimental Study of Two Methods of Instruction for Mastering Multiplication Facts at the Third-Grade Level," Dissertation Abstracts, XXVIII: 390-A, 1967.

18. Hartung, Maurice L. "The Role of Experience in the Learning of Mathematics," The Arithmetic Teacher, XVIII (May, 1971), 279-295.

19. Henderson, Kenneth B. "Concepts," National Council of Teachers of Mathematics: 33rd Yearbook- The Teaching of Secondary School Mathematics, 1970, 166-195.

20. Hinkelman, Emmet Arthur. "A Study of the Principles Governing Fractions Known by 5th and 6th Grade Children," Educational Administration and Supervision, XLII (March, 1956), 153-161.

21. John, Lenore. "The Effect of Using the Long Division Form in Teaching Division by One-Digit Numbers," Elementary School Journal, XXX (May, 1930), 675-692.

22. Kaplan, Abraham. The Conduct of Inquiry. San Francisco: Chandler Publishing Company, 1964.

23. Kolesnik, Thoedore S. "Illustrating the Multiplication and Division of Common Fractions," The Arithmetic Teacher, X (May, 1963), 268-271.

24. Lachman, Roy. "The Model in Theory Construction," Psychological Review, LXVII (1960), 113-129.

25. Lindquist, Everet Franklin. Design and Analysis of Experiments in Psychology and Education. Boston: Houghton Mifflin Co, 1953.

26. Lord, Fredric M., and Melvin R. Novick. Statistical Thoeries of Mental Test Scores. Reading, Mass.: Addison-Wesley Publishing, 1968.

27. Mac Schell, Leo. "Two Aspects of Introductory Multiplication: the Array and the Distributive Property," Dissertation Abstracts, XXV: 5161, 1965.

28. Matthews, Warren N. "Teaching Comparison of Common Fractions," The Arithmetic Teacher, XV (March, 1968), 271-273.

29. Myers, Jerome L. Fundamentals of Experimental Design. Boston: Allyn and Bacon, 1966.

30. Nelson, Diane, and Marvin N. Nelson. "Pegboard Multiplication of a Fraction by a Fraction," The Arithmetic Teacher, XVI (February, 1969), 142-144.

31. Phenix, Philip H. Philosophy of Education. New York: Henry Holt and Co., 1958.

32. Rector, Robert Earl. "The Relative Effectiveness of Four Strategies for Teaching Mathematical Concepts," Dissertation Abstracts, XXIX: 520-A, 1968.

33. Romberg, Thomas A. "A Note on Multiplying Fractions," The Arithmetic Teacher, XV (March, 1968), 263-270.

34. Scandura, Joseph M. "Learning Verbal and Symbolic Statements of Mathematical Rules," Journal of Educational Psychology, LVII (1967), 356-364.

35. _____. "New Directions for Theory and Research in Rule Learning," Acta Psychologica, XXVIII (1968), 301-321.

36. _____. "Precision in Research in Mathematics Learning: The Emerging Field of Psychomathematics," Journal of Research in Science Teaching, IV (1966), 253-273.

37. _____, and John Durnin. "Extra-Scope Transfer in Learning Mathematical Strategies," Journal of Educational Psychology, LIX (1968), 350-354.

38. _____, Ernest Woodward, and Frank Lee. "Rule Generality and Consistency in Mathematics Learning," American Educational Research Journal, IV (1967), 303-319.

39. Schrankler, William. "A Study of the Effectiveness of Four Methods for Teaching Multiplication of Whole Numbers in Grade Four," Dissertation Abstracts, XXVII: 4055, 1967.

40. Scott, Lloyd. "A Study of Teaching Division through the Use of Two Algorisms," School, Science, and Mathematics, LXIII (December, 1963), 739-752.

41. Shumway, Richard J. "Negative Instances and Mathematical Concept Formation: A Preliminary Study," Journal for Research in Mathematics Education, II (May, 1971), 218-227.

42. Smith, B. Othanel, Milton Meiux, Jerrold Coombs, Graham Nuthall, and Robert Precians. A Study of the Strategies of Teaching. Urbana: Bureau of Educational Research, University of Illinois, 1967.

43. Spooner, George A., and Richard L. Mentzer. Introduction to Number Systems. Englewood Cliffs: Prentice-Hall, Inc., 1968.

44. Tatsuoka, Maurice M. Multivariate Analysis. New York: John Wiley and Sons, Inc., 1971.

45. Vance, J.H. "The Effects of a Mathematics Laboratory Program in Grades 7 and 8- an Experimental Study." Unpublished doctoral dissertation, University of Alberta, 1969.

46. Willer, David. Scientific Sociology. Englewood Cliffs: Prentice-Hall, Inc., 1967.

47. Wolfe, Richard Edgar. "Strategies of Justification Used in the Classroom by Teachers of Secondary School Mathematics," School, Science, and Mathematics, LXXII (April, 1972), 334-338.

48. Wright, David J. "Groups and Experimental Units in Educational Research." Papre presented at the American Educational Research Association Annual Meeting, Los Angeles, 1969.

APPENDIX A

INSTRUCTIONAL MATERIAL

Multiplication of a Mixed Number by a Fraction

Outline for approach p:

> WE WILL NOW LEARN A WAY TO MULTIPLY A MIXED NUMBER BY A WHOLE NUMBER. REMEMBER THAT A MIXED NUMBER IS ONE LIKE $2\frac{1}{2}$ WHERE WE HAVE A WHOLE PART AND A FRACTIONAL PART LESS THAN 1. WE ALL REMEMBER THAT WHOLE NUMBERS ARE NUMBERS LIKE 1,2,3.
>
> BUT TO DO THIS, WE MUST FIRST DO SOME REVIEWING.
>
> I WILL CALL OUT SOME MULTIPLICATION QUESTIONS.

Ask the following questions:

6 x 2
7 x 5
9 x 8
2 x 8
9 x 6

If there is much difficulty in answering these questions on the part of the class as a whole, then draw diagrams to illustrate the problems.

For example, to illustrate:

6 x 2

```
x x
x x
x x
x x
x x
x x
```

5 x 5

```
x x x x x
x x x x x
x x x x x
x x x x x
x x x x x
```

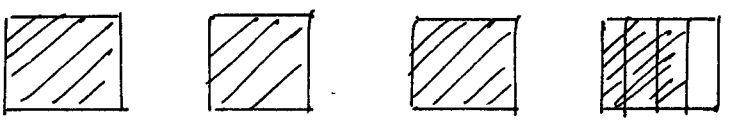
Notice that the first number represents the number of rows and the second number represents the number of columns.

BECAUSE WE ARE DEALING WITH FRACTIONS, WE HAD BETTER REMEMBER SOME OF OUR FRACTION WORK, AS WELL.

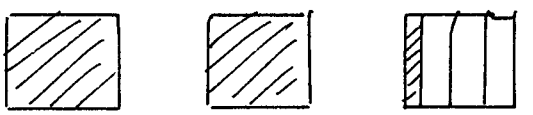
Go over items 15-19 on the pretest. Read each item, and ask for an answer. Discuss, in particular, that the name of a shaded region depends on the number of parts of the same size into which the figure is drawn and the number of these parts that are shaded in.

Then review mixed numbers in the following manner:

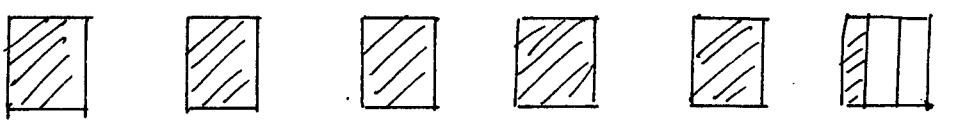
SUPPOSE I HAVE $3\frac{3}{4}$. THAT MEANS THAT I HAVE 3 WHOLES AND $\frac{3}{4}$ OF ANOTHER WHOLE. THAT IS, I HAVE A PICTURE LIKE:



IF I HAVE $2\frac{1}{4}$, I DRAW A PICTURE LIKE:



IF I HAVE $5\frac{1}{3}$, I DRAW A PICTURE LIKE:

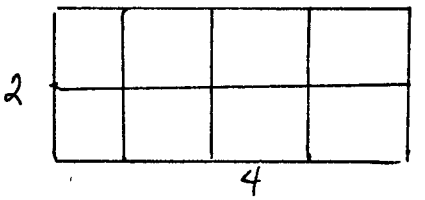


Have the students think about how to draw diagrams representing $6\frac{1}{2}$; $2\frac{3}{5}$; $1\frac{1}{3}$.

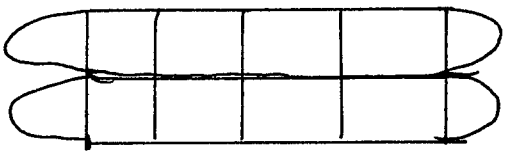
Ask one student to come to the board to show each of these to the class.

THERE IS ONE MORE PIECE OF REVIEWING TO BE DONE. WE WOULD LIKE TO REVIEW SOME IDEAS ABOUT AREA. REMEMBER THAT THE AREA OF A RECTANGLE TELLS ME THE NUMBER OF UNIT SQUARES THAT FIT ONTO THE RECTANGLE. FOR EXAMPLE, THE AREA OF A RECTANGLE WHICH IS 2 IN. BY 4 IN.

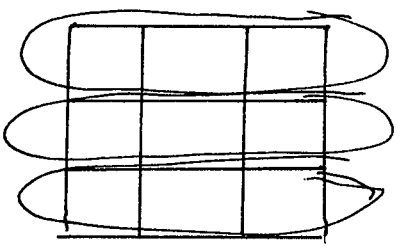
IS 8 SQ. IN. SINCE I CAN DRAW A DIAGRAM LIKE:



NOTICE THAT I COULD LOOK AT THIS DIAGRAM AS COUNTING 2 GROUPS OF 4 SQUARES EACH, WHICH IS ANOTHER WAY OF SAYING 2 x 4:

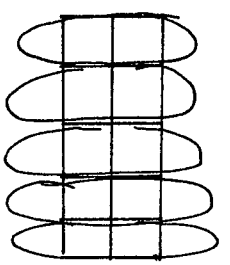


LET'S DO THE SAME FOR A RECTANGLE WHICH IS 3 IN. BY 3 IN. I CAN DRAW:



AND I SEE THAT THE AREA IS 3 x 3 = 9 SQ. IN.

IF MY RECTANGLE WERE 5 IN. BY 2 IN., I WOULD DRAW:



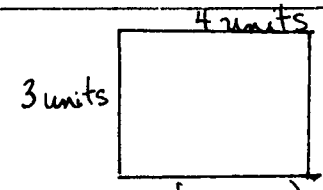
AND THE AREA IS 10 SQ. IN. = 5 x 2 SQ. IN.

Ask the students to consider what the areas of rectangles with dimensions 5 by 8, 3 by 9, and 8 by 4 would be. Explain that they may always go back to a diagram to find the solutions. Call on three individuals to get the answers to these three questions.

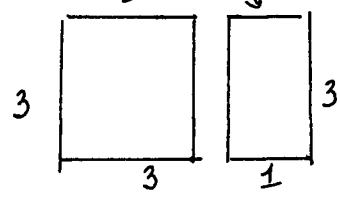
Finally, ask whether the area of a rectangle changes if it is broken into

smaller pieces and their areas totalled to get the full original area. If there is any hesitation here, discuss to the degree you deem necessary.

Draw the rectangle below on the board:



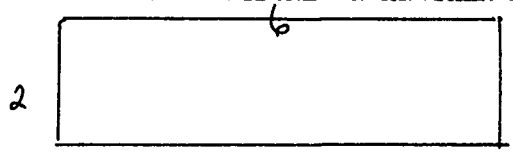
SUPPOSE I BREAK THIS RECTANGLE INTO 2 PIECES, LIKE SO:



NOTICE THAT TO FIND THE AREA OF THE FIRST RECTANGLE, I CAN JUST TOTAL UP THE AREAS OF THE SMALLER RECTANGLES:

$3 \times 3 + 3 \times 1 = 3 \times 4.$

LET'S FOLLOW THE SAME PROCEDURE ON ANOTHER RECTANGLE. I START WITH:



HOW CAN I BREAK IT UP TO GET THE SAME TOTAL AREA?

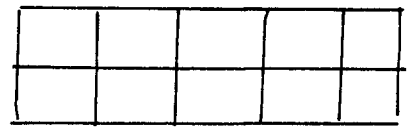
Accept any suggested break up and check the hypothesis that the areas of the two smaller pieces do total the area of the larger piece.

** Here the student will learn how to find products of whole numbers and fractions with numerators of one, by finding the areas of appropriate rectangles, such as one with dimensions 6 by $\frac{1}{3}$ or 4 by $\frac{1}{5}$. **

I WANT TO BE ABLE TO FIND ANSWERS TO MULTIPLICATION QUESTIONS WHERE ONE OF THE NUMBERS TO BE MULTIPLIED IS A FRACTION. BEFORE, WE NOTICED THAT AREAS OF RECTANGLES COULD BE FOUND BY MULTIPLYING

LENGTH BY WIDTH. REVERSING THIS IDEA, WE MIGHT SUPPOSE THAT MULTIPLYING TWO NUMBERS IS THE SAME AS FINDING AREAS OF RECTANGLES WITH THESE NUMBERS AS SIDE LENGTHS.

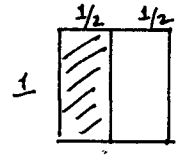
FOR EXAMPLE, THE ANSWER TO 2×5 COULD BE FOUND BY FINDING THE AREA OF A RECTANGLE WITH WIDTH 2 AND LENGTH 5.



NOW LET'S APPLY THIS SAME IDEA TO FRACTIONS.

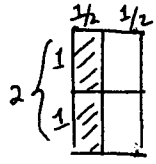
SUPPOSE I WANT TO MULTIPLY A WHOLE NUMBER BY $\frac{1}{2}$. THAT MEANS I WANT TO FIND THE AREA OF A RECTANGLE WITH DIMENSIONS- THAT WHOLE NUMBER BY $\frac{1}{2}$.

SUPPOSE MY RECTANGLE HAS DIMENSIONS 1 BY $\frac{1}{2}$. I WOULD DRAW:



NOTICE THAT I STARTED WITH A 1 BY 1 SQUARE, SO IT HAS AREA 1. THEREFORE, SINCE I JUST USE $\frac{1}{2}$ OF THE RECTANGLE, THE AREA WOULD BE $\frac{1}{2}$, OR ONE HALF, WHICH CAN BE WRITTEN AS $1 \times \frac{1}{2}$ BECAUSE OF THE CONNECTION BETWEEN MULTIPLICATION AND AREA.

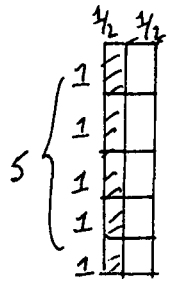
IF I WERE TRYING TO FIND THE SOLUTION TO $2 \times \frac{1}{2}$, I WOULD DRAW A RECTANGLE OF DIMENSIONS 2 BY $\frac{1}{2}$, LIKE SO:



THEN I MIGHT SAY, I HAVE 2 HALVES, OR $\frac{2}{2}$, OR $2 \times \frac{1}{2}$.

LET'S FIND THE AREA OF A 5 BY $\frac{1}{2}$ RECTANGLE TO FIND $5 \times \frac{1}{2}$.

I DRAW A RECTANGLE WITH DIMENSIONS 5 BY $\frac{1}{2}$, LIKE SO:

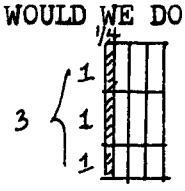


THEN I COUNT: I HAVE 5 HALVES, OR $\frac{5}{2}$, OR $5 \times \frac{1}{2}$.

Suggested end of day 1.

Take special note to point out to the students that we are using multiplication notation to represent the area of rectangles just as we did when we found that the areas of rectangles with whole number sides could be found by taking the products of the dimensions.

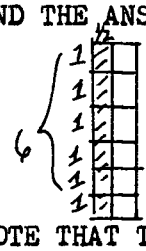
WHAT WOULD WE DO TO FIND $3 \times \frac{1}{4}$? WE WOULD DRAW A RECTANGLE LIKE:



AND NOTE THAT THE AREA IS 3 FOURTHS = $\frac{3}{4}$ $= 3 \times \frac{1}{4}$.

THEREFORE, $3 \times \frac{1}{4} = \frac{3}{4}$.

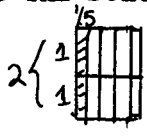
TO FIND THE ANSWER TO $6 \times \frac{1}{2}$, WE WOULD DRAW:



AND NOTE THAT THE AREA IS 6 HALVES, OR $\frac{6}{2}$, OR $6 \times \frac{1}{2}$.

THEREFORE, $6 \times \frac{1}{2} = \frac{6}{2}$.

TO FIND THE SOLUTION TO $2 \times \frac{1}{5}$, I WOULD DRAW A RECTANGLE:



AND NOTE THAT THE AREA SHADED IN IS 2 FIFTHS, OR $\frac{2}{5}$, OR $2 \times \frac{1}{5}$.

THEREFORE, $2 \times \frac{1}{5} = \frac{2}{5}$.

Ask the students to consider the answers to:

$6 \times \frac{1}{5}$

$7 \times \frac{1}{6}$

$8 \times \frac{1}{3}$

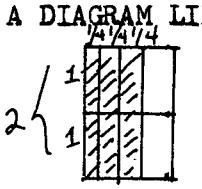
Explain that they may always go back to the diagram to find the solution. Have three individuals come to the board to show the answers to these three questions.

** Here the student will learn how to find products of whole numbers and proper fractions (fractions with numerators smaller than the denominators), by finding the areas of appropriate rectangles, such as one with dimensions 6 by $\frac{2}{3}$ or one with dimensions 4 by $\frac{3}{5}$. **

NOW THAT WE KNOW HOW TO FIND AREAS OF RECTANGLES WHERE ONE OF THE DIMENSIONS IS A FRACTION LIKE $\frac{1}{2}$ OR $\frac{1}{3}$ OR $\frac{1}{4}$, LET'S TRY TO EXTEND THIS TO FRACTIONS WHERE THE NUMERATOR IS NOT 1, LIKE $\frac{2}{3}$ OR $\frac{4}{5}$ OR $\frac{3}{4}$.

SUPPOSE WE ARE TRYING TO FIND $2 \times \frac{3}{4}$. THAT MEANS WE WANT TO FIND THE AREA OF A RECTANGLE WITH DIMENSIONS 2 BY $\frac{3}{4}$. SO, JUST AS BEFORE, WE DRAW A RECTANGLE OF DIMENSIONS 2 BY 1, BUT ONLY MARK IN THE AREA IN WHICH WE ARE INTERESTED.

SO WE DRAW A DIAGRAM LIKE THE FOLLOWING ONE:



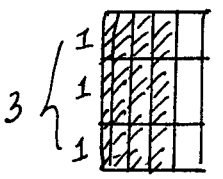
NOTICE THAT WE ARE NOW TRYING TO FIND THE AREA OF A RECTANGLE WHICH IS OF DIMENSIONS 2 BY $\frac{3}{4}$. SO WE HAVE 3 COLUMNS NOW, EACH OF AREA $2 \times \frac{1}{4}$. I WOULD LIKE TO RENAME THIS AREA:

I HAVE $\frac{2}{4} + \frac{2}{4} + \frac{2}{4} = \frac{6}{4} = \frac{3 \times 2}{4}$ SHADED IN.

NOTICE THAT IF I COUNT FOURTHS, I HAVE 2×3 FOURTHS, OR 3×2 FOURTHS. SO, $2 \times \frac{3}{4} = \frac{2 \times 3}{4} = \frac{6}{4}$.

LET'S USE THIS PROCEDURE TO FIND $3 \times \frac{4}{5}$.

I FIND THE AREA OF A RECTANGLE OF DIMENSIONS 3 BY $\frac{4}{5}$, SO I DRAW A RECTANGLE OF DIMENSIONS 3 BY 1, AND SHADE IN THE $\frac{4}{5}$, LIKE:



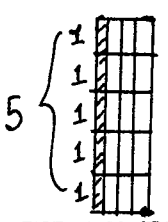
THEN MY AREA IS $\frac{3}{5} + \frac{3}{5} + \frac{3}{5} + \frac{3}{5} = \frac{12}{5} = \frac{3 \times 4}{5}$.

THEREFORE, THE PRODUCT $3 \times \frac{4}{5} = \frac{12}{5}$.

NOTICE THAT IN MY DIAGRAM, 12 RECTANGLES, EACH OF AREA $\frac{1}{5}$ (FOUR COLUMNS EACH WITH THREE FIFTHS) WERE SHADED IN.

SUPPOSE I WERE TRYING TO FIND $5 \times \frac{3}{4}$. WHAT WOULD YOU PREDICT THAT THE ANSWER IS? ACCORDING TO THE PATTERN FROM THE PAST TWO PROBLEMS, IT WOULD SEEM THAT THE ANSWER SHOULD BE (3×5) FOURTHS, OR $\frac{15}{4}$.

IF WE DRAW THE DIAGRAM, WE SEE THAT OUR PREDICTION WAS CORRECT. WE HAVE 3 COLUMNS EACH OF AREA $(5 \times \frac{1}{4})$ AND A TOTAL OF 3×5 RECTANGLES EACH OF AREA $\frac{1}{4}$ SHADED IN.



CONSIDER THE ANSWERS TO THE FOLLOWING 3 QUESTIONS:

$8 \times \frac{3}{7}$

$5 \times \frac{3}{4}$

$3 \times \frac{7}{8}$

Ask one student to explain the answer to each of these questions by first giving the answer and then drawing a diagram to verify it on the board.

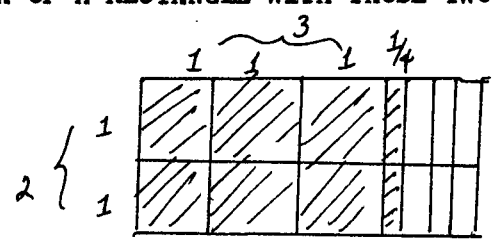
Suggested end of day 2.

** Here the student will learn how to find products of whole numbers and mixed numbers by finding the areas of appropriate rectangles with dimensions such as 6 by $2\frac{1}{7}$, or 3 by $3\frac{1}{4}$. **

NOW THAT WE KNOW HOW TO FIND PRODUCTS OF WHOLE NUMBERS AND WHOLE NUMBERS AND WHOLE NUMBERS AND FRACTIONS, LET'S COMBINE THESE TO FIND PRODUCTS OF WHOLE NUMBERS AND MIXED NUMBERS, SUCH AS $2 \times 3\frac{1}{2}$.

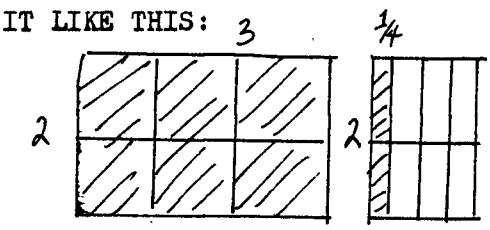
SUPPOSE I WANT THE ANSWER TO THE MULTIPLICATION QUESTION: $2 \times 3\frac{1}{4}$. I REALIZE THAT TO FIND THE ANSWER, I CAN DRAW A RECTANGLE WITH DIMENSIONS 2 BY $3\frac{1}{4}$. AS BEFORE, MULTIPLYING TWO NUMBERS IS THE SAME AS FINDING THE AREA OF A RECTANGLE WITH THOSE TWO NUMBERS AS SIDE LENGTHS.

THEREFORE, I DRAW:



IT WOULD BE EASY IF I COULD SPLIT UP THE PROBLEM INTO FINDING THE AREAS OF PARTS OF RECTANGLES, THOUGH. HOW MIGHT I SPLIT THIS RECTANGLE TO MAKE THE COMPUTATION EASIER?

ONE WAY WOULD BE TO SPLIT IT LIKE THIS:



THEN THE AREA OF THE 2 BY 3 RECTANGLE WOULD BE $2 \times 3 = 6$ AND THE AREA OF THE 2 BY $\frac{1}{4}$ RECTANGLE WOULD BE $2 \times \frac{1}{4} = \frac{2}{4}$.

I MIGHT WRITE:

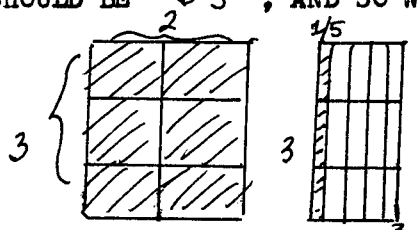
$$2 \times 3\tfrac{1}{4} = 2 \times 3 + 2 \times \tfrac{1}{4} = 6 + \tfrac{2}{4} = 6\tfrac{2}{4} .$$

NOTICE THAT THE WHOLE NUMBER PART OF THE ANSWER CAME FROM MULTIPLYING THE WHOLE NUMBER 2 BY THE WHOLE NUMBER 3 AND THE FRACTIONAL PART BY MULTIPLYING THE WHOLE NUMBER 2 BY THE FRACTION $\frac{1}{4}$.

LET'S TRY THIS PROCEDURE ON A FEW MORE PROBLEMS.

SUPPOSE WE WANT TO FIND $3 \times 2\frac{1}{5}$.

WHAT DO YOU PREDICT THE ANSWER WILL BE? ACCORDING TO WHAT WE GUESSED BEFORE, THE WHOLE NUMBER PART OF THE ANSWER SHOULD COME FROM 3×2 AND THE FRACTIONAL PART FROM $3 \times \frac{1}{5}$. THEREFORE OUR ANSWER SHOULD BE $6\frac{3}{5}$, AND SO WE DRAW THE DIAGRAM TO CHECK:

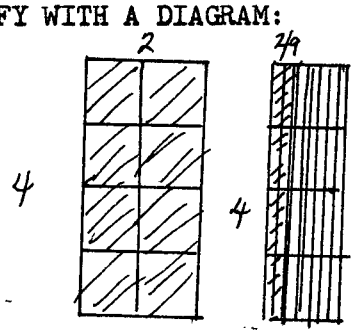


AND WE DO HAVE A TOTAL AREA OF $6 + \frac{3}{5} = 6\frac{3}{5}$.

WHAT WOULD BE THE ANSWER TO $4 \times 2\frac{2}{9}$? IT SHOULD BE WHAT?

Expect the answer $8\frac{8}{9}$.

LET'S VERIFY WITH A DIAGRAM:



Have the students work on:

$3 \times 4\frac{2}{7}$
$2 \times 2\frac{3}{8}$
$4 \times 3\frac{1}{5}$

Ask three students to come to the board and show their answers with a diagrammatic explanation for each.

Hand students worksheet 1 to complete.

Suggested end of day 3.

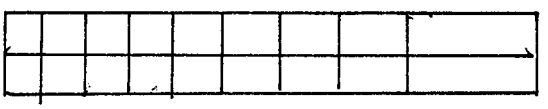
Multiplication of a Mixed Number by a Fraction

Outline for approach P:

NOW THAT WE KNOW HOW TO MULTIPLY A MIXED NUMBER BY A WHOLE NUMBER, WE ARE GOING TO LEARN HOW TO MULTIPLY A MIXED NUMBER BY A FRACTION.

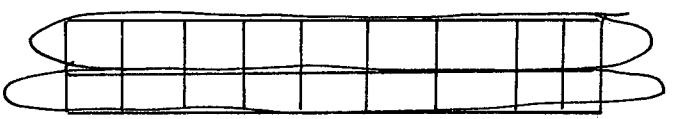
BUT, AGAIN, WE JUST WANT TO REMIND OURSELVES OF A FEW THINGS BEFORE WE GO ON.

LET'S REVIEW QUICKLY AGAIN SOME IDEAS ABOUT AREA. REMEMBER THAT THE AREA OF A RECTANGLE TELLS ME THE NUMBER OF SQUARE UNITS THAT FIT INTO THE RECTANGLE. FOR EXAMPLE, TO FIND THE AREA OF A RECTANGLE WHICH IS 2 UNITS BY 9 UNITS, I DRAW:

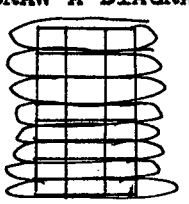


AND SEE THAT THE AREA IS 18 SQ. UNITS.

NOTICE THAT I CAN VIEW THIS AS 2 x 9 SQ. UNITS LIKE SO:

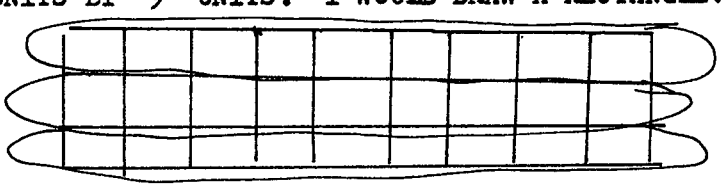


SO, IF I HAVE A RECTANGLE WHICH HAS DIMENSIONS 8 UNITS BY 3 UNITS, I CAN DRAW A DIAGRAM SUCH AS:



AND VERIFY THAT THE AREA IS 8 x 3 SQ. UNITS.

LET'S TRY ONE MORE AREA QUESTION. SUPPOSE MY RECTANGLE HAS DIMENSIONS OF 3 UNITS BY 9 UNITS. I WOULD DRAW A RECTANGLE:



AND THE AREA IS INDEED 3 x 9 SQ. UNITS.

Ask the students to consider the following products as area questions:

9 x 3

7 x 2

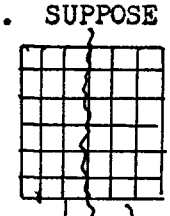
5 x 8

Ask three students to come to the board to diagrammatically explain their answers.

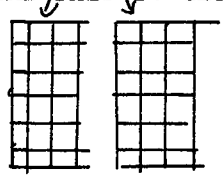
WHAT HAPPENS IF I TRY TO FIND THE AREA OF A RECTANGLE BY BREAKING IT UP INTO SMALLER RECTANGLES AND TOTALLING UP THE AREAS OF THESE? WILL THE TOTAL FROM THESE RECTANGLES BE THE SAME AS THE AREA OF THE ORIGINAL RECTANGLE? WHAT'S YOUR GUESS?

Expect some kind of answer here.

LET'S CHECK THIS. SUPPOSE I START WITH A 6 BY 6 RECTANGLE.



I COULD CUT IT LIKE THIS TO GET TWO SMALLER RECTANGLES:



WHAT'S THE AREA OF THE ORIGINAL RECTANGLE? WHAT'S THE TOTAL AREA OF THE TWO SMALLER RECTANGLES? DOES THIS SEEM REASONABLE?

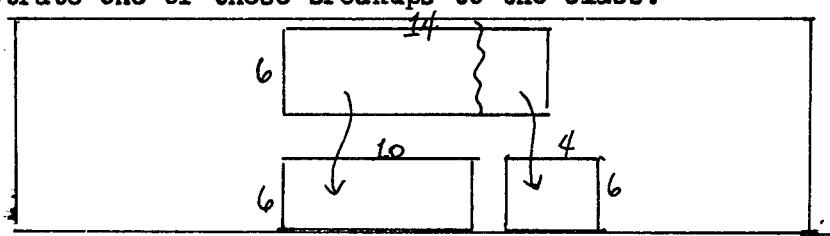
Expect the answers for the three areas and the sum of 36,18 and 18.

Also expect the students to believe in the reasonability of the premise.

> HOW WOULD YOU USE THIS PROCEDURE TO HELP YOU TO FIND THE AREA OF A RECTANGLE OF DIMENSIONS 6 BY 14 ?

Allow the students to suggest any breakup of the 14 so as to minimize the difficulty of the computation. For example, to break up to a 6 by 10 and a 6 by 4 situation, or a 6 by 7 and 6 by 7.

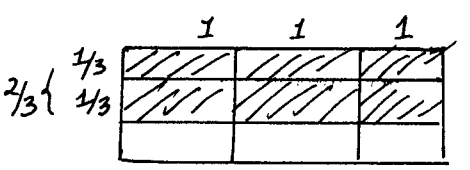
Demonstrate one of these breakups to the class.



NOW LET'S GET BACK TO MULTIPLYING WITH FRACTIONS.

EARLIER WE LEARNED HOW TO FIND AREAS OF RECTANGLES WITH DIMENSIONS LIKE 3 BY $\frac{2}{3}$. THIS WAS TO SOLVE PROBLEMS LIKE $3 \times \frac{2}{3}$. NOW WE WOULD LIKE TO SOLVE PROBLEMS WHERE THE NUMBERS ARE TURNED AROUND, PROBLEMS LIKE $\frac{2}{3} \times 3$.

JUST AS BEFORE, LET'S START WITH THE FIRST NUMBER STANDING FOR THE DIMENSION OF THE "VERTICAL" SIDE OF THE RECTANGLE AND THE SECOND NUMBER THE DIMENSION OF THE "HORIZONTAL" SIDE. SO, TO SHOW $\frac{2}{3}$ BY 3, I DRAW:



NOTICE, I CAN START BY SETTING UP RECTANGLES WHICH HAVE A DIMENSION OF 1 FOR THE FRACTIONAL LENGTH AND JUST USE PART OF THE RECTANGLE.

TO FIND THIS PRODUCT, I FIND THE AREA OF THE SHADED REGION OF THE RECTANGLE. NOTE THAT I HAVE 3 COLUMNS EACH OF AREA $\frac{2}{3} \times 1 = \frac{2}{3}$. AND SO I HAVE A TOTAL AREA OF $3 \times \frac{2}{3} = \frac{6}{3}$; THIS IS THE SAME ANSWER AS $\frac{2}{3} \times 3$, WHICH WE DID BEFORE.

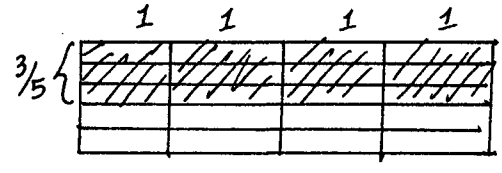
WE CAN CHECK ON THE DIAGRAM.

I HAVE (2×3) RECTANGLES, EACH OF AREA $\frac{1}{3}$, SO I HAVE A TOTAL AREA OF $6 \times \frac{1}{3} = \frac{6}{3}$, AS WE SAID.

LET'S FIND THE PRODUCT: $\frac{3}{5} \times 4$.

I WANT TO FIND THE AREA OF A RECTANGLE WITH DIMENSIONS $\frac{3}{5}$ BY 4.

I DRAW:

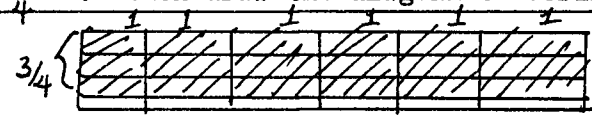


NOTICE THAT I HAVE 4 COLUMNS EACH OF AREA $\frac{3}{5} \times 1 = \frac{3}{5}$, SO I HAVE A TOTAL AREA OF $4 \times \frac{3}{5} = \frac{12}{5}$.

THEREFORE, $\frac{3}{5} \times 4 = \frac{12}{5}$.

WHAT WOULD BE THE PRODUCT: $\frac{3}{4} \times 6$?

Expect the answer $\frac{18}{4}$. Then draw the diagram to verify.



I HAVE 6 COLUMNS EACH OF AREA $\frac{3}{4}$, SO I HAVE A TOTAL AREA OF $\frac{18}{4}$.

Ask the students to find these products:

$\frac{3}{7} \times 2$

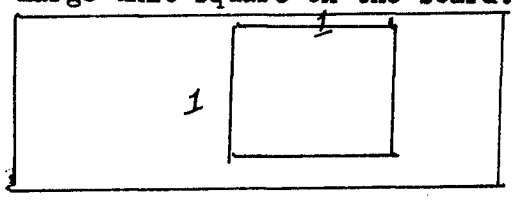
$\frac{4}{8} \times 3$

$\frac{1}{5} \times 4$

Explain that they may always refer back to a diagram to make certain. Have three students come to the board to diagrammtically explain their answers.

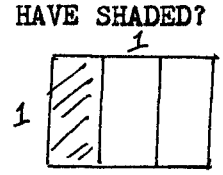
** Here the students will learn to find the products of unit fractions by finding the area of appropriate rectangles; he will do problems such as $\frac{1}{2} \times \frac{1}{3}$ or $\frac{1}{5} \times \frac{1}{6}$. **

Draw a large unit square on the board.



Label the sides as indicated.

WHAT HAPPENS IF I CUT THE SQUARE LIKE THIS? WHAT ARE THE DIMENSIONS OF THE RECTANGLES I HAVE SHADED? NOTICE THAT THE ORIGINAL SQUARE HAD AREA 1.

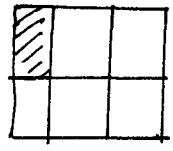


Expect the answer 1 by $\frac{1}{3}$.

WHAT IS THE AREA OF EACH OF THESE RECTANGLES?

Expect the answer $1/3$.

WHAT HAPPENS IF I ALSO CUT THE SQUARE THIS WAY?



WHAT ARE THE DIMENSIONS OF THE NEW SMALLER RECTANGLES FORMED?

Expect the answer $\frac{1}{2}$ by $\frac{1}{3}$.

NOW THE AREA OF EACH OF THESE SMALLER RECTANGLES CAN BE REPRESENTED BY A MULTIPLICATION STATEMENT, NAMELY $\frac{1}{2} \times \frac{1}{3}$.
WHAT IS THE AREA OF EACH OF THESE SMALLER RECTANGLES? NOTICE THAT I KNOW THE AREA OF THE ENTIRE SQUARE WAS 1 SQUARE UNIT, SO THE AREA OF EACH OF THESE RECTANGLES MUST BE A FRACTION OF THAT AND THE FRACTION IS DETERMINED BY THE NUMBER OF RECTANGLES THERE ARE IN THE FIGURE.

HOW MANY RECTANGLES ARE THERE OF DIMENSIONS $\frac{1}{2}$ BY $\frac{1}{3}$?

Expect the answer: 6.

THEREFORE, EACH OF THE RECTANGLES HAS WHAT AREA?

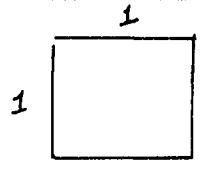
Expect the area: $\frac{1}{6}$.

SINCE WE ALREADY SAID THIS AREA COULD BE REPRESENTED IN A MULTIPLICATION WAY AS $\frac{1}{2} \times \frac{1}{3}$, THAT MEANS

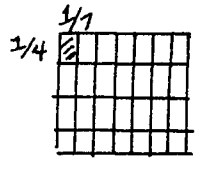
$$\underbrace{\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}}$$.

HOW WOULD WE FIND $\frac{1}{4} \times \frac{1}{7}$?

WE DRAW A RECTANGLE WITH DIMENSIONS 1 BY 1, LIKE SO:



AND CUT:



HOW MANY SMALL RECTANGLES ARE THERE IN THE UNIT SQUARE? WHAT IS THE TOTAL AREA OF EACH?

Expect the answers 28 and $\frac{1}{28}$, respectively.

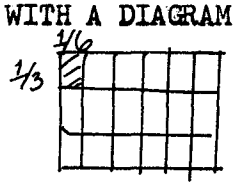
SINCE EACH AREA CAN ALSO BE REPRESENTED AS $\frac{1}{4} \times \frac{1}{7}$, THAT MEANS

THAT $\frac{1}{4} \times \frac{1}{7} = \frac{1}{28}$.

WHAT DO YOU THINK $\frac{1}{3} \times \frac{1}{6}$ WILL BE?

Expect the answer $\frac{1}{18}$.

LET'S CHECK WITH A DIAGRAM. I CAN DRAW:

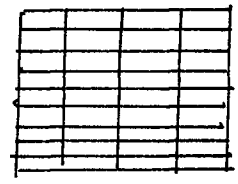


AND EACH RECTANGLE, WHICH IS OF DIMENSIONS $\frac{1}{3}$ BY $\frac{1}{6}$ DOES HAVE AREA $\frac{1}{18}$ SINCE THERE ARE 18 OF THEM IN THE UNIT SQUARE.

WHAT ABOUT $\frac{1}{9} \times \frac{1}{4}$?

Expect the answer $\frac{1}{36}$.

LET'S CHECK. I CAN DRAW:



AND I SEE THAT THERE ARE 36 SMALL RECTANGLES, SO EACH HAS AREA $\frac{1}{36}$. BUT SINCE THE DIMENSIONS OF EACH ARE $\frac{1}{9}$ BY $\frac{1}{4}$, THEN

$$\frac{1}{9} \times \frac{1}{4} = \frac{1}{36} .$$

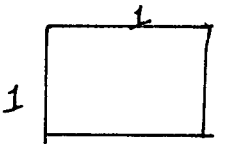
DO YOU NOTICE THAT IN EACH OF THESE PROBLEMS, IF I MULTIPLY $\frac{1}{\square}$ BY $\frac{1}{\triangle}$, I GET A FIGURE WITH $\square \times \triangle$ SMALL RECTANGLES IN IT, EACH OF AREA, THEREFORE, $\frac{1}{\square \times \triangle}$? CHECK BACK TO OUR DIAGRAMS FOR $\frac{1}{2} \times \frac{1}{3}$, $\frac{1}{4} \times \frac{1}{7}$, AND $\frac{1}{9} \times \frac{1}{4}$.

Suggested end of day 4.

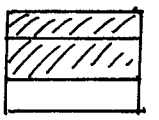
** Here the student will learn how to find products of any two proper fractions by finding the areas of appropriate rectangles; he will do problems

such as $\frac{4}{5} \times \frac{2}{3}$ or $\frac{2}{5} \times \frac{3}{7}$. **

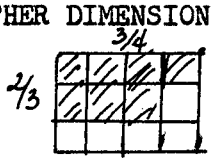
SUPPOSE WE WANT TO MULTIPLY $\frac{2}{3}$ BY $\frac{3}{4}$. WE KNOW THAT WE CAN FIND THE AREA OF A RECTANGLE WITH THESE DIMENSIONS TO SOLVE OUR PROBLEM. SO LET'S START WITH A SQUARE AGAIN.



IF WE WANT ONE OF THE DIMENSIONS TO BE $\frac{2}{3}$, WE CUT IT SO:



IF WE WANT THE OTHER DIMENSION TO BE $\frac{3}{4}$, WE CUT IT SO:



THEN THE AREA OF THE SHADED REGION IS THE AREA WE WANT.

HOW MANY OF THE SMALL RECTANGLES ARE SHADED IN?

Expect the answer 6.

NOTICE THAT THE NUMBER IS 2 x 3 SINCE THERE ARE 2 ROWS EACH OF 3 RECTANGLES SHADED IN.

WHAT IS THE AREA OF EACH OF THESE RECTANGLES?

Expect the answer $\frac{1}{12}$.

WE ALREADY LEARNED THAT SINCE THERE ARE 3 x 4 OF THESE RECTANGLES ALTOGETHER, EACH HAS AREA $\frac{1}{12}$.

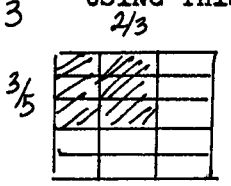
THEREFORE, WE HAVE 2 x 3 PIECES EACH OF AREA $\frac{1}{12}$, SO WE HAVE A TOTAL AREA OF $\frac{6}{12}$.

NOTICE THAT THE NUMERATOR OF MY ANSWER TELLS ME THE NUMBER OF

SHADED PIECES AND THE DENOMINATOR TELLS ME THE TOTAL NUMBER OF PIECES IN THE FIGURE.

LET'S FIND $\frac{3}{5} \times \frac{2}{3}$ USING THIS APPROACH:

I DRAW:



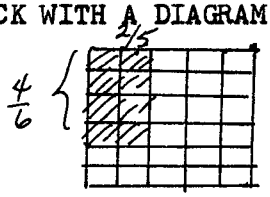
THEN THE NUMBER OF SHADED SQUARES IS 3×2. EACH OF THESE SQUARES HAS AREA $\frac{1}{15}$ SINCE THERE ARE 5×3 OF THEM IN THE ENTIRE UNIT SQUARE. SO I HAVE A TOTAL AREA OF $\frac{6}{15}$ SHADED IN.

THEREFORE, $\frac{3}{5} \times \frac{2}{3} = \frac{6}{15}$.

WHAT WOULD $\frac{4}{6} \times \frac{2}{5}$ BE?

Expect the answer: $\frac{8}{30}$.

LET'S CHECK WITH A DIAGRAM:



WE DO HAVE 4×2 SQUARES EACH OF AREA $\frac{1}{30}$ SHADED IN, SO A TOTAL AREA OF $\frac{8}{30}$.

NOTICE THAT THE NUMERATOR OF MY PRODUCT IS THE PRODUCT OF MY NUMERATORS ($8 = 4 \times 2$) JUST AS THE DENOMINATOR OF MY PRODUCT WAS THE PRODUCT OF MY DENOMINATORS ($30 = 6 \times 5$).

Ask the students to find the following products:

$\frac{3}{4} \times \frac{1}{6}$

$\frac{2}{5} \times \frac{1}{3}$

$\frac{4}{7} \times \frac{2}{5}$

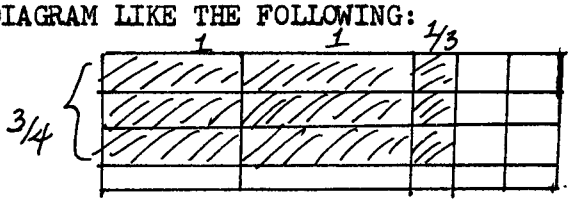
Have three students come to the board to diagram their answers.

** Here the student will learn to find products of fractions and mixed numbers by finding the areas of appropriate rectangles with dimensions like $\frac{1}{2}$ by $3\frac{1}{3}$ or $\frac{2}{3}$ by $4\frac{1}{5}$. **

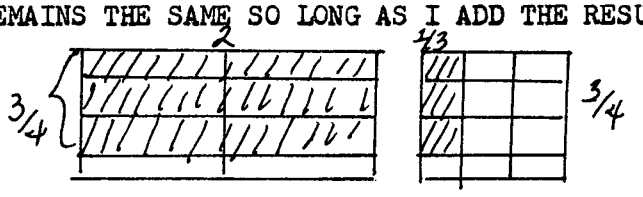
NOW THAT WE KNOW HOW TO FIND AREAS OF RECTANGLES LIKE $\frac{3}{4}$ BY 2 AND $\frac{3}{4}$ BY $\frac{1}{3}$, WE ARE READY TO FIND AREAS OF RECTANGLES LIKE $\frac{3}{4}$ BY $2\frac{1}{3}$ IN ORDER TO FIND PRODUCTS LIKE $\frac{3}{4} \times 2\frac{1}{3}$.

SUPPOSE I WANT TO FIND THE SOLUTION TO: $\frac{3}{4} \times 2\frac{1}{3}$.

I CAN DRAW A DIAGRAM LIKE THE FOLLOWING:



RECALLING THE REVIEW DONE EARLIER ABOUT SPLITTING UP RECTANGLES WITHOUT CHANGING AREA, I REALIZE THAT I CAN SPLIT THIS RECTANGLE INTO TWO PARTS SO AS TO MAKE THE COMPUTATION EASIER. ONE RECTANGLE WILL HAVE DIMENSIONS $\frac{3}{4}$ BY 2 AND THE OTHER DIMENSIONS $\frac{3}{4}$ BY $\frac{1}{3}$ AND THE AREA REMAINS THE SAME SO LONG AS I ADD THE RESULTS.



I CAN EASILY FIND THE AREAS OF EACH OF THESE SMALLER RECTANGLES. THE AREA OF THE FIRST RECTANGLE OF DIMENSIONS $\frac{3}{4}$ BY 2 I GET BY MULTIPLYING $\frac{3}{4}$ BY 2.

I KNOW THAT $\frac{3}{4} \times 2 = \frac{6}{4}$ WHICH I CAN VERIFY WITH THE DIAGRAM.

TO GET THE AREA OF THE OTHER RECTANGLE WITH DIMENSIONS $\frac{3}{4}$ BY $\frac{1}{3}$, I MULTIPLY $\frac{3}{4}$ BY $\frac{1}{3}$ AND GET $\frac{3}{4} \times \frac{1}{3} = \frac{3}{12}$ WHICH I CAN ALSO VERIFY WITH A DIAGRAM.

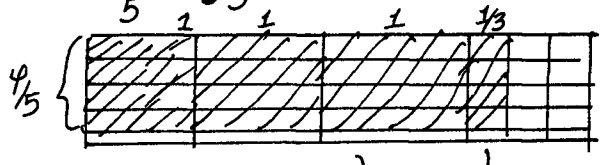
SO I CAN TOTAL THEM TO GET THE AREA OF THE ORIGINAL RECTANGLE AND

I FIND THAT

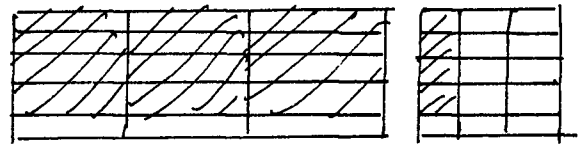
$$\frac{3}{4} \times 2\frac{1}{3} = \frac{6}{4} + \frac{3}{12} .$$

WE WILL LEAVE OUR ANSWERS IN THIS FORM TO SHOW SEPARATELY THE FRACTION TIMES THE WHOLE NUMBER PART AND THE FRACTION TIMES THE FRACTION PART.

LET'S FIND $\frac{4}{5} \times 3\frac{1}{3}$ USING AREAS. I CAN DRAW:



WHICH I CAN SPLIT LIKE SO:



THE AREA OF THE FIRST RECTANGLE IS $\frac{4}{5} \times 3 = \frac{12}{5}$.

THE AREA OF THE SECOND RECTANGLE IS $\frac{4}{5} \times \frac{1}{3} = \frac{4}{15}$.

SO MY TOTAL AREA, AND THE PRODUCT $\frac{4}{5} \times 3\frac{1}{3}$ IS $\frac{12}{5} + \frac{4}{15}$.

THAT IS,

$$\frac{4}{5} \times 3\frac{1}{3} = \frac{4}{5} \times (3 + \frac{1}{3}) = \frac{12}{5} + \frac{4}{15} .$$

Ask the students to find these products:

$\frac{3}{4} \times 2\frac{2}{3}$

$\frac{4}{5} \times 3\frac{1}{5}$

$\frac{6}{9} \times 1\frac{5}{6}$

$\frac{4}{7} \times 7\frac{1}{3}$

Ask four students to come to the board and diagrammatically explain their answers. Suggest strongly that they do their breaking of the rectangles so as to get a fraction by a whole and a fraction by a fraction for their pieces.

Suggested end of day 5.

Multiplication of a Mixed Number by a Fraction

Outline for approach a:

WE WILL NOW LEARN A WAY TO MULTIPLY A MIXED NUMBER BY A WHOLE NUMBER. REMEMBER THAT A MIXED NUMBER IS ONE LIKE $2\frac{1}{2}$ WHERE WE HAVE A WHOLE NUMBER PART AND A FRACTIONAL PART LESS THAN 1. WE ALL REMEMBER THAT WHOLE NUMBERS ARE NUMBERS LIKE 1,2,3.

BUT TO DO THIS, WE MUST FIRST DO SOME REVIEWING.
I WILL CALL OUT SOME MULTIPLICATION QUESTIONS.

Ask the following questions:

6 x 2
7 x 5
9 x 8
2 x 8
9 x 6

If there is much difficulty in answering these questions on the part of the class as a whole, then apply an argument involving the distributive principle like the following one:

For example, to illustrate:

$$6 \times 2 = 3 \times 2 + 3 \times 2 = 6 + 6 = 12$$

$$5 \times 5 = 2 \times 5 + 2 \times 5 + 1 \times 5 = 10 + 10 + 5 = 25 .$$

Do <u>not</u> use diagrams to illustrate these.

If the student has trouble in breaking up the first number into only two addends, like 6 into 3+3, allow him to use more addends which are smaller, such as 6 as 2+2+2.

BECAUSE WE ARE DEALING WITH FRACTIONS, WE HAD BETTER REMEMBER SOME OF OUR FRACTION WORK, AS WELL.

Go over items 20-25 on the pretest. Read each item, and ask for an answer. Discuss, in particular, that a fraction can be renamed in terms of a multiplication statement: e.g.

$$\frac{4}{7} = \frac{1}{7} + \frac{1}{7} + \frac{1}{7} + \frac{1}{7} = 4 \times \frac{1}{7}$$

$$\frac{3}{6} = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 3 \times \frac{1}{6}$$

Then review mixed numbers in the following manner:

SUPPOSE I HAVE $3\frac{3}{4}$. THAT MEANS THAT I HAVE 3 WHOLES AND $\frac{3}{4}$ OF ANOTHER WHOLE. THAT IS, I CAN REWRITE $3\frac{3}{4} = 3 + \frac{3}{4}$.
IF I HAVE $2\frac{1}{4}$, I CAN REWRITE THIS AS $2 + \frac{1}{4}$.
IF I HAVE $5\frac{1}{3}$, I CAN REWRITE THIS AS $5 + \frac{1}{3}$.

Have the students write addition expressions to represent

$6\frac{1}{2}$; $2\frac{3}{5}$; $1\frac{3}{5}$.

Ask one student to answer aloud each of these questions.

THERE ARE ONLY ONE OR TWO MORE THINGS TO BE REVIEWED.
ONE OF THESE IS THE SPECIAL PROPERTY OF MULTIPLICATION BY 1.

Go over items 31-34 on the pretest. In particular, have the students note that any number times one is that number, even if that number is a fraction.

ANOTHER IMPORTANT PROPERTY OF MULTIPLICATION, BESIDES MULTIPLYING BY ONE, IS THE COMBINATION OF MULTIPLICATION AND ADDITION TO MAKE MULTIPLICATION EASIER.

SUPPOSE I AM TRYING TO FIND THE ANSWER TO 3×14. ONE WAY IS TO FIND $3 \times 7 + 3 \times 7$, WHICH IS MUCH EASIER TO DO.

THEN $3 \times 14 = 3 \times 7 + 3 \times 7$

$= 21 + 21$

$= 42.$

OR

$3 \times 14 = (3 \times 4) + (3 \times 4) + (3 \times 6)$

$= 12 + 12 + 18$

$= 42.$

WHAT WOULD BE A WAY TO FIND 8×16 BY FINDING AND ADDING SMALLER PRODUCTS?

Accept any suggested breakup of 16 which would make the problem easier, for example: $8 \times 10 + 8 \times 6$ or $8 \times 8 + 8 \times 8$.

Go over items 38-40 on the pretest. Read each item, and ask for an answer. Point out the use of the distributive principle in each case, for example, in item 38, the (42+38) was broken up.

** Here the student will learn how to find products of whole numbers and fractions with numerators of one by appropriately renaming the fractions; he will do problems such as $6 \times \frac{1}{3}$ or $4 \times \frac{1}{5}$. **

I WANT TO BE ABLE TO FIND ANSWERS TO MULTIPLICATION QUESTIONS WHERE ONE OF THE NUMBERS TO BE MULTIPLED IS A FRACTION. SUPPOSE I WANT TO MULTIPLY A WHOLE NUMBER BY $\frac{1}{2}$. THAT MEANS I WANT TO FIND ANOTHER NAME FOR THAT WHOLE NUMBER TIMES $\frac{1}{2}$.

SUPPOSE THE PROBLEM IS $1 \times \frac{1}{2}$. I KNOW THAT ONE TIMES ANYTHING IS THAT THING ITSELF, SO

$$\underbrace{1 \times \frac{1}{2} = \frac{1}{2}}.$$

IF I WERE TRYING TO FIND THE SOLUTION TO $2 \times \frac{1}{2}$, I WOULD WRITE:

$2 \times \frac{1}{2} = \frac{1}{2} + \frac{1}{2} = \frac{2}{2}$, SO

$2 \times \frac{1}{2} = \frac{2}{2}$.

LET'S FIND $5 \times \frac{1}{2}$.

$5 \times \frac{1}{2} = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{5}{2}$, SO

$5 \times \frac{1}{2} = \frac{5}{2}$.

Take special note to point out to the students that we are using multiplication notation as repeated addition and renaming fractions accordingly. Especially point out that this is no different than what we did with whole numbers, for example, $3 \times 4 = 4+4+4$.

Suggested end of day 1.

WHAT WOULD WE DO TO FIND $3 \times \frac{1}{4}$? WE WOULD REWRITE THIS AS:

$3 \times \frac{1}{4} = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$.

THEREFORE, $3 \times \frac{1}{4} = \frac{3}{4}$.

TO FIND THE ANSWER TO $6 \times \frac{1}{2}$, WE WOULD REWRITE THIS AS:

$6 \times \frac{1}{2} = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{6}{2}$

THEREFORE, $6 \times \frac{1}{2} = \frac{6}{2}$.

TO FIND THE SOLUTION TO $2 \times \frac{1}{5}$, WE WOULD REWRITE THIS AS:

$2 \times \frac{1}{5} = \frac{1}{5} + \frac{1}{5}$

THEREFORE, $2 \times \frac{1}{5} = \frac{2}{5}$.

Make sure that the students see that in each case, the addition can be performed to get the resulting fraction.

Ask the students to consider the answers to:

$6 \times \frac{1}{5}$

$7 \times \frac{1}{6}$

$8 \times 1/3$

Explain that they may always go back to the addition problem to find the solution. Have three individuals give their answers to these three questions. Do <u>not</u> use diagrams to show the answers.

** Here the student will learn how to find products of whole numbers and proper fractions (fractions with numerators smaller than the denominators) again by renaming the product as a fraction; he will do questions such as $6 \times \frac{2}{3}$ or $4 \times \frac{3}{5}$. **

NOW THAT WE KNOW HOW TO FIND PRODUCTS WHERE ONE OF THE NUMBERS TO BE MULTIPLIED IS A FRACTION SUCH AS $\frac{1}{2}$ OR $\frac{1}{3}$ OR $\frac{1}{4}$, LET'S TRY TO EXTEND THIS TO FRACTIONS WHERE THE NUMERATOR IS NOT ONE, SUCH AS $\frac{2}{3}$ OR $\frac{4}{5}$ OR $\frac{3}{4}$.

SUPPOSE WE ARE TRYING TO FIND, FOR EXAMPLE, $2 \times \frac{3}{4}$. THAT MEANS THAT WE WANT TO RENAME $2 \times \frac{3}{4}$ AS A FRACTION.

BUT $2 \times \frac{3}{4} = \frac{3}{4} + \frac{3}{4} = \frac{3+3}{4} = \frac{2\times 3}{4} = \frac{6}{4}$.

THEREFORE, $2 \times \frac{3}{4} = \frac{2\times 3}{4} = \frac{6}{4}$.

NOTICE I COULD GO BACK TO: $2 \times \frac{3}{4} = (\frac{1}{4} + \frac{1}{4} + \frac{1}{4}) + (\frac{1}{4} + \frac{1}{4} + \frac{1}{4})$

LET'S USE THIS PROCEDURE TO FIND $3 \times \frac{4}{5}$.

I REWRITE:

$$3 \times \frac{4}{5} = \frac{4}{5} + \frac{4}{5} + \frac{4}{5} = \frac{4+4+4}{5} = \frac{3\times 4}{5} = \frac{12}{5} .$$

THEREFORE, $3 \times \frac{4}{5} = \frac{3\times 4}{5} = \frac{12}{5}$.

AGAIN I COULD GO BACK TO ADDING FRACTIONS WITH NUMERATORS OF ONE TO VERIFY:

$$3 \times \frac{4}{5} = 3 \times (\frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5}) = \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5}$$

SUPPOSE I WERE TRYING TO FIND $5 \times \frac{3}{4}$. WHAT WOULD YOU PREDICT THAT THE ANSWER IS? ACCORDING TO THE PATTERN FROM THE PAST TWO PROBLEMS, IT WOULD SEEM THAT THE ANSWER SHOULD BE 3×5 FOURTHS, OR $\frac{15}{4}$. IF WE DO THE ADDITION TO CHECK, WE SEE THAT OUR PREDICTION WAS CORRECT. WE HAVE:

$$5 \times \frac{3}{4} = \frac{3}{4} + \frac{3}{4} + \frac{3}{4} + \frac{3}{4} + \frac{3}{4} = \frac{5 \times 3}{4} = \frac{15}{4} .$$

CONSIDER THE ANSWERS TO THE FOLLOWING 3 QUESTIONS:

$8 \times \frac{3}{7}$

$5 \times \frac{3}{4}$

$3 \times \frac{7}{8}$

Ask one student to explain the answer to each of these questions by first giving the answer and then writing the addition statement to verify.

Suggested end of day 2.

** Here the student will learn how to find products of whole numbers and mixed numbers by renaming the mixed numbers as sums and using the distributive principle of multiplication over addition; he will do problems such as $6 \times 2\frac{1}{7}$, or $3 \times 3\frac{1}{4}$. **

NOW THAT WE KNOW HOW TO FIND PRODUCTS OF WHOLE NUMBERS AND WHOLE NUMBERS AND WHOLE NUMBERS AND FRACTIONS, LET'S COMBINE THESE TO FIND PRODUCTS OF WHOLE NUMBERS AND MIXED NUMBERS, SUCH AS $2 \times 3\frac{1}{4}$.

SUPPOSE I WANT THE ANSWER TO THE MULTIPLICATION QUESTION: $2 \times 3\frac{1}{4}$. I REALIZE THAT TO FIND THE ANSWER, I MUST RENAME THIS SOMEHOW. I KNOW THAT I CAN REWRITE $3\frac{1}{4}$ AS $3 + \frac{1}{4}$.

THEN, $2 \times 3\frac{1}{4}$ WOULD BE THE SAME AS

$2 \times 3 + 2 \times \frac{1}{4}$ (JUST AS $2 \times 7 = 2 \times 3 + 2 \times 4$)

SO I GET

$$2 \times 3\tfrac{1}{4} = 2 \times 3 + 2 \times \tfrac{1}{4} = 6 + \tfrac{2}{4} = 6\tfrac{2}{4}.$$

NOTICE THAT THE WHOLE NUMBER PART OF MY ANSWER CAME FROM MULTIPLYING THE WHOLE NUMBER 2 BY THE WHOLE NUMBER 3 AND THE FRACTIONAL PART BY MULTIPLYING THE WHOLE NUMBER 2 BY THE FRACTIONAL PART $\frac{1}{4}$.

LET'S TRY THIS PROCEDURE ON ANOTHER QUESTION.

SUPPOSE WE WANT TO FIND $3 \times 2\frac{1}{5}$.

WHAT DO YOU PREDICT THE ANSWER WILL BE? ACCORDING TO WHAT WE GUESSED BEFORE, THE WHOLE NUMBER PART OF OUR ANSWER WILL COME FROM (3 x 2) AND THE FRACTIONAL PART FROM ($3 \times \frac{1}{5}$). THEREFORE OUR ANSWER SHOULD BE $6\frac{3}{5}$, AND WE WRITE THE ADDITION STATEMENT TO CHECK:

$$3 \times 2\tfrac{1}{5} = 3 \times 2 + 3 \times \tfrac{1}{5} = 6 + \tfrac{3}{5} = 6\tfrac{3}{5}.$$

WE GET THE TOTAL DESIRED.

WHAT WOULD BE THE ANSWER TO $4 \times 2\frac{2}{9}$? IT SHOULD BE WHAT?

Expect the answer $8\frac{8}{9}$.

LET'S VERIFY WITH AN ADDITION STATEMENT:

$$4 \times 2\tfrac{2}{9} = 4 \times (2 + \tfrac{2}{9}) = 4 \times 2 + 4 \times \tfrac{2}{9} = 8 + \tfrac{8}{9} = 8\tfrac{8}{9}.$$

Have the students work on:

$3 \times 4\frac{2}{7}$

$2 \times 2\frac{3}{8}$

$4 \times 3\frac{1}{5}$

Ask three students to come to the board to show their answers with appropriate addition statements for each.

Hand students worksheet 1 to complete.

Suggested end of day 3.

Multiplication of a Mixed Number by a Fraction

Outline for approach A:

NOW THAT WE KNOW HOW TO MULTIPLY A MIXED NUMBER BY A WHOLE NUMBER, WE ARE GOING TO LEARN HOW TO MULTIPLY A MIXED NUMBER BY A FRACTION.

BUT, AGAIN, WE JUST WANT TO REMIND OURSELVES OF A FEW THINGS BEFORE WE GO ON.

LET'S REVIEW QUICKLY AGAIN THE PROPERTY OF MULTIPLICATION BY 1. IF I MULTIPLY A NUMBER BY 1, IT DOES NOT CHANGE THAT NUMBER, EVEN IF THAT NUMBER IS A FRACTION. SO THAT

$1 \times \frac{2}{3} = ?$

Expect the answer $\frac{2}{3}$.

SIMILARLY, IF I HAVE A NUMBER I CAN ALWAYS RENAME IT BY MULTIPLYING IT BY 1. FOR EXAMPLE, ANOTHER NAME FOR $7 = 1 \times 7$.

USING THE PROPERTY OF 1, WHAT WOULD BE SOME OTHER NAMES FOR:

6

5

$\frac{3}{5}$

Expect the answers: 1×6 , 1×5, $1 \times \frac{3}{5}$.

ONE MORE IMPORTANT PIECE OF REVIEWING IS ANOTHER FORM OF RENAMING. I CAN ALWAYS RENAME A FRACTION IN TERMS OF A MULTIPLICATION STATEMENT AND VICE VERSA. FOR EXAMPLE, I CAN REWRITE

$\frac{5}{7}$ AS $\frac{1}{7} + \frac{1}{7} + \frac{1}{7} + \frac{1}{7} + \frac{1}{7}$ AND SO AS $5 \times \frac{1}{7}$, 5 GROUPS OF $\frac{1}{7}$

OR I CAN RENAME $\frac{3}{4}$ AS $3 \times \frac{1}{4}$.

HOW CAN I RENAME $3 \times \frac{1}{5}$ AS A FRACTION?

Expect the answer $\frac{3}{5}$.

HOW CAN I RENAME $\frac{2}{7}$ AS A MULTIPLICATION STATEMENT?

Expect the answer $2 \times \frac{1}{7}$.

Ask the students to rename the following fractions in multiplication form:

$\frac{4}{6}$; $\frac{3}{8}$; $\frac{5}{10}$.

Have three students explain their answers.

FINALLY, LET'S GO OVER THE RELATIONSHIP BETWEEN MULTIPLICATION AND ADDITION. OFTEN, TO FIND THE ANSWER TO A DIFFICULT MULTIPLICATION QUESTION, I COMBINE MULTIPLICATION AND ADDITION. FOR EXAMPLE, TO FIND 34×502, I MIGHT FIND

$34 \times 500 + 34 \times 2$ AND ADD THEM TO FIND 34×502.

I CAN ALSO USE THIS PRINCIPLE WITH FRACTIONS.

TO FIND $7 \times 2\frac{1}{3}$, I CAN RENAME THE MIXED NUMBER PART AS $2 + \frac{1}{3}$, AND FIND EACH OF THE PRODUCTS SEPARATELY AND ADD.

HOW COULD YOU SPLIT THE MIXED NUMBER IN EACH OF THESE FOR PURPOSES OF EASIER MULTIPLICATION?

$6 \times 4\frac{1}{5}$

$8 \times 3\frac{2}{21}$

$3 \times 4\frac{3}{10}$

Expect the breakup of the mixed number into a whole part and a fractional part.

NOW LET'S GET BACK TO MULTIPLICATION OF FRACTIONS.

BEFORE, WE LEARNED HOW TO FIND PRODUCTS OF WHOLES BY FRACTIONS, LIKE $3 \times \frac{2}{3}$. NOW WE WOULD LIKE TO SOLVE QUESTIONS WHERE THE NUMBERS ARE TURNED AROUND, PROBLEMS LIKE $\frac{2}{3} \times 3$.

JUST AS BEFORE, LET'S TRY SOME RENAMING.

$\frac{2}{3}$ CAN BE RENAMED IN A MULTIPLICATION WAY AS $2 \times \frac{1}{3}$.

THEREFORE, $\frac{2}{3} \times 3 = (2 \times \frac{1}{3}) \times 3$.

SINCE I CAN MULTIPLY NUMBERS IN ANY ORDER, I CAN REWRITE THIS AS

$$
\begin{aligned}
& 2 \times (\tfrac{1}{3} \times 3) \\
&= 2 \times (3 \times \tfrac{1}{3}) \\
&= (2 \times 3) \times \tfrac{1}{3} \\
&= 6 \times \tfrac{1}{3} \\
&= \tfrac{6}{3}.
\end{aligned}
$$

SO, $\frac{2}{3} \times 3 = \frac{6}{3}$.

LET'S FIND THE PRODUCT: $\frac{3}{5} \times 4$.

WELL, $\frac{3}{5}$ CAN BE RENAMED AS $3 \times \frac{1}{5}$.

SO,

$$
\begin{aligned}
& \tfrac{3}{5} \times 4 \\
&= (3 \times \tfrac{1}{5}) \times 4 \\
&= (3 \times 4) \times \tfrac{1}{5} \\
&= 12 \times \tfrac{1}{5}
\end{aligned}
$$

THEREFORE, $\frac{3}{5} \times 4 = \frac{12}{5}$.

WHAT WOULD BE THE PRODUCT: $\frac{3}{4} \times 6$?

Expect the answer $\frac{18}{4}$. Then write the renaming to verify:

$$\frac{3}{4} \times 6 = 3 \times \frac{1}{4} \times 6 = 3 \times 6 \times \frac{1}{4} = 18 \times \frac{1}{4}.$$

Point out to the students that they could use the order principle to find

the answers, but that they will use renaming here just to get used to it in situations where they don't have as much option. For example, they could solve $4 \times \frac{3}{5} = \frac{3}{5} \times 4$, but they would, instead, use the whole renaming process.

Ask the students to find these products:

$\frac{3}{7} \times 2$

$\frac{4}{8} \times 3$

$\frac{1}{5} \times 4$

Explain that they may always go back to a renaming statement to verify their answers.

** Here the students will learn to find the products of unit fractions, such as $\frac{1}{2} \times \frac{1}{3}$ or $\frac{1}{5} \times \frac{1}{6}$. **

SUPPOSE I WANT TO FIND THE ANSWER TO A MULTIPLICATION QUESTION WHERE BOTH OF THE NUMBERS TO BE MULTIPLIED ARE FRACTIONS. FOR EXAMPLE, WE MIGHT TRY $\frac{1}{2} \times \frac{1}{3}$.
WHAT DO YOU SUGGEST THE ANSWER MIGHT BE ?

Wait for a response of $\frac{1}{6}$ before proceeding.

LET'S SEE IF THIS IS REASONABLE. WHAT DO WE KNOW ABOUT $\frac{1}{6}$?
ONE THING WE KNOW IS THAT $\frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 6 \times \frac{1}{6} = 1$ (IS 1). SO, WE KNOW THAT $6 \times \frac{1}{6} = 1$. THEREFORE, $\frac{1}{6}$ IS A NUMBER I CAN MULTIPLY BY 6 TO GET 1.

WILL THERE BE ANY OTHER NUMBERS I CAN MULTIPLY BY 6 TO GET 1?
IF THERE WERE, THEY WOULD HAVE TO EQUAL $\frac{1}{6}$ SINCE THAT IS EXACTLY WHAT WE MEAN BY $\frac{1}{6}$, NAMELY THAT $6 \times \frac{1}{6} = 1$.

THEN, IF $6 \times (\frac{1}{2} \times \frac{1}{3}) = 1$, THAT WOULD MEAN THAT

$\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$.

SINCE $\frac{1}{6}$ WAS THE ONLY NUMBER I COULD MULTIPLY BY 6 TO GET 1.

BUT, $6 \times (\frac{1}{2} \times \frac{1}{3})$

$= 3 \times 2 \times (\frac{1}{2} \times \frac{1}{3})$

$= (3 \times \frac{1}{3}) \times (2 \times \frac{1}{2})$

$= 1 \times 1$

$= 1$

SINCE I CAN MOVE THE NUMBERS AROUND WITHOUT CHANGING THE PRODUCT.

NOTICE THAT ALL I DID WAS TO DECIDE WHETHER $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$

WAS TO SEE IF $6 \times (\frac{1}{2} \times \frac{1}{3}) = 1$.

HOW WOULD WE FIND $\frac{1}{4} \times \frac{1}{7}$?

WE CHECK TO SEE IF $\frac{1}{4} \times \frac{1}{7} = \frac{1}{28}$ BY SEEING IF $28 \times (\frac{1}{4} \times \frac{1}{7})$ IS 1. BUT,

$28 \times (\frac{1}{4} \times \frac{1}{7})$

$= 7 \times 4 \times (\frac{1}{4} \times \frac{1}{7})$

$= (7 \times \frac{1}{7}) \times (4 \times \frac{1}{4})$

$= 1 \times 1$

$= 1$

SO, $28 \times (\frac{1}{4} \times \frac{1}{7}) = 1$, AND SO,

$\frac{1}{4} \times \frac{1}{7} = \frac{1}{28}$.

WHAT DO YOU THINK $\frac{1}{3} \times \frac{1}{6}$ WILL BE?

Expect the answer $\frac{1}{18}$.

LET US CHECK THIS:

IF THIS IS SO, THEN $18 \times (\frac{1}{3} \times \frac{1}{6}) = 1$.

BUT, $18 \times (\frac{1}{3} \times \frac{1}{6})$

$= 6 \times 3 \times (\frac{1}{3} \times \frac{1}{6})$

$= (6 \times \frac{1}{6}) \times (3 \times \frac{1}{3})$

$= 1 \times 1$

$= 1.$

SO, $18 \times (\frac{1}{3} \times \frac{1}{6}) = 1$, SO $\underbrace{\frac{1}{3} \times \frac{1}{6}} = \frac{1}{18}$.

DO YOU NOTICE THAT IN EACH OF THESE PROBLEMS, IF I MULTIPLY $\frac{1}{\square}$ BY $\frac{1}{\triangle}$, I GET AS AN ANSWER A FRACTION $\frac{1}{\square \times \triangle}$. THIS SEEMS REASONABLE SINCE IN EACH CASE IF I MULTIPLY $\square \times \triangle$ BY $\frac{1}{\square} \times \frac{1}{\triangle}$, I GOT $(\square \times \frac{1}{\square}) \times (\triangle \times \frac{1}{\triangle}) = 1 \times 1 = 1.$

Suggested end of day 4.

** Here the student will learn how to find products of any two proper fractions such as $\frac{4}{5} \times \frac{2}{3}$ or $\frac{2}{5} \times \frac{3}{7}$. **

SUPPOSE WE WANT TO MULTIPLY $\frac{2}{3}$ BY $\frac{3}{4}$. WE KNOW THAT WE CAN DO SOME RENAMING TO FIND THE ANSWER.

$\frac{2}{3}$ CAN BE RENAMED AS $2 \times \frac{1}{3}$ AND $\frac{3}{4}$ AS $3 \times \frac{1}{4}$.

THEREFORE,

$\frac{2}{3} \times \frac{3}{4}$

$= (2 \times \frac{1}{3}) \times (3 \times \frac{1}{4})$

SINCE I CAN MULTIPLY IN ANY ORDER THIS IS THE SAME AS

$(2 \times 3) \times (\frac{1}{3} \times \frac{1}{4})$

WE ALREADY KNOW THAT $\frac{1}{3} \times \frac{1}{4}$ IS $\frac{1}{12}$.

WE ALSO KNOW THAT $(2 \times 3) \times \frac{1}{12} = \frac{6}{12}$.

THEREFORE, $\underbrace{\frac{2}{3} \times \frac{3}{4} = \frac{6}{12}}$.

LET'S FIND $\frac{3}{3} \times \frac{2}{3}$ USING THIS APPROACH:

I WRITE:

$$\frac{3}{5} \times \frac{2}{3}$$
$$= (3 \times \frac{1}{5}) \times (2 \times \frac{1}{3})$$
$$= (3 \times 2) \times (\frac{1}{5} \times \frac{1}{3})$$
$$= (3 \times 2) \times \frac{1}{15}$$
$$= \frac{6}{15} .$$

THEREFORE, $\frac{3}{5} \times \frac{2}{3} = \frac{6}{15}$.

WHAT WOULD $\frac{4}{6} \times \frac{2}{5}$ BE?

Expect the answer $\frac{8}{30}$.

LET'S CHECK WITH A RENAMING:

$$\frac{4}{6} \times \frac{2}{5}$$
$$= (4 \times \frac{1}{6}) \times (2 \times \frac{1}{5})$$
$$= (4 \times 2) \times (\frac{1}{6} \times \frac{1}{5})$$
$$= (4 \times 2) \times \frac{1}{30}$$
$$= \frac{8}{30} .$$

Ask the students to find the following products:

$\frac{3}{4} \times \frac{1}{6}$

$\frac{2}{5} \times \frac{1}{3}$

$\frac{4}{7} \times \frac{2}{5}$

Have three students come to the board to show the renaming that leads to the answer.

Point out to the students that the numerator of their answer comes from the numerator of the two fractions they are multiplying since they are collecting numerators in their renaming tasks just as they are collecting denominators in terms of the fractions $\frac{1}{\square} \times \frac{1}{\triangle}$.

** Here the student will learn how to find products of fractions and mixed numbers such as $\frac{1}{2} \times 3\frac{1}{3}$ or $\frac{2}{3} \times 4\frac{1}{5}$. **

NOW THAT WE KNOW HOW TO FIND PRODUCTS LIKE $\frac{3}{4} \times 2$ AND $\frac{3}{4} \times \frac{1}{3}$, WE ARE READY TO FIND PRODUCTS LIKE $\frac{3}{4} \times 2\frac{1}{3}$.

SUPPOSE I WANT TO FIND THE ANSWER TO : $\frac{3}{4} \times 2\frac{1}{3}$.

I CAN WRITE THIS AS:

$$\frac{3}{4} \times (2 + \frac{1}{3})$$

$$= \frac{3}{4} \times 2 + \frac{3}{4} \times \frac{1}{3}$$

NOTICE THAT I AM SPLITTING THE MIXED NUMBER HERE JUST AS I DID WHEN I WAS FINDING THE PRODUCTS OF WHOLE NUMBERS AND MIXED NUMBERS. THEN I CAN FIND THE PRODUCTS OF THE INDIVIDUAL PARTS AND TOTAL THEM. THE FIRST PRODUCT IS $\frac{3}{4} \times 2 = \frac{6}{4}$ WHICH I CAN VERIFY BY RENAMING.

THE SECOND PRODUCT I CAN FIND IS $\frac{3}{4} \times \frac{1}{3} = \frac{3}{12}$ WHICH I CAN VERIFY BY RENAMING.

THEREFORE, I CAN TOTAL UP TO GET THE PRODUCT $\frac{3}{4} \times 2\frac{1}{3}$ AND I FIND THAT $\frac{3}{4} \times 2\frac{1}{3} = \frac{6}{4} + \frac{3}{12}$.

WE WILL LEAVE OUR ANSWERS IN THIS FORM SINCE WE HAVE NOT BEEN TOO CONCERNED ABOUT REDUCING FRACTIONS OR COMBINING THEM FROM SEPARATE ANSWERS.

LET'S FIND $\frac{4}{5} \times 3\frac{1}{3}$ USING THIS METHOD:

I CAN RENAME THIS AS:

$$\frac{4}{5} \times 3\frac{1}{3} = \frac{4}{5} \times 3 + \frac{4}{5} \times \frac{1}{3}$$

THEN, $\frac{4}{5} \times 3 = \frac{12}{5}$

SINCE $\frac{4}{5} \times 3$

$= (4 \times \frac{1}{5}) \times 3$

$= (4 \times 3) \times \frac{1}{5}$

$= 12 \times \frac{1}{5} = \frac{12}{5}$

AND $\frac{4}{5} \times \frac{1}{3} = \frac{4}{15}$

SINCE:

$\frac{4}{5} \times \frac{1}{3}$

$= (4 \times \frac{1}{5}) \times (1 \times \frac{1}{3})$

$= (4 \times 1) \times (\frac{1}{5} \times \frac{1}{3})$

$= 4 \times \frac{1}{15}$

$= \frac{4}{15}$.

SO THE TOTAL PRODUCT IS: $\frac{4}{5} \times 3\frac{1}{3} = \frac{12}{5} + \frac{4}{15}$.

Ask the students to find these products:

$\frac{3}{4} \times 2\frac{2}{3}$

$\frac{4}{5} \times 3\frac{1}{5}$

$\frac{6}{9} \times 1\frac{5}{6}$

$\frac{4}{7} \times 7\frac{1}{3}$

Ask four students to come to the board and use a renaming argument to explain their answers. Suggest strongly that they do their breaking up of the mixed number into a whole piece and a proper fractional piece as was suggested here.

Suggested end of day 5.

Changing a Fraction to a Decimal

Outline for approach p:

> WE ARE GOING TO LEARN A METHOD FOR WRITING FRACTIONS IN A NEW WAY, BUT TO DO THIS, WE MUST FIRST DO SOME REVIEWING ABOUT FRACTIONS AND WE MUST LEARN THIS NEW SYSTEM OF WRITING NUMBERS-AS DECIMALS.

Go over items 1-5 on the pretest. In particular, have students realize that the name of the shaded region in a figure depends on the numberof parts of the same size into which the figure is drawn and the number of these parts that are shaded. For each question, read the question and ask for the correct answer. This review should go quickly, particularly since the students worked with fractions in the previous unit.

> SUPPOSE I WANT TO FIND THE SIZE OF EACH GROUP OF OBJECTS WHICH RESULTS FROM SHARING AN ORIGINAL GROUP AMONG 3 PEOPLE FAIRLY. WHAT OPERATION DO I PERFORM?

Expect the answer: division.

> LET'S SEE. I WANT TO PUT, SAY, 27 MARBLES INTO 3 PILES, SO THAT EACH OF 3 BOYS GETS THE SAME NUMBER OF MARBLES. HOW MANY DOES EACH BOY GET?

Expect the answer 9.

> EACH BOY GETS 9 SINCE I CAN KEEP GIVING EACH BOY 1 AT A TIME UNTIL I USE UP ALL THE MARBLES. AFTER GIVING EACH BOY 1, I HAVE USED UP 3 AND HAVE 24 LEFT. THEN I COULD GIVE EACH ANOTHER, USING

UP ANOTHER 3 AND LEAVING ONLY 21. I COULD KEEP GOING AND COUNT AT THE END. ANOTHER WAY WOULD BE TO GIVE EACH BOY, SAY, 3, USING UP 9, AND LEAVING ONLY 18. THEN I MIGHT GIVE EACH BOY 5, USING UP 15 MORE FOR A TOTAL OF 24, LEAVING ONLY 3 MORE. EACH BOY GETS 1 MORE SO THAT HE GETS A TOTAL OF 9. LET'S WRITE THESE TWO METHODS OUT:

```
3 )27 |
   3  | 1
  24  |
   3  | 1
  21  |
   3  | 1
  18  |
  ...  |
       |
       | 9
```

```
3 )27 |
   9  | 3
  18  |
  15  | 5
   3  |
   3  | 1
   0  | 9
```

SO THERE ARE MANY WAYS TO SOLVE THIS QUESTION, BUT THE OBJECT IS TO DIVIDE 27 BY 3 TO SEE HOW MANY OBJECTS EACH BOY GETS. I CAN ALWAYS USE DIVISION TO FIND THE ANSWER TO A PROBLEM INVOLVING SHARING OBJECTS EVENLY AMONG A CERTAIN GROUP OF PEOPLE.

Go over items 6-10 on the pretest. Read each item, and have a student answer it. Show how the division idea is involved in each.

To remind students of the division algorithm for the wholes, give them the problem $2639 \div 13$. Note that to find the solution, they must keep taking chunks of 13's out of 2639 until they can find the size of each of the 13 groups in 2639.

SPEAKING OF DIVISION, SUPPOSE I WANT TO SOLVE $2639 \div 13$? HOW DO I BEGIN? I SET UP THE QUESTION, LIKE SO:

$13\overline{)2639}$ AND I KEEP TAKING CHUNKS OUT OF 2639 TO FILL UP MY 13 PILES AND COUNT THE TOTAL NUMBER IN EACH OF THESE PILES AT THE END.

I CAN TAKE OUT 13 GROUPS OF SIZE 200 FIRST, LEAVING 39 OBJECTS.

```
13)2639
   2600 | 200
     39 |
     39 |   3
      0 | 203
```

THEN I CAN PUT 3 MORE OBJECTS IN EACH PILE TO USE THEM ALL UP, SO, $2639 \div 13 = 203$.

I MIGHT HAVE SOLVED THIS MANY OTHER WAYS: FOR INSTANCE, I MIGHT HAVE TAKEN OUT 100 AND THEN ANOTHER 100 AND THEN 3.

Allow the students to solve these two division questions and have two students show their results to the class:

$9\overline{)288}$ AND $20\overline{)4300}$

** Here the student will learn some of the features of the decimal system.**

AS WE SAID BEFORE, WE ARE GOING TO WRITE FRACTIONS IN A NEW WAY. THESE NEW SYMBOLS WILL BE CALLED DECIMALS, SO WE WILL LEARN SOMETHING ABOUT THE SYSTEM OF DECIMALS.

LET'S REVIEW PLACE VALUE FOR THE WHOLE NUMBERS. WE READ 345 AS "THREE HUNDREDS AND FORTY-FIVE" BECAUSE OF THE SYSTEM WE HAVE FOR WRITING NUMBERS. WE CAN TELL BY HOW THE NUMBER IS WRITTEN WHETHER THE 3 STANDS FOR 3 HUNDREDS OR 3 TENS OR 3 ONES. NOTICE THAT FOR EACH PLACE, THE VALUE OF THE PLACE TO THE RIGHT IS $\frac{1}{10}$ THE VALUE OF THE PLACE TO THE LEFT. FOR EXAMPLE, 1 IS

$\frac{1}{10}$ OF 10, 10 IS $\frac{1}{10}$ OF 100, 100 IS $\frac{1}{10}$ OF 1000.

SO IT SEEMS REASONABLE THAT THERE SHOULD BE A PLACE TO THE RIGHT OF THE ONES SO THAT THE VALUE OF THE PLACE IS $\frac{1}{10}$ OF 1, NAMELY $\frac{1}{10}$. SURELY, $\frac{1}{10}$ IS $\frac{1}{10}$ OF 1; THAT'S WHAT WE MEAN BY $\frac{1}{10}$.

SO SUPPOSE I START WITH THE NUMBER $463\frac{6}{10}$. IN MY NEW SYSTEM, I WOULD LIKE TO WRITE THIS WITHOUT ANY FRACTIONS. IF I WRITE IT AS 463 6, I NEED A MARKER TO TELL ME THE DIFFERENCE BETWEEN THIS NUMBER AND THE NUMBER "FOUR THOUSAND SIX HUNDRED THIRTY-SIX". THIS MARKER, CALLED THE DECIMAL POINT, APPEARS BETWEEN THE ONES COLUMN AND THE TENTHS COLUMN. THEREFORE, I WRITE THIS NUMBER AS 463.6 .

HOW WOULD YOU WRITE $45\frac{2}{10}$ AS A DECIMAL?

Expect the answer: 45.2 .

Have the students write decimals for each of the following. Ask a student to read each of his decimal equivalents.

$27\frac{3}{10}$

$376\frac{1}{10}$

$4\frac{8}{10}$

NOTICE THAT IF I HAVE A WHOLE NUMBER, I CAN ALSO WRITE THIS AS A DECIMAL. FOR EXAMPLE, I CAN WRITE THE WHOLE NUMBER $4 = 4 + \frac{0}{10}$ AS 4.0 . HOW MIGHT I WRITE 82 AS A DECIMAL?

Expect the answer: 82.0 .

NOW JUST AS WE CAN WRITE A DECIMAL FOR A MIXED NUMBER OR FRACTION, WE MIGHT START WITH A DECIMAL AND TRY TO FIGURE OUT ITS FRACTIONAL

OR MIXED NUMBER EQUIVALENT.

SUPPOSE WE START WITH THE DECIMAL 42.9 . HOW COULD WE WRITE THIS IN FRACTIONAL FORM?

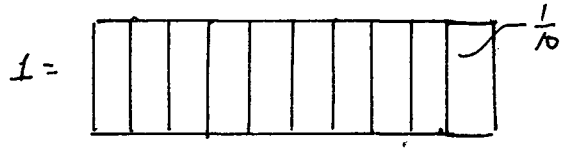
Expect the answer: $42\frac{9}{10}$.

ONE POSSIBLE ANSWER IS AS: $42\frac{9}{10}$. SURELY THIS IS JUST WHAT WE HAVE BEEN DISCUSSING. BUT OTHER POSSIBILITIES ALSO EXIST. JUST AS 428 CAN BE READ EITHER AS " FOUR HUNDRED TWENTY EIGHT ONES" OR "FORTY-TWO TENS AND EIGHT ONES", WE MIGHT SUSPECT THAT 42.8 CAN BE READ IN SEVERAL WAYS. ONE WAY IS 42 ONES AND 8 TENTHS, AS YOU SUGGESTED. BUT ANOTHER MIGHT BE 428 TENTHS. LET'S SEE IF THIS MAKES SENSE.

SUPPOSE I HAVE 42 ONES. EACH 1 IS HOW MANY TENTHS?

Expect the answer: 10 tenths.

WE CAN DRAW A DIAGRAM TO SEE THIS.



THEN, IF EACH 1 IS $\frac{1}{10}$, 42 ONES = 42 x 1 = 42 x (10 TENTHS)

= 420 TENTHS = $\frac{420}{10}$.

FOR EXAMPLE, 2 = 2 x 1 = 20 TENTHS = $\frac{20}{10}$.

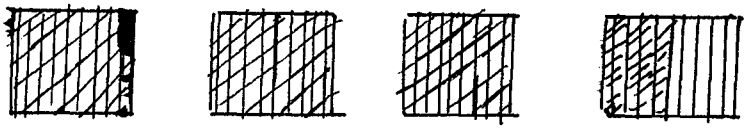
THEN SINCE I HAD 42.8, I HAVE $\frac{420}{10} + \frac{8}{10} = \frac{428}{10}$ = 428 TENTHS.

LET'S READ SOME OTHER DECIMALS SEVERAL WAYS.

HOW MIGHT I READ 3.5 ?

Expect the answer: $3\frac{5}{10}$ or $\frac{35}{10}$.

ONE WAY IS AS THE MIXED NUMBER $3\frac{5}{10}$; ANOTHER SHOULD BE 35 TENTHS SINCE EACH WHOLE IS 10 TENTHS, THEN 3 ONES = 3 x (10 TENTHS) = 30 TENTHS AND 30 TENTHS + 5 TENTHS = 35 TENTHS.



RREAD EACH OF THESE IN TWO WAYS:

36.9

1.5

28.4

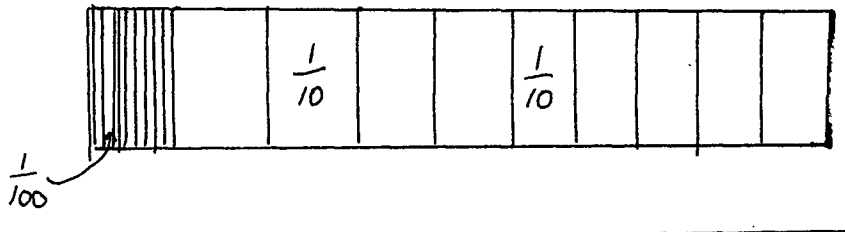
Ask a few students to read each of these- once as a mixed number and once as an improper fraction.

Suggested end of day I.

NOW THAT WE HAVE CONSIDERED THE POSSIBILITY OF A TENTHS PLACE, WE OUGHT TO CONSIDER WHAT OTHER COLUMNS WE MIGHT HAVE EVEN FURTHER TO THE RIGHT. JSUT AS $1 = 10 \times \frac{1}{10}$, WE MUST FIND A 'PLACE VALUE' TO THE RIGHT SO THAT 10x THAT VALUE = $\frac{1}{10}$.
WHAT NUMBER CAN I MULTIPLY BY 10 TO GET $\frac{1}{10}$?

Expect the answer: $\frac{1}{100}$.

$\frac{1}{100}$ WILL DO, SINCE I CAN MAKE A DRAWING LIKE THE FOLLOWING:



SIMILARLY, THE NEXT PLACE TO THE RIGHT WOULD BE THE THOUSANDTHS SINCE $\frac{1}{100} = 10 \times \frac{1}{1000}$.

THEREFORE, THE DECIMAL 45.82 WOULD BE READ AS "FORTY-FIVE ONES, 8 TENTHS, AND 2 HUNDREDTHS."

HOWEVER, AS BEFORE, WE CAN FIND SEVERAL OTHER WAYS OF READING THIS DECIMAL. FOR EXAMPLE, I MIGHT READ THIS AS "FORTY-FIVE ONES AND EIGHTY-TWO HUNDREDTHS." LET'S SEE IF THIS MAKES SENSE: 45.82 MEANS FORTY-FIVE ONES AND EIGHT TENTHS AND TWO HUNDREDTHS. BUT WE KNOW THAT $\frac{1}{10} = \frac{10}{100}$. THAT IS HOW WE KNEW THAT THE HUNDREDTHS COLUMN APPEARED TO THE RIGHT OF THE TENTHS.

SO, $\frac{8}{10} = 8 \times \frac{1}{10} = 8 \times \frac{10}{100} = 80$ HUNDREDTHS AND SO $\frac{80}{100} + \frac{2}{100} = \frac{82}{100}$ AND WE CAN READ THE ANSWER AS FORTY-FIVE ONES AND EIGHTY-TWO HUNDREDTHS. ANOTHER POSSIBLE WAY TO READ THE DECIMAL 45.82 WOULD BE "FOR THOUSAND FIVE HUNDRED EIGHTY TWO HUNDREDTHS"= $\frac{4582}{100}$. WE KNOW THAT 45.8 CAN BE READ AS $\frac{458}{10}$ AND SINCE EACH TENTH IS TEN HUNDREDTHS, WE KNOW THAT

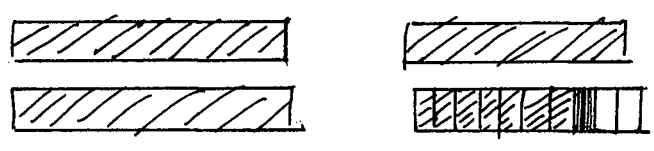
$$\frac{458}{10} = \frac{458 \times 10}{100} = \frac{4580}{100}$$

AND THAT $45.82 = \frac{4580}{100} + \frac{2}{100} = \frac{4582}{100}$. HOW COULD YOU READ 3.78?

Expect the answer: either "3 ones and 7 tenths and 8 hundredths" or "3 ones and 78 hundredths" or "378 hundredths".

NOTICE THAT WE CAN DRAW DIAGRAMS TO ILLUSTRATE EACH OF THESE READINGS:

e.g. $3 \times 1 + 7 \times \frac{1}{10} + 8 \times \frac{1}{100}$



Have the students read each of the following decimals first as mixed numbers and then as hundredths.

54.29

36.21

8.09

4.23

Have students read out their answers.

NOW SUPPOSE WE WANT TO WRITE SOME MIXED NUMBERS OR FRACTIONS AS DECIMALS; WE MIGHT WISH, FOR EXAMPLE, TO WRITE $13\frac{81}{100}$ AS A DECIMAL. WHAT WOULD IT BE?

Expect the answer: 13.81 .

WRITE EACH OF THE FOLLOWING AS A DECIMAL:

$5\frac{2}{100}$

$6\frac{13}{100}$

$28\frac{14}{100}$

Have the students who read out their answers explain in terms of either a diagram or an explanation of their renaming .

SIMILARLY, I MIGHT READ AND WRITE DECIMALS INTO THE THOUSANDTHS. HOW WOULD I READ 34.529?

Expect the answers: 34 ones and 5 tenths and 2 hundredths and 9 thousandths, or 34 ones and 529 thousandths, or 34,529 thousandths. If not all these answers come up, mention the ones that did not. Point out the analogy to the hundredths situation.

IN PARTICULAR, I CAN REWRITE A WHOLE NUMBER, SAY 4, AS TENTHS OR

HUNDREDTHS OR THOUSANDTHS. THAT IS, 4 = 4.0 = 4.00 = 4.000, SO 4 CAN BE READ AS 4, 40 TENTHS, 400 HUNDREDHTS, OR 4000 THOUSANDTHS.

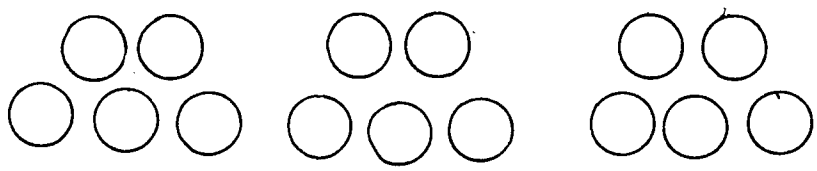
Hand students worksheet 1 to complete.

** Here the students will be introduced to the idea of dividing decimals by wholes.**

SUPPOSE WE WOULD LIKE TO FIND THE ANSWER TO $1.5 \div 3$?

WE KNOW THAT $15 \div 3$ MEANS TO SEPARATE 15 INTO 3 GROUPS OF THE SAME SIZE AND THEN FIND THE SIZE OF EACH OF THESE GROUPS.

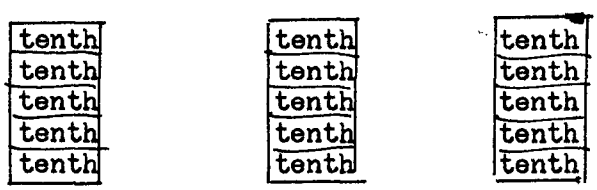
THIS DIAGRAM WOULD SHOW WHAT WE MEAN:



WELL, WE SHOULD MEAN A SIMILAR THING BY $1.5 \div 3$. WE SHOULD SEPARATE 1.5 INTO 3 GROUPS SO THAT EACH HAS THE SAME SIZE AND OUR ANSWER IS THE ANSWER TO $1.5 \div 3$.

WE ALREADY KNOW THAT 1.5 CAN BE READ AS $\frac{15}{10}$ OR 15 TENTHS.

SO TO DIVIDE 15 TENTHS BY 3, I MERELY PUT THE 15 TENTHS INTO 3 GROUPS AND FIND THE SIZE OF EACH GROUP. I CAN DRAW:

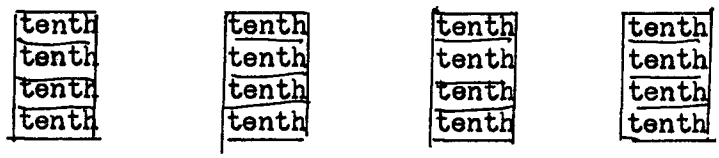


NOTICE THAT THIS WAS THE SAME AS PUTTING 15 OBJECTS OF ANY SORT INTO 3 GROUPS AND THEN REMEMBERING THE TYPE OF OBJECT BEING DEALT WITH. THAT IS, THE ANSWER WAS $(15 \div 3)$ TENTHS.

LET'S FIND $1.6 \div 4$. WHAT DO YOU EXPECT THE ANSWER TO BE?

Expect the answer: 4 tenths or .4 .

LET'S DRAW A PICTURE. $1.6 = \frac{16}{10} = 16$ TENTHS. THEN, TO PUT 16 TENTHS INTO 4 GROUPS, I DRAW:



AND WE SEE THAT INDEED I HAVE 4 TENTHS = .4 IN EACH OF THE GROUPS.

Have the students find the answers to:

$4.5 \div 9$

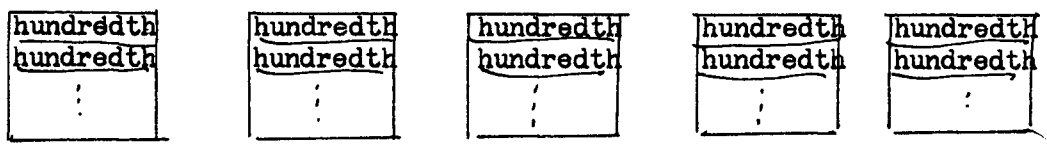
$5.6 \div 8$

$4.8 \div 6$

Ask students to read out their answers. If there is any difficulty with any of these, draw a diagram as was done for $1.6 \div 4$ to illustrate.

Suggested end of day 2.

WHAT WOULD WE DO IF WE WERE TRYING TO FIND $1.25 \div 5$? THAT MEANS THAT I WANT TO DIVIDE 125 HUNDREDTHS INTO 5 GROUPS AND FIND THE SIZE OF EACH GROUP. I CAN BEGIN A DRAWING LIKE SO:



NOTICE THAT WHAT I AM REALLY DOING IS DIVIDING 125 BY 5 AND THEN REMEMBERING THAT I AM DEALING WITH HUNDREDTHS.

Ask the students to find answers to:

$.25 \div 5$

$1.25 \div 25$

$2.00 \div 4$

Have three students explain their answers. If there is any difficulty, draw diagrams as indicated above.

NOW WE CAN WRITE THIS SCHEME IN A WAY THAT WE USUALLY USE TO WRITE DIVISION QUESTIONS.

FOR EXAMPLE, TO FIND $4.2 \div 6$, I WRITE:

$6\overline{)4.2} \rightarrow 6\overline{)42 \text{ TENTHS}}$ AND TO FIND THE ANSWER; I WRITE:

```
6)42 TENTHS
  42 TENTHS   |  7 TENTHS
   0 TENTHS   |  7 TENTHS =  .7
```

HOW COULD I WRITE OUT THIS QUESTION: $4.5 \div 9$? I MIGHT WRITE:

```
9)4.5 -> 9)45 TENTHS
          45 TENTHS   |  5 TENTHS
           0 TENTHS   |  5 TENTHS =  .5
```

HOW COULD WE WRITE $.36 \div 6$? WE WRITE:

```
6)._36 -> 6)36 HUNDREDTHS
           36 HUNDREDTHS   |  6 HUNDREDTHS
            0 HUNDREDTHS   |  6 HUNDREDTHS =  .06
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Hand students worksheet 2 to complete.

** Here the student will now leave decimals for a while to discover that another interpretation for a fraction is division. That is, $\frac{4}{3} = 4 \div 3$. This will tie in to decimals when he discovers that

$\frac{1}{2} = 1 \div 2$, and so to get the decimal for $\frac{1}{2}$, he divides 2 into 1. **

NOW AS WE SAID BEFORE, THE WHOLE IDEA OF INTRODUCING DECIMALS WAS TO EVENTUALLY SHOW YOU A WAY TO WRITE FRACTIONS AS DECIMALS. BUT WE HAVE FORGOTEN ALL ABOUT FRACTIONS. LET US RETURN TO THIS SUBJECT TEMPORARILY.

SUPPOSE I HAVE THE FRACTION $\frac{12}{4}$. LET US SEE WHAT WHOLE NUMBER THIS IS ANOTHER NAME FOR. DO YOU KNOW ALREADY?

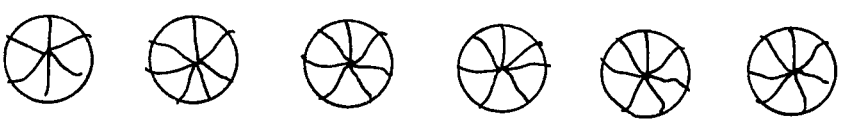
Expect the answer 3. In any case, go on with the following discussion.

$\frac{12}{4}$ TELLS ME THAT I HAVE TWELVE PIECES EACH OF SIZE $\frac{1}{4}$. ONE WAY TO FIND THE NUMBER OF WHOLES THERE ARE IN $\frac{12}{4}$ IS TO COLLECT FOURTHS INTO WHOLES UNTIL I HAVE USED ALL 12 PIECES. I COULD DRAW:

AND SO I SEE THAT $\frac{12}{4} = 3$.

LET'S DO THE SAME WITH $\frac{42}{7}$. WHAT SHOULD I DRAW AND WHAT WHOLE NUMBER DO I GET AS EQUAL TO $\frac{42}{7}$?

Expect the answer 6 with the drawing:



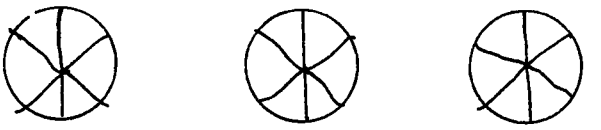
If they do not draw this diagram, you draw it for them.

NOTICE THAT $\frac{42}{7} = 6$.

NOW WE CAN DO THE SAME FOR $\frac{18}{6}$. WHAT IS THE WHOLE NUMBER FOR $\frac{18}{6}$?

Expect the answer: 3.

Draw:

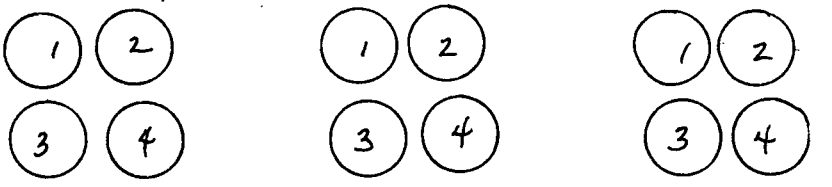


MY DRAWING INDICATES THAT 3 IS THE CORRECT ANSWER.

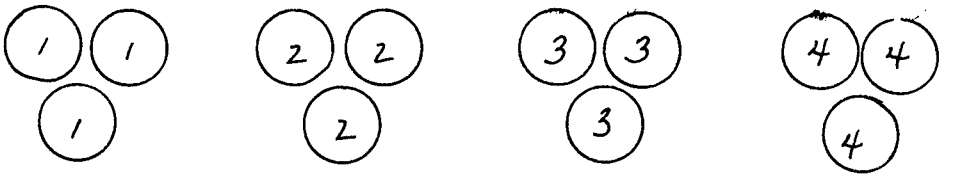
NOTICE THAT EACH TIME I WAS FINDING HOW MANY GROUPS OF THE DENOMINATOR THERE WERE IN THE NUMERATOR (GROUPS OF 4 IN 12, GROUPS OF 7 IN 42, AND GROUPS OF 6 IN 18).

WE ALREADY KNOW THAT FINDING THE NUMBER OF GROUPS OF ONE NUMBER IN ANOTHER IS THE SAME AS DIVIDING THAT SECOND NUMBER BY THE FIRST. BUT FINDING THE NUMBER OF GROUPS OF ONE NUMBER IN ANOTHER (LIKE THE NUMBER OF FOURS IN TWELVE) IS THE SAME AS FINDING THE SIZE OF EACH OF THAT NUMBER OF GROUPS IN THE SECOND NUMBER(LIKE THE SIZE OF EACH OF FOUR GROUPS OF TWELVE OBJECTS TOTAL).

FOR EXAMPLE, TO SHOW $12 \div 4$, I MIGHT DRAW:



OR I MIGHT DRAW:



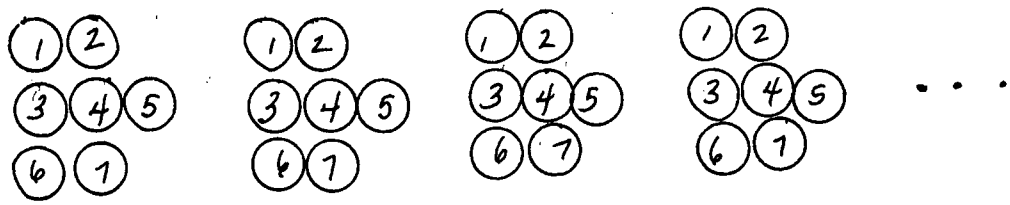
IN EITHER CASE, I GET THE SAME ANSWER, NAMELY 3.

SO SINCE $\frac{12}{4}$ WAS FOUND BY FINDING THE NUMBER OF GROUPS OF 4 IN 12, OR DIVIDING 12 BY 4, I COULD ALSO FIND $\frac{12}{4}$ BY FINDING THE SIZE

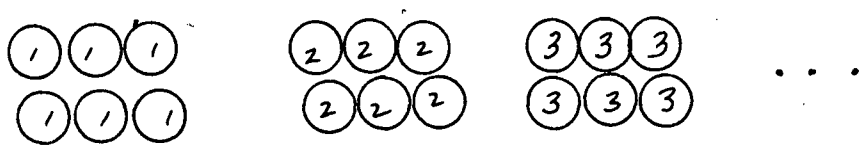
OF EACH OF 4 GROUPS OF EQUAL SIZE TO SHARE ALL 12 OBJECTS.

LET'S CHECK TO SEE IF FINDING $\frac{42}{7}$ BY DIVIDING 42 BY 7 TO FIND THE NUMBER OF SEVENS IN FORTY-TWO IS THE SAME AS FINDING THE SIZE OF EACH OF SEVEN EQUAL GROUPS INTO WHICH 42 OBJECTS ARE DIVIDED.

I CAN DRAW (TO SHOW HOW MANY 7'S IN 42):



I CAN DRAW THE DIAGRAM HERE TO SHOW THE SIZE OF EACH OF THE 7 EQUAL GROUPS INTO WHICH 42 OBJECTS CAN BE DIVIDED.



SO, $\frac{42}{7} = 42 \div 7$.

SIMILARLY, $\frac{18}{6} = 3 = 18 \div 6$ SINCE I WAS FINDING THE NUMBER OF 6'S IN 18, WHICH IS THE SAME AS FINDING THE SIZE OF EACH OF 6 EQUAL GROUPS AMONG WHICH 18 OBJECTS CAN BE DIVIDED.

FOR EACH OF THESE FRACTIONS, GIVE A DIVISION SENTENCE THAT WOULD GET A WHOLE NUMBER NAME FOR THE FRACTION:

$\frac{16}{8}$

$\frac{15}{3}$

$\frac{14}{2}$

Ask 3 students to read out their answers and then diagrammatically explain them.

Suggested end of day 3.

** Here the student will learn that another name for a fraction can always be found by dividing the numerator by the denominator, even when the division is not even. **

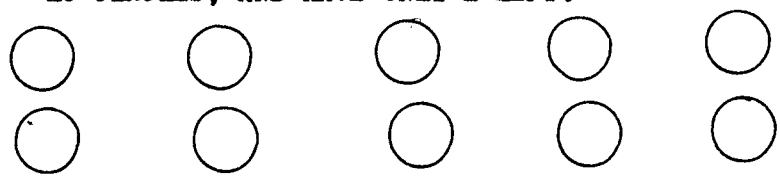
NOW WE HAVE SEEN THAT FOR SOME FRACTIONS, ANOTHER WAY OF FINDING A NAME FOR THE FRACTION IS TO DIVIDE THE DENOMINATOR INTO THE NUMERATOR. WE WOULD LIKE TO FIND OUT IF THIS IS ALWAYS TRUE.

SUPPOSE WE START WITH $\frac{11}{5}$. IF OUR RULE WORKS, $\frac{11}{5}$ SHOULD BE THE SAME AS $11 \div 5$: THAT IS, IF I TAKE 11 OBJECTS AND PUT THEM IN 5 EQUAL GROUPS, THE SIZE OF EACH GROUP SHOULD BE $11 \div 5$, OR $\frac{11}{5}$.

SUPPOSE WE TAKE 11 OBJECTS, SAY, 11 CIRCLES. I WANT TO PUT THEM IN 5 GROUPS. WE CAN SET UP 5 PILES.
THEN WE CAN KEEP PUTTING CIRCLES ON THE PILES EVENLY UNTIL ALL THE CIRCLES ARE USED UP.

TO BEGIN, LET US PUT ONE CIRCLE ON EACH PILE.

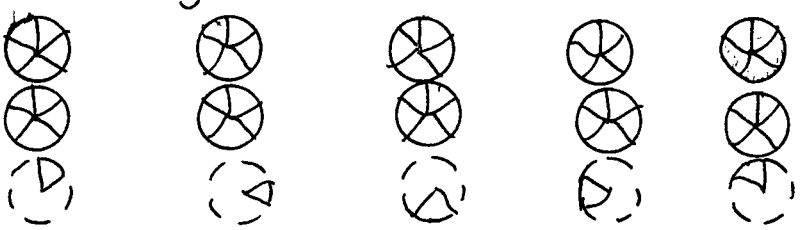
THEN I STILL HAVE 6 CIRCLES LEFT.
SO I CAN PUT ANOTHER CIRCLE ON EACH PILE AND I HAVE USED UP $5+5 = 10$ CIRCLES, AND HAVE ONLY 1 LEFT.



SINCE I ONLY HAVE ONE MORE CIRCLE TO PUT ONTO ALL 5 PILES EVENLY, I MUST SPLIT IT UP INTO 5 EQUAL PARTS. THAT MEANS, EACH PART

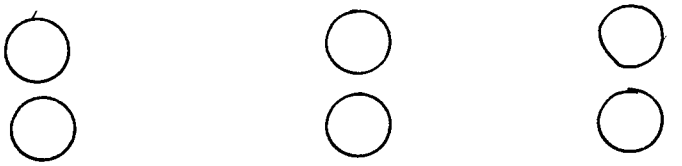
BECOMES $\frac{1}{5}$. SO, ON EACH PILE, I CAN PUT $2\frac{1}{5}$ CIRCLES.

SO, $11 \div 5 = 2\frac{1}{5}$, BUT $2\frac{1}{5}$ IS ANOTHER NAME FOR $\frac{11}{5}$ SINCE I CAN SPLIT EACH ONE WHOLE CIRCLE INTO 5 PARTS BEFORE I COUNT UP THE TOTAL SIZE OF EACH PILE, AND I SEE THAT EACH PILE HAS $\frac{5}{5} + \frac{5}{5} + \frac{1}{5} = \frac{11}{5}$ ON IT.

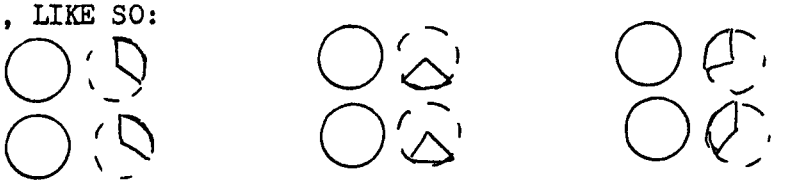


LET'S TRY THE SAME IDEA WITH $\frac{8}{3}$. IF OUR RULE WORKS, WE SHOULD EXPECT THAT IF WE PUT 8 CIRCLES ONTO 3 EQUAL PILES, EACH PILE WILL BE OF SIZE $\frac{8}{3}$.

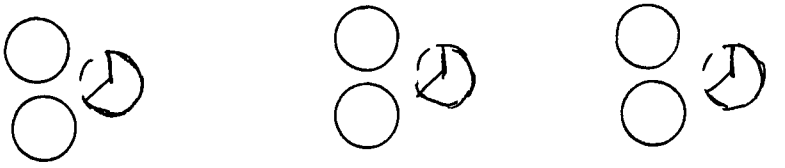
I CAN START WITH PUTTING 2 OBJECTS ON EACH PILE, AND I HAVE USED UP 6 AND HAVE 2 LEFT.



THEN, I CAN CUT EACH OF THE REMAINING TWO CIRCLES INTO THIRDS AND SEPARATELY SPLIT EACH OF THEM UP AND PUT THEM EVENLY ON THE PILES, LIKE SO:



AND THEN



THEN I COUNT UP AND SEE THAT ON EACH PILE, I HAVE $8 \div 3 = 2\frac{2}{3}$ CIRCLES. BUT I CAN THEN SPLIT EACH WHOLE INTO 3 PIECES, AND RENAME MY GROUPS AS BEING OF SIZE $\frac{3}{3} + \frac{3}{3} + \frac{2}{3} = \frac{8}{3}$.

CAN ANYONE SHOW ME WHY THEY THINK $\frac{5}{4} = 5 \div 4$?

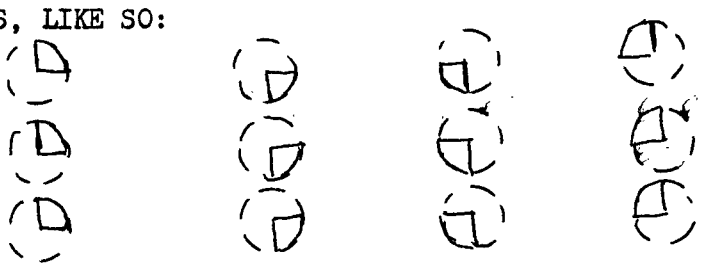
Expect someone to draw 4 piles and try to split 5 objects evenly among them, like so;

If no one does this, you draw the diagram.

Notice that each group has $1\frac{1}{4} = \frac{4}{4} + \frac{1}{4} = \frac{5}{4}$ as its size.

CAN ANYONE TELL ME WHY THEY THINK $\frac{3}{4} = 3 \div 4$?

I CAN DRAW 4 PILES AND TRY TO PUT MY CIRCLES EVENLY ON THEM. BUT SINCE I ONLY HAVE 3 CIRCLES, I CAN'T EVEN PUT A WHOLE ONE ON EACH PILE. WHAT I CAN DO IS SPLIT EACH OF THE THREE WHOLES INTO FOURTHS AND PUT ONE FOURTH OF EACH ON EACH OF THE 4 PILES, LIKE SO:



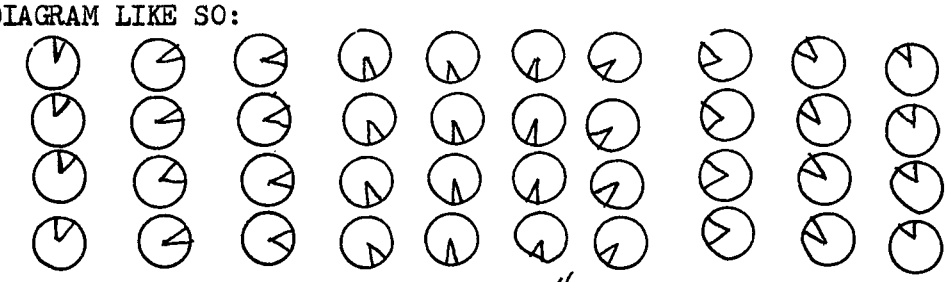
OBVIOUSLY, THE SIZE OF EACH PILE IS $\frac{3}{4}$.

Hand students worksheet 3 to complete.

Suggested end of day 4.

** Here the student will finally tie together his work on decimals and fractions to find the decimal equivalent to a fraction. **

NOW THAT WE KNOW HOW TO REWRITE A FRACTION AS A DIVISION QUESTION, WE CAN PROCEED TO TIE TOGETHER SOME OF OUR DECIMAL IDEAS WITH THIS ONE. LET'S LOOK AT $\frac{4}{10}$. WE KNOW $\frac{4}{10} = 4 \div 10$ BY DRAWING A DIAGRAM LIKE SO:



ON THE OTHER HAND, WE ALSO KNOW THAT $\frac{4}{10}$ CAN BE WRITTEN AS A DECIMAL AS .4 . IT WOULD BE INTERESTING TO BE ABLE TO SHOW DIRECTLY THAT $4 \div 10 = .4$.

WELL, $4 \div 10$ CAN BE WRITTEN $10\overline{)4}$. BUT IF I TRY TO DO THIS DIVISION, I ONLY GET AN ANSWER OF 0 WITH REMAINDER 4, WHICH DOES NOT HELP ME MUCH.

SO I MIGHT TRY TO RENAME $4 = 4.0 = \frac{40}{10} = 40$ TENTHS.

THEN, $4 \div 10$ CAN BE WRITTEN:

$$10\overline{)4} \rightarrow 10\overline{)4.0} \rightarrow 10\overline{)\begin{array}{r} 40 \text{ TENTHS} \\ \underline{40 \text{ TENTHS}} \\ 0 \text{ TENTHS} \end{array}} \quad \begin{array}{r} 4 \text{ TENTHS} \\ \hline 4 \text{ TENTHS} \end{array} = .4$$

WE HAVE DIRECTLY CONVERTED, THEN, $\frac{4}{10}$ INTO A DECIMAL BY USING DIVISION.

LET US TRY CONVERTING $\frac{6}{10}$ TO A DECIMAL. WE KNOW THAT OUR RESULT SHOULD BE .6 . BUT WE CAN TRY TO FIND THIS THROUGH DIVISION:

$$10\overline{)6} \rightarrow 10\overline{)6.0} \rightarrow 10\overline{)\begin{array}{r}60 \text{ TENTHS}\\ \underline{60 \text{ TENTHS}}\\ 0 \text{ TENTHS}\end{array}}\ \begin{array}{r}6 \text{ TENTHS}\\ \underline{6 \text{ TENTHS}}\end{array} = .6 \ .$$

WE SHOULD NOW TRY TO FIND DECIMALS WE MIGHT NOT ALREADY KNOW. SUPPOSE WE WANT THE DECIMAL FOR $\frac{1}{2}$.

$\frac{1}{2} = 1 \div 2$, SO WE WRITE:

$$2\overline{)1} \rightarrow 2\overline{)1.0} \rightarrow 2\overline{)\begin{array}{r}10 \text{ TENTHS}\\ \underline{10 \text{ TENTHS}}\\ 0 \text{ TENTHS}\end{array}}\ \begin{array}{r}5 \text{ TENTHS}\\ \underline{5 \text{ TENTHS}}\end{array} = .5 \ .$$

SO THE DECIMAL FOR $\frac{1}{2} = .5$.

THIS MAKES SENSE SINCE $.5 = \frac{5}{10} = \frac{1}{2}$.

NOTICE THAT I CONVERTED THE 1 INTO 1.0 ONLY BECAUSE I WOULD NOT GET ANYWHERE BY DIVIDING 1 BY 2 IN THAT FORM. I WOULD ONLY HAVE GOTTEN 0, REMAINDER 1.

NOW LET US TRY ANOTHER FRACTION. SUPPOSE WE START WITH $\frac{3}{4}$ AND WE WANT THE DECIMAL FOR THIS.

WE WRITE $\frac{3}{4} = 3 \div 4$, SO WE WRITE:

$$4\overline{)3} \rightarrow 4\overline{)3.0} \rightarrow 4\overline{)\begin{array}{r}30 \text{ TENTHS}\\ \underline{28 \text{ TENTHS}}\\ 2 \text{ TENTHS}\end{array}}\ \begin{array}{r}7 \text{ TENTHS}\\ \underline{7 \text{ TENTHS}}\end{array}$$

SO I GET A REMAINDER WHICH I AM NOT SURE HOW TO HANDLE. ONE POSSIBILITY IS TO CHANGE 3 INTO 3.00 INSTEAD OF 3.0 . THIS SEEMS REASONABLE; I CHANGED IT FROM 3 TO 3.0 ONLY TO AVOID THE PROBLEMS OF A REMAINDER, SO I MIGHT TRY TO GO ONE STEP FURTHER.

$$4\overline{)3.00} \rightarrow 4\overline{)\begin{array}{r}300 \text{ HUNDREDTHS}\\ \underline{280 \text{ HUNDREDTHS}}\\ 20 \text{ HUNDREDTHS}\\ \underline{20 \text{ HUNDREDTHS}}\\ 0 \text{ HUNDREDTHS}\end{array}}\ \begin{array}{r}70 \text{ HUNDREDTHS}\\ \\ \underline{5 \text{ HUNDREDTHS}}\\ 75 \text{ HUNDREDTHS}\end{array} = .75$$

SO THE DECIMAL EQUIVALENT FOR $\frac{3}{4} = .75$.

WE WILL NOT ALWAYS GET RID OF THE REMAINDER BY STOPPING AT THE TENTHS PLACE OR EVEN THE HUNDREDTHS OR THOUSANDTHS, BUT WE CAN KEEP TRYING PLACES UNTIL WE EITHER STOP OR ARE CONVINCED WE CANNOT STOP GETTING REMAINDERS.

LET'S TRY ONE MORE QUESTION WITH HUNDREDTHS TOGETHER. WE CAN GET THE DECIMAL FOR $\frac{2}{25}$.

$$2 \div 25 \rightarrow 25\overline{)2} \rightarrow 25\overline{)2.00} \rightarrow 25\overline{)\begin{array}{r} 200 \text{ HUNDREDTHS} \\ \underline{200 \text{ HUNDREDTHS}} \\ 0 \text{ HUNDREDTHS} \end{array}} \begin{array}{l} \\ \underline{8 \text{ HUND.}} \\ 8 \text{ HUND.} \end{array}$$

$$= .08$$

SO, $\frac{2}{25} = .08$.

Hand students worksheet 4 to complete.

<u>Suggested end of day 5.</u>

Changing a Fraction to a Decimal

Outline for approach P:

> WE HAVE NOW HAD A LITTLE EXPERIENCE CONVERTING FRACTIONS TO DECIMALS. BUT SO FAR WE HAVE ONLY USED A FEW DENOMINATORS, LIKE HALVES, AND FOURTHS. NOW WE WILL LEARN TO WORK WITH OTHER DENOMINATORS AS WELL.
>
> AGAIN, WE MUST DO A BIT OF REVIEWING FIRST.
> WHAT DO WE MEAN BY THE FRACTION $\frac{3}{7}$? WHAT DOES THE NUMERATOR, 3, TELL US?

Expect the answer: the number of sevenths we are considering.

> WHAT DOES THE DENOMINATOR, 7, TELL US?

Expect the answer: the whole was divided into 7 equal parts.

> TO DRAW THIS, WE MIGHT DRAW SOMETHING LIKE: 
>
> WHAT DIAGRAM WOULD YOU DRAW TO SHOW $\frac{6}{8}$?

Have one student come to the board to draw his diagram. Repeat this procedure with $\frac{1}{3}$ and $\frac{2}{5}$.

> BECAUSE WE HAVE AIREADY LEARNED THAT IN CHANGING FRACTIONS TO EQUIVALENT DECIMALS, DIVISION IS INVOLVED, LET US QUICKLY GO OVER SOME IDEAS ABOUT DIVISION.
>
> SUPPOSE I HAD A PROBLEM WHERE I EANTED TO DIVIDE 81 PEANUTS AMONG 9 PEOPLE. NOTICE THAT I WOULD USE DIVISION TO SOLVE THIS PROBLEM

AND WOULD SAY THAT IF I WANT TO SPLIT 81 PEANUTS INTO 9 EQUAL GROUPS, EACH PERSON WOULD GET $81 \div 9 = 9$ PEANUTS.

HOW WOULD I SOLVE THE PROBLEM OF SPLITTING 56 CANDIES EVENLY AMONG 8 PEOPLE?

Expect the answer: divide 56 by 8.

WHAT WOULD YOU DO TO SOLVE EACH OF THESE:

TO SPLIT 72 CANDIES AMONG 9 CHILDREN

TO SPLIT 18 PENCILS AMONG 6 CHILDREN

TO SPLIT 15 OBJECTS INTO 3 EVEN PILES

Have three students read out their answers to each of these. Make sure that the students are aware that division solves the type of problem which requires even grouping of a number of objects.

** Here the student will review the idea that $\frac{a}{b} = a \div b$ using examples with denominators other than 2,4,5,10,20, or 50. **

WE HAVE ALREADY SEEN THAT FOR SOME FRACTIONS, ANOTHER WAY OF FINDING A NAME FOR THE FRACTION IS TO DIVIDE THE DENOMINATOR INTO THE NUMERATOR. LET'S SEE IF THIS SAME RULE CAN BE APPLIED TO THE FRACTIONS I AM ABOUT TO PRESENT AND LET'S REVIEW WHY THIS RULE HOLDS.

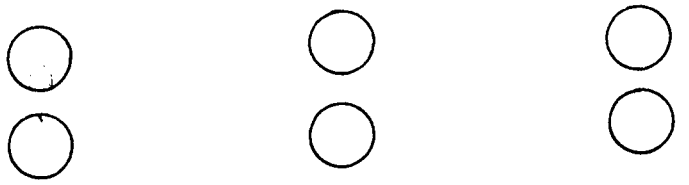
SUPPOSE I HAVE THE FRACTION $\frac{7}{3}$. IF THE RULE WORKS, $\frac{7}{3}$ SHOULD BE THE SAME AS $7 \div 3$.

SUPPOSE WE LOOK AT $7 \div 3$. $7 \div 3$ TELLS ME THAT I SHOULD TRY TO DIVIDE 7 OBJECTS INTO 3 EQUAL GROUPS AND FIND THE SIZE OF

EACH GROUP. WE WANT THAT SIZE TO WORK OUT TO BE $\frac{7}{3}$.

LET US SET UP 3 PILES TO REPRESENT OUR 3 GROUPS AND DISTRIBUTE 7 CIRCLES EVENLY ON THESE 3 PILES.

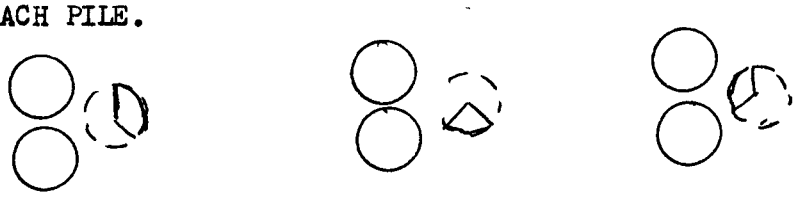
I START BY PUTTING ONE CIRCLE ON EACH PILE, USING 3 AND LEAVING 4. THEN, I CAN PUT ONE MORE CIRCLE ON EACH PILE, USING UP 3 MORE AND NOW LEAVING ONLY ONE CIRCLE.



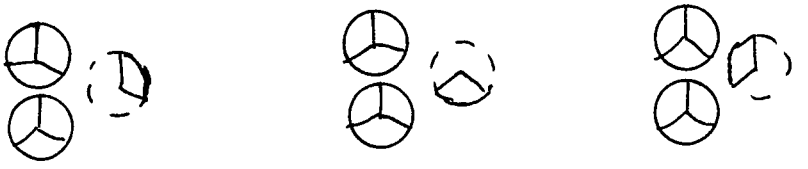
BUT TO PUT THE SAME AMOUNT ON ALL 3 PILES, I MUST SPLIT THE ONE CIRCLE THAT IS LEFT INTO 3 EQUAL PARTS. EACH PART IS WHAT PART OF THE CIRCLE?

Expect the answer: $\frac{1}{3}$.

THEN I PUT $\frac{1}{3}$ MORE CIRCLE ON EACH PILE AND COUNT UP THE SIZE OF EACH PILE.



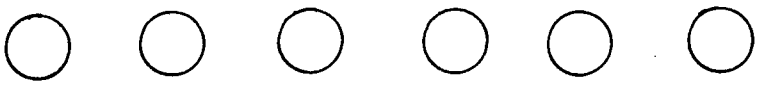
NOTICE, THEN, THAT $7 \div 3 = 2\frac{1}{3}$, BUT IF I WANT TO EXPRESS THIS ANSWER AS AN IMPROPER FRACTION RATHER THAN AMIXED NUMBER, I CAN SPLIT UP EACH CIRCLE ON EACH PILE INTO THIRDS AND COUNT THE SIZE TOTALLY IN THIRDS.



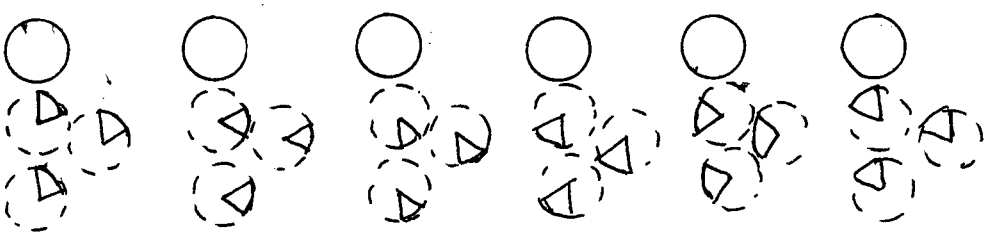
AND I SEE THAT $7 \div 3 = \frac{3}{3} + \frac{3}{3} + \frac{1}{3} = \frac{7}{3}$.

LET'S USE THIS SAME APPROACH TO SOLVE THE PROBLEM OF WHETHER $\frac{9}{6} = 9 \div 6$. IF I WANT TO FIND THE SOLUTION TO $9 \div 6$, I DISTRIBUTE 9 CIRCLES EVENLY ON 6 PILES AND FIND THE SIZE OF EACH OF THESE PILES. THIS SIZE = $9 \div 6$.

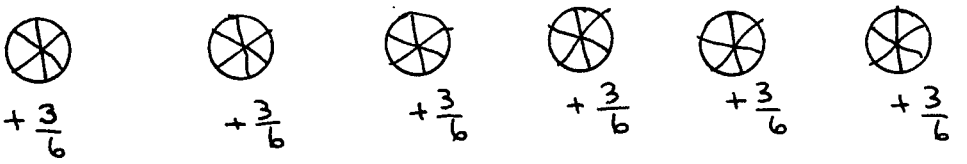
I SET UP 6 PILES. I CAN PUT 1 CIRCLE ON EACH PILE, USING UP 6, LEAVING 3.



THEN I CANNOT PUT ANOTHER WHOLE CIRCLE ON EACH PILE, SINCE I ONLY HAVE 3 LEFT AND NEED 6 TO PUT ONE ON EACH PILE. BUT WHAT I CAN DO IS TO TAKE EACH OF MY LEFTOVER CIRCLES AND SPLIT IT EVENLY INTO 6 PARTS. THEN I CAN PUT 1 OF EACH OF THESE PARTS ON EACH PILE, AND SO DO THE SAME FOR EACH OF THE 3 CIRCLES.



NOTICE, THEN, THAT $9 \div 6 = 1\frac{3}{6}$. BUT IF I WANT TO EXPRESS THIS ANSWER AS AN IMPROPER FRACTION RATHER THAN A MIXED NUMBER, I CAN SPLIT UP EACH CIRCLE ON EACH PILE INTO SIXTHS AND COUNT THE SIZE TOTALLY IN SIXTHS.



AND I SEE THAT $9 \div 6 = \frac{6}{6} + \frac{3}{6} = \frac{9}{6}$.

IT DOES SEEM TO BE TRUE THAT ANY FRACTION CAN BE EXPRESSED AS THE QUOTIENT FROM DIVIDING ITS NUMERATOR BY ITS DENOMINATOR.

Suggested end of day 6.

Hand students worksheet 5 to complete.

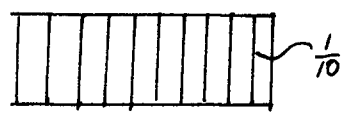
** Here the student will learn to convert fractions to decimals even when the decimals are non-terminating by approximating to three places. **

SUPPOSE WE NOW TRY TO CONVERT $\frac{1}{3}$ TO A DECIMAL.

TO FIND THE DECIMAL FOR $\frac{1}{3}$, WE KNOW THAT WE CAN FIND THE ANSWER TO $1 \div 3$. BUT TO FIND THE ANSWER TO $1 \div 3$, REMEMBER THAT WE TRIED TO EXPRESS 1 AS A DECIMAL SO THAT WE COULD USE OUR USUAL DIVISION PROCEDURES AND AVOID FRACTIONAL REMAINDERS.

SO, WE SAY, $1 = 1 + \frac{0}{10} = 1.0 = 10$ TENTHS.

IF YOU REMEMBER, 1.0 = 10 TENTHS SINCE WE CAN DRAW A DIAGRAM LIKE THIS ONE:



NOW, TO DIVIDE 10 TENTHS BY 3, WE WRITE:

$$3\overline{)\begin{array}{r} 10 \text{ TENTHS} \\ \underline{9 \text{ TENTHS}} \\ 1 \text{ TENTH} \end{array}} \quad \frac{3 \text{ TENTHS}}{3 \text{ TENTHS}} = .3$$

AND WE SEE THAT WE STILL HAVE A REMAINDER. HOWEVER, WE DO KNOW NOW THAT $\frac{1}{3}$ IS SOMEWHERE AROUND 3 TENTHS = .3. WE OFTEN WRITE THIS AS $\frac{1}{3} \approx .3$. THIS MEANS THAT .3 IS ALMOST EQUAL TO $\frac{1}{3}$.

WHEN THIS SAME THING HAPPENED TO USE WHEN WE WERE TRYING TO FIND THE DECIMAL TO $\frac{1}{4}$, WE TRIED WRITING 1 = 1.00; THAT IS, WE WENT TO THE HUNDREDTHS PLACE.

SO, 1 = 1.00 = 100 HUNDREDTHS (SINCE 10 TENTHS = 10 x 10 HUNDREDTHS)

AND WE DIVIDE:

```
3 ) 100 HUNDREDTHS
     90 HUNDREDTHS   30 HUNDREDTHS
     10 HUNDREDTHS
      9 HUNDREDTHS    3 HUNDREDTHS
      1 HUNDREDTH    33 HUNDREDTHS  = .33
```

AGAIN, WE HAVE A REMAINDER, BUT NOW WE HAVE A BETTER <u>APPROXIMATION</u> TO $\frac{1}{3}$, AND WE WRITE $\frac{1}{3} \approx .33$. THIS IS CALLED A TWO-PLACE APPROXIMATION, WHEREAS .3 WAS CALLED THE ONE-PLACE APPROXIMATION. THIS IS BECAUSE WE HAVE APPROXIMATED THE SECOND PLACE AFTER THE DECIMAL POINT (THE HUNDREDTHS).

WELL, SINCE WE STILL HAD A REMAINDER, WE MIGHT TRY WRITING 1 IN YET ANOTHER WAY. WHAT DO YOU SUGGEST?

Expect the answer: as 1.000 = 1000 thousandths.

LET US TRY WRITING 1 AS 1000 THOUSANDTHS.

WE GET:

```
3 ) 1  ->  3 ) 1000 THOUSANDTHS
                900 THOUSANDTHS   300 THOUSANDTHS
                100 THOUSANDTHS
                 90 THOUSANDTHS    30 THOUSANDTHS
                 10 THOUSANDTHS
                  9 THOUSANDTHS     3 THOUSANDTHS
                  1 THOUSANDTH    333 THOUSANDTHS = .333
```

AND, AGAIN, WE STILL HAVE A REMAINDER BUT A STILL BETTER APPROXIMATION FOR $\frac{1}{3}$.

HERE, WE SEE $\frac{1}{3} \approx .333$, A THREE-PLACE APPROXIMATION.

NOTICE THAT THE FIRST TIME WE DID THE DIVISION, WE FOUND THAT $\frac{1}{3}$ WAS BETWEEN .3 AND .4 SINCE WE HAD .3 WITH A REMAINDER UNDER A TENTH, THE NEXT TIME THAT IT WAS BETWEEN .33 AND .34 SINCE WE HAD .33 WITH A REMAINDER UNDER A HUNDREDTH, AND THIS TIME THAT IT IS BETWEEN .33 AND .334 .

DO YOU THINK WE SHOULD TRY TO WRITE 1 = 1.0000 = 10000 TEN-THOUSANDTHS?

WHAT DO YOU THINK WILL HAPPEN?

Expect the answer: we will get an approximation of .3333, but not an exact answer. Suggest this if the answer is not forthcoming.

SOMETIMES IT TURNS OUT THAT WE NEVER GET AN EXACT DECIMAL ANSWER FOR A FRACTION NAME. FOR EXAMPLE, HERE WITH $\frac{1}{3}$, WE WILL KEEP GETTING BETTER APPROXIMATIONS- FROM .3 TO .33, FROM .33 TO .333, FROM .333 TO .3333, ETC., BUT NEVER AN EXACT ANSWER. THE ANSWER CANNOT BE FOUND, THEREFORE, AS TENTHS OR HUNDREDTHS OR THOUSANDTHS.

Suggested end of day 7.

LET US LOOK AT THIS A LITTLE BIT. REMEMBER WHEN WE WERE TRYING TO FIND THE DECIMAL FOR $\frac{1}{2}$. WE FOUND OUT THAT IT WAS $.5 = \frac{5}{10}$, AND WE ALREADY KNEW THAT $\frac{1}{2}$ COULD BE WRITTEN IN TENTHS AS $\frac{5}{10}$. WHEN WE FOUND THE DECIMAL FOR $\frac{3}{4}$, WE FOUND THAT IT WAS .75, OR 75 HUNDREDTHS ($\frac{75}{100}$) AND WE KNOW THAT WE CAN REDUCE $\frac{75}{100}$ TO GET $\frac{3}{4}$. BUT CAN YOU WRITE $\frac{1}{3}$ AS AN EVEN NUMBER OF TENTHS? AS AN EVEN NUMBER OF HUNDREDTHS? AS AN EVEN NUMBER OF THOUSANDTHS?

IT TURNS OUT THAT WE CANNOT, AND THAT IS WHY WE MUST BE SATISFIED WITH ONLY AN APPROXIMATE DECIMAL FOR $\frac{1}{3}$.

NOTICE, HOWEVER, THAT THERE IS A PATTERN TO THE ANSWERS WE GET. WE STARTED WITH .3, THEN GOT .33, THEN .333. IT TURNS OUT THAT EVEN THOUGH WE CANNOT GET AN EXACT DECIMAL, WE CAN ALWAYS FIND A PATTERN.

LET'S TRY TO GET SOME OTHER DECIMAL EQUIVALENTS. AND WE CAN DECIDE, IN ADVANCE, THAT EVEN IF WE CANNOT GET AN EXACT ANSWER, WE WILL USUALLY BE HAPPY WITH A <u>3-PLACE</u> APPROXIMATION.

LET'S GET THE DECIMAL FOR $\frac{2}{9}$.

WE START BY WRITING 2 = 2.0 = 20 TENTHS.

THEN, WE WRITE:

```
9 )20 TENTHS
   18 TENTHS   |  2 TENTHS
    2 TENTHS   |  2 TENTHS = .2
```

WE STILL HAVE A REMAINDER, BUT A ONE-PLACE APPROXIMATION FOR $\frac{2}{9} \approx .2$.

LET'S TRY GETTING A TWO-PLACE APPROXIMATION.

```
9 )2.00  →  9 )200 HUNDREDTHS
               180 HUNDREDTHS   |  20 HUNDREDTHS
                20 HUNDREDTHS
                18 HUNDREDTHS   |   2 HUNDREDTHS
                 2 HUNDREDTHS   |  22 HUNDREDTHS = .22
```

AGAIN, WE DID NOT GET AN EXACT ANSWER, BUT OUR TWO-PLACE APPROXIMATION FOR $\frac{2}{9}$ IS .22 . THIS TELLS US THAT IS BETWEEN .22 AND .23 .

IF WE TRY TO GET THE THREE-PLACE APPROXIMATION, WHAT DO YOU GUESS IT WILL BE?

Expect the answer: .222.

WE CAN CHECK:

```
9)2.000  →  9)2000 THOUSANDTHS
              1800 THOUSANDTHS      200 THOUSANDTHS
              ----
               200 THOUSANDTHS
               180 THOUSANDTHS       20 THOUSANDTHS
               ---
                20 THOUSANDTHS
                18 THOUSANDTHS        2 THOUSANDTHS
                --                   ---
                 2 THOUSANDTHS      222 THOUSANDTHS = .222
```

AND YOU WERE CORRECT.

AGAIN, NOTICE THAT THERE WAS A PATTERN, WE GOT .2, THEN .22, THEN .222 .

BUT THE PATTERN IS NOT ALWAYS THIS NICE. LET US TRY ANOTHER QUESTION. LET'S FIND THE DECIMAL FOR $\frac{3}{11}$.

LET US GO DIRECTLY TO THE 3-PLACE APPROXIMATION. WE CANNOT LOSE ANYTHING BY THIS APPROACH, SINCE WE CAN JUST READ OFF THE 1 AND 2 PLACE APPROXIMATIONS FROM IT. FOR EXAMPLE, THE 3-PLACE APPROXIMATION FOR $\frac{2}{9}$ = .222 TELLS ME THAT THE ONE-PLACE APPROXIMATION IS .2 AND THE TWO-PLACE IS .22 .

NOW TO GET BACK TO $\frac{3}{11}$. I WANT TO REWRITE 3 = 3.000 = 3000 THOUSANDTHS.

THEN,

```
11 ⟌3.000  →  11 ⟌3000 THOUSANDTHS
                 2200 THOUSANDTHS    200 THOUSANDTHS
                  800 THOUSANDTHS
                  770 THOUSANDTHS     70 THOUSANDTHS
                   30 THOUSANDTHS
                   22 THOUSANDTHS      2 THOUSANDTHS
                    8 THOUSANDTHS    272 THOUSANDTHS
                                       = .272
```

SO I SEE THAT $\frac{3}{11} \approx .272$.

IT TURNS OUT THAT IF I HAD USED 1 = 1.0000INSTEAD OF 1.000, I WOULD HAVE FOUND THAT THE FOUR-PLACE APPROXIMATION IS .2727 .

CAN ANYONE GUESS WHAT THE NEXT DIGIT WOULD BE FOR THE FIVE-PLACE APPROXIMATION?

Expect the answer: 2.

ACTUALLY THE PATTERN IS 27 REPEATING.

For each of these problems, have the students find the three-place approximation to the fraction's decimal equivalent. Make sure that the students who show their answers also explain why the numerator was converted into thousandths as it was.

$\frac{2}{7}$

$\frac{3}{13}$

$\frac{4}{11}$

Suggested end of day 8.

Changing a Fraction to a Decimal

Outline for approach a:

> WE ARE GOING TO LEARN A METHOD FOR WRITING FRACTIONS IN A NEW WAY, BUT TO DO THIS, WE MUST FIRST DO SOME REVIEWING ABOUT FRACTIONS AND WE MUST LEARN THIS NEW SYSTEM OF WRITING NUMBERS- AS DECIMALS.

Go over items 21-25 on the pretest. In particular, have students realize that a fraction can always be written as a multiplication expression. For example, $\frac{4}{5} = \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} = 4 \times \frac{1}{5}$. For each question on the pretest listed (21-25), read the question and ask for the correct answer. This review should go quickly, particularly since the students worked with fractions in the previous unit.

> SUPPOSE I WANT TO FIND THE SOLUTION TO THE PROBLEM $3 \times \square = 12$. I HAVE A MULTIPLICATION SENTENCE TO SOLVE, BUT BECAUSE I AM MISSING ONE OF THE MULTIPLIERS, RATHER THAN THE PRODUCT, THE OPERATION I PERFORM IS DIVISION. HERE I WOULD DIVIDE 12 BY 3, AND MY SOLUTION IS WHAT?

Expect the answer: $\square = 4$.

> THEREFORE, I CAN TURN ANY MULTIPLICATION STATEMENT INTO A DIVISION STATEMENT AND VICE VERSA. FOR EXAMPLE, HOW DO I TURN THE FOLLOWING INTO A MULTIPLICATION STATEMENT:
>
> $12 \div 3 = \square$.

Expect the answer: $\square \times 3 = 12$.

HOW DO I TURN THIS MULTIPLICATION STATEMENT INTO A DIVISION ONE:

$3 \times \square = 27$

Expect the answer: $27 \div \square = 3$ or $27 \div 3 = \square$.

Go over items 16-20 on the pretest and make sure the students see the relationship between multiplication and division. Read each question and have a student provide the answer.

NOTICE THAT IN SOLVING $27 \div 3$, I DID NOT REALLY NEED TO GO THROUGH A LONG DIVISION PROCESS, SINCE I QUICKLY KNEW THE ANSWER AS A FACT. BUT SUPPOSE MY QUESTION WERE THE FOLLOWING: $13 \times \square = 2639$. WHAT WOULD I HAVE TO DO TO SOLVE THIS QUESTION?

Expect the answer: divide 2639 by 13.

WE CAN ACTUALLY DO THE DIVISION HERE. I BEGIN BY SETTING UP THE QUESTION, LIKE SO:

$13\overline{)2639}$ AND I KEEP TAKIND CHUNKS OUT OF 2639 IN GROUPS OF 13 AND COUNT UP THE NUMBER OF CHUNKS OF 13 FINALLY TAKEN OUT TO USE UP THE 2639 ENTIRELY.

I CAN TAKE OUT 13 CHUNKS EACH OF SIZE 200 FIRST, LEAVING 39 . THEN I CAN TAKE OUT 3 MORE CHUNKS OF 13 TO USE UP ALL 2639.

$$\begin{array}{r|l} 13\overline{)2639} & \\ \underline{2600} & 200 \\ 39 & \\ \underline{39} & \underline{3} \\ & 203 \end{array}$$

THEREFORE, IF $13 \times \square = 2639$, $\square = 203$.

NOTICE THAT I COULD HAVE USED THIS SAME TECHNIQUE TO SOLVE $\square \times 13 = 2639$, SINCE WE KNOW THAT THE ORDER IN WHICH WE MULTIPLY NEVER CHANGES THE ANSWER.

Allow the students to solve these two division questions and have two students show their results to the class;

$9\overline{)288}$ AND $20\overline{)4300}$

** Here the student will learn some of the features of the decimal system. **

AS WE SAID BEFORE, WE ARE GOING TO WRITE FRACTIONS IN A NEW WAY, THESE NEW SYMBOLS WILL BE CALLED DECIMALS, SO WE WILL LEARN SOMETHING ABOUT THE SYSTEM OF DECIMALS.

LET'S REVIEW PLACE VALUE FOR THE WHOLE NUMBERS. WE READ 345 AS " THREE HUNDREDS AND FORTY-FIVE" BECAUSE OF THE SYSTEM WE HAVE FOR WRITING NUMBERS. WE CAN TELL BY HOW THE NUMBER IS WRITTEN WHETHER THE 3 STANDS FOR 3 HUNDREDS OR 3 TENS OR 3 ONES.

NOTICE THAT FOR EACH PLACE, THE VALUE OF THE PLACE TO THE RIGHT IS $\frac{1}{10}$ THE VALUE OF THE PLACE TO THE LEFT. FOR EXAMPLE, 1 IS $\frac{1}{10}$ OF 10, SINCE $10 \times 1 = 10$; 10 IS $\frac{1}{10}$ OF 100, SINCE $10 \times 10 = 100$, 100 IS $\frac{1}{10}$ OF 1000, SINCE $10 \times 100 = 1000$.

SO IT SEEMS REASONABLE THAT THERE SHOULD BE A PLACE TO THE RIGHT OF THE ONES SO THAT THE VALUE OF THE PLACE IS $\frac{1}{10}$ OF 1,

NAMELY $\frac{1}{10}$. $\frac{1}{10}$ IS $\frac{1}{10}$ OF 1 SINCE $10 \times \frac{1}{10} = 1$.
SO SUPPOSE I START WITH THE NUMBER $463\frac{6}{10}$. IN MY NEW SYSTEM, I WOULD LIKE TO WRITE THIS WITHOUT ANY FRACTIONS. IF I WRITE IT AS 463 6, I NEED A MARKER TO TELL ME THE DIFFERENCE BETWEEN THIS NUMBER AND THE NUMBER "FOUR THOUSAND SIX HUNDRED THIRTY-SIX". THIS MARKER, CALLED THE DECIMAL POINT, APPEARS BETWEEN THE ONES COLUMN AND THE TENTHS COLUMN; THEREFORE, I WRITE THIS NUMBER AS 463.6 .

HOW WOULD YOU WRITE $45\frac{2}{10}$ AS A DECIMAL?

Expect the answer: 45.2 .

Have the students write decimals for each of the following. Ask a student to read each of his decimal equivalents.

$227\frac{3}{10}$

$376\frac{1}{10}$

$4\frac{8}{10}$

NOTICE THAT IF I HAVE A WHOLE NUMBER, I CAN ALSO WRITE THIS AS A DECIMAL. FOR EXAMPLE, I CAN WRITE THE WHOLE NUMBER $4 = 4 + \frac{0}{10}$ AS 4.0 . HOW MIGHT I WRITE 82 AS A DECIMAL?

Expect the answer: 82.0 .

NOW JUST AS WE CAN WRITE A DECIMAL FOR A MIXED NUMBER OR FRACTION, WE MIGHT START WITH A DECIMAL AND TRY TO FIGURE OUT ITS FRACTIONAL OR MIXED NUMBER EQUIVALENT.

SUPPOSE WE START WITH THE DECIMAL 42.8 . HOW COULD WE WRITE THIS IN FRACTIONAL FORM?

Expect the answer: $42\frac{8}{10}$.

ONE POSSIBLE ANSWER IS AS: $42\frac{8}{10}$. SURELY THIS IS JUST WHAT WE HAVE BEEN DISCUSSING. BUT OTHER POSSIBILITIES ALSO EXIST. JUST AS 428 CAN BE READ EITHER AS "FOUR HUNDRED TWENTY-EIGHT ONES" OR AS "FORTY-TWO TENS AND EIGHT ONES", WE MIGHT SUSPECT THAT 42.8 CAN BE READ IN SEVERAL WYAS. ONE WAY IS AS "FORTY-TWO ONES AND EIGHT TENTHS" AS YOU SUGGESTED. BUT ANOTHER MIGHT BE AS "FOUR HUNDRED TWENTY-EIGHT TENTHS". LET'S SEE IF THIS MAKES SENSE:

SUPPOSE I HAVE 42 ONES. EACH 1 IS HOW MANY TENTHS?

Expect the answer: 10 tenths.

WE CAN RENAME $1 = 10 \times \frac{1}{10}$ TO CHECK THIS,

THEN, IF EACH 1 IS $10 \times \frac{1}{10}$, 42 ONES = 42 x 1 CAN BE RENAMED:

$$
\begin{aligned}
& 42 \times 1 \\
= \; & 42 \times (10 \times \tfrac{1}{10}) \\
= \; & (42 \times 10) \times \tfrac{1}{10} \\
= \; & 420 \times \tfrac{1}{10} \\
= \; & \tfrac{420}{10} \quad .
\end{aligned}
$$

THEN, SINCE I HAD 42.8, I HAVE $\frac{420}{10} + \frac{8}{10} = \frac{428}{10}$ = 428 TENTHS.

LET'S READ SOME OTHER DECIMALS SEVERAL WAYS.

HOW MIGHT I READ 3.5?

Expect the answer: $3\frac{5}{10}$ or $\frac{35}{10}$.

ONE WAY IS AS THE MIXED NUMBER $3\frac{5}{10}$. ANOTHER WOULD BE 35 TENTHS, SINCE I CAN RENAME 3.5 AS 3 ONES + 5 TENTHS.

3 ONES = 3×1

$= 3 \times (10 \times \frac{1}{10})$

$= (3 \times 10) \times \frac{1}{10}$

$= 30 \times \frac{1}{10}$ = 30 TENTHS.

THEN, 3 ONES + 5 TENTHS = 30 TENTHS + 5 TENTHS = 35 TENTHS.

READ EACH OF THESE IN TWO WAYS:

36.9

1.5

28.4

Ask a few students to read each of these- once as a mixed number and once as an improper fraction in tenths.

Suggested end of day 1.

NOW THAT WE HAVE CONSIDERED THE POSSIBILITY OF A TENTHS PLACE, WE OUGHT TO CONSIDER WHAT OTHER COLUMNS WE MIGHT HAVE EVEN FURTHER TO THE RIGHT. JUST AS $1 = 10 \times \frac{1}{10}$, WE MUST FIND A PLACE VALUE SO THAT 10x THAT VALUE = $\frac{1}{10}$.

WHAT NUMBER CAN I MULTIPLY BY 10 TO GET $\frac{1}{10}$?

Expect the answer: $\frac{1}{100}$.

$\frac{1}{100}$ WILL DO, SINCE $10 \times \frac{1}{100} = \frac{10}{100} = \frac{1}{10}$.

SIMILARLY, THE NEXT PLACE TO THE RIGHT WOULD BE THOUSANDTHS SINCE $\frac{1}{100} = 10 \times \frac{1}{1000}$.

THEREFORE, THE DECIMAL 45.82 WOULD BE READ AS "FORTY-FIVE ONES, EIGHT TENTHS AND TWO HUNDREDTHS".

HOWEVER, AS BEFORE, WE CAN FIND SEVERAL OTHER WAYS OF READING THIS DECIMAL. FOR EXAMPLE, I MIGHT READ THIS AS "FORTY-FIVE ONES AND 82 HUNDREDTHS". LET'S SEE IF THIS MAKES SENSE:

45.82 MEANS FORTY-FIVE ONES AND EIGHT TENTHS AND TWO HUNDREDTHS. BUT WE KNOW THAT $\frac{1}{10} = \frac{10}{100}$. THAT IS HOW WE KNEW THAT THE HUNDREDTHS COLUMN APPEARED TO THE RIGHT OF THE TENTHS.

SO, $\frac{8}{10} = 8 \times \frac{10}{100} = \frac{80}{100}$ = 80 HUNDREDTHS

AND $\frac{80}{100} + \frac{2}{100} = \frac{82}{100}$ = 82 HUNDREDTHS.

ANOTHER POSSIBLE WAY TO READ THE DECIMAL 45.82 WOULD BE "FOUR THOUSAND FIVE HUNDRED EIGHTY-TWO HUNDREDTHS" = $\frac{4582}{100}$.

WE KNOW THAT 45.8 CAN BE READ AS $\frac{458}{10}$ AND SINCE EACH TENTH = $\frac{10}{100}$

WE KNOW THAT $\frac{458}{10} = \frac{458 \times 10}{100} = \frac{4580}{100}$,

AND THAT $45.82 = \frac{4580}{100} + \frac{2}{100} = \frac{4582}{100}$.

HOW COULD YOU READ 3.78?

Expect the answers: either "three ones and seven tenths and eight hundredths" or "three ones and seventy-eight hundredths" or "378 hundredths".

NOTICE THAT WE CAN RENAME APPROPRIATELY TO ILLUSTRATE EACH OF THESE READINGS.

(a) 3.78 = 3 ONES + 7 TENTHS + 8 HUNDREDTHS (BY DEFINITION)

(b) $3.78 = 3 \text{ ONES} + 7 \times (10 \times \frac{1}{100}) + 8 \times \frac{1}{100}$

$= 3 \text{ ONES} + (7 \times 10) \times \frac{1}{100} + 8 \times \frac{1}{100}$

$= 3 \text{ ONES} + 70 \times \frac{1}{100} + 8 \times \frac{1}{100}$

$= 3 \text{ ONES} + 78 \times \frac{1}{100}$

= 3 ONES + 78 HUNDREDTHS

(c) 3.78 = 3 ONES + 78 HUNDREDTHS

= 3 x 1 + 78 HUNDREDTHS

= 3 x $(100 \times \frac{1}{100})$ + 78 HUNDREDTHS

= (3 x 100) x $\frac{1}{100}$ + 78 HUNDREDTHS

= 300 x $\frac{1}{100}$ + 78 HUNDREDTHS

= 300 HUNDREDTHS + 78 HUNDREDTHS

= 378 HUNDREDTHS

Have the students read each of the following decimals first as mixed numbers and then as hundredths.

54.29

36.21

8.09

4.23

Have students read out their answers. If there is any difficulty, have them rename as above illustrated:

NOW SUPPOSE WE WANT TO WRITE SOME MIXED NUMBERS OR FRACTIONS AS DECIMALS. WE MIGHT WISH, FOR EXAMPLE, TO WRITE $13\frac{81}{100}$ AS A DECIMAL. WHAT WOULD IT BE?

Expect the answer: 13.81.

WRITE EACH OF THE FOLLOWING AS A DECIMAL:

$5\frac{2}{100}$

$6\frac{13}{100}$

$28\frac{14}{100}$

Have the students who read out their answers explain in terms of renaming

how they got the correct answer.

> SIMILARLY, I MIGHT READ AND WRITE DECIMALS INTO THE THOUSANDTHS. HOW WOULD I READ 34.529?

Expect the answers: 34 ones and 5 tenths and 2 hundredths and 9 thousandths, or 34 ones and 529 thousandths, or 34,529 thousandths. If not all these answers come up, mention the ones that did not. Point out the analogy to the hundredths situation.

> IN PARTICULAR, I CAN REWRITE A WHOLE NUMBER, SAY 4, AS TENTHS OR HUNDREDTHS OR THOUSANDTHS. THAT IS, $4 = 4.0 = 4.00 = 4.000$, SO 4 CAN BE READ AS 4, 40 TENTHS, 400 HUNDREDTHS, OR 4000 THOUSANDTHS.

Hand students worksheet 1 to complete.

** Here the students will be introduced to the idea of dividing decimals by wholes.**

> SUPPOSE WE WOULD LIKE TO FIND THE ANSWER TO $1.5 \div 3$.
>
> WE KNOW THAT IF $1.5 \div 3 = \square$, THEN $\square \times 3 = 1.5$.
>
> WE HAVE NEVER DONE ANY MULTIPLICATION WITH DECIMALS, SO WE MIGHT SIMPLIFY OUR PROBLEM BY REWRITING $1.5 = \frac{15}{10} = 15 \times \frac{1}{10}$.
>
> WE KNOW THAT $1.5 = \frac{15}{10}$ FROM OUR PREVIOUS DISCUSSION AND WE KNOW THAT $\frac{15}{10} = 15 \times \frac{1}{10}$ SINCE $\frac{15}{10} = \underbrace{\frac{1}{10} + \frac{1}{10} + \frac{1}{10} \ldots + \frac{1}{10}}_{\text{(15 TIMES)}}$
>
> THEN I AM LOOKING FOR A NUMBER TO MULTIPLY BY 3 TO GET 15 TENTHS.
>
> NOTICE THAT IF I WRITE THIS AS:
>
> $\square \times 3 = 15 \times \frac{1}{10}$, I MAY HAVE TO REPLACE $\square$ WITH

A PRODUCE RATHER THAN JUST A SINGLE NUMBER.

FOR EXAMPLE, TO SOLVE $3 \times \square = 3 \times 5 \times 4$, I REPLACE $\square$ WITH 5×4.

WE CAN DO THE SAME THING HERE.

IF $\square \times 3 = 15 \times \frac{1}{10}$, WE SEE THAT WE CAN REPLACE $\square$ BY $5 \times \frac{1}{10}$.

THEN, $\square \times 3 = (5 \times \frac{1}{10}) \times 3 = (5 \times 3) \times \frac{1}{10} = 15 \times \frac{1}{10} = 1.5$.

SO, $(5 \times \frac{1}{10}) \times 3 = 5$ TENTHS $\times 3 = .5 \times 3 = 1.5$.

NOTICE THAT THIS WAS THE SAME AS FINDING WHAT I HAD TO MULTIPLY 3 BY TO GET 15 AND THEN REMEMBERING TO MULTIPLY THAT ANSWER BY $\frac{1}{10}$.

SO THE ANSWER WAS $(15 \div 3)$ TENTHS.

LET'S FIND $1.6 \div 4$. WHAT DO YOU EXPECT THE ANSWER TO BE?

Expect the answer: 4 tenths or .4 .

LET'S WRITE A MULTIPLICATION SENTENCE.

$\square \times 4 = 1.6 = 16$ TENTHS,

$4 \times 4 = 16$, SO 4 TENTHS $\times 4 = 16$ TENTHS $= 1.6$.

THEREFORE, $\square = 4$ TENTHS $= .4$.

Have the students find answers to:

$4.5 \div 9$

$5.6 \div 8$

$4.8 \div 6$

Ask students to read out their answers. If there is any difficulty with any of these, rewrite as multiplication sentences to find the solution.

Suggested end of day 2.

WHAT WOULD WE DO IF WE WERE TRYING TO FIND $1.25 \div 5$? THAT MEANS THAT I WANT TO FIND WHAT NUMBER TO MULTIPLY 5 BY TO GET 125 HUNDREDTHS. SO WHAT I AM REALLY DOING IS SOLVING:

$\square \times 5 = 125$ HUNDREDTHS.

NOTICE THAT WHAT I AM REALLY DOING IS DIVIDING 125 BY 5 AND THEN REMEMBERING THAT I AM DELAING WITH HUNDREDTHS.

Ask the students to find answers to:

$.25 \div 5$

$1.25 \div 25$

$2.00 \div 4$

Have three students explain their answers. If there is any difficulty, rename as multiplication sentences as illustrated above.

NOW WE CAN WRITE THIS SCHEME IN A WAY THAT WE USUALLY USE TO WRITE DIVISION QUESTIONS.

FOR EXAMPLE, TO FIND $4.2 \div 6$, I WRITE:

$6\overline{)4.2} \rightarrow 6\overline{)42 \text{ TENTHS}} \rightarrow$ AND TO FIND THE ANSWER:

I WRITE:

$$\begin{array}{r} 6\overline{)42 \text{ TENTHS}} \\ \underline{42 \text{ TENTHS}} \\ 0 \text{ TENTHS} \end{array} \quad \begin{array}{r} 7 \text{ TENTHS} \\ \hline 7 \text{ TENTHS} = .7 \end{array}$$

HOW COULD I WRITE OUT THIS PROBLEM: $4.5 \div 9$? I MIGHT WRITE:

$$9\overline{)4.5} \rightarrow 9\overline{)\begin{array}{r} 45 \text{ TENTHS} \\ \underline{45 \text{ TENTHS}} \\ 0 \text{ TENTHS} \end{array}} \quad \begin{array}{r} 5 \text{ TENTHS} \\ \hline 5 \text{ TENTHS} = .5 \end{array}$$

HOW COULD WE WRITE $.36 \div 6$? I MIGHT WRITE:

$$6\overline{).36} \rightarrow 6\overline{)36 \text{ HUNDREDTHS}}$$

$$\begin{array}{r} 36 \text{ HUNDREDTHS} \\ \underline{36 \text{ HUNDREDTHS}} \\ 0 \text{ HUNDREDTHS} \end{array} \quad \begin{array}{r} 6 \text{ HUNDREDTHS} \\ \underline{6 \text{ HUNDREDTHS}} \\ 6 \text{ HUNDREDTHS} \end{array} = .06$$

Hand students worksheet 2 to complete.

** Here the student will now leave decimals for a while to discover that another interpretation for a fraction is division. That is, $\frac{4}{3} = 4 \div 3$. This will tie in to decimals when he discovers that $\frac{1}{2} = 1 \div 2$, and so to get the decimal for $\frac{1}{2}$, he divides 2 into 1. **

NOW AS WE SAID BEFORE, THE WHOLE IDEA OF INTRODUCING DECIMALS WAS TO EVENTUALLY SHOW YOU A WAY TO WRITE FRACTIONS AS DECIMALS. BUT WE HAVE FORGOTTEN ALL ABOUT FRACTIONS. LET US RETURN TO THIS SUBJECT TEMPORARILY.

SUPPOSE I HAVE THE FRACTION $\frac{12}{4}$. LET US SEE WHAT WHOLE NUMBER THIS IS ANOTHER NAME FOR. DO YOU KNOW ALREADY?

Expect the answer:3. In any case, go on with the following discussion.

TELLS ME THAT I HAVE $12 \times \frac{1}{4}$. I CAN RENAME THIS:

$$\begin{aligned} & 12 \times \tfrac{1}{4} \\ &= (3 \times 4) \times \tfrac{1}{4} \\ &= 3 \times (4 \times \tfrac{1}{4}) \\ &= 3 \times 1 \\ &= 3 \end{aligned}$$

BUT: $4 \times \frac{1}{4} = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1.$

AND SO I SEE THAT $\frac{12}{4} = 3$.

LET'S DO THE SAME WITH $\frac{42}{7}$. WHAT SHOULD I WRITE FOR MY RENAMING

AND WHAT WHOLE NUMBER DO I GET EQUAL TO $\frac{42}{7}$?

Expect the students to suggest:

$$= 42 \times \frac{1}{7}$$
$$= (6 \times 7) \times \frac{1}{7}$$
$$= 6 \times (7 \times \frac{1}{7})$$
$$= 6 \times 1$$
$$= 6$$

If they do not suggest this, you rename for them.

NOTICE THAT $\frac{42}{7} = 6$.

NOW WE CAN DO THE SAME FOR $\frac{18}{6}$. WHAT IS THE WHOLE NUMBER FOR $\frac{18}{6}$?

Expect the answer: 3.

Rename:

$$\frac{18}{6} = 18 \times \frac{1}{6}$$
$$= (3 \times 6) \times \frac{1}{6}$$
$$= 3 \times (6 \times \frac{1}{6})$$
$$= 3 \times 1$$
$$= 3$$

MY RENAMING INDICATES THAT 3 IS THE CORRECT ANSWER.

NOTICE THAT EACH TIME I WAS FINDING A WAY OF WRITING THE NUMERATOR AS A PRODUCT WHERE ONE OF THE MULTIPLIERS WAS THE DENOMINATOR: $4 \times \square = 12$ FOR $\frac{12}{4}$, $7 \times \square = 42$ FOR $\frac{42}{7}$, AND $6 \times \square = 18$ FOR $\frac{18}{6}$.

WE AIREADY KNOW THAT FINDING WHAT NUMBER TO MULTIPLY ONE BY TO GET ANOTHER IS THE SAME AS DIVIDING THAT SECOND NUMBER BY THE FIRST.

SO, IN FINDING THE WHOLE NUMBER FOR $\frac{42}{7}$, FOR EXAMPLE, I WAS FINDING OUT HOW TO WRITE $42 = \square \times 7$, OR $42 = 7 \times \square$. AND THAT ANSWER WAS THE WHOLE NUMBER FOR $\frac{42}{7}$. BUT THAT ANSWER IS ALSO THE ANSWER TO $42 \div 7 = \square$ OR $42 \div \square = 7$.

FOR EACH OF THESE FRACTIONS, GIVE A DIVISION SENTENCE THAT WOULD GET A WHOLE NUMBER NAME FOR THE FRACTION:

$\frac{16}{8}$

$\frac{15}{3}$

$\frac{14}{2}$

Ask 3 students to read out their answers and then explain them by renaming in a multiplication sentence.

Suggested end of day 3.

** Here the student will learn that another name for a fraction can always be found by dividing the numerator by the denominator, even when the division is not even. **

NOW WE HAVE SEEN THAT FOR SOME FRACTIONS, ANOTHER WAY OF FINDING A NAME FOR THE FRACTION IS TO DIVIDE THE DENOMINATOR INTO THE NUMERATOR. WE WOULD LIKE TO FIND OUT IF THIS IS ALWAYS TRUE.

SUPPOSE WE START WITH $\frac{11}{5}$. IF OUR RULE WORKS, $\frac{11}{5}$ SHOULD BE THE SAME AS $11 \div 5$; THAT IS, IF I FIND THE SOLUTION TO $\square \times 5 = 11$, THAT NUMBER $\square$ SHOULD BE $\frac{11}{5}$.

LET'S DO THIS.

SUPPOSE WE WRITE: $\square \times 5 = 11$. IF $\square = \frac{11}{5}$, THEN IT SHOULD BE TRUE THAT $\frac{11}{5} \times 5 = 11$.

BUT, $\frac{11}{5} \times 5$

$= (11 \times \frac{1}{5}) \times 5$

$= 11 \times (5 \times \frac{1}{5})$

$= 11 \times 1$

$= 11$

SO, $\frac{11}{5} \times 5 = 11$, AND, THEREFORE, $11 \div 5 = \frac{11}{5}$.

LET'S TRY THE SAME IDEA WITH $\frac{8}{3}$. IF OUR RULE WORKS, WE SHOULD EXPECT THAT THE SOLUTION TO $\square \times 3 = 8$ SHOULD BE $\frac{8}{3}$.

WELL, $\frac{8}{3} \times 3$

$= (8 \times \frac{1}{3}) \times 3$

$= 8 \times (3 \times \frac{1}{3})$

$= 8 \times 1$

$= 8.$

SO, $\square \times 3 = 8$ AND SO $8 \div 3 = \frac{8}{3}$.

CAN ANYONE TELL ME WHY THEY THINK $\frac{5}{4} = 5 \div 4$?

Expect someone to suggest: If $\square \times 4 = 5$, then $\square = \frac{5}{4}$ since

$\frac{5}{4} \times 4$

$= (5 \times \frac{1}{4}) \times 4$

$= 5 \times (4 \times \frac{1}{4})$

$= 5 \times 1$

$= 5$

If no one does this, you write out this renaming scheme.

CAN ANYONE TELL ME WHY THEY THINK $\frac{3}{4} = 3 \div 4$?

I CAN RENAME $3 \div 4 = \square$ INTO $\square \times 4 = 3$.

THEN, $\frac{3}{4} \times 4$

$= \frac{3}{4} \times 4$

$= (3 \times \frac{1}{4}) \times 4$

$= 3 \times (4 \times \frac{1}{4})$

$= 3 \times 1$

$= 3$

SO, $\square = 3 \div 4 = \frac{3}{4}$.

Hand students worksheet 3 to complete.

Suggested end of day 4.

NOW THAT WE KNOW HOW TO RENAME A FRACTION AS A DIVISION QUESTION, WE CAN PROCEED TO TIE TOGETHER SOME OF OUR DECIMAL IDEAS WITH THIS ONE.

LET'S LOOK AT $\frac{4}{10}$. WE KNOW THAT $\frac{4}{10} = 4 \div 10$ BECAUSE IF $4 \div 10 = \square$, THEN $\square \times 10 = 4$, AND

$\square \times 10 = (4 \times \frac{1}{10}) \times 10 = 4 \times (10 \times \frac{1}{10}) = 4 \times 1 = 4$,

SO, $\frac{4}{10} \times 10$ DOES EQUAL 4, AND THUS, $\frac{4}{10} = 4 \div 10$.

ON THE OTHER HAND, WE ALSO KNOW THAT $\frac{4}{10}$ CAN BE WRITTEN AS A DECIMAL AS .4 . IT WOULD BE INTERESTING TO BE ABLE TO SHOW DIRECTLY THAT $4 \div 10 = .4$.

WELL, $4 \div 10$ CAN BE WRITTEN AS $10 \overline{)4}$. BUT IF I TRY TO DO THIS DIVISION, I ONLY GET AN ANSWER OF 0 WITH REMAINDER 4, WHICH DOES NOT HELP ME MUCH.

SO I MIGHT TRY TO RENAME $4 = 4.0 = \frac{40}{10} = 40$ TENTHS.

REMEMBER, $4.0 = 4 \times 1 = 4 \times (10 \times \frac{1}{10}) = (4 \times 10) \times \frac{1}{10} =$ 40 TENTHS.

THEN, $4 \div 10$ CAN BE WRITTEN:

$$10\overline{)4} \rightarrow 10\overline{)4.0} \rightarrow 10\overline{)\begin{array}{r} 40 \text{ TENTHS} \\ \underline{40 \text{ TENTHS}} \\ 0 \text{ TENTHS} \end{array}} \quad \begin{array}{r} 4 \text{ TENTHS} \\ \overline{4 \text{ TENTHS}} = .4 \end{array}$$

WE HAVE DIRECTLY CONVERTED, THEN, $\frac{4}{10}$ INTO A DECIMAL BY USING DIVISION.

LET US TRY CONVERTING $\frac{6}{10}$ TO A DECIMAL. WE KNOW THAT OUR RESULT SHOULD BE .6 . BUT WE CAN TRY TO FIND THIS THROUGH DIVISION:

$$10\overline{)6} \rightarrow 10\overline{)6.0} \rightarrow 10\overline{)\begin{array}{r} 60 \text{ TENTHS} \\ \underline{60 \text{ TENTHS}} \\ 0 \text{ TENTHS} \end{array}} \quad \begin{array}{r} 6 \text{ TENTHS} \\ \overline{6 \text{ TENTHS}} = .6 \end{array}$$

WE SHOULD NOW TRY TO FIND DECIMALS WE MIGHT NOT ALREADY KNOW. SUPPOSE WE WANT THE DECIMAL FOR $\frac{1}{2}$.

$\frac{1}{2} = 1 \div 2$, SO WE WRITE:

$$2\overline{)1} \rightarrow 2\overline{)1.0} \rightarrow 2\overline{)\begin{array}{r} 10 \text{ TENTHS} \\ \underline{10 \text{ TENTHS}} \\ 0 \text{ TENTHS} \end{array}} \quad \begin{array}{r} 5 \text{ TENTHS} \\ \overline{5 \text{ TENTHS}} = .5 \end{array}$$

SO, THE DECIMAL FOR $\frac{1}{2} = .5$.

THIS MAKES SENSE SINCE $.5 = \frac{5}{10} = \frac{1}{2}$.

NOTICE THAT I CONVERTED THE 1 INTO 1.0 ONLY BECAUSE I WOULD NOT GET ANYWHERE BY DIVIDING 1 BY 2 IN THAT FORM. I WOULD ONLY HAVE GOTTEN 0, REMAINDER 1.

NOW LET US TRY ANOTHER FRACTION. SUPPOSE WE START WITH $\frac{3}{4}$ AND WE WANT THE DECIMAL FOR THIS.

WE WRITE $\frac{3}{4} = 3 \div 4$, SO WE WRITE:

$$4\overline{)3} \rightarrow 4\overline{)3.0} \rightarrow 4\overline{)30 \text{ TENTHS}}$$

30 TENTHS	
28 TENTHS	7 TENTHS
2 TENTHS	7 TENTHS

SO I GET A REMAINDER WHICH I AM NOT SURE HOW TO HANDLE. ONE POSSIBILITY IS TO CHANGE 3 INTO 3.00 INSTEAD. THIS SEEMS REASONABLE. I CHANGED IT FROM 3 TO 3.0 ONLY TO AVOID THE PROBLEMS OR A REMAINDER, SO I MIGHT TRY TO GO ONE STEP FURTHER.

$$4\overline{)3.00} \rightarrow 4\overline{)300 \text{ HUNDREDTHS}}$$

300 HUNDREDTHS	
280 HUNDREDTHS	70 HUNDREDTHS
20 HUNDREDTHS	
20 HUNDREDTHS	5 HUNDREDTHS
0 HUNDREDTHS	75 HUNDREDTHS = .75

SO THE DECIMAL EQUIVALENT FOR $\frac{3}{4}$ = .75 .

WE WILL NOT ALWAYS GET RID OF THE REMAINDER BY STOPPING AT THE TENTHS PLACE, OR EVEN THE HUNDREDTHS OR THOUSANDTHS, BUT WE CAN KEEP TRYING PLACES UNTIL WE EITHER STOP OR ARE CONVINCED WE CANNOT STOP GETTING REMAINDERS.

LET'S TRY ONE MORE QUESTION WITH HUNDREDTHS TOGETHER. WE CAN GET THE DECIMAL FOR $\frac{2}{25}$.

$$\frac{2}{25} = 25\overline{)2} \rightarrow 25\overline{)2.0} \rightarrow 25\overline{)2.00}$$

$$25\overline{)200 \text{ HUNDREDTHS}}$$

200 HUNDREDTHS	
200 HUNDREDTHS	8 HUNDREDTHS
0 HUNDREDTHS	8 HUNDREDTHS = .08 .

SO, $\frac{2}{25}$ = .08. THIS SEEMS CLEAR SINCE .08 = $\frac{8}{100}$ = $\frac{2}{25}$.

Hand students worksheet 4 to complete.

Suggested end of day 5.

Changing a Fraction to a Decimal

Outline for approach A:

WE HAVE NOW HAD A LITTLE EXPERIENCE CONVERTING FRACTIONS TO DECIMALS. BUT SO FAR WE HAVE ONLY USED A FEW DENOMINATORS, LIKE HALVES, AND FOURTHS. NOW WE WILL LEARN TO WORK WITH OTHER DENOMINATORS, AS WELL.

AGAIN, WE MUST DO A BIT OF REVIEWING FIRST.

WHAT DO WE MEAN BY THE FRACTION $\frac{3}{7}$? WE CAN MEAN

$$\frac{3}{7} = \frac{1}{7} + \frac{1}{7} + \frac{1}{7} = 3 \times \frac{1}{7}$$

HOW COULD YOU WRITE $\frac{6}{8}$ AS A MULTIPLICATION EXPRESSION?

Expect the answer: $6 \times \frac{1}{8}$

Repeat this procedure with $\frac{1}{3}$ and $\frac{2}{5}$.

BECAUSE WE HAVE ALREADY LEARNED THAT IN CHANGING FRACTIONS TO EQUIVALENT DECIMALS, DIVISION IS INVOLVED, LET US QUICKLY GO OVER SOME IDEAS ABOUT DIVISION.

SUPPOSE I WANT TO SOLVE THE MULTIPLICATION QUESTION: $8 \times \square = 32$.

WHAT DO I DO TO SOLVE THIS?

Expect the answer: divide 32 by 8.

HOW WOULD YOU SOLVE $\square \times 8 = 56$?

Expect the answer: divide 56 by 8.

NOTICE THAT I DIVIDED NO MATTER WHETHER I WAS SOLVING $8 \times \square = 32$ OR $\square \times 8 = 32$. THE ORDER OF THE MULTIPLICATION DID NOT MATTER,

SO DIVISION SOLVED BOTH PROBLEMS EQUALLY WELL.

THE IMPORTANT THING TO NOTICE IS THAT IF AMULTIPLICATION QUESTION IS MISSING A FACTOR, I PERFORM DIVISION TO FIND THAT FACTOR OUT.

IN TERMS OF MULTIPLICATION STATEMENTS, IT IS IMPORTANT FOR WHAT WE ARE GOING TO DO TO REALIZE HOW THE FORM OF A MULTIPLICATION STATEMENT CAN BE CHANGED WITHOUT CHANGING ITS VALUE.

FOR EXAMPLE, I CAN CHANGE $4 \times 5 \times 9$ TO READ $5 \times 4 \times 9$ WITHOUT CHANGING ITS VALUE SINCE THE ORDER OF MULTIPLICATION DOES NOT MATTER.

I CAN ALSO MULTIPLY THAT EXPRESSION BY 1 WITHOUT CHANGING IT.

SO, $4 \times 5 \times 9 = 4 \times 5 \times 9 \times 1$.

BESIDES THAT, I CAN EVEN MULTIPLY IT BY FORMS OF 1 OTHER THAN JUST 1. FOR EXAMPLE, WE KNOW THAT $\frac{2}{2} = 2 \times \frac{1}{2} = 1$, SO WE CAN RENAME:

$$4 \times 5 \times 9 = 4 \times 5 \times 9 \times (2 \times \frac{1}{2})$$

RENAME EACH OF THESE IN SEVERAL WAYS:

7×6

9×7

4×7

Ask several students to read their answers. Make sure some renaming is based on the order principle and some on the multiplication by one or forms of one principle.

** Here the student will review the idea that $\frac{a}{b} = a \div b$ using examples with denominators other than 2,4,5,10,20, or 50. **

WE HAVE ALREADY SEEN THAT FOR SOME FRACTIONS, ANOTHER WAY OF FINDING A NAME FOR THE FRACTION IS TO DIVIDE THE DENOMINATOR INTO THE NUMERATOR. LET'S SEE IF THIS SAME RULE CAN BE APPLIED TO THE FRACTIONS I AM ABOUT TO PRESENT AND LET'S REVIEW WHY THIS RULE HOLDS.

SUPPOSE I HAVE THE FRACTION $\frac{7}{3}$. IF THE RULE WORKS, $\frac{7}{3}$ SHOULD BE THE SAME AS $7 \div 3$.

SUPPOSE WE LOOK AT $7 \div 3$. IF $7 \div 3 = \square$, THEN I KNOW THAT $\square \times 3 = 7$. WE WANT TO SEE IF $\square$ CAN BE $\frac{7}{3}$.

LET US MULTIPLY $\frac{7}{3}$ BY 3:

$$\frac{7}{3} \times 3$$
$$= (7 \times \frac{1}{3}) \times 3$$
$$= 7 \times (3 \times \frac{1}{3}) \qquad \text{BUT } 3 \times \frac{1}{3} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = \frac{3}{3} = 1$$
$$= 7 \times 1$$
$$= 7.$$

SO, BY RENAMING, I FIND THAT $\frac{7}{3} \times 3$ DOES EQUAL 7, AND SO

$$7 \div 3 = \frac{7}{3} .$$

LET US USE THIS SAME APPROACH TO SOLVE THE PROBLEM OF WHETHER $\frac{9}{6} = 9 \div 6$. IF I WANT TO FIND THE SOLUTION TO $9 \div 6$, I WRITE:

IF $9 \div 6 = \square$, THEN $\square \times 6 = 9$.

THEN,

$$\frac{9}{6} \times 6$$
$$= (9 \times \frac{1}{6}) \times 6$$
$$= 9 \times (6 \times \frac{1}{6})$$
$$= 9 \times 1$$
$$= 9$$

SO, BY RENAMING, I FIND THAT $\frac{9}{6} \times 6 = 9$, SO $9 \div 6 = \frac{9}{6}$.

IT DOES SEEM TO BE TRUE THAT ANY FRACTION CAN BE EXPRESSED AS THE QUOTIENT FROM DIVIDING ITS NUMERATOR BY ITS DENOMINATOR.

Suggested end of day 6.

Hand students worksheet 5 to complete.

** Here the student will learn to convert fractions to decimals even when the decimals are non-terminating by approximating to 3 places. **

SUPPOSE WE NOW TRY TO CONVERT $\frac{1}{3}$ TO A DECIMAL.

TO FIND THE DECIMAL FOR $\frac{1}{3}$, WE KNOW THAT WE CAN FIND THE ANSWER TO $1 \div 3$. BUT TO FIND THE ANSWER TO $1 \div 3$, REMEMBER THAT WE TRIED TO EXPRESS 1 AS A DECIMAL SO THAT WE COULD USE OUR USUAL DIVISION PROCEDURE AND AVOID FRACTIONAL REMAINDERS.

SO, WE SAY, $1 = 1 + \frac{0}{10} = 1.0 = 10$ TENTHS.

IF YOU REMEMBER, 1.0 = 10 TENTHS SINCE;

$$1 = 10 \times \frac{1}{10} = 10 \text{ TENTHS.}$$

NOW, TO DIVIDE 10 TENTHS BY 3, WE WRITE:

$$\begin{array}{r|l} 3\overline{)\ 10 \text{ TENTHS}} & \\ \underline{9 \text{ TENTHS}} & \underline{3 \text{ TENTHS}} \\ 1 \text{ TENTH} & 3 \text{ TENTHS} = .3 \end{array}$$

AND WE SEE THAT WE STILL HAVE A REMAINDER. HOWEVER, WE DO KNOW NOW THAT $\frac{1}{3}$ IS SOMEWHERE AROUND 3 TENTHS = .3. WE OFTEN WRITE THIS AS $\frac{1}{3} \approx .3$. THIS MEANS THAT .3 IS ALMOST EQUAL TO $\frac{1}{3}$.

WHEN THIS SAME THING HAPPENED TO US WHEN WE WERE TRYING TO

FIND THE DECIMAL TO $\frac{3}{4}$, WE TRIED WRITING 1 = 1.00; THAT IS, WE WENT TO THE HUNDREDTHS PLACE.

SO, 1 = 1.0 = 100 HUNDREDTHS (SINCE 1 = 100 x $\frac{1}{100}$ = 100 HUNDREDTHS). AND WE DIVIDE:

```
3)100 HUNDREDTHS
   90 HUNDREDTHS  | 30 HUNDREDTHS
   10 HUNDREDTHS
    9 HUNDREDTHS  |  3 HUNDREDTHS
    1 HUNDREDTH   | 33 HUNDREDTHS = .33
```

AGAIN, WE HAVE A REMAINDER, BUT NOW WE HAVE A BETTER <u>APPROXIMATION</u> TO $\frac{1}{3}$, AND WE WRITE $\frac{1}{3} \approx .33$. THIS IS CALLED A TWO-PLACE APPROXIMATION, WHEREAS .3 WAS CALLED THE ONE-PLACE APPROXIMATION. THIS IS BECAUSE WE HAVE APPROXIMATED THE SECOND PLACE AFTER THE DECIMAL POINT (THE HUNDREDTHS).

WELL, SINCE WE STILL HAVE A REMAINDER, WE MIGHT TRY WRITING 1 IN YET ANOTHER WAY. WHAT DO YOU SUGGEST?

Expect the answer as 1.000 = 1000 thousandths.

LET US TRY WRITING 1 AS 1000 THOUSANDTHS (1 = 1000 x $\frac{1}{1000}$). WE GET:

```
3)1.000 →  3)1000 THOUSANDTHS
              900 THOUSANDTHS  | 300 THOUSANDTHS
              100 THOUSANDTHS
               90 THOUSANDTHS  |  30 THOUSANDTHS
               10 THOUSANDTHS
                9 THOUSANDTHS  |   3 THOUSANDTHS
                1 THOUSANDTH   | 333 THOUSANDTHS = .333
```

AND, AGAIN, WE STILL HAVE A REMAINDER BUT A STILL BETTER APPROXIMATION FOR $\frac{1}{3}$. HERE, WE SEE $\frac{1}{3} \approx .333$, A THREE-PLACE APPROXIMATION.

NOTICE THAT THE FIRST TIME WE DID THE DIVISION, WE FOUND THAT $\frac{1}{3}$ WAS BETWEEN .3 AND .4 SINCE WE HAD .3 WITH A REMAINDER UNDER ONE TENTH, THE NEXT TIME THAT IT WAS BETWEEN .33 AND .34 SINCE WE HAD .33 WITH A REMAINDER UNDER ONE HUNDREDTH, AND THIS TIME THAT IT WAS BETWEEN .333 AND .334.

DO YOU THINK WE SHOULD TRY TO WRITE 1 = 1.0000 = 10,000 TEN-THOUSANDTHS? WHAT DO YOU THINK WILL HAPPEN?

Expect the answer: we will get an approximation of .3333, but not an exact answer. Suggest this if the answer is not forthcoming.

SOMETIMES IT TURNS OUT THAT WE NEVER GET AN EXACT DECIMAL FOR A FRACTION NAME. FOR EXAMPLE, HERE WITH $\frac{1}{3}$, WE WILL KEEP GETTING BETTER APPROXIMATIONS FROM .3 TO .33 TO .333 TO .3333 TO .33333, ETC., BUT NEVER AN EXACT ANSWER. THE ANSWER CANNOT BE FOUND, THEREFORE, AS TENTHS OR HUNDREDTHS OR THOUSANDTHS.

~~Suggested end of day 7.~~

LET US LOOK AT THIS A LITTLE BIT. REMEMBER WHEN WE WERE TRYING TO FIND THE DECIMAL FOR $\frac{1}{2}$. WE FOUND THAT IT WAS $.5 = \frac{5}{10}$, AND WE ALREADY KNOW THAT $\frac{1}{2}$ COULD BE WRITTEN IN TENTHS AS $\frac{5}{10}$. WHEN WE FOUND THE DECIMAL FOR $\frac{3}{4}$, WE FOUND THAT IT WAS .75, OR 75 HUNDREDTHS ($\frac{75}{100}$) AND WE KNOW THAT WE CAN REDUCE $\frac{75}{100}$ TO GET $\frac{3}{4}$. BUT CAN YOU WRITE $\frac{1}{3}$ AS AN EVEN NUMBER OF TENTHS? AS AN EVEN NUMBER OF HUNDREDTHS? AS AN EVEN NUMBER OF THOUSANDTHS? IT TURNS OUT THAT WE CANNOT, AND THAT IS WHY WE MUST BE SATISFIED WITH ONLY AN APPROXIMATE DECIMAL FOR $\frac{1}{3}$.

NOTICE, HOWEVER, THAT THERE IS A PATTERN TO THE ANSWERS WE GET. WE STARTED WITH .3, THEN GOT .33, THEN .333. IT TURNS OUT THAT EVEN THOUGH WE CANNOT GET AN EXACT DECIMAL, WE CAN ALWAYS FIND A PATTERN.

LET'S TRY TO GET SOME OTHER DECIMAL EQUIVALENTS, AND WE CAN DECIDE, IN ADVANCE, THAT EVEN IF WE CANNOT GET AN EXACT ANSWER, WE WILL USUALLY BE HAPPY WITH A <u>3-PLACE</u> APPROXIMATION.

LET'S GET THE DECIMAL FOR $\frac{2}{9}$.

WE START BY WRITING 2 = 2.0 = 20 TENTHS, SINCE 2 = 2 x 1 $= 2 \times (10 \times \frac{1}{10}) = (2 \times 10) \times \frac{1}{10} = 20 \times \frac{1}{10}$.

THEN WE WRITE:

```
9)20 TENTHS
  18 TENTHS
  ---------
   2 TENTHS
```

$\frac{2 \text{ TENTHS}}{2 \text{ TENTHS}} = .2$

WE STILL HAVE A REMAINDER, BUT A ONE-PLACE APPROXIMATION FOR $\frac{2}{9}$ IS .2 .

LET'S TRY GETTING A TWO-PLACE APPROXIMATION.

```
9)2.00 → 9)200 HUNDREDTHS
           180 HUNDREDTHS   20 HUNDREDTHS
           ---------------
            20 HUNDREDTHS
            18 HUNDREDTHS    2 HUNDREDTHS
            -------------   ------------
             2 HUNDREDTHS   22 HUNDREDTHS = .22
```

AGAIN, WE DID NOT GET AN EXACT ANSWER, BUT OUR TWO-PLACE APPROXIMATION IS $\frac{2}{9} \approx .22$. THIS TELLS US THAT $\frac{2}{9}$ IS BETWEEN .22 AND .23.

IF WE TRY TO GET THE THREE-PLACE APPROXIMATION, WHAT DO YOU GUESS

IT WILL BE?

Expect the answer; .222.

WE CAN CHECK:

```
9)2.000 → 9)2000 THOUSANDTHS
           1800 THOUSANDTHS | 200 THOUSANDTHS
            200 THOUSANDTHS |
            180 THOUSANDTHS |  20 THOUSANDTHS
             20 THOUSANDTHS |
             18 THOUSANDTHS |   2 THOUSANDTHS
              2 THOUSANDTHS | 222 THOUSANDTHS = .222
```

AND YOU WERE CORRECT.

AGAIN, NOTICE THAT THERE WAS A PATTERN. WE GOT .2, THEN .22, THEN .222.

BUT THE PATTERN IS NOT ALWAYS THIS NICE. LET US TRY ANOTHER QUESTION. LET'S FIND THE DECIMAL FOR $\frac{3}{11}$.

LET US GO DIRECTLY TO THE 3-PLACE APPROXIMATION. WE CANNOT LOSE ANYTHING BY THIS APPROACH, SINCE WE CAN JUST READ OFF THE 1 AND 2 PLACE APPROXIMATIONS FROM IT. FOR EXAMPLE, THE 3-PLACE APPROXIMATION FOR $\frac{2}{9}$ = .222 TELLS ME THAT THE 1-PLACE APPROXIMATION IS .2 AND THE 2-PLACE IS .22.

NOW TO GET BACK TO $\frac{3}{11}$. I WANT TO RENAME 3 = 3.000 = 3000 THOUSANDTHS. THEN ,

```
11)3.000 → 11)3000 THOUSANDTHS
              2200 THOUSANDTHS | 200 THOUSANDTHS
               800 THOUSANDTHS |
               770 THOUSANDTHS |  70 THOUSANDTHS
                30 THOUSANDTHS |
                22 THOUSANDTHS |   2 THOUSANDTHS
                 8 THOUSANDTHS | 272 THOUSANDTHS = .272
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SO, I SEE THAT $\frac{3}{11} \approx .272$.

IT TURNS OUT THAT IF I HAD USED 1 = 1.0000 INSTEAD OF 1.000, I WOULD HAVE FOUND THAT THE FOUR-PLACE APPROXIMATION IS .2727.

CAN ANYONE GUESS WHAT THE NEXT DIGIT WOULD BE FOR THE FIVE-PLACE APPROXIMATION?

Expect the answer:2.

ACTUALLY THE PATTERN IS 27 REPEATING.

For each of these problems, have the students find the three-place approximation to the fraction's decimal equivalent. Make sure that the students who show their answers on the board also explain why the numerator was converted into thousandths as it was.

$\frac{2}{7}$

$\frac{3}{13}$

$\frac{4}{11}$

Suggested end of day 8.

Comparison of Fractions Using the Cross-Product

Outline for approach p:

> WE WILL NOW LEARN A WAY TO DECIDE WHETHER A FRACTION IS MORE THAN OR LESS THAN A GIVEN WHOLE NUMBER, FOR EXAMPLE, WHETHER $\frac{8}{3}$ IS GREATER THAN OR LESS THAN 4.
>
> BUT TO DO THIS, WE MUST FIRST DO SOME REVIEWING.
>
> I WILL CALL OUT SOME MULTIPLICATION QUESTIONS.

Ask the following questions:

6 x 2
7 x 5
9 x 8
2 x 8
9 x 6

If there is much difficulty in answering these questions on the part of the class as a whole, then draw diagrams to illustrate the problems. For example, to illustrate:

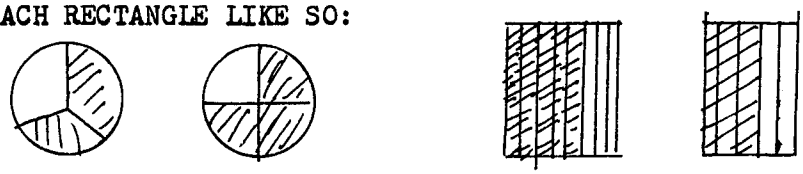
6 x 2	5 x 5
x x	x x x x x
x x	x x x x x
x x	x x x x x
x x	x x x x x
x x	x x x x x
x x	

Notice that the first number represents the number of rows and the second number represents the number of columns.

> BECAUSE WE ARE DEALING WITH FRACTIONS, WE HAD BETTER REMEMBER SOME OF OUR FRACTION WORK, AS WELL.

Go over items 15-19 on the pretest. Read each item, and ask for an answer. Discuss, in particular, that the name of a shaded region depends on the number of parts of the same size into which the figure is drawn and the number of these parts that are shaded in.

LOOK AT THESE PICTURES. EACH CIRCULAR REGION REPRESENTS THE NUMBER 1. EACH RECTANGULAR REGION REPRESENTS THE NUMBER 1 ALSO. NOTICE THAT THE CIRCLES ARE THE SAME SIZE AND THE RECTANGLES ARE THE SAME SIZE. NOW LET US SHADE PART OF EACH CIRCLE AND PART OF EACH RECTANGLE LIKE SO:



WHAT FRACTIONS WOULD WE WRITE UNDER THESE SHADED PICTURES?

Expect the answers: $\frac{2}{3}, \frac{3}{4}, \frac{5}{8}, \frac{3}{5}$.

THE SHADED CIRCLES SHOW US THAT $\frac{2}{3}$ IS LESS THAN $\frac{3}{4}$ BECAUSE THE SHADED REGION IN THE FIRST CIRCLE IS LESS THAN THE SHADED REGION IN THE SECOND CIRCLE.

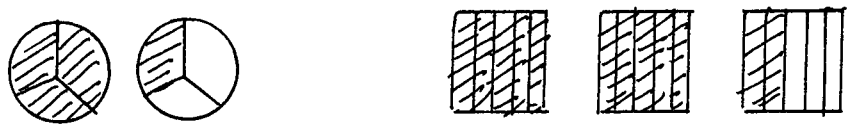
THE SHADED REGION IN THE FIRST RECTANGLE IS GREATER THAN THE SHADED REGION IN THE SECOND RECTANGLE. THIS SHOWS US THAT $\frac{5}{8}$ IS GREATER THAN $\frac{3}{5}$. WE CAN WRITE $\frac{2}{3} < \frac{3}{4}$ AND $\frac{5}{8} > \frac{3}{5}$.

IT WOULD NOT BE SO EASY TO USE THESE PICTURES TO DECIDE WHETHER $\frac{2}{3}$ IS GREATER THAN $\frac{3}{5}$. CAN YOU SEE WHY IT IS HARD TO COMPARE FRACTIONS WHEN WE USE DIFFERENT SHAPES TO REPRESENT 1?

FOR THE NEXT LITTLE WHILE WE SHALL TRY TO FIND A WAY TO COMPARE

ANY TWO FRACTIONS FOR SIZE. WE SHALL USE PICTURES AT FIRST, BUT LATER WE SHALL BE ABLE TO DO IT WITHOUT PICTURES TO HELP US.

NEXT WE SHALL LOOK AT THESE PICTURES OF FRACTIONS. CAN YOU SEE HOW THEY ARE DIFFERENT FROM THE ONES WE LOOKED AT BEFORE?



IN THE FIRST PICTURES WE LOOKED AT, EACH OF THE FRACTIONS REPRESENTED A NUMBER THAT WAS <u>LESS THAN 1</u>. IN THESE PICTURES, EACH FRACTION REPRESENTS A NUMBER THAT IS GREATER THAN 1. THE FRACTION WE SHOULD USE FOR THE SHADED CIRCULAR REGIONS IS $\frac{4}{3}$ BECAUSE THERE ARE THREE THIRDS SHADED IN THE FIRST CIRCLE AND ONE MORE THIRD IN THE SECOND CIRCLE. THREE THIRDS AND ONE MORE THIRD IS FOUR THIRDS.

NOW LOOK AT THE RECTANGLES. THE FRACTION THAT WE SHOULD USE TO REPRESENT THE SHADED RECTANGULAR REGIONS IS $\frac{12}{5}$ BECAUSE THERE ARE FIVE FIFTHS SHADED IN THE FIRST RECTANGLE AND ANOTHER FIVE FIFTHS IN THE SECOND AND TWO MORE FIFTHS IN THE THIRD RECTANGLE. FIVE FIFTHS AND FIVE MORE FIFTHS AND TWO FIFTHS IS TWELVE FIFTHS.

CAN YOU TELL FROM THESE PICTURES WHETHER $\frac{4}{3}$ IS GREATER THAN $\frac{12}{5}$ OR WHETHER $\frac{4}{3}$ IS LESS THAN $\frac{12}{5}$? EVEN THOUGH ONE PICTURE USES CIRCLES AND THE OTHER USES RECTANGLES, IT IS EASY TO TELL THAT $\frac{4}{3}$ IS GREATER THAN $\frac{12}{5}$ BECAUSE $\frac{4}{3}$ IS LESS THAN 2 WHEREAS 2 IS LESS THAN $\frac{12}{5}$. WE HAVE $\frac{4}{3} < 2$ AND $2 < \frac{12}{5}$, SO $\frac{4}{3} < \frac{12}{5}$. NOTICE THAT IF ONE FRACTION IS GREATER THAN ANOTHER, IF WE REVERSE THE ORDER IN WHICH WE WRITE THE COMPARISON SENTENCE, WE GET

A SENTENCE INVOLVING A LESS THAN COMPARISON.

FOR EXAMPLE, WE CAN SAY $\frac{1}{2} < \frac{3}{5}$ OR $\frac{3}{5} > \frac{1}{2}$.

BE CAREFUL THAT THE OPEN END OF THE $>$ SYMBOL IS POINTING TOWARD THE GREATER NUMBER.

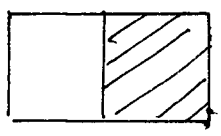
Ask the students to draw diagrams to determine the greater of the two fractions in each of the following pairs. Discuss the answers.

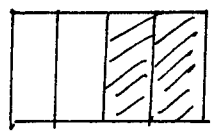
$\frac{3}{4}$ or $\frac{1}{2}$

$\frac{2}{5}$ or $\frac{3}{6}$

$\frac{4}{9}$ or $\frac{1}{3}$

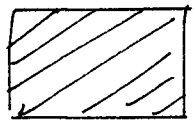
NOW WE CAN ALL FIND MANY <u>DIFFERENT</u> FRACTIONS TO REPRESENT THE <u>SAME</u> SHADED REGION IN A FIGURE: FOR INSTANCE, IN BOTH FIGURES BELOW WE HAVE SHADED EXACTLY THE SAME AREA, BUT WE SHOULD PROBABLY WRITE $\frac{1}{2}$ FOR THE FIRST PICTURE AND WE SHOULD PROBABLY WRITE $\frac{2}{4}$ FOR THE SECOND PICTURE.

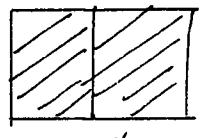


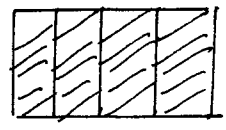


WE SAY THAT THE TWO SHADED REGIONS ARE EQUIVALENT, WE CAN CALL THE TWO FRACTIONS EQUIVALENT FRACTIONS AND WE WRITE: $\frac{1}{2} = \frac{2}{4}$.

IN THE SAME WAY THERE ARE MANY DIFFERENT EQUIVALENT FRACTIONS TO REPRESENT THE NUMBER 1. THE DIAGRAMS BELOW ARE ALL EQUIVALENT. THEY SHOW US THAT WE CAN THINK OF 1 BY USING THE FRACTION $\frac{1}{1}$ OR THE FRACTION $\frac{2}{2}$ OR THE FRACTION $\frac{4}{4}$.







WE CAN WRITE $\frac{1}{1} = \frac{2}{2} = \frac{4}{4}$. CAN YOU THINK OF AT LEAST FOUR NEW FRACTIONS, ALL OF WHICH ARE EQUIVALENT TO $\frac{1}{1}$? CAN YOU

DRAW DIAGRAMS TO SHOW THIS EQUIVALENCE?

Expect answers like $\frac{4}{4}$ or $\frac{5}{5}$ or $\frac{6}{6}$ or $\frac{10}{10}$, etc.

** Here the student will learn how to compare a given fraction with 1. He will, in particular, learn that any fraction whose numerator is smaller than its denominator is less than 1. **

I WANT TO BE ABLE TO COMPARE FRACTIONS TO WHOLE NUMBERS, SO I MIGHT AS WELL START BY COMPARING FRACTIONS WITH 1.

HOW DO I TELL IF A FRACTION LIKE $\frac{5}{3}$ IS GREATER THAN 1 OR LESS THAN 1 ? WELL, IF $\frac{5}{3}$ IS GREATER THAN 1 THAT WOULD MEAN THAT $\frac{5}{3}$ COVERS MORE AREA THAN 1 IN DIAGRAMS WHERE THE SAME UNIT IS USED IN BOTH PICTURES.

I MIGHT DRAW DIAGRAMS LIKE SO:

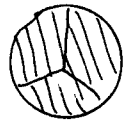
(NOTICE THAT THE CIRCLES ARE ALL THE SAME SIZE.)

HOWEVER, ANOTHER WAY TO COMPARE THE TWO NUMBERS MIGHT BE TO CHANGE 1 INTO THIRDS, AND THEN COMPARE THE CHANGED NAME FOR 1 TO $\frac{5}{3}$.

HOW DO I GET AN EQUIVALENT FRACTION TO 1 IN THIRDS? REMEMBER WHAT WE JUST DID BEFORE TO GET NAMES FOR 1.

Expect the answer: cut 1 circle into 3 pieces.

WE CAN TAKE OUR ONE UNIT AND SPLIT IT INTO THREE PIECES. THEN WE SAY THAT ANOTHER NAME FOR 1 IS $\frac{3}{3}$.



WE KNOW THAT $\frac{5}{3} > \frac{3}{3}$ SINCE 5 PIECES EACH OF THE SAME SIZE HAS TO BE GREATER THAN ONLY 3 OF THOSE PIECES.

HOW CAN WE COMPARE $\frac{2}{4}$ AND 1 FOR SIZE? ONE WAY WOULD BE TO DRAW DIAGRAMS LIKE THIS:

BECAUSE THE FIRST DIAGRAM HAS A SMALLER SHADED AREA THAN THE SECOND DIAGRAM, WE KNOW THAT $\frac{2}{4}$ IS LESS THAN 1 AND WE CAN WRITE $\frac{2}{4} < 1$.

IT MIGHT BECOME EVEN CLEARER IF WE CHANGED THE SECOND DIAGRAM A BIT AND INSTEAD DREW AN EQUIVALENT DIAGRAM FOR IT, LIKE THIS:

IT IS VERY EASY NOW TO TELL WHICH IS GREATER. ALL THAT WE NEED TO DO IS TO COMPARE THE NUMERATORS OF THE FRACTIONS. THE NUMERATORS ARE 2 AND 4. BECAUSE 2 IS LESS THAN 4, WE KNOW THAT $\frac{2}{4}$ MUST BE LESS THAN $\frac{4}{4}$, OR 1.

NOTICE THAT IN EACH CASE WE CONVERTED 1 TO A FRACTION WITH THE SAME DENOMINATOR AS THE OTHER FRACTION. WHEN WE HAD $\frac{5}{3}$ FOR THE FRACTION, WE CHANGED 1 TO $\frac{3}{3}$; WHEN WE HAD $\frac{2}{4}$, WE CHANGED 1 TO $\frac{4}{4}$.

THEN ALL WE HAD TO DO WAS TO COMPARE NUMERATORS. WE CHECKED TO SEE IF THE NUMERATOR OF THE FRACTION OTHER THAN 1 WAS GREATER OR LESS THAN THE NUMERATOR OF THE FRACTION WHICH WAS EQUIVALENT TO 1.

BUT WHAT IS THE NUMERATOR OF THE FRACTION EQUIVALENT TO 1 IN EACH

OF THOSE CASES?

Expect the answer: three or four, or the same as the denominators.

IN BOTH CASES, THAT NUMERATOR WAS EQUAL TO THE DENOMINATOR OF THE FRACTION EQUIVALENT TO 1 AND ALSO THE DENOMINATOR OF THE OTHER FRACTION. THIS MAKES SENSE SINCE FIRST, WE PICKED THE DENOMINATOR OF THE FRACTION EQUIVALENT TO 1 TO MAKE IT HAVE THE SAME DENOMINATOR AS THE OTHER FRACTION, AND SECOND, THE NUMERATOR OF THE FRACTION EQUALS ITS DENOMINATOR WHEN THE FRACTION IS ANOTHER NAME FOR 1.

SO IF I AM COMPARING THE NUMERATOR OF THE ORIGINAL FRACTION WITH THE NUMERATOR OF THE NEW FRACTION THAT IS EQUIVALENT TO 1, THAT IS THE SAME AS COMPARING THE NUMERATOR OF THE ORIGINAL FRACTION TO ITS DENOMINATOR.

LET'S CHECK. TO COMPARE $\frac{5}{3}$ WITH 1, WE CHANGED 1 INTO THIRDS AND COMPARED $\frac{5}{3}$ TO $\frac{3}{3}$ OR 5 TO 3.
I SAID $5 > 3$, AND, THEREFORE, $\frac{5}{3} > 1$.

TO COMPARE $\frac{2}{4}$ WITH 1, I CHANGED 1 INTO $\frac{4}{4}$ AND COMPARED $\frac{2}{4}$ TO $\frac{4}{4}$ OR 2 TO 4.
I SAID $2 < 4$, AND, THEREFORE, $\frac{2}{4} < 1$.

LET'S TRY THIS PROCEDURE WITH COMPARING $\frac{6}{5}$ TO 1. DO YOU EXPECT THAT $\frac{6}{5} > 1$ OR $\frac{6}{5} < 1$?

Expect the answer: $\frac{6}{5} > 1$.

TO COMPARE $\frac{6}{5}$ WITH 1, I COULD EITHER:

(a) FIND AN EQUIVALENT FRACTION TO 1 WITH DENOMINATOR 5 AND COMPARE $\frac{6}{5}$ TO THAT EQUIVALENT FRACTION, OR

(b) I COULD COMPARE THE NUMERATOR OF THE FRACTION, 6, TO THE DENOMINATOR, 5. SINCE $6 > 5$, THEN $\frac{6}{5} > 1$.

BE SURE TO UNDERSTAND THAT ALTHOUGH I AM USING A SLIGHTLY DIFFERENT METHOD IN (a) AND (b), I AM STILL COMPARING THE SAME TWO NUMBERS AND WILL ALWAYS GET THE SAME RESULTS FOR THE COMPARISON.

THAT IS, HERE, IF I USED PLAN (a), I WOULD SAY $\frac{5}{5} = 1$ AND THEN $\frac{6}{5} > \frac{5}{5}$ SINCE $6 > 5$. THEREFORE, $\frac{6}{5} > 1$.

Ask the students to use each of techniques a and b to find out which fraction is greater in each of the following pairs. If any student tries to answer in terms of whether the fraction is proper or improper, explain that he is correct, but that you would like him to try to show that your techniques also work.

$\frac{3}{8}$, 1

$\frac{7}{5}$, 1

$\frac{5}{2}$, 1

$\frac{3}{6}$, 1

Ask four students to come to the board to show both of their explanations for each of the four questions.

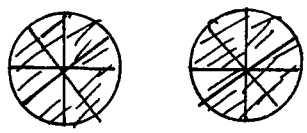
Suggested end of day 1.

** Here the student will learn how to compare fractions to whole numbers other than 1 by first learning a way to rename these whole numbers as equivalent fractions. **

NOW THAT WE KNOW HOW TO COMPARE FRACTIONS TO 1, WE WOULD LIKE TO SEE IF WE COULD COMPARE THESE FRACTIONS TO OTHER WHOLE NUMBERS. JUST AS BEFORE, WE MIGHT TRY TO FIND FRACTION NAMES FOR THESE WHOLE NUMBERS SO THAT IT MIGHT BE EASIER TO COMPARE THEM WITH THE FRACTIONS WE ARE GIVEN. SO NOW LET'S SEE HOW GOOD WE ARE AT RENAMING WHOLE NUMBERS OTHER THAN 1 AS FRACTIONS.

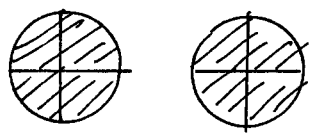
SUPPOSE WE WANT ANOTHER NAME FOR 2. ONE WAY IS TO DRAW A PICTURE.

FOR EXAMPLE, TO FIND ANOTHER NAME FOR 2, I MIGHT DRAW:



$$2 = \frac{8}{8} + \frac{8}{8} = \frac{16}{8}$$

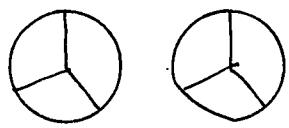
OR I MIGHT DRAW:



$$2 = \frac{4}{4} + \frac{4}{4} = \frac{8}{4}$$

WHAT OTHER FRACTION COULD WE GET BY MARKING THE DRAWING DIFFERNTLY?

Accept any answer where each whole is cut into the same number of parts, e.g.



WHAT DO YOU NOTICE ABOUT THE NUMERATORS OF OUR ANSWERS AS COMPARED TO THE DENOMINATORS?

Expect the answer that each numerator is twice as much as the denominator. If no one says this, go on with the following anyway.

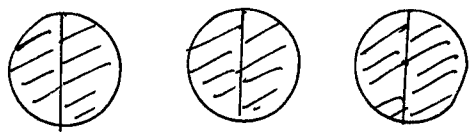
IN EACH SITUATION, WE HAD SOMETHING LIKE $\frac{9}{9} + \frac{9}{9}$ AND SO OUR ANSWER WAS $\frac{18}{9} = \frac{2 \times 9}{9}$ OR $\frac{10}{10} + \frac{10}{10} = \frac{20}{10} = \frac{2 \times 10}{10}$.
OUR NUMERATOR WAS TWICE OUR DENOMINATOR FOR FRACTIONS EQUIVALENT

TO TWO.

WHAT WOULD YOU PREDICT ABOUT THE NUMERATOR AS COMPARED TO THE DENOMINATOR IF WE WROTE FRACTIONS FOR 3 INSTEAD OF 2?

Expect the answer: the numerator will be three times as large.

LET'S CHECK. ONE WAY TO GET A FRACTION EQUIVALENT TO 3 IS TO DRAW 3 WHOLES AND DIVIDE EACH ONE UP THE SAME WAY. FOR EXAMPLE, I MIGHT DRAW:



THEN MY EQUIVALENT FRACTION NAME FOR 3 IS $\frac{2}{2} + \frac{2}{2} + \frac{2}{2} = \frac{6}{2}$.

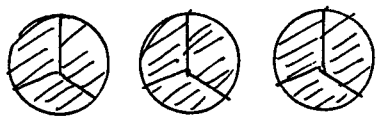
NOTICE THAT THE NUMERATOR IS 3 TIMES THE DENOMINATOR.

DOES ANYONE ELSE HAVE ANOTHER FRACTION NAME FOR 3?

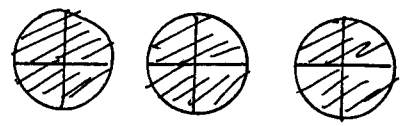
Expect an answer like: $\frac{9}{3}$ or $\frac{12}{4}$ or $\frac{15}{5}$.

LET'S SEE IF WE CAN CHECK THIS WITH A DIAGRAM.

Draw, e.g., for $3 = \frac{9}{3}$



for $3 = \frac{12}{4}$

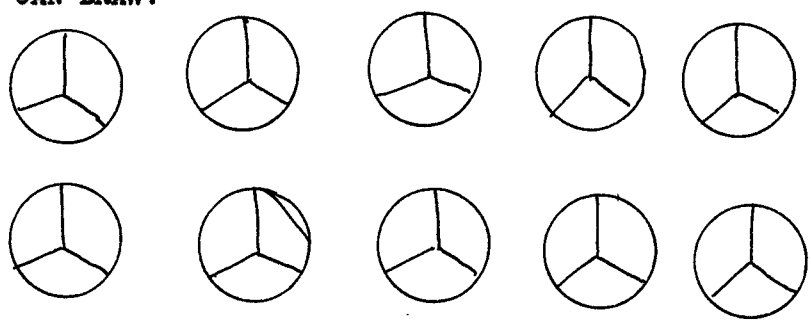


SUPPOSE I AM TRYING TO FIND A FRACTION EQUIVALENT TO 10. WHAT DO YOU SUPPOSE WE MIGHT GUESS ABOUT THE RELATIONSHIP BETWEEN THE NUMERATOR AND THE DENOMINATOR?

Expect the answer: the numerator will be 10 times as large as the denominator.

LET'S SEE IF WE WERE RIGHT. SUPPOSE WE WANT THE DENOMINATOR TO BE 3. SINCE WE PREDICTED THAT THE NUMERATOR WILL BE 10 TIMES AS LARGE AS THE DENOMINATOR, WE SHOULD GET A FRACTION $\frac{30}{3}$ FOR OUR ANSWER.

WE CAN DRAW:



Ask the students to work out equivalent fractions for each of the following whole numbers so that the fractions have the given denominators.

4 WITH DENOMINATOR 5

7 WITH DENOMINATOR 4

5 WITH DENOMINATOR 2

8 WITH DENOMINATOR 3

Ask a student to come to the board for each problem showing his answer and the accompanying diagram. The teacher can draw the diagram outlines in advance to save time.

Then have the students try to solve the following questions. If they have trouble, allow them to go back to a diagram.

$3 = \frac{\square}{4}$

$6 = \frac{\square}{8}$

$5 = \frac{\square}{3}$

$9 = \frac{\square}{2}$

Have a student read out his answers and make sure that everyone under-

stands how he got these correct answers.

Suggested end of day 2.

** Here the student will learn how to compare a fraction with any whole number, for example, to compare $\frac{8}{3}$ with 4. **

> WE ALREADY KNOW HOW TO COMPARE A FRACTION WITH 1. NOW WE WILL TRY TO COMPARE FRACTIONS TO OTHER WHOLE NUMBERS.
> SUPPOSE I WANT TO COMPARE $\frac{16}{3}$ TO 5.
> AS BEFORE, IF WE TURN 5 INTO THIRDS, IT WILL BE EASIER TO COMPARE IT TO $\frac{16}{3}$ BECAUSE IT IS VERY SIMPLE TO COMPARE FRACTIONS WITH THE SAME DENOMINATOR.
> FOR EXAMPLE, WHICH IS LARGER, $\frac{5}{8}$ OR $\frac{3}{8}$?

Expect the answer $\frac{5}{8}$.

> THIS IS SO SINCE 5 PIECES OF A CERTAIN SIZE WILL ALWAYS BE GREATER THAN ONLY 3 PIECES OF THAT SIZE.
> WHAT ABOUT $\frac{7}{2}$ OR $\frac{2}{2}$?

Expect the answer $\frac{7}{2}$.

> THIS IS SO SINCE 7 PIECES OF THE SAME SIZE WILL ALWAYS BE GREATER THAN ONLY 2 PIECES OF THAT SIZE.
>
> THE SAME PRINCIPLE APPLIES NO MATTER WHAT THE DENOMINATOR. IF TWO FRACTIONS HAVE THE SAME DENOMINATOR, WE NEED ONLY COMPARE THEIR NUMERATORS TO DECIDE WHICH IS GREATER.

If you feel that the students are having difficulty with this idea, draw diagrams like;

for $\frac{7}{2}$ and $\frac{5}{2}$

and so show them that 7 pieces of the same size would be greater than 5 of them.

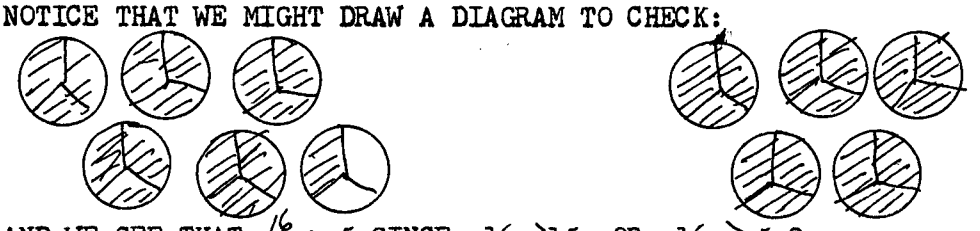
NOW LET'S GET BACK TO OUR PROBLEM OF COMPARING $\frac{16}{3}$ TO 5. SUPPOSE I WANT TO CHANGE 5 TO AN EQUIVALENT FRACTION WITH DENOMINATOR 3. WHAT DO I GET?

Expect the answer: $\frac{15}{3}$.

NOTICE THAT I COULD WRITE THIS AS $\frac{5x3}{3}$ SINCE WE ALL KNOW THAT THE NUMERATOR IS 5 TIMES THE DENOMINATOR.
THEN, WHICH IS GREATER, $\frac{16}{3}$ OR $\frac{5x3}{3}$?

Expect the answer: $\frac{16}{3}$.

NOTICE THAT WE MIGHT DRAW A DIAGRAM TO CHECK:



AND WE SEE THAT $\frac{16}{3} > 5$ SINCE $16 > 15$, OR $16 > 5x3$.

SUPPOSE WE WANT TO COMPARE $\frac{5}{3}$ TO 4. WHAT EQUIVALENT FRACTION FOR 4 WOULD MAKE THIS PROBLEM EASIER?

Expect the answer: $\frac{12}{3}$.

NOTICE THAT THIS EQUIVALENT FRACTION TO 4 HAS DENOMINATOR 3 AND NUMERATOR 4 x 3. THEN WHICH IS GREATER, $\frac{5}{3}$ OR $\frac{4x3}{3}$?

Expect the answer: $\frac{4x3}{3}$.

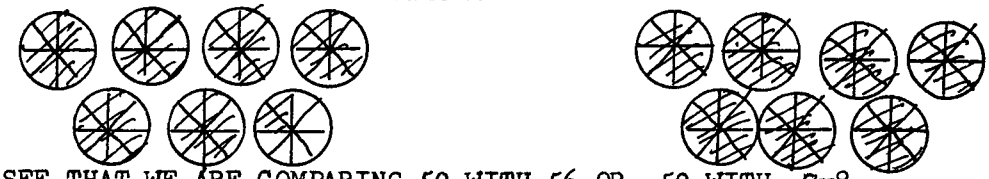
WE CAN DRAW A DIAGRAM TO CHECK:

AND WE SEE THAT $\frac{5}{3} < 4$ SINCE $5 < 12$ OR $5 < 4 \times 3$.

TO COMPARE $\frac{50}{8}$ TO 7, WHAT DO YOU SUGGEST DOING?

Expect the answer: Change 7 into $\frac{56}{8}$ and then compare 50 with 56.

NOTICE THAT WE CAN DRAW THE DIAGRAM:



AND SEE THAT WE ARE COMPARING 50 WITH 56 OR 50 WITH 7×8.

Ask the students to compare:

$\frac{19}{3}$ TO 7

$\frac{24}{7}$ TO 5

$\frac{65}{8}$ TO 8

Ask three students to come to the board to show their solutions as diagrams. For each problem, be sure to restate the answer so that it can be seen as comparing the first fraction's numerator with the product of its denominator and the whole number, e.g. for the first problem, as 19 compared to 3×7.

WHEN WE EXAMINE WHAT WE HAVE DONE SO FAR, WE SEE THAT WE ALWAYS CHANGED THE WHOLE NUMBER TO AN EQUIVALENT FRACTION WITH DENOMINATOR EQUAL TO THE DENOMINATOR OF THE GIVEN FRACTION. THEN WE COMPARED THE NUMERATOR OF THE GIVEN FRACTION TO THE NUMERATOR OF THIS FRACTION EQUIVALENT TO THE WHOLE NUMBER.

HOWEVER, BECAUSE OF THE WAY OF GETTING EQUIVALENT FRACTIONS TO WHOLES, WE SEE THAT THIS IS EXACTLY THE SAME AS COMPARING THE NUMERATOR OF THE GIVEN FRACTION WITH THE PRODUCT OF THE DENOMINATOR OF THAT FRACTION AND THE WHOLE NUMBER.

FOR EXAMPLE, TO COMPARE $\frac{16}{3}$ WITH 5, WE COMPARED 16 TO 5x3.
TO COMPARE $\frac{5}{3}$ TO 4, WE COMPARED 5 TO 4x3.

$\frac{16}{3}$ TO 5 = $\frac{16}{3}$ TO $\frac{5x3}{3}$

$\frac{5}{3}$ TO 4 = $\frac{5}{3}$ TO $\frac{4x3}{3}$

SO THAT TO COMPARE ANY FRACTION TO A WHOLE NUMBER, I CAN FIND WHOLE NUMBERS TO COMPARE INSTEAD__ JUST AS IN COMPARING $\frac{16}{3}$ TO 5, I COMPARED THE WHOLE NUMBERS 16 AND 15 OR 16 AND 5x3.

For each of the following, ask the students to write a whole number inequality they could solve to find the answer to the question of which of the two numbers in the pair is greater.

$\frac{25}{6}$ AND 4

$\frac{14}{5}$ AND 3

$\frac{19}{4}$ AND 5

Have three students come to the board to show their answers. They can diagram to show the validity if necessary. Expect answers:

$25 > 6x4$		$25 > 4x6$
$14 < 5x3$		$14 < 3x5$
$19 < 4x5$	or	$19 < 5x4$

Be sure to stress the importance of not changing around the numbers in filling in the inequality sign. For example, to fill in $\frac{16}{3} \sim 5$, I say $16 > 5x3$, so $\frac{16}{3} > 5$; but to fill in $5 \sim \frac{16}{3}$, I say $5x3 < 16$, so $5 < \frac{16}{3}$.
Hand worksheet 1 to the students to complete.

Suggested end of day 3.

Comparison of Fractions Using the Cross-Product

Outline for approach P:

NOW THAT WE KNOW HOT TO COMPARE A FRACTION TO A WHOLE NUMBER, WE ARE GOING TO LEARN HOW TO COMPARE TWO FRACTIONS.

BUT, AGAIN, WE JUST WANT TO REMIND OURSELVES OF A FEW THINGS BEFORE WE GO ON.

FIRST OF ALL, LET'S ALL REMEMBER WHAT WE MEAN BY A FRACTION LIKE $\frac{4}{5}$.

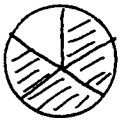
WHAT DOES THE TOP-4- TELL US?

Expect the answer- the number of parts that we are considering(shaded).

WHAT DOES THE BOTTOM-5- TELL US?

Expect the answer- the number of equal parts into which a whole is divided.

SO, TO DRAW A DIAGRAM SHOWING $\frac{4}{5}$, WE MIGHT DRAW:



HOW WOULD YOU DRAW A DIAGRAM REPRESENTING $\frac{3}{4}$?

Have a student come to the board to show his diagram.

WHAT WOULD A DIAGRAM FOR $\frac{4}{6}$ BE LIKE?

Have another student come to the board to show his diagram.

WELL, THIS SEEMS REASONABLY EASY, SO LET'S GO ON TO THE NEXT BIT OF REVIEWING.

HOW CAN WE TELL, USING DIAGRAMS, WHICH OF TWO FRACTIONS IS LARGER?

Expect the answer- the larger fraction covers more area.

SINCE WE ARE LOOKING AT AMOUNT OF AREA COVERED, TELL ME WHICH IS GREATER IN EACH OF THESE PAIRS:

(a)  (b) 

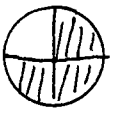
(a)  (b) 

Expect the answers: (a), (b).

NOTICE THAT WE DID NOT EVEN HAVE TO KNOW THE NAMES FOR THE SHADED AREAS TO TELL WHICH FRACTION REPRESENTATION WOULD BE GREATER. BUT SUPPOSE WE START OUT WITH FRACTIONS AND WANT TO DRAW DIAGRAMS IN ORDER TO HELP US DO THE COMPARISON.

FOR EXAMPLE, SUPPOSE WE START BY COMPARING $\frac{2}{3}$ AND $\frac{3}{4}$.

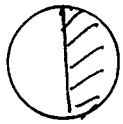
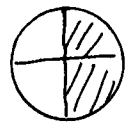
WE WOULD DRAW:

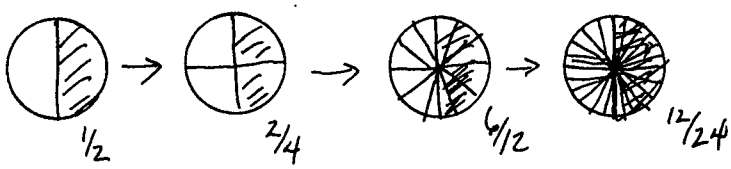
AND SEE THAT $\frac{2}{3} < \frac{3}{4}$.

Ask the students to decide which of $\frac{5}{7}$ or $\frac{2}{3}$ is greater by drawing a diagram. Have one student come to the board to show his answer. Then repeat the procedure for comparing $\frac{1}{2}$ and $\frac{4}{5}$ and $\frac{3}{6}$ and $\frac{2}{5}$.

FINALLY, WE WANT TO REMEMBER THAT NOT ONLY ARE FRACTIONS GREATER THAN OR LESS THAN OTHERS; THEY ARE SOMETIMES EQUAL, OR EQUIVALENT. REMEMBER THAT EQUIVALENT FRACTIONS ARE FRACTIONS WHICH COVER THE SAME AREA. FOR EXAMPLE, $\frac{1}{2} = \frac{2}{4}$ SINCE I CAN

DRAW:  AND CUT TO GET  .

NOTICE THAT I CAN ALWAYS GET MORE EQUIVALENT FRACTIONS BY CUTTING UP THE PIECES OF MY DIAGRAM. FOR EXAMPLE,



Ask the students to find other names of the following by cutting:

$\frac{1}{3}$

$\frac{2}{5}$

$\frac{1}{4}$

Accept any answers for students which demonstrate cutting of all pieces in the original diagram in the same manner, e.g. all are cut in half, or all are cut in thirds, etc.

Suggested end of day 4.

** Here the student will review the idea that two fractions with the same denominator can be compared by examining their numerators, e.g. $\frac{6}{3} > \frac{4}{3}$ since $6 > 4$. **

I WANT TO BE ABLE TO COMPARE ANY TWO FRACTIONS, BUT I MIGHT AS WELL START BY COMPARING FRACTIONS WITH THE SAME DENOMINATOR.

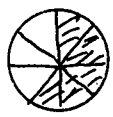
SUPPOSE I WANT TO COMPARE $\frac{7}{4}$ WITH $\frac{5}{4}$. THIS IS FAIRLY EASY. ALL I HAVE TO DO IS TO DRAW A DIAGRAM SHOWING $\frac{7}{4}$ AND ANOTHER SHOWING $\frac{5}{4}$ AND SEEING WHICH COVERS MORE AREA.

IN THE FIRST CASE, I AM TAKING SEVEN PIECES EACH OF A CERTAIN SIZE WHEREAS, IN THE SECOND CASE, I AM TAKING ONLY FIVE PIECES OF THAT SIZE. BECAUSE WE ARE TAKING MORE OF THE SAME SIZE PIECES, WE CAN SAY THAT <u>SEVEN</u> FOURTHS IS GREATER THAN <u>FIVE</u> FOURTHS AND WRITE $\frac{7}{4} > \frac{5}{4}$.

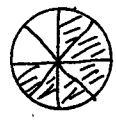
WHAT IF WE WERE TO COMPARE $\frac{5}{8}$ TO $\frac{6}{8}$? WHICH IS GREATER?

Expect the answer $\frac{6}{8}$.

LET'S USE A DIAGRAM TO CHECK:



5 EIGHTHS



6 EIGHTHS

BECAUSE $5 < 6$, WE CAN SEE THAT $\frac{5}{8} < \frac{6}{8}$.

Ask the students to name the greater fraction in each of these pairs.

$\frac{3}{9}$ AND $\frac{5}{9}$

$\frac{4}{6}$ AND $\frac{3}{6}$

$\frac{5}{3}$ AND $\frac{6}{3}$

Have three students come to the board to draw diagrams to explain their answers.

** Here the student will learn to generate equivalent fractions for two given ones with a common denominator as the product of the original two denominators; for example, for $\frac{3}{4}$ and $\frac{5}{6}$, to produce $\frac{18}{24}$ and $\frac{20}{24}$ as equivalent fractions for purposes of comparison. **

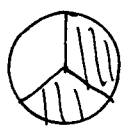
NOW THAT WE ALREADY KNOW A BIT ABOUT COMPARING SOME FRACTIONS, LET US TRY TO SEE IF WE CAN USE THIS SAME TECHNIQUE ON ANY FRAC-

IS IT EASY TO DECIDE WHETHER $\frac{2}{3} > \frac{3}{4}$? CAN WE MAKE IT EASIER?
SUPPOSE WE COULD REWRITE BOTH OF THESE FRACTIONS SO THAT THEY BOTH HAVE THE SAME DENOMINATOR. THAT WOULD MAKE IT A LOT EASIER.

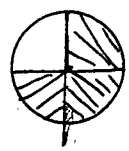
WE ALREADY KNOW THAT A WAY TO GET EQUIVALENT NAMES FOR A FRACTION IS TO DRAW DIAGRAMS AND PRETEND TO CUT THEM.
FOR EXAMPLE, TO CHANGE $\frac{2}{3}$ INTO ANOTHER NAME, I MIGHT CUT LIKE:

AND SEE THAT $\frac{2}{3} = \frac{4}{6}$.

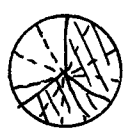
NOW SUPPOSE I WERE TRYING TO COMPARE $\frac{2}{3}$ WITH $\frac{3}{4}$.
ONE WAY TO COMPARE THESE IS TO FIND OTHER NAMES FOR EACH SO THAT THE TWO NEW NAMES HAVE THE SAME DENOMINATOR.
FIRST WE COULD DRAW



AND



BUT IF I HAD DIVISIONS OF 12 INSTEAD OF 3 AND 4 FOR EACH CIRCLE, I COULD EASILY COUNT THE NUMBER OF SUBDIVISIONS REPRESENTED BY EACH OF THE FRACTIONS $\frac{2}{3}$ AND $\frac{3}{4}$ IN TERMS OF TWELFTHS AND COMPARE THESE LIKE SO:

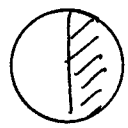


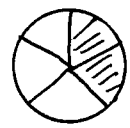
NOW NOTICE THAT THE $\frac{2}{3}$ BECOMES $\frac{8}{12}$ AND THE $\frac{3}{4}$ BECOMES $\frac{9}{12}$.
THIS SEEMS REASONABLE, SINCE IF I CUT EACH THIRD INTO FOURTHS, THEN I GET A TOTAL OF 12 PIECES, AND SO $\frac{2}{3}$ BECOMES $\frac{2 \times 4}{12}$. SIMILARLY IF I CUT EACH FOURTH INTO THREE, I GET A TOTAL OF 12 PIECES, AND

SO $\frac{3}{4}$ BECOMES $\frac{3 \times 3}{12}$.

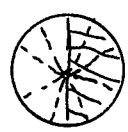
NOTICE THAT IF I AM COMPARING A FRACTION WITH DENOMINATOR 3 TO ONE OF DENOMINATOR 4, I CAN CUT EACH OF THE THIRDS INTO 4 PIECES, EACH OF THE FOURTHS INTO 3 PIECES, AND AN EXACT NUMBER OF TWELFTHS TO REPRESENT EACH OF THE ORIGINAL FRACTIONS.

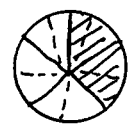
SUPPOSE I WANT TO COMPARE $\frac{1}{2}$ WITH $\frac{2}{5}$. WHAT EQUIVALENT FRACTIONS SHOULD I USE? WE CAN BEGIN BY DRAWING THESE DIAGRAMS:





THEN I CAN CUT EACH OF THE HALVES IN THE FIRST DIAGRAM INTO 5 PIECES TO GET A TOTAL OF 10 PIECES AND EACH OF THE FIFTHS IN THE SECOND DIAGRAM INTO 2 PIECES TO GET A TOTAL OF 10 PIECES, LIKE SO:





THEN, $\frac{1}{2} = \frac{5}{10}$ AND $\frac{2}{5} = \frac{4}{10}$, AND WE COULD EASILY COMPARE $\frac{5}{10}$ AND $\frac{4}{10}$.

AGAIN, NOTICE THAT I CUT EACH FIFTH INTO 2 PIECES AND EACH HALF INTO 5 PIECES, SO THAT BOTH DIAGRAMS HAD THE SAME NUMBER OF SUBDIVISIONS AND SO THAT I COULD EASILY COUNT THE NUMBER OF THESE SUBDIVISIONS EQUAL TO EACH $\frac{1}{2}$ AND EQUAL TO EACH $\frac{1}{5}$.

WHAT DO YOU EXPECT TO BE THE DENOMINATOR OF FRACTIONS EQUIVALENT TO $\frac{1}{2}$ AND $\frac{2}{3}$ SO AS TO MAKE THEM EASIER TO COMPARE?

Expect the answer: 6.

WE CAN DRAW:

AND SEE THAT, INDEED, $\frac{1}{2}$ CAN BE REWRITTEN AS $\frac{3}{6}$ AND $\frac{2}{3}$ AS $\frac{4}{6}$.

Ask the students to attempt to rewrite each of these pairs of fractions as equivalent fractions with the same denominator.

$\frac{3}{4}$ AND $\frac{2}{3}$

$\frac{4}{5}$ AND $\frac{2}{3}$

$\frac{2}{6}$ AND $\frac{1}{2}$

Have three students show their work on the board. If any students use denominators other than the products of the given ones, explain that they are correct (if they are), but ask them to use the product as a guarantee that the procedure will work.. For example, for $\frac{1}{2}$ and $\frac{3}{4}$, we might use the denominator 4, but we could certainly not do this for $\frac{1}{5}$ and $\frac{3}{7}$; point out that we can <u>always</u> use the product.

Be sure that the students realize that the product of the two denominators can always be used since we are insuring that each of the small pieces into which each of the diagrams is divided can be written as an integral number of the new subdivisions, and therefore the total fractions we start with can also be written as integral multiples.

** Here the student will learn that $\frac{a}{b} > \frac{c}{d}$ only when a x d > b x c, for example, $\frac{3}{6} > \frac{2}{5}$ only because 3 x 5 > 2 x 6. **

NOW WE ALREADY REALIZE THAT AN EASIER WAY TO COMPARE TWO FRACTIONS TO DECIDE WHICH IS GREATER IS TO RENAME THEM AS FRACTIONS WITH A COMMON DENOMINATOR AND WE HAVE PRACTICED REWRITING

EQUIVALENT FRACTIONS.

SUPPOSE I WANT TO COMPARE $\frac{1}{2}$ WITH $\frac{4}{6}$ TO FIND WHICH IS GREATER. I CAN REWRITE THESE AS EQUIVALENT FRACTIONS WITH THE SAME DENOMINATOR. WHAT FRACTIONS WOULD I USE?

Expect the answers: $\frac{6}{12}$, $\frac{8}{12}$.

I CAN CHECK THIS WITH A DIAGRAM:

THEN, I NOTICE THAT I AM COMPARING $\frac{6}{12}$ WITH $\frac{8}{12}$.
WHICH IS GREATER?

Expect the answer: $\frac{8}{12}$.

BUT WHERE DID THE 6 IN $\frac{6}{12}$ COME FROM? IT CAME FROM 1 x 6, THE ONE PIECE OUT OF THE TWO THAT I STARTED WITH (FROM $\frac{1}{2}$) WAS CUT INTO 6 PIECES. WHERE DID THE 8 COME FROM? IT CAME FROM 4x2, THE FOUR PIECES OUT OF THE 6 (FROM $\frac{4}{6}$) THAT I STARTED WITH WERE CUT INTO 2 PIECES EACH.
WHY WERE THE ORIGINAL HALVES CUT INTO 6 PIECES EACH?
BECAUSE THE OTHER DENOMINATOR WAS 6.
WHY WERE THE ORIGINAL SIXTHS CUT INTO 2 PIECES EACH?

Expect the answer: Because the other denominator was 2.

SO, IN COMPARING $\frac{1}{2}$ AND $\frac{4}{6}$, I AM REALLY COMPARING 6 AND 8, OR 1x6 AND 4x2.
TO COMPARE $\frac{1}{2}$ AND $\frac{4}{6}$, I SAY $1x6 < 4x2$, SO $\frac{1}{2} < \frac{4}{6}$.

SUPPOSE I WANT TO COMPARE $\frac{1}{2}$ WITH $\frac{3}{5}$ TO FIND WHICH IS GREATER. I CAN RENAME THESE AS EQUIVALENT FRACTIONS WITH THE SAME DENOMINATOR. WHAT FRACTIONS WOULD I USE?

Expect the answers: $\frac{5}{10}$ and $\frac{6}{10}$.

I CAN CHECK WITH A DIAGRAM:

THEN I NOTICE THAT I AM COMPARING $\frac{5}{10}$ WITH $\frac{6}{10}$.
WHICH IS GREATER?

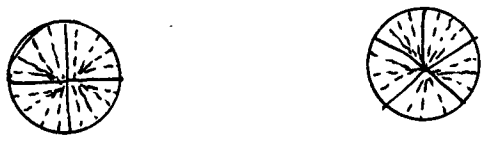
Expect the answer $\frac{6}{10}$.

BUT IN COMPARING $\frac{5}{10}$ AND $\frac{6}{10}$, I AM REALLY COMPARING 5 AND 6, OR 1x5 AND 3x2. REMEMBER, THAT 1x5 TELLS ME THAT I ORIGINALLY HAD 1 PIECE WHICH WAS SUBDIVIDED INTO 5, AND THE 3x2 THAT I ORIGINALLY HAD 3 PIECES EACH SUBDIVIDED INTO 2. THE SUBDIVISIONS ARE NOW ALL EQUAL SO I ONLY NEED COUNT THE NUMBERS OF EACH TO TELL WHICH FRACTION IS GREATER.
SINCE $1x5 < 3x2$, THEN $\frac{1}{2} < \frac{3}{5}$.

WHICH OF THESE TWO FRACTIONS IS GREATER: $\frac{3}{4}$ OR $\frac{2}{5}$? HOW CAN WE TELL?

Expect the answer: change both into twentieths and then see if the numerator of the first is greater than the one of the second. Or, expect: multiply 3x5 and compare it to 4x2.

NOW WE HAVE TWO ALTERNATIVES. WE COULD CONVERT BOTH FRACTIONS TO TWENTIETHS AND COMPARE LIKE SO:



OR ELSE WE MIGHT SIMPLY NOTICE THAT THIS IS EXACTLY THE SAME AS COMPARING 3x5 WITH 4x2, SINCE 3x5 TELLS ME I ORIGINALLY HAD 3 PIECES, EACH OF WHICH WAS CUT INTO FIFTHS AND SO NOW I HAVE 3x5, AND 2x4, THAT I ORIGINALLY HAD 2 PIECES, EACH OF WHICH WAS CUT INTO FOURTHS, AND SO I NOW HAVE 2x4 PIECES, ALL OF THESE OF THE SAME SIZE. NOTICE THIS IS THE SAME AS COMPAING THE NUMERATORS IN THE ABOVE METHOD.

CAN YOU SEE WHY THIS IS CALLED THE CROSS-PRODUCT RULE FOR COMPARING FRACTIONS? I TAKE THE PRODUCT ACROSS THE FRACTIONS AND COMPARE THESE: $\frac{3}{4} \times \frac{2}{5}$

FOR EXAMPLE, IN $\frac{3}{4}$ AND $\frac{5}{6}$,

I COMPARED 3x6 WITH 4x5 WHICH COMES FROM $\frac{3}{4} \times \frac{5}{6}$.

Ask the students to find the larger in each of these pairs by using the cross product rule. Have a student show each of these on the board diagrammatically explaining why the rule works.

$\frac{3}{5}$	OR	$\frac{2}{4}$
$\frac{3}{9}$	OR	$\frac{4}{5}$
$\frac{1}{5}$	OR	$\frac{2}{6}$
$\frac{1}{3}$	OR	$\frac{3}{5}$

Suggested end of day 5.

Comparison of Fractions Using the Cross-Product

Outline for approach a:

> WE WILL NOW LEARN A WAY TO DECIDE WHETHER A FRACTION IS MORE THAN OR LESS THAN A GIVEN WHOLE NUMBER. FOR EXAMPLE, WHETHER $\frac{8}{3}$ IS GREATER THAN OR LESS THAN 4.
>
> BUT TO DO THIS, WE MUST FIRST DO SOME REVIEWING.
>
> I WILL CALL OUT SOME MULTIPLICATION QUESTIONS.

Ask the following questions:

6 x 2
7 x 5
9 x 8
2 x 8
9 x 6

If there is much difficulty in answering these questions on the part of the class as a whole, then apply an argument involving the distributive principle like the following one:

To illustrate:

$$6 \times 2 = 3 \times 2 + 3 \times 2 = 6 + 6 = 12$$

$$5 \times 5 = 2 \times 5 + 2 \times 5 + 1 \times 5 = 10 + 10 + 5 = 25$$

Do not use diagrams to illustrate these.

If the student has trouble in breaking up the first number into only two addends, like 6 into 3+3, allow him to use more addends which are smaller, such as 6 into 2+2+2.

> BECAUSE WE ARE DEALING WITH FRACTIONS, WE HAD BETTER REMEMBER SOME OF OUR FRACTION WORK, AS WELL.

Go over items 27-32 on the pretest. Read each item and ask for an answer. Discuss, in particular, that a fraction can be renamed in terms of a multiplication statement:

$$\frac{4}{6} = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 4 \times \frac{1}{6}$$

$$\frac{3}{5} = \frac{1}{5} + \frac{1}{5} + \frac{1}{5} = 3 \times \frac{1}{5}$$

ONE OF THE PROPERTIES OF MULTIPLICATION WITH WHICH WE ARE FAMILIAR IS THIS: IF WE MULTIPLY ANY NUMBER BY ONE, WE DON'T CHANGE IT. FOR EXAMPLE, $1 \times 8 = 8$.

THIS WORKS WITH FRACTIONS ALSO. FOR INSTANCE, $1 \times \frac{2}{3} = \frac{2}{3}$.

BUT 1 HAS MANY NAMES. FOR INSTANCE, $1 = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 4 \times \frac{1}{4}$.

SO, IF $1 \times \frac{2}{3} = \frac{2}{3}$, THEN $4 \times \frac{1}{4} \times \frac{2}{3} = \frac{2}{3}$.

NOTICE THAT IT DID NOT MATTER WHAT NAME OF 1 I CHOSE TO USE; I COULD MULTIPLY BY THAT NUMBER AND NOT CHANGE THE OTHER NUMBER IN THE PRODUCT.

WE HAVE ALREADY NOTICED THAT ANOTHER WAY OF WRITING A FRACTION IS AS A MULTIPLICATION EXPRESSION. IN PARTICULAR, A FRACTION LIKE $\frac{6}{8} = 6 \times \frac{1}{8}$ SINCE IT IS THE SAME AS $\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$ WHICH IS <u>SIX</u> ONE-EIGHTHS. SIMILARLY

$$\frac{5}{4} = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 5 \times \frac{1}{4} \quad \text{OR}$$

$$\frac{8}{2} = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 8 \times \frac{1}{2}.$$

SO IF WE WANT TO FIND EVEN MORE NAMES FOR THESE FRACTIONS INVOLVING MULTIPLICATION, WHAT WE MIGHT DO IS TO MULTIPLY ANY OF THESE NAMES BY SOME FORM OF 1. FOR EXAMPLE, WE MIGHT WRITE:

$$\frac{3}{6} = 3 \times \frac{1}{6} \times 1$$

$$\text{OR} \quad = 3 \times \frac{1}{6} \times \left(2 \times \frac{1}{2}\right)$$

$$\text{OR} \quad = 3 \times \frac{1}{6} \times \left(4 \times \frac{1}{4}\right)$$

THIS WILL BE USEFUL LATER WHEN WE ARE LOOKING FOR OTHER NAMES FOR FRACTIONS TO MAKE IT EASIER TO COMPARE THEM.

ANOTHER QUESTION WE MIGHT ASK ABOUT FRACTIONS IS HOW WE ARE GOING TO DECIDE WHICH OF TWO FRACTIONS IS THE GREATER IF THEY HAVE THE SAME DENOMINATOR.

LET'S INVESTIGATE THIS.

Go over items 33-35 on the pretest. Point out that in comparing, e.g. 5x9 with 4x9, we know that 5x9 is greater, since multiplication as repeated addition implies that $5x9 = 9+9+9+9+9$, whereas

$4x9 = 9+9+9+9$ only.

NOW, JUST AS $\underline{83} \times 59 > \underline{80} \times 59$ OR $\underline{78} \times 23 > \underline{43} \times 23$,

WE CAN MAKE STATEMENTS ABOUT FRACTIONS.

SUPPOSE I AM COMPARING $\frac{5}{6}$ TO $\frac{4}{6}$. I CAN REWRITE THESE AS:

$\frac{5}{6} = 5 \times \frac{1}{6}$

$\frac{4}{6} = 4 \times \frac{1}{6}$

CERTAINLY, 5 TIMES A QUANTITY IS GREATER THAN 4 TIMES THAT SAME QUANTITY, SO , $\frac{5}{6} > \frac{4}{6}$.

SO TO COMPARE FRACTIONS WITH THE SAME DENOMINATOR, WE CAN REWRITE THEM AS MULTIPLICATION EXPRESSIONS AND THEN COMPARE.

FOR EXAMPLE, WHICH IS GREATER: $\frac{8}{7}$ OR $\frac{9}{7}$?

Expect the answer $\frac{9}{7}$, since $\frac{8}{7} = 8 \times \frac{1}{7}$ while $\frac{9}{7} = 9 \times \frac{1}{7}$.

IF ONE FRACTION IS GREATER THAN ANOTHER, AS $\frac{9}{7}$ IS IN COMPARISON WITH $\frac{8}{7}$, THEN WE WRITE: $\frac{9}{7} > \frac{8}{7}$.

NOTICE THAT IF ONE FRACTION IS GREATER THAN ANOTHER, IF WE REVERSE THE ORDER IN WHICH WE WRITE THE COMPARISON SENTENCE, WE GET A SENTENCE INVOLVING A "LESS THAN" COMPARISON.

FOR EXAMPLE, WE CAN SAY $\frac{9}{7} > \frac{8}{7}$ OR $\frac{8}{7} < \frac{9}{7}$.

BE CAREFUL THAT THE OPEN END OF THE $>$ SYMBOL IS POINTING TOWARD THE GREATER NUMBER.

WHICH FRACTION IN EACH OF THESE PAIRS IS LARGER? WHY?

$\frac{3}{8}$ OR $\frac{5}{8}$

$\frac{7}{9}$ OR $\frac{6}{9}$

$\frac{8}{3}$ OR $\frac{9}{3}$

Expect answers: $\frac{5}{8}$; $\frac{7}{9}$; $\frac{9}{3}$.

WE ALL KNOW THAT I CAN FIND A LOT OF FRACTION NAMES TO REPRESENT A GIVEN FRACTION.

FOR EXAMPLE, THE FRACTION $\frac{1}{2}$ CAN ALSO BE REPRESENTED AS $\frac{2}{4}$: $\frac{1}{2} = \frac{2}{4}$.

IN THE SAME WAY THERE ARE MANY DIFFERENT EQUIVALENT FRACTIONS OR EXPRESSIONS TO REPRESENT THE NUMBER 1.

WE KNOW THAT $\frac{1}{2} + \frac{1}{2} = \frac{2}{2} = 1$ SINCE WHAT WE MEAN BY ONE-HALF IS THAT TWO OF THEM MAKE ONE. WE KNOW THAT $\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{4}{4} = 1$ SINCE WHAT WE MEAN BY ONE-FOURTH IS THAT FOUR OF THEM MAKE ONE.

CAN YOU THINK OF AT LEAST FOUR MORE NEW FRACTIONS, ALL OF WHICH ARE EQUIVALENT TO 1? CAN YOU WRITE THEM AS MULTIPLICATION EXPRESSIONS?

Expect the answers like: $\frac{5}{5}$ or $\frac{6}{6}$ or $\frac{10}{10}$; $5 \times \frac{1}{5}$ or $6 \times \frac{1}{6}$.

** Here the student will learn how to compare a given fraction with 1. He will, in particular, learn that any fraction whose numerator is smaller than its denominator is less than 1. **

I WANT TO BE ABLE TO COMPARE FRACTIONS TO WHOLE NUMBERS, SO I MIGHT AS WELL START BY COMPARING FRACTIONS WITH 1.

HOW DO I TELL IF A FRACTION LIKE $\frac{5}{3}$ IS GREATER THAN 1 OR LESS THAN 1 ?

WELL, IF $\frac{5}{3}$ IS GREATER THAN 1, THAT WOULD MEAN THAT I COULD CHANGE 1 INTO THIRDS AND THEN COMPARE THE CHANGED NAME FOR 1 WITH $\frac{5}{3}$.

HOW DO I GET AN EQUIVALENT FRACTION TO 1 IN THIRDS? REMEMBER WHAT WE MENTIONED BEFORE ABOUT GETTING NAMES FOR 1.

Expect the answer: $3 \times \frac{1}{3} = \frac{3}{3}$.

WE CAN TAKE OUR ONE UNIT AND WRITE IT IN THIRDS. WE KNOW THAT 1 IS THREE THIRDS SO THAT $3 \times \frac{1}{3} = \frac{3}{3} = 1$. WE SAY THAT ANOTHER NAME FOR 1 IS $3 \times \frac{1}{3}$.

WE KNOW THAT $\frac{5}{3} > \frac{3}{3}$ SINCE 5x SOMETHING HAS TO BE GREATER THAN ONLY 3 x THAT SOMETHING.

HOW CAN WE COMPARE $\frac{2}{4}$ AND 1 FOR SIZE?

IT MIGHT BE CLEARER IF WE CHANGED THE ONE INTO A FRACTION INVOLVING FOURTHS.

IT IS VERY EASY NOW TO TELL WHICH IS GREATER. ALL THAT WE NEED TO DO IS TO COMPARE THE NUMERATORS OF THE FRACTIONS. THE NUMERATORS ARE 2 AND 4. BECAUSE 2 IS LESS THAN 4, WE KNOW THAT

$\frac{2}{4}$ MUST BE LESS THAN $\frac{4}{4}$.

THAT IS, $2 \times \frac{1}{4}$ IS LESS THAN $4 \times \frac{1}{4}$.

NOTICE THAT IN EACH CASE, WE CONVERTED 1 TO A FRACTION WITH THE SAME DENOMINATOR AS THE OTHER FRACTION. WHEN WE HAD $\frac{5}{3}$ FOR THE FRACTION, WE CHANGED 1 TO $\frac{3}{3}$. WHEN WE HAD $\frac{2}{4}$, WE CHANGED 1 TO $\frac{4}{4}$.

THEN ALL WE HAD TO DO WAS TO COMPARE NUMERATORS. WE CHECKED TO SEE IF THE NUMERATOR OF THE FRACTION OTHER THAN 1 WAS GREATER OR LESS THAN THE NUMERATOR OF THE FRACTION WHICH WAS EQUIVALENT TO 1.

BUT WHAT IS THE NUMERATOR OF THE FRACTION EQUIVALENT TO 1 IN EACH OF THOSE CASES?

Expect the answer: three or four, or the same as the denominator.

IN BOTH CASES, THAT NUMERATOR WAS EQUAL TO THE DENOMINATOR OF THE FRACTION EQUIVALENT TO 1 AND ALSO TO THE DENOMINATOR OF THE OTHER FRACTION. THIS MAKES SENSE SINCE FIRST, WE PICKED THE DENOMINATOR OF THE FRACTION EQUIVALENT TO 1 TO MAKE IT HAVE THE SAME DENOMINATOR AS THE OTHER FRACTION, AND SECOND, THE NUMERATOR OF A FRACTION EQUALS ITS DENOMINATOR WHEN THE FRACTION IS ANOTHER NAME FOR 1.

SO IF I AM COMPARING THE NUMERATOR OF THE ORIGINAL FRACTION WITH THE NUMERATOR OF THE NEW FRACTION THAT IS EQUIVALENT TO 1, THAT IS THE SAME AS COMPARING THE NUMERATOR OF THE ORIGINAL FRACTION TO ITS DENOMINATOR.

LET'S CHECK, TO COMPARE $\frac{5}{3}$ WITH 1, WE CHANGED 1 INTO THIRDS AND COMPARED $5 \times \frac{1}{3}$ TO $3 \times \frac{1}{3}$ OR 5 TO 3.
I SAID $5 > 3$, AND, THEREFORE, $\frac{5}{3} > 1$.

TO COMPARE $\frac{2}{4}$ WITH 1, I CHANGED 1 INTO $\frac{4}{4}$ AND COMPARED $2 \times \frac{1}{4}$ WITH $4 \times \frac{1}{4}$ OR 2 TO 4.
I SAID $2 < 4$, AND, THEREFORE, $\frac{2}{4} < 1$.

LET'S TRY THIS PROCEDURE WITH COMPARING $\frac{6}{5}$ TO 1. DO YOU EXPECT THAT $\frac{6}{5} > 1$ OR $\frac{6}{5} < 1$?

Expect the answer: $\frac{6}{5} > 1$.

TO COMPARE $\frac{6}{5}$ WITH 1, I COULD EITHER:

(a) FIND AN EQUIVALENT FRACTION TO 1 WITH DENOMINATOR 5 AND COMPARE $\frac{6}{5}$ TO THAT EQUIVALENT FRACTION, OR

(b) I COULD COMPARE THE NUMERATOR OF THE FRACTION, 6, TO THE DENOMINATOR, 5. SINCE $6 > 5$, THEN $\frac{6}{5} > 1$.

BE SURE THAT YOU UNDERSTAND THAT ALTHOUGH I AM USING A SLIGHTLY DIFFERENT METHOD IN (a) AND (b), I AM STILL COMPARING THE SAME TWO NUMBERS AND WILL ALWAYS GET THE SAME ANSWER FOR THE COMPARISON.

THAT IS, HERE, IF I USED PLAN (a), I WOULD SAY $\frac{5}{5} = 1$, AND THEN $6 \times \frac{1}{5} > 5 \times \frac{1}{5}$ SINCE $6 > 5$. THEREFORE, $\frac{6}{5} > 1$.

Ask the students to use each of techniques a and b to find out which fraction is greater in each of the following pairs. If any student tries to answer in terms of whether the fraction is proper or improper, explain that he is correct, but that you would like him to try to show that your

techniques also work.

$\frac{3}{8}$, 1

$\frac{7}{5}$, 1

$\frac{5}{2}$, 1

$\frac{3}{6}$, 1

Ask four students to come to the board to show both of their explanations for each of the four questions.

Suggested end of day 1.

** Here the student will learn how to compare fractions to whole numbers other than 1 by first learning a way to rename those whole numbers as equivalent fractions. **

NOW THAT WE KNOW HOW TO COMPARE FRACTIONS TO 1, WE WOULD LIKE TO SEE IF WE COULD COMPARE THESE FRACTIONS TO OTHER WHOLE NUMBERS. JUST AS BEFORE, WE MIGHT TRY TO FIND FRACTION NAMES FOR THESE WHOLE NUMBERS SO THAT IT MIGHT BE EASIER TO COMPARE THEM WITH THE FRACTIONS WE ARE GIVEN. SO NOW LET'S SEE HOW GOOD WE ARE AT RENAMING WHOLE NUMBERS OTHER THAN 1 AS FRACTIONS.

SUPPOSE WE WANT ANOTHER NAME FOR 2. ONE WAY IS TO RENAME 2 AS 2 x 1 AND USE ANOTHER NAME FOR 1.

A review of the review on this topic might be appropriate here.

FOR EXAMPLE, I MIGHT WRITE:

$$2 = 2 \times (8 \times \tfrac{1}{8})$$
$$= (2 \times 8) \times \tfrac{1}{8}$$
$$= 16 \times \tfrac{1}{8} = \tfrac{16}{8} .$$

ON THE OTHER HAND, I MIGHT WRITE INSTEAD:

$$2 = 2 \times 1$$
$$= 2 \times (3 \times \tfrac{1}{3})$$
$$= (2 \times 3) \times \tfrac{1}{3}$$
$$= 6 \times \tfrac{1}{3} \quad = \quad \tfrac{6}{3} \quad .$$

WHAT OTHER RENAMINGS FOR 1 MIGHT I USE?

Accept any answer where the one is rewritten as a number times one over that number, e.g. $4 \times \frac{1}{4}$ or $6 \times \frac{1}{6}$.

WHAT DO YOU NOTICE ABOUT THE NUMERATORS OF OUR ANSWERS AS COMPARED TO THE DENOMINATORS?

Expect the answer that each numerator is twice as much as the denominator. If no one says this, go on with the following anyway.

IN EACH SITUATION, WE HAD SOMETHING LIKE $2 = 2 \times (9 \times \frac{1}{9})$ $= (2 \times 9) \times \frac{1}{9}$ OR $2 = 2 \times (10 \times \frac{1}{10}) = (2 \times 10) \times \frac{1}{10}$. THEREFORE, OUR NUMERATOR WAS TWICE OUR DENOMINATOR.

WHAT WOULD YOU PREDICT ABOUT THE NUMERATOR AS COMPARED TO THE DENOMINATOR IN FRACTIONS EQUIVALENT TO 3?

Expect the answer: the numerator will be three times as large.

LET'S CHECK. ONE WAY TO GET A FRACTION EQUIVALENT TO 3 IS TO REWRITE 3 AS 3x1 AND THEN RENAME 1. FOR EXAMPLE, I MIGHT WRITE:

$$3 = 3 \times 1$$
$$= 3 \times (2 \times \tfrac{1}{2})$$
$$= (3 \times 2) \times \tfrac{1}{2}$$
$$= 6 \times \tfrac{1}{2} \quad = \quad \tfrac{6}{2} \quad .$$

NOTICE THAT THE NUMERATOR IS 3 TIMES THE DENOMINATOR.

DOES ANYONE ELSE HAVE ANOTHER FRACTION NAME FOR 3?

Expect an answer like: $\frac{9}{3}$ or $\frac{12}{4}$ or $\frac{15}{5}$.

LET'S SEE IF WE CAN CHECK THESE BY RENAMING.

Write, e.g., for $3 = \frac{9}{3}$

$$3 = 3 \times 1$$
$$= 3 \times (3 \times \tfrac{1}{3})$$
$$= (3 \times 3) \times \tfrac{1}{3}$$
$$= 9 \times \tfrac{1}{3} = \tfrac{9}{3}$$

for $3 = \frac{12}{4}$

$$3 = 3 \times 1$$
$$= 3 \times (4 \times \tfrac{1}{4})$$
$$= (3 \times 4) \times \tfrac{1}{4}$$
$$= 12 \times \tfrac{1}{4} = \tfrac{12}{4}$$

SUPPOSE I AM TRYING TO FIND A FRACTION EQUIVALENT TO 10. WHAT DO YOU SUPPOSE WE MIGHT GUESS ABOUT THE RELATIONSHIP BETWEEN THE NUMERATOR AND THE DENOMINATOR?

Expect the answer: the numerator will be 10 times as large as the denominator.

LET'S SEE IF WE WERE RIGHT. SUPPOSE WE WANT THE DENOMINATOR TO BE 3. SINCE WE PREDICTED THAT THE NUMERATOR WILL BE 10 TIMES AS LARGE AS THE DENOMINATOR, WE SHOULD GET A FRACTION $\frac{30}{3}$ FOR OUR ANSWER.

WE CAN WRITE:

$$10 = 10 \times 1$$
$$= 10 \times (3 \times \tfrac{1}{3})$$
$$= (10 \times 3) \times \tfrac{1}{3}$$
$$= 30 \times \tfrac{1}{3} = \tfrac{30}{3} .$$

Ask the students to work out equivalent fractions for each of the following

whole numbers so that the fractions have the given denominator.

4 WITH DENOMINATOR 5
7 WITH DENOMINATOR 4
5 WITH DENOMINATOR 2
8 WITH DENOMINATOR 3.

Ask a student to come to the board for each problem showing his answer and the accompanying renaming.

Then have students try to solve the following questions. If they have trouble, allow them to go back to a renaming argument.

$3 = \frac{\square}{4}$
$6 = \frac{\square}{8}$
$5 = \frac{\square}{3}$
$9 = \frac{\square}{2}$

Have a student read out his answers and make sure that everyone understands how to get the correct answers.

Sugggested end of day 2.

** Here the student will learn how to compare a fraction with any whole number, for example, to compare $\frac{8}{3}$ with 4. **

WE ALREADY KNOW HOW TO COMPARE A FRACTION WITH 1. NOW WE WILL TRY TO COMPARE FRACTIONS TO OTHER WHOLE NUMBERS. SUPPOSE I WANT TO COMPARE $\frac{16}{3}$ TO 5.
AS BEFORE, IF WE TURN 5 INTO THIRDS, IT WILL BE EASIER TO COMPARE IT TO $\frac{16}{3}$ SINCE IT IS VERY SIMPLE TO COMPARE FRACTIONS WITH THE SAME DENOMINATOR.

THE SAME PRINCIPLE APPLIES NO MATTER WHAT THE DENOMINATOR. IF TWO FRACTIONS HAVE THE SAME DENOMINATOR, WE NEED ONLY COMPARE THEIR NUMERATORS TO DECIDE WHICH IS GREATER.

If you feel that the students are still having difficulty with this idea, go back to questions like: $\frac{5}{8}$ versus $\frac{7}{8}$:

$$\frac{5}{8} = 5 \times \frac{1}{8} = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$$

$$\frac{7}{8} = 7 \times \frac{1}{8} = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$$

so, $7 \times \frac{1}{8} > 5 \times \frac{1}{8}$.

NOW LET'S GET BACK TO OUR PROBLEM OF COMPARING $\frac{16}{3}$ TO 5. SUPPOSE I WANT TO CHANGE 5 TO AN EQUIVALENT FRACTION WITH DENOMINATOR 3. WHAT DO I GET?

Expect the answer: $\frac{15}{3}$.

NOTICE THAT I COULD WRITE THIS AS $\frac{5 \times 3}{3}$ SINCE WE ALL KNOW THAT THE NUMERATOR IS 5 TIMES THE DENOMINATOR.
THEN, WHICH IS GREATER, $\frac{16}{3}$ OR $\frac{5 \times 3}{3}$?

Expect the answer: $\frac{16}{3}$.

NOTICE THAT WE MIGHT GO THROUGH A RENAMING ARGUMENT TO CHECK:

$$\begin{aligned} 5 &= 5 \times 1 \\ &= 5 \times (3 \times \tfrac{1}{3}) \\ &= (5 \times 3) \times \tfrac{1}{3} \\ &= 15 \times \tfrac{1}{3} = \tfrac{15}{3} . \end{aligned}$$

AND SO WE SEE THAT $\frac{16}{3} > 5$ SINCE $16 > 15$, OR $16 > 5 \times 3$.

SUPPOSE WE WANT TO COMPARE $\frac{5}{3}$ TO 4. WHAT EQUIVALENT FRACTION FOR 4 WOULD MAKE THIS PROBLEM EASIER?

Expect the answer: $\frac{4x3}{3}$.

> NOTICE THAT THIS EQUIVALENT FRACTION TO 4 HAS DENOMINATOR 3 AND NUMERATOR 4x3. THEN, WHICH IS GREATER, $\frac{5}{3}$ OR $\frac{4x3}{3}$?

Expect the answer: $\frac{4x3}{3}$.

> WE CAN RENAME 4 TO CHECK:
>
> $4 = 4 \times 1$
>
> $= 4 \times (3 \times \frac{1}{3})$
>
> $= (4 \times 3) \times \frac{1}{3}$
>
> $= 12 \times \frac{1}{3} = \frac{12}{3}$.
>
> THEN, $\frac{5}{3} < 4$ SINCE $5 < 12$ OR $5 < 4x3$.
>
> TO COMPARE $\frac{50}{8}$ TO 7, WHAT DO YOU SUGGEST DOING?

Expect the answer: Change 7 into $\frac{56}{8}$ and then compare 50 with 56.

> NOTICE THAT WE CAN RENAME 7:
>
> $7 = 7 \times 1$
>
> $= 7 \times (8 \times \frac{1}{8})$
>
> $= (7 \times 8) \times \frac{1}{8}$
>
> $= 56 \times \frac{1}{8} = \frac{56}{8}$.
>
> AND WE SEE THAT WE ARE COMPARING 50 WITH 56 OR 50 WITH 7x8.

Ask the students to compare:

> $\frac{19}{3}$ TO 7
>
> $\frac{24}{7}$ TO 5
>
> $\frac{65}{8}$ TO 8

Ask three students to come to the board to show their solutions as

renamings. For each problem, be sure to restate the answer so that it can be seen as comparing the first fraction's numerator with the product of its denominator and the whole number, e.g. for the first problem, as 19 compared to 3x7.

WHEN WE EXAMINE WHAT WE HAVE DONE SO FAR, WE SEE THAT WE ALWAYS CHANGED THE WHOLE NUMBER TO AN EQUIVALENT FRACTION WITH DENOMINATOR EQUAL TO THE DENOMINATOR OF THE GIVEN FRACTION. THEN WE COMPARED THE NUMERATOR OF THE GIVEN FRACTION TO THE NUMERATOR OF THIS FRACTION EQUIVALENT TO THE WHOLE NUMBER.

HOWEVER, BECAUSE OF THE WAY OF GETTING EQUIVALENT FRACTIONS TO WHOLES, WE SEE THAT THIS IS EXACTLY THE SAME AS COMPARING THE NUMERATOR OF THE GIVEN FRACTION WITH THE PRODUCT OF THE DENOMINATOR OF THAT FRACTION AND THE WHOLE NUMBER.

FOR EXAMPLE, TO COMPARE $\frac{16}{3}$ TO 5, WE COMPARED 16 TO 5x3.
TO COMPARE $\frac{5}{3}$ TO 4, WE COMPARED 5 TO 4x3.

$$\frac{16}{3} \text{ TO } 5 = \frac{16}{3} \text{ TO } \frac{5\times3}{3}$$

$$\frac{5}{3} \text{ TO } 4 = \frac{5}{3} \text{ TO } \frac{4\times3}{3}$$

SO THAT TO COMPARE ANY FRACTION TO A WHOLE NUMBER, I CAN FIND WHOLE NUMBERS TO COMPARE INSTEAD.

JUST AS IN COMPARING $\frac{16}{3}$ TO 5, I COMPARED THE WHOLE NUMBERS 16 AND 15 OR 16 AND 5x3.

For each of the following, ask the students to write a whole number inequality they could solve to find the answer to the question of which of the two numbers in the pair is greater.

$\frac{25}{6}$ AND 4

$\frac{14}{5}$ AND 3

$\frac{19}{4}$ AND 5

Have three students come to the board and show their answers. They can rename to show the validity. Expect answers:

$25 > 6x4$		$25 > 4x6$
$14 < 5x3$		$14 < 3x5$
$19 < 4x5$	or	$19 < 5x4$

Be sure to stress the importance of not changing around the numbers in filling in the inequality sign. For example, to fill in $\frac{16}{3} \sim 5$, I say $16 > 5x3$, so $\frac{16}{3} > 5$; but to fill in $5 \sim \frac{16}{3}$, I say $5x3 < 16$, so $5 < \frac{16}{3}$.

Hand out worksheet 1 to the students to complete.

Suggested end of day 3.

Comparison of Fractions Using the Cross-Product

Outline for approach A:

NOW THAT WE KNOW HOW TO COMPARE A FRACTION TO A WHOLE NUMBER, WE ARE GOING TO LEARN HOW TO COMPARE TWO FRACTIONS.

BUT, AGAIN, WE JUST WANT TO REMIND OURSELVES OF A FEW THINGS BEFORE WE GO ON.

FIRST OF ALL, LET'S ALL REMEMBER WHAT WE MEAN BY A FRACTION LIKE $\frac{4}{5}$.
ONE MEANING OF $\frac{4}{5}$ IS $\frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} = 4 \times \frac{1}{5}$.

HOW COULD I INTERPRET $\frac{3}{6}$ AS A MULTIPLICATION EXPRESSION?

Expect the answer: $3 \times \frac{1}{6}$.

HOW COULD I INTERPRET $\frac{4}{3}$ AS A MULTIPLICATION EXPRESSION?

Expect the answer: $4 \times \frac{1}{3}$.

WELL, THIS SEEMS REASONABLY EASY, SO LET'S GO ON TO THE NEXT BIT OF REVIEWING.

HOW CAN WE TELL, USING A MULTIPLICATION EXPRESSION, WHICH OF TWO FRACTIONS WITH THE SAME DENOMINATOR IS GREATER?
LET'S LOOK AT THIS PROBLEM.

SUPPOSE I WERE COMPARING 8x9 WITH 7x9. WHICH IS GREATER?

Expect the answer: 8 x9.

NOW, JUST AS 8x9 IS GREATER THAN 7x9 OR 78x4 IS GREATER THAN 54 x 4, WE CAN MAKE STATEMENTS ABOUT FRACTIONS.

SUPPOSE I AM COMPARING $\frac{6}{3}$ TO $\frac{5}{3}$. IC AN REWRITE THESE AS:

$\frac{6}{3} = 6 \times \frac{1}{3}$

$\frac{5}{3} = 5 \times \frac{1}{3}$

CERTAINLY, 6 TIMES A QUANTITY IS GREATER THAN ONLY 5 TIMES THAT QUANTITY, SO $\frac{6}{3} > \frac{5}{3}$.

SO TO COMPARE FRACTIONS WITH THE SAME DENOMINATOR, WE CAN RE-NAME THEM AS MULTIPLICATION EXPRESSIONS AND THEN COMPARE.

Ask the students to decide which of $\frac{3}{5}$ or $\frac{4}{5}$ is greater by renaming as multiplication expressions. Have one student explain his answer. Then repeat the procedure for comparing $\frac{1}{4}$ and $\frac{2}{4}$ and $\frac{7}{18}$ and $\frac{9}{18}$.

FINALLY, WE WANT TO EXAMINE SOMETHING ABOUT FINDING FRACTIONS EQUIVALENT TO A GIVEN ONE.

THERE ARE A LOT OF WAYS OF CHANGING THE EXPRESSION FOR A NUM-BER WITHOUT CHANGING THE ACTUAL NUMBER. FOR EXAMPLE, I CAN REWRITE:

7x9x6 AS 7x6x9.

THIS IS A CHANGE, BUT IS STILL ANOTHER NAME FOR THE SAME NUMBER. ANOTHER THING I CAN DO IS TO MULTIPLY A NUMBER BY 1. FOR EXAMPLE, $8 = 8 \times 1$.

SOMETIMES, WE USE A MORE COMPLICATED FORM OF 1, LIKE PERHAPS,

$= 2 \times \frac{1}{2}$, SO I MIGHT RENAME $8 = 8 \times (2 \times \frac{1}{2})$.

SUPPOSE I WANT AN EQUIVALENT TO $\frac{11}{3}$:

I CAN REWRITE THIS AS: $\frac{11}{3} = 11 \times \frac{1}{3}$

OR AS $\frac{11}{3} = 11 \times \frac{1}{3} \times (2 \times \frac{1}{2})$.

Ask the students to find other names of the following by renaming in

terms of multiplication by some form of 1.

$\frac{1}{3}$

$\frac{2}{5}$

$\frac{1}{4}$

Accept any answers which demonstrate renaming of 1 as a number times one over that number, e.g. by multiplication by $2 \times \frac{1}{2}$ or $5 \times \frac{1}{5}$.

Suggested end of day 4.

** Here the student will review the idea that two fractions with the same denominator can be compared by examining their numerators, e.g. $\frac{6}{3} > \frac{4}{3}$ since $6 > 4$. **

I WANT TO BE ABLE TO COMPARE ANY TWO FRACTIONS, BUT I MIGHT AS WELL START BY COMPARING FRACTIONS WITH THE SAME DENOMINATOR.

SUPPOSE I WANT TO COMPARE $\frac{7}{4}$ WITH $\frac{5}{4}$. THIS IS FAIRLY EASY. ALL I HAVE TO DO IS TO REWRITE EACH AS A MULTIPLICATION EXPRESSION AND SEE WHICH IS GREATER.

$$\frac{7}{4} = 7 \times \frac{1}{4}$$

$$\frac{5}{4} = 5 \times \frac{1}{4}$$

IN THE FIRST CASE, I AM TAKING SEVEN GROUPS OF A CERTAIN QUANTITY (ONE-FOURTH), WHEREAS IN THE SECOND CASE, I AM TAKING ONLY FIVE GROUPS OF THAT SAME QUANTITY. BECAUSE WE ARE TAKING MORE OF THE SAME 'OBJECTS', I CAN TELL THAT <u>SEVEN</u> ONE-FOURTHS IS GREATER THAN <u>FIVE</u> ONE-FOURTHS AND WE CAN WRITE .

WHAT IF I WERE TO COMPARE $\frac{5}{8}$ TO $\frac{6}{8}$? WHICH IS GREATER?

Expect the answer: $\frac{6}{8}$.

LET'S RENAME TO CHECK:

$$\frac{5}{8} = 5 \times \frac{1}{8}$$

$$\frac{6}{8} = 6 \times \frac{1}{8}$$

BECAUSE $5 < 6$, WE CAN SEE THAT $\frac{5}{8} < \frac{6}{8}$.

Ask the students to name the greater fraction in each of these pairs.

$\frac{3}{9}$ AND $\frac{5}{9}$

$\frac{4}{6}$ AND $\frac{3}{6}$

$\frac{5}{3}$ AND $\frac{6}{3}$

Have three students come to the board and rename to explain their answers.

** Here the student will learn to generate equivalent expressions for two given fractions such that the new expressions have a common term, for example, for $\frac{3}{4}$ and $\frac{5}{6}$, to produce $18 \times (\frac{1}{4} \times \frac{1}{6})$ and $20 \times (\frac{1}{4} \times \frac{1}{6})$ as equivalent expressions for purposes of comparison. **

NOW THAT WE ALREADY KNOW A BIT ABOUT COMPARING SOME FRACTIONS, LET US TRY TO SEE IF WE CAN USE THIS SAME TECHNIQUE ON ANY FRACTIONS. IS IT EASY TO DECIDE WHETHER $\frac{2}{3} > \frac{3}{4}$? CAN WE MAKE IT EASIER? SUPPOSE WE COULD REWRITE BOTH OF THESE FRACTIONS SO THAT THEY BOTH HAVE THE SAME DENOMINATOR. THAT WOULD MAKE IT A LOT EASIER.

WE ALREADY KNOW THAT A WAY TO GET EQUIVALENT NAMES FOR A FRACTION IS TO REWRITE IT AS A MULTIPLICATION EXPRESSION AND THEN MULTIPLY THAT BY SOME FORM OF ONE.

FOR EXAMPLE, TO CHANGE $\frac{2}{3}$ INTO ANOTHER NAME, I MIGHT REWRITE AS:

$$\frac{2}{3} = 2 \times \frac{1}{3} = 2 \times \frac{1}{3} \times (2 \times \frac{1}{2}) = (2 \times 2) \times (\frac{1}{3} \times \frac{1}{2})$$

AND SEE THAT $\frac{2}{3} = 4 \times (\frac{1}{3} \times \frac{1}{2})$.

NOW SUPPOSE I WERE TRYING TO COMPARE $\frac{2}{3}$ WITH $\frac{3}{4}$.

ONE WAY TO COMPARE THESE IS TO FIND OTHER NAMES FOR EACH SO THAT THE TWO NEW NAMES HAVE A COMMON PART WITH DIFFERENCE IN THE MULTIPLIER ONLY.

SO, I REWRITE:

$$= 2 \times \frac{1}{3} \qquad \frac{3}{4} = 3 \times \frac{1}{4} \quad .$$

SINCE I CAN MULTIPLY BY ANY FORM OF ONE AND NOT CHANGE A NUMBER, I CAN FIND OTHER NAMES FOR $\frac{2}{3}$ AND $\frac{3}{4}$, LIKE SO:

$$= 2 \times \frac{1}{3} \qquad = 3 \times \frac{1}{4}$$

$$= 2 \times \frac{1}{3} \times (4 \times \frac{1}{4}) \qquad = 3 \times \frac{1}{4} \times (3 \times \frac{1}{3})$$

$$= 2 \times 4 \times (\frac{1}{3} \times \frac{1}{4}) \qquad = 3 \times 3 \times (\frac{1}{3} \times \frac{1}{4})$$

$$= 8 \times (\frac{1}{3} \times \frac{1}{4}) \qquad = 9 \times (\frac{1}{3} \times \frac{1}{4})$$

NOW NOTICE THAT THE $\frac{2}{3}$ IS REWRITTEN AS $8 \times (\frac{1}{3} \times \frac{1}{4})$ AND THE $\frac{3}{4}$ AS $9 \times (\frac{1}{3} \times \frac{1}{4})$. ALL I DID WAS TO MULTIPLY EACH BY A FORM OF 1. I MULTIPLIED THE $\frac{2}{3}$ BY $4 \times \frac{1}{4}$ AND THE $\frac{3}{4}$ BY $3 \times \frac{1}{3}$. IN BOTH EXPRESSIONS, I GET A TERM WHICH INCLUDES A $\frac{1}{3} \times \frac{1}{4}$.

NOTICE THAT IF I AM COMPARING A FRACTION WITH DENOMINATOR 3 TO ONE OF DENOMINATOR 4, I CAN MULTIPLY THE FIRST FRACTION BY $4 \times \frac{1}{4}$ AND THE SECOND BY $3 \times \frac{1}{3}$, AND THE SAME EXPRESSION $\frac{1}{3} \times \frac{1}{4}$ OCCURS IN EACH EXPRESSION.

SUPPOSE I WANT TO COMPARE $\frac{1}{2}$ WITH $\frac{2}{5}$. WHAT EQUIVALENT EXPRESSIONS SHOULD I USE? WE CAN BEGIN BY REWRITING:

$$\frac{1}{2} = 1 \times \frac{1}{2} \qquad \frac{2}{5} = 2 \times \frac{1}{5}$$

THEN I CAN MULTIPLY EACH OF THESE BY A FORM OF 1, SO AS TO GET EQUIVALENT FRACTIONAL PARTS IN EACH. THE FIRST I MULTIPLY BY

$5 \times \frac{1}{5}$ AND THE SECOND BY $2 \times \frac{1}{2}$, LIKE SO:

$= 1 \times \frac{1}{2} \times (5 \times \frac{1}{5})$ $\qquad$ $= 2 \times \frac{1}{5} \times (2 \times \frac{1}{2})$

$= (1 \times 5) \times (\frac{1}{2} \times \frac{1}{5})$ $\qquad$ $= (2 \times 2) \times (\frac{1}{2} \times \frac{1}{5})$

$= 5 \times (\frac{1}{2} \times \frac{1}{5})$ $\qquad$ $= 4 \times (\frac{1}{2} \times \frac{1}{5})$

THEN, $\frac{1}{2} = 5 \times (\frac{1}{2} \times \frac{1}{5})$ AND $\frac{2}{5} = 4 \times (\frac{1}{2} \times \frac{1}{5})$ AND WE COULD EASILY COMPARE $\frac{1}{2}$ AND $\frac{2}{5}$.

AGAIN, NOTICE THAT I MULTIPLY THE FIRST EXPRESSION BY $5 \times \frac{1}{2}$ AND THE SECOND BY $2 \times \frac{1}{2}$ SO THAT BOTH EXPRESSIONS HAVE A TERM $(\frac{1}{2} \times \frac{1}{5})$ AND ONLY DIFFER IN THE NUMBER OF THESE THAT WERE THERE.

WHAT DO YOU EXPECT TO BE THE COMMON EXPRESSION IN FINDING EQUIVALENTS TO $\frac{1}{2}$ AND $\frac{2}{3}$ SO AS TO MAKE COMPARISON EASIER?

Expect the answer: $\frac{1}{2} \times \frac{1}{3}$.

WE CAN RENAME:

$= 1 \times \frac{1}{2}$ $\qquad$ $= 2 \times \frac{1}{3}$

$= 1 \times \frac{1}{2} \times (3 \times \frac{1}{3})$ $\qquad$ $= 2 \times \frac{1}{3} \times (2 \times \frac{1}{2})$

$= (1 \times 3) \times (\frac{1}{2} \times \frac{1}{3})$ $\qquad$ $= (2 \times 2) \times (\frac{1}{2} \times \frac{1}{3})$

$= 3 \times (\frac{1}{2} \times \frac{1}{3})$ $\qquad$ $= 4 \times (\frac{1}{2} \times \frac{1}{3})$

AND SEE THAT, INDEED, $\frac{1}{2}$ CAN BE REWRITTEN AS $3 \times (\frac{1}{2} \times \frac{1}{3})$ AND $\frac{2}{3}$ AS $4 \times (\frac{1}{2} \times \frac{1}{3})$.

Ask the students to attempt to rewrite each of these pairs of fractions as expression which are equivalent to those fractions but with a common part.

$\frac{3}{4}$ AND $\frac{2}{3}$

$\frac{4}{5}$ AND $\frac{2}{3}$

$\frac{2}{6}$ AND $\frac{1}{2}$

Have three students show their work on the board.

Make sure the students realize that we examine the denominator of the other fraction to get the name for 1 to be used in renaming a fraction.

** Here the student will learn that $\frac{a}{b} > \frac{c}{d}$ only when $a \times d > b \times c$, for example, $\frac{3}{6} > \frac{2}{5}$ only because $3 \times 5 > 2 \times 6$. **

NOW THAT WE ALREADY REALIZE THAT AN EASIER WAY TO COMPARE TWO FRACTIONS TO DECIDE WHICH IS LARGER IS TO RENAME THEM AS MULTIPLICATION EXPRESSIONS WITH A COMMON PART AND WE HAVE PRACTICED RENAMING FRACTIONS, WE CAN GO ON TO COMPARING THEM.

SUPPOSE I WANT TO COMPARE $\frac{1}{2}$ WITH $\frac{4}{6}$ TO FIND WHICH IS GREATER. I CAN RENAME THESE AS MULTIPLICATION EXPRESSIONS WITH A COMMON PART. WHAT WILL THESE EXPRESSIONS BE?

Expect the answers: $1 \times 6 \times (\frac{1}{2} \times \frac{1}{6})$ and $4 \times 2 \times (\frac{1}{2} \times \frac{1}{6})$.

I CAN CHECK THIS:

$= 1 \times \frac{1}{2}$	$= 4 \times \frac{1}{6}$
$= 1 \times \frac{1}{2} \times (6 \times \frac{1}{6})$	$= 4 \times \frac{1}{6} \times (2 \times \frac{1}{2})$
$= (1 \times 6) \times (\frac{1}{2} \times \frac{1}{6})$	$= (4 \times 2) \times (\frac{1}{2} \times \frac{1}{6})$
$= 6 \times (\frac{1}{2} \times \frac{1}{6})$	$= 8 \times (\frac{1}{2} \times \frac{1}{6})$

THEN, I NOTICE THAT I AM COMPARING 6 WITH 8. WHICH IS GREATER?

Expect the answer: 8.

BUT WHERE DID THE 6 IN $6 \times (\frac{1}{2} \times \frac{1}{6})$ COME FROM? IT CAME FROM 1×6, THE ONE FROM THE ORIGINAL EXPRESSION FOR $\frac{1}{2}$ AND THE 6 FROM MULTIPLYING THAT BY $6 \times \frac{1}{6}$. WHERE DID THE 8 COME FROM? IT

CAME FROM 4x2, THE 4 FROM THE ORIGINAL EXPRESSION FOR $\frac{4}{6}$ AND THE 2 FROM MULTIPLYING THAT BY $2 \times \frac{1}{2}$.
NOTICE THAT THE FRACTION WITH 2 AS DENOMINATOR WAS MULTIPLIED BY $6 \times \frac{1}{6}$, SINCE THE OTHER DENOMINATOR WAS 6.
WHY WAS THE FRACTION WITH 6 AS DENOMINATOR MULTIPLIED BY 2?

Expect the answer: because the other denominator was 2.

SO, IN COMPARING $\frac{1}{2}$ AND $\frac{4}{6}$, I AM REALLY COMPARING 6 AND 8, OR 1x 6 AND 4 x 2.

TO COMPARE $\frac{1}{2}$ AND $\frac{4}{6}$,I SAY $1 \times 6 < 4 \times 2$, SO $\frac{1}{2} < \frac{4}{6}$.

SUPPOSE I WANT TO COMPARE $\frac{1}{2}$ WITH $\frac{3}{5}$ TO FIND WHICH IS GREATER.
I CAN RENAME THESE BY EQUIVALENT EXPRESSIONS WITH A COMMON PART.
WHAT EXPRESSIONS WOULD I USE?

Expect the answers: $1 \times 5 \times (\frac{1}{2} \times \frac{1}{5})$ and $3 \times 2 \times (\frac{1}{2} \times \frac{1}{5})$

I CAN CHECK:

$$= 1 \times \frac{1}{2} \qquad\qquad = 3 \times \frac{1}{5}$$
$$= 1 \times \frac{1}{2} \times (5 \times \frac{1}{5}) \qquad\qquad = 3 \times \frac{1}{5} \times (2 \times \frac{1}{2})$$
$$= (1 \times 5) \times (\frac{1}{2} \times \frac{1}{5}) \qquad\qquad = (3 \times 2) \times (\frac{1}{2} \times \frac{1}{5})$$

THEN, I NOTICE THAT I AM COMPARING 1x5 WITH 4x2. WHICH IS GREATER?

Expect the answer: 4x2.

REMEMBER THAT 1x5 TELLS ME THAT I ORIGINALLY HAD A MULTIPLE OF 1 OF SOME FRACTION AND THEM MULTIPLIED THAT BY $5 \times \frac{1}{5}$ AND THE 3x2 TELLS ME THAT I ORIGINALLY HAD 3 TIMES SOME FRACTION AND

THEN MULTIPLIED BY $2 \times \frac{1}{2}$. THEN BOTH EXPRESSIONS NOW HAVE A PART ($\frac{1}{2} \times \frac{1}{5}$). SO I NEED ONLY COUNT HOW MANY OF THESE I HAVE FOR EACH.

SINCE $1 \times 5 < 3 \times 2$, THEN $\frac{1}{2} < \frac{3}{5}$.

WHICH OF THESE TWO FRACTIONS IS GREATER: $\frac{3}{4}$ OR $\frac{2}{5}$?

Expect the answer: multiply the first by $5 \times \frac{1}{5}$ and the second by $4 \times \frac{1}{4}$, or else compare 3x5 with 2x4.

NOW WE HAVE TWO ALTERNATIVES. WE COULD CONVERT BOTH FRACTIONS TO EQUIVALENT EXPRESSIONS AND COMPARE THEM LIKE SO:

$= 3 \times \frac{1}{4}$	$= 2 \times \frac{1}{5}$
$= 3 \times \frac{1}{4} \times (5 \times \frac{1}{5})$	$= 2 \times \frac{1}{5} \times (4 \times \frac{1}{4})$
$= (3 \times 5) \times (\frac{1}{4} \times \frac{1}{5})$	$= (2 \times 4) \times (\frac{1}{4} \times \frac{1}{5})$

OR ELSE WE MIGHT SIMPLY NOTICE THAT THIS IS EXACTLY THE SAME AS COMPARING 3x5 WITH 2x4, SINCE 3x5 TELLS ME I ORIGINALLY HAD 3 TIMES A FRACTION AND THEN MULTIPLED IT BY $5 \times \frac{1}{5}$ AND THE 2x4 THAT I ORIGINALLY HAD 2 x ANOTHER FRACTION AND THEN MULTIPLIED IT BY $4 \times \frac{1}{4}$. NOW EACH EXPRESSION HAS SOME MULTIPLE OF ($\frac{1}{4} \times \frac{1}{5}$).

CAN YOU SEE WHY THIS IS CALLED THE CROSS-PRODUCT RULE FOR COMPARING FRACTIONS. I TAKE THE PRODUCT ACROSS THE FRACTIONS AND COMPARE THESE. FOR EXAMPLE, IN $\frac{3}{4}$ AND $\frac{5}{6}$,
I COMPARE 3×6 WITH 5×4 WHICH COMES FROM $\frac{3}{4} \times \frac{5}{6}$.

Ask the students to find the larger in each of these paris by using the cross product rule. Have a student show each of these on the board diagrammatically explaining why the rule works.

$\frac{3}{5}$	OR	$\frac{2}{7}$
$\frac{3}{8}$	OR	$\frac{4}{5}$
$\frac{1}{5}$	OR	$\frac{2}{6}$
$\frac{1}{3}$	OR	$\frac{3}{5}$

Suggested end of day 5.

Finding the Square Root of a Fraction

Outline for approach p:

> WE ARE GOING TO LEARN A WAY TO EXPRESS ANY FRACTION AS THE PRODUCT OF TWO EQUAL FRACTIONS (OR APPROXIMATELY SO). FOR EXAMPLE, WE WILL LEARN THAT JUST AS 16 CAN BE EXPRESSED AS THE PRODUCT OF TWO EQUAL FACTORS, NAMELY 4 x 4, $\frac{16}{9}$ CAN BE EXPRESSED AS THE PRODUCT OF TWO IDENTICAL FRACTIONS, NAMELY $\frac{16}{9} = \frac{4}{3} \times \frac{4}{3}$.
>
> BUT FIRST WE MUST DO A BIT OF REVIEWING.
>
> I AM GOING TO ASK YOU SOME MULTIPLICATION QUESTIONS.

Ask the following:

8 x 5
7 x 9
4 x 6
3 x 3
8 x 8

If there is much difficulty in answering these questions on the part of the class as a whole, then draw diagrams to illustrate the problems. For example, to illustrate:

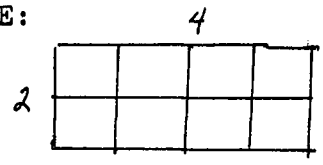
3 x5	4 x 4
x x x x x	x x x x
x x x x x	x x x x
x x x x x	x x x x
	x x x x

Notice that the first number represents the number of rows and the second number represents the number of columns.

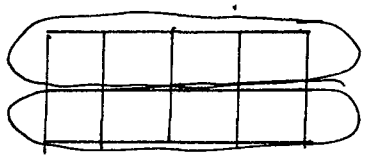
Go over items 11-15 on the pretest. In particular, have students realize

that the name of the shaded region in a figure depends on the number of parts of the same size into which the figure is drawn and the number of these parts that are shaded in. For each question, read the question and ask for the correct answer. This review should go quickly, particularly since the students worked with fractions in the previous unit.

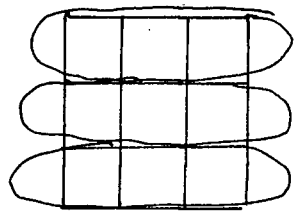
WE WOULD NOW LIKE TO REVIEW SOME IDEAS ABOUT AREA. REMEMBER THAT THE AREA OF A RECTANGLE TELLS ME THE NUMBER OF UNIT SQUARES THAT FIT ONTO THE RECTANGLE. FOR EXAMPLE, THE AREA OF A RECTANGLE WHICH IS 2 IN. BY 4 IN. IS 8 SQ. IN., SINCE I CAN DRAW A DIAGRAM LIKE:



NOTICE THAT I COULD LOOK AT THIS DIAGRAM AS COUNTING 2 GROUPS OF 4 SQUARES EACH, WHICH IS ANOTHER WAY OF SAYING 2 x 4:

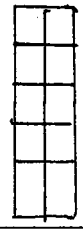


LET'S DO THIS FOR A RECTANGLE WHICH IS 3 IN. BY 3 IN. (A SQUARE). I CAN DRAW:



AND I SEE THAT THE AREA IS 3 x 3 = 9 SQ. IN.

IF MY RECTANGLE IS 5 IN. BY 2 IN., I WOULD DRAW:



AND THE AREA IS 10 SQ. IN. = 5 x 2 SQ. IN.

Ask the students to consider what the areas of rectangles with dimensions 5 by 8, 3 by 9, and 8 by 4 would be. Explain that they may always go back to a diagram to find the solutions. Call on three individuals to get the answers to these three questions.

FINALLY WE WANT TO REMEMBER HOW TO DIVIDE, BECAUSE SOME OF OUR WORK WILL INVOLVE DIVISION.

SUPPOSE I WANT TO SOLVE $2639 \div 13$. HOW DO I BEGIN?

I SET UP THE QUESTION, LIKE SO:

$13\overline{)2639}$

AND I KEEP TAKING CHUNKS OF 13 OUT OF 2639 UNTIL I HAVE USED UP ALL OF THE 2639. MY ANSWER IS THE NUMBER OF CHUNKS I TOOK OUT.

SO I CAN TAKE 200 CHUNKS OF 13 OUT FIRST, LEAVING ONLY 39.

```
13)2639
   2600 | 200
     39 |
     39 |  3
      0 | 203
```

THEN I CAN TAKE OUT 3 MORE CHUNKS OF 13 AND USE ALL OF THE 2639.

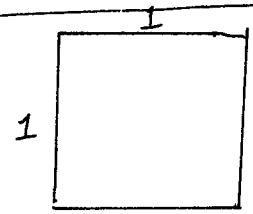
SO, $2639 \div 13 = 203$.

Allow the students to solve these two division questions and have two students show their results to the class:

$9\overline{)288}$ AND $20\overline{)4300}$

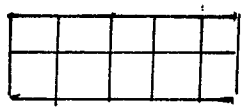
** Here the student will learn how to multiply fractions in order that he might find answers to problems like $\frac{\square}{\triangle} \times \frac{\square}{\triangle} = \frac{16}{9}$. **

Draw a large unit square on the board.



I WANT TO BE ABLE TO FIND ANSWERS TO MULTIPLICATION QUESTIONS WHERE ONE OF THE NUMBERS TO BE MULTIPLIED IS A FRACTION. BEFORE, WE NOTED THAT AREAS OF RECTANGLES COULD BE FOUND BY MULTIPLYING LENGTH BY WIDTH. REVERSING THIS IDEA, WE MIGHT SUPPOSE THAT MULTIPLYING TWO NUMBERS IS THE SAME AS FINDING AREAS OF RECTANGLES WITH THESE NUMBERS AS SIDE LENGTHS.

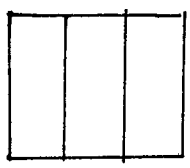
FOR EXAMPLE, THE ANSWER TO 2 x 5 COULD BE FOUND BY FINDING THE AREA OF A RECTANGLE WITH WIDTH 2 AND LENGTH 5.



NOW LET'S APPLY THIS SAME IDEA TO FRACTIONS.

SUPPOSE I WANT TO MULTIPLY A FRACTION BY ANOTHER FRACTION. THAT MEANS I WANT TO FIND THE AREA OF A RECTANGLE WITH THESE TWO FRACTIONS AS SIDE LENGTHS.

WHAT HAPPENS IF I CUT THE SQUARE (ON THE BOARD) LIKE THIS? WHAT ARE THE DIMENSIONS OF THE RECTANGLES I FORM? I NOTICE THAT I STARTED WITH A 1 BY 1 SQUARE, SO IT HAS AREA 1.

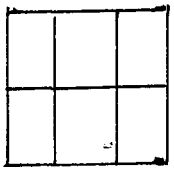


Expect the answer: 1 by $\frac{1}{3}$.

WHAT IS THE AREA OF EACH OF THESE RECTANGLES?

Expect the answer: $\frac{1}{3}$.

WHAT HAPPENS IF I ALSO CUT THE SQUARE THIS WAY?



WHAT ARE THE DIMENSIONS OF THE NEW SMALLER RECTANGLES FORMED?

Expect the answer: $\frac{1}{2}$ by $\frac{1}{3}$.

NOW THE AREA OF EACH OF THESE SMALLER RECTANGLES CAN BE REPRESENTED BY A MULTIPLICATION STATEMENT, NAMELY $\frac{1}{2} \times \frac{1}{3}$.

WHAT IS THE AREA OF EACH OF THESE SMALLER RECTANGLES? NOTICE THAT I KNOW THE AREA OF THE ENTIRE SQUARE IS 1 SQUARE UNIT, SO THE AREA OF EACH OF THESE RECTANGLES MUST BE A FRACTION OF THAT AND THE FRACTION IS DETERMINED BY THE NUMBER OF RECTANGLES THERE ARE IN THE FIGURE.

HOW MANY RECTANGLES ARE THERE OF DIMENSIONS $\frac{1}{2} \times \frac{1}{3}$?

Expect the answer: 6.

THEREFORE, EACH OF THE RECTANGLES HAS WHAT AREA?

Expect the answer: $\frac{1}{6}$.

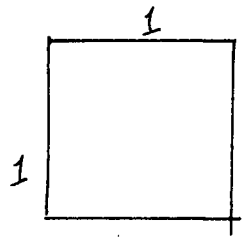
SINCE WE ALREADY SAID THIS AREA COULD BE REPRESENTED IN A MULTIPLICATION WAY AS $\frac{1}{2} \times \frac{1}{3}$, THAT MEANS

$$\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}.$$

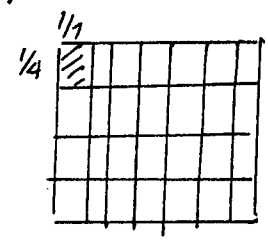
Suggested end of day 1.

HOW WOULD WE FIND $\frac{1}{4} \times \frac{1}{7}$?

WE DRAW A RECTANGLE WITH DIMENSIONS 1 BY 1, LIKE SO:



AND CUT IT:



HOW MANY SMALL RECTANGLES ARE THERE IN THE UNIT SQUARE? WHAT IS THE AREA OF EACH?

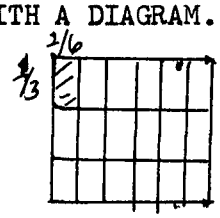
Expect the answers; 28 and $\frac{1}{28}$.

SINCE EACH AREA CAN ALSO BE REPRESENTED AS $\frac{1}{4} \times \frac{1}{7}$, THAT MEANS THAT $\frac{1}{4} \times \frac{1}{7} = \frac{1}{28}$.

WHAT DO YOU THINK $\frac{1}{3} \times \frac{1}{6}$ WILL BE?

Expect the answer: $\frac{1}{18}$.

LET US CHECK WITH A DIAGRAM. I CAN DRAW:

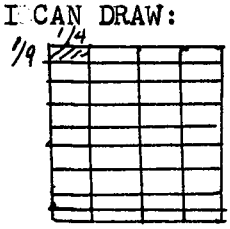


AND EACH RECTANGLE, WHICH IS OF DIMENSIONS $\frac{1}{3}$ BY $\frac{1}{6}$ DOES HAVE AREA $\frac{1}{18}$ SINCE THERE ARE 18 OF THEM IN THE UNIT SQUARE.

WHAT ABOUT $\frac{1}{9} \times \frac{1}{4}$?

Expect the answer: $\frac{1}{36}$.

LET'S CHECK. I CAN DRAW:



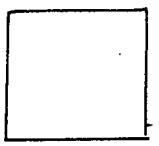
AND I SEE THAT THERE ARE 36 SMALL RECTANGLES, SO EACH HAS AREA $\frac{1}{36}$.

BUT SINCE THE DIMENSIONS OF EACH ARE $\frac{1}{9}$ BY $\frac{1}{4}$, THEN

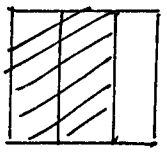
$$\frac{1}{9} \times \frac{1}{4} = \frac{1}{36} .$$

DO YOU NOTICE THAT IN EACH OF THESE QUESTIONS, IF I MULTIPLY $\frac{1}{\square}$ BY $\frac{1}{\triangle}$, I GET A FIGURE WITH $\square \times \triangle$ SMALL RECTANGLES IN IT, EACH OF AREA, THEREFORE, $\frac{1}{\square \times \triangle}$? CHECK BACK TO OUR DIAGRAMS FOR $\frac{1}{2} \times \frac{1}{3}$, $\frac{1}{4} \times \frac{1}{7}$, AND $\frac{1}{9} \times \frac{1}{4}$.

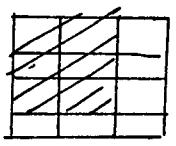
NOW SUPPOSE WE WANT TO MULTIPLY $\frac{2}{3}$ BY $\frac{3}{4}$. WE KNOW THAT WE CAN FIND THE AREA OF A RECTANGLE WITH THESE DIMENSIONS TO SOLVE OUR QUESTION. SO LET'S START OUT WITH A SQUARE AGAIN.



IF WE WANT ONE OF THE DIMENSIONS TO BE $\frac{2}{3}$, WE CUT IT SO:



IF WE WANT THE OTHER DIMENSIONS TO BE $\frac{3}{4}$, WE CUT IT SO:



THEN THE AREA OF THE SHADED REGION IS THE AREA I WANT.

HOW MANY OF THE SMALL RECTANGLES ARE SHADED IN?

Expect the answer: 6.

NOTICE THAT THE NUMBER IS 2 x 3 SINCE THERE ARE 2 ROWS EACH OF 3 RECTANGLES SHADED IN.

WHAT IS THE AREA OF EACH OF THESE RECTANGLES?

Expect the answer: $\frac{1}{12}$.

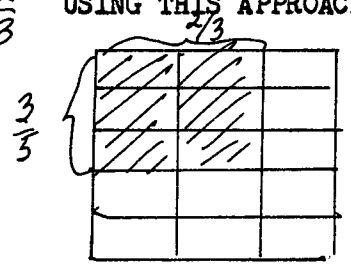
WE HAVE ALREADY LEARNED THAT SINCE THERE ARE 3x4 OF THESE RECTNAGLES ALTOGETHER, EACH HAS AREA $\frac{1}{12}$.

THEREFORE, WE HAVE 2x3 PIECES EACH OF AREA $\frac{1}{12}$, SO WE HAVE A TOTAL AREA OF $\frac{6}{12}$.

NOTICE THAT THE NUMERATOR OF MY ANSWER TELLS ME THE NUMBER OF SHADED PIECES AND THE DENOMINATOR TELLS ME THE NUMBER OF THE PIECES ALTOGETHER IN THE FIGURE.

LET'S FIND $\frac{3}{5} \times \frac{2}{3}$ USING THIS APPROACH:

I DRAW:



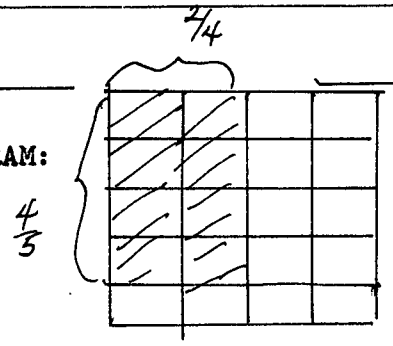
THEN THE NUMBER OF SHADED SQUARES IS 3 x 2. EACH OF THESE SQUARES HAS AREA $\frac{1}{15}$ SINCE THERE ARE 5 x 3 OF THEM IN THE ENTIRE UNIT SQUARE. SO I HAVE A TOTAL AREA OF $\frac{6}{15}$.

THEREFORE, $\frac{3}{5} \times \frac{2}{3} = \frac{6}{15}$.

WHAT WOULD $\frac{4}{5} \times \frac{2}{4}$ BE?

Expect the answer: $\frac{8}{20}$.

LET'S CHECK WITH A DIAGRAM:



WE DO HAVE 4 x 2 SQUARES EACH OF AREA $\frac{1}{20}$ SHADED IN, SO A TOTAL AREA OF $\frac{8}{20}$.

Ask the students to find the following products:

$$\frac{3}{4} \times \frac{1}{3}$$

$$\frac{6}{9} \times \frac{2}{4}$$

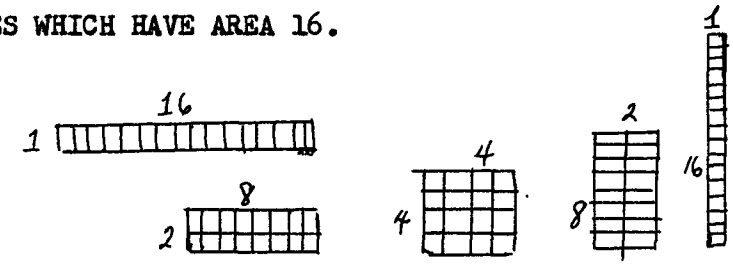
$$\frac{5}{8} \times \frac{3}{4}$$

Have three students come to the board and diagrammtically explain their answers.

Suggested end of day 2.

** Here the student will learn that the square root of a number is that number which I can multiply by itself to get the given number; for example, the square root of 9 is 3 since 3 x 3 = 9. **

SUPPOSE I WANT TO FIND ALL THE NUMBERS WHICH GO INTO 16 EVENLY. ALL I HAVE TO DO IS TO DRAW ALL THE RECTANGLES WITH WHOLE NUMBER SIDES WHICH HAVE AREA 16.



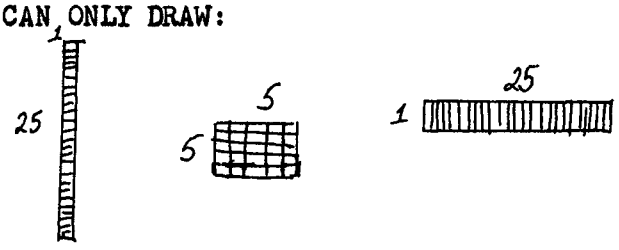
FROM THESE, I CAN SEE THAT 1,2,4,8, AND 16 ARE THOSE NUMBERS I WAS LOOKING FOR.

BUT MORE THAN THAT, I CAN SEE THAT AMONG MY DRAWINGS OF RECTANGLES, ONLY ONE IS A SQUARE, NAMELY THE 4 BY 4 SQUARE. BECAUSE OF THIS PROPERTY, 4 IS CALLED THE <u>SQUARE ROOT</u> OF 16. WE WRITE THIS: $4 = \sqrt{16}$.

SO, BECAUSE 4 HAS THE PROPERTY THAT 4 TIMES ITSELF IS 16, 4 IS THE SIDE OF A SQUARE WITH AREA OF 16, AND 4 x 4 = 16 MEANS THAT

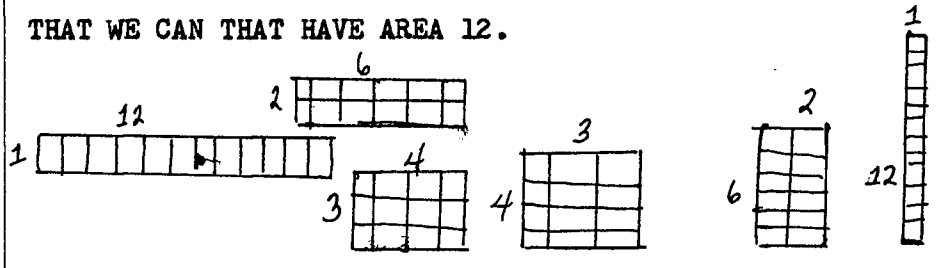
$4 = \sqrt{16}$.

LET'S TRY TO FIND THE SQUARE ROOT OF 25. WE TRY DRAWING ALL THE RECTANGLES WE CAN WITH AREA 25 AND WHOLE NUMBER SIDE LENGTHS. WE CAN ONLY DRAW:



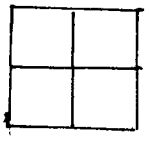
ONLY ONE OF THESE IS A SQUARE, NAMELY THE RECTANGLE WHICH IS 5 BY 5. THEREFORE, I SAY THAT $5 = \sqrt{25}$ AND INDEED 5 x ITSELF IS 25.

SOME NUMBERS DO NOT HAVE WHOLE NUMBER SQUARE ROOTS. FOR EXAMPLE, TAKE 12. LET US DRAW ALL THE RECTANGLES WITH WHOLE NUMBER SIDES THAT WE CAN THAT HAVE AREA 12.



NOTICE THAT NONE OF THESE IS A SQUARE.

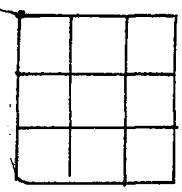
WE ALREADY KNOW MANY SQUARE ROOTS. FOR EXAMPLE, THE SQUARE ROOT OF 4 IS 2, SINCE 2 x 2 = 4 AND WE CAN DRAW:



WHAT IS THE SQUARE ROOT OF 9?

Expect the answer: 3.

AND WE CAN DRAW:



Ask the students to call out the square roots of the following:

36

49

100

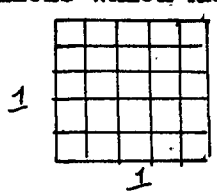
For each, draw the diagram of the square on the board, so that it is apparent that a square with side the square root of these numbers has the correct area.

** Here the student will learn to apply the idea of square root to fractions. **

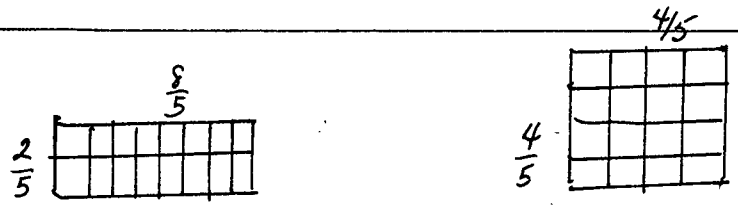
SUPPOSE WE ARE LOOKING FOR THE SQUARE ROOT OF A FRACTION, LIKE $\frac{16}{25}$. THAT MEANS THAT WE ARE LOOKING FOR THE DIMENSIONS OF THAT RECTANGLE WITH AREA $\frac{16}{25}$ THAT HAPPENS TO BE A SQUARE.

TO DRAW RECTANGLES WITH AREA $\frac{16}{25}$, WE WOULD BE LIKELY TO START OUT WITH A FIGURE LIKE THE ONE BELOW AND PICK 16 RECTANGLES OF IT WHICH WOULD HELP US FORM A SQUARE.

WE START WITH PIECES WHICH ARE 25THS.



AND TRY TO ARRANGE THEM VARIOUS WAYS TO TRY TO GET A SQUARE.

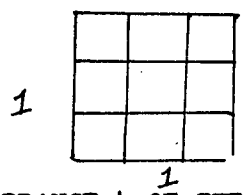


NOTICE THAT WE DO GET A SQUARE WHEN THE 16 PIECES ARE ARRANGED IN A 4 BY 4 ARRAY. BUT SINCE EACH PIECE STARTED OUT WITH SIDES $\frac{1}{5}$ BY $\frac{1}{5}$, THAT MEANS THE DIMENSIONS OF MY SQUARE WITH AREA $\frac{16}{25}$ ARE $\frac{4}{5}$ BY $\frac{4}{5}$, SO $\underbrace{\frac{4}{5} \times \frac{4}{5} = \frac{16}{25}}$.

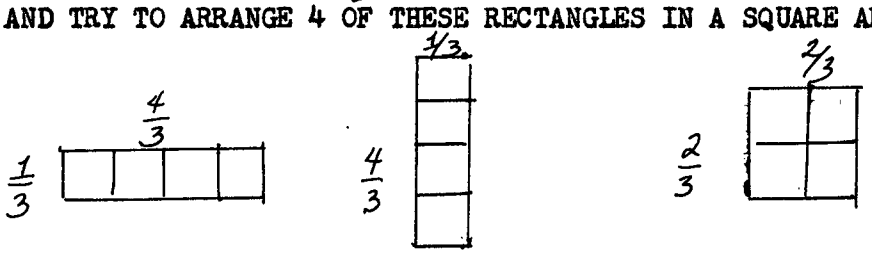
THIS SEEMS REASONABLE; WHEN I MULTIPLY $\frac{4}{5} \times \frac{4}{5}$, I GET $\frac{4 \times 4}{5 \times 5}$.

SUPPOSE WE WERE TRYING TO FIND THE SQUARE ROOT OF $\frac{4}{9}$.

WE MIGHT START WITH A DIAGRAM LIKE SO:



AND TRY TO ARRANGE 4 OF THESE RECTANGLES IN A SQUARE ARRANGEMENT.



AND WE SEE THAT A 2 BY 2 ARRAY OF THESE SQUARES WILL DO. SINCE EACH PIECE STARTED OUT WITH SIDES $\frac{1}{3}$ BY $\frac{1}{3}$, THAT MEANS THE DIMENSIONS OF MY SQUARE WITH AREA $\frac{4}{9}$ ARE $\frac{2}{3}$ BY $\frac{2}{3}$, SO

$$\frac{4}{9} = \frac{2}{3} \times \frac{2}{3} .$$

WHEN I MULTIPLY $\frac{2}{3} \times \frac{2}{3}$, I GET $\frac{2 \times 2}{3 \times 3}$.

NOTICE THAT IN EACH OF THE TWO EXAMPLES SO FAR, THE NUMERATOR OF THE SQUARE ROOT FRACTION WAS THE SQUARE ROOT OF THE NUMERATOR, AND THE DENOMINATOR OF THE SQUARE ROOT FRACTION THE SQUARE ROOT

OF THE DENOMINATOR. IN $\frac{4}{5} = \sqrt{\frac{16}{25}}$, WE NOTICE THAT $4 = \sqrt{16}$ AND $5 = \sqrt{25}$. AND IN $\frac{2}{3} = \sqrt{\frac{4}{9}}$, WE NOTICE THAT $2 = \sqrt{4}$ AND $3 = \sqrt{9}$.

COULD THIS JUST BE COINCIDENCE? LET US CHECK.

SUPPOSE WE DO FIND THE SQUARE ROOT OF A FRACTION, LIKE $\frac{\square}{\triangle}$.

SUPPOSE THIS SQUARE ROOT TURNS OUT TO BE $\frac{\square}{\bigcirc}$.

THAT MEANS THAT $\frac{\square}{\bigcirc} \times \frac{\square}{\bigcirc} = \frac{\square}{\triangle}$.

BUT NOW DO WE MULTIPLY FRACTIONS? TO MULTIPLY $\frac{\square}{\bigcirc} \times \frac{\square}{\bigcirc}$, WE GET $\frac{\square \times \square}{\bigcirc \times \bigcirc}$. THEREFORE, $\frac{\square \times \square}{\bigcirc \times \bigcirc} = \frac{\square}{\triangle}$, SO THAT WE MIGHT AS WELL JUST FIND THE SQUARE ROOTS OF THE NUMERATOR AND THE DENOMINATOR SEPARATELY IF WE CAN, AND THEN FORM THE APPROPRIATE FRACTION AFTER WE ARE DONE.

WHAT WOULD THE FOLLOWING SQUARE ROOTS BE? LET US CHECK THEM BY USING THE AREA ARGUMENT AND THEN THE SEPARATE NUMERATOR AND DENOMINATOR ONE.

$\frac{9}{16}$

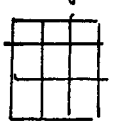
$\frac{4}{25}$

$\frac{25}{36}$

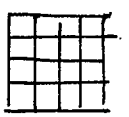
Ask students to come to the board to show their arguments. See to it that the students understand that they can always go back to the area argument with whole numbers by attacking the numerator and denominator separately, but that the strictly fraction method will always work.

For example, with $\frac{9}{16}$, you might show;

$\sqrt{9} = 3$



$\sqrt{16} = 4$



So, $\sqrt{\frac{9}{16}} = \frac{\sqrt{9}}{\sqrt{16}} = \frac{3}{4}$.

Do not go into the possibilities of renaming fractions and then finding the square roots- as, for example, renaming $\frac{9}{36}$ as $\frac{1}{4}$ before finding the square root.

Suggested end of day 3.

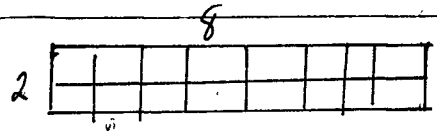
** Here the student will learn that division can provide a technique for finding the square root of a whole number, and thus, later, the square root of a fraction. **

WE HAVE ALREADY SEEN THAT IT IS EASY TO FIND THE SQUARE ROOTS OF SOME NUMBERS, LIKE 81 OR 100, OR 36. BUT WHAT WILL WE DO WHEN FACED WITH THE PROBLEM OF FINDING THE SQUARE ROOT OF, SAY, 576. WE DO NOT YET HAVE ANY METHOD, AND WE CANNOT TELL BY JUST LOOKING. SO WE WILL NOW GO BACK TO TRY TO DEVELOP A METHOD. AFTER ALL, IT MIGHT TAKE A PRETTY LONG TIME TO FIGURE OUT HOW TO DRAW ALL THE RECTANGLES WITH WHOLE NUMBER SIDES WHICH HAVE AREA 576.

WE OUGHT TO START WITH AN EASIER PROBLEM FIRST JUST TO DEMONSTRATE THE TECHNIQUE AND THAT IT WORKS. SO LET US SAY THAT WE START BY LOOKING FOR THE SQUARE ROOT OF 16. WE ALL KNOW ALREADY THAT 4 IS OUR ANSWER, BUT LET US PRETEND THAT WE DO NOT KNOW THIS AND PROCEED TO FIND IT.

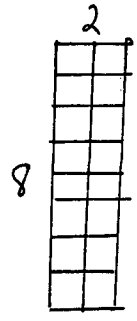
IN ORDER TO FIND THE SQUARE ROOT OF 16, I AM LOOKING FOR A RECTANGLE WITH AREA 16 WHICH HAPPENS TO BE A SQUARE. SINCE I DO NOT KNOW IN ADVANCE WHAT THE SIZE OF THIS SQUARE IS IN TERMS OF SIDE LENGTH, I MIGHT MAKE ANY GUESS. SUPPOSE MY FIRST GUESS IS 8.

I AM NOW GOING TO DRAW A RECTANGLE WITH LENGTH 8 AND AREA 16.

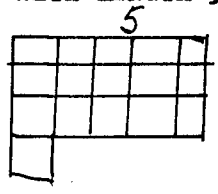


I SEE THAT THIS IS NOT A SQUARE. I NEED MORE HEIGHT AND LESS LENGTH.

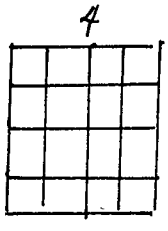
SO I REARRANGE MY 16 SQUARES SO THAT THE HEIGHT IS A NUMBER BETWEEN 2 AND 8. IT OBVIOUSLY MUST BE MORE THAN 2, SINCE I WANT A HIGHER RECTANGLE. THE REASON IT SHOULD BE LESS THAN 8 IS THAT IF I CHOOSE A HEIGHT OF 8, IT'S JUST LIKE TURNING THE SKINNY RECTANGLE ON ITS SIDE AND THEN I WOULD HAVE IT TOO HIGH FOR ITS WIDTH.



SUPPOSE I CHOOSE MY NUMBER BETWEEN 2 AND 8 TO BE 5. NOW I TRY TO DRAW A RECTANGLE WITH LENGTH 5 AND AREA 16.



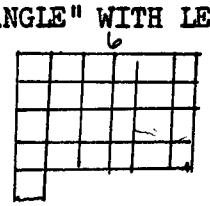
UNFORTUNATELY, I DON'T EVEN GET A RECTANGLE, MUCH LESS A SQUARE. I DO NOTICE, HOWEVER, THAT IT IS A LITTLE TOO LONG FOR ITS HEIGHT, SO I TRY A NUMBER FOR ITS LENGTH WHICH IS SOMEWHAT SMALLER THAN 5, BUT MORE THAN 3, WHICH IS THE APPROXIMATE HEIGHT NOW. I MIGHT TRY 4.



AND I DO GET A SQUARE WITH AREA 16. THEREFORE, 4, THE SIDE LENGTH, IS THE SQUARE ROOT OF 16.

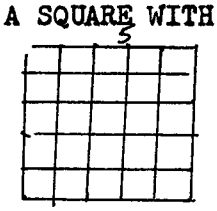
WHAT WOULD I DO TO FIND THE SQUARE ROOT OF 25? I MIGHT MAKE A FIRST GUESS OF 6.

I NOW DRAW A "RECTANGLE" WITH LENGTH 6 AND AREA 25.



I DO NOT GET A RECTANGLE, BUT I DO SEE THAT I NEED MORE HEIGHT AND LESS LENGTH. SINCE THE HEIGHT IS NOT APPROXIMATELY 4, I CHOOSE A NUMBER BETWEEN 6 AND 4, SAY, 5.

I NOW TRY TO DRAW A SQUARE WITH SIDE 5 AND AREA 25, AND I SUCCEED:



THEREFORE, $5 = \sqrt{25}$.

BUT TO GET BACK TO FINDING THE SQUARE ROOT OF 576. IT WOULD TAKE MUCH TOO LONG TO HAVE TO KEEP DRAWING 576 LITTLE SQUARES, SO LET US EXAMINE WHAT WE DID PURELY NUMERICALLY IN THE PREVIOUS TWO CASES.

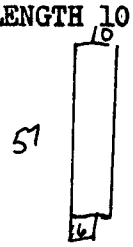
I FIRST MADE A FIRST GUESS FOR THE SQUARE ROOT. SUPPOSE MY GUESS WERE 10. I WOULD THEN ATTEMPT TO DRAW A RECTANGLE WITH LENGTH 10 AND AREA 576. WHAT WOULD THE WIDTH OF THE RECTANGLE BE, OR AT LEAST, HOW WOULD I FIND IT? REMEMBER THAT I GET THE AREA OF A RECTANGLE BY MULTIPLYING ITS LENGTH AND ITS WIDTH.

Expect: divide 576 by 10.

SINCE 10 x WIDTH = 576, THEN THE WIDTH = 576 ÷ 10, AND I PERFORM DIVISION:

```
10 ) 576
     500   50
      76
      70    7
       6   57
```

AND I DO NOT GET AN EXACT RECTANGLE, BUT I GET A SHAPE APPROXIMATELY OF LENGTH 10 AND WIDTH 57.

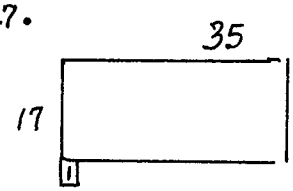


NOW I SEE THAT I NEED MORE LENGTH AND LESS HEIGHT OR WIDTH, SO I CHOOSE A NEW SIDE LENGTH BETWEEN 10 AND 57, SAY, HALFWAY BETWEEN THE TWO, ABOUT 35.

I NOW DIVIDE 576 BY 35 TO FIND THE NEW WIDTH AND GET:

```
35 ) 576
     350   10
     226
     225    7
       1   17
```

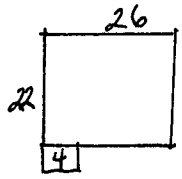
AND FIND THAT I NOW HAVE APPROXIMATELY A RECTANGLE OF LENGTH 35 AND WIDTH 17.



THIS IS STILL NOT A SQUARE, SO I TRY A LENGTH BETWEEN 17 AND 35, SAY, 26. THEN, IF THE LENGTH IS 26, THE WIDTH OF THE NEW RECTANGLE WITH AREA 576 WOULD BE 576 ÷ 26 =

```
26 ) 576 |
     520 | 20
     ---
      56 |
      52 | 2
     ---
       4 | 22
```

SO I NOW HAVE ALMOST A RECTANGLE WITH LENGTH 26 AND WIDTH 22.



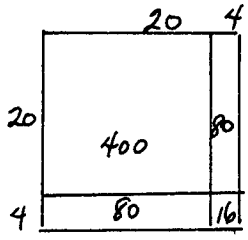
THIS IS STILL NOT A SQUARE, SO I TRY A LENGTH BETWEEN 22 AND 26, SAY 24. THEN, IF THE LENGTH IS 24, THE WIDTH = 576 ÷ 24--

```
24 ) 576 |
     480 | 20
     ---
      96 |
      96 | 4
     ---
       0 | 24
```

AND FINALLY, THE LENGTH EQUALS THE WIDTH = 24.

SO, 576 = 24.

WE CAN DRAW A DIAGRAM TO CHECK THIS:



400 + 80 + 80 + 16 = 576

SO NOTICE THAT WHAT WE HAVE DONE IS TO TRY TO FORCE OUR RECTANGLES INTO SQUARES. NOTICE THAT I CAN MAKE ANY FIRST GUESS, BUT I SAVE MYSELF TIME IF I CAN MAKE A GOOD FIRST GUESS.

FOR EXAMPLE, IF I REALIZE THAT 10 = $\sqrt{100}$ AND 20 = $\sqrt{400}$ AND 30 = $\sqrt{900}$, I WOULD REALIZE THAT SINCE 576 IS BETWEEN 400 AND 900, I COULD TELL THAT $\sqrt{576}$ IS BETWEEN 20 AND 30, AND MY FIRST GUESS MIGHT HAVE BEEN 25.

SUPPOSE I TRY TO FIND $\sqrt{1369}$.

I KNOW THAT $30 = \sqrt{900}$ AND $40 = \sqrt{1600}$, SO I KNOW THAT I WANT TO CHOOSE A NUMBER BETWEEN 30 AND 40, SAY, 35.

THEN IF THE LENGTH OF MY RECTANGLE IS 35, I GET THE WIDTH BY $1369 \div 35$:

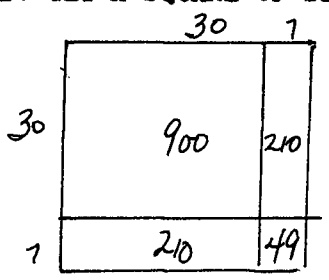
```
35 ) 1369 )
     1050 )  30
      319
      315 )   9
        4 )  39
```

AND I GET A "RECTANGLE" OF LENGTH 35 AND WIDTH OVER 39.

SO I NEED MORE LENGTH. I CHOOSE A NUMBER BETWEEN 35 AND 39, SAY, 37. IF THE LENGTH IS 37, THE WIDTH IS $1369 \div 37$:

```
37 ) 1369 )
     1110 )  30
      259
      259 )   7
        0 )  37
```

AND I DO GET A SQUARE OF SIDE 37 WITH AREA 1369.



$900 + 210 + 210 + 49 = 1369$

Suggested end of day 4.

Hand students worksheet 1 to complete.

** The student will now apply his technique for finding square roots to finding square roots of fractions. **

SUPPOSE I AM LOOKING FOR THE SQUARE ROOT OF A FRACTION, SAY $\sqrt{\frac{576}{4}}$.

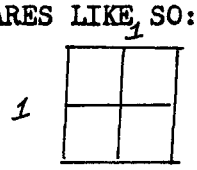
WHAT CAN I DO? IF YOU RECALL, WE LEARNED BEFORE THAT WE COULD SEPARATELY FIND THE SQUARE ROOTS OF NUMERATOR AND DENOMINATOR AND FORM A FRACTION OUT OF THEM WHICH WOULD BE THE APPROPRIATE SQUARE ROOT.

WE JUST FOUND BEFORE THAT $\sqrt{576} = 24$ AND WE KNOW THAT $\sqrt{4} = 2$, SO WHAT DO WE KNOW ABOUT $\sqrt{\frac{576}{4}}$?

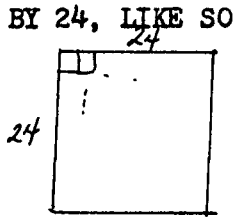
Expect the answer: $\sqrt{\frac{576}{4}} = \frac{24}{2}$.

INDEED, IF WE MULTIPLY: $\frac{24}{2} \times \frac{24}{2} = \frac{576}{4}$.

IF WE WISH TO DRAW A DIAGRAM OF THIS, WE SEE THAT WE COULD START OUT WITH 576 SQUARES EACH OF AREA $\frac{1}{4}$, FORMED BY CUTTING UP UNIT SQUARES LIKE SO:



THEN I COULD ARRANGE THESE 576 SQUARES (EACH OF AREA $\frac{1}{4}$) INTO AN ARRAY OF 24 BY 24, LIKE SO:



AND I DO HAVE 576 BLOCKS ARRANGED IN A SQUARE AND SINCE EACH BLOCK HAS AREA $\frac{1}{4}$, I HAVE A TOTAL AREA OF $\frac{576}{4}$.

Ask the students to find the square roots of the following fractions by finding the square roots of numerators and denominators separately, but then verifying with a diagram.

If the students have not seemed to adjust to the procedure of finding the square root, do another problem like the following one:

To find $\sqrt{625}$, I figure out that the square root must be between 20 and 30 since $20 = \sqrt{400}$ and $30 = \sqrt{900}$, and $\sqrt{626}$ is between 20 and 30. My first guess might be 24. Then I divide 24 into 625:

$$\begin{array}{r|l} 24\,\overline{)\,625} & \\ \underline{480} & 20 \\ 145 & \\ \underline{144} & \underline{6} \\ 1 & 26 \end{array}$$

I get a width of 26 for a length of 24. So, if I try a length between the two, say, 25, I get a new width. When I divide 625 by 25, I do get 25, so I have a square and have found the square root of 625 to be 25.

Hand students worksheet 2 to complete.

Suggested end of day 5.

Finding the Square Root of a Fraction

Outline for approach P:

WE HAVE ALREADY LEARNED HOW TO FIND THE SQUARE ROOTS OF SOME FRACTIONS AND WHOLE NUMBERS, BUT AS WE MENTIONED ALREADY, THERE ARE SOME NUMBERS LIKE 12, WHICH DON'T HAVE WHOLE NUMBER SQUARE ROOTS. WE ARE GOING TO LEARN HOW TO DEAL WITH THESE NUMBERS, TOO.

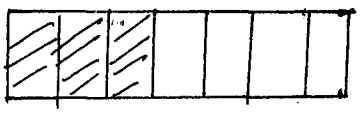
AGAIN, WE MUST DO A BIT OF REVIEWING FIRST.
WHAT DO WE MEAN BY THE FRACTION $\frac{3}{7}$? WHAT DOES THE NUMERATOR, 3, TELL US?

Expect the answer: the number of sevenths we are considering.

WHAT DOES THE DENOMINATOR, 7, TELL US?

Expect the answer: that the whole was divided into 7 equal parts.

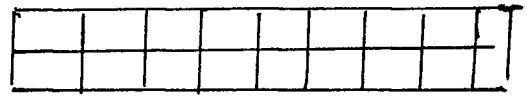
TO DRAW THIS, WE MIGHT DRAW SOMETHING LIKE:



WHAT DIAGRAM WOULD YOU SUGGEST TO SHOW $\frac{6}{8}$?

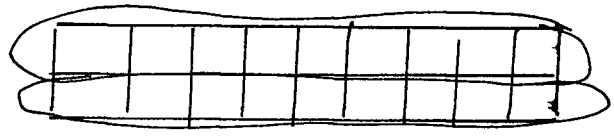
Have one student come to the board to draw his diagram. Repeat this procedure with $\frac{3}{5}$ and $\frac{2}{4}$.

LET US REVIEW QUICKLY AGAIN SOME IDEAS ABOUT AREA. REMEMBER THAT THE AREA OF A RECTANGLE TELLS ME THE NUMBER OF SQUARE UNITS THAT FIT ONTO THE RECTANGLE. FOR EXAMPLE, TO FIND THE AREA OF A RECTANGLE WHICH IS 2 UNITS BY 9 UNITS, I DRAW:

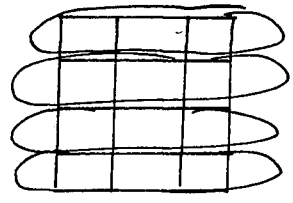


AND SEE THAT THE AREA IS 18 SQ. UNITS.

NOTICE THAT I CAN VIEW THIS AS 2 x 9 SQ. UNITS LIKE SO:

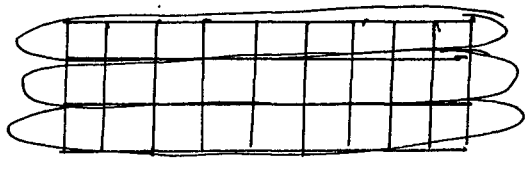


SO, IF I HAVE A RECTANGLE WHICH HAS DIMENSIONS 4 UNITS BY 3 UNITS, I CAN DRAW A DIAGRAM LIKE:



AND VERIFY THAT THE AREA = 4 x 3 SQ. UNITS.

LET'S TRY ONE MORE AREA QUESTION. SUPPOSE MY RECTANGLE HAS DIMENSIONS OF 3 UNITS BY 9 UNITS. I WOULD DRAW:



AND THE AREA IS INDEED 3x 9 SQ. UNITS.

Ask the students to consider the following products as area questions:

9 x 3
7 x 2
5 x 8

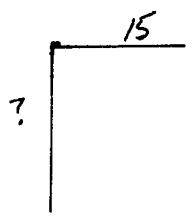
Ask three students to come to the board to diagramatically explain their answers.

** Here the students will review the idea that the square root of a whole is found by repeated division in order to find a square with the given area. **

IF YOU RECALL, WE HAVE AIREADY LEARNED THAT IN ORDER TO FIND THE SQUARE ROOT OF A NUMBER, I AM LOOKING FOR ANOTHER NUMBER WHICH I CAN MULTIPLY BY ITSELF TO GET THE NUMBER I AM GIVEN. FOR EXAMPLE, $4 = \sqrt{16}$, AND, AS YOU MIGHT RECALL, THIS IS THE SAME PROBLEM AS FINDING A RECTANGLE WITH THAT GIVEN AREA, 16, WHICH HAPPENS TO BE A SQUARE. THIS IS SO SINCE IF A RECTANGLE HAS THE GIVEN AREA, ITS SIDES CAN BE MULTIPLIED TOGETHER TO GET THAT NUMBER AND FOR A SQUARE THE SIDES WOULD BE EQUAL.

YOU MIGHT ALSO RECALL THAT WE PERFORMED DIVISION WHEN THE NUMBER WE WERE TRYING TO FIND THE SQUARE ROOT OF WAS LARGE IN ORDER TO MAKE THE PROBLEM EASIER.

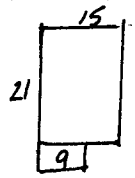
FOR EXAMPLE, TO FIND $\sqrt{324}$, I MIGHT SAY:
SINCE $20 = \sqrt{400}$ AND $10 = \sqrt{100}$, THEN, $\sqrt{324}$ MUST BE BETWEEN 20 AND 10. LET US GUESS 15. SO, I FORM A RECTANGLE WITH LENGTH 15 AND AREA 324 AND TRY TO FIND THE WIDTH TO SEE IF IT IS 15.



IN ORDER TO FIND THE WIDTH, I DIVIDE 324 BY 15, AND GET:

$$\begin{array}{r|l} 15 & 324 \\ & \underline{300} \quad 20 \\ & 24 \\ & \underline{15} \quad \underline{1} \\ & 9 \quad 21 \end{array}$$

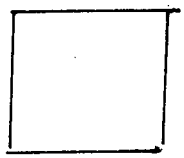
AND I SEE THAT I DO NOT QUITE GET A RECTANGLE, BUT MY SHAPE HAS AN APPROXIMATE RECTANGULAR SHAPE WITH LENGTH 15 AND WIDTH 21.



OBVIOUSLY, THIS IS NOT SQUARE. IT NEEDS MORE LENGTH AND LESS WIDTH, SO I CHOOSE AS MY NEXT GUESS FOR THE LENGTH A NUMBER BETWEEN 15 AND 21, SAY, 18.

```
18 ) 324 |
     180 | 10
     144 |
     144 |  8
       0 |  18
```

AND I FIND THAT 18 IS THE SQUARE ROOT.



AND WHEN I CALCULATE THE AREA OF THIS SQUARE, I GET 324

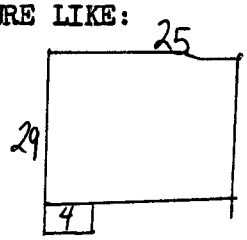
	10	8
10	100	80
8	80	64

100 + 80 + 80 + 64 = 324

LET US TRY TO FIND THE SQUARE ROOT OF 729 BY THIS TECHNIQUE. I MIGHT MAKE A FIRST GUESS FOR THE LENGTH OF A NUMBER BETWEEN 20 AND 30 SINCE 20 = $\sqrt{400}$ AND 30 = $\sqrt{900}$ AND 729 IS BETWEEN 400 AND 900. SUPPOSE MY GUESS IS 25. IF THE LENGTH OF THE RECTANGLE IS 25, AND THE AREA IS 729, I GET THE WIDTH BY DIVIDING:

```
25 ) 729 |
     500 | 20
     229 |
     200 |  8
      29 |
      25 |  1/29
       4 |
```

AND I GET A FIGURE LIKE:



THIS IS NOT A SQUARE. IT NEEDS MORE LENGTH AND LESS WIDTH. SO I CHOOSE ANOTHER GUESS FOR THELENGTH BETWEEN 25 AND 29, SAY, 27. TO GET THE WIDTH, I DIVIDE:

```
27 ) 729 )
     540 )  20
     189
     189 )  7/27
       0 )
```

AND SO THE LENGTH EQUALS THE WIDTH IS 27, AND I CAN DRAW THE SQUARE TO VERIFY MY RESULT.

Hand students worksheet 3 to complete.

Suggested end of day 6.

** Here the students will learn about approximating square roots of wholes. **

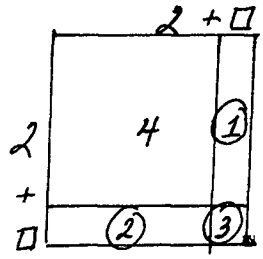
SUPPOSE WE NOW WANT TO FIND $\sqrt{5}$. WE KNOW THAT 2 IS TOO SMALL, SINCE $2 \times 2 = 4$ AND 3 IS TOO LARGE SINCE $3 \times 3 = 9$. SO WE HAVE A PROBLEM WE HAD NOT PREVIOUSLY FACED.

ONE TECHNIQUE WE MIGHT USE IS TO CONVERT 5 TO A FRACTION AND TRY TO FIND A FRACTION WITH A PERFECT SQUARE ROOT NEAR IT. FOR EXAMPLE, WE KNOW THAT $5 = \frac{125}{25}$, AND $\frac{125}{25}$ IS CLOSE TO $\frac{121}{25}$, WHICH HAS A PERFECT SQUARE ROOT, NAMELY $\frac{11}{5}$. BUT THIS MIGHT PROVE VERY LONG AND TEDIOUS IF WE DON'T FIND THE RIGHT FRACTION RIGHT

AWAY.

LET'S TRY TO DEVELOP A METHOD.

SUPPOSE WE TRY TO DRAW A SQUARE WITH AREA OF 5. WE MIGHT DRAW SOMETHING LIKE:



WHERE SECTIONS 1, 2 AND 3 HAVE A TOTAL AREA OF 1 UNIT. HOWEVER, WE DO NOT KNOW THE SIZE OF THE □ WHICH EXTENDS FROM 2.

WHAT IS THE AREA OF SECTION 1?

Expect the answer: 2 x □ .

WHAT IS THE AREA OF SECTION 2?

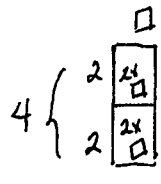
Expect the answer: 2 x □ .

WHAT IS THE AREA OF SECTION 3?

Expect the answer: □ x □ .

NOW, IN ORDER OF SIZE, SECTIONS 1 AND 2 COME BEFORE 3, SO LET US PRETEND TEMPORARILY THAT THEY MAKE UP MOST OF THE 1 UNIT THAT SECTIONS 1, 2, AND 3 MAKE UP ALTOGETHER.

SO, THE TOTAL AREA OF SECTIONS 1 AND 2 IS (2 x □) + (2 x □) IS ABOUT 1. BUT, (2 x □) + (2 x □) IS ANOTHER WAY OF WRITING 4 x □, SINCE I CAN DRAW:



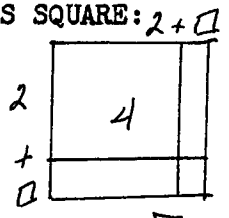
NOW, IF $4 \times \square$ IS ABOUT 1, WHAT IS $\square$?

Expect the answer: about $\frac{1}{4}$.

NOW SUPPOSE WE PRETEND THAT $\square$ IS $\frac{1}{4}$. WHAT IS THE AREA OF SECTION 3- IN OTHER WORDS, WHAT IS $\square \times \square$?

Expect the answer: $\frac{1}{16}$.

SINCE THIS NUMBER IS LESS THAN $\frac{1}{2}$, WE CAN SAY THAT WE HAVE USED UP MOST OF THE EXTRA 1 UNIT OF AREA THAT IS NOT IN THE 2 BY 2 SECTION OF THIS SQUARE:



AND SO WE CAN SAY THAT $\sqrt{5}$ IS ABOUT $2\frac{1}{4}$. WE WRITE THIS

$\sqrt{5} \approx 2\frac{1}{4}$.

WE MIGHT HAVE USED DIVISION PROCEDURE AGAIN, AS WELL.

SO OUR FIRST GUESS MIGHT HAVE BEEN 2. THEN WE WOULD DIVIDE 5 BY 2, AND GET :

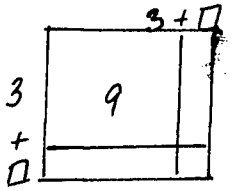
$$\begin{array}{r|l} 2\overline{)5} & \\ \frac{4}{1} & \frac{2}{1} \end{array}$$

UNFORTUNATELY, THIS GETS US INTO DIVISION WITH FRACTIONS SINCE OUR NEXT GUESS WOULD BE A NUMBER BETWEEN 2 AND $2\frac{1}{2}$. AND SO, WE WILL AVOID THE DIVISION TECHNIQUE FOR THE TIME BEING.

SUPPOSE WE WANT TO FIND $\sqrt{13}$.

WE KNOW THAT THE ANSWER MUST BE BETWEEN 3 AND 4, SINCE $3 \times 3 = 9$ AND $4 \times 4 = 16$ AND 13 IS BETWEEN 9 AND 16.

SO, WE AGAIN DRAW A SQUARE LIKE SO:



WE HAVE USED UP 9 UNITS OF THE 13 IN THE 3 x 3 SQUARE, LEAVING US WITH 4 MORE UNITS DISTRIBUTED IN SECTIONS 1, 2, AND 3.

NOW WHAT IS THE AREA OF SECTION 1?

Expect the answer: 3 x □ .

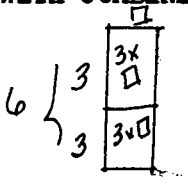
WHAT IS THE AREA OF SECTION 2?

Expect the answer: 3 x □.

WHAT IS THE AREA OF SECTION 3?

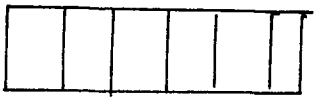
Expect the answer: □ x □.

AGAIN, WE WILL GO FROM LARGEST TO SMALLEST. IF WE TEMPORARILY IGNORE SECTION 3, WE HAVE USED UP MOST OF THE 4 UNITS IN SECTIONS 1 AND 2 WITH COMBINED AREA: 3 x □ + 3 x □ = 6 x □ .

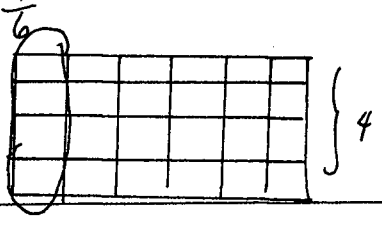


IF 6 x □ = 4, WHAT IS □ ?

WE KNOW THAT 6 x $\frac{1}{6}$ = 1 BY DRAWING:



SO THAT 6 x $\frac{4}{6}$ = 4



AND CERTAINLY WE NEED TO MULTIPLY $\frac{1}{6}$ BY SOMETHING 4 TIMES AS GREAT IN ORDER TO GET 4 INSTEAD OF 1.

THEN, IF WE GUESS $\square = \frac{4}{6}$, WE FIND THAT THE NEGLECTED AREA IN SECTION 3 IS $\square \times \square = \frac{4}{6} \times \frac{4}{6} = \frac{16}{36}$ WHICH IS LESS THAN $\frac{1}{2}$ (WHICH IS $\frac{18}{36}$) AND SO WE SAY THAT

$$\sqrt{13} \approx 3\ 4/6 .$$

SUPPOSE WE WANT TO FIND $\sqrt{37}$. IF WE DID NOT KNOW IN ADVANCE TO TRY A NUMBER NEAR 6, WE MIGHT USE THE DIVISION PROCEDURE TO CLOSE IN ON A NUMBER BETWEEN 6 AND 7.

FOR EXAMPLE, WE MIGHT DO:

GUESS THE LENGTH OF THE SQUARE WITH AREA 37 TO BE 4.

THEN THE WIDTH = $37 \div 4$

$$\begin{array}{r|l} 4\overline{)37} & \\ \underline{36} & 9 \\ 1 & 9 \end{array}$$

SO WE TRY A NUMBER BETWEEN 4 AND 9 FOR THE LENGTH , SAY, 7.

SO THE WIDTH = $37 \div 7$

$$\begin{array}{r|l} 7\overline{)37} & \\ \underline{35} & 5 \\ 2 & 5 \end{array}$$

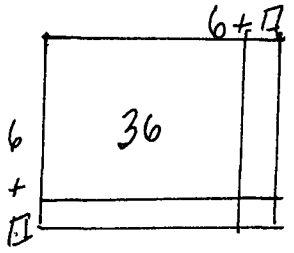
SO WE STILL DID NOT GET A SQUARE. WE NOW TRY A NUMBER BETWEEN 5 AND 7, SAY, 6.

IF THE LENGTH = 6, THE WIDTH = $37 \div 6$,

$$\begin{array}{r|l} 6\overline{)37} & \\ \underline{36} & 6 \\ 1 & 6 \end{array}$$

AND WE ARE BACK TO THE POINT WHERE OUR DIVISION PROCESS CANNOT HELP US. HOWEVER, WE DO KNOW FOR SURE THAT $\sqrt{37}$ IS BETWEEN 6 AND 7.

SO, LET US DRAW:

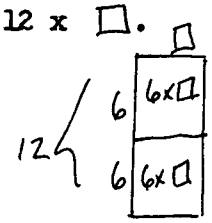


WHAT ARE THE AREAS OF EACH OF THE SECTIONS 1, 2, AND 3?

Expect the answers: 6 x □, 6 x □, and □ x □ .

SINCE SECTION 3 IS SMALLEST WE MIGHT IGNORE IT TEMPORARILY.

WE SEE THAT SECTIONS 1 AND 2 HAVE A TOTAL AREA OF (6 x □) + (6 x □) = 12 x □.



SINCE THERE IS ONLY ONE UNIT OF THE 37 NOT IN THE 6 BY 6 SQUARE, WE HAVE TO ASSUME THAT 12 x □ IS ABOUT 1. WHAT VALUE IS □ ?

Expect the answer: about $\frac{1}{12}$.

THEN WHAT IS THE AREA OF SECTION 3?

Expect the answer: □ x □ = $\frac{1}{144}$.

THIS IS CERTAINLY SMALLER THAN $\frac{1}{2}$, SO WE CAN ASSUME THAT

$\sqrt{37} \approx 6\frac{1}{12}$.

NOTICE THAT IF I HAD BEEN TRYING TO FIND 38, I WOULD HAVE DONE THE SAME THING UNTIL I SAID THAT 12 x □ IS ABOUT 1. THEN I

WOULD HAVE SAID THAT 12 x $\square$ IS ABOUT 2, AND SO $\square$ IS ABOUT $2 \times \frac{1}{12} = \frac{2}{12}$. I STILL WOULD HAVE CHECKED $\square \times \square$ TO SEE THAT IT WAS SMALLER THAN $\frac{1}{2}$, AND SINCE $\frac{2}{12} \times \frac{2}{12}$ IS SMALLER THAN $\frac{1}{2}$, I WOULD HAVE ACCEPTED THIS APPROXIMATION.

Suggested end of day 7.

Ask the students to work out the approximate square roots for the following and then ask students to come to the board to show their work.

8
18
50
67

** Here the students will learn how to approximate the square roots of fractions whose denominators are perfect squares, but whose numerators are not. **

NOW SUPPOSE I WANT TO FIND THE APPROXIMATE SQUARE ROOT FOR $\sqrt{\frac{5}{4}}$. I HAVE ALREADY LEARNED THAT I CAN SEPARATELY ATTACK THE NUMERATORS AND DENOMINATORS AND PUT TOGETHER THESE SQUARE ROOTS TO FORM THE SQUARE ROOT FRACTION.

SUPPOSE WE DO THAT HERE.

WE FOUND THAT $\sqrt{5} \approx 2\frac{1}{4}$ FROM OUR PREVIOUS WORK AND WE KNOW THAT $\sqrt{4} = 2$.

SO THE APPROXIMATE SQUARE ROOT OF $\frac{5}{4}$ IS $\frac{2\frac{1}{4}}{2}$.

SUPPOSE WE WERE TRYING TO FIND $\sqrt{\frac{13}{4}}$.

WE KNOW THAT $\sqrt{13} \approx 3\frac{4}{6}$ AND $\sqrt{4} = 2$, SO $\sqrt{\frac{13}{4}} \approx \frac{3\frac{4}{6}}{2}$.

LET'S DO THE SAME THING FOR $\sqrt{\frac{37}{4}}$.

WE CAN WRITE THAT THE APPROXIMATE SQUARE ROOT OF $\sqrt{\frac{37}{4}}$ IS $\frac{6\frac{1}{12}}{2}$.

Have the students find approximate square roots for each of the following: have a few students show their work on the board.

$\sqrt{\frac{32}{9}}$

$\sqrt{\frac{47}{25}}$

$\sqrt{\frac{28}{49}}$

Suggested end of day 8.

Finding the Square Root of a Fraction

Outline for approach a:

WE ARE GOING TO LEARN A WAY TO EXPRESS ANY FRACTION AS THE PRODUCT OF TWO EQUAL FRACTIONS (OR APPROXIMATELY SO). FOR EXAMPLE, WE WILL LEARN THAT JUST AS 16 CAN BE EXPRESSED AS THE PRODUCT OF TWO EQUAL FACTORS, NAMELY, 4 x 4, $\frac{16}{9}$ CAN BE EXPRESSED AS THE PRODUCT OF TWO IDENTICAL FRACTIONS, NAMELY $\frac{16}{9} = \frac{4}{3} \times \frac{4}{3}$.

BUT FIRST WE MUST DO A LITTLE BIT OF REVIEWING.

I AM GOING TO ASK YOU SOME MULTIPLICATION QUESTIONS.

Ask the folllowing:

8 x 5

7 x 9

4 x 6

3 x 3

8 x 8

If there is much difficulty in answering these questions on the part of the class as a whole, then apply an argument involving the distributive principle like the following one:

To illustrate:

$$3 \times 5 = 2 \times 5 + 1 \times 5 = 10 + 5 = 15$$

$$4 \times 4 = 2 \times 4 + 2 \times 4 = 8 + 8 = 16$$

Do not use diagrams to illustrate these.

If the students have trouble in breaking up the first number into only two addends, like 6 into 3+3, allow them to use more addends which are

smaller, such as 6 as 2+2+2.

Go over items 29-33 on the pretest. In particular, have students see the relationship between a fraction and a multiplication expression. That is, a fraction like $\frac{3}{7}$ can be written as $3 \times \frac{1}{7}$ or $\frac{4}{5}$ as $4 \times \frac{1}{5}$. Go back to the repeated addition explanation to show this if it is necessary. For each question, read the question and ask for the correct answer. This review should go quickly because the students worked with fractions in the previous unit.

> SUPPOSE I WANT TO FIND THE SOLUTION TO THE QUESTION $3 \times \square = 12$. I HAVE A MULTIPLICATION SENTENCE TO SOLVE, BUT BECAUSE I AM MISSING ONE OF THE MULTIPLIERS, RATHER THAN THE PRODUCT, THE OPERATION I PERFORM IS DIVISION. HERE I WOULD DIVIDE 12 BY 3, AND MY SOLUTION IS WHAT?

Expect the answer: $\square = 4$.

> THEREFORE, I CAN TURN ANY MULTIPLICATION STATEMENT INTO A DIVISION STATEMENT AND VICE VERSA. FOR EXAMPLE, HOW DO I TURN THE FOLLOWING INTO A MULTIPLICATION STATEMENT:
>
> $12 \div 3 = \square$.

Expect the answer: $\square \times 3 = 12$.

> HOW DO I TURN THIS MULTIPLICATION STATEMENT INTO A DIVISION ONE: $3 \times \square = 27$?

Expect the answer: $27 \div \square = 3$ or $27 \div 3 = \square$.

Go over items 21-25 on the pretest and make sure the students see the

relationship between multiplication and division. Read each question and have a student provide the answer.

> NOTICE THAT IN SOLVING $27 \div 3$, I DID NOT REALLY NEED TO GO THROUGH A LONG DIVISION PROCESS, SINCE I QUICKLY READ OFF THE ANSWER AS A FACT. BUT SUPPOSE I HAD A QUESTION LIKE:
>
> $13 \times \square = 2639$.
>
> WHAT WOULD I DO TO SOLVE THIS QUESTION?

Expect the answer: divide 2639 by 13.

> HOW MIGHT I BEGIN?
>
> I SET UP THE QUESTION, LIKE SO:
>
> $13\overline{)2639}$ AND I KEEP TAKING CHUNKS OF 13 OUT OF 2639 UNTIL I HAVE USED ALL OF THE 2639. MY ANSWER IS THE NUMBER OF CHUNKS I TOOK OUT.
>
> SO I CAN TAKE 200 CHUNKS OF 13 OUT FIRST, LEAVING ONLY 39.
>
> $$\begin{array}{r|l} 13\overline{)2639} & \\ \underline{2600} & 200 \\ 39 & \\ \underline{39} & \underline{3} \\ 0 & 203 \end{array}$$
>
> THEN I TAKE OUT 3 MORE CHUNKS OF 13, AND USE ALL OF THE 2639.
>
> SO, $2639 \div 13 = 203$.

Allow the students to solve these two division questions and have two students show their results to the class:

> $9\overline{)288}$ AND $20\overline{)4300}$

** Here the student will learn how to multiply fractions in order that he might find answers to problems like $\frac{\square}{\triangle} \times \frac{\square}{\triangle} = \frac{16}{9}$. **

SUPPOSE I WANT TO FIND THE ANSWER TO A MULTIPLICATION QUESTION WHERE BOTH OF THE NUMBERS TO BE MULTIPLIED ARE FRACTIONS. FOR EXAMPLE, WE MIGHT TRY $\frac{1}{2} \times \frac{1}{3}$.

WHAT DO YOU SUGGEST THE ANSWER MIGHT BE?

Wait for a response of $\frac{1}{6}$ before proceeding.

LET'S SEE IF THIS IS REASONABLE.

IF $\frac{1}{6}$ IS ANOTHER NAME FOR $\frac{1}{2} \times \frac{1}{3}$, THEN SINCE

$\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6} = \frac{6}{6} = 6 \times \frac{1}{6} = 1$, THAT WOULD MEAN THAT

$6 \times (\frac{1}{2} \times \frac{1}{3}) = 1.$

IF THIS IS SO, WE KNOW OUR PRODUCT IS $\frac{1}{6}$ SINCE THERE IS ONLY ONE NUMBER I CAN MULTIPLY BY 6 TO GET 1 AND THAT NUMBER IS $\frac{1}{6}$.

WELL,

$6 \times (\frac{1}{2} \times \frac{1}{3})$

$= 3 \times 2 \times (\frac{1}{2} \times \frac{1}{3})$

$= (3 \times \frac{1}{3}) \times (2 \times \frac{1}{2})$ BUT $2 \times \frac{1}{2} = 3 \times \frac{1}{3} = 1.$

$= 1 \times 1$

$= 1$

NOTICE THAT ALL I DID WAS TO DECIDE WHETHER $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$ BY SEEING IF $6 \times (\frac{1}{2} \times \frac{1}{3}) = 1.$

Suggested end of day 1.

HOW WOULD WE FIND $\frac{1}{4} \times \frac{1}{7}$?

WE CHECK TO SEE IF $\frac{1}{4} \times \frac{1}{7} = \frac{1}{28}$ BY SEEING IF $28 \times (\frac{1}{4} \times \frac{1}{7})$ EQUALS 1. BUT,

$$
\begin{aligned}
& 28 \times (\tfrac{1}{4} \times \tfrac{1}{7}) \\
&= 4 \times 7 \times (\tfrac{1}{4} \times \tfrac{1}{7}) \\
&= (4 \times \tfrac{1}{4}) \times (7 \times \tfrac{1}{7}) \\
&= 1 \times 1 \\
&= 1
\end{aligned}
$$

SO, $28 \times (\frac{1}{4} \times \frac{1}{7}) = 1$, AND, SO, $\frac{1}{4} \times \frac{1}{7} = \frac{1}{28}$.

WHAT DO YOU THINK $\frac{1}{3} \times \frac{1}{6}$ WILL BE?

Expect the answer: $\frac{1}{18}$.

LET US CHECK THIS:

IF THIS IS SO, THEN $18 \times (\frac{1}{3} \times \frac{1}{6}) = 1$,

BUT,

$$
\begin{aligned}
& 18 \times (\tfrac{1}{3} \times \tfrac{1}{6}) \\
&= 6 \times 3 \times (\tfrac{1}{3} \times \tfrac{1}{6}) \\
&= (6 \times \tfrac{1}{6}) \times (3 \times \tfrac{1}{3}) \\
&= 1 \times 1 \\
&= 1
\end{aligned}
$$

SO, $18 \times (\frac{1}{3} \times \frac{1}{6}) = 1$, SO, $\frac{1}{3} \times \frac{1}{6} = \frac{1}{18}$.

DO YOU NOTICE THAT IN EACH OF THESE PROBLEMS, IF I MULTIPLY $\frac{1}{\square}$ BY $\frac{1}{\triangle}$, I GET AS AN ANSWER A FRACTION $\frac{1}{\square \times \triangle}$. THIS SEEMS REASONABLE SINCE IN EACH CASE IF I MULTIPLIED $\frac{1}{\square} \times \frac{1}{\triangle}$ BY $\square \times \triangle$, I GOT $(\square \times \frac{1}{\square}) \times (\triangle \times \frac{1}{\triangle}) = 1 \times 1 = 1$, AND, SO, $\frac{1}{\square} \times \frac{1}{\triangle} = \frac{1}{\square \times \triangle}$.

SUPPOSE WE WANT TO MULTIPLY $\frac{2}{3}$ BY $\frac{3}{4}$. WE KNOW THAT WE CAN DO SOME RENAMING TO FIND THIS ANSWER.

$\frac{2}{3}$ CAN BE RENAMED AS $2 \times \frac{1}{3}$ AND $\frac{3}{4}$ AS $3 \times \frac{1}{4}$.

THEREFORE,

$$\frac{2}{3} \times \frac{3}{4}$$
$$= (2 \times \frac{1}{3}) \times (3 \times \frac{1}{4})$$

SINCE I CAN MULTIPLY IN ANY ORDER, THIS IS THE SAME AS:

$$(2 \times 3) \times (\frac{1}{3} \times \frac{1}{4})$$
$$= 6 \times \frac{1}{12}$$
$$= \frac{6}{12}.$$

SO, $\frac{2}{3} \times \frac{3}{4} = \frac{6}{12}$.

LET'S FIND $\frac{3}{5} \times \frac{2}{3}$ USING THIS APPROACH:

I RENAME:

$$\frac{3}{5} \times \frac{2}{3}$$
$$= (3 \times \frac{1}{5}) \times (2 \times \frac{1}{3})$$
$$= (3 \times 2) \times (\frac{1}{5} \times \frac{1}{3})$$
$$= 6 \times \frac{1}{15}$$
$$= \frac{6}{15}.$$

THEREFORE, $\frac{3}{5} \times \frac{2}{3} = \frac{6}{15}$.

WHAT WOULD $\frac{4}{5} \times \frac{2}{4}$ BE?

Expect the answer: $\frac{8}{20}$.

LET'S CHECK BY RENAMING:

$$\frac{4}{5} \times \frac{2}{4}$$
$$= (4 \times \frac{1}{5}) \times (2 \times \frac{1}{4})$$
$$= (4 \times 2) \times (\frac{1}{5} \times \frac{1}{4})$$

$$= 8 \times \frac{1}{20}$$
$$= \frac{8}{20}.$$

Ask the students to find the following products:

$$\frac{3}{4} \times \frac{1}{3}$$
$$\frac{6}{9} \times \frac{2}{4}$$
$$\frac{5}{8} \times \frac{3}{4}$$

Have three students come to the board to show the renaming that leads to the answer.

Suggested end of day 2.

** Here the student will learn that the square root of a number is that number which I can multiply by itself to get the given number; for example, the squre root of 9 is 3 since 3 x 3 = 9. **

SUPPOSE I WANT TO FIND ALL THE NUMBERS WHICH GO INTO 16 EVENLY. ALL I HAVE TO DO IS LIST ALL THE MULTIPLICATION FACTS I KNOW FOR 16.

16 = 16 x 1
= 8 x 2
= 4 x 4
= 2 x 8
= 1 x 16

NOTICE THAT FROM THESE, I CAN SEE THAT 1,2,4,8 AND 16 ARE THOSE NUMBERS I WAS LOOKING FOR.

BUT MORE THAN THAT, I CAN SEE THAT AMONG THOSE LIST OF FACTS, THERE WAS ONE FACT WHERE I MULTIPLIED A NUMBER BY ITSELF TO GET 16.

THAT NUMBER WAS 4. BECAUSE OF THIS SPECIAL PROPERTY, 4 IS CALLED THE <u>SQUARE ROOT</u> OF 16. WE WRITE THIS $4 = \sqrt{16}$.

LET'S TRY TO FIND THE SQUARE ROOT OF 25. WE LIST ALL THE MULTIPLICATION FACTS WE KNOW FOR 25:

$$\begin{aligned} 25 &= 25 \times 1 \\ &= 5 \times 5 \\ &= 1 \times 25 \end{aligned}$$

AMONG THESE FACTS, WE FIND ONE WHERE A NUMBER IS MULTIPLIED BY ITSELF TO GET 25. THIS NUMBER IS 5. SO, $5 = \sqrt{25}$.

SOME NUMBERS DO NOT HAVE WHOLE NUMBER SQUARE ROOTS. FOR EXAMPLE, TAKE 12. LET US WRITE ALL THE MULTIPLICATION FACTS FOR 12.

$$\begin{aligned} 12 &= 12 \times 1 \\ &= 6 \times 2 \\ &= 4 \times 3 \\ &= 3 \times 4 \\ &= 2 \times 6 \\ &= 1 \times 12 \end{aligned}$$

NONE OF THESE FACTS PROVIDE US WITH A SQUARE ROOT FOR 12.

WE ALREADY KNOW MANY SQUARE ROOTS. FOR EXAMPLE, THE SQUARE ROOT OF 4 IS 2, SINCE $2 \times 2 = 4$.

WHAT IS THE SQUARE ROOT OF 9?

Expect the answer: 3.

Ask the students to call out the square roots of the following:

36

49

100

** Here, the student will learn to apply the idea of square root to fractions. **

SUPPOSE WE ARE LOOKING FOR THE SQUARE ROOT OF A FRACTION, LIKE $\frac{16}{25}$.

THAT MEANS THAT WE ARE LOOKING FOR ANOTHER FRACTION TO MULTIPLY BY ITSELF TO GET $\frac{16}{25}$.

AGAIN, WE LIST MANY MULTIPLICATION SENTENCES FOR $\frac{16}{25}$.

WE MIGHT LIST:

$$= 16 \times \frac{1}{25}$$
$$= 8 \times (2 \times \frac{1}{25})$$
$$= (8 \times \frac{1}{5}) \times (2 \times \frac{1}{5})$$
$$= (4 \times \frac{1}{5}) \times (4 \times \frac{1}{5})$$

AND SO ON. NOTICE THAT THE LIST COULD GET PRETTY LONG.

BUT WE SEEMD TO HAVE FOUND AN ANSWER, NAMELY $4 \times \frac{1}{5} = \frac{4}{5}$.

INDEED, $\frac{4}{5} \times \frac{4}{5} = \frac{16}{25}$.

SUPPOSE WE WERE TRYING TO FIND THE SQUARE ROOT OF $\frac{4}{9}$.

WE MIGHT START WITH SOME MULTIPLICATION SENTENCES:

$$= 4 \times \frac{1}{9}$$
$$= 2 \times (2 \times \frac{1}{9}) = 2 \times \frac{2}{9}$$
$$= (2 \times \frac{1}{3}) \times (2 \times \frac{1}{3}) = \frac{2}{3} \times \frac{2}{3}$$

AND SO ON.

WE SEEMED TO HAVE FOUND AN ANSWER, NAMELY $2 \times \frac{1}{3} = \frac{2}{3}$.

INDEED, $\frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$.

NOTICE IN EACH OF THE TWO EXAMPLES SO FAR, THE NUMERATOR OF THE SQUARE ROOT FRACTION WAS THE SQUARE ROOT OF THE NUMERATOR, AND THE DENOMINATOR OF THE SQUARE ROOT FRACTION WAS THE SQUARE ROOT OF THE DENOMINATOR.

IN $\frac{4}{5} = \sqrt{\frac{16}{25}}$, WE NOTICE THAT $4 = \sqrt{16}$ AND $5 = \sqrt{25}$.

IN $\frac{2}{3} = \sqrt{\frac{4}{9}}$, WE NOTICE THAT $2 = \sqrt{4}$ AND $3 = \sqrt{9}$.

COULD THIS JUST BE COINCIDENCE? LET US CHECK.

SUPPOSE WE DO FIND THE SQUARE ROOT OF A FRACTION, LIKE $\frac{\square}{\triangle}$.

SUPPOSE THIS SQUARE ROOT TURNS OUT TO BE $\frac{\square}{\hexagon}$.

THAT MEANS THAT $\frac{\square}{\hexagon} \times \frac{\square}{\hexagon} = \frac{\square}{\triangle}$.

BUT HOW DO WE MULTIPLY FRACTIONS? TO MULTIPLY $\frac{\square}{\hexagon} \times \frac{\square}{\hexagon}$,

WE GET $\frac{\square \times \square}{\hexagon \times \hexagon}$. THEREFORE, $\frac{\square \times \square}{\hexagon \times \hexagon} = \frac{\square}{\triangle}$. SO THAT WE MIGHT AS WELL JUST FIND THE SQUARE ROOTS OF THE NUMERATOR AND DENOMINATOR SEPARATELY AND THEN FORM THE APPROPRIATE FRACTION AFTER WE ARE DONE.

WHAT WOULD THE FOLLOWING SQUARE ROOTS BE? LET US CHECK THEM BY MULTIPLYING THEM OUT AS FRACTIONS:

$\frac{9}{16}$

$\frac{4}{25}$

$\frac{25}{36}$

Ask students to read out their answers. See to it that the students understand that they can always go back to finding the square roots of whole numbers by attacking the numerator and denominator separately, but that the strictly fraction method will always work. For example, with $\frac{9}{16}$, you might show:

$\sqrt{16} = 4$ since $4 \times 4 = 16$ and $\sqrt{9} = 3$ since $3 \times 3 = 9$

So, $\sqrt{\frac{9}{16}} = \frac{3}{4}$.

Do not go into the possibilities of renaming fractions and then finding the square roots- as, for example, renaming $\frac{9}{36}$ as $\frac{1}{4}$ before finding the square root.

Suggested end of day 3.

** Here the student will learn that division can provide a technique for finding the square root of a whole number, and thus, later, the square root of a fraction. **

WE HAVE ALREADY SEEN THAT IT IS EASY TO FIND THE SQUARE ROOTS OF SOME NUMBERS, LIKE 81 OR 100 OR 36. BUT WHAT WILL WE DO WHEN FACED WITH THE PROBLEM OF FINDING THE SQUARE ROOT OF, SAY, 576. WE DO NOT YET HAVE ANY METHOD AND WE CANNOT TELL BY JUST LOOKING, SO WE WILL NOW GO BACK TO TRY TO DEVELOP A METHOD. AFTER ALL, IT MIGHT TAKE A PRETTY LONG TIME TO LIST ALL THE FACTS FOR 576.

WE OUGHT TO START WITH AN EASIER PROBLEM FIRST JUST TO DEMONSTRATE THE TECHNIQUE AND THAT IT WORKS. SO LET US SAY THAT WE START BY LOOKING FOR THE SQUARE ROOT OF 16. WE ALL KNOW ALREADY THAT 4 IS OUR ANSWER, BUT LET US PRETEND THAT WE DO NOT KNOW THIS AND PROCEED TO FIND IT.

IN ORDER TO FIND THE SQUARE ROOT OF 16, I AM LOOKING FOR A FACT FOR 16 SO THAT I MULTIPLY A NUMBER BY ITSELF TO GET 16. SINCE I DO NOT KNOW IN ADVANCE WHAT THAT NUMBER IS, I MIGHT MAKE ANY GUESS. SUPPOSE MY FIRST GUESS IS 8.

I KNOW THAT $16 = 8 \times \square$ CAN BE SOLVED BY $\square = 2$,
THAT IS, $8 \times 2 = 16$.
NOW IS EE THAT 8 IS TOO LARGE AS A SQUAREROOT GUESS AND 2 IS TOO SMALL, SO I MUST FIND A NUMBER BETWEEN THEM TO TRY TO MULTIPLY BY ITSELF TO GET 16. I CAN SEE THAT THE NUMBER MUST BE BETWEEN 2 AND 8, SINCE IF I LIST ALL THE FACTS FOR A NUMBER, LIKE 16,

AS ONE FACTOR GETS LARGER, THE OTHER GETS SMALLER.

16 x 1

8 x 2

4 x 4

ETC.

OTHERWISE, IF BOTH GET LARGER, MY ANSWER WOULD BE OVER 16, AND IF BOTH GOT SMALLER, MY PRODUCT WOULD BE UNDER 16.

SUPPOSE I CHOOSE MY NUMBER BETWEEN 2 AND 8 TO BE 5. NOW I TRY TO FIND $\square$ SO THAT $5 \times \square = 16$.
I CERTAINLY DON'T GET A WHOLE NUMBER ANSWER, BUT I KNOW MY ANSWER IS BETWEEN 3 AND 4. SO, NOW I HAVE THAT 16 IS ABOUT 5 x 3.

SINCE $5 \neq 3$, I CHOOSE A NUMBER BETWEEN THESE TWO TO TRY FOR THE SQUARE ROOT. HOW ABOUT 4?
4 WORKS, SINCE $4 \times 4 = 16$. SO, 4 IS THE SQUARE ROOT OF 16.

WHAT WOULD I DO TO FIND THE SQUARE ROOT OF 25? I MIGHT MAKE A FIRST GUESS OF 6.
I TRY TO FIND $\square$ SO THAT $\square \times 6 = 25$. BUT $\square$ IS NOT A WHOLE NUMBER, BUT IS A NUMBER BETWEEN 4 AND 5. SO NOW I HAVE THAT 25 IS ABOUT 6 x 4.

SINCE $6 \neq 4$, I CHOOSE A NUMBER BETWEEN THESE TWO TO TRY FOR THE SQUARE ROOT.
HOW ABOUT 5?
5 WORKS, SINCE $5 \times 5 = 25$, SO, $5 = \sqrt{25}$.

BUT TO GET BACK TO FINDING THE SQUARE ROOT OF 576. IT WOULD TAKE MUCH TOO LONG TO LIST ALL THE FACTS FOR IT, SO LET US

EXAMINE WHAT WE DID PURELY NUMERICALLY IN THE PREVIOUS TWO CASES.

I FIRST MADE A FIRST GUESS FOR THE SQUARE ROOT. SUPPOSE MY GUESS NOW WERE 10. I WOULD THEN ATTEMPT TO FIND $\square$ SO THAT $10 \times \square = 576$. HOW DO I FIND $\square$?

Expect: divide 576 by 10.

SINCE $10 \times \square = 576$, THEN $\square = 576 \div 10$ AND I PERFORM THE DIVISION:

```
10 ) 576 |
     500 | 50
      76 |
      70 |  7
       6 | 57
```

AND I FIND THAT $\square$ IS ABOUT 57, SO 576 IS ABOUT 10×57.

NOW I SEE THAT 10 AND 57 ARE NOT EQUAL, SO I CHOOSE A NUMBER BETWEEN THEM AS MY NEXT GUESS FOR THE SQUARE ROOT. I MIGHT CHOOSE A NUMBER ABOUT HALFWAY BETWEEN 10 AND 57, SAY, 35.

I NOW DIVIDE 576 BY 35 TO FIND THE SOLUTION TO $\square \times 35 = 576$, TO SEE IF $\square = 35$.

```
35 ) 576 |
     350 | 10
     226 |
     225 |  7
       1 | 17
```

AND I FIND THAT 576 IS ABOUT 35×17.

BUT THIS IS NOT EXACT, AND $35 \neq 17$, SO I MAKE ANOTHER GUESS BETWEEN THESE TWO, SAY, 26.

THEN, TO SOLVE $\square \times 26 = 576$ TO SEE IF $\square = 26$, I DIVIDE:

```
26 ) 576 )
     520 | 20
      56 |
      52 | 2
       4 | 22
```

SO I STILL DO NOT GET AN EXACT ANSWER BUT I FIND THAT 576 IS ABOUT 22 x 26. SO I MAKE ANOTHER GUESS FOR THE SQUARE ROOT BETWEEN 22 AND 26, SAY, 24.

THEN, IF $\square$ x 24 = 576, I SEE IF $\square$ = 24.

```
24 ) 576 )
     480 | 20
      96 |
      96 | 4
       0 | 24
```

AND $\square$ = 24, SO 24 x 24 = 576 AND 24 = $\sqrt{576}$.

NOTICE THAT WHAT WE HAVE DONE SO FAR IS TO TRY TO FIND A FACTOR $\square$ SO THAT $\square$ x $\square$ = 576 BY REALIZING THAT IF I MAKE A GUESS, SAY, $\triangle$, AND THEN DIVIDE 576 BY $\triangle$ TO GET AN ANSWER, IF THAT ANSWER IS $\triangle$, I HAVE FOUND THE SQUARE ROOT. IF IT IS NOT, I GET A NEW $\triangle$ BY FINDING A NUMBER BETWEEN $\triangle$ AND THE ANSWER I GOT. NOTICE THAT I CAN MAKE ANY FIRST GUESS, BUT I SAVE MYSELF TIME IF I CAN MAKE A GODD FIRST GUESS.

FOR EXAMPLE, IF I REALIZE THAT 10 = $\sqrt{100}$ AND 20 = $\sqrt{400}$ AND 30 = $\sqrt{900}$, I WOULD REALIZE THAT SINCE 576 IS BETWEEN 400 AND 900, I COULD TELL THAT $\sqrt{576}$ IS BETWEEN 20 AND 30, AND MY FIRST GUESS MIGHT HAVE BEEN 25.

SUPPOSE I TRY TO FIND $\sqrt{1369}$.
I KNOW THAT 30 = $\sqrt{900}$ AND 40 = $\sqrt{1600}$, SO I KNOW THAT I WANT TO CHOOSE A NUMBER BETWEEN 30 AND 40, SAY, 35.

THEN, IF I WRITE: $\square \times 35 = 1369$, I SOLVE TO SEE IF $\square = 35$.

```
35) 1369 )
    1050 | 30
     319 |
     315 |  9
       4 | 39
```

AND I FIND THAT I DO NOT GET AN EXACT ANSWER, BUT 1369 IS ABOUT 39 x 35. SO MY NEXT GUESS FOR THE SQUARE ROOT IS A NUMBER BETWEEN 39 AND 35, SAY, 37.

```
37 ) 1369 )
     1110 | 30
      259 |
      259 |  7
        0 | 37
```

AND $37 \times 37 = 1369$, SO $37 = \sqrt{1369}$.

Suggested end of day 4.

Hand students worksheet 1 to complete.

** The students will now apply his technique for finding square roots to finding square roots of fractions. **

SUPPOSE I AM LOOKING FOR THE SQUARE ROOT OF A FRACTION, SAY $\sqrt{\frac{576}{4}}$. WHAT CAN I DO? IF YOU RECALL, WE LEARNED BEFORE THAT WE COULD SEPARATELY FIND THE SQUARE ROOTS OF NUMERATOR AND DENOMINATOR AND FORM A FRACTION OUT OF THEM WHICH WOULD BE THE APPROPRIATE SQUARE ROOT.

WE JUST FOUND BEFORE THAT $\sqrt{576} = 24$ AND WE KNOW THAT $\sqrt{4} = 2$. SO WHAT DO WE KNOW ABOUT $\sqrt{\frac{576}{4}}$?

Expect the answer: $\sqrt{\frac{576}{4}} = \frac{24}{2}$.

INDEED, $\frac{24}{2} \times \frac{24}{2} = \frac{576}{4}$.

IF WE WISH TO MULTIPLY OUT, WE SEE THAT

$$\frac{24}{2} \times \frac{24}{2} = \frac{24 \times 24}{2 \times 2}$$

NOTICE THAT WE MIGHT ALSO WRITE THIS:

$$\frac{576}{4} = 576 \times \frac{1}{4}$$
$$= 24 \times 24 \times \frac{1}{2} \times \frac{1}{2}$$
$$= (24 \times \frac{1}{2}) \times (24 \times \frac{1}{2})$$

SO, $24 \times \frac{1}{2} = \sqrt{576 \times \frac{1}{4}}$.

Ask the students to find the square roots of the following fractions by finding the square roots of numerators and denominators separately, but then verifying with a renaming of the fraction in a multiplicative way (as above).

$$\frac{1369}{4}$$
$$\frac{4}{576}$$
$$\frac{81}{1089}$$

If the students have not seemed to adjust to the procedure of finding the square root, do another problem like the following one:

To find $\sqrt{625}$, I figure that the square root must be between 20 and 30, since $20 = \sqrt{400}$ and $30 = \sqrt{900}$, and 626 is between 400 and 900. My first guess might be 24. Then, I divide:

$$\begin{array}{r|l} 24\overline{)625} & \\ \underline{480} & 20 \\ 145 & \\ \underline{144} & \underline{6} \\ 1 & 26 \end{array}$$

and get $\square$ is about 26 in $24 \times \square = 625$. So, I try a new square root guess between 26 and 24, 25. I divide 625 by 25, get 25, and realize that I have found the square root.

Hand students worksheet 2 to complete.

Suggested end of day 5.

Finding the Square Root of a Fraction

<u>Outline for approach A</u>:

> WE HAVE ALREADY LEARNED HOW TO FIND THE SQUARE ROOTS OF SOME FRACTIONS AND WHOLE NUMBERS, BUT AS WAS MENTIONED EARLIER, THERE ARE SOME NUMBERS LIKE 12 WHICH DON'T HAVE WHOLE NUMBER SQUARE ROOTS, WE ARE GOING TO LEARN HOW TO DEAL WITH THESE NUMBERS, TOO, NOW.
>
> AGAIN, WE MUST DO A BIT OF REVIEWING FIRST.
>
> WHAT DO WE MEAN BY THE FRACTION $\frac{3}{7}$? WE CAN MEAN
>
> $\frac{3}{7} = \frac{1}{7} + \frac{1}{7} + \frac{1}{7} = 3 \times \frac{1}{7}$.
>
> HOW COULD YOU WRITE $\frac{6}{8}$ AS A MULTIPLICATION EXPRESSION?

Expect the answer: $6 \times \frac{1}{8}$.

Repeat this procedure with $\frac{3}{5}$ and $\frac{2}{4}$.

> BECAUSE WE HAVE ALREADY LEARNED THAT IN FINDING SQUARE ROOTS, DIVISION WAS INVOLVED, LET US QUICKLY GO OVER SOME IDEAS ABOUT DIVISION.
>
> SUPPOSE I WANT TO SOLVE THE MULTIPLICATION QUESTION: $8 \times \square = 32$.
> WHAT DO I DO TO SOLVE THIS?

Expect the answerL divide 32 by 8.

> HOW WOULD I SOLVE $\square \times 8 = 56$?

Expect the answer: divide 56 by 8.

IT IS IMPORTANT TO SEE THAT IF A MULTIPLICATION QUESTION IS MISSING A FACTOR, I PERFORM DIVISION TO FIND THAT FACTOR.

ONE FINAL THING TO RECALL IS THAT IN MULTIPLYING, MULTIPLICATION CAN BE DONE IN ANY ORDER AND FURTHERMORE, ANY NUMBER CAN BE MULTIPLIED BY 1 OR ANY FORM OF 1 WITHOUT CHANGING ITS VALUE.

FOR EXAMPLE, ANOTHER EXPRESSION FOR 5×8 WOULD BE 8×5 AND YET ANOTHER WOULD BE $8 \times 5 \times 1$ AND YET ANOTHER $8 \times 5 \times \frac{2}{2}$ $= 8 \times 5 \times (2 \times \frac{1}{2})$.

RENAME EACH OF THESE IN SEVERAL WAYS:

8×7

$4 \times 3 \times 5$

9×2

Ask a few students to read out their answers to these.

** Here the students will review the idea that the square root of a whole is found by repeated division in order to find a number to multiply by itself to get the given number.**

IF YOU RECALL, WE HAVE ALREADY LEARNED THAT IN ORDER TO FIND THE SQUARE ROOT OF A NUMBER, I AM LOOKING FOR ANOTHER NUMBER WHICH I CAN MULTIPLY BY ITSELF TO GET THE NUMBER I AM GIVEN. FOR EXAMPLE, $4 = \sqrt{16}$. AND, AS YOU MIGHT ALSO RECALL, THIS IS THE PROBLEM OF GOING THROUGH A LIST OF MULTIPLICATION FACTS FOR THAT NUMBER AND PICKING OUT THAT FACT WHERE THE SAME NUMBER OCCURS AS BOTH FACTORS.

YOU MAY REMEMBER THAT WE PERFORMED DIVISION WHEN THE NUMBER WE WERE TRYING TO FIND THE SQUARE ROOT OF WAS LARGE IN ORDER TO

MAKE THE PROBLEM EASIER.

FOR EXAMPLE, TO FIND $\sqrt{324}$, I MIGHT SAY:

SINCE $20 = \sqrt{400}$ AND $10 = \sqrt{100}$, THEN $\sqrt{324}$ MUST BE BETWEEN 10 AND 20. LET US GUESS 15. SO, I SOLVE $15 \times \square = 324$ TO SEE IF $\square = 15$. TO DO THIS, I DIVIDE 324 BY 15,

```
15 ) 324 |
     300 | 20
      24 |
      15 |  1
       9 | 21
```

AND I SEE THAT $\square$ IS SLIGHTLY OVER 21. SO, 324 IS ABOUT 15×21. I WANT TO INCREASE MY GUESS FROM 15, SINCE BY INCREASING THAT I WILL ALSO DECREASE 21 AND MAYBE THEY WILL MEET. SO I GUESS A NUMBER BETWEEN THEM, SAY, 18. I SOLVE $18 \times \square = 324$ TO SEE IF $\square = 18$.

```
18 ) 324 |
     180 | 10
     144 |
     144 |  8
       0 | 18
```

AND $\square = 18$, SO $18 \times 18 = 324$ AND $\sqrt{324} = 18$.

LET'S TRY TO FIND THE SQUARE ROOT OF 729 BY THIS TECHNIQUE. I MIGHT MAKE A GUESS FOR THE FIRST FACTOR OF A NUMBER BETWEEN 20 AND 30 SINCE $20 = \sqrt{400}$ AND $30 = \sqrt{900}$ AND 729 IS BETWEEN 400 AND 900. SUPPOSE MY GUESS IS 25. IF $25 \times \square = 729$, I WANT TO KNOW IF $\square = 25$.

$= 729 \div 25$

$$
\begin{array}{r|l}
25\overline{)729} & \\
\underline{500} & 20 \\
229 & \\
\underline{200} & 8 \\
29 & \\
\underline{25} & \underline{1} \\
4 & 29
\end{array}
$$

AND I FIND THAT $\square$ IS ABOUT 29 SO 729 IS ABOUT 25×29.

NOW I GUESS A NUMBER BETWEEN 25 AND 29 AS THE FIRST FACTOR, SAY, 27. I SOLVE $27 \times \square = 729$, TO SEE IF $\square = 27$.

$$
\begin{array}{r|l}
27\overline{)729} & \\
\underline{540} & 20 \\
189 & \\
\underline{189} & \underline{7} \\
0 & 27
\end{array}
$$

AND $27 \times 27 = 729$, SO $729 = 27$.

Hand students worksheet 3 to complete.

Suggested end of day 6.

** Here the students will learn about approximating square roots of wholes. **

SUPPOSE WE NOW WANT TO FIND $\sqrt{5}$. WE KNOW THAT 2 IS TOO SMALL, SINCE $2 \times 2 = 4$ AND 3 IS TOO LARGE SINCE $3 \times 3 = 9$. SO WE HAVE A PROBLEM WE HAD NOT PREVIOUSLY FACED.

ONE TECHNIQUE WE MIGHT USE IS TO CONVERT 5 TO A FRACTION AND TRY TO FIND A FRACTION NEAR IT WITH A PERFECT SQUARE ROOT NEAR IT. FOR EXAMPLE, WE KNOW THAT $5 = 5 \times 1 = 5 \times (25 \times \frac{1}{25})$ $= 125 \times \frac{1}{25}$ IS CLOSE TO $121 \times \frac{1}{25}$ WHICH HAS A PERFECT SQUARE ROOT OF $11 \times \frac{1}{5} = \frac{11}{5}$. BUT THIS MIGHT PROVE VERY LONG AND

TEDIOUS IF WE DON'T FIND THE RIGHT FRACTION RIGHT AWAY.

LET'S TRY TO DEVELOP A METHOD.

SUPPOSE WE TRY TO FIND $\sqrt{5}$.

WE KNOW THAT THE SOLUTION IS SOMETHING BETWEEN 2 AND 3. LET US CALL IT $2+\square$, WHERE $\square$ IS SMALLER THAN 1.

THEN, SINCE $2+\square$ IS THE SQUARE ROOT OF 5, WE HAVE

$$(2+\square) \times (2+\square) = 5.$$

HOW CAN WE MULTIPLY THESE NUMBERS OUT?

WELL, JUST AS $4 \times (5+6) = 4 \times 5 + 4 \times 6$, WE CAN CONSIDER:

$$\underline{(2+\square)} \times (2+\square) = \underline{(2+\square)} \times 2 + \underline{(2+\square)} \times \square.$$

THEN WE CAN REWRITE EACH OF THE EXPRESSIONS ON THE RIGHT:

$$\underline{(2+\square)} \times 2 = 2 \times 2 + \square \times 2 \quad \text{AND}$$

$$\underline{(2+\square)} \times \square = 2 \times \square + \square \times \square \quad .$$

Notice that the use of the distributive law here is very involved and it may be necessary to go back to consideration of simpler examples, like $2 \times (4+5) = 2 \times 4 + 2 \times 5$ or $\underline{(2+3)} \times (4+5) = \underline{(2+3)} \times 4 + \underline{(2+3)} \times 5$ and then distribute again. Try to use underlining to indicate the common term as above and make sure that the entire first expression is viewed as the common term the first time the distributive property is used, but as the usual addends the second time through. That is, the first time $(2+\square)$ on the left is treated as a united quantity which travels together in order to break up the right hand expression, but afterwards it too is broken up.

THEN, WE HAVE THAT $(2 \times 2) + (2 \times \square) + (\square \times 2) + (\square \times \square) = 5.$

SINCE $2 \times 2 = 4$, THAT MEANS THAT THE THREE REMAINING EXPRESSIONS MAKE UP THE ONE LEFT $(5-4)$, THAT IS:

$(2 \times \square) + (\square \times 2) + (\square \times \square) = 1.$

SUPPOSE WE PRETEND TEMPORARILY THAT WE CAN IGNORE THE $\square \times \square$ PART. THEN WE HAVE $(2 \times \square) + (\square \times 2) = 1.$

BUT $2 \times \square = \square \times 2$, SO WE HAVE

$(2 \times \square) + (2 \times \square) = 1.$

$(2 \times \square) + (2 \times \square) = 4 \times \square$ USING THE DISTRIBUTIVE PRINCIPLE.

SO, $4 \times \square = 1$, SO $\square$ IS $\frac{1}{4}$.

IF WE ASSUME $\square$ IS $\frac{1}{4}$, THEN WE NOTICE THAT $\square \times \square$ IS ONLY WHAT NUMBER?

Expect the answer: $\frac{1}{16}$.

SINCE THIS NUMBER, $\square \times \square = \frac{1}{16}$, IS LESS THAN $\frac{1}{2}$, WE CAN SAY THAT WE HAVE USED UP MOST OF THE 1 (5 - 4) IN THE OTHER EXPRESSIONS BESIDES THE $\square \times \square$ AND SO WE SAY THAT 5 IS ABOUT $2\frac{1}{4}$. WE WRITE THIS: $\sqrt{5} \approx 2\frac{1}{4}$.

WE MIGHT HAVE USED THE DIVISION PROCEDURE AGAIN, AS WELL. SO OUR FIRST GUESS WOULD HAVE BEEN 2. THEN WE WOULD DIVIDE 5 BY 2, AND GET:

$$\begin{array}{r|l} 2 & \sqrt{5} \\ & \underline{4} \\ & 1 \end{array} \quad \begin{array}{l} \\ \underline{2} \\ 2 \end{array}$$

UNFORTUNATELY, THIS GETS US INTO DIVISION WITH FRACTIONS SINCE OUR NEXT CHOICE WOULD BE A NUMBER BETWEEN 2 AND $2\frac{1}{2}$, AND SO WE WILL AVOID THE DIVISION TECHNIQUE FOR THE TIME BEING.

SUPPOSE WE WANT TO FIND $\sqrt{13}$.

WE KNOW THAT THE ANSWER MUST BE BETWEEN 3 AND 4, SINCE $3 \times 3 = 9$ AND $4 \times 4 = 16$, AND 13 IS BETWEEN 9 AND 16.

SO, AGAIN, WE ASSUME THAT THE SQUARE ROOT IS A LITTLE OVER 3, SAY, $3+\square$. THEN, BECAUSE IT IS A SQUARE ROOT:

$$(3+\square) \times (3+\square) = 13.$$

WE CAN RENAME:

$$\underline{(3+\square)} \times (3+\square) = \underline{(3+\square)} \times 3 + \underline{(3+\square)} \times \square .$$
$$= (3 \times 3) + (\square \times 3) + (3 \times \square) + (\square \times \square)$$
$$= 9 + (3 \times \square) + (3 \times \square) + (\square \times \square)$$

THEN, IF $9 + (3 \times \square) + (3 \times \square) + (\square \times \square) = 13$,

$$(3 \times \square) + (3 \times \square) + (\square \times \square) = 13 - 9 = 4.$$

AGAIN, LET US IGNORE THE $\square \times \square$ TEMPORARILY, IF WE DO, WE GET

$(3 \times \square) + (3 \times \square)$ IS ABOUT 4.

BUT, $(3 \times \square) + (3 \times \square) = 6 \times \square$

AND IF $6 \times \square = 4$, THEN $\square = 4 \times \frac{1}{6}$ SINCE

$$6 \times (4 \times \tfrac{1}{6}) = 4 \times (6 \times 1/6) = 4 \times 1 = 4.$$

SO, BY IGNORING $\square \times \square$, WE GET $\square = \frac{4}{6}$.

THEN, IF WE INCLUDE $\square \times \square$, WE GET $\square \times \square = \frac{4}{6} \times \frac{4}{6} = \frac{16}{36}$.

BUT SINCE THIS IS LESS THAN $\frac{1}{2}$, WE DO NOT WORRY ABOUT IT, AND SAY THAT WE ARE CLOSE ENOUGH TO BE SATISFIED.

WE WRITE:

$$\sqrt{13} \approx 3\frac{4}{6}$$

SUPPOSE WE WANT TO FIND $\sqrt{37}$. IF WE DID NOT KNOW IN ADVANCE TO TRY A NUMBER NEAR 6, WE MIGHT USE OUR DIVISION PROCEDURE TO CLOSE IN ON A NUMBER BETWEEN 6 AND 7.

FOR EXAMPLE, WE MIGHT DO:

$4 \times \square = 37$ TO SEE IF $\square = 4$.

WE DIVIDE:

$$\begin{array}{r|l} 4\overline{)37} & \\ \underline{36} & \underline{9} \\ 1 & 9 \end{array}$$

AND WE FIND THAT $\square$ IS ABOUT 9. SO NOW WE TRY A NUMBER FOR THE SQUARE ROOT BETWEEN 4 AND 9, LIKE 7.

SO WE SOLVE: $7 \times \square = 37$ TO SEE IF $\square = 7$.

$$\begin{array}{r|l} 7\overline{)37} & \\ \underline{35} & \underline{5} \\ 2 & 5 \end{array}$$

AND WE FIND THAT $\square$ IS ABOUT 5. SO NOW WE TRY A NUMBER BETWEEN 7 AND 5, SAY, 6.

$$\begin{array}{r|l} 6\overline{)37} & \\ \underline{36} & \underline{6} \\ 1 & 6 \end{array}$$

AND WE ARE BACK TO THE POINT WHERE OUR DIVISION PROCESS CANNOT HELP US. HOWEVER, WE DO KNOW FOR SURE NOW THAT $\square$ SITS BETWEEN 6 AND 7.

SO WE GO BACK TO OUR PROCEDURE OF BEFORE.

WE KNOW THAT $37 = 6 + \square$.

SO, $(6 + \square) \times (6 + \square) = 37$

BUT, $\underline{(6 + \square)} \times (6 + \square)$

$= \underline{(6 + \square)} \times 6 + \underline{(6 + \square)} \times \square$

$= (6 \times 6) + (\square \times 6) + (6 \times \square) + (\square \times \square)$

$= 36 + (\square \times 6) + (6 \times \square) + (\square \times \square)$

$= 36 + (6 \times \square) + (6 \times \square) + (\square \times \square)$

THEN, $36 + (6 \times \square) + (6 \times \square) + (\square \times \square) = 37$,

SO, $(6 \times \square) + (6 \times \square) + (\square \times \square) = 37 - 36 = 1$.

TEMPORARILY IGNORING THE $\square \times \square$, WE FIND THAT

$(6 \times \square) + (6 \times \square)$ IS ABOUT 1.

BUT, $(6 \times \square) + (6 \times \square) = 12 \times \square$, SO $\square$ IS ABOUT $\frac{1}{12}$.

IF WE THEN CALCULATE $\square \times \square$, WE GET $\frac{1}{144}$ WHICH IS MUCH SMALLER THAN $\frac{1}{2}$, SO WE ARE NOT TOO FAR OFF, AND WE CAN SAY THAT

$\sqrt{37} \approx 6\frac{1}{12}$.

NOTICE THAT IF I HAD BEEN TRYING TO FIND $\sqrt{38}$, I WOULD HAVE DONE THE SAME THING UNTIL I SAID THAT $12 \times \square$ IS ABOUT 1. THEN I WOULD HAVE SAID THAT $12 \times \square$ IS ABOUT 2, AND SO $\square$ IS ABOUT $2 \times \frac{1}{12} = \frac{2}{12}$. I STILL WOULD HAVE CHECKED TO SEE THAT $\square \times \square$ WAS SMALLER THAN $\frac{1}{2}$ AND SINCE $\frac{2}{12} \times \frac{2}{12}$ IS SMALLER THAN $\frac{1}{2}$, I WOULD HAVE ACCEPTED THIS APPROXIMATION. ACTUALLY, IT'S A LITTLE SMALL, SINCE IT'S THE SQUARE ROOT OF $38 + \frac{4}{144}$.

Suggested end of day 7.

Ask the students to work out the approximate square roots for each of the following and then ask students to come to the board to show their work.

8
18
50
67

** Here the students will learn how to approximate the square roots of fractions whose denominators are perfect squares, but whose numerators

are not. **

NOW SUPPOSE I WANT TO FIND THE APPROXIMATE SQUARE ROOT OF $\frac{5}{4}$.
I HAVE ALREADY LEARNED THAT I CAN SEPARATELY ATTACK THE NUMERATORS AND DENOMINATORS AND PUT TOGETHER THESE SQUARE ROOTS TO FORM THE SQUARE ROOT FRACTION.

SUPPOSE WE DO THAT HERE.
WE FOUND THAT $\sqrt{5} \approx 2\frac{1}{4}$ FROM OUR PREVIOUS WORK AND WE KNOW THAT $\sqrt{4} = 2$.
SO THE APPROXIMATE SQUARE ROOT OF $\frac{5}{4}$ IS $\frac{2\frac{1}{4}}{2}$.

SUPPOSE I AM TRYING TO FIND $\sqrt{\frac{13}{4}}$.
WE KNOW THAT $\sqrt{13} \approx 3\frac{4}{6}$ AND $\sqrt{4} = 2$, SO $\sqrt{\frac{13}{4}} \approx \frac{3\frac{4}{6}}{2}$.

LET'S DO THE SAME FOR $\sqrt{\frac{37}{4}}$.
WE CAN WRITE THAT THE APPROXIMATE $\sqrt{\frac{37}{4}}$ IS $\frac{6\frac{1}{12}}{2}$.

Have the students find approximate square roots for each of the following have a few students show their work on the board.

$\sqrt{\frac{32}{9}}$

$\sqrt{\frac{47}{25}}$

$\sqrt{\frac{28}{49}}$

Suggested end of day 8.

APPENDIX B

CRITERION TESTS--

PRETESTS

COMPUTATION TESTS

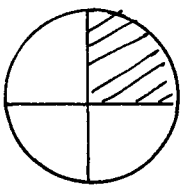
EXTENSION TESTS

Multiplication of a Mixed Number by a Fraction-Pretest

Read each question. Answer the question in the space provided on the hand side of the page.

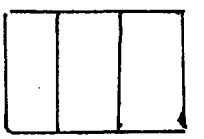
1. $6 \times 3 = \square$	1. $\square =$ _____
2. $2 \times 9 = \square$	2. $\square =$ _____
3. $8 \times 4 = \square$	3. $\square =$ _____
4. $6 \times 8 = \square$	4. $\square =$ _____
5. $7 \times 5 = \square$	5. $\square =$ _____
6. $9 \times 7 = \square$	6. $\square =$ _____
7. $6 \times 7 = \square$	7. $\square =$ _____
8. $8 \times 9 = \square$	8. $\square =$ _____
9. $6 \times 6 = \square$	9. $\square =$ _____
10. $8 \times 8 = \square$	10. $\square =$ _____
11. $3\frac{1}{2} = 3 + \frac{\square}{\triangle}$	11. $\frac{\square}{\triangle} =$ _____
12. $6\frac{1}{4} = \square + \frac{1}{4}$	12. $\square =$ _____
13. $29\frac{5}{9} = \square + \frac{5}{9}$	13. $\square =$ _____
14. $\square\frac{6}{8} = 5 + \frac{6}{8}$	14. $\square =$ _____

15. Give a fraction name to the shaded portion below:

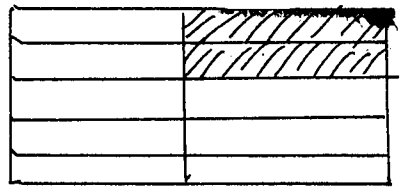


15. _____

16. Shade in $\frac{1}{3}$ of the diagram on the right.

16. 

17. Give a fraction name to the shaded portion below:



17. _____

18. Draw a diagram with $\frac{2}{3}$ shaded in on the line at the right.

18. _____

19. Draw a diagram with $\frac{3}{4}$ shaded in on the line at the right.

19. _____

20. $\square \times \frac{1}{5} = 1$ — 20. $\square =$ _____

21. $\square \times \frac{1}{8} = 1$ — 21. $\square =$ _____

22. $6 \times \frac{\square}{\triangle} = 1$ — 22. $\frac{\square}{\triangle} =$ _____

23. $\frac{3}{2} = 3 \times \frac{\square}{\triangle}$ — 23. $\frac{\square}{\triangle} =$ _____

24. $\frac{8}{9} = \square \times \frac{1}{9}$ — 24. $\square =$ _____

25. $\frac{16}{5} = \square \times \frac{1}{5}$ — 25. $\square =$ _____

26. What is the length of a rectangle with a width of 2 feet and an area of 6 x 2 sq. ft.?

26. _____

27. What is the width of a rectangle with a length of 8 feet and an area of 8 x 3 sq. ft.?

27. _____

28. What is the area of a rectangle with width 3 ft. and length 5 ft.?

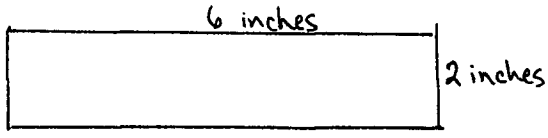
28. _____

29. What operation would you use to find the area of a rectangle with length 89 ft. and width 38 ft.?

(Would you add, subtract, multiply, or divide?)

29. _____

30. Using the idea that to find area, you find the number of one-unit squares in a figure, show how you would find the area of the figure below:



30. □ = _____

31. $1 \times 6 = \square$ — 31. □ = _____

32. $\square \times 8 = 8$ — 32. □ = _____

33. $\square \times 239 = 239$ — 33. □ = _____

34. $1 \times \frac{1}{4} = \frac{\square}{\triangle}$ — 34. $\frac{\square}{\triangle}$ = _____

35. $33 \times 24 \times 51 = 24 \times \square \times 33$ — 35. □ = _____

36. $56 \times 19 \times \square = 19 \times \triangle \times 48$ — 36. □ = _____ ; △ = _____

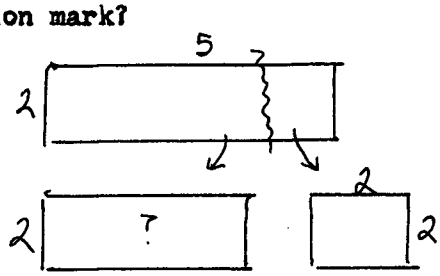
37. $27 \times 56 \times 65 \times 41 = 27 \times \square \times 56 \times 41$ — 37. □ = _____

38. $55 \times (\square + 38) = (55 \times 42) + (55 \times 38)$ — 38. □ = _____

39. $\square \times (23 + 872) = 8950$ — 39. □ = _____

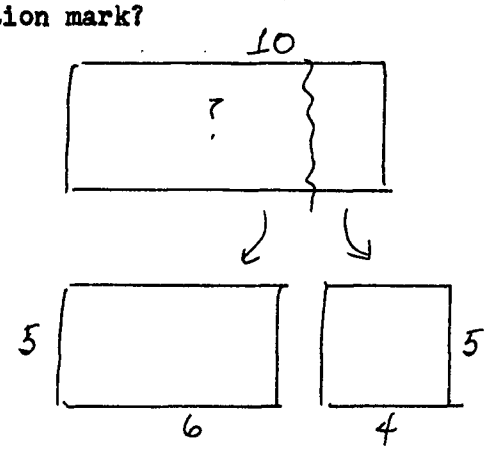
40. $6 \times (84 + \square) = (6 \times 84) + (6 \times 156)$ — 40. □ = _____

41. What is the area of the figure below marked with the question mark?



41. _____

41. What is the area of the figure below marked with the question mark?



42. _____

Multiplication of a Fraction by a Mixed Number

Read each question. Do your work in the space provided below the question. Then write your answer on the line to the right. There are 24 questions.

1. $2 \times \frac{6}{7} =$

1._____

2. $4 \times \frac{3}{8} =$

2._____

3. $6 \times \frac{8}{9} =$

3._____

4. $5 \times \frac{5}{7} =$

4._____

5. $9 \times \frac{8}{11} =$

5._____

6. $7 \times \frac{8}{9} =$

6._____

7. $2 \times 2\frac{3}{8} =$

7._____

8. $4 \times 2\frac{2}{13} =$

8._____

9. $8 \times 4\frac{3}{25} =$

9._____

10. $6 \times 5\frac{4}{29} =$

10._____

11. $9 \times 6\frac{5}{70} =$

11._____

12. $7 \times 7\frac{6}{50} =$

12._____

13. $\frac{2}{7} \times \frac{1}{2} =$

13._____

14. $\frac{2}{3} \times \frac{8}{9} =$

14._____

15. $\frac{4}{6} \times \frac{5}{6} =$

15._____

16. $\frac{4}{5} \times \frac{7}{8} =$

16._____

17. $\frac{8}{9} \times \frac{8}{9} =$

17._____

18. $\frac{5}{6} \times \frac{9}{10} =$

18._____

19. $\frac{1}{2} \times 8\frac{6}{9} =$

19._____

20. $\frac{2}{3} \times 7\frac{4}{5} =$

20._____

21. $\frac{4}{5} \times 7\frac{4}{6} =$

21._____

22. $\frac{3}{4} \times 8\frac{7}{9} =$

22._____

23. $\frac{8}{9} \times 8\frac{6}{7} =$

23._____

24. $\frac{7}{9} \times 7\frac{8}{9} =$

24. _____

Multiplication of a Mixed Number by a Fraction-Extended

Instructions:

Read each question thoroughly. Do your work in the space provided below the question, but write your answer on the lines to the right of the questions. For those questions which are divided into parts (a),(b),(c), and (d), answer only part (d) on the line to the right and do all other work in the space below the question.

1. If you know that $\frac{1}{81} \times \frac{1}{63} = \frac{1}{5103}$, you could also tell that $\frac{2}{81} \times \frac{1}{63} = \frac{\square}{\triangle}$. What is $\frac{\square}{\triangle}$?

1. $\frac{\square}{\triangle} =$ _____

2. If you know that $\frac{53}{69} \times \frac{48}{73} = \frac{2544}{5037}$, find values for $\square$ and $\triangle$ so that $\frac{48}{69} \times \frac{53}{\triangle} = \frac{\square}{5037}$.

2. $\square =$ _____ ; $\triangle =$ _____

3. If you know that $\frac{13}{64} \times \frac{12}{15} = \frac{156}{960}$, you could also tell that $\frac{13}{64} \times \frac{12}{10 \times 15} = \frac{\square}{\triangle}$. What is $\frac{\square}{\triangle}$?

3. $\frac{\square}{\triangle} =$ _____

4. If you know that $\frac{13}{5} \times \frac{4}{27} = \frac{52}{135}$, you could also tell that $\frac{13 \times 10}{5} \times \frac{4}{27} = \frac{\square}{\triangle}$. What is $\frac{\square}{\triangle}$?

4. $\frac{\square}{\triangle} =$ _____

5. Write the following as a multiplication expression:

$2\frac{3}{5} + 2\frac{3}{5} + 2\frac{3}{5} + 2\frac{3}{5}$

5. _____

6. Write the following as strictly a multiplication expression (not involving addition). Do not compute the answer.

$$\frac{5}{6} \times 2\frac{4}{9} + \frac{5}{6} \times \frac{5}{9}$$

6._____

7. Write the following as strictly a multiplication expression (not involving addition). Do not compute the answer.

$$\frac{3}{4} \times 2\frac{1}{4} + \frac{3}{4} \times 2\frac{3}{4}$$

7._____

8. Find the answers to a,b, and c. <u>Write the answer to (d) on the line to the right</u>. The first three answers are only meant to help you with the answer to (d).

($\triangle$ is a whole number).

(a) $\frac{3}{4} \times \frac{4}{3} = \triangle$

(b) $\frac{8}{9} \times \frac{9}{8} = \triangle$

(c) $\frac{5}{6} \times \frac{6}{5} = \triangle$

(d) $\frac{879}{432} \times \frac{432}{879} = \square$

8. $\square$ = _____

9. Find the answers to a,b, and c. <u>Write the answer to (d) on the line to the right</u>. ($\square$ is a whole number).

(a) $3 \times 2\frac{1}{3} = \square$

(b) $4 \times 2\frac{1}{4} = \square$

(c) $9 \times 2\frac{1}{9} = \square$

(d) $\boxdot \times 2\frac{1}{\boxdot} = 17$

9. $\boxdot$ = _____

10. If $3 \times 4\frac{\square}{9} = 12\frac{6}{9}$, what is $\square$?

10. $\square =$ _____

11. If $5 \times \square\frac{\triangle}{8} = 20\frac{5}{8}$, what are $\square$ and $\triangle$?

11. $\square =$ _____ ; $\triangle =$ _____

12. If $\square \times 3\frac{5}{16} = 9\frac{15}{16}$, what is $\square$?

12. $\square =$ _____

13. If $\frac{7}{2} \times \square\frac{1}{4} = \frac{35}{2} + \frac{7}{8}$, what is $\square$?

13. $\square =$ _____

14. If $\frac{2}{3} \times 5\frac{\square}{\triangle} = \frac{10}{3} + \frac{8}{15}$, what is $\frac{\square}{\triangle}$?

14. $\frac{\square}{\triangle} =$ _____

15. How would you use the rule for multiplying a fraction by a mixed number to multiply a mixed number by a mixed number- for example, to multiply: $2\frac{1}{4} \times 3\frac{1}{5}$. Show all of your steps in the space below and write the answer on the line to the right.

15. _____

16. Find: $\frac{\frac{1}{2}}{2} \times 24\frac{2}{5}$

16. _____

17. Find: $\dfrac{\frac{2}{3}}{4} \times 16\frac{9}{11}$

17._____

18. Find the answers to a, b, and c. <u>Write the answer to (d) on the line to the right.</u> The first three answers are only to help you with (d).

(a)
$$\begin{array}{r} 2\tfrac{1}{3} \\ \times\ 3\tfrac{1}{4} \\ \hline 6 = 3 \times 2 \\ 1 = 3 \times \tfrac{1}{3} \\ 2/4 = 2 \times 1/4 \\ +\ 1/12 = 1/3 \times 1/4 \\ \hline \square\ \frac{\triangle}{\bigcirc} \end{array}$$

(b)
$$\begin{array}{r} 2\tfrac{1}{3} \\ \times 4\tfrac{1}{5} \\ \hline 8 = 4 \times 2 \\ 4/3 = 4 \times 1/3 \\ 2/5 = 2 \times 1/5 \\ +\ 1/15 = 1/3 \times 1/5 \\ \hline \square\ \frac{\triangle}{\bigcirc} \end{array}$$

(c)
$$\begin{array}{r} 3\tfrac{1}{5} \\ \times\ 1\tfrac{4}{6} \\ \hline 3 = 1 \times 3 \\ 1/5 = 1 \times 1/5 \\ 12/6 = 3 \times 4/6 \\ +\ 4/30 = 1/5 \times 4/6 \\ \hline \square\ \frac{\triangle}{\bigcirc} \end{array}$$

(d)
$$\begin{array}{r} 4\tfrac{1}{3} \\ \times\ 11\tfrac{1}{5} \\ \hline \triangle\ \frac{\square}{\bigcirc} \end{array}$$

18. $\triangle\ \frac{\square}{\bigcirc}$ = _____

19. What is $(6 \times \frac{2}{9}) \times 5\frac{1}{3}$?

19._____

20. What is $(\frac{1}{3} \times \frac{1}{2}) \times 2\frac{5}{8}$?

20._____

21. What is $\frac{3}{4} \times (\frac{2}{3} \times 3)$?

21._____

22. What is $\left(4 \times \frac{1}{3} \times \frac{1}{5}\right) \times 2\frac{1}{6}$?

22. _____

23. If $\left(3 \times \frac{1}{2}\right) \times 3\frac{1}{4} = \frac{\square}{\triangle} \times 9\frac{3}{4}$, what is $\frac{\square}{\triangle}$?

23. $\frac{\square}{\triangle}$ = _____

24. If $\left(6 \times \frac{\square}{\triangle}\right) \times 5\frac{1}{3} = \frac{30}{3} + \frac{6}{9}$, what is $\frac{\square}{\triangle}$?

24. $\frac{\square}{\triangle}$ = _____

25. If $\left(4 \times \frac{2}{3}\right) \times \triangle\frac{\square}{\bigcirc} = \frac{16}{3} + \frac{8}{15}$, what is $\triangle\frac{\square}{\bigcirc}$?

25. $\triangle\frac{\square}{\bigcirc}$ = _____

26. Find values for $\square$ and $\triangle$ so that

$$4 \times \frac{\square}{3} \times 2\frac{5}{\triangle} = \frac{40}{3} + \frac{25}{27} .$$

26. $\square$ = _____ ; $\triangle$ = _____

For the next four questions, write your answer in the space below the question.

27. Susie has suggested another rule for multiplying fractions. To multiply, for example, $\frac{2}{3} \times \frac{4}{5}$, she doubles the first fraction's numerator and doubles the second fraction's denominator and then multiplies.

$$\frac{4}{3} \times \frac{4}{10} = \frac{16}{30} \text{ , so } \frac{2}{3} \times \frac{4}{5} = \frac{16}{30} .$$

She claims that her answer is equivalent to the one which the teacher gets by multiplying by the method taught in class. ($\frac{2}{3} \times \frac{4}{5} = \frac{8}{15}$) Will she always get equivalent answers to yours? Why do you think she should? Use $\frac{3}{4} \times \frac{2}{9}$ as an example.

28. Johnny has suggested another rule for multiplying fractions. To multiply, for example, $\frac{2}{3} \times \frac{4}{5}$, he finds the answer to $\frac{4}{3} \times \frac{2}{5}$. He switches the numerators of the fractions and then multiplies. He claims that his answer is equivalent to the one which the teacher gets by the method taught in class. Will he always get equivalent answers to yours? Why do you think he should? Use $\frac{3}{4} \times \frac{2}{9}$ as an example.

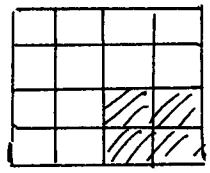
29. Sam has suggested a rule for multiplying two fractions, too. To find the answer to, for example, $\frac{2}{3} \times \frac{4}{5}$, Sam finds the answer to $\frac{3}{2} \times \frac{5}{4}$ and then turns the fraction upside down. In other words, he turns both fractions upside down, multiplies, and then turns his answer upside down. He claims that his answer is the same as the one the teacher gets by the method taught in class. Will he always get the same answer as yours? Why do you think he should? Use $\frac{3}{4} \times \frac{2}{9}$ as an example.

30. Judy has suggested a way to check multiplication of fraction answers. She says that one can be sure that $\frac{a}{b} \times \frac{c}{d} = \frac{\square}{\triangle}$ if $\square \div c = a$ and $\triangle \div d = b$. For example, it is true that $\frac{3}{5} \times \frac{2}{9} = \frac{6}{45}$ since $6 \div 2 = 3$ and $45 \div 9 = 5$. Can you be sure if your answer is right by checking with Judy's method? Why do you think so?

Changing Fractions to Decimals-Pretest

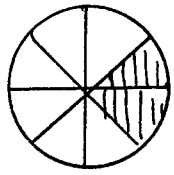
Read each question. Answer the question in the space provided on the right hand side of the page.

1. Give a fraction name to the shaded portion below:



1._____

2. Give a fraction name to the shaded portion below:



2._____

3. Shade in $\frac{1}{5}$ of the diagram on the right.

3._____

4. Draw a diagram with $\frac{3}{4}$ shaded in on the line to the right.

4._____

5. Draw a diagram with $\frac{2}{6}$ shaded in on the line to the right.

5._____

6. What equation would you write to describe the action in sharing 27 marbles among 3 people.

6._____

7. What equation would you write to describe the action in sharing 18 cupcakes among 6 children?

7._____

8. What problem would you write to describe the action in sharing 24 pencils among 8 students?

8._____

9. Draw a diagram below showing why $8 \div 4 = 2$.

9._____

10. Draw a diagram showing why $12 \div \square = 4$. Use the space below for the drawing and find the value of $\square$.

10._____

11. Divide:

$11 \overline{)425}$

11._____

12. Divide:

$20 \overline{)5834}$

12._____

13. Divide:

$$7\overline{)2859}$$

13._____

14. Divide:

$$25\overline{)4632}$$

14._____

15. Divide:

$$6\overline{)384}$$

15._____

16. Write the following multiplication statement as a division statement:

$$532 \times 18 = 9576$$

16._____

17. Write the following division statement as a multiplication statement:

$$492 \div 123 = 4$$

17._____

18. Write the following division statement as a multiplication statement:

$$\square \div 6 = \triangle$$

18._____

19. Write the following multiplication statement as a division statement:

$$\square \times 6 = \triangle$$

19._____

20. Write the following multiplication statement as a division statement:

$$\square \times \triangle = \bigcirc$$

20._____

21. $\square \times \frac{1}{9} = 1$ — 21. $\square =$ _____

22. $8 \times \frac{\square}{\triangle} = 1$ — 22. $\frac{\square}{\triangle} =$ _____

23. $\frac{14}{9} = 14 \times \frac{\square}{\triangle}$ — 23. $\frac{\square}{\triangle} =$ _____

24. $\frac{3}{7} = \square \times \frac{1}{7}$ — 24. $\square =$ _____

25. $\frac{38}{6} = \square \times \frac{1}{6}$ — 25. $\square =$ _____

26. $74 \times 83 \times 77 = \square \times 74 \times 77$ — 26. $\square =$ _____

27. $312 \times 25 \times \square \times 87 = 43 \times \triangle \times 25 \times 312$ — 27. $\square =$ _____ ; $\triangle =$ _____

28. $45 \times 213 \times 87 = \square \times 203 \times 87$ 28. $\square =$ _____

29. $1 \times 576 = \square$ 29. $\square =$ _____

30. $\square \times (374 + 596) = 374 + 596$ 30. $\square =$ _____

Changing a Fraction to a Decimal

Read each question. Do your work in the space provided below the question. Then write the decimal equivalent for the given fraction on the line to the right of the page. There are 18 questions.

1. $\frac{2}{5} =$

1._____

2. $\frac{3}{4} =$

2._____

3. $\frac{1}{2} =$

3._____

4. $\frac{3}{8} =$

4._____

5. $\frac{7}{10} =$

5._____

6. $\frac{13}{25} =$

6._____

7. $\frac{17}{20} =$

7._____

8. $\frac{37}{50} =$

8._____

9. $\frac{2}{3} \approx$

9._____

10. $\frac{5}{6} \approx$

10._____

11. $\frac{5}{7} \approx$

11._____

12. $\frac{3}{9} \approx$

12._____

13. $\frac{3}{7} \approx$

13._____

14. $\frac{8}{9} \approx$

14._____

15. $\frac{5}{11} \approx$

15._____

16. $\frac{6}{13} \approx$

16._____

17. $\frac{9}{14} \approx$

17._____

18. $\frac{15}{17} \approx$

18._____

Changing a Fraction to a Decimal-Extended

Instructions:

Read each question thoroughly. Do your work in the space provided below the question, but write your answers on the lines to the right of the question. For those questions which are divided into parts (a),(b),(c), and (d), answer only part (d) on the line to the right and do all other work in the space below the question. If a □ or a △ appear in a question, the value of the □ or △ is the same throughout the question. Do not compute this value.

1. If you know that the decimal for $\frac{4}{3} \approx 1.333$, you can say that the decimal for $2 \times \frac{4}{3} \approx$ ___?___.

1. ?= _____

2. For each part, find the decimal equivalent required. Write the answer to (d) on the line to the right. The first three answers are only meant to help you with the answer to (d).

   (a) If you know that the decimal for $\frac{4}{6} \approx .666$, you can also tell that the decimal for $\frac{4}{10 \times 6} \approx$ _____.

   (b) If you know that the decimal for $\frac{3}{5} = .600$, you can also tell that the decimal for $\frac{3}{10 \times 5} =$ _____.

   (c) If you know that the decimal for $\frac{6}{7} \approx .857$, you can also tell that the decimal for $\frac{6}{10 \times 7} \approx$ _____.

   (d) If you know that the decimal for $\frac{6}{\square} \approx .295$, you can also tell that the decimal for $\frac{6}{10 \times \square} \approx$ ___?___.

2. ?= _____

3. For each part, find the decimal equivalent required.
<u>Write the answer to (d) on the line to the right.</u>
Do all other work below.

(a) If you know that the decimal for $\frac{4}{6} \approx .666$, you can also tell that the decimal for $\frac{10 \times 4}{6} \approx$ _____.

(b) If you know that the decimal for $\frac{3}{5} = .600$, you can also tell that the decimal for $\frac{10 \times 3}{5} =$ _____.

(c) If you know that the decimal for $\frac{6}{7} \approx .857$, you can also tell that the decimal for $\frac{10 \times 6}{7} \approx$ _____.

(d) If you know that the decimal for $\frac{6}{\square} \approx .295$, you can also tell that the decimal for $\frac{10 \times 6}{\square} \approx$ ___?___.

3. ? = _____

4. If you know that the decimal for $\frac{7}{25} = .28$, you can also tell that the decimal for $\frac{7}{25 \times .2} =$ ___?___.

4. ? = _____

5. If you know that the decimal for $\frac{3}{50} = .06$, and the decimal for $\frac{9}{50} = .18$, then you can also tell that the decimal for $\frac{3+9}{50} = \frac{12}{50} =$ ___?___.

5. ? = _____

6. If you know that the decimal for $\frac{9}{15} = .600$ and the decimal for $\frac{7}{15}$ .467, then you can also tell that the decimal for $\frac{9-7}{15} = \frac{2}{15} =$ ___?___

6. ? = _____

7. If you know that the decimal for $\frac{17}{16} = 1.0625$, then you can also tell that the decimal for $\frac{1}{16}$ = ___?___.

7. ?= _____

8. If you know that the decimal for $\frac{8}{14} \approx .571$ and that the decimal for $\frac{7}{14} = .5$, then you can also tell that the decimal for $\frac{8-7}{14} = \frac{1}{14} \approx$ ___?___

8. ?= _____

9. For each part, find the decimal equivalent required.
<u>Write the answer to (d) on the line to the right.</u>
Show all other work below.

(a) If you know that the decimals for $\frac{4}{9} \approx .444$ and $\frac{2}{9} \approx .222$, then you can also tell that .666 is the approximate decimal for what fraction $\frac{\square}{\triangle}$?

(b) If you know that the decimals for $\frac{3}{16} \approx .187$ and $\frac{8}{16} = .500$, then you can also tell that .687 is the approximate decimal for what fraction $\frac{\square}{\triangle}$?

(c) If you know that the decimals for $\frac{15}{20} = .750$ and $\frac{4}{20} = .200$, then you can also tell that .950 is the approximate decimal for what fraction $\frac{\square}{\triangle}$?

(d) If you know that the decimals for $\frac{37}{\square} = .074$ and $\frac{12}{\square} = .024$, then .098 is the approximate decimal for what fraction $\frac{\triangle}{\square}$? Do not find the value of $\square$.

9. $\frac{\triangle}{\square}$ = _____

10. If you know that the decimal for $\frac{2}{3} \approx .666$, then you could also tell that $3 \times \square \approx 2$, where $\square$ is a decimal?

10. $\square \approx$ _____

11. If you know that the decimal for $\frac{2}{5} = .400$, then you could also tell that $5 \times \square = 2$, where $\square$ is a decimal?

11. $\square =$ _____

12. If the decimal for $\frac{\square}{11} \approx .45$, what whole number is $\square$?

12. $\square =$ _____

13. If the decimal for $\frac{9}{\square} \approx .81$, what whole number is $\square$?

13. $\square =$ _____

14. If the decimal for $\frac{8}{\square} \approx .88$ and the decimal for $\frac{5}{\square} \approx .55$, then .33 is the approximate decimal for what fraction $\frac{\triangle}{\square}$? Do not find the value of $\square$.

14. $\frac{\triangle}{\square} =$ _____

15. If the decimal for $\frac{2}{\square}$ = .04, what whole number is $\square$?

15._____

16. Suppose the decimals for $\frac{\square}{50}$ = .56 and for $\frac{\triangle}{50}$ =.48. If the decimal (.56-.48) = .08 is the decimal equivalent for the fraction $\frac{\nabla}{50}$, what can you say about the relationship between $\square$, $\triangle$, and ∇ .

16._____

17. If you know that the decimal for $\frac{\square}{\triangle}$ =.89, then what decimal can I multiply by $\triangle$ to get $\square$?

17._____

18. How would you use the rule for changing a fraction to a decimal to find the decimal for a mixed number, like $2\frac{1}{2}$ in two different ways. Show all of your work in the space below and write the decimal equivalent on the line to the right.

18._____

19. Find the decimal equivalent for $\frac{2}{0.3}$. Notice that the denominator is (0.3) (or 3 tenths).

19._____

20. Find the decimal equivalent for $\frac{8}{0.2}$.

20._____

21. Find the decimal equivalent for $\frac{0.7}{2}$.

21._____

22. Find the decimal equivalent for $\frac{\frac{1}{2}}{4}$.

22._____

23. Find the decimal equivalent for $\frac{\frac{1}{3}}{2}$.

23._____

24. Find the decimal equivalent for $\frac{\frac{4}{9}}{2}$.

24._____

25. Find the decimal equivalent for $\dfrac{\frac{3}{5}}{\frac{8}{10}}$.

25._____

26. Find the decimal equivalent for $\dfrac{\frac{4}{6}}{\frac{5}{3}}$.

26._____

<u>For the next four questions, write your answer in the space below the question.</u>

27. Susie has suggested another rule for finding the decimal equivalent for a fraction $\frac{a}{b}$. She multiplies each of the numerator and denominator by ten, and then she divides the denominator into the numerator. For example, to find the decimal for $\frac{3}{5}$, she divides:

$$50\overline{)30} \rightarrow 50\overline{)30.0} \rightarrow 50\overline{)300 \text{ tenths}} \quad \frac{6 \text{ tenths}}{6 \text{ tenths}} = .6$$

$$\begin{array}{r} 300 \text{ tenths} \\ \underline{300 \text{ tenths}} \\ 0 \text{ tenths} \end{array}$$

She claims that her answer is alwasy correct, and the same as the one the teacher gets by using the method taught in class. Will Susie always get the right answer? Why do you think so? Show how Susie would find the decimal for $\frac{5}{8}$ and explain why her method seems to work.

28. Johnny has suggested another method for finding the decimal equivalent for a fraction when the numerator is larger than the denominator. He subtracts the denominator from the numerator and then divides this number by the denominator. For example, to find the decimal for $\frac{6}{4}$, he writes: $6 - 4 = 2$, so I divide 2 by 4.

$$4\overline{)\,2} \rightarrow 4\overline{)\,2.0} \rightarrow 4\overline{)\,20 \text{ tenths}}$$

$$\begin{array}{r|l} \underline{20 \text{ tenths}} & \underline{5 \text{ tenths}} \\ 0 \text{ tenths} & 5 \text{ tenths} = .5. \end{array}$$

Then he <u>adds one</u> to the answer, to get $.5 + 1 = 1.5$. He claims that his method always works for problems where the numerator is larger than the denominator. Is he correct? Show how Johnny would find the decimal for $\frac{8}{5}$ and explain why his method seems to work.

29. Sam's rule for finding the decimal equivalent to a given fraction is very much like Susie's (in problem 27), but instead of multiplying numerator and denominator by 10, he multiplies each by 100 and then divides. For example, to find the decimal for $\frac{3}{5}$, he divides:

$$500\overline{)300} \rightarrow 500\overline{)300.0} \longrightarrow 500\overline{)\begin{array}{r|l} 3000 \text{ tenths} & \\ \underline{3000 \text{ tenths}} & \underline{6 \text{ tenths}} \\ 0 \text{ tenths} & 6 \text{ tenths} = .6 \end{array}}$$

He claims that his answer is always correct. Will Sam always get the correct answer? Why do you think so? Show how Sam would find the decimal for $\frac{5}{8}$ and explain why his method seems to work.

30. Judy's method for finding the decimal for a fraction is very much like Johnny's, only hers works for any fraction. She adds the denominator to the numerator and then divides this number by the denominator. For example, to find the decimal for $\frac{6}{4}$, she writes: $6+4 = 10$, so :

$$4\overline{)10} \rightarrow 4\overline{)10.0} \longrightarrow 4\overline{)\begin{array}{r|l} 100 \text{ tenths} & \\ \underline{100 \text{ tenths}} & \underline{25 \text{ tenths}} \\ 0 \text{ tenths} & 25 \text{ tenths} = 2.5 \end{array}}$$

Then she <u>subtracts one</u> from this answer, to get $2.5-1 = 1.5$.

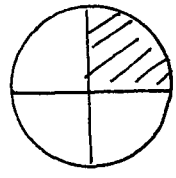
She claims that her method always works. Show how Judy would find the decimal for $\frac{5}{8}$ and explain why her method seems to work.

Comparing Fractions- Pretest

Read each question. Answer the question in the space provided on the right hand side of the page.

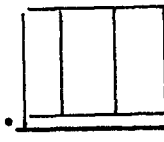
1. 6 x 3 = □	1. □ = _____
2. 2 x 9 = □	2. □ = _____
3. 8 x 4 = □	3. □ = _____
4. 6 x 8 = □	4. □ = _____
5. 7 x 5 = □	5. □ = _____
6. 9 x 7 = □	6. □ = _____
7. 6 x 7 = □	7. □ = _____
8. 8 x 9 = □	8. □ = _____
9. 6 x 6 = □	9. □ = _____
10. 8 x 8 = □	10. □ = _____

11. Give a fraction name to the shaded portion below:

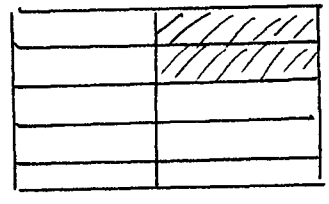


11. _____

12. Shade in $\frac{1}{3}$ of the diagram on the right.

12. 

13. Give a fraction name to the shaded portion below:



13. _____

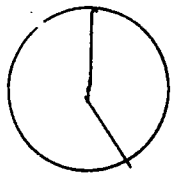
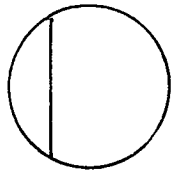
14. Draw a diagram with $\frac{2}{3}$ shaded in on the line at the right.

14._____

15. Draw a diagram with $\frac{3}{4}$ shaded in on the line at the right.

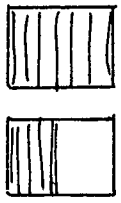
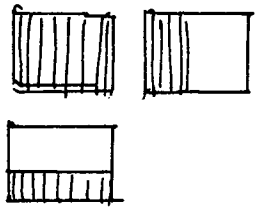
15._____

16. Is the fraction representing area (a) more than, less than, or the same as that representing area (b)?

(a)  (b) 

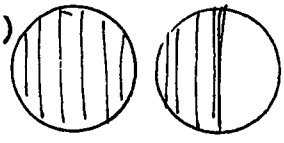
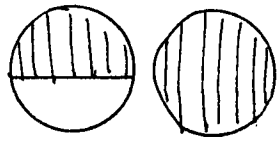
16._____

17. Is the fraction representing area (a) more than, less than, or the same as that representing area (b)?

(a)  (b) 

17._____

18. Is the fraction representing area (a) more than, less than, or the same as that representing area (b)?

(a)  (b) 

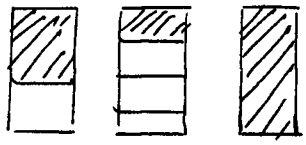
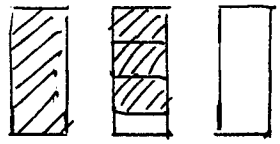
18._____

19. Is the fraction representing area (a) more than, less than, or the same as that representing area (b)?

(a) (b)

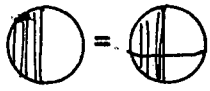
19._____

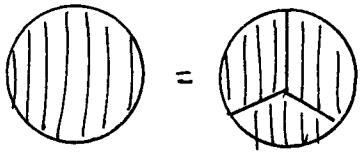
20. Is the fraction representing area (a) more than, less than, or the same as that representing area (b)?

(a)  (b) 

20._____

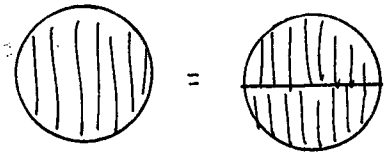
21. What statement does the diagram below suggest?

(For example,  suggests $\frac{1}{2} = \frac{2}{4}$).



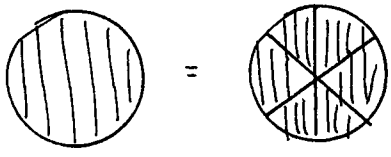
21._____

22. What statement does the diagram below suggest?



22._____

23. What statement does the diagram below suggest?



23._____

24. $33 \times 24 \times 51 = 24 \times \square \times 33$ 24. $\square =$ _____

25. $56 \times 19 \times 48 = 19 \times \square \times 48$ 25. $\square =$ _____

26. $27 \times 56 \times 65 \times 41 = 27 \times \square \times 56 \times 41$ 26. $\square =$ _____

27. $\square \times \frac{1}{5} = 1$ 27. $\square =$ _____

28. $\square \times \frac{1}{8} = 1$ 28. $\square =$ _____

29. $6 \times \frac{\square}{\triangle} = 1$ 29. $\frac{\square}{\triangle} =$ _____

30. $\frac{3}{2} = 3 \times \frac{\square}{\triangle}$ 30. $\frac{\square}{\triangle} =$ _____

31. $\frac{8}{9} = \square \times \frac{1}{9}$ 31. $\square =$ _____

32. $\frac{16}{5} = \square \times \frac{1}{5}$ 32. $\square =$ _____

33. Which is larger, 44×13 or 52×13? 33._____

34. Which is larger, 69×158 or 32×158? 34._____

35. Which is larger, $15 \times \frac{1}{4}$ or $16 \times \frac{1}{4}$? 35._____

Comparing Fractions

Read each question. Do your work in the space provided below the question. Then write the greater of the two fractions or whole numbers on the line to the right. There are 27 questions.

1. $\frac{18}{5}$ or 3

1._____

2. $\frac{25}{4}$ or 2

2._____

3. $\frac{29}{10}$ or 2

3._____

4. $\frac{33}{7}$ or 5

4._____

5. $\frac{28}{8}$ or 4

5._____

6. $\frac{27}{5}$ or 5

6._____

7. $\frac{45}{7}$ or 7

7._____

8. $\frac{68}{9}$ or 8

8._____

9. $\frac{52}{5}$ or 10

9._____

10. $\frac{3}{4}$ or $\frac{5}{6}$

10._____

11. $\frac{1}{2}$ or $\frac{5}{9}$

11._____

12. $\frac{4}{8}$ or $\frac{2}{3}$

12._____

13. $\frac{6}{9}$ or $\frac{3}{6}$

13._____

14. $\frac{4}{6}$ or $\frac{6}{7}$

14._____

15. $\frac{5}{9}$ or $\frac{4}{6}$

15._____

16. $\frac{7}{8}$ or $\frac{8}{9}$

16._____

17. $\frac{9}{10}$ or $\frac{7}{9}$

17._____

18. $\frac{6}{7}$ or $\frac{6}{8}$

18._____

19. $\frac{7}{4}$ or $\frac{4}{2}$

19._____

20. $\frac{6}{2}$ or $\frac{9}{2}$

20._____

21. $\frac{3}{2}$ or $\frac{4}{3}$

21._____

22. $\frac{10}{3}$ or $\frac{7}{4}$

22._____

23. $\frac{5}{3}$ or $\frac{8}{7}$

23._____

24. $\frac{4}{3}$ or $\frac{10}{9}$

24._____

25. $\frac{7}{5}$ or $\frac{9}{8}$

25._____

26. $\frac{8}{6}$ or $\frac{10}{7}$

26._____

27. $\frac{8}{6}$ or $\frac{9}{5}$

27._____

Comparison of Fractions-Extended

Instructions:

Read each question thoroughly. Do your work in the space provided below the question, but write your answers on the lines to the right of the questions. For those questions which are divided into parts (a),(b),(c), and (d), answer only part (d) on the line to the right and do all other work in the space below the question. $\square$ is the same number in both fractions. Do not find a value for $\square$.

1. Which is greater: $\frac{\square}{37}$ or $\frac{\square}{38}$?

1._____

2. Which is greater: $\frac{87}{\square}$ or $\frac{88}{\square}$?

2._____

3. For each, find which is greater. Write the answer to (d) on the line to the right. The first three answers are only meant to help you with the answer to (d).

   (a) $\frac{2}{3}$ or $\frac{1}{2}$

   (b) $\frac{3}{4}$ or $\frac{4}{5}$

   (c) $\frac{7}{8}$ or $\frac{6}{7}$

   (d) $\frac{\square}{\square+1}$ or $\frac{\square-1}{\square}$

3._____

4. For each, find which is greater. <u>Write the answer to (d) on the line to the right.</u> The first three answers are only meant to help you with the answer to (d).

(a) $\frac{2}{3}$ or $\frac{2}{5}$

(b) $\frac{3}{17}$ or $\frac{3}{28}$

(c) $\frac{13}{53}$ or $\frac{13}{78}$

(d) $\frac{\square}{834}$ or $\frac{\square}{729}$

4._____

5. For each, find which is greater. <u>Write the answer to (d) on the line to the right.</u> The first three answers are only meant to help you with the answer to (d).

(a) $\frac{1}{14}$ or $\frac{2}{14}$

(b) $\frac{21}{4}$ or $\frac{33}{4}$

(c) $\frac{40}{58}$ or $\frac{33}{58}$

(d) $\frac{6}{\square}$ or $\frac{3}{\square}$

5._____

6. For each, find which is greater. <u>Write the answer to (d) on the line to the right.</u> The first three answers are only meant to help you with the answer to (d).

(a) $\frac{33-1}{33}$ or $\frac{33}{33+1}$

(b) $\frac{33}{33+9}$ or $\frac{33-9}{33}$

(c) $\frac{33-18}{33}$ or $\frac{33}{33+18}$

(d) $\frac{33}{33+\square}$ or $\frac{33-\square}{33}$

6._____

7. If $\frac{384}{529} > \frac{384}{\square}$, what can you say about $\square$?

7._____

8. If $483 \times 25 > 365 \times 18$, which is greater:

$\frac{2 \times 483}{365}$ or $\frac{2 \times 18}{25}$

8._____

9. If $6 \times 18 > 13 \times 8$, which is greater:

$\frac{6}{13}$ Or $\frac{8}{18}$

9._____

10. If $17 \times 11 < 14 \times 15$, which is greater:

$\frac{14}{17}$ or $\frac{11}{15}$

10._____

11. If $82 \times 15 > 63 \times \square$, which is greater:

$\frac{82}{\square}$ or $\frac{63}{15}$

11._____

12. If $8 \times (5+\square) > 6 \times 3$, which is greater:

$\frac{5+\square}{6}$ or $\frac{3}{8}$

12._____

13. If $425 \times 211 < 310 \times 316$, which is greater:

$$\frac{425 - 310}{310} \quad \text{or} \quad \frac{316 - 211}{211}$$

13._____

14. If $6 \times (3 - \square) > 2 \times 5$, which is greater:

$$\frac{6}{5} \quad \text{or} \quad \frac{2}{3-}$$

14._____

15. Use the rule for comparing fractions to compare two whole numbers, 5 and 7. Show all of your steps in the space below and write the greater of the two numbers on the line to the right. (Hint: Write them as fractions first.)

15._____

16. Use the rule for comparing fractions to compare the two mixed numbers, $2\frac{1}{3}$ and $2\frac{2}{5}$. Show all of your work in the space below and write the greater of the two numbers on the line to the right.

16._____

17. Use the rule for comparing fractions to compare $2\frac{1}{5}$ to $20/9$. Write the larger on the line to the right.

17._____

18. Use the rule for comparing fractions to compare $3\frac{2}{3}$ to $\frac{31}{9}$. Write the larger on the line to the right.

18._____

19. If $22 \times \square = 83 \times \triangle$, is $\frac{22}{83}$ more than, less than, or equal to $\frac{\square}{\triangle}$?

19._____

20. You know that $\frac{22}{\square} = \frac{83}{\triangle}$ if $22 \times \triangle =$ _____?

20._____

21. You know that $\frac{\square}{\triangle} = \frac{\sqrt{\ }}{\hexagon}$ if _____ = _____?

21._____

22. Arrange these three fractions from largest to smallest.

$\frac{3}{5}$ $\frac{19}{30}$ $\frac{7}{12}$

22._____
Largest, , Sm.

23. Arrange these three fractions from largest to smallest:

$\frac{4}{9}$ $\frac{2}{3}$ $\frac{6}{11}$

23._____
Largest, ,Sm.

24. Arrange these three fractions from largest to smallest:

$\frac{8}{3}$ $\frac{25}{10}$ $\frac{25}{9}$

24._____
Largest, ,Sm

25. Find three fractions less than $\frac{2}{5}$.

25._____

26. Arrange these three fractions from largest to smallest:

$\frac{3}{13}$ $\frac{6}{25}$ $\frac{14}{52}$

26._____
Largest, ,Sm.

<u>For the next four questions, write the answer in the space below the question.</u>

27. Susie has suggested another rule for comparing fractions. To decide for example, if $\frac{2}{3} > \frac{4}{9}$, she compares $\frac{2}{4}$ to $\frac{3}{9}$. If $\frac{2}{4} > \frac{3}{9}$, then $\frac{2}{3} > \frac{4}{9}$. She switches the first denominator with the second numerator and then compares these to decide if the original first fraction is greater than the original second fraction. She claims that her answer is always the same as the one which the teacher gets by her method. Will Susie always get the right answer? Why do you think she should? Show how Susie would compare $\frac{7}{11}$ and $\frac{5}{8}$. Explain her method and why it works.

28. Johnny has suggested a method for comparing fractions when the numerator of the second goes into the numerator of the first exactly and the denominator of the second goes into the denominator of the first exactly. For example, his method works for comparing $\frac{15}{16}$ and $\frac{3}{4}$ since 3 goes into 15 exactly and so does 4 into 16. Johnny decided if the first numerator divided by the second is greater than the first denominator divided by the second, then the first fraction is greater than the second. For example, $15 \div 3 = 5$ and $16 \div 4 = 4$ and $5 > 4$, so $\frac{15}{16} > \frac{3}{4}$. Does Johnny's method always work for these kinds of problems. Explain why, using the problem of comparing $\frac{24}{25}$ and $\frac{4}{5}$ as an example.

29. Sam says that $\frac{a}{b} > \frac{c}{d}$ whenever $\frac{1}{a \times d} < \frac{1}{b \times c}$ and only then, so to compare, for example, $\frac{2}{3}$ and $\frac{4}{9}$, he discovers that $\frac{1}{2 \times 9} < \frac{1}{3 \times 4}$, so he concludes that the first fraction is greater than the second. Does Sam 's method always work for comparing fractions? Explain what Sam would do and why he is correct by comparing $\frac{7}{11}$ and $\frac{5}{8}$ by his method.

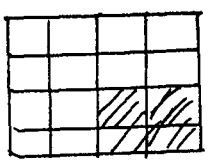
30. Judy says that she can tell if $\frac{a}{b} > \frac{c}{d}$ right away. If $a > (b \times c) \div d$, then the first fraction is larger. For example, to compare $\frac{2}{3}$ and $\frac{4}{9}$, she says $2 > (4 \times 3) \div 9$, since $(4 \times 3) \div 9$ is $12 \div 9 = 1\frac{3}{9}$. Therefore, the first fraction is greater. Does Judy's method alwasy work for comparing fractions? Explain what Judy would do and why she is correct by comparing $\frac{7}{11}$ and $\frac{5}{8}$ by her method.

Finding the Square Root of a Fraction-Pretest

Read each question. Answer the question in the space provided on the right hand side of the page.

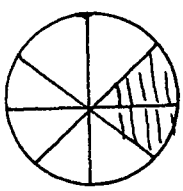
1. $8 \times 7 = \square$	1. $\square =$ _____
2. $4 \times 9 = \square$	2. $\square =$ _____
3. $6 \times 5 = \square$	3. $\square =$ _____
4. $9 \times 2 = \square$	4. $\square =$ _____
5. $7 \times 5 = \square$	5. $\square =$ _____
6. $6 \times 8 = \square$	6. $\square =$ _____
7. $4 \times 5 = \square$	7. $\square =$ _____
8. $9 \times 8 = \square$	8. $\square =$ _____
9. $6 \times 7 = \square$	9. $\square =$ _____
10. $7 \times 7 = \square$	10. $\square =$ _____

11. Give a fraction name to the shaded portion below:



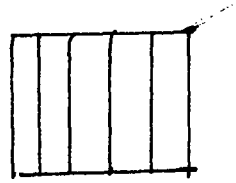
11. _____

12. Give a fraction name to the shaded portion below:



12. _____

13. Shade in $\frac{1}{5}$ of the diagram on the right.



13. _____

14. Draw a diagram with $\frac{3}{4}$ shaded in on the line to the right.

14._____

15. Draw a diagram with $\frac{2}{6}$ shaded in on the line to the right.

15._____

16. Write the following multiplication statement as a division statement:

$$532 \times 18 = 9576$$

16._____

17. Write the following division statement as a multiplication statement:

$$492 \div 123 = 4$$

17._____

18. Write the following division statement as a multiplication statement:

$$\square \div 6 = \triangle$$

18._____

19. Write the following multiplication statement as a division statement:

$$\square \times 6 = \triangle$$

19._____

20. Write the following multiplication statement as a division statement:

$$\square \times \triangle = \nabla$$

20. _____

21. $\square \times \frac{1}{9} = 1$ — 21. $\square =$ _____

22. $8 \times \frac{\square}{\triangle} = 1$ — 22. $\frac{\square}{\triangle} =$ _____

23. $\frac{14}{9} = 14 \times \frac{\square}{\triangle}$ — 23. $\frac{\square}{\triangle} =$ _____

24. $\frac{3}{7} = \square \times \frac{1}{7}$ — 24. $\square =$ _____

25. $\frac{38}{6} = \square \times \frac{1}{6}$ — 25. $\square =$ _____

26. $74 \times 83 \times 77 = \square \times 74 \times 77$ — 26. $\square =$ _____

27. $312 \times 25 \times \square \times 87 = 43 \times \triangle \times 25 \times 312$ — 27. $\square =$; $\triangle =$ _____

28. $45 \times 203 \times 87 = \square \times 203 \times 87$ — 28. $\square =$ _____

29. $1 \times 576 = \square$ — 29. $\square =$ _____

30. $\square \times (374+596) = 374+596$ — 30. $\square =$ _____

31. What is the length of a rectangle with a width of 2 ft. and an area of 6 x 2 sq. ft.?

31. _____

32. What is the width of a rectangle with a length of 8 ft. and an area of 8 x 3 sq. ft.?

32. _____

33. What is the area of a rectangle with a width of 3 ft. and a length of 5 ft.?

33. _____

34. What operation would you use to find the area of a rectangle with length 89 ft. and width 38 ft.? (Would you add, subtract, multiply or divide?)

34._____

35. Using the idea that to find area, you find the number of one-unit squares in a figure, show how you would find the area of the figure below:

35._____

36. Divide:

$11\overline{)425}$

36._____

37. Divide:

$20\overline{)5834}$

37._____

38. **Divide:**

$$7\overline{)2859}$$

38._____

39. **Divide:**

$$25\overline{)4632}$$

39._____

40. **Divide:**

$$6\overline{)384}$$

40._____

Finding the Square Root of a Fraction

Read each question. Do your work in the space provided below the question. Then write the square root of the fraction on the line to the right of the page. There are 15 questions.

1. $\sqrt{\frac{1089}{4}}$

1._____

2. $\sqrt{\frac{2764}{9}}$

2._____

3. $\sqrt{\frac{961}{16}}$

3._____

4. $\sqrt{\frac{1849}{25}}$

4._____

5. $\sqrt{\frac{3025}{16}}$

5._____

6. $\sqrt{\frac{1521}{9}}$

6._____

7. $\sqrt{\frac{676}{16}}$

7._____

8. $\sqrt{\frac{361}{25}}$

8._____

9. $\sqrt{\frac{1369}{9}}$

9._____

10. $\sqrt{\frac{10}{4}}$

10._____

11. $\sqrt{\frac{38}{9}}$

11._____

12. $\sqrt{\frac{37}{16}}$

12._____

13. $\sqrt{\frac{22}{25}}$

13._____

14. $\sqrt{\frac{46}{16}}$

14._____

15. $\sqrt{\frac{93}{25}}$

15._____

Finding the Square Root of a Fraction-Extended

Read each question thoroughly. Do your work in the space provided below the question, but write your answers on the lines to the right of the questions. For those questions which are divided into parts (a),(b),(c), and (d), answer only part (d) on the line to the right and do all other work in the space below the question. If a $\square$ or $\triangle$ appear in a question, the value of the $\square$ or $\triangle$ is the same throughout the question.

1. If you know that $\sqrt{\frac{169}{225}} = \frac{13}{15}$, then you can also tell that $\sqrt{\frac{4 \times 169}{225}} = \frac{\square}{\triangle}$. What is $\frac{\square}{\triangle}$?

1. $\frac{\square}{\triangle} =$ _____

2. If you know that $\sqrt{\frac{256}{25}} = \frac{16}{5}$, then you can also tell that $\sqrt{\frac{256}{25 \times 9}} = \frac{\square}{\triangle}$. What is $\frac{\square}{\triangle}$?

2. $\frac{\square}{\triangle} =$ _____

3. If you know that $\sqrt{\frac{9}{16}} = \frac{3}{4}$ and $\sqrt{\frac{25}{49}} = \frac{5}{7}$, you can also tell that $\sqrt{\frac{9 \times 25}{16 \times 49}} = \frac{\square}{\triangle}$. What is $\frac{\square}{\triangle}$?

3. $\frac{\square}{\triangle} =$ _____

4. For each part, find the square root required. <u>Write the answer to (d) on the line to the right</u>. The first three questions are only meant to help you with the answer to (d).

(a) If you know that $\sqrt{\frac{9}{16}} = \frac{3}{4}$, you can also tell that $\sqrt{\frac{16}{9}} = \frac{\square}{\triangle}$.

(b) If you know that $\sqrt{\frac{16}{25}} = \frac{4}{5}$, you can also tell that $\sqrt{\frac{25}{16}} = \frac{\square}{\triangle}$.

(c) If you know that $\sqrt{\frac{81}{100}} = \frac{9}{10}$, you can also tell that $\sqrt{\frac{100}{81}} = \frac{\square}{\triangle}$.

(d) If you know that $\sqrt{\frac{x}{y}} = \frac{8}{5}$, you can also tell that $\sqrt{\frac{y}{x}} = \frac{\triangle}{\square}$.

4. $\frac{\triangle}{\square} =$ _____

5. If you know that $\sqrt{\frac{8}{9}} \approx \frac{2\frac{3}{4}}{3}$, you can also tell that $\sqrt{\frac{4 \times 8}{9}} \approx \frac{\square}{\triangle}$. What is $\frac{\square}{\triangle}$?

5. $\frac{\square}{\triangle} =$ _____

6. If you know that $\sqrt{\frac{15}{25}} \approx \frac{3\frac{7}{8}}{5}$, you can also tell that $\sqrt{\frac{15}{25 \times 100}} \approx \frac{\square}{\triangle}$. What is $\frac{\square}{\triangle}$?

6. $\frac{\square}{\triangle} =$ _____

7. If you know that $\sqrt{\frac{8}{9}} \approx \frac{2\frac{3}{4}}{3}$, you can also tell that $\sqrt{\frac{9 \times 8}{9}} \approx \frac{\square}{\triangle}$. What is $\frac{\square}{\triangle}$?

7. $\frac{\square}{\triangle} \approx$ _____

8. If you know that $\sqrt{\frac{16}{25}} = \frac{4}{5}$ and $\sqrt{\frac{49}{64}} = \frac{7}{8}$, you can also tell that $\sqrt{\frac{16 \times 64}{25 \times 49}} = \frac{\square}{\triangle}$. What is $\frac{\square}{\triangle}$?

8. $\frac{\square}{\triangle}$ = _____

9. For each part, find the square root required. <u>Write the answer to (d) only on the line to the right</u>.

(a) $\sqrt{\frac{1}{9}} = \frac{1}{\square} \times \sqrt{9}$

(b) $\sqrt{\frac{1}{25}} = \frac{1}{\square} \times \sqrt{25}$

(c) $\sqrt{\frac{1}{100}} = \frac{1}{\square} \times \sqrt{100}$

(d) $\sqrt{\frac{1}{\triangle}} = \frac{1}{1000} \times \sqrt{1000}$

9. $\triangle$ = _____

10. $\sqrt{\frac{\square}{81}} = \frac{5}{9}$.

10. $\square$ = _____

11. $\sqrt{\frac{49}{\square}} = \frac{7}{8}$

11. $\square$ = _____

12. $\sqrt{\frac{\square}{4} \times \frac{2}{9}} = \frac{2}{6}$

12. $\square$ = _____

13. How do a and b compare if $\sqrt{\frac{a}{b}}$ is greater than 1?

13._____

14. If you know that $\sqrt{\frac{8}{9}} \approx \frac{2\frac{3}{4}}{3}$ and $\sqrt{\frac{11}{25}} \approx \frac{\frac{\square}{\triangle}}{5}$

then you can also tell that

$$\left(\frac{2\frac{3}{4}}{3} \times \frac{\frac{\square}{\triangle}}{5}\right) \times \left(\frac{2\frac{3}{4}}{3} \times \frac{\frac{\square}{\triangle}}{5}\right) = \frac{\square}{\triangle}.$$

What is $\frac{\square}{\triangle}$?

14. $\frac{\square}{\triangle}$ = _____

15. If you know that $\sqrt{\frac{\square \times 6}{9}} = \frac{24}{3}$, what is $\square$?

15._____

16. Find values for $\square$ and $\triangle$ so that $\sqrt{\frac{8 \times \square}{\triangle}} = \frac{12}{5}$.

16._____

17. If you know that $\sqrt{\frac{\triangle}{49}} = \frac{161}{7}$ and $\sqrt{\frac{81}{\square}} = \frac{9}{87}$, then you can also tell that

$$\frac{161}{7} \times \frac{9}{87} \times \frac{161}{7} \times \frac{9}{87} = \frac{\bigcirc}{\triangle}.$$

Do not find values for $\square$ and $\triangle$. Express $\frac{\bigcirc}{\triangle}$ in terms of $\square$ and $\triangle$.

17._____

18. Find a fraction $\approx \sqrt{\frac{4}{5}}$.

18._____

19. Find a fraction $\approx \sqrt{\frac{5}{8}}$.

19._____

20. Using $\sqrt{\frac{18}{1}}$ as an example, show how you would use the rule for finding the square root of fractions to find square roots of wholes.

20._____

21. How would you use the rule for finding the square roots of fractions to find the square root of a mixed number, like $1\frac{9}{16}$. Show all of your steps and write the actual square root on the line to the right.

21._____

22. Find $\sqrt{\dfrac{\frac{16}{25}}{49}}$

22._____

23. Find $\sqrt{\dfrac{\frac{4}{25}}{64}}$

23._____

24. Find $\sqrt{\dfrac{\frac{8}{16}}{2}}$

24._____

25. Find $\sqrt{\dfrac{\frac{81}{25}}{49}}$

25._____

26. Find $\sqrt{\dfrac{\frac{16}{100}}{25}}$

26._____

<u>For the next four questions, write the answer in the space below the question.</u>

27. Susie has suggested another method for finding square roots of fractions. She says that if she wants to find $\sqrt{\frac{a}{b}}$, she finds $\sqrt{\frac{4 \times a}{b}}$ and then multiplies the answer by $\frac{1}{2}$.

For example, to find $\sqrt{\frac{16}{25}}$, she says that $4 \times 16 = 64$; then, $\sqrt{\frac{64}{25}} = \frac{8}{5}$ and $\frac{1}{2} \times \frac{8}{5} = \frac{8}{10}$.

She claims that her answer is always correct and equivalent to the one the teacher gets by using the method taught in class. Will Susie always get an answer equivalent to the teacher's if she follows her directions correctly? Show how Susie would find $\sqrt{\frac{36}{100}}$ using her method and explain why it seems to work.

28. Johnny says that another way to find $\sqrt{\frac{a}{b}}$ is to find $\sqrt{a \times b}$ and then find $\frac{a}{\sqrt{a \times b}}$. For example, to find $\sqrt{\frac{16}{25}}$, he says $16 \times 25 = 400$ and $\sqrt{400} = 20$. Therefore, $\sqrt{\frac{16}{25}} = \frac{16}{20}$.

He claims that he always gets an answer equivalent to the teacher's using the method taught in class. Is he correct? Show how Johnny would find $\sqrt{\frac{36}{100}}$ and explain why his method seems to work.

29. Sam says that he has still another way to find $\sqrt{\frac{a}{b}}$. He says $\sqrt{\frac{a}{b}} = \frac{\sqrt{a \times b}}{b}$. For example, to find $\sqrt{\frac{16}{25}}$, he says $16 \times 25 = 400$ and $\sqrt{400} = 20$. Therefore, $\sqrt{\frac{16}{25}} = \frac{20}{25}$.

He claims that his method is correct. Show how Sam would find $\sqrt{\frac{36}{100}}$ and explain why his method seems to work.

30. To find $\sqrt{\frac{a}{b}}$, Judy finds $\sqrt{\frac{a}{4 \times b}}$ and then multiplies her answer by 2. For example, to find $\sqrt{\frac{16}{25}}$, she says $25 \times 4 = 100$ and then $\sqrt{\frac{16}{100}} = \frac{4}{10}$ and $2 \times \frac{4}{10} = \frac{8}{10}$. So, $\sqrt{\frac{16}{25}} = \frac{8}{10}$.

She claims that her method is correct. Show how Judy would find $\sqrt{\frac{36}{100}}$ and explain why her method seems to work.

APPENDIX C

STUDENT WORKSHEETS

Worksheet 1

For each question, show all of your work for the problem beneath it.

1. $4 \times \frac{1}{5} =$

2. $3 \times \frac{1}{7} =$

3. $2 \times \frac{1}{5} =$

4. $3 \times \frac{4}{5} =$

5. $6 \times \frac{2}{5} =$

6. $8 \times \frac{3}{5} =$

7. $2 \times 3\frac{3}{8} =$

8. $3 \times 2\frac{1}{4} =$

9. $4 \times 3\frac{2}{13} =$

Worksheet 1

Write each of these in decimal form:

1. $56\frac{2}{10}$ 1._____
2. $4\frac{3}{100}$ 2._____
3. $5\frac{16}{100}$ 3._____
4. $\frac{35}{10}$ 4._____
5. $\frac{14}{10}$ 5._____
6. $\frac{285}{10}$ 6._____
7. $\frac{365}{100}$ 7._____
8. $1\frac{9}{100}$ 8._____

Write each of these as both a mixed number or whole number and as an improper fraction.

9. 4.25 9._____
10. 5.8 10._____
11. 16.14 11._____
12. 4.3 12._____
13. 218.1 13._____
14. 6.0 14._____
15. 15.1 15._____

Worksheet 2

Find each answer by dividing. Show all work next to the question. Put the answer on the line at the right.

1. $4.9 \div 7$

1._____

2. $.49 \div 7$

2._____

3. $3.6 \div 6$

3._____

4. $3.30 \div 11$

4._____

5. $.25 \div 25$

5._____

6. $3.00 \div 50$

6._____

Worksheet 3

Express each fraction below as the answer to a division question. Show all of your work, and explain why your answer is correct either by diagramming or renaming.

1. $\frac{3}{7} =$

2. $\frac{2}{9} =$

3. $\frac{15}{9} =$

4. $\frac{17}{5} =$

5. $\frac{36}{6} =$

6. $\frac{49}{8} =$

7. $\frac{15}{4} =$

8. $\frac{16}{3} =$

Worksheet 4

Convert each fraction to its decimal equivalent. Show all of your work below the question and put the decimal answer on the line at the right.

1. $\frac{3}{10}$

1._____

2. $\frac{2}{4}$

2._____

3. $\frac{6}{10}$

3._____

4. $\frac{4}{5}$

4._____

5. $\frac{16}{50}$

5._____

Worksheet 5

Convert each of these fractions to a division question and show why this was correct.

1. $\frac{8}{9}$

2. $\frac{13}{5}$

3. $\frac{16}{4}$

4. $\frac{36}{8}$

Worksheet 1

Show all your work for each question in the space below the question. For each, write a whole number statement that you might be led to in the comparison.

Example: For comparing $\frac{6}{3}$ and 3, I would compare 6 and 3 x 3, so since $6 < 3 \times 3$, then $\frac{6}{3} < 3$.

Compare:

1. $\frac{18}{4}$ and 4.

2. $\frac{13}{5}$ and 3.

3. $\frac{21}{4}$ and 5.

4. $\frac{36}{5}$ and 8.

Worksheet 1

Use division to find the following square roots. Show your work below the question.

1. $\sqrt{676}$ =

2. $\sqrt{8100}$ =

3. $\sqrt{2025}$ =

4. $\sqrt{169}$ =

Worksheet 2

Find the following square roots. Show all work.

1. $\sqrt{\frac{81}{4}} =$

2. $\sqrt{\frac{1089}{9}} =$

3. $\sqrt{\frac{841}{16}} =$

4. $\sqrt{\frac{1156}{25}} =$

5. $\sqrt{\frac{676}{36}} =$

Worksheet 3

Find the square roots of each by dividing. Do your work by the question.

1. $\sqrt{389}$ =

2. $\sqrt{576}$ =

3. $\sqrt{1024}$ =

4. $\sqrt{961}$ =