POLYHEDRAL STUDIES ON SCHEDULING AND ROUTING PROBLEMS

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Abstract

During the last decade, there have been major advances in solving a class of large-scale real world combinatorial optimization problems. Such problems are formulated as *Travelling Salesman Problems* (TSP), some involving up to thousands of cities. These achievements, mainly due to the use of so called *polyhedral techniques*, have established the importance of the polyhedral study for various combinatorial optimization problems.

This thesis studies polyhedral structures of two well known combinatorial problems: (i) *precedence constrained single machine scheduling* and (ii) TSP, both *Symmetric TSP* (STSP) and *Asymmetric TSP* (ATSP). These problems are of both theoretical interest and practical importance. Better knowledge of the polyhedral descriptions of these problems may facilitate the polyhedral study of more complex scheduling and routing problems.

For the scheduling problem, we present two classes of *facetial inequalities*, which suffice to describe the linear system of the scheduling problem when the precedence constraints are series-parallel. We also propose a cutting plane procedure based on these *facet cuts*. The computational results show the procedure yields feasible schedules with relative deviations from the optimum less than 0.25% on the average and less than 1% in the empirical worst case.

For TSPs, we explore a Hamiltonian path approach to the polyhedral study. We propose various facet extension techniques for deriving large classes of facets from known facets. In the STSP case, we propose new clique lifting results. In the ATSP case, we develop a *Tree Composition* method, which generates all *non-spanning clique tree* facetial inequalities.

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Chapter 1

Introduction

Over the past decade, we have witnessed major advances in solving a class of large-scale real world combinatorial optimization problems. Such problems are formulated as *Travelling Salesman Problems* (TSP), some involving up to thousands of cities, see Padberg and Rinaldi [49], and Kolata [31]. These achievements are largely due to the use of so called *polyhedral techniques*, which employ polyhedral descriptions of the underlying problems. The successes have motivated in recent years extensive studies on polyhedral structures and computations for various combinatorial optimization problems.

The present thesis studies polyhedral structures of two well known combinatorial problems, namely, *precedence constrained single machine scheduling* in **Chapter 2**, and TSP, both *Symmetric TSP* (STSP) and *Asymmetric TSP* (ATSP) in **Chapters 4** and **5**, respectively. These problems are of both practical importance and theoretical interest, because (a) they are very simple, yet fundamental, versions of machine scheduling and vehicle routing problems, which often arise in production and distribution management; (b) they are *strongly NP-hard*, see Garey and Johnson [22]. Better knowledge of the polyhedral descriptions of these problems may thence facilitate the polyhedral study of more complex scheduling and routing problems.

To test the facetial results derived in **Chapter 2**, we perform polyhedral computations for the precedence constrained single machine scheduling problem. **Chapter 3** introduces a cutting plane procedure which uses effective cuts (i.e., *facet-defining inequalities*) derived in **Chapter 2**. Computational results show that the solution procedure

consistently yields feasible solutions with a guaranteed high accuracy over a broad range of instances. The procedure generates feasible schedules with percent deviation from the optimum less than 0.25% on the average and less than 1% in the empirical worst case.

The rest of the chapter is organized as follow. Section 1.1 briefly reviews terminology and notation used for the polyhedral methods. Sections 1.2 and Section 1.3 summarize the motivations and main results in this thesis for the scheduling problem and Travelling Salesman Problems, respectively.

1.1 Preliminaries

In this section, we briefly review the terminology and notation which are necessary for our exposition. For a more detailed treatment of the theory of combinatorial polyhedra, we refer to Grötschel and Padberg [28].

Let R denote the real line and R^n the n-dimensional real vector space. For $x^1, \cdots, x^k \in R^n$ and $\lambda_1, \cdots, \lambda_k \in R$, the vector $x \in R^n$ with $x = \lambda_1 x^1 + \cdots + \lambda_k x^k$ is called a *linear combination* of the vectors $x^1, \cdots, x^k$. If in addition, $\lambda_1 + \cdots + \lambda_k = 1$ (resp., all λ_i are nonnegative and $\lambda_1 + \cdots + \lambda_k = 1$), then x is called an *affine* (resp., *convex*) combination of the vectors $x^1, \cdots, x^k$.

If $\emptyset \neq S \subseteq R^n$, then the set of all convex combinations of finitely many vectors in S is called the *convex hull* of S and is denoted by $\text{conv}(S)$.

A nonempty set $S \subseteq R^n$ is called *linearly* (resp., *affinely) independent*, if for every finite set $\{x^1, x^2, \cdots, x^k\} \subseteq S$, the equations $\lambda_1 x^1 + \cdots + \lambda_k x^k = 0$ (resp., $\lambda_1 x^1 + \cdots + \lambda_k x^k = 0$ and $\lambda_1 + \cdots + \lambda_k = 0$) imply $\lambda_i = 0$, $i = 1, \cdots, k$; otherwise S is called *linearly* (resp., *affinely) dependent.*

The *rank* (resp., *affine rank*) of a set $S \in R^n$ is the cardinality of the largest linearly (resp., affinely) independent subset of S, and the *dimension* of S, denoted by $\dim(S)$, is

the affine rank of S minus one. A set $S \subseteq R^n$ is called *full-dimensional* if $\dim(S) = n$. The notion of affine rank or independence is useful in the theory of polyhedra because it is invariant under translations of the origin.

To keep notation to a minimum, we denote by ax the scalar product of any vectors $a, x \in R^n$. A set $H \in R^n$ is said to be a *halfspace* if there exist a vector $a \in R^n$ and a scalar $a_0 \in R$ such that $H = \{x \in R^n : ax \leq a_0\}$. Then H is the halfspace defined by the inequality $ax \leq a_0$. An inequality $ax \leq a_0$ is called *valid* with respect to a set $S \subseteq R^n$ if S is contained in the halfspace defined by $ax \leq a_0$. A valid inequality $ax \leq a_0$ for S is called *tight*, or *supporting*, if $ax = a_0$ for some $x \in S$. An inequality $ax \leq a_0$ for S is called an *implicit equation* if $ax = a_0$ holds for all $x \in S$.

A *polyhedron* is the intersection of finitely many halfspaces. A bounded polyhedron is called a *polytope*. For any polyhedron $\mathcal{P} \subseteq R^n$, there exist an (m, n)-matrix A and a vector $b \in R^m$ such that $\mathcal{P}$ can be represented in the form

$$\mathcal{P} = \{x \in R^n : Ax \leq b\}.$$

A subset F of a polyhedron $\mathcal{P}$ is called a *face* of $\mathcal{P}$ if there exists a supporting inequality $ax \leq a_0$ with respect to $\mathcal{P}$ such that $F = \{x \in \mathcal{P} : ax = a_0\}$. We say that the inequality $ax \leq a_0$ *defines* F or is *face-defining* for $\mathcal{P}$. A *facet* F of $\mathcal{P}$ is any face of $\mathcal{P}$ with $\dim F = \dim \mathcal{P} - 1$. A valid inequality $ax \leq a_0$ is *facet-defining*, or *facet-inducing*, if the induced face $F = \{x \in \mathcal{P} : ax = a_0\}$ contains exactly $\dim \mathcal{P}$ affinely independent points.

A valid inequality $fx \leq f_0$ for $\mathcal{P}$ is said to *dominate* $cx \leq c_0$ if any $x \in \mathcal{P}$ satisfying $cx = c_0$ also satisfies $fx = f_0$. Any two valid inequalities $fx \leq f_0$ and $cx \leq c_0$ are *equivalent* w.r.t. $\mathcal{P}$ if they dominate each other. It is well-known (e.g., [28]) that two valid inequalities for $\mathcal{P}$ are equivalent iff one can be obtained by multiplying the other with a positive scalar and adding a linear combination of the implicit equations for $\mathcal{P}$.

The *polyhedral technique* is a Linear Programming (LP) approach to combinatorial optimization problems. The feasible solution set to any combinatorial problem can be represented as a collection of points in an appropriate space R^n. To apply the polyhedral method, we first need to study the *polyhedral structure*, i.e., a linear description, of the convex hull $\mathcal{P}$ of the feasible solution set. We then devise cutting plane procedures using known valid inequalities which define faces (of high dimension, preferably facets) of $\mathcal{P}$ as cutting planes. The procedures often yield very tight lower (resp., upper) bounds if it is a minimization (resp., maximization) problem. Such bounds can then be used in subsequent *Branch and Bound* or *Branch and Cut* procedures if an exact optimum is sought.

1.2 Precedence constrained single machine scheduling

We consider the following *single machine nonpreemptive scheduling problem.* Each of n jobs, available for processing at time zero, is to be processed without interruption on a single machine. Associated with each job j is its processing time p_j and its linear deferral (delay) cost rate w_j [53], both of which are independent of job sequencing. The machine can process only one job at a time. Precedence constraints may be assumed between jobs. A *schedule* is specified by the job completion times C_j. A schedule is *feasible* if it satisfies all precedence constraints.

Our objective is to find a feasible schedule so as to minimize the *total weighted completion time*, i.e., the sum of the deferral costs w_jC_j. In a production process, such costs w_jC_j may measure the cost of work-in-process inventory for each job j. For the treatment of other objectives for single machine scheduling, such as minimizing makespan, weighted tardiness, the number of late jobs, etc., interested readers are referred to an excellent survey by Lawler *et al* [36].

When there are no precedence constraints, the problem can be solved, in $O(n \log n)$ computational steps, by sequencing jobs in nonincreasing order of ratios w_j/p_j [54]. Special precedence relations, such as *rooted trees*, *parallel chains*, etc., were considered by Sidney [53]. Lawler [33] considered *series-parallel* precedence relation which subsumes all previous special cases, and derived an elegant $O(n \log n)$ optimization procedure. With general precedence constraints, the problem becomes *strongly NP-hard* even if the weights or processing times are assumed to be zero or one [33,37]. This indicates that the existence of a polynomially bounded optimization algorithm is highly unlikely.

For the problems with general constraints, Potts [50] summarized various solution methods up to 1985, and presented an integer programming formulation, where a Branch and Bound solution procedure is applied using lower bounds from a Lagrangian Relaxation. Recently, Van de Velde (1990) proposed a dual decomposition method based on another Lagrangian Relaxation [56]. These methods have difficulties in solving large problems because of weak lower bounds. The reported computational results for these methods involve problems of size only up to 100 jobs.

Balas [3] pioneered the study of scheduling polyhedra on a closely related job shop scheduling problem, allowing for multiple machines, release dates and sequence-dependent changeover times. In his work, precedence constraints only arise between jobs processed on different machines. Balas gives some fundamental results about scheduling polyhedra and partial characterizations thereof. Queyranne [52] studied single machine scheduling polyhedra, and derived the class of facet-defining inequalities for the scheduling polyhedra without precedence constraints. He also proposed an $O(n \log n)$ separation algorithm for this class of inequalities.

Other polyhedral studies of scheduling problems explored alternate formulations. Dyer and Wolsey [16] used a mixed-integer programming formulation for the single machine problem with release dates. They also compared alternate relaxations in terms of

the resulting lower bounds for that problem. Sousa and Wolsey [55] used a large-scale 0-1 programming formulation for the single machine problem with release dates, deadlines and a general separable objective function. More recently, Wolsey [57] considered extended mixed-integer programming formulations, and derived very tight lower bounds, for the precedence constrained scheduling problem. His formulations involve $O(n^2)$ variables, mostly binary, and $O(n^2)$ or $O(n^3)$ constraints.

In this thesis, we consider a formulation of the precedence constrained single scheduling problem involving only n variables, one per job. In **Chapter 2**, we propose three classes of facet defining inequalities for the scheduling problem. We show that two classes suffice to define the scheduling polyhedra when the constraints are series-parallel. The first class, introduced by Queyranne [52], consists of inequalities that result from *parallel composition*, and are thus called *parallel inequalities.* The inequalities in the second class result from *series compositions* and are called *series inequalities.*

In **Chapter 3**, we develop a cutting plane algorithm for the scheduling problem. The Cutting plane methods, based on facet defining inequalities, have proven to be highly successful in solving large scale TSP problems [49]. In contrast, the cutting plane approaches have not been well explored in the area of scheduling [56]. This motivates the present computational study. The proposed algorithm includes *separation procedures* for the above *facet cuts.* The separation for the class of parallel inequalities takes $O(n \log n)$ computational steps. The separation for a subclass of series inequalities, called *simple series inequalities*, requires $O(n^2)$ computational steps. The algorithm is tested on an IDM 386/25 Personal Computer. We use a random sample of several hundred problems with up to several hundreds of jobs. The computational results show that this algorithm consistently yields tight lower bounds and feasible schedules with maximum percent deviations from the optimum less than 0.25 % on the average, and less than 1 % in the empirical worst case.

These results indicate that the polyhedral approach has potential for solving large-size constrained scheduling problems. These tight bounds can be used for other optimization procedures, such as Branch and Bound or Branch and Cut methods if an optimal schedule is sought. The ultimate goal of this research will be to apply similar polyhedral techniques to more complex scheduling problems.

1.3 Travelling Salesman Problems

Travelling Salesman Problems (TSPs) arise when seeking a shortest tour, starting from a home city, visiting each given city exactly once and then returning to the home city. If the travel distance (or cost) from one city to another equals the return distance, the problem is called a *Symmetric Travelling Salesman* (STS) problem; otherwise it is an *Asymmetric Travelling Salesman* (ATS) problem. For more than a century, TSP has been one of the most well-known mathematical problems and is notoriously difficult.

TSPs are of both theoretical interest and practical importance. They are known to be so-called *strongly NP*-hard, see Garey and Johnson [22]. Therefore the existence of a polynomial optimization algorithm is highly unlikely. On the other hand, many practical problems in production and distribution can be formulated as TSPs. For example, such problems arise in the fabrication of circuit boards, where lasers must drill tens to hundreds of thousands of holes in a board. Deciding an optimal order to drill these holes is a travelling salesman problem. TSPs also represent the simple versions of vehicle routing problems. Kearny [30] reported that annual distribution costs in the United States are about 400 billion dollars. We refer to [35] for a thorough discussion of, and motivation for, the study of Travelling Salesman problems.

For convenience, we consider STS and ATS problems in *complete graphs* on any given node set V'. A feasible solution to the STS (resp., ATS) problem is a Hamiltonian

circuit C on V', uniquely determined by its corresponding 0-1 *incidence vector* y^C, where $y_e^C = 1$ if edge e is used in C and zero otherwise. The *STS polytope* $STSP(V')$ (resp., *ATS polytope* $ATSP(V')$) is the convex hull of all incidence vectors of feasible solutions to the STS (resp., ATS) problem on V'. In addition to being interesting mathematical objects in themselves, these polytopes are worthy of study because corresponding valid linear inequalities, in particular those that are facet-defining, play an important role in formulating and solving Travelling Salesman problems.

There have been major advances in solving many large scale real world Travelling Salesman problems over the past decade. The most notable one is the resolution of the 2392-city problem by Padberg and Rinaldi [49]. (See also Kolata [31] for a recent report on some ongoing work.) They developed an effective *Branch and Cut* procedure which uses some simple families of facet defining inequalities, namely the *subtour elimination constraints* and *simple clique tree inequalities.* Their success has motivated extensive study of polyhedral structure of TSP. For more details, we refer to Grötschel and Padberg [28].

To establish facetial results for a given polyhedron, *indirect proofs* (or *polyhedral proofs*) [28] are often used. Such proofs are particularly effective if the polyhedron is full dimensional, because an inequality defining any given facet is unique up to a positive multiple. However, this is not the case for TSPs. Since every solution to TSPs satisfies *degree constraint equations*, i.e., the restrictions that the travelling salesman enters each city and leaves it exactly once, the resulting polytopes are far from being full dimensional. This considerably complicates indirect proofs for TSP polyhedral results [27,28], [29].

In recent years, two major research directions have been followed for the polyhedral study of TSPs. The first one is to reformulate TSPs by relaxing the degree constraints: the *monotone relaxation* [28] (resp., *graphical relaxation* [13],[10]) assumes that each city is visited at most once (resp., at least once). The resulting polytopes contain

TS polytopes as faces, and have full dimension. The second approach employs *clique-lifting* [27],[38],[41], [6] for extending TSP facets, and *composition* [41] techniques for extending STSP facets. Using these approaches, large classes of facet defining inequalities have been obtained. However, the derivations of these results still require very complicated and lengthy proofs, see [27,28], [29], [41].

The primary motivation for the present work is twofold. First, as the search for TSP facets becomes increasingly difficult, a simple, alternate approach is proposed. Second, while new TSP facets emerge from time to time, stronger but easily applicable facet generating operations are of particular interest.

This thesis explores a *projective approach* to the polyhedral study of Travelling Salesman (TS) problems. Suppose that we ignore the visit of the home city, and seek a shortest *path* that visits each city exactly once with arbitrary (and distinct) initial and final cities. This is called a *Shortest Hamiltonian Path* (SHP) problem. The associated polytope, the *Hamiltonian Path* (HP) *Polytope*, is the convex hull of incidence vectors of all Hamiltonian paths. First, we observe that there is a one-to-one correspondence between solutions of TSP and those of HP problems. Next, we note that in the SHP problem, all degree constraints are relaxed, because the end cities are not fixed in advance. These observations lead to a projective approach to the polyhedral study of TSPs. More precisely, we show that an HP polytope is a projection of a TS polytope. Two important properties are: (a) *near-full dimensionality*: HP polytopes are of full dimension minus one; (b) *polyhedral equivalence*: the polyhedral structures of the HP polytope and the STS polytope are isomorphic. Property (a) greatly facilitates the study of HP polytopes, and property (b) enables one to directly transfer facetial results from HP polytopes to TS polytopes.

In **Chapter 4**, we study Symmetric Travelling Salesman (STS) polytopes. Using the projective approach, we first derive various *Clique-Lifting* techniques to extend known STS facets. We then demonstrate how these lifting results generalize previous results.

Finally, we show that all facets with $0-1$ coefficients are *clique-liftable* with respect to any node.

In **Chapter 5**, we study Asymmetric Travelling Salesman (ATS) polytopes. As Balas and Fischetti [6] have shown that all ATS facets can be clique-lifted, we consider another type of facet extension technique, *facet composition.* In particular, we work on *symmetric inequalities* (i.e., those inequalities $ax \leq a_0$ with coefficients satisfying $a_{ij} = a_{ji}$ for all i and j). By this facet composition technique, we show that large known classes of facets of STS polytopes also induce facets of ATS polytopes. On the other hand, all symmetric facet defining inequalities for ATS polytopes are easily shown to induce facets of the STS polytopes [18]. As a result, new facets of STS polytopes can also be derived in this fashion.

Chapter 2

Single Machine Scheduling: Polyhedral Structure

2.1 Introduction

A set $N = \{1, 2, \cdots, n\}$ of jobs is to be processed without interruption on a single machine. All jobs are released at time $t = 0$. At any instant, the machine can process at most one job. With each job $j \in N$ is associated a known positive processing time p_j. Precedence constraints are represented by an acyclic digraph $G(N) = (N, A(N))$, where $(i, j) \in A(N)$ denotes that job i must be processed before job j.

We use the job completion times $C = (C_1, \cdots, C_n)^t$ to describe *schedules* of interest. Thus, by letting I^N denote the index set of the jobs in N that can be sequenced first (i.e., those jobs without predecessors), the collection of all *feasible schedules* can be represented as follows:

$$\begin{aligned} T(N) = \; & \{C \in R^N : \; C_i \geq p_i, \; \forall i \in I^N; \; C_j - C_i \geq p_j, \; \forall (i,j) \in A(N); \\ & C_j - C_i \geq p_j \vee C_i - C_j \geq p_i, \; \forall i, j \in N, \; i \neq j, \; (i,j) \notin A(N)\} \end{aligned}$$

We note that $T(N)$ is a disjunctive set.

Most of the research on this subject to date deals with optimization versions of the job sequencing problem with some objective function $f(C)$ over $T(N)$. For general precedence relations on the job set, the optimization problem has been shown to be NP-hard, see [33,37]. However, if the precedence constraints are *series-parallel* and the objective function $f(C)$ admits of a *string interchange* relation (linear functions are special cases), an optimal sequence can be found in $O(n \log n)$ time, see Lawler [34].

In this chapter, we study the facial structure of the *scheduling polyhedron* $P(N)$ defined as follows:

$$P(N) = \text{conv}(T(N))$$

In the next section, we introduce notation, definitions and preliminary algebraic results. We present some classes of valid inequalities for the scheduling polyhedron $P(N)$ with general precedence constraints in section 2.3, and a partial description of its facial structure in section 2.4. Three classes of valid inequalities used in this description are associated with series composition, parallel composition and induced Z-subgraphs, respectively. The class of inequalities used in Queyranne [52] is a special case of the parallel composition class. In section 2.5, we derive the unique minimal linear system defining $P(N)$, consisting of the first two classes of facet-inducing inequalities, if the precedence constraints are series-parallel. We also show that facet-inducing inequalities of the third class, associated with Z-induced subgraphs, arise whenever the precedence constraints are not series-parallel.

2.2 Definitions and preliminaries

Graph $G(S) = (S, A(S))$ is said to be a *subgraph* of $G(N)$ induced by S if $S \subseteq N$ and $A(S) = \{(i,j) \in A(N) : i \in S, j \in S\}$. Similarly, we define the collection of all feasible *subschedules* induced by job set $S \subseteq N$:

$$\begin{aligned} T(S) = \quad & \{C \in R^S : \; C_i \geq p_i, \; \forall i \in I^S; \; C_j - C_i \geq p_j, \; \forall (i,j) \in A(S); \\ & C_j - C_i \geq p_j \vee C_i - C_j \geq p_i, \; \forall i,j \in S, \; i \neq j, \; (i,j) \notin A(S)\}. \end{aligned}$$

Note now that $G(S)$ specifies precedence relations on job set S and the scheduling polyhedron induced by S is

$$P(S) = \text{conv}(T(S)).$$

Node i has *precedence over* node j (or job i must be processed before job j), denoted by $i \to j$, if there is a directed path from i to j in $G(N) = (N, A(N))$. Similarly, for any disjoint subsets S_1, S_2 of N, $S_1 \to S_2$ iff $i \to j$ for all $i \in S_1$ and $j \in S_2$. For any $S \subseteq N$, $A(S)$ is said to be *transitively reduced* if $(i,j) \in A(S)$ implies that (i,j) is the unique directed path from i to j in $G(S)$. For any $S \subseteq N$, $\bar{A}(S)$ is said to be *transitively closed* if

$$\{(i,k),(k,j)\} \subseteq \bar{A}(S) \Rightarrow (i,j) \in \bar{A}(S).$$

Note that if $\bar{A}_1(S)$ and $\bar{A}_2(S)$ are two transitively closed arc sets containing $A(S)$, then so is $\bar{A}_1(S) \cap \bar{A}_2(S)$. Therefore, we may define the transitive closure of $A(S)$ as the smallest transitively closed set $\bar{A}(S)$ satisfying $\bar{A}(S) \supseteq A(S)$. Unless otherwise noted, we use $A(S)$ (resp., $\bar{A}(S)$) to represent the transitively reduced arc set (resp., the transitive closure) of $A(S)$.

A set $S \subseteq N$ is said to be *Z-inducing* if $|S| \geq 4$ and there exists a numbering of the nodes in S such that $S = \{1,2,3,...,k\}$ and $\{(i,j) \in \bar{A}(N) : i,j \in \{1,2,3,4\}\} = \{(1,2),(3,2),(3,4)\}$. Note that a graph $G(N)$ *contains a Z-subgraph* if there is a Z-inducing subset $S \subseteq N$. Note also that a graph is *series-parallel* if and only if it contains no Z-subgraph, see Lawler [34].

Graph $G(S)$ is *series-decomposable* if there exists a partition (S_1, S_2) of S such that $S_1 \to S_2$, in which case, we may write $S = S_1 \to S_2$.

A schedule $C \in R^S$ is a *permutation schedule* on S if $C \in T(S)$ has no inserted idle times.

Subset $S \subseteq N$ is an *initial set* of N if $k \to j$ and $j \in S$ imply $k \in S$.

Subset $S \subseteq N$ is a *terminal set* of N if $i \to k$ and $i \in S$ imply $k \in S$.

Subset $S \subseteq N$ is an *intermediate set* of N if $i \to k \to j$, $i \in S$ and $j \in S$ imply $k \in S$.

Note that the intersection of any collection of initial (resp., terminal; resp., intermediate) sets is an initial (resp., terminal; resp., intermediate) set. Thus, for any $S \subseteq N$, let

$$\begin{aligned} I(S) &= \bigcap\{I : S \subseteq I,\ I \text{ is initial}\}, \\ T(S) &= \bigcap\{T : S \subseteq T,\ T \text{ is terminal}\}, \\ Q(S) &= \bigcap\{Q : S \subseteq Q,\ Q \text{ is intermediate}\}. \end{aligned}$$

We call $I(S)$ (resp., $T(S)$, $Q(S)$) the *smallest* initial (resp., terminal, intermediate) set containing S.

The following three propositions establish some properties of the subsets I, T and Q of N.

Proposition 2.1 *$S \subset N$ is an initial set of N iff $N \setminus S$ is a terminal set of N.*

Proposition 2.2 *If $S \subseteq N$ is an initial or a terminal set of N, then S is an intermediate set of N.*

Proofs of **Propositions 2.1** and **2.2** are straightforward, and therefore omitted.

Proposition 2.3 *Assume*

1. *Q is an intermediate set of N; and*
2. *$I(Q)$ is the smallest initial set of N containing Q.*

Then, Q is a terminal set of $I(Q)$.

Proof. By contradiction, suppose that Q is not a terminal set of $I(Q)$, i.e., there exists some $j \in Q$ such that $j \to k$ for some $k \in I(Q) \setminus Q$. We claim that k has no successors in Q since otherwise assumption (1) implies $k \in Q$ which is a contradiction. Let V be

the set containing k and all its successors. Thus, $I(Q) \setminus V$ is another initial set of N containing Q which contradicts assumption (2). □

For $S \subseteq N$, we let $a_S \in R^S$ be the subvector of $a \in R^N$ whose components are those in a indexed by S. Let (S_1, S_2) be a partition of S. Then by $C = (a_{S_1}, b_{S_2})$, where $a, b \in R^S$, we mean that

$$C_k = \begin{cases} a_k, & \text{if } k \in S_1; \\ b_k, & \text{if } k \in S_2. \end{cases}$$

For arbitrary vectors $u, v \in R^N$ and subset $S \subseteq N$, we let

$$u(S) \doteq \sum_{j \in S} u_j, \qquad u^2(S) \doteq \sum_{j \in S} u_j^2,$$

$$u(S)^2 \doteq (\sum_{j \in S} u_j)^2, \qquad u * v(S) \doteq \sum_{j \in S} u_j v_j.$$

Define also

$$g(S) \doteq (p(S)^2 + p^2(S))/2,$$

$$g'(S) \doteq (p(S)^2 - p^2(S))/2.$$

Proposition 2.4 *Let $w_k \in R$, $k = 1, ..., n$. Let $a^i \in R^N$, $i = 1, ..., n = |N|$, be n affinely independent vectors satisfying*

$$\sum_{k=1}^{n} w_k a_k^i = w_0, \quad \forall i, \tag{2.1}$$

and let $S = \{k : w_k \neq 0\}$.

Then,

1. *There are precisely $s = |S|$ affinely independent vectors in $\{a_S^i : i = 1, ..., n\}$; and*
2. *There are precisely $l = |V| + 1$ affinely independent vectors in $\{a_V^i : i = 1, ..., n\}$ for all $V \subseteq N$ such that $S \not\subseteq V$.*

Proof.

(1) Let M be an n by $n+1$ matrix whose ith row is $((a^i)', 1)$; M_S be an n by $s+1$ submatrix of M whose ith row is $((a^i_S)', 1)$; and M_V be an n by l submatrix of M whose ith row is $((a^i_V)', 1)$.

(2.1) implies that the rank of M is n (denoted by $r(M) = n$). Thus, there is a unique (up to multiples) solution to the system

$$Mx = 0, \tag{2.2}$$

namely, by (2.1), the solution

$$x^0_k = \begin{cases} w_k, & \text{if } k \in S; \\ -w_0, & \text{if } k = n+1; \\ 0, & \text{otherwise.} \end{cases}$$

Since $r(M_S) < s$ implies that $r(M) < n$ and $r(M_S) > s$ implies that x^0 does not solve (2.2), we must have $r(M_S) = s$. Hence, (1) follows.

(2) Obviously, $r(M_V) \leq l$. We claim that $r(M_V) \geq l$ since otherwise we have $M_V z = 0$ for some $z \neq 0$. Using this z, we can construct another solution, linearly independent of x^0, to (2.2) which is a contradiction with (2.1). Hence, (ii) follows. □

Corollary 2.5 *The facet-inducing inequalities for a full dimensional polyhedron $P \subseteq R^N$ are unique up to some positive multiples.*

Proof. Since $P \subseteq R^N$ is full dimensional, there exists no equation satisfied by all points $x \in P \subseteq R^N$. The result then follows from Theorems 2.5 and 2.6 in Pulleyblank [51]. □

Proposition 2.4 and **Corollary 2.5** are used in Section 4 for proving that certain valid inequalities are facet-inducing.

2.3 Three classes of valid inequalities

To make the cutting plane method a success, valid inequalities, especially those inducing facets, play a key role. In this section, we present some classes of valid inequalities for $P(N)$. Of particular interest are those valid inequalities which, under certain conditions, are facet-inducing. **Proposition** 2.7 and **Proposition** 2.8 give three classes of valid inequalities which play an important role in the sequel. We begin with **Proposition** 2.6 which gives a necessary condition for an inequality to be *valid* for $P(N)$.

Proposition 2.6 *If*

$$\sum_{k \in N} w_k C_k \geq w_0$$

is a valid inequality for $P(N)$, then $w(N) \geq 0$.

Proof. Suppose that $w(N) < 0$. Let $C \in P(N)$ be a schedule and $y \in R$, and let

$$\hat{C} = C + y\mathbf{1},$$

where $\mathbf{1} \in R^n$ is a vector of one's.

Clearly, $\hat{C} \in P(N)$ for all $y \geq 0$. But $\sum_{k \in N} w_k \hat{C}_k < w_0$ for y sufficiently large, which produces a contradiction. □

We call a valid inequality for $P(N)$ a *positive-sum* (resp., *zero-sum*) valid inequality if $w(N) > 0$ (resp., $w(N) = 0$).

We now present three classes of valid inequalities for $P(N)$. The first class was introduced in Queyranne [52]. The other two are new, arising from the precedence constraints.

Proposition 2.7 *(i) For all $S \subseteq N$,*

$$p * C(S) \geq g(S) \tag{2.3}$$

is a valid inequality for $P(N)$. Moreover, inequality (2.3) is tight if S is an initial set of N.

(ii) For any $S_1 \subseteq N, S_2 \subseteq N$ such that $S_1 \to S_2$,

$$\begin{aligned} F(S_1, S_2) \doteq \quad & -p(S_2)p * C(S_1) + p(S_1)p * C(S_2) \\ & -p(S_2)g(S_1) - p(S_1)g(S_2) + p(S_2)p^2(S_1) \quad \geq 0 \end{aligned} \tag{2.4}$$

is a valid inequality of $P(N)$. Moreover, inequality (2.4) is tight if $S = S_1 \cup S_2$ is an intermediate set of N.

Proof. It follows from Theorem 3.1 of Queyranne [52] that (2.3) is a valid inequality for $P(N)$. Next, observe that if S is initial, then any permutation schedule in $T(N)$ with sequence $(S, N \setminus S)$ is tight for (2.3). This completes the proof for (i)

(ii) Note that for any $C \in T(N)$, there exists a time t such that

(a) all jobs indexed by S_1 are completed before or at time t, and

(b) all jobs indexed by S_2 start after or at time t.

Using the results in (i), we have (a) implies

$$p(S_2) \sum_{i \in S_1} p_i(t - C_i + p_i) \geq g(S_1)p(S_2), \tag{2.5}$$

and (b) implies

$$p(S_1) \sum_{j \in S_2} p_j(C_j - t) \geq g(S_2)p(S_1). \tag{2.6}$$

Adding (2.5) to (2.6) yields (2.4).

Next, observe that if S is intermediate, then for some $t \geq 0$, there exists a feasible permutation schedule tight for both (2.5) and (2.6), therefore tight for (2.4). □

Suppose now that N is Z-inducing. This structure is used to generate the third class of valid inequalities studied here. Because N is Z-inducing, there exists a numbering of

nodes in N such that

$$\{(i,j) \in \bar{A}(N) : \forall i,j \in \{1,2,3,4\}\} = \{(1,2),(3,2),(3,4)\}.$$

For any $Q \subseteq \{i \in N : \{(3,i),(i,2)\} \subseteq \bar{A}(N), \{(1,i),(i,1),(4,i),(i,4)\} \cap \bar{A}(N) = \emptyset\}$, let

$S \doteq \{1,2,3,4\} \cup Q$, $S' \doteq S \setminus \{3\}$, $S'' \doteq S \setminus \{2\}$, $S'_1 \doteq \{1,3\} \cup Q$, and $S'_2 \doteq \{2,4\} \cup Q$.

Note that Q may be empty.

Proposition 2.8 *The following inequalities*

$$\begin{aligned} -p(S')p * C(S'_1) \;+\; & [p(\{1\} \cup Q)p(S) + p_2p_3]C_2 + p_3p_4C_4 \\ \geq\; & p(S')g'(\{1,2,3\} \cup Q) + p_3p_4^2, \end{aligned} \tag{2.7}$$

$$\begin{aligned} -p_1p_2C_1 \;-\; & [p(\{4\} \cup Q)p(S) + p_2p_3]C_3 + p(S'')p * C(S'_2) \\ \geq\; & p(S'')g(\{2,4\} \cup Q) + p_1p_2p_3 \end{aligned} \tag{2.8}$$

are valid for $P(N)$. Moreover, they are tight if S is intermediate.

Proof. Define
$T^{31} \doteq \{C \in T(N) : C_1 - C_3 \geq p_1\}$ and $T^{13} \doteq \{C \in T(N) : C_3 - C_1 \geq p_3\}$. Clearly, $T(N) = T^{31} \cup T^{13}$.

Claim 1. (2.7) is valid for T^{31}.

Proof. Note that the definition of T^{31} implies that $\{3\} \rightarrow S'$. Using **Proposition** 2.7(ii) with $S_1 = \{3\}$ and $S_2 = S'$, the inequality

$$-p(S')p_3C_3 + p_3(p * C(S')) \geq p_3g(S') \tag{2.9}$$

is valid for T^{31}. Using **Proposition** 2.7(ii) again, but with $S_1 = \{1\} \cup Q$ and $S_2 = \{2\}$, the inequality

$$p(S)[-p_1C_1 - p * C(Q) + p(\{1\} \cup Q)C_2] \geq p(S)g'(\{1,2\} \cup Q) \tag{2.10}$$

is valid for T^{31}. The sum of (2.9) and (2.10) yields (2.7).

This completes the proof for **Claim 1**.

Claim 2. (2.7) is valid for T^{13}.

Proof. Note that for T^{13}, we have $\{1\} \rightarrow \{3\} \rightarrow \{Q, 2, 4\}$. The following two inequalities

$$p_1 p(S')(-C_1 + C_2) \geq p_1 p(S')(p_2 + p_3 + p(Q)), \tag{2.11}$$

$$p_1 p_3(-C_3 + C_2) \geq p_1 p_3(p_2 + p(Q)), \tag{2.12}$$

directly follow from the induced precedence relation $\{1\} \rightarrow \{3\} \rightarrow Q \rightarrow \{2\}$. Using **Proposition** 2.7(ii) with $S_1 = \{3\}$ and $S_2 = S_2'$, the inequality

$$-p(S_2')p_3 C_3 + p_3 p * C(S_2') \geq p_3 g(S_2') \tag{2.13}$$

valid for T^{13}. Also, using **Proposition** 2.7(ii) with $S_1 = Q$ and $S_2 = \{2\}$, the inequality

$$-p(S)p * C(Q) + p(S)p(Q)C_2 \geq p(S)g'(\{2\} \cup Q) \tag{2.14}$$

is valid for T^{13}. The sum of (2.11) to (2.14) yields (2.7). This completes the proof for **Claim 2**.

Claim 1 and **Claim 2** imply that (2.7) is valid for $T(N)$, and therefore for $P(N)$.

Similarly, we can show (2.8) is a valid inequality for $P(N)$ by considering

$$T^{24} = \{C \in T(N) : C_4 - C_2 \geq p_4\} \text{ and } T^{42} = \{C \in T(N) : C_2 - C_4 \geq p_2\}.$$

Next, observe that if S is intermediate, then any feasible schedule with uninterrupted subsequence (3,1,Q,4,2) (no idle time, no other jobs in between) is tight for (2.7) and (2.8).

The proof is complete. □

We call the inequality (2.3) *parallel-inequality*, (2.4) *series-inequality*, and the inequalities (2.7) and (2.8) *Z-inequalities*, respectively.

2.4 Structure of the precedence constrained scheduling polyhedra

In this section, we study the facial structure of $P(N)$. We begin with a discussion of some properties of the faces of $P(N)$. We then show under which conditions the valid inequalities, introduced in the previous section, are facet-inducing for $P(N)$. The proofs of all lemmata in this section are found in section 2.6.

Lemma 2.9 *Let $n = |N|$. A valid inequality*

$$\sum_{i \in N} w_i C_i \geq w_0 \tag{2.15}$$

is $(n-b-1)$-dimensional face-inducing for $P(N)$, where $b = 0, 1, 2, ..., n-1$, if and only if there are precisely $n-b$ affinely independent feasible schedules $C^j \in P(N), j = 1, ..., n-b$, all tight for (2.15).

Note. The above lemma is implicitly assumed in Balas [3], proof of Theorem 4.4.

With **Lemma** 2.9, we may use *solutions* or *schedules* interchangeably in the statements about faces of scheduling polyhedra.

Lemma 2.10 *Set S is not series-decomposable if and only if there exist at least $|S|$ affinely independent permutation schedules in $T(S)$.*

The above two lemmata are used in studying the dimension of the face of $P(N)$.

We are now in a position to study the facial structure of $P(N)$.

Theorem 2.11 (Face-Lifting Theorem) *Let*

$$\sum_{i \in S} w_i C_i \geq w_0, \tag{2.16}$$

where $C \in R^S$, be an $(s-b-1)$-dimensional face-inducing inequality of $P(S)$, where $0 \leq b < s$.

(1) If S is an initial set of N, then (2.16) is an $(n-b-1)$-dimensional face-inducing inequality for $P(N)$.

(2) If S is an intermediate set of N with $w(S)=0$, then (2.16) is a zero-sum $(n-b-1)$-dimensional face-inducing inequality for $P(N)$.

Proof. Let $s=|S|$ and $w_i=0$, for all $i \in N \setminus S$. Let

$$F_S = \{C \in P(S) : \sum_{i \in S} w_i C_i = w_0\}, \quad S \subseteq N. \tag{2.17}$$

First, note that since $C \in T(N) \Rightarrow C_S \in T(S)$, any valid inequality for $P(S)$ induces a valid inequality for $P(N)$, and hence so does (2.16): $\sum_{i \in S} w_i C_i \geq w_0$ holds for all $C \in P(N)$.

Next, let $P(S)$ be represented by a minimal linear system, that is,

$$P(S) = \{C \in R^S : \sum_{i \in S} a_{hi} C_i \geq a_{h0}, h \in I\}, \tag{2.18}$$

where I is an index set of all facet-inducing inequalities. For face F_S, let

$$I_e = \{h \in I : \sum_{i \in S} a_{hi} C_i = a_{h0}, \forall C \in F_S\}. \tag{2.19}$$

Face F_S is an $(s-b-1)$-dimensional face of $P(S)$ if and only if matrix A_e with rows $a_{h.}$, $h \in I_e$, has rank $b+1$. As observed before, the inequalities

$$\sum_{i \in S} a_{hi} C_i \geq a_{h0}, \quad \forall h \in I_e,$$

are also valid inequalities for $P(N)$. Moreover, these inequalities are tight for all $C \in F_N$. Therefore, F_N is at most $(n-b-1)$-dimensional. We are done if we find $(n-b)$ affinely independent schedules in F_N.

(1) *S is initial.* Since F_S is $(s-b-1)$-dimensional, there exist $s-b$ affinely independent feasible schedules $C_S^j \in F_S$, for all $j=1,...,s-b$.

Let $C_{N\setminus S} \in T(N \setminus S)$, and $u = \max\{(C_S^j)_k : k \in S, j = 1, ..., s-b\}$. For all $j = 1, ..., s-b$, let

$$(C_N^j)_i = \begin{cases} (C_S^j)_i, & \text{if } i \in S; \\ (C_{N\setminus S})_i + u, & \text{if } i \in N \setminus S. \end{cases}$$

Since S is an initial set of N, by **Proposition** 2.1, $N \setminus S$ is a terminal set of N. It follows that

$$C_N^j \in T(N), j = 1, ..., s-b.$$

Now, define

$$C^j = \begin{cases} C_N^j, & \text{if } 1 \le j \le s-b; \\ C_N^1 + \sum_{k=j+b}^{n} \mathbf{e}_{\pi(k)}, & \text{if } s-b < j \le n-b; \end{cases}$$

where π is the permutation induced by C_N^1.

It is easy to verify that $C^j \in F_N, j = 1, ..., n-b$, and that they are affinely independent.

(2) *S is intermediate.* Since F_S is $(s-b-1)$ dimensional, there are $s-b$ affinely independent schedules, $C_S^j \in F_S$, $j = 1, 2, ..., s-b$.

Let $U = \{k : k \in N \setminus S, \exists i \in S, k \to i\}$, $V = N \setminus (S \cup U)$, and $u = |U|$.

Let $C_U \in T(U)$, $C_V \in T(V)$.

Let $m_1 = \max\{(C_U)_i : i \in U\}$, $m_2 = m_1 + \max\{(C_S^j)_i : i \in S, j = 1, ..., s-b\}$.

Define the following schedules for $P(N)$.

For $j = u+1, u+2, ..., u+s-b$,

$$(C_N^j)_i = \begin{cases} (C_U)_i, & \text{if } i \in U; \\ m_1 + (C_S^{j-u})_i, & \text{if } i \in S; \\ m_2 + (C_V)_i, & \text{if } i \in V. \end{cases}$$

For $j = 1, 2, ..., u$, and $j = u+s-b+1, u+s-b+2, ..., n-b$,

$$C_N^j = C_N^{u+1} + \sum_{k=\alpha(j)}^{n} \mathbf{e}_{\pi(k)}.$$

where π is the permutation induced by C_N^{u+1} and

$$\alpha(j) = \begin{cases} j, & \text{if } 1 \leq j \leq u; \\ j+b, & \text{otherwise.} \end{cases}$$

It can be easily verified that schedules $C_N^j \in F_N$, $j = 1, ..., n-b$, are affinely independent.

Therefore, we have constructed $(n-b)$ affinely independent schedules in F_N in either case.

Note that if U (or V) is empty, the above proof is simplified. This completes the proof of the lifting theorem. □

We mentioned in the previous section that valid inequalities (2.3), (2.4), (2.7) and (2.8) are tight under certain conditions. We study below the dimension of the face they induce under those conditions. Before showing this, we need some intermediate results.

Lemma 2.12 *Let*

$$\sum_{k \in N} w_k C_k \geq w_0 \tag{2.20}$$

be facet-inducing for $P(N)$ and let S be an initial set of N such that for some $i \in S, w_i \neq 0$ and $w(S) = w(N)$. Then $w_k = 0$, for all $k \in N \setminus S$.

Lemma 2.13 *Let*

$$\sum_{k \in N} w_k C_k \geq w_0 \tag{2.21}$$

induce a d-dimensional face of $P(N)$, where $0 \leq d \leq n-1$, and let Q be the smallest set containing $S = \{i \in N : w_i \neq 0\}$, which is initial if $w(N) > 0$; intermediate if $w(N) = 0$.

Then, for every schedule $C \in T(N)$ tight for (4.5.1), $C_Q - t\mathbf{1}_Q$, for some $t \geq 0$ ($t = 0$ if $w(N) > 0$), is a permutation schedule in $T(Q)$, if any of the following conditions holds:

(i) there exists no proper initial set Q_0 of Q such that $w(Q_0) = w(N)$; or

(ii) $d = n-1$.

Lemma 2.13 asserts that, under the stated conditions, any schedule C tight for (2.21) has the jobs in Q processed consecutively, that is, without any other jobs (jobs not in Q) or idle time inserted in between.

Theorem 2.14 and **Theorem** 2.16 relate the dimension of a face to the number of series compositions in an appropriate set containing all jobs with nonzero coefficients in the corresponding inequality. In **Theorem** 2.14, we consider positive-sum inequalities, while in **Theorem** 2.16, we consider zero-sum inequalities.

Theorem 2.14 *Let*

$$\sum_{i \in N} w_i C_i \geq w_0 \tag{2.22}$$

be a positive-sum tight valid inequality for $P(N)$. Let S be the smallest initial set containing $\hat{S} = \{i \in N : w_i \neq 0\}$. Assume

(i) there exists no proper initial set S_0 of S such that $w(S_0) = w(N)$; and

(ii) $S = S_1 \to S_2 \to \cdots \to S_{b+1}$, where $0 \leq b \leq n-1$ is the total number of series-compositions in S.

Then, $F_N = \{C \in P(N) : w * C(S) = w_0\}$ is at most $(n-b-1)$-dimensional.

Proof. Note that $b = 0$ iff S is not series-decomposable, and thus, (ii) implies that (2.22) is not facet-inducing if $b > 0$.

By condition (i), **Lemma** 2.13 implies that for any $C \in T(N) \cap F_N$, C_S is a permutation schedule in $T(S)$, and therefore C is tight for all series-inequalities (2.4) induced by the b pairs of sets, $(S_1 \to S_2)$, $(S_2 \to S_3)$, ..., $(S_b \to S_{b+1})$, (cf. **Proposition** 2.7).

Assume that we have a finite $P(N)$-defining linear system $a_h C \geq a_{h0}$, $h \in I$, including (2.22) as the first inequality, and the inequalities (2.4) induced by $(S_{i-1} \to S_i)$ as the i-th inequality for $i = 2, ..., b+1$. Note now that for all $C \in F_N$,

$$a_h C = a_{h_0}, \quad h = 1, ..., b+1.$$

It suffices to show that a_h, $h = 1, ..., b+1$, are linearly independent.

Let $\hat{A}_e$ be the matrix whose rows are a_h, $h = 1, ..., b+1$. For any $y \in R^{b+1}$ such that $y^t \hat{A}_e = \mathbf{0}_N^t$, we have

$$0 = y^t \hat{A}_e \mathbf{1}_N = y^t (w(N), 0, ..., 0)^t = y_1 w(N)$$

and therefore $y_1 = 0$ since $w(N) > 0$. Further, observe that any square submatrix of $\hat{A}_e$ with columns corresponding to exactly one job from each set $S_i, i = 1, ..., b$, is a diagonal matrix with nonzero diagonal elements. Thus, rows $2, ..., b+1$ of $\hat{A}_e$ are linearly independent, which implies that $y_2 = ... = y_{b+1} = 0$. It follows that $\hat{A}_e$ itself is of rank $b+1$. Therefore, face F_N is of dimension at most $(n-b-1)$. the proof is complete. □

The following theorem yields the dimension of the face induced by a parallel inequality (2.3).

Theorem 2.15 *For any initial set $S \subseteq N$,*

$$p * C(S) \geq g(S) \tag{2.23}$$

is an $(n-b-1)$-dimensional face-inducing inequality for $P(N)$, where $0 \leq b \leq n-1$ is the total number of series-compositions in S.

Proof. Let $S = S_1 \to S_2 \to ... \to S_{b+1}$.

Note that (2.23) satisfies condition (i) of **Theorem** 2.14, and therefore F_N is at most $(n-b-1)$-dimensional. This further implies, by the **Face-Lifting Theorem**, that $F_S = \{C \in P(S) : p * C(S) = g(S)\}$ is at most $(s-b-1)$-dimensional. To show that F_S is exactly $(s-b-1)$-dimensional, we construct $s-b$ affinely independent schedules in F_S as follows.

By **Lemma** 2.10, there exist $s_k = |S_k|$ affinely independent permutation schedules $\bar{C}_{S_k}^h$, $h = 1, ..., s_k$, in $T(S_k)$, for $k = 1, 2, ..., b+1$.

Let $s_0 = m_0 = 0$, $z_0 = 1$, and $s = |S| = \sum_{k=1}^{b+1} s_k$. Recursively define $m_k = p(S_k) + m_{k-1}$, $z_k = z_{k-1} + s_{k-1} - 1$, for $k = 1, ..., b+1$. Define $s - b$ schedules $C^j \in T(S)$ as follows: For $j = 1, 2, ..., s-b$, $k = 1, 2, ..., b+1$, let

$$C^j_{S_k} = \begin{cases} \bar{C}^{j-z_k}_{S_k} + m_{k-1}, & \text{if } z_k < j \leq z_k + s_k; \\ \bar{C}^1_{S_k} + m_{k-1}, & \text{otherwise.} \end{cases}$$

It is easy to verify that $C^j \in T(S) \cap F_S$. Moreover, all $C^j - C^1$, $j = 2, 3, ..., s-b$ are linearly independent. This implies that all C^js are affinely independent, and therefore F_S is $(s-b-1)$-dimensional. Since S is initial, by the **Face-Lifting Theorem**, F_N is $(n-b-1)$-dimensional. □

We now turn to the dimension of the face induced by a zero-sum inequality.

Theorem 2.16 *Let*

$$\sum_{i \in N} w_i C_i \geq w_0 \tag{2.24}$$

be a zero-sum tight valid inequality for $P(N)$. Let S be the smallest intermediate set containing $\hat{S} = \{i \in N : w_i \neq 0\}$. Assume that

(i) there exists no proper initial set S_0 of S such that $w(S_0) = 0$;

(ii) $S = S_1 \to S_2 \to, \cdots, \to S_{b+1}$, where $0 \leq b \leq n-1$ is the total number of series-compositions in S.

Then

$$F_N = \{C \in P(N) : w * C(S) = w_0\}$$

is at most d-dimensional, where $d = \min\{n-1, n-b\}$.

Proof. By condition (i), **Lemma** 2.13 implies that $\forall C \in T(N) \cap F_N$, $C_S - t\mathbf{1}_S$, for some $t \geq 0$, is a permutation schedule in $T(S)$, and therefore C is tight for series-inequalities (2.4) induced by the b pairs of sets, $(S_1 \to S_2)$, $(S_2 \to S_3)$, ..., $(S_b \to S_{b+1})$. Moreover, it

can be easily verified that $\hat{A}_e$ (defined the same as in **Theorem** 2.14) is of rank at least $\max\{1, b\}$ (cf. **Theorem** 2.14.). It follows that F_N is at most d-dimensional. □

The following theorem yields the dimension of the face induced by a series-inequality (2.4).

Theorem 2.17 *For any S_1, S_2 such that $S_1 \to S_2$ and $S = S_1 \cup S_2$ is an intermediate set of N,*

$$\begin{aligned} F(S_1, S_2) \doteq \; & -(p(S_2))p * C(S_1) + (p(S_1))p * C(S_2) \\ & -p(S_2)g(S_1) - p(S_1)g(S_2) + p(S_2)p^2(S_1) \geq 0 \end{aligned} \tag{2.25}$$

is a $(n-b)$-dimensional face-inducing inequality for $P(N)$, and $b \leq n-1$ is the total number of the series-compositions in S.

Proof. Let $s = |S|$.

Note that (2.25) satisfies condition (i) of **Theorem** 2.16, and therefore F_N is at most $(n-b)$-dimensional, since $b \geq 1$. This further implies, by the **Face-Lifting Theorem**, that $F_S = \{C \in P(S) : F(S_1, S_2) = 0\}$ is at most $(s-b)$-dimensional. To show that F_S is exactly $(s-b)$-dimensional, we construct $s-b+1$ affinely independent schedules in F_S as follows.

Construct $s-b$ affinely independent feasible permutation schedules $C^j \in T(S)$, $j = 1, ..., s-b$ in the same way as we do in **Theorem** 2.15 and let $C^{s-b+1} = C^1 + \mathbf{1}_S$. Observe that $C^j \in T(S) \cap F_S$, $j = 1, ..., s-b+1$, and we show below that they are affinely independent.

For any $v \in R^{s-b+1}$ such that $\sum_{j=1}^{s-b+1} v_j = 0$ and $\sum_{j=1}^{s-b+1} v_j C^j = \mathbf{0}_S$, we have

$$p^t \sum_{j=1}^{s-b+1} v_j C^j = \sum_{j=1}^{s-b+1} v_j g(S) + v_{s-b+1} p(S) = 0$$

implying $v_{s-b+1} = 0$, where $p^t \in R^s$ is the processing time vector for job set S. Moreover, since $C^j, j = 1, ..., s-b$, are affinely independent, we have $v_j = 0, j = 1, ..., s-b$, which implies that $C^j, j = 1, ..., s-b+1$, are affinely independent. Therefore F_S is $(s-b)$-dimensional. Since S is intermediate, by the **Face-Lifting Theorem**, F_N is $(n-b)$-dimensional. □

Theorem 2.18 below is one of the main results of this chapter. It implies that all facet-inducing inequalities other than parallel inequalities (2.3) must be zero-sum inequalities.

Theorem 2.18 *Let*

$$\sum_{j \in N} w_j C_j \geq w_0 \tag{2.26}$$

be a valid inequality of $P(N)$, and $S = \{i \in N : w_i \neq 0\}$ be its support. *Then this inequality is positive-sum facet-inducing for $P(N)$, if and only if*

(i) S is an initial set of N;

(ii) S is not series-decomposable; and

(iii) (2.26) is a positive multiple of (2.3).

Proof. *Sufficiency* trivially follows from **Theorem** 2.15.

Necessity: Let S_0 be the smallest initial set with $w(S_0) = w(S)$.

Since (2.26) is facet-inducing, there exist n affinely independent schedules $C^j \in T(N)$, $j = 1, ..., n$, tight for (2.26). By **Lemma** 2.13, $\forall j$, $C^j_{S_0} \in T(S_0)$ is a permutation schedule. Thus, $C^j, j = 1, ..., n$, are tight for the valid inequality $p * C(S_0) \geq g(S_0)$. Hence, by **Corollary** 2.5, both (i), i.e., $S_0 = S$, and (iii) follow. Moreover, by **Proposition** 2.4, we have $s = |S|$ affinely independent feasible permutation schedules in $\{C^j_S : j = 1, ..., n\}$, which, by **Lemma** 2.10, implies (ii). The proof is complete. □

Theorem 2.19 below characterizes which series-inequalities are facet-inducing for $P(N)$.

Theorem 2.19 *Let*

$$\sum_{k \in N} w_k C_k \geq w_0 \tag{2.27}$$

be a valid inequality for $P(N)$. Let S be the smallest intermediate set containing $\hat{S} = \{i \in N : \quad w_i \neq 0\}$. Assume $S = (S_1 \rightarrow S_2)$.

Then, (2.27) is zero-sum facet-inducing for $P(N)$ if and only if

(i) $\hat{S}$ is an intermediate set of N, i.e., $S = \hat{S}$;

(ii) neither S_1 nor S_2 is itself series-decomposable; and

(iii) (2.27) is a positive multiple of (2.4).

Proof.

Sufficiency trivially follows from **Theorem** 2.17.

Necessity Since (2.27) is facet-inducing, we have n affinely independent schedules

$$C^j \in F_N = \{C \in P(N) : w * C(S) = w_0\}, \quad j = 1, ..., n.$$

Lemma 2.13 implies that $\forall j$, $C_S^j - t_j \mathbf{1}_S \in T(S)$, for some $t_j \geq 0$, are permutation schedules, therefore all $C^j s$ are tight for the series-inequality defined by $(S_1 \rightarrow S_2)$. By **Corollary** 2.5, (i) and (iii) follow.

We show (ii) holds by contradiction.

Suppose (ii) does not hold. Then, $b \geq 2$. Further, Using (i) ,(iii), and **Theorem** 2.17, we derive a contradiction: $\dim(F_N) = n - b < n - 1$. The proof is complete. □

Thus, **Theorem** 2.18 characterizes all positive facet-inducing inequalities for $P(N)$ and **Theorem** 2.19 characterizes one class of zero-sum facet-inducing inequalities — the series-inequalities. The following theorem characterizes another class of zero-sum facet-inducing inequalities — the Z-inequalities.

Assume that N is Z-inducing. Let S, S'_1, S'_2,S', S'', and Q be defined as in Section 3, just before **Proposition** 2.8.

Theorem 2.20 *Inequalities (2.7) and (2.8) are facet-inducing for $P(N)$ if and only if S is an intermediate set of N.*

Proof. If either (2.7) or (2.8) is facet-inducing, then by **Lemma** 2.13, any feasible schedule in $T(N)$ tight for either inequality has the jobs in S processed consecutively without inserted idle times. This implies that S is intermediate.

Conversely, since (2.7) and (2.8) are valid for $P(N)$ by **Proposition** 2.8, it suffices to construct n affinely independent feasible schedules tight for (2.7) and (2.8).

Let $s = |S|$, $q = |Q|$.

Let $C^1 \in T(S)$ be a feasible permutation schedule with sequence $(1, 3, Q, 2, 4)$.

For $j = 2, ..., q+2$, let C^j be the permutation schedule with job 1 in the jth-position and the rest of the jobs in the same sequence as in C^1. Further, let C^{q+3} be the feasible permutation schedule with sequence $(3, 4, 1, Q, 2)$, and $C^s = C^{q+4} = C^1 + \mathbf{1}_S$.

Since $C^j - C^1$, $j = 2, ..., s$, are linearly independent, schedules $C^1, ..., C^s$, are affinely independent. Next, we can check that C^{q+3} is tight for all inequalities (2.11), (2.12), (2.13) and (2.14), and all other schedules are tight for (2.9) and (2.10). Therefore, these schedules are tight for (2.7). Finally, since S is intermediate, we can use the **Face-lifting Theorem** to lift these schedules and construct n affinely independent schedules in $T(N)$ tight for (2.7).

The construction of the tight schedules for (2.8) is similar. The proof is complete. □

From **Propositions** 2.7, 2.8, and **Theorems** 2.15, 2.17 and 2.20, we have the following corollary.

Corollary 2.21 *(1) Parallel inequality (2.3) is face-inducing if and only if S is initial; and*

(2) Zero-sum valid inequalities (2.4), (2.7) and (2.8) are face-inducing if and only if S is intermediate.

2.5 Facet Lifting

In this section, we show that, when a job set S results from series or parallel composition of two subsets S_1 and S_2, all facets of $P(S)$ can be derived. Some facets of $P(S)$ are in fact obtained by lifting the facets of $P(S_1)$ or $P(S_2)$. These results further lead to a minimal linear system sufficient for scheduling polyhedra when the precedence constraints are series-parallel.

Lemma 2.22 (Facet-Lowering Lemma) *Let*

$$\sum_{k \in S} w_k C_k \geq w_0 \tag{2.28}$$

be a facet-inducing inequality for $P(S)$ and let $S_0 \subseteq S$ and $S_0 \supseteq \{i \in N : w_i \neq 0\}$. Then $w * C(S_0) \geq w_0$ is a facet-inducing inequality for $P(S_0)$, if S_0 is

(i) an initial set of S with $w(S) > 0$; or

(ii) an intermediate set of S with $w(S) = 0$.

Proof. Either (i) or (ii) implies that for any $\hat{C} \in T(S_0)$, there exists $C \in T(S)$ such that $C_{S_0} - t\mathbf{1}_{S_0} = \hat{C}$, for some $t \geq 0$ (Note. $t = 0$ if $w(S) > 0$). Thus, $\forall \hat{C} \in T(S_0)$, $w * \hat{C}(S_0) = w * C(S_0) \geq w_0$. Therefore, (2.28) is also valid for $P(S_0)$. Further, by **Proposition** 2.4, we have $|S_0|$ affinely independent schedules in $\{C^j_{S_0} : j = 1, ..., s\}$, tight for $w * C(S_0) \geq w_0$, where $C^j, j = 1, ..., s$ are s affinely independent schedules in $T(S)$, tight for (2.28). The proof is complete. □

Set S is said to be the *parallel composition* of S_1 and S_2, denoted by $S = (S_1 || S_2)$, if $S = S_1 \cup S_2$ and neither $i \to j$ nor $j \to i$ for all $i \in S_1$ and $j \in S_2$.

The following theorem characterizes all facets resulting from parallel composition.

Theorem 2.23 *The facet-inducing inequalities for $P(S)$, where $S = (S_1||S_2)$, are precisely the following:*

(i) all facet-inducing inequalities for $P(S_1)$,

(ii) all facet-inducing inequalities for $P(S_2)$, and

(iii) all parallel inequalities $p * C(I_1 \cup I_2) \geq g(I_1 \cup I_2)$,

where I_1 and I_2 are any nonempty initial sets of S_1 and S_2, respectively.

Proof. Since S_1, S_2, and $I_1 \cup I_2$ are initial set of S, any of (i), (ii) (by the **Face-Lifting Theorem**), and (iii) (by **Theorem** 2.18) is sufficient to define a facet of $P(S)$.

Conversely, let

$$\sum_{k \in S} w_k C_k \geq w_0 \tag{2.29}$$

be facet-inducing for $P(S)$ and $\hat{S} = \{i \in S : w_i \neq 0\}$.

If $\hat{S} \subseteq S_1$ (resp., $\hat{S} \subseteq S_2$), by the **Facet-Lowering Lemma**, (2.29) has representation (i) (resp., (ii)).

If both $\hat{S} \cap S_1$ and $\hat{S} \cap S_2$ are nonempty, we claim that $w(S) > 0$ since otherwise $w(S) = 0$ implies, by **Lemma** 2.12, that $w(S_i) < 0$ for some $i \in \{1, 2\}$, and (2.29) is not valid since S_i is a terminal set of S. Since (5.2.1) is positive facet-inducing, by **Theorem** 2.18, $\hat{S}$ is an initial set of S. By Letting $I_1 = \hat{S} \cap S_1$ and $I_2 = \hat{S} \cap S_2$, (2.29) has representation (iii). The proof is complete. □

The following theorem characterizes all facets resulting from series composition.

Theorem 2.24 *The facet-inducing inequalities for $P(S)$, where $S = (S_1 \rightarrow S_2)$, are precisely the following:*

(i) all facet-inducing inequalities for $P(S_1)$,

(ii) all zero-sum facet-inducing inequalities for $P(S_2)$, and

(iii)

$$\begin{aligned} &-(p(I))p*C(T)+(p(T))p*C(I) \\ &\geq p(I)g(T)+p(T)g(I)-p(I)p^2(T), \end{aligned} \tag{2.30}$$

where I and T satisfy the following two conditions:

(a) T is a terminal set of S_1 and I is an initial set of S_2; and

(b) neither T nor I is itself series-decomposable.

Proof First, we need to show any of (i), (ii) and (iii) is sufficient to define a facet of $P(S)$.

(1) Since S_1 is an initial (therefore, intermediate) set of S, by the **Face-Lifting Theorem** , any facet of $P(S_1)$ induces a facet of $P(S)$.

(2) Since S_2 is an intermediate set of S, by the **Face-Lifting Theorem** , any zero-sum facet of $P(S_2)$ induces a zero-sum facet of $P(S)$.

(3) By **Theorem** 2.19, the given conditions imply that (2.30) induces a zero-sum facet for $P(S)$.

Conversely, let

$$\sum_{k\in S} w_k C_k \geq w_0, \quad C \in R^S, \tag{2.31}$$

be facet-inducing for $P(S)$ and let $\hat{S} = \{i \in S : w_i \neq 0\}$. We need to consider only two cases:

(I) If (2.31) is a positive-sum facet, then by **Theorem** 2.18, $\hat{S}$ is an initial set of S and is not series-decomposable which also implies $\hat{S} \subseteq S_1$. Further, by the **Facet-Lowering Lemma**, (2.31) has representation (i).

(II) If (2.31) is a zero-sum facet of $P(S)$, then we need to consider the following two cases:

(1) neither $\hat{S} \cap S_1$ nor $\hat{S} \cap S_2$ is empty.

Since $S = (S_1 \rightarrow S_2)$, we have $\hat{S} \cap S_1 \rightarrow \hat{S} \cap S_2$. By **Theorem** 2.19, (2.31) has representation (iii) with $T = \hat{S} \cap S_1$ and $I = \hat{S} \cap S_2$.

(2) one of $\hat{S} \cap S_1$ and $\hat{S} \cap S_2$ is empty. Observe that both S_1 and S_2 are intermediate set of S, by the **Facet-Lowering Lemma**, (2.31) has one of the representations (i) and (ii). The proof is complete. □

As a straightforward consequence of **Theorem** 2.23 and **Theorem** 2.24, we have the following.

Corollary 2.25 *The following is the unique (up to positive multiples) minimal linear system sufficient to define $P(N)$ with series-parallel precedence constraints:*

(i) $p * C(S) \geq g(S)$, for all non-series-decomposable initial set S of N; and

(ii)

$$\begin{aligned} &-(p(S_2))p * C(S_1) + (p(S_1))p * C(S_2) \\ &\quad \geq p(S_2)g(S_1) + p(S_1)g(S_2) - p(S_2)p^2(S_1), \end{aligned}$$

for all non-series-decomposable sets S_1 and S_2 such that $S = S_1 \rightarrow S_2$ is an intermediate set of N.

Given a scheduling polyhedron $P(N)$ subject to series-parallel precedence constraints, **Corollary** 2.25 gives a complete characterization of all facets of $P(N)$. These facets can be generated by repeatedly applying the rules stated in **Theorem** 2.23 and **Theorem** 2.24, starting from the leaves (singleton subsets of N) and following the compositions specified by any series-parallel decomposition tree of N (see Lawler [34], for details on the series-parallel decomposition trees).

If the precedence relations on the job set are not series-parallel, the facet-inducing inequalities of the scheduling polyhedron $P(N)$ are not all of parallel or series inequalities, as shown by the following theorem.

Theorem 2.26 *All facet-inducing inequalities for $P(N)$ are in the form of either (2.3) or (2.4) if and only if $G(N)$ is series-parallel.*

Proof. Sufficiency follows from **Corollary** 2.25.

Necessity: If $G(N)$ is not series-parallel, it is well known that N is Z-inducing (see Lawler [34]).

Let $S \subseteq N$ be a minimal intermediate Z-inducing set.

Renumber the jobs such that $S = \{1, 2, 3, ..., k\}$,

$$\{(i,j) \in \bar{A}(N) : \forall i, j \in \{1,2,3,4\}\} = \{(1,2),(3,2),(3,4)\},$$

and $Q = \{5, ..., k\}$. Note that Q may be empty..

Since S is a minimal intermediate Z-inducing set, $\{1,2\}$, $\{3,4\}$ and $\{3,2\} \cup Q$ are intermediate sets of N. Moreover, for all $j \in Q$, we have $(3,j) \in \bar{A}(N)$ and $(j,2) \in \bar{A}(N)$. By **Theorem** 2.20, Z-inequality (2.7) or (2.8) induced by S is a facet-inducing inequality for $P(N)$. The proof is complete. □

Note that, for series-parallel precedence constraints, facets are uniquely defined by the index set of their nonzero coefficients. However, **Theorems** 2.20 and 2.26 show that this is never the case when the precedence constraints are not series-parallel.

2.6 Proofs of the four lemmata

Proof of Lemma 2.9. Let $S \doteq \{i : w_i \neq 0\}$. and $F \doteq \{C \in P(N) : w * C(S) = w_0\}$.

First, F is an $(n-b-1)$-dimensional face of $P(N)$ if and only if there are precisely $n-b$ affinely independent points $\hat{C}^k \in F, k = 1, 2, ..., n-b$. Let $C^j \in T(N), j = 1, 2, ..., m$ be a minimal collection of affinely independent feasible schedules such that each $\hat{C}^k$ can be represented by a positive convex combination of $C^j s$. Then, we have $\forall k$,

$$w_0 = w * \hat{C}^k(S) = \sum_{h \in S} w_h \sum_{j=1}^{m} v_{kj} C_h^j = \sum_{j=1}^{m} v_{kj} w * C^j(S) \geq w_0,$$

where $v_{kj} \geq 0, \forall j$, and $\sum_{j=1}^{m} v_{kj} = 1$, which implies that $C^j \in F$, $j = 1, ..., m$.

Thus, $C^j s$ also span the linear hull of F. Hence, $\dim(F) = n - b - 1$ if and only if there are precisely $n - b$ affinely independent schedules in F. The proof is complete. □

Proof of Lemma 2.10. Let $s = |S|$.

Sufficiency. Suppose S is series-decomposable, i.e. $S = S_1 \rightarrow S_2$.

Let $C^j \in R^s$, $j = 1, ..., l$, be a maximal collection of affinely independent permutation schedules in $T(S)$. Clearly, all $C^j s$ are tight for both (2.3) and series-inequality (2.4), whose coefficients are linearly independent. It follows that $l \leq s + 1 - 2 = s - 1$.

Necessity. Suppose S is not series-decomposable.

By induction on $|S|$. For $|S| = 1$, the *necessity* trivially holds. Suppose it holds for all $|S| \leq s - 1$.

Let $S' = S \setminus \{i\}$, where $\{i\}$ is a terminal set in S. We have two cases:

(1) S' itself is not series-decomposable. Then, by the induction hypothesis, there are $s - 1$ affinely independent permutation schedules in $T(S')$, namely, $\hat{C}^j, j = 1, ..., s - 1$. Observe that there exists some terminal set $\{k\}$ of S' such that job k can be sequenced after job i, since otherwise $S = (S' \rightarrow \{i\})$.

Define

$$C_h^j = \begin{cases} \hat{C}_h^j, & \text{if } h \in S'; \\ p(S), & \text{if } h = i, \end{cases}$$

for all $j = 1, ..., s - 1$, and let C^s be the permutation schedule in $T(S)$ such that job k is sequenced last and the rest of the jobs are sequenced the same order as in C^1. We easily verify that $C^1, ..., C^s$ are affinely independent permutation schedules in $T(S)$.

(2) S' is series-decomposable, i.e., $S' = (S_1 \rightarrow S_2)$.

First, observe that $S_1 \cup \{i\}$ is not series-decomposable, since otherwise $S_1 \cup \{i\} = (S'' \rightarrow S''')$ with S''' containing terminal job i, and thus, $S = S'' \rightarrow (S''' \cup S_2)$, a contradiction. Therefore, by the induction hypothesis, there are $|S_1| + 1$ affinely independent

permutation schedules in $T(S_1 \cup \{i\})$, namely, $C^j_{S_1\cup\{i\}}$, $j = 1, ..., |S_1| + 1$.

Let $C_{S_2} \in T(S_2)$ be any permutation schedule and define

$$C^j_h = \begin{cases} (C^j_{S_1\cup\{i\}})_h, & \text{if } h \in S_1 \cup \{i\}; \\ (C_{S_2})_h + p(S_1 \cup \{i\}), & \text{if } h \in S_2, \end{cases}$$

for $j = 1, ..., |S_1| + 1$.

Let π be the permutation induced by C^1.

For $|S_1| + 2 \leq j \leq s$, let C^j be a permutation schedule with job i sequenced at the $\pi(j)$-th position and the rest of the jobs are sequenced in the same order as in C^1.

Observe that no job $k \in S_2$ has precedence over job i, since otherwise $S = S_1 \rightarrow (S_2 \cup \{i\})$, a contradiction. Therefore, all C^j, $j = 1, ..., s$, are feasible permutation schedules in $T(S)$. Moreover, since $C^j - C^1$, $j = 2, 3, ..., s$, are linearly independent, $C^j, j = 1, 2, ..., s$ are affinely independent.

In either case, we can construct s affinely independent permutation schedules in $T(S)$. The proof is complete. □

Proof of Lemma 2.12.

(1) Since (2.20) is facet-inducing, there are n affinely independent schedules, $C^j \in T(N)$, $j = 1, ..., n$, satisfying:

$$\sum_{k\in N} w_k C^j_k = w_0.$$

(2) We claim that

$$\sum_{k\in S} w_k C^j_k = v_j = v,$$

for all $j = 1, ..., n$, and prove it by contradiction.

Suppose not. Then, there exist some h, l such that $v_h > v_l$. Let

$$\hat{C}_i = \begin{cases} (C^l_S)_i, & \text{if } i \in S; \\ (C^h_{N\setminus S})_i + u, & \text{if } i \in N \setminus S. \end{cases}$$

Since S is an initial set of N, $\hat{C} \in T(N)$ for u sufficiently large. Moreover,

$$\sum_{k \in N} w_k \hat{C}_k = v_l + w_0 - v_h < w_0,$$

which contradicts the fact that (2.20) is a valid inequality for $P(N)$.

From (1), (2), and **Corollary** 2.5, it follows that $w_0 = v$ and $w_k = 0$ for all $k \in N \setminus S$. The proof is complete. □

Proof of Lemma 2.13 Let X be any initial set of N and $\bar{X} = N \setminus X$ such that $X \cap S \neq \emptyset$ and $\bar{X} \cap S \neq \emptyset$. Observe that $\bar{X}$ is a terminal set of N, we have $w(\bar{X}) \geq 0$ since otherwise (2.21) is not valid. Using condition (i), it is trivial that $w(\bar{X}) > 0$. Using condition (ii), we cannot have $w(\bar{X}) = 0$ since otherwise we have $w(X) = w(N)$ and by **Lemma** 2.12, $w_i = 0, \forall i \in \bar{X}$. This further implies $\exists k \in S, w_k = 0$, a contradiction. Therefore, we have $w(\bar{X}) > 0$ in either case.

Next, by contradiction, suppose $\exists C \in T(N)$ tight for (2.21) such that $C_Q - t\mathbf{1}_Q$, for any $t \geq 0$, is not a permutation schedule in $T(Q)$. Then, C represents a feasible schedule such that either there exists some idle time between the completion times of some jobs in Q, or jobs in Q are not sequenced consecutively. Let $I(Q)$ be the smallest initial set containing Q. Let π be a permutation induced by C. For $w(N) > 0$, we note that

(a) $I(Q) = Q$, by definition of Q and $I(Q)$;

(b) $C_{\pi(1)} = p_{\pi(1)}$, since otherwise $\hat{C} \doteq C - (C_{\pi(1)} - p_{\pi(1)})\mathbf{1}_N \in T(N)$, but $w * \hat{C}(N) = w_0 - w(N) < w_0$, a contradiction. Hence, there is no idle time before $C_{\pi(1)}$ for $w(N) > 0$.

Define

$$\delta(x) = \begin{cases} 1, & \text{if } x > 0; \\ 0, & \text{if } x \leq 0. \end{cases}$$

By **Proposition** 2.3, Q is a terminal set of $I(Q)$. Let $M_c = \max\{C_k : k \in I(Q)\}$ and

define a schedule $\hat{C}$ as follows:

$$\hat{C}_i = \begin{cases} C_i, & \text{if } i \in I(Q) \setminus Q; \\ C_i + M_c(1 - \delta(w(N))), & \text{if } i \in Q; \\ C_i + 2M_c, & \text{if } i \in N \setminus I(Q). \end{cases}$$

Observe that $\hat{C} \in T(N)$ and is also tight for (2.21) but has some idle time inserted between the completion times of some jobs in Q. (Note: For $w(N) > 0$, none of the completion times of jobs in Q change, i.e., $C_Q = \hat{C}_Q$. Nevertheless, if C_Q contains any idle time, so does $\hat{C}_Q$; else some other job is sequenced between jobs in Q and thus, by definition, $\hat{C}_Q$ contains idle time). Since Q is the smallest (intermediate if $w(N) = 0$; or initial if $w(N) > 0$) set containing S, there exists an idle time interval $(t, t+u)$, for some $t, u > 0$ (for $w(N) > 0$, cf. Note (*b*) above), such that some jobs in S are processed before t and some jobs in S are processed after $t + u$.

Let $X = \{i \in N : \hat{C}_i \leq t\}$ $\bar{X} = \{i \in N : \hat{C}_i > t + u\}$. Clearly, $N = X \cup \bar{X}$ and both $S \cap \bar{X}$ and $S \cap X$ are nonempty.

Define $\bar{C} = \hat{C} - u \sum_{i \in \bar{X}} \mathbf{e}_i$.

It is clear that $\bar{C} \in T(N)$. Since we have before $w(\bar{X}) > 0$,

$$\sum_{i \in N} w_i \bar{C}_i = w_0 - w(\bar{X})u < w_0,$$

a contradiction. This completes the proof for the lemma. □

Chapter 3

Single Machine Scheduling: Polyhedral Computations

This chapter presents a cutting plane procedure, using *facet cuts* derived in Chapter 2. In addition, extensive computational results as well as their analysis are provided.

3.1 Introduction

For the last thirty-five years, since the work of Smith [54], the single machine scheduling problem has been one of the challenging research topics in combinatorial optimization. The problem with general precedence constraints was shown to be strongly NP-hard by Lawler [33] and Lenstra and Rinnooy Kan [37]. For the series-parallel precedence constraints, an elegant $O(n \log n)$ algorithm was derived by Lawler [33]. A discussion of the algorithms (up to 1985) for the problem with general constraints can be found in Potts [50]. Very tight formulations with $O(n^2)$ binary variables and $O(n^3)$ constraints are studied by Wolsey [57]. A new Lagrangian-based lower bounding method with $O(n)$ continuous variables (and one Lagrangian multiplier per nonredundant precedence constraint) is introduced in Van de Velde [56].

In spite of these research efforts, one is still unable to solve even medium-sized problems. The reported largest solved problem instances involve no more than 100 jobs, see Potts [50], where Branch-and-Bound methods were used for obtaining optimal solutions. The reason why the methods do not apply to the larger problems may be a combination of (1) the number $O(n^2)$ of binary variables in the integer programming formulations, and (2) weak lower bounds. To a large extent, the present polyhedral approach overcomes

both difficulties.

In this chapter, a linear programming based cutting plane algorithm is developed for the scheduling problem. The proposed algorithm works on a *compact* linear formulation with only n variables, one per job. Two families of facet-defining inequalities, the parallel inequalities and the so-called *simple series inequalities*, are used in the cutting plane procedure. Extensive computational experiments are conducted on an IDM 386/25 Personal Computer with a sample of 280 problems of sizes 30 to 160 jobs. The average computation time for the 160-job problems is less than 20 minutes. The results also show that the percent gap in objective function value between the feasible solution derived from the cutting plane algorithm and the linear programming lower bound is less than 0.25 percent on the average, and less than 1 percent in the empirical worst case. A detailed computational report can be found in Section 3.4.

3.2 A cutting plane procedure

The proposed cutting plane algorithm consists of three stages. It begins by solving an initial linear programming (LP) problem. This initial LP formulation defines a polyhedron containing the set of feasible schedules. The second stage is an iterative process: at each iteration, it generates (1) an optimal solution to the current LP relaxation, which serves as a lower bound (LB) ; (2) possibly a valid inequality or a *cut*, violated by the LP solution; (3) a heuristic schedule which serves as an upper bound (UB). The cuts are successively added to the LP formulation. This process shrinks the feasible region and yields increasingly tighter bounds on the value of the optimal schedules. The iteration terminates when either the current UB equals LB, (hence the corresponding schedule is optimal) or when the LP solution cannot be cut off by any other available cuts. At the third stage, a heuristic is applied to the incumbent schedule for possible improvement

(if it is not proven optimal), and then a final solution report is produced. For a general discussion on cutting plane methods, the reader is referred to the text by Nemhauser and Wolsey [45].

3.2.1 Initial Formulation

The proper choice of an initial formulation is important in cutting plane algorithms. We first include all the following simple facets (3.1) and (3.2):

$$\begin{array}{lll} \min & \sum_{j\in N} w_j C_j & \\ st. & C_j \geq p_j, & \text{all } \{j\} \text{ initial} \qquad (3.1) \\ & C_j - C_i \geq p_j, & \text{all nonredundant } (i,j). \qquad (3.2) \end{array}$$

The inclusion of all these facets ensures that the LP solutions obtained from stage 2 induce feasible *sequences.*

Computational experiments have shown that this initial formulation is satisfactory in most cases, except for problem instances with very few precedence relations. From our computational experience, we found that, when only a few precedence constraints are present, the initial formulation yields very weak bounds. As a result, too many cuts, and thus excessive computation time, are needed to reduce the gap between UB and LB.

To speed up the computation for these sparsely precedence-constrained problems, we extend the initial formulation by including an additional set of cuts (most of them known to be facet-defining). The set of cuts consists of $n-1$ parallel inequalities obtained as follows. We first use a heuristic method to find a reasonably good feasible sequence, say, $(\pi(1), \pi(2), ..., \pi(n))$, see Section 3.2.3 for details. Let $S_i \doteq \{\pi(1), \pi(2), ..., \pi(i)\}$, $i = 2, 3, ..., n$. Note that these sets S_i necessarily induce initial sets. Then, we add to the original initial formulation the following parallel inequalities:

$$p * C(S_i) \geq g(S_i), \qquad i = 1, 2, ..., n.$$

Recall from Chapter 2 that the inequality induced by set S_i is facet-defining, unless the set S_i is series-decomposable. The computational experience confirms the effectiveness of this method for sparse precedence constraints.

3.2.2 Cutting Planes and Separation Algorithms

Two types of cuts are used in the separation routine: parallel cuts and simple series cuts. Recall that the parallel cuts take on the following form:

$$p * C(S) \geq g(S), \quad \text{where } S \text{ is a nonempty initial set.} \tag{3.3}$$

Queyranne [52] proposed an $O(n \log n)$ separation procedure for the scheduling problem with no precedence constraints. Note that the procedure is still valid for the precedence constrained scheduling problems: the LP solution at each iteration necessarily induces a feasible sequence, because of the inclusion of all non-redundant precedence constraints in the initial formulation. Thus, any set $S = \{\pi(1), \cdots, \pi(i)\}$ identified by the algorithm below is necessarily an initial set.

Parallel Cut Separation:
INPUT: Integer n; processing time vector p; completion time vector C.
RESULT: a subset $S \subseteq \{1, \ldots, n\}$ maximizing the violation
$\Gamma(S) \doteq g(S) - p * C(S)$ of (3.3).

begin
1. Sort $N = \{1, \ldots, n\}$ as $N = (\pi(1), \ldots, \pi(n))$
   in nondecreasing order of the C_j coefficients;
2. Let $S' := \emptyset$; $\rho := 0$; $\gamma' := 0$; $\gamma := 0$; $S := \emptyset$;
3. **For** $i := 1$ **to** n **do**
   begin
   $k := \pi(i)$; $S' := S' \cup \{k\}$; $\rho := \rho + p_k$; $\gamma' := \gamma' + p_k(\rho - C_k)$;
   if $\gamma' > \gamma$ **then begin** $\gamma := \gamma'$; $S := S'$ **end**
   end

end.

Note that in steps 2 and 3, $\rho = P(S')$ and $\gamma' = \Gamma(S')$. Note also that except for

the $O(n\log(n))$ sort in Step 1, the algorithm runs in linear time. See Queyranne [52] for details.

We now relate the lower bound obtained after adding all (relevant) parallel cuts, to the lower bound obtained by Van de Velde [56]. We may write the precedence constrained scheduling problem as follows.

$$Z^{opt} = \min \quad \textstyle\sum_{j\in N} w_j C_j \tag{3.4}$$

$$st. \quad C_j - C_i \geq p_j, \qquad \text{all nonredundant } (i,j) \tag{3.5}$$

$$C \in Q \tag{3.6}$$

where Q is the set of all feasible schedules.

Let B be the arc-node incidence matrix of $G = (N, A(N))$. Consider the following linear program:

$$LB_P = \min \quad wC \tag{3.7}$$

$$st. \quad BC \geq b \tag{3.8}$$

$$p * C(S) \geq g(S), \quad \text{all } S \subseteq N \tag{3.9}$$

where b is an appropriate right-hand-side, so (3.8) is just (3.5) written in matrix form. As (3.9) is a relaxation of (3.6) (cf. Queyranne [52]), problem (3.7)– (3.9) is a relaxation of problem (3.4)– (3.6). So $LB_P \leq Z^{opt}$. In effect, LB_P is precisely the lower bound obtained by the above cutting plane method after all parallel cuts are exhausted.

Van de Velde's lower bound is obtained by relaxing precedence constraints (3.5) with Lagrangian multipliers $\lambda_{ij} \geq 0$. For any vector $\lambda \geq 0$ of such Lagrangian multipliers, the subproblem in this approach is the following unconstrained scheduling problem

$$\min\{wC + \lambda(b - BC) : C \in Q\} = \min\{(w - \lambda B)C : C \in Q\} - \lambda b, \tag{3.10}$$

which is solved by Smith's rule in $O(n\log n)$ computational steps [54]. The best possible lower bound LB_R, which can be obtained using this approach, is the optimum value of

the following problem:

$$\begin{aligned} LB_R &= \max_{\lambda \geq 0}\{\min\{wC + \lambda(b - BC) : C \in Q\}\} && (3.11)\\ &= \max_{\lambda \geq 0}\{\min\{wC + \lambda(b - BC) : C \in \text{conv } Q\}\} && (3.12)\\ &= \max_{\lambda \geq 0}\{\min\{wC + \lambda(b - BC) : p * C(S) \geq g(S), \forall S \subseteq N\}\} && (3.13)\\ &= \min\{wC : BC \geq b, p * C(S) \geq g(S), \forall S \subseteq N\} && (3.14) \end{aligned}$$

where (3.12) follows since the objective function in (3.11) is linear in C; (3.13) follows from Queyranne [52]; (3.14) follows from the Geoffrion's theory of Lagrangian Relaxation [24]. Thus, we have shown the following result:

Theorem 3.1 *The optimal lower bound in Van de Velde's Lagrangian Relaxation approach is equal to the lower bound LB_P obtained by the cutting plane approach using only the parallel cuts.*

Note that Van de Velde does not solve (3.11) exactly, but only approximately using an ascent method for improving the Lagrangian multipliers. Therefore, his lower bound is generally weaker than LB_P.

The second type of cuts is a subclass of series cuts, the *simple series cuts*, where one of the subsets, S_1 or S_2 in $S_1 \rightarrow S_2$, is a singleton. So, simple series cuts are of the form

$$-p(S_2)C_u + p * C(S_2) \geq g(S_2), \quad \text{where } \{u\} \rightarrow S_2, \tag{3.15}$$

or

$$-p * C(S_1) + p(S_1)C_v \geq g'(S_1 \cup \{v\}), \quad \text{where } S_1 \rightarrow \{v\}. \tag{3.16}$$

For simplicity, we denote the simple series inequalities by $\Gamma(S_1, S_2) \leq 0$, where S_1 (resp., S_2) is singleton if it is the cut (3.15) (resp., (3.16).), and $\Gamma(S_1, S_2)$ denotes the amount by which the cut is violated.

We now develop an $O(n^2)$ separation procedure for the series cuts. To do so, we rewrite the above inequalities in the following equivalent forms: for (3.15),

$$\sum_{j \in S_2} p_j(C_j - C_u) \geq g(S_2), \quad \text{where } \{u\} \to S_2, \tag{3.17}$$

and, for (3.16),

$$\sum_{i \in S_1} p_i(t_v - t_i) \geq g(S_1), \quad \text{where } S_1 \to \{v\}. \tag{3.18}$$

where $t_i = C_i - p_i$ is the *start time* of job $i \in N$,

Inequality (3.17) can be viewed as a parallel inequality with respect to redefined "completion times" $C_j - C_u$ and restricted to those jobs j that are successors of given job u. For inequality (3.18), one may imagine that all jobs in S_1 start backward from time t_v, and thus have "completion times" $t_v - t_i$. Therefore, it is also in the form of a parallel cut, restricted to the set of jobs that are predecessors of job v. For convenience, we call $\{u\} \to S_2$ a *fan-out structure*, and $S_1 \to \{v\}$ a *fan-in structure.*

A simple extension of the Parallel Cut Separation leads to a separation procedure for the simple series cuts. For every $u = \pi(1), \pi(2), \cdots, \pi(n-2)$, apply the Parallel Cut Separation to the set of jobs that are successors of u, using $C_j - C_u$ as completion times. This produces a fan-out structure that maximizes the violation of (3.17). A similar procedure applies to (3.18). In the end, a most violated simple series cut is identified. Thus, finding a most violated simple series inequality of either fan-in or fan-out structure requires $O(n^2)$ computational steps. This Simple Series Cut Separation algorithm is described in details on the following page (Recall that $\bar{A}(N)$ denotes the transitive closure of $A(N)$).

Simple Series Cut Separation:
INPUT: Integer n; processing time vector p; completion time vector C.
RESULT: Find disjoint subsets S_1 and S_2, such that $S_1 \longrightarrow S_2$,
one of them is a singleton set, and $\Gamma(S_1, S_2)$ is maximized.

begin

1. Sort the job set as $N = (\pi(1), \ldots, \pi(n))$
   in nondecreasing order of the completion times C_j;
   and as $N = (\pi'(1), \ldots, \pi'(n))$
   in nonincreasing order of the start times t_j;
2. [FORWARD SEARCH]
   Let $S' := \emptyset$; $\rho := 0$; $\gamma' := 0$; $\gamma_1 := 0$; $S_2 := \emptyset$;
3. **For** $h := 1$ **to** $n - 2$ **do**
   For $j := h + 1$ **to** n **do**
   if $(\pi(h), \pi(j)) \in \bar{A}(N)$ **then**
   begin
   $i := \pi(h)$; $k := \pi(j)$; $S' := S' \cup \{k\}$;
   $\rho := \rho + p_k$; $\gamma' := \gamma' + p_k(\rho - C_k + C_i)$;
   if $\gamma' > \gamma_1$ **then begin** $\gamma_1 := \gamma'$; $S_2 := S'$, $u := i$ **end**
   end;
4. [BACKWARD SEARCH]
   Let $S' := \emptyset$; $\rho := 0$; $\gamma' := 0$; $\gamma_2 := 0$; $S_1 := \emptyset$;
5. **For** $h := n$ **step** -1 **to** 2 **do**
   For $j := h - 1$ **step** -1 **to** 1 **do**
   if $(\pi(j), \pi(h)) \in \bar{A}(N)$ **then**
   begin
   $i := \pi'(h)$; $k := \pi'(j)$; $S' := S' \cup \{k\}$;
   $\rho := \rho + p_k$; $\gamma' := \gamma' + p_k(\rho - t_i + t_k)$;
   if $\gamma' > \gamma_2$ **then begin** $\gamma_2 := \gamma'$; $S_1 := S'$, $v := i$ **end**
   end
6. Retrieve inequality $\Gamma(S_1, S_2) \leq 0$ from
   (3) if $\gamma_1 > \gamma_2$ and from (5) otherwise.

end.

3.2.3 Heuristics for the Upper Bounds

Cutting plane algorithms often allow the easy derivation of good feasible solutions. This is the case for the precedence-constrained scheduling problem studied here. The following heuristic procedures are used in our algorithm.

The first heuristic is a procedure for obtaining an initial feasible sequence. Recall that this initial sequence is used to establish a tighter LP formulation than the original one. The heuristic is constructive, starting with the whole job set $S = N$ and the given precedence graph G. Then, repeat the following process until the set S is empty: remove from S an initial job i (relative to current graph G) with largest ratio w_i/p_i; update $S \leftarrow S \setminus \{i\}$ and G. The sequence in which the jobs have been removed is, by construction, a feasible sequence.

This simple heuristic has proven to be effective when the precedence graph is sparse. In particular, it yields an optimal schedule when $A(N)$ is empty (Smith [54]). There are other heuristics for obtaining feasible schedules, such as Morton and Dharan's Tree Optimal heuristic [39]. Sophisticated heuristics may not be very useful in our algorithm. We briefly tested the Tree Optimal heuristic solution against the feasible schedules induced by the LP solutions at stage 2 on several 100-job problem instances. We found that the cutting plane procedure consistently generated better feasible schedules, and generally used less CPU time, than the Tree Optimal heuristic.

The second heuristic derives a feasible sequence from an LP solution by employing a so-called *priority rule*. At each iteration, that is, when a cut is added, an optimal solution C to the current LP relaxation is obtained. Since all precedence constraints are included in the initial LP formulation, the solution C necessarily induces a feasible sequence in which job i is processed before job j whenever $C_i \leq C_j$.

We then try to improve the feasible sequence produced by the above heuristic. This

involves *1-OPT* type interchanges. First, we do a backward search: for $i = n-1, ...2, 1$, find a set S of jobs immediately following, but not constrained by, job i and with largest ratio $w(S)/p(S)$; if the largest ratio $w(S)/p(S)$ is larger than w_i/p_i, then update the sequence by interchanging i and S. We then perform a similar forward search, trying to exchange job j with a set of jobs immediately preceding it. We call this procedure *One-Pass* 1-OPT. A *Full 1-OPT* is to repeat One-Pass 1-OPT until no further improvement can be made.

3.2.4 The Cutting Plane Procedure

The whole cutting plane procedure is summarized as follows.

Step 0: Read data. Start the timing subroutine.
Step 1: Compute the transitive reduction and transitive closure of the precedence digraph. (The former is used for setting up an initial formulation, and the latter for the simple series cut separation.)
Step 2: Set up the initial LP formulation and solve this LP problem.
Step 3: (a) Generate a heuristic feasible schedule from the current LP solution; store the schedule if its cost is the best found so far; go to Step 5 if UB=LB, where UB is the cost of the best schedule found so far, and LB is the cost of the current LP solution.
(b) Apply the Parallel Cut Separation. Go to (d) if a violated parallel cut is found;
(c) Apply the Simple Series Cut Separation. Go to **Step 4** if no violated series cut is found;
(d) Add the violated cut and reoptimize the extended LP to update the LP solution and its cost LB; Go to (a).
Step 4: Apply the Full 1-OPT heuristic to the best feasible schedule for final improvement;
Step 5: Stop the timing subroutine; Produce a solution report; Halt.

Note that **Step 3** is an iterative phase, the most time-consuming part of the algorithm.

The algorithm, written in FORTRAN, uses the XMP Linear Programming Library as the LP optimizer in Steps 2 and 3. To facilitate the use of the subroutines in XMP, an equivalent *start time* formulation is used. Since start times $t_j = C_j - p_j$, for all j, the formulation replaces all trivial constraints $C_j \geq p_j$ by the nonnegativity constraints on all t_j variables. The solution report, however, is presented in terms of completion times for comparison with other solution methods. At the present, the algorithm is coded for problems of size up to 400 jobs. Extending the problem size requires changing only a few statements.

3.3 Computational Experience

In this section, we report our computational results on a fairly large sample of randomly generated problems. We evaluate the algorithm based on the computation time, and the *percent gap* $r \doteq 100(UB - LB)/LB$. For example, if a feasible schedule C with value UB satisfies $r \leq 0.3$, then it is guaranteed, even without knowing the optimal value Z^{opt}, that the schedule cannot deviate from the optimal schedule by more than 0.3 percent, since

$$(UB - Z^{opt})/Z^{opt} \leq (UB - LB)/LB \leq 0.003.$$

To illustrate how the cutting plane procedure works, we begin with two simple numerical examples.

EXAMPLE 1. Consider Potts' 10-job example [50].

Job i	1	2	3	4	5	6	7	8	9	10
p_i	6	9	1	3	9	5	7	7	6	2
w_i	2	5	9	6	5	4	9	3	8	5

Table 3.1: Data for the 10-job example

The precedence relations can be represented by the following nonredundant arc set

$$A(N) = \{(1,6),(1,7),(2,4),(2,5),(3,8),(5,9),(6,10),(7,9),(8,10)\}.$$

After solving the initial LP program, which includes all constraints of types (3.1) and (3.2), an LP solution of value 1526 is obtained. In addition, the heuristic gives a feasible schedule of value 1530. Table 3.2 summarizes the execution of the algorithm, where i denotes the iteration index in Step 3, and the second column represents the cut type, "P" for a parallel cut and "SS" for a simple series cut.

i	Cut type	Violated facet cut	LB	UB	% Gap
1	1	$6C_1 + 9C_2 + C_3 + 3C_4 + 9C_5 + 7C_7 + 6C_9 \geq 987$	1526.	1530.	4.
2	P	$9C_2 + C_3 + 3C_4 \geq 130$	1526.7	1530.	3.3
3	SS	$-6C_1 - 9C_2 - 9C_5 - 7C_7 + 31C_9 \geq 543$	1527.2	1530.	2.8
4	SS	$-6C_1 - 5C_6 - 7C_8 + 18C_{10} \geq 143$	1527.5	1530.	2.5
5	SS	$-6C_1 - 9C_5 - 7C_7 + 22C_9 \geq 291$	1528.0	1530.	2.
6	SS	$-5C_6 - 7C_8 + 12C_{10} \geq 59$	1530.	1530.	0.

Table 3.2: Cutting planes and the corresponding solutions

Observe that after adding two parallel cuts and four series cuts, the problem is solved to optimality without any branch and bound. Note also that Potts' lower bound, obtained after sixteen iterations, is 1516, weaker than our *initial* LP value 1526.

EXAMPLE 2. Consider Wolsey's 30-job example [57].

Job i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
p_i	57	45	50	76	73	82	51	19	82	94	98	15	52	78	75
	4	44	89	80	74	43	9	32	2	43	56	27	45	68	63
w_i	9	3	7	1	6	8	6	4	3	6	7	8	9	5	3
	5	5	2	5	1	5	1	8	7	7	7	2	5	9	2

Table 3.3: Data for the 30-job problem

The nonredundant precedence relations can be represented by the following arc set.

$$
\begin{array}{lllll}
A(N) = \{ & (1,2), & (2,6), & (2,7), & (3,8), & (3,9), \\
 & (3,13), & (4,7), & (4,9), & (5,8), & (5,19), \\
 & (6,9), & (6,12), & (7,8), & (7,10), & (7,13), \\
 & (8,11), & (9,18), & (9,25), & (10,12), & (11,15), \\
 & (11,16), & (11,17), & (12,17), & (12,25), & (13,16), \\
 & (13,17), & (13,19), & (14,20), & (14,21), & (14,23), \\
 & (15,18), & (15,20), & (17,18), & (17,22), & (18,21), \\
 & (18,23), & (19,20), & (19,21), & (19,28), & (21,27), \\
 & (21,29), & (21,30), & (22,26), & (23,24), & (24,28), \\
 & (24,30), & (25,28), & (26,28), & (26,29), & (26,30) \}
\end{array}
$$

Table 3.4 gives a summary of the solution report for each iteration. The third column represents the cumulative computation time (IDM 386/25). Note that the initial lower bound (LB) is 56,495.18. After five parallel cuts were added, no additional violated parallel cut is found, and the current LB of 119,329.04 is the best possible lower bound LB_R that can be obtained by Van de Velde's Lagrangian relaxation approach. The algorithm exhausted all parallel and simple series cuts at iteration 31, with a total of 6 parallel cuts and 25 series cuts. At this point, $LB = 121,031.9$ and $UB = 121,757$ with the percent gap below 0.6 percent. The cumulative computation time is 12.74 seconds.

Iter	Cut type	Cumul.time	LB	UB	% Gap
0		1.37	56495.1757	123568	118.72310
1	P	1.53	71642.9524	123568	72.47754
2	P	1.81	103696.0370	121799	17.45772
3	P	2.52	116799.2382	121799	4.28065
4	P	2.85	119231.7078	121799	2.15320
5	P	3.02	119329.0365	121799	2.06988
6	SS	3.35	119422.0401	121799	1.99039
7	SS	3.68	119611.9931	121799	1.82842
8	SS	3.95	119623.1333	121799	1.81893
9	SS	4.28	119703.7625	121799	1.75035
10	SS	4.66	119867.9137	121799	1.61101
11	SS	4.99	119959.1578	121799	1.53372
12	SS	5.32	120062.2191	121799	1.44657
13	SS	5.65	120364.2442	121799	1.19201
14	P	5.87	120425.5471	121799	1.14050
15	SS	6.31	120638.8663	121799	.96166
16	SS	6.64	120674.8199	121799	.93158
17	SS	6.97	120811.7984	121799	.81714
18	SS	7.30	120848.5473	121799	.78648
19	SS	7.69	120881.8021	121757	.72401
20	SS	8.07	120904.6679	121757	.70496
21	SS	8.40	120914.2967	121757	.69694
22	SS	8.78	120932.7068	121757	.68161
23	SS	9.11	120941.9816	121757	.67389
24	SS	9.50	120955.3635	121757	.66275
25	SS	9.88	120981.0002	121757	.64142
26	SS	10.27	121001.3894	121757	.62446
27	SS	10.60	121017.5399	121757	.61104
28	SS	10.98	121026.4292	121757	.60365
29	SS	11.53	121031.6281	121757	.59932
30	SS	11.92	121031.7969	121757	.59918
31	SS	12.30	121031.8770	121757	.59912

Table 3.4: Solutions from the cutting plane procedure

Job	Schedule (optimal)	Schedule (at iter 31)	LP Solution (final)	LP Solution (at iter 5)
1	57	57	57.0000	57.0000
3	107	107	107.0000	107.0000
2	152	152	234.1205	382.1086
6	234	234	443.1205	527.1086
4	310	310	310.1205	382.1086
7	361	361	361.1205	433.1086
13	413	413	413.1205	485.1086
10	507	599	537.1205	527.1086
12	522	614	552.1205	542.1086
5	595	486	434.1205	433.1086
8	614	505	453.1205	452.1086
11	712	712	623.4538	550.1086
16	716	716	627.4538	554.1086
17	760	760	757.1067	594.1086
22	769	769	766.1067	603.1086
26	825	825	822.1067	659.1086
9	907	907	963.5766	1092.1319
25	950	950	1006.5766	1135.1319
14	1028	1028	1041.5766	1181.1319
15	1103	1103	1116.5766	1092.1319
18	1192	1192	1205.5766	1181.1319
23	1224	1224	1237.5766	1213.1319
24	1226	1226	1239.5766	1215.1319
19	1306	1306	1260.1559	1181.1319
21	1349	1349	1323.5428	1224.1319
29	1417	1417	1391.5428	1292.1319
28	1462	1462	1365.8683	1260.1319
27	1489	1489	1489.0000	1489.0000
30	1552	1552	1552.0000	1552.0000
20	1626	1626	1626.0000	1626.0000

Table 3.5: The optimal schedule and the relevant solutions

At the final upper bound improvement stage, we first applied a Full 1-OPT but failed to improve the upper bound. Then we tried to interchange a 2-job subsequence and its adjacent subsequence (cf. Table 3.5, the interchange of subsequences 5,8 and 10,12), and obtained a feasible schedule with the objective value 121,559, and 113,214 with respect to start times. This has been shown to be optimal by Wolsey [57]. Relative to 121,559, the percent gaps of Van de Velde's lower bound and our lower bound are about 1.9% and 0.47%, respectively. The optimal schedule as well as relevant heuristic and LP solutions are given in Table 3.5. Note that the completion times for the first two jobs and the last three jobs in Table 3.5 are identical for all columns. This may suggest a possible decomposition method for further improving the algorithm.

We now test the algorithm on a sample of randomly generated problems of size n, ranging from $n = 30$ to 160 jobs. For each job i, an integer job processing time p_i was obtained from the uniform distribution [1,100], and an integer weight w_i from the uniform distribution [1,10]. The precedence graph G was induced by specifying the *probability* P with which each arc (i, j) with $i < j$ was included. For each n, we generated two problems for each of the P values $.001, .02, .04, .06, .08, .10, .15, .20, .30$ and $.50$. In total, this produces 280 test problems.

Table 3.6 presents a summary of the computational results. For each n, the second column (ACT) and the third column (MCT) provide respectively the average and maximum computation time, in seconds, over 20 problems. The fourth column (OPT) represents the number of problems (out of 20) which were solved to proven optimality (i.e., when $LB = UB$). In total, 47 out of 280 problems were provably solved to optimality. The fifth column (AGAP) gives the average percent gap. The sixth column (MGAP) gives the maximum percent gaps and the last column gives the value of P for the problem giving this value MGAP.

n	ACT	MCT	# OPT	AGAP	MGAP	P
30	6.94	14.06	10	.16	.95	.10
40	19.50	75.79	7	.10	.74	.30
50	36.08	72.61	4	.11	.62	.15
60	65.00	133.19	4	.18	.54	.15
70	109.46	214.05	2	.20	.61	.15
80	129.68	246.56	2	.29	.53	.10
90	225.16	562.22	3	.23	.95	.10
100	290.33	530.80	2	.21	.51	.20
110	414.27	1614.70	3	.21	.88	.10
120	466.18	1449.15	2	.35	.96	.06
130	605.82	1888.78	2	.19	.49	.15
140	680.39	1769.81	2	.30	.87	.08
150	851.16	2074.26	2	.30	.67	.08
160	1137.32	3513.58	2	.29	.66	.15

Table 3.6: A summary of the results for 280 problems

Table 3.7 gives a comparison of our computational results with those of Van de Velde (1990). For each value of P, Van de Velde tested his Lagrangian relaxation method on a sample of 45 problems (5 problems each for every $n = 20, 30, \cdots, 100$). The resulting average percent gaps $AGAP^V$ are given in the second column of Table 3.7. We used a sample of 28 problems (2 problems each for every $n = 30, 40, \cdots, 160$). The third and fourth columns represent the average percent gaps and the maximum percent gaps, respectively, obtained by our cutting plane procedure. Clearly, the cutting plane approach outperforms the Lagrangian approach for all P values in terms of the resulting percent gaps. (If Van de Velde's sample of problems was used, we expect even better results, because problems with n less than 60 tend to yield smaller gaps for our approach, see Table 3.6.) These results are due to: (a) a much tighter bound from the cutting plane procedure, see **Theorem 3.1**; and (b) a more effective heuristic solution method. The last column provides the number of problems (out of 28) with the feasible solutions proven to be optimal.

P	$AGAP^V$	AGAP	MGAP	# OPT
.001	0.007	0	0	28
.02	0.069	0.035	0.467	9
.04	0.248	0.098	0.347	4
.06	0.675	0.248	0.962	2
.08	1.076	0.329	0.872	1
.10	1.518	0.408	0.952	0
.15	2.024	0.409	0.794	0
.20	2.113	0.303	0.860	1
.30	3.733	0.263	0.756	0
.50	4.116	0.154	0.359	2

Table 3.7: Computational results with respect to P values

Potts [50] observed that problems with P between .04 to .1 are more difficult for his approach. For our cutting plane procedure, the problem difficulty may depend on both the gap and the computation time. Our computational results in Table 3.7 show that the percent gaps are not very sensitive to P. Problems with P between .06 to .3 have slightly larger percent gaps (up to 0.96%). On the other hand, more computation time is usually required for problems with P values around .02. In terms of *order strength*, that is, the ratio of the number of arcs in the transitive closure to the maximum number $n(n-1)/2$ of possible arcs, larger percent gaps tend to occur for problems with order strength between .4 and .8.

In Section 3.4, we present the detailed computational results for 280 problems. Computation Times (CT), in seconds, are given in column 7. In column 4, Ord.Sth. denotes the order strength of the precedence graph. Column 5 represents the number of arcs in the transitive reduction of the digraph. Column 6 indicates the number of cuts used in **Step 3** of the cutting plane procedure. Note that for almost all of the instances with $P = 0.001$, only one cut has been used. This explains that our heuristic procedure in the initial formulation finds a very good basis when the precedence graph is very

sparse. (From the computational experiments, we found that those problems are most time-consuming if no heuristic is used in the initial formulation.)

This computational study shows that the cutting plane approach based on facet defining cuts is effective for solving the precedence constrained job sequencing problems. Because of the LP formulation with n variables and tight lower bounds obtained, the algorithm has potential for solving large-scale problems to optimality with the aid of a Branch and Bound or Branch and Cut procedure. To further improve the algorithm, the following are among the possible improvements:

- a better heuristic for feasible schedules, especially for the initial formulation;
- separation routines for more complex facet classes (see Chapter 2); and
- a Branch and Cut procedure at the final stage.

3.4 Computational Results for 280 problems

#Jobs	Name	P	Ord.Sth.	# Arcs	# Cuts	CT	% Gap	Statistic
30	1	.001	.00460	2	1	2.36	.00000	AGAP
	2	.001	.00000	0	1	2.42	.00000	.1589
	3	.020	.03678	11	2	3.13	.00000	
	4	.020	.03448	9	1	2.36	.00000	ACT
	5	.040	.07356	22	2	3.57	.00000	6.938
	6	.040	.03218	10	2	2.97	.00000	
	7	.060	.06667	23	4	3.74	.00000	
	8	.060	.06207	21	7	4.95	.00000	
	9	.080	.11954	29	6	4.62	.21058	
	10	.080	.14483	32	22	8.34	.03627	
	11	.100	.17931	34	24	9.01	.32752	
	12	.100	.30575	42	26	10.82	.95170	
	13	.150	.38621	50	24	10.16	.34220	
	14	.150	.48736	54	24	11.09	.26927	
	15	.200	.57701	52	16	6.26	.00000	
	16	.200	.55632	45	18	9.40	.08065	
	17	.300	.62299	59	29	14.06	.41234	
	18	.300	.69425	54	20	9.72	.42745	
	19	.500	.88046	46	23	10.88	.12030	
	20	.500	.91035	45	20	8.90	.00000	
40	1	.001	.00000	0	1	3.90	.00000	AGAP
	2	.001	.00128	1	1	3.90	.00000	.0958
	3	.020	.03590	23	9	8.63	.00000	
	4	.020	.01795	12	1	3.90	.00000	ACT
	5	.040	.09615	34	14	11.53	.00462	19.4955
	6	.040	.04231	27	10	9.01	.00000	
	7	.060	.18718	55	33	17.42	.04461	
	8	.060	.14231	32	17	17.30	.09125	
	9	.080	.12308	54	45	24.00	.00000	
	10	.080	.24359	52	36	19.28	.02760	
	11	.100	.30641	61	33	18.89	.07638	
	12	.100	.21410	66	28	16.53	.23652	
	13	.150	.42180	73	26	15.60	.08634	
	14	.150	.53590	72	35	22.13	.14889	
	15	.200	.63974	74	36	21.36	.09571	
	16	.200	.56923	83	28	18.51	.20624	
	17	.300	.78718	80	26	16.87	.02668	
	18	.300	.79487	74	27	36.69	.74286	
	19	.500	.90256	62	55	75.79	.12927	
	20	.500	.93333	66	37	28.67	.00000	

Table 3.8: Computational results for $n = 30, 40$

#Jobs	Name	P	Ord.Sth.	# Arcs	# Cuts	CT	% Gap	Statistic
50	1	.001	.00000	0	1	5.71	.00000	AGAP
	2	.001	.00082	1	1	5.93	.00000	.1069
	3	.020	.02122	20	1	6.26	.00000	
	4	.020	.03429	28	5	10.10	.00646	ACT
	5	.040	.07429	50	56	43.72	.00000	36.076
	6	.040	.12408	62	43	29.44	.06609	
	7	.060	.13061	67	56	43.28	.02668	
	8	.060	.13796	65	56	41.80	.04592	
	9	.080	.25633	80	42	30.75	.09488	
	10	.080	.22531	76	49	36.74	.05686	
	11	.100	.28082	84	39	28.23	.11125	
	12	.100	.40082	74	47	36.47	.15972	
	13	.150	.54857	108	51	42.62	.25105	
	14	.150	.62939	83	58	51.63	.62363	
	15	.200	.73714	87	55	49.10	.29168	
	16	.200	.70612	99	48	44.10	.25035	
	17	.300	.83918	79	30	26.20	.01573	
	18	.300	.83347	90	58	63.93	.11547	
	19	.500	.95510	67	44	72.61	.02199	
	20	.500	.95020	75	48	52.89	.00095	
60	1	.001	.00113	2	1	8.96	.00000	AGAP
	2	.001	.00169	3	1	10.16	.00000	.1754
	3	.020	.06497	46	26	36.47	.00000	
	4	.020	.02260	33	40	64.98	.00000	ACT
	5	.040	.06328	64	66	63.39	.00242	64.996
	6	.040	.08362	70	73	69.70	.02342	
	7	.060	.19209	88	55	50.69	.11333	
	8	.060	.22712	96	72	69.04	.47417	
	9	.080	.32090	94	67	65.36	.14805	
	10	.080	.31808	107	52	53.28	.16912	
	11	.100	.49378	118	63	69.32	.36744	
	12	.100	.44350	113	54	62.67	.24633	
	13	.150	.61582	119	48	56.03	.53784	
	14	.150	.62429	120	69	86.34	.54094	
	15	.200	.71073	122	61	81.84	.15449	
	16	.200	.69661	111	84	112.76	.22352	
	17	.300	.84746	106	60	81.94	.10690	
	18	.300	.85254	116	78	133.19	.29882	
	19	.500	.94237	99	60	79.64	.01473	
	20	.500	.94915	89	35	44.16	.08674	

Table 3.9: Computational results for $n = 50, 60$

#Jobs	Name	P	Ord.Sth.	# Arcs	# Cuts	CT	% Gap	Statistic
70	1	.001	.00041	1	1	12.36	.00000	AGAP
	2	.001	.00041	1	1	12.03	.00000	.1951
	3	.020	.03064	43	57	108.31	.00975	
	4	.020	.02650	39	25	105.08	.46689	ACT
	5	.040	.11511	87	103	139.84	.00377	109.46
	6	.040	.08861	94	81	106.22	.10148	
	7	.060	.17143	112	86	108.64	.01781	
	8	.060	.22485	119	76	94.86	.09505	
	9	.080	.34576	147	65	84.80	.18851	
	10	.080	.31387	139	58	68.76	.34954	
	11	.100	.42774	163	79	108.86	.19217	
	12	.100	.56522	138	70	98.21	.20274	
	13	.150	.64306	138	106	169.00	.61276	
	14	.150	.60373	148	66	82.71	.23170	
	15	.200	.82650	132	82	138.85	.34946	
	16	.200	.76563	145	42	97.98	.40806	
	17	.300	.89151	139	78	131.76	.27388	
	18	.300	.87992	140	75	196.91	.18546	
	19	.500	.95694	111	57	214.05	.09142	
	20	.500	.96398	105	40	109.91	.12117	
80	1	.001	.00158	5	1	17.31	.00000	AGAP
	2	.001	.00063	2	1	16.09	.00000	.2870
	3	.020	.04620	75	46	110.13	.10601	
	4	.020	.04620	60	19	47.79	.03868	ACT
	5	.040	.06709	107	120	221.19	.00306	129.68
	6	.040	.17690	125	95	132.75	.17918	
	7	.060	.28196	168	113	175.32	.22464	
	8	.060	.22722	160	64	117.37	.36282	
	9	.080	.46171	170	71	109.24	.25996	
	10	.080	.34240	170	65	98.75	.71447	
	11	.100	.36266	183	61	99.08	.15841	
	12	.100	.46772	185	78	126.77	.53486	
	13	.150	.72468	174	83	157.80	.45760	
	14	.150	.70949	168	84	165.54	.41877	
	15	.200	.78798	162	83	154.67	.39464	
	16	.200	.72089	206	76	146.54	.38601	
	17	.300	.91994	132	19	70.74	.75583	
	18	.300	.85886	158	83	246.56	.43060	
	19	.500	.95285	122	39	171.69	.23316	
	20	.500	.95601	136	82	208.22	.08160	

Table 3.10: Computational results for $n = 70, 80$

#Jobs	Name	P	Ord.Sth.	# Arcs	# Cuts	CT	% Gap	Statistic
90	1	.001	.00050	2	1	22.36	.00000	AGAP
	2	.001	.00050	2	1	20.38	.00000	.2290
	3	.020	.03071	72	140	562.22	.00000	
	4	.020	.03845	87	94	310.38	.01671	ACT
	5	.040	.11261	156	107	192.23	.08633	225.14
	6	.040	.10786	140	102	206.74	.13070	
	7	.060	.24145	177	92	162.97	.11381	
	8	.060	.25368	171	97	167.64	.26182	
	9	.080	.36654	205	70	131.55	.47914	
	10	.080	.41973	182	92	208.00	.54811	
	11	.100	.50737	200	78	175.93	.95225	
	12	.100	.56105	207	74	161.43	.35829	
	13	.150	.72584	216	108	298.19	.59628	
	14	.150	.70212	205	98	215.64	.32440	
	15	.200	.81248	202	103	244.26	.18701	
	16	.200	.76405	207	81	221.79	.16728	
	17	.300	.87965	169	96	265.67	.13002	
	18	.300	.87590	194	114	321.92	.20084	
	19	.500	.97129	129	73	284.40	.00510	
	20	.500	.96404	145	82	329.00	.02184	
100	1	.001	.00081	4	1	24.71	.00000	AGAP
	2	.001	.00101	5	1	25.65	.00000	.2095
	3	.020	.04525	99	60	199.71	.05779	
	4	.020	.03030	89	99	530.80	.01623	ACT
	5	.040	.10101	162	148	389.54	.08459	290.33
	6	.040	.09677	170	122	269.68	.05475	
	7	.060	.23899	213	106	222.06	.17924	
	8	.060	.30929	209	129	265.18	.06981	
	9	.080	.43495	218	105	218.50	.30606	
	10	.080	.45394	228	110	243.71	.25238	
	11	.100	.57737	239	126	418.37	.30853	
	12	.100	.50202	233	109	250.13	.37136	
	13	.150	.73960	209	126	372.78	.43368	
	14	.150	.75414	218	141	449.40	.48349	
	15	.200	.83778	200	90	320.55	.19209	
	16	.200	.79859	217	126	381.19	.51074	
	17	.300	.91515	188	67	376.02	.27127	
	18	.300	.92222	178	87	461.76	.18564	
	19	.500	.96606	164	56	232.22	.12527	
	20	.500	.97030	157	53	154.68	.28807	

Table 3.11: Computational results for $n = 90, 100$

#Jobs	Name	P	Ord.Sth.	# Arcs	# Cuts	CT	% Gap	Statistic
110	1	.001	.00050	3	1	30.43	.00000	AGAP
	2	.001	.00100	6	1	30.05	.00000	.2101
	3	.020	.04087	110	317	1614.70	.00000	
	4	.020	.05588	129	222	813.66	.00948	ACT
	5	.040	.26539	201	113	289.51	.07890	414.27
	6	.040	.17831	205	102	245.90	.05274	
	7	.060	.31793	271	94	233.10	.21554	
	8	.060	.32127	266	114	288.90	.46806	
	9	.080	.44671	285	102	245.14	.19019	
	10	.080	.48791	276	97	256.06	.24626	
	11	.100	.66806	241	156	459.94	.23973	
	12	.100	.58415	294	108	348.67	.88693	
	13	.150	.75113	268	109	308.46	.24279	
	14	.150	.75663	256	109	488.46	.29757	
	15	.200	.84304	232	116	456.71	.40698	
	16	.200	.84871	237	132	397.77	.21956	
	17	.300	.90142	223	34	166.75	.41074	
	18	.300	.92410	206	115	540.19	.04542	
	19	.500	.97448	173	86	570.29	.13094	
	20	.500	.96931	162	91	500.76	.06043	
120	1	.001	.00140	10	1	42.85	.00000	AGAP
	2	.001	.00084	6	119	768.46	.00000	.3466
	3	.020	.05574	154	205	909.18	.04924	
	4	.020	.03529	126	265	1449.15	.00363	ACT
	5	.040	.28123	261	124	352.18	.14214	466.18
	6	.040	.23193	236	136	494.98	.16975	
	7	.060	.36807	279	124	389.59	.96183	
	8	.060	.41583	300	111	311.48	.19506	
	9	.080	.52507	304	104	327.85	.55736	
	10	.080	.52563	305	104	567.11	.40434	
	11	.100	.64160	282	111	334.94	.27584	
	12	.100	.62661	304	121	422.11	.70647	
	13	.150	.74776	313	132	475.77	.68209	
	14	.150	.80924	257	50	237.55	.69801	
	15	.200	.84566	258	75	524.92	.76291	
	16	.200	.85294	267	105	463.24	.33794	
	17	.300	.92031	229	52	234.37	.29232	
	18	.300	.91681	239	50	301.60	.37252	
	19	.500	.98039	176	38	183.45	.22883	
	20	.500	.97521	186	68	532.78	.09273	

Table 3.12: Computational results for $n = 110, 120$

#Jobs	Name	P	Ord.Sth.	# Arcs	# Cuts	CT	% Gap	Statistic
130	1	.001	.00143	12	45	264.53	.00000	AGAP
	2	.001	.00119	10	13	114.96	.00000	.1888
	3	.020	.04568	168	296	1726.70	.03171	
	4	.020	.04770	153	312	1888.78	.05270	ACT
	5	.040	.17197	291	145	445.51	.11493	605.82
	6	.040	.13787	274	118	390.14	.06434	
	7	.060	.41145	321	68	291.05	.44764	
	8	.060	.42922	302	76	300.77	.34043	
	9	.080	.52009	344	103	332.69	.25293	
	10	.080	.56947	301	129	445.23	.25269	
	11	.100	.68515	322	130	527.62	.33697	
	12	.100	.64782	329	103	471.09	.37895	
	13	.150	.75826	332	149	627.36	.49034	
	14	.150	.79165	322	115	407.88	.17745	
	15	.200	.87871	265	115	589.68	.15636	
	16	.200	.86094	292	110	524.70	.11167	
	17	.300	.94252	231	123	1051.00	.12904	
	18	.300	.90912	288	170	773.95	.17247	
	19	.500	.97865	194	77	485.10	.07439	
	20	.500	.96923	219	76	457.75	.19034	
140	1	.001	.00113	11	1	64.15	.00000	AGAP
	2	.001	.00082	8	124	1180.68	.00000	.2956
	3	.020	.05704	181	304	1769.81	.02924	
	4	.020	.06495	199	259	1342.71	.07154	ACT
	5	.040	.22354	332	150	571.00	.28460	680.39
	6	.040	.19168	297	139	581.77	.14907	
	7	.060	.46218	326	187	647.36	.31147	
	8	.060	.46578	321	134	525.03	.31678	
	9	.080	.54820	354	153	643.45	.40014	
	10	.080	.52795	368	149	601.05	.87223	
	11	.100	.69239	372	113	567.71	.36579	
	12	.100	.68612	325	152	775.33	.58323	
	13	.150	.77359	354	113	527.67	.09663	
	14	.150	.82867	304	40	217.95	.79355	
	15	.200	.84327	347	154	858.60	.14770	
	16	.200	.86218	312	32	257.88	.86032	
	17	.300	.94132	278	162	1052.65	.11478	
	18	.300	.93731	291	59	361.58	.15913	
	19	.500	.97718	225	87	593.25	.13079	
	20	.500	.97811	218	55	468.07	.22498	

Table 3.13: Computational results for $n = 130, 140$

#Jobs	Name	P	Ord.Sth.	# Arcs	# Cuts	CT	% Gap	Statistic
150	1	.001	.00107	12	119	1112.90	.00000	AGAP
	2	.001	.00125	14	125	1194.52	.00000	.3038
	3	.020	.05145	223	344	2074.26	.00365	
	4	.020	.05414	219	268	1613.16	.00618	ACT
	5	.040	.20546	354	151	585.62	.34732	851.16
	6	.040	.22407	326	156	537.89	.14711	
	7	.060	.48054	388	157	671.85	.49276	
	8	.060	.41682	422	117	583.81	.20610	
	9	.080	.53127	437	154	692.94	.66706	
	10	.080	.61494	405	156	847.50	.63095	
	11	.100	.69942	388	105	653.78	.71995	
	12	.100	.69110	389	124	682.55	.21691	
	13	.150	.80143	370	81	443.47	.38478	
	14	.150	.83400	338	151	934.89	.22556	
	15	.200	.88143	326	209	1729.00	.34891	
	16	.200	.88608	330	60	449.67	.58260	
	17	.300	.93709	279	51	430.12	.42886	
	18	.300	.94622	280	116	1051.93	.15777	
	19	.500	.97888	236	58	296.93	.14922	
	20	.500	.98201	231	46	436.44	.35984	
160	1	.001	.00087	11	1	112.16	.00000	AGAP
	2	.001	.00126	15	129	2280.40	.00000	.2889
	3	.020	.06022	233	289	2005.44	.01548	
	4	.020	.05299	220	402	3513.58	.01715	ACT
	5	.040	.26832	386	160	633.40	.20351	1137.32
	6	.040	.24772	369	150	591.05	.24819	
	7	.060	.38585	474	142	625.27	.34809	
	8	.060	.48884	429	112	738.20	.54318	
	9	.080	.58530	426	111	625.88	.39843	
	10	.080	.61580	423	152	921.66	.42387	
	11	.100	.71069	394	176	950.65	.72379	
	12	.100	.73538	396	107	698.43	.43946	
	13	.150	.81942	406	210	1544.83	.66236	
	14	.150	.83561	392	166	1100.38	.34659	
	15	.200	.89285	357	75	628.79	.41888	
	16	.200	.89222	345	191	1288.22	.24187	
	17	.300	.93789	327	158	1277.35	.19612	
	18	.300	.93931	325	86	1132.02	.21530	
	19	.500	.98608	223	56	384.31	.17023	
	20	.500	.98019	251	113	1694.40	.16632	

Table 3.14: Computational results for $n = 150, 160$

Chapter 4

Hamiltonian Path and Symmetric Travelling Salesman Problem

This chapter introduces a Hamiltonian Path (HP) approach for studying symmetric travelling salesman polytopes (STSPs). By studying HP polytopes, we develop general *clique-lifting* results, all based on simple conditions, for deriving large new classes of STSP facets. These results apply to all known non-trivial STSP facets, and generalize clique-lifting results of Maurras [38], Grötschel and Padberg [27] and Naddef and Rinaldi [41].

4.1 Introduction

Let $K(V') = (V', E(V'))$ be a complete undirected graph, where the node set V' represents a set of cities. A feasible solution to the Symmetric Travelling Salesman (STS) problem is a Hamiltonian cycle C on V', uniquely determined by its corresponding 0-1 *incidence vector* $y \in R^{E(V')}$ (where $y_e = 1$ if edge e is used in C and zero otherwise). This chapter deals with the *STS polytope* $STSP(V')$, the convex hull of incidence vectors y of all feasible solutions to the STS problem.

The facial structure of STS polytopes (STSPs) has been the object of considerable research over the past two decades. We refer to Grötschel and Padberg [28] for an excellent survey (up to 1985). Since then, there have been many advances in studying STS polytopes and in solving large-scale STS problems. Large classes of STSP valid inequalities were proposed by Boyd and Cunningham [7,8], Cornuéjols, *et al.* [13], Fleischmann [19,20,21], Naddef and Rinaldi [41,42,44], Naddef [40]. Some of these inequalities,

such as *clique-tree* [29], *chain* (introduced in [47], and proven to be facet-defining independently by S. Boyd and M. Hartmann, cf. [8]), *ladder* [8], *crown* and *path-tree* [42,43] inequalities are known to be facet-defining. The study of STSP facets is also of practical importance: Padberg and Rinaldi [49] have successfully demonstrated the use of facet-defining inequalities, such as clique-tree inequalities, in solving large-scale, real-world STS problems (with over 2000 cities) to optimality. For small STS polytopes, Naddef and Rinaldi [43] present a list of inequalities defining 133,707 distinct facets of $STSP(V')$ when $|V'| = 8$, including nonnegativity constraints, sub-tour elimination, ladder, chain, crown, and path inequalities. Boyd and Cunningham [8] have proven that part of this list entirely describes $STSP(V')$ with $|V'| \leq 7$. Christof *et al* [15] use a computer code to derive a complete description of $STSP(V')$ with $|V'| = 8$, where in addition to Naddef and Rinaldi's list, three classes of new facets are discovered. For the case $|V'| = 9$, we introduce a new facet in **Figure 4.3**, illustrating the incompleteness of all the above known facet classes for $|V'| \geq 9$.

This chapter uses a *projective approach* to the polyhedral study of symmetric travelling salesman problems. Consider the node set $V \doteq V' \setminus \{h\}$, where $h \in V'$ is any given *projection node.* Let $HP(V)$ denote the *Hamiltonian Path Polytope* (HPP), i.e., the convex hull of incidence vectors x of all Hamiltonian paths (with free endnodes) on V. Using a one-to-one correspondence between Hamiltonian cycles on V' and Hamiltonian paths on $V \doteq V' \setminus \{h\}$, we show that $HP(V)$ is a *projection* of $STSP(V')$ onto the subspace $R^{E(V)}$ of $R^{E(V')}$. More importantly, $HP(V)$ exhibits three fundamental properties for this projective approach: (a) $HP(V)$ is *near-full dimensional,* with a single implicit equation $\sum_{e \in E(V)} x_e = |V| - 1$; (b) a move from one vertex $x \in HP(V)$ to another one may involve only one interchange of components in x, whereas such a move involves at least two interchanges in $y \in STSP(V')$ (the so-called *2-opt interchange*); (c) *polyhedral*

equivalence: the polyhedral structures of $HP(V)$ and $STSP(V')$ are isomorphic. Properties (a) and (b) greatly facilitate the indirect proofs of any facetial results for $HP(V)$, and property (c) implies immediately that the corresponding facetial results also hold for $STSP(V')$. In contrast, additional hard work is required for transferring facetial results, derived from monotone [28] or graphical relaxations [13], to corresponding results for $STSP(V')$.

We study extensions of STSP facets by clique-lifting. For ATS polytopes, Balas and Fischetti [6] prove a general clique-lifting result that applies to all nontrivial facets of ATS polytopes. For STS polytopes, it is unknown whether such a powerful lifting result holds. Maurras (1975), Grötschel and Padberg (1979), and Naddef and Rinaldi (1989) present various sufficient conditions for clique-lifting of STSP facets. The early lifting results employs *simple conditions* (verifiable by inspection or in linear time) but their applications seem restricted. Naddef and Rinaldi's clique-lifting result applies to large classes of known STSP facets, but the condition seems difficult to verify in general (we do not know whether there exists a polynomial-time algorithm to verify their condition). In this paper, we propose a rather general clique-lifting result (with *at least two* new nodes added) that applies to any nontrivial STSP facet w.r.t. any given node. (In the sequel, "w.r.t." stands for "with respect to".) We also show, by the projective approach, that Naddef and Rinaldi's sufficient condition for *node-cloning* (i.e., *clique lifting* with *exactly one* new node added) is in fact *necessary*. As the condition is difficult to verify, we propose three simplified node-cloning results, all based on conditions that can be verified either by inspection or by a procedure of time complexity linear in the number of edges. The first result seems applicable to all known nontrivial facets (such as clique-tree, chain, ladder, crown, path and path tree, etc.) w.r.t. any node. The second result generalizes the clique-lifting results in Maurras [38], Grötschel and Padberg [27]. The third result shows that any node can be cloned in any facet-defining *rank inequality* (i.e., inequality

$ay \leq a_0$ with zero-one coefficients).

The rest of the chapter is organized as follows. Section 4.2 gives basic terminology and notation. Section 4.3 introduces the projective approach and the polyhedral equivalence of $HP(V)$ and $STSP(V')$, as well as other related results. In Section 4.4, we propose a clique-lifting result (with clique size greater than two) for all nontrivial STSP facets and prove that Naddef and Rinaldi's clique-lifting condition is necessary. In Section 4.5, we present our simple sufficient conditions for node-cloning. Section 4.6 offers concluding remarks.

4.2 Definition and Notation

Our notation and terminology mostly follow those in [6],[28],[41].

(1). For any node set V with $n \doteq |V| \geq 3$, let $K(V) \doteq (V, E(V))$ denote the complete undirected graph with node set V. Note that $|E(V)| = n(n-1)/2$. For any subsets S_1 and S_2 of V, let

$$E(S_1 : S_2) \doteq \{(i,j) \in E(V) : i \in S_1, j \in S_2\}.$$

When $S = S_1 = S_2$, denote $E(S : S)$ by $E(S)$, i.e., the collection of all edges in $E(V)$ with both endnodes in S. For any $S \subseteq V$, denote the complete graph on node set S by $K(S) \doteq (S, E(S))$. For any node $v \in V$, let $\delta(v) \doteq E(\{v\} : V \setminus \{v\})$ denote the *star* of v, i.e., the set of all edges in $E(V)$ incident to v.

(2). A *partial graph* $(S, \widetilde{E})$ is a subgraph of $K(S)$ with $\widetilde{E} \subseteq E(S)$. The *incidence vector* of partial graph $(V, \widetilde{E})$ of $K(S)$ is the vector $\chi \in R^{E(S)}$ such that $\chi_e = 1$ if $e \in \widetilde{E}$ and $\chi_e = 0$ otherwise. For any vector $\psi \in R^{E(S)}$ and $\widetilde{E} \subseteq E(S)$, let

$$\psi(\widetilde{E}) \doteq \sum_{e \in \widetilde{E}} \psi_e. \tag{4.1}$$

A *Hamiltonian cycle* C on S is a connected partial graph $(S, \widetilde{E})$ such that

$$|\widetilde{E} \cap \delta(v)| = 2, \quad \text{for all } v \in S.$$

Without risking confusion, a Hamiltonian cycle C with edge set

$$\widetilde{E} = \{(v_{\sigma(i)}, v_{\sigma(i+1)}) : i = 1, ..., s-1\} \cup \{(v_{\sigma(s)}, v_{\sigma(1)})\}$$

on node set $S = \{v_1, ..., v_s\} \subseteq V$ may be represented by edge set $\widetilde{E}$ alone or, equivalently, by the circular permutation $\langle v_{\sigma(1)} ... v_{\sigma(s)} \rangle$ of the nodes.

A *Hamiltonian path* P on $S = \{v_1, ..., v_s\} \subseteq V$ is a connected partial graph $(S, \hat{E})$ such that for some distinct nodes $i, j \in S$, $(S, \hat{E} \cup \{i, j\})$ defines a Hamiltonian cycle on S. Note that the endnodes of P are not specified in advance, i.e., we consider here paths with *free endnodes.* Hamiltonian path P may be represented by its edge set $\{(v_i, v_{i+1}) : i = 1, ..., s-1\}$ or, equivalently, by its corresponding node permutation $(v_1 ... v_s)$ with *endnodes* v_1 and v_s. Note that if $|S| = 1$, the first form is an empty set, whereas the second form reduces to a single node (v_1). When a path is written in the form $(u ... vp ... q)$, we always allow $u = v$ or $p = q$ or both.

For any node set S, the *STS polytope* $STSP(S)$ and *HP polytope* $HP(S)$ are defined, respectively, as follows.

$$STSP(S) \doteq \text{conv}\{y^C \in R^{E(S)} : C \text{ is a Hamiltonian cycle on } S\},$$

$$HP(S) \doteq \text{conv}\{x^P \in R^{E(S)} : P \text{ is a Hamiltonian path on } S\}.$$

(3). To any inequality $cx \leq c_0$ with $c \geq 0$ and $x \in R^{E(S)}$, we associate its *support graph*:

$$G_c(S) \doteq (S, E_c), \quad \text{where } E_c \doteq \{e \in E(S) : c_e > 0\}.$$

The *complement graph* $\overline{G}_c(S) = (S, \overline{E_c})$, where $\overline{E_c} \doteq E(S) \setminus E_c$, is called the *zero-graph* w.r.t. $cx \leq c_0$.

(4). Let $ay \leq a_0$ and $cx \leq c_0$ be two valid inequalities for $STSP(S)$ and $HP(S)$, respectively. We say that C is an *a-tight cycle* (resp., P is a *c-tight path*) on S if C is a Hamiltonian cycle on S and $ay^C = a_0$ (resp., if P is a Hamiltonian path on S and $cx^P = c_0$).

4.3 Polyhedral Equivalence: HPP v.s. STSP

The STS polytope (STSP) is far from being full dimensional because every Hamiltonian cycle satisfies a degree constraint equation at each node. In contrast, the HP formulation relaxes all degree constraints. This observation suggests that $HP(V)$ be more nearly full dimensional. It leads to the development of a projective approach to the study of travelling salesman problems. A similar projective approach has been used for the equipartition polytope by Conforti, Rao and Sassano [12].

Let V' be a node set with $|V'| = n+1$, and for some fixed node $h \in V'$, let $V \doteq V' \backslash \{h\}$. Consider two complete undirected graphs:

$$K(V) = (V, E(V)) \quad \text{and} \quad K(V') = (V', E(V')).$$

That is, $K(V)$ is obtained from $K(V')$ by eliminating node $h \in V'$ and all incident edges. Let $STSP(V')$ be the STS polytope defined on $K(V')$, and $HP(V)$ be the HP polytope defined on $K(V)$. Observe that P is a Hamiltonian path on V with endnodes u and v iff $C \doteq P \cup \{(h,u),(h,v)\}$ is a Hamiltonian cycle on V'. Thus, the incidence vector $x^P \in R^{E(V)}$ is a *projection* of the incidence vector $y^C \in R^{E(V')}$ onto subspace $R^{E(V)}$ of $R^{E(V')}$. It follows that the polytope $HP(V)$ is a projection of $STSP(V')$. In what follows, we assume that all incidence vectors y on $K(V')$ are indexed as: $y = (y_1, ..., y_m, y_{m+1}, ..., y_{m'})^t$, where the first $m \doteq n(n-1)/2$ elements correspond to the edge set $E(V)$.

Let A_n be a node-edge incidence matrix of $K(V)$. Observe that the incidence vectors $y^C \in R^{E(V')}$ of all Hamiltonian cycles C on V' and their projections $x^P \in R^{E(V)}$ satisfy the following identity:

$$y^C_{uv} \equiv \begin{cases} x^P_{uv}, & (u,v) \in E(V); \\ 2 - \sum_{e \in E(V)} (A_n)_{ve} x^P_e, & (u,v) \in E(h:V). \end{cases} \tag{4.2}$$

Clearly, the above equation defines an affine transformation from $HP(V)$ to $STSP(V')$. The following result shows that this transformation is affine-rank preserving.

Lemma 4.1 *Let $B_y \doteq [y^1, ..., y^l]$ be any matrix such that all column vectors y^j are incidence vectors of Hamiltonian cycles on $K(V')$. Let $B_x \doteq [x^1, ..., x^l]$, where x^j is the projection of y^j into $R^{E(V)}$, $j = 1, ..., l$. Then the number of affinely independent column vectors in B_y is equal to the number of affinely independent column vectors in B_x.*

Proof: From equation (4.2), it follows

$$\text{aff.rank}\begin{pmatrix} B_y \end{pmatrix} = \text{rank}\begin{pmatrix} \mathbf{1}^t \\ B_y \end{pmatrix} \geq \text{rank}\begin{pmatrix} \mathbf{1}^t \\ B_x \end{pmatrix}$$

$$\geq \text{rank}\left[\begin{pmatrix} 1 & \mathbf{0}^t \\ \mathbf{0} & I \\ \mathbf{2} & -A_n \end{pmatrix}\begin{pmatrix} \mathbf{1}^t \\ B_x \end{pmatrix}\right] = \text{rank}\begin{pmatrix} \mathbf{1}^t \\ B_y \end{pmatrix} = \text{aff.rank}\begin{pmatrix} B_y \end{pmatrix},$$

where "aff.rank" denotes the affine rank of the column set, "rank" denotes the linear rank, $\mathbf{0}$ (resp., $\mathbf{1}$) is a vector of zeroes (resp., ones) and I is an identity matrix, all of conformable dimension. Hence, equality must hold throughout. □

The dimension of $HP(V)$ could now be derived from that of $STSP(V')$ using **Lemma 4.1**. In fact, the direct derivation of $\dim(HP(V))$ given below is very simple. Thus, we obtain a much simpler proof than any known proof for $\dim(STSP(V'))$ (see [28], section 4.1, for two other proofs).

Theorem 4.2 *The dimension of $HP(V)$ is:*

$$\dim(HP(V)) = \frac{n(n-1)}{2} - 1, \quad \textit{where } n = |V| \geq 3. \tag{4.3}$$

Proof. Since every Hamiltonian path P on V must satisfy implicit equation $x^P(E(V)) = n-1$ for $HP(V)$, we have

$$\dim(HP(V)) \leq \frac{n(n-1)}{2} - 1. \tag{4.4}$$

Let $fx = f_0$ be any equation satisfied by all incidence vectors $x \in HP(V)$. For any distinct $e_1, e_2 \in E(V)$, there exists a Hamiltonian cycle C in $K(V)$ containing e_1 and e_2. Since $P^1 = C \setminus e_1$ and $P^2 = C \setminus e_2$ are two Hamiltonian paths in $K(V)$,

$$0 = fx^{P^1} - fx^{P^2} = f_{e_2} - f_{e_1}.$$

This shows that f_e is constant for all $e \in E(V)$, and equation $fx = f_0$ must be a multiple of implicit equation $x(E(V)) = n-1$. □

Remark 4.3 By **Theorem 4.2**, two valid inequalities $cx \leq c_0$ and $c'x \leq c'_0$ for $HP(V)$ are equivalent iff there exist two scalars $\alpha > 0$ and β such that $c'_0 = \alpha c_0 + (n-1)\beta$ and $c'_e = \alpha c_e + \beta$ for all $e \in E(V)$. This equivalence greatly simplifies indirect proofs of facetial results for $HP(V)$. □

Corollary 4.4 *The dimension of $STSP(V')$ is:*

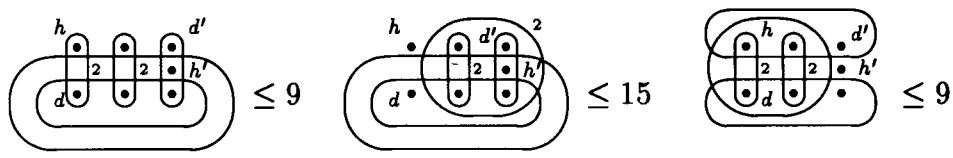
$$\dim(STSP(V')) = \frac{n(n+1)}{2} - (n+1). \tag{4.5}$$

Proof. From **Lemma 4.1** and **Theorem 4.2**, it follows that

$$\dim(STSP(V')) = \dim(HP(V)) = \frac{n(n-1)}{2} - 1 = \frac{n(n+1)}{2} - (n+1). \quad \square$$

Remark 4.5 D. Naddef points out that such dimensionality results are equivalent to characterizations of the constant TSP (see [25], section 2). We therefore also obtain as a corollary a simple proof of the theorem of Berenguer (theorem 1 in Gilmore et al. [25]). □

Closed form representations for inequalities $ay \leq a_0$ are used in the figures hereafter. For each inequality with nonnegative coefficients, a collection of node subsets with positive weights (numbers) are given. Weights are indicated next to the corresponding sets and all set weights of value one are omitted for simplicity. Then, each coefficient a_{ij} equals the total weight of all the subsets which contain both nodes i and j.



(a) An Envelope (b) h-normal form (c) h'-normal form

Figure 4.1: Equivalent envelope inequalities

Figure 4.1 presents such representations. For instance, in Fig. 4.1(a), $a_0 = 9$, $a_{dh} = 2$, $a_{d'h'} = a_{dh'} = 1$ and $a_{hd'} = a_{hh'} = a_{dd'} = 0$, etc.

Now, we turn to relationships between valid inequalities for $STSP(V')$ and those for $HP(V)$. Recall that $V' \setminus V = \{h\}$. Consider two inequalities $ay \leq a_0$ and $cx \leq c_0$, defined on $STSP(V')$ and $HP(V)$, respectively, and satisfying:

$$a_0 = c_0; \quad a_e = c_e \geq 0, \quad \forall e \in E(V); \quad \text{and} \quad a_e = 0, \quad \forall e \in E(h : V).$$

Then, $ay \leq a_0$ is said to have an *isolated node* h (cf. Fig. 4.1(b)). Inequality $cx \leq c_0$ is the *h-restriction* of $ay \leq a_0$, and conversely, $ay \leq a_0$ is an *extension* of $cx \leq c_0$.

By **Lemma 4.1** and **Theorem 4.2**, we have immediately:

Theorem 4.6 (Polyhedral Equivalence) *For any integer k such that $0 \leq k \leq n(n-1)/2 - 1$, the inequality $ay \leq a_0$ with isolated node h defines a k-dimensional face of $STSP(V')$ iff its h-restriction $cx \leq c_0$ defines a k-dimensional face of $HP(V)$.*

The above theorem, along with **Remark 4.3**, provides a *projective approach* to the study of STS polytopes. This theorem also suggests the following unique representation of an STSP valid inequality w.r.t. a given node $h \in V'$.

h-normal form. An inequality $ay \leq a_0$ for $STSP(V')$ is in *normal form* w.r.t. a given node $h \in V'$ (h-normal form, for short) if

(C1) $a_e = 0$ for all $e \in E(\{h\} : V' \setminus \{h\})$;

(C2) $a \geq 0$ and $a_e = 0$ for some $e \in E(V' \setminus \{h\})$;

(C3) $\min\{a_e > 0 : e \in E(V')\} = 1$.

Inequality $cx \leq c_0$ for $HP(V)$ is in *normal form* if its extension $ay \leq a_0$ for $STSP(V \cup \{h\})$, where $h \notin V$, is in h-normal form. See **Figure** 4.1 for examples.

Remark 4.7 Balas and Fischetti [6] use a similar notion, *h-canonical form*, to uniquely represent ATSP inequalities. We use this notion here for STSP inequalities with a different scaling in (C3). Such forms can also be obtained by applications of Remark 4.2 of Grötschel and Padberg [27]. □

Any valid inequality $ay \leq a_0$ for $STSP(V')$ admits a *unique* h-normal form $a'y \leq a'_0$ w.r.t. any given node $h \in V'$. This can be obtained by the following h-normalization procedure.

<u>h-Normalization procedure</u>

Input: A valid inequality $ay \leq a_0$ for $STSP(V')$, where $|V'| = n + 1$,
and any given *projection node* $h \in V'$.

Output: An equivalent valid inequality $a'y \leq a'_0$ for $STSP(V')$ in h-normal form.

Step 1: h-Projection. Adding to $ay \leq a_0$ the following linear combination of degree constraints:

$$\sum_{v \in V' \setminus \{h\}} -a_{vh} y(\delta(v)) = \sum_{v \in V' \setminus \{h\}} -2a_{vh}$$

yields $\bar{a}y \leq \bar{a}_0$. Note that after the projection, $\bar{a}_e = 0$ for all $e \in \delta(h)$.

Step 2: Shifting. Adding to $\bar{a}y \leq \bar{a}_0$ the implicit equation $\lambda y(E(V' \setminus \{h\})) = \lambda(n-1)$, for $STSP(V')$, where

$$\lambda = -\min\{\bar{a}_e : e \in E(V' \setminus \{h\})\},$$

yields $\tilde{a}y \leq \tilde{a}_0$. Note that after shifting, $\tilde{a}_0 = \bar{a}_0 + \lambda(n-1)$, $\tilde{a} \geq 0$, and $\tilde{a}_e = \bar{a}_e + \lambda$ if $e \in E(V' \setminus \{h\})$ and zero otherwise. Note also that for some $e \in E(V' \setminus \{h\})$, $\tilde{a}_e = 0$.

Step 3: Scaling. The *scaling operation* is defined by

$$(a', a'_0) \doteq (\tilde{a}, \tilde{a}_0)/\gamma,$$

where $\gamma = \min\{\tilde{a}_e > 0 : e \in E(V')\}$.

For instance, Fig. 4.1(a) represents inequality $ay \leq a_0$. After the normalization procedures w.r.t. nodes h and h', we obtain its h-normal form in Fig. 4.1(b) and h'-normal form in Fig. 4.1(c), respectively. Observe that in this example, the h-normal form in Fig. 4.1(b) is obtained by simply *complementing* set $\{h, d\}$: adding to $ay \leq a_0$ the following implicit equations for $STSP(V')$:

$$-y(\delta(v)) = -2, \forall v \in \{h, d\}; \quad y(\delta(v)) = 2, \forall v \in V' \setminus \{h, d\}. \tag{4.6}$$

To obtain the h'-normal form in Fig. 4.1(c), we need two successive set complementations.

h-projected inequality, or h-projection. Let $a'y \leq a'_0$ be the h-normal form of $ay \leq a_0$ for $STSP(V')$, and let $cx \leq c_0$ be the h-restriction of $a'y \leq a'_0$. Then $cx \leq c_0$ for $HP(V' \setminus \{h\})$ is an *h-projected inequality* or *h-projection* of $ay \leq a_0$.

Two important observations immediately follow from the above procedure. First, the normalization procedure requires $O(E(V'))$ computational steps. Thus, we obtain a linear time algorithm to check whether or not two valid inequalities define the same face (facet) of $STSP(V')$. Second, the coefficients of the h-normal form $a'y \leq a'_0$ satisfy: $a' \geq 0$, $a'_e = 0$ for all $e \in \delta(h)$ and $a'_e = 0$ for some $e \in E(V' \setminus \{h\})$. These edges correspond to linearly independent columns of the node-edge incidence matrix of $K(V')$. By Lemma 6.3 in [29], it follows that $a'y \leq a'_0$ is in *support-reduced form*, and therefore $a'y \leq a'_0$ is facet-defining for the monotone STS polytope $MTSP(V')$ if it defines a nontrivial facet of $STSP(V')$.

Remark 4.8 A similar h-projection technique was also introduced independently by Araque [2] in his polyhedral study of vehicle routing problems (VRP) with single depot and variable fleet size. Consider a VRP with one vehicle and a single depot node h. Then, the coefficients of $\bar{a}$ in Step 1 of the h-**Normalization procedure** satisfy $-\bar{a}_{ij} = a_{hi} + a_{hj} - a_{ij} \equiv s_{ij}$ for all $(i,j) \in E(V' \setminus \{h\})$, where s_{ij}, as observed by Araque [2], is exactly the *saving* defined by Clarke and Wright [11]. □

From results in [28] and the polyhedral equivalence, we obtain simple classes of facet-defining inequalities for $HP(V)$.

Proposition 4.9 *Let $\delta(v) \doteq E(\{v\} : V \setminus \{v\})$ for all $v \in V$, and let $n \doteq |V|$. The following is a system of facet-defining inequalities for $HP(V)$, where $n \geq 4$, no two of which are equivalent.*

(a) $x_e \geq 0, \qquad \forall e \in E(V);$

(b) $x(\delta(v)) \leq 2, \qquad \forall v \in V;$

(c) $x(E(W)) \leq |W| - 1, \qquad \forall W \subseteq V$ *and* $2 \leq |W| \leq n-1.$

Note that for $|W| = n-1$, (c) can be restated as the *lower degree constraint* $x(\delta(v)) \geq 1, \{v\} = V \setminus W$ for $HP(V)$. Accordingly, constraints (b) are called the *upper degree constraints.* Note also that, for all $v \in V$, the degree constraint $x(\delta(v)) \geq 1$ (resp., $x(\delta(v)) \leq 2$) is equivalent to $x_{vh} \leq 1$ (resp., $x_{vh} \geq 0$) for $STSP(V \cup \{h\})$, where $h \notin V$. For $W = \{i, j\}$, (c) is equivalent to the simple upper bound constraint $x_{ij} \leq 1$ for $HP(V)$. Constraints (b) and (c) are in normal form. The normal form for the nonnegativity constraints (a) is

$$x(E(V) \setminus e) \leq n-1, \qquad \forall e \in E(V).$$

In accordance with the definition of STSP trivial facets, we also define HPP *trivial facets.*

STSP Trivial Facets. Inequality $ay \leq a_0$ defines a *trivial facet* of $STSP(V')$ if it is equivalent to $y_e \geq 0$ for some $e \in E(V')$. (cf. [26,29])

HPP Trivial Facets. Inequality $cx \leq c_0$ defines a *trivial facet* of $HP(V)$ if it is equivalent either to $x_e \geq 0$ (with normal form: $x(E(V) \setminus \{e\}) \leq |V| - 1$) for some $e \in E(V')$ or to $x(\delta(v)) \leq 2$ for some $v \in V$. (cf. **Proposition 4.9**(a),(b).)

The next lemma exhibits an important property of HPP nontrivial facet-defining inequality. This property will be frequently (and sometimes implicitly) used in the proofs of our polyhedral results.

Lemma 4.10 *Let* $cx \leq c_0$ *be in normal form and define a nontrivial facet of* $HP(V)$. *Then*

(1) for every edge $e \in E(V)$, there exists a c-tight path P on V such that $e \in P$; and
(2) for every node $v \in V$, there exists a c-tight path P on V such that v is one of its endnodes.

Proof: (1) follows since otherwise $cx \leq c_0$ is equivalent to $x_e \geq 0$ for some $e \in E(V)$.

(2) holds since otherwise $cx \leq c_0$ is equivalent to $x(\delta(v)) \leq 2$ for some $v \in V$. □

4.4 Clique-Lifting

In this section, we discuss general clique-lifting results for extending known STSP facets. First, we review related notions and results. Then, we show that any nontrivial STSP facet can be extended to a *lifted* facet by replacing any node with a clique of size greater than two. Finally, we prove a *necessary* and sufficient condition for *node-cloning* (i.e., clique-lifting of size two), extending Naddef and Rinaldi's *zero-lifting* result in [41].

Definition 4.A(**c-Connectedness**, adapted from Naddef and Rinaldi [41]).
Nodes $u, v \in V$ are *c-adjacent* w.r.t. the valid inequality $cx \leq c_0$ for $HP(V)$ if there exists a c-tight path P on V such that u and v are the endnodes of P. A subset $U \subseteq V$ of nodes is *c-connected* if for any pair of nodes $p, q \in U$, there exists a *c-adjacent node sequence* $u_1, u_2, ..., u_l \in U$ with $u_1 \doteq p$ and $u_l \doteq q$ such that u_i and u_{i+1} are c-adjacent for every $i = 1, ..., l-1$.

This c-connectedness condition is equivalent to the *Partition Condition*, **(PC)** for short, as shown below.

Lemma 4.11 (Partition Condition) *Let $ay \leq a_0$ be a valid inequality for $STSP(V')$ and $cx \leq c_0$ be its h-projection for $HP(V)$, where $V = V' \setminus \{h\}$. Then the following statements are equivalent.*
(1) V is c-connected.

(2) For every proper partition $\{U, U'\}$ *of* V*, there exists a c-tight path* $(u...u')$ *on* V *such that* $u \in U$ *and* $u' \in U'$.

(3) **(PC)**: *For every* proper partition $\{U, U'\}$ *of* V*, there exists an a-tight cycle* C *on* V' *containing a 3-node subchain* (uhu') *with* $u \in U$ *and* $u' \in U'$.

Proof: If there exists a proper partition $\{U, U'\}$ violating statement (2), V is clearly not c-connected. Conversely, let $(u'...v')$ be a c-tight path on V and initially let $U \doteq \{u', v'\}$. Observe that U is c-connected. Then, repeat the following procedure.

(A) Using statement (2), there exists a c-tight path $(u...v)$ on V for some $u \in U$ and $v \in U' \doteq V \setminus U$. Hence, $U \cup \{v\}$ is c-connected. Replace U by $U \cup \{v\}$. Repeat (A) until $U = V$.

This shows that statements (1) and (2) are equivalent. Finally, note that by polyhedral equivalence, statements (2) and (3) are equivalent. □

The c-connectedness property plays an important role in deriving our main lifting results. A simple sufficient condition for this property to hold w.r.t. a nontrivial facet-defining HPP inequality $cx \leq c_0$ is given below.

Lemma 4.12 (Sufficient Condition for c-Connectedness) *Let* $cx \leq c_0$ *define a non-trivial facet of* $HP(V)$ *with* $c \geq 0$*, and let* $\overline{G}_c(V)$ *be its zero-graph (cf. (3) in Section 2). Suppose that the set* $U \subseteq V$ *induces a connected subgraph* $\overline{G}_c(U)$ *of* $\overline{G}_c(V)$*. Then* U *is c-connected.*

Proof: Since $cx \leq c_0$ is nontrivial, for any (u, v) with $c_{uv} = 0$, there exists a c-tight path $(p...uv...q)$ on V containing (u, v). Thus, $(u...pq...v)$ is also a c-tight path, showing u and v are c-adjacent. Further, since $\overline{G}_c(U)$ is connected, there exists a c-adjacent node sequence connecting any pair of nodes $u, v \in U$, Hence, U is c-connected. □

Definition 4.B (Clones, Balas and Fischetti [6]).

A pair of nodes h and t in V' are called *clones*, or form a *clone pair*, w.r.t. $ay \leq a_0$ if

(C1) $a_{hv} = a_{tv}$ for all $v \in V' \setminus \{h, t\}$, and

(C2) $a_{ht} = \max\{a_{uh} + a_{vh} - a_{uv} : u, v \in V' \setminus \{h, t\}, u \neq v\}$.

An inequality $ay \leq a_0$ for $STSP(V')$ is called *primitive* (or *simple* [41]) if there exists no clone pair. In **Figure 4.2**, (a) represents a primitive inequality, where in (b), h' and s' (resp., h and s) form a clone pair.

Definition 4.C (Clique-Lifting, Zero-Lifting and Node-Cloning).
Let $ay \leq a_0$ be a valid inequality for $STSP(V')$. For any given $h \in V'$, define *clique weight*

$$\xi \doteq \max\{a_{ih} + a_{jh} - a_{ij} : i, j \in V' \setminus \{h\}, i \neq j\}.$$

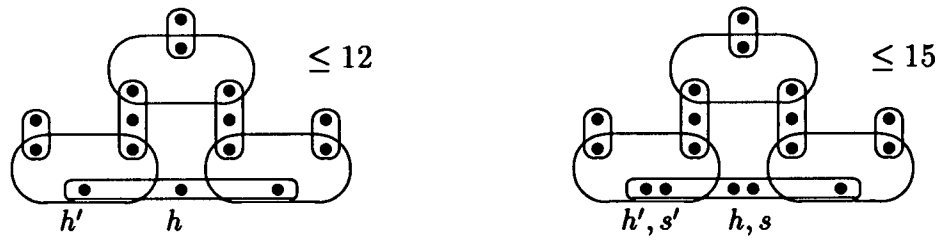
Consider any node set S such that $S \cap V' = \emptyset$. We say that the inequality $a^*y^* \leq a_0^*$ for $STSP(V' \cup S)$ is obtained by *clique-lifting of* $ay \leq a_0$ w.r.t. to node h if $a_0^* = a_0 + \xi|S|$, and

$$a_{uv}^* \doteq \begin{cases} a_{uv}, & u \in V',\ v \in V'; \\ a_{hv}, & u \in S,\ v \in V' \setminus \{h\}; \\ \xi, & u \in S \cup \{h\},\ v \in S \cup \{h\}. \end{cases} \tag{4.7}$$

It is a *zero-lifting* if $\xi = 0$. It is called *node-cloning* if $|S| = 1$. The facet-defining inequality $ay \leq a_0$ is *clique-liftable* (resp., *node-clonable*) w.r.t. h if the lifted inequality $a^*y^* \leq a_0^*$ is facet-defining for any nonempty S (resp., singleton set S).

Clique-Lifting can be viewed as replacing h by the clique $(\{h\} \cup S, E(\{h\} \cup S))$ with clique weight ξ. Fig. 4.2(b) represents the lifted inequality obtained by two successive node-cloning of the inequality in Fig. 4.2(a) w.r.t. nodes h' and h, respectively.

Clique-Lifting, equivalent to $|S|$ successive node-clonings on h, yields a valid STSP inequality, as shown by the following two lemmata.



(a) A primitive bipartition inequality (b) A lifted bipartition inequality

Figure 4.2: Facet-defining bipartition inequalities

Lemma 4.13 (Property of a Node-Cloned Inequality) *Let $ay \leq a_0$ be an inequality for $STSP(V')$, and let nodes h, t be a clone pair w.r.t. $ay \leq a_0$. If the inequality $a^*y^* \leq a_0^*$ for $STSP(V' \cup \{s\})$ is obtained by node-cloning on h, then $a_{hs}^* = a_{ts}^* = a_{ht}$.*

Proof: Since h and t form a clone pair, node-cloning of $ay \leq a_0$ w.r.t. h yields

$$\begin{aligned} a_{hs}^* &= \max\{a_{ih} + a_{jh} - a_{ij} : i, j \in V' \setminus \{h\}, i \neq j\} \\ &= \max\{a_{ht}, \max\{a_{ht} + a_{hv} - a_{tv} : v \in V' \setminus \{h, t\}\}\} = a_{ht}. \end{aligned}$$

Similarly, we show that $a_{ts}^* = a_{ht}$. □

Lemma 4.14 (Validity of Clique-Lifting) *Let $ay \leq a_0$ be any valid inequality for $STSP(V')$, and let the inequality $a^*y^* \leq a_0^*$ be obtained by clique-lifting of $ay \leq a_0$ w.r.t. any given node h. Then $a^*y^* \leq a_0^*$ is valid for $STSP(V' \cup S)$.*

Proof: Consider first $S = \{s\}$. If there exists an a^*-tight cycle $C^* \doteq \langle usv...r \rangle$ on $V' \cup \{s\}$ such that $a^*(C^*) > a_0^*$, then $C \doteq \langle uv...r \rangle$ is a Hamiltonian cycle on V' with

$$a(C) = a^*(C^*) - a_{us} - a_{vs} + a_{uv} = a^*(C^*) - a_{uh} - a_{vh} + a_{uv} > a_0^* - a_{hs}^* = a_0,$$

a contradiction. Hence, node-cloning yields a valid inequality. Using induction on $|S|$, and **Lemma 4.13**, the proof is complete. □

Node-cloning or clique-lifting are equivalent to *zero-lifting* of an STSP inequality $by \geq b_0$ in *tight triangular form.* Recall the definition of this form below.

Definition 4.D (TT-Form, Naddef and Rinaldi [41]).
Inequality $by \geq b_0$ is a *tight triangular inequality,* or in *tight triangular form* (TT-inequality, or TT-form, for short) if

(T1) for all distinct nodes $u, v, w \in V'$, the triangular inequality $b_{uv} \leq b_{uw} + b_{wv}$ holds; and

(T2) for any $w \in V'$, there exists a pair of nodes $u, v \in V' \setminus \{w\}$ such that $b_{uv} = b_{uw} + b_{wv}$.

Note that any STSP inequality has a unique (up to positive scaling) TT-representation.

Proposition 4.15 (Characterizations of Primitive Inequalities) *Let $ay \leq a_0$ be any valid inequality for $STSP(V')$. For any node $h \in V'$, let $a^h y \leq a_0^h$ be the h-normal form of $ay \leq a_0$ and $G_{a^h}(V')$ be its support graph. Then the following statements are equivalent:*

(a) $ay \leq a_0$ is a primitive inequality.

(b) There exists no clone pair w.r.t. $ay \leq a_0$.

(c) The support graph $G_{a^h}(V')$ contains exactly one isolated node h for any $h \in V'$.

(d) The coefficients of the TT-form $by \geq b_0$ of $ay \leq a_0$ are all positive.

Proof: (a) $\Longleftrightarrow$ (b) by definition. To prove (b) $\Longleftrightarrow$ (c), observe that by the uniqueness property of h-normal form, there exists a clone pair $h,t \in V'$ w.r.t. $ay \leq a_0$ iff both h and t are isolated nodes in $G_{a^h}(V')$. The latter is equivalent to $b_{ht} = 0$, proving (c) $\Longleftrightarrow$ (d). □

From this proposition, it follows that node-cloning of $ay \leq a_0$ w.r.t. node h is equivalent to *zero-lifting* of its TT-form $by \geq b_0$ w.r.t. h (or, equivalently, node-cloning of $-by \leq -b_0$ with weight $\xi \equiv 0$).

We now present the first main result of this section.

Theorem 4.16 (Clique-Lifting with $|S| \geq 2$) *Let $ay \leq a_0$ be any nontrivial facet-defining inequality for $STSP(V')$. Then, inequality $a^*y^* \leq a_0^*$ obtained by clique-lifting of $ay \leq a_0$ w.r.t. any node h is facet-defining for $STSP(V' \cup S)$, whenever $|S| \geq 2$.*

Proof. By **Lemma 4.14**, $a^*y^* \leq a_0^*$ is valid. Let $V \doteq V' \setminus \{h\}$.

Let $cx \leq c_0$ and $c^*x^* \leq c_0^*$ be the h-projections of $ay \leq a_0$ and $a^*y^* \leq a_0^*$, respectively. By **Lemma 4.13**, **Lemma 4.14** and **Proposition 4.15**, $c^*x^* \leq c_0^*$ is a valid inequality for $HP(V \cup S)$ with $c_0^* = c_0$; $c_e^* = c_e$ for all $e \in E(V)$ and zero otherwise. Let $fx^* \leq f_0$ be a facet-defining inequality for $HP(V \cup S)$ that dominates $c^*x^* \leq c_0^*$.

(a) Since $cx \leq c_0$ is nontrivial, for any distinct nodes $i, j \in S$ and $v \in V$, let $(u...v)$ be a c-tight path on V and $(ij...k)$ be a Hamiltonian path on S. The Hamiltonian cycle $C \doteq \langle ij...ku...v \rangle$ on $V \cup S$ satisfies $c^*(C) = c_0^*$. Observe that $C \setminus (i,j)$ and $C \setminus (i,v)$ are c^*-tight paths, and hence $f_{iv} = f_{ij}$. Repeating (a) shows that there exists $\beta \geq 0$ such that $f_e = \beta$ for all $e \in E(S) \cup E(S : V)$.

(b) For any c-tight path P on V, there exists an extended c^*-tight path P^* on $V \cup S$ containing P. Using (a), it follows that

$$f_0 = f(P^*) = f(P) + \beta|S| \text{ for all } c\text{-tight path } P \text{ on } V.$$

Since $cx \leq c_0$ is facet-defining, there exist two scalars $\alpha \geq 0$ and γ such that $f_e = \alpha c_e + \gamma$ for all $e \in E(V)$. Next, since $cx \leq c_0$ is in normal form, there exists $e = (p,q) \in E(V)$ such that $c_e = 0$. Let $(u...pq...v)$ be a c-tight path on V and let $(ij...k)$ be a Hamiltonian path on S. Comparing the two extended c^*-tight paths $(ij...ku...pq...v)$ and $(q...vij...ku...p)$ on $V \cup S$ yields $\gamma = f_{pq} = f_{vi} = \beta$. By **Remark 4.3**, $fx^* \leq f_0$ is equivalent to $c^*x^* \leq c_0^*$.

The proof is complete by polyhedral equivalence. □

For any nontrivial facet-defining inequality $ay \leq a_0$ for $STSP(V')$ in h-normal form, the above lifting result immediately implies that it also induces a facet of $STSP(V' \cup S)$ for all $|S| \geq 2$.

We now turn to node-cloning. The following general node-cloning result is due to Naddef and Rinaldi [41], and an alternate (simpler) proof is provided below for completeness.

Proposition 4.17 (Sufficient Condition for Node-Cloning) *Let $ay \leq a_0$ define a nontrivial facet for $STSP(V')$, and let $cx \leq c_0$ be its h-projected inequality for $HP(V)$, where $V \doteq V' \setminus \{h\}$. If the condition* **PC** *stated in* **Lemma 4.11** *is satisfied, then $a^*y^* \leq a_0^*$, obtained by node-cloning of $ay \leq a_0$ w.r.t. h, is facet-defining for $STSP(V' \cup \{s\})$.*

Proof: Note first that V is c-connected by **Lemma 4.11**. Now, let $c^*x^* \leq c_0^*$ be an h-projected inequality of $a^*y^* \leq a_0^*$ and $fx^* \leq f_0$ be a facet-defining inequality for $HP(V \cup \{s\})$ that dominates $c^*x^* \leq c_0^*$. If nodes $u, v \in V$ are c-adjacent, then there exists a c-tight path $(u...v)$ on V. Comparing the c^*-tight paths $P^1 \doteq (u...vs)$ and $P^2 \doteq (su...v)$ yields $f_{us} = f_{vs}$. Since V is c-connected, we deduce that there exists some scalar β such that $\beta = f_{vs}$ for all $v \in V$. Next, since $cx \leq c_0$ defines a facet of $HP(V)$ and every c-tight path $P = (u...v)$ on V satisfies:

$$f(P) = f(P \cup (v, s)) - f_{vs} = f_0 - \beta,$$

it follows that for some scalars $\alpha \geq 0$ and γ, we have $f_e = \alpha c_e + \gamma$ for all $e \in E(V)$. The rest of the proof is similar to the last part of the proof of **Theorem 4.16**: since $cx \leq c_0$ is in normal form, there exists an edge $(p, q) \in E(V)$ with $c_{pq} = 0$. Let $(u...pq...v)$ be a c-tight path on V. Comparing the c^*-tight paths $(u...pq...vs)$ and $(q...vsu...p)$ on $V \cup \{s\}$ yields $\gamma = f_{pq} = f_{us} = \beta$. Hence, $c^*x^* \leq c_0^*$ is equivalent to $fx^* \leq f_0$. The proof is complete by polyhedral equivalence. □

The next result implies that if node h is a clone to some other node, then $ay \leq a_0$ is node-clonable w.r.t. node h.

Corollary 4.18 (Lifting of Clone Pairs) *Let $ay \leq a_0$ define a nontrivial facet of $STSP(V')$ with $a \geq 0$, and suppose that nodes h and t are clones w.r.t. $ay \leq a_0$. Then the inequality $a^*y^* \leq a_0^*$ for $STSP(V' \cup \{s\})$ obtained by node-cloning of $ay \leq a_0$ w.r.t. h (or t) is facet-defining.*

Proof: Let $cx \leq c_0$ be the h-projected inequality for $HP(V)$ w.r.t. $ay \leq a_0$, where $V \doteq V' \setminus \{h\}$. Observe that the support graph $G_c(V)$ contains an isolated node t, and thus its complement, the zero-graph $\overline{G}_c(V)$ is connected. By **Lemma 4.12**, V is c-connected. The proof is complete by **Proposition 4.17**. □

As an application of **Corollary 4.18**, consider Theorem 4.12 (i) in Grötschel and Padberg [27]. As any two nodes in their set Z form a clone-pair, their result follows directly from **Corollary 4.18**.

The next theorem shows that Naddef and Rinaldi's node-cloning condition in **Proposition 4.17** is also necessary.

Theorem 4.19 (Necessary and Sufficient Condition for Node-Cloning) *Let $ay \leq a_0$ be any nontrivial facet-defining inequality for $STSP(V')$. Let $a^*y^* \leq a_0^*$ be an inequality for $STSP(V' \cup \{s\})$ obtained by node-cloning of $ay \leq a_0$ w.r.t. node h. Then $a^*y^* \leq a_0^*$ is facet-defining if and only if the following Partition Condition holds:*
*(**PC**) for every proper partition $\{U, U'\}$ of the node set $V' \setminus \{h\}$, there exists an a-tight cycle C on V' containing a 3-node subchain (uhu') with $u \in U$ and $u' \in U'$.*

Proof: Let $cx \leq c_0$ and $c^*x^* \leq c_0^*$ be the h-projected inequalities of $ay \leq a_0$ and $a^*y^* \leq a_0^*$, respectively. Let $V \doteq V' \setminus \{h\}$.
Sufficiency trivially follows from **Proposition 4.17**.

Necessity. Suppose that (**PC**) does not hold. Then by **Lemma 4.11**(2), there exists a proper partition $\{U, U'\}$ of V such that no c-tight path on V exists with one endnode in U and the other in U'.

Consider any c^*-tight path P^* on $V \cup \{s\}$. Note that P^* has one of the following forms: $(su...v)$; or $(u...psq...v)$. Observe that $c(P^* \cap E(V)) = c_0$. For the former form, either $u, v \in U$ or $u, v \in U'$; and for the latter, either $u, v, p, q \in U$ or $u, v, p, q \in U'$. (Otherwise $P^* \cap E(V)$ can be extended to a c-tight path on V with one endnode in U and the other in U', contradicting the supposition above.) In either case, the incidence vector x^* of P^* satisfies the equality $2x^*(E(U \cup \{s\})) + x^*(E(U : U')) = 2|U|$. It follows that the incidence vectors of all c^*-tight paths P^* satisfy this equation, which is linearly independent of two other linearly independent equations: $c^*x^* = c_0^*$ and $x^*(E(V \cup \{s\})) = |V|$ (Note that by **Proposition 4.15**(c), $c_e^* = 0$ for all $e \in E(\{s\} : V)$ since h, s is a clone pair w.r.t. $a^*y^* \leq a_0^*$). This implies that $c^*x^* \leq c_0^*$, and thus $a^*y^* \leq a_0^*$ (by polyhedral equivalence), is not facet-defining. □

As already observed by Naddef and Rinadi [41], such node-cloning operations are repeatedly applicable. Large classes of STSP facets can be obtained in this fashion. This observation is justified by the following theorem.

Theorem 4.20 (Repeated Node-Cloning and Clique-Lifting) *Let $ay \leq a_0$ be any nontrivial facet-defining inequality for $STSP(V')$. Suppose that:*

(a) $ay \leq a_0$ is node-clonable w.r.t. node $h \in V'$; and

(b) the inequality $a^*y^* \leq a_0^*$ for $STSP(V^*)$, obtained by clique-lifting of $ay \leq a_0$ w.r.t. any node $t \in V'$ where $V^* \doteq V' \cup S$, is facet-defining.

Then $a^*y^* \leq a_0^*$ is node-clonable w.r.t. any node in $\{h, t\} \cup S$.

Proof: Since any pair of nodes in $\{t\} \cup S$ forms a clone pair, by **Corollary 4.18**, $a^*y^* \leq a_0^*$

is node-clonable w.r.t. any node in $\{t\} \cup S$. If $t = h$, then we are done. Else we show that $a^*y^* \leq a_0^*$ is node-clonable w.r.t. h, by verifying condition **(PC)**.

Consider any proper partition $\{U, U'\}$ of $V^* \setminus \{h\}$. Define a partition $\{Q, Q'\}$ of $V' \setminus \{h\}$ corresponding to the following two possible cases:

Case (1). If neither U nor U' is contained in S, define $Q \doteq U \setminus S$ and $Q' \doteq U' \setminus S$.

Case (2). W.l.o.g. assume that $S \subseteq U$, and define $Q \doteq \{t\}$ and $Q' \doteq V' \setminus \{h, t\}$.

By (a),(b) and **Theorem 4.19**, there exists an a-tight cycle C on V' containing subchain (uhu') with $u \in Q$ and $u' \in Q'$ (condition **(PC)**). In either case, we obtain an a^*-tight cycle C^* on V^* by replacing node t in C with a Hamiltonian path $(t...s)$ on $\{t\} \cup S$ so that this C^* contains $(uh'u')$ with $u \in U$ and $u' \in U'$. (Note that for Case (2), $u = t$.)

Hence condition **(PC)** holds w.r.t. node h. □

The condition of the above theorem is weaker than the condition of Theorem 4.9 in Naddef and Rinadi [41], where it is required that *every* node in V' be node-clonable. Consider a proper subset $U \subset V'$, and suppose that a nontrivial facet-defining inequality $ay \leq a_0$ for $STSP(V')$ is node-clonable w.r.t. every node in U. Let $a^*y^* \leq a_0^*$ for $STSP(V^*)$ be obtained by repeated clique-lifting of $ay \leq a_0$ w.r.t. every node $v \in V' \setminus U$, such that each such node v is replaced by a clique of size greater than two. By **Theorem 4.16**, $a^*y^* \leq a_0^*$ is facet-defining. Then **Theorem 4.20** implies that $a^*y^* \leq a_0^*$ is node-clonable w.r.t. *any node in* V^*, but Theorem 4.9 does not.

4.5 Simple Conditions for Node-Cloning

As condition **(PC)** may be difficult to verify, we present in this section three simple sufficient conditions for node-cloning. The first one may be checked in linear time and seems to apply to all known STSP facets. The second condition, which is verified by

visual inspection, generalizes clique-lifting results in Maurras [38], and Grötschel and Padberg [27]. The third condition allows cloning any node in a facet-defining inequality with 0-1 coefficients (a so-called *rank inequality*). We do not know of any STSP facet that cannot be lifted, based on these simple conditions.

Recall that the *zero-graph* $\overline{G_c}$ is the complement of the support graph G_c associated with an HPP inequality $cx \leq c_0$.

Let $ay \leq a_0$ be any inequality for $STSP(V')$.

Definition 4.E (Condition $\mathcal{B}(h, d; \omega)$). The inequality $ay \leq a_0$ is said to satisfy *condition* $\mathcal{B}(h, d; \omega)$ with $h, d \in V'$ and scalar $\omega \geq 1$ if $ay \leq a_0$ is in h-normal form with h-restriction denoted by $cx \leq c_0$, and if there exists a partition $\{h, d, U, U'\}$ of V' such that:

B1. $U = \{v \in V' \setminus \{h, d\} : c_{dv} > 0\}$ and $U' = \{v \in V' \setminus \{h, d\} : c_{dv} = 0\}$;

B2. if $U \neq \emptyset$, then $c_e \leq \omega$ for all $e \in E(d : U)$ and $c_e \geq \omega$ for all $e \in E(U)$.

Note that $ay \leq a_0$ in h-normal form ensures the existence of a partition $\{h, d, U, U'\}$ satisfying **(B1)** and $U' \neq \emptyset$, because $a_e = c_e = 0$ for some $e \in E(V' \setminus \{h\})$.

Let $ay \leq a_0$ be a *star inequality* [21], and let $a'y \leq a'_0$ be its h-normal form w.r.t. any node h. Then $a'y \leq a'_0$ satisfies condition $\mathcal{B}(h, d; \omega)$ for some properly chosen d. (The h-normal forms of the star inequalities are easily obtained by appropriately complementing *circles* and *spikes*, cf. equation (4.6) and **Figure** 4.1.) For instance, Fig. 4.1(a) represents the smallest non-comb star inequality. It is easily verified that the inequality in Fig. 4.1(b) (resp., Fig. 4.1(c)) satisfies condition $\mathcal{B}(h, d; 2)$ (resp., $\mathcal{B}(h', d'; 1)$). Thus, the following lifting result applies to all facet-defining star inequalities w.r.t. any node.

Other known facet-defining inequalities, such as clique-tree, ladder, chain, and crown inequalities, etc., can also be shown to satisfy condition $\mathcal{B}(h, d; \omega)$ w.r.t. any node h. In fact, we do not even know of any STSP facet-defining inequality whose h-normal form

violates condition $\mathcal{B}(h, d; \omega)$.

The following theorem deals with node-cloning of an STSP facet-defining inequality in h-normal form w.r.t. the isolated node h. This lifting is a special case of zero-lifting.

Theorem 4.21 (1-Node Zero-Lifting Theorem) *Let $ay \leq a_0$ be a nontrivial facet defining inequality for $STSP(V')$ satisfying condition $\mathcal{B}(h, d; \omega)$. Then the inequality $a^*y^* \leq a_0^*$ for $STSP(V' \cup \{s\})$, obtained by node-cloning of $ay \leq a_0$ w.r.t. h, is facet-defining.*

Proof: Let $V \doteq V' \setminus \{h\}$, let $cx \leq c_0$ denote the h-restriction of $ay \leq a_0$. Let $\{U, U'\}$ be a partition of $V' \setminus \{h, d\}$ as required in condition $\mathcal{B}(h, d; \omega)$.

First, using (**B1**), $U' \cup \{d\}$ induces a connected subgraph of the zero-graph $\overline{G_c}(V)$, and thus by **Lemma 4.12**, $U' \cup \{d\}$ is c-connected. Next, since $cx \leq c_0$ is nontrivial, for any $w \in U$, there exists a c-tight path on V of the form $P \doteq (w...t)$. If $t \in U' \cup \{d\}$, then w is c-adjacent to some node in $U' \cup \{d\}$. Otherwise $t \in U$ and P has the form $(w...udv...t)$. This implies that $u \in U$ (since otherwise $c_{ud} = 0$ by (**B1**) and Hamiltonian path $P' \doteq (u...wt...vd)$ satisfies $c(P') > c_0$), and therefore by (**B2**), we have $c_{ut} \geq c_{ud}$. Hence, $(w...ut...vd)$ is a c-tight path on V and w is c-adjacent to d. It follows that V is c-connected. The proof is complete by **Proposition 4.17** and polyhedral equivalence. □

In some applications, it is more convenient to apply node-cloning to an STSP facet-defining inequality, *not* in h-normal form. In the sequel, we discuss sufficient conditions for such node-clonings. These conditions can be checked *directly* by visual inspection, using closed form representations.

Definition 4.F (**w-regular node**). Node $h \in V'$ is ω-regular w.r.t. $ay \leq a_0$ if $\omega > 0$ and there exists a proper partition $\{h, Q, Q'\}$ of V' such that $a_{hv} = \omega$ for all $v \in Q$ and $a_{hv} = 0$ for all $v \in Q'$.

Theorem 4.22 (ω-Regular Node-Cloning) *Let $ay \leq a_0$ define any nontrivial facet of $STSP(V')$ with $a \geq 0$. Suppose that node h is ω-regular w.r.t. the partition $\{h, Q, Q'\}$. Let $V \doteq V' \setminus \{h\}$. Suppose also that one of the following conditions is satisfied.*
(i) $Q = \{d\}$; there exists a node $v' \in Q'$ such that $a_{dv'} = 0$ and $a_{dv} \leq \omega$ for all $v \in Q' \setminus \{v'\}$.
(ii) For all $e \in E(Q) \neq \emptyset$, $a_e \geq \omega$; there exists a node $d \in Q$ such that $a_{dv} \leq \omega$ for all $v \in V \setminus \{d\}$.
(iii) There exists an edge $e = (i, j) \in E(Q)$ such that $a_e = 0$ and $a_e \leq \omega$ for all $e \in E(Q) \cup E(Q : Q')$.

Then the inequality $a^*y^* \leq a_0^*$, obtained by node-cloning of $ay \leq a_0$ w.r.t. node h, defines a facet of $STSP(V' \cup \{s\})$.

Proof: Let $a'y \leq a'_0$ be the h-normal form of $ay \leq a_0$ and $cx \leq c_0$ be the h-restriction of $a'y \leq a'_0$. Note that $a^*y^* \leq a_0^*$ is equivalent to the inequality obtained by node-cloning of $a'y' \leq a'_0$ w.r.t. h. We thus need only prove that $a'y' \leq a'_0$ is node-clonable w.r.t. h. Note also that $V = Q \cup Q'$.

Case (i). Note first that clique weight $\xi = a^*_{hs} = a_{hd} + a_{hv'} - a_{dv'} = \omega$. Define $U \doteq \{v \in V \setminus \{d\} : c_{dv} > 0\}$ and $U' \doteq \{v \in V \setminus \{d\} : c_{dv} = 0\}$. Then $a'y' \leq a'_0$ satisfies condition $\mathcal{B}(h, d; \omega)$.

Case (ii). Note first that clique weight $\xi = a^*_{hs} = a_{hd} + a_{hv} - a_{dv} = \omega$ for any $v \in Q \setminus \{d\}$. Observe that, in this case, $a'y' \leq a'_0$ may be obtained by complementing set $Q \cup \{h\}$, cf. equation (4.6). Thus,

$$a'_e = \begin{cases} a_e - \omega, & e \in E(Q \cup \{h\}); \\ a_e + \omega, & e \in E(Q'); \\ a_e & \text{otherwise.} \end{cases} \tag{4.8}$$

Also observe that (ii) implies $a_{dv} = \omega$, and thus $c_{dv} = 0$, for all $v \in Q \setminus \{d\}$. Further by (ii), we have that $c_{dv} \leq \omega$ for all $v \in Q'$, and $c_e \geq \omega$ for all $e \in E(Q')$. Hence, condition

$\mathcal{B}(h, d; \omega)$ also holds w.r.t. $a'y' \leq a'_0$ and a partition $\{h, d, U, U'\}$, where $U \subseteq Q'$.

The proof for **Cases (i) and (ii)** is complete by **Theorem 4.21**.

Case (iii). Note that clique weight $\xi = a^*_{hs} = a_{hi} + a_{hj} - a_{ij} = 2\omega$. Also observe that:

(O1) $c_e \leq \omega$ for all $e \in E(Q)$; and

(O2) $\omega \leq c_e \leq 2\omega$ for all $e \in E(Q : Q')$, and $c_e \geq 2\omega$ for all $e \in E(Q')$.

(1) We show first that Q is c-connected. It suffices to show that for any proper partition $\{Z', Z''\}$ of Q, there exists a c-tight path on V with one endnode in Z' and the other in Z''. Fix $w \in Q'$ and let $P \doteq (w...v)$ be a c-tight path on V. Consider the following two possible cases.

(1.a) If $v \in Q'$, then by (O2), $c_{wv} \geq 2\omega$. Therefore P contains no edge e^* with $c_{e^*} < 2\omega$ (since otherwise $c(P \cup \{(w, v)\} \setminus \{e^*\}) > c_0$). Therefore, by (O1), P contains no edge in $E(Q)$, and P has the general form $(w'...z'p...qz''...w'')$, where $\{w', w''\} = \{w, v\}$, $z' \in Z'$, $z'' \in Z''$ and $p, q \in Q'$. It follows from (O1) and (O2) that $c_{w'p} + c_{qw''} \geq 2\omega \geq c_{z'p} + c_{qz''}$. Thus $(z'...w'p...qw''...z'')$ is a c-tight path with $z' \in Z'$ and $z'' \in Z''$.

(1.b) If $v \in Q$, assume w.l.o.g. $v \in Z'$. Then P has the general form $(w...z''u...v)$ for some $z'' \in Z''$. By (O1) and (O2), we have $c_{wu} \geq c_{z''u}$ for either $u \in Q$ or $u \in Q'$. Thus, $(z''...wu...v)$ is a c-tight path with $v \in Z'$ and $z'' \in Z''$.

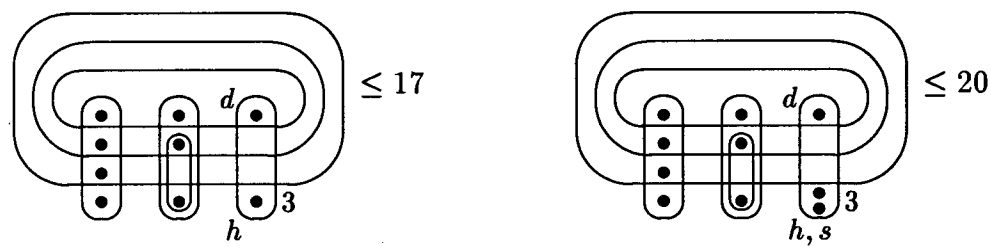
(1.a) and (1.b) shows that Q is c-connected.

(2) We now show that every node $w \in Q'$ is c-adjacent to some node in Q. In doing so, we consider a c-tight path on V: $P \doteq (w...v)$ as in (1) above. If $v \in Q$, we are done for node w. Otherwise, $v \in Q'$, and P has the general form $(w...pz...v)$ for some $z \in Q$. As in (1.a), P contains no edge in $E(Q)$, and thus $p \in Q'$. Hence, by (O1) and (O2), we have $c_{pv} \geq c_{pz}$, implying that $(w...pv...z)$ is a c-tight path on V.

From (1) and (2), it follows that V is c-connected. The proof of **Case (iii)** is complete by **Proposition 4.17** and by polyhedral equivalence. □

We first observe that Maurras' clique-lifting result (Prop.1, pp.188, [38]) is a special case of **Theorem 4.22**(ii) (and also a special case of **Theorem 4.24** below). Next, **Theorem 4.22** (i) and (ii) generalize Theorem 4.12 (ii) in Grötschel and Padberg [27] (Using $Z \doteq \{h\}$). Finally, observe that **Theorem 4.22** (iii) implies Theorem 5.10 in [27].

As examples, we apply ω-regular node-cloning to two new facets in Fig. 4.2(a) and Fig. 4.3(a). (These inequalities were proven by the authors to be facet-defining, using the projective approach). Applying **Theorem 4.22** (ii) and (iii) successively to nodes h and h' in Fig. 4.2(a), we obtain a new STSP facet in Fig. 4.2(b). Applying **Theorem 4.22** (i) to node h in Fig. 4.3(a) yields a new facet of $STSP(V')$ with $|V'| = 10$ in Fig. 4.3(b). These new facets cannot be generated by the clique-lifting results of Maurras, or Grötschel and Padberg.



(a) A primitive binested inequality (b) A lifted binested inequality

Figure 4.3: Facet-defining binested inequalities

Now, we make use of the above result to derive our third main lifting result, *rank inequality lifting.* We first show an intermediate result.

Proposition 4.23 (Zero-Lifting for 0-1-Inequalities in h-normal form) *Let $ay \leq a_0$ define a nontrivial facet of $STSP(V')$ in h-normal form. Assume that $a_e \in \{0,1\}$ for all $e \in E(V)$, where $V \doteq V' \setminus \{h\}$. Then the inequality $a^*y^* \leq a_0^*$, obtained by node-cloning (zero-lifting) of $ay \leq a_0$ w.r.t. h, defines a facet of $STSP(V' \cup \{s\})$.*

Proof: Let $cx \leq c_0$ be its h-restriction on $HP(V)$. By **Proposition 4.17**, it suffices to show that V is c-connected.

Suppose that V is not c-connected. Then by **Lemma 4.11**(2), there exists a proper partition $\{U, U'\}$ of V such that there exists no c-tight path on V with one endnode in U and the other in U'. If $c_{uu'} = 0$ for some $(u, u') \in E(U : U')$, then let $(p...uu'...q)$ be a c-tight path containing (u, u') (since $cx \leq c_0$ is nontrivial). As $c_{pq} \geq 0 = c_{uu'}$, $(u...pq...u')$ is another c-tight path, contradicting the supposition. Therefore, we must have $c_e = 1$ for all $e \in E(U : U')$. Assume w.l.o.g. that $|U| \leq |U'|$. Since $cx \leq c_0$ is nontrivial, for any $u \in U$, there exists a c-tight path $P \doteq (u...v)$ on V. If $v \in U'$, a contradiction is immediate. So consider $v \in U$. Since $|U| \leq |U'|$, P contains at least one edge in $E(U')$. Thus, P has the general form $(u...pq...v)$ with $p, q \in E(U')$. A contradiction is produced by defining a c-tight path $(u...pv...q)$ (since $c_{pv} = 1 \geq c_{pq}$). This shows V is c-connected. □

Recall (pp.282, [28]) that an STSP *rank inequality* is a valid STSP inequality $ay \leq a_0$ with $a_e \in \{0, 1\}$ for all e.

Theorem 4.24 (Node-Cloning for Rank Inequalities) *Let $ay \leq a_0$ be a facet-defining rank inequality for $STSP(V')$. Then the inequality $a^*y^* \leq a_0^*$, obtained by node-cloning of $ay \leq a_0$ w.r.t any node h, is facet-defining for $STSP(V' \cup \{s\})$.*

Proof: Let $V \doteq V' \setminus \{h\}$, $Z \doteq \{v \in V : (h, v) \in E'\}$ and $W \doteq V \setminus Z$. If $Z = \emptyset$, the result holds immediately by **Proposition 4.23**. If $W = \emptyset$, we add the degree constraint $-y(\delta(h)) = -2$, and thus **Proposition 4.23** also applies. If both of the above sets are nonempty, then node h is 1-regular and the result follows by **Theorem 4.22** (by (iii) if $a_e = 0$ for some $e \in E(Q)$; and by (i) or (ii) otherwise). □

With the above result and **Theorem 4.20**, one may obtain large new classes of

STSP facets from rank inequalities. For instance, these liftings can repeatedly apply to rank inequalities of Fig. 4.2(a), the Petersen graph, primitive chains, clique-trees, etc. Thus, to show that a *general* clique-tree is facet-defining, it suffices to consider the corresponding *primitive* clique-tree. Another consequence is that lifted rank inequalities are again clique-liftable w.r.t. any node.

In conclusion, we remark that the Hamiltonian path approach facilitates the STSP polyhedral study. By exploiting the near-full dimensional property of $HP(V)$ and the polyhedral equivalence between $HP(V)$ and $STSP(V')$, we obtain simpler polyhedral proofs for STSP facetial results, as well as stronger lifting results for extending STSP facets than the results from the other approaches. This approach also applies to:

- simple composition methods for extending STSP facets.
- the polyhedral study of asymmetric travelling salesman problem. See **Chapter 5.**
- the polyhedral study of other related problems, such as vehicle routing problems [2],[14].

Chapter 5

Symmetric Inequalities for Asymmetric Travelling Salesman Polytopes

In this Chapter, the Hamiltonian Path approach developed in **Chapter 4** extends to the polyhedral study of *Asymmetric Travelling Salesman* (ATS) polytopes. By a *Tree Composition* technique, we obtain a large new class of symmetric facet-defining inequalities for ATS polytopes.

5.1 Introduction

Let $D(V') = (V', A(V'))$ be a complete directed graph, where the node set V' represents a set of cities. A feasible solution to the Asymmetric Travelling Salesman (ATS) problem is a Hamiltonian cycle C on V', uniquely determined by its corresponding 0-1 *incidence vector* $y \in R^{A(V')}$ (where $y_e = 1$ if edge e is used in C and zero otherwise). This chapter studies the *ATS polytope* $ATSP(V')$, the convex hull of incidence vectors y of all feasible solutions to the ATS problem.

Over the past decade, substantial knowledge has been gained on the facial structure of ATS polytopes. Grötschel and Padberg [28] introduced several classes of valid inequalities, such as the D_k^+, D_k^-, $C3$, $C2$ inequalities, and proved some of these to be facet-defining. Fischetti [17] later proved that all these, as well as the *directed version* of Comb inequalities, are facet defining, except for the 6-node comb. Balas [4], and Balas and Fischetti [5,6] used relaxation methods and a general node lifting technique to derive large new classes of facet-defining inequalities, called *CAT, FDA* and *SD inequalities.* Very recently, Fischetti [18] has shown by an inductive proof technique that the directed

version of the Clique Tree Inequalities, is also facet-defining for ATS polytopes.

Grötschel and Padberg [28] addressed some interesting relationships between STS and ATS polytopes. To any given valid STSP inequality

$$\sum\{a'_{ij}y'_{ij} : i,j \in V', i < j\} \leq a'_0, \tag{5.1}$$

which we write in short $a'y' \leq a'_0$, corresponds a unique *directed version*

$$\sum\{a_{ij}y_{ij} : i,j \in V', i \neq j\} \leq a_0, \tag{5.2}$$

in short $ay \leq a_0$, for the corresponding ATS polytope, where $a_0 = a'_0$, and $a_{ij} = a_{ji} = a'_{ij}$ for all $i,j \in V'$, $i < j$. Thus, we say that an ATSP inequality $ay \leq a_0$ is a *symmetric inequality* if it is the directed version of some STSP inequality $a'y' \leq a'_0$. Conversely, any symmetric ATSP inequality $ay \leq a_0$ yields an *undirected version* $a'y' \leq a'_0$ for the corresponding STS polytope. It is easy to observe that an inequality $a'y' \leq a'_0$ is an STSP *valid* inequality if and only if its directed version $ay \leq a_0$ is an ATSP valid inequality. An important question raised by Grötschel and Padberg [28] is "whether or not the directed version $ay \leq a_0$ of an STSP *facet-defining* inequality $a'y' \leq a'_0$ is also ATSP facet-defining." Conversely, as observed in Fischetti [18], the undirected versions of all symmetric ATSP facet-defining inequalities induce facets of STS polytopes. The study of symmetric ATSP facet-defining inequalities thus has some theoretical importance. To date, the largest known class of symmetric ATSP facet-defining inequalities is the directed version of the famous Clique Tree class, which subsumes all previously known symmetric ATSP facet-defining inequalities, see Fischetti [18].

There are three major motivations for the present work. First, and as mentioned above, in addition to its theoretical interest, the study of symmetric ATSP inequalities is of practical importance: we expect that the cutting plane-based solution procedures that have been so successful in solving large scale STS problems will be adapted to ATS

problems. In this respect, we feel that the symmetric ATSP facet-defining inequalities deserve further study, in particular because STSP *separation* heuristics also apply to ATSP symmetric facets. Second, large classes of valid inequalities, such as Naddef and Rinaldi's Regular Path Trees, Boyd and Cunningham's Ladders, and Padberg and Hong's Chains, etc., were shown to be STSP facet-defining in recent years [42]. It is of interest to know when the directed versions of these large classes are also ATSP facet defining. Finally, known classes of STSP facet-defining inequalities were extended by various *composition methods*, see Naddef and Rinaldi [44]. A composition approach is also introduced here for ATS polytopes and is used to generate large new classes of symmetric ATSP facet-defining inequalities.

Two major difficulties arise when *directly* studying ATS polytopes: (i) these are quite far from being full dimensional polytopes, due to the *degree constraints*; and (ii) Hamiltonian cycles and circuits are rather cumbersome objects to work with. These difficulties complicate the (indirect) proofs that stated valid inequalities are facet-defining. In an *indirect proof* that valid inequality $cx \leq c_0$ is facet-defining for a given polyhedron $\mathcal{P}$, one aims at showing that any valid inequality $fx \leq f_0$ that *dominates* $cx \leq c_0$ (i.e., such that $fx = f_0$ whenever $cx = c_0$, for all $x \in \mathcal{P}$) must be *equivalent* to $cx \leq c_0$. This amounts to showing that $fx \leq f_0$ can be obtained as a linear combination of *implicit equations* for $\mathcal{P}$ (equations satisfied by all $x \in \mathcal{P}$) and a positive multiple of $cx \leq c_0$. If there are *several* linearly independent implicit equations for $\mathcal{P}$, as is the case for STS and ATS polytopes, the indirect proofs tend to be difficult. In addition, the cumbersomeness of Hamiltonian cycles or circuits manifests itself when trying to determine the structure of such dominating inequalities $fx \leq f_0$. The components of f are usually determined by comparing *c-tight* solutions (Hamiltonian cycles or circuits whose incidence vector x satisfies $cx = c_0$), which differ from each other in as few components as possible. Indeed,

two distinct Hamiltonian circuits differ by at least six edges (the so-called *3-opt interchange*). The resulting conditions on the components of vector f then involve at least six components and are thus quite unwieldy.

Various relaxation methods were proposed to overcome these difficulties, most notably (i) the *Monotone ATSP* approach (see Grötschel and Padberg [28] for a recent survey), and Fischetti [17]) and *Graphical ATSP* approach introduced by Chopra and Rinaldi [10]. The corresponding objects, *edge-subsets* of Hamiltonian cycles/circuits, and *tours*, respectively, are much more manageable than Hamiltonian cycles/circuits. Unfortunately, it is not easy to convert MATSP and GATSP facetial results into corresponding ATSP results.

The Hamiltonian Path approach, introduced in **Chapter 4**, can be extended to the polyhedral study of ATS polytopes. This approach uses an *Asymmetric Hamiltonian Path* (AHP) *formulation* and thus also relaxes the degree constraints. For any arbitrary node $h \in V'$, let $V \doteq V' \setminus \{h\}$. Let A (resp., A') denote the edge set of the complete digraph on node set V (resp., V'). The *Asymmetric Hamiltonian Path polytope* $AHP(V) \subseteq R^A$ is the convex hull of incidence vectors of all Hamiltonian paths (with free endpoints) on node set V. We show that polytope $AHP(V)$ is a *projection* of $ATSP(V')$ onto subspace R^A of $R^{A'}$. This projection has two fundamental properties: $AHP(V)$ has full dimension minus one (with a single implicit equation $\sum_e x_e = |V| - 1$); and the *polyhedral structures* of $AHP(V)$ and $ATSP(V')$ are equivalent. We are led to prefer the AHP approach for studying ATS polytopes to the MATSP or GATSP relaxations, for three main reasons which follow mostly from the above two properties. First, the (indirect) proofs for the AHP polyhedral results are almost as easy as in the full dimensional case, because the single implicit equation is easily dealt with. Second, all polyhedral results for $AHP(V)$ trivially extend to $ATSP(V')$, due to their polyhedral equivalence. Finally, we found that Hamiltonian paths are much easier to work with than Hamiltonian circuits, in particular

when comparing solutions in indirect proofs, as two distinct Hamiltonian paths may differ in a single pair of edges.

The present paper is organized as follows. In Section 5.2, we introduce definitions and notation. In Section 5.3, we establish the main properties of $ATSP(V')$ and its projection $AHP(V)$, and then develop a projective approach for the ATS problem. In Section 5.4, we show that some known classes of STSP facet-defining inequalities, such as *Path, Wheelbarrow, Envelope, Ladder* and *Chain Inequalities*, induce symmetric ATSP and AHP facet-defining inequalities. The derivations are based on the projective approach, along with a **Symmetric Dominance Lemma**. In Section 5.5, we propose a *Tree Composition* of symmetric inequalities. This composition generates a large class of symmetric facet-defining inequalities for ATS polytopes, called *ATSP Tree Inequalities*, which generalize the directed versions of non-spanning Clique Tree Inequalities and Regular Path Tree Inequalities. Furthermore, the proposed Tree Composition may also be used to derive a large new class of STSP facet-defining Tree Inequalities. **Figures 5.1- 5.5** herein present examples of new ATSP facets. The proofs of the main results in Sections 5.4 and 5.5 are provided in Sections 5.6 and 5.7, respectively.

5.2 Definitions and Notation

Most of the notation and terminology below follow from [5] and [28].

Let $D(V) = (V, A)$ denote the complete (loop-free) digraph on $n = |V|$ nodes, where

$$A = \{(i,j) : i \in V, j \in V \setminus \{i\}\}. \tag{5.3}$$

Hereafter, we assume $n \geq 3$. Digraph $D(V)$ contains $m \doteq |A| = n(n-1)$ edges. For any two subsets V_1, V_2 of V, define

$$(V_1 : V_2) \doteq \{(i,j) \in A : i \in V_1, j \in V_2\}. \tag{5.4}$$

If $V_1 = V_2$, $(V_1 : V_1)$ is denoted by $A(V_1)$. Thus, $A = A(V)$. A singleton set $\{v\}$ is denoted by v whenever convenient without causing confusion. Let $\delta^+(v) = (v : V)$ and $\delta^-(v) = (V : v)$ denote the *out-star* and *in-star* of node v, respectively.

For complete undirected graph $(V, E(V))$ and any subsets $V_1, V_2 \subseteq V$, the edge sets $E(V_1 : V_2) \doteq \{(i,j) \in E(V) : i \in V_1, j \in V_2\} = E(V_2 : V_1)$ and $E(V_1) \doteq E(V_1 : V_1)$ are similarly defined.

A *partial graph* $(S, \tilde{A})$ of $D(V)$ is a digraph with $S \subseteq V$ and $\tilde{A} \subseteq A$. The *incidence vector* of the partial graph $(V, \tilde{A})$ of $D(V)$ is the vector $y \in R^A$ such that $y_e = 1$ if $e \in \tilde{A}$ and $y_e = 0$ otherwise. For any vector $\chi \in R^A$ and $\tilde{A} \subseteq A$, let

$$\chi(\tilde{A}) \doteq \sum_{e \in \tilde{A}} \chi_e. \tag{5.5}$$

For simplicity, let

$$\chi(S : T) \doteq \chi((S : T)) \quad \text{and} \quad \chi(S) \doteq \chi(A(S)), \quad \forall S, T \subseteq V. \tag{5.6}$$

For any $S \subseteq V$, a *Hamiltonian circuit* C on S is a connected partial graph $(S, \tilde{A})$ such that

$$|\tilde{A} \cap \delta^+(v)| = |\tilde{A} \cap \delta^-(v)| = 1 \text{ for all } v \in S.$$

By definition, the incidence vector y^C of any Hamiltonian circuit C on V thus satisfies the following *degree constraints*:

$$y^C(\delta^+(v)) = y^C(\delta^-(v)) = 1, \quad \text{for all } v \in V.$$

Without risking confusion, a Hamiltonian circuit C with edge set

$$\tilde{A} = \{(v_s, v_1)\} \cup \{(v_i, v_{i+1}) : i = 1, ..., s-1\}$$

on node set $S = \{v_1, ..., v_s\} \subseteq V$ may be represented by edge set $\tilde{A}$ alone or, equivalently, by the circular permutation $\langle v_1 ... v_s \rangle$ of the nodes in S.

A *Hamiltonian path* P on $S = \{v_1, ..., v_s\} \subseteq V$ is a connected partial graph $(S, \hat{A})$ such that for some distinct $i, j \in S$, $(S, \hat{A} \cup \{i, j\})$ defines a Hamiltonian circuit on S. Note that the endnodes of P are not specified in advance, i.e., we consider here P with *free endnodes.* Hamiltonian path P may be represented as the edge set $\{(v_i, v_{i+1}) : i = 1, ..., s-1\}$ or, equivalently, as its corresponding node permutation $(v_1 ... v_s)$. Note that if $|S| = 1$, the first form is an empty set, whereas the second form reduces to a single node (v_1). Note also that whenever a subchain of a Path is written in the form $u...v$, we allow the subchain to be a singleton, i.e., $u = v$.

For any node set S, the *ATS polytope* $ATSP(S)$ (resp., *AHP polytope* $AHP(S)$) is the convex hull of the incidence vectors of all Hamiltonian circuits (resp., Hamiltonian paths) on the complete (loop-free) digraph $(S, A(S))$:

$$ATSP(S) \doteq \text{conv}\{y^C \in R^{A(S)} : C \text{ is a Hamiltonian circuit on } S\},$$

$$AHP(S) \doteq \text{conv}\{x^P \in R^{A(S)} : P \text{ is a Hamiltonian path on } S\}.$$

Recall that $STSP(S)$ (resp., $HP(S)$) is the convex hull of incidence vectors of the undirected Hamiltonian circuits (resp., undirected Hamiltonian paths) in a complete undirected graph $K(S) = (S, E)$. Note that the number of variables defining polytope $STSP(S)$ (resp., $HP(S)$) is half of the number of variables defining its asymmetric counterpart $ATSP(S)$ (resp., $AHP(S)$).

Let $ay \leq a_0$ and $cx \leq c_0$ be any valid inequalities for $ATSP(S)$ and $AHP(S)$, respectively. An *a-tight* circuit C (resp., a *c-tight path* P) on S is a Hamiltonian circuit (resp., a Hamiltonian path) on S such that $ay^C = a_0$ (resp., $cx^P = c_0$).

5.3 The ATS Polytope and its Projection

For any fixed node $h \in V'$, let $V \doteq V' \setminus \{h\}$ and $n \doteq |V|$. Consider two complete (loop-free) digraphs:

$$D(V) = (V, A) \quad \text{and} \quad D(V') = (V', A'),$$

where $A' = A \cup \{(v,h),(h,v) : v \in V\}$. That is, $D(V)$ is obtained from $D(V')$ by eliminating node $h \in V'$ and all incident edges. Let $ATSP(V')$ be the ATS polytope defined on $D(V')$, and $AHP(V)$ be the AHP polytope defined on $D(V)$. Observe that P is a Hamiltonian path on V from node u to node v iff $C \doteq P \cup \{(h,u),(v,h)\}$ is a Hamiltonian circuit on V'. Thus, incidence vector $x^P \in R^A$ is a *projection* of incidence vector $y^C \in R^{A'}$ onto subspace R^A of $R^{A'}$. It follows that polytope $AHP(V)$ is a projection of $ATSP(V')$. Moreover, there is a one-to-one correspondence between the vertices of $ATSP(V')$ and those of $AHP(V)$. This correspondence will be shown to be affine-rank preserving.

Associate two $n \times n(n-1)$ matrices, $\mathcal{A}^-$ and $\mathcal{A}^+$, to complete digraph $D(V)$ as follows. For all $v \in V$ and $e \in A$, let $\mathcal{A}^-_{ve} = 1$ if $e \in (V : v)$; i.e., if v is the *head* of edge e; and zero otherwise. Similarly, let $\mathcal{A}^+_{ve} = 1$ if $e \in (v : V)$, i.e., if v is the *tail* of edge e; and zero otherwise. Thus, $\mathcal{A}^-$ (resp., $\mathcal{A}^+$) is the *tail* (resp., *head*) node-edge incidence matrix of $D(V)$. By definition, the incidence vector $y^C \in R^{A'}$ of any Hamiltonian circuit C on V' and its projection $x^P \in R^A$ satisfy the following identity:

$$y^C_{uv} = \begin{cases} x^P_{uv}, & (u,v) \in A; \\ 1 - \sum_{e \in A} \mathcal{A}^-_{ve} x^P_e, & (u,v) \in (h : V); \\ 1 - \sum_{e \in A} \mathcal{A}^+_{ue} x^P_e, & (u,v) \in (V : h). \end{cases} \tag{5.7}$$

Throughout the paper, we assume, unless otherwise noted, that

- $x \in R^A$ is a projection of $y \in R^{A'}$; and

- $y = (y_{e_1}, .., y_{e_m}, y_{hv_1}, ..., y_{hv_n}, y_{v_1h}, ..., y_{v_nh})^t$, where $m \doteq n(n-1)$, $e_l \in A$ for all $l = 1, ..., m$, and $v_l \in V$ for all $l = 1, ..., n$.

Lemma 5.1 *Let $B_y \doteq [y^1, ..., y^l]$ be any matrix such that all columns y^j are incidence vectors of Hamiltonian circuits on $D(V')$. Let matrix $B_x \doteq [x^1, ..., x^l]$, where x^j is the projection of y^j onto R^A, $j = 1, ..., l$. Then, the number of affinely independent columns in B_y is equal to the number of affinely independent columns in B_x.*

Proof. From (5.7), it follows

$$\begin{aligned}
\operatorname{rank}\begin{pmatrix} \mathbf{1}^t \\ B_y \end{pmatrix} &\geq \operatorname{rank}\begin{pmatrix} \mathbf{1}^t \\ B_x \end{pmatrix} \\
&\geq \operatorname{rank}\left[\begin{pmatrix} 1 & \mathbf{0}^t \\ \mathbf{0} & I \\ \mathbf{1} & -\mathcal{A}^- \\ \mathbf{1} & -\mathcal{A}^+ \end{pmatrix}\begin{pmatrix} \mathbf{1}^t \\ B_x \end{pmatrix}\right] = \operatorname{rank}\begin{pmatrix} \mathbf{1}^t \\ B_y \end{pmatrix}.
\end{aligned}$$

where $\mathbf{0}$ (resp., $\mathbf{1}$) is a vector of zeroes (resp., ones) and I is an identity matrix, all of conformable dimension. Hence, equality must hold throughout.

This completes the proof. □

Theorem 5.2 *The dimension of $AHP(V)$ is*

$$\dim(AHP(V)) = n(n-1) - 1, \tag{5.8}$$

where $n = |V| \geq 2$.

Proof. For $n = 2, 3, 4$, equation (5.8) follows from enumerating all possible corresponding Hamiltonian paths. Let $n \geq 5$. First, observe that for any Hamiltonian path P on V, we have $\sum_{e \in A} x_e^P = n - 1$. Therefore,

$$\dim(AHP(V)) \leq n(n-1) - 1.$$

Next, let $fx = f_0$ be any equality satisfied by the incidence vectors x of all Hamiltonian paths on V. Let e and e' be any two edges in $A(V)$ with distinct endnodes. Then, there exists a Hamiltonian circuit C on V containing both e and e'. Comparing Hamiltonian paths $C \setminus e$ and $C \setminus e'$ yields:

$$0 = fx^{C\setminus e} - fx^{C\setminus e'} = f_{e'} - f_e.$$

As $n \geq 5$, repeating the above argument to all pairs of non-adjacent edges implies that $fx \leq f_0$ is a multiple of $x(V) = n - 1$. □

The dimension of the ATS polytope is described in Grötschel and Padberg [28]. The following is an alternative and simple derivation of this result.

Corollary 5.3 *The dimension of $ATSP(V')$ is*

$$\dim(ATSP(V')) = n(n+1) - 2(n+1) + 1, \tag{5.9}$$

where $|V'| = n + 1 \geq 3$.

Proof. Follows directly from **Lemma 5.1** and **Theorem 5.2** by noting that

$$n(n-1) - 1 = n(n+1) - 2(n+1) + 1. \qquad \square$$

Now, we turn to relationships between valid inequalities for $ATSP(V')$ and for $AHP(V)$. Consider any two inequalities $ay \leq a_0$ and $cx \leq c_0$, defined on $ATSP(V')$ and $AHP(V)$, respectively, and satisfying:

$$a_0 = c_0; \quad a_e = c_e \geq 0, \quad \forall e \in A; \quad \text{and} \quad a_e = 0, \quad \forall e \in (h:V) \cup (V:h).$$

We then say that inequality $ay \leq a_0$ has *an isolated node*, that $ay \leq a_0$ is an *extension* of $cx \leq c_0$, and that $cx \leq c_0$ is a *restriction* of, or a *projected inequality* for, $ay \leq a_0$ w.r.t. node h. By **Lemma 5.1** and **Theorem 5.2**, we have immediately:

Theorem 5.4 (Polyhedral Equivalence) *For any k such that $0 \leq k \leq n(n-1)-1$, inequality $ay \leq a_0$ with an isolated node defines a k-dimensional face of $ATSP(V')$ iff its restriction $cx \leq c_0$ defines a k-dimensional face of $AHP(V)$.*

The above result may also be derived by an application of Remark 4.2 of [27].

By appropriate scaling, the inequality $ay \leq a_0$ with an isolated node is in *h-canonical* form and its corresponding restriction $cx \leq c_0$ is a AHP *canonical* inequality.

h-canonical form (Balas and Fischetti [5])

An inequality $ay \leq a_0$ for $ATSP(V')$ is in *canonical form* with respect to a given node $h \in V'$ (h-canonical form, for short) if

(C1) $a_{vh} = a_{hv} = 0$ for all $v \in V = V' \setminus h$;

(C2) $a \geq 0$ and $a_e = 0$ for some $e \in A = A(V)$;

(C3) the coefficients a_e are relatively prime integers.

AHP Canonical Inequality Inequality $cx \leq c_0$ for $AHP(V)$ is a *canonical inequality*, or *in canonical form*, if the coefficients c_e satisfy conditions (C2) and (C3) above.

It is shown in [5] that any valid inequality for $ATSP(V')$ has a unique h-canonical form. This *uniqueness property* also follows directly from **Theorems 5.2** and **5.4** by observing that conditions (C2) and (C3) may be imposed on the restriction $cx \leq c_0$ by using the unique implicit equation $x(V) = n-1$ for $AHP(V)$, followed by an appropriate positive scaling.

Following the results in [28] and the polyhedral equivalence, *trivial facets* of ATSP and AHP polytopes are defined as follows:

Trivial Facets Inequality $ay \leq a_0$ (resp., $cx \leq c_0$) defines a *trivial facet* of $ATSP(V')$ (resp., $AHP(V)$) if it is equivalent to $y_e \geq 0$ (resp., $x_e \geq 0$) for some $e \in A'$ (resp., $e \in A(V)$).

From now on, we will work with symmetric inequalities, in canonical form for $AHP(V)$, and in h-canonical form for $ASTP(V')$, i.e., with an isolated node h readily identified, for the following reasons. First, the projections of these inequalities $ay \leq a_0$ with respect to isolated node h are in fact the restrictions $cx \leq c_0$ of $ay \leq a_0$. Consequently, the "structural" definitions of inequalities, such as Comb, Clique Tree, etc., $ay \leq a_0$ for $ATSP(V')$ directly apply to $cx \leq c_0$ for $AHP(V' \setminus h)$ as well. Next, it follows immediately from **Theorem 5.4** that $ay \leq a_0$ defines a facet of $ATSP(V')$ iff $cx \leq c_0$ defines a facet of $AHP(V' \setminus h)$. So, no additional work is needed to recover one inequality from the other. Finally, the ATSP facet composition results are relatively easier to obtain, as is the case for *2-SUM* GTSP facet compositions of Naddef and Rinaldi [44]. The proposed ATSP facet composition in Section 5.4 requires that all composing inequalities be in h-canonical forms. The facet-defining inequalities given as examples in Figures hereafter are for ATS polytopes and are all in h-canonical form, unless otherwise noted.

5.4 Symmetric Facet-defining Inequalities

In this section, we study three classes of symmetric inequalities: *Regular Star*, *Chain* and *Ladder* Inequalities. Of particular interest are these inequalities in "nice" h-canonical form, i.e., with an isolated node. Their undirected versions are known to be facet-defining for STS polytopes. For STS polytopes, the inequalities in the first class was introduced as the *Path Inequalities*, along with variations, the *Wheelbarrow* and *Bicycle Inequalities*, by Cornuéjols *et al.* [13]. They were called *Regular Star Inequalities* by Fleischmann [21] and shown to be STSP facet-defining by Naddef and Rinaldi [42]. The second class was introduced by Padberg and Hong [47] and shown to be STSP facet-defining by M. Hartmann and by S. Boyd (cf. Naddef and Rinaldi [41]). The Ladder Inequality was introduced and proven to be STSP facet-defining by Boyd and Cunningham [8]. We will

show by projection that the Path, Wheelbarrow, Envelope, Ladder and Chain inequalities induce facets of ATS polytopes as well.

A *closed form representation* for inequalities $ay \leq a_0$ is used in the Figures hereafter, where a collection of node subsets with positive weights (numbers) are given. Then, each coefficient a_{ij} equals the total weight of all the subsets which contain both nodes i and j. All set weights of value one are omitted for simplicity. Whenever necessary, nodes are labeled with letters or *italic* numbers. A *hollow* circle g denotes an *optional node*, i.e., node g may or may not be present. If the right hand side of an inequality with node g is different from that without g, then the former is also indicated in parentheses. **Figure 5.1** gives such a representation. Some of the resulting coefficients in the represented ATSP inequality are

$$a_{b_1^i b_3^i} = 2, \quad a_{b_1^j b_2^j} = 4, \quad a_{b_3^i b_5^k} = 4, \quad a_{b_3^i b_1^k} = 0, \quad \text{etc.}$$

and the right hand side $a_0 = 44$ if node g is absent, $a_0 = 48$ if present.

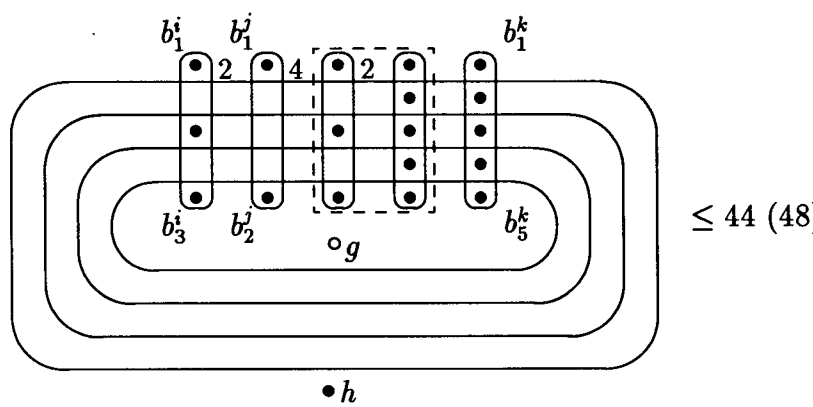


Figure 5.1: An ATSP Regular Star Inequality

Balas and Fischetti [6] recently proposed a powerful *node lifting* result for ATSP facets,

called *Clique-Lifting*. This result implies that a class of ATSP inequalities are facet-defining if all the *primitive inequalities* in this class are facet-defining (cf. Fischetti [17] for the definition of a primitive inequality). Therefore, it suffices to consider ATSP inequalities in primitive form. In the sequel, all inequalities are introduced in primitive form.

We now introduce primitive *Star Inequalities* for $ATSP(V')$. **Figure 5.1** presents an example. Given are proper subsets of V' as follows: N *nested circles*, $A_1 \supset A_2 \supset ... \supset A_N$; and an odd number $s \geq 3$ of disjoint *spikes*, $B^1, B^2, ..., B^s$, where, for all i, B^i consists of nodes $b^i_j, j = 1, ..., n_i$ such that $b^i_{n_i} \in A_N$ and $b^i_1 \in V' \setminus A_1$. Moreover, assume that for every i, inequality $j < l$ implies that there is no q such that $b^i_j \in A_q$ and $b^i_l \notin A_q$. Let $M(v)$ denote the number of circles A_q containing node v. With each spike B^i associate a *spike weight* N_i such that $N_i \geq M(b^i_{j+1}) - M(b^i_j)$ for all $1 \leq j \leq n_i - 1$ (cf. Fleischmann [21] for details). Then, a *primitive Star Inequality* $ay \leq a_0$ for $ATSP(V')$ is defined by

$$\sum_{q=1}^{N} y(A_q) + \sum_{i=1}^{s} N_i y(B^i) \leq \sum_{q=1}^{N} |A_q| + \sum_{i=1}^{s} N_i(|B^i| - 1) - \frac{(s+1)N}{2}. \tag{5.10}$$

Let $a'y' \leq a'_0$ be the undirected version of inequality (5.10) for $STSP(V')$. By adding to $a'y' \leq a'_0$ proper multiples of the following valid equations (obtained from the degree constraints):

$$2y'(S) + y'(\delta(S)) = 2|S|, \quad \text{for all} \quad S \in \{A_1, ..., A_N, B^1, ..., B^s\},$$

we obtain

$$\sum_{q=1}^{N} y'(\delta(A_q)) + \sum_{i=1}^{s} N_i y'(\delta(B^i)) \geq 2\sum_{q=1}^{N} N_q + (s+1)N, \tag{5.11}$$

which is the *Star Inequality* for $STSP(V')$ in Fleischmann's form [21].

When $N_i = M(b^i_{j+1}) - M(b^i_j)$ for all i and j, inequality (5.10) represents a *Regular* Star Inequality. If $N = 1$ and all spike weights $N_i = 1$, inequality (5.10) reduces to

a Comb Inequality [28]. Hence, the Regular Star class properly generalizes the Comb class. **Figure 5.1** presents a Regular Star Inequality with four circles and five spikes, i.e., a *5-Star* (equivalently, a Path Inequality with five "paths"), in h-canonical form. It reduces to a 3-Star if the spikes in the dash-box are removed. By definition, the spike weights for B^i, B^j and the leftmost spike in the dash-box are 2, 4 and 2, respectively. Recall that all other set weights are equal to one. A *Path Inequality* includes both nodes h and g, whereas a *Wheelbarrow Inequality* (resp., *Bicycle Inequality*) excludes node g (resp., excludes both nodes h and g). If node g exists, then the right hand side needs to be properly adjusted according to (5.10).

If a Regular Star is not a Bicycle Inequality, its restriction $cx \leq c_0$ is in canonical form for $AHP(V)$, and reads:

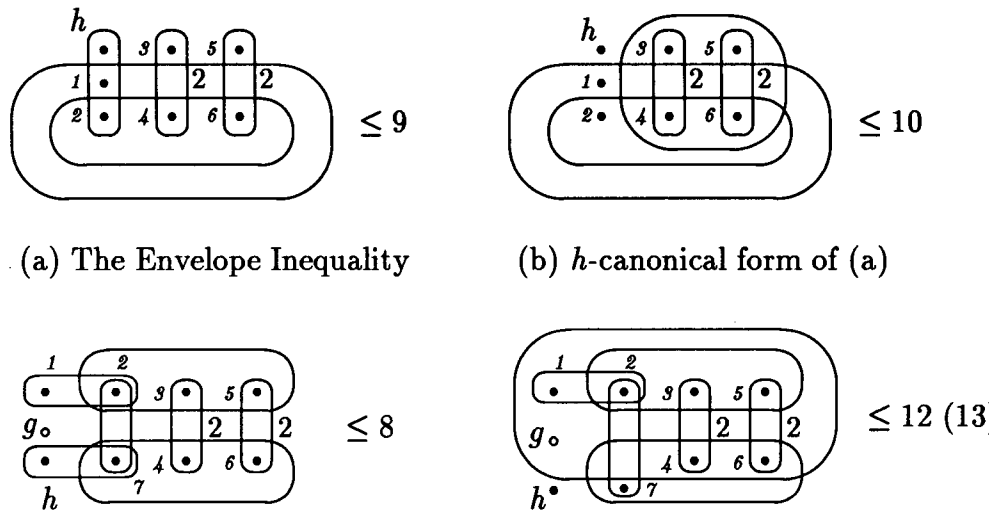
$$\sum_{q=1}^{N} x(A_q) + \sum_{i=1}^{s} N_i x(B^i) \leq \sum_{q=1}^{N} |A_q| + \sum_{i=1}^{s} N_i(|B^i| - 1) - \frac{(s+1)N}{2}. \tag{5.12}$$

Theorem 5.5 *Primitive Path and Wheelbarrow Inequalities (resp., their restrictions) are facet-defining for $ATSP(V')$ (resp., $AHP(V)$).*

The proof of **Theorem 5.5** is given in Section 6.

The ATSP Bicycle Inequalities do not have such nice h-canonical forms. The smallest non-comb Bicycle Inequality, also called the *Envelope Inequality* by Boyd and Cunningham, is given in Fig. 5.2(a). An h-canonical form of the Envelope Inequality is presented in Fig. 5.2(b). Fig. 5.2(c) represents the Ladder Inequality. Recall that node g is optional. Fig. 5.2(d) is its h-canonical form. Note that these h-canonical forms are obtained by replacing all non-empty subsets containing node h with their complements. In Fig. 5.2(b) and (d), *italic* numbers are used to label the nodes for convenience in proving the next theorem. (The other numbers are set weights.) If node g exists, an alternative simple h-canonical form for Fig. 5.2(c) is Fig. 5.2(c) itself by exchanging nodes h and g. We choose

the form in Fig. 5.2(c), because it is convenient for showing that the Ladder Inequality is facet-defining for $ATSP(V')$ for $|V'| = 8, 9$.



(a) The Envelope Inequality (b) h-canonical form of (a)

(c) The Ladder Inequality (d) h-canonical form of (c)

Figure 5.2: Symmetric facets and their projections

Theorem 5.6 *The following statements are true:*

1. *The Envelope Inequality defines a facet of $ATSP(V')$, where $|V'| = 7$.*
2. *The Ladder Inequality defines a facet of $ATSP(V')$, where $|V'| \geq 8$.*

A proof of **Theorem 5.6** is contained in Section 5. We remark that the Envelope Inequality is one of the two smallest non-trivial, non-subtour elimination, primitive symmetric facet-defining inequalities for $ATSP(V')$ with $|V'| = 7$. (Otherwise, the undirected version of some primitive symmetric ATSP facet defining inequality would induce a facet of $STSP(V')$ with $|V'| = 7$, contradicting a theorem by Boyd and Cunningham [8].) The other one is a 7-node Wheelbarrow Inequality.

Note that Clique Lifting may then be applied to any node of these ATSP facet-defining Envelope and Ladder Inequalities. Other extensions of the above results to general Bicycle and Ladder Inequalities (without isolated nodes) are left for the interested reader, using proofs similar to those of **Theorems 5.5** and **5.6**.

We now consider *primitive* ATSP *Chain Inequalities* with an isolated node h. Given are any positive integers p and k such that $k - p$ is a positive even number. Let R and S_i for $i = 0, ..., k$, be proper subsets of V' satisfying

(i) $S_i \cap S_j = \emptyset$ for all $i, j = 1, ..., k$ $(i \neq j)$;

(ii) $S_i = \{s_i\}$ for all $i = 1, ..., p$;

(iii) $S_i \setminus S_0 = \{b_1^i\}$ and $S_0 \cap S_i = \{b_2^i\}$ for all $i = p+1, ..., k$;

(iv) $R \doteq \{r_1, ..., r_p\} \subset S_0$ and $R \cap S_i = \emptyset$ for all $i = 1, ..., k$; and

(v) $V' \setminus \bigcup_{i=0}^{k} S_i = \{h\}$.

Let $T \doteq \{s_1, ..., s_p\}$.

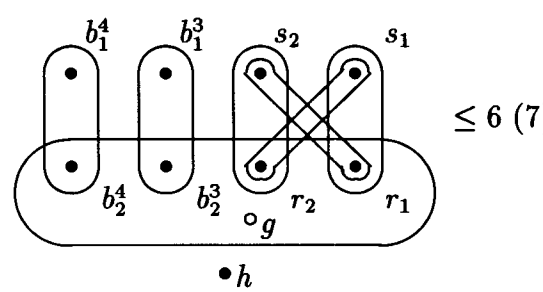


Figure 5.3: A 9-node ATSP Chain Inequality

From the general Chain Inequality (see [47] for definition), we obtain a primitive

Chain Inequality for $ATSP(V')$:

$$\sum_{i=0}^{k} y(S_i) + y(R:T) + y(T:R) \leq |S_0| + \frac{1}{2}(k+p-2). \tag{5.13}$$

For $p = 1$, this reduces to a Comb Inequality. If $k - p > 0$ is odd, then it can be shown that the inequality is still valid but not facet-defining, see Padberg and Hong [47]. The smallest non-comb Chain Inequality in h-canonical form is presented in **Figure 5.3**.

Theorem 5.7 *Primitive Chain Inequalities defined in (5.13) with an isolated node (resp., their restrictions) are facet-defining for $ATSP(V')$ (resp., $AHP(V)$).*

This theorem is also proved in Section 5.

General Chain Inequalities are then shown to be ATSP facet-defining by Clique-Lifting on any node.

The proofs of **Theorems 5.5-5.7** are based on the following two auxiliary results. The first one follows from the known STSP facetial results and the polyhedral equivalence of $STSP(V')$ and $HP(V)$ in **Chapter 1**.

Proposition 5.8 *The undirected versions of the symmetric inequalities $ay \leq a_0$ and their projections $cx \leq c_0$ stated in* **Theorems 5.5-5.7** *define facets of $STSP(V')$ and $HP(V)$, respectively.*

Lemma 5.9 (Symmetric Dominance Lemma) *Let $cx \leq c_0$ be a* symmetric *inequality for $AHP(V)$ whose undirected version $c'x' \leq c'_0$ defines a facet of $HP(V)$. If there exists a* symmetric *facet-defining inequality $fx \leq f_0$ for $AHP(V)$ which dominates $cx \leq c_0$, then $cx \leq c_0$ defines a facet of $AHP(V)$.*

Proof. If $fx \leq f_0$ is symmetric, then it has an undirected version $f'x' \leq f'_0$ for $HP(V)$. Observe that $f'x' \leq f'_0$ also dominates the HP facet-defining inequality $c'x' \leq c'_0$, and

therefore, they are equivalent. Because the equality set of $HP(V)$ reduces to $x'(E(V)) = n-1$, there exist scalars $\alpha > 0$ and β such that $f'_e = \alpha c'_e + \beta$ for all $e \in E(V)$ and $f'_0 = \alpha c'_0 + (n-1)\beta$. It follows that $f_e = \alpha c_e + \beta$ for all $e \in A(V)$. Thus, $cx \leq c_0$ is equivalent to $fx \leq f_0$, hence facet-defining. □

Thus, the proofs in Section 5 amount to showing that for each restriction $cx \leq c_0$ of the proposed h-canonical ATSP inequalities, there exists a symmetric AHP facet-defining inequality that dominates $cx \leq c_0$. The result then immediately translates into ATSP term by the polyhedral equivalence of **Theorem 5.4**.

5.5 Composing Symmetric Inequalities

In this section, we propose a *Tree Composition* of ATSP symmetric inequalities. Regular Star, Ladder and Chain Inequalities, each with an isolated node, are possible building blocks for the composition. Tree Composition may be used to generate a large new class of ATSP facet-defining inequalities, called *ATSP Tree Inequalities*, for ATS polytopes, generalizing the directed versions of Regular Path Tree, and thus all non-spanning Clique Tree, Inequalities with an isolated node.

The application of Tree Composition also extends to facets of STS polytopes. The composition is similar in spirit to Naddef and Rinaldi's *2-SUM* composition. However, the conditions required here for each composing inequality are different and much easier to apply, mostly by visual inspection. Furthermore, large classes of STSP facet-defining Tree Inequalities resulting from 2-SUM composition, such as *Regular Path Trees*, may also be obtained by the present Tree Composition. Consequently, this Tree Composition may be used to obtain large new classes of tree-like facets of STS polytopes as well.

Definition 5.A We say that an inequality $ay \leq a_0$ for $ATSP(V')$ satisfies *Condition* $\mathcal{S}(h, S, Z; d; \omega; \eta)$ if there exists a proper partition $\{h, S, Z\}$ of V', with $|S| \geq 2$; a node

$d \in S$, called a *tip*; and two positive scalars ω and η, such that the following conditions hold:

1. $a_e \geq 0, \quad \forall e \in A(V')$;
2. $a_e = 0, \quad \forall e \in (h : S \cup Z) \cup (S \cup Z : h) \cup (d : Z) \cup (Z : d)$;
3. if $|S| \geq 3$, then $a_e \geq \eta\omega, \quad \forall e \in A(S \setminus d)$; and
4. $a_{dv} = a_{vd} = \omega, \quad \forall v \in S \setminus d$.

An inequality $ay \leq a_0$ for $ATSP(V')$ is said to be an *ATSP Tree Inequality* with respect to a subset S of V', called a *bud*, if the following two conditions hold:

(S1) $ay \leq a_0$ is a symmetric valid inequality for $ATSP(V')$; and

(S2) $ay \leq a_0$ satisfies Condition $\mathcal{S}(h, S, Z; d; \omega; 1)$.

An ATSP Tree Inequality $ay \leq a_0$ with respect to bud S is said to be a *composable* ATSP Tree Inequality if, in addition to (S2), it satisfies the following three conditions:

(S1') $ay \leq a_0$ defines a non-trivial symmetric *facet* of $ATSP(V')$;

(S3) for every $v \in S \setminus d$, there exists an a-tight Hamiltonian circuit C on V', called an *$\mathcal{S}$-circuit*, of the form $\langle z...uv...qdh \rangle$ with $a_{uv} = 0$; and

(S4) there exists a Hamiltonian circuit C_d on $V' \setminus d$, called a *$\mathcal{U}$-circuit*, such that $a(C_d) = a_0$.

Conditions (S3) and (S4) are sometimes hard to verify and may be removed by strengthening (S2):

(S2') There exist two *disjoint* subsets S and S' of V' such that $ay \leq a_0$ satisfies both Conditions $\mathcal{S}(h, S, Z; d; \omega; 2)$ and $\mathcal{S}(h, S', Z'; d'; \omega'; 2)$.

Proposition 5.10 *(S1'),(S2')* $\Longrightarrow$ *(S3),(S4) w.r.t. both* (S, d) *and* (S', d').

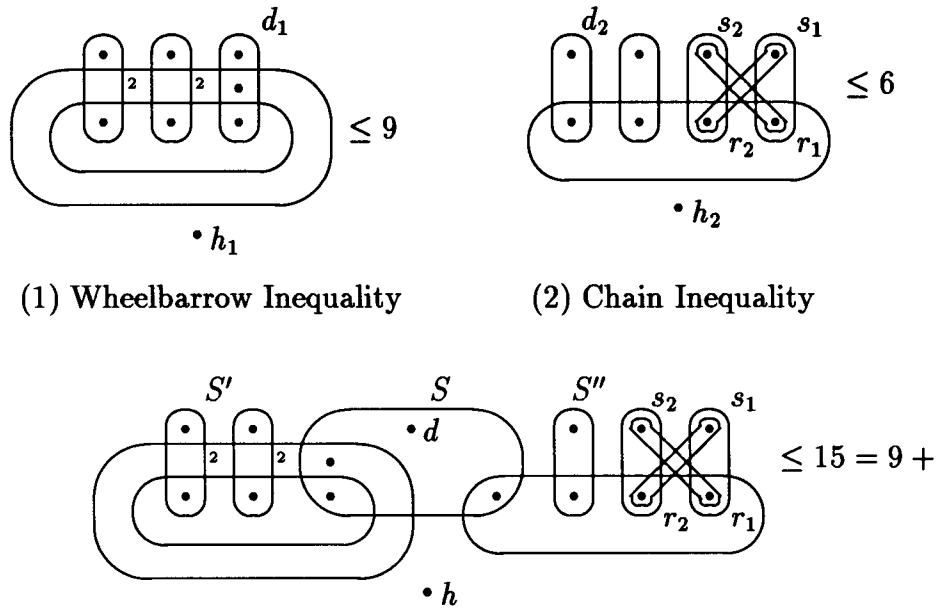
A proof is provided in Section 6.

By **Proposition 5.10**, conditions (S1') and (S2') are sufficient for recognizing composable ATSP Tree Inequalities by inspection. Path and Wheelbarrow Inequalities are composable ATSP Tree Inequalities with respect to any spike as a bud. Clique Tree, Ladder and Chain Inequalities, each with an isolated node, are composable ATSP Tree Inequalities with respect to any pendant tooth as a bud. For some cases, however, the weaker conditions (S1'),(S3) and (S4) may apply, while (S2') fails. For instance, the Chain Inequality in **Figure 5.3** is also a composable ATSP Tree Inequality relative to bud $S \doteq \{s_1, r_1, r_2\}$ with tip $d \doteq s_1$ and $\omega = 1$, in which case condition (S2') fails but (S2) holds, since $a_{r_1 r_2} = 1 < 2\omega$. However, (S3) and (S4) hold: for $v = r_1$ and $v = r_2$, the $\mathcal{S}$-circuits are $\langle b_1^3 b_2^3 b_2^4 b_1^4 v s_2 r_2 d h\rangle$ and $\langle b_1^3 b_2^3 b_2^4 b_1^4 v s_2 r_1 d h\rangle$, respectively. The $\mathcal{U}$-circuit is $C_d \doteq \langle b_1^4 b_2^4 r_1 s_2 r_2 b_2^3 b_1^3 h\rangle$.

Let $a^1 y^1 \leq a_0^1$ (resp., $a^2 y^2 \leq a_0^2$) be an ATSP Tree Inequality for $ATSP(V_1')$ (resp., $ATSP(V_2')$) with respect to bud S_1 (resp., bud S_2). Assume that $V_1' \cap V_2' = \{d_1, h_1\} = \{d_2, h_2\}$, $d = d_1 = d_2$ and $h = h_1 = h_2$, where $d_1 \in S_1$ and $d_2 \in S_2$ are tips. Assume also w.l.o.g. that $\omega = \omega_1 = \omega_2$ (by scaling). Define $V' \doteq V_1' \cup V_2'$, $S \doteq S_1 \cup S_2$ and construct inequality $ay \leq y_0$ for $ATSP(V')$ by the following *Tree Composition*:

$$a_e \doteq \begin{cases} a_e^i, & e \in A(V_i' \setminus h_i), \quad i = 1, 2; \\ \omega, & e \in A(S) \setminus \{A(S_1) \cup A(S_2)\}; \\ 0, & \text{otherwise}; \end{cases} \tag{5.14}$$

and $a_0 = a_0^1 + a_0^2$. This composition is illustrated in **Figure 5.4**. After composition, two buds containing their corresponding tips d_1 and d_2 are combined into a single bud S in Fig. 5.4(3), and d_1 and d_2 (resp., nodes h_1 and h_2), are merged into a single tip d (resp., node h).



(1) Wheelbarrow Inequality (2) Chain Inequality

(3) An ATSP Tree Inequality

Figure 5.4: A Tree Composition

Theorem 5.11 (Tree Composition Theorem) *Let $a^1y^1 \leq a_0^1$ and $a^2y^2 \leq a_0^2$ be ATSP Tree Inequalities for $ATSP(V_1')$ and $ATSP(V_2')$ w.r.t. buds S_1 and S_2, respectively, and let $ay \leq a_0$ be obtained by the Tree Composition (5.14). Then*

(i) $ay \leq a_0$ is a valid inequality for $ATSP(V')$, where $V' = V_1' \cup V_2'$.

(ii) if, furthermore, the composing inequalities are composable ATSP Tree Inequalities w.r.t. buds S_1 and S_2, then the following statements hold:

(A) The composed inequality $ay \leq a_0$ is facet-defining for $ATSP(V')$.

(B) The composed inequality $ay \leq a_0$ is a composable ATSP Tree Inequality for $ATSP(V')$ with respect to the composed bud $S = S_1 \cup S_2$.

(C) If $a^1y^1 \leq a_0^1$ is a composable Tree for $ATSP(V_1)$ with respect to another bud S' with tip d', satisfying $S' \cap S_1 = \emptyset$, then, the composed inequality $ay \leq a_0$ is a composable ATSP

Tree Inequality for $ATSP(V')$ with respect to bud S'.

The proof is given in Section 6.

Statements (B) and (C) ensure that the Tree Composition procedure can be repeatedly applied so as to construct a large class of *tree-like* facet-defining inequalities for ATSP polytopes. For instance, **Theorem 5.11** implies that Fig. 5.4(3) represents an ATSP Tree Inequality with respect to any one of buds S, S' and S'', along with a corresponding tip (S by Statement (B); the other cases by Statement (C)). Therefore, the resulting inequality can be further used for Tree composition with respect to any bud. By repeating Tree Compositions, a large class of *tree-like* facet-defining inequalities may then be constructed for ATS polytopes.

Finally, we show how the Tree Composition generates the ATSP version of a subclass of the *Path Trees* (See **Figure 5.5**), which were introduced as generalizations of the well-known clique trees in [44] and proven to be STSP facet-defining in [42].

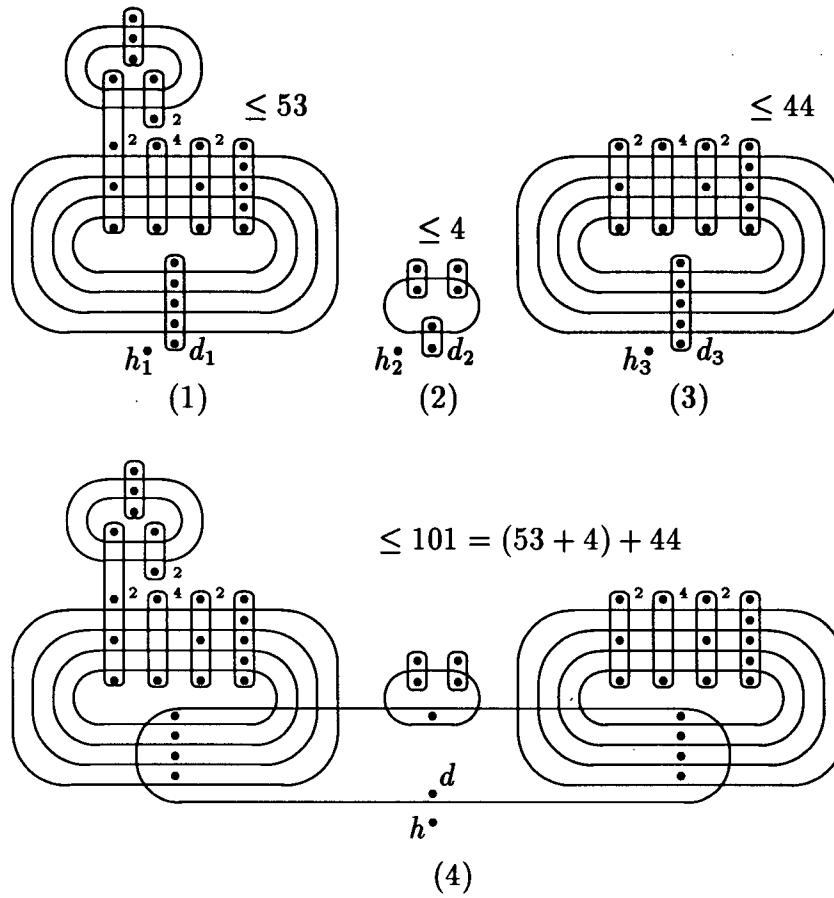


Figure 5.5: A Path Tree obtained from the Tree Compositions

Write inequality (5.12) in the following equivalent form:

$$\sum_{q=1}^{N} y(A_q) + \sum_{i=1}^{s} N_i y(B^i) \leq \sum_{q=1}^{N} |A_q| + \sum_{i=1}^{s} N_i(|B^i| - 1) - \frac{1}{2}\sum_{q=1}^{N}(k_q + 1), \tag{5.15}$$

where k_q denotes the number of spikes intersected by circle q. By (S1') and (S2'), a Regular Star with an isolated node (i.e., Path or Wheelbarrow) is a composable ATSP Tree Inequality with respect to any spike as a bud. For $i = 1, 2$, let $a^i y^i \leq a_0^i$ be such a Regular Star Inequality with an isolated node h_i in form (5.15) for $ATSP(V_i)$. Then, the Tree Composition can be applied to these inequalities and the resulting inequality $ay \leq a_0$ can be easily shown to be in form (5.15) as well. It then follows by repeating the

Tree Composition that the Path Tree Inequalities in form (5.15) are ATSP facet-defining as well.

Figure 5.5 illustrates how a Path Tree facet-defining inequality for the ATS polytopes may be built by repeated Tree Compositions. Fig. 5.5(1) represents a Path Tree Inequality after one application of the Tree Composition on the spikes with weight two. After two other applications of the composition on the spikes with weight one, a larger Path Tree facet-defining Inequality for the ATS polytope is obtained as shown in Fig. 5.5(4). It also follows that the undirected version of this inequality defines a facet of the corresponding STS polytope.

5.6 Proofs of the Main Results in Section 5.4

We show in this section that the proposed classes of symmetric inequalities $cx \leq c_0$ in section 4 define facets of AHP polytopes. For each of the following proofs, we implicitly assume that $fx \leq f_0$ are facet-defining inequalities that dominate $cx \leq c_0$. By **Proposition 5.8** and the **Symmetric Dominance Lemma**, it suffices to show that their dominating facet-defining inequalities $fx \leq f_0$ are symmetric.

We will use the following notation in the sequel. For a subchain $P = (v_1 ... v_l)$ of any Hamiltonian path on V, define $\overleftarrow{P} \doteq (v_l ... v_1)$ to be the same subchain in reverse order. Given any inequality $fx \leq f_0$ for $AHP(V)$, define the *f-length* of P by

$$f(P) = f(v_1 ... v_l) \doteq \sum_{i=1}^{l-1} f_{v_i v_{i+1}}, \quad 1 \leq l \leq n. \tag{5.16}$$

Note that P may be a single edge, or even a single node; in the latter case $f(P) \doteq 0$. Path P is *f-symmetric* if $f(P) = f(\overleftarrow{P})$. If $p = f(P)$, define $\overleftarrow{p} \doteq f(\overleftarrow{P})$. By definition, for any symmetric inequality $cx \leq c_0$, all edges $e \in A$ are c-symmetric.

Proof of Theorem 5.5: Define

$$\alpha \doteq f(b_1^j b_1^k), \quad \overleftarrow{\alpha} \doteq f(b_1^k b_1^j), \quad \beta \doteq f(b_{n_i}^i b_{n_j}^j), \quad \overleftarrow{\beta} \doteq f(b_{n_j}^j b_{n_i}^i).$$

Define also $P_l \doteq (b_1^l ... b_{n_l}^l)$ for all $l = 1, ..., s$.

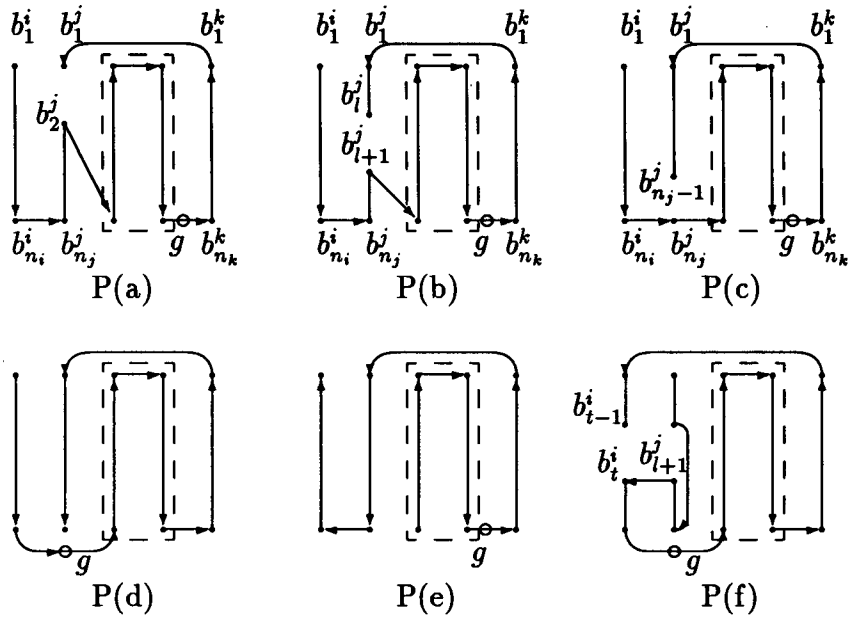


Figure 5.6: c-tight paths

Given in **Figure 5.6** are six c-tight paths on V (cf. **Figure 5.1**, disregarding h and inequality (5.12)), where the dashed box contains an even (possibly zero) number of spikes and a fixed Hamiltonian chain on these spikes. The node g in **Figure 5.6** or in the following given paths and chains may or may not be present.

(i) Comparing path $P = P(a)$ with path $P' = P(a) \cup (b_1^j, b_1^i) \setminus (b_1^k, b_1^j)$, i.e.,

$$0 = fx^P - fx^{P'} = f_{b_1^k b_1^j} - f_{b_1^j b_1^i},$$

yields $f_{b_1^j b_1^i} = \overleftarrow{\alpha}$. Replacing i, j, k in $P(a)$ by k, i, j, we obtain as above that $f_{b_1^i b_1^k} = \overleftarrow{\alpha}$. Similarly, we show by reversing these paths that

$$f_{b_1^i b_1^j} = f_{b_1^j b_1^k} = f_{b_1^k b_1^i} = \alpha. \tag{5.17}$$

(ii) Let $P(l)$ be obtained from $P(b)$ by replacing (b_1^k, b_1^j) and $(b_1^j ... b_l^j)$ with (b_1^j, b_1^i) and $(b_l^j ... b_1^j)$, respectively, for $l = 2, ..., n_j - 1$. Let $P(n_j)$ be obtained from $P(d)$ by replacing (b_1^k, b_1^j) and P_j with (b_1^j, b_1^i) and $\overleftarrow{P}_j$, respectively. By (i), and by comparing $P(n_j)$ with $P(d)$, $P(l)$ with $P(b)$ for $l = 2, ..., n_j - 1$, it follows that the following $n_j - 1$ linearly independent equations hold:

$$\sum_{t=1}^{l-1} f_{b_t^j b_{t+1}^j} = \sum_{t=1}^{l-1} f_{b_{t+1}^j b_t^j}, \quad l = 2, ..., n_j.$$

Therefore, edges (b_t^j, b_{t+1}^j) are f-symmetric for all $1 \leq t \leq n_j - 1$ and $1 \leq j \leq s$, and $f(P_l) = f(\overleftarrow{P}_l)$ for all $1 \leq l \leq s$.

(iii) Let $\gamma = f_{b_{n_j-1}^j n_j} = f_{b_{n_j}^j n_j - 1}$, due to (ii). From path $P(e)$, we construct:

$$P \doteq (...b_{n_k}^k ... b_1^k b_1^i ... b_{n_i}^i b_{n_j}^j ... b_1^j),$$

and from path $P(c)$:

$$P' \doteq (b_1^j ... b_{n_j}^j ... b_{n_k}^k ... b_1^k b_1^i ... b_{n_i}^i).$$

By (i) and (ii), and by Comparing P with $P(e)$ and P' with $P(c)$, respectively, it follows

$$\alpha + \beta = \overleftarrow{\alpha} + \overleftarrow{\beta}, \quad \alpha + \gamma = \overleftarrow{\alpha} + \beta.$$

Further, comparing $\overleftarrow{P'}$ with $\overleftarrow{P}(c)$ yields $\overleftarrow{\alpha} + \gamma = \alpha + \overleftarrow{\beta}$. From these three equations, we obtain $\alpha = \overleftarrow{\alpha}$ and $\beta = \overleftarrow{\beta}$. Moreover, if node g exists, i.e., for Path Inequalities, define

$$P \doteq P(d) \cup \overleftarrow{P}_i \cup (b_{n_j}^j, b_{n_i}^i) \setminus \{P_i \cup (b_{n_i}^i, g)\}.$$

By (i), (ii), (iii), and by comparing this P with $P(d)$ and $\overleftarrow{P}$ with $\overleftarrow{P}(d)$ respectively, it follows

$$f_{b_{n_i}^i g} = f_{g b_{n_i}^i} = \beta, \quad \forall 1 \leq i \leq s.$$

Since the above hold for any i, j, k, it follows that $A(\{b_1^1, ..., b_1^s\}) \cup A(\{g, b_{n_1}^1, ..., b_{n_s}^s\})$ contains only f-symmetric edges.

(iv) For any j such that $n_j \geq 3$, construct the following c-tight path:

$$P' \doteq (b_1^i ... b_{n_i}^i g ... b_{n_k}^k ... b_1^k b_1^j ... b_l^j b_{n_j}^j ... b_{l+1}^j).$$

Observe that by (ii) and (iii), all edges in P' are f-symmetric except edge $(b_l^j, b_{n_j}^j)$, therefore comparing P' with $\overleftarrow{P'}$ yields that $(b_l^j, b_{n_j}^j)$ is also f-symmetric for $1 \leq l \leq n_j - 2$. Further, if $n_j \geq 4$, from $P(d)$, construct the following c-tight path:

$$P \doteq (b_1^i ... b_{n_i}^i g ... b_{n_k}^k ... b_1^k b_1^j ... b_t^j b_{l+1}^j ... b_{n_j}^j b_l^j ... b_{t+1}^j).$$

Observe that all edges in P have been shown to be f-symmetric except edge (b_t^j, b_{l+1}^j). Therefore, comparing P with $\overleftarrow{P}$ implies that (b_t^j, b_{l+1}^j) is f-symmetric for all $1 \leq t < l < n_j$. It follows that all edges in $A(\{b_1^j, ..., b_{n_j}^j\})$ are f-symmetric for all $1 \leq j \leq s$.

(v) For any pair of nodes b_t^i and b_{l+1}^j such that $1 \leq t \leq n_i - 1$, $1 \leq l \leq n_j - 1$, $i \neq j$ and $M(b_t^i) \leq M(b_{l+1}^j)$ (cf. page 110 for the definition of $M(*)$), define c-tight paths: for $t \geq 2$, $P(t, l+1) \doteq P(f)$; otherwise $t = 1$, we have

$$P(t, l+1) \doteq (b_1^j ... b_l^j b_{n_j}^j ... b_{l+1}^j b_1^i ... b_{n_i}^i ... g ... b_{n_k}^k ... b_1^k).$$

Observe that $P(t, l+1)$ contains exactly one edge (b_t^i, b_{l+1}^j) not shown to be f-symmetric, therefore comparing $P(t, l+1)$ with $\overleftarrow{P}(t, l+1)$ yields (b_t^i, b_{l+1}^j) as f-symmetric.

For all b_t^i and b_{l+1}^j such that $M(b_t^i) > M(b_{l+1}^j)$, exchange i and j, and then repeat (v).

If node g exists, define c-tight paths

$$P_j(t) \doteq (b_1^i ... b_{n_i}^i ... b_{n_k}^k ... b_1^k b_1^j ... b_t^j g b_{n_j}^j ... b_{t+1}^j), \quad 1 \leq t \leq n_j - 1.$$

Observe also that for any $1 \leq t \leq n_j - 1$, the edges in $P_j(t)$ are shown to be f-symmetric except for the edge (b_t^j, g). By comparing $P_j(t)$ with $\overleftarrow{P}_j(t)$, it follows that edges (b_t^j, g) are f-symmetric for all t and j.

From (iii), (iv) and (v), it follows that $fx \leq f_0$ is a symmetric inequality. The proof is complete. □

Proof of Theorem 5.6: This proof is conducted by successive augmentations of a set E_f of f-symmetric edges until $E_f = A(V)$. The set E_f has the property that $e \in E_f$ implies $\overleftarrow{e} \in E_f$. For any subset $E' \subseteq A(V)$, define the following operation

$$E' \bigoplus E_f : \quad \text{replace } E_f \text{ with } \quad E_f \cup \{e \in A(V) : \text{ either } e \text{ or } \overleftarrow{e} \text{ is in } E'\}.$$

The proof consists of the following three phases.

(i) An initial set $E_f \subseteq A(V)$ of f-symmetric edges with the *same* f-length is constructed.

(ii) A sequence of pairwise c-tight path comparisons is conducted and E_f is augmented at each step. We denote this by "$P, P' \Longrightarrow$ *something*", meaning that comparing c-tight paths P and P' implies "*something*". Moreover, the implication may implicitly use the previous results and obvious symmetries. The f-length of the underlined part in P has been shown to be equal to that in P'.

(iii) Select any edge $e \in A(V) \setminus E_f$ and construct a c-tight path P on V such that $e \in P$ and $P \setminus e \subseteq E_f$. Thus, $P, \overleftarrow{P} \Longrightarrow \{e\} \bigoplus E_f$. Repeating (iii) until $E_f = A(V)$.

<u>**Envelope:**</u> Let $\alpha \doteq f_{12}$, $\beta \doteq f_{34}$, $\overleftarrow{\beta} \doteq f_{43}$, $\gamma \doteq f_{24}$, $\overleftarrow{\gamma} \doteq f_{42}$, $\xi \doteq f_{56}$ and $\overleftarrow{\xi} \doteq f_{65}$.

(i) Let $C^1 = \langle 124356 \rangle$ and $C^2 = \langle 214356 \rangle$.

$C^1 \setminus (1,2)$, $C^1 \setminus (3,5) \Longrightarrow f_{12} = f_{35}$; $C^2 \setminus (2,1)$, $C^2 \setminus (3,5) \Longrightarrow f_{21} = f_{35}$.

Hence, $f_{21} = f_{12} = \alpha$ and let $E_f \doteq \{(1,2),(2,1)\}$.

Observe that for any $e \in E_1 \doteq \{(3,5)\} \cup (\{1\},\{4,6\})$, there exists $e' \in E_f$ and a Hamiltonian cycle C on V such that $e, e' \in C$ and $cx^C = c_0 + 1$. (For $e = (3,5)$, let $e' = (1,2)$ and $C = C^1$ above; for $e = (1,4)$, let $e' = (2,1)$ and $C = C^2$ above; for $e = (1,6)$, let $e' = (2,1)$ and $\langle 216534 \rangle$.) Thus, comparing $C \setminus e$ and $C \setminus e'$, and then

$\overleftarrow{C} \setminus \overleftarrow{e}$ and $\overleftarrow{C} \setminus \overleftarrow{e'}$ implies that e is f-symmetric. Hence, set $E_f = \{e\} \bigoplus E_f$. It follows by repeating the above that $E_f = E_1 \bigoplus E_f$. Moreover, $f_e = \alpha$ for all $e \in E_f$.

(ii) $(\underline{4356}\ \underline{21})$, $(\underline{12}\ \underline{4356}) \Longrightarrow f_{62} = f_{24} = \gamma$.

$(2\underline{156}\ \underline{43})$, $(\underline{156}\underline{243}) \Longrightarrow f_{64} = 2\gamma - \alpha$, due to the above and (i).

Reversing the above two pairs of paths, we have

$f_{26} = f_{42} = \overleftarrow{\gamma}$; and $f_{46} = 2\overleftarrow{\gamma} - \alpha$, respectively.

Using the above results, we have the following implications:

$(\underline{12}6\underline{43}\underline{5})$, $(\underline{53}4\underline{62}\underline{1}) \Longrightarrow \overleftarrow{\gamma} + 2\gamma - \alpha + \overleftarrow{\beta} = \gamma + 2\overleftarrow{\gamma} - \alpha + \beta$; and

$(5\underline{61}243)$,$(34\underline{21}\underline{6}5) \Longrightarrow \xi + \gamma + \overleftarrow{\beta} = \overleftarrow{\xi} + \overleftarrow{\gamma} + \beta$.

It follows that $\xi = \overleftarrow{\xi}$. By symmetry, $\beta = \overleftarrow{\beta}$. Hence, $\gamma = \overleftarrow{\gamma}$ and

$$\{(3,4),(5,6),(4,6),(2,4),(2,6)\} \bigoplus E_f.$$

(iii) For edges $(1,3)$,$(1,5)$,$(2,3)$,$(2,5)$,$(3,6)$,$(4,5)$, construct, respectively, the corresponding desired c-tight paths

$$(213465),(215643),(123465),(125643),(124365),(345621).$$

□

Ladder: Let $\alpha = f_{47}$ and $E_f = \emptyset$.

(i) Let $C^1 = \langle 7216534 \rangle$ and $C^2 = \langle 7432156 \rangle$.

$C^1 \setminus (7,2)$, $C^1 \setminus (4,7) \Longrightarrow f_{72} = f_{47} = \alpha$; it follows $f_{72} = f_{67} = \alpha$ by exchanging nodes $3,4$ with $5,6$.

$C^2 \setminus (7,4)$, $C^2 \setminus (6,7) \Longrightarrow f_{74} = f_{67}$. Hence, $f_{74} = f_{47} = f_{67} = f_{76} = \alpha$ and $\{(4,7),(6,7)\} \bigoplus E_f$.

Observe that for any $e \in E_1 \doteq \{(7,2)\} \cup (\{1\},\{3,4,5,6\})$, there exists $e' \in E_f$ and a Hamiltonian cycle C on V such that $e, e' \in C$ and $cx^C = c_0 + 1$ (either C^1 or C^2 above).

Thus, comparing $C \setminus e$ and $C \setminus e'$, $\overleftarrow{C} \setminus \overleftarrow{e}$ and $\overleftarrow{C} \setminus \overleftarrow{e'}$ implies that e is f-symmetric. Hence, $\{e\} \oplus E_f$. Repeating the above yields $E_1 \oplus E_f$. Moreover, $f_e = \alpha$ for all $e \in E_f$.

(ii) $(\underline{7432156})$, $(\underline{7432165}) \Longrightarrow \{(5,6),(3,4)\} \oplus E_f$.

$(\underline{12743}\ \underline{56})$, $(\underline{12765}\ \underline{34}) \Longrightarrow \{(3,5)\} \oplus E_f$.

$(\underline{72134}\ \underline{65})$, $(\underline{72156}\ \underline{43}) \Longrightarrow \{(4,6)\} \oplus E_f$.

$(\underline{72}\ \underline{13465})$, $(\underline{56431}\ \underline{27}) \Longrightarrow \{(1,2)\} \oplus E_f$.

$(\underline{743}\ \underline{2156})$, $(\underline{6512}\ \underline{347}) \Longrightarrow \{(2,3),(2,5)\} \oplus E_f$.

(iii) For edges $(1,7)$,$(2,4)$,$(2,6)$,$(3,6)$,$(3,7)$,$(4,5)$,$(5,7)$, construct, respectively, the corresponding desired c-tight paths

$$(5643217),(1243567),(1265347),(7436521),(1256437),(1234567),(1234657).$$

This completes the proof for $|V'| = 8$. If node g exists, revise the above proof as follows:

- for every path or cycle used in (i), (ii) and (iii) above, replace subchains (12) and (21) by $(g12)$ and $(21g)$, respectively;
- in (i), replace E_1 by $\{(7,2),(1,g)\} \cup (\{g\},\{3,4,5,6\})$;
- in (iii), replace $(1,7)$ by $(g,7)$, and add:" for edges $(2,g),(1,3),(1,4),(1,5)$, $(1,6),(1,7)$, construct, respectively, the following desired c-tight paths $(12g43567)$,$(7213465g)$,$(7214356g)$, $(7215643g)$,$(7216534g)$,$(g5643217)$."

This shows that the result holds for the case $|V'| = 9$. Finally, apply Clique-Lifting on the isolated node g to complete the proof. □

Proof of Theorem 5.7: Let $cx \leq c_0$ be the restriction of inequality (5.13), and let $fx \leq f_0$ be a facet-defining inequality for $AHP(V)$ that dominates $cx \leq c_0$. Note

that by definition, for any $t, q \in \{1, ..., p\}$, there exists a "zig-zag" Hamiltonian path $\hat{P} = (r_q ... s_t)$ on $R \cup T$ such that $c(\hat{P}) = 2p - 1$ (cf. Fig. 5.7(1)). Let $(b^k_{n_k} ... b^k_1)$ and $(b^k_1 ... b^k_{n_k})$ denote two fixed subchains $(r_q ... s_t)$ and $(s_t ... r_q)$, respectively. Let $n_i = 2$ for all i. Thus, following (i), (ii), (iii) and (v) in the proof of **Theorem 5.5**, where $n_i = 2$ for all i, and the subchains $\overleftarrow{P}_k$ and P_k are replaced by $(r_q \cdots s_t)$ and $(s_t \cdots r_q)$, respectively, we show that all edges in $A \setminus A(R \cup T)$ are f-symmetric. Construct the four c-tight paths on V, see **Figure 5.7**.

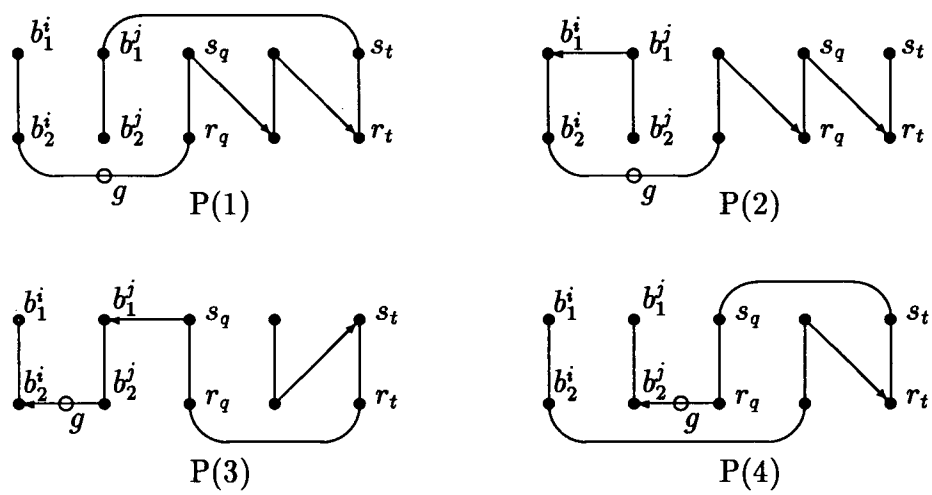


Figure 5.7: c-tight paths for the Chain Inequality

For any $1 \leq q \leq p$, we obtain from path $P(1)$ a c-tight path

$$P' \doteq P(1) \cup (b^j_2 r_q g b^i_2 b^i_1) \setminus (b^i_1 b^i_2 g r_q s_q).$$

Thus, comparing $P(1)$ with P', and $\overleftarrow{P}(1)$ with $\overleftarrow{P'}$, respectively, yields $f_{s_q r_q} = f_{r_q s_q}$.

For any $1 \leq q < t \leq p$, we obtain from path $P(2)$ a c-tight path

$$P'' \doteq P(2) \cup \{(r_t, b^j_2), (s_t, r_t)\} \setminus \{(s_q, r_t), (r_t, s_t)\}.$$

As above, comparing $P(2)$ with P'', and $\overleftarrow{P}(2)$ with $\overleftarrow{P''}$, respectively, yields $f_{s_q r_t} = f_{r_t s_q}$.

Finally, observe that all edges in paths P(3) and P(4) have been shown to be f-symmetric except edges (r_q, r_t) and (s_q, s_t). Thus, comparing $P(3)$ with $\overleftarrow{P}(3)$, and $P(4)$ with $\overleftarrow{P}(4)$, respectively, implies that edges (r_q, r_t) and (s_q, s_t) are f-symmetric. It follows that $fx \leq f_0$ is symmetric. □

5.7 Proofs of the Main Results in Section 5.5

Proof of Proposition 5.10: Let $d \in S$ and $u \doteq d' \in S'$ be the tips. For any $v \in S \setminus d$, we have $a_{uv} = 0$ by (S2'). By (S1'), construct an a-tight circuit C containing (u, v). Thus, C contains a subchain $(uv...qd)$. First, observe that $a_{qd} = \omega$; otherwise replacing the subchain in C with $(uq...vd)$ yields a circuit with a-length at least $a_0 + \omega$. So, C is in the form of either $\langle v...h...qdp...u\rangle$ or $\langle v...qdp...h...u\rangle$. If $a_{dp} = 0$, then construct the desired $\mathcal{S}$-circuit by moving h in C to the position between d and p; else, $a_{dp} = \omega$, and moving d in C to the position immediately before h yields the desired $\mathcal{S}$-circuit, since $a_{qd} + a_{dp} = 2\omega \leq a_{pq}$ (due to (S2')) and $ay \leq a_0$ is valid for $ATSP(V')$. This shows that (S3) holds.

To show that (S4) also holds, we first claim that there exists an a-tight circuit C on V' of the form $\langle u...pdq\rangle$ with either (i) $p, q \in Z \cup h$ or (ii) $p, q \in S$. To prove this claim, let $n \doteq |V' \setminus h|$ and let $cx \leq c_0$ be the restriction of $ay \leq a_0$ w.r.t. node h. By (S1') and polyhedral equivalence, there are $n(n-1) - 1$ affinely independent incidence vectors y of a-tight circuits on V' (resp., incidence vectors x of c-tight paths on $V' \setminus h$). If the claim is false, then these incidence vectors y satisfy equations $y(d : S \setminus d) + y(S \setminus d : d) = 1$, $ay = a_0$ and $y(V' \setminus h) = n - 1$. Observe that the above three equations are in "h-canonical form", and their restrictions $x(d : S \setminus d) + x(S \setminus d : d) = 1$, $cx = c_0$ and $x(V' \setminus h) = n - 1$ are three linearly independent equations on $AHP(V' \setminus h)$. (Note that $c_e \geq 2\omega$ for all $e \in A(S \setminus d)$,

due to (S2').) Further, these equations are satisfied by $\dim(AHP(V' \setminus h)) = n(n-1)-1$ affinely independent incidence vectors x above, a contradiction. Hence, the claim holds.

Construct $C_d \doteq \langle u...pq \rangle$. Note that $a_{pd} = a_{dq} = 0$ for case (i); $a_{pq} \geq 2\omega = a_{pd} + a_{dq}$ for case (ii). Therefore, C_d is the desired $\mathcal{U}$-circuit. □

To prove the **Tree Composition Theorem**, we need the following four auxiliary results.

Let $ay \leq a_0$ be a composable ATSP Tree Inequality with respect to a bud S for $ATSP(V')$ and let $\{h, S, Z\}$ be the corresponding partition. Define $V \doteq V' \setminus h$. Then, the restriction $cx \leq c_0$ of $ay \leq a_0$ w.r.t. h is called a *composable Tree Component* for $AHP(V)$. Define the following special paths w.r.t. $cx \leq c_0$ and tip d:

$\mathcal{Z}$-path: A c-tight path $(t...vd...s)$ w.r.t. $v \in Z$ such that

$$c_e = 0, \forall e \in (\{t, v\} : \{d, s\}).$$

$\mathcal{S}$-path: A c-tight path $(z...uv...qd)$ w.r.t. $v \in S \setminus d$ such that

$$c_{zv} = c_{uv} = 0 \text{ and } c_e \geq \omega \text{ for all } e \text{ used in subchain } (v...qd).$$

$\mathcal{U}$-path: A Hamiltonian path P_d on $V \setminus d$ such that

$$c(P_d) = c_0 \text{ and both endnodes in } Z.$$

Lemma 5.12 *Let $cx \leq c_0$ be a composable Tree Component for $AHP(V)$. Then,*

1. *for every $v \in Z$, there exists a $\mathcal{Z}$-path $(t...vd...s)$;*
2. *for every $v \in S \setminus d$, there exists an $\mathcal{S}$-path $(z...uv...qd)$; and*
3. *there exists $\mathcal{U}$-path P_d on $V \setminus d$.*

Proof. Note first that by polyhedral equivalence and (S1), $cx \leq c_0$ defines a non-trivial facet of $AHP(V)$.

(1) For every $v \in Z$, there exists a c-tight path containing (v, d) of the form $(t...vd...s)$. If there is an edge $e \in (\{t, v\}, \{d, s\})$ with $c_e > 0$, say $e = (v, s)$, w.l.o.g., then a Hamiltonian path $P \doteq (t...vs...d)$ yields $cx^P > c_0$, a contradiction.

(2) For every $v \in S \setminus d$, by (S3), there exists an a-tight $\mathcal{S}$-circuit $\langle z...uv...qdh \rangle$ with $a_{uv} = 0$. Remove h from this circuit to obtain an $\mathcal{S}$-path $(z...uv...qd)$. (Note that $c_{uv} = 0$ holds by definition and $c_{zv} = 0$ follows since $(u...zv...qd)$ is also a c-tight path. The last condition follows by contradiction: if for some e in subchain $(v...qd)$, $c_e < \omega$, replacing e with (d, v) and then connecting an endnode of the resulting subchain with node u of $(u...z)$ yields a Hamiltonian path P with $c(P) > c_0$.)

(3) By (S4), there exists a $\mathcal{U}$-circuit C_d. Then, $\mathcal{U}$-path P_d is obtained by removing h from C_d. If one of the endnodes in P_d is in S, then connecting this endnode with d yields a Hamiltonian path P on V with $c(P) = c_0 + \omega$, a contradiction. □

Lemma 5.13 (3-Path Lemma) *Let $cx \leq c_0$ and $fx \leq f_0$ be two valid inequalities for $AHP(V)$. Assume the former inequality is symmetric and dominated by the latter. Let U_1, U_2, U_3 be a partition of V and let $P_i \doteq (u_i...v_i)$ be a Hamiltonian path on U_i, $i = 1, 2, 3$. Define*

$$E_p \doteq \{P_i, \overleftarrow{P}_i : i = 1, 2, 3\}, \quad E_0 \doteq \bigcup_{i \neq j} (\{u_i, v_i\} : \{u_j, v_j\}).$$

If $c_e = 0$ for all $e \in E_0$ and $\sum_{i=1}^{3} c(P_i) = c_0$, then

(i) $f(P_i) = f(\overleftarrow{P}_i)$ for $i = 1, 2, 3$; and

(ii) f_e is a constant for all $e \in E_0$.

Proof. (i) From P_1, P_2, P_3, we obtain two Hamiltonian circuits on V:

$$C \doteq \langle u_1...v_1u_2...v_2u_3...v_3 \rangle, \quad C' \doteq \langle u_1...v_1v_2...u_2u_3...v_3 \rangle, \tag{5.18}$$

such that $c(C) = c(C') = c_0$. Comparing c-tight paths $C \setminus (v_3, u_1)$ and $C \setminus (v_1, u_2)$ yields $f_{v_3 u_1} = f_{v_1 u_2}$; comparing $C' \setminus (v_3, u_1)$ with $C' \setminus (v_1, v_2)$ yields

$$f_{v_1 v_2} = f_{v_3 u_1} = f_{v_1 u_2}. \tag{5.19}$$

Further, comparing

$$P \doteq (u_3 ... v_3 u_1 ... v_1 u_2 ... v_2) \quad \text{and} \quad P' \doteq (u_3 ... v_3 u_1 ... v_1 v_2 ... u_2)$$

yields

$$0 = f(P) - f(P') = f_{v_1 u_2} + f(P_2) - f_{v_1 v_2} - f(\overleftarrow{P}_2) = f(P_2) - f(\overleftarrow{P}_2).$$

Similarly, we show that P_1 and P_3 are f-symmetric. Hence, (i) holds.

(ii) Observe that for any $e_0 \in E_0$, there exists a Hamiltonian circuit $C \subseteq E_p \cup E_0$ on V such that $e_0 \in C$ and $c(C) = c_0$. W.l.o.g., assume that C has form (5.18) and $e_0 = (v_1, u_2)$. Comparing the c-tight paths $C \setminus (v_1, u_2)$, $C \setminus (v_2, u_3)$ and $C \setminus (v_3, u_1)$ yields

$$f_{v_1 u_2} = f_{v_2 u_3} = f_{v_3 u_1} = f(C) - f_0.$$

Similarly, using $\overleftarrow{C}$, we obtain

$$f_{u_2 v_1} = f_{u_3 v_2} = f_{u_1 v_3} = f(\overleftarrow{C}) - f_0.$$

Further, by (i) and $f(C \setminus (v_1, u_2)) = f(\overleftarrow{C} \setminus (u_2, v_1))$, we have

$$0 = \sum_{i=1}^{3} f(P_i) + 2f_{v_1 u_2} - \sum_{i=1}^{3} f(\overleftarrow{P}_i) - 2f_{u_2 v_1} = 2f_{v_1 u_2} - 2f_{u_2 v_1}.$$

Hence, all edges $e_0 \in E_0$ are f-symmetric.

Since all elements in $E_0 \cup E_p$ are shown to be f-symmetric, repeating (5.19) for all permutations of indices of u_1, u_2, u_3 and exchanging the roles of the u and v nodes imply that (ii) holds. The proof is complete. □

Lemma 5.14 *The composed inequality $ay \leq a_0$ for $ATSP(V')$ defined in (5.14) is valid for $ATSP(V')$.*

Proof. It suffices to show that its undirected version $a'y \leq a_0$ is valid for $STSP(V')$.

Let $S_1' \doteq S_1 \setminus d_1$ and $S_2' \doteq S_2 \setminus d_2$. An edge e is called an ω-edge if $e \in E(d : S_1' \cup S_2') \cup E(S_1' : S_2')$. Let C^* be any solution to

$$\max\{a'y^C : C \text{ is a Hamiltonian cycle}\} = a_0^*.$$

Let $C = C^*$.

(i) [ω-Edge Reduction] Let $E(V_1' : V_2 \setminus d_2)$ be the set of all *crossing edges*, and let $E(S_1 : S_2')$ be the set of all *ω-crossing edges.*

For any Hamiltonian cycle C_0 on V', let $N_\omega(C_0) \doteq |C_0 \cap E(S_1 : S_2')|$ denote the number of ω-crossing edges used in C_0. If there exists a pair (u, v) and (s, t) of ω-crossing edges in C such that some subchain connecting (u, v) and (s, t) in C contains an odd number of crossing edges, then transform C to Hamiltonian cycle C' as follows:

$$C \doteq \langle d...uv...st...r\rangle \longrightarrow C' \doteq \langle d...us...vt...r\rangle. \tag{5.20}$$

Since $a'_{us} + a'_{vt} \geq 2\omega$, we have $a_0^* = a'(C) \leq a'(C') = a_0^*$, and $N_\omega(C') = N_\omega(C) - 2$. Let $C = C'$. Observe that such a reduction is always possible whenever $|N_\omega(C)| > 2$. Hence, we repeat (i) until $N_\omega(C) \leq 2$.

(ii) Exchange sets S_1' and S_2' in (i) and then apply (i).

(iii) First, observe that C contains at most one edge in $E_\omega \doteq E(S_1' : S_2')$. Otherwise, C contains exactly two edges $(u, v), (t, s) \in E_\omega$. Further, by (i) and (ii), C contains the subchain (pdq) with $a'_{pd} = a'_{dq} = 0$, where d is the common tip. Replacing subchains (uv) and (pdq) in C with (udv) and (pq), respectively, we obtain a Hamiltonian cycle C' on V' such that $a'(C') \geq a'(C) + \omega > a_0^*$, a contradiction.

Now, let $P_1 = C \cap E(V_1')$ and $P_2 = C \cap E(V_2')$. Note that P_1 and P_2 are subsets of some Hamiltonian cycles C_1 on V_1' and C_2 on V_2', respectively. If C contains no edges in E_ω, then

$$a'(C^*) = a'(C) = a'(P_1) + a'(P_2) \leq a'(C_1) + a'(C_2) \leq a_0^1 + a_0^2 = a_0.$$

Otherwise, C contains exactly one edge $(u, v) \in E_\omega$ with $u \in S_1'$ and $v \in S_2'$, and a subchain (pdq). Let $P_1' \doteq P_1 \cup (u, d)$ and $P_2' \doteq P_2 \cup (d, v)$. If $p, q \in V_1'$ (resp., $p, q \in V_2'$), then, P_2' (resp., P_1') is a subset of some Hamiltonian cycle on V_2' (resp., V_1'). Else, w.l.o.g. assume $q \in V_1'$ and $p \in V_2'$, and observe also that either P_1' is a subset of some Hamiltonian cycle on V_1' or P_2' is on V_2', since otherwise both P_1' and P_2' contain some *sub-cycles* covering d, and so does $P_1 \cup P_2 \cup (u, v) \subset C$, which contradicts that C is a Hamiltonian cycle on V'. If P_1' is contained in some Hamiltonian cycle C_1' on V_1', then

$$a'(C^*) = a'(C) = a'(P_1') + a'(P_2) \leq a'(C_1') + a'(C_2) \leq a_0^1 + a_0^2 = a_0;$$

else the above holds by exchanging indices 1 and 2. □

Let inequality $ay \leq a_0$ for $ATSP(V')$ be obtained by the Tree Composition (5.14) of two ATSP Tree Inequalities $a^1y^1 \leq a_0^1$ and $a^2y^2 \leq a_0^2$, relative to buds S_1 and S_2, respectively, and let $\{h_1, S_1, Z_1\}$ and $\{h_2, S_2, Z_2\}$ be the corresponding partitions. Define $V_i \doteq V_i' \setminus h_i$, $i = 1, 2$, and $V \doteq V' \setminus h$. Recall that $V_1 \cap V_2 = \{d\} = \{d_1\} = \{d_2\}$, where d is the common tip. The restrictions $cx \leq c_0$, $c^1x^1 \leq c_0^1$ and $c^2x^2 \leq c_0^2$ of $ay \leq a_0$, $a^1y^1 \leq a_0^1$ and $a^2y^2 \leq a_0^2$, w.r.t. h, h_1 and h_2, are called *Tree Components*, for $AHP(V)$, $AHP(V_1)$ and $AHP(V_2)$, respectively. Then, we say that $cx \leq c_0$ is obtained by *AHP Tree Composition* of two Tree Components $c^1x^1 \leq c_0^1$ and $c^2x^2 \leq c_0^2$. Recall that the restriction of a composable ATSP Tree Inequality is a composable Tree Component. See **Figure 5.4** for an example of the AHP Tree Composition, disregarding nodes h_1, h_2, h.

Theorem 5.15 *Inequality $cx \leq c_0$ obtained by AHP Tree Composition of two composable Tree Components $c^1x^1 \leq c_0^1$ and $c^2x^2 \leq c_0^2$ defines a facet of $AHP(V)$.*

Proof: The validity of $cx \leq c_0$ follows by **Lemma 5.14** and the polyhedral equivalence. Let $fx \leq f_0$ be a facet-defining inequality that dominates $cx \leq c_0$.

Let $\{V_1, V_2 \setminus d_2\}$ be the partition of V, and by **Lemma 5.12**(3), let $P_2 = (u_2...v_2)$ be a $\mathcal{U}$-path on $V_2 \setminus d_2$ with $c(P_2) = c_0^2$ and $u_2, v_2 \in Z_2$.

(i) For every $v_1 \in Z_1$, by **Lemma 5.12**(1), we construct a $\mathcal{Z}$-path $(t_1...v_1d...s_1)$ on V_1, and let $P_1 = (t_1...v_1)$ and $P_3 = (d...s_1)$; for every $v_1 \in S_1 \setminus d_1$, by **Lemma 5.12**(2), we construct an $\mathcal{S}$-path $(z_1...u_1v_1...q_1d)$ on V_1, and let $P_1 = (z_1...u_1)$ and $P_3 = (v_1...q_1d)$. Note that for all $v \in V_1 \setminus d_1$, we obtain P_1, P_2 and P_3 satisfying the conditions in **Lemma 5.13**, and therefore we have $f_e = f_{u_2d}$ for all $e \in (\{u_2, v_2\} : V_1) \cup (V_1 : \{u_2, v_2\})$ and $f(P_2) = f(\overleftarrow{P}_2)$. This further implies that for any c^1-tight path P' on V_1, there exists a c-tight path P on V containing P' and P_2 such that

$$f_0 = fx^P = f(P') + f(P_2) + f_{u_2d}.$$

Since $c^1x^1 \leq c_0^1$ is facet-defining for $AHP(V_1)$ and the above equality holds for all c^1-tight paths P', there exist real numbers $\alpha_1 > 0$ and β_1 such that $f_e = \alpha_1 c_e + \beta_1$ for all $e \in A(V_1)$.

By partitioning V into $\{V_2, V_1 \setminus d_1\}$, we show by symmetry that for some real numbers $\alpha_2 > 0$ and β_2, $f_e = \alpha_2 c_e + \beta_2$ holds for all $e \in A(V_2)$.

(ii) For any $v_2 \in Z_2$, construct a $\mathcal{Z}$ path $(t_2...v_2d...s_2)$ on V_2. As in (i), for every $v_1 \in Z_1$, construct a $\mathcal{Z}$-path $(t_1...v_1d...s_1)$ on V_1; and for every $v_1 \in S_1 \setminus d$, we construct an $\mathcal{S}$-path $(z_1...u_1v_1...q_1d)$ on V_1. As required by the conditions for **Lemma 5.13**, we obtain for the former case

$$P_1 \doteq (t_1...v_1), \quad P_2 \doteq (t_2...v_2), \quad P_3 \doteq (q_1...d...s_2),$$

and for the latter case,

$$P_1 \doteq (z_1 ... u_1), \quad P_2 \doteq (t_2 ... v_2), \quad P_3 \doteq (v_1 ... q_1 d ... s_2).$$

Further, since $c_{z_1 v_1} = c_{v_2 s_2} = 0$, by (i) and **Lemma 5.13**, we have

$$f_{v_2 v_1} = f_{v_1 v_2} = f_{v_2 s_2} = f_{z_1 v_1} = \beta_1 = \beta_2 = \beta.$$

By symmetry, it follows $f_e = \beta$ for all $e \in A(V) \setminus \{A(V_1) \cup A(V_2) \cup A(S)\}$.

(iii) For any $v_i \in S_i \setminus d$, construct an $\mathcal{S}$-path $(z_i ... u_i v_i ... q_i d)$, $i = 1, 2$. Recall that $(v_i ... q_i d)$ uses only edges e with $c_e \geq \omega$ and $c_{q_i d} = \omega$ by definition. Possibly $q_i = v_i$.) Comparing the following two c-tight paths:

$$P' \doteq (z_1 ... u_1 z_2 ... u_2 v_2 ... q_2 d q_1 ... v_1), \quad P'' \doteq (z_1 ... u_1 z_2 ... u_2 d q_2 ... v_2 v_1 ... q_1)$$

yields $0 = f(P') - f(P'') = \beta + f_{dq_1} - \beta - f_{v_2 v_1} = \alpha_1 \omega + \beta - f_{v_2 v_1}$. Further, comparing $\overleftarrow{P'}$ and $\overleftarrow{P''}$ yields $f_{v_1 v_2} = \alpha_1 \omega + \beta$, due to (i) and (ii).

By exchanging V_1 and V_2, we show also $f_{v_1 v_2} = f_{v_2 v_1} = \alpha_2 \omega + \beta$. It follows that $\alpha \doteq \alpha_1 = \alpha_2$,

$$f_e = \alpha\omega + \beta, \quad e \in (S_1, S_2) \cup (S_2, S_1).$$

(i), (ii) and (iii) imply that $f_e = \alpha c_e + \beta$ for all $e \in A(V)$. □

Now, we are in a position to prove the Tree Composition Theorem.

Proof of the Tree Composition Theorem: Part (i) of the theorem is proved in **Lemma 5.14**. For part (ii), Statement (A) follows from **Theorem 5.15**, by polyhedral equivalence.

For statements (B) and (C), (S2) holds by inspection. We need to examine conditions (S3) and (S4) for each case.

For (B), Condition (S3) follows by constructing, for every $v_1 \in S_1 \setminus d$, $\mathcal{S}$-circuit $\langle u_2...v_2z_1...u_1v_1...q_1dh\rangle$ on V', where $(u_2...v_2)$ is a $\mathcal{U}$-path on $V_2 \setminus d$ and $(z_1...u_1v_1...q_1d)$ is an $\mathcal{S}$-path on V_1. Similarly, we obtain a requisite $\mathcal{S}$-circuit for every $v_2 \in S_2 \setminus d$.

(S4) holds by exhibiting $\mathcal{U}$-circuit $\langle u_1...v_1u_2...v_2h\rangle$ on $V' \setminus d$, where $(u_i...v_i)$ is a $\mathcal{U}$-path on $V_i \setminus d_i$, $i = 1, 2$.

For (C), Condition (S3) follows by constructing, for every $v_1 \in S' \setminus d'$, $\mathcal{S}$-circuit $\langle u_2...v_2z_1...u_1v_1...q_1d'h\rangle$ on V', where $(u_2...v_2)$ is a $\mathcal{U}$-path on $V_2 \setminus d$ and $(z_1...u_1v_1...q_1d')$ is an $\mathcal{S}$-path on V_1.

(S4) holds by exhibiting $\mathcal{U}$-circuit $\langle u_1...v_1u_2...v_2h\rangle$ on $V' \setminus d'$, where $(u_1...v_1)$ (resp., $(u_2...v_2)$) is a $\mathcal{U}$-path on $V_1 \setminus d'$ (resp., $V_2 \setminus d$).

This completes the proof of the theorem. □

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Appendix A

Generating the Sample of 280 scheduling problems

This appendix contains all relevant information to generate the 280 scheduling problems used in the computational experiments in **Chapter 3**.

A problem generator, coded in FORTRAN, is presented. The problem generator uses the portable random number generators in [9]. To randomly generate an instance of the scheduling problem, one needs to specify five input parameters: the number n of jobs, the maximum processing time ($IPMAX = 100$ in this study), the maximum job weight ($IWMAX = 10$ in this study), the P-value, i.e., the probability of including each arc (i, j) with $i < j$ (before the transitive closure is taken), and the seed number for the random generator. The output consists of three parts: (1) the number of jobs, and the number of arcs in the transitive reduction of the precedence graph; (2) the integer job processing times and weights; and (3) the arcs e, represented by their head nodes ($TO(e)$) and tail nodes ($FROM(e)$), in the transitive reduction.

Tables A.1 and A.2 provide the seed numbers used for generating the 280 problems. These numbers are produced as follows. For each n, one arbitrarily chooses a seed number for the first problem in the group of 20 problems. Then, the random number produced immediately after the completion of the previous instance is used as the seed number for the next problem. This process continues till all instances in the group are produced. Thus, the 280 instances can be easily generated in 14 runs of the problem generator, using a DO loop with relevant parameters in it (note that the P values vary within each group).

```
C    PROGRAM FOR GENERATING THE SINGLE-MACHINE PRECEDENCE CONSTRAINED
C    JOB SEQUENCING PROBLEMS. (USING MICROSOFT FORTRAN 5.0)
C     PRECEDENCE MATRIX, PROCESSING TIMES AND JOB WEIGHTS.
      COMMON NP(200,200),P(200),W(200)
      INTEGER FROM(3000),TO(3000)
C     INPUT
C     VARIABLE  FORMAT  DEFINITION
C     ------------------------------------------------------
C     N         I5      NO. OF JOBS
C     IPMAX     I5      MAXIMUM PROCESSING TIME
C     IWMAX     I5      MAXIMUM JOB WEIGHT
C     PROB      F5.3    P VALUE FOR THE PRECEDENCE GRAPH
C     NSEED     I10     SEED FOR THE RANDOM NUMBER GENERATOR
C
C     THE FOLLOWING 5 STATEMENTS MAY BE CHANGED IF
C     A DIFFERENT FORTRAN COMPILER IS USED.
      WRITE(*,*) 'Input data file name?'
      OPEN(UNIT=23,FILE=' ')
      OPEN(UNIT=24,FILE='dg.log')
      WRITE(*,*) 'Output data file name?'
      OPEN(UNIT=25,FILE=' ')
C
      READ(23,100) N,IPMAX,IWMAX,PROB,NSEED
  100 FORMAT(3I5,F5.3,I10)
      NPOS=0
      NAME=NAME+1
      DO 33 I=1,N
      DO 33 J=1,N
   33 NP(I,J)=0
      DO 10 I=1,N
        P(I) = IVUNIF(NSEED,IPMAX)*1.0
        W(I) = IVUNIF(NSEED,IWMAX)*1.0
        IA=I+1
        DO 20 J=IA,N
        PP=UNIF(NSEED)
        IF(PP.GE.PROB) GOTO 20
        NP(I,J)=1
        NP(J,I)=1
        NPOS=NPOS+1
   20   CONTINUE
   10 CONTINUE
```

```
C     FIND THE TRANSITIVE CLOSURE AND REDUCTION OF THE PRECEDENCE DIGRAPH.
      CALL TRANSCR(N,NPOSA)
C
C     COMPUTE THE ORDER STRENGTH R
      NPOSTO=NPOS+NPOSA
      R=(NPOSTO*1.0)/(N*(N-1)*1.0/2.0)
      WRITE(24,101) N,NC,IPMAX,IWMAX,PROB,R,NSEED
  101 FORMAT(4I5,2X,F5.3,2X,F7.6,2X,I10)
      N1=N-1
      NC=0
      DO 61 J=1,N1
        JA=J+1
        DO 62 I=JA,N
          IF(NP(I,J).LT.1) GOTO 62
          NC=NC+1
          FROM(NC)=J
          TO(NC)=I
   62   CONTINUE
   61 CONTINUE
C    NC IS THE NUMBER OF ARCS IN THE TRANSITIVE CLOSURE
      WRITE(25,63) N,NC
   63 FORMAT(3I5)
      DO 140 I=1,N
       WRITE(25,150) P(I),W(I)
  150  FORMAT(2F10.2)
  140 CONTINUE
      DO 143 I=1,NC
       WRITE(25,153) FROM(I),TO(I)
  153 FORMAT(2I5)
  143 CONTINUE
      STOP
      END
```

```
      SUBROUTINE TRANSCR(NOP,NPOSA)
      COMMON NP(200,200),P(200),W(200)
C
C     FIND THE TRANSITIVE CLOSURE AND REDUCTION
C     OF THE GIVEN DIGRAPH REPRESENTED BY NP(I,J).
C
C     INPUT
C     -----
C     THE PRECEDENCE GRAPH NP
C
C     OUTPUT
C     ------
C     THE UPPER TRIANGULAR OF NP: THE TRANSITIVE CLOSURE
C     THE LOWER TRIAGULAR OF NP: THE TRANSITIVE REDUCTION
C     NPOSA: NO. OF ADDITIONAL ARCS USED IN CONSTRUCTING THE
C            TRANSITIVE CLOSURE
C
      NPOSA=0
C
      NT=NOP-2
      DO 10 III=1,NT
       I = NT-III+1
       I1=I+1
       I2=I+2
       DO 20 J=I2,NOP
        J0=J-1
        DO 30 K=I1,J0
         IF (NP(I,K).EQ.0 .OR. NP(K,J).EQ. 0) GOTO 30
          IF (NP(I,J).NE.1) NPOSA=NPOSA+1
          NP(I,J)=1
          NP(J,I)=0
   30   CONTINUE
   20  CONTINUE
   10 CONTINUE
      RETURN
      END
```

```
      FUNCTION UNIF(IX)
C
C   PORTABLE RANDOM NUMBER GENERATOR IMPLEMENTING THE RECURSION:
C      IX = 16807 * IX MOD (2**(31) - 1)
C   USING ONLY 32 BITS, INCLUDING SIGN.
C
CC  REFERENCE: "A GUIDE TO SIMULATION", P. BRATLEY,
C              B.L. FOX AND L.E. SCHRAGE, 1983,
C              Springer-Verlag New York Inc.
C
C   SOME COMPILERS REQUIRE THE DECLARATION:
C      INTEGER*4 IX, K1
C
C   INPUT:
C     IX = INTEGER GREATER THAN 0 AND LESS THAN 2147483647
C
C   OUTPUTS:
C     IX = NEW PSEUDORANDOM VALUE,
C     UNIF = A UNIFORM FRACTION BETWEEN 0 AND 1
C
      K1 = IX/127773
      IX = 16807*(IX - K1*127773) - K1 * 2836
      IF ( IX .LT. 0 ) IX = IX + 2147483647
      UNIF = IX*4.656612875E-10
      RETURN
      END
```

```
      FUNCTION IVUNIF(IX,N)
C
C   PORTABLE CODE TO GENERATE AN INTEGER UNIFORM OVER
C   THE INTERVAL 1,N USING INVERSION
C
CC  REFERENCE: "A GUIDE TO SIMULATION", P. BRATLEY,
C              B.L. FOX AND L.E. SCHRAGE, 1983,
C              Springer-Verlag New York Inc.
C
C   INPUTS:
C     IX = RANDOM NUMBER SEED
C     N = THE UPPER BOUND OF THE UNIFORM INTEGER
C     M = LARGEST VALUE FOR IX + 1
C
C   AUXILIARY ROUTINE:
C     UNIF
C
C   OUTPUTS:
C     IX = NEW PSEUDORANDOM VALUE,
C     IVUNIF = A UNIFORM INTEGER BETWEEN 1 AND N
C
C   THE RANDOM NUMBER GENERATOR IS ASSUMED TO USE THIS MODULUS
      DATA M/2147483647/
C
      JUNK = UNIF(IX)
C
C   WE WANT INTEGER PART OF IX/(M/N) USING INFINITE PRECISION.
C   FORTRAN TRUNCATION UNDERSTATES M/N, THUS, FORTRAN MAY
C   OVERSTATES WHEN IT COMPUTES IX/(M/N). SO DECREASE JTRY
C   UNTIL JTRY <= IX/(M/N), I.E. M/IX <= N/JTRY.
C
      JTRY = IX/(M/N)
      IF(JTRY .LE. 0) GOTO 500
      GOTO 200
  100 JTRY=JTRY-1
```

```
C
C  NOW CHECK IF M/IX <= N/JTRY
C
  200 I = M
      J = IX
      K = N
      L = JTRY
C
C  MORE GENERALLY, IS I/J <= K/L, WHERE I>=J, K>=L
C  IF THE INTEGER PARTS DIFFER THE QUESTION IS ANSWERED.
C
  300 IJI = I/J
      KLI = K/L
      IF( IJI .LT. KLI ) GOTO 500
      IF( IJI .GT. KLI ) GOTO 100
C
C  MUST LOOK AT THE REMAINDERS; IS IJR/J <= KLR/L
C
      IJR = I - IJI * J
      IF ( IJR .LE. 0 ) GOTO 500
      KLR = K - KLI * L
      IF ( KLR .LE. 0 ) GOTO 100
C
C  STILL NOT RESOLVED; REPHRASE IN STANDARD FORM,
C  I.E. IS L/KLR <= J/IJR?
C
      I = L
      K = J
      J = KLR
      L = IJR
      GOTO 300
C
C   NOW ADD 1 TO PUT IN THE RANGE 1,N
C
  500 IVUNIF = JTRY + 1
      RETURN
      END
```

		n=30	n=40	n=50	n=60
1	.001	8913445	47920113	19026	85617
2	.001	755493537	1617675166	1715650722	1664517790
3	.020	1809917341	1175781217	106619163	299964548
4	.020	1725092278	1058321011	946083291	1527627754
5	.040	1822638204	1046809820	1851494651	589620248
6	.040	1180075628	1112508140	35270033	305479182
7	.060	150617066	1562075571	890663124	13878902
8	.060	1548235839	678966767	2111398339	359148491
9	.080	373360250	1078420215	1642491473	554424631
10	.080	1157536406	234794403	1253273425	73982478
11	.100	94545423	1482029273	114463185	313191047
12	.100	1103566408	1539623401	1192173437	829336955
13	.150	206050178	2111547401	1080955147	1401168127
14	.150	1524763277	525300461	411821081	1088629889
15	.200	581095035	455447345	289919440	685524764
16	.200	1260330676	1553388927	1817182748	1898463718
17	.300	2069131884	2064816761	1622539993	2139336862
18	.300	930436326	849467010	228036027	49606783
19	.500	575844519	1762170024	765120123	1988304799
20	.500	1424549194	1310705738	1588399166	646765699

		n=70	n=80	n=90	n=100
1	.001	76815623	4375178	350983217	2593
2	.001	1201897528	1611982795	1728743609	1468695453
3	.020	1630262163	697923001	236415471	1096690315
4	.020	1221144650	247554449	2056300857	1630069622
5	.040	18421799	439147435	668603022	1583467229
6	.040	417497068	234303362	873157283	782036086
7	.060	758772556	1679599250	1467748577	1798123704
8	.060	107891396	1669397446	1764811933	2043023199
9	.080	829805974	1558045213	387567746	486855177
10	.080	1807075147	1150686510	95713841	1991158124
11	.100	345975380	985796663	428865251	1426983424
12	.100	999548725	1837490202	427903940	71993197
13	.150	698081069	1344833772	349430181	133879335
14	.150	599774400	2128144389	1808998354	1365624620
15	.200	776266564	967140344	964298149	217047395
16	.200	2084721901	514443436	2089434947	1362135656
17	.300	984818336	2013815749	383679383	544616922
18	.300	1482079848	1460738242	757904826	2019009597
19	.500	1692157679	1748668138	728441454	513041050
20	.500	338192698	1739693055	1709478887	387185455

Table A.1: Seeds for the instances with $n = 30, 40, 50, 60, 70, 80, 90, 100$

		n=110	n=120	n=130
1	.001	578	327819	7456673
2	.001	872858166	1896871566	2090383671
3	.020	1966911261	1879863869	613433542
4	.020	2035263084	1751592867	2002449486
5	.040	1077304999	927231622	1046463762
6	.040	1367175068	1907798662	818258585
7	.060	1810714895	1892363717	1970034241
8	.060	1350354151	1351713937	1274437629
9	.080	1145775358	1788610971	1473953588
10	.080	415529128	795452143	62622184
11	.100	1992218419	1633349205	2013606980
12	.100	120830678	1207063493	731735072
13	.150	115769795	302181295	1371935740
14	.150	392329467	278166407	830555894
15	.200	278009491	1189966678	646486032
16	.200	517310160	350462138	814653809
17	.300	1625666109	66273535	1787943512
18	.300	1168319371	710696277	162859647
19	.500	1497262303	1172431752	2131368145
20	.500	81858226	134821562	1849154095

		n=140	n=150	n=160
1	.001	87485	1782437	7142
2	.001	347175951	153399920	661277793
3	.020	2040294373	1372129829	2105052591
4	.020	1031163234	451762956	1604032548
5	.040	1248974721	1730677699	1335044189
6	.040	1161785585	922885587	259379126
7	.060	147857977	1942371325	1562102629
8	.060	559322792	615429360	1301809084
9	.080	1142323614	2101567237	1795967001
10	.080	1994794676	1030042377	1083362231
11	.100	236427843	397005716	500851712
12	.100	309642129	2127352365	809794613
13	.150	628556917	185713150	1888704444
14	.150	67336514	499856450	623919773
15	.200	1546694624	747760362	266082890
16	.200	1908158645	1210383192	2086462989
17	.300	34474010	96454607	284799641
18	.300	601131006	799671032	1756198218
19	.500	405094013	2130125375	1023000068
20	.500	1210698212	1194121298	597523231

Table A.2: Seeds for the instances with $n = 110, 120, 130, 140, 150, 160$