THE EFFECT OF INSTRUCTION IN MODULAR ARITHMETIC ON
THE ABILITY OF GRADE 6 STUDENTS TO DIVIDE
FRACTIONS AND GIVE A RATIONAL EXPLANATION OF THE PROCESS

by

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B.A., University of British Columbia, 1956

A THESIS SUBMITTED IN PARTIAL FULFILMENT OF
THE REQUIREMENTS FOR THE DEGREE OF
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of
Elementary Education

We accept this thesis as conforming to the required standard

THE UNIVERSITY OF BRITISH COLUMBIA
August, 1973
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Division
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Date August 27, 1973
Abstract

The problem under investigation in this study was to find out what relationship a unit in modular arithmetic might have to Grade 6 pupils' skill in computing the division of fractions and to their understanding of the mathematical basis of the algorithm. It was hypothesized that a unit in modular arithmetic would aid in developing skill in computing and understanding of the algorithm. The study was conducted with a sample of 58 Grade 6 students from the same school. The subjects were assigned to two treatment groups. Both groups received a review of fraction concepts at the beginning of the study. Following this, one group was taught modular arithmetic while the other group reviewed adding and subtracting of fractions. Then both groups were taught multiplication and division of fractions. Following the instruction period, both groups were tested for ability to compute division of fractions. To test understanding of the division of fractions algorithm, an interview inventory test was administered to all subjects in both groups. A statistical analysis of the data from these tests revealed no support for the hypotheses. The conclusion was that teaching modular arithmetic to the Grade 6 pupils participating in the study did not appear to improve their ability to compute division of fractions nor their understanding of the mathematical basis of the division of fractions.
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CHAPTER I

NATURE OF THE STUDY

Introduction

Elementary school mathematics educators, whether they be classroom teachers or curriculum designers, face a common problem in deciding what balance to maintain between computational efficiency and an understanding of mathematical concepts. Emphasis has varied in the past few decades. Prior to 1960, arithmetic programs tended to stress computational efficiency and the social uses of arithmetic. During the 1960's, new elementary school programs laid greater stress on the understanding of mathematical concepts, with less emphasis on computational skills.

At the present time the search continues for an acceptable balance between skills and understanding as some educators question the rigorous approach of the 1960's. Elementary school mathematics programs continue to be revised in attempts to maximize both skills and understanding.¹ This study was prompted by the apparent need to investigate ways of achieving this balance.

Statement of the Problem

The problem under investigation in this study is stated

¹ British Columbia Department of Education, Mathematics, Primary - Years 1-3, Years 7-8, 1972, pp.1-2.
as follows:

Will Grade 6 students who are taught a unit on modular arithmetic show a greater understanding of the mathematical basis of division of fractions and a greater skill in performing the computation of division of fractions than students who are not taught modular arithmetic?

Uses of the Term "Fraction"

The term "fraction" is a potential source of misunderstanding in elementary mathematics. In introducing the concept of fractional numbers, Marks, Purdy, and Kenny point out that, "Accurate use of language dealing with fractional numbers and fractions frequently requires so much verbiage that it may stand in the way of clarity."\(^2\) This section will attempt to indicate how the term "fraction" is used in this study.

There are three common usages of the term "fraction": fraction as a symbol, fraction as a number pair, and fraction as a number.\(^3\) All of these three usages appear at one point or another in this study.

The fraction as a symbol consists of three parts, a top numeral called a numerator, a bottom numeral called a denominator, and a bar between them. Students review the fraction


as a symbol in the introductory lessons of this study.

The idea of the fraction as a number pair is introduced in the primary grades and the fraction symbol is eventually introduced as a convenient notation for recording the number pair. The number pair can represent part of a whole or a subset of a set. Students review fraction as a number pair in the introductory lessons of this study.

The idea of fraction as a number is introduced in the early intermediate grades. A choice is usually made at this point whether to introduce the term "rational number", the term "fractional number", or to continue to use the term "fraction". In this study, the choice was to use "fraction" rather than to introduce other terms. The term "rational number" was not familiar to the students nor was the term "fractional number".

The term "fraction" as used in the statement of the problem in an earlier section of this chapter refers to fraction as a number. Fractions as numbers are most often associated with the rational numbers. However, in the elementary school the term "fractions" generally refers to the set of non-negative rational numbers. This is the meaning of "fractions" in this study.

Suitability of the Grade Level

Grade 6 is often considered to be an appropriate grade level in which to introduce division of fractions. In the
primary grades students learn that a fraction is a symbol for an ordered pair of natural numbers with some physical referents. Students then learn to generate sets of equivalent fractions and from these sets to develop an intuitive notion of rational numbers. Next the comparison of rational numbers is introduced and students are taught to find if one fraction is greater than, less than, or equal to another fraction. The order in which students usually learn operations with rational numbers is addition, subtraction, multiplication, and division. Thus students at the Grade 6 level, the level chosen for this study, are generally considered able to learn division of fractions.

Importance of Fractions in the Elementary School

Opinions vary as to the priority which should be given to instruction in fractions in the elementary school mathematics program. Price, for example, suggests that there is diminishing value in teaching fractions because the operations (especially division) have little practical value in daily life and, furthermore, the metric system will soon make the fraction an anachronism. He asks, "Can you think of one specific example in the real world in which division of one fraction by another is necessary?" and suggests the answer, "The fractions that we teach are just not fractions of the real world." If we convert to the metric system, he

argues, we shall have to be concerned only with tenths and
decimal computation takes care of these.

Brueckner and Grossnickle, in their 1953 edition of
Making Arithmetic Meaningful ⁵, advised teachers to teach the
pupil to invert the divisor and multiply even though he may
not understand the mathematical basis of the operation. The
teacher should on no account attempt to rationalize the
process because it has no social significance.

Despite efforts to improve the mathematical correctness
of elementary programs in the past decade, the foregoing
viewpoint may exist today in more sophisticated forms.

Commenting on this, Botts says,

... we may be fostering a new kind of rote learning
when we prompt teachers to insist that students always
use the approved terms and notations - not because they
make the subject any clearer, but because the teacher
has been led to believe that this is what's "right" and
important to mathematicians. ⁶

Marks, Purdy, and Kenny point out that fractions are
valuable in their own right and that there are many everyday
problems that are impossible to solve if only whole numbers
are used. They illustrate with the case of three boys wish­
ing to share two apples equally. This problem can be solved
with a knife, "... but to express the answer mathematically
demands using numbers other than whole numbers." ⁷ Decimal

⁵ Leo J. Brueckner and Foster E. Grossnickle, Making Arith­

⁶ Botts, op.cit., p.220.

⁷ Marks, Purdy, Kenny, op.cit., p.189.
notation may be used to express these numbers but this may not be the ideal way to introduce the concept to children. Even though metrication is imminent in Canada, there will still be a need to use fractions other than tenths, particularly halves, thirds, and fourths.

Riess argues that a utilitarian viewpoint is not sufficient. In *A New Rationale for the Teaching of Fractions* she states,

> Today we know that hardly any other aspect of arithmetic has so much to contribute to the child's capacity for abstract reasoning and relational thinking as a thorough understanding of the concept of fractions.

She contrasts this with the reality that,

> ... no other phase in elementary mathematics presently contributes so little to the child's appreciation of number and is so unpopular with pupils and teachers alike as the study of fractions.

Riess goes on to point out that attempts have been made to relieve this drabness by teaching "useful" fractions but she questions whether it is right to carry pragmatic teaching beyond the first few grades. She suggests that,

> A program based on undistorted life experiences of the child has its place in the first grades, provided it does not block a fundamentally different phase of development by being carried on too long. In other words, one of the crucial problems in teaching fractions is to strike the right balance between teaching them as a useful tool and helping the child extend his concept of number in a mathematical sense, without losing contact with the child's own level of thought and interest.

---


9 Ibid., p.107.
The Division Algorithm

There are three division of fractions algorithms in common use; the common denominator method, the complex fraction method, and the inverse operations method. The three methods are illustrated here.

Common denominator method

\[
\frac{3}{4} + \frac{2}{7} = \frac{21}{28} + \frac{8}{28} = \frac{29}{8}
\]

Complex fraction method

\[
\frac{3}{4} + \frac{2}{7} = \frac{3}{4} \cdot \frac{7}{2} = \frac{21}{8}
\]

Inverse operations method

\[
\frac{3}{4} + \frac{2}{7} = n
\]

\[
n \cdot \frac{2}{7} = \frac{3}{4}
\]

\[
n \cdot \frac{2}{7} \cdot \frac{7}{2} = \frac{3}{4} \cdot \frac{7}{2}
\]

\[
n \cdot 1 = \frac{3}{4} \cdot \frac{7}{2}
\]

\[
n = \frac{21}{8}
\]

Some studies have been done to evaluate the three methods. Bidwell examines the three methods in the light of modern learning theory and in particular the theories of Gagné and Ausubel. By combining Gagné's suggestion of the need for hierarchies of required prior concepts with Ausubel's idea of advanced organizers, which are prior learned materials designed to maximize efficient learning, Bidwell attempts to establish
the degree of meaningfulness in each method. He concludes that,

The inverse operations approach clearly utilizes a dominant generalization in mathematics, the concept of inverse operations. Specifically, the method uses the property that multiplication and division are inverse operations. The student has already used this property with addition and subtraction of whole numbers. 10

Bidwell conducted this study to compare the three methods of learning division of fractional numbers and concluded that the students were able to perform the inverse operations method with less error and better understanding and integration of concepts than with the other two methods.

Ingersoll conducted an experiment to test the efficiency of two methods of division which employ the multiplicative inverse. These correspond to the two methods which Bidwell labelled the "complex fraction method" and the "inverse operations method". Ingersoll omitted the common denominator method from his study because he failed to find evidence that it is superior to other methods. Ingersoll found that,

Typically, arithmetic texts which have emphasized the inversion principle . . . have tended to employ one of two methods. The first uses complex fractions and the multiplicative identity element and the other uses the associative property of multiplication and the multiplicative identity element . . . . 11

It should be noted that Ingersoll uses the term "inversion"
to refer to the use of the reciprocal, not to inverse operations.

Ingersoll's study was conducted along the following lines. All subjects were given an introductory program which stressed the development of the concept of the reciprocal. Then one group was placed on Program A (Associative) while the second group was placed on Program CF (Complex Fractions). Program A developed the reciprocal principle in division using the associative property of multiplication and the multiplicative identity element. It emphasized the operation allowing relocation of parentheses in the expression 
\[(A \times B) \times C = A \times (B \times C).\]
Program CF developed the reciprocal principle in division by emphasizing the identity elements of multiplication and division and complex fractions. A third program, Program R, was made up of a random selection of items from the other two programs. On a test of achievement overall results favoured the complex fraction approach.

There is some support for the view that none of the three methods of division of fractions referred to previously can be really meaningful to children of this age group. Bates believes that,

Most of the concepts needed for an understanding of division using fractions are 'secondary' concepts - generalizations which are based upon previous abstractions, that is, on 'primary' concepts. 12

He suggests that there is a real need to bridge the gap between the physical world and the abstract secondary concepts of reciprocal and inverse operations. Furthermore, he is not convinced that the gap can be bridged. In reaching the rote process of invert and multiply,

... the intermediate thought of translating division by \( n \) into multiplication by the multiplicative inverse of \( n \) depends on a concept which I suspect is not acquirable, at present, by many children of elementary school age. 13

It was the purpose of this study to see if modular arithmetic can help to bridge the gap.

A search of the literature reveals that there are arguments in favour of each of several methods of dividing fractions. Thus the choice of algorithm must be made on the basis of suitability for this study. The method chosen was the inverse operation method. An outline of this method showing the mathematical properties involved is given here.

**Question:** \( \frac{a}{b} + \frac{c}{d} = \frac{e}{f} \)

**Step 1:** \( \frac{a}{b} = \frac{c}{d} \times \frac{e}{f} \)  
**Definition of division**

**Step 2:** \( \frac{d}{c} \times \frac{a}{b} = \frac{d}{c} \times (\frac{c}{d} \times \frac{e}{f}) \)  
**Cancellation property**

**Step 3:** \( \frac{d}{c} \times \frac{a}{b} = (\frac{d}{c} \times \frac{c}{d}) \times \frac{e}{f} \)  
**Associative property of multiplication**

**Step 4:** \( \frac{a}{b} \times \frac{d}{c} = 1 \times \frac{e}{f} \)  
**Property of reciprocals;** \( \frac{d}{c} \times \frac{c}{d} = 1 \)

**Step 5:** \( \frac{a}{b} \times \frac{d}{c} = \frac{e}{f} \)  
**Multiplicative identity;** \( 1 \times \frac{e}{f} = \frac{e}{f} \)

**Step 6:** \( \frac{a}{b} + \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} \)  
**Transitive property of the relation = .**

It was important to ensure that the division of fractions method used in this study did not contain any concepts which could not be developed readily during the course of instruction in division of fractions. Many of the subjects in this study had used the *Seeing Through Arithmetic (1st. Canadian edition)*\(^{14}\) program in previous school years, as far as was known from a study of their school records. This might suggest the use of the complex fraction method inasmuch as it is the one used in this edition of *Seeing Through Arithmetic*. However, an examination of the algorithm yielded no compelling evidence that it was the only one that could follow from the concepts developed in the *Seeing Through Arithmetic (1st. Canadian edition)* program.

Modular Arithmetic

One group of subjects studied a unit on modular arithmetic in this study. Since modular arithmetic is not a familiar topic in elementary school mathematics, it is discussed in some detail in this section.

There seems to be a fairly wide divergence of opinion on the value of modular arithmetic in the elementary school. Spitzer argues that because few elementary mathematics programs include modular arithmetic, this is an indication of its lack of worth. Of the few programs that include a section

on modular arithmetic he says, "The producers . . . may have concluded that inclusion of the topic will be beneficial either to their prestige or sales." 15 On the other hand, Howard and Dumas feel that the topic has importance in introducing certain characteristics of the mathematical system such as the properties of closure, associativity, commutativity, and particularly inverse elements. 16 Westcott and Smith believe that "... because a modular system operates within limits, it makes basic mathematical principles less complex to demonstrate to children." 17 Mueller states that,

Properly slanted, work with modular arithmetic with an emphasis upon pattern can arouse the elementary student's mathematical imagination and intuition and, in the process, provide a possible foretaste of the thrill of scholarly discovery . . . . at an elementary level, modular arithmetic provides finite, miniature number systems which may be explored rather thoroughly. 18

The term modular arithmetic refers to number systems which use a finite number of units and then repeat themselves. It is sometimes called "clock" arithmetic because it is cyclic in nature and because the ordinary clock provides a physical representation of such a system. The analogy can


be extended to include modular systems with any number of elements so that for instructional purposes the expressions "modulo seven clock" or "seven hour clock", for example, are used.

While the regular 12-hour clock is useful for introducing the idea of modular arithmetic, it is not suitable for illustrating the mathematical properties of division which are central to this study. The properties of modular number systems vary depending upon the number of units in the system. In systems having prime-numbered moduli there is a unique quotient for every number in the system for any non-zero divisor. This property is essential in order to demonstrate the inverse relationship between division and multiplication using the table of multiplication for the system. Non-prime moduli do not have unique quotients for every divisor; there is more than one quotient for some divisions.

Although multiplication and division are the operations used in this study, it was reasoned that students should be introduced to addition and subtraction as well. Using the clock analogy, operations in modular arithmetic are defined as follows.

Addition is defined as a clockwise movement of the hand, starting from the numeral representing the first addend and moving through the number of points representing each successive addend. The sum is represented by the final numeral to which the hand points.
Subtraction is defined as a counter-clockwise movement of the hand, starting from the numeral representing the minuend and moving through a number of points representing the subtrahend. The difference is represented by the final numeral to which the hand points.

Multiplication is defined in terms of successive addition. The hand is started at 0 and moved in a clockwise direction. One factor represents the size of each move and the other factor represents the number of moves. The product is represented by the final numeral to which the hand points.

Division is defined on the clock as repeated subtraction. The hand is placed at the numeral representing the dividend and moved in a counter-clockwise direction. The size of each move is determined by the divisor and the number of moves required in order for the hand to rest at 0 represents the quotient.

By performing these four operations in a system having a prime-number modulus, complete tables of addition, subtraction, multiplication, and division can be constructed. These tables in turn can be used to show the inverse relationship between addition and subtraction and between multiplication and division. Other properties which can be developed once the operations are learned include the multiplicative inverse, the associative and commutative properties of multiplication, and the identity element for multiplication.
Most of the properties needed to explain division of fractions are developed through the elementary school grades by the use of whole numbers. However, the multiplicative inverse, or reciprocal, is not a property of whole numbers and is usually introduced to students in terms of fractions immediately prior to its use in division of fractions. Modular arithmetic provides an opportunity to introduce this property to students in another system.

The table below illustrates some properties in terms of fractions and modular arithmetic.

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<tr>
<th></th>
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<th>Modular Arithmetic</th>
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<tr>
<td>Reciprocal</td>
<td>( \frac{2}{5} \times \frac{5}{2} = 1 )</td>
<td>( 3 \times 2 = 1 \pmod{5} )</td>
</tr>
<tr>
<td>Multiplicative identity element</td>
<td>( \frac{7}{8} \times 1 = \frac{7}{8} )</td>
<td>( 4 \times 1 = 4 \pmod{5} )</td>
</tr>
<tr>
<td>Commutative property</td>
<td>( \frac{1}{4} \times \frac{2}{3} = \frac{2}{3} \times \frac{1}{4} )</td>
<td>( 3 \times 4 = 4 \times 3 \pmod{5} )</td>
</tr>
<tr>
<td>Associative property</td>
<td>( \frac{1}{4} \times (\frac{5}{6} \times \frac{3}{4}) = (\frac{1}{4} \times \frac{5}{6}) \times \frac{3}{4} )</td>
<td>((3 \times 2) \times 4 = 3 \times (2 \times 4) \pmod{5})</td>
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**Related Studies**

Standard bibliographical sources were studied and no research studies could be located which had attempted to relate a study of modular arithmetic to operations with fractions. Lyda and Tayler made a study of the effect of modular
arithmetic on student's understanding of the decimal numeration system. No statistically significant evidence was obtained although the authors concluded that, "When pupils are given instruction in modular arithmetic some growth occurs in their understanding of the decimal numeration system."  

Statement of the Hypotheses

The following hypotheses were tested in this study.

1. Students who are taught modular arithmetic will show a significantly better ability to divide fractions, as measured by a test of computation, than students who are not taught modular arithmetic.

2. Students who are taught modular arithmetic will show a significantly better understanding of division of fractions, as measured by an interview test, than students who are not taught modular arithmetic.

CHAPTER II

DESIGN AND PROCEDURE

Design

The basic design of this investigation is that of a two group study. The groups are identified and defined as follows.

T-1 (Review) Group - Grade 6 students who received instruction based on the British Columbia curriculum guide for arithmetic, including a review of addition and subtraction of fractions, prior to being taught multiplication and division of fractions.

T-2 (Modular Arithmetic) Group - Grade 6 students who received instruction based on the British Columbia curriculum guide for arithmetic, with the exception that a review of addition and subtraction of fractions was replaced by a unit on modular arithmetic in which specific mathematical properties relating to multiplication and division were emphasized.

Sample

The sample consisted of all Grade 6 students in Queen's Park School, Penticton, British Columbia. This is an urban public school of about 475 students which is situated within School District #15. The pupils are drawn from an area of

---

lower socio-economic status within the community. However the community as a whole does not have areas of extremely high or low socio-economic status. All the students were white.

There were 68 pupils, 36 boys and 32 girls, in Grade 6 at the beginning of the school term. The mathematical background of the pupils varied since 25 of the pupils had attended the same school since Grade 1 while 43 had attended one or more other schools. No attempt was made to trace the mathematical experience of this latter group.

Measures of Population Variables

Intelligence

All students in Grade 6 in September, 1972 were administered the Otis-Lennon Mental Ability Test, Elementary II, Form A\textsuperscript{2} during the second week of school. This is a widely used test of general ability in which,

Emphasis is placed upon measuring the pupil's facility in reasoning and in dealing abstractly with verbal, symbolic, and configural content, sampling a broad range of cognitive abilities. \textsuperscript{3}

Arithmetic computation

All students in Grade 6 in September, 1972 were administered the British Columbia Test, Arithmetic Computation, V-VI, Form A\textsuperscript{4} during the second week of school. This is essentially


\textsuperscript{3} \textit{---------}, Manual For Administration, Otis-Lennon Mental Ability Test, Elementary, Form J, 1967, p.4.

\textsuperscript{4} British Columbia Department of Education, Division of Tests, Standards, and Research, British Columbia Test, Arithmetic Computation, V-VI, Form A, 1951.
a duplication of the Stanford Achievement Test. The items on the test are based on the work for the grades preceding the ones given in the title.

**Arithmetic reasoning**

All students in Grade 6 in December, 1972 were administered the British Columbia Test, Arithmetic Reasoning, VI-VII, Form A. This test is also based on the Stanford Achievement Test. Again the problems are based on the work for the grade preceding the ones given in the title.

**Formation of Groups**

The T-1 (Review) and T-2 (Modular Arithmetic) Groups were selected in the following manner. The results of the Otis-Lennon Mental Ability Test and the British Columbia Test, Arithmetic Computation were tabulated and these steps taken.

1. Students were listed in rank order of I.Q. as obtained from the Otis-Lennon Test. Starting with the first pair of students in this list, the first student in each pair was assigned to A group if a coin toss produced a head and to B group if a coin toss produced a tail. The second student was assigned to the other group. The same procedure was followed with each pair of students in the list until all were placed.

2. The students within each group, A and B, were then listed in rank order of scores on the Arithmetic Computation

---

5 [Footnote], British Columbia Test, Arithmetic Reasoning, VI-VII, Form A, 1951.
Test. Each group was further divided to form A-1, A-2, B-1, and B-2 groups as follows. Starting with the first pair of students in A group, the first student in the pair was assigned to A-1 if a coin toss produced a head and to A-2 if a coin toss produced a tail. The second student in the pair was assigned to the other group. The same procedure was followed with each pair of students in the A group. B-1 and B-2 groups were formed in a similar manner from B group.

3. The four groups thus obtained were re-formed into two groups as follows. A-1 was to be combined with B-1 if a coin toss produced a head; A-1 was to be combined with B-2 if a coin toss produced a tail. Since the toss produced a head, the A-1 and B-1 groups were combined to form one instructional group and A-2 and B-2 were combined to form the other group.

4. The two treatment groups were formed by a similar method of coin tossing. The A-1 B-1 group was assigned the Modular Arithmetic Treatment (T-2) and the A-2 B-2 group was assigned the Fraction Review Treatment (T-1).

The means and standard deviations of the I.Q. scores for each group are shown in Table I.

<table>
<thead>
<tr>
<th>TABLE I</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>MEANS AND STANDARD DEVIATIONS OF I.Q. SCORES FOR TREATMENT GROUPS</strong></td>
</tr>
<tr>
<td>Mean</td>
</tr>
<tr>
<td>T-1 Group</td>
</tr>
<tr>
<td>T-2 Group</td>
</tr>
</tbody>
</table>
The means and standard deviations of the Arithmetic Computation Test scores are shown in Table II.

**TABLE II**

MEANS AND STANDARD DEVIATIONS OF ARITHMETIC COMPUTATION SCORES FOR TREATMENT GROUPS

<table>
<thead>
<tr>
<th>Group</th>
<th>Mean</th>
<th>Std. Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>T-1 Group</td>
<td>14.4</td>
<td>4.9</td>
</tr>
<tr>
<td>T-2 Group</td>
<td>13.2</td>
<td>5.9</td>
</tr>
</tbody>
</table>

During the period of the study, two members of the T-1 Group and five members of the T-2 Group left the school. Five students who entered the school at various times during the study were included in the instruction but not in the study. Attendance was judged to be satisfactory if a student missed no more than an average of one period out of six. One member of the T-2 Group missed the entire period of instruction on modular arithmetic and had to be omitted from the study. Otherwise, one student was omitted from the T-1 Group and no students were omitted from the T-2 Group because of poor attendance.

The ages of all students taking part in the study were verified from school records. The normal September age range for students in Grade 6 is 10 years 9 months to 11 years 8 months. One year beyond this range was not considered unusual for Grade 6. One child aged 13-9 was omitted from
the final sample because of her age. Two other students aged 13-2 and 13-4 were retained in the final sample because no record was available of when they first had entered school. Both had a non-English background. The mean age and range of each group is shown in Table III.

<table>
<thead>
<tr>
<th>TABLE III</th>
</tr>
</thead>
<tbody>
<tr>
<td>MEAN AGES AND RANGES FOR TREATMENT GROUPS</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Mean age (months)</th>
<th>Range (months)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>T-1 Group</strong></td>
<td>137.3</td>
</tr>
<tr>
<td><strong>T-2 Group</strong></td>
<td>135.5</td>
</tr>
</tbody>
</table>

No attempt was made to control the sex variable in the formation of the treatment groups. Composition of the final sample is shown in Table IV.

<table>
<thead>
<tr>
<th>TABLE IV</th>
</tr>
</thead>
<tbody>
<tr>
<td>COMPOSITION OF THE TREATMENT GROUPS</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Boys</th>
<th>Girls</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>T-1 Group</strong></td>
<td>13</td>
<td>18</td>
</tr>
<tr>
<td><strong>T-2 Group</strong></td>
<td>17</td>
<td>10</td>
</tr>
<tr>
<td>Total</td>
<td>30</td>
<td>28</td>
</tr>
</tbody>
</table>
Instructional Procedure

The following schedule of instruction was used in this study. During the first three days of the 1972-73 school year (September 8, 9, 10) the two Grade 6 classes remained intact and the time was used as an orientation period. Classroom teachers restricted arithmetic instruction to a small amount of review of basic facts and operations with whole numbers.

On the next two school days (September 13 and 14) measures of population variables were administered and the instructional groups were formed.

During the period September 15 to December 1, both groups were taught by the same classroom teacher (not the experimenter). Topics covered in this period were geometry, measures of perimeter, area and volume, division with two and three digit divisors, signs of inequality, and the distributive law. Problem solving involved whole number computation only. No work or review was done with fractions.

Commencing December 4, instruction of both groups, T-1 (Review) Group and T-2 (Modular Arithmetic) Group, was undertaken by the experimenter. The following schedule was followed.

December 4 - 22: Both groups reviewed these topics - concept of fraction as a part of a whole and a subset of a set; review of terms such as proper, improper, mixed numeral, denominator, numerator; equivalent fractions, including lower terms, higher terms, lowest terms; finding common denominators by several methods. An outline of these lessons is contained in Appendix A.
Instruction in this section and in succeeding sections was by various visual and manipulative devices such as a magnet board and an overhead projector. Practice was supplied by a 24-page booklet of pencil and paper exercises. Also each lesson commenced with a five-minute review of basic number facts and questions selected from measurement, geometry, and whole number computation.

At the end of the unit the Fraction Concepts Test was administered to both groups. A copy of the Fraction Concepts Test is included in Appendix E. This three week period served to ensure a common background of knowledge of fractions for students in both groups and also served to familiarize students with a different instructor. One class period during this time was used to administer the B.C. Arithmetic Test (Reasoning) to both groups.

Following a Christmas vacation period of eleven days, T-1 Group studied a unit on addition and subtraction of fractions for a period of twelve lessons. This was essentially a review of Grade 5 instruction although no assumption was made that the students remembered how to add or subtract fractions. No pre-test on these skills was given to the group. An outline of lessons is included in Appendix B. Practice was supplied by a 14-page booklet of pencil and paper exercises.

During this same period of January 3-19, 1973 T-2 Group studied a unit on modular arithmetic designed to develop an understanding of some mathematical properties which have an application to the division of fractions. An outline of the
lessons and a copy of the 17-page booklet of practice exercises is included in Appendix C. In this unit, instruction was supplemented by commercially prepared teaching tapes.  

Both groups continued to receive an identical five-minute drill and review session at the beginning of each lesson. The drills included review of fractional ideas developed in the review unit but did not include any of the work that either group was currently doing.

At the end of this twelve-lesson period, T-1 group was given the Adding and Subtracting Fractions Test. This was a computation test containing fifty items prepared by the experimenter. A copy of the Adding and Subtracting Fractions Test is included in Appendix E.

At the same time, T-2 Group was given the Modular Arithmetic Test. This was a test on the ability to compute with modular arithmetic and also on the understanding of some mathematical properties of this system. The test was prepared by the experimenter and contained 49 items. A copy of the Modular Arithmetic Test is included in Appendix E.

Following these units, both groups received instruction in multiplication and division of fractions for a period of seventeen lessons. An outline of these lessons is included in Appendix D. After ten lessons on multiplication, the Multiplication of Fractions Test was given to both groups. This was a computation test prepared by the experimenter and contained 34 items. A copy of the test is included in Appendix E.

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6 Wollensak Teaching Tape, Mathematics, Clock Arithmetic: Problems C-3504, 1968.
At the end of a further seven lessons, the Division of Fractions Test was given to both groups. This computation test was a criterion measure in this study. A description of how the Division of Fractions Test was developed is contained in a later section and a copy of the test is included in Appendix E.

After the completion of the above schedule of instruction, the two groups were returned to the original classroom teacher for instruction in other topics in the Grade 6 program. In the first three school days following the end of instruction by the experimenter, an individual interview was conducted with each child by the experimenter to determine the child's understanding of division of fractions. A copy of the Interview Test and an explanation of how it was administered is included in Appendix E.

Also on the first day following the end of instruction by the experimenter, the T-1 Group was re-tested with the Adding and Subtracting Fractions Test. At the same time, the T-2 Group was given the Adding and Subtracting Fractions Test. The T-1 Group had received instruction in adding and subtracting fractions while the T-2 Group received instruction in modular arithmetic. The T-2 Group had no review of adding or subtracting fractions since Grade 5.

Control of Variables

The experimental treatment in this study was the unit of instruction on modular arithmetic. The lessons in this treatment were designed to assist the students in the T-2 Group to
understand the mathematical properties which they would later need to understand division of fractions.

None of the subjects taking part in the study were told the experimental nature of the study. It was felt that the teaching arrangements were close enough to normal to be accepted by the students without query. The experimenter was known to the students and had taught arithmetic to a Grade 6 group in the school in the previous year. Thus a precedent had been set for him to teach Grade 6 students.

The teacher variable was controlled by both groups being taught by the same teacher. However, this meant that the time of day for the arithmetic lesson had to be different for the two groups. The times chosen for instruction in arithmetic were 9:05 - 9:55 a.m. and 10:45 - 11:35 a.m. The T-1 and T-2 Groups were assigned to a time for instruction by the toss of a coin. The T-2 Group was instructed from 9:05 to 9:55 each day of the week and the T-1 Group was instructed from 10:45 to 11:35 each day of the week. The 9:05 time was immediately after school opening and the 10:45 time followed a 15-minute recess so that neither lesson followed immediately upon a lesson in some other subject. No attempt was made to counterbalance the effect of this variable by reversing the times of instruction for each group.

Development of Tests

The criterion measures in this study consisted of two tests, the Division of Fractions Test for computation and
the Interview Test for understanding. Both tests were
designed by the experimenter.

**Division of Fractions Test**

In order to develop a pool of questions which would be
representative of the computational skills required for the
division of fractions, the following combinations of divi-
dends, divisors, and quotients were considered.

\[
\text{Fraction} \quad \frac{\text{Fraction}}{\text{Whole number}} + \frac{\text{Whole number}}{\text{Mixed numeral}} = \frac{\text{Whole number}}{\text{Mixed numeral}}
\]

There are 17 possible combinations of the above if whole
number divided by whole number is omitted. In each case,
the quotients could be arrived at directly in lowest terms or
it could be necessary for the students to re-write the quotient
in lowest terms.

Using the above classification, it was possible to identify
34 types of questions. A decision was made to use no denomin-
ator larger than 12 in any dividend or divisor. One question
of each type was written by the experimenter to create a pool of
34 questions. This was judged, in the experience of the exper-
imenter, to be too many questions for Grade 6 students to
complete in one sitting. From this pool, 24 questions were
selected at random.

Students in Grade 6 are usually expected to express
fractional answers in lowest terms. It was reasoned that
students who had not reviewed addition and subtraction of
fractions might understand division of fractions but not be
adept at writing answers in lowest terms. For this reason two sets of scores were collected for the Division of Fractions Test. One set was designated Correct answers, that is, correct answers expressed in lowest terms. The other set was designated Partial answers, that is, answers which were correct in every respect except not being expressed in lowest terms.

Using the test scores of students included in the two treatment groups, a reliability coefficient of .90 was obtained using the K-R 21 formula. 7

Content validity was established for this test by submitting it to a panel of five experienced elementary teachers. Each agreed that in his opinion the test was a valid measure of division of fractions for Grade 6.

**Interview Test**

An individual interview inventory was devised by the experimenter after Gray's model. The purpose of such an instrument is to evaluate pupil progress toward acquiring mathematical understanding.

... the technique consists of facing a child with a problem or example, letting him find a solution, then challenging or questioning him to elicit his highest level of understanding of the process. 8

The specific purpose of the interview test in this study was to determine whether students performed the computation


of division of fractions solely by rote memory or whether they displayed a rational understanding of the process.

In determining rational understanding, Brownell's definition of "meaningful habituation" was used as a guide. Brownell lists these criteria for meaningful habituation:
1) a correct answer given at once and with apparent confidence,
2) inability to convince the child that his answer was incorrect by mentioning other possibilities, and 3) the child's success in defending the chosen answer.

There were four items on the test. Three of these items were chosen from the 24 items on the Division of Fractions Test. The error count on a pilot form of the test administered to a group of Grade 7 students was used to select items of varying difficulty. A fourth item which was not included in the Division of Fractions Test was chosen to be the first item on the Interview Test.

A questioning and scoring technique was developed and tested with Grade 7 students. The subjects were each given a test blank containing the questions and were allowed to use a pencil to find the answers. Upon the completion of each question, the examiner noted the results on a separate scoring blank. A copy of the test blank, scoring blank, and directions for administering are included in Appendix E. The test was administered to 58 students and required an average of nine minutes to administer.

A scoring technique was devised to record the desired information. Student responses were classified as Correct, Partial (not in lowest terms), or Wrong. Correct and Partial answers were classified as Rational or Rote. A Rational response was one in which the student used the inverse relationship of multiplication and division and the reciprocal of the divisor to explain the division process. It was not necessary for the student to use the term "reciprocal". A Rote response was one given by a student who could give no reason for the process except, perhaps, that this is what he had been told to do. If the student made no attempt to perform the operation, this fact was noted on the score sheet. Also, if the student used any other algorithm except the one taught, this fact was noted.

It was not possible to make any statistical tests of the reliability of the Interview Test in this study.

Additional tests

Four additional tests were constructed by the experimenter.

The Fraction Concepts Test was a test of understanding which required no computation. The items in the test were constructed to measure the student's understanding of the fractional concepts taught in the review unit at the beginning of the study.

The Adding and Subtracting Fractions Test was a computation test containing items representative of the
types of adding and subtracting questions reviewed with the T-1 Group.

The Modular Arithmetic Test contained items intended to measure computation with modular arithmetic and also the student's understanding of the mathematical principles chosen to be illustrated during the unit of instruction.

The Multiplication of Fractions Test was a computation test containing items representative of the types of multiplication questions taught during the unit of instruction.

These tests were judged to have content validity because they contained items similar to material taught.

Using the test scores of the students in the treatment groups taking the tests, the following reliability coefficients were obtained using the K-R 21 formula:

- Fraction Concepts Test .876
- Adding and Subtracting Fractions Test .880
- Multiplication of Fractions Test .958
- Modular Arithmetic Test .867

Statistical Treatment

Procedure

The data from the tests were treated as follows. The means, adjusted means, and the standard errors were calculated using the BMDX82 computer program and the analysis

10 Thorndike and Hagen, loc.cit.
of covariance was done using the BMD05V computer program at the University of British Columbia Computing Centre. In this study, I.Q. scores, Arithmetic Computation scores, Arithmetic Reasoning scores, and Fraction Concepts scores were used as covariates. Subjects were divided into high and low I.Q. groups on the basis of I.Q. scores divided at the median.

Chi square tests were used to test for significance of differences of numbers of subjects falling in the different response categories for the items on the Interview Test. In cases where cell sizes are small, the usual method of computing the chi square would give an over-estimate of the true value and, as a result, some hypotheses might be rejected which in fact should not have been. Therefore to correct for this possibility, where cell sizes were less than 5, the Yates correction for small sample sizes was employed. Whenever this correction was used, the fact was noted in the table.

**Decision rule**

In the statistical testing procedure, the null hypothesis was rejected whenever the probability of committing a Type I error was equal to or less than .05.

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12 Ibid., p.543 - 555.
CHAPTER III

ANALYSIS OF THE DATA

The data were analyzed to test the hypotheses set forth in Chapter I and re-stated here.

1) Students who are taught modular arithmetic will show a significantly better ability to divide fractions, as measured by a test of computation, than students who are not taught modular arithmetic.

2) Students who are taught modular arithmetic will show a significantly better understanding of division of fractions, as measured by an interview test, than students who are not taught modular arithmetic.

Findings From the Division of Fractions Test

The means, adjusted means, and standard errors for the Division of Fractions Test (Correct answers only) are presented in Table V.

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Adj. Mean</th>
<th>Std. Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sex</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Boys</td>
<td>15.97</td>
<td>15.59</td>
<td>1.31</td>
</tr>
<tr>
<td>Girls</td>
<td>18.64</td>
<td>19.05</td>
<td>1.35</td>
</tr>
<tr>
<td>I.Q.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>High</td>
<td>18.33</td>
<td>16.53</td>
<td>1.53</td>
</tr>
<tr>
<td>Low</td>
<td>16.11</td>
<td>18.04</td>
<td>1.61</td>
</tr>
<tr>
<td>Treatment</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>T-1</td>
<td>18.84</td>
<td>18.32</td>
<td>1.49</td>
</tr>
<tr>
<td>T-2</td>
<td>15.45</td>
<td>16.04</td>
<td>1.66</td>
</tr>
</tbody>
</table>
The significance of the differences of the adjusted mean scores for Correct responses only was tested by means of a 2(Treatment) by 2(Sex) by 2(I.Q.) analysis of covariance. A summary of the analysis of covariance is presented in Table VI.

TABLE VI

SUMMARY OF THE ANALYSIS OF COVARIANCE OF DIVISION OF FRACTIONS (CORRECT ANSWERS ONLY) SCORES WITH I.Q., ARITHMETIC COMPUTATION, ARITHMETIC REASONING, AND FRACTION CONCEPTS HELD CONSTANT

<table>
<thead>
<tr>
<th>Source</th>
<th>df</th>
<th>Mean Square</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Treatment</td>
<td>1</td>
<td>25.30</td>
<td>1.13</td>
</tr>
<tr>
<td>Sex</td>
<td>1</td>
<td>133.20</td>
<td>5.93*</td>
</tr>
<tr>
<td>I.Q.</td>
<td>1</td>
<td>2.64</td>
<td>0.12</td>
</tr>
<tr>
<td>Treatment x Sex</td>
<td>1</td>
<td>5.03</td>
<td>0.22</td>
</tr>
<tr>
<td>Treatment x I.Q.</td>
<td>1</td>
<td>34.83</td>
<td>1.55</td>
</tr>
<tr>
<td>Sex x I.Q.</td>
<td>1</td>
<td>44.22</td>
<td>1.97</td>
</tr>
<tr>
<td>Treatment x Sex x I.Q.</td>
<td>1</td>
<td>0.59</td>
<td>0.03</td>
</tr>
<tr>
<td>Error (within)</td>
<td>1</td>
<td>22.45</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>53</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

* p < .05
The means, adjusted means, and standard errors for the Division of Fractions Test (Correct and Partially correct answers) are presented in Table VII.

**TABLE VII**
MEANS, ADJUSTED MEANS, AND STANDARD ERRORS OF THE DIVISION OF FRACTIONS TEST (CORRECT AND PARTIAL ANSWERS)

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Adj. Mean</th>
<th>Std. Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sex</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Boys</td>
<td>16.86</td>
<td>16.30</td>
<td>1.33</td>
</tr>
<tr>
<td>Girls</td>
<td>19.43</td>
<td>20.03</td>
<td>1.38</td>
</tr>
<tr>
<td>I.Q.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>High</td>
<td>18.80</td>
<td>17.21</td>
<td>1.57</td>
</tr>
<tr>
<td>Low</td>
<td>17.36</td>
<td>19.07</td>
<td>1.65</td>
</tr>
<tr>
<td>Treatment</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>T-1</td>
<td>19.45</td>
<td>19.08</td>
<td>1.53</td>
</tr>
<tr>
<td>T-2</td>
<td>16.55</td>
<td>16.99</td>
<td>1.70</td>
</tr>
</tbody>
</table>

The significance of the differences of the adjusted mean scores for the Correct and Partially correct responses was tested by means of a 2(Treatment) by 2(Sex) by 2(I.Q.) analysis of covariance. A summary of the analysis of covariance is presented in Table VIII.
### TABLE VIII
SUMMARY OF THE ANALYSIS OF COVARIANCE OF DIVISION OF FRACTIONS (CORRECT AND PARTIAL ANSWERS) SCORES WITH I.Q., ARITHMETIC COMPUTATION, ARITHMETIC REASONING, AND FRACTION CONCEPTS HELD CONSTANT

<table>
<thead>
<tr>
<th>Source</th>
<th>df</th>
<th>Mean Square</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Treatment</td>
<td>1</td>
<td>15.64</td>
<td>0.65</td>
</tr>
<tr>
<td>Sex</td>
<td>1</td>
<td>148.21</td>
<td>6.19*</td>
</tr>
<tr>
<td>I.Q.</td>
<td>1</td>
<td>1.98</td>
<td>0.08</td>
</tr>
<tr>
<td>Treatment x Sex</td>
<td>1</td>
<td>4.13</td>
<td>0.17</td>
</tr>
<tr>
<td>Treatment x I.Q.</td>
<td>1</td>
<td>33.93</td>
<td>1.42</td>
</tr>
<tr>
<td>Sex x I.Q.</td>
<td>1</td>
<td>16.10</td>
<td>0.67</td>
</tr>
<tr>
<td>Treatment x Sex x I.Q.</td>
<td>1</td>
<td>8.85</td>
<td>0.37</td>
</tr>
<tr>
<td>Error (within)</td>
<td>46</td>
<td>23.96</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>53</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

* \( p < .05 \)

From Table VI and VIII it can be seen that in both cases, using Correct answers and Correct and Partially correct answers, the only significant difference was that due to sex, favouring girls over boys. Since the difference due to treatments (T-1 compared with T-2) was not significant, the first hypothesis was not supported. Since intelligence test scores were included as one of the covariates, no significant difference due to intelligence was expected. Intelligence was included as part of the analysis so that possible interaction between I.Q. and Treatment or Sex might be revealed.
Findings From the Interview Test

The response categories in the Interview Test are abbreviated throughout the analysis. A list of abbreviations used in the analysis is given here for convenience.

C - Correct answer
P - Partially correct answer (not in lowest terms)
W - Wrong answer
N - No attempt to answer
R - Rational response
Ro - Rote response

The data from the Interview Test were grouped according to the number of students in each treatment group giving different types of answers to each of the four test items. Table IX shows the number of students giving C,P,W, and N answers.

<table>
<thead>
<tr>
<th>Response category</th>
<th>Test item 1 ( (N=31) )</th>
<th>Test item 2 ( (N=31) )</th>
<th>Test item 3 ( (N=31) )</th>
<th>Test item 4 ( (N=31) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>30</td>
<td>26</td>
<td>29</td>
<td>18</td>
</tr>
<tr>
<td>P</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>W</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>N</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table IX
THE NUMBER OF T-1 AND T-2 SUBJECTS IN C,P,W, AND N CATEGORIES FOR EACH OF THE FOUR INTERVIEW TEST ITEMS
From the distribution of scores in Table IX, the differences between treatment groups in the number of students obtaining Correct answers for each test item were tested for significance by means of the chi square test. Table X shows the results of the chi square tests. (To be significant at the .05 level, chi square in these analyses was equal to or greater than 3.841.)

**TABLE X**

COMPARISON OF THE TREATMENT GROUPS IN THE NUMBER OF SUBJECTS GIVING CORRECT ANSWERS TO EACH ITEM IN THE INTERVIEW TEST

<table>
<thead>
<tr>
<th>Test item</th>
<th>T-1 Group (N=31)</th>
<th>T-2 Group (N=27)</th>
<th>$\chi^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Number of correct answers</td>
<td>Number of correct answers</td>
<td></td>
</tr>
<tr>
<td>1. (\frac{3}{4} + \frac{1}{4} = )</td>
<td>30</td>
<td>26</td>
<td>1</td>
</tr>
<tr>
<td>2. (\frac{9}{16} + \frac{3}{8} = )</td>
<td>29</td>
<td>18</td>
<td>6.861</td>
</tr>
<tr>
<td>3. (4\frac{1}{6} + 5 = )</td>
<td>29</td>
<td>17</td>
<td>8.174</td>
</tr>
<tr>
<td>4. (\frac{3}{4} + 1\frac{2}{3} = )</td>
<td>24</td>
<td>14</td>
<td>4.198</td>
</tr>
</tbody>
</table>

1 $f_o - f_e$ too small to apply Yates correction.
Also from the distribution of scores in Table IX, the differences between the treatment groups in the total number of students obtaining Correct and Partial answers for each test item were tested for significance by means of the chi square test. Table XI shows the results of the chi square tests.

**TABLE XI**

**COMPARISON OF THE TREATMENT GROUPS IN THE NUMBER OF SUBJECTS GIVING CORRECT AND PARTIAL ANSWERS TO EACH ITEM IN THE INTERVIEW TEST**

<table>
<thead>
<tr>
<th>Test item</th>
<th>T-1 Group (N=31)</th>
<th>T-2 Group (N=27)</th>
<th>( \chi^1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Number of C and P answers</td>
<td>Number of C and P answers</td>
<td></td>
</tr>
<tr>
<td>1. ( \left(\frac{3}{4} + \frac{1}{4} = \right) )</td>
<td>30</td>
<td>27</td>
<td>01</td>
</tr>
<tr>
<td>2. ( \left(\frac{9}{16} + \frac{3}{8} = \right) )</td>
<td>29</td>
<td>22</td>
<td>.927</td>
</tr>
<tr>
<td>3. ( \left(4\frac{1}{6} + 5 = \right) )</td>
<td>29</td>
<td>18</td>
<td>6.861</td>
</tr>
<tr>
<td>4. ( \left(3\frac{3}{4} + 1\frac{2}{3} = \right) )</td>
<td>24</td>
<td>16</td>
<td>2.188</td>
</tr>
</tbody>
</table>

1 Yates correction applied.

From Table X it can be seen that there were significant differences in the number of students scoring Correct answers in Test Items 2, 3, and 4. From Table XI it can be seen that there was a significant difference in Item 3 for students scoring Correct and Partial answers. In each case
the difference favoured the T—1 (Review) Group. Since there were no differences favouring the T—2 (Modular Arithmetic) Group in any test item regardless of how it was scored, the first hypothesis was not supported.

However, the chief purpose of the Interview Test was to obtain a measure of understanding. For this purpose, each Correct and Partial answer was categorized during the scoring of the test as being a Rational or a Rote response. Table XII shows the number of Rational and Rote responses in each treatment group for each test item.

**TABLE XII**

THE NUMBER OF T—1 AND T—2 SUBJECTS GIVING RATIONAL AND ROTE RESPONSES TO EACH ITEM IN THE INTERVIEW TEST

<table>
<thead>
<tr>
<th>Test item</th>
<th>Test item</th>
<th>Test item</th>
<th>Test item</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>Response</td>
<td>T—1</td>
<td>T—2</td>
<td>T—1</td>
</tr>
<tr>
<td>R</td>
<td>3</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>Ro</td>
<td>27</td>
<td>23</td>
<td>25</td>
</tr>
<tr>
<td></td>
<td>25</td>
<td>16</td>
<td>22</td>
</tr>
</tbody>
</table>

From the distribution of scores in Table XII the differences between treatment groups in the number of students scoring in the Rational category were tested by means of the chi square test. Table XIII shows the results of the chi square tests.
### TABLE XIII

**COMPARISON OF THE TREATMENT GROUPS IN THE NUMBER OF SUBJECTS GIVING RATIONAL RESPONSES TO EACH ITEM IN THE INTERVIEW TEST**

<table>
<thead>
<tr>
<th>Test item</th>
<th>T-1 Group</th>
<th>T-2 Group</th>
<th>$\chi^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $(\frac{3}{4} + \frac{1}{4} = )$</td>
<td>(N=30) 3</td>
<td>(N=27) 4</td>
<td>.027 1</td>
</tr>
<tr>
<td>2. $(\frac{9}{16} + \frac{3}{8} = )$</td>
<td>(N=29) 4</td>
<td>(N=22) 4</td>
<td>.006</td>
</tr>
<tr>
<td>3. $(4\frac{1}{6} + 5\frac{2}{3} = )$</td>
<td>(N=29) 4</td>
<td>(N=18) 2</td>
<td>.--- 2</td>
</tr>
<tr>
<td>4. $(\frac{3}{4} + \frac{2}{3} = )$</td>
<td>(N=24) 2</td>
<td>(N=16) 2</td>
<td>.---</td>
</tr>
</tbody>
</table>

1 Yates correction used.

2 $f_o - f_e$ too small in Items 3 and 4 to apply Yates correction.

From this table it can be seen that there were no significant differences in the number of subjects giving Rational responses to any test item. Since there was no observed difference due to treatment, the second hypothesis was not supported.
Findings From the Modular Arithmetic Test

The T-2 (Modular Arithmetic) Group was the only group to receive the Modular Arithmetic Test and therefore no comparison of Groups (T-1 with T-2) could be made. However, as a matter of interest the means and standard deviations of the Modular Arithmetic Test for the total T-2 Group and the T-2 Group divided by sex and intelligence are presented in Table XIV.

TABLE XIV
MEANS AND STANDARD DEVIATIONS OF THE MODULAR ARITHMETIC TEST SCORES FOR THE TOTAL T-2 GROUP AND THE T-2 GROUP DIVIDED BY SEX AND INTELLIGENCE

<table>
<thead>
<tr>
<th></th>
<th>N</th>
<th>Mean</th>
<th>Std. Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total T-2 Group</td>
<td>27</td>
<td>33.5</td>
<td>8.4</td>
</tr>
<tr>
<td>Sex</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Boys</td>
<td>17</td>
<td>35.1</td>
<td>7.2</td>
</tr>
<tr>
<td>Girls</td>
<td>10</td>
<td>30.9</td>
<td>10.0</td>
</tr>
<tr>
<td>Intelligence^1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>High</td>
<td>15</td>
<td>35.9</td>
<td>9.7</td>
</tr>
<tr>
<td>Low</td>
<td>12</td>
<td>30.5</td>
<td>5.3</td>
</tr>
</tbody>
</table>

^1 Intelligence classified as High or Low I.Q. scores divided at the median of the combined T-1 and T-2 Groups (104.5).
Additional Findings

Since it is possible that success in computation of division of fractions could be related to the student's ability to compute multiplication of fractions, the scores from the Multiplication of Fractions Test were also examined. The means, adjusted means, and standard errors for the Multiplication of Fractions Test are presented in Table XV.

<table>
<thead>
<tr>
<th>TABLE XV</th>
</tr>
</thead>
<tbody>
<tr>
<td>MEANS, ADJUSTED MEANS, AND STANDARD ERRORS OF THE MULTIPLICATION OF FRACTIONS TEST</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Sex</th>
<th>Mean</th>
<th>Adj. Mean</th>
<th>Std. Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boys</td>
<td>23.17</td>
<td>22.21</td>
<td>2.05</td>
</tr>
<tr>
<td>Girls</td>
<td>26.00</td>
<td>27.21</td>
<td>2.12</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>I.Q.</th>
<th>Mean</th>
<th>Adj. Mean</th>
<th>Std. Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>High</td>
<td>25.97</td>
<td>21.32</td>
<td>2.29</td>
</tr>
<tr>
<td>Low</td>
<td>23.01</td>
<td>27.30</td>
<td>2.37</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Treatment</th>
<th>Mean</th>
<th>Adj. Mean</th>
<th>Std. Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>T-1</td>
<td>27.87</td>
<td>27.21</td>
<td>1.97</td>
</tr>
<tr>
<td>T-2</td>
<td>20.70</td>
<td>21.66</td>
<td>2.22</td>
</tr>
</tbody>
</table>

The significance of the differences of the adjusted mean scores was tested by means of a 2(Treatment) by 2(Sex) by 2(I.Q.) analysis of covariance. A summary of the analysis of covariance is presented in Table XVI.
TABLE XVI

SUMMARY OF THE ANALYSIS OF COVARIANCE OF THE MULTIPLICATION OF FRACTIONS TEST SCORES WITH I.Q., ARITHMETIC COMPUTATION, ARITHMETIC REASONING, AND FRACTION CONCEPTS HELD CONSTANT

<table>
<thead>
<tr>
<th>Source</th>
<th>df</th>
<th>Mean Square</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Treatment</td>
<td>1</td>
<td>320.14</td>
<td>6.08*</td>
</tr>
<tr>
<td>Sex</td>
<td>1</td>
<td>109.32</td>
<td>2.08</td>
</tr>
<tr>
<td>I.Q.</td>
<td>1</td>
<td>137.46</td>
<td>2.61</td>
</tr>
<tr>
<td>Treatment x Sex</td>
<td>1</td>
<td>54.06</td>
<td>1.03</td>
</tr>
<tr>
<td>Treatment x I.Q.</td>
<td>1</td>
<td>117.62</td>
<td>2.23</td>
</tr>
<tr>
<td>Sex x I.Q.</td>
<td>1</td>
<td>44.32</td>
<td>0.84</td>
</tr>
<tr>
<td>Treatment x Sex x I.Q.</td>
<td>1</td>
<td>0.28</td>
<td>0.01</td>
</tr>
<tr>
<td>Error (within)</td>
<td>46</td>
<td>52.64</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>53</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

* p < .05

From Table XVI it can be seen that there was a significant difference between the treatment groups, favouring the T-1 (Review) Group over the T-2 (Modular Arithmetic) Group. It would appear from this finding that at the time instruction in division of fractions was begun, students in the T-1 Group were better able to compute multiplication of fractions than students in T-2 Group.
The ability of the two treatment groups to add and subtract fractions was examined also. Although the T-2 Group had received no review of adding and subtracting fractions since the previous school year, they were given the Adding and Subtracting Fractions Test after completing the program of multiplication and division of fractions. At the same time, the T-1 Group were re-tested with this same test.

The means, adjusted means, and standard errors of the Adding and Subtracting Fractions Test are presented in Table XVII.

**TABLE XVII**

**MEANS, ADJUSTED MEANS, AND STANDARD ERRORS OF THE ADDING AND SUBTRACTING OF FRACTIONS TEST**

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Adj. Mean</th>
<th>Std. Error</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Sex</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Boys</td>
<td>13.16</td>
<td>12.35</td>
<td>2.68</td>
</tr>
<tr>
<td>Girls</td>
<td>14.46</td>
<td>18.20</td>
<td>2.77</td>
</tr>
<tr>
<td><strong>I.Q.</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>High</td>
<td>18.87</td>
<td>16.69</td>
<td>2.83</td>
</tr>
<tr>
<td>Low</td>
<td>11.96</td>
<td>13.55</td>
<td>2.98</td>
</tr>
<tr>
<td><strong>Treatment</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>T-1</td>
<td>25.06</td>
<td>24.62</td>
<td>2.57</td>
</tr>
<tr>
<td>T-2</td>
<td>3.81</td>
<td>4.31</td>
<td>2.89</td>
</tr>
</tbody>
</table>

The significance of the differences of the adjusted mean scores was tested by means of a 2(Treatment) by 2(Sex) by 2(I.Q.) analysis of covariance. A summary of the analysis of covariance is presented in Table XVIII.
TABLE XVIII
SUMMARY OF THE ANALYSIS OF COVARIANCE OF THE ADDING AND SUBTRACTING OF FRACTIONS TEST SCORES WITH I.Q., ARITHMETIC COMPUTATION, ARITHMETIC REASONING, AND FRACTION CONCEPTS HELD CONSTANT

<table>
<thead>
<tr>
<th>Source</th>
<th>df</th>
<th>Mean Square</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Treatment</td>
<td>1</td>
<td>4887.84</td>
<td>57.09 *</td>
</tr>
<tr>
<td>Sex</td>
<td>1</td>
<td>1.29</td>
<td>0.01</td>
</tr>
<tr>
<td>I.Q.</td>
<td>1</td>
<td>72.72</td>
<td>0.85</td>
</tr>
<tr>
<td>Treatment x Sex</td>
<td>1</td>
<td>78.93</td>
<td>0.92</td>
</tr>
<tr>
<td>Treatment x I.Q.</td>
<td>1</td>
<td>506.52</td>
<td>5.92 *</td>
</tr>
<tr>
<td>Sex x I.Q.</td>
<td>1</td>
<td>75.57</td>
<td>0.88</td>
</tr>
<tr>
<td>Treatment x Sex x I.Q.</td>
<td>1</td>
<td>3.04</td>
<td>0.04</td>
</tr>
<tr>
<td>Error (within)</td>
<td>46</td>
<td>85.62</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>53</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

* p < .05

From Table XVIII it can be seen that there was a significant difference between the treatment groups (T-1 compared with T-2) in favour of the T-1 Group, as might be expected. It can also be seen that there was a significant difference due to interaction of method with I.Q.. Subjects had been divided into high and low I.Q. groups on the basis of intelligence test scores divided at the median (104.5). The adjusted mean scores of the Adding and Subtracting of
Fractions Test for the high and low intelligence groups within each treatment group are presented in Table XIX.

**TABLE XIX**
ADJUSTED MEAN ADDING AND SUBTRACTING OF FRACTIONS TEST SCORES FOR T-1 AND T-2 GROUPS CLASSIFIED AS HIGH AND LOW I.Q. SCORES DIVIDED AT THE MEDIAN, 104.5.

<table>
<thead>
<tr>
<th>Intelligence Grouping</th>
<th>High</th>
<th>Low</th>
</tr>
</thead>
<tbody>
<tr>
<td>T-1 Group</td>
<td>30.1</td>
<td>19.5</td>
</tr>
<tr>
<td>Treatment</td>
<td></td>
<td></td>
</tr>
<tr>
<td>T-2 Group</td>
<td>3.2</td>
<td>5.7</td>
</tr>
</tbody>
</table>

From Table XIX it can be seen that the difference due to interaction of treatment with I.Q. favoured the low I.Q. group within T-2 Group. However the scores for the T-2 Group were so low that it is not likely that any inference can be drawn from the interaction.
CHAPTER IV

DISCUSSION AND CONCLUSIONS

The problem under discussion in this study was whether Grade 6 students who were taught modular arithmetic would show greater skill in performing the computation of division of fractions and greater understanding of the mathematical basis of the division of fractions than students who were not taught modular arithmetic. The research hypotheses were based on the belief that Grade 6 students who were taught modular arithmetic would show greater skill and understanding.

An analysis of the results of the Division of Fractions Test which was used as a measure of computational skill revealed no support for the first hypothesis, which was that modular arithmetic would improve skill in computation of division of fractions.

An analysis of the results of the Interview Test which was used as a measure of understanding revealed no support for the second hypothesis, which was that modular arithmetic would improve understanding of the mathematical basis of division of fractions.

Discussion

Difference due to sex

Although difference due to sex was not under investigation in this study, the difference between the adjusted
mean scores for boys and girls in the Division of Fractions Test seems worth noting. The adjusted mean score for girls was higher than the adjusted mean score for boys. The difference was significant at the .05 level.

Teaching division of fractions

The small number of rational responses from both groups in the Interview Test may be supporting evidence that the concepts involved in the division of fractions algorithm may be too abstract for children of this age group, thereby supporting Bates' contention. Otherwise the unit on modular arithmetic seems to be unrelated to student's ability to compute division of fractions.

Multiplication of fractions

The analysis of covariance of the Multiplication of Fractions Test revealed a significant difference between the adjusted mean scores of the treatment groups, favouring the T-1 (Review) Group over the T-2 (Modular Arithmetic) Group. No reason can be given at this point for this difference. However, the relationship between multiplication and division of fractions might be a topic for further study.

Adding and subtracting fractions

The significant difference in adding and subtracting fractions between the T-1 and T-2 Groups in favour of the T-1 Group was to be expected since the T-2 Group had had no review of the topic since the previous school year. However, the poor retention of skill in adding and subtracting

1 Thomas Bates, "The Road to Inverse and Multiply", The Arithmetic Teacher, 15:348, April 1968.
fractions on the part of the T-2 Group would seem to indicate that the substitution of an alternative program (modular arithmetic for adding and subtracting fractions) causes students to suffer in skill development. There was a significant interaction between Treatment and I.Q. favouring the low I.Q. group within the T-2 Group. However, the low scores of the T-2 Group in adding and subtracting fractions makes it unlikely that an inference can be drawn from this result as far as the T-2 Group is concerned.

**Instructional time**

A total of forty-three instructional periods were used in this study. This is probably more time than is normally devoted to the review and teaching of fractions in Grade 6. Thus any implications that this study might have for the teaching of division of fractions in Grade 6 might be influenced by the extensive nature of the instruction.

However, not enough time may have been devoted to instruction in modular arithmetic. A greater effect on the T-2 Group might have been observed if more periods of instruction had been given.

**Interview Test**

There is one aspect of the Interview Test that seems to be of sufficient importance to mention here. This experimenter came to the conclusion that knowing the students as well as he did may have been more of a disadvantage than an advantage. While it helped to put students at ease, it also seemed to put the interviewer in the position of sub-consciously
anticipating responses. It might be a better technique for the interviewer to be someone unknown to the student.

Limitations of the study

Several limitations of this study should be noted. The students all came from the same school and the sample size was small. This limitation was offset to some extent by the experimenter being free to create two instructional groups at random from the sample. The conclusions from this study will relate only to other groups with the same characteristics.

The time of day of instruction may have had some effect on the results of the study. The T-2 Group was taught first period in the day while the T-1 Group was taught after a morning recess. It may be that this recreational period served to stimulate the T-1 Group mentally.

Suggestions for further study

It would be of interest to replicate this study with a group of students who had a stronger background of mathematical understanding. The mathematics program which many of these students had followed in their elementary school years did not stress mathematical properties such as associativity, commutativity, and the identity elements. Had the students in the T-2 Group been more familiar with these concepts, modular arithmetic may have been of more help to them, particularly in understanding the concept of the reciprocal.
It is suggested that with further refinement, the interview test technique could reveal useful information about the level of mathematical understanding which children possess in other areas of elementary school mathematics.

The poor retention of adding and subtracting fractions skills on the part of the T-2 Group suggests the need for further studies to determine if substituting modular arithmetic for a review of adding and subtracting fractions does, in fact, inhibit pupil growth in the latter skill.

Conclusions

1. Teaching modular arithmetic to the Grade 6 students participating in this study did not appear to improve their ability to compute division of fractions.

2. Teaching modular arithmetic to the Grade 6 students participating in this study did not appear to improve their understanding of the mathematical basis of the division of fractions.
BIBLIOGRAPHY


APPENDIX A

OUTLINE OF LESSONS FOR REVIEW OF FRACTION CONCEPTS

T-1 AND T-2 GROUPS
LESIONS FOR REVIEW OF FRACTION CONCEPTS

Appropriate practice exercises were provided for these lessons by means of a 24-page booklet of exercises prepared by the experimenter.

Lesson 1:

Objective: To review the concept of fraction as part of a whole and subset of a set.

Lesson: Overhead transparencies of geometric shapes separated into halves, thirds, fourths, fifths, sixths, and eighths. Counters on overhead projector to show subsets of a set. Students to have an opportunity to manipulate devices. Stress equal parts and equal subsets. Vocabulary - numerator, denominator, proper, improper, mixed numeral.

Lesson 2:

Objective: To continue review of concept of fraction.

Lesson: Continue activities of Lesson 1. Use practice exercises from Lesson 1 for discussion.

Lesson 3:

Objective: To review equivalent fractions.

Lesson: Use paper folding to show equivalent fractions which are part of a whole. Use multi-coloured counters on overhead projector to show equivalent fractions which are subsets of a set. Vocabulary - lower terms, higher terms, lowest terms, equivalent.
Review of Fraction Concepts

Lesson 4:

Objective: To review equivalent fractions; building sets of equivalent fractions; changing to lower and higher terms.

Lesson: Practice with folding paper and magnet board using fractional parts. Also illustrate \( \frac{1}{2} \times \frac{2}{2} = \frac{2}{4} \) and \( \frac{2}{4} \div \frac{2}{2} = \frac{1}{2} \).

Lesson 5:

Objective: To review equivalent fractions; writing fractions in lowest terms.

Lesson: Use blocks on overhead projector to illustrate

\[
\begin{array}{c}
\frac{4}{8} \text{ black} \\
\frac{1}{2} \text{ black}
\end{array}
\]

Give similar examples and practice on blackboard.

Lesson 6:

Objective: To review finding the missing term in a pair of equivalent fractions.

Lesson: Learn first to find by inspection what number both numerator and denominator of the complete fraction in a pair of fractions can be multiplied or divided by.

Lesson 7:

Objective: To review equivalent fractions; find if a pair of fractions are equivalent.

Lesson: Use concrete materials (fractional parts) to find if, for example, \( \frac{3}{4} = \frac{6}{8} \). Review Lesson 6 for inspection method.
Review of Fraction Concepts

Lesson 8:

**Objective**: Review cross multiply test to find missing term in a pair of equivalent fractions.

**Lesson**: Illustrate cross multiply test. Give blackboard practice.

Lesson 9:

**Objective**: To review comparison of size of fractions.

**Lesson**: Build several lines of a fraction chart as a class exercise. Show on an overhead transparency how to use the chart to compare size of fractions.

\[
\begin{align*}
\frac{1}{2} & \quad \frac{2}{4} \quad \frac{3}{6} \quad \frac{4}{8} \quad \frac{5}{10} \\
\frac{1}{3} & \quad \frac{2}{6} \quad \frac{3}{9} \quad \frac{4}{12} \quad \frac{5}{15}
\end{align*}
\]

Build charts in the practice booklet.

Lesson 10:

**Objective**: To review finding the denominator common to two or more fractions, using the chart.

**Lesson**: Use an overhead transparency of the fraction chart to show how to locate a denominator common to two or more fractions.

Lesson 11:

**Objective**: To review finding a common denominator for two or more fractions by multiplying denominators.
Review of Fraction Concepts

Lesson 11 (continued):

Lesson: Find a common denominator for $\frac{1}{2}$, $\frac{1}{3}$. Show by an array that $2 \times 3$ gives a product that can be divided by both 2 and 3. Thus the product can be a common denominator for these two fractions.

Lesson 12:

Objective: To review lowest common denominator for two or more fractions.

Lesson: Discuss methods which can be used: a) by inspection - Can the denominator of one fraction be a common denominator for the pair? b) use a chart, c) use the product of denominators.

Lesson 13:

Objective: To review writing a mixed numeral in the form of an improper fraction and vica versa.

Lesson: Give students practice on the magnet board with situations such as
APPENDIX B

OUTLINE OF LESSONS FOR ADDING AND SUBTRACTING FRACTIONS

T-1 GROUP ONLY
LESSONS FOR ADDING AND SUBTRACTING FRACTIONS

Appropriate practice exercises were provided for these lessons by means of a 14-page booklet of exercises prepared by the experimenter.

Lesson 1:

Objective: To review pre-vacation study of fraction concepts.

Lesson: Use an overhead projector and transparencies and a magnet board to review briefly - fractions as a part of a whole and a sub-set of a set; equivalent fractions; missing terms in a pair of equivalent fractions.

Lesson 2:

Objective: Review mixed numerals and improper fractions.

Lesson: Give students an opportunity to perform a number of manipulations on the magnet board, such as:

\[
5 \text{ fourths} \quad 0 \quad \frac{1}{4} \quad 0
\]

and

\[
2\frac{1}{3} \quad 0 
\]

Lesson 3:

Objective: Pupils learn to add and subtract fractions with like denominators.
Lessons For Adding and Subtracting Fractions

Lesson 3 (continued):

Lesson: Use the overhead projector and opaque fractional parts to illustrate the following.

\[
\frac{1}{3} + \frac{1}{3} = \frac{2}{3} \quad \frac{3}{4} - \frac{1}{4} = \frac{2}{4}
\]

Use other similar examples.

Lesson 4:

Objective: To provide practice for pupils in adding and subtracting fractions with like denominators.

Lesson: Have pupils illustrate addition and subtraction with the overhead projector and magnet board with fractional parts. Use questions such as:

\[
\frac{1}{8} + \frac{5}{8} = \quad \frac{2}{3} + \frac{2}{3} = \quad \frac{7}{8} - \frac{3}{8} =
\]

Lesson 5:

Objective: To review finding a common denominator for two or more fractions.


Lesson 6:

Objective: To use the concept of common denominator to add two fractions with unlike denominators.

Lesson: Use the magnet board to illustrate several examples of simple addition such as: \( \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \)

Lesson 7:

Objective: To use the concept of common denominator to subtract two fractions with unlike denominators.
Lessons For Adding and Subtracting Fractions

Lesson 7 (continued):

Lesson: Use the magnet board to illustrate several examples of subtraction such as: \( \frac{3}{4} - \frac{1}{2} = \frac{3}{4} - \frac{2}{4} = \frac{1}{4} \)
and \( \frac{2}{3} - \frac{1}{2} = \frac{4}{6} - \frac{3}{6} = \frac{1}{6} \).

Lesson 8:

Objective: To give students practice in adding and subtracting fractions with unlike denominators.

Lesson: Have pupils illustrate on magnet board with fractional parts how to perform several sample questions.

Lesson 9:

Objective: Pupils learn to add mixed numerals.

Lesson: Review a) finding common denominators, b) rewriting mixed numerals where the fractional part is improper, \( \frac{7}{4} = \frac{3}{4} \). Use the magnet board to demonstrate questions such as: \( \frac{3}{4} + \frac{3}{4} = \frac{6}{4} = \frac{4}{2} = \frac{4}{2} \).

Lesson 10:

Objective: Pupils learn to subtract mixed numerals.

Lesson: Review: a) finding common denominators, b) rewriting mixed numerals in the subtrahend, \( \frac{3}{8} = \frac{11}{8} \).

Illustrate on the magnet board, such as:
\( \frac{5}{8} - \frac{7}{8} = \frac{13}{8} - \frac{7}{8} = \frac{1}{8} = \frac{1}{8} \).

Lesson 11:

Objective: To give pupils practice in adding and subtracting mixed numerals.
Lessons For Adding and Subtracting Fractions

Lesson 11 (continued):

*Lesson*: Have pupils illustrate several examples on the overhead projector and magnet board using fractional parts.

Lesson 12:

*Objective*: Pupils learn to add three fractions with unlike denominators.

*Lesson*: Use the magnet board to illustrate a simple example such as:

\[
\frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{6}{12} + \frac{4}{12} + \frac{3}{12} = \frac{13}{12}
\]
APPENDIX C

OUTLINE OF LESSONS AND PRACTICE EXERCISES
FOR MODULAR ARITHMETIC

T-2 GROUP ONLY
LESSONS IN MODULAR ARITHMETIC

Lesson 1:

Objective: To introduce to students the idea of arithmetic operations on a clock, starting with addition.

Lesson: Introduce clock arithmetic with a problem such as: a) It is 9 o'clock now. In 2 hours time we will have a film. What time will we have a film? $9 + 2 = 11$
b) It is 9 o'clock now. In 6 hours time you will be going home. What time will you go home? $9 + 6 = 3$

Use an overhead transparency of a clock. Show that 12 on the clock acts the same a 0 in ordinary arithmetic when adding. Vocabulary - modular, mod 12.

Equipment: Give students materials necessary to construct a small cardboard clock for computation.

Practice: Exercises 1, 2, 3, and 4 on page 1 of practice booklet. Complete the mod 12 adding table on page 2. Use this table to do exercise 5 on page 3.

Lesson 2:

Objective: To introduce multiplication in clock arithmetic.

Lesson: Introduce multiplication with a problem such as: A truck driver takes 2 hours to deliver a load and return to the warehouse. If he starts at noon and has 3 loads to deliver, what time will he finish? $3 \times 2 = 6$

Equipment: Individual clocks.
Modular Arithmetic

Lesson 2 (continued):

Practice: Exercises 6,7, and 8 on page 3 of practice booklet.

Lesson 3:

Objective: To show 12 as 0 in mod 12 multiplication.
Lesson: Use exercises 7 and 8 on page 3 of the practice booklet as examples. What number in each question in exercise 8 acts the same as 12 in each question in exercise 7? Elicit 0.
Equipment: Make sets of cards to use as aids.
Practice: Fill in the multiplication table on page 4 using the cards as aids. Do exercise 9 on page 5.

Lesson 4:

Objective: To show addition in another module besides 12, namely 7; to show subtraction.
Lesson: Demonstrate the 7-clock on the overhead projector. Use week as an example.
Equipment: Make small individual clocks.
Practice: Complete adding (mod 7) table on page 5 of the practice booklet. Exercises 10,11,12, and 13 on page 6. Project all exercises on the overhead projector for corrections.

Lesson 5:

Objective: To show multiplication in another module besides 12, namely 7.
Modular Arithmetic

Lesson 5 (continued):

Lesson: Use 7-clock transparency to demonstrate. Have students practice with individual clocks. Use cards for practice.

Practice: Fill in multiplication table (mod 7) on page 7. Exercise 14 on page 7, exercises 15, 16, 17, and 18 on page 8 of the practice booklet.

Lesson 6:

Objective: To show addition in another module, namely 5.

Lesson: Use a 5-clock transparency to demonstrate. Give students materials to construct a hand model of 5-clock. Use cards for practice.

Practice: Complete the mod 5 addition table on page 9. Exercises 19 and 20 on page 9.

Lesson 7:

Objective: Review addition in modular arithmetic.

Lesson: Wollensak Teaching Tape R-3501

Practice: Wollensak Worksheet for Clock Arithmetic, R-3501.

Lesson 8:

Objective: Review subtraction in modular arithmetic.

Lesson: Wollensak Teaching Tape R-3502

Practice: Wollensak Worksheet for Clock Arithmetic, R-3502.


Modular Arithmetic

Lesson 9:

**Objective**: Review multiplication in modular arithmetic.

**Lesson**: Wollensak Teaching Tape R-3503.

**Practice**: Wollensak worksheet for Clock Arithmetic, R-3503.

Lesson 10:

**Objective**: To show division in clock arithmetic.

**Lesson**: a) Introduce division on a clock by re-wording the problem in Lesson 2. A truck driver takes 2 hours to deliver a load and return to the warehouse. If he starts work at noon and stops work at 6 p.m., how many loads can he deliver? $6 \div 2 = 3$. Show this on the clock by starting at 6 and moving back 2 hours at a time until arriving at 12.

b) Division on the multiplication table - show that $3 \div 2 = n$ means $3 = 2 \times n$. On the table of mod 5 multiplication, the product 3 can be $2 \times 4$ or $4 \times 2$. Thus, $n = 4$.

**Practice**: Exercise 21 on page 16.

Lesson 11:

**Objective**: To show some properties of multiplication in modular arithmetic and compare them with ordinary arithmetic.

**Lesson**: Use exercises 15, 16, 17, and 18 on page 8 of the practice booklet as examples for discussion. Have students locate a pattern and make a statement about each.

---

Modular Arithmetic

Lesson 11:

Ex. 15 - You notice that when you multiply a number by 1, the answer is the number you multiplied. Is this true in ordinary arithmetic (except for 0)? (yes)

Ex. 16 - Does it matter how you group three numbers when you multiply? Is this true in ordinary arithmetic? (yes)

Ex. 17 - Does it matter in what order you multiply numbers in mod 7? Is this true in ordinary arithmetic? (yes)

Ex. 18 - Is there some other number in the mod 7 system that you can multiply any number (except 0) in the mod 7 system by to get the product 1? (yes) Is this true in ordinary arithmetic? (no)

Practice: No written exercise in this lesson.

Lesson 12:

Objective: To apply properties discussed in Lesson 11 to another module, mod 5.

Lesson: Review the commutative and associative properties of multiplication, the identity element for multiplication, and the reciprocal. (These terms not used with the students; they are not familiar with them.)

Practice: Exercises 22, 23, 24, and 25 on page 17.
PRACTICE EXERCISES

MODULAR ARITHMETIC
1. a) What time is 5 hours after 1 o'clock? _______
b) What time is 4 hours after 3 o'clock? _______
c) What time is 4 hours after 11 o'clock? _______
d) What time is 8 hours after 9 o'clock? _______
e) What time is 11 hours after 4 o'clock? _______

2. Name the following sums on the "12 clock".
   a) 3 + 5 f) 8 + 9
   b) 7 + 2 g) 11 + 5
   c) 9 + 9 h) 10 + 9
   d) 10 + 12 i) 12 + 9
   e) 11 + 11 j) 10 + 11

3. Find the sum on the 12 clock for each question. Write the sum on the line.
   a) 12 + 12 = _______ e) 9 + 12 = _______
   b) 5 + 12 = _______ f) 8 + 12 = _______
   c) 10 + 12 = _______ g) 7 + 12 = _______
   d) 6 + 12 = _______ h) 1 + 12 = _______

4. Answer these questions.
   a) When you add 12 to a number in clock arithmetic, the sum is ________.
   b) In clock arithmetic, 12 is the same as ______ in ordinary arithmetic.
Practice Exercises - Modular Arithmetic

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Practice Exercises - Modular Arithmetic

5. Use your addition table to find the answers to the following questions in mod 12.

a) \( 4 + \_\_\_ = 10 \)  
f) \( \_\_ + 8 = 2 \)

b) \( 9 + \_\_\_ = 11 \)  
g) \( \_\_ + 7 = 3 \)

c) \( 5 + \_\_\_ = 0 \)  
h) \( \_\_ + 5 = 5 \)

d) \( 9 + \_\_\_ = 2 \)  
i) \( \_\_ + 11 = 2 \)

e) \( 4 + \_\_\_ = 1 \)  
j) \( \_\_ + 8 = 6 \)

6. Use your 12 clock to do the following multiplication.

a) \( 6 \times 2 \)  
f) \( 7 \times 9 \)

b) \( 3 \times 3 \)  
g) \( 11 \times 4 \)

c) \( 4 \times 3 \)  
h) \( 7 \times 10 \)

d) \( 7 \times 7 \)  
i) \( 6 \times 10 \)

e) \( 8 \times 6 \)  
j) \( 11 \times 7 \)

7. Find the answer to each question by doing multiplication on the 12 clock.

\[
\begin{array}{ccc}
4 \times 12 = \_\_\_ & \quad & 9 \times 12 = \_\_\_ \\
2 \times 12 = \_\_\_ & \quad & 1 \times 12 = \_\_\_ \\
3 \times 12 = \_\_\_ & \quad & 12 \times 12 = \_\_\_ \\
8 \times 12 = \_\_\_ & \quad & 10 \times 12 = \_\_\_ \\
5 \times 12 = \_\_\_ & \quad & 6 \times 12 = \_\_\_
\end{array}
\]

8. Use ordinary multiplication to find the answer to each question below.

\[
\begin{array}{ccc}
4 \times 0 = \_\_\_ & \quad & 9 \times 0 = \_\_\_ & \quad & 2 \times 0 = \_\_\_ \\
1 \times 0 = \_\_\_ & \quad & 3 \times 0 = \_\_\_ & \quad & 12 \times 0 = \_\_\_ \\
8 \times 0 = \_\_\_ & \quad & 10 \times 0 = \_\_\_ & \quad & 5 \times 0 = \_\_\_
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Practice Exercises - Modular Arithmetic

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Practice Exercises - Modular Arithmetic

9. Use your mod 12 multiplication table to answer these questions.

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a) \(3 \times \_ = 3\)  
f) \(7 \times \_ = 1\)  
b) \(2 \times 4 = \_\)  
g) \(9 \times 5 = \_\)  
c) \(11 \times \_ = 4\)  
h) \(5 \times \_ = 11\)  
d) \(6 \times 7 = \_\)  
i) \(11 \times \_ = 3\)  
e) \(10 \times 11 = \_\)  
j) \(7 \times \_ = 9\)
Practice Exercises - Modular Arithmetic

10. Add these in the mod 7 system.
   a) $3 + 4 = $   e) $2 + 0 = $   i) $5 + 4 = $
   b) $2 + 0 = $   f) $1 + 1 = $   j) $6 + 3 = $
   c) $1 + 6 = $   g) $4 + 4 = $   k) $5 + 5 = $
   d) $6 + 1 = $   h) $0 + 6 = $   l) $6 + 6 = $

11. Find $n$ in each question (mod 7).
   a) $3 + n = 6$   e) $n + 5 = 3$
   b) $5 + n = 2$   f) $4 + n = 0$
   c) $4 + n = 3$   g) $n + 6 = 0$
   d) $n + 6 = 4$   h) $3 + n = 0$

12. Look at the mod 7 addition table and answer the following questions.
   Is $4 + 5$ equal to $5 + 4$ ?
   Is $3 + 4$ equal to $4 + 3$ ?
   Is $5 + 6$ equal to $6 + 5$ ?
   Is $2 + 4$ equal to $4 + 2$ ?
   Is $5 + 0$ equal to $0 + 5$ ?
   Is $5 + 3$ equal to $3 + 5$ ?

13. Use your mod 7 addition table to find answers to these questions.
   a) $6 - 2 = $   f) $4 - 6 = $   k) $0 - 4 = $
   b) $5 - 3 = $   g) $5 - 6 = $   l) $2 - 3 = $
   c) $4 - 1 = $   h) $1 - 5 = $   m) $3 - 5 = $
   d) $6 - 0 = $   i) $3 - 6 = $   n) $0 - 5 = $
   e) $4 - 3 = $   j) $4 - 2 = $   o) $0 - 6 = $
14. Use your mod 7 multiplication table to find the answers to these questions.

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- $3 \times 4 = 6 \times 5$
- $4 \times 5 = 2 \times 3$
- $3 \times 6 = 4 \times 4$
- $6 \times 5 = 2 \times 6$
Practice Exercises - Modular Arithmetic

15. Use your mod 7 multiplication table to find the answer to each question.

\[
\begin{align*}
3 \times 1 &= \_ \_ \\
6 \times 1 &= \_ \_ \\
4 \times 1 &= \_ \_ \\
5 \times 1 &= \_ \_ \\
1 \times 1 &= \_ \_ \\
2 \times 1 &= \_ \_ \\
\end{align*}
\]

16. Use your table, or your clock, or your cards to find the answers to these questions in mod 7.

\[
\begin{align*}
(3 \times 4) \times 2 &= \_ \_ \\
3 \times (4 \times 2) &= \_ \_ \\
(4 \times 5) \times 2 &= \_ \_ \\
4 \times (5 \times 2) &= \_ \_ \\
(3 \times 6) \times 4 &= \_ \_ \\
3 \times (6 \times 4) &= \_ \_ \\
(4 \times 3) \times 5 &= \_ \_ \\
4 \times (3 \times 5) &= \_ \_ \\
(6 \times 5) \times 4 &= \_ \_ \\
6 \times (5 \times 4) &= \_ \_ \\
(2 \times 5) \times 6 &= \_ \_ \\
2 \times (5 \times 6) &= \_ \_ \\
\end{align*}
\]

17. Use your mod 7 multiplication table to answer these questions.

Is \(3 \times 4\) equal to \(4 \times 3\) ?
Is \(5 \times 6\) equal to \(6 \times 5\) ?
Is \(3 \times 6\) equal to \(6 \times 3\) ?
Is \(2 \times 4\) equal to \(4 \times 2\) ?
Is \(4 \times 5\) equal to \(5 \times 4\) ?

18. Use your mod 7 multiplication table to answer these questions.

\[
\begin{align*}
1 \times \_ \_ &= 1 \\
4 \times \_ \_ &= 1 \\
2 \times \_ \_ &= 1 \\
5 \times \_ \_ &= 1 \\
3 \times \_ \_ &= 1 \\
6 \times \_ \_ &= 1 \\
\end{align*}
\]
NAME _________________________________

A. _________________________________

B. 4 + 2 = 6 (mod 8) or 4 + 2 = 6 (mod 8)

C. 4 + 6 = 2 (mod 8) or 4 + 6 = 2 (mod 8)

D. 3 + 3 + 2 = ______ (mod 8)

E. 5 + 4 + 3 = ______ (mod 8)

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CLOCK ARITHMETIC: ADDITION R-3501

G. $3 + 4 = \underline{\hspace{2cm}}$ (mod 5)

H. $4 + 4 + 3 = \underline{\hspace{2cm}}$ (mod 5)

1. $3 + 3 = \underline{\hspace{2cm}}$ (mod 4)

2. $3 + 3 = \underline{\hspace{2cm}}$ (mod 5)

3. $3 + 3 = \underline{\hspace{2cm}}$ (mod 6)

4. $4 + \underline{\hspace{2cm}} = 7$ (mod 8)

5. $4 + \underline{\hspace{2cm}} = 0$ (mod 8)

6. $3 + \underline{\hspace{2cm}} = 2$ (mod 8)

7. $5 + \underline{\hspace{2cm}} = 3$ (mod 8)

8. $7 + \underline{\hspace{2cm}} = 4$ (mod 8)

9. $2 + 6 + \underline{\hspace{2cm}} = 7$ (mod 8)

10. $4 + 2 = \underline{\hspace{2cm}}$ (mod 5)

11. $3 + 5 = \underline{\hspace{2cm}}$ (mod 6)

12. $8 + 7 = \underline{\hspace{2cm}}$ (mod 9)

13. $3 + 2 = \underline{\hspace{2cm}}$ (mod 4)

14. $2 + \underline{\hspace{2cm}} = 0$ (mod 7)

15. \underline{\hspace{2cm}} + 4 = 3 (mod 5)

16. \underline{\hspace{2cm}} + \underline{\hspace{2cm}} = 2$ (mod 6)

17. \underline{\hspace{2cm}} + \underline{\hspace{2cm}} = \underline{\hspace{2cm}}$ (mod 5)
# WOLLENSAK TEACHING TAPE WORKSHEET

## CLOCK ARITHMETIC: SUBTRACTION R-3502

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### A. [Diagram of a clock with numbers 0 to 7]

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### B. 4-2 = 2 (mod 8) or 4-2 = 2 (mod 8)

### C. 4-6 = 6 (mod 8) or 4-6 = 6 (mod 8)

### D. 5-3 = ___ (mod 8)

### G. 6-4 = ___ (mod 8)

### E. 5-6 = ___ (mod 8)

### H. 2-2 = ___ (mod 8)

### F. 0-2 = ___ (mod 8)

### I. 2-5 = ___ (mod 8)

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CLOCK ARITHMETIC: SUBTRACTION R-3502

J.

K. 3 - 2 = ____ (mod 4)

L. 2 - 3 = ____ (mod 4)

1. 0 - 3 = ____ (mod 4) 4. 6 - ____ = 4 (mod 8)
2. 1 - 2 = ____ (mod 4) 5. 6 - ____ = 7 (mod 8)
3. 3 - 3 = ____ (mod 4) 6. 2 - ____ = 3 (mod 4)

7. 5 - ____ = 3 (mod 8) 9. ____ - 1 = 3 (mod 4)
8. 0 - ____ = 1 (mod 4) 10. ____ - 7 = 4 (mod 8)

11. ____ - 2 = 3 (mod 4) 12. ____ - 6 = 5 (mod 8)

SUPPLEMENTARY PROBLEMS

13. 3 - 0 = ____ (mod 5) 18. ____ - 4 = 1 (mod 5)
14. 0 - 4 = ____ (mod 5) 19. ____ - 3 = 5 (mod 6)
15. 1 - 3 = ____ (mod 4) 20. ____ - 4 = 4 (mod 8)
16. 3 - 4 = ____ (mod 6) 21. 3 - ____ = 1 (mod 4)
17. 4 - 5 = ____ (mod 6) 22. 1 - ____ = 3 (mod 4)
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CLOCK ARITHMETIC: MULTIPLICATION R-3503

Name ________________________________

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CLOCK ARITHMETIC: MULTIPLICATION R-3503

B. \(7 \times 6 = \underline{\text{}}\, (\text{mod 8})\)  
D. \(3 \times 5 = \underline{\text{}}\, (\text{mod 8})\)  
C. \(5 \times 3 = \underline{\text{}}\, (\text{mod 8})\)  
E. \(6 \times 4 = \underline{\text{}}\, (\text{mod 8})\)

F. \(4 \times 5 = \underline{\text{}}\, (\text{mod 3})\)  
I. \(2 \times 8 = \underline{\text{}}\, (\text{mod 4})\)  
G. \(4 \times 5 = \underline{\text{}}\, (\text{mod 4})\)  
J. \(5 \times 3 = \underline{\text{}}\, (\text{mod 6})\)  
H. \(4 \times 5 = \underline{\text{}}\, (\text{mod 15})\)  
K. \(8 \times 3 = \underline{\text{}}\, (\text{mod 7})\)

SUPPLEMENTARY PROBLEMS

1. \(8 \times 5 = \underline{\text{}}\, (\text{mod 6})\)  

2. \(7 \times 6 = \underline{\text{}}\, (\text{mod 9})\)  

3. \(9 \times 3 = \underline{\text{}}\, (\text{mod 12})\)  

4. \(4 \times 7 = \underline{\text{}}\, (\text{mod 11})\)  

5. \(3 \times 6 = \underline{\text{}}\, (\text{mod 7})\)  

6. \(8 \times 7 = \underline{\text{}}\, (\text{mod 9})\)  

7. \(2 \times 3 = \underline{\text{}}\, (\text{mod 4})\)
**Practice Exercises - Modular Arithmetic**

19. Use your mod 5 addition table to answer these questions.

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3 + 2 = ____  
2 + 3 = ____  
4 + 3 = ____  
3 + 4 = ____  
1 + 3 = ____  
3 + 1 = ____  
4 + 1 = ____  
1 + 4 = ____

20. Use the mod 5 addition table to find the answers to these questions.

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4 - 2 = ____  
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0 - 4 = ____
Practice Exercises - Modular Arithmetic

21. Use your mod 7 multiplication table to answer these questions.

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6 \div 3 = N \quad 4 \div 5 = N
4 \div 2 = N \quad 3 \div 6 = N
5 \div 1 = N \quad 4 \div 6 = N
6 \div 4 = N \quad 3 \div 2 = N
2 \div 3 = N \quad 6 \div 5 = N
Practice Exercises - Modular Arithmetic

22. Use your mod 5 multiplication table to answer these questions.
   \[ 1 \times \_ = 1 \quad 2 \times \_ = 2 \quad 3 \times \_ = 3 \]

23. Use your mod 5 multiplication table.
   \[ (2 \times 3) \times 4 = \_ \quad 2 \times (3 \times 4) = \_ \]
   \[ (3 \times 4) \times 1 = \_ \quad 3 \times (4 \times 1) = \_ \]
   \[ (0 \times 3) \times 4 = \_ \quad 0 \times (3 \times 4) = \_ \]
   \[ (1 \times 3) \times 2 = \_ \quad 1 \times (3 \times 2) = \_ \]

24. Use your mod 5 multiplication table.
   \[ 3 \times 4 = \_ \quad 4 \times 3 = \_ \]
   \[ 2 \times 3 = \_ \quad 3 \times 2 = \_ \]
   \[ 4 \times 1 = \_ \quad 1 \times 4 = \_ \]
   \[ 0 \times 2 = \_ \quad 2 \times 0 = \_ \]

25. Use your mod 5 multiplication table to answer these questions.
   \[ 4 \div 2 = N \quad 3 \div 2 = N \]
   \[ 3 \div 1 = N \quad 1 \div 3 = N \]
   \[ 4 \div 3 = N \quad 3 \div 3 = N \]
   \[ 2 \div 4 = N \quad 2 \div 3 = N \]
   \[ 1 \div 4 = N \quad 4 \div 4 = N \]
APPENDIX D

OUTLINE OF LESSONS FOR MULTIPLICATION OF FRACTIONS

OUTLINE OF LESSONS AND SELECTED PRACTICE PAGES FOR
DIVISION OF FRACTIONS

T-1 AND T-2 GROUPS
LESSONS FOR MULTIPLICATION OF FRACTIONS

Appropriate practice exercises were provided for these lessons by means of a 12-page booklet of exercises prepared by the experimenter.

Lesson 1:

Objective: To introduce multiplication of fractions using regions as models (unit numerators only).

Lesson: Use overhead transparencies with overlays to explain samples in practice booklet. Example:

\[
\frac{1}{3} \times \frac{1}{4} = \frac{1}{12}
\]

Lesson 2:

Objective: To illustrate how a number line can be used as a model for multiplication of fractions.

Lesson: Use the overhead transparency of a number line to illustrate an example such as: \(5 \times \frac{1}{4}\)

Lesson 3:

Objective: To illustrate multiplication of fractions using fractional parts and using regions as models, (numerators greater than 1)

Lesson: Use a measuring cup and liquid to illustrate the example in the practice booklet; use a magnet board to illustrate examples such as \(5 \times \frac{2}{3}\). Use an overhead transparency with an overlay to illustrate \(\frac{1}{3} \times \frac{3}{4}\) using regions.
Lessons For Multiplication of Fractions

Lesson 4:

Objective: To further illustrate multiplication of a fraction by a whole number.

Lesson: Use an overhead transparency to illustrate \( \frac{1}{3} \) of 4 and \( \frac{3}{4} \) of 3. Show that, for example, \( \frac{1}{2} \) of 6 can be 6 halves or 3 wholes.

Lesson 5:

Objective: To relate multiplication of fractions to finding area of a rectangle.

Lesson: Review finding the area of a rectangle, using whole numbers. Re-name whole numbers with fractional names.

Lesson 6:

Objective: To show that multiplication of fractions is commutative and associative. (These terms not used.)

Lesson: Give pairs of questions such as:
\[
\frac{1}{4} \times \frac{1}{3} = \_\_\_\_ \quad \frac{1}{3} \times \frac{1}{4} = \_\_\_\_ \quad \text{and}
\]
\[
\frac{1}{2} \times \left( \frac{1}{3} \times \frac{1}{4} \right) = \_\_\_\_ \quad \left( \frac{1}{2} \times \frac{1}{3} \right) \times \frac{1}{4} = \_\_\_\_
\]
and have students compare the answers in each pair.

Lesson 7:

Objective: To show students a short cut in multiplying fractions.

Lesson: Illustrate that in \( \frac{2}{3} \times \frac{3}{5} \), the denominator 3 in \( \frac{2}{3} \) and the numerator 3 in \( \frac{3}{5} \) can each be divided by 3 before multiplication takes place.
Lessons For Multiplication of Fractions

Lesson 8:

**Objective**: To give students practice in multiplying fractions.

**Lesson**: Review several examples briefly and illustrate with concrete aids.

Lesson 9:

**Objective**: To illustrate multiplication of mixed numerals.

**Lesson**: Review re-writing mixed numerals as improper fractions. Stress that this must be done before multiplying two mixed numerals. (At least, at this introductory stage.)

Lesson 10:

**Objective**: To illustrate the reciprocal.

**Lesson**: Use examples such as \( \frac{1}{4} \times \frac{4}{4} = 1 \) and have students supply the missing numerator and denominator. Review that a whole number, such as 6, can be written \( \frac{6}{1} \) and that the reciprocal of 6 is \( \frac{1}{6} \).
LESSONS FOR DIVISION OF FRACTIONS

Lesson 1:

Objective: To illustrate the division of groups and of objects.

Lesson: Use overhead transparencies and a magnet board with fractional parts to illustrate examples A to I on pages 1 and 2 of the practice booklet. Have students cut and fold strip of paper from Exercise 3 on page 3 to find the answer.

Lesson 2:

Objective: To illustrate the inverse relationship of multiplication and division.

Lesson: Use examples similar to the questions on page 4 of the practice booklet, starting with whole number examples. Show that if $7 \times 5 = 35$, then $35 \div 5 = 7$ and $35 \div 7 = 5$; similarly, if $\frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$, then $\frac{1}{12} \div \frac{1}{3} = \frac{1}{4}$ and $\frac{1}{12} \div \frac{1}{4} = \frac{1}{3}$.

Lesson 3:

Objective: To review the reciprocal, the commutative and associative properties of multiplication, and the identity element for multiplication. (These terms are not used with the students.)

Lesson: Review examples of each property and illustrate. Give the exercises on page 5 and 6 in the booklet.
Lessons For Division of Fractions

Lesson 4:

Objective: To teach the division of fractions algorithm.

Lesson: Explain to the students that they should look for mathematical properties within the algorithm that they have reviewed. Use an overhead transparency to colour code the reciprocal throughout the algorithm. Remind students that mixed numerals must be written as improper fractions.

Lesson 5:

Objective: To review division of fractions.

Lesson: Review the algorithm completely. Stress that mixed numerals must be written in the form of improper fractions. Review the reciprocal of whole numbers.

Lesson 6:

Objective: To further review division of fractions.


Lesson 7:

Objective: To further review division of fractions.

A. In numerals: 6 + 3 = □
In words: How many 3's in 6?
Answer: 2

B. In numerals: \( \frac{1}{4} \div \frac{1}{8} = □ \)
In words: How many \( \frac{1}{8} \)'s in \( \frac{1}{4} \)?
Answer: 

C. In numerals: 1 ÷ \( \frac{1}{3} \) = □
In words: How many \( \frac{1}{3} \)'s in 1?
Answer: 

D. In numerals: \( 1\frac{1}{2} ÷ \frac{1}{2} = □ \)
In words: How many \( \frac{1}{2} \)'s in \( 1\frac{1}{2} \)?
Answer: 

E. In numerals: 3 ÷ \( 1\frac{1}{2} \) = □
In words: How many \( 1\frac{1}{2} \)'s in 3?
Answer: 

F. In numerals: \( \frac{3}{4} ÷ \frac{1}{8} = □ \)
In words: How many \( \frac{1}{8} \)'s in \( \frac{3}{4} \)?
Answer: 

Those were easy. Try these.

G. In numerals: $4 \div \frac{2}{3} = \square$
In words: How many $\frac{2}{3}$'s in 4?
Answer: \[
\begin{array}{c}
\frac{2}{3} \\
\frac{2}{3} \\
\frac{2}{3} \\
4
\end{array}
\]

H. In numerals: $4\frac{1}{2} \div 3 = \square$
In words: How many 3's in $4\frac{1}{2}$?
Answer: \[
\begin{array}{c}
4\frac{1}{2}
\end{array}
\]

I. In numerals: $\frac{1}{8} \div \frac{1}{4} = \square$
In words: How many $\frac{1}{4}$'s in $\frac{1}{8}$?
or What part of $\frac{1}{4}$ is $\frac{1}{8}$?
Answer: \[
\begin{array}{c}
\frac{1}{8}
\end{array}
\]
This one is harder to picture.

J. In numerals: \( \frac{5}{8} \div \frac{5}{6} = \square \)

In words: How many \( \frac{5}{6} \) in \( \frac{5}{8} \)?

Or What part of \( \frac{5}{6} \) is \( \frac{5}{8} \)?

Cut out this portion.
Fold in two lengthwise.
Fold in two again.

The double shaded part is what fraction of the shaded part?

\( \frac{5}{8} \div \frac{5}{6} = \)

Look at these equations. Can you figure out a way of dividing two fractions to get the correct answer?

\[
\begin{align*}
\frac{1}{4} \div \frac{1}{8} &= 2 \\
1\frac{1}{2} \div \frac{1}{2} &= 3 \\
\frac{3}{4} \div \frac{1}{8} &= 6 \\
4\frac{1}{2} \div 3 &= 1\frac{1}{2} \\
1 \div \frac{1}{3} &= 3 \\
3 \div 1\frac{1}{2} &= 2 \\
4 \div \frac{2}{3} &= 6 \\
\frac{1}{8} \div \frac{1}{4} &= \frac{1}{2}
\end{align*}
\]
<table>
<thead>
<tr>
<th>Example</th>
<th>$7 \times 5 = 35$</th>
<th>$35 \div 5 = 7$</th>
<th>$35 \div 7 = 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Example</td>
<td>$\frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$</td>
<td>$\frac{1}{12} \div \frac{1}{3} = \frac{1}{4}$</td>
<td>$\frac{1}{12} \div \frac{1}{4} = \frac{1}{3}$</td>
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<td></td>
<td>$4 \times \frac{1}{5} = \frac{4}{5} \div 4 = -$</td>
<td>$\frac{4}{5} \div \frac{1}{5} = 6$</td>
<td>$54 \div 6 = 9$</td>
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<tr>
<td></td>
<td>$9 \times = 54$</td>
<td>$54 \div 6 = 9$</td>
<td>$54 \div 6 = 6$</td>
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<tr>
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<td>$2 \times 1\frac{1}{2} = 3 \div 2 = 1\frac{1}{2}$</td>
<td>$3 \div 1\frac{1}{2} = 2$</td>
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<td>$\frac{7}{9} \times \frac{3}{5} = \frac{7}{15} \div \frac{3}{5} = \frac{7}{15} \div \frac{3}{5} = \frac{7}{8}$</td>
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<td>$\frac{5}{8} \times 3 = \frac{5}{8} \times 3 = \frac{5}{8} \times 3 = \frac{5}{8}$</td>
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<td>$1\frac{1}{4} \times \frac{3}{4} = \frac{15}{16} \div 1\frac{1}{4} = \frac{15}{16} \div 1\frac{1}{4} = 3\frac{1}{4}$</td>
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<tr>
<td></td>
<td>$100 \times \frac{2}{100} = 7 \div 100 = 7 \div 100 = 7 \div \frac{2}{100} = 7$</td>
<td>$7 \div 100 = 7$</td>
<td>$7 \div \frac{2}{100} = 7$</td>
</tr>
</tbody>
</table>
A.
\[
\begin{align*}
\frac{2}{3} \times \frac{3}{2} &= \frac{6}{6} = 1 \\
\frac{4}{5} \times \frac{5}{4} &= 1 \\
\frac{9}{12} \times \frac{12}{9} &= 1 \\
\frac{3}{8} \times \frac{8}{3} &= 1 \\
\frac{9}{16} \times \frac{16}{9} &= 1 \\
\frac{25}{2} \times \frac{2}{25} &= 1 \\
\frac{5}{8} \times \frac{40}{40} &= 1 \\
\frac{13}{5} \times \frac{65}{65} &= 1 \\
\frac{7}{10} \times \frac{10}{7} &= 1 \\
\frac{9}{16} \times \frac{16}{9} &= 1 \\
\frac{37}{4} \times \frac{4}{37} &= 1 \\
\frac{17}{3} \times \frac{3}{17} &= 1 \\
\frac{9}{13} \times \frac{13}{9} &= 1 \\
8 \times \frac{1}{8} &= 1 \\
\frac{7}{8} \times \frac{8}{7} &= 1 \\
2 \frac{1}{2} \times \frac{2}{5} &= 1 \\
1 \frac{3}{4} \times \frac{4}{3} &= 1 \\
8 \frac{1}{2} \times \frac{2}{8} &= 1 \\
1 \frac{1}{2} \times \frac{1}{1} &= 1 \\
5 \frac{1}{4} \times \frac{4}{5} &= 1 \\
\frac{1}{10} \times 50 &= 1 \\
50 \times \frac{1}{10} &= 1
\end{align*}
\]
A. Example

$$\left( \frac{1}{4} \times \frac{1}{3} \right) \times \frac{1}{2} = n$$
$$\frac{1}{12} \times \frac{1}{2} = \frac{1}{24}$$

$$\left( \frac{3}{5} \times \frac{2}{3} \right) \times \frac{1}{5} = n$$
$$\frac{3}{5} \times \frac{2}{3} \times \frac{1}{5} = n$$

$$\frac{3}{5} \times \frac{1}{5} = \frac{3}{25}$$

$$\left( \frac{5}{6} \times \frac{2}{7} \right) \times \frac{1}{2} = n$$
$$\frac{5}{6} \times \frac{2}{7} \times \frac{1}{2} = n$$

$$\frac{5}{6} \times \frac{1}{2} = \frac{5}{12}$$

$$\left( \frac{3}{4} \times \frac{1}{5} \right) \times \frac{2}{3} = n$$
$$\frac{3}{4} \times \frac{1}{5} \times \frac{2}{3} = n$$

$$\frac{3}{4} \times \frac{1}{5} = \frac{3}{20}$$

B.

$$9 \times 1 = 9$$
$$12 \times 1 = 12$$
$$100 \times 1 = 100$$

$$\frac{1}{2} \times 1 = \frac{1}{2}$$
$$2\frac{1}{2} \times 1 = 2\frac{1}{2}$$
$$65 \times 1 = 65$$

C.

$$1 = \frac{6}{6}$$
$$1 = \frac{12}{12}$$
$$1 = \frac{40}{40}$$
$$1 = \frac{100}{100}$$

$$1 = \frac{3}{3}$$
$$1 = \frac{16}{16}$$
$$1 = \frac{8}{8}$$
$$1 = \frac{7}{7}$$

D.

$$9 \times \frac{6}{6} = 9$$
$$12 \times \frac{8}{7} = 12\frac{8}{7}$$
$$100 \times \frac{12}{25} = 100\frac{12}{25}$$

$$\frac{1}{2} \times \frac{10}{10} = \frac{10}{25}$$
$$2\frac{1}{2} \times \frac{25}{25} = 2\frac{25}{25}$$
$$65 \times \frac{99}{99} = 65\frac{99}{99}$$
APPENDIX E

TESTS CONSTRUCTED FOR AND USED IN THIS STUDY

1. FRACTION CONCEPTS
2. ADDING AND SUBTRACTING FRACTIONS
3. MODULAR ARITHMETIC
4. MULTIPLICATION OF FRACTIONS
5. DIVISION OF FRACTIONS
6. INTERVIEW TEST
FRACTION CONCEPTS TEST

1. a) What fraction of A is shaded?  
   b) What fraction of A is not shaded?  
   c) What fraction of the objects is shaded?  
   d) What fraction of the objects is not shaded?  
   e) What fraction of C is shaded?  
   f) What fraction of C is not shaded?  
   g) What fraction of the objects is shaded?  
   h) What fraction of the objects is not shaded?  

2. a) Is \( \frac{1}{5} \) of this object shaded?  
   b) Are \( \frac{3}{8} \) of these objects shaded?  

3. a) Circle the proper fractions.  
   \[ \frac{3}{8}, \frac{16}{5}, \frac{7}{7}, \frac{9}{10}, \frac{14}{18} \]  
   b) Circle the improper fractions.  
   \[ \frac{9}{7}, \frac{10}{10}, \frac{2}{1}, \frac{6}{12}, \frac{99}{100} \]  
   c) Write a mixed numeral.  

4. Look at the fraction chart and write a fraction telling what part of each bar is shaded.

5. For each fraction, write an equivalent fraction in lowest terms.

\[
\frac{12}{16} = \quad \frac{21}{24} = \quad \frac{42}{14} = \quad \frac{9}{21} = \quad \frac{75}{100} =
\]

6. Circle the fractions that are equivalent to \( \frac{3}{8} \).

\[
\frac{9}{24} \quad \frac{13}{18} \quad \frac{15}{40} \quad \frac{8}{3} \quad \frac{39}{104}
\]

7. Circle the numbers which can be common denominators for \( \frac{2}{3} \) and \( \frac{3}{4} \).

\[
8 \quad 12 \quad 18 \quad 24 \quad 36
\]
Fraction Concepts Test

8. Show how to use the cross multiply test to find if each pair of fractions below is an equivalent pair.
   a) \( \frac{6}{9}, \frac{8}{12} \)   yes ___ no ___ (check one)
   b) \( \frac{7}{8}, \frac{6}{7} \)   yes ___ no ___ (check one)

9. Find a common denominator for each set of fractions.
   \( \frac{3}{7}, \frac{3}{4} \)   \( \frac{1}{2}, \frac{2}{3}, \frac{3}{4} \)   \( \frac{3}{50}, \frac{8}{75} \)

10. Find the lowest common denominator for each set of fractions.
    \( \frac{7}{12}, \frac{1}{3} \)   \( \frac{1}{5}, \frac{2}{3}, \frac{5}{6} \)

11. Change each improper fraction to a mixed numeral. Change each mixed numeral to an improper fraction.
    \( \frac{19}{3} = \)   \( \frac{72}{5} = \)   \( \frac{23}{16} = \)   \( \frac{97}{5} = \)

12. Put the correct sign between each pair of fractions.
    Use the signs = , < , >
    \( \frac{2}{3}, \frac{4}{6}, \frac{9}{10}, \frac{7}{8}, \frac{7}{2}, \frac{41}{2}, \frac{1}{16}, \frac{1}{12} \)
ADDING AND SUBTRACTING FRACTIONS TEST

Part I

A. \( \frac{2}{7} + \frac{3}{7} = \)

B. \( \frac{5}{6} + \frac{5}{6} = \)

C. \( \frac{1}{12} + \frac{1}{3} = \)

D. \( \frac{15}{16} + \frac{7}{8} = \)

E. \( \frac{3}{7} + \frac{2}{3} = \)

F. \( \frac{3}{4} + \frac{7}{10} = \)

G. \( \frac{3}{8} + 9\frac{7}{8} = \)

H. \( \frac{11}{18} + 11\frac{1}{9} = \)

I. \( 2\frac{9}{16} + \frac{7}{8} = \)

J. \( \frac{11}{15} + 5\frac{9}{10} = \)

K. \( 21\frac{5}{6} + 13\frac{1}{6} = \)

L. \( 63\frac{3}{16} + 11\frac{5}{8} = \)

M. \( 6\frac{1}{2} + 1\frac{7}{8} = \)

N. \( 3\frac{1}{7} + 8\frac{1}{2} = \)

O. \( 9\frac{5}{12} + 2\frac{7}{8} = \)

P. \( \frac{3}{4} + \frac{3}{4} + \frac{3}{4} = \)

Q. \( \frac{19}{20} + \frac{3}{5} + \frac{1}{4} = \)

R. \( \frac{3}{16} + \frac{1}{8} + \frac{3}{8} = \)
Adding and Subtracting Fractions Test

S. \( \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \)  
T. \( 4\frac{1}{4} + \frac{7}{12} + \frac{5}{8} = \)

U. \( 40 + \frac{11}{16} = \)  
V. \( 18\frac{5}{9} + 6 = \)

W. \( \frac{2}{5} + \underline{7\frac{2}{5}} \)  
X. \( \frac{6}{5} + \underline{\frac{1}{3}} \)

Y. \( \frac{2}{9} + \underline{\frac{5}{9}} + \frac{1}{9} \)  
Z. \( 16\frac{3}{10} + \underline{\frac{1}{10}} \)

Part II

A. \( \frac{8}{9} - \frac{7}{9} = \)  
B. \( \frac{7}{8} - \frac{1}{8} = \)

C. \( \frac{15}{16} - \frac{1}{4} = \)  
D. \( \frac{19}{20} - \frac{1}{5} = \)

E. \( \frac{7}{10} - \frac{1}{4} = \)  
F. \( \frac{11}{16} - \frac{1}{12} = \)

G. \( 20\frac{1}{3} - \frac{2}{3} = \)  
H. \( 33\frac{1}{3} - \frac{5}{6} = \)

I. \( 15\frac{11}{12} - \frac{7}{8} = \)  
J. \( 2\frac{1}{5} - 1\frac{1}{4} = \)
Adding and Subtracting Fractions Test

K. \(13\frac{7}{8} - 4\frac{7}{8} = \)

L. \(8\frac{9}{16} - 8\frac{1}{16} = \)

M. \(16\frac{7}{10} - 3\frac{1}{2} = \)

N. \(20\frac{1}{12} - 19\frac{1}{3} = \)

O. \(20\frac{4}{5} - 16\frac{1}{3} = \)

P. \(12\frac{1}{3} - 3\frac{3}{5} = \)

Q. \(10\frac{7}{12} - 8 = \)

R. \(15 - 7\frac{5}{6} = \)

S. \(8 - \frac{9}{20} = \)

T. \(7\frac{1}{8} - \frac{5}{8} = \)

U. \(15\frac{11}{15} - \frac{4}{15} = \)

V. \(\frac{9}{9} - \frac{2}{9} = \)

W. \(9\frac{2}{3} - 4\frac{3}{5} = \)

X. \(\frac{15}{16} - \frac{19}{24} = \)
MODULAR ARITHMETIC

TEST

1. Use your mod 7 clock to find the answers to these questions. Watch the signs.
   \[3 + 3 = \quad 0 + 4 = \quad 5 + 6 = \]
   \[6 - 4 = \quad 0 - 3 = \quad 2 - 6 = \]
   \[2 \times 3 = \quad 4 \times 4 = \quad 6 \times 5 = \]

2. True or False?
   a) \[3 \times 1 = 3 \text{ (mod 5)}\]
   b) \[4 \times 5 = 5 \times 4 \text{ (mod 8)}\]
   c) \[2 \times 0 = 0 \text{ (mod 12)}\]
   d) \[5 \times 6 = 3 \times 3 \text{ (mod 7)}\]
   e) \[4 \times 2 = 8 \text{ (mod 7)}\]
   f) \[(2 \times 3) \times 4 = 2 \times (3 \times 4) \text{ (mod 5)}\]
   g) \[4 \times 2 = 1 \text{ (ordinary arithmetic)}\]
   h) \[11 \times 3 = 3 \times 11 \text{ (mod 12)}\]
   i) \[6 \times 9 = 9 \times 6 \text{ (ordinary arithmetic)}\]
   j) \[7 \times 7 = 1 \text{ (mod 8)}\]

3. a) In ordinary arithmetic, what is the answer when you multiply a number by 0 ?

   b) In mod 5 arithmetic, what is the answer when you multiply a number by 0 ?

   c) In ordinary arithmetic, what is the answer when you multiply a number by 1 ?
Modular Arithmetic Test

3.  d) In mod 5 arithmetic, what is the answer when you multiply a number by 1 ? __________

e) In ordinary arithmetic, can you multiply any whole number (except 1) by another whole number and get the product 1 ? ______

f) In mod 5 arithmetic, can you multiply every number (except 0) by some other whole number to get the product "1" ? ______

Prove your answer to (f) by trying to find the missing numbers in each question below.

\[
\begin{align*}
1 \times ____ &= 1 \pmod{5} & 3 \times ____ &= 1 \pmod{5} \\
2 \times ____ &= 1 \pmod{5} & 4 \times ____ &= 1 \pmod{5}
\end{align*}
\]

4. Use your cards to find the missing numbers.

a) \(2 \times 5 = ____ \pmod{6}\) \hspace{1cm} g) \(6 + 4 + 5 = ____ \pmod{8}\)

b) \(3 + 4 + 3 = ____ \pmod{5}\) \hspace{1cm} h) \(2 - 4 = ____ \pmod{5}\)

c) \(9 - 6 = ____ \pmod{12}\) \hspace{1cm} i) \(7 \times 0 = ____ \pmod{11}\)

d) \(11 + 11 = ____ \pmod{12}\) \hspace{1cm} j) \(4 \times ____ = 2 \pmod{5}\)

e) \(4 \times 6 = ____ \pmod{8}\) \hspace{1cm} k) \(6 + ____ = 0 \pmod{12}\)

f) \(5 \times 5 = ____ \pmod{6}\)

5. Find the missing numbers.

\[
\begin{align*}
4 \times 3 &= ____ \pmod{5} & 3 \times 2 &= ____ \pmod{5} & 4 \times 5 &= ____ \pmod{7} \\
2 + 4 &= ____ \pmod{5} & 1 \div 3 &= ____ \pmod{5} & 6 \div 4 &= ____ \pmod{7} \\
2 \div 3 &= ____ \pmod{5} & 1 \div 2 &= ____ \pmod{5} & 6 \div 5 &= ____ \pmod{7}
\end{align*}
\]
MULTIPLICATION OF FRACTIONS

TEST

1. \[ \frac{1}{4} \times \frac{3}{5} = \]

2. \[ \frac{3}{4} \times \frac{2}{3} = \]

3. \[ \frac{3}{16} \times 3 = \]

4. \[ \frac{1}{16} \times 4 = \]

5. \[ \frac{2}{7} \times 14 = \]

6. \[ \frac{2}{3} \times 8 = \]

7. \[ \frac{3}{4} \times 10 = \]

8. \[ \frac{1}{3} \times \frac{1}{4} = \]

9. \[ \frac{5}{12} \times 2 = \]

10. \[ \frac{3}{7} \times \frac{1}{3} = \]

11. \[ \frac{2}{3} \times 2\frac{1}{5} = \]

12. \[ \frac{2}{7} \times 6\frac{3}{4} = \]

13. \[ 5 \times \frac{3}{16} = \]

14. \[ 8 \times \frac{1}{16} = \]

15. \[ 12 \times \frac{2}{3} = \]

16. \[ 11 \times \frac{4}{5} = \]
17. $16 \times \frac{5}{6} = \phantom{18.} 18. \phantom{16} \times \frac{1}{3} = \\
19. \phantom{16} \times \frac{3}{8} = \phantom{18.} 20. \phantom{16} \times \frac{3}{4} = \\
21. \frac{1}{3} \times 8 = \phantom{18.} 22. \phantom{16} \times \frac{3}{16} = \\
23. \frac{7}{8} \times 16 = \phantom{18.} 24. \phantom{16} \times \frac{1}{4} = \\
25. \frac{4}{2} \times \frac{5}{7} = \phantom{18.} 26. \phantom{16} \times 15 = \\
27. \frac{1}{5} \times 7 = \phantom{18.} 28. \phantom{16} \times 12 = \\
29. \frac{1}{2} \times \frac{1}{3} = \phantom{18.} 30. \phantom{16} \times \frac{1}{5} = \\
31. \frac{3}{4} \times \frac{7}{15} = \phantom{18.} 32. \phantom{16} \times \frac{1}{1} = 1 \\
33. \phantom{16} \times \frac{9}{16} = \phantom{18.} 34. \phantom{16} \times \frac{1}{2} = 1
DIVISION OF FRACTIONS TEST

1. \( 8 \div \frac{2}{5} = \)

2. \( \frac{41}{6} \div 5 = \)

3. \( \frac{3}{8} \div \frac{2}{5} = \)

4. \( \frac{1}{3} \div 3 = \)

5. \( 3 \div \frac{2}{5} = \)

6. \( \frac{2}{3} \div \frac{1}{12} = \)

7. \( \frac{12}{3} \div \frac{3}{4} = \)

8. \( \frac{3}{8} \div \frac{1}{3} = \)

9. \( \frac{9}{16} \div \frac{3}{8} = \)

10. \( 4 \div \frac{41}{2} = \)

11. \( \frac{42}{3} \div \frac{2}{3} = \)

12. \( \frac{5}{6} \div 5 = \)
Division of Fractions Test

13. \( \frac{4}{5} \div 2\frac{2}{3} = \)

14. \( \frac{7}{8} \div 4 = \)

15. \( 8\frac{1}{2} \div 2\frac{5}{6} = \)

16. \( 14 \div 2\frac{1}{3} = \)

17. \( 3\frac{3}{4} \div 1\frac{2}{3} = \)

18. \( 4\frac{1}{8} \div 3\frac{3}{4} = \)

19. \( 12 \div 7\frac{1}{5} = \)

20. \( 1\frac{1}{2} \div 1\frac{2}{3} = \)

21. \( 14\frac{2}{5} \div 12 = \)

22. \( \frac{4}{9} \div 6 = \)

23. \( 1 \div \frac{5}{6} = \)

24. \( \frac{7}{16} \div 10\frac{1}{2} = \)
Directions for Administering and Scoring

1. Present the child with a copy of the Student's Sheet and a pencil. Explain that the examiner would like to find out how well he understands division of fractions. Assure him that this is not for his report card and try to put him at ease. Tell him that you will ask him to do some examples and then will ask him a few questions about what he does.

2. Fill in the data at the top of the interview blank and the scoring sheet. Be sure that students write both first and last names. Encircle his group on the scoring sheet, T-1 or T-2.

3. Point to the first example and ask the child to find the answer. Tell him that he may do any necessary work on the paper.

4. When the child has completed each example, challenge him with these questions.
   a) Are you sure that this is the correct answer? What if I said that the answer was ( Ques. 1- \( \frac{3}{16} \), Ques. 2- \( \frac{1}{2} \), Ques. 3 - \( 20\frac{5}{6} \), Ques. 4 - 2 )? How could you prove me wrong?
   b) Why did you change the divide sign to a multiply sign? Why did you write ( \( \frac{4}{1} \), \( \frac{8}{3} \), \( \frac{1}{5} \), \( \frac{3}{5} \) ) instead of ( \( \frac{1}{4} \), \( \frac{3}{8} \), 5 , \( 1\frac{2}{3} \) )? Do you know why it is important to do this?
c) Can you give any reason for doing these last two steps together?

d) Do you know any other way to get the answer to this question? If the student does not understand the question as first put, re-phrase it without indicating the answer sought. Questioning should continue until each child has had full opportunity to demonstrate his understanding. The examiner should be careful not to use a leading question that might suggest the correct answer.

5. Record each response for each example on the interviewer's blank according to the code. Explain any of the subject's answers about which there is uncertainty. If the subject appears to know how to work an example but gets the wrong answer, urge him to check his work.

Response categories and scoring procedure

1. Circle the appropriate code letters.

C - Correct  W - Wrong  P - Partially correct

"Partially correct" - the subject gets the correct answer but does not write it in lowest terms. For example, Ques. 1 - $\frac{12}{4}$, Ques. 2 - $\frac{72}{48}$ or $\frac{3}{2}$, Ques. 3 - $\frac{25}{30}$, Ques. 4- $\frac{45}{20}$ or $\frac{9}{4}$, or similar answers.

2. R - Rational response. This response is recorded only with a C or P response. A Rational response requires an explanation of why the "invert and multiply" procedure is used. This includes an understanding of the concept
of the reciprocal but not necessarily the use of the term "reciprocal," and the idea that division and multiplication are inverse operations.

3. Ro - Rote response using the procedure "invert the divisor and multiply". This response could occur with a C or P response. No further explanation is required from the student.

4. O - Other. The subject can tell some other way of finding the answer. This might be intuitively or by some other algorithm such as the common denominator method.

5. N - No attempt to do a question.
INTERVIEW TEST

Student's Sheet

Name __________________________ Date ________ Group: T-1  T-2

1. \( \frac{3}{4} \div \frac{1}{4} = \)

2. \( \frac{9}{16} \div \frac{3}{8} = \)

3. \( 4\frac{1}{6} \div 5 = \)

4. \( 3\frac{3}{4} \div 1\frac{2}{3} = \)
INTERVIEW TEST

Scoring Sheet

Name ______________________ Date ______ Group: T-1 T-2

1. \[ \frac{3}{4} + \frac{1}{4} = \] 1. C W P
   R
   Ro
   O N

2. \[ \frac{9}{16} \div \frac{3}{8} = \] 2. C W P
   R
   Ro
   O N

3. \[ 4\frac{1}{6} \div 5 = \] 3. C W P
   R
   Ro
   O N

4. \[ 3\frac{3}{4} \div 1\frac{2}{3} = \] 4. C W P
   R
   Ro
   O N