DERIVATION AND ANALYSIS OF COMPATIBLE TREE TAPER AND VOLUME ESTIMATING SYSTEMS

by

JULIEN PIERRE DEMAERSCHALK
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Department of Forestry

The University of British Columbia

Vancouver 8, Canada

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Supervisor: Dr. D. D. Munro

ABSTRACT

Compatible taper and volume equations give identical estimates of total volume of trees.

Two basically opposite techniques for the construction of compatible systems of estimating tree taper (decrease in diameter with increase in height) and volume were derived and examined statistically. In the first method compatible taper equations are derived from volume equations fitted on tree volume data. In the second method compatible volume equations are derived from taper equations fitted on tree taper data. Both systems have been tested for bias in the estimation of diameter inside bark at any height, height for any diameter, section volume and total tree volume. In addition to conventional estimates for all trees, classes representing each fifth of the D^2H range were used. No method gave completely satisfactory results for the equations tested. However, a few equations in both systems appear to be sufficiently unbiased to be useful for many purposes.

All tests were repeated on data where butt flare measurements were eliminated. Taper equations on these adjusted data showed much less bias over most of the length of the tree bole.

Weighting techniques did not produce any significant improvement. Use of non-linear techniques made a small difference in some cases.

Meyer's correction factor of the logarithmic volume equation was tested and found to be unnecessary.

A good relationship which existed between coefficients from taper and volume equations and form is thought to be useful in certain applications.

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SELECTED SYMBOLS

Symbol	Page first used	Meaning
$<$	54	less than;e.g. $2 < 3$
$\sum$	19	sum of
$\int$	37	integral notation
π	37	3.14159
F	16	Douglas-fir (*Pseudotsuga menziesii* (Mirb.) Franco)
C	16	western redcedar (*Thuja plicata* Donn.)
S	16	spruce (*Picea glauca* (Moench) Voss, *P. Engelmanni* Parry and *P. mariana* (Mill.) B.S.P.)
B	16	balsam (*Abies amabilis* (Dougl.) Forbes and *A. grandis* (Dougl.) Lindl.)
A	16	aspen (*Populus tremuloides* Michx.)
Cot	16	cottonwood (*Populus trichocarpa* Torr. and Gray)
Pl	16	lodgepole pine (*Pinus contorta* Dougl.)
Pw	16	white pine (*Pinus monticola* Dougl.)
D	5	diameter breast height,outside bark, in inches
H	5	total height of tree,in feet
V	5	total volume,inside bark,in cubic feet

Symbol	Page first used	Meaning
V_j, V_{ij}	19	individual volume observation
$\widehat{V}_j, \widehat{V}_{ij}$	19	individual predicted volume
B'	6	basal area, outside bark, of tree in square feet at 4.5 feet above the ground $= 3.14159\ D^2/(4\ (144))$
D_{16}	7	diameter, in inches, at the small end of the first 16 foot log
D_{25}	7	diameter, in inches, at 25 feet height
d	8	diameter inside bark, in inches, at any given height or distance from the tip
d_1	35	diameter inside bark, in inches, at one foot height
$d_{4.5}$	8	diameter inside bark, in inches, at breast height
h	11	height above the ground, in feet
$\underline{l}$	8	distance from the tip, in feet
l_1, l_2	52	lower and upper limit of tree section expressed as distances from the tip
AFQ	15	absolute form quotient = ratio of diameter at half the height between breast height and total height and diameter at breast height, all measured inside bark
CFF	7	cylindrical form factor $= V\ /(B'H)$

Symbol	Page first used	Meaning
a,b,c,e,f,g	7	volume or taper equation coefficients
$b_o, b_1, \ldots, b_5$	5	volume or taper equation coefficients
c_1, c_2	8	taper equation constants
p,q,...,u	37	free parameters of volume-based taper equations
k	52	$4(144)/\Pi = 183.3466$
log	5	logarithm to base ten
ln	8	logarithm to base e (e = 2.71828)
m	19	number of independent variables in a regression
n	19	number of observations in a given size class
N	19	total number of observations for a given species
R	18	multiple correlation coefficient
σ^2	31	population variance of the dependent variable in regression analysis
MB_c	19	approximated mean bias of the variable of interest for a given size class
MB_t	21	approximated mean bias of the variable of interest over all size classes
SE_c	19	approximated standard error of estimate of the variable of interest for a given size class

Symbol	Page first used	Meaning
SE_t	19	approximated standard error of estimate of the variable of interest over all size classes
SE_E	14	standard error of estimate of the variable used as dependent variable in fitting the regression equation

1. Introduction

A taper and volume system is here defined as compatible when integration of the taper equation yields the same total volume as given by the volume equation.The most important benefit of a compatible model is that consistent results are obtained in taper and volume analyses.The user of a taper equation is confused when confronted by a taper analysis that,upon summation of the section volumes,yields a different total volume from that obtained in a volume analysis.

Taper and volume data should not be considered independently but should be analysed as mathematically dependent quantities. This leads us to another advantage of compatible systems namely that appropriate models are suggested through the existing knowledge of volume models.

There are two basically different techniques which can be used to obtain compatible systems of taper and volume.One technique involves fitting a taper equation on taper data and deriving from it,by integration,a volume prediction system.The other technique is more or less the opposite;a volume equation is fitted on the volume data and from it a compatible taper equation is derived.One is a taper-based system while the other is volume-based.

A good tree taper and volume estimating system should be unbiased in the prediction of diameter at any height,height for any diameter,volume of any section and total volume.

The objectives of this study are to:

1.demonstrate with many examples how volume equations can be converted to taper equations and vice versa.

2.find out how equations should be tested and analysed to detect the biases involved in diameter estimation and study their effects on height,section volume and total volume predictions.

3.compare both techniques for several species in order to see which technique is best.

A flowchart of the derivation of the equation systems is given in figure 1.

Taper and volume equations can be fitted by several statistical methods.Conventional least squares,weighted least squares and non-linear least squares procedures are all examined in this study.

Butt flare often causes significant bias in taper and volume estimation.How much this bias can be reduced by fitting the equations on data adjusted for butt flare is investigated.Adjusting for butt flare is compared with the results obtained from ignoring the observations below breast height.

Some species are apparently similar in form.The possibility of combining species of similar form is examined and the resulting loss in precision and accuracy is assessed.

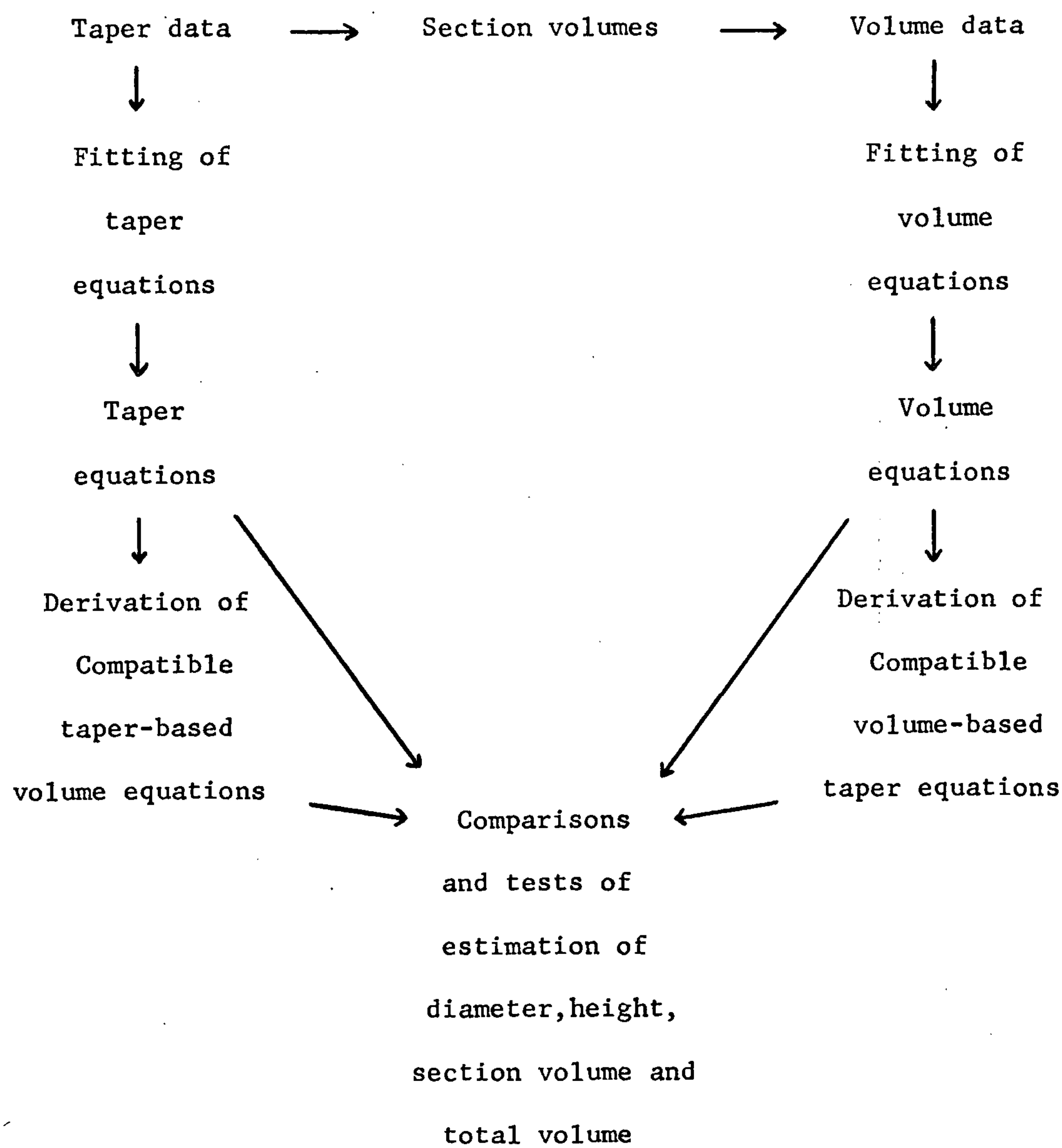
For some equations the use of theoretically derived correction factors has been advised to correct for bias.The usefulness of these correction factors is evaluated.

Possible relationships between some equation coefficients and tree form are examined.Some practical applications are discussed.

Theoretical derivations of all volume-based taper equations and all taper-based volume equations as well as the formulae for the computation of heights and section volumes are explained in detail.

Figure 1.

Flowchart of the Derivation of the Equations



2. Literature review

There are few topics in forestry which have been discussed for so many years and by so many authors as have taper, form and volume of trees. Extensive reviews are given in most forest mensuration books (Chapman and Meyer, 1953; Loetsch and Haller, 1964; Prodan, 1965; Avery, 1967; Husch *et al.*, 1972).

Only the contributions most important to this thesis are reviewed here. Significant aspects of some publications are discussed in more detail in later sections of the thesis.

No review is made of the different tree form theories (nutritional, mechanistic, water conductive, hormonal and pipe model) and biological relationships between tree or stand characteristics and tree form are not considered here. Interesting discussions about these topics were given by Gray (1943; 1944; 1956), Newnham (1958), Larson (1963), Heger (1965a), Shinozaki *et al.* (1965) and Doerner (1965).

Tree volume equations have been discussed in the literature since the early 19th century. Since then many different functions have been proposed.

Schumacher and Hall (1933) proposed the following logarithmic volume equation:

$$\log V = b_o + b_1 \log D + b_2 \log H$$

This equation is tested extensively in later sections.

Meyer (1944) suggested a volume-diameter ratio equation:

$$V / D = b_o + b_1 D + b_2 H + b_3 DH$$

A similar equation was preferred by Stoate (1945):

$$V = b_o + b_1 D^2 + b_2 H + b_3 D^2H$$

but he argued that using only D^2H was almost as good as any other equation.

A comprehensive comparison of volume equations was made by Spurr (1952) who decided that the combined variable volume equation:

$$V = b_o + b_1 D^2H$$

was one of the most promising. Since then the fact that the relationship between V and D^2H could not be expressed by a single linear regression over the full range of data has been recognized often (Smith and Ker, 1957; Smith and Breadon, 1964; Myers and Edminster, 1972).

Honer (1965) developed a new volume equation:

$$V = D^2 / (b_o + b_1 / H)$$

which was fitted by a linear least squares procedure as:

$$D^2 / V = b_o + b_1 / H$$

Tarif tables, developed in Britain by Hummel (1955), which provide a "local volume table" for each particular stand, were extensively discussed by Turnbull and Hoyer (1965). They also have developed a tarif-based procedure for estimation of diameters and volumes of 16 foot logs to facilitate studies of growth and yield.

Zaharov (1965) studied the linear relationship between form height and total height. This is basically the same as the equation:

$$V/B' = b_o + b_1 H$$

which was tested by Smith and Munro (1965) and Christie (1970).

After a trial of Hohenadl's method, Heger (1965b) concluded that it was an efficient and accurate means of both stem form and stem volume estimation.

Newnham (1967) proposed a modification to the combined variable volume equation :

$$V = b_o + b_1 D^a H^b$$

where a and b were the coefficients of log D and log H in the logarithmic volume equation.

A suitable form factor equation to predict volume was given by Van Laar (1968):

$$CFF = b_o + b_1 \log D + b_2 \log H + b_3 \log D_{25}$$

From their studies on Ponderosa pine[1],Hazard and Berger (1972) concluded that direct calculation by use of the optical dendrometer (Barr and Stroud,model FP-15) appears to be an improvement over volume tables.

Van Laar was not the first one to include a third independent factor.Mesavage and Girard (1946) used as a third measure the ratio D_{16}/D.Naslund (1947) introduced the crown ratio as a third explanatory variable in the prediction of volume,as a substitution for the form point.

Ilvessalo (1947) and Van Soest (1959) preferred as a third measure the diameter at twenty feet above the ground.Van Soest concluded, however,that the diameter at thirty percent of the height was best. The same idea was shared by Pollanschutz (1966).

Schmid *et al.*(1971) used the diameter outside bark at 6-9 meters as an additional variable.

Not only total volume prediction but also the distribution of volume within the tree were studied.

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[1]The tree names are given with the corresponding Latin names in Appendix 1.

Speidel (1957) used graphical techniques to relate the percentage of total volume to the percentage of total tree height.

Log position volume tables,based on hand drawn harmonized taper curves,were developed by Fligg and Breadon (1959).

Honer and Sayn-Wittgenstein (1963) stressed the need for a mathematical tree volume expression which would yield tree and stand volumes based on D and H (form estimates optional) for any demanded stump height and top diameter.Fulfilling this need Honer (1964;1965b) proposed three mathematical models to express the volume distribution over the tree stem.These models describe well the distribution of volume and can be used to estimate volume to any standard of utilization when applied to an estimate of total volume.

Burkhart et al.(1971) developed a technique to predict proportions of tree volume by log positions.Separate prediction equations were fitted for each peeler log.

Because the most complete information concerning the form of a tree can be given by means of a taper equation,taper curve or a taper table (Meyer,1953) many authors have concentrated their efforts on this problem.Taper equations,describing the tree profile,have been developed since the beginning of this century.

Hojer (1903) was the first to propose a mathematical equation to describe the stem profile:

$$d/d_{4.5} = c_1 \ln ((c_2 + \underline{1}(100/H))/c_2)$$

The constants c_1 and c_2 were defined for each form class.

Jonson (1910;1911;1926-27) introduced a new constant into the equation of Hojer in order to obtain better results.These taper equations were compiled independently of tree species.The form class, which had to be known, was usually measured or estimated by the "form point" approach. A good description of how these taper equations are constructed was given by Claughton-Wallin (1918).

Jonson's "absolute form quotient" was mentioned as an excellent expression of taper or stem form (Claughton-Wallin and McVicker, 1920).

His investigations on many species led Behre (1923;1927;1935) to present a new taper equation which seemed to be more consistent with nature:

$$d/D = (\underline{l}/H)/(b_o + b_1\ \underline{l}/H)$$

This equation is discussed in more detail in later sections. Some transformations of the Behre equations were given by Bruce (1972).

Petterson (1927) suggested two separate logarithmic curves to describe the stem profile, one for the main stem and the other for the top portion. Heijbel (1928) tried a combination of three different equations for different portions of the stem.

Matte (1949) described the stem profile above breast height by the function:

$$(d/D)^2 = b_o(\underline{l}/H)^2 + b_1(\underline{l}/H)^3 + b_2(\underline{l}/H)^4$$

The coefficients were found to be related to D and H.

Bruce and Schumacher (1950) mentioned that the best check of a taper table is a check of a volume table derived therefrom.

Graphical techniques were used by Duff and Burstall (1955) to develop an integrated system of taper and volume.

In order to get a prediction system in which D could be estimated from diameter measurements at different heights, Breadon (1957) fitted butt-taper equations on plotted averages.

The following taper equation was tested by Osumi (1959):

$$d/D = b_o(\underline{l}/H) + b_1(\underline{l}/H)^2 + b_2(\underline{l}/H)^3$$

This equation is tested in later sections.

Giurgiu (1963) proposed as a taper equation a 15th degree polynomial of d/D as a function of $\underline{l}/H$.

Prodan (1965) suggested the taper equation:

$$d/D = (\underline{l}/H)^2/(b_o + b_1(\underline{l}/H) + b_2(\underline{l}/H)^2)$$

Some work has been carried out on tree taper curves using multivariate methods (Fries,1965).Fries and Matern (1965) agreed that models expressed in mathematical functions have a considerable advantage over those given only as tables or graphs.They found polynomials to be the appropriate expressions if certain restrictions were imposed.After comparison of multivariate and other methods for analysis of tree taper,Kozak and Smith (1966) concluded that the use of simple functions, sorting and graphical methods is adequate for many uses in operations research.

Kuusela (1965) used diameter at 0.1 of the height as the basic diameter in a system in which nine regression equations were used for each form factor to predict proportional diameters at different heights.

According to Grosenbaugh (1954;1966),who introduced some new tree measurement concepts,polynomial analysis may rationalize observed variation in form after measurement but it does not promise more efficient estimation procedures.

An extensive study of thickness and percentage of bark (Smith and Kozak,1967) made it possible to convert outside bark measurements of form and taper to inside bark values.

An integrated system of taper and volume equations for red alder was provided by Bruce et al.(1968).This taper equation,which contained very high powers of the term 1/(H - 4.5),was conditioned so as to provide a constant double bark thickness ratio at breast height.This equation is also tested later.Referring to the dependence of the taper measurements taken on the same tree,they mentioned that no widely applicable solution is available.They gave also some good reasons why prediction of the square of diameter might be preferred.

After Munro (1968) had discussed the estimation of upper stem diameters from a function involving D,h/H and $(h/H)^2$,the following taper equation was proposed by Kozak et al.(1969a,b):

$$(d/D)^2 = b_o + b_1 (h/H) + b_2 (h/H)^2$$

To make the diameters inside bark zero at the top,the least squares solution was conditioned by imposing the restraint $b_o + b_1 + b_2 = 0$. For spruce and western redcedar an additional condition was necessary to prevent negative diameters near the top.These taper equations were later converted into volume equations and point sampling factors (Demaerschalk,1971a).In their forest inventory program,Kozak and Munro (personal communications) use a system of separately fitted taper and volume equations.Volumes computed from the taper equation are adjusted to the volumes given by the volume function,by application of a proportional percentage correction on each tree section.

After it was demonstrated (Demaerschalk,1971b;1972a) that the following logarithmic taper equation :

$$\log d = b_o + b_1 \log D + b_2 \log l + b_3 \log H$$

could be derived from a logarithmic volume equation and vice versa, it soon became clear that this technique of deriving taper equations from existing volume equations could be applied to many volume functions (Demaerschalk,1973).

According to Smith and Kozak (1971) best results will come from use of locally derived equations for estimation of upper bole diameters.

The studies of slash pine by Bennett and Swindel (1972) included the development of a new interesting equation for the prediction of taper above breast height:

$$d = b_o D(\underline{1}/(H - 4.5)) + b_1(\underline{1} (h - 4.5)) + b_2 H(\underline{1} (h - 4.5)) + b_3(\underline{1} (h - 4.5))(H + h + 4.5)$$

This equation is discussed in more detail in later sections.

A general method to convert taper and volume equations from one unit system to another was described in detail (Demaerschalk,1972b).

In fitting taper and volume equations,the assumptions of the regression analysis often are not met.Heterogeneity of variances seems to be one of the most serious problems.Being more compatible with the homogeneity of variance requirement,the logarithmic volume equation was favoured by many investigators.A correction factor for the bias,introduced by the equation,was developed (Meyer,1938;1944; Baskerville,1972).

Weighting,another way to correct for heterogeneous variances, has been studied by many authors.Meyer (1953) described the form factor method of preparing volume tables as the proper method of weighting the tree volume residuals.This approach was followed by Evert (1969) who developed and tested several form factor equations,one of which is described in later sections.

Cunia (1964;1965;1968) proposed weighted least squares to overcome the difficulty of obtaining an equally good fit for all sections of the tree volume curve.

Many authors came to the conclusion that the variance of volume is directly related to the square of the quantity D^2H (Munro,1964; Haack,1963;Gregory and Haack,1964;Evert,1969;Smalley and Beck,1971). This would agree with the form factor approach.

Gerrard (1966) settled on an exponential relationship between the variance of volume and D and H.

Smith and Munro (1965) found no wholly satisfactory method of weighting or transformation.

Non-linear fitting with weighting of the volume equation :

$$V = b_o D^{b_1} H^{b_2}$$

was recommended by Moser and Beers (1969) who also found that the variance of volume was exponentially related to D^2H.This equation would retain the statistical advantages of lack of bias and overcome the shortcomings of the logarithmic volume equation by weighting (Husch et al.,1972).

Comparison and testing the functions for precision and accuracy is still a difficult task.Freese (1960) recommended the chi-square test to check the accuracy.

To compare different volume equations,Furnival (1961) developed an index employing the concept of maximum likelihood.

After comparison of several absolute and relative standard errors of estimate of tree volume,Hejjas (1967) concluded that none could by itself indicate the best equation.The standard error of estimate (SE_E) and possibly the sum or mean of absolute deviations should always be calculated.

Williams (1972) studied the effects some violations of the assumptions might have on the outcome of the regression analysis. He concluded that if errors are involved in the measurement of the independent variables, these might seriously bias the regression coefficients and SE_E.

3. Data

The data used in this study consists of a sample of 752 tree records for eight species or species groups taken from the British Columbia Forest Service (B.C.F.S.) data bank. It represents a subsample of the data used by Kozak et al. (1969a) in their taper studies. These eight species or species groups were chosen because of their homogeneous distribution of trees over the D-H range. For each "species" the sample was taken so as to give a good representation of the widest possible range of D,H and absolute form quotient (AFQ). Combinations of D,H and form, represented by only a few trees or near the extremes of the data bank records were avoided.

The observations from each tree are:

-diameters inside bark and outside bark at one foot, 4.5 feet and at each tenth of the height above breast height, to the nearest tenth of an inch.

-total height to the nearest tenth of a foot.

A summary of the data is given for each species in table I. Only information about D,H, form and double bark thickness has been studied. It has been shown that average effects of age, site, crown class and similar factors are small in relation to those of D,H, form and bark thickness (Demaerschalk and Smith, 1972).

Table I

Averages and Range of Data

species or species groups	number of trees	D (inches)			H (feet)			D^2H		$(AFQ)^2$		$\frac{d^2_{4.5}}{D^2}$	V (cubic feet)		
		min	ave	max	min	ave	max	min	max	min	max	mean	min	ave	max
F - coastal	65	5.3	15.2	25.8	52.4	104.4	159.9	1640	85096	.36	.62	.74	3.4	62.5	166.9
C - coastal	63	5.9	14.8	28.2	31.1	76.4	127.3	1083	78803	.19	.60	.88	3.3	44.2	139.2
S - interior	91	5.1	13.1	21.3	37.4	89.4	134.5	1215	57165	.25	.59	.88	2.9	43.9	120.9
B - coastal	71	5.9	18.1	33.6	32.5	107.6	164.5	1131	174824	.30	.62	.91	2.7	110.3	369.2
A	111	4.7	9.8	16.3	41.5	76.2	101.8	1085	25247	.24	.64	.84	2.4	21.1	53.3
Cot	109	5.1	8.9	15.1	44.8	73.2	106.9	1306	20880	.19	.57	.82	2.4	14.6	43.9
Pl	152	5.1	12.3	20.4	42.6	79.4	123.8	1340	43668	.26	.72	.91	3.6	34.3	96.2
Pw	90	5.8	15.7	23.4	45.4	105.4	147.1	1802	71855	.28	.57	.90	4.5	68.9	157.4

4. Volume-based systems of tree taper and volume estimation

4.1. Volume equations

4.1.1. Fitting volume equations

A selection of volume equations was made. Some of these are used extensively, others rarely. The following equations were considered for testing:

1. logarithmic volume equation (Schumacher and Hall, 1933)

   $\log V = b_o + b_1 \log D + b_2 \log H$

2. logarithmic combined variable volume equation (Spurr, 1952)

   $\log V = b_o + b_1 \log (D^2H)$

3. Honer's volume equation (Honer, 1965a)

   $D^2/V = b_o + b_1/ H$

4. combined variable volume equation (Spurr, 1952)

   $V = b_o + b_1 D^2H$

5. weighted combined variable volume equation

   $V /(D^2H) = b_o + b_1/(D^2H)$

6. comprehensive combined variable equation (Spurr, 1952; Gerrard, 1966)

   $V = b_o + b_1 D + b_2 H + b_3 DH + b_4 D^2 + b_5 D^2H$

7. weighted comprehensive equation

   $V /(D^2H) = b_o + b_1/(DH) + b_2/ D^2 + b_3/ D + b_4/ H + b_5/(D^2H)$

8. combined variable equation without intercept

   $V = b_o D^2H$

9. volume over basal area as a function of height (Smith and Munro, 1965; Christie, 1970)

   $V / B' = b_o + b_1H$

10. V / B' as a function of H and H^2 (Christie,1970;Demaerschalk and Smith,1972)

$$V / B' = b_o + b_1 H + b_2 H^2$$

11. Meyer's volume-diameter ratio equation (Meyer,1944)

$$V / D = b_o + b_1 H + b_2 D + b_3 DH$$

12. cylindrical form factor equation (Evert,1969)

$$V /(D^2H) = b_o + b_1(H /(H - 4.5))^2$$

The numbering system of all taper and volume equations is given in Appendix 2.The tree volumes were computed by Smalian's formula,except the tip section which was considered having a form factor 0.4 and the section below one foot which was considered as a cylinder.

All these volume equations were first fitted by a linear least squares procedure.

Plottings of the variances of volume showed that there was a linear relationship between the variance of volume and $(D^2H)^2$,favouring the use of weighted volume equations 5,7 and 12.

In fitting these equations,the underlying assumptions of the regression analysis were usually not met.For many of these equations the assumed linear model is not correct.Variances are seldom homogeneous.In some cases,the violations of the assumptions are minor.

The coefficients of the equations as well as the standard errors of estimate (SE_E) and the coefficients of determination (R^2100) are given for all species in Appendix 3.

4.1.2. Tests of total volume estimation

The results in Appendix 3 are far from sufficient to judge the effectiveness of each equation in estimating the total volume. The SE_E's are merely an overall measure of variation of the data and are of little use when variances are not homogeneous or when the linear model is biased. Moreover they cannot be compared directly if the dependent variables differ. The volume estimation was therefore tested in a more detailed manner. For each equation an approximated standard error of estimate (SE_c) and mean bias (MB_c) was computed for each fifth of the range of D^2H within each species. D^2H is more or less linearly related to volume and was therefore used as a measure to break down the data into size classes. In order to have in each D^2H class a minimum number of five trees, only five size classes could be used. A summary of the number of trees and the mean volume for each size class and for all species is given in table II.

For each size class the SE_c was computed in the following way:

$$SE_c = \left(\sum_{j=1}^{n}(V_j - \hat{V}_j)^2 / n\right)^{\frac{1}{2}}$$

What is considered as mean bias (MB_c) is a measure of lack of fit and computed for each size class as the mean of the residuals:

$$MB_c = \sum_{j=1}^{n}(V_j - \hat{V}_j) / n$$

An approximation of the overall standard error of estimate in terms of volume was computed as:

$$SE_t = \left(\sum_{i=1}^{5}\sum_{j=1}^{n_i}(V_{ij} - \hat{V}_{ij})^2 / (N - m - 1)\right)^{\frac{1}{2}}$$

An overall measure of bias was computed for each equation as the

Table II

Number of Trees and Mean Volume for All Size Classes and for All Species

Number of trees for the species

Size class	D	C	S	B	A	Cot	Pl	Pw
1	28	35	36	36	50	54	68	25
2	10	11	21	18	20	22	33	18
3	11	6	19	6	15	15	26	22
4	5	6	8	5	17	12	17	12
5	11	5	7	6	9	6	8	13
Total	65	63	91	71	111	109	152	90

Mean volume for the species (in cubic feet)

Size class	D	C	S	B	A	Cot	Pl	Pw
1	16	17	13	35	7	6	12	19
2	49	48	40	121	20	15	34	50
3	84	70	64	196	33	23	54	76
4	119	101	92	254	41	31	68	111
5	145	124	109	328	47	39	88	140
Total	63	44	44	110	21	15	34	69

mean of all residuals:

$$MB_t = \sum_{i=1}^{5} \sum_{j=1}^{n_i} (V_{ij} - \hat{V}_{ij}) \ / \ N$$

A tabulation of all standard errors of estimate and biases for all volume equations and all species is given in table III. Some conclusions which can be drawn from a comparison of these results will now be discussed.

For most species the best volume equations seem to be 1,2,6,7 and 11. The logarithmic volume equation 1 is slightly negatively biased when calculated without adjustments. However, the bias is very small for most species (-1.50 to +1.04 cu. ft. for Douglas-fir) and the total bias is negligible (-0.16 cu. ft. for Douglas-fir). The combined variable logarithmic volume equation 2 comes very close to equation 1, but is definitely not as good because of the conditioning of the powers of D and H. Honer's volume equation 3 is good for some species but for others the SE_c's and MB_c's are for some sizes much larger. The combined variable volume equation 4 always has an overall zero bias because it was fitted with V as dependent variable. This is, however, a good example of an equation in which the linear model does not hold (see figure 2). The real relationship between V and D^2H is a curve. A straight line overestimates the smallest, underestimates the middle and overestimates again the largest size classes. The comprehensive combined variable volume equation 6, which assumes an additional significant effect of D, H, D^2 and interaction D-H, gives good results for all the species. Although the difference between the overall SE_t and MB_t for equations 4 and 6 may not seem important, the differences for the individual classes are substantial.

Figure 2

Relationships between Dependent and Independent

Variables for Some Volume Equations

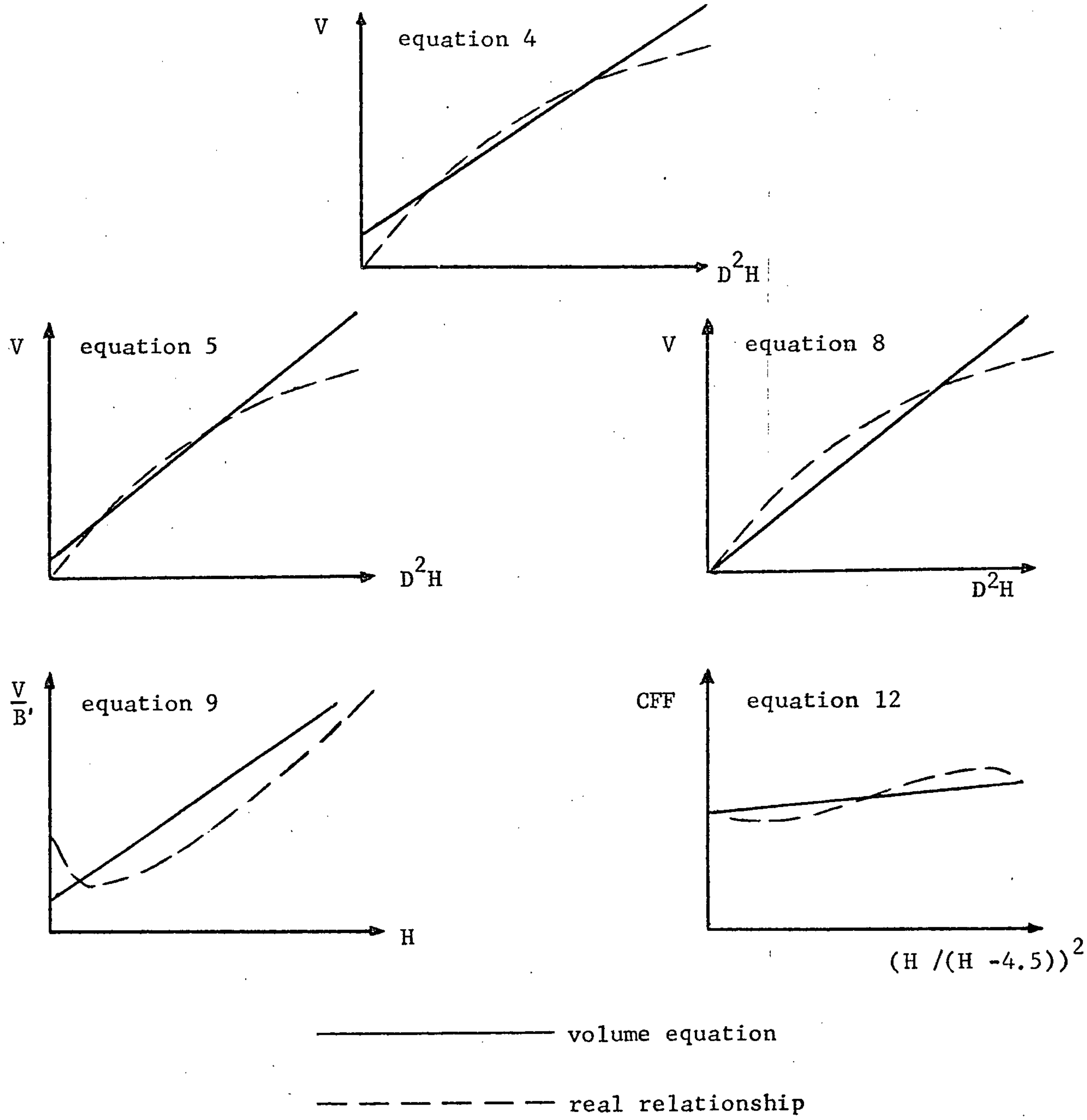


Table III

Total Volume Estimation Tests of Volume-Based Linear Volume Equations

Douglas-fir

size class	1	2	3	4	5	6	7	8	9	10	11	12
	SE_c(in cubic feet) of volume for volume equations											
1	1.46	1.70	1.86	2.04	1.71	1.41	1.42	2.07	1.77	1.84	1.57	1.81
2	4.35	3.93	3.91	3.93	4.00	4.54	4.50	4.43	4.00	4.12	4.67	3.98
3	7.31	9.29	10.54	9.55	9.24	6.90	6.97	9.85	9.49	8.89	6.94	9.69
4	9.64	8.19	8.47	8.19	7.98	10.37	10.09	7.99	8.24	8.40	9.91	8.09
5	13.74	16.51	18.74	16.51	18.03	13.38	13.48	16.68	18.82	17.76	13.63	18.49
SE_t	7.38	8.47	9.51	8.55	8.97	7.47	7.48	8.70	9.32	8.94	7.41	9.23
	MB_c(in cubic feet) of volume											
1	-0.02	0.07	-0.03	0.95	-0.20	0.00	0.06	-1.13	-0.16	-0.25	0.01	-0.04
2	-0.36	-0.23	-0.18	-0.71	-0.36	0.10	-0.05	-2.06	0.13	-0.89	-0.30	0.03
3	-0.95	-2.55	-3.31	-3.37	-1.43	0.10	-0.26	-3.94	-1.03	-1.51	-0.39	-1.67
4	-1.50	-1.20	-1.07	-1.74	2.05	-1.32	-1.34	-1.41	2.77	2.72	-1.37	1.75
5	1.04	2.36	2.86	2.41	7.65	0.41	0.90	3.45	8.46	9.42	1.26	7.05
MB_t	-0.16	-0.13	-0.20	0.00	1.07	0.00	0.02	-1.00	1.42	1.31	0.00	1.03

Western redcedar

size class	1	2	3	4	5	6	7	8	9	10	11	12
	SE_c(in cubic feet) of volume for volume equations											
1	1.62	1.65	2.12	2.45	1.87	1.78	1.69	4.09	2.09	2.09	1.79	2.03
2	6.16	6.00	5.57	6.28	6.57	6.64	6.19	7.76	5.74	5.79	6.39	5.97
3.	11.90	12.18	13.52	13.19	11.98	10.81	11.61	13.94	12.99	13.03	11.50	12.67
4	9.16	9.60	13.67	9.49	17.39	8.50	8.80	9.82	13.26	13.63	8.79	14.35
5	11.12	11.68	16.12	12.37	25.50	9.75	10.46	14.28	17.16	17.16	10.61	19.50
SE_t	6.43	6.55	8.11	7.00	10.34	6.22	6.41	8.11	8.15	8.29	6.35	8.72
	MB_c(in cubic feet) of volume											
1	0.01	-0.01	-0.21	1.19	-0.70	0.23	0.31	-3.65	-0.13	-0.09	0.31	-0.29
2	-0.27	-0.13	0.89	-2.19	1.13	0.65	0.32	-4.81	1.08	1.36	-0.19	1.40
3	-3.78	-3.89	-3.84	-6.36	0.36	-2.47	-3.43	-7.54	-2.72	-2.49	-3.60	-1.52
4	1.64	1.60	2.08	0.64	14.54	0.62	0.76	2.52	6.11	5.74	1.10	9.06
5	2.23	2.17	2.67	3.33	22.02	-0.83	0.39	7.25	8.85	7.93	1.25	13.22
MB_t	-0.07	-0.07	0.06	0.00	2.98	0.00	0.01	-2.77	1.14	1.13	0.00	1.85

Table III (continued)

Spruce

size class	SE_c (in cubic feet) of volume for volume equations 1	2	3	4	5	6	7	8	9	10	11	12
1	1.36	1.36	1.38	1.70	1.43	1.40	1.33	1.81	1.38	1.31	1.33	1.39
2	3.05	3.12	3.30	3.25	3.07	3.07	3.02	3.62	3.27	3.29	3.10	3.18
3	6.88	6.75	6.56	6.84	6.79	6.66	6.84	6.85	6.61	6.53	6.69	6.67
4	5.50	5.52	5.73	5.73	5.99	5.78	5.67	5.66	5.66	5.92	5.76	5.78
5	8.78	8.95	9.18	8.58	11.78	7.50	7.57	9.37	9.99	8.90	7.83	10.90
SE_t	4.70	4.67	4.70	4.73	5.20	4.57	4.60	4.92	4.83	4.69	4.56	5.00
	MB_c (in cubic feet) of volume											
1	-0.21	-0.22	-0.31	0.74	-0.40	-0.21	-0.08	-1.09	-0.22	-0.14	0.05	-0.32
2	-0.55	-0.59	-0.64	-0.95	-0.51	0.38	0.26	-1.94	-0.57	-0.06	0.09	-0.46
3	-0.21	-0.16	-0.02	-1.02	0.89	0.59	0.19	-1.23	0.25	0.55	-0.09	0.68
4	-0.73	-0.70	-0.75	-1.68	1.94	-2.08	-1.93	-0.98	0.15	-0.86	-2.01	1.12
5	4.33	4.59	4.92	3.76	8.76	0.72	1.32	5.19	6.25	4.42	2.03	7.58
MB_t	0.02	0.03	0.04	0.00	0.76	0.00	0.00	-0.82	0.33	0.31	0.00	0.59

Balsam

size class	SE_c (in cubic feet) of volume for volume equations 1	2	3	4	5	6	7	8	9	10	11	12
1	3.71	4.17	4.40	5.40	4.49	3.89	3.57	7.02	4.65	4.20	3.69	4.51
2	9.64	10.79	13.01	10.64	13.01	9.36	9.47	12.09	12.40	13.62	9.47	13.00
3	16.86	17.23	20.18	15.89	24.53	16.13	16.58	15.73	18.44	18.31	16.48	22.53
4	6.32	6.98	12.98	6.62	23.33	6.95	6.53	6.62	10.34	8.91	6.35	18.99
5	14.04	17.62	27.17	17.60	38.87	12.17	15.70	18.75	24.28	24.47	14.22	33.85
SE_t	8.79	9.79	12.88	9.84	16.66	8.61	9.13	10.91	11.83	12.10	8.76	14.99
	MB_c (in cubic feet) of volume											
1	-0.64	-1.17	-1.53	0.73	-1.88	0.29	-0.11	-5.07	-1.48	-0.77	-0.27	-1.74
2	1.05	1.28	3.77	-2.49	4.81	-1.06	1.38	-5.40	1.89	3.79	0.16	4.28
3	6.26	7.00	12.51	2.41	19.22	3.96	5.17	2.27	9.53	7.54	5.15	16.40
4	0.38	2.57	10.74	-1.21	22.16	-0.21	-1.86	0.57	7.62	1.12	-0.32	17.69
5	-4.84	3.06	17.54	1.69	34.27	-2.34	-8.43	6.15	14.02	2.63	-3.76	28.00
MB_t	0.09	0.76	3.48	0.00	6.35	0.00	-0.11	-3.19	2.26	1.51	0.00	5.20

Table III (continued)

Aspen

size class	SE$_c$(in cubic feet) of volume for volume equations 1	2	3	4	5	6	7	8	9	10	11	12
1	0.54	0.51	0.54	0.60	0.51	0.62	0.53	0.60	0.51	0.50	0.57	0.51
2	2.02	2.11	2.09	2.13	2.09	2.04	1.95	2.14	2.11	2.11	1.97	2.09
3	1.83	2.01	1.78	2.20	1.77	1.59	1.76	2.22	1.71	1.73	1.79	1.65
4	4.57	4.43	4.38	4.49	4.38	4.61	4.55	4.45	4.38	4.39	4.54	4.38
5	2.91	5.13	5.82	4.82	5.79	2.27	3.32	5.07	5.99	6.01	3.36	6.10
SE$_t$	2.31	2.59	2.67	2.59	2.67	2.28	2.36	2.63	2.70	2.72	2.34	2.71

size class	MB$_c$(in cubic feet) of volume 1	2	3	4	5	6	7	8	9	10	11	12
1	-0.04	-0.09	-0.19	0.25	-0.11	-0.11	-0.02	-0.30	-0.11	-0.10	-0.01	-0.13
2	0.40	-0.03	0.02	-0.08	0.07	0.63	0.25	-0.35	0.13	0.17	0.32	0.16
3	-1.20	-1.45	-1.14	-1.70	-1.13	-0.41	-0.94	-1.74	-1.03	-1.05	-0.99	-0.94
4	-0.99	-0.58	0.00	-0.94	-0.02	-0.63	-0.59	-0.78	0.13	0.07	-0.75	0.26
5	1.76	3.83	4.74	3.38	4.69	1.09	2.54	3.76	4.91	4.84	2.41	5.07
MB$_t$	-0.09	-0.02	0.15	0.00	0.19	0.00	0.03	-0.25	0.25	0.25	0.00	0.29

Cottonwood

size class	SE$_c$(in cubic feet) of volume for volume equations 1	2	3	4	5	6	7	8	9	10	11	12
1	0.57	0.59	0.59	0.67	0.57	0.58	0.56	0.62	0.56	0.57	0.65	0.56
2	0.90	1.11	1.00	1.14	1.06	0.93	0.89	1.22	0.99	0.98	0.93	1.00
3	1.14	1.97	1.75	2.02	1.95	0.99	1.08	2.05	1.86	1.77	1.11	1.87
4	3.35	4.00	4.06	3.89	4.09	3.12	3.24	3.92	4.14	4.12	3.32	4.14
5	3.06	4.43	4.95	3.91	4.68	2.69	2.67	4.07	5.01	5.07	2.95	4.96
SE$_t$	1.52	1.97	2.01	1.90	2.01	1.43	1.46	1.93	2.05	2.04	1.52	2.04

size class	MB$_c$(in cubic feet) of volume 1	2	3	4	5	6	7	8	9	10	11	12
1	-0.04	-0.09	-0.20	0.25	-0.08	-0.05	-0.03	-0.17	-0.12	-0.13	0.03	-0.12
2	-0.24	-0.39	-0.32	-0.42	-0.25	0.14	0.08	-0.62	-0.17	-0.24	-0.16	-0.17
3	-0.15	-0.36	0.08	-0.75	-0.12	0.23	0.12	-0.75	0.17	0.17	-0.18	0.14
4	0.10	0.76	1.45	-0.01	1.09	-0.28	-0.17	0.20	1.52	1.58	-0.10	1.47
5	1.31	2.32	3.60	1.14	2.76	-0.14	0.13	1.57	3.46	3.75	0.96	3.37
MB$_t$	-0.01	0.04	0.21	0.00	0.17	0.00	0.01	-0.20	0.29	0.29	0.00	0.27

Table III (continued)

Lodgepole pine

size class	SE$_c$(in cubic feet) of volume for volume equations 1	2	3	4	5	6	7	8	9	10	11	12
1	1.06	1.07	1.18	1.28	1.12	1.07	1.07	1.66	1.22	1.17	1.08	1.20
2	1.95	2.08	2.27	2.28	2.11	1.88	1.90	2.80	2.67	2.26	1.94	2.20
3	3.82	4.18	4.82	4.28	4.30	3.71	3.69	4.32	4.71	4.93	3.90	4.70
4	5.92	6.02	7.08	5.98	7.02	5.87	5.85	6.19	7.16	7.27	5.89	7.39
5	5.76	5.72	6.41	5.84	7.23	5.87	6.05	6.26	6.79	6.51	5.75	7.29
SE$_t$	3.11	3.22	3.70	3.31	3.62	3.11	3.12	3.55	3.73	3.78	3.14	3.82
	MB$_c$(in cubic feet) of volume											
1	-0.01	-0.01	-0.24	0.47	-0.23	0.09	0.03	-1.23	-0.11	-0.11	0.09	-0.18
2	-0.47	-0.61	-0.48	-1.11	-0.68	-0.11	0.09	-1.96	-0.46	-0.26	-0.40	-0.37
3	-0.40	-0.40	0.60	-0.92	0.69	-0.36	-0.17	-0.88	0.76	0.86	-0.44	1.09
4	1.43	1.56	3.31	1.45	3.96	1.01	0.92	2.17	3.79	3.55	1.34	4.33
5	-0.35	-0.18	2.40	0.50	4.09	-1.31	-2.15	2.03	3.28	2.59	-0.51	4.16
MB$_t$	-0.04	-0.04	0.39	0.00	0.53	0.00	-0.01	-0.78	0.58	0.58	0.00	0.73

White pine

size class	SE$_c$(in cubic feet) of volume for volume equations 1	2	3	4	5	6	7	8	9	10	11	12
1	1.89	1.93	2.03	2.20	2.04	1.87	1.85	2.58	2.06	2.01	1.91	2.07
2	3.73	3.69	3.76	3.83	3.90	3.67	3.70	4.61	3.78	3.71	3.65	3.80
3	4.19	4.19	4.50	4.16	4.15	4.25	4.20	4.19	4.40	4.68	4.23	4.36
4	9.15	8.95	8.84	8.96	10.05	8.84	8.98	9.08	9.18	8.93	8.94	9.59
5	9.65	9.48	9.37	9.48	10.13	9.44	9.50	9.59	9.30	9.34	9.47	9.50
SE$_t$	5.81	5.70	5.72	5.74	6.13	5.80	5.84	5.95	5.76	5.79	5.76	5.90
	MB$_c$(in cubic feet) of volume											
1	0.09	-0.03	-0.31	0.74	-0.52	0.31	0.15	-1.64	-0.24	-0.14	0.12	-0.41
2	-0.68	-0.78	-0.85	-1.36	-1.30	-0.81	-0.56	-2.82	-0.97	-0.42	-0.57	-0.99
3	0.15	0.32	0.75	-0.33	0.93	0.11	0.43	-0.96	0.80	1.28	0.46	1.08
4	1.71	2.03	2.61	2.19	5.10	2.21	2.07	2.70	3.42	2.93	2.12	4.36
5	-2.34	-2.18	-2.14	-1.01	3.07	1.70	-2.34	0.31	-0.31	-2.22	-2.17	1.27
MB$_t$	-0.19	-0.13	-0.03	0.00	0.95	0.00	-0.03	-0.85	0.35	0.26	0.00	0.72

In the case of Douglas-fir, for example, difference in size class bias is as high as -3.47 cu. ft.

Volume equation 8, which is identical to equation 4 but with zero intercept, assumes a constant CFF for all trees:

$$CFF = 183.3466\ b_o$$

Because of the real relationship between V and D^2H, this equation underestimates always the smaller sizes and overestimates the larger. The overall bias is always negative (see figure 2).

The volume equations 9 and 10 with V/B' as a function of H and H^2 are mostly negatively biased for the small, and positively biased for the larger, size classes. The bias can be large. The linear model clearly does not hold because V/B', which normally decreases when H decreases, increases again when H becomes small (see figure 2). The functions 9 and 10 assume a linear relationship between CFF and 1/H or between CFF and 1/H and H.

Meyer's volume equation 11 performs very well. Overall bias is always zero and bias of the individual size classes is very small (-1.37 cu. ft. maximum for Douglas-fir).

The cylindrical form factor volume equation 12 has the same pattern of bias as equation 9, due to a wrong assumption of the linear model (see figure 2).

Weighting was tested in equations 5 and 7. Equation 5, which is the weighted form of equation 4, has an intercept which is always smaller and a slope which is always steeper than in 4 (this comparison of the coefficients is made after transformation of 5 to the same function as 4). This shifting is caused by giving more weight to the smaller trees and results in an overestimation, sometimes very large (e.g. +7.65 cu. ft. for Douglas-fir),of the larger size classes and an underestimation of the small sizes (see figure 2).Overall bias is always positive.Weighting has a negative effect here because the assumption of a linear model is not met.Weighting has,generally speaking,no effect in 7,compared with 6.

It can be seen in these tests that if two equations differ in SE_c for a given size class it usually is due to a difference in bias,MB_c.The square of SE_c consists of two components.One component is a measure of the variation of the data (pure error),the other component is a measure of the square of the bias (lack of fit). Although various methods of data collection and analysis can result in errors which appear to be very large,in relation to the biases discussed here,such errors should be reducible by further sampling. Bias is of great importance because it can lead to consistently high or low estimates which may be undetectable in conventional inventory methods.

Because in these studies of taper and volume estimation it's most important to minimize any systematic bias,most emphasis in the following tests will be put on the amount of bias.

4.1.3. Non-linear fitting of volume equations

Volume equations 1 and 3 were transformed into a non-linear form:

13. $V = 10^{b_o} D^{b_1} H^{b_2}$

14. $V = D^2/(b_o + b_1/ H)$

and fitted by a non-linear least squares procedure (UBC BMDX85 computer program) with and without weighting. The same weights were used as in equations 5 and 7. The coefficients of these equations are given in Appendix 3.2. and the results of the total volume estimation tests in table IV.

To start the iteration in the non-linear least squares procedure, the values of the corresponding linear equations were used as first approximations of the coefficients.

In case of weighting, the coefficient b_1 in equation 13 is usually larger and b_2 usually smaller than the corresponding coefficients in 1. Without weighting it is just the reverse. Although there is a systematic difference between the coefficients of 1 and 13, the differences are very small in the case of weighting. This is because taking the logarithm is in itself a weighting.

Generally, the non-linear equation 13 gives higher estimates than the linear, and weighting gives higher estimates than non-weighting. For some species, e.g. western redcedar and cottonwood, the unweighted form of 13 seems to be superior, for others both weighted and unweighted are as good as 1, but not better.

In case of no weighting, the coefficient b_o in equation 14 is usually smaller and b_1 much bigger than the corresponding coefficients in 3. With weighting 14 and 3 are very similar. This could be expected

Table IV

Total Volume Estimation Tests of Volume-Based Non-Linear Volume Equations

size class	D	C	S	B	A	Cot	Pl	Pw
<u>equation 13</u>		MB_c (in cubic feet) of volume for the species						
1	0.15	0.77	0.57	-1.30	-0.03	0.16	0.25	-0.08
2	-0.07	-0.15	0.14	0.05	0.62	-0.04	-0.23	-0.80
3	-0.09	-3.37	-0.18	6.21	-0.71	-0.08	-0.45	0.48
4	-1.44	0.54	-2.17	0.93	-0.83	-0.36	1.06	2.30
5	0.61	0.11	1.87	-4.63	1.58	0.44	-1.14	-2.00
MB_t	0.03	0.14	0.17	-0.45	0.01	0.04	0.05	-0.05
<u>equation 13(w)</u>			MB_c (in cubic feet) of volume					
1	0.02	0.08	-0.17	-0.56	-0.02	-0.03	0.02	0.17
2	-0.17	0.26	-0.40	1.59	0.43	-0.19	-0.38	-0.52
3	-0.62	-3.06	0.09	7.34	-0.93	-0.06	-0.22	0.29
4	-0.84	3.08	-0.27	1.90	-0.77	0.27	1.68	1.89
5	1.97	4.14	4.96	-2.58	2.21	1.55	-0.02	-2.08
MB_t	0.15	0.42	0.22	0.66	0.00	0.06	0.08	-0.04
<u>equation 14</u>			MB_c (in cubic feet) of volume					
1	-2.32	-2.08	0.02	-3.72	-0.56	-0.47	-0.91	-0.59
2	-4.18	-2.17	-0.46	-3.03	-0.58	-0.93	-1.62	-1.26
3	-4.53	-5.94	-0.33	2.84	-1.72	-0.78	-0.75	0.38
4	-1.44	1.64	-2.15	-0.64	-0.66	0.32	1.98	2.57
5	4.77	3.83	2.81	3.28	3.88	2.25	1.30	-1.75
MB_t	-1.71	-1.64	-0.14	-2.18	-0.38	-0.37	-0.60	-0.23
<u>equation 14(w)</u>			MB_c (in cubic feet) of volume					
1	-0.00	0.04	-0.21	-1.20	-0.12	-0.13	-0.11	-0.19
2	0.42	1.77	-0.33	3.45	0.14	-0.22	-0.26	-0.56
3	-1.41	-2.52	0.48	10.24	-1.00	0.17	0.87	1.18
4	1.93	4.29	-0.05	6.94	0.16	1.55	3.60	3.19
5	6.96	5.18	5.78	11.98	4.95	3.65	2.67	-1.48
MB_t	1.15	0.91	0.38	2.63	0.26	0.29	0.59	0.33

because 3 is nearly completely homogeneous in variance. The weighted equation gives for all species higher estimates than the unweighted.

4.1.4. Correction factor for the logarithmic equation

Because the antilog of the mean of the logarithms of some values is smaller than the arithmetic mean of these values, it is generally assumed that the logarithmic volume equation 1 gives an overall underestimation of volume.

If the normality and homogeneous variance assumptions are assumed to be correct in the linear logarithmic volume equation 1, then it can be shown that the correction factor, to be applied to correct for the underestimation, is

$$10^{1.1513\,\sigma^2} \qquad \text{(for logarithm base 10)}$$

σ^2 is estimated by the square of SE_E of log V (see table V). This factor is often called Meyer's correction factor (Meyer, 1938; 1944).

Although the logarithmic volume equation 1 gives for most species (6 out of 8) an overall underestimation, this negative bias is extremely small. The largest negative overall bias is -0.19 cu. ft. For two species (spruce and balsam) there is an average overestimation. Application of the correction factor gives, of course, higher estimates but results for most species (5 out of 8) in a larger absolute bias. This means that the correction factor is too big. This may be explained by the fact that the assumptions, under which the correction factor is derived, do not hold in practise. It is doubtful if the normality assumption holds and there may be slight departures from the linear model. (See table VI for the results of the volume estimation test.)

Table V

Meyer's Correction Factors for the Logarithmic Volume Equation 1

species	SE_E of log V	Meyer's correction factor
Douglas-fir	0.037144	1.0037
Western redcedar	0.049756	1.0066
Spruce	0.038673	1.0040
Balsam	0.035998	1.0035
Aspen	0.037248	1.0037
Cottonwood	0.038152	1.0039
Lodgepole pine	0.033634	1.0030
White pine	0.032335	1.0028

Table VI

Total Volume Estimation Bias after Application of Meyer's Correction Factor

equation 1 size class	MB_c (in cubic feet) of volume for the species D	C	S	B	A	Cot	Pl	Pw
1	0.04	0.12	-0.16	-0.52	-0.01	-0.02	0.02	0.14
2	-0.18	0.05	-0.40	1.47	0.48	-0.18	-0.37	-0.54
3	-0.65	-3.35	0.04	6.95	-0.90	-0.06	-0.24	0.36
4	-1.07	2.31	-0.36	1.26	-0.84	0.22	1.64	2.03
5	1.58	3.06	4.78	-3.73	1.94	1.46	-0.08	-1.96
MB_t	0.06	0.22	0.19	0.47	-0.01	0.05	0.07	0.01

4.1.5.Volume equations for combinations of species

Covariance tests were carried out to check if several species could be combined in one equation without a loss in precision and accuracy.The tests were done with volume equation 1 and using the following combinations of species:

combination number	combined species
1.	all eight species
2.	Douglas-fir,spruce and balsam
3.	Douglas-fir,spruce,balsam and western redcedar
4.	aspen and cottonwood
5.	lodgepole pine and white pine

The coefficients of the equations are in Appendix 3.3. and the biases for individual and combined species are given in table VII.

Because of the differences in bark thickness and form,the combination of species in the same equation was unsuccessful and results in large bias except for the combination of the two broad-leaved species and the combination of the two pines,for which only the intercept is significantly different.

Table VII

Total Volume Estimation Bias for

Combinations of Species

equation 1

MB_c (in cubic feet) of volume for the species

	D			C		S			B		
combination	1	2	3	1	3	1	2	3	1	2	3
size class											
1	1.84	1.95	2.09	-0.37	-0.29	-0.46	-0.33	-0.18	-3.88	-3.85	-3.96
2	5.17	5.08	4.71	3.60	2.48	-1.26	-1.20	-1.31	-5.50	-6.37	-8.98
3	7.01	6.38	5.44	3.91	1.87	-1.24	-1.39	-2.11	-1.46	-3.79	-9.61
4	12.40	11.19	8.36	19.19	14.41	-2.13	-2.65	-4.09	-6.62	-10.08	-18.57
5	19.25	17.47	13.13	27.06	20.27	2.68	1.87	-0.51	-7.26	-12.27	-25.53
MB_t	6.99	6.52	5.41	4.77	3.43	-0.71	-0.79	-1.21	-4.57	-5.64	-8.56

MB_c (in cubic feet) of volume for the species

	A		Cot		Pl		Pw	
combination	1	4	1	4	1	5	1	5
size class								
1	0.00	-0.49	0.75	0.32	-1.08	-0.11	-0.84	0.55
2	-0.13	-0.65	1.20	0.57	-2.54	-0.80	-2.19	0.05
3	-1.91	-2.37	1.69	1.03	-2.84	-0.89	-1.33	0.89
4	-1.50	-2.44	3.04	1.81	-1.10	0.82	0.47	2.29
5	2.26	0.32	4.83	3.47	-2.89	-1.13	-3.40	-1.99
MB_t	-0.33	-1.01	1.45	0.81	-1.80	-0.34	-1.42	0.40

4.1.6. Volume equations for data adjusted for butt flare

The previous fittings and tests were carried out on the data exactly as recorded. Later on, however, taper equations will be fitted on both the original data and data adjusted for butt flare. The adjusting consists of replacing the original observation of diameter inside bark at one foot by the diameter at breast height outside bark. This results in a strongly reduced butt flare which might allow a much better fit of the taper equations. To make it possible to compare these taper equations fitted on adjusted taper data with volume-based taper equations, these volume equations too should be fitted on volumes adjusted for butt flare. Therefore all volume equations have been fitted on the adjusted data as well. The coefficients for equations 1 and 4 are in Appendix 3.4. and the results from the total volume estimation tests are in table VIII.

Volumes of the adjusted data are smaller for most species. This results in volume equations giving lower estimates than the non-adjusted equations. Adjusting the data does not mean very much for Douglas-fir and cottonwood where about half of the trees have a D bigger than the d_1. It means much more for species with an important butt flare, like cedar, where adjusting may reduce the total volume by as much as 14 cu. ft.

Table VIII

Total Volume Estimation Tests of Volume-Based Volume Equations for Data Adjusted for Butt Flare

	equation 1				equation 4			
	SE_c (in cubic feet) of volume for the species							
size class	D	C	A	P1	D	C	A	P1
1	1.45	1.54	0.49	1.03	2.02	2.32	0.62	1.26
2	4.39	5.63	1.80	1.93	4.01	5.62	1.97	2.30
3	7.02	9.94	1.78	3.75	9.24	11.20	2.31	4.24
4	9.33	7.85	3.74	5.60	8.45	8.72	3.55	5.84
5	13.45	12.32	2.68	5.71	16.19	13.64	4.35	5.78
SE_t	7.20	5.99	1.99	3.02	8.41	6.62	2.67	3.26
	equation 1				equation 4			
	MB_c (in cubic feet) of volume for the species							
size class	D	C	A	P1	D	C	A	P1
1	-0.05	-0.07	-0.05	-0.02	0.91	0.95	0.27	0.46
2	-0.34	0.03	0.36	-0.40	-0.77	-1.67	-0.20	-1.09
3	-0.90	-3.32	-1.10	-0.48	-3.32	-5.62	-1.89	-0.99
4	-0.77	1.60	-0.34	1.57	-1.06	0.98	-0.39	1.66
5	0.86	1.16	1.28	-0.74	2.18	2.62	2.82	0.25
MB_t	-0.14	-0.10	-0.06	-0.04	0.00	0.00	0.00	0.00

4.2. Volume-based taper equations

4.2.1. Derivation of compatible taper equations from volume equations

The reasoning process by which a compatible taper equation is derived from a volume equation is based on the premise that total volume estimates, based on integration of the taper equation, must be identical to those given by the existing volume equation.

This means that :

$$\int_0^H (\Pi\, d^2/(4(144)))\ d\underline{l} = \text{Volume Function (let's call it VF)}$$

or alternatively :

$$\Pi\, d^2 H/(4(144)) = VF$$

The value of d^2 can be calculated specifically as:

$$d^2 = 4(144)\ VF\ /(\Pi\, H)$$

From here a more generalized taper function can be derived. Using the taper data, the values of the unknown parameters (free parameters) of these taper functions can be derived by a least squares procedure so as to minimize the SE_E of diameter inside bark.

The taper functions derived from the volume equations are the following:

1^t. $d = a\ D^b\ \underline{l}^c\ H^e$

2^t. $d = a\ D^b\ \underline{l}^c\ H^e$

3^t. $d = (a\ D^2\underline{l}^p/(b\ H^{p+1} + c\ H^p))^{\frac{1}{2}}$

4^t. $d = (a\ \underline{l}^p/\ H^{p+1} + b\ D^2\underline{l}^q/\ H^q)^{\frac{1}{2}}$

5^t. $d = (a\ \underline{l}^p/\ H^{p+1} + b\ D^2\underline{l}^q/\ H^q)^{\frac{1}{2}}$

6^t. $d = (a\ \underline{l}^p/\ H^{p+1} + b\ D\ \underline{l}^q/\ H^{q+1} + c\ \underline{l}^r/\ H^r + e\ D\ \underline{l}^s/\ H^s + f\ D^2\underline{l}^t/\ H^{t+1} + g\ D^2\underline{l}^u/\ H^u)^{\frac{1}{2}}$

$$7^t.\ d = (a\ \underline{1}^p /\ H^{p+1} + b\ D\ \underline{1}^q /\ H^{q+1} + c\ \underline{1}^r /\ H^r + e\ D\ \underline{1}^s /\ H^s + f\ D^2\underline{1}^t /\ H^{t+1} + g\ D^2\underline{1}^u /\ H^u)^{\frac{1}{2}}$$

$$8^t.\ d = a\ D(\underline{1}\ /\ H)^{p/2}$$

$$9^t.\ d = (a\ D^2\underline{1}^p /\ H^{p+1} + b\ D^2\underline{1}^q /\ H^q)^{\frac{1}{2}}$$

$$10^t.\ d = (a\ D^2\underline{1}^p /\ H^{p+1} + b\ D^2\underline{1}^q /\ H^q + c\ D^2\underline{1}^r /\ H^{r-1})^{\frac{1}{2}}$$

$$11^t.\ d = (a\ D\ \underline{1}^p /\ H^{p+1} + b\ D\ \underline{1}^q /\ H^q + c\ D^2\underline{1}^r /\ H^{r+1} + e\ D^2\underline{1}^s /\ H^s)^{\frac{1}{2}}$$

$$12^t.\ d = (a\ D^2\underline{1}^p /\ H^p + b\ D^2\underline{1}^q H^{2-q}/(H - 4.5)^2)^{\frac{1}{2}}$$

Throughout the text and appendixes these volume-based taper equations have the same number as the volume equations from which they are derived,except that a subscript "t" is added to distinguish them as taper equations (see Appendix 2).

The derivation of these taper equations,the formulae to compute height and section volume as well as the meaning of all these coefficients is given in Appendix 4.

The coefficients p,q,...,u are called here "free parameters" and a,b,c,e,f and g are coefficients whose value is based on the volume equation coefficients $b_o, b_1, \ldots, b_5$ and on the values of the free parameters.

It was first attempted to fit all these taper equations on the taper data by a non-linear least squares procedure in order to minimize the SE_E of d.This was carried out for most functions and most species with satisfactory results.For some equations,however, the derivation became troublesome (equations 10^t,11^t and 12^t) and sometimes practically impossible as was the case for equations 6^t and 7^t.Difficulties are caused by too many negative coefficients in the volume equations resulting in negative diameters.No suitable set of values for the free parameters can be found to make d positive for all heights. After removing these troublesome coefficients, the taper equations were often conditioned such that they become useless for estimating d. Also, conditioning of these taper equations meant that they lost their compatability with the volume equations from which they are derived. Thus, although the derivation of a taper equation from a given volume function may be theoretically possible, practically a meaningful derivation may be impossible.

The fact that no useful taper equation can be derived can be considered a weak point for a volume function if a compatible system of taper and volume is desired.

It was also tested for taper equations, with two or more free parameters to be estimated, how many of these parameters could be kept constant without any significant loss. These tests showed clearly that optimizing only one parameter, while the others are kept constant with appropriate values, does not result in any significant loss in precision and accuracy. This greatly facilitates the estimation procedure.

A summary of the values of the free parameters of these volume-based taper equations as well as the SE_E of d is given in table IX. The taper equations with only one free parameter ($1^t, 2^t, 3^t$ and 8^t) had a p-value usually ranging from 1.3 to 2.0. It may seem unusual that each species has almost exactly the same p-value for each of the four equations. This may be explained by the fact that, after the unimportant equation terms are eliminated, they have almost identical forms.

For the equations with two free parameters ($4^t, 5^t, 9^t$ and 12^t) the parameter p was usually kept constant with value 1. The estimated values of the other parameter q are closely related to the parameter values in

Table IX

Taper Equations Derived from the Linear Volume Equations and Their Standard Errors of Estimate

Parameter values of the taper equations

species	1^t	2^t	3^t	4^t		5^t		8^t	9^t	
	p	p	p	p	q	p	q	p	p	q
D	1.3	1.3	1.3	1.0	1.3	1.0	1.3	1.3	1.0	1.3
C	2.0	2.0	2.0	1.0	2.1	1.0	2.0	2.0	1.0	3.4
S	1.6	1.6	1.6	1.0	1.7	1.0	1.6	1.7	1.0	1.8
B	1.6	1.6	1.6	1.0	1.7	1.0	1.6	1.6	1.0	1.8
A	1.5	1.5	1.5	1.0	1.5	1.0	1.5	1.5	1.0	1.5
Cot	1.6	1.6	1.6	1.0	1.6	1.0	1.6	1.6	-1.0	1.5
Pl	1.4	1.4	1.4	1.0	1.5	1.0	1.4	1.4	1.0	1.5
Pw	1.5	1.5	1.5	1.0	1.6	1.0	1.5	1.5	1.0	1.7

Parameter values of the taper equations

species	10^t			11^t				12^t	
	p	q	r	p	q	r	s	p	q
D	1.0	1.8	1.0	-1.0	1.0	-1.0	1.2	1.0	1.3
C	1.0	2.5	1.0	-1.0	1.0	-1.0	5.6	(-	-)[2]
S	-1.0	1.5	-1.0	-1.0	1.0	-1.0	2.2	1.0	1.9
B	-1.0	1.4	-1.0	-1.0	1.0	-1.0	1.7	-1.0	1.5
A	-1.0	1.4	-1.0	-1.0	1.0	-1.0	1.4	1.5	1.5
Cot	1.0	1.8	1.0	-1.0	1.0	-1.0	1.6	(-	-)[2]
Pl	1.0	1.4	-1.0	-1.0	1.0	-1.0	1.6	-1.0	1.4
Pw	-1.0	1.4	-1.0	-1.0	1.0	-1.0	1.7	(-	-)[2]

[2] no useful taper equation could be derived

Table IX (continued)

species	SE_E(in inches) of d for the taper equations 1^t	2^t	3^t	4^t	5^t	8^t	9^t	10^t	11^t	12^t
D	1.04	1.06	1.09	1.10	1.06	1.09	1.07	1.07	1.74	1.07
C	1.97	1.98	2.00	2.01	1.95	2.14	1.80	1.89	1.70	( )[3]
S	1.43	1.43	1.42	1.45	1.41	1.45	1.41	2.10	1.56	1.40
B	2.09	2.08	2.07	2.21	2.06	2.17	2.06	3.32	2.30	2.17
A	1.10	1.11	1.11	1.12	1.11	1.11	1.11	1.22	1.60	1.11
Cot	0.51	0.53	0.54	0.55	0.54	0.54	0.64	0.55	1.58	( )[3]
Pl	0.89	0.89	0.89	0.91	0.88	0.93	0.89	1.13	0.91	0.94
Pw	1.24	1.24	1.24	1.25	1.22	1.26	1.22	1.78	1.36	( )[3]

[3] no useful taper equation could be derived

Table X

Taper Equations Derived from the Non-Linear Volume Equations and Their Standard Errors of Estimate

species	Parameter values p of the equations 13^t	$13^t(w)$	14^t	$14^t(w)$	SE_E(in inches) of d for the equations 13^t	$13^t(w)$	14^t	$14^t(w)$
D	1.3	1.3	1.3	1.3	1.05	1.04	1.13	1.08
C	2.0	2.0	2.0	2.0	1.99	1.97	2.04	1.99
S	1.6	1.6	1.6	1.6	1.44	1.42	1.43	1.42
B	1.6	1.6	1.6	1.6	2.10	2.09	2.13	2.08
A	1.5	1.5	1.5	1.5	1.10	1.10	1.11	1.11
Cot	1.6	1.6	1.6	1.6	0.52	0.51	0.54	0.54
Pl	1.4	1.4	1.4	1.4	0.89	0.88	0.91	0.89
Pw	1.5	1.5	1.5	1.5	1.24	1.23	1.24	1.23

the equations with only one parameter.

The derivation of taper equation 9^t was a problem in the case of cottonwood which had a negative b_o value in volume equation 9.

For equation 10^t,useful functions could be derived for four species.A meaningful derivation was impossible for the other species, having negative b_o and b_2 coefficients in volume equation 10.

In equation 11^t,two terms had to be eliminated for all species. Except for Douglas-fir and cottonwood,the results were still reasonable.

The derivation of 12^t was useful for some species but did not work for others (western redcedar,cottonwood and white pine).

Looking at the SE_E's of d,the results are almost identical for most taper equations (ranging,for Douglas-fir,from 1.04 to 1.10 inches for nine equations out of ten).

Taper equations were also derived from the non-linear volume equations 13,13(w),14 and 14(w),whose results are given in table X. The values of the parameters of the weighted non-linear volume equations are the same as for the unweighted equations and the SE_E's of d are identical or differ only by insignificant amounts (maximum 0.05 inches).

In all previous taper equations d was used as dependent variable. To check what differences occur when other dependent variables are used,taper equation 1^t was fitted in four different ways for Douglas-fir and aspen.As can be seen in table XI the values of the parameters for different fittings sometimes differ (maximum difference is 0.3), but the differences in standard error of estimate are minor (maximum 0.08 inches).This,however,does not mean that it is immaterial which p-value is used.

From comparisons of the SE_E's of these taper equations, it would seem as if more or less identical taper equations can be derived from volume equations which differ substantially. This may seem contradictory because, after all, if a volume equation is biased then the taper equation, which integrates to the same volume, should be also biased. And there are large differences in bias of the different volume equations. This illustrates the fact that the SE_E of d is a poor measure of comparison in those cases where there is bias and variances are heterogeneous. Different equations may have an identical SE_E of d, but their pattern of bias of under- and overestimation for the different size classes may be quite different depending on the functional form of the equation or which p-value is used.

The taper equations are also used to estimate total tree volumes and volumes of particular tree sections. Different patterns of bias in diameter estimation will result in a different bias of volume depending on the size of the tree and the height where the bias occurs. Volumes of tree sections are obtained by integrating the taper equation between the appropriate limits. These limits of integration may be defined in different ways. They can be expressed as section heights in which case the integration is straightforward. If the limits are given as section diameters, then the corresponding section heights should first be estimated by solving the taper equation for the heights, before the integration can be carried out. This last procedure introduces another source of error section volume estimation. If the diameter for a given height is biased, then so will be the height for a given diameter.

To determine the overall performance of the taper equations, they will not only be tested for bias in diameter estimation but also for bias in section volume estimation with known heights, for bias in section heights and for bias in section volume estimation with unknown heights. The bias in total volume estimation has already been tested while testing the volume equations.

Table XI

Taper Equation 1^t Derived with Different Dependent Variables

	Douglas-fir		Aspen	
dependent variable	parameter p	SE_t (inches) of d	parameter p	SE_t (inches) of d
d	1.30	1.04	1.50	1.10
log d	1.30	1.04	1.55	1.11
d^2	1.40	1.08	1.80	1.18
d^2/ D^{b_1}	1.35	1.06	1.65	1.13

4.2.2. Tests of diameter estimation

To test the diameter estimation of the taper equations, approximate standard errors of estimate SE_c and estimations of the bias MB_c of d are computed for all size classes and for the different heights within each size class. The heights of interest are 1 foot, 4.5 feet and each tenth of the height between breast height and tree top. The standard errors of estimate and the estimations of bias are computed in the same way as was done for the total volume estimation of the volume equations.

A comparison of the overall SE_E's of d has been made already in the previous section.

As an example of a complete diameter estimation test, all results for one equation and one species are given in Table XII. It would be too lengthy to reproduce here all the results for all equations and all species. Some tables of results and examples of output, useful for comparison, are given and all important features are discussed.

For reasons mentioned in a previous section, most emphasis will be on bias rather than on standard error of estimate.

To compare the performance of different taper equations on the same species, all taper equations were tested on Douglas-fir, aspen and cottonwood. The results concerning the total bias for Douglas-fir are presented in table XIII with some results for aspen.

To compare the performance of the same taper equations on different species, four equations were tested on all species. Results for equations 1^t and 8^t are in table XIV.

Table XII

Example of a Diameter Estimation Test of a

Volume-Based Taper Equation

Test of taper equation 1[t] for Douglas-fir

Mean d (in inches) at heights

size class	1'	4.5'	.1H	.2H	.3H	.4H	.5H	.6H	.7H	.8H	.9H	1 H
1	9.4	8.2	7.6	7.2	6.8	6.3	5.8	5.0	4.3	3.2	1.9	0.0
2	15.2	13.0	12.0	11.1	10.6	9.9	9.0	8.0	6.8	5.5	3.2	0.0
3	18.5	15.6	14.3	13.5	12.5	11.7	10.8	9.6	8.0	6.1	3.4	0.0
4	23.1	18.5	16.6	15.6	14.8	14.0	12.9	11.5	9.5	7.2	3.8	0.0
5	24.2	20.4	18.1	17.1	16.0	14.9	13.9	12.3	10.1	7.8	4.3	0.0
tot.	15.4	13.0	11.9	11.2	10.5	9.8	9.0	8.0	6.7	5.1	2.9	0.0

SE_c (in inches) of diameter

size class	1'	4.5'	.1H	.2H	.3H	.4H	.5H	.6H	.7H	.8H	.9H	1 H
1	0.9	0.5	0.6	0.5	0.4	0.5	0.5	0.5	0.5	0.5	0.4	0.0
2	2.0	0.8	0.8	0.8	0.6	0.6	0.7	0.8	0.8	1.1	0.3	0.0
3	2.8	0.6	0.8	0.8	0.8	0.8	1.0	1.1	1.1	0.9	0.6	0.0
4	4.3	0.7	1.3	1.2	0.8	1.0	0.9	1.3	1.1	1.1	0.8	0.0
5	3.5	0.6	1.5	1.3	1.1	0.9	1.1	1.3	1.0	1.1	1.3	0.0
SE_t	2.4	0.6	0.9	0.8	0.7	0.7	0.8	0.9	0.8	0.9	0.7	0.0

MB_c (in inches) of diameter

size class	1'	4.5'	.1H	.2H	.3H	.4H	.5H	.6H	.7H	.8H	.9H	1 H
1	-0.5	0.4	0.4	0.2	0.0	-0.1	-0.3	-0.3	-0.3	-0.1	0.1	0.0
2	-1.4	0.6	0.6	0.5	0.1	-0.2	-0.4	-0.5	-0.6	-0.8	-0.1	0.0
3	-2.1	0.5	0.7	0.5	0.2	-0.1	-0.5	-0.7	-0.6	-0.4	0.2	0.0
4	-3.8	0.5	1.2	0.8	0.2	-0.3	-0.8	-1.0	-0.8	-0.5	0.6	0.0
5	-3.1	0.4	1.3	0.9	0.5	-0.0	-0.6	-0.8	-0.6	-0.4	0.4	0.0
MB_t	-1.6	0.5	0.7	0.5	0.2	-0.1	-0.4	-0.5	-0.5	-0.3	0.2	0.0

Table XIII

Total Diameter Bias of Volume-Based Taper Equations

for Douglas-fir and Aspen

Douglas-fir

taper eq.	MB_t of diameter (in inches) at heights 1'	4.5'	.1H	.2H	.3H	.4H	.5H	.6H	.7H	.8H	.9H	1 H
1^t(p=1.3)	-1.6	0.5	0.7	0.5	0.2	-0.1	-0.4	-0.5	-0.5	-0.3	0.2	0.0
1^t(p=1.4)	-1.3	0.7	0.9	0.6	0.2	-0.2	-0.6	-0.7	-0.7	-0.6	-0.1	0.0
2^t	-1.6	0.5	0.7	0.5	0.2	-0.1	-0.4	-0.5	-0.5	-0.3	0.2	0.0
3^t	-1.6	0.4	0.7	0.5	0.2	-0.1	-0.4	-0.5	-0.5	-0.3	0.2	0.0
4^t	-1.5	0.5	0.8	0.6	0.3	-0.0	-0.3	-0.4	-0.4	-0.2	0.3	0.0
5^t	-1.5	0.5	0.8	0.5	0.2	-0.1	-0.4	-0.5	-0.4	-0.3	0.2	0.0
8^t	-1.8	0.3	0.5	0.3	0.0	-0.3	-0.5	-0.6	-0.6	-0.4	0.1	0.0
9^t	-1.5	0.5	0.8	0.5	0.3	-0.1	-0.3	-0.4	-0.4	-0.2	0.3	0.0
10^t	-1.5	0.6	0.8	0.5	0.2	-0.1	-0.4	-0.5	-0.4	-0.1	0.5	0.0
11^t	-0.0	2.0	2.3	2.1	1.8	1.5	1.1	1.0	1.0	1.0	1.3	0.0
12^t	-1.5	0.5	0.8	0.5	0.3	-0.1	-0.4	-0.5	-0.4	-0.3	0.2	0.0
13^t	-1.6	0.5	0.8	0.5	0.2	-0.1	-0.4	-0.5	-0.5	-0.3	0.2	0.0
13^t (w)	-1.6	0.5	0.7	0.5	0.2	-0.1	-0.4	-0.5	-0.5	-0.3	0.2	0.0
14^t	-2.0	0.1	0.4	0.1	-0.1	-0.4	-0.7	-0.7	-0.7	-0.5	0.1	0.0
14^t (w)	-1.5	0.6	0.8	0.6	0.3	-0.1	-0.4	-0.5	-0.4	-0.3	0.2	0.0

Aspen

taper eq.	MB_t of diameter (in inches) at heights 1'	4.5'	.1H	.2H	.3H	.4H	.5H	.6H	.7H	.8H	.9H	1 H
1^t(p=1.5)	-1.4	0.6	0.6	0.3	0.1	-0.1	-0.3	-0.4	-0.4	-0.2	0.2	0.0
1^t(p=1.6)	-1.2	0.8	0.7	0.4	0.1	-0.2	-0.4	-0.6	-0.5	-0.3	0.0	0.0
1^t(p=1.7)	-1.1	0.9	0.8	0.4	0.1	-0.2	-0.5	-0.7	-0.7	-0.5	-0.1	0.0
1^t(p=1.8)	-0.9	1.1	0.9	0.5	0.1	-0.3	-0.6	-0.9	-0.9	-0.7	-0.3	0.0

Table XIV

Total Diameter Bias of Equations 1^t and 8^t

Equation 1^t

MB_t of diameter (in inches) at heights

species	1'	4.5'	.1H	.2H	.3H	.4H	.5H	.6H	.7H	.8H	.9H	1 H
D	-1.6	0.5	0.7	0.5	0.2	-0.1	-0.4	-0.5	-0.5	-0.3	0.2	0.0
C	-3.7	0.9	1.7	1.1	0.4	-0.1	-0.6	-0.9	-1.2	-1.0	-0.6	0.0
S	-2.8	0.7	0.9	0.6	0.2	-0.2	-0.4	-0.5	-0.4	-0.3	-0.2	0.0
B	-3.5	0.9	1.5	0.8	0.2	-0.3	-0.6	-0.9	-1.0	-0.8	-0.4	0.0
A	-1.4	0.6	0.6	0.3	0.1	-0.1	-0.3	-0.4	-0.4	-0.2	0.2	0.0
Cot	-0.3	0.4	0.3	0.1	-0.1	-0.2	-0.2	-0.2	-0.0	0.1	0.2	0.0
Pl	-1.5	0.4	0.5	0.4	0.2	-0.0	-0.2	-0.4	-0.3	-0.2	0.1	0.0
Pw	-2.7	0.5	1.0	0.6	0.1	-0.3	-0.5	-0.5	-0.5	-0.3	-0.0	0.0

Equation 8^t

MB_t of diameter (in inches) at heights

species	1'	4.5'	.1H	.2H	.3H	.4H	.5H	.6H	.7H	.8H	.9H	1 H
D	-1.8	0.3	0.5	0.3	0.0	-0.3	-0.5	-0.6	-0.6	-0.4	0.1	0.0
C	-4.7	0.0	0.9	0.3	-0.2	-0.7	-1.1	-1.3	-1.5	-1.2	-0.7	0.0
S	-2.7	0.7	0.9	0.4	0.0	-0.4	-0.7	-0.7	-0.7	-0.6	-0.4	0.0
B	-4.1	0.3	1.0	0.3	-0.3	-0.7	-0.9	-1.2	-1.2	-0.9	-0.5	0.0
A	-1.5	0.5	0.6	0.3	0.0	-0.2	-0.3	-0.5	-0.4	-0.2	0.2	0.0
Cot	-0.4	0.3	0.2	0.0	-0.2	-0.2	-0.2	-0.2	-0.1	0.1	0.2	0.0
Pl	-1.7	0.1	0.3	0.2	-0.0	-0.2	-0.4	-0.5	-0.4	-0.3	0.0	0.0
Pw	-2.9	0.3	0.8	0.5	-0.0	-0.4	-0.6	-0.6	-0.5	-0.3	-0.1	0.0

Most equations have basically the same pattern of bias (see figure 3).The stump diameter (at one foot) is always underestimated. The lower 30 to 40% of the tree is overestimated and the upper part is again underestimated,except for the extreme top section.
The pattern of bias is more or less the same for all species.The size of the bias differs slightly from one equation to another.Equations which seem to give less underestimation in the upper part usually have more overestimation in the lower part and vice versa.

These results do not explain by themselves why there are these large differences in bias of total volume in the volume equations from which these taper equations are derived.It is therefore necessary to examine the distribution of the bias over the different size classes.It is here that the differences among the taper equations become apparent.

Taper equation 1^t has more or less the same pattern of bias for all size classes.Equation 4^t overestimates almost the entire profile of the smallest and largest trees,but gives much lower estimates than 1^t for the middle sized trees.This is due to the fact that the combined variable volume equation 4 overestimates total volume of smallest and largest trees and underestimates the middle sized trees.Equation 5^t is more negatively biased for the smaller trees and more positively biased for the larger trees.This could be expected,knowing the bias of the weighted combined variable volume equation 5.

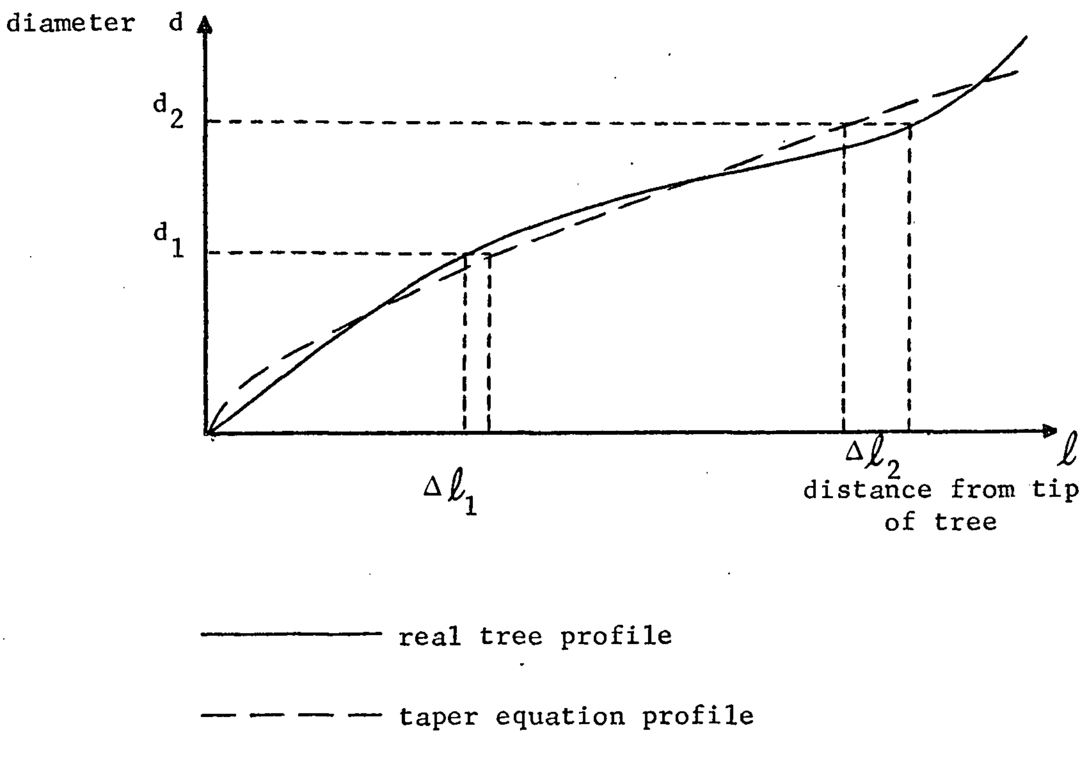
The way the bias of taper estimation differs from one equation to another and from one size class to another can be explained by the bias of total volume of the corresponding volume equations.

The bias of diameter,added over all size classes,is fairly

Figure 3

Pattern of Bias in Diameter and Height Estimation

for Volume-Based Taper Equations



d_i , i = 1,2 a given diameter

$\Delta\ell_i$, i = 1,2 bias of distance for d_i

constant, but there is a substantial difference among the taper equations in the way the pattern of bias changes from one size class to another. In table XII, the pattern of bias for the given example is constant for all size classes. Table XV shows an example where the pattern of bias changes from one size class to another. In this case, the equation looks excellent, total bias being nearly zero, however, there is substantial bias in each size class. The positive bias in one class is eliminated by the negative bias in another.

The absolute bias at the different heights is for most species fairly small (seldom larger than one inch, except at one foot). The bias at one foot may be larger due to butt flare.

Results are usually poor for the taper equations which are conditioned (e.g. equation 11^t).

The taper equations, derived from the non-linear volume equations, are sometimes similar to the linear equations (equation 13^t), sometimes different (equation 14^t). Weighting of the non-linear equations makes no difference for equation $13^t(w)$, but increases the bias in the lower part and decreases the bias in the upper part of the tree for equation $14^t(w)$.

Fitting taper equation 1^t with d^2 or d^2/D^{b1} as dependent variable causes more overestimation in the lower tree and more underestimation in the upper tree. Using d^2 gives more weight to diameters which were already overestimated.

Table XV

Pattern of Diameter Bias of Equation 3^t for Cottonwood

size class	MB_c (in inches) of diameter at heights 1'	4.5'	.1H	.2H	.3H	.4H	.5H	.6H	.7H	.8H	.9H	1 H
1	-0.2	0.2	0.1	-0.0	-0.2	-0.2	-0.2	-0.1	-0.0	0.1	0.2	0.0
2	-0.3	0.3	0.2	-0.1	-0.2	-0.2	-0.3	-0.3	-0.1	0.2	0.2	0.0
3	-0.3	0.7	0.6	0.1	-0.1	-0.2	-0.3	-0.3	-0.2	0.0	0.1	0.0
4	-0.5	0.7	0.7	0.3	0.1	0.0	0.0	-0.0	0.1	0.4	0.5	0.0
5	-0.8	1.1	0.9	0.7	0.5	0.4	0.3	0.3	0.3	0.3	-0.0	0.0
MB_t	-0.3	0.4	0.3	0.1	-0.1	-0.1	-0.2	-0.1	-0.0	0.1	0.2	0.0

4.2.3. Tests of section volume estimation with known heights

When the heights of the section are known, the volume is computed by a straightforward integration of the taper equation between these two heights:

$$V_s = \int_{\underline{l}_2}^{\underline{l}_1} (d^2/\ k)\ d\underline{l}$$

The section volume equations are worked out for all volume-based taper functions in Appendix 4.

In this case the pattern of bias of section volume will be the same as for diameter. The absolute value of the bias will be more or less linearly related to the bias of squared diameter and section length.

Tests, identical to those for diameter, were carried out for section volumes. The section volumes of interest are the section below 4.5 feet and each tenth of the bole above breast height.

There was no need to repeat these tests for all equations and species since the way equations and species compare with each other remains the same as for diameter estimation.

To get an indication of absolute values of bias which may be involved, a complete example of a test is given in table XVI. Several equations are compared for the same species in table XVII.

Between most equations, the differences in bias, added over all size classes, are minor (usually much less than 0.5 cu. ft.), but differences in bias for individual size classes may be important (1 cu. ft. and more). They lead to the large differences in bias of total volume. The bias for the largest size class is compared for a few equations for Douglas-fir in table XVIII.

Part of the bias is due to the different ways volumes are computed. The volume observations are based on Smalian's formula, which assumes a paraboloid form. The top section was assumed to have a cylindrical form factor of 0.4 while the stump was taken as a cylinder. Even if the taper equation would give an exact estimation of the tree profile, there would be a discrepancy between observed and predicted value. This discrepancy would only be important for top and bottom.

The actual values of the bias are fairly small (usually less than 1 cu. ft.). Bias increases towards the lower part of the tree.

Table XVI

Example of a Section Volume Estimation Test with Known Heights of a Volume-Based Taper Equation

Test of taper equation 1^t for Douglas-fir

size class	Mean V_s (in cubic feet) at heights										
	<4.5'	.1H	.2H	.3H	.4H	.5H	.6H	.7H	.8H	.9H	1 H
1	2.11	2.78	2.44	2.20	1.94	1.66	1.34	1.00	0.66	0.32	0.08
2	5.16	8.56	7.35	6.47	5.78	4.93	3.97	3.02	2.12	1.13	0.23
3	7.52	15.15	13.05	11.48	9.96	8.57	7.09	5.33	3.64	1.69	0.33
4	11.40	21.32	17.98	16.04	14.34	12.51	10.33	7.73	5.00	2.35	0.43
5	12.84	26.76	22.33	19.72	17.26	14.98	12.42	9.24	5.95	2.93	0.59
tot.	6.02	11.25	9.55	8.45	7.43	6.42	5.28	3.95	2.59	1.28	0.26
size class	SE_c (in cubic feet) of section volume										
1	0.26	0.42	0.40	0.27	0.27	0.29	0.29	0.30	0.17	0.10	0.04
2	0.94	1.07	1.01	0.87	0.65	0.64	0.67	0.61	0.61	0.41	0.04
3	1.51	1.33	1.41	1.25	1.21	1.31	1.39	1.26	0.93	0.49	0.12
4	2.72	2.22	2.51	1.92	1.61	1.73	1.85	1.75	1.32	0.69	0.22
5	2.38	2.75	3.35	2.78	2.11	1.94	2.20	1.94	1.38	1.01	0.38
SE_t	1.44	1.49	1.72	1.42	1.14	1.12	1.23	1.11	0.82	0.53	0.18
size class	MB_c (in cubic feet) of section volume										
1	-0.10	0.31	0.25	0.09	-0.04	-0.13	-0.15	-0.14	-0.10	-0.02	0.01
2	-0.52	0.83	0.78	0.44	-0.04	-0.30	-0.40	-0.44	-0.44	-0.26	0.01
3	-0.98	1.23	1.12	0.57	0.05	-0.51	-0.88	-0.85	-0.57	-0.19	0.06
4	-2.31	2.01	2.22	1.14	-0.06	-1.00	-1.44	-1.29	-0.81	-0.15	0.17
5	-1.97	2.26	2.80	1.65	0.50	-0.68	-1.38	-1.26	-0.77	-0.23	0.13
MB_t	-0.80	1.01	1.06	0.57	0.06	-0.38	-0.62	-0.58	-0.40	-0.13	0.05

Table XVII

Bias of Section Volume Estimation with Known Heights of Volume-Based Taper Equations for Douglas-fir and Aspen

Douglas-fir

taper eq.	MB_t (in cubic feet) of section volume at heights <4.5'	.1H	.2H	.3H	.4H	.5H	.6H	.7H	.8H	.9H	1 H
1^t(p=1.3)	-0.80	1.01	1.06	1.57	0.06	-0.38	-0.62	-0.58	-0.40	-0.13	0.05
1^t(p=1.4)	-0.58	1.43	1.30	0.66	0.03	-0.50	-0.80	-0.79	-0.60	-0.29	-0.01
2^t	-0.80	1.01	1.06	0.57	0.07	-0.38	-0.62	-0.58	-0.40	-0.13	0.05
3^t	-0.77	0.99	1.05	0.56	0.06	-0.39	-0.63	-0.59	-0.40	-0.13	0.05
4^t	-0.77	0.98	1.05	0.57	0.08	-0.36	-0.59	-0.55	-0.37	-0.11	0.06
5^t	-0.70	1.24	1.26	0.74	0.21	-0.26	-0.52	-0.51	-0.35	-0.10	0.06
8^t	-0.88	0.84	0.92	0.45	-0.04	-0.46	-0.68	-0.63	-0.43	-0.15	0.05
9^t	-0.69	1.25	1.29	0.78	0.25	-0.21	-0.48	-0.46	-0.31	-0.07	0.07
10^t	-0.66	1.28	1.23	0.68	0.14	-0.31	-0.54	-0.47	-0.24	0.04	0.16
11^t	0.34	3.63	3.55	2.90	2.21	1.55	1.07	0.83	0.69	0.58	0.32
12^t	-0.70	1.20	1.24	0.73	0.21	-0.26	-0.52	-0.50	-0.34	-0.09	0.06
13^t	-0.79	1.04	1.09	0.60	0.09	-0.36	-0.61	-0.57	-0.39	-0.13	0.05
13^t (w)	-0.78	1.07	1.11	0.61	0.10	-0.35	-0.60	-0.57	-0.39	-0.13	0.05
14^t	-0.98	0.71	0.81	0.35	-0.12	-0.53	-0.73	-0.66	-0.45	-0.16	0.04
14^t (w)	-0.67	1.26	1.28	0.76	0.22	-0.26	-0.52	-0.51	-0.35	-0.11	0.06

Aspen

taper eq.	MB_t (in cubic feet) of section volume at heights <4.5'	.1H	.2H	.3H	.4H	.5H	.6H	.7H	.8H	.9H	1 H
1^t(p=1.5)	-0.55	0.57	0.44	0.19	-0.01	-0.14	-0.21	-0.21	-0.14	-0.04	0.01
1^t(p=1.8)	-0.26	0.93	0.62	0.23	-0.08	-0.28	-0.39	-0.40	-0.29	-0.14	-0.03

Table XVIII

Section Volume Bias of Largest Size Class of Several Taper Equations for Douglas-fir

taper eq.	MB_c (in cubic feet) of section volume of largest size class at heights <4.5'	.1H	.2H	.3H	.4H	.5H	.6H	.7H	.8H	.9H	1 H
1^t	-1.97	2.26	2.80	1.65	0.50	-0.68	-1.38	-1.26	-0.77	-0.23	0.13
3^t	-1.74	2.60	3.10	1.90	0.70	-0.51	-1.25	-1.16	-0.71	-0.20	0.13
4^t	-1.83	2.47	2.99	1.82	0.65	-0.54	-1.26	-1.16	-0.70	-0.18	0.14
5^t	-1.42	3.55	3.92	2.60	1.29	-0.04	-0.88	-0.89	-0.53	-0.10	0.16
8^t	-1.73	2.73	3.20	1.99	0.78	-0.45	-1.21	-1.13	-0.69	-0.19	0.14
9^t	-1.41	3.58	3.98	2.68	1.39	0.07	-0.77	-0.78	-0.43	-0.03	0.19

4.2.4. Tests of height estimation

The height h of a given diameter is computed by estimating first its distance from the tip, l, and then subtracting this from the total height H of the tree. So, although the title of this section is about the height, the following tests and discussions will center around the distance l from the tip.

Some taper equations (e.g. 1^t, 3^t and 8^t) can simply be transformed into an equation predicting l as a function of d, D and H. These examples were described in Appendix 4. Volume-based taper equations with more than one free parameter usually can not be transformed this way. In these cases, distances must be estimated by an iteration procedure. Two methods of iteration were tested, the "Binary chop" and the Newton-Raphson method. A cone approach was used to give a first approximation. The Newton-Raphson method was found to be the most appropriate.

The pattern of bias for distance estimation is exactly the opposite of the pattern for diameter. Distances from the tip of the tree will be underestimated when diameters are overestimated and vice versa (see figure 3).

Of interest in these tests were the distances for the diameters at one foot, 4.5 feet and for the diameters at each tenth of the height. Tables XIX and XX contain a summary of some of the results. They indicate that bias can be very large. Biases of more than five feet are common. Bias of the distance for the diameter at one foot is very large (in many cases more than 20 feet) due to butt flare.

Table XIX

Example of a Distance Estimation Test

of a Volume-Based Taper Equation

Test of taper equation 1^t for Douglas-fir

Mean $\underline{1}$ (in feet) for diameters at heights

size class	1'	4.5'	.1H	.2H	.3H	.4H	.5H	.6H	.7H	.8H	.9H	1 H
1	75.2	71.7	64.6	57.5	50.4	43.3	36.2	29.1	22.0	14.9	7.8	0.0
2	104.5	101.0	90.9	80.9	70.8	60.7	50.7	40.6	30.5	20.5	10.4	0.0
3	128.3	124.8	112.3	99.8	87.3	74.9	62.4	49.9	37.5	25.0	12.5	0.0
4	131.7	128.2	115.5	102.7	89.9	77.2	64.4	51.7	38.9	26.1	13.4	0.0
5	136.3	132.8	119.5	106.3	93.0	79.8	66.6	53.3	40.1	26.8	13.6	0.0
tot.	103.4	99.9	89.9	80.0	70.1	60.1	50.2	40.2	30.3	20.4	10.4	0.0

SE_c (in feet) of distance

size class												
1	11.6	6.1	7.2	5.4	4.5	5.1	5.0	5.2	5.0	3.5	2.3	0.0
2	23.1	9.6	8.6	8.7	7.1	5.9	6.8	7.4	6.6	9.2	1.6	0.0
3	33.3	6.9	8.9	8.2	7.8	7.5	9.7	9.7	8.8	6.3	2.8	0.0
4	42.8	6.7	11.6	11.9	7.2	8.5	7.3	9.4	7.8	7.6	4.0	0.0
5	34.8	6.2	13.1	11.3	9.3	7.4	8.6	9.9	7.0	6.1	5.1	0.0
SE_t	25.9	6.9	9.3	8.3	6.8	6.4	7.4	7.7	6.6	6.0	3.1	0.0

MB_c (in feet) of distance

size class												
1	7.1	-5.2	-5.3	-2.4	-0.3	1.5	3.0	2.6	2.7	0.8	-0.6	0.0
2	16.3	-6.7	-6.9	-5.9	-1.5	2.1	3.3	4.5	4.6	5.7	0.5	0.0
3	25.8	-5.7	-8.2	-4.6	-1.9	1.9	5.5	7.1	5.6	3.4	-0.7	0.0
4	38.8	-5.1	-11.1	-8.3	-3.0	2.0	6.0	7.1	5.3	2.8	-2.6	0.0
5	30.7	-3.6	-11.6	-7.6	-4.1	0.5	4.8	6.5	4.4	2.7	-1.5	0.0
MB_t	18.1	-5.2	-7.5	-4.6	-1.6	1.5	4.0	4.7	4.0	2.5	-0.8	0.0

A bias in diameter estimation of one tenth of an inch causes a bias of approximately one foot in distance estimation. This simple rule is sufficient to get an indication of the bias involved.

Table XX

Bias of Distance Estimation of Some Volume-Based Taper Equations

Douglas-fir

taper eq.	MB_t (in feet) of the distance for diameters at heights 1'	4.5'	.1H	.2H	.3H	.4H	.5H	.6H	.7H	.8H	.9H	1 H
1^t	18.1	-5.2	-7.5	-4.6	-1.6	1.5	4.0	4.7	4.0	2.0	-0.8	0.0
3^t	19.1	-4.3	-6.7	-4.0	-1.0	2.1	4.5	5.1	4.3	2.7	-0.6	0.0
8^t	21.7	-2.3	-4.9	-2.2	0.6	3.5	5.8	6.1	5.1	3.2	-0.4	0.0

Taper equation 1^t

species	MB_t (in feet) of the distance for diameters at heights 1'	4.5'	.1H	.2H	.3H	.4H	.5H	.6H	.7H	.8H	.9H	1 H
D	18.1	-5.2	-7.5	-4.6	-1.6	1.5	4.0	4.7	4.0	2.0	-0.8	0.0
A	14.4	-6.2	-6.0	-3.1	-0.8	1.1	2.3	3.7	3.1	1.4	-0.7	0.0
Cot	3.1	-3.8	-2.6	-0.7	1.2	1.5	1.6	1.5	0.4	-0.7	-1.1	0.0

4.2.5. Tests of section volume estimation with unknown heights

Section limits may be given in terms of diameter. In that case, the section heights must be estimated first before the integration can be carried out. This procedure of section volume estimation is subject to two sources of error, namely bias in diameter estimation and bias in the estimation of both distances from the tip.

The same tests as for section volume estimation with known heights were repeated here and some results are given in table XXI.

Bias in diameter and bias in distance are always of the opposite sign. They may partially eliminate each others' effect on section volume, thus producing a fairly good final result.

Bias in distance usually has more weight than bias in diameter. This results in a pattern of bias similar to the one for distance estimation and opposite to the one of section volume with known heights.

The absolute bias in the upper parts of the tree is comparable with the bias which occurred when heights were known. Bias is larger in the lower parts of the tree. Results are useless for the lower 10% of the tree because of the large bias in distance estimation at butt flare.

One other reason why this kind of estimation is still reasonable, in spite of the large bias in section distances, is the fact that usually both section distances are biased in the same way and more or less by the same amounts.

Table XXI

Bias of Section Volume Estimation with Unknown Heights

of a Volume-Based Taper Equation

Test of taper equation 1^t for Douglas-fir

size class	MB_c (in cubic feet) of section volume at heights .2H	.3H	.4H	.5H	.6H	.7H	.8H	.9H	1 H
1	-0.95	-0.85	-0.53	-0.36	0.08	-0.04	0.18	0.08	0.01
2	-0.66	-2.89	-2.13	-0.49	-0.45	0.03	-0.32	0.68	0.04
3	-3.91	-2.45	-2.92	-2.50	-0.90	0.62	0.60	0.60	0.03
4	-5.32	-6.34	-6.08	-3.17	-1.14	1.18	0.93	1.12	-0.02
5	-7.10	-5.29	-5.98	-4.88	-1.59	1.62	0.65	0.83	0.15
MB_t	-2.78	-2.60	-2.53	-1.72	-0.54	0.46	0.31	0.47	0.02

Taper equation 1^t

species	MB_t (in cubic feet) of section volume at heights .2H	.3H	.4H	.5H	.6H	.7H	.8H	.9H	1 H
D	-2.78	-2.60	-2.53	-1.72	-0.54	0.46	0.31	0.47	0.02
A	-1.38	-0.96	-0.79	-0.33	-0.33	0.10	0.14	0.13	0.01
Cot	-0.79	-0.47	-0.09	-0.04	-0.01	0.09	0.06	0.00	0.01

4.2.6. Tests of volume-based taper equations for data adjusted for butt flare

Taper equations were also derived from some of the volume equations fitted on the adjusted data. The values of the free parameters for equations 1^t and 4^t are given in table XXII, together with their SE_E's of d. The parameter values are smaller than for the non-adjusted data. Smaller parameter values are related to a better tree form. In later sections this is investigated further.

All previously described tests were repeated and the results for equation 1^t are given in table XXIII.

The partial elimination of butt flare improves the estimation of taper. The overall SE_E's of d are decreased by about 50% and the equations are much less biased. There is less overestimation in the lower part of the tree and less underestimation in the upper part. For some species there is a slight increase in bias in the top section (maximum 0.3 inches).

Because of the improvement in the estimation of diameter, there is also an improvement in section volume and height estimation. Only for Douglas-fir and cottonwood are results similar to the unadjusted data, for reasons mentioned in section 4.1.6.

Table XXII

Volume-Based Taper Equations for Data Adjusted for Butt Flare and Their Standard Errors of Estimate

	parameter values of the equations			SE_E of d (in inches) for equations	
	1^t	4^t		1^t	4^t
species	p	p	q		
D	1.3	1.0	1.3	0.92	0.98
C	1.6	1.0	1.8	1.01	1.11
S	1.4	1.0	1.4	0.65	0.75
B	1.4	1.0	1.4	0.89	1.16
A	1.3	1.0	1.4	0.59	0.63
Cot	1.6	1.0	1.6	0.47	0.51
Pl	1.3	1.0	1.3	0.56	0.61
Pw	1.4	1.0	1.4	0.66	0.69

Table XXIII

Tests of Volume-Based Taper Equation 1^t for Data Adjusted for Butt Flare

	MB_t(in inches)of diameter at heights											
species	1'	4.5'	.1H	.2H	.3H	.4H	.5H	.6H	.7H	.8H	.9H	1 H
D	-1.5	0.4	0.7	0.5	0.2	-0.1	-0.4	-0.5	-0.5	-0.3	0.2	0.0
C	-0.7	-0.4	0.8	0.5	0.2	-0.1	-0.3	-0.4	-0.5	-0.3	0.0	0.0
S	-0.4	0.0	0.4	0.2	-0.0	-0.2	-0.3	-0.2	-0.1	0.1	0.2	0.0
B	-0.3	0.0	0.9	0.4	-0.0	-0.3	-0.4	-0.5	-0.5	-0.2	0.2	0.0
A	-0.4	0.1	0.3	0.1	-0.0	-0.1	-0.2	-0.2	-0.1	0.2	0.5	0.0
Cot	-0.2	0.3	0.3	0.0	-0.1	-0.2	-0.2	-0.2	-0.0	0.1	0.2	0.0
Pl	-0.2	0.0	0.2	0.2	0.0	-0.1	-0.2	-0.2	-0.2	-0.0	0.3	0.0
Pw	-0.4	0.1	0.6	0.4	-0.0	-0.3	-0.4	-0.4	-0.3	-0.0	0.2	0.0

Table XXIII (continued)

MB_t(in cu. ft.) of section volume (with known heights) at height

species	4.5'	.1H	.2H	.3H	.4H	.5H	.6H	.7H	.8H	.9H	1 H
D	-0.66	0.98	1.04	0.55	0.05	-0.39	-0.63	-0.59	-0.40	-0.13	0.05
C	-0.67	-0.07	0.75	0.41	0.14	-0.06	-0.17	-0.20	-0.16	-0.06	-0.00
A	-0.11	0.18	0.18	0.05	-0.06	-0.11	-0.14	-0.11	-0.03	0.04	0.04
Cot	0.02	0.23	0.11	-0.02	-0.08	-0.09	-0.08	-0.06	-0.02	0.00	0.00

MB_t(in feet) of distance for diameters at height

species	1'	4.5'	.1H	.2H	.3H	.4H	.5H	.6H	.7H	.8H	.9H	1 H
D	16.5	-5.2	-7.5	-4.6	-1.6	1.6	4.0	4.7	4.0	2.5	-0.7	0.0
C	4.6	2.2	-5.0	-3.0	-1.0	0.6	1.9	2.3	2.8	1.6	0.1	0.0
A	4.3	-1.9	-3.3	-1.2	0.3	1.3	1.6	2.3	1.1	-0.8	-2.3	0.0
Cot	1.5	-3.6	-2.5	-0.3	1.4	1.6	1.7	1.6	0.6	-0.7	-1.1	0.0

MB_t(in cu. ft.) of section volume (with unknown heights) at heights

species	.2H	.3H	.4H	.5H	.6H	.7H	.8H	.9H	1 H
D	-2.78	-2.60	-2.53	-1.72	-0.53	0.47	0.32	0.47	0.04
C	-1.65	-1.22	-0.85	-0.63	-0.13	-0.15	0.12	0.16	0.01
A	-1.23	-0.60	-0.52	-0.07	-0.17	0.22	0.17	0.11	0.00
Cot	-0.79	-0.48	-0.09	-0.04	-0.01	0.09	0.06	0.00	0.01

5. Taper-based systems of tree taper and volume estimation

5.1. Taper equations

5.1.1. Fitting taper equations

The taper equations, which were derived from the volume equations, can all be fitted on the taper data without being conditioned by the volume functions. Until recently (Demaerschalk, 1971b; 1972a; 1973), taper and volume studies have been considered as two essentially separate fields. Therefore most of these taper equations have never been tried in the past. Yet, many other promising forms of taper functions were developed. Some of these are tested in this study, together with a few functions whose form was derived from the previous volume equations.

The following selection of taper equations was made:

I. logarithmic taper equation (Demaerschalk, 1971b)

$$\log d = b_o + b_1 \log D + b_2 \log \underline{1} + b_3 \log H \quad \text{(compares with } 1^t\text{)}$$

II. equation developed by Kozak, Munro and Smith (1969a)

$$(d \;/\; D)^2 = b_o + b_1(h \;/\; H) + b_2(h \;/\; H)^2$$

III. Bennett and Swindel's taper equation (1972)

$$d = b_o D \,\underline{1}\, /(H - 4.5) + b_1(\underline{1}(h - 4.5)) + b_2 H \,\underline{1}\, (h - 4.5) + b_3(\underline{1}(h - 4.5))(H + h + 4.5)$$

IV. equation with same form as equation 4^t

$$(d \;/\; D)^2 = b_o \,\underline{1}^p/(D^2 H^{p+1}) + b_1(\underline{1} \;/\; H)^q$$

V. Matte's taper equation (1949)

$$(d \;/\; D)^2 = b_o(\underline{1} \;/\; H)^2 + b_1(\underline{1} \;/\; H)^3 + b_2(\underline{1} \;/\; H)^4$$

VI. equation proposed by Osumi (1959)

$$(d \;/\; D) = b_o(\underline{1} \;/\; H) + b_1(\underline{1} \;/\; H)^2 + b_2(\underline{1} \;/\; H)^3$$

VII. taper equation developed by Bruce, Curtis and Vancoevering (1968)

$$(d / D)^2 = b_o X^{3/2} + b_1(X^{3/2} - X^3)\, D\, (10^{-2}) + b_2(X^{3/2} - X^3)\, H\, (10^{-3}) + b_3(X^{3/2} - X^{32})\, H\, D\, (10^{-5}) + b_4(X^{3/2} - X^{32})\, H^{1/2}\, (10^{-3}) + b_5(X^{3/2} - X^{40})\, H^2\, (10^{-6})$$

where $X = (\underline{1} / (H - 4.5))$

VIII. equation with same form as equation 8[t]

$$(d / D)^2 = b_o(\underline{1} / H)^p$$

IX. equation with same form as equation 9[t]

$$(d / D)^2 = b_o\ \underline{1}^p / H^{p+1} + b_1(\underline{1} / H)^q$$

X. Behre's taper equation (1923) in conditioned form

$$(d / D) = (\underline{1} / H)/(b_o + b_1\ (\underline{1} / H))$$

XI. Behre's taper equation with condition $b_o + b_1 = 1$

$$((\underline{1} / H)/(d / D) - 1) = b_1\ ((\underline{1} / H) - 1)$$

All taper equations are conditioned such that the diameter at the top of the tree is zero. Taper equation II had to be severely conditioned for western redcedar in order to avoid negative diameters.

In equation VII, the b_o coefficient is conditioned to be equal to the mean $(d_{4.5} / D)^2$ for each species. Plottings indicated this ratio to be constant throughout the range of observations.

Plottings of dependent over independent variables showed again that often the assumptions of the regression analysis were not met (wrong model, heterogeneous variances). Observations from the same tree are not independent.

Taper equation I, the analogue of 1[t], will later on be fitted in several different ways.

The coefficients of the equations, the SE_E's and the coefficients of determination are summarized in Appendix 5.1.

For some equations, the coefficients of determination vary considerably (from 38% up to 80% for equation VII) from one species to another. All independent variables are not always significant (equations IV and VII). The condition which Matte (1949) posed on his taper equation V, namely that $b_o + b_1 + b_2 = 1$, to make d = D for $\underline{1}$ = H, seems acceptable for Douglas-fir and cottonwood. For the other species, double bark thickness and butt flare are such that this condition does not hold. The condition in XI that $b_o + b_1 = 1$ looks reasonable, except for aspen and Douglas-fir.

The p and q values in equations IV, VIII and IX are the ones obtained in the derivation of the corresponding equations from the volume functions.

There are some consistent differences between the coefficients of these taper equations and these derived from the volume functions. In taper equation I, b_1 is always smaller and b_3 always larger than the corresponding coefficients in 1^t and 2^t. In taper equation IV, b_o is smaller and b_1 is larger than in 4^t and 5^t. The b_o coefficient in equation VIII is always larger than in 8^t. In equation IX, b_o is usually smaller and b_1 larger than in 9^t. These differences seem minor but will prove to be very important in the estimation of total volume.

The b_1 coefficient in equation IV, b_o coefficient in VIII and the b_1 coefficient in IX are very similar and closely related to double bark thickness. This will be discussed in more detail in a later section.

These taper equations will now be tested for the estimation of diameter, section volume with known heights, height and section volume with unknown heights. Volume equations are derived from them and tested for total volume estimation.

The formulae to compute diameter,height,section volume and total volume are derived for each taper equation in Appendix 6.

In future discussions,these taper equations I to XI will be called the "taper-based" taper equations while the taper equations 1^t to 14^t,derived from the volume functions,are called the "volume-based" taper equations,unless it is clear from the context which equations are meant.

5.1.2.Tests of diameter estimation

These taper-based taper equations were subjected to the same tests as were carried out on the volume-based taper equations.Some results of the tests of diameter estimation appear in tables XXIV and XXV.

Although the standard errors of d are similar for most equations, some seem to give consistently better results than others.The better ones seem to be equation III,V,VI and VII.But before the bias has been considered,no conclusion can be drawn from this.

The volume-based taper equations had all the same pattern of bias.This is not the case here (see figure 4).Equations I,II,IV,VIII and IX have essentially the same pattern as before,although equations II and VIII overestimate a larger portion of the bigger trees. Figure 4 gives only an idea about the kind of bias (positive or negative),not about the size of the bias.

Equation X has the same pattern for the smaller trees,but over-estimates the complete tree profile of the bigger trees.Equation XI, the conditioned form of X,has a bias even worse than X.

Figure 4

Patterns of Bias in Diameter Estimation

of Taper-Based Taper Equations

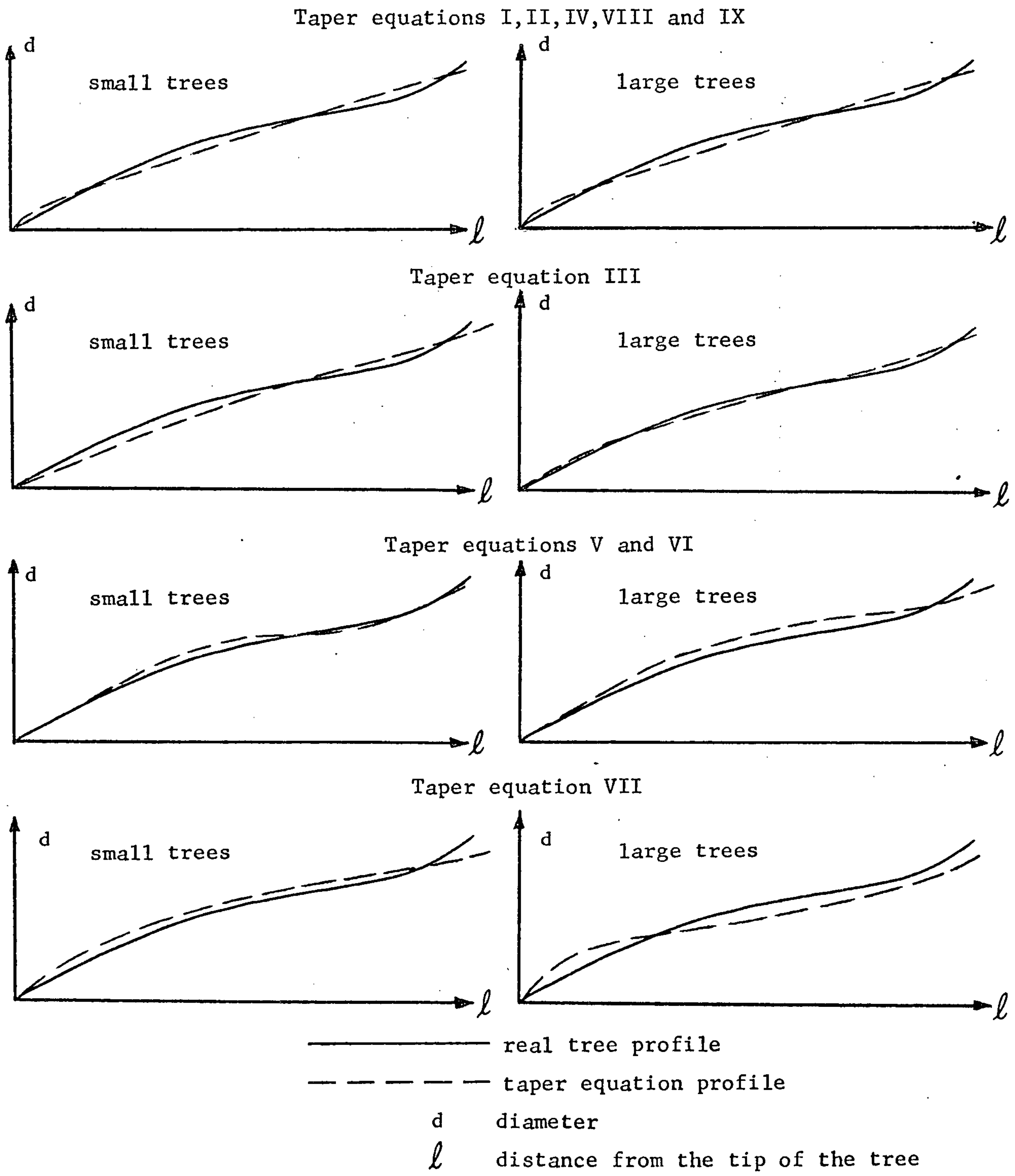


Table XXIV

Diameter Estimation Test of Taper-Based Taper Equations

species	I	II	III	IV	V	VI	VII	VIII	IX	X	XI
	SE_t (in inches) of diameter for equations										
D	1.04	1.08	0.97	1.07	0.93	0.94	0.90	1.08	1.07	1.16	1.54
C	2.06	2.08	1.67	2.01	1.88	1.73	1.31	2.08	1.85	2.20	2.17
S	1.44	1.40	1.24	1.41	1.25	1.23	1.16	1.41	1.42	1.68	1.51
B	2.11	2.05	1.76	2.10	1.86	1.86	1.57	2.08	2.10	2.28	2.24
A	1.12	1.11	1.02	1.11	1.03	1.03	1.01	1.11	1.11	1.78	1.13
Cot	0.55	0.57	0.49	0.53	0.53	0.52	0.67	0.53	0.54	0.55	0.54
Pl	0.90	0.89	0.74	0.89	0.79	0.79	0.71	0.90	0.89	1.05	0.99
Pw	1.22	1.21	1.04	1.23	1.07	1.05	0.89	1.23	1.24	1.42	1.32

Douglas-fir

taper eq.	1'	4.5'	.1H	.2H	.3H	.4H	.5H	.6H	.7H	.8H	.9H	1 H
	MB_t (in inches) of diameter at heights											
I	-1.4	0.6	0.9	0.6	0.3	-0.0	-0.3	-0.4	-0.4	-0.3	0.2	0.0
II	-1.2	0.8	1.0	0.6	0.3	-0.0	-0.3	-0.4	-0.3	-0.0	0.6	0.0
III	-1.1	0.7	0.6	0.2	0.0	-0.1	-0.2	-0.1	-0.1	-0.2	-0.1	0.0
IV	-1.4	0.7	0.9	0.7	0.4	0.0	-0.3	-0.4	-0.4	-0.3	0.3	0.0
V	-0.6	0.9	0.3	-0.1	-0.1	0.1	0.2	0.3	0.3	0.1	0.0	0.0
VI	-0.9	0.8	0.5	0.1	-0.0	-0.1	-0.0	0.1	0.2	0.1	0.1	0.0
VII	-0.2	0.0	-0.2	-0.1	-0.1	-0.1	-0.1	0.0	0.3	0.6	1.2	0.0
VIII	-1.3	0.7	1.0	0.7	0.4	0.1	-0.3	-0.4	-0.4	-0.2	0.2	0.0
IX	-1.4	0.7	0.9	0.7	0.4	0.1	-0.2	-0.4	-0.3	-0.2	0.3	0.0
X	-2.2	-0.1	0.5	0.5	0.5	0.3	0.1	0.0	-0.0	-0.2	-0.1	0.0
XI	-0.2	1.8	2.1	1.8	1.4	0.9	0.4	0.0	-0.3	-0.6	-0.4	0.0

Taper equation I

species	1'	4.5'	.1H	.2H	.3H	.4H	.5H	.6H	.7H	.8H	.9H	1 H
	MB_t (in inches) of diameter at heights											
D	-1.4	0.6	0.9	0.6	0.3	-0.0	-0.3	-0.4	-0.4	-0.3	0.2	0.0
C	-4.5	0.3	1.4	1.0	0.6	0.3	-0.0	-0.2	-0.4	-0.2	0.0	0.0
S	-2.8	0.7	1.0	0.6	0.3	0.0	-0.2	-0.2	-0.2	-0.1	0.1	0.0
B	-3.8	0.7	1.4	0.9	0.4	0.1	-0.2	-0.4	-0.4	-0.1	0.1	0.0
A	-1.2	0.8	0.8	0.5	0.2	-0.1	-0.3	-0.5	-0.4	-0.2	0.1	0.0
Cot	-0.1	0.6	0.4	0.1	-0.1	-0.2	-0.3	-0.3	-0.2	-0.1	0.0	0.0
Pl	-1.4	0.4	0.6	0.5	0.2	0.0	-0.1	-0.3	-0.3	-0.2	0.1	0.0
Pw	-2.6	0.7	1.1	0.8	0.3	-0.1	-0.3	-0.3	-0.3	-0.1	0.1	0.0

Equations V and VI show a new pattern of bias.This bias is very large for some species (e.g. for western redcedar there is for several sections a bias of more than 3 inches).The smaller trees have some underestimation in the lower half and near the top.

Taper equation III underestimates the total top half of the smaller trees.For the bigger trees the underestimated area shifts to the middle,leaving top and bottom overestimated.

Equation VII,which has the smallest standard error of estimate of d of all equations,has quite a surprising pattern of bias.The profile of the smaller trees is overestimated.Other size classes have 60 to 70% of their lower profile underestimated and have a large overestimation near the top.The fit near the base of the tree is fairly good, because the taper curve is forced through the diameter inside bark at breast height.

When only total bias is considered,some taper equations (e.g. V, III,VI,VII and X),whose pattern of bias is of the opposite sign for different size classes,look very attractive,although they may be worse than others.

The equations III,V,VI and VII have a sigmoid form.This characteristic does not seem to be sufficient to produce a good taper function.

The results for the logarithmic taper equation I do not differ very much from the ones for equation 1^t,the taper equation derived from the logarithmic volume equation (e.g. for Douglas-fir maximum difference is 0.2 inches).This suggests that two different procedures,to fit basically the same taper equation,may produce similar final results with regard to the diameter estimation.The same could be said for taper equations VIII and 8^t.Equations 1^t and 8^t have for most species a better fit in the lower part of the tree,while I and VIII are better in the upper part.For example,for Douglas-fir equation 8^t is for some lower sections 0.5 inches less biased but for some upper sections 0.2 inches more biased than equation VIII.

Most taper equations show a similar performance for all species. Some equations (e.g. equations V and VI) are fairly good for some species and unacceptable for others (e.g. for western redcedar an overestimation of 4 inches in the upper part of the larger size classes).

To check if the kind of dependent variable used could make any significant difference,taper equation I was fitted in three different ways.The three dependent variables are log d,d and d^2.The regression coefficients are given in Appendix 5.2. and results from the diameter estimation tests are in table XXVI.These results indicate that,by giving more weight to d,b_o and b_1 increases and b_3 decreases.This was the case for both species for which the test was done (Douglas-fir and cottonwood).Although the SE_t's of d are almost identical,the different methods cause some differences in bias.For Douglas-fir, using log d or d as dependent variable is better for diameter estimation (for some sections the bias for d^2 is 0.3 inches larger than for log d or d).Using d^2 as dependent variable causes more overestimation in the lower tree and more underestimation in the upper tree. For cottonwood,d or d^2 seems to be best (for some sections d or d^2 is 0.3 inches less biased than log d).So,the effect of the kind of dependent variable used may differ from one species to another.

Table XXV

Comparison of Diameter Bias of Volume-Based and Taper-Based

Taper Equations for Douglas-fir

taper eq.	MB_t (in inches) of diameter at heights 1'	4.5'	.1H	.2H	.3H	.4H	.5H	.6H	.7H	.8H	.9H	1 H
1^t	-1.6	0.5	0.7	0.5	0.2	-0.1	-0.4	-0.5	-0.5	-0.3	0.2	0.0
I	-1.4	0.6	0.9	0.6	0.3	-0.0	-0.3	-0.4	-0.4	-0.3	0.2	0.0
4^t	-1.5	0.5	0.8	0.6	0.3	-0.0	-0.3	-0.4	-0.4	-0.2	0.3	0.0
IV	-1.4	0.7	0.9	0.7	0.4	0.0	-0.3	-0.4	-0.4	-0.3	0.3	0.0
8^t	-1.8	0.3	0.5	0.3	0.0	-0.3	-0.5	-0.6	-0.6	-0.4	0.1	0.0
VIII	-1.3	0.7	1.0	0.7	0.4	0.1	-0.3	-0.4	-0.2	-0.2	0.2	0.0

Table XXVI

Diameter Bias of Equation I Fitted with

Different Dependent Variables

dependent variable	MB_t (in inches) of diameter at heights 1'	4.5'	.1H	.2H	.3H	.4H	.5H	.6H	.7H	.8H	.9H	1 H
Douglas-fir												
log d	-1.4	0.6	0.9	0.6	0.3	-0.0	-0.3	-0.4	-0.4	-0.3	0.2	0.0
d	-1.5	0.5	0.8	0.6	0.3	0.0	-0.3	-0.4	-0.3	-0.2	0.3	0.0
d^2	-1.2	0.8	1.0	0.6	0.2	-0.2	-0.5	-0.7	-0.7	-0.6	-0.1	0.0
Cottonwood												
log d	-0.1	0.6	0.4	0.1	-0.1	-0.2	-0.3	-0.3	-0.2	-0.1	0.0	0.0
d	-0.3	0.3	0.3	0.1	-0.1	-0.1	-0.2	-0.1	-0.0	0.1	0.2	0.0
d^2	-0.3	0.4	0.3	0.1	-0.1	-0.1	-0.2	-0.2	-0.0	0.1	0.2	0.0

5.1.3. Tests of section volume estimation with known heights

Section volume estimation was only tested for equations which proved reasonable for diameter estimation. Osumi's taper equation VI and equation IX were not tested because of their similarity to Matte's equation V and equation IV respectively. Some of the results of the tests on equations I,II,III,IV,V and VIII are presented in table XXVII. The functions to compute section volumes are given in Appendix 6.

It was easy to find a good relationship between the diameter bias and the bias in height (0.1 inch of bias in diameter corresponds roughly with 1 foot bias in height). Such a good relationship does not exist for section volume because the bias of a section is based on the bias of two diameters and depends on the size of these diameters themselves.

The pattern of bias and all the differences between the equations and the species are, of course, similar to the ones for diameter. Table XXVIII features a comparison between volume-based and corresponding taper-based taper equations. This again shows how the volume-based equations fit the tree better in the lower part but are more biased than the taper-based equations in the upper part of the tree.

Some taper equations (e.g. equation V) are positively biased over most of the length of the tree profile. This will result in a large overestimation of total volume.

Table XXVII

Bias of Section Volume Estimation with Known Heights

of Taper-Based Taper Equations

Douglas-fir

taper eq.	MB_t (in cubic feet) of section volume at heights 4.5'	.1H	.2H	.3H	.4H	.5H	.6H	.7H	.8H	.9H	1 H
I (log d)	-0.64	1.41	1.40	0.85	0.29	-0.20	-0.49	-0.49	-0.34	-0.11	0.05
I (d_2)	-0.73	1.23	1.30	0.81	0.30	-0.15	-0.41	-0.40	-0.25	-0.04	0.09
I (d^2)	-0.51	1.61	1.45	0.78	0.13	-0.43	-0.75	-0.76	-0.58	-0.28	-0.01
II	-0.45	1.74	1.59	0.96	0.36	-0.14	-0.39	-0.34	-0.14	0.12	0.20
III	-0.41	1.07	0.49	0.00	-0.19	-0.23	-0.16	-0.03	-0.00	-0.04	-0.02
IV	-0.58	1.52	1.50	0.95	0.38	-0.12	-0.42	-0.43	-0.30	-0.08	0.07
V	-0.08	1.27	0.36	0.00	0.11	0.28	0.38	0.38	0.20	-0.00	-0.03
VIII	-0.51	1.70	1.66	1.08	0.49	-0.04	-0.36	-0.39	-0.27	-0.07	0.07

Taper equation I

species	MB_t (in cubic feet) of section volume at heights 4.5'	.1H	.2H	.3H	.4H	.5H	.6H	.7H	.8H	.9H	1 H
D	-0.64	1.41	1.40	0.85	0.29	-0.20	-0.49	-0.49	-0.34	-0.11	0.05
C	-2.96	0.68	1.33	0.85	0.46	0.16	-0.03	-0.13	-0.13	-0.06	-0.01
S	-1.50	1.15	1.12	0.63	0.20	-0.10	-0.20	-0.18	-0.10	-0.03	0.01
B	-3.28	2.39	3.07	1.64	0.60	0.02	-0.38	-0.50	-0.30	-0.07	0.03
A	-0.43	0.77	0.58	0.28	0.05	-0.11	-0.21	-0.23	-0.15	-0.06	-0.00
Cot	0.07	0.40	0.21	0.03	-0.07	-0.11	-0.12	-0.10	-0.06	-0.03	-0.01
Pl	-0.63	0.55	0.63	0.42	0.17	-0.02	-0.14	-0.18	-0.13	-0.05	0.01
Pw	-1.37	1.60	1.68	0.86	0.13	-0.24	-0.34	-0.28	-0.16	-0.04	0.02

Douglas-fir

taper eq.	MB_c (in cu. ft.) of section volume of largest size class at heights 4.5'	.1H	.2H	.3H	.4H	.5H	.6H	.7H	.8H	.9H	1 H
I	-1.59	3.31	3.69	2.39	1.09	-0.21	-1.04	-1.02	-0.63	-0.16	0.14
II	-0.82	4.92	4.85	3.23	1.74	0.33	-0.50	-0.44	0.01	0.46	0.51
III	-0.76	2.75	1.24	-0.14	-0.56	-0.69	-0.51	-0.00	0.22	0.05	-0.03
IV	-1.15	4.26	4.53	3.12	1.72	0.31	-0.62	-0.70	-0.41	-0.04	0.18
V	0.05	3.92	1.86	0.88	1.08	1.31	1.37	1.32	0.84	0.17	-0.04
VIII	-0.94	4.82	5.01	3.53	2.06	0.58	-0.41	-0.55	-0.31	0.01	0.19

Table XXVIII

Comparison of Bias of Section Volume Estimation with Known Heights of Volume-Based and Taper-Based Taper Equations for Douglas-fir

taper eq.	MB_t (in cubic feet) of section volume at heights 4.5'	.1H	.2H	.3H	.4H	.5H	.6H	.7H	.8H	.9H	1 H
1^t	-0.80	1.01	1.06	0.57	0.06	-0.38	-0.62	-0.58	-0.40	-0.13	0.05
I	-0.64	1.41	1.40	0.85	0.29	-0.20	-0.49	-0.49	-0.34	-0.11	0.05
4^t	-0.77	0.98	1.05	0.57	0.08	-0.36	-0.59	-0.55	-0.37	-0.11	0.06
IV	-0.58	1.52	1.50	0.95	0.38	-0.12	-0.42	-0.43	-0.30	-0.08	0.07
8^t	-0.88	0.84	0.92	0.45	-0.04	-0.46	-0.68	-0.63	-0.43	-0.15	0.05
VIII	-0.51	1.70	1.66	1.08	0.49	-0.04	-0.36	-0.39	-0.27	-0.07	0.07

5.1.4. Tests of height estimation

There is no need to repeat all tests for all the equations and all the species since it is known that 0.1 inch of bias in diameter corresponds with 1 foot bias in height.

The results of equations I,II and VIII for Douglas-fir,western redcedar,aspen and cottonwood are summarized in table XXIX.

As could be expected from the results of the diameter estimation, these equations are less biased in the upper portion of the tree,but more biased in the lower portion.

Differences in bias of 6 feet or more may exist between the different derivation methods.

Table XXIX

Bias of Distance Estimation of Some

Taper-Based Taper Equations

	Douglas-fir											
taper eq.	MB_t(in feet) of the distance for diameters at heights 1'	4.5'	.1H	.2H	.3H	.4H	.5H	.6H	.7H	.8H	.9H	1 H
I	15.4	-7.3	-9.2	-6.2	-3.0	0.4	3.0	3.9	3.4	2.2	-0.9	0.0
II	12.7	-7.3	-8.7	-8.7	-5.4	-2.2	1.0	3.5	3.9	2.7	-2.9	0.0
VIII	15.3	-7.3	-9.2	-6.2	-3.0	0.3	2.9	3.8	3.3	2.0	-0.9	0.0
species	Taper equation I											
D	15.4	-7.3	-9.2	-6.2	-3.0	0.4	3.0	3.9	3.4	2.2	-0.9	0.0
C	27.6	-2.1	-8.3	-5.9	-3.4	-1.4	0.2	1.1	2.0	1.3	0.0	0.0
A	11.0	-7.8	-7.3	-3.4	-1.5	0.6	2.1	3.6	3.3	1.8	-0.4	0.0
Cot	0.3	-5.5	-3.9	-1.3	0.9	1.6	2.2	2.4	1.7	0.5	-0.1	0.0

5.1.5. Tests of section volume estimation with unknown heights

The results of equation I for Douglas-fir, western redcedar, aspen and cottonwood are given in table XXX.

The kind of bias in section volume estimation, with unknown heights, is the same as for the estimation of distance from the tip of the tree. The results are very similar to those for the volume-based taper equations.

Table XXX

Bias of Section Volume Estimation with Unknown Heights of a Taper-Based Taper Equation

	Taper equation I								
	MB_t(in cubic feet) of section volume at heights								
species	.2H	.3H	.4H	.5H	.6H	.7H	.8H	.9H	1 H
D	-2.99	-2.77	-2.67	-1.85	-0.66	0.34	0.24	0.43	0.03
C	-1.97	-1.50	-1.06	-0.79	-0.28	-0.24	0.06	0.12	0.01
A	-1.49	-1.04	-0.88	-0.40	-0.38	0.06	0.18	0.12	0.01
Cot	-0.94	-0.62	-0.22	-0.13	-0.07	0.06	0.05	0.00	0.01

5.1.6. Taper-based taper equations for data adjusted for butt flare

All taper equations I to XI were fitted on the adjusted data as well, for four species. Coefficients, SE_E's and coefficients of determination for some equations are summarized in Appendix 5.3.

Because butt flare is largely eliminated, the SE_E's are usually much smaller and the coefficients of determination larger. This is also true for the so called sigmoid taper curves of which some are expected to account for butt flare.

The same tests on diameter, height and section volume estimation were repeated on these adjusted equations. Results are given in table XXXI. Despite a slight occasional increase in bias in the top section, all equations have greatly improved after butt flare was eliminated. The greatest improvement is, as expected, in the lower part of the tree and for those species with the largest butt swell.

These results are very similar to the ones for the volume-based taper equations for adjusted data.

Table XXXI

Tests of Taper-Based Taper Equations for

Data Adjusted for Butt Flare

SE_t(in inches) of diameter for the equations

species	I	II	III	IV	V	VI	VII	VIII	IX	X	XI
D	0.91	0.97	0.85	0.95	0.82	0.83	0.78	0.97	0.96	1.04	1.48
C	1.03	1.27	0.95	1.25	1.22	1.18	0.89	1.32	1.09	1.27	1.39
A	0.69	0.66	0.53	0.65	0.57	0.58	0.67	0.66	0.64	1.72	0.66
Cot	0.52	0.54	0.44	0.49	0.49	0.48	0.64	0.49	0.50	0.50	0.51

Taper equation I

MB_t(in inches) of diameter at heights

species	1'	4.5'	.1H	.2H	.3H	.4H	.5H	.6H	.7H	.8H	.9H	1 H
D	-1.3	0.6	0.9	0.6	0.3	-0.0	-0.3	-0.4	-0.4	-0.3	0.2	0.0
C	-0.8	-0.4	0.8	0.5	0.2	-0.0	-0.2	-0.3	-0.4	-0.2	0.1	0.0
A	0.1	0.6	0.6	0.3	0.0	-0.2	-0.3	-0.5	-0.4	-0.2	0.1	0.0
Cot	0.1	0.6	0.4	0.1	-0.1	-0.2	-0.3	-0.3	-0.2	-0.1	0.0	0.0

Taper equation I

MB_t(in cu. ft.) of section volume (with known heights) at heights

species	4.5'	.1H	.2H	.3H	.4H	.5H	.6H	.7H	.8H	.9H	1 H
D	-0.50	1.39	1.38	0.84	0.28	-0.22	-0.50	-0.50	-0.35	-0.11	0.05
C	-0.74	-0.18	0.68	0.37	0.13	-0.05	-0.14	-0.17	-0.14	-0.04	0.00
A	0.16	0.55	0.41	0.16	-0.04	-0.17	-0.24	-0.24	-0.16	-0.06	0.00
Cot	0.13	0.38	0.20	0.01	-0.08	-0.11	-0.12	-0.10	-0.06	-0.03	-0.01

Douglas-fir

MB_t (in feet) of the distance for diameters at heights

taper eq.	1'	4.5'	.1H	.2H	.3H	.4H	.5H	.6H	.7H	.8H	.9H	1 H
I	13.8	-7.2	-9.2	-6.1	-3.0	0.4	3.0	3.9	3.4	2.2	-0.8	0.0
II	12.1	-7.0	-8.5	-5.3	-2.2	1.1	3.4	3.8	2.6	0.4	-3.0	0.0
VIII	14.0	-7.1	-9.0	-6.0	-2.9	0.4	3.0	3.9	3.4	2.1	-0.9	0.0

Taper equation I

MB_t(in cu. ft.)of section volume (with unknown heights)at height

species	.2H	.3H	.4H	.5H	.6H	.7H	.8H	.9H	1 H
D	-2.98	-2.76	-2.66	-1.84	-0.66	0.35	0.25	0.43	0.03
C	-1.52	-1.12	-0.78	-0.58	-0.10	-0.13	0.13	0.15	0.01
A	-1.41	-0.95	-0.81	-0.33	-0.33	0.11	0.14	0.13	0.01
Cot	-0.93	-0.48	-0.21	-0.12	-0.06	0.06	0.06	0.00	0.02

5.2. Taper-based volume equations

5.2.1. Derivation of compatible volume equations from taper equations

Volume equations were derived from taper equations I to XI by integration of the taper equation over the total length of the tree:

$$V = \int_0^H (d^2/\ k)\ d\underline{l}$$

The integrations are shown for each equation in Appendix 6. The volume functions derived from the taper equations have the same number as the taper equations from which they are derived, except that a subscript "v" is added to distinguish them as volume equations. They are the following:

I^v. $\log V = a + b \log D + c \log H$

II^v. $V = a\ D^2H$

III^v. $V = (a^2\ H^3/3 + b^2\ H^5/5 + c^2\ H^7/7 + 2ab\ H^4/4 + 2ac\ H^5/5 + 2bc\ H^6/6)/k$

where a, b and c are functions of D, H and the coefficients of equation III.

IV^v. $V = a + b\ D^2H$

V^v. $V = a\ D^2H$

VI^v. $V = a\ D^2H$

VII^v. $V = D^2(a\ X_1 + b\ X_2 + c\ X_3 + e\ X_4)/k$

where $X_1 = H^{5/2}/((H - 4.5)^{3/2}5/2)$ $\quad X_3 = H^{33}/((H - 4.5)^{32}33)$

$X_2 = H^4/((H - 4.5)^3 4)$ $\quad X_4 = H^{41}/((H - 4.5)^{40}41)$

and a, b, c and e are functions of D, H and the coefficients of equation VII.

$VIII^v$. $V = a\ D^2H$

$IX^v.\ V / B' = a + b\ H$

$X^v.\ V = a\ D^2H$

$XI^v.\ V = a\ D^2H$

The coefficients a,b,c and e can be computed directly from the coefficients of the taper-based taper equations according to the formulae given in Appendix 6.

The equations I^v to XI^v are called "taper-based" volume equations while the volume equations 1 to 14,fitted on the volume data,are called "volume-based" volume equations,unless it is clear from the context which equations are meant.

All taper-based volume equations,except III^v and VII^v,have a functional form identical to some of the volume-based volume equations. To compare these corresponding equations,the coefficients of some taper-based volume equations are given in table XXXII.

The functional forms of these volume equations are very important as they reveal the built-in assumptions about the cylindrical form factors.

Not less than six taper equations (equations II,V,VI,VIII,X and XI) integrate to the combined variable volume equation without intercept (compares with volume equation 8).These taper equations assume a constant cylindrical form factor for all trees which is computed as:

CFF = 183.3466 a

All volume equations,derived from these six taper equations,have coefficients which are larger than the coefficient in equation 8. The coefficients of II^v,V^v,$VIII^v$ and XI^v are very similar.Equation VI^v is always bigger than equation 8,but consistently smaller than the other four equations.

Table XXXII

Taper-Based Volume Equation Coefficients

	Coefficients of the volume equations				
	I^v			IV^v	
species	a	b	c	a	b 10^2
D	-2.950510	1.736276	1.281513	0.436084	0.200044
C	-2.346608	1.493320	1.144144	0.989094	0.216252
S	-2.752611	1.732854	1.214090	0.036790	0.236367
B	-2.864758	1.518873	1.416063	0.169288	0.236882
A	-3.606290	1.431840	1.817881	-0.127365	0.241152
Cot	-3.676927	1.563550	1.757194	-0.026213	1.208321
Pl	-2.822723	1.580697	1.350776	0.464814	0.236496
Pw	-2.553785	1.779785	1.090640	0.448075	0.230738

	Coefficients of the volume equations				
	II^v	V^v	VI^v	$VIII^v$	X^v
species	a 10^2	a 10^2	a 10^2	a 10^2	a 10^2
D	0.204565	0.202589	0.201616	0.204169	0.201624
C	0.229419	0.228051	0.220935	0.229425	0.214935
S	0.234827	0.236009	0.232074	0.236797	0.218765
B	0.244619	0.244346	0.240604	0.243075	0.240822
A	0.236419	0.236834	0.232532	0.238427	0.311075
Cot	0.211062	0.210420	0.208322	0.207725	0.200125
Pl	0.246868	0.245626	0.243868	0.247618	0.237819
Pw	0.233643	0.233513	0.230895	0.238621	0.227153

Comparing the coefficients of I^V with equation 1, the intercept is smaller and the other coefficients are larger. Differences are for some species important.

The intercepts of equation IV^V are for all species smaller than in equation 4 and the slope coefficients are bigger. This will result in lower total volume estimates for the smallest trees and higher estimates for the larger size classes. The same can be said for equations IX^V and 9.

Some taper equations (equations III and VII) result in fairly complicated volume equations whose functional form can not be compared with any other volume equation.

5.2.2. Tests of total volume estimation

Total volume estimation was tested for all taper-based volume equations I^V to XI^V and for all species. These tests are important as they will show what kind of bias may be expected if volume equations are derived from taper equations.

A summary of the bias of the equations is given for some species in table XXXIII.

From the comparison of the coefficients in the previous section it could be expected that most of these taper-based volume equations would be more biased than the volume-based volume equations 1 to 14. Most equations have an overall overestimation for most species. Equation VII^V largely underestimates volume, due to the fact that 60 to 70% of the lower bole is underestimated by taper equation VII.

Table XXXIII

Bias in Total Volume Estimation of

Taper-Based Volume Equations

Douglas-fir MB_c (in cubic feet) of volume for equations

size class	I^v	$I^v(d)$	$I^v(d^2)$	II^v	III^v	IV^v	V^v	VI^v	VII^v	$VIII^v$	IX^v	X^v	XI^v
1	0.2	0.2	-0.1	-0.0	0.3	0.1	-0.2	-0.2	-2.3	-0.0	0.2	-0.2	2.5
2	0.8	0.8	-0.1	1.4	0.9	0.7	0.9	0.6	-8.1	1.3	1.2	0.6	8.9
3	2.0	1.7	-0.2	1.8	0.1	0.4	1.0	0.6	-18.0	1.7	0.8	0.7	14.8
4	2.4	2.5	0.6	7.2	-0.5	4.9	6.0	5.4	-26.2	7.0	5.5	5.4	26.4
5	6.0	6.3	4.2	14.3	1.6	11.2	12.7	12.0	-31.2	14.0	11.9	12.0	38.4
MB_t	1.7	1.8	0.6	3.5	0.5	2.5	2.9	2.5	-12.6	3.4	2.9	2.5	13.5

Western redcedar MB_c (in cubic feet) of volume for equations

size class	I^v	II^v	III^v	IV^v	V^v	VI^v	VII^v	$VIII^v$	IX^v	X^v	XI^v
1	0.6	-0.1	-0.4	-0.1	-0.2	-0.7	-1.4	-0.1	-0.4	-1.2	-1.0
2	-0.2	6.5	2.0	4.4	6.2	4.5	-4.5	6.5	1.0	3.1	3.6
3	-3.3	8.9	-1.8	5.3	8.4	5.9	-13.0	8.9	-2.3	3.9	4.7
4	1.1	29.6	5.0	23.1	28.8	24.8	-20.7	29.6	7.5	21.4	22.7
5	1.0	41.5	6.7	33.0	40.5	35.4	-31.3	41.5	11.1	31.1	32.8
MB_t	0.2	8.1	1.0	6.0	7.7	6.1	-7.3	8.1	1.3	4.8	5.3

Balsam MB_c (in cubic feet) of volume for equations

size class	I^v	II^v	III^v	IV^v	V^v	VI^v	VII^v	$VIII^v$	IX^v	X^v	XI^v
1	1.0	-0.8	0.1	-1.7	-0.8	-1.3	-5.5	-1.0	-1.9	-1.3	-1.0
2	5.0	11.4	5.1	7.4	11.3	9.2	-20.7	10.6	4.9	9.4	10.7
3	12.5	31.1	9.1	24.1	30.8	27.4	-42.2	29.6	18.4	27.6	29.6
4	7.1	37.7	1.8	28.6	37.3	32.9	-69.0	35.8	20.7	33.1	35.8
5	-1.3	54.9	2.8	42.9	54.4	48.6	-96.4	52.4	32.3	48.9	52.4
MB_t	3.2	12.4	2.5	8.7	12.3	10.4	-24.6	11.6	6.0	10.5	11.6

Most equations have more or less the same pattern of bias. The smaller trees are underestimated and the larger size classes are overestimated.Most taper-based taper equations overestimated a large portion of the tree profile of the larger size classes.This causes an overestimation in total volume which is so important that results often become useless.

While the logarithmic volume equation 1 underestimates the volume of most sizes,equation I^V overestimates most size classes.

In the diameter estimation,fitting equation I with d as dependent variable was superior for Douglas-fir.For total volume estimation d^2 seems to be slightly better.

Only equations I^V and III^V give reasonable results for volume estimation for all species and can compete in performance with the best volume-based volume equations.

5.2.3.Tests of total volume estimation for data adjusted for butt flare

Taper equations,fitted on adjusted data,were converted in the same way to taper-based volume equations.Total volume estimation tests are given in table XXXIV.

Although adjusting results in some improvement,bias for most equations is still considerable.Adjusting for butt flare is not sufficient to produce relatively unbiased taper-based volume equations, except for I^V and III^V.

Table XXXIV

Bias in Total Volume Estimation of Taper-Based Volume Equations

for Data Adjusted for Butt Flare

Douglas-fir size class	MB$_c$ (in cubic feet) of volume for equations										
	I^v	II^v	III^v	IV^v	V^v	VI^v	VII^v	$VIII^v$	IX^v	X^v	XI^v
1	0.2	-0.1	0.1	-0.0	-0.2	-0.3	-2.5	-0.1	0.2	-0.3	2.4
2	0.8	1.3	0.8	0.5	0.8	0.6	-8.3	1.1	1.1	0.7	9.0
3	2.0	2.1	0.2	0.2	1.2	0.9	-17.4	1.7	0.3	1.0	15.1
4	3.1	8.1	0.1	5.1	6.8	6.4	-24.3	7.6	5.4	6.5	27.4
5	5.7	14.4	1.2	10.4	12.8	12.2	-29.5	13.7	10.4	12.3	38.6
MB$_t$	1.8	3.6	0.4	2.2	2.9	2.7	-12.2	3.3	2.5	2.8	13.6

Western redcedar size class	MB$_c$ (in cubic feet) of volume for equations										
	I^v	II^v	III^v	IV^v	V^v	VI^v	VII^v	$VIII^v$	IX^v	X^v	XI^v
1	-0.4	-2.2	-0.8	-2.0	-0.7	-0.9	-1.8	-2.2	-0.7	-0.8	0.0
2	-0.2	-0.1	1.1	-3.0	4.6	3.9	-3.2	-0.1	-2.2	4.4	7.0
3	-3.5	-1.0	-1.9	-5.9	5.8	4.8	-8.0	-1.0	-8.8	5.5	9.3
4	-0.1	12.1	3.0	2.9	23.2	21.7	-6.9	12.1	-6.3	22.8	29.1
5	-1.4	18.0	2.9	6.0	32.2	30.2	-11.3	18.0	-8.9	31.7	39.5
MB$_t$	-0.3	1.2	0.1	-1.5	5.7	5.1	-3.8	1.2	-2.9	5.5	8.0

Aspen size class	MB$_c$ (in cubic feet) of volume for equations										
	I^v	II^v	III^v	IV^v	V^v	VI^v	VII^v	$VIII^v$	IX^v	X^v	XI^v
1	0.1	-0.1	-0.1	-0.3	-0.2	-0.2	-0.8	-0.4	-0.2	2.0	-0.0
2	1.3	0.4	0.3	-0.9	0.2	0.1	-2.9	-0.6	-0.7	7.0	0.7
3	0.2	-0.5	-1.3	-2.6	-0.8	-1.0	-6.4	-2.0	-2.5	9.9	-0.0
4	0.3	1.7	-0.3	-1.1	1.3	1.1	-6.6	-0.3	-1.0	15.2	2.3
5	0.3	5.7	1.9	2.0	5.2	4.9	-5.5	3.2	2.3	22.7	6.4
MB$_t$	0.4	0.7	-0.1	-0.7	0.5	0.4	-3.2	-0.4	-0.5	7.7	1.0

6. Additional aspects of both systems and possible ways to improve them

6.1. Taper equations on data above breast height

Adjusting the observation at one foot does not completely eliminate the butt flare. To check if taper equations could be further improved by eliminating the observation at one foot, two taper equations (equation I and V) were fitted on the data of Douglas-fir and western redcedar. I is a non-sigmoid and V is a sigmoid taper equation.

The data with the observation at one foot eliminated are called the "reduced data".

Results of some diameter estimation tests are given in table XXXV. Compared with the taper equations on adjusted data, there is no significant improvement. Adjusting, as applied before, seems to be as efficient in reducing the bias as eliminating the observation below breast height.

Table XXXV

Diameter Bias of Equations I and V

for Reduced Data

Taper equation I											
	MB_t (in inches) of diameter at heights										
species	4.5'	.1H	.2H	.3H	.4H	.5H	.6H	.7H	.8H	.9H	1 H
D	0.4	0.7	0.4	0.2	-0.1	-0.4	-0.5	-0.4	-0.3	0.2	0.0
C	-0.5	0.7	0.4	0.2	-0.1	-0.3	-0.3	-0.4	-0.2	0.1	0.0
Taper equation V											
	MB_t (in inches) of diameter at heights										
species	4.5'	.1H	.2H	.3H	.4H	.5H	.6H	.7H	.8H	.9H	1 H
D	0.1	0.2	0.1	0.2	0.2	0.2	0.1	0.0	-0.2	-0.2	0.0
C	-0.1	0.6	0.3	0.3	0.4	0.5	0.6	0.5	0.5	0.3	0.0

6.2. Relation between coefficients and form

Because of their functional form, some taper and volume equations assume a constant cylindrical form factor for all trees and therefore do not account for variation in form. Even the so called "variable form" taper and volume equations seldom account for the whole range of variation in form. The equation coefficients define a mean profile for all trees or a mean profile for all trees of a given D and H class and vary according to this mean form. This introduces a bias for all trees having a form different from the mean form.

If there exists a relationship between the form and the values of the coefficients, then this relationship could be used to account for a wider variation in form and as such reduce the bias of the equations.

In the following sections, the relationship between the coefficients and the form of the trees is examined for taper-based taper equations and for taper-based and volume-based volume functions.

6.2.1. Relation between taper-based taper equation coefficients and form

The following two taper equations

I. $\log d = b_o + b_1 \log D + b_2 \log \underline{l} + b_3 \log H$

VIII. $\log(d / D) = b_o + b_1 \log(\underline{l} / H)$

were fitted on the different classes of squared absolute form quotient for western redcedar and cottonwood. These two species have a wide range of form classes represented by a sufficient number of trees. Coefficients are given in table XXXVI.

Table XXXVI

Relation between Taper-Based Taper Equation

Coefficients and Form

species	AFQ^2 class	number of trees	Coefficients of equation I: b_o	b_1	b_2	b_3
C	0.2	8	0.145080	0.792272	0.936968	-0.887739
	0.3	20	0.158145	0.869953	0.858106	-0.856013
	0.4	22	-0.085453	0.866069	0.792294	-0.655524
	0.5	13	0.092535	0.900508	0.740062	-0.709870
Cot	0.3	31	-0.435447	0.778065	0.940845	-0.587400
	0.4	61	-0.247975	0.884045	0.848057	-0.650687
	0.5	17	-0.084730	0.955490	0.784375	-0.706669

species	AFQ^2 class	Coefficients of equation VIII: b_o 10^2	b_1
C	0.2	-2.648940	0.935387
	0.3	0.281354	0.857257
	0.4	2.212020	0.791830
	0.5	4.713390	0.739204
Cot	0.3	0.547316	0.941292
	0.4	1.018130	0.848170
	0.5	2.190320	0.784569

species	AFQ^2 class	real mean AFQ^2	Mean AFQ^2 of each form class estimated by: overall equation I	separate equations I
C	0.2	0.2191	0.3101	0.2481
	0.3	0.3066	0.3277	0.3164
	0.4	0.4021	0.3678	0.3769
	0.5	0.5114	0.3899	0.4428

For both equations and both species there is a good relationship between coefficients and form, although the relationship for equation VIII is better than for equation I.

To check what kind of improvement may be expected from the use of this relationship, the mean $(AFQ)^2$ was computed for the trees in each $(AFQ)^2$ class. The computation was done once with the overall taper equation I and once with the separate taper equations for each $(AFQ)^2$ class. The results are in table XXXVI.

Taper equation I, which may be considered a variable form taper equation, accounts for only part of the variation in form. Use of the relationship between the taper-based taper equation coefficients and form improves the estimation system significantly.

In equation VIII, the relationship is even better and identical for both species. As the squared absolute form quotient increases, b_o increases and b_1 decreases. This relationship can be justified in a theoretical way. Taper equation VIII is nothing else than

$$(d\ /\ D)^2 = 10^{2b_o}\ (\underline{1}\ /\ H)^{2b_1}$$

If double bark thickness would be zero and D measured at ground level we would have the following equation

$$(d\ /\ D)^2 = (\underline{1}\ /\ H)^{2b_1'}$$

For a neiloid form, b_1' would be 1.5, for a cone 1.0 and for a paraboloid 0.5. The b_1' value decreases with improving form. This is exactly what happens with the b_1 coefficient in equation VIII for the different form classes. For western redcedar, b_1 went down from 0.94 to 0.74 with increasing form.

The cylindrical form factor of a tree as defined by equation

VIII is computed as

$$CFF = 10^{2b_o} /(2\ b_1 + 1)$$

which again shows that an increase of b_o and a decrease of b_1 means an improvement of the tree form.

6.2.2. Relation between volume-based taper equation parameters and form

In most volume-based taper equations, the free parameters p and q can be compared with the $2b_1'$ values in equation VIII (as defined in the previous section). This explains why for most species p and q ranged from 1.3 to 2.0. A paraboloid would be represented by 1.0 and a cone by 2.0.

The relationship found in the previous section suggests the existence of a similar relationship for the free parameters. This has been tested for taper equations 1^t and 8^t for the same species (see table XXXVII). The results show a strong negative correlation between the parameter value and the form. The parameter values for each form class are almost identical for both equations.

The mean AFQ^2 has been computed for each form class using once an overall and second separate parameter values (see table XXXVII).

Although the use of the relationship between form and parameter value does not make the system perfect, it improves it a great deal.

Table XXXVII

Relation between Volume-Based Taper Equation

Parameters and Form

	Optimum parameter values for				
	western redcedar			cottonwood	
AFQ^2 class	equation 1^t p	equation 8^t p	AFQ^2 class	equation 1^t p	equation 8^t p
0.2	2.5	2.5	0.3	1.8	1.8
0.3	2.1	2.1	0.4	1.6	1.6
0.4	1.7	1.8	0.5	1.4	1.4
0.5	1.4	1.5			

		Western redcedar Mean AFQ^2 of each form class estimated by			
AFQ^2 class	real mean AFQ^2	equation 1^t overall p	equation 1^t separate p's	equation 8^t overall p	equation 8^t separate p's
0.2	0.2191	0.2713	0.2175	0.2530	0.2030
0.3	0.3066	0.2813	0.2696	0.2519	0.2414
0.4	0.4021	0.3053	0.3451	0.2481	0.2694
0.5	0.5114	0.3224	0.4125	0.2372	0.2920

6.2.3. Relation between volume-based volume equation coefficients and form

Some volume equations, like the logarithmic volume equation 1 and the combined variable volume equation 4 are considered to be variable form equations because the cylindrical form factors of the trees, as defined by these equations, change with D and H.

The CFF, as defined by equation 1, is equal to

$$CFF = 10^{b_o}\, k\, D^{b_1 - 2}\, H^{b_2 - 1}$$

where b_1 is usually less than 2 and b_2 usually larger than 1 (except for western redcedar) such that the CFF decreases with increasing D and increases with increasing H.

The CFF, as defined by equation 4, is equal to

$$CFF = b_o\, k/(D^2H) + b_1\, k$$

where both, b_o and b_1, are usually positive, such that the CFF decreases with both increasing D and H.

Notice that the assumptions which the volume functions make about the CFF may be contradictory. The fact that a variable form volume equation defines different CFF values for different values of D and H does not necessarily mean that they account for the full range of variation in form factors; neither does it mean that these CFF-D-H relationships are the real ones. Volume equations are often significantly biased. Because $V\text{-}D^2H$ is not a linear relationship, the intercept of equation 4 is more a measure of the range of the observations. But it is this intercept which will define how the CFF will vary according to D and H.

Furthermore,these equations still assume that the form within the same D and H class is constant.This is not true.

To check if there is a clearcut relationship between volume equation coefficients and form,equations 1 and 4 were fitted on the different CFF classes of western redcedar (see table XXXVIII). In both equations,coefficients and form are highly correlated.

For each form factor class the mean CFF was computed,first by using only the overall volume equation and second by using separate volume equations for each form class.The results of this test are in table XXXVIII.

In case only one overall volume equation is used,equation 1 accounts only for part of the variation in form while equation 4 gives highly biased results.Only by using the relationship between coefficients and form factor can the whole range of form be accounted for in an unbiased way.

6.2.4.Use of the relation between coefficients and form

Is it of any use to know these relationships between coefficients and form if form factors or form quotients are impossible or,for many reasons,not practical to measure on each tree?

First of all a distinction must be made between selecting an appropriate set of coefficients for each individual tree and selecting an appropriate set for a group of trees.The word "group" must be interpreted in a broad sense.It could be an age group,a provenance or trees subject to a particular thinning or fertilization regime or trees in a particular forest region etc.

Table XXXVIII

Relation between Volume-Based Volume Equation

Coefficients and Form

Western redcedar

CFF class	number of trees	Coefficients of equation 1 b_o	b_1	b_2	equation 4 b_o	$b_1 \; 10^2$
0.3	17	-2.550800	1.751960	1.056900	3.963690	0.161048
0.4	31	-2.465400	1.880370	0.968841	3.113030	0.194225
0.5	15	-2.273020	1.973080	0.849701	0.824420	0.254421

CFF class	real mean CFF	Mean CFF of each form class estimated by overall equation 1	separate equations 1	overall equation 4	separate equations 4
0.3	0.318	0.348	0.317	0.343	0.318
0.4	0.405	0.407	0.404	0.441	0.426
0.5	0.501	0.452	0.501	0.578	0.504

As far as individual trees are concerned, the taper and volume estimation might be improved without taking any more measurements, if a good correlation could be found between form factor and a function of D and H (the only two variables measured). The relationship between AFQ^2 and the variables D, H, D / H and D^2H was tested (by plotting) for Douglas-fir, western redcedar and cottonwood. A relationship exists between form and some of these variables, but only a small amount of the variation is accounted for. Some relationships are of the opposite kind for different species. A good correlation is lacking.

Therefore, if one wants to select the appropriate set of coefficients for each individual tree, one will reduce the bias of taper and volume by only a small amount if form factor is predicted from D and H functions.

Taking additional measurements for a better prediction of form will ensure a greatly reduced bias but may only be of importance in special circumstances, e.g. research studies of thinning, fertilization, etc.

Selecting the appropriate coefficients may prove to be more valuable for groups of trees if there really exists an important difference in mean form. Form may, for some species, be closely related to age. A better form may result from particular climatic conditions. Some provenances may have a significantly different bark thickness. Thinning and fertilization may result in a different tree profile.

Even if these factors are responsible for only a small amount of variation in form in the population, then, selecting the appropriate coefficients based on the mean form of the group, will reduce the individual tree bias only slightly, but may nearly eliminate the overall bias of the group.

7. Discussion, summary and suggestions

After having seen all these results some conclusions may be drawn. They are an expression of how one's own degrees of belief about particular assumptions may have been changed, or have remained unchanged, and are therefore subjective.

Some people would have had different opinions about most assumptions because of different past experiences. Others would have used different criteria to select the data or might have used the same criteria but would have selected other data. Different people formulate the problems or assumptions in a different way and test them differently. People with the same prior opinions might have changed their minds in a different way after having seen the same results.

Therefore, people, interested in these problems, should know why the study was done, how it was done, what the results look like and what the current opinion of the investigator is. They are then free to make up their own mind.

The objectives of this study were explained in the introduction. How it was done and what the results are can be found throughout the whole text. The way the degrees of belief of the investigator have been modified or have remained unchanged will now be summarized.

Many of the volume equations tested give reasonable results for some species but only a few (equations 1,2,6,7 and 11) are relatively unbiased for all species.

All the volume equations studied could be converted theoretically into compatible volume-based taper equations, but the conversion was impracticable for some of them (equations 6,7 and 11).

Most volume-based taper equations give very similar results but differ in the way the pattern of bias changes from one size class to another.Usually,the taper equations derived from the best volume equations also perform best for diameter estimation.

None of the taper-based taper equations is without any systematic bias.Sigmoid taper curves may look more attractive but still fail in solving the problem of bias.

All the taper-based taper equations could be transformed into compatible taper-based volume equations.Many of these volume equations are simple and similar in form.Most are very much biased for total volume estimation,especially for the larger size classes.

The volume-based system can be fairly unbiased for total volume and good for diameter estimation.The best performing equation here seems to be the logarithmic volume equation 1.

For the taper-based systems,the equations which gave best results for most species are the logarithmic equation I and Bennett-Swindel's taper equation III.

The taper-based systems are usually more biased for total volume and not better than the volume-based systems for diameter estimation.

Therefore,if a compatible estimating system for tree taper and volume is desired,a volume-based system looks more promising.

In selecting a particular system or a particular function,one also has to take into account the costs of analysis and introduction of the new system.There may be no need to start off with a completely new system if one already has a satisfactory volume equation from which a suitable taper equation can be derived.

Transformation of a taper equation into a height estimating system may be theoretically correct but often involves large biases.

Although biased, section volume estimation is adequate for several equations. If the bias is small and similar in all size classes, a correction may be applied. The correction will be positive for some sections and negative for others according to the kind of bias involved. When the heights are unknown, section volume estimation should be avoided in the lower parts of the tree.

Conventional methods of equation testing (SE_E, R^2, etc) often are inadequate to compare the effectiveness of different functions. It is recommended that extensive tests of bias of diameter, height, section volume and total volume be carried out before adopting any particular system of tree taper and volume.

Taper equations which do not account for butt flare should not use the observations below breast height.

Application of weighting procedures in fitting taper or volume equations seldom seems to be efficient. It has little or no effect on bias if the model is correct and it often makes things worse if the model is wrong.

Non-linear fitting makes a small difference in some cases and none in other cases. It should be tested in each particular application.

For the data analysed herein, in which extreme values have been eliminated, there does not seem to be any need for using Meyer's correction factor for logarithmic equations. However, more investigation is needed to find out how violations of the assumptions affect the effectiveness of the correction factor.

In fitting the equations,some species,similar in form,may be combined in the same system without too much loss.However,not too much will be gained unless adjustments are made for bark thickness and form.

Whether or not the findings in this study apply to unusually shaped and large trees requires further investigation.

Some of these taper and volume systems may be improved further. Separate equations for small and large trees may be effective in case the linear model assumption is not met.An improvement may also result from different conditioning of the coefficients.Addition of other variables may reduce the bias significantly.Use of different taper equations for lower and upper tree bole could be an alternative to sigmoid taper equations.

There exists a good relationship between coefficients of some equations and form.This relationship is interesting from both a theoretical and a practical point of view.

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APPENDIX 1

Common Names and Latin Names of the

Tree Species and Species Groups [4]

1. Trembling Aspen (*Populus tremuloides* Michx.)
2. Coast Balsam Species (*Abies amabilis* (Dougl.) Forbes and *A. grandis* (Dougl.) Lindl.)
3. Western Redcedar (*Thuja plicata* Donn.)
4. Black Cottonwood (*Populus trichocarpa* Torr. and Gray)
5. Douglas-fir (*Pseudotsuga menziesii* (Mirb.) Franco)
6. Lodgepole Pine (*Pinus contorta* Dougl.)
7. Ponderosa Pine (*Pinus ponderosa* Laws.)
8. Slash Pine (*Pinus elliottii* Engelm. var. *elliottii*)
9. Western White Pine (*Pinus monticola* Dougl.)
10. Interior Spruce Species (*Picea glauca* (Moench) Voss, *P. Engelmanni* Parry and *P. mariana* (Mill.) B.S.P.)

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[4] Based on Appendix I from Browne (1962), except numbers 7 and 8.

Appendix 2

Numbering of the Volume and Taper Equations

1. Volume-based systems

Name	Number of volume equation	Number of volume-based taper equation
logarithmic volume equation	1	1^t
logarithmic combined variable	2	2^t
Honer's volume equation	3	3^t
combined variable volume equation	4	4^t
weighted combined variable	5	5^t
comprehensive combined variable	6	6^t
weighted comprehensive	7	7^t
combined variable (zero intercept)	8	8^t
V / B' as function of H	9	9^t
V / B' as function of H and H^2	10	10^t
Meyer's volume-diameter ratio equation	11	11^t
cylindrical form factor volume equation	12	12^t
non-linear form of logarithmic, no weighting	13	13^t
non-linear form of logarithmic, with weighting	13(w)	13^t(w)
non-linear form of Honer's equation, no weighting	14	14^t
non-linear form of Honer's equation, with weighting	14(w)	14^t(w)

2. Taper-based systems

Name	Number of taper equation	Number of taper-based volume equation
logarithmic taper equation	I	I^v
taper equation of Kozak, Munro and Smith	II	II^v
Bennett and Swindel's taper equation	III	III^v
equation with same form as 4^t	IV	IV^v
Matte's taper equation	V	V^v
Osumi's taper equation	VI	VI^v
taper equation of Bruce, Curtis and Vancoevering	VII	VII^v
equation with same form as 8^t	VIII	$VIII^v$
equation with same form as 9^t	IX	IX^v
unconditioned Behre' taper equation	X	X^v
conditioned Behre's taper equation	XI	XI^v

Appendix 3

Summary of the Volume-Based Volume Equations

3.1. Linear volume equations

1. $\log V = b_o + b_1 \log D + b_2 \log H$

species	b_o	b_1	b_2	SE_E 10	R^2 10^2
D	-2.83276	1.76116	1.20365	0.37144	99.4
C	-2.20396	1.68684	0.94454	0.49756	98.6
S	-2.54178	1.88818	1.01215	0.38673	99.3
B	-2.60213	1.69759	1.17051	0.35998	99.6
A	-3.00096	1.72263	1.33892	0.37248	99.2
Cot	-3.25913	1.73206	1.44513	0.38152	98.9
Pl	-2.48880	1.78377	1.05510	0.33634	99.2
Pw	-2.51801	1.82359	1.04051	0.32335	99.3

2. $\log V = b_o + b_1 \log (D^2H)$

species	b_o	b_1	SE_E 10	R^2 10^2
D	-2.52827	0.960198	0.40079	99.3
C	-2.12342	0.869511	0.49560	98.6
S	-2.48746	0.962877	0.38542	99.3
B	-2.34958	0.937439	0.38904	99.5
A	-2.57462	0.983941	0.40855	99.0
Cot	-2.69274	1.002760	0.43874	98.5
Pl	-2.35261	0.934576	0.34784	99.2
Pw	-2.40040	0.944235	0.32372	99.3

3. $D^2 / V = b_o + b_1 / H$

species	b_o	b_1	SE_E	R^2 10^2
D	0.697910	430.855	0.48836	90.2
C	2.830720	252.549	0.82766	66.3
S	0.616650	383.407	0.49396	91.7
B	0.635343	355.833	0.45566	93.6
A	-0.879354	547.995	0.84125	80.9
Cot	-0.033198	436.402	0.62789	87.8
Pl	0.814179	349.554	0.56406	86.0
Pw	0.911824	345.288	0.34874	90.4

4. $V = b_o + b_1 D^2H$

species	b_o	$b_1\ 10^2$	SE_E	$R^2\ 10^2$
D	2.43215	0.186177	8.55	97.2
C	5.85424	0.168317	7.00	96.5
S	2.19154	0.211675	4.73	98.0
B	6.79594	0.206315	9.84	99.0
A	0.68262	0.221711	2.59	97.3
Cot	0.55679	0.199651	1.90	97.0
Pl	2.15687	0.218430	3.31	98.2
Pw	2.88724	0.211811	5.74	98.3

5. $V/(D^2H) = b_o + b_1/(D^2H)$

species	$b_o\ 10^2$	b_1	$SE_E\ 10^3$	$R^2\ 10^2$
D	0.195342	0.544671	0.18985	14.1
C	0.200187	1.578370	0.30334	41.1
S	0.224735	0.371562	0.21057	10.6
B	0.230976	0.770271	0.26654	15.4
A	0.230173	0.089940	0.21312	1.0
Cot	0.211162	-0.090401	0.20058	0.9
Pl	0.230832	0.855657	0.20327	31.1
Pw	0.221252	0.890530	0.18189	26.2

6. $V = b_o + b_1 D + b_2 H + b_3 DH + b_4 D^2 + b_5 D^2H$

species	b_o	b_1	b_2	$b_3\ 10$	$b_4\ 10$	$b_5\ 10^2$	SE_E	$R^2\ 10^2$
D	8.57299	-0.324102	-0.228254	0.308613	-0.712229	0.149274	7.47	98.0
C	10.73580	-0.858671	-0.371924	0.562510	-0.051280	0.026351	6.22	97.4
S	15.28650	-3.747810	-0.172540	0.439281	2.052750	-0.011020	4.57	98.2
B	-34.39410	6.715050	0.108075	-0.203816	-2.960250	0.339907	8.60	99.3
A	9.44231	-0.533039	-0.298294	0.474012	-1.007390	0.135440	2.28	98.0
Cot	10.37870	-1.430180	-0.268870	0.535666	-0.537087	0.064625	1.43	98.3
Pl	2.13527	-0.114786	-0.109543	0.226315	-0.166353	0.146681	3.11	98.4
Pw	-6.58185	-0.121157	0.185972	-0.110171	0.555434	0.199308	5.80	98.3

7. $V/(D^2H) = b_o + b_1/(D\ H) + b_2/D^2 + b_3/D + b_4/H + b_5/(D^2H)$

species	$b_o\ 10^2$	b_1	b_2	$b_3\ 10$	$b_4\ 10$	b_5	SE_E	$R^2\ 10^2$
D	0.177184	0.262031	-0.139823	0.204433	-0.854369	3.33407	0.1684	36.7
C	0.079447	-0.074431	-0.130385	0.272067	0.068316	1.63112	0.2500	62.6
S	0.079118	-1.685210	-0.123082	0.293750	0.876817	7.20132	0.2031	20.6
B	0.166709	0.305371	-0.137910	0.266846	-0.594730	0.63753	0.1968	56.6
A	0.349691	1.478920	0.018987	-0.055435	-1.728240	-4.03959	0.1954	19.8
Cot	0.045153	-1.844980	-0.202850	0.470614	0.114538	9.39339	0.1716	30.2
Pl	0.060537	-1.628170	-0.190995	0.411507	0.541317	8.89101	0.1827	45.8
Pw	0.138884	-1.399970	0.008645	0.129167	0.759510	3.57925	0.1683	39.7

8. $V = b_o\ D^2H$

species	$b_o\ 10^2$	SE_E	$R^2\ 10^2$
D	0.190636	8.63	97.1
C	0.181876	8.05	95.3
S	0.218613	4.89	97.8
B	0.213506	10.84	98.8
A	0.226414	2.61	97.3
Cot	0.204669	1.92	96.9
Pl	0.227784	3.54	97.9
Pw	0.218347	5.92	98.1

9. $V / B' = b_o + b_1\ H$

species	b_o	b_1	SE_E	$R^2\ 10^2$
D	2.09803	0.345823	4.06856	87.0
C	12.52330	0.225925	3.95751	61.8
S	4.44776	0.368000	3.45596	87.7
B	9.38033	0.339157	4.62709	86.2
A	0.50554	0.419175	3.00959	86.7
Cot	-2.86449	0.423589	2.71700	84.8
Pl	5.62314	0.371020	3.03212	86.4
Pw	8.14278	0.337024	3.44886	83.3

10. $V / B' = b_o + b_1 H + b_2 H^2$

species	b_o	b_1	$b_2\ 10^3$	SE_E	$R^2\ 10^2$
D	12.01110	0.141068	0.971455	4.00404	87.6
C	10.01420	0.293838	-0.424102	3.98295	61.9
S	-3.85863	0.578065	-1.216960	3.39840	88.2
B	-2.63995	0.607556	-1.326650	4.38988	87.8
A	-2.08107	0.494368	-0.512147	3.02087	86.8
Cot	1.93972	0.286102	0.942181	2.72106	84.8
Pl	0.92035	0.498520	-0.806094	3.02151	86.6
Pw	-0.37319	0.525672	-0.978412	3.40892	83.8

11. $V / D = b_o + b_1 H + b_2 D + b_3 D H$

species	b_o	$b_1\ 10^2$	$b_2\ 10$	$b_3\ 10^2$	SE_E	$R^2\ 10^2$
D	-0.294880	1.27424	-0.491841	0.179870	3.56637	96.8
C	-0.551858	2.09796	0.257830	0.086887	3.27701	94.1
S	-0.560030	0.98764	0.592112	0.138466	2.74893	96.7
B	-0.260299	1.75939	-0.425959	0.185516	3.84807	98.3
A	0.032511	0.64967	-0.729600	0.254262	1.82375	96.4
Cot	-0.201713	0.97272	-0.637007	0.217405	1.32613	96.5
Pl	-0.268547	1.02856	0.121876	0.169301	1.99489	97.1
Pw	-0.265554	0.66074	0.340717	0.167891	3.06261	96.7

12. $V /(D^2H) = b_o + b_1 (H /(H - 4.5))^2$

species	$b_o\ 10^3$	$b_1\ 10^2$	$SE_E\ 10^4$	$R^2\ 10^2$
D	0.054422	0.177392	1.94184	10.1
C	-3.713480	0.517358	2.85293	47.9
S	0.520494	0.158726	2.09424	11.6
B	-0.273878	0.240028	2.58145	20.7
A	2.274120	0.004383	2.14215	0.0
Cot	3.568640	-0.129582	1.96237	5.0
Pl	-0.167796	0.229431	2.22406	17.5
Pw	-1.247120	0.321505	1.82124	26.0

3.2. Non-linear volume equations

13 and 13 (w). $V = 10^{b_o} D^{b_1} H^{b_2}$

species	without weighting b_o	b_1	b_2	with weighting b_o	b_1	b_2
D	-2.9079	1.6948	1.2809	-2.8271	1.7712	1.1958
C	-2.3220	1.4743	1.1437	-2.1874	1.7196	0.9173
S	-2.2681	1.8669	0.8898	-2.5375	1.8954	1.0067
B	-2.7860	1.6615	1.2797	-2.5969	1.7079	1.1624
A	-3.1407	1.6608	1.4462	-2.9562	1.7507	1.3013
Cot	-3.3495	1.5947	1.5667	-3.2324	1.7481	1.4235
P1	-2.4479	1.7385	1.0616	-2.4746	1.7936	1.0427
Pw	-2.3236	1.9296	0.8823	-2.5434	1.8071	1.0633

14 and 14 (w). $V = D^2/(b_o + b_1/ H)$

species	without weighting b_o	b_1	with weighting b_o	b_1
D	-0.90764	645.36	0.44330	451.51
C	1.53950	391.04	2.69910	254.86
S	1.10420	343.61	0.61799	379.79
B	0.45065	403.54	0.84351	331.44
A	-0.64754	501.53	0.07047	424.91
Cot	-1.22320	595.87	-0.69806	530.35
P1	0.38200	401.47	0.88161	341.01
Pw	0.74006	366.73	0.92371	341.49

3.3. Volume equation 1 for combinations of species

combination number	coefficients of equation 1 b_o	b_1	b_2
1.	-2.55388	1.89858	1.00835
2.	-2.50380	1.88952	0.98854
3.	-2.50416	1.82087	1.02751
4.	-3.06473	1.78035	1.32984
5.	-2.45284	1.80574	1.02162

3.4. Volume equations for data adjusted for butt flare

1. $\log V = b_o + b_1 \log D + b_2 \log H$

species	b_o	b_1	b_2	SE_E 10	R^2 10^2
D	-2.80967	1.76364	1.19068	0.37131	99.4
C	-2.26455	1.66945	0.97222	0.48639	98.7
S	-2.63154	1.81916	1.08910	0.40659	99.2
B	-2.57132	1.67844	1.16149	0.34646	99.6
A	-2.99624	1.69457	1.34460	0.33833	99.3
Cot	-3.23852	1.73268	1.43303	0.35945	99.0
Pl	-2.50795	1.75606	1.07620	0.33217	99.2
Pw	-2.50698	1.83438	1.02347	0.32528	99.3

4. $V = b_o + b_1 D^2H$

species	b_o	b_1 10^2	SE_E	R^2 10^2
D	2.50881	0.185490	8.41	97.3
C	5.30481	0.158568	6.62	96.5
S	2.57052	0.200707	4.80	97.7
B	7.06665	0.198745	10.08	98.9
A	0.85918	0.211975	2.27	97.7
Cot	0.60635	0.197921	1.88	97.0
Pl	2.21598	0.212704	3.26	98.1
Pw	2.77376	0.206884	5.80	98.1

Appendix 4

Derivation of Compatible Taper Equations from Volume Equations and the Functions to Estimate Height and Section Volume

1^t. Derivation of taper equation from volume equation 1

Integration of the taper equation over the total length of the tree must yield volume equation 1

$$\int_0^H (d^2/k)\, d\underline{l} = 10^{b_o} D^{b_1} H^{b_2}$$

or $$d^2 H / k = 10^{b_o} D^{b_1} H^{b_2}$$

$$d^2 = k\, 10^{b_o} D^{b_1} H^{b_2-1}$$

where b_o, b_1 and b_2 are the coefficients of volume equation 1.

A more general and useful taper equation is

$$d^2 = k\,(p+1)\, 10^{b_o} D^{b_1}\, \underline{l}^p/(H^{p-b_2+1})$$

or $$d = (k\, 10^{b_o}(p+1))^{\frac{1}{2}} D^{b_1/2}\, \underline{l}^{p/2} H^{(b_2-p-1)/2}$$

which is the same as

$$d = a\, D^b\, \underline{l}^c\, H^e \qquad (1^t)$$

where $a = (k\, 10^{b_o}(p+1))^{\frac{1}{2}}$ $\qquad c = p/2$

$b = b_1/2$ $\qquad e = (b_2 - p - 1)/2$

and where p is the only free parameter.

Other ways of writing this taper equation are

$$d / D^b = a\, \underline{l}^c / H^{-e}$$ which looks very familiar

and $$\log d = \log a + b \log D + c \log \underline{l} + e \log H$$

which is the logarithmic form of equation 1^t.

The height of a given diameter can be computed by the following transformation of equation 1^t:

$$\underline{l} = (d / (a\, D^b H^e))^{1/c}$$

where a, b, c and e have same meaning as in equation 1^t.

Volume of a given section can be computed as

$$V_s = \int_{\underline{l}_2}^{\underline{l}_1} (d^2/\ k)\ d\underline{l} = a^2\ D^{2b}\ H^{2e}(\underline{l}_1^{2c+1} - \underline{l}_2^{2c+1})/(k\ (2\ c + 1)) =$$
$$= 10^{b_o}\ D^{b_1}\ H^{(b_2-p-1)}\ (\underline{l}_1^{p+1} - \underline{l}_2^{p+1})$$

where $\underline{l}_1$ and $\underline{l}_2$ are respectively the lower and upper distances from the tip of the tree.

2^t. Taper equation from volume equation 2.

Integration of the taper equation over total tree length must yield volume equation 2

$$\int_0^H (d^2/\ k)\ d\underline{l} = 10^{b_o}\ D^{2b_1}\ H^{b_1}$$

This equation leads to the same type of taper equation as 1^t, however, some coefficients will be different:

$$d = a\ D^b\ \underline{l}^c\ H^e \qquad (\ 2^t\ )$$

where $a = (10^{b_o}\ k\ (p + 1))^{\frac{1}{2}}$ $\qquad$ $c = p\ /\ 2$

$b = b_1$ $\qquad$ $e = (b_1 - p - 1)/\ 2$

and p is the free parameter.

Height for a given diameter is estimated by

$$\underline{l} = (d\ /(a\ D^b\ H^e))^{1/c}$$

and section volume can be computed as

$$V_s = a^2\ D^{2b}\ H^{2e}\ (\underline{l}_1^{2c+1} - \underline{l}_2^{2c+1})/(k\ (2\ c + 1))$$
$$= 10^{b_o}\ D^{2b_1}\ H^{(b_1-p-1)}\ (\underline{l}_1^{p+1} - \underline{l}_2^{p+1})$$

3^t. Taper equation from volume equation 3

Integration of taper equation must yield volume equation 3

$$\int_0^H (d^2/\,k)\;d\underline{l} = D^2/(b_o + b_1/\,H)$$

or $\quad d^2H\,/\,k = D^2/(b_o + b_1/\,H) \quad\longrightarrow\quad d^2 = D^2\,k\,/(b_o\,H + b_1)$

A more general equation is

$$d^2 = k\,(p+1)\,D^2\,\underline{l}^p/(b_o\,H^{p+1} + b_1\,H^p)$$

or $\quad d = (a\,D^2\,\underline{l}^p/(b\,H^{p+1} + c\,H^p))^{\frac{1}{2}} \qquad (3^t)$

where $a = (k\,(p+1))$ $\qquad b = b_o$ $\qquad c = b_1$

and p is the free parameter.

Height of a given diameter is computed as

$$\underline{l} = (d^2(b\,H^{p+1} + c\,H^p)/(a\,D^2))^{1/p}$$

and section volume as

$$V_s = D^2(\underline{l}_1^{p+1} - \underline{l}_2^{p+1})/(b\,H^{p+1} + c\,H^p)$$

4^t. Taper equation from volume equation 4

Integration of taper equation yields volume equation 4

$$\int_0^H (d^2/\,k)\;d\underline{l} = b_o + b_1\,D^2H$$

or $\quad d^2\,H\,/\,k = b_o + b_1\,D^2H \quad\longrightarrow\quad d^2 = k\,b_o/\,H + k\,b_1\,D^2$

or more generally as

$$d^2 = k\,b_o(p+1)\,\underline{l}^p/\,H^{p+1} + k\,b_1(q+1)\,D^2\,\underline{l}^q/\,H^q$$

or $\quad d = (a\,\underline{l}^p/\,H^{p+1} + b\,D^2\,\underline{l}^q/\,H^q)^{\frac{1}{2}} \qquad (4^t)$

where $a = k\,b_o(p+1)$ $\qquad b = k\,b_1(q+1)$

and p and q are the two free parameters.

Volume of a tree section is computed as

$$V_s = b_o(\underline{l}_1^{p+1} - \underline{l}_2^{p+1})/\,H^{p+1} + b_1\,D^2(\underline{l}_1^{q+1} - \underline{l}_2^{q+1})/\,H^q$$

5^t. Taper equation from volume equation 5

The only difference with equation 4 is that the intercept b_o, in equation 5, corresponds with the slope coefficient in equation 4 and vice versa, so that

$$d = (a\ \underline{l}^p /\ H^{p+1} + b\ D^2\ \underline{l}^q /\ H^q)^{\frac{1}{2}} \qquad (5^t)$$

with $a = k\ b_1(p + 1)$ $\qquad$ $b = k\ b_o(q + 1)$

and p and q are the free parameters.

The section volume is given by

$$V_s = b_1(\underline{l}_1^{p+1} - \underline{l}_2^{p+1}) /\ H^{p+1} + b_o\ D^2\ (\underline{l}_1^{q+1} - \underline{l}_2^{q+1}) /\ H^q$$

6^t. Taper equation from volume equation 6

$$\int_0^H (d^2/\ k)\ d\underline{l} = b_o + b_1\ D + b_2\ H + b_3\ D\ H + b_4\ D^2 + b_5\ D^2 H$$

$$d^2 = k\ (b_o + b_1\ D + b_2\ H + b_3\ D\ H + b_4\ D^2 + b_5\ D^2 H) /\ H$$

A general taper equation is

$$d = (a\ \underline{l}^p /\ H^{p+1} + b\ D\ \underline{l}^q /\ H^{q+1} + c\ \underline{l}^r /\ H^r + e\ D\ \underline{l}^s /\ H^s + f\ D^2\ \underline{l}^t /\ H^{t+1} + g\ D^2\ \underline{l}^u /\ H^u)^{\frac{1}{2}}$$

where $a = k\ (p + 1)\ b_o$ $\qquad$ $c = k\ (r + 1)\ b_2$ $\qquad$ $f = k\ (t + 1)\ b_4$

$b = k\ (q + 1)\ b_1$ $\qquad$ $e = k\ (s + 1)\ b_3$ $\qquad$ $g = k\ (u + 1)\ b_5$

and p, q, ..., g are the free parameters.

Section volume is computed as

$$\begin{aligned} V_s = {} & b_o(\underline{l}_1^{p+1} - \underline{l}_2^{p+1}) /\ H^{p+1} + b_1\ D\ (\underline{l}_1^{q+1} - \underline{l}_2^{q+1}) /\ H^{q+1} + \\ & b_2(\underline{l}_1^{r+1} - \underline{l}_2^{r+1}) /\ H^r + b_3\ D\ (\underline{l}_1^{s+1} - \underline{l}_2^{s+1}) /\ H^s + \\ & b_4\ D^2(\underline{l}_1^{t+1} - \underline{l}_2^{t+1}) /\ H^{t+1} + b_5\ D^2(\underline{l}_1^{u+1} - \underline{l}_2^{u+1}) /\ H^u \end{aligned}$$

7^t. Taper equation from volume equation 7

After multiplication by D^2H, equation 7 has exactly the same form as volume equation 6. Therefore, taper equation 7^t has the same form as equation 6^t; the only differences will be in the coefficients which are:

$a = k\ (p + 1)\ b_5 \qquad c = k\ (r + 1)\ b_2 \qquad f = k\ (t + 1)\ b_4$

$b = k\ (q + 1)\ b_1 \qquad e = k\ (s + 1)\ b_3 \qquad g = k\ (u + 1)\ b_o$

To make the section volume equation of 6^t applicable for 7^t, replace only b_o by b_5 and vice versa.

8^t. Taper equation from volume equation 8

$$\int_0^H (d^2/\ k)\ d\underline{1} = b_o\ D^2H \qquad \longrightarrow \qquad d^2\ H\ /\ k = b_o\ D^2H$$

$d^2 = k\ b_o\ D^2$ and more general $d^2 = k\ (p + 1)\ b_o\ D^2\underline{1}^p/\ H^p$

or $\quad d = a\ D\ (\underline{1}\ /\ H)^{p/2}$ $\qquad (\ 8^t\ )$

where $a = (k\ (p + 1)\ b_o)^{\frac{1}{2}}$

The height equation is

$$\underline{1} = (d^2\ H^p/(a^2\ D^2))^{1/p}$$

and the section volume equation is

$$V_s = b_o\ D^2\ (\underline{1}_1^{p+1} - \underline{1}_2^{p+1})/\ H^p$$

9^t. Taper equation from volume equation 9

$$\int_0^H (d^2/\ k)\ d\underline{1} = b_o\ B' + b_1\ H\ B' \qquad \longrightarrow \qquad d^2\ H\ /\ k = b_o\ B' + b_1\ H\ B'$$

$d^2 = k\ (b_o\ B'\ /\ H + b_1\ B')$ and a more general equation is

$$d = (a\ D^2\ \underline{1}^p/\ H^{p+1} + b\ D^2\ \underline{1}^q/\ H^q)^{\frac{1}{2}} \qquad (\ 9^t\ )$$

where $a = (p + 1)\ b_o \qquad\qquad b = (q + 1)\ b_1$

and p and q are the two free parameters.

Section volume is given by

$$V_s = b_o\ B'(\underline{1}_1^{p+1} - \underline{1}_2^{p+1})/\ H^{p+1} + b_1\ B'(\underline{1}_1^{q+1} - \underline{1}_2^{q+1})/\ H^q$$

10^t. Taper equation from volume equation 10

$$\int_0^H (d^2/\ k)\ d\underline{1} = b_o\ B' + b_1\ B' H + b_2\ B' H^2$$

$$d^2 = k\ (b_o\ B'/\ H + b_1\ B' + b_2\ B' H) \quad \text{or more general}$$

$$d = (a\ D^2\ \underline{1}^p/\ H^{p+1} + b\ D^2\ \underline{1}^q/\ H^q + c\ D^2\ \underline{1}^r/\ H^{r-1})^{\frac{1}{2}} \qquad (10^t)$$

where $a = (p + 1)\ b_o$ $\quad b = (q + 1)\ b_1$ $\quad c = (r + 1)\ b_2$

The section volume equation is

$$V_s = b_o\ B'(\underline{1}_1^{p+1} - \underline{1}_2^{p+1})/\ H^{p+1} + b_1\ B'(\underline{1}_1^{q+1} - \underline{1}_2^{q+1})/\ H^q + b_2\ B'(\underline{1}_1^{r+1} - \underline{1}_2^{r+1})/\ H^{r-1}$$

11^t. Taper equation from volume equation 11

$$\int_0^H (d^2/\ k)\ d\underline{1} = b_o\ D + b_1\ D\ H + b_2\ D^2 + b_3\ D^2 H$$

$$d^2 = k\ (b_o\ D\ /\ H + b_1\ D + b_2\ D^2/\ H + b_3\ D^2) \quad \text{and more general}$$

$$d = (a\ D\ \underline{1}^p/\ H^{p+1} + b\ D\ \underline{1}^q/\ H^q + c\ D^2\ \underline{1}^r/\ H^{r+1} + e\ D^2\ \underline{1}^s/\ H^s)^{\frac{1}{2}} \quad (11^t)$$

where $a = k\ (p + 1)\ b_o$ $\qquad c = k\ (r + 1)\ b_2$

$b = k\ (q + 1)\ b_1$ $\qquad e = k\ (s + 1)\ b_3$

and p,q,r and s are four free parameters.

Section volume is given by

$$V_s = b_o\ D\ (\underline{1}_1^{p+1} - \underline{1}_2^{p+1})/\ H^{p+1} + b_1\ D\ (\underline{1}_1^{q+1} - \underline{1}_2^{q+1})/\ H^q + b_2\ D^2(\underline{1}_1^{r+1} - \underline{1}_2^{r+1})/\ H^{r+1} + b_3\ D^2(\underline{1}_1^{s+1} - \underline{1}_2^{s+1})/\ H^s$$

12^t. Taper equation from volume equation 12

$$\int_0^H (d^2/\ k)\ d\underline{l} = b_o\ D^2 H + b_1\ D^2 H\ (H\ /(H - 4.5))^2$$

$$d^2 = k\ (b_o\ D^2 + b_1\ D^2 (H\ /(H - 4.5))^2) \qquad \text{or more general}$$

$$d = (a\ D^2\ \underline{l}^p /\ H^p + b\ D^2\ \underline{l}^q\ H^{2-q}/(H - 4.5)^2)^{\frac{1}{2}} \qquad (12^t)$$

where $a = k\ (p + 1)\ b_o$ $\qquad$ $b = k\ (q + 1)\ b_1$

and p and q are the two free parameters.

The section volume equation is

$$V_s = b_o\ D^2(\underline{l}_1^{p+1} - \underline{l}_2^{p+1})/\ H^p + b_1\ D^2\ H^{2-q}(\underline{l}_1^{q+1} - \underline{l}_2^{q+1})/(H - 4.5)^2$$

Appendix 5

Summary of the Taper-Based Taper Equations

5.1. Linear taper equations

I. $\log d = b_o + b_1 \log D + b_2 \log \underline{1} + b_3 \log H$

species	b_o	b_1	b_2	b_3	$SE_E\ 10^2$	$R^2\ 10^2$
D	-0.162157	0.868138	0.653171	-0.512414	4.71777	97.0
C	0.169313	0.746660	0.821086	-0.749014	7.55771	93.6
S	-0.044339	0.866427	0.757857	-0.650812	5.35571	96.6
B	-0.106822	0.759437	0.721269	-0.513237	5.60663	96.4
A	-0.467402	0.715920	0.779922	-0.370981	7.05527	94.2
Cot	-0.488623	0.781775	0.865773	-0.487176	6.23952	95.8
P1	-0.090332	0.790349	0.696068	-0.520680	4.89761	96.4
Pw	0.051129	0.889893	0.735208	-0.689888	4.54512	97.2

II. $(d / D)^2 = b_o + b_1(h / H) + b_2(h / H)^2$

species	b_o	b_1	b_2	$SE_E\ 10$	$R^2\ 10^2$
D	0.871680	-1.23628	0.364600	0.950599	88.6
C	1.261930	-2.52386	1.261930	2.320710	75.3
S	1.193263	-2.18970	0.996437	1.788320	81.7
B	1.168674	-1.98361	0.814936	1.926460	78.4
A	1.119859	-1.87857	0.758711	2.137020	72.3
Cot	0.989716	-1.63695	0.647234	0.724694	94.9
P1	1.111171	-1.72886	0.617689	1.119800	90.2
Pw	1.129284	-1.94681	0.817526	1.342570	87.6

III. $d = b_o D \underline{1} /(H - 4.5) + b_1 \underline{1} (h - 4.5) + b_2 H \underline{1} (h - 4.5) + b_3 \underline{1} (h - 4.5)(H + h + 4.5)$

species	b_o	$b_1\ 10^3$	$b_2\ 10^4$	$b_3\ 10^4$	SE_E	$R^2\ 10^2$
D	0.90353	1.33319	-0.276904	0.145493	0.97	96.5
C	1.04891	0.46416	-0.851405	0.495464	1.67	92.1
S	1.05401	0.92629	-0.446678	0.249172	1.24	94.1
B	1.06210	1.24688	-0.400974	0.214041	1.76	94.7
A	1.00922	0.89985	-0.522060	0.312830	1.02	92.5
Cot	0.93994	0.35499	-0.109357	0.089113	0.49	97.5
P1	1.01177	1.97007	-0.618876	0.315387	0.74	97.1
Pw	1.02971	1.21326	-0.367131	0.192799	1.04	96.5

IV. $(d / D)^2 = b_o \, \underline{1}^p/(D^2 \, H^{p+1}) \; + \; b_1(\underline{1} / H)^q$

species	b_o	b_1	p	q	SE_E 10	R^2 10^2
D	159.909	0.84360	1.0	1.3	0.954973	88.5
C	362.694	1.22912	1.0	2.1	2.287740	76.2
S	13.491	1.17010	1.0	1.7	1.810280	81.2
B	62.077	1.17265	1.0	1.7	1.948000	77.9
A	-46.704	1.10536	1.0	1.5	2.153190	71.9
Cot	-9.612	0.99307	1.0	1.6	0.727191	94.8
P1	170.444	1.08402	1.0	1.5	1.132630	90.0
Pw	164.306	1.09993	1.0	1.6	1.361990	87.2

V. $(d / D)^2 = b_o(\underline{1} / H)^2 + b_1(\underline{1} / H)^3 + b_2(\underline{1} / H)^4$

species	b_o	b_1	b_2	SE_E 10	R^2 10^2
D	4.62922	-8.7816	5.11886	0.825042	91.5
C	6.22618	-14.6201	9.98878	1.862160	84.3
S	5.44968	-11.4127	7.34664	1.539780	86.4
B	5.67746	-11.4947	7.14594	1.742120	82.4
A	5.35872	-10.7451	6.67131	2.004280	75.7
Cot	2.92868	-4.1791	2.27177	0.691878	95.3
P1	5.11474	-9.5667	5.68553	0.976763	92.6
Pw	5.30943	-10.7156	6.68614	1.094080	91.8

VI. $(d / D) = b_o(\underline{1} / H) + b_1(\underline{1} / H)^2 + b_2(\underline{1} / H)^3$

species	b_o	b_1	b_2	SE_E 10	R^2 10^2
D	2.28893	-3.021.5	1.69490	0.539458	94.5
C	2.30077	-3.64133	2.54321	1.027900	89.2
S	2.22228	-2.96840	1.87881	0.785964	92.7
B	2.34849	-3.10336	1.86728	0.863131	90.8
A	2.21314	-2.75188	1.62288	0.891086	89.9
Cot	1.56226	-1.02407	0.45500	0.522427	96.3
P1	2.34731	-2.95912	1.70004	0.601989	94.9
Pw	2.25931	-2.95878	1.79786	0.628728	94.9

$$\text{VII.}\ (d\ /\ D)^2 = b_o X^{3/2} + b_1(X^{3/2} - X^3)\ D\ (10^{-2}) + b_2(X^{3/2} - X^3)\ H\ (10^{-3}) + b_3(X^{3/2} - X^{32})\ H\ D\ (10^{-5}) + b_4(X^{3/2} - X^{32})\ H^{1/2}(10^{-3}) + b_5(X^{3/2} - X^{40})\ H^2(10^{-6})$$

species	b_o	b_1	b_2	b_3	b_4	b_5	SE_E	$R^2\ 10^2$
D	0.73450	-1.99546	3.53279	0.51737	2.36775	-9.0379	0.073476	52.9
C	0.88447	-2.78067	2.79508	0.70021	15.68650	-37.1231	0.137140	77.5
S	0.88363	-0.64805	1.74957	-8.91048	8.93185	-15.1479	0.125641	67.8
B	0.90870	-2.00471	4.04452	3.34638	14.21590	-27.8347	0.139566	60.4
A	0.84443	-0.90189	0.89915	-3.71850	14.50930	-24.5373	0.187546	37.8
Cot	0.81910	-3.03294	0.63127	5.09090	4.15988	-11.8752	0.077595	40.9
Pl	0.90531	-3.59780	4.66746	-3.49117	8.41048	-13.4578	0.078120	66.6
Pw	0.89597	-2.03845	1.85461	11.03750	13.88770	-33.4201	0.075477	80.2

$$\text{VIII.}\ (d\ /\ D)^2 = b_o\ (\underline{1}\ /\ H)^p$$

species	b_o	p	SE_E 10	$R^2\ 10^2$
D	0.86098	1.3	0.964611	88.3
C	1.26193	2.0	2.330710	75.3
S	1.17223	1.7	1.809440	81.2
B	1.15874	1.6	1.944430	78.0
A	1.09287	1.5	2.153450	71.9
Cot	0.99023	1.6	0.727003	94.8
Pl	1.08960	1.4	1.142300	89.8
Pw	1.09376	1.5	1.379860	86.9

$$\text{IX.}\ (d\ /\ D)^2 = b_o\ \underline{1}^p /\ H^{p+1} + b_1(\underline{1}\ /\ H)^q$$

species	b_o	b_1	p	q	SE_E 10	$R^2\ 10^2$
D	4.19968	0.81415	1.0	1.3	0.960627	88.4
C	21.08250	1.09720	1.0	3.4	2.134400	79.3
S	2.32421	1.15975	1.0	1.8	1.802380	81.4
B	5.52793	1.13070	1.0	1.8	1.951150	77.8
A	-4.40072	1.16083	1.0	1.5	2.151000	72.0
Cot	-3.96910	1.03407	1.0	1.5	0.719214	94.9
Pl	6.41902	1.01592	1.0	1.5	1.131260	90.0
Pw	7.39517	1.04899	1.0	1.7	1.357030	87.3

X. $(d / D) = (\underline{1} / H)/(b_o + b_1(\underline{1} / H))$ fitted as $D / d = b_1 + b_o(H / \underline{1})$

species	b_o	b_1	SE_E	R^2 10^2
D	0.465224	0.690940	0.367501	91.4
C	0.662119	0.354711	0.862773	80.9
S	0.516681	0.556082	0.423853	91.8
B	0.504741	0.511545	0.517858	86.9
A	0.726016	0.051026	0.992652	81.1
Cot	0.740751	0.290296	0.958412	81.5
P1	0.457406	0.591305	0.453220	88.1
Pw	0.509568	0.541935	0.364708	92.8

XI. $((\underline{1} / H)/(d / D) - 1) = b_1 ((\underline{1} / H) - 1)$

species	b_1	SE_E	R^2 10^2
D	0.431930	0.113473	69.8
C	0.314941	0.160535	24.4
S	0.423523	0.098748	60.9
B	0.473939	0.102240	65.2
A	0.410182	0.120581	50.1
Cot	0.254496	0.116287	31.6
P1	0.501599	0.087244	74.4
Pw	0.446054	0.087263	70.7

5.2. Taper equation I fitted with different dependent variables

dependent variable	Douglas-fir b_o	b_1	b_2	b_3	SE_t of d (inches)
log d	-0.162157	0.868138	0.653171	-0.512414	1.04
d_2	-0.136230	0.883840	0.632780	-0.515890	1.03
d^2	-0.096128	0.906790	0.701590	-0.613660	1.08

dependent variable	Cottonwood b_o	b_1	b_2	b_3	SE_t of d (inches)
log d	-0.488623	0.781775	0.865773	-0.487176	0.55
d_2	-0.276700	0.863640	0.792930	-0.576870	0.51
d^2	-0.175560	0.904020	0.800010	-0.657910	0.51

5.3. Taper equations for data adjusted for butt flare

I. $\log d = b_o + b_1 \log D + b_2 \log \underline{1} + b_3 \log H$

species	b_o	b_1	b_2	b_3	SE_E 10	R^2 10^2
D	-0.150252	0.868583	0.653324	-0.518731	0.460612	97.1
C	0.157124	0.747335	0.785345	-0.718886	0.661530	94.7
A	-0.462614	0.704369	0.763453	-0.356531	0.666486	94.6
Cot	-0.481989	0.780574	0.863673	-0.488747	0.619915	95.8

II. $(d / D)^2 = b_o + b_1(h / H) + b_2(h / H)^2$

species	b_o	b_1	b_2	SE_E 10	R^2 10^2
D	0.86687	-1.21924	0.352370	0.780893	92.0
C	1.03041	-2.06082	1.030410	1.065230	88.2
A	0.96139	-1.32943	0.368035	0.642552	95.4
Cot	0.97231	-1.57660	0.604290	0.589740	96.4

IV. $(d / D)^2 = b_o \ \underline{1}^p/(D^2 H^{p+1}) + b_1 (\underline{1} / H)^q$

species	b_o	b_1	p	q	SE_E 10	R^2 10^2
D	190.755	0.837508	1.0	1.3	0.779648	92.0
C	594.564	0.957025	1.0	2.1	0.949124	90.7
A	108.710	0.966915	1.0	1.5	0.697680	94.6
Cot	11.225	0.975905	1.0	1.6	0.595081	96.3

V. $(d / D)^2 = b_o(\underline{1} / H)^2 + b_1(\underline{1} / H)^3 + b_2(\underline{1} / H)^4$

species	b_o	b_1	b_2	SE_E 10	R^2 10^2
D	4.57828	-8.62166	5.00122	0.636191	94.7
C	3.64827	-6.28611	3.67434	0.845534	92.6
A	4.02690	-6.45690	3.43559	0.593059	96.1
Cot	2.77992	-3.70044	1.91081	0.559366	96.8

Appendix 6

Derivation of Compatible Volume Equations from Taper Equations and the Functions to Estimate Diameter, Height and Section Volume

I^V. Derivation of a volume equation from taper equation I

Diameter is estimated from taper equation I as

$$d = 10^{b_o} D^{b_1} \underline{l}^{b_2} H^{b_3}$$

and distance from the tip for any diameter is computed as

$$\underline{l} = (10^{-b_o} D^{-b_1} d H^{-b_3})^{1/b_2}$$

If $\underline{l}_1$ and $\underline{l}_2$ are respectively the lower and upper distance from the tip, then the volume of that section is obtained by

$$V_s = \int_{\underline{l}_2}^{\underline{l}_1} (d^2/k)\, d\underline{l} = 10^{2b_o} D^{2b_1} H^{2b_3} (\underline{l}_1^{2b_2+1} - \underline{l}_2^{2b_2+1})/(k(2b_2 + 1))$$

Total volume is derived from section volume by taking $\underline{l}_1 = H$ and $\underline{l}_2 = 0$, then

$$V = 10^{2b_o} D^{2b_1} H^{2b_2+2b_3+1} /(k (2 b_2 + 1))$$

or $\log V = a + b \log D + c \log H$ $\qquad (I^V)$

where $a = \log (10^{2b_o} /(k (2 b_2 + 1)))$ $\qquad b = 2 b_1$

$c = 2 b_2 + 2 b_3 + 1$

II^V. Volume equation from taper equation II

The diameter equation is

$$d = D (b_o + b_1(h / H) + b_2(h / H)^2)^{\frac{1}{2}}$$

and the height equation is

$$h = H (-b_1 -(b_1^2 - 4 b_2(b_o - (d / D)^2))^{\frac{1}{2}})/(2 b_2)$$

Section volume is computed as

$$V_s = D^2(b_o(h_2 - h_1) + b_1(h_2^2 - h_1^2)/(2\ H) + b_2(h_2^3 - h_1^3)/(3\ H^2))/\ k$$

where h_1 and h_2 are respectively the lower and upper height of the section.

The total volume is

$$V = a\ D^2 H \qquad (\ II^v\ )$$

where $a = (b_o + b_1/\ 2 + b_2/\ 3)/\ k$

Taking into account the first conditioning,"a" can be written as

$$a = (-b_1/2\ -2b_2/3)/\ k$$

After the second conditioning:

$$a = b_3/(3\ k)$$

III^v.Volume equation from taper equation III

The diameter equation is the taper equation itself.

Section volume is obtained after a lengthy,but straightforward,integration

$$V_s = (a^2(\underline{l}_1^3 - \underline{l}_2^3)/\ 3 + b^2(\underline{l}_1^5 - \underline{l}_2^5)/\ 5 + c^2(\underline{l}_1^7 - \underline{l}_2^7)/\ 7 +$$
$$2\ a\ b\ (\underline{l}_1^4 - \underline{l}_2^4)/\ 4 + 2a\ c\ (\underline{l}_1^5 - \underline{l}_2^5)/\ 5 + 2\ b\ c\ (\underline{l}_1^6 - \underline{l}_2^6)/\ 6)/\ k$$

and the volume equation is

$$V = (a^2\ H^3/\ 3 + b^2\ H^5/\ 5 + c^2\ H^7/\ 7 + 2ab\ H^4/\ 4 + 2ac\ H^5/\ 5 +$$
$$2bc\ H^6/\ 6)/\ k \qquad (\ III^v\ )$$

where $a = b_1 D\ /(H - 4.5) + b_1 H - 4.5\ b_1 + b_2 H^2 - 4.5\ b_2 H +$
$$b_3(2H^2 - 4.5\ H - (4.5)^2)$$

$$b = -b_1 - b_2 H - 3b_3 H$$

$$c = b_3$$

IV^V. Volume equation from taper equation IV

$$d = D\,(b_o \underline{l}^p/(D^2H^{p+1}) + b_1(\underline{l}\ /\ H)^q)^{\frac{1}{2}}$$

$$V_s = D^2(b_o(\underline{l}_1^{p+1} - \underline{l}_2^{p+1})/((p+1)D^2H^{p+1}) + b_1(\underline{l}_1^{q+1} - \underline{l}_2^{q+1})/((q+1)H^q))/k$$

$$V = a + b\,D^2H \qquad (IV^V)$$

where $a = b_o/((p+1)\,k)$ $\qquad$ $b = b_1/((q+1)\,k)$

V. Volume equation from taper equation V

$$d = D\,(b_o(\underline{l}\ /\ H)^2 + b_1(\underline{l}\ /\ H)^3 + b_2(\underline{l}\ /\ H)^4)^{\frac{1}{2}}$$

$$V_s = D^2(b_o(\underline{l}_1^3 - \underline{l}_2^3)/(3H^2) + b_1(\underline{l}_1^4 - \underline{l}_2^4)/(4H^3) + b_2(\underline{l}_1^5 - \underline{l}_2^5)/(5H^4))/k$$

$$V = a\,D^2H \qquad (V^V)$$

where $a = (b_o/\ 3 + b_1/\ 4 + b_2/\ 5)/\ k$

VI. Volume equation from equation VI

$$d = D\,(b_o\underline{l}\ /\ H + b_1(\underline{l}\ /\ H)^2 + b_2(\underline{l}\ /\ H)^3)$$

$$V_s = D^2(b_o^2(\underline{l}_1^3 - \underline{l}_2^3)/(3H^2) + b_1^2(\underline{l}_1^5 - \underline{l}_2^5)/(5H^4) + b_2^2(\underline{l}_1^7 - \underline{l}_2^7)/(7H^6) +$$
$$2b_ob_1(\underline{l}_1^4 - \underline{l}_2^4)/(4H^3) + 2b_ob_2(\underline{l}_1^5 - \underline{l}_2^5)/(5H^4) +$$
$$2b_1b_2(\underline{l}_1^6 - \underline{l}_2^6)/(6H^5))/\ k$$

$$V = a\,D^2H \qquad (VI^V)$$

where $a = (b_o^2/\ 3 + b_1^2/\ 5 + b_2^2/\ 7 + 2b_ob_1/\ 4 + 2b_ob_2/\ 5 + 2b_1b_2/\ 6)/\ k$

VII^V. Volume equation from equation VII

$$d = D\,(b_o X^{3/2} + b_1 (X^{3/2} - X^3)\,D\,(10^{-2}) + b_2 (X^{3/2} - X^3)\,H\,(10^{-3}) + b_3 (X^{3/2} - X^{32})\,H\,D\,(10^{-5}) + b_4 (X^{3/2} - X^{32})\,H^{1/2}(10^{-3}) + b_5 (X^{3/2} - X^{40})\,H^2 (10^{-6}))^{\frac{1}{2}}$$

where $X = \underline{l}\ /(H - 4.5)$

$$V_s = D^2(((\underline{l}_1^{5/2} - \underline{l}_2^{5/2})/((H - 4.5)^{3/2} 5/2)(b_o + b_1 D\ 10^{-2} + b_2 H\ 10^{-3} + b_3 D\ H\ 10^{-5} + b_4 H^{1/2}\ 10^{-3} + b_5 H^2 10^{-6}) + ((\underline{l}_1^4 - \underline{l}_2^4)/(4(H - 4.5)^3)) (-b_1 D\ 10^{-2} - b_2 H\ 10^{-3}) + ((\underline{l}_1^{33} - \underline{l}_2^{33})/(33(H - 4.5)^{32}))(-b_3 HD\ 10^{-5} - b_4 H^{1/2} 10^{-3}) + ((\underline{l}_1^{41} - \underline{l}_2^{41})/(41(H - 4.5)^{40}))(-b_5 H^2\ 10^{-6}))/\ k$$

Volume equation VII^V is derived from here by substituting H for $\underline{l}_1$ and 0 for $\underline{l}_2$.

$VIII^V$. Volume equation from equation VIII

$$d = D\,(b_o(\underline{l}\ /\ H)^p)^{\frac{1}{2}}$$

$$\underline{l} = H\,(d^2/(b_o D^2))^{1/p}$$

$$V_s = D^2(b_o(\underline{l}_1^{p+1} - \underline{l}_2^{p+1})/((p + 1)\ H^p))/\ k$$

$$V = a\ D^2 H \qquad (VIII^V)$$

where $a = b_o/(k\ (p + 1))$

IX^v. Volume equation from taper equation IX

$$d = D\,(b_o\,\underline{l}^p / H^{p+1} + b_1(\underline{l} / H)^q)^{\frac{1}{2}}$$

$$V_s = D^2(b_o(\underline{l}_1^{p+1} - \underline{l}_2^{p+1})/((p + 1)\,H^{p+1}) + b_1(\underline{l}_1^{q+1} - \underline{l}_2^{q+1})/((q + 1)H^q))/\,k$$

$$V\,/\,B = a + b\,H \qquad (\,IX^v\,)$$

where $a = b_o/(p + 1)$ $\qquad$ $b = b_1/(q + 1)$

X^v. Volume equation from taper equation X

$$d = D\,(\underline{l}\,/\,H)/(b_o + b_1(\underline{l}\,/\,H))$$

$$\underline{l} = b_o d\,H\,/(D - b_1 d)$$

$$V_s = D^2((\underline{l}_1 - \underline{l}_2) - 2b_o H\,\ln((b_o + b_1\underline{l}_1/\,H)/(b_o + b_1\underline{l}_2/\,H))/\,b_1 + b_o^2\,H\,(1/(b_o + b_1\underline{l}_2/\,H) - 1/(b_o + b_1\underline{l}_1/\,H))/\,b_1))/(k\,b_1^2)$$

$$V = a\,D^2 H \qquad (\,X^v\,)$$

where $a = (1 + 2b_o\ln(b_o/(b_o + b_1))/\,b_1 + b_o^2(-1/(b_o + b_1) + 1/b_o)/b_1)/(kb_1^2)$

XI^v. Volume equation from taper equation XI

The same equations as for equation X can be applied here, using the relationship $b_o = 1 - b_1$