INVESTMENT UNDER RISK TOLERANCE CONSTRAINTS AND NON-CONCAVE UTILITY FUNCTIONS: IMPLICIT RISKS, INCENTIVES AND OPTIMAL STRATEGIES

by

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Abstract

The objective of this thesis is to contribute in the understanding of both the induced behavior and the underlying risks of a decision maker who is rewarded through option-like compensation schemes or who is subject to risk tolerance constraints.

In the first part of the thesis we consider a risk averse investor who maximizes his expected utility subject to a risk tolerance constraint expressed in terms of the risk measure known as Conditional Value-at-Risk. We study some of the implicit risks associated with the optimal strategies followed by this investor. In particular, embedded probability measures are uncovered using duality theory and used to assess the probability of surpassing a loss threshold defined by the risk measure known as Value-at-Risk. Using one of these embedded probabilities, we derive a measure of the financial cost of hedging the loss exposure associated to the optimal strategies, and we show that, under certain assumptions, it is a coherent measure of risk.

In the second part of the thesis, we analyze the investment decisions that managers undertake when they are paid with option-like compensation packages. We consider two particular cases:

- We study the optimal risk taking strategies followed by a fund manager who is paid through a relatively general option-like compensation scheme. Our analysis is developed in a continuous-time framework that permits to obtain explicit formulas. These are used first to analyze the incentives induced by this type of compensation schemes and, second, to establish criterions to determine appropriate parameter values for these compensation packages in order to induce specific manager's behaviors.
- We consider a hedge fund manager who is paid through a simple option-like compensation scheme and whose investment universe includes options. We analyze the nature of

the optimal investment strategies followed by this manager. In particular, we establish explicit optimal conditions for option investments in terms of embedded martingale measures that are derived using duality theory. Our analysis in this case is developed in a discrete-time framework, which allows to consider incomplete markets and fat-tailed distributions -such as option return distributions- in a much simpler manner than in a continuous-time framework.

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Preface

The development of the foundations of Portfolio Theory in the nineteen fifties by Harry Markowitz marked an important breakthrough in Economics and Finance. In his seminal paper "Portfolio Selection", published in 1952, Markowitz assembled for the first time a theory of choice under certainty that was capable of explaining the observed diversification behavior of investors in the financial markets. This theory was the first mathematical formulation of diversification (Rubinstein (2002)), yet simple enough to be understood and applied. Therefore, this seminal work, together with other early important contributions by Tobin (1958), Markowitz (1959), Sharpe (1964), Lintner (1965), Samuelson (1969), Hakasson (1970), Fama (1970), Merton (1969, 1973), among others, opened the door to a torrent of research that has extended, enriched and applied this theory in many ways, with no sign of diminishing its momentum yet.

Each investment decision has an associated risk. Variance from the mean rate of return is the measure that Markowitz used to quantify and include risk in his basic portfolio model. He used it because variance captures, in a relatively simple manner, the interrelations among the different securities's returns included in the portfolio's investment universe. The degree of these return interrelations -measured by the covariances- was precisely what explained, under Markowitz's model, the investor's diversification behavior. However, as Markowitz himself recognized since the very beginning (Markowitz (1959)), variance has several inconveniences as a risk measure. For instance, variance does not distinguish between values that are above or below their expected mean, by the same amount. This is contrary to the notion that risk quantification should be more concerned about values below the average than about those above the average. Therefore, since the early days of portfolio theory there has been a permanent interest of proposing other risk measures. Notable initial efforts in such issue include those done by Stone (1973) and Fishburn (1977), among others. At the beginning of the nineties, J.P. Morgan popularized the use of the Value-at-Risk (VaR) measure, given its easy interpretation (Jorion (2000b)). However, this measure has serious drawbacks. Probably the most important one is that it can penalize diversification. That is, the risk of a portfolio constructed by investing our wealth equally among two financial positions can be strictly larger, as measured by VaR, than the average risk of investing our money in any of these two positions (e.g., Föllmer and Scheid (2002)). This counterintuitive situation in which VaR might incur has led to the recent development of the concept of "Coherent Measures of Risk" (Artzner, et al. (1999)). This concept established a benchmark for differentiating between "good" and "bad" risk measures (Ortobelli, et al. (2005)) and motivated the definition of other related classes of risk measures such as: Expected Bounded Risk Measures (Rockafellar, et al. (2002b)), Spectral Measures of Risk (Acerbi (2002)), and Conditional Drawdown Risk Measures (Chekhlov et al (2005)). Nonetheless, it is important to emphasize that besides its *desirable* properties, the proper use of a risk measure depends importantly on the context where it is used. In particular, in Portfolio Theory, optimal investment decisions are typically consistent with the solution of an "expected utility maximization problem" (e.g., Ortobelli, et al. (2005)), where this consistency is usually characterized through the concept of stochastic dominance (Hadar and Rusell (1969), Rothschild and Stiglitz (1970)).

Conditional Value-at-Risk [1] (CVaR), defined as the expected loss given that the loss has exceeded VaR (refer to Rockafellar and Uryasev (2002a) for a formal and general definition), is currently the most popular member of the class of coherent risk measures. This is mainly because of its closed relation with VaR, nowadays the most used risk measure in financial markets, but also significantly because of its *nice* mathematical properties. For instance, CVaR is -for general loss distributions- a convex function and, moreover, it can be determined as the solution of a minimization convex problem (Rockafellar and Uryasev (2002a)).

[1] For continuous loss distributions, CVaR is also known as Mean Excess Loss, Mean or Expected Shortfall (e.g., Acerbi and Tasche (2002)), or Tail-Value-at-Risk (Artzner, et al. (1999)).

Therefore, its inclusion in portfolio optimization, either in the objective function or in the set of constraints, is rather appealing. For example, consider a typical portfolio problem in which the investor wants to minimize risk subject to an expected return restriction and, probably, to other linear constraints (e.g., non-short-selling constraints), where risk is measure in terms of CVaR. Hence, since CVaR is convex, the problem has a unique solution and, given the linearity of the constraints, there exists many efficient algorithms to obtain the optimal solution (e.g., Bazaraa, et al. (1993)). Convexity of CVaR has been also used to derive equivalent formulations of generic portfolio problems that consider CVaR in the objective function or constraints (Krokhmal, et al. (2001)).

Chapter 1 of this Ph.D. thesis considers a risk averse investor who maximizes his expected utility subject to a risk tolerance constraint expressed in terms of CVaR. This chapter studies some of the implicit risks associated with the optimal strategies followed by this investor. In particular, embedded probability measures are uncovered using duality theory and used to assess the probability of surpassing a loss threshold defined by the VaR measure. Using one of these embedded probabilities, we derive a measure of the financial cost of hedging the loss exposure associated to the optimal strategies, and we show that, under certain assumptions, it is a coherent measure of risk.

The variety of financial problems addressed from Markowitz's seminal work has reached enormous proportions in both the academic and practitioner's arenas. Indeed, it is not presumptuous to say that Portfolio Theory laid the groundwork for the Mathematical Theory of Finance. One of the major applications of Portfolio Theory is precisely the issue that motivated it in the first place: the understanding of investors' behavior. This issue has recently attracted the attention of both academics and finance professionals in relation to the *compensation packages* that many managers and executives receive nowadays. For instance, the interest in executive compensation has grown significantly in the last fifteen years because of two main reasons. First, the escalation and recent decline of the average executive compensation amount. Second, the replacement of base salaries by stock options as the single largest component of executive compensation (Hall and Murphy (2003)). On the other hand, managerial compensation has captured the interest of many investors and academics, especially after the nearly catastrophic bankrupt of LTCM (Long Term Capital Management) hedge fund in 1998 which, according to the U.S. Federal Reserve (Jorion (2000a)), jeopardized the world economy. Hedge Fund managers, like some other managers and traders, are compensated with option-like schemes.

Option-like compensation packages for both, executives and managers, are designed, in principle, to align interests between investors and executives or managers. Therefore, it is crucial to understand the induced incentives derived from these compensation mechanisms. Chapters 2 and 3 of this Ph.D. thesis analyze the investment decisions that managers undertake when they are paid with this type of compensation packages. Chapter 3 studies the optimal risk taking strategies followed by a fund manager who is paid through a relatively general option-like compensation scheme. This chapter is developed in a continuous-time framework that permits to obtain explicit formulas. These are used first to analyze the induced incentives by this type of compensation schemes and, second, to establish criterions to determine appropriate parameter values for these compensation packages in order to induce specific manager's behaviors. Chapter 2 focuses on the particular case of a hedge fund manager who is paid through a simple option-like compensation scheme and whose investment universe includes options. This chapter analyzes the nature of the optimal investment strategies followed by this manager. In particular, it establishes explicit optimal conditions for option investments in terms of embedded martingale measures that are derived using duality theory. Given the inclusion of options in the manager's investment universe, Chapter 2 is developed in a discrete-time framework that allows to consider incomplete markets, in which options are not necessarily replicable, and *fat-tailed* distributions -such as option return distributions- in a much simpler manner than in a continuous-time framework.

Overall, the objective of this Ph.D. thesis is to contribute in the understanding of both the induced behavior and the underlying risks of a decision maker who is rewarded through option-like compensation schemes or who is subject to risk tolerance constraints. Although related, each chapter is intended to be self-contained.

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Dedication

I dedicate this thesis to God, who has blessed me in so many ways; to my precious wife, who has given me her entire love and unconditional support, regardless of the personal and professional sacrifices that she has had to make; to my three wonderful kids, María, Sebastián and Isabel, who have brought so much meaning and happiness to my life; to my parents, whose love and support have never been apart from me regardless the distance and the challenging situations; to my brother and sister, who have always given me their love, respect and their unconditional support.

Chapter 1

Living on the Edge: How risky is it to operate at the limit of the tolerated risk?

1.1 Introduction

As financial markets have grown in complexity and volume of operation, risk management has become an increasingly crucial issue. Therefore, different risk measures have been proposed to evaluate the risk of financial positions, particularly to evaluate the risk of positions associated with investment portfolios. For a long time the variance has been a standard to evaluate the risk of such portfolios. Therefore, it is a typical element included in models that look for optimal portfolios such as the mean-variance approach (Markowitz (1952)), that targets a portfolio with the best mean-variance tradeoff. However, the variance has several drawbacks. For instance, it has the disadvantage of not distinguising between losses and profits. To overcome such drawbacks, alternative risk measures have been proposed. The most popular of these, and indeed the standard in many financial markets worldwide, is Value at Risk (VaR). This risk measure evaluates the maximal loss of a financial position over a time horizon and with a certain level of confidence. Yet, while VaR penalizes only for the losses, it could also penalize for the diversification of positions. That is, the VaR measure could indicate a risk increment of a portfolio, even if the volatility of the portfolio is reduced through the diversification of its positions. Therefore, VaR is said to be an *incoherent* measure of risk. Hence, alternative *coherent* risk measures, in the sense of Artzner, et al. (1999), have been proposed to be used instead of or as a complement to VaR. One of the measures within this class of coherent measures of risk is the one known as Conditional Value at Risk (CVaR), which is the expected loss in the event that the loss has surpassed the standard VaR threshold (Jorion (2000b)). This chapter studies the implicit risks of the strategies followed by an investor who maximizes the expected value of his final wealth, subject to budget and risk tolerance constraints expressed in terms of coherent risk measures such as CVaR.

This chapter undercovers embedded probabilities that are implicit in the optimal strategies followed by the type of investor considered. Key relationships between these embedded probabilities are obtained and used to evaluate the implicit risks associated with the optimal strategies whenever these involve operating at the limit of the tolerated risk. This evaluation is done, under each embedded probability, in terms of the probability assessment of surpassing a certain VaR threshold and, more importantly, in terms of a risk measure, in the sense of Föllmer H. and Shied (2002), which is a coherent measure of risk (Artzner, et al. (1999)), under certain conditions.

This chapter is organized as follows: Section 2 describes the decision space framework, while Section 3 specifies the basic model that constitutes the starting point for the analysis of this chapter. Section 4 develops the way in which risk management is considered and how it is incorporated into the basic model to derive the model over which the main analysis of this chapter is done. In Section 5, duality is applied to the model of Section 4 to undercover (three) embedded probabilities within this model, and then it is used again to establish one of the main results of the chapter (Theorem 1.5.1), which describes a general relationship between these embedded probabilities. Other interesting, and useful, consequences of Theorem 1.5.1 are also established in Section 5. Section 6 defines a measure that attempts to evaluate the financial cost of hedging the loss exposure associated to the optimal strategies followed by the type of investor considered in the model of Section 4. This measure, named *Expected Shortfall Hedge*, is proved to be a monetary measure of risk (Föllmer H. and Shied (2002)) and, under certain conditions, a coherent measure of risk (Artzner, et al. (1999)). Section 7 concludes and proposes two main directions for further research.

1.2 Decision Space Framework

1.2.1 Probability Space

The probability space uses a *scenario tree structure* that models all possible scenarios or states (represented by nodes of a tree) of the market over a finite number of discrete time periods $t = 0, \dots, T$. The scenario tree structure is such that every possible state is the consequence of a unique sequence (trajectory) of states (events). This is convenient for assigning probabilities to each of the tree scenario nodes.

Assume that every node $n \in N_t$, where N_t denotes the set of all nodes at time t, has a unique parent denoted by $a(n) \in N_{t-1}$ and a set of child nodes $C(n) \subseteq N_{t+1}$. Defining a probability measure (i.e. assigning probabilities to each node of the tree), say P, consists of assigning (strictly) positive weights p_n to each leaf node $n \in N_T$ such that $\sum_{n \in N_T} p_n = 1$ and then recursively computing the remaining node probabilities

$$p_n = \sum_{m \in C(n)} p_m \ , \ \forall\, n \in N_t, \ \forall\, t = T-1, \dots, 0.$$

Let Ω be the set of possible trajectories or sequences of events (from time 0 to the end of period T) in the *scenario tree*, then (Ω, P) defines a *sample space.* Every node $n \in N_t$ has a unique history up to time t and a unique set of possible future trajectories. N_t induces a unique set of histories up to time t, say F_t, and a partition of Ω. The collection of sets $\{F_t\}_{t=0,\dots,T}$ satisfy $F_t \subset F_{t+1}$ for $t = 0, \dots, T-1$. The triplet (Ω, F_T, P) forms a *probability space.* Hence, for a probability space (Ω, F_T, P), the conditional probability of state (event) m given that n occurs ($m \in C(n)$) is $\left(\frac{p_m}{p_n}\right)$, and if $\{X_t\}_{t=0,\dots,T}$ is a discrete stochastic process defined on (Ω, F_T, P), then $E^P[X_t] = \sum_{n \in N_t} X_n p_n$; and $E^P[X_{t+1}|F_t] = \sum_{m \in C(n)} \left(\frac{p_m}{p_n}\right) X_m$ is random variable taking values over the nodes $n \in N_t$.

Definition 1.2.1 (Martingales) *Let* $\{Z_t\}_{t=0,\ldots,T}$ *be a discrete stochastic process defined on* (Ω, F_T, P). *If there exists a probability measure* Q *such that*

$$Z_t = E^Q\left[Z_{t+1}|F_t\right] \ , \ t = 0, \ldots, T-1$$

then $\{Z_t\}_{t=0,\ldots,T}$ *is called a martingale under* Q*, or simply a* Q*-martingale, and* Q *is called a martingale measure for the process* $\{Z_t\}_{t=0,\ldots,T}$.

1.2.2 Financial Market

The market consists of $I+1$ tradable securities $i = 0, \ldots, I$ whose prices at node n are $S_n = (S_n^0, \ldots, S_n^I)$. Assume that one of the securities, the *numeraire*, always has a strictly positive value and, without loss of generality, assume it is security 0. Define discount factors $\beta_n = \frac{1}{S_n^0}$ $\forall\ n \in N_t$, and the discounted price (relative to the *numeraire*) $Z_n^i \equiv \beta_n S_n^i$, $\forall\ i = 0, \ldots, I$ where $Z_n^0 = 1$ $\forall\ n \in N_t$, $\forall\ t = 0, \ldots, T$. The market can be complete or incomplete.

Let θ_n^i be the amount of security i held by the investor in state $n \in N_t$. Thus, the portfolio value in state $n \in N_t$ is

$$Z_n \cdot \theta_n \equiv \sum_{i=0}^{I} Z_n^i \cdot \theta_n^i \ .$$

We assume a class of investors that do not influence the prices of any security and trade at every time-step based on historical information up to time t.

Arbitrage

Arbitrage refers to the opportunity of making a sure profit out of nothing (usually through the purchase and sale of assets). In our framework, arbitrage reduces to finding a portfolio with zero initial value whose terminal values, obtained through self-financing strategies, are nonnegative for any scenario and for which at least one of those is strictly positive and has a positive probability of occurring.

Definition 1.2.2 (Arbitrage) *There is arbitrage if there exists a strategy*

$$\{\theta_n\}_{n \in N_t, 0 \leq t \leq T} \ ,$$

where θ_n^i is the quantity of security i held in state $n \in N_t$, such that

$$\begin{aligned}
&Z_0 \cdot \theta_0 = 0 \\
&Z_n \cdot \left(\theta_n - \theta_{a(n)}\right) = 0 \ , \ \forall \ n \in N_t \ , \ \forall \ t = 1, \ldots, T. \\
&Z_n \cdot \theta_n \geq 0 \ , \ \forall \ n \in N_T \ , \ \textit{and} \\
&P\left\{Z_m \cdot \theta_m > 0\right\} > 0 \ , \ \textit{for some } m \in N_T \ .
\end{aligned}$$

Assume that there is no arbitrage opportunity in our financial market. This assumption guarantees, under this framework, the existence of a martingale measure for $\{Z_t\}_{t=0,\ldots,T}$ (King (2002), Theorem 2.2).

Completeness

Completeness of a financial market refers to the capability of replicating any sequence of payoffs. That is, a financial market is said to be complete if given a sequence of payoffs, it is always possible to construct a portfolio whose values coincide with such sequence of payoffs, for each possible scenario.

Definition 1.2.3 (Completeness) *The financial market is said to be complete if and only if given a sequence of payoffs* $\{c_n\}_{n \in N_t, 0 \leq t \leq T}$ *, there exists a strategy*

$$\{\theta_n\}_{n \in N_t, 0 \leq t \leq T} ,$$

where θ_n^i *is the quantity of security* i *held in state* $n \in N_t$*, such that*

$$Z_n \cdot \theta_n = c_n, \ \forall \ n \in N_t, \ t = 0, ..., T.$$

A financial market is said to be incomplete if it is not complete. We do not make any assumption regarding the completeness of the market. Therefore, all the results developed in this chapter applied to both complete and incomplete markets.

1.3 Basic Model

Consider the model

$$
\begin{array}{lll}
Max_{\theta} & \sum_{n\in N_T} U\left(Z_n \cdot \theta_n\right) p_n & \\
s.t. & & \\
& Z_0 \cdot \theta_0 & = \ \beta_0 W_0 \\
& Z_n \cdot \left(\theta_n - \theta_{a(n)}\right) & = \ -\beta_n L_n \ , \ \ \forall\, n \in N_t \ \ \forall\, t = 1, \ldots, \ T
\end{array}
\tag{1.1}
$$

where $U(\cdot)$ is assumed to be a strictly increasing concave function and $W_0 \geq 0$ is the initial wealth. This model applies to several relevant cases including

- self-financing strategies, when $L_n = 0$ for all $n \in N_t$, $\forall\, t = 1, \ldots, T$. This case is relevant to study, for example, the problem of finding an arbitrage (let $W_0 = 0$ and $U(x)$ be the identity function); and
- the case of strategies that consider the payment of liabilities, when $L_n > 0$, and/or the inclusion of future endowments, when $L_n < 0$. An example of this case is the problem faced by the writer of a contingent claim who receives a payment of F_0 $(= W_0)$ monetary units an who is obliged to pay an amount of F_n $(= L_n)$ in scenario $n \in N_t$, for $t = 1, \ldots, T$.

1.4 Risk Management Modelling

Risk management is incorporated to Model (1.1) through the addition of a *risk tolerance* constraint that specifies the maximum *risk* tolerated, where risk is measured in terms of CVaR. This risk constraint can be established by the investor or by a supervisory/regulatory institution, such as a central bank. In the case that this constraint is established only by investor's risk concerns, it is natural to think that these risk concerns can be, in theory, included in the investor's utility function. However, in practice, determining the precise form of the investor's utility function is rather difficult. Therefore, the inclusion of this risk constraint in Model (1.1) can be seen as a practical way of approximating the *true* investor's utility function.

Conditional Value at Risk (CVaR)

Let $l(x, y)$ a loss function, where x represents a decision and y symbolizes the future (stochastic) values of relevant variables (e.g. interest rate). Then, the Conditional Value at Risk associated with the loss function l and a level of confidence α, which we denote by $\Phi_\alpha(x)$, is defined as the mean of the α-tail distribution of the loss function $l(x, \cdot)$. This distribution is defined as

$$\Psi_\alpha(x, \eta) = \begin{cases} 0 & \text{if } \eta < VaR_\alpha(x) \\ \frac{[\Psi(x,\eta)-\alpha]}{1-\alpha} & \text{if } \eta \geq VaR_\alpha(x) \end{cases}$$

where $\Psi(x, \eta)$ is the distribution of the loss function $l(x, y)$, i.e.

$$\Psi(x, \eta) = P\{l(x, y) \leq \eta\} \ , \tag{1.2}$$

and $VaR_\alpha(x)$ is the Value at Risk associated with decision x at a confidence level α. Value at Risk (VaR) is a popular measure of risk which aids to define a threshold for the maximum loss associated with a loss function $l(x, y)$. $VaR_\alpha(x)$ is defined as

$$VaR_\alpha(x) = Inf\{\eta | \Psi(x, \eta) \geq \alpha\}$$

where $\Psi(x, \eta)$ is defined in (1.2). Conditional Value at Risk (CVaR) is a measure of risk with several important properties not shared by VaR (Pflug (2000), Rockafellar and Uryasev (2000), Rockafellar and Uryasev (2002)). For instance,

- if the loss function $l(x, y)$ is convex (sublinear) with respect to x then $\Phi_\alpha(x)$ is convex (sublinear), and
- $\Phi_\alpha(x)$ behaves continuously and has left and right derivatives with respect to the level of confidence $\alpha \in (0, 1)$.

The convexity of a risk measure is an appealing property since it means that, under such risk measure, diversification does not increase the risk. Continuity, on the other hand, assures that slight changes on the level of confidence imply slight changes on the risk of the underlying position. These two properties are a direct consequence of the following useful way of expressing $\Phi_\alpha(x)$,

$$\Phi_\alpha(x) = Min_{\ \eta}\ F_\alpha(x, \eta) \tag{1.3}$$

where $F_\alpha(x, \eta) = \eta + \frac{1}{1-\alpha} E\left[(l(x, y) - \eta)^+\right]$ and $(x)^+ = max\,\{x, 0\}$ (Rockafellar and Uryasev (2002)). Formula (1.3) is useful not only for modelling CVaR within optimal investment problems (as described below), but also for obtaining VaR (for the same level of confidence) simultaneously since $VaR_\alpha(x) \in Argmin_{\ \eta}\ F_\alpha(x, \eta)$ (Rockafellar and Uryasev (2002)).

Risk Tolerance Constraint

The *risk tolerance* constraint that is considered for Model (1.1) is

$$\Phi_\alpha(\theta) \leq RT \ , \tag{1.4}$$

where RT is the maximum tolerated loss that the investor is expecting to face if the loss surpasses the Value at Risk threshold and whose amount could be directly established by the investor or indirectly set by a supervisory institution. RT is presumably, although not necessarily, chosen in such a way that it is not redundant with respect to the utility function $U(\cdot)$, i.e. that restriction (1.4) effectively constraints the set of optimal solutions of Model (1.1). For instance, the investor could solve Model (1.1) to obtain an optimal strategy θ^{**} with an associated conditional value at risk of $\Phi_\alpha(\theta^{**})$. Hence, if $\Phi_\alpha(\theta^{**})$ is not affordable by the investor and/or not allowed by a relevant supervisory institution, RT should be equal to a value strictly less than $\Phi_\alpha(\theta^{**})$.

Constraint (1.4) is, by (1.3), equivalent to

$$Min_{\ \eta} \ F_\alpha(\theta, \eta) \leq RT$$

or,

$$F_\alpha(\theta, \eta) \leq RT \ , \ \text{ for some } \eta \in \Re$$

i.e.,

$$\eta + \frac{1}{1-\alpha} E\left[(l(\theta, y) - \eta)^{+}\right] \leq RT \text{ , for some } \eta \in \Re. \tag{1.5}$$

Constraint (1.5) reduces to

$$\eta + \frac{1}{1-\alpha} \sum_{n \in N_T} (l(\theta, Z_n) - \eta)^{+} p_n \leq RT \text{ , for some } \eta \in \Re, \tag{1.6}$$

which, within any optimization problem, can be modelled as

$$\begin{array}{rcl} \eta + \left(\frac{1}{1-\alpha}\right) \sum_{n \in N_T} s_n p_n & \leq & RT \\ -s_n + l(\theta, y_n) - \eta & \leq & 0 \text{ , } \forall\, n \in N_T \\ -s_n & \leq & 0 \text{ , } \forall\, n \in N_T \end{array} \tag{1.7}$$

since $(s_n)_{n \in N_T}$ satisfying (1.7) and such that $s_n > (l(\theta, y_n) - \eta)^{+}$ for some $n \in N_T$ is never better than $(\tilde{s}_n)_{n \in N_T}$, where $\tilde{s}_n = (l(\theta, y_n) - \eta)^{+}$ $\forall\, n \in N_T$. Clearly, $(\tilde{s}_n)_{n \in N_T}$ satisfies (1.7), and hence (1.6), but also in a less restrictive manner, thus allowing for a broader set of feasible solutions and therefore leading to a better, or at least not worse, objective function value.

Within the context of portfolio management, a particular relevant case for the loss function $l(\cdot, \cdot)$ is $l(\theta, Z_n) = -(Z_n \cdot \theta_n - \beta_0 W_0)$ for $n \in N_T$. The model is

$$\begin{array}{lll} Max_{\,\eta,\theta,s} & \sum_{n \in N_T} U(Z_n \cdot \theta_n)\, p_n & \\ s.t. & & \\ & Z_0 \cdot \theta_0 & = \quad \beta_0 W_0 \\ & Z_n \cdot \left(\theta_n - \theta_{a(n)}\right) & = \quad -\beta_n L_n \text{ , } \quad \forall\, n \in N_t \text{ , } \forall\, t = 1, \ldots, T \\ & \eta + \left(\frac{1}{1-\alpha}\right) \sum_{n \in N_T} s_n p_n & \leq \quad RT \\ & -s_n - (Z_n \cdot \theta_n - \beta_0 W_0) - \eta & \leq 0 \quad \text{, } \forall\, n \in N_T \\ & -s_n & \leq 0 \quad \text{, } \forall\, n \in N_T \end{array} \tag{1.8}$$

and whose dual (Zangwill (1969)) is

$$
\begin{array}{rrcl}
Min_{\ \theta,\mathbf{x},\mathbf{y}} & \multicolumn{3}{l}{\sum_{n\in N_T}\left(U(Z_n\cdot\theta_n)-U'(Z_n\cdot\theta_n)(Z_n\cdot\theta_n)\right)p_n} \\
& \multicolumn{3}{l}{+(RT-\beta_0 W_0)x+\left(y_0\beta_0 W_0-\sum_{t=1}^{T}\sum_{n\in N_t} y_n\beta_n L_n\right)} \\
s.t. & & & \\
& U'\left(Z_n\cdot\theta_n\right)p_n+x_n-y_n & = & 0\ ,\ \forall\, n\in N_T \\
& y_n Z_n-\sum_{m\in C(n)} y_m Z_m & = & 0\ ,\ \forall\, n\in N_t, \\
& & & \forall\, t=0,\ldots,T-1. \\
& x_n-x\left(\frac{p_n}{1-\alpha}\right) & \leq & 0\ ,\ \forall\, n\in N_T \\
& x-\sum_{n\in N_T} x_n & = & 0 \\
& -x & \leq & 0 \\
& -x_n & \leq & 0\ ,\ \forall\, n\in N_T\ .
\end{array}
\qquad (1.9)
$$

where:

1. y_0 corresponds to the initial portfolio's value constraint (first restriction of the primal problem);

2. $\mathbf{y}\equiv(y_n)_{n\in N_t,\ t\in\{1,\ldots,T\}}$ is associated with the re-balancing conditions over the portfolio's value at each period (second set of restrictions of the primal problem), and

3. $\mathbf{x}\equiv\left(x,(x_n)_{n\in N_T}\right)$ corresponds to the risk tolerance constraint (specified by the third and the last two sets of restrictions of the primal problem).

1.5 Embedded Probabilities

The dual problem contains embedded probability measures that yield different assessments of the probability of surpassing the VaR threshold. These probability measures and their relationship are described in the following propositions.

Proposition 1.5.1 (Embedded Probabilities) *Let $(\theta^*, \mathbf{x}^*, \mathbf{y}^*)$ be an optimal solution of the dual problem (1.9), where* $\mathbf{x}^* = \left(x^*, (x_n^*)_{n \in N_T}\right)$ *and* $\mathbf{y}^* = (y_n^*)_{n \in N_t,\ t=0,\dots,T}$. *Then,*

i) $Q^* \equiv (q_n^*)_{n \in N_T}$, *where* $q_n^* = \left(\frac{y_n^*}{y_0^*}\right)$ $\forall\, n \in N_T$, *defines a martingale measure of* $\{Z_t\}_{t=0,\dots,T}$ *in* (Ω, F).

ii) If $x^* > 0$, *i.e. if the risk tolerance constraint is binding for the optimal solution, then* $\hat{P} \equiv (\hat{p}_n)_{n \in N_T}$, *where* $\hat{p}_n = \left(\frac{x_n^*}{x^*}\right)$ $\forall\, n \in N_T$, *defines a probability measure on* (Ω, F).

iii) $\bar{P} \equiv (\bar{p}_n)_{n \in N_T}$, *where* $\bar{p}_n = \frac{U'(Z_n \cdot \theta_n^*) p_n}{\sum_{n \in N_T} U'(Z_n \cdot \theta_n^*) p_n}$ $\forall\, n \in N_T$, *defines a probability measure in* (Ω, F).

Proof:

i) From (1.9),

$$\begin{aligned} y_n^* &= U'\left(Z_n \cdot \theta_n^*\right) p_n + x_n^* \\ y_0^* &= \textstyle\sum_{n \in N_T} y_n^* \end{aligned}$$

$U(\cdot)$ strictly increasing and $x_n^* \geq 0$ implies $y_n^* > 0\ \forall\, n \in N_T$ and hence, $y_0^* = \sum_{n \in N_T} y_n^* > 0$. Therefore,

1) $q_n^* = \frac{y_n^*}{y_0^*} > 0\ \forall\, n \in N_T$;

2) $1 = \sum_{n \in N_T} \left(\frac{y_n^*}{y_0^*}\right) = \sum_{n \in N_T} q_n^*$, and

3) $Z_n = \sum_{m \in C(n)} \left(\frac{q_m^*}{q_n^*}\right) Z_m\ \forall\, n \in N_t\ ,\ \forall\, t = 0, \dots, T-1$.. That is,

$$E^{Q^*}[Z_{t+1} | F_t] = Z_t\ ,\ \forall\, t = 0, \dots, T-1.$$

Hence, $\{Z_t\}_{t=0,\ldots,T}$ is a Q^*-martingale.

ii) By hypothesis and from (1.9),

1) $\hat{p}_n = \left(\frac{x_n^*}{x^*}\right) \geq 0$, $\forall\ n \in N_T$; and

2) $1 = \sum_{n\in N_T}\left(\frac{x_n^*}{x^*}\right) = \sum_{n\in N_T}\hat{p}_n$. Hence, $(\hat{p}_n)_{n\in N_T}$ defines a probability measure in (Ω, F).

iii) $U(\cdot)$ strictly increasing implies $U'(Z_n \cdot \theta_n)p_n > 0$ for all $n \in N_T$. Therefore,

1) $\bar{p}_n = \frac{U'(Z_n\cdot\theta_n^*)p_n}{\sum_{n\in N_T} U'(Z_n\cdot\theta_n^*)p_n} \geq 0$, $\forall\ n \in N_T$; and

2) $1 = \sum_{n\in N_T}\frac{U'(Z_n\cdot\theta_n^*)p_n}{\sum_{n\in N_T} U'(Z_n\cdot\theta_n^*)p_n} = \sum_{n\in N_T}\bar{p}_n$.

Q.E.D.

Remark 1.5.1 *Q^* is not necessarily the unique martingale measure of $\{Z_t\}_{t=0,\ldots,T}$, since our framework allows for incomplete markets. Indeed, for any feasible solution* $\mathbf{y}$ *of Problem* (1.8)*, $Q \equiv \left(\frac{y_n}{y_0}\right)_{n\in N_T}$ defines a martingale measure of $\{Z_t\}_{t=0,\ldots,T}$.*

These three probability measures, Q^*, $\hat{P}$, and $\bar{P}$, have special and, in general, different meanings. Q^* is a martingale measure and hence linked to arbitrage pricing. $\bar{P}$ is a measure depending directly on the preferences of the investor and standard for models such as Model (1.1). $\hat{P}$ is shown later on to be a conditional distribution.

We are particularly interested in comparing and interpreting the different assessments of the event of surpassing the VaR threshold, when using an optimal strategy, under Q^*, $\hat{P}$, and $\bar{P}$. We define the latter event in terms of the (primal) variables of Problem (1.8).

Definition 1.5.1 *Let* $(\eta^*, \theta^*, \mathbf{s}^*)$ *be an optimal solution of problem (1.8). Define*

$$A = \{n \in N_T | s_n^* > 0\} \ ,$$

i.e., A is the event of surpassing the VaR threshold.

Before assesing the probability of A under Q^*, $\hat{P}$, and $\bar{P}$, we state some basic relations between VaR and CVaR within the context of Problem (1.8) and describe precisely which conditional distribution $\hat{P}$ is.

Proposition 1.5.2 *Let* $(\eta^*, \theta^*, \mathbf{s}^*)$ *be an optimal solution of problem (1.8), and A as defined in 1.5.1. Then,*

i) If $A = \emptyset$ *then* $\Phi_\alpha(\theta^*) = VaR_\alpha(\theta^*)$.

ii) If $A \neq \emptyset$ *then* $\Phi_\alpha(\theta^*) > VaR_\alpha(\theta^*)$. *Moreover,*

if $\sum_{n \in A} \left(\frac{p_n}{1-\alpha}\right) \leq 1$ *then*

$$\Phi_\alpha(\theta^*) = VaR_\alpha(\theta^*) + E^{\tilde{P}}[s_T^*] \tag{1.10}$$

where $\tilde{P}$ *is any probability measure in* (Ω, F) *that satisfies* $\tilde{p}_n = \frac{p_n}{1-\alpha}$ *for all* $n \in A$.

Proof: Observe that

$$\begin{aligned}
\Phi_\alpha(\theta^*) &= \eta^* + \textstyle\sum_{n \in A} \left(\frac{p_n}{1-\alpha}\right) s_n^* \\
&= VaR_\alpha(\theta^*) + \textstyle\sum_{n \in A} \left(\frac{p_n}{1-\alpha}\right) s_n^* \\
&= \begin{cases} VaR_\alpha(\theta^*) \ \text{if } A = \emptyset \\ VaR_\alpha(\theta^*) + k, \ \text{for } k > 0, \text{ if } A \neq \emptyset \end{cases}
\end{aligned}$$

Under the hypothesis $\sum_{n\in A}\left(\frac{p_n}{1-\alpha}\right) \le 1$, it is clear that there exists a probability measure $\tilde{P}$ such that $\tilde{p}_n = p_n/(1-\alpha)$, for all[1] $n \in A$. Therefore,

$$\begin{aligned}
\Phi_\alpha(\theta^*) &= VaR_\alpha(\theta^*) + \textstyle\sum_{n\in A}\tilde{p}_n s_n^* \\
&= VaR_\alpha(\theta^*) + \textstyle\sum_{n\in A}\tilde{p}_n s_n^* + \sum_{n\notin A}\tilde{p}_n s_n^* \\
&= VaR_\alpha(\theta^*) + \textstyle\sum_{n\in N_T}\tilde{p}_n s_n^* \\
&= VaR_\alpha(\theta^*) + E^{\tilde{P}}\left[s_T^*\right] .
\end{aligned}$$

Q.E.D.

The next two propositions show that, under certain conditions, $\hat{P}$ not only satisfies the conditions in Proposition 1.5.2 *ii)*, and hence (1.10) holds under $\hat{P}$, but also coincides with the α-tail distribution of the loss function

$$l(\theta, Z_n) = -(Z_n \cdot \theta_n - \beta_0 W_0) .$$

Proposition 1.5.3 *If $x^* > 0$, i.e. if the risk tolerance constraint is binding for the optimal solution, and $A \neq \emptyset$ then*

i) $\Phi_\alpha(\theta^*) = RT = VaR_\alpha(\theta^*) + E^{\hat{P}}\left[s_T^*\right]$.

ii) If $|A| \ge 2$ then

$$\left(\frac{\hat{p}_{n_i}}{\hat{p}_{n_j}}\right) = \left(\frac{p_{n_i}}{p_{n_j}}\right) \ \forall\ n_i, n_j \in A .$$

[1] For instance, let $\tilde{p}_n = p_n/(1-\alpha)$ for all $n \in A$ and, if $A \neq N_T$,

$$\tilde{p}_n = \frac{1 - \sum_{n\in A}\left(\frac{p_n}{1-\alpha}\right)}{|N_T \setminus A|}$$

for $n \in N_T \setminus A$.

Proof: i) From Proposition 1.5.1, $\hat{P} = (\hat{p}_n)_{n \in N_T}$ is a probability measure and thus it is satisfied

$$\sum_{n \in A} \hat{p}_n \leq 1 \, . \tag{1.11}$$

From the dual problem (1.9) and complementarity,

$$x_n^* = x^* \left(\frac{p_n}{1-\alpha} \right) \quad \forall \, n \in A \, ,$$

which, under the assumption $x^* > 0$, implies

$$\hat{p}_n = \frac{x_n^*}{x^*} = \frac{p_n}{1-\alpha} \quad \forall \, n \in A \, . \tag{1.12}$$

Therefore, from (1.11) and (1.12),

$$\sum_{n \in A} \hat{p}_n = \sum_{n \in A} \left(\frac{p_n}{1-\alpha} \right) \leq 1 \, .$$

Thus, the hypothesis of Proposition 1.5.2 ii) is satisfied and hence

$$\Phi_\alpha \left(\theta^* \right) = VaR_\alpha \left(\theta^* \right) + E^{\hat{P}} \left[s_T^* \right] .$$

Finally, from the assumption $x^* > 0$ and applying the property of complementarity to the corresponding primal restriction leads to

$$\Phi_\alpha \left(\theta^* \right) = RT \, .$$

ii) Let $n_i, n_j \in A$. Then, from (1.12)

$$\hat{p}_{n_i} = \frac{p_{n_i}}{1-\alpha} \text{ and } \hat{p}_{n_j} = \frac{p_{n_j}}{1-\alpha} \ .$$

Therefore,

$$\frac{\hat{p}_{n_i}}{\hat{p}_{n_j}} = \frac{p_{n_i}}{p_{n_j}} \ .$$

Q.E.D.

Proposition 1.5.4 *If $x^* > 0$, i.e. if the risk tolerance constraint is binding for the optimal solution, then $\hat{P}$ is the α-tail loss distribution.*

<u>Proof</u>:

From Proposition 1.5.1, $\hat{P}$ is a probability measure, and from (1.12) $\hat{P}$ satisfies that $\hat{p}_n = \frac{p_n}{1-\alpha}$ $\forall\ n \in A$. Let $n \in N_T \setminus A$. Hence, by definition of A, $s_n^* = 0$. If $\hat{p}_n > 0$ then, by definition of $\hat{P}$, $x_n^* > 0$ and thus, by the application of the complementarity property to the corresponding primal

$$0 = s_n^* = -\left(Z_n \cdot \theta_n^* - \beta_0 W_0\right) - \eta^* = -\left(Z_n \cdot \theta_n^* - \beta_0 W_0\right) - VaR_\alpha\left(\theta^*\right) \ .$$

Therefore,

$$-\left(Z_n \cdot \theta_n^* - \beta_0 W_0\right) = VaR_\alpha\left(\theta^*\right) \ .$$

That is, $\hat{p}_n > 0$ for $n \in N_T \setminus A$ implies that scenario n corresponds to the VaR scenario. Thus, for any other scenario $n \in N_T \setminus A$ such that $-(Z_n \cdot \theta_n^* - \beta_0 W_0) \neq VaR_\alpha(\theta^*)$ it must be satisfied that $\hat{p}_n = 0$. Hence,

$$\hat{p}_n = \begin{cases} \frac{p_n}{1-\alpha} & \text{if } n \in A. \\ 1 - \frac{1}{1-\alpha} \sum_{n \in A} p_n = \frac{\left(\sum_{n \in N_T \setminus A} p_n\right) - \alpha}{1-\alpha} & \text{if } n \in N_T \setminus A \text{ and} \\ & -(Z_n \cdot \theta_n^* - \beta_0 W_0) = VaR_\alpha(\theta^*). \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, $\hat{P}$ is the α-tail loss distribution.

Q.E.D.

We now obtain a lower bound of the probability of surpassing VaR under $\hat{P}$. This is useful to compare the assessment of the probability of such event under the different embedded probabilities.

Corollary 1.5.1 *If $x^* > 0$, i.e. if the risk tolerance constraint is binding for the optimal solution, then*

$$\hat{P}(A) > 1 - \frac{p_{VaR}}{(1-\alpha)}$$

where $p_{VaR} = p_n$ and $n \in N_T$ is such that $-(Z_n \cdot \theta_n^* - \beta_0 W_0) = VaR_\alpha(\theta^*)$.

Proof: By definition of VaR,

$$\alpha > \sum_{n \in N_T \setminus A} p_n - p_{VaR} \,.$$

Hence,

$$
\begin{aligned}
1 &= \tfrac{1-\alpha}{1-\alpha} \\
&< \tfrac{p_{VaR}-\left(\sum_{n\in N_T\setminus A} p_n\right)+1}{1-\alpha} \\
&= \left(\tfrac{\sum_{n\in A} p_n}{1-\alpha}\right) + \tfrac{p_{VaR}}{1-\alpha} \\
&= \hat{P}(A) + \tfrac{p_{VaR}}{1-\alpha}
\end{aligned}
$$

where the last equality follows from Proposition 1.5.4. Thus

$$1 < \hat{P}(A) + \frac{p_{VaR}}{1-\alpha} \tag{1.13}$$

The result follows from (1.13).

Q.E.D.

The main result that establishes the relations among the embedded probabilities (Q^*, $\hat{P}$, and $\bar{P}$) and P is presented in the following theorem.

Theorem 1.5.1 (Relations among the embedded probabilities)

$$\left(1-\frac{x^*}{y_0^*}\right)\bar{P}(A) + \left(\frac{x^*}{y_0^*}\right)\left(\frac{P(A)}{1-\alpha}\right) = Q^*(A) \tag{1.14}$$

Proof: From Problem (1.9)

$$U'\left(Z_n \cdot \theta_n^*\right) p_n + x_n^* = y_n^* \ , \ \forall \ n \in N_T \tag{1.15}$$

and hence

$$U'(Z_n \cdot \theta_n)\, p_n + x_n^* = y_n^*\ , \ \forall\, n \in A\ .$$

Therefore,

$$\sum_{n \in A} U'(Z_n \cdot \theta_n)\, p_n + \sum_{n \in A} x_n^* = \sum_{n \in A} y_n^*$$

so

$$\left(\frac{1}{y_0^*}\right) \sum_{n \in A} U'(Z_n \cdot \theta_n)\, p_n + \left(\frac{1}{y_0^*}\right) \sum_{n \in A} x_n^* = \sum_{n \in A} \frac{y_n^*}{y_0^*} = Q^*(A)\ . \tag{1.16}$$

By complementarity,

$$x_n^* = x^* \frac{p_n}{1-\alpha}\ , \quad \forall\, n \in A\ .$$

Thus

$$\sum_{n \in A} x_n^* = \left(\frac{x^*}{1-\alpha}\right) \sum_{n \in A} p_n = \left(\frac{x^*}{1-\alpha}\right) P(A)\ . \tag{1.17}$$

Hence, combining (1.16) and (1.17), leads to

$$\left(\frac{1}{y_0^*}\right) \sum_{n \in A} U'(Z_n \cdot \theta_n)\, p_n + \left(\frac{x^*}{y_0^*}\right) \left(\frac{P(A)}{1-\alpha}\right) = \sum_{n \in A} \frac{y_n^*}{y_0^*} = Q^*(A)\ . \tag{1.18}$$

Now, from (1.15) and dual problem (1.9)

$$y_0^* = \sum_{n \in N_T} y_n^* = \sum_{n \in N_T} U'\left(Z_n \cdot \theta_n^*\right) p_n + \sum_{n \in N_T} x_n^* = \sum_{n \in N_T} U'\left(Z_n \cdot \theta_n^*\right) p_n + x^* .$$

Thus,

$$\left(\frac{1}{y_0^*}\right) \sum_{n \in N_T} U'\left(Z_n \cdot \theta_n^*\right) p_n = \left(1 - \frac{x^*}{y_0^*}\right) . \tag{1.19}$$

Therefore, combining (1.18) and (1.19),

$$\left(1 - \frac{x^*}{y_0^*}\right) \left(\frac{\sum_{n \in A} U'\left(Z_n \cdot \theta_n^*\right) p_n}{\sum_{n \in N_T} U'\left(Z_n \cdot \theta_n^*\right) p_n}\right) + \left(\frac{x^*}{y_0^*}\right) \left(\frac{P(A)}{1-\alpha}\right) = Q^*(A)$$

which, by definition of $\bar{P}$, implies

$$\left(1 - \frac{x^*}{y_0^*}\right) \bar{P}(A) + \left(\frac{x^*}{y_0^*}\right) \left(\frac{P(A)}{1-\alpha}\right) = Q^*(A) .$$

Q.E.D.

Whether the risk tolerance constraint is binding or not and/or whether the investor or decision maker is risk-neutral or not, Theorem 1.5.1 allows us to compare the embedded probabilities and P with each other. The following corollaries and lemmas establish these comparisons.

Corollary 1.5.2 *If* $x^* = 0$, *i.e. if the* **risk tolerance constraint is not binding,** *then* $Q^*(A) = \bar{P}(A)$.

Proof: The result is a direct consequence of applying condition $x^* = 0$ to Equation (1.14).

Q.E.D.

Corollary 1.5.3 *If $U(x) = x$ for all x in its domain, i.e. if the* **investor is risk-neutral,** *then $\bar{P}(A) = P(A)$ and hence,*

$$\left(1 - \frac{x^*}{y_0^*}\right) P(A) + \left(\frac{x^*}{y_0^*}\right)\left(\frac{P(A)}{1-\alpha}\right) = Q^*(A) \ . \tag{1.20}$$

Proof: Observe that

$$\bar{P}(A) = \frac{\sum_{n\in A} U'\left(Z_n \cdot \theta_n\right) p_n}{\sum_{n\in N_T} U'\left(Z_n \cdot \theta_n\right) p_n} = \frac{\sum_{n\in A} p_n}{\sum_{n\in N_T} p_n} = \sum_{n\in A} p_n = P(A)$$

where the second inequality is due to the risk-neutrality assumption on the investor preferences. The result follows from substituting the above expression in Equation (1.14).

Q.E.D.

Lemma 1.5.1 *If $U(x) = x$ for all x in the domain of $U(\cdot)$, i.e. if the* **investor is risk-neutral,** *then $Q^*(A) \geq P(A)$. Moreover, the latter* **inequality is strict** *if $x^* > 0$, i.e. if the* **risk tolerance constraint is binding**.

Proof: From Equation (1.20)

$$\begin{aligned} Q^*(A) &= P(A)\left[1 - \tfrac{x^*}{y_0^*} + \tfrac{x^*}{y_0^*(1-\alpha)}\right] \\ &= P(A)\left[1 + \left(\tfrac{x^*}{y_0^*}\right)\left(\tfrac{\alpha}{1-\alpha}\right)\right] \\ &\geq P(A) \ . \end{aligned}$$

Finally, the latter inequality is strict if $x^* > 0$.

Q.E.D.

Corollary 1.5.4 *If* $x^* > 0$, *i.e. if the* **risk tolerance constraint is binding** *for the optimal solution, then*

$$\left(1 - \frac{x^*}{y_0^*}\right) \bar{P}(A) + \left(\frac{x^*}{y_0^*}\right) \hat{P}(A) = Q^*(A) \ .$$

Proof: From Equation (1.12) $\hat{p}_n = \frac{p_n}{1-\alpha}\ \forall\ n \in A$. Therefore,

$$\hat{P}(A) = \sum_{n \in A} \hat{p}_n = \sum_{n \in A} \frac{p_n}{1-\alpha} = \frac{\sum_{n \in A} p_n}{1-\alpha} = \frac{P(A)}{1-\alpha} \ .$$

Then, by Theorem 1.5.1 the result follows.

Q.E.D.

Lemma 1.5.2 *If* $x^* > 0$ *,i.e. if the* **risk tolerance constraint is binding** *for the optimal solution and* $p_{VaR} \leq \left(1 - \bar{P}(A)\right)(1-\alpha)$, *then* $Q^*(A) > \bar{P}(A)$.

Proof: From Corollaries 1.5.4 and 1.5.1,

$$\begin{array}{rcl} Q^*(A) & = & \left(1 - \frac{x^*}{y_0^*}\right) \bar{P}(A) + \left(\frac{x^*}{y_0^*}\right) \hat{P}(A) \\ & > & \left(1 - \frac{x^*}{y_0^*}\right) \bar{P}(A) + \left(\frac{x^*}{y_0^*}\right) \left(1 - \frac{p_{VaR}}{1-\alpha}\right) \\ & > & Min\left\{\bar{P}(A), 1 - \frac{p_{VaR}}{1-\alpha}\right\} \end{array} .$$

Hence, if condition $p_{VaR} \leq \left(1 - \bar{P}(A)\right)(1-\alpha)$ is satisfied then

$$Min\left\{\bar{P}(A), 1-\frac{p_{VaR}}{1-\alpha}\right\} = \bar{P}(A) \ .$$

Therefore $Q^*(A) > \bar{P}(A)$.

Q.E.D.

Corollary 1.5.5 *If $U(x) = x$ for all x in its domain, i.e. if the investor is risk-neutral, and $x^* > 0$, i.e. if the* **risk tolerance constraint is binding** *for the optimal solution, then*

$$\left(1-\frac{x^*}{y_0^*}\right) P(A) + \left(\frac{x^*}{y_0^*}\right) \hat{P}(A) = Q^*(A) \ .$$

<u>Proof</u>: This follows from Corollaries 1.5.3 and 1.5.4.

Q.E.D.

In summary, if the investor is risk-neutral or the risk tolerance constraint is binding and the probability of losing an amount equal to VaR is small enough, then the probability of surpassing VaR is larger or equal under Q^* than under P or $\bar{P}$ respectively. Otherwise, the probability of surpassing VaR under Q^* is a convex combination of the corresponding probability assessments of $\bar{P}$ and $\hat{P}$. Moreover, the probability of surpassing VaR is *strictly* larger under Q^* than under P or $\bar{P}$ if the risk tolerance constraint is binding and, respectively, the investor is risk neutral or the probability of losing an amount equal to VaR is small enough. Therefore, if the risk tolerance constraint is binding, $Q^*(A)$ is in some sense a measure of how risky it is to operate at the limit of the tolerated risk. Indeed, $Q^*(A)$ has the interpretation

$$Q^*(A) = E^{Q^*}[1_A] \ , \tag{1.21}$$

where

$$1_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{otherwise} \end{cases} .$$

Hence, since Q^* is a martingale measure, $Q^*(A)$ can be interpreted as *a no-arbitrage price* of a binary option (Hull (2003)) that provides one monetary unit in the event of a loss greater than VaR. Therefore, *if the market is complete, $Q^*(A)$ can be regarded as the cost of hedging one monetary unit of loss beyond the VaR threshold.* The latter interpretation of $Q^*(A)$ serves as a motivation to define an implicit risk measure for the optimal strategies of Problem (1.8). This risk measure is described in the following section.

1.6 Implicit Risks

The assessment of the probability of surpassing the VaR threshold under the different embedded probabilities provides us with an indication of the implicit risks associated with an optimal strategy for Problem (1.8). However, these embedded probabilities do not directly define measures of risk in the sense of quantifying the required capital to hedge for the implicit risks of such optimal strategy. From this perspective of capital requirement, a measure of risk is defined (Föllmer and Shied (2002)) as follows:

Definition 1.6.1 (Monetary Measure of Risk) *Let $\rho : \mathcal{X} \rightarrow \Re$ be a mapping, where $\mathcal{X}$ is a class of the financial positions. Then, ρ is said to be a monetary measure of risk in $\mathcal{X}$, if it satisfies the following two conditions:*

- *(Monotonicity) Let $X, Y \in \mathcal{X}$. Then, $X \leq Y$ a.s. implies $\rho(X) \geq \rho(Y)$.*
- *(Translation Invariance) Let $X \in \mathcal{X}$ and $m \in \Re$. Then,*

$$\rho(X + m) = \rho(X) - m \ .$$

Within the framework of Problem (1.8), the class of financial positions that we consider is the set of the feasible investment strategies of (1.8) for different values of risk tolerance parameter RT in a relevant convex set S (e.g. $S = \left[0, \bar{RT}\right]$, where $\bar{RT} \leq RT_0 = \phi_\alpha(\theta^{**})$ and θ^{**} is an optimal strategy for Problem (1.1)). That is, we consider

$$\mathcal{X} = \{\theta(RT)|RT \in S \text{ and } \theta \text{ is a feasible solution of Problem (1.8)}\}$$

which, under the linearity of the restrictions in (1.8) and the convexity of S, is a convex set.

VaR and CVaR are monetary measures of risk in the sense of Definition 1.6.1 (Föllmer and Shied (2002)). Indeed, VaR and its multiples are in practice capital requirements of some sort for many financial institutions (Jorion (2000b)). However, VaR has the disadvantage of not providing information about the extent of the losses beyond the threshold amount that it defines. On the other hand, CVaR does quantify losses beyond the VaR threshold and hence a capital requirement equal to CVaR should cover, at least, for the expected shortfall beyond VaR. Nevertheless, CVaR could be much larger than VaR and so it could be very costly as a capital requirement. Nonetheless, in principle, there is no need to require for the whole expected shortfall, CVaR - VaR, but rather for the premium of an insurance, or a hedge, that will ensure for the payment of the expected shortfall in case the loss goes beyond the VaR threshold. In other words, it would be enough to require the price of a binary call option whose underlying is the loss of the portfolio and which pays (CVaR - VaR) with a strike price equal to VaR. From observation (1.21), and denoting CVaR and VaR (as before) by $\Phi_\alpha(\theta)$ and $VaR_\alpha(\theta)$ respectively, for a feasible strategy θ, it follows that, under the assumption of completeness of the market, such binary option would cost

$$Q^*(A)\left(\Phi_\alpha(\theta) - VaR_\alpha(\theta)\right) \ . \tag{1.22}$$

Therefore, a *natural* capital requirement that takes into account losses beyond the VaR threshold is

$$VaR_\alpha(\theta) + Q^*(A)\left(\Phi_\alpha(\theta) - VaR_\alpha(\theta)\right)$$

or equivalently,

$$\left(1 - Q^*(A)\right) VaR_\alpha(\theta) + Q^*(A)\Phi_\alpha(\theta) \ . \tag{1.23}$$

The latter measure, called the *Expected Shortfall Hedge* (ESH), is a monetary risk measure (See Proposition 1.6.1).

When the market is incomplete Q^* is only one of several martingale measures of the discount price process $\{Z_t\}_{t=0,\dots,T}$. Hence, (1.22) is just one possible no-arbitrage price of a binary option that pays the expected shortfall loss beyond the VaR threshold. However, the election of a different martingale measure Q does not change most of the results that are next obtained for ESH. For instance, if other martingale measure Q is chosen, the associated expected shortfall hedge ESH_α^Q, i.e. expression (1.23) but with Q instead of Q^*, will still be a monetary measure of risk and will posses the basic properties described in Proposition 1.6.2. Nonetheless, it is important to emphasize that under the generality of allowing for incompleteness of the market, ESH_α would not necessarily *superhedge* for the expected shortfall, i.e. it would not necessarily be enough to provide for a payoff of at least the expected shortfall in *all* possible scenarios. Therefore, in this case, the decision maker should consdider Q^* and the other martingale measures in his analysis (see Föllmer and Shied (2002) for a detailed treatment on this issue).

Proposition 1.6.1 *Let* $ESH_\alpha(\theta) = VaR_\alpha(\theta) + Q^*(A)\left(\Phi_\alpha(\theta) - VaR_\alpha(\theta)\right)$*, for* $\alpha \in (0,1)$*. Then,* ESH_α *is a monetary measure of risk.*

Proof: Let θ_1, $\theta_2 \in \mathcal{X}$.

- Monotonicity: Assume that $\theta_1 \geq \theta_2$ *a.s.*. Then, given that VaR and Φ_α are monetary measures of risk, it is satisfied that

$$\begin{aligned} VaR_\alpha(\theta_1) &\leq VaR_\alpha(\theta_2) \\ \Phi_\alpha(\theta_1) &\leq \Phi_\alpha(\theta_2) \end{aligned}.$$

  Since $Q^*(A) \in [0,1]$, then $(1 - Q^*(A)) \in [0,1]$ and so

$$\begin{aligned} ESH_\alpha(\theta_1) &= (1 - Q^*(A))\, VaR_\alpha(\theta_1) + Q^*(A)\Phi_\alpha(\theta_1) \\ &\leq (1 - Q^*(A))\, VaR_\alpha(\theta_2) + Q^*(A)\Phi_\alpha(\theta_2) \\ &= ESH_\alpha(\theta_2)\ . \end{aligned}$$

- Translation Invariance: Let $m \in \Re$. Hence,

$$\begin{aligned} ESH_\alpha(\theta_1 + m) &= (1 - Q^*(A))\, VaR_\alpha(\theta_1 + m) + Q^*(A)\Phi_\alpha(\theta_1 + m) \\ &= (1 - Q^*(A))\,(VaR_\alpha(\theta_1) - m) + Q^*(A)\,(\Phi_\alpha(\theta_1) - m) \\ &= (1 - Q^*(A))\, VaR_\alpha(\theta_1) + Q^*(A)\Phi_\alpha(\theta_1) - m \\ &= ESH_\alpha(\theta_1) - m\ , \end{aligned}$$

where the second equality follows because VaR and CVaR are monetary measures of risk.

Q.E.D.

The essence of proving that ESH is a monetary risk measure relies on expressing it as a convex combination of VaR and CVaR. Indeed, from substituting VaR and CVaR by any other pair of monetary measures of risk in the proof of Proposition 1.6.1, it can be concluded that any convex combination of monetary measures of risk is also a monetary measure of risk. Therefore,

$$\left(1 - \bar{P}(A)\right) VaR_\alpha(\theta) + \bar{P}(A)\Phi_\alpha(\theta)$$

and

$$\left(1 - \hat{P}(A)\right) VaR_\alpha(\theta) + \hat{P}(A)\Phi_\alpha(\theta)$$

are also monetary measures of risk though these measures do not share the *hedging* meaning of ESH. Moreover, the definition of ESH as a weighted average of VaR and CVaR also implies that ESH inherits certain properties shared by both VaR and CVaR. For instance, VaR and CVaR are positive homogeneous for any loss distribution and therefore, ESH is also positive homogeneous. Furthermore, although VaR is not a convex risk measure for any loss distribution, ESH is convex whenever VaR is convex. The latter claims are formalized in the next proposition.

Proposition 1.6.2 (Basic Properties of ESH) *Let* $\alpha \in (0,1)$. *Then,*

i) ESH_α *is positive homogeneous in* $\mathcal{X}$ *for all* $\alpha \in (0,1)$.

ii) If VaR_α *is convex in* $\mathcal{X}$, *for* $\alpha \in (0,1)$, *then* ESH_α *is convex in* $\mathcal{X}$.

Proof: i) Let $\lambda \geq 0$, $\alpha \in (0,1)$ and $\theta \in \mathcal{X}$. Then,

$$\begin{aligned} ESH_\alpha(\lambda\theta) &= (1 - Q^*(A))\, VaR_\alpha(\lambda\theta) + Q^*(A)\Phi_\alpha(\lambda\theta) \\ &= (1 - Q^*(A))\, [\lambda VaR_\alpha(\theta)] + Q^*(A)\, [\lambda\Phi_\alpha(\theta)] \\ &= \lambda\, [(1 - Q^*(A))\, VaR_\alpha(\theta) + Q^*(A)\Phi_\alpha(\theta)] \\ &= \lambda ESH_\alpha(\theta) \end{aligned}$$

where the second equality is due to the fact that VaR and CVaR are positive homogeneous (Föllmer and Shied (2002)).

ii) Since the loss function $l(\theta, Z_n) = -(Z_n \cdot \theta_n - \beta_0 W_0)$ for $n \in N_T$ is convex in θ, then $CVaR_\alpha$ is convex in θ (Corollary 11 in Rockafellar and Uryasev (2002)). Therefore,

$$\begin{aligned} ESH_\alpha\left(\lambda\theta + (1-\lambda)\theta'\right) &= (1 - Q^*(A))\, VaR_\alpha\left(\lambda\theta + (1-\lambda)\theta'\right) \\ &\quad + Q^*(A)\Phi_\alpha\left(\lambda\theta + (1-\lambda)\theta'\right) \\ &\leq (1 - Q^*(A))\left[\lambda VaR_\alpha(\theta) + (1-\lambda) VaR_\alpha(\theta')\right] \\ &\quad + Q^*(A)\left[\lambda\Phi_\alpha(\theta) + (1-\lambda)\Phi_\alpha(\theta')\right] \\ &= \lambda\, [(1 - Q^*(A))\, VaR_\alpha(\theta) + \; Q^*(A)\Phi_\alpha(\theta)] \\ &\quad +(1-\lambda)\left[(1 - Q^*(A))\, VaR_\alpha(\theta') + \; Q^*(A)\Phi_\alpha(\theta')\right] \\ &= \lambda ESH_\alpha(\theta) + (1-\lambda) ESH_\alpha(\theta') \end{aligned}$$

for any $\lambda \in (0,1)$ and $\theta, \theta' \in \mathcal{X}$. Hence, ESH_α is convex.

Q.E.D.

Proposition 1.6.2 states that ESH is, at least, as *appealing* as VaR in the sense that ESH is positive homogeneous, as is VaR, and convex whenever VaR is convex. Within the context of the optimal investment Problem (1.8), ESH is convex for a broader class of loss distributions than VaR. This is shown in the next proposition, for which we require first the following preliminary result:

Lemma 1.6.1 *Let $\alpha \in (0,1)$ and assume the following:*

1. *The investor of Problem (1.8) is risk neutral with a maximum tolerated risk equal to its initial wealth, i.e. $RT = \beta_0 W_0$.*
2. $Q = \hat{P}$.

If the corresponding dual problem (1.9) has a solution $(\mathbf{x}, \mathbf{y})$ such that

$$x_n \neq 0 \ , \ \forall \, n \in N_T \ , \tag{1.24}$$

and,

$$\frac{y_n}{y_{\bar{n}}} \geq \frac{p_n}{p_{\bar{n}}} \ , \ \forall \, n, \bar{n} \in N_T \tag{1.25}$$

then there exists an optimal solution $(\mathbf{x}^*, \mathbf{y}^*)$ of Problem (1.9) that satisfies

$$\frac{x^*}{y_0^*} = \left(\frac{1}{\alpha}\right)\left[\left(\frac{1-\alpha}{1-\alpha^+}\right)^2\left(\frac{1-\beta^+}{1-\beta}\right)\right] + \left(1 - \frac{1}{\alpha}\right) \tag{1.26}$$

for some β, $\beta^+ \in [\alpha, 1)$ and $\beta^+ \geq \beta$.

Proof: Let $(\mathbf{x}, \mathbf{y})$ be an optimal solution of Problem (1.9) that satisfies conditions (1.24) and (1.25), under assumptions *1* and *2*. If $(\mathbf{x}, \mathbf{y})$ satisfies (1.26) then lemma's claim holds. Otherwise, we will construct, departing from $(\mathbf{x}, \mathbf{y})$, another solution $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ of Problem (1.9) that does satisfy (1.26) for some β, $\beta^+ \in [\alpha, 1)$ and $\beta^+ \geq \beta$.

Let $\beta = \alpha$ and $\beta^+ \in [\alpha, 1)$. The idea is to *scale* $\mathbf{x}$ and $\mathbf{y}$ in such a way that the derived pair $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ is feasible, optimal and satisfies condition (1.26). That is, we find $\delta > 0$ and $\gamma_n > 0$, for all $n \in N_T$, such that

$$\tilde{x}_n \equiv \gamma_n x_n \ , \ \tilde{y}_n \equiv \delta y_n \ , \ \forall \ n \in N_T \ , \tag{1.27}$$

is a solution of (1.9) and satisfies (1.26). To show this, we primarily focus on the first set of constraints of (1.9) and condition (1.26). The pair $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$, as defined in (1.27), satisfies such conditions if and only if

$$p_n + \gamma_n x_n = \delta y_n \ , \ \forall \ n \in N_T \ , \tag{1.28}$$

and,

$$\frac{\sum_{n \in N_T} \gamma_n x_n}{\delta y_0} = \left(\frac{1}{\alpha}\right) \left[\frac{1-\alpha}{(1-\alpha^+)^2} \left(1 - \beta^+\right)\right] + \left(1 - \frac{1}{\alpha}\right) \ , \tag{1.29}$$

where we recall that β has been chosen equal to α. The set of conditions defined by (1.28) and (1.29) form a system of $|N_T| + 1$ linear equations with $|N_T| + 1$ variables. This system always has a solution. For instance, let

$$m(\beta^+) \equiv \left(\frac{1}{\alpha}\right)\left[\frac{1-\alpha}{(1-\alpha^+)^2}(1-\beta^+)\right] + \left(1-\frac{1}{\alpha}\right) .$$

Therefore, from (1.28) and (1.29), and provided that $m(\beta^+) \neq 1$, we obtain

$$\delta = \frac{1}{y_0\,(1-m(\beta^+))}$$

from which, using (1.28),

$$\begin{aligned} \gamma_n &= \tfrac{\delta y_n - p_n}{x_n} \\ &= \tfrac{y_n - p_n y_0 \left(1-m(\beta^+)\right)}{x_n y_0 (1-m(\beta^+))} \end{aligned} ,$$

where δ and γ_n are not necessarily positive unless the value of β^+ is chosen appropriately. Later on we establish conditions on β^+ under which $m(\beta^+) \neq 1$, and δ and γ_n's are strictly positive.

Therefore, $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$, as defined in (1.27), satisfies by construction condition (1.26) and the first set of constraints of Problem (1.9). The vector $\tilde{\mathbf{y}}$ clearly satisfies the second set of constraints of Problem (1.9) and if we define

$$\tilde{x} \equiv \sum_{n \in N_T} \tilde{x}_n$$

then, $\tilde{x}$ satisfies the last three sets of constraints of Problem (1.9). Hence, the only constraint of Problem (1.9) that remains to be verified by $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ is

$$\tilde{x}_n \leq \tilde{x}\left(\frac{p_n}{1-\alpha}\right) \;\forall\, n \in N_T . \tag{1.30}$$

To prove (1.30) recall that, by assumption, $\mathbf{x}$ is optimal and thus it must be satisfied that

$$x_n \leq x\left(\frac{p_n}{1-\alpha}\right), \forall\, n \in N_T\ .$$

Let $\bar{n} \in N_T$. Therefore,

$$\begin{aligned}
\gamma_{\bar{n}} x_{\bar{n}} &\leq \left[\tfrac{\gamma_{\bar{n}} x}{\tilde{x}}\right] \tilde{x}\left(\tfrac{p_{\bar{n}}}{1-\alpha}\right) \\
&= \left[\frac{\gamma_{\bar{n}}}{\left(\frac{\tilde{x}}{x}\right)}\right] \tilde{x}\left(\tfrac{p_{\bar{n}}}{1-\alpha}\right) \\
&= \left[\frac{\gamma_{\bar{n}}}{\sum_{n\in N_T} \gamma_n\left(\frac{x_n}{x}\right)}\right] \tilde{x}\left(\tfrac{p_{\bar{n}}}{1-\alpha}\right) \\
&= \left[\frac{1}{\sum_{n\in N_T}\left(\frac{\gamma_n}{\gamma_{\bar{n}}}\right)\left(\frac{x_n}{x}\right)}\right] \tilde{x}\left(\tfrac{p_{\bar{n}}}{1-\alpha}\right)\ .
\end{aligned}$$

We claim that assumption (1.25) implies that

$$\begin{aligned}
\tfrac{\gamma_n}{\gamma_{\bar{n}}} &= \left(\tfrac{x_{\bar{n}}}{x_n}\right)\left[\tfrac{\delta y_n - p_n}{\delta y_{\bar{n}} - p_{\bar{n}}}\right] \\
&\geq \left(\tfrac{x_{\bar{n}}}{x_n}\right)\left(\tfrac{y_n}{y_{\bar{n}}}\right)
\end{aligned} \tag{1.31}$$

for all $n, \bar{n} \in N_T$. If the latter inequality holds then

$$\begin{aligned}
\textstyle\sum_{n\in N_T}\left(\frac{\gamma_n}{\gamma_{\bar{n}}}\right)\left(\frac{x_n}{x}\right) &\geq \textstyle\sum_{n\in N_T}\left(\frac{x_{\bar{n}}}{x}\right)\left(\frac{y_n}{y_{\bar{n}}}\right) \\
&= \left(\tfrac{x_{\bar{n}}}{x}\right)\left(\tfrac{1}{y_{\bar{n}}}\right)\textstyle\sum_{n\in N_T} y_n \\
&= \frac{\left(\frac{x_{\bar{n}}}{x}\right)}{\left(\frac{y_{\bar{n}}}{y_0}\right)} \\
&= \left(\frac{\hat{p}_{\bar{n}}}{q_{\bar{n}}}\right) \\
&= 1
\end{aligned}$$

where the last equality is due to the assumption $Q = \hat{P}$. Hence,

$$\left[\frac{1}{\sum_{n\in N_T}\left(\frac{\gamma_n}{\gamma_{\bar{n}}}\right)\left(\frac{x_n}{x}\right)}\right] \leq 1 \,.$$

Therefore,

$$\begin{aligned}\gamma_{\bar{n}}x_{\bar{n}} &\leq \left[\frac{1}{\sum_{n\in N_T}\left(\frac{\gamma_n}{\gamma_{\bar{n}}}\right)\left(\frac{x_n}{x}\right)}\right]\tilde{x}\left(\frac{p_n}{1-\alpha}\right) \\ &\leq \tilde{x}\left(\frac{p_n}{1-\alpha}\right)\end{aligned} \,.$$

To prove (1.31) it is enough to show that

$$\frac{\delta y_n - p_n}{\delta y_{\bar{n}} - p_{\bar{n}}} \geq \left(\frac{y_n}{y_{\bar{n}}}\right) \tag{1.32}$$

since $x_n > 0$, for all $n \in N_T$. Observe that

$$\frac{dm(\beta^+)}{d\beta^+} = -\frac{1-\alpha}{\alpha(1-\alpha^+)^2} < 0 \quad , \ \forall \ \ \beta^+ \in \Re \,,$$

and so,

$$\frac{d\left(\frac{1}{1-m(\beta^+)}\right)}{d\beta^+} = \frac{\frac{dm(\beta^+)}{d\beta^+}}{\left(1-m(\beta^+)\right)^2} < 0 \,.$$

Therefore, $m(\beta^+)$ and $\frac{1}{1-m(\beta^+)}$ are strictly decreasing functions for all $\beta^+ \in \Re$. If β^+ is restricted to the interval $(1 - \frac{\left(1-\alpha^+\right)^2}{1-\alpha}, 1-(1-\alpha^+)^2)$ then

$$0 < m(\beta^+) \equiv \left(\frac{1}{\alpha}\right)\left[\frac{1-\alpha}{(1-\alpha^+)^2}(1-\beta^+)\right] + \left(1-\frac{1}{\alpha}\right) < 1 \,.$$

That is, the function m is bounded if $\beta^+ \in (1 - \frac{(1-\alpha^+)^2}{1-\alpha}, 1 - (1-\alpha^+)^2)$. Thus, as β^+ approaches $\left(1 - \frac{(1-\alpha^+)^2}{1-\alpha}\right)$ from the right, the limit of $m(\beta^+)$ exists and equals

$$\lim_{\beta^+ \to \left(1 - \frac{(1-\alpha^+)^2}{1-\alpha}\right)^+} m(\beta^+) = 1 \ .$$

This implies that

$$\lim_{\beta^+ \to \left(1 - \frac{(1-\alpha^+)^2}{1-\alpha}\right)^+} \delta(\beta^+) = \frac{1}{y_0\,(1 - m(\beta^+))} = \infty \ . \tag{1.33}$$

Therefore,

$$\lim_{\beta^+ \to \left(1 - \frac{(1-\alpha^+)^2}{1-\alpha}\right)^+} \frac{\delta(\beta^+) y_n - p_n}{\delta(\beta^+) y_{\bar{n}} - p_{\bar{n}}} = \frac{y_n}{y_{\bar{n}}} \tag{1.34}$$

On the other hand, assumption (1.25) implies

$$\frac{\partial \left(\frac{\delta(\beta^+) y_n - p_n}{\delta(\beta^+) y_{\bar{n}} - p_{\bar{n}}} \right)}{\partial \beta^+} = \frac{\frac{\partial \delta(\beta^+)}{\partial \beta^+}}{\left(\delta(\beta^+) y_{\bar{n}} - p_{\bar{n}}\right)^2} \left[y_n p_{\bar{n}} - y_{\bar{n}} p_n \right] \geq 0 \ , \ \forall\, n \in N_T \ . \tag{1.35}$$

That is, $\frac{\delta(\beta^+) y_n - p_n}{\delta(\beta^+) y_{\bar{n}} - p_{\bar{n}}}$ is a monotone increasing function under assumption (1.25). Hence, (1.35) and (1.34) imply (1.32) (See Appendix 1).

To complete the proof, we now establish appropriate conditions on β^+ to guarantee the nonnegativity of δ and the γ_n's as well as the condition $m(\beta^+) \neq 1$. Recall that $\beta^+ \in (1 - \frac{(1-\alpha^+)^2}{1-\alpha}, 1 - (1-\alpha^+)^2) \subseteq [\alpha, 1)$ implies $0 < m(\beta^+) < 1$ and so

$$\delta = \frac{1}{y_0 \left(1 - m(\beta^+)\right)} > 0 \; .$$

However, the nonnegativity of δ does not ensure that the γ_n's are nonnegative too. Further restrictions on β^+ need to be imposed.

Recall that $\gamma_n = (\delta y_n - p_n)/x_n$, for all $n \in N_T$. Hence, by (1.33), β^+ can be chosen sufficiently close to $\left(1 - \frac{(1-\alpha^+)^2}{1-\alpha}\right)$, but above it and less than $1 - (1-\alpha^+)^2$, such that we assure that all the γ_n's are positive. That is, there exists $\tilde{\beta} \in \left(1 - \frac{(1-\alpha^+)^2}{1-\alpha}, 1 - (1-\alpha^+)^2\right)$ such that if $\beta^+ \in \left(1 - \frac{(1-\alpha^+)^2}{1-\alpha}, \tilde{\beta}\right]$ then

$$\gamma_n = \frac{\delta(\beta^+) y_0 - p_n}{x_n} \geq 0 \quad , \; \forall \, n \in N_T .$$

Therefore, if $\beta^+ \in \left(1 - \frac{(1-\alpha^+)^2}{1-\alpha}, \tilde{\beta}\right]$, then $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ is a feasible solution of (1.9). We now prove that $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ is also optimal.

By assumption, the investor is risk neutral and the maximum tolerated risk is the initial wealth. Hence, the dual objective function of (1.9) associated with $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ reduces to

$$\delta \left[\tilde{y}_0 \beta_0 W_0 - \sum_{t=1}^{T} \sum_{n \in N_t} \tilde{y}_n \beta_n L_n \right] \; .$$

Thus, since multiplying the objective function by a positive constant does not alter the point at which the optimal is reached, $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ must be an optimal dual solution associated with θ.

Q.E.D.

Under the assumptions of Lemma 1.6.1, there are multiple optimal solutions.

Proposition 1.6.3 (Convexity of ESH) *Suppose that the assumptions of Lemma* 1.6.1 *hold, and assume that the cardinality of the sample space* $|\Omega| = |N_T|$ *is such that given* $\theta_1, \theta_2 \in \mathcal{X}$, $\lambda \in (0,1)$ *and* $\alpha \in (0,1)$, *there exists a probability measure* P', *defined over* Ω, *that satisfies*

$$\begin{aligned} VaR_{\alpha,P'}(\theta) &= VaR_{\alpha,P}(\theta) \ , \ \textit{for } \theta = \theta_1, \theta_2, \lambda\theta_1 + (1-\lambda)\theta_2 \\ \Phi^{+}_{\alpha,P'}(\theta) &= \Phi^{+}_{\alpha,P}(\theta) \ , \ \textit{for } \theta = \theta_1, \theta_2, \lambda\theta_1 + (1-\lambda)\theta_2 \end{aligned}$$

Let $(\mathbf{x}^*, \mathbf{y}^*)$ *be a solution of the dual problem (*1.9*) that satisfies condition (*1.26*), and let ESH be the expected shortfall measure associated with this solution. Then, ESH is a convex measure of risk in* $\mathcal{X}$.

<u>Proof</u>: By definition of ESH (see Equation (1.23)),

$$ESH_\alpha(\theta) = (1 - Q^*(A))\, VaR_\alpha(\theta) + Q^*(A)\Phi_\alpha(\theta) \ , \ \forall \ \theta \in \mathcal{X} \ , \tag{1.36}$$

where Q^* is defined as in Proposition 1.5.1. On the other hand, Φ_α can be expressed as (See Proposition 6 in Rockafellar and Uryasev (2002))

$$\Phi_\alpha(\theta) = \left(\frac{\alpha^+ - \alpha}{1-\alpha}\right) VaR_\alpha(\theta) + \left(\frac{1-\alpha^+}{1-\alpha}\right) \Phi^+_\alpha(\theta) \quad , \ \forall \, \theta \in \mathcal{X} \ . \tag{1.37}$$

where $\Phi^+_\alpha(\theta) = E^P\left[l(\theta,y) | l(\theta,y) > VaR_\alpha(\theta)\right]$ and $\alpha^+ = \Psi(\theta, VaR_\alpha(\theta))$. Therefore, substituting (1.37) into (1.36) leads to

$$
\begin{aligned}
ESH_\alpha(\theta) &= \left[1 - Q^*(A)\left(\tfrac{1-\alpha^+}{1-\alpha}\right)\right] VaR_\alpha(\theta) + \\
& \left[Q^*(A)\left(\tfrac{1-\alpha^+}{1-\alpha}\right)\right] \Phi_\alpha^+(\theta) \ , \ \forall\, \theta \in \mathcal{X} \ .
\end{aligned}
\tag{1.38}
$$

The essence of the proof relies on showing that, under the conditions stated in the proposition, given $\theta_1, \theta_2 \in \mathcal{X}$ and $\lambda \in (0,1)$, there exist a probability measure $P^{'}$ and a level of confidence β such that

1. $Q^*(A)\left(\frac{1-\alpha^+}{1-\alpha}\right) = \left(\frac{1-\beta^+}{1-\beta}\right) \ \forall\, \theta \in \mathcal{X}$, and

2. $$
\begin{aligned}
VaR_{\alpha,P}(\theta) &= VaR_{\beta,P^{'}}(\theta) \\
CVaR^+_{\alpha,P}(\theta) &= CVaR^+_{\beta,P^{'}}(\theta)
\end{aligned}
$$

where $\beta^+ = \Psi^{P^{'}}(\theta, VaR_\beta(\theta))$ and $\theta \in \{\theta_1, \theta_2, \lambda\theta_1 + (1-\lambda)\theta_2\}$. If these two conditions are satisfied, then

$$
\begin{aligned}
ESH_{\alpha,P}(\theta) &= \left[1 - Q^*(A)\left(\tfrac{1-\alpha^+}{1-\alpha}\right)\right] VaR_{\alpha,P}(\theta) \\
&+ \left[Q^*(A)\left(\tfrac{1-\alpha^+}{1-\alpha}\right)\right] \Phi^+_{\alpha,P}(\theta) \\
&= \left(\tfrac{\beta^+-\beta}{1-\beta}\right) VaR_{\beta,P^{'}}(\theta) + \left(\tfrac{1-\beta^+}{1-\beta}\right) \Phi^+_{\beta,P^{'}}(\theta) \\
&= \Phi_{\beta,P^{'}}(\theta)
\end{aligned}
$$

for $\theta = \theta_1, \theta_2, \lambda\theta_1 + (1-\lambda)\theta_2$ and, therefore,

$$
\begin{aligned}
ESH_{\alpha,P}(\lambda\theta_1 + (1-\lambda)\theta_2) &= \Phi_{\beta,P^{'}}(\lambda\theta_1 + (1-\lambda)\theta_2) \\
&\leq \lambda\Phi_{\beta,P^{'}}(\theta_1) + (1-\lambda)\Phi_{\beta,P^{'}}(\theta_2) \\
&= \lambda ESH_{\alpha,P}(\theta_1) + (1-\lambda)ESH_{\alpha,P}(\theta_2)
\end{aligned}
\ ,
$$

where the last *inequality* is due to the convexity of CVaR (See Corollary 11 in Rockafellar and Uryasev (2002)). Thus, given that θ_1, θ_2, and λ are arbitrary, we conclude that $ESH_{\alpha,P}$

is convex in $\mathcal{X}$. We now prove claims 1 and 2 to complete the proof.

1. Lemma 1.6.1 assures the existence of $(\mathbf{x}^*, \mathbf{y}^*)$. By assumption, $(\mathbf{x}^*, \mathbf{y}^*)$ satisfies

$$\frac{x^*}{y_0^*} = \left(\frac{1}{\alpha}\right)\left[\left(\frac{1-\alpha}{1-\alpha^+}\right)^2\left(\frac{1-\beta^+}{1-\beta}\right)\right] + \left(1-\frac{1}{\alpha}\right) \tag{1.39}$$

for some $\beta,\ \beta^+ \in [\alpha, 1)$ and $\beta^+ \geq \beta$. From Lemma 1.5.1,

$$\begin{aligned} Q^*(A) &= P(A)\left[1 + \left(\tfrac{x^*}{y_0^*}\right)\left(\tfrac{\alpha}{1-\alpha}\right)\right] \\ &= (1-\alpha^+)\left[1 + \left(\tfrac{x^*}{y_0^*}\right)\left(\tfrac{\alpha}{1-\alpha}\right)\right] . \end{aligned} \tag{1.40}$$

Substitution of (1.39) into (1.40) leads to Condition 1.

2. Given $\theta_1,\ \theta_2 \in \mathcal{X}$, and $\lambda \in (0,1)$ we need to show that there exists a probability measure $P' \equiv \left(p_n'\right)_{n\in N_T}$ such that

   (a) $VaR_{\alpha,P}(\theta) = VaR_{\alpha,P'}(\theta)$, for $\theta = \theta_1, \theta_2, \lambda\theta_1 + (1-\lambda)\theta_2$.

   (b) $\Phi^+_{\alpha,P}(\theta) = \Phi^+_{\alpha,P'}(\theta)$, for $\theta = \theta_1, \theta_2, \lambda\theta_1 + (1-\lambda)\theta_2$.

   Conditions (a) and (b) determine a system of equations for every triplet $(\theta_1, \theta_2, \lambda)$. Therefore, if the sample space is *large enough*, as it is appropriately assumed in the statement of the Proposition, we can always find a set of values $(p_n')_{n\in N_T}$ that define a probability measure which solves this system of equations.

Q.E.D.

The convexity of ESH relies on the idea that, under certain assumptions, the Expected Shortfall Hedge of a given strategy, with a pre-specified level of confidence, is equivalent to

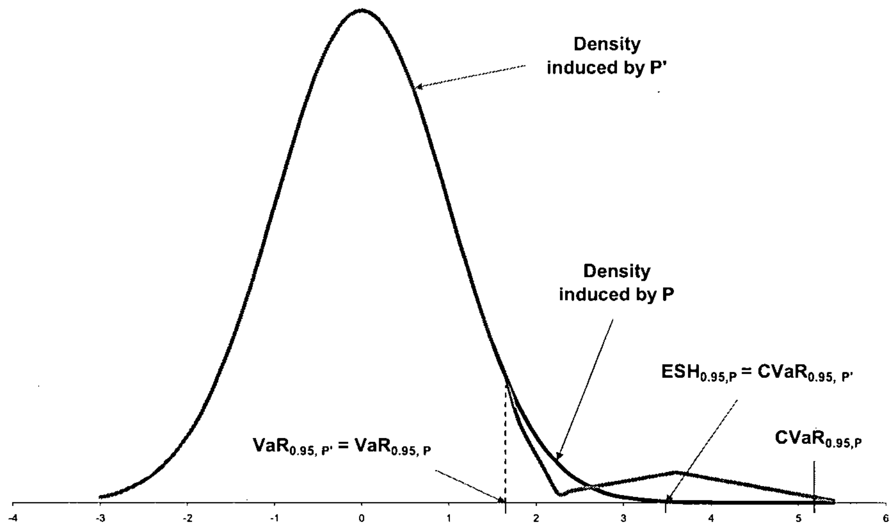

Figure 1.1: **Convexity of ESH**. This graph shows a density P (gray line) and a density P' (black line) such that the ESH, under P, coincides with the CVaR, under P', for a confidence level of $\alpha = 0.95$ and where P' is obtained by *distorting appropriately* the tail of the P density.

the Conditional Value at Risk of the same strategy but under a different probability measure and, *possibly*, a different level of confidence. Although, in principle, the alternative probability measures under which the ESH is a CVaR could be quite different from the original distribution, we should focus on those which tend to be close to the original distribution given that this distribution should contain important historical and expert information. In particular, given that ESH and CVaR are risk measures for extreme losses, we should concentrate on those alternative distributions that are *as close as possible* to the original distribution for losses up to the VaR threshold and therefore, differ from the original distribution only on the *tail* area. (see Figure 1.1).

From Propositions 1.6.2 and 1.6.3 we know that, under certain conditions, ESH is both convex and positive homogeneous. Therefore, ESH is a *coherent* measure of risk in the sense of Artzner et al (1999). Next corollary states this fact.

Corollary 1.6.1 (Coherence of ESH) *Suppose that the assumptions of Proposition* 1.6.3 *hold. Then, ESH is a coherent measure of risk in* $\mathcal{X}$.

Proof: A direct consequence of Propositions 1.6.2 and 1.6.3.

Q.E.D.

1.7 Conclusions and Further Research

The motivation for this research was to assess the implicit risks of the strategies followed by an investor who maximizes the expected value of his final wealth, subject to budget and risk tolerance constraints. A discrete-time framework that allows to consider incomplete markets and distributions other than the normal distribution, including the typically observed *fat-tailed* distributions, is used. Risk tolerance constraints are considered in terms of a maximum tolerated loss, which is measured using a coherent and stable measure of risk such as the Conditional Value at Risk (CVaR). Implicit Risks are uncovered in terms of the likelihood of surpassing the standard Value at Risk (VaR) threshold. For instance, three embedded probability measures are determined and compared with each other when assessing the probability of surpassing the VaR threshold. One of these embedded probability measures is a martingale measure and, hence, it is used to quantify the cost of hedging losses beyond the VaR threshold. In fact, it is proved that if this hedging cost is visualized as a capital requirement, then the mapping that associates this financial cost to a given strategy is a monetary measure of risk. Moreover, this monetary measure of risk, which we denominate *Expected Shortfall Hedge* (ESH), is proved to be a coherent measure of risk for a broader class of loss distributions than those for which VaR is coherent. The other two embedded probability measures are related to the particular utility function of the decision maker and the risk tolerance constraint. Finally, it is also proved that if the risk tolerance constraint is binding, then the probability of surpassing the VaR threshold and the cost of hedging losses beyond the VaR threshold are strictly larger under the embedded probabilities than under the *natural* probability measure. Therefore, this research particularly emphasizes the importance of considering these implicit risks, especially when the selected strategy operates at the limit of the tolerated risk.

Further research can be done in at least two main directions. A first direction should be in the line of extending the results obtained in this chapter by considering other convex risk measures and/or other type of loss functions and/or dynamic risk measures. A second and broader direction should be based on the observation that the inclusion of a convex risk measure, such as CVaR, in a stochastic programming model *induces* another implicit risk measure, such as ESH. Therefore, a problem to address is whether, in general, the inclusion of convex risk measures in a stochastic programming model always induces implicit risk measures. The *inverse* problem is also of interest. That is, given a risk measure, is there an associated stochastic programming model that induces such risk measure?

Chapter 2

The Duality of Option Investment Strategies for Hedge Funds

2.1 Introduction

Hedge Funds are private investment partnerships that attempt to obtain superior risk-adjusted returns in any market condition for their mostly wealthy investors. Although hedge funds have long existed[1], it was until the collapse of the Long-Term Capital Management (LTCM) hedge fund in 1998 that the influence and role that these institutions play in the financial markets was fully realized (e.g. de Brouwer (2001)). The high-profile (four billion dollars) failure of LTCM jeopardized several large financial institutions and, according to the U.S. Federal Reserve, the world economy.(e.g. Jorion (2000a)).

The LTCM event attracted immediately the attention of academics, who were interested in understanding well what actually happened and which was the real nature of these financial institutions. Early studies on hedge funds, such as Fung and Hsieh (1997), Eichengreen, et al. (1998), Fung and Hsieh (1999), Ackermann, et al. (1999) and Brown, et al. (1999), among the most important, were particularly difficult to realize since, at that time, information about hedge funds, both qualitative and quantitative, was not freely available for the general investment public. This is because hedge funds deal with *sophisticated* investors, such as wealthy and institutional investors, for which standard disclosure information regulations do not apply.

Over the past six years, the number of hedge funds has doubled and the assets under management by the hedge fund industry have grown exponentially, being one of the main reasons of this phenomena the significant increment of asset allocation to hedge funds by institutional investors (Fung and Hsieh (2006)).

[1]The term *hedge fund* describes the "hedge" against risk that some of these partnerships aim for through their strategies. The first official hedge fund was founded in 1949 by Alfred Winslow Jones in the United States. For more on the history of hedge funds, see Eichengreen, et al. (1998). Early hedge funds, such as the Chest Fund at King's College (Cambridge) which was managed by J.M. Keynes from 1927 - 1945, are discussed in Ziemba (2003), and Ziemba (2007).

Academic research on hedge funds has been mainly focused on their performance, their strategies and the risk associated with such strategies. Performance of hedge funds is typically measure in absolute terms with respect to the risk free rate and they normally use a wide variety of dynamic investment strategies.

Hedge fund performance is perhaps the main research topic given the popular claim that hedge funds obtain superior risk-adjusted returns. To verify this claim, many related studies do some kind of empirical analysis based on a performance model, which is typically a generalization or an extension of Sharpe's Performance Model (Sharpe (1992)). Sharpe proposes a simple linear regression model to explain portfolio returns in terms of suitable chosen *factor* variables and finds that his model explains reasonably well mutual fund returns. However, Fung and Hsieh (1997), when studying the nature of the hedge fund strategies, apply Sharpe's model to a large database of hedge fund returns and observe that Sharpe's model does not explain satisfactorily these returns. Therefore, several extensions of Sharpe's model have been proposed for such matter. Most of these extensions are based on the work of Glosten and Jagannathan (1994), who propose a general performance model based on a contingent claim perspective and which, in practical terms, reduces to add call-option payoff terms to Sharpe's model. Recently, Agarwal and Naik (2004) extend a bit further Glosten and Jagannathan's model and propose to include also put-option payoff terms. They find that this model fits well observed hedge fund returns. Overall, performance studies conclude that hedge funds consistently outperform mutual funds but not standard market indices.

Explaining the performance of hedge funds is intimately related to understanding their strategies. Therefore, studies that focus on performance also provide insights about their strategies and vice versa. For example, Agarwal and Naik (2004) conclude from the good fit of their performance model, the nonlinear option-like nature of hedge fund returns, which are characteristics that have been deduced by others that were primarily studying hedge fund strategies (e.g. Fund and Hsieh (1997)).

Describing the strategies followed by hedge funds is not an easy task. This is due to the lack of information, given the sui generis nature of hedge funds as financial entities for *sophisticated* investors, and the wide variety of strategies that hedge funds actually follow. However, there are some common characteristics that are well known about their strategies. For instance, most hedge funds use dynamic investment strategies that very often involve leverage and which have a low correlation with standard asset classes (Fung and Hsieh (1999, 2006)). Nonetheless, there exists great variety of investment styles. Consultants classify hedge funds according to self-described styles. Among the most common are *Long/short equity* funds, which are typically exposed to a long-short portfolio of equities with a long bias, *Event driven* funds, that specialize in trading corporate events, such as merger transactions or corporate restructuring, and *Equity market neutral* funds, which usually trade long-short portfolios of equities with little directional exposure to the stock market. Nevertheless, there are alternative classifications. For example, Fung and Hsieh (1997), using a principal component approach based on returns alone, provide a quantitative classification of hedge fund styles.

One of the popular beliefs about hedge funds is that these financial institutions do not have systematic risk. There are few studies that address this issue. This is because hedge fund managers usually diversify their fund's performance across a variety of strategies. Hence, risk analysis has focused on specific hedge fund style strategies. For example, Fung and Hsieh (2001) study the risk associated with the strategy style known as *trend following* and Mitchell and Pulvino (2001) analyze risk associated with the *risk-arbitrage* strategy style. Recently, Agarwal and Naik (2004) have proposed a general approach to characterize the risk of any hedge fund strategy. These studies conclude that hedge fund strategies do have systematic risk, and moreover, that in some cases, such as equity-oriented hedge fund strategies, payoffs resemble a short position in a put option on the market index. Therefore, risk management of hedge fund strategies is a crucial topic that needs to be addressed further.

Instead of trying to describe or deduce the strategies that hedge funds actually follow, other academics have theoretically derived the strategies that hedge fund managers should implement, given some preference model framework, and analyze the risks associated with such strategies. Hedge fund managers are typically compensated with an incentive fee formed by a fixed compensation plus a variable payment that is equal to a predetermined percentage of the positive profits over a specified benchmark. In other words, hedge fund managers are compensated with a fixed compensation plus a percentage of the payoff of a call option, whose underlying in the fund value. Carpenter (2000) solves explicitly for the first time, in a continuous-time framework, the dynamic investment problem of a risk averse manager who is compensated with a call option on the assets he controls, such as a hedge fund manager. Carpenter finds that the optimal strategy implies that the compensation option ends either deep in or deep out of the money. That is, the hedge fund manager implements a policy that leads to a fund value that is either well above or well below the benchmark. Carpenter also finds that the volatility associated with the optimal strategy can be strictly below the volatility of the policy followed by the same manager if he were trading his own account. This latter feature has questioned, in general, the ability of this type of option-like mechanisms to induce specific risk profiles. This topic is one of the main issues discussed in Chapter Three of this Ph.D. Thesis.

Cadenillas, et al. (2004) extend Carpenter's (2000) model, in a continuous-time framework, to study the case in which the manager chooses, in addition to the volatility of the portfolio of the assets that he controls, his level of effort. Carlson and Lazrak (2005) complement Cadenillas, et al. (2004) analysis by investigating the case in which the manager decides on the leverage instead of the effort level. Recently, Panageas and Westerfield (2006) obtain, also in a continuous-time framework, the optimal portfolio choice of a hedge fund manager who is compensated with a *high-water-mark* contract, where the benchmark is the last recorded maximum fund value. Unlike previous related studies, they assume that the horizon time is indefinite or infinite. They find that under such assumption the optimal portfolio will place a constant fraction in a mean-variance portfolio and the rest in a riskless asset. This implies that even risk-neutral investors will not invest unboundedly in risky assets, contrary to what previous studies with finite horizons would imply (e.g. Carpenter (2000)). Therefore, they conclude that risk-seeking incentives of option-like compensation schemes rely more on the horizon than in the convexity of the compensation scheme.

This chapter studies the nature of the optimal strategies followed by a hedge fund manager who includes stock index options in his investment universe. Our main motivations are, on one hand, that payment benchmarks for hedge fund managers are typically established in reference to a certain stock index and, on the other hand, that previous studies on optimal strategies use probability frameworks that do not allow to consider the well known skewed and fat-tailed features of option return distributions. Therefore, we consider a discrete-time framework that, in contrast to typical used continuous-time frameworks, can be easily adapted to consider not only these type of distributions, but also incomplete markets.

We consider a general risk averse hedge fund manager who is compensated with a fixed salary and predetermined percentage of the net profits with respect to a benchmark. Using duality theory, we obtain explicit theoretical conditions under which it is optimal for this manager to invest in stock index options. These conditions establish *pricing thresholds* for the stock index options, in terms of embedded martingale measures that are linked to the preferences of the hedge fund manager. We derive these optimal investment conditions for different benchmark policies and risk management considerations. The numerical valuation of these optimal conditions is relatively easy to obtain, given that all the models used involved optimization problems with linear constraints, for which there are well established algorithms to solve them (e.g. Bazarra, et al. (1993)). We illustrate our results with some examples.

This chapter is organized as follows: Section 2 describes the decision space framework used for our models. Section 3 presents our basic model, establishes explicit optimal investment conditions for such model, and develops a detailed example that shows the application of these conditions. Section 4 extends these results for a broader class of utility functions. Section 5 illustrates the risk incentives induced by the manager's compensation scheme. Section 6 generalizes the results of Section 4 for multiple periods of fee payments and different policies to determine the benchmark. Section 7 considers more advanced models, and Section 8 concludes.

2.2 Decision Space Framework

Our framework is based on King (2002) and uses three elements: a probability space, a financial market, and a class of investors.

Probability Space

The probability space uses a *scenario tree structure* that models all possible scenarios or states (represented by nodes of a tree) of the market over a finite number of discrete time periods $t = 0, \ldots, T$. The scenario tree structure is such that every possible state is the consequence of a unique sequence (trajectory) of states (events). That is, every node $n \in N_t$, where N_t denotes the set of all nodes at time t, has a unique parent, denoted by $a(n) \in N_{t-1}$, although with a set on possibly many child nodes, denoted by $C(n) \subseteq N_{t+1}$. This is convenient for assigning probabilities to each of the tree scenario nodes. Defining a probability measure P (i.e., assigning probabilities to each node of the tree) in this type of scenario tree consists of assigning weights $p_n > 0$ to each leaf node $n \in N_T$ with $\sum_{n \in N_T} p_n = 1$ and then recursively computing the remaining node probabilities via

$$p_n = \sum_{m \in C(n)} p_m, \ \forall \ n \in N_t, \ t = T-1, \ldots, 0.$$

Let Ω be the set of possible trajectories or sequence of events (from time 0 to the end of period T) in the *scenario tree*, then (Ω, P) defines a *sample space*. Every node $n \in N_t$ has a unique history up to time t and a unique set of possible future trajectories. Therefore, N_t induces a unique set of histories up to time t, say F_t, and a partition of Ω. The collection of sets $\{F_t\}_{t=0,\ldots,T}$ satisfy $F_t \subseteq F_{t+1}$ for $t = 0, \ldots, T-1$. The triplet (Ω, F_T, P) forms a *probability space*. For the probability space (Ω, F_T, P), the conditional probability of state (event) m given that n occurs, where $m \in C(n)$, is $\left(\frac{p_m}{p_n}\right)$, and if $\{X_t\}_{t=0,\ldots,T}$ is a discrete stochastic process defined on our probability space, then $E^P[X_t] = \sum_{n \in N_t} X_n p_n$ and $E^P[X_{t+1}|F_t] = \sum_{m \in C(n)} \left(\frac{p_m}{p_n}\right) X_m$.

Martingales

Definition 2.2.1 *Let $\{Z_t\}_{t=0,\ldots,T}$ be a stochastic process defined in (Ω, F_T). If there exists a probability measure Q such that*

$$Z_t = E^Q[Z_{t+1}|N_t], \ t = 0, \ldots, T-1$$

then the stochastic process $\{Z_t\}_{t=0,\ldots,T}$ is called a martingale under Q, and Q is called a martingale measure for the process $\{Z_t\}_{t=0,\ldots,T}$.

Martingales are useful in the financial context, for example, to determine if a price process is *fair* in the sense that at any time the price equates its expected future value (under a probability measure Q), i.e., if the price process is a *martingale* (under Q).

Financial Market and Investors

The market has $I+1$ tradable securities $i = 0, \ldots, I$, whose prices at each node n are denoted by the vector $S_n = (S_n^0, \ldots, S_n^I)$. We assume that one of the securities, the *numeraire*, has always a strictly positive value and without loss of generality we assume that it is security 0. This *numeraire* defines the discount factors $\beta_n = (\frac{1}{S_n^0})$ and the discounted prices (relative to the *numeraire*) $Z_n^i \equiv \beta_n S_n^i$ for $i = 0, \ldots, I$. The price of the *numeraire* in any state equals one. The market may be either complete or incomplete.

Let θ_n^i be the amount of security i held by the investor in state $n \in N_t$. Thus, the portfolio value in state $n \in N_t$ is $Z_n \cdot \theta_n \equiv \sum_{i=0}^{I} Z_n^i \cdot \theta_n^i$.

We assume that investors do not influence the security prices of any security and trade at every time-step based on information up to time t.

Arbitrage

Arbitrage is the opportunity of making a sure profit with no risk (usually through the purchase and sale of assets). Here, arbitrage is to find a portfolio with zero initial value whose terminal values, obtained through self-financing strategies, are nonnegative for all scenarios and at least one of those terminal values is strictly positive with a positive probability. In other words, there is an arbitrage if there exists a sequence of portfolio holdings $\{\theta_n\}_{n \in N_t,\ 0 \leq t \leq T}$ such that

$$
\begin{aligned}
&Z_0 \cdot \theta_0 = 0 \\
&Z_n \cdot \left(\theta_n - \theta_{a(n)}\right) = 0 \ \ \forall\, n \in N_t \ , \ \forall\, t = 1, ..., T \\
&Z_n \cdot \theta_n \geq 0 \ \ \forall\, n \in N_T \ \text{, and} \\
&P\left\{Z_m \cdot \theta_m > 0\right\} > 0 \text{ for some } m \in N_T
\end{aligned}
$$

We assume that there is *no arbitrage* in our financial market. This assumption guarantees the existence of martingale measures for $\{Z_t\}_{t=0,...,T}$ (King (2002), Theorem 2.2), which are used to express optimal conditions for option investments for models presented below.

2.3 The Basic Problem

We consider a hedge fund manager who controls assets with a current value of W_0 and who receives compensation of a fixed fee f and an incentive fee of a percentage α of the positive profit, if any, of the hedge fund manager's portfolio with respect to a previously specified benchmark, usually based on a stock index, over a T-period horizon. The hedge fund manager can invest in any of $I + 1$ securities and can buy or sell European stock index options. Our *basic problem* is that faced by a hedge fund manager determining the investment strategy that maximizes his expected fee.

We focus on two disjoint cases: either the hedge fund manager is only allowed to *buy* stock index options or he is only allowed to *sell* stock index options. We make the following assumptions:

- The hedge fund manager only considers strategies that do not lead to negative values of his portfolio. Hence, $Z_n \cdot \theta_n \geq 0$, for all $n \in N_T$.
- The manager can re-balance his portfolio periodically. However, there are no inflows or outflows of capital during $[0, T]$. hence, the hedge fund manager only considers

self-financing strategies that satisfy

$$Z_n \cdot \left(\theta_n - \theta_{a(n)}\right) = 0, \text{ for all } n \in N_T, \ \forall\, t = 1, \ldots, T.$$

This restriction is relaxed in subsequent sections.

- If the hedge fund manager takes a position in stock index options at time 0, then he maintains such position until time T.

2.3.1 Buying Options

We consider the set of strategies in which the hedge fund manager is allowed to invest in any of the $I+1$ securities and to buy (European) Options (vanilla Calls and Puts) whose underlying is a Stock Index, which is assumed to be the benchmark. The manager's objective is to maximize his expected fee (we study more complex objective functions in subsequent sections). The optimization investment problem, without the constant term $f\left(\sum_{n\epsilon N_T} \beta_n p_n\right)$, which does not influence the optimal solution, is

$$\begin{array}{l}
Max_{\ \theta,\epsilon_0,s} \ \ \alpha\left[\sum_{n\in N_T} \beta_n s_n p_n\right] \\
s.t. \\
Z_0 \cdot \theta_0 + \beta_0\left(\epsilon_0 \cdot V_0\right) = \beta_0 W_0 \\
Z_n \cdot \left(\theta_n - \theta_{a(n)}\right) = 0 \ , \ \ \forall\ n \in N_t \ , \ \forall\ t = 1, \ldots,\ T \\
Z_n \cdot \theta_n - \beta_n s_n + \beta_n\left(\epsilon_0 \cdot V_n\right) = \beta_n B_n \ , \ \ \forall\ n \in N_T \\
Z_n \cdot \theta_n \geq 0, \ s_n \geq 0 \ , \ \forall\ n \in N_T \\
\epsilon_0 \geq \mathbf{0} \ ,
\end{array} \tag{2.1}$$

where $\epsilon_0 \equiv \left[\epsilon_0^C, \epsilon_0^P\right]$, $V_0 \equiv \left[C_0(1+tc^C), P_0(1+tc^P)\right]$, $V_n \equiv [C_n, P_n]$, $\mathbf{0} \equiv [0,0]$, and

s_n	:	The surplus over the benchmark in scenario $n \in N_T$.
ϵ_0^C (ϵ_0^P)	:	Quantity of Call (Put) options that are purchased at time 0.
C_0 (P_0)	:	Call (Put) *price* at time 0.
C_n (P_n)	:	Call (Put) *payoff* in scenario $n \in N_T$.
tc^C (tc^P)	:	Transaction Costs when buying Calls (Puts) at time 0. (as a percentage of the option's value).
W_0	:	Initial Value of the Hedge Fund Manager's portfolio.
B_n	:	Benchmark value in scenario $n \in N_T$.

The dual problem of (2.1), which provides insights into the nature of the optimal strategies for the hedge fund manager, is (see Appendix 2)

$$\begin{aligned}
& Min_{\;y_0,\mathbf{y},\mathbf{x},\lambda,\mu,\eta_C,\eta_P} \;\; y_0\beta_0 W_0 - \textstyle\sum_{n\in N_T} x_n\beta_n B_n \\
& s.t. \\
& \alpha p_n + \mu_n - x_n = 0, \;\forall\; n \in N_T \\
& \lambda_n + x_n - y_n = 0, \;\forall\; n \in N_T \\
& (y_n Z_n - \textstyle\sum_{m\in C(n)} y_m Z_m) = 0, \;\forall\; n \in N_t, \;\forall\; t = 0,\dots,T-1 \\
& \textstyle\sum_{n\in N_T} x_n\beta_n C_n - y_0\beta_0 C_0(1+tc^C) + \eta_C = 0 \\
& \textstyle\sum_{n\in N_T} x_n\beta_n P_n - y_0\beta_0 P_0(1+tc^P) + \eta_P = 0 \\
& \lambda_n \geq 0, \;\mu_n \geq 0, \;\forall\; n \in N_T \\
& \eta_C \geq 0, \;\eta_P \geq 0\;.
\end{aligned} \tag{2.2}$$

The first set of restrictions corresponds to the positive profits, if any, over the Benchmark. The second set of restrictions is associated with the final value of the hedge fund manager's portfolio. The third set of restrictions is consequence of the self-financing strategies constraint. The last two constraints correspond to the purchase of options.

Observe from the first two sets of restrictions of the dual that

$$x_n = \mu_n + \alpha p_n \geq \alpha p_n > 0\;, \forall\; n \in N_T\;,$$

and $y_n = x_n + \lambda_n \geq x_n$, $\forall\, n \in N_T$. Thus,

$$y_n \geq x_n \geq \alpha p_n > 0 \,. \tag{2.3}$$

The third set of restrictions of the dual problem and the fact that $Z_n^0 = 1$ for all $n \in N_t$, $\forall\ t = 1, \ldots, T$, imply that

$$y_n = \sum_{m \in C(n)} y_m \ , \ \forall\, n \in\ N_t \ , \ \forall\, t = 0, \ldots, T-1. \tag{2.4}$$

Hence,

$$y_0 = \sum_{n \in N_T} y_n \ . \tag{2.5}$$

Therefore, from (2.3) and (2.5),

$$y_0 \geq \sum_{n \in N_T} x_n \ . \tag{2.6}$$

These preliminary observations of the dual problem are key for its analysis and interpretation.

Analysis and Interpretation of the Dual Problem

1. Let y_0^* and $(x_n^*)_{n \in N_T}$ be the optimal values for the linear programming dual variables y_0 and $(x_n)_{n \in N_T}$. These values represent, respectively, the marginal change of the optimal expected fee of the hedge fund manager with respect to the initial hedge fund portfolio's value W_0, and the possible benchmark values $(B_n)_{n \in N_T}$.

2. The last two restrictions of the dual problem are equivalent to

$$y_0 \;\geq\; \textstyle\sum_{n \in N_T} x_n(\frac{\beta_n}{\beta_0})(\frac{C_n}{C_0})(1+tc^C)^{-1} \tag{2.7}$$

$$y_0 \;\geq\; \textstyle\sum_{n \in N_T} x_n(\frac{\beta_n}{\beta_0})(\frac{P_n}{P_0})(1+tc^P)^{-1}. \tag{2.8}$$

The expressions on the right hand side of these two inequalities can be interpreted as a (discounted) *weighted average return of the Stock Index Options.*

3. From (2.6) - (2.8), it must be satisfied that

$$\begin{aligned} y_0 \;\geq\; & Max\Big\{\textstyle\sum_{n \in N_T} x_n(\frac{\beta_n}{\beta_0})(\frac{C_n}{C_0})(1+tc^C)^{-1}\,, \\ & \textstyle\sum_{n \in N_T} x_n(\frac{\beta_n}{\beta_0})(\frac{P_n}{P_0})(1+tc^P)^{-1}, \sum_{n \in N_T} x_n\Big\} \end{aligned}$$

Therefore, the optimal value of y_0, say y_0^*, must satisfy

$$\begin{aligned} y_0^* \;=\; & Max\Big\{\textstyle\sum_{n \in N_T} x_n^*(\frac{\beta_n}{\beta_0})(\frac{C_n}{C_0})(1+tc^C)^{-1}\,, \\ & \textstyle\sum_{n \in N_T} x_n^*(\frac{\beta_n}{\beta_0})(\frac{P_n}{P_0})(1+tc^P)^{-1}, \sum_{n \in N_T} x_n^*\Big\} \end{aligned}$$

where $(x_n^*)_{n \in N_T}$ are the optimal values of $(x_n^*)_{n \in N_T}$. Hence, if

$$y_0^* = \sum_{n \in N_T} x_n^* \left(\frac{\beta_n}{\beta_0}\right)\left(\frac{C_n}{C_0}\right)(1+tc^C)^{-1}$$

or

$$y_0^* = \sum_{n \in N_T} x_n^* \left(\frac{\beta_n}{\beta_0}\right)\left(\frac{P_n}{P_0}\right)(1+tc^P)^{-1}\,,$$

then, from 2, y_0^* *can be interpreted as a (discounted) weighted average return of Stock Index Options.* Otherwise, if

$$y_0^* = \sum_{n \in N_T} x_n^* , \tag{2.9}$$

then, by (2.3) and (2.5), $x_n^* = y_n^*$, for all $n \in N_T$, and hence, $\lambda_n^* = 0$, for all $n \in N_T$. Thus, by linear complementarity, $Z_n \cdot \theta_n^* > 0$ for all $n \in N_T$. That is, if condition (2.9) is satisfied, then the hedge fund manager aims for a strictly positive value of his security's portfolio for every possible final scenario. Therefore, given the *no arbitrage* assumption, the initial value of the security's portfolio, $Z_0 \cdot \theta_0^*$, must be strictly positive. In other words, if condition (2.9) holds, then the hedge fund manager must invest in the security's portfolio.

4. The dual objective function can be expressed as

$$y_0 \beta_0 W_0 - \sum_{n \in N_T} x_n \beta_n B_n = y_0 \left(\beta_0 W_0 - \sum_{n \in N_T} \left(\frac{x_n}{y_0} \right) \beta_n B_n \right) .$$

Therefore, the optimal dual value,

$$y_0^* \left(\beta_0 W_0 - \sum_{n \in N_T} \left(\frac{x_n^*}{y_0^*} \right) \beta_n B_n \right) ,$$

can be interpreted, using 2 and 3, as a weighted average return over the remaining capital derived from subtracting the weighted average,

$$\sum_{n \in N_T} \left(\frac{x_n^*}{y_0^*} \right) \beta_n B_n ,$$

from the current value of the hedge fund manager's portfolio.

5. From equations (2.3)-(2.5)

$$q_n \equiv \frac{y_n}{y_0} > 0 , \ \forall \, n \in N_T$$

defines a probability measure Q. Furthermore, the third set of restrictions of the dual problem (2.2) implies that $\{q_n\}_{n \in N_T}$ defines a martingale measure for the (discounted) price process $\{Z_t\}_{t=0,\ldots,T}$. Moreover, if the final values of the portfolio ($Z_n \cdot \theta_n$, $n \in N_T$), are unrestricted, then $x_n = y_n$ for all $n \in N_T$ (see the Appendix 2 for the proof), and, thus, the dual objective function is

$$y_0 \left(\beta_0 W_0 - \sum_{n \in N_T} \left(\frac{y_n}{y_0} \right) \beta_n B_n \right) = y_0 \left(\beta_0 W_0 - E^Q \left[\beta_T B_T \right] \right). \tag{2.10}$$

Hence, the optimal dual value,

$$y_0^* \left(\beta_0 W_0 - E^{Q^*} \left[\beta_T B_T \right] \right) ,$$

can be interpreted as a *weighted return over the remaining capital derived from subtracting the expected value of the benchmark (under Q^*) at time T from the current value of the hedge fund manager's portfolio.*

6. From equations (2.3) and (2.5), it follows that $y_0 \geq \alpha$. Therefore, based on the interpretations of y_0^* argued in Items 4 and 5, the *optimal hedge fund manager's expected fee must be at least α percent of the remaining capital derived from subtracting a weighted average value (expected value, under the no-bankruptcy assumption made in Item 5) of the benchmark from the current value of the hedge fund manager's portfolio.*

7. Finally, equation (2.3) provides us with a set of lower bounds for the dual variables $\{y_n\}_{n \in N_T}$, and, hence, with a set of (scaled) lower bounds for the martingale probabilities $\{q_n\}_{n \in N_T}$ defined in Item 5. That is $y_0 \; q_n \geq \alpha p_n$, $\forall \; n \in N_T$.

We summarize the previous analysis in Table 1.

Table 1

Expression	Interpretation	Bounds
y_0^*	1. Marginal change of the optimal expected fee with respect to W_0, 2. Weighted Average Return.	$y_0^* \geq \alpha$
x_n^*	Marginal change of the optimal expected fee with respect to the benchmark value B_n, $n \in N_T$.	αp_n
$Q \equiv (q_n)_{n \in N_T}$, $q_n = \left(\frac{y_n}{y_0}\right)$	Martingale Measure of the discounted price process $(Z_t)_{t=0,\dots,T}$.	$q_n \geq \alpha \left(\frac{p_n}{y_0}\right)$
$y_0^* \left(\beta_0 W_0 - \sum_{n \in N_T} \left(\frac{x_n^*}{y_0^*}\right) \beta_n B_n\right)$	Weighted average return over the remaining capital derived from subtracting an average of the possible benchmark values at T from the current value of the hedge fund manager's portfolio	$\alpha \left(\beta_0 W_0\right)$ $-\alpha \sum_{n \in N_T} \left(\frac{x_n^*}{y_0^*}\right) \beta_n B_n$

Example 1

Description:

Consider a static one-period horizon problem with two equally likely scenarios, n_1 and n_2, and a financial market with two securities: a bond (security 0, the *numeraire*) and a stock (security 1). We assume that the hedge fund manager receives, at time T, a variable incentive fee of 20% over the positive profit of the portfolio he controls, with respect to a benchmark B. This benchmark is established as the capital that would be accumulated if the current portfolio value W_0 would be invested at a rate of return equal to the return gained by the stock index SI plus a fixed return κ. That is, the benchmark B can take the two following possible values:

$$
\begin{aligned}
B_{n_1} &= \left(\frac{SI_{n_1}}{SI_0} + \kappa\right) W_0 \ , \text{ in scenario } n_1, \text{ and} \\
B_{n_2} &= \left(\frac{SI_{n_2}}{SI_0} + \kappa\right) W_0 \ , \text{ in scenario } n_2,
\end{aligned}
$$

where SI_0 is the current value of the stock index and SI_{n_j} represents the stock index value in scenario j, $j = 1, 2$.

Finally, we assume that there is an option market on the stock index SI. The basic problem is then

$$
\begin{aligned}
& Max_{\ \theta,\epsilon_0,s} \quad (0.2)\left[\beta_{n_1} s_{n_1} p_{n_1} + \beta_{n_2} s_{n_2} p_{n_2}\right] \\
& s.t. \\
& \theta_0^0 + Z_0^1 \theta_0^1 + \beta_0 \epsilon_0^C C_0 + \beta_0 \epsilon_0^P P_0 = \beta_0 W_0 \\
& \left(\theta_{n_1}^0 - \theta_0^0\right) + Z_{n_1}^1 \left(\theta_{n_1}^1 - \theta_0^1\right) = 0 \\
& \left(\theta_{n_2}^0 - \theta_0^0\right) + Z_{n_2}^1 \left(\theta_{n_2}^1 - \theta_0^1\right) = 0 \\
& \theta_{n_1}^0 + Z_{n_1}^1 \theta_{n_1}^1 - \beta_{n_1} s_{n_1} + \beta_{n_1} \epsilon_0^C C_{n_1} + \beta_{n_1} \epsilon_0^P P_{n_1} = \beta_{n_1} B_{n_1} \\
& \theta_{n_2}^0 + Z_{n_2}^1 \theta_{n_2}^1 - \beta_{n_2} s_{n_2} + \beta_{n_2} \epsilon_0^C C_{n_2} + \beta_{n_2} \epsilon_0^P P_{n_2} = \beta_{n_2} B_{n_2} \\
& \theta_{n_1}^0 + Z_{n_1}^1 \theta_{n_1}^1 \geq 0, \ \theta_{n_2}^0 + Z_{n_2}^1 \theta_{n_2}^1 \geq 0 \\
& \epsilon_0^C \geq 0, \ \epsilon_0^P \geq 0 \ , \\
& s_{n_1} \geq 0, \ s_{n_2} \geq 0 \ .
\end{aligned}
\tag{2.11}
$$

where $\beta_{n_j} = (S_{n_j}^0)^{-1} = (S_0^0(1+r))^{-1}$, $j = 1, 2$; r is the one-period (fixed) interest rate; C_{n_j} (P_{n_j}) is the Call (Put) payoff under scenario n_j, $j = 1, 2$; and C_0 (P_0) is the current price of the Call (Put) option.

We assume that the current price of the options on the Stock Index SI correspond to the *no-arbitrage* prices

$$
\begin{aligned}
C_0 &= \left(\tfrac{1}{1+r}\right)\left[\tilde{q} C_{n_1} + (1-\tilde{q}) C_{n_2}\right] \quad , \quad \text{and} \\
P_0 &= \left(\tfrac{1}{1+r}\right)\left[\tilde{q} P_{n_1} + (1-\tilde{q}) P_{n_2}\right] \quad ,
\end{aligned}
$$

where

$$\tilde{q} = \frac{SI_0(1+r) - SI_{n_2}}{SI_{n_1} - SI_{n_2}} \quad . \tag{2.12}$$

Suppose that the current security prices are $S_0^1 = 90$ and $S_0^0 = 10$, and that $W_0 = 500$, $SI_0 = 100$, $r = 5\%$ and $\kappa = 2\%$, with the following scenarios:

Scenario n_1:

$S_{n_1}^0 = (1+r)S_0^0 = 10.5$, $S_{n_1}^1 = 108$, and $SI_{n_1} = 115$. Therefore,

$$B_{n_1} = \left(\frac{SI_{n_1}}{SI_0} + \kappa\right) W_0 = 585.$$

Scenario n_2:

$S_{n_2}^0 = (1+r)S_0^0 = 10.5$, $S_{n_2}^1 = 67.5$, and $SI_{n_2} = 70$. Therefore,

$$B_{n_2} = \left(\frac{SI_{n_2}}{SI_0} + \kappa\right) W_0 = 360.$$

Thus,

$$\tilde{q} = \frac{100(1+0.05) - 70}{115 - 70} = 0.7778$$

and hence, considering an exercise price of 80 for both options, it holds that $C_0 = 25.926$ and $P_0 = 2.116$.

Solution:

The optimal strategy for the hedge fund manager is

$$\begin{array}{ll} \theta_0^{0,*} = -92.857 \quad , & \theta_0^{1,*} = 14.444 \\ \epsilon_0^{C,*} = 0 \quad , & \epsilon_0^{P,*} = 60.75 \end{array}$$

which means that the manager borrows cash, at a rate of interest $r = 5\%$, takes a long position in the risky security S^1, and buys put options. This leads to the expected variable fee

$$\begin{aligned} 0.2\left[\beta_{n_1} s_{n_1} p_{n_1} + \beta_{n_2} s_{n_2} p_{n_2}\right] &= \left(\tfrac{0.2}{10.5}\right)\left[(0)\left(\tfrac{1}{2}\right) + (247.501)\left(\tfrac{1}{2}\right)\right] \\ &= 2.35715 \ . \end{aligned}$$

The Dual Problem and its Interpretation:

The optimal solution of the dual problem of (2.11) is

$$\begin{array}{lll} y_0^* = 0.45, & y_1^* = 0.3, & x_1^* = 0.3, \\ & y_2^* = 0.15, & x_2^* = 0.1 \ . \end{array}$$

Then

$$\begin{aligned} y_0^*\left[\beta_0 W_0 - \left(\tfrac{x_1^*}{y_0^*}\beta_{n_1} B_{n_1} + \tfrac{x_2^*}{y_0^*}\beta_{n_2} B_{n_2}\right)\right] &= (0.45)\left[\left(\tfrac{500}{10}\right) - \left(\tfrac{1}{10.5}\right)\left(\left\{\tfrac{0.3}{0.45}\right\}(585) + \left\{\tfrac{0.1}{0.45}\right\}(360)\right)\right] \\ &= (0.45)\left[50 - 44.762\right] \\ &= 2.35715 \\ &= 0.2\left[\beta_{n_1} s_{n_1} p_{n_1} + \beta_{n_2} s_{n_2} p_{n_2}\right] \ . \end{aligned}$$

Thus, the optimal expected variable fee for the hedge fund manager is approximately[2] 45% ($= y_0^*$) over the remaining capital derived from subtracting the expected (discounted)

[2] $\left[\left(\frac{x_1^*}{y_0^*}\right)\beta_{n_1} B_{n_1} + \left(\frac{x_2^*}{y_0^*}\right)\beta_{n_2} B_{n_2}\right] = 44.762 \approx 48.571 = \left[\left(\frac{y_1^*}{y_0^*}\right)\beta_{n_1} B_{n_1} + \left(\frac{y_2^*}{y_0^*}\right)\beta_{n_2} B_{n_2}\right] = E^{Q^*}\left[\beta_T B_T\right]$

benchmark value

$$\begin{aligned}\left[\left(\frac{y_1^*}{y_0^*}\right)\beta_{n_1}B_{n_1}+\left(\frac{y_2^*}{y_0^*}\right)\beta_{n_2}B_{n_2}\right] &= E^{Q^*}\left[\beta_T B_T\right]\\ &= 48.571\ ,\end{aligned}$$

from the current (discounted) hedge fund manager portfolio $\beta_0 W_0 = (1/10)500 = 50$. Note that Q^* does not coincide with the probability measure $\tilde{Q} \equiv (\tilde{q}, 1-\tilde{q})$, where $\tilde{q}$ is defined in (2.12).

Moreover, observe that

$$\begin{aligned}\beta_0 P_0 &= \left(\frac{2.116}{10}\right)\\ &= \left(\frac{1}{10.5}\right)\left[\frac{0.3}{0.45}(0)+\frac{0.1}{0.45}(10)\right]\\ &= \left[\left(\frac{x_1^*}{y_0^*}\right)(\beta_{n_1})(P_{n_1})(1+tc^P)^{-1}+\left(\frac{x_2^*}{y_0^*}\right)(\beta_{n_2})(P_{n_2})(1+tc^P)^{-1}\right]\ .\end{aligned}$$

This latter condition is expected to hold since optimal purchase of (Put) Stock Index Options, i.e., $\epsilon_0^{P,*} > 0$, implies, by linear complementarity,

$$\begin{aligned}\beta_0 P_0 &= \textstyle\sum_{n\in N_T}\left(\frac{x_n^*}{y_0^*}\right)(\beta_n)(P_n)(1+tc^P)^{-1}\\ &\leq \textstyle\sum_{n\in N_T}\left(\frac{y_n^*}{y_0^*}\right)(\beta_n)(P_n)(1+tc^P)^{-1}\\ &= E^{Q^*}\left[\beta_T P_T\right](1+tc^P)^{-1}.\end{aligned}\tag{2.13}$$

where the first inequality follows from (2.3). Therefore, from (2.13), we have proved

Proposition 2.3.1 *If $\epsilon_0^{C,*} > 0$, then $\beta_0 C_0(1+tc^C) = \sum_{n\in N_T}\left(\frac{x_n^*}{y_0^*}\right)\beta_n C_n \leq E^{Q^*}\left[\beta_T C_T\right]$, where $Q^* = \left\{\frac{y_n^*}{y_0^*}\right\}_{n\in N_T}$. Analogously, if $\epsilon_0^{P,*} > 0$ then,*

$$\beta_0 P_0(1+tc^P) = \sum_{n\in N_T}\left(\frac{x_n^*}{y_0^*}\right)\beta_n P_n \leq E^{Q^*}\left[\beta_T P_T\right].$$

The optimal investment condition for the purchase of stock index options given in Proposition 2.3.1 can be restated as the Stock Index Options must not be overpriced. This is proved in

Corollary 2.3.1 *If* $\epsilon_0^{C,*} > 0$ $\left(\epsilon_0^{P,*} > 0\right)$, *then*

$$\beta_0 C_0(1 + tc^C) \leq Max_{Q \in \mathcal{Q}}\ E^Q\left[\beta_T C_T\right]\ ,\left(\beta_0 P_0(1 + tc^P) \leq Max_{\{Q \in \mathcal{Q}\}}\ E^Q\left[\beta_T P_T\right]\right)\ .$$

where $\mathcal{Q}$ *denotes the set of martingale measures of* $\{C_t\}_{t=0,\ldots,T}$ $\left(\{P_t\}_{t=0,\ldots,T}\right)$.

Proof: The result follows directly from Proposition 2.3.1 and the obvious fact that

$$Max_{Q \in \mathcal{Q}}\ E^Q\left[\beta_T C_T\right] \geq E^{Q^*}\left[\beta_T C_T\right].$$

Q.E.D.

Within this framework King (2002, page 550) proves that the fair value of the contingent claim V_t at time T is

$$Max_{Q \in \mathcal{Q}}\ E^Q\left[\beta_T V_T\right]\ ,$$

where V_t is the contingent's claim price at time t, and $\mathcal{Q}$ denotes the set of martingale measures of $\{V_t\}_{t=0,\ldots,T}$. Hence, Corollary 2.3.1 states that it is optimal to purchase Stock Index Options if these are not overpriced.

In summary, if the hedge fund manager is allowed to purchase Stock Index Options, then the optimal strategy leads to a percentage (of at least α*) over the remaining capital derived* *from subtracting the average value of the Benchmark from the current value of the hedge fund manager's portfolio and will include the purchase of options if these are not overpriced (with respect to the no-arbitrage price under Q^*).*

2.3.2 Selling Options

We concentrate now on the set of strategies where the hedge fund manager is allowed to sell Stock Index Options. The manager's objective is again to maximize his expected variable fee. The optimal investment problem is

$$\begin{array}{ll} Max_{\,\theta,\bar{\epsilon}_0^C,\,\bar{\epsilon}_0^P,s} \;\; \alpha\left[\sum_{n\in N_T}\beta_n s_n p_n\right] & \\ s.t. & \\ Z_0\cdot\theta_0-\bar{\epsilon}_0^C\beta_0\bar{C}_0(1+tc^C)-\bar{\epsilon}_0^P\beta_0\bar{P}_0(1+tc^P)=\beta_0W_0 & \\ Z_n\cdot(\theta_n-\theta_{a(n)})=0\;,\quad\forall\, n\in N_t,\;\forall\, t=1,\ldots,\;T & \qquad (2.14) \\ Z_n\cdot\theta_n-\beta_n s_n-\bar{\epsilon}_0^C\beta_nC_n-\bar{\epsilon}_0^P\beta_nP_n=\beta_nB_n\;,\quad\forall\, n\in N_T & \\ Z_n\cdot\theta_n\geq 0,\;\; s_n\geq 0\;,\;\forall\, n\in N_T & \\ \bar{\epsilon}_0^C\geq 0,\;\;\bar{\epsilon}_0^P\geq 0\quad. & \end{array}$$

where $\bar{\epsilon}_0^C$ ($\bar{\epsilon}_0^P$) is the amount of Stock Index Call (Put) Options that are sold and $\bar{C}_0$ ($\bar{P}_0$) is the current (unitary) price of such Call (Put) options. As in the case of purchasing Stock Index Options, the dual of (2.14) provides insights into the structure of the optimal strategies for the hedge fund manager. The feasibility of (2.14) is discussed in Section 2.3.4). The dual (see Appendix 2 for its derivation) of problem (2.14) is

$$
\begin{aligned}
& Min_{\ y_0,\mathbf{y},\mathbf{x},\lambda,\mu,\eta_C,\eta_P} \ \ y_0\beta_0 W_0 - \textstyle\sum_{n\in N_T} x_n\beta_n B_n \\
& s.t. \\
& \alpha p_n + \mu_n - x_n = 0, \ \forall \ n \in N_T \\
& \lambda_n + x_n - y_n = 0, \ \forall \ n \in N_T \\
& (y_n Z_n - \textstyle\sum_{m\in C(n)} y_m Z_m) = 0, \ \forall \ n \in N_t, \ \forall \ t = 0,\ldots,T-1 \\
& y_0\beta_0\bar{C}_0(1+tc^C) - \textstyle\sum_{n\in N_T} x_n\beta_n C_n + \eta_C = 0 \\
& y_0\beta_0\bar{P}_0(1+tc^P) - \textstyle\sum_{n\in N_T} x_n\beta_n P_n + \eta_P = 0 \\
& \lambda_n \geq 0, \ \mu_n \geq 0, \ \forall \ n \in N_T \\
& \eta_C \geq 0, \ \eta_P \geq 0 \ .
\end{aligned}
\tag{2.15}
$$

Analysis and Interpretation of the Dual Problem

The difference between the dual problems (2.2) and (2.15) is only in the last two restrictions. Therefore, we expect that most of the interpretations derived for purchasing options are valid for the case of selling options. For instance, interpretations 1, 4, 5, 6 and 7 of Section 2.3.1 apply exactly or almost exactly in the same manner. Interpretations 2 and 3 do not have a straight forward interpretation in this case although they do have the following corresponding counterparts:

1. (2') The last two restrictions of the dual problem are equivalent to

$$
\begin{aligned}
y_0 &\leq \textstyle\sum_{n\in N_T}[(\frac{\beta_n}{\beta_0})(\frac{C_n}{C_0})(1+tc^C)^{-1}]x_n \ . \\
y_0 &\leq \textstyle\sum_{n\in N_T}[(\frac{\beta_n}{\beta_0})(\frac{P_n}{P_0})(1+tc^P)^{-1}]x_n \ .
\end{aligned}
\tag{2.16}
$$

   The right hand side of each of the previous restrictions can be interpreted as *weighted average returns of Stock Index Options.*

2. (4') Equations (2.16) and (2.6) imply that

$$\sum_{n\in N_T} x_n \;\le\; y_0 \;\le\; Min\{\sum_{n\in N_T}[(\tfrac{\beta_n}{\beta_0})(\tfrac{C_n}{C_0})(1+tc^C)^{-1}]x_n,$$
$$\sum_{n\in N_T}[(\tfrac{\beta_n}{\beta_0})(\tfrac{P_n}{P_0})(1+tc^P)^{-1}]x_n\}\ .$$

Therefore, optimality implies $y_0^* = \sum_{n\in N_T} x_n^*$. Hence, $x_n^* = y_n^*$ and, thus, $\lambda_n^* = 0$ for all $n \in N_T$. By linear complementarity,

$$Z_n \cdot \theta_n^* > 0\ ,\ \forall\ n \in N_T\ .$$

Hence, given the assumption of no arbitrage, the initial value of the security's portfolio, $Z_0 \cdot \theta_0^*$, must be strictly positive. That is, in the case of selling Stock Index Options, the optimal strategy is to always invest in the security's portfolio. The reason is that if the manager sells options, he must, at least partially, hedge his position by investing in the security's portfolio; otherwise, he needs to invest in his attempt to beat the benchmark. Therefore, in any case, the hedge fund manager must invest in the security's portfolio.

We now state necessary optimality conditions for the sale of options.

Proposition 2.3.2 *If $\bar{\epsilon}_0^{C,*} > 0$, then $\beta_0\bar{C}_0(1+tc^C) = E^{Q^*}[\beta_T C_T]$, where $Q^* = \left\{\frac{y_n^*}{y_0^*}\right\}_{n\in N_T}$. Analogously, if $\bar{\epsilon}_0^{P,*} > 0$ then,*

$$\beta_0\bar{P}_0(1+tc^P) = E^{Q^*}[\beta_T P_T]\ ,$$

i.e., selling calls or puts on a Stock Index is optimal if these are fairly priced (under Q^*).

<u>Proof:</u> Assume that $\bar{\epsilon}_0^{C,*} > 0$. Then, by linear complementarity,

$$y_0^*\beta_0\bar{C}_0(1+tc^C) = \sum_{n\in N_T}(\beta_n C_n)x_n^*\ . \tag{2.17}$$

Optimality implies $y_0^* = \sum_{n \in N_T} x_n^*$ and, hence, $x_n^* = y_n^*$, $\forall\, n \in\ N_T$. Therefore,

$$\sum_{n \in N_T} (\beta_n C_n) x_n^* = \sum_{n \in N_T} (\beta_n C_n) y_n^* \ . \tag{2.18}$$

Thus, combining equations (2.17) and (2.18) yields $y_0^* \beta_0 \bar{C}_0 (1 + tc^C) = \sum_{n \in N_T} (\beta_n C_n) y_n^*$. Hence,

$$\begin{aligned} \beta_0 \bar{C}_0 (1 + tc^C) &= \textstyle\sum_{n \in N_T} (\beta_n C_n) \left(\frac{y_n^*}{y_0^*} \right) \\ &= E^{Q^*} \left[\beta_T C_T \right] \ . \end{aligned}$$

The case of selling put options is proved in an analogous manner.

Q.E.D.

In summary, if the hedge fund manager is allowed to sell Stock Index Options, then the optimal strategy leads to a percentage (of at least α *%) over the remaining capital derived from subtracting the average value of the Benchmark from the current value of the hedge fund manager's portfolio. Moreover, the optimal strategy will always involve an investment in the security's portfolio and will include the sale of options only if these are fairly priced (with respect to the no-arbitrage price under* Q^**).*

Recall that in this case the optimal strategy for hedge fund managers involves an investment in the security's portfolio as a way of hedging, at least partially, the associated risk with the sale of the Stock Index Option.

Risk Management is discussed in Section 2.7. However, as motivation, we present a simple modification of model (2.14) that considers this issue.

Suppose that the hedge fund manager, who is usually a major investor of the hedge fund, agrees with the investors to consider strategies that could reduce, at least in a proportion δ, the expected payoff of the Stock Index Options that he decides to sell. Therefore, the hedge fund manager considers the *modified* objective function

$$Max_{\theta,\bar{\epsilon}_0^C,\bar{\epsilon}_0^P,s} \ \alpha \left[\sum_{n \in N_T} \beta_n s_n p_n \right] - \delta \left[\sum_{n \in N_T} \beta_n \left(\bar{\epsilon}_0 \cdot V_n \right) p_n \right] , \tag{2.19}$$

where $\sum_{n \in N_T} \beta_n \left(\bar{\epsilon}_0 \cdot V_n \right) p_n$ represents the expected payoff of the sold options and $\delta \in [0,1]$ is the proportion of such payoff that is desired to be minimized. In this case, the manager will sell Stock Index Options if such options are overpriced. Proposition 2.3.3 proves that the overpricing of the Stock Index Options is a necessary optimal condition for the sale of such options and also states by how much these options must be overpriced.

Proposition 2.3.3 *Consider model (2.14) but with the objective function (2.19). Then,* $\bar{\epsilon}_0^{C,*} > 0$ *implies*

$$\beta_0 \bar{C}_0 (1 + tc^C) = E^{Q^*} \left[\beta_T C_T \right] + \left(\frac{\delta}{y_0^*} \right) E^P \left[\beta_T C_T \right] ,$$

where $Q^* = \left\{ \frac{y_n^*}{y_0^*} \right\}_{n \in N_T}$.

Proof: The addition of the term $\delta \left[\sum_{n \in N_T} \left(\bar{\epsilon}_0 \cdot V_n \right) p_n \right]$ to the objective function of model (2.14) implies that the dual restriction,

$$y_0 \beta_0 \bar{C}_0 (1 + tc^C) - \sum_{n \in N_T} x_n \beta_n C_n \leq 0 ,$$

is replaced by $y_0\beta_0\bar{C}_0(1+tc^C) - \sum_{n\in N_T} x_n\beta_n C_n - \delta \sum_{n\in N_T} (\beta_n C_n)\, p_n \leq 0$.

Therefore, $\bar{\epsilon}_0^{C,*} > 0$ implies, by linear complementarity,

$$\beta_0\bar{C}_0(1+tc^C) = \sum_{n\in N_T} \left(\frac{x_n^*}{y_0^*}\right) \beta_n C_n + \left(\frac{\delta}{y_0^*}\right) \left[\sum_{n\in N_T} \beta_n C_n p_n\right] .$$

Recall from the proof of Proposition 2.3.2 that optimality implies $x_n^* = y_n^* \;\; \forall\, n \in N_T$. Hence,

$$\begin{aligned}
\beta_0\bar{C}_0(1+tc^C) &= \textstyle\sum_{n\in N_T} \left(\frac{y_n^*}{y_0^*}\right) \beta_n C_n + \left(\frac{\delta}{y_0^*}\right) \left[\sum_{n\in N_T} \beta_n C_n p_n\right] \\
&= E^{Q^*}\left[\beta_T C_T\right] + \left(\frac{\delta}{y_0^*}\right) E^P\left[\beta_T C_T\right] \; .
\end{aligned}$$

Q.E.D.

This result shows not only how risk management considerations can be included but also how to extract relevant information through the use of the duality framework.

2.3.3 Buying and Selling Options

The analysis of the two previous subsections allow us to study the nature of the optimal strategies for the entire *basic problem* in a straight forward manner. For instance, the optimization *model for the basic problem* is

$$\begin{aligned}
& Max_{\;\theta,\epsilon_0,\bar{\epsilon}_0,s} \;\; \alpha \left[\textstyle\sum_{n\in N_T} \beta_n s_n p_n\right] \\
& s.t. \\
& Z_0 \cdot \theta_0 + \beta_0 \left(\epsilon_0 \cdot V_0 - \bar{\epsilon}_0 \cdot \bar{V}_0\right) = \beta_0 W_0 \\
& Z_n \cdot (\theta_n - \theta_{a(n)}) = 0 \;, \quad \forall\, n \in N_t \;\; \forall\, t = 1, \ldots, \; T \\
& Z_n \cdot \theta_n - \beta_n s_n + \beta_n \left(\epsilon_0 - \bar{\epsilon}_0\right) \cdot V_n = \beta_n B_n \;, \quad \forall\, n \in N_T \\
& Z_n \cdot \theta_n \geq 0, \;\; s_n \geq 0 \;, \; \forall\, n \in N_T \\
& \epsilon_0 \geq \mathbf{0}, \;\; \bar{\epsilon}_0 \geq \mathbf{0} \;,
\end{aligned} \tag{2.20}$$

where $\epsilon_\mathbf{0} \equiv \left[\epsilon_0^C, \epsilon_0^P\right]$, $\bar{\epsilon}_\mathbf{0} \equiv \left[\bar{\epsilon}_0^C,\ \bar{\epsilon}_0^P\right]$, $V_n \equiv [C_n, P_n]\ \forall\ n \in N_T$, and

$$V_0 \equiv \left[(1+tc^C)C_0, (1+tc^P)P_0\right],\ \bar{V}_0 \equiv \left[(1+tc^C)\bar{C}_0, (1+tc^P)\bar{P}_0\right].$$

The dual is

$$\begin{aligned}
&Min_{\ y_0,\mathbf{y},\mathbf{x}}\ \ y_0\beta_0 W_0 - \textstyle\sum_{n\in N_T} x_n\beta_n B_n \\
&s.t. \\
&\alpha p_n - x_n \le 0,\ \forall\ n \in N_T \\
&x_n - y_n \le 0,\ \forall\ n \in N_T \\
&(y_n Z_n - \textstyle\sum_{m\in C(n)} y_m Z_m) = 0,\ \forall\ n \in N_t,\ \forall\ t = 0,\ldots,T-1 \\
&\textstyle\sum_{n\in N_T} x_n\beta_n C_n - y_0\beta_0 C_0(1+tc^C) \le 0 \\
&\textstyle\sum_{n\in N_T} x_n\beta_n P_n - y_0\beta_0 P_0(1+tc^P) \le 0 \\
&y_0\beta_0\bar{C}_0(1+tc^C) - \textstyle\sum_{n\epsilon N_T} x_n\beta_n C_n \le 0 \\
&y_0\beta_0\bar{P}_0(1+tc^P) - \textstyle\sum_{n\in N_T} x_n\beta_n P_n \le 0.
\end{aligned} \tag{2.21}$$

Comparing the dual of the basic problem with the dual problems (2.2) and (2.15), we observe that the dual problem (2.21) has only more restrictions than the other two dual problems. This follows because the model for the basic problem can be derived from any of the primal models (2.1) and (2.14) by simply adding terms into two of their set restrictions (the initial and the final set of restrictions). Therefore, necessary optimality conditions for the purchase and sale of Stock Index Options should be similar to those obtained in Propositions 2.3.1 and 2.3.2 since these conditions are derived from the application of linear complementarity to the dual restrictions shown in

Proposition 2.3.4 *i) Purchase Conditions: If* $\epsilon_0^{C,*} > 0$ $\left(\epsilon_0^{P,*} > 0\right)$*, then*

$$\beta_0 C_0(1+tc^C) \le E^{Q^*}\left[\beta_T C_T\right]\ ,\left(\beta_0 P_0(1+tc^P) \le E^{Q^*}\left[\beta_T P_T\right]\right)\ .$$

ii) Sale Conditions: If $\bar{\epsilon}_0^{C,*} > 0 \ \left(\bar{\epsilon}_0^{P,*} > 0\right)$, *then*

$$\beta_0 \bar{C}_0(1+tc^C) \leq E^{Q^*}\left[\beta_T C_T\right] \ , \left(\beta_0 \bar{P}_0(1+tc^P) \leq E^{Q^*}\left[\beta_T P_T\right]\right) \ .$$

<u>Proof:</u>

i) Assume $\epsilon_0^{C,*} > 0$. By linear complementarity,

$$\beta_0 C_0(1+tc^C) = \sum_{n \in N_T} \left(\frac{x_n^*}{y_0^*}\right) \beta_n C_n \ ,$$

but, $x_n^* \leq y_n^* \ \ \forall \ n \in N_T$. So $\beta_0 C_0(1+tc^C) \leq E^{Q^*}\left[\beta_T C_T\right]$.

ii) Assume $\bar{\epsilon}_0^{C,*} > 0$. By linear complementarity,

$$y_0^* \beta_0 \bar{C}_0(1+tc^C) = \sum_{n \in N_T} (\beta_n C_n) x_n^* \ .$$

Therefore, $\beta_0 \bar{C}_0(1+tc^C) = \sum_{n \in N_T} (\beta_n C_n) \left(\frac{x_n^*}{y_0^*}\right)$.

But, $x_n^* \leq y_n^*$, $\forall \ n \in N_T$. Hence,

$$\beta_0 \bar{C}_0(1+tc^C) \leq \sum_{n \in N_T} (\beta_n C_n) \left(\frac{y_n^*}{y_0^*}\right) = E^{Q^*}\left[\beta_T C_T\right] \ .$$

In general, $\beta_0 \bar{C}_0(1+tc^C) \neq E^{Q^*}\left[\beta_T C_T\right]$, since feasibility of the dual problem implies

$$y_0^* \ = \ Max_{n \in N_T} \left\{ \sum_{n \in N_T} x_n^*, \sum_{n \in N_T} x_n^* \left(\frac{\beta_n}{\beta_0}\right) \left(\frac{C_n}{C_0}\right), \sum_{n \in N_T} x_n^* \left(\frac{\beta_n}{\beta_0}\right) \left(\frac{P_n}{P_0}\right) \right\} \ .$$

Q.E.D.

2.3.4 Feasibility of the Basic Problem

We now establish conditions for feasibility of the optimization problems (2.1), (2.14), and (2.20).

Feasibility of Model (2.1):

Model (2.1) is feasible if and only if the following inequality holds:

$$\begin{aligned}\beta_0 W_0 \;\geq\; & Min_{\;\theta,\epsilon_0}\; Z_0\cdot\theta_0+\beta_0\left(\epsilon_0\cdot V_0\right)\\ & s.t.\\ & Z_n\cdot\left(\theta_n-\theta_{a(n)}\right)=0\;,\quad\forall\; n\in N_t\;,\;\forall\; t=1,\ldots,\;T\\ & Z_n\cdot\theta_n+\beta_n\epsilon_0\cdot V_n=\beta_n B_n,\;\forall\; n\in N_T\\ & \epsilon_0\geq\mathbf{0},\;Z_n\cdot\theta_n\geq 0\;\forall\; n\in N_T.\end{aligned} \tag{2.22}$$

Model (2.1) is feasible if and only if the current value of the hedge fund portfolio, $\beta_0 W_0$, is no less than the minimal amount needed to replicate the benchmark through self-financing strategies derived from the investment in the security's portfolio and the purchase of Stock Index Options.

The dual of the optimization problem on the right hand side of inequality (2.22) is

$$\begin{aligned}& Max_{\;\mathbf{x},\mathbf{q}}\;\textstyle\sum_{n\in N_T} x_n\beta_n B_n\\ & s.t.\\ & q_n Z_n=\textstyle\sum_{m\in C(n)} q_m Z_m,\forall\; n\in N_t\;,\;\forall\; t=1,\ldots,\;T.\\ & q_n\geq 0,\;\forall\; n\in N_T\\ & q_0=1\\ & \beta_0 V_0\leq\textstyle\sum_{n\in N_T} x_n\beta_n V_n\\ & q_n\geq x_n\geq 0,\;\forall\; n\in N_T\;.\end{aligned} \tag{2.23}$$

The first three restrictions imply that $Q \equiv \{q_n\}_{n \in N_T}$ is a martingale measure of $\{Z_t\}_{t=0,\ldots,T}$. Hence, recalling that $\mathcal{Q}$ represents the set of all the martingale measures of $\{Z_t\}_{t=0,\ldots,T}$, then the dual problem (2.23) becomes

$$\begin{aligned}
& Max_{\ \mathbf{x},\mathbf{q}} \ \textstyle\sum_{n \in N_T} x_n \beta_n B_n \\
& s.t. \\
& q_n \geq x_n \geq 0, \ \forall \ n \in N_T \\
& \beta_0 V_0 \leq \textstyle\sum_{n \in N_T} x_n \beta_n V_n \\
& \{q_n\}_{n \in N_T} = Q \subseteq \mathcal{Q}
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
& Max_{\ \mathbf{x},\mathbf{q}} \ \textstyle\sum_{n \in N_T} q_n \beta_n B_n \\
& s.t. \\
& \beta_0 V_0 \leq \textstyle\sum_{n \in N_T} q_n \beta_n V_n \\
& \{q_n\}_{n \in N_T} = Q \subseteq \mathcal{Q}
\end{aligned}$$

Therefore, we have proved

Proposition 2.3.5 *Model (2.1) is feasible if and only if*

$$\begin{aligned}
\beta_0 W_0 \ \geq \ & Max_{Q \in \mathcal{Q}} \ E^Q \left[\beta_T B_T\right] \\
& s.t. \\
& \beta_0 V_0 \leq E^Q \left[\beta_T V_T\right].
\end{aligned} \tag{2.24}$$

That is, Model (2.1) is feasible if and only if the current value of the hedge fund manager's portfolio is no less than the maximum expected value of the (discounted) benchmark over all the martingale measures of the (discounted) price process $\{Z_t\}_{t=0,\ldots,T}$ under which the Stock Index Options are *overpriced* with respect to its current market value V_0.

The feasibility of Model (2.1) requires that the optimization problem on the right hand side of the inequality (2.24) is bounded. Therefore, feasibility of Model (2.1) demands the feasibility of Model (2.24) which needs the existence of a martingale measure of the (discounted) price process, say Q, such that $\beta_0 V_0 \leq E^Q [\beta_T V_T]$.

Feasibility of Model (2.1) implies the existence of a martingale measure Q under which the current market value of the Stock Index Option, V_0, is not greater than the corresponding no-arbitrage price under Q. This condition is consistent with the result of Proposition 2.3.1 in which the purchase of a Stock Index Options is optimal if its current price is not greater than the no-arbitrage price under the specific martingale measure Q^* defined in Proposition 2.3.1.

The analysis of the feasibility of Model (2.14) is completely analogous to the case of Model (2.1). The necessary and sufficient conditions for the feasibility of Model (2.14) are in

Proposition 2.3.6 *Model (2.14) is feasible if and only if*

$$\begin{aligned} \beta_0 W_0 \;\geq\; & Max_{Q \in \mathcal{Q}}\, E^Q [\beta_T B_T] \\ & s.t. \\ & \beta_0 \bar{V}_0 \geq E^Q [\beta_T V_T] . \end{aligned} \tag{2.25}$$

Hence, Model (2.14) is feasible if and only if the current value of the hedge fund manager's portfolio is at least equal to the maximum expected value of the (discounted) benchmark over all the martingale measures of the (discounted) price process $\{Z_t\}_{t=0,\ldots,T}$ under which the Stock Index Options are underpriced with respect to the current market value $\bar{V}_0$.

As with Model (2.1), the feasibility of Model (2.14) requires the existence of a martingale measure Q of the discounted price process $\{Z_t\}_{t=0,\ldots,T}$, under which the current (sale) price of the current value of the Stock Index Option is not less than the corresponding no-arbitrage price under Q. This condition is consistent with the optimal sale condition stated in Proposition 2.3.2.

The analysis of the feasibility of Model (2.20) is analogous and straightforward from the results established in Propositions 2.3.5 and 2.3.6.

Proposition 2.3.7 *Model (2.20) is feasible if and only if*

$$\begin{aligned} \beta_0 W_0 \ \geq \ & Max_{Q \in \mathcal{Q}} \ E^Q \left[\beta_T B_T \right] \\ & s.t. \\ & \beta_0 \bar{V}_0 \geq E^Q \left[\beta_T V_T \right] \geq \beta_0 V_0. \end{aligned} \tag{2.26}$$

Therefore, the basic problem is feasible if and only if the current value of the hedge fund manager's portfolio is at least equal to the maximum expected value of the (discounted) benchmark over all the martingale measures of the (discounted) price process $\{Z_t\}_{t=0,\ldots,T}$ under which the corresponding no-arbitrage price of the Stock Index Options is within the bid-ask price range $[V_0, \bar{V}_0]$.

2.4 Other Utility Functions

We now extend the analysis of the previous section to the more general class of utility functions $\mathcal{U}$ formed by the set of functions that are twice differentiable, concave, and strictly increasing. This class of utility functions is standard to incorporate risk aversion (Pratt (1964)). Second order differentiability assures that $U(\cdot)$ and $U'(\cdot)$ are continuous, and, hence, if the feasible region is bounded, it guarantees that the maximum is achieved.

Let $U(\cdot)$ be a utility function belonging to the class of functions $\mathcal{U}$. Consider the basic problem with

$$\sum_{n \in N_T} U\left(\left[f + \alpha s_n\right] \beta_n\right) p_n$$

replacing $f\left[\sum_{n \in N_T} \beta_n p_n\right] + \alpha\left[\sum_{n \in N_T} \beta_n s_n p_n\right]$ as the objective function. That is, the optimization problem is

$$\begin{array}{ll} Max_{\ \theta, \epsilon_0, \mathbf{s}} & \sum_{n \in N_T} U\left(\left[f + \alpha s_n\right] \beta_n\right) p_n \\ s.t. & \\ Z_0 \cdot \theta_0 + \beta_0 \left(\epsilon_0 \cdot V_0 - \bar{\epsilon}_0 \cdot \bar{V}_0\right) = \beta_0 W_0 & \\ Z_n \cdot \left(\theta_n - \theta_{a(n)}\right) = 0 \ , \ \forall\, n \in N_t \ , \ \forall\, t = 1, \ldots, \ T \ . & \\ Z_n \cdot \theta_n - \beta_n s_n + \beta_n \left(\epsilon_0 - \bar{\epsilon}_0\right) \cdot V_n = \beta_n B_n \ , \ \forall\, n \in N_T & \\ Z_n \cdot \theta_n \geq 0, \ s_n \geq 0 \ , \ \forall\, n \in N_T & \\ \epsilon_0 \geq \mathbf{0}, \ \bar{\epsilon}_0 \geq \mathbf{0} & \end{array} \tag{2.27}$$

The dual of this problem provides us with information about the optimal investment strategies. From standard nonlinear programming theory (e. g., Zangwill (1969), pp. 47 - 52), the dual of (2.27) is

$$\begin{array}{l} Min_{\ y_0, \mathbf{y}, \mathbf{x}} \ \left[\left(y_0 \beta_0 W_0 - \sum_{n \in N_T} x_n \beta_n B_n\right) + \sum_{n \in N_T} \left(U\left(\left[f + \alpha s_n\right] \beta_n\right) - \alpha s_n U'\left(\left[f + \alpha s_n\right] \beta_n\right)\right) p_n\right] \\ s.t. \\ \alpha p_n U'\left(\left[f + \alpha s_n\right] \beta_n\right) - x_n \leq 0, \ \forall\, n \in N_T \\ x_n - y_n \leq 0, \ \forall\, n \in N_T \\ \left(y_n Z_n - \sum_{m \epsilon C(n)} y_m Z_m\right) = 0, \ \forall\, n \in N_t \ , \ \forall\, t = 0, \ldots, T-1 \ . \\ \sum_{n \in N_T} x_n \beta_n V_n - \beta_0 V_0 y_0 \leq \mathbf{0} \\ -\sum_{n \in N_T} x_n \beta_n V_n + \beta_0 \bar{V}_0 y_0 \leq \mathbf{0} \ . \end{array} \tag{2.28}$$

Except for the first restriction, the rest of the dual restrictions do not depend on the utility function U, therefore, most of the results and interpretations developed in Subsection

2.3.3 are preserved. For instance, y_0^* still satisfies the condition,

$$y_0^* = Max\left\{\sum_{n\in N_T} x_n^*, \sum_{n\in N_T} x_n^*\left(\frac{\beta_n}{\beta_0}\right)\left(\frac{C_n}{C_0}\right), \sum_{n\in N_T} x_n^*\left(\frac{\beta_n}{\beta_0}\right)\left(\frac{P_n}{P_0}\right)\right\}, \qquad (2.29)$$

and, hence, y_0^* is still interpreted as an average return. Other important results are also maintained. For example,

$$q_n \equiv \left(\frac{y_n}{y_0}\right), \ \forall\, n \in N_T ,$$

still defines a martingale measure of the (discounted) price process $\{Z_t\}_{t=0,\ldots,T}$. Nonnegativity of y_n is guaranteed since U is strictly increasing. For instance, the first two sets of restrictions of (2.28) imply that

$$y_n \geq x_n \geq \alpha p_n U'\left([f+\alpha s_n]\,\beta_n\right) > 0, \ \forall\, n \in N_T . \qquad (2.30)$$

The dual objective function is the sum of the *dual objective function of the basic problem,*

$$y_0\beta_0 W_0 - \sum_{n\in N_T} x_n\beta_n B_n ,$$

and the nonlinear term $\sum_{n\in N_T}\left(U\left([f+\alpha s_n]\,\beta_n\right) - \alpha s_n U'\left([f+\alpha s_n]\,\beta_n\right)\right)p_n$.

The dual objective function of the the basic problem has a key role in the understanding of the optimal strategies for the hedge fund manager. It tells us that the manager should obtain the *best* return out of the remaining capital derived from subtracting the expected benchmark value, under a specific martingale measure Q^*, from the hedge fund manager's portfolio value. Therefore, martingale measures are crucial for evaluating *appropriately* the

Benchmark. We inherit such importance of the martingale measures since the current dual objective function includes the dual objective function of the basic problem. Hence, it is important to understand the parameters that define such martingale measures. In particular, it is useful to comprehend the way the utility function $U(\cdot)$, the variable fee percentage α, and the *natural* or *given* probabilities $(p_n)_{n\in N_T}$ determine these martingale measures. Equation (2.30), a generalization of equation (2.3), provides understanding of the relationship between such parameters and the martingale measures. Equation (2.30) implies that

$$y_0\, q_n \geq \alpha p_n U'\left([f+\alpha s_n]\,\beta_n\right),\ \forall\, n \in N_T\ . \tag{2.31}$$

Hence,

$$\left\{\alpha p_n U'\left([f+\alpha s_n]\,\beta_n\right)\right\}_{n\in N_T}$$

defines a set of (scaled) lower bounds for the martingale measures $Q \equiv (q_n)_{n\in N_T}$. The value of each of these lower bounds depends on a combination of three factors: α, $(p_n)_{n\in N_T}$, and the marginal utility of hedge fund manager's compensation. This dependency of the lower bounds with respect to these three factors has the following characteristics

- Each of these factors has, *ceteris paribus*, a monotonically increasing relationship with the corresponding lower bounds; that is, the higher the value of any of the factors, the higher the corresponding lower bound value.
- The first two factors, α and $(p_n)_{n\in N_T}$, <u>do not depend</u> on the particular strategy followed by the hedge fund manager.
- Given a feasible strategy, two lower bounds for two different scenarios have the same value if they have equivalent tradeoff between *natural* (or given) probability and marginal

utility of the hedge fund manager's compensation fee. That is, given a feasible strategy (θ, ϵ, s), two scenarios n_i and n_j have the same lower bound if

$$p_{n_i} U' \left([f + \alpha s_{n_i}] \beta_{n_i} \right) = p_{n_j} U' \left([f + \alpha s_{n_j}] \beta_{n_j} \right) .$$

This *tradeoff condition* implies, given the concavity of U, that the larger the surplus on a particular scenario is with respect to the surplus of an equivalent scenario, the higher the probability of occurrence of such a scenario.

Although $y_0 q_n$, for $n \in N_T$, does not necessarily equate its corresponding lower bound, the previous analysis describes the behavior of the forces that determine q_n, $n \in N_T$. The optimal value y_0^* is the *best* return from the remaining capital $\beta_0 W_0 - E^{Q^*} [\beta_T B_T]$.

For the basic problem, y_0^* must be at least α. We now obtain the corresponding lower bound of y_0^* for the current case. Again, we use Equation (2.30) and that $y_0 = \sum_{n \in N_T} y_n$ is also satisfied. Therefore, assuming that $(\theta^*, \epsilon^*, s^*)$ is an optimal solution, then

$$y_0^* \geq \alpha \sum_{n \in N_T} p_n U' \left([f + \alpha s_n^*] \beta_n \right) .$$

The lower bound of y_0^* depends, in general, on the marginal utility of the optimal surplus. If we consider a risk-neutral hedge fund manager, as in the basic problem case, this lower bound reduces to α.

To conclude this section, we state necessary conditions under which it is optimal to invest in options.

Proposition 2.4.1 *(i) If* $\epsilon_0^{C,*} > 0$ $\left(\epsilon_0^{P,*} > 0\right)$, *then*

$$\beta_0 C_0 \left(1 + tc^C\right) \leq E^{Q^*}\left[\beta_T C_T\right] \ , \ \left(\beta_0 P_0 \left(1 + tc^P\right) \leq E^{Q^*}\left[\beta_T P_T\right]\right) \ ,$$

and, if (ii) $\bar{\epsilon}_0^{C,*} > 0$ $\left(\bar{\epsilon}_0^{P,*} > 0\right)$, *then*

$$\beta_0 \bar{C}_0 \left(1 + tc^C\right) \leq E^{Q^*}\left[\beta_T C_T\right] \ , \ \left(\beta_0 \bar{P}_0 \left(1 + tc^P\right) \leq E^{Q^*}\left[\beta_T P_T\right]\right) \ ,$$

where C_0 $(\bar{C}_0)$ *and* P_0 $(\bar{P}_0)$ *denote the Call and Put bid (ask) prices, and* $Q^* \equiv \left\{\frac{y_n^*}{y_0^*}\right\}$ *is a martingale measure for the process* $\{Z_t\}_{t=0,\ldots,T}$, *that is, buying or selling options is optimal if these are not overpriced under* Q^*.

Proof:

(i) Assume $\epsilon_0^{C,*} > 0$. Then, by complementarity,

$$\sum_{n \in N_T} x_n^* \beta_n C_n = \beta_0 C_0 \left(1 + tc^C\right) y_0^* \ .$$

But $y_n^* \geq x_n^*$ $\forall$ $n \in N_T$, hence $\sum_{n \in N_T} y_n^* \beta_n C_n \geq \sum_{n \in N_T} x_n^* \beta_n C_n \geq \beta_0 C_0 \left(1 + tc^C\right) y_0^*$. Therefore,

$$E^{Q^*}\left[\beta_T C_T\right] = \sum_{n \in N_T} q_n^* \beta_n C_n \equiv \sum_{n \in N_T} \frac{y_n^*}{y_0^*} \beta_n C_n \geq \beta_0 C_0 \left(1 + tc^C\right) \ .$$

The proofs of the purchase of Stock Index Options case and analogous.

Q.E.D.

Proposition 2.4.2 proves that if the purchase and the sale of Stock Index Options are optimal in a simultaneous manner, then the sale and the purchase prices must be equal.

Proposition 2.4.2 *(i) If $\epsilon_0^{C,*} > 0$ and $\bar{\epsilon}_0^{C,*} > 0$, then $\bar{C}_0 = C_0$.*

(ii) Analogously, if $\epsilon_0^{P,*} > 0$ and $\bar{\epsilon}_0^{P,*} > 0$, then $\bar{P}_0 = P_0$.

Proof:

(i) Assume $\epsilon_0^{C,*} > 0$ and $\bar{\epsilon}_0^{C,*} > 0$. Then, by complementarity,

$$\beta_0 \bar{C}_0 \left(1 + tc^C\right) = \sum_{n \in N_T} x_n^* \beta_n C_n = \beta_0 C_0 \left(1 + tc^C\right) \ .$$

Q.E.D.

The latter result can also be obtained from the feasibility of (2.28) and the assumption $\bar{V}_0 \geq V_0$. Feasibility of (2.28) implies $\beta_0 \bar{V}_0 y_0 \leq \sum_{n \in N_T} x_n \beta_n V_n \leq \beta_0 V_0 y_0$, which combined with $\bar{V}_0 \geq V_0$ yields $\bar{V}_0 = V_0$.

Next section solves Problem (2.27) for two particular examples to illustrate the risk incentives induced by the manager's compensation scheme, for a specific utility function.

2.5 Risk Incentives

In the previous sections we have characterized, using duality theory, the nature of the optimal strategies followed by the hedge fund manager, when stock index options are included in his investment universe. For instance, we have proved that the purchase or sale of such options is optimal if these are, respectively, underpriced or not overpriced, and established explicit *pricing thresholds* to determine the underpricing or overpricing of these options. Moreover, we have shown that these pricing thresholds depend on the variable fee percentage and the utility function that describes the preferences of the manager.

In this section, we illustrate the connection of the variable fee (α) and the risk aversion level (γ) of the manager with the optimal strategy, in terms of the risk incentives induced by the combination of α and γ. We do this in the following manner. We consider a specific risk averse hedge fund manager that solves Problem (2.27), for particular market and investment conditions, and measure the variability of his compensation with respect to its optimal expected value. The variability of the manager's optimal compensation is directly linked to the variability of the final value of the portfolio. Therefore, the higher the variability of the optimal manager's compensation, the riskier we consider the strategy that leads to such compensation.

Our risk incentives study focuses on two particular cases. First, we analyze the situation in which the manager is only allowed to buy options. Second, we study the more general case in which the manager is permitted to buy and sell options. In both cases, we study the risk incentives induced for different degrees of risk aversion and distinct variable compensation fees.

2.5.1 Buying Options

Consider the static one-period horizon problem described in Example 1 of Section 2.3.1, but we now suppose that the hedge fund manager is risk averse and his preferences are described by the utility function

$$U(x) = \frac{x^{1-\gamma}}{1-\gamma}$$

where $\gamma > 0$ represents his level of risk aversion and U belongs to the class of utility functions $\mathcal{U}$, described in Section 2.4. That is, the hedge fund manager solves the problem

$$
\begin{aligned}
& Max_{\ \theta,\epsilon_0,s} \quad \frac{\left([f+\alpha s_{n_1}]\beta_{n_1}\right)^{1-\gamma}}{1-\gamma} p_{n_1} + \frac{\left([f+\alpha s_{n_2}]\beta_{n_2}\right)^{1-\gamma}}{1-\gamma} p_{n_2} \\
& s.t. \\
& \theta_0^0 + Z_0^1\theta_0^1 + \beta_0\epsilon_0^C C_0 + \beta_0\epsilon_0^P P_0 = \beta_0 W_0 \\
& \left(\theta_{n_1}^0 - \theta_0^0\right) + Z_{n_1}^1\left(\theta_{n_1}^1 - \theta_0^1\right) = 0 \\
& \left(\theta_{n_2}^0 - \theta_0^0\right) + Z_{n_2}^1\left(\theta_{n_2}^1 - \theta_0^1\right) = 0 \\
& \theta_{n_1}^0 + Z_{n_1}^1\theta_{n_1}^1 - \beta_{n_1}s_{n_1} + \beta_{n_1}\epsilon_0^C C_{n_1} + \beta_{n_1}\epsilon_0^P P_{n_1} = \beta_{n_1}B_{n_1} \\
& \theta_{n_2}^0 + Z_{n_2}^1\theta_{n_2}^1 - \beta_{n_2}s_{n_2} + \beta_{n_2}\epsilon_0^C C_{n_2} + \beta_{n_2}\epsilon_0^P P_{n_2} = \beta_{n_2}B_{n_2} \\
& \theta_{n_1}^0 + Z_{n_1}^1\theta_{n_1}^1 \geq 0,\ \theta_{n_2}^0 + Z_{n_2}^1\theta_{n_2}^1 \geq 0 \\
& \epsilon_0^C \geq 0,\ \epsilon_0^P \geq 0\ , \\
& s_{n_1} \geq 0,\ s_{n_2} \geq 0\ .
\end{aligned}
\tag{2.32}
$$

where $W_0 = 500$, and

Variable	Value at $t = 0$	Value in scenario n_1	Value in scenario n_2
S^1	90	108	67.5
S^0	10	10.5	10.5
SI	100	115	70
B	-	585	360
C	25.926	35	0
P	2.116	0	10

where $\beta_{n_j} = (S_{n_j}^0)^{-1} = \left(S_0^0(1+r)\right)^{-1}$, $j = 1, 2$; $r = 5\%$ is the one-period (fixed) interest rate, and the fixed salary f is assumed to be 0.5% of the current fund capital ($W_0 = 500$), i.e. $f = 2.5$.

We solved Problem (2.32) for different levels γ of risk aversion and distinct values of the variable fee percentage α. For all the cases that we consider, the optimal investment strategy followed by the hedge fund manager is the same:

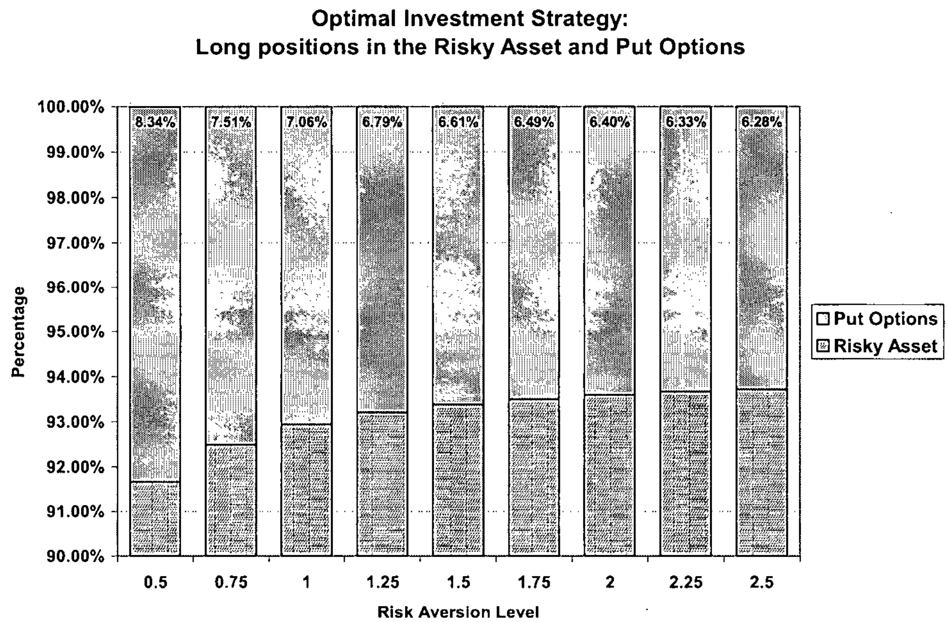

Figure 2.1: **Optimal Strategy**. This graph shows the optimal long positions in the risky security and the put options, as a percentage of the sum of the amount borrowed and the initial capital W_0.

- Borrow money from the bank, at an interest rate r,
- Take a long position in the risky asset S^1, and
- Buy put options.

This strategy is somewhat natural. This is because, on one hand, the expected returns of the risky asset and a long position in put options are both larger than the interest rate r and, on the other hand, the expected return of a long position in call options is significantly smaller than a long position in put options. Figure 2.1 shows the optimal long positions in the risky security and the put options, as a percentage of the sum of the amount borrowed and the initial capital W_0, for different levels of risk aversion. As it can be observed, although both percentages are relatively stable across the levels of risk aversion considered, the higher the manager's risk aversion, the smaller the percentage invested in put options.

In order to study the risk incentives induced by the compensation scheme as γ and α vary, we use the standard deviation of the hedge fund manager's optimal compensation,

$$\sigma^* = \sqrt{\left[f + \alpha s^*_{n_1} - \mu^*\right]^2 p_{n_1} + \left[f + \alpha s^*_{n_2} - \mu^*\right]^2 p_{n_2}} \ ,$$

where $s^*_{n_i}$ is the optimal surplus value for scenario $i = 1, 2$ and

$$\mu^* = \left[f + \alpha s^*_{n_1}\right] p_{n_1} + \left[f + \alpha s^*_{n_2}\right] p_{n_2} \ ,$$

as a measure of risk.

We analyze the effect of the risk aversion level by solving Problem (2.32) for different values of γ, given a fixed variable fee percentage α of 20%. Figure 2.2 shows the graph of σ^*, as a function of γ, for the values of γ considered. From this graph we can observe that the higher the risk aversion level, the smaller the risk incentives induced by the compensation scheme. Moreover, this behavior holds when $\sigma^*(\gamma)$ is scaled by the corresponding expected compensation $\mu^*(\gamma)$. This means, in particular, that the higher the risk aversion, the higher the expected compensation per unit of risk. In other words, as the manager's risk aversion increases, he tends to target for a more efficient mean-risk relationship.

We study the impact of the variable fee percentage α in the same manner that we analyzed the effect of the risk aversion level. That is, we solved Problem (2.32) for different values of α, given a fixed risk aversion level γ. Figure 2.3 plots several curves of σ^*, as a function of α, for different values of γ. From these curves, we conclude that (i) managers with the same risk aversion will have higher risk incentives for higher variable fee percentages α,

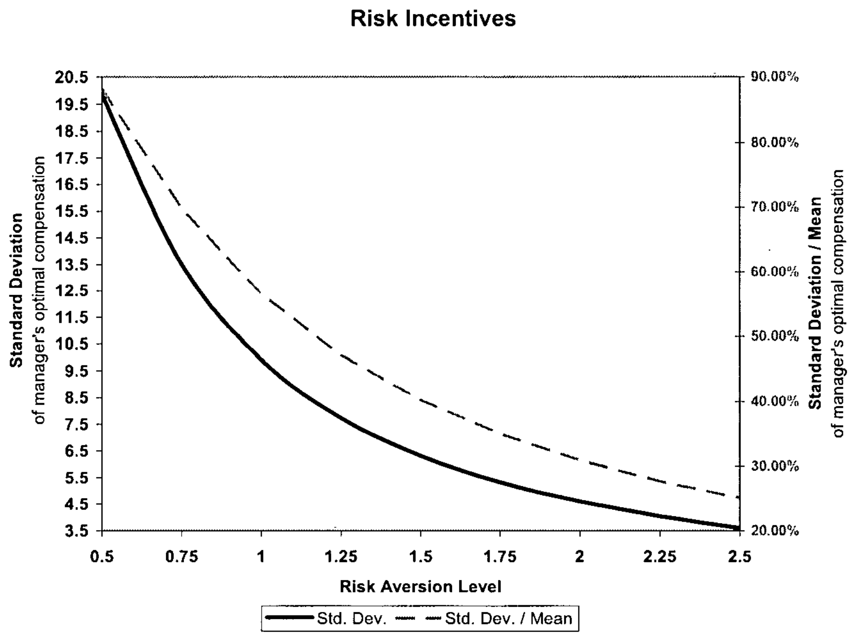

Figure 2.2: **Risk Aversion Effect**. This graph shows the effect of the risk aversion level on the risk induced by the compensation scheme.

and (ii) given a fixed α, the higher the the risk aversion level, the smaller the risk incentives induced by the compensation scheme.

2.5.2 Buying and Selling Options

Consider the example developed in 2.5.1, but we now allow the manager to sell options too. Therefore, in this case the hedge fund manager solves for Problem (2.32) but using

$$\theta_0^0 + Z_0^1\theta_0^1 + \beta_0\left(\epsilon_0^C - \bar{\epsilon}_0^C\right)C_0 + \beta_0\left(\epsilon_0^P - \bar{\epsilon}_0^P\right)P_0 = \beta_0 W_0$$

as the initial budget constraint and

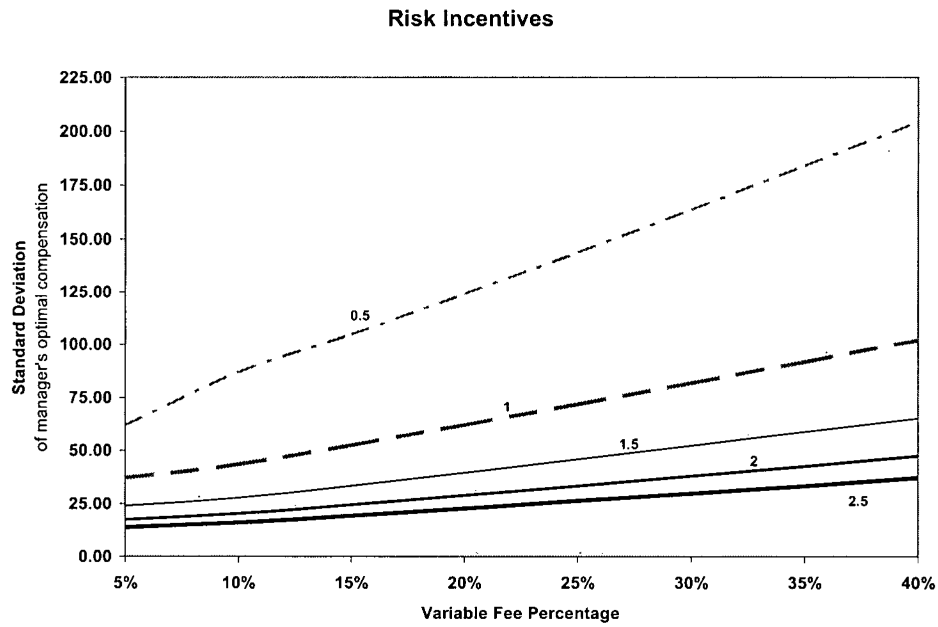

Figure 2.3: **Variable Fee Effect**. This graph shows the effect of the variable fee percentage on the risk induced by the compensation scheme, for different levels of risk aversion.

$$
\begin{aligned}
\theta^0_{n_1} + Z^1_{n_1}\theta^1_{n_1} - \beta_{n_1}s_{n_1} + \beta_{n_1}\left(\epsilon^C_0 - \bar{\epsilon}^C_0\right)C_{n_1} + \beta_{n_1}\left(\epsilon^P_0 - \bar{\epsilon}^P_0\right)P_{n_1} &= \beta_{n_1}B_{n_1} \\
\theta^0_{n_2} + Z^1_{n_2}\theta^1_{n_2} - \beta_{n_2}s_{n_2} + \beta_{n_2}\left(\epsilon^C_0 - \bar{\epsilon}^C_0\right)C_{n_2} + \beta_{n_2}\left(\epsilon^P_0 - \bar{\epsilon}^P_0\right)P_{n_2} &= \beta_{n_2}B_{n_2}
\end{aligned}
$$

as the surplus constraints.

We solved the modified optimization problem for different levels γ of risk aversion and distinct values of the variable fee percentage α. For any combination of (γ, α) considered, the optimal strategy is the following:

- Borrow money from the bank, at an interest rate r,
- Take a long position in the risky asset S^1, and

- Buy put options.

- Sell call options.

The unique difference between the case in which the only the purchase of options is allowed and the current case is that now the manager also sells call options. This is because (i) taking a long position in put options is significantly more profitable than taking a long position in call options, (ii) the sale of call options can be *hedged* by taking a long position in the risky asset, and (iii) the risky asset generates a higher expected return than the interest rate r. Therefore, the manager *borrows* extra money by selling call options, and uses this money to buy more put options and to take a longer position in the risky asset. The latter, in order to hedge the sale of call options.

The risk incentives results are similar to those obtained in 2.5.1. That is, managers with the same risk aversion will have higher risk incentives for higher variable fee percentages α, and given a fixed α, the higher the the risk aversion level, the smaller the risk incentives induced by the compensation scheme. However, σ^* values are significantly higher than in previous case. These results are expected since the manager has now a broader set of investment possibilities, which implies that the optimal strategy leads to a higher expected compensation and thus, to a higher risk too.

We compare the risk incentives results with those of 2.5.1 by scaling the standard deviations of the manager's compensation by its corresponding mean. For instance, Figure 2.4 shows σ^*/μ^* for different values of γ. It can be observed that, in relative terms, the manager takes higher risks when he is allowed to buy and sell options.

Chapter 3 of this Ph.D. Thesis treats in depth, among other issues, the risk incentives induced by more general option-like compensation schemes.

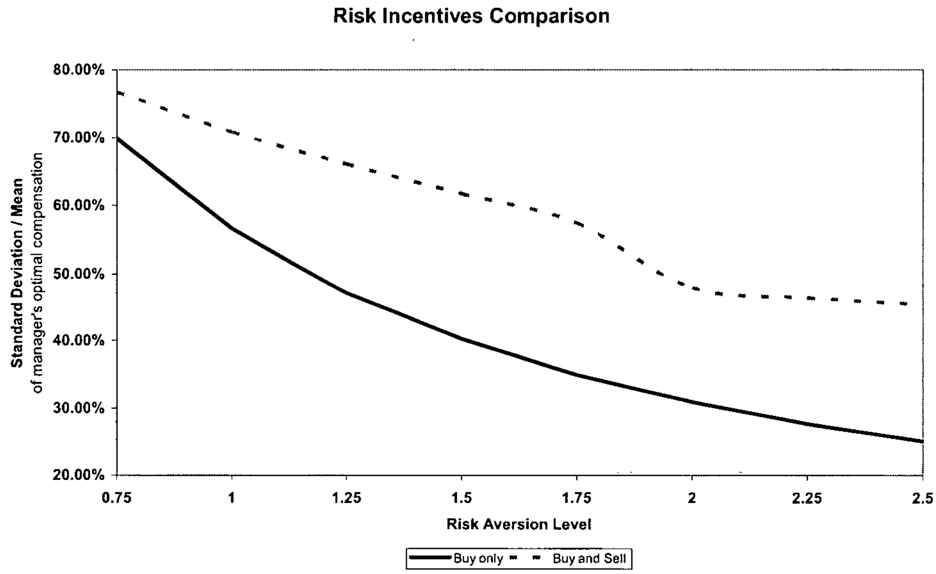

Figure 2.4: **Risk Aversion Effect: Buying versus Buying-and-Selling.** This graph compares the effect of the risk aversion level on the risk induced by the compensation scheme, for the two sets of investment conditions considered in this section.

2.6 Multiple Monitoring Dates

All the models considered in the previous sections assume that the performance of the hedge fund manager is measured, or monitored, at the end of the planning horizon. In this section, we relax this assumption and study the nature of the optimal strategies when the hedge fund manager is faced with multiple dates in which his performance, and hence his compensation, is evaluated. It is shown that the nature of such optimal strategies depends on the mechanism or policy to determine the benchmark. We focus on two policies: Fixed or Stock Index Based Benchmarks and *High Water Marks* Benchmarks. The first policy refers to the common practice of determining the benchmark based on a preestablished fixed return (such as zero) or on a Stock Index (e.g., return obtained by S&P500 over the planning horizon). In the second policy the benchmark is the maximum historical value of the portfolio. We proceed in two phases. In phase one, we study the case of two monitoring dates, and, then, we study the general case in phase two.

2.6.1 Two-Period Monitoring Case

Consider two monitoring dates T_1 and T_2, where $T_1 < T_2$, in which the hedge fund portfolio's value is measured against its benchmark. Suppose that the hedge fund manager follows self-financing strategies between monitoring dates and that a proportion q (with $q \leq 100\%$) of the profit over the benchmark is subtracted from the hedge fund portfolio's value at T_1. This amount should cover *at least* the variable fee of the hedge fund manager, i.e. $q \geq \alpha$, and it could possibly include some revenue payments to the investors. We further assume that the rebalancing of the investment positions does not occur at T_1 but a period later, $T_1 + 1$. Finally, we consider a utility function U that is time-additive, i. e.

$$U\left(\beta_{T_1}[f + \alpha s_{T_1}], \beta_{T_2}[f + \alpha s_{T_2}]\right) = U_1\left(\beta_{T_1}[f + \alpha s_{T_1}]\right) + U_2\left(\beta_{T_2}[f + \alpha s_{T_2}]\right) \ ,$$

where each U_i $(i = 1, 2)$ belongs to the class of functions $\mathcal{U}$ that is defined in the previous section.

Fixed or Stock Index Based Benchmarks

We assume that the benchmark that applies is either fixed (e. g. 0 %) or based on a Stock Index (e. g. 2 % above a certain Stock Index return). The model in this case is

$$
\begin{array}{l}
Max_{\ \theta,\epsilon_0,s} \quad \left[\sum_{n\in N_{T_1}} U_1\left([f+\alpha s_n^1]\beta_n\right)p_n + \sum_{n\in N_{T_2}} U_2\left([f+\alpha s_n^2]\beta_n\right)p_n\right] \\
s.t. \\
Z_0\cdot\theta_0 + \beta_0\left(\epsilon_0\cdot V_0 - \bar{\epsilon}_0\cdot\bar{V}_0\right) = \beta_0 W_0 \\
Z_n\cdot\left(\theta_n - \theta_{a(n)}\right) = 0\ ,\ \forall\, n\in\ N_t\ \ \forall\, t=1,\ldots,\ T_1 \\
Z_n\cdot\theta_n - \beta_n s_n^1 + \beta_n\left(\epsilon_0 - \bar{\epsilon}_0\right)\cdot V_n^0 = \beta_n B_n^1\ ,\ \forall\, n\in N_{T_1} \\
Z_n\cdot\theta_n + \beta_n\left(\epsilon_{T_1+1}\cdot V_{T_1+1,n} - \bar{\epsilon}_{T_1+1}\cdot\bar{V}_{T_1+1,n}\right) - \\
\left[Z_n\cdot\theta_{a(n)} + \beta_n\left(\epsilon_0-\bar{\epsilon}_0\right)\cdot V_{a(n)}^0\left(\frac{\beta_{a(n)}}{\beta_n}\right) - q\beta_n s_{a(n)}^1\left(\frac{\beta_{a(n)}}{\beta_n}\right)\right] = 0,\ \forall\, n\in N_{T_1+1} \\
Z_n\cdot\left(\theta_n - \theta_{a(n)}\right) = 0\ ,\ \ \forall\, n\in\ N_t\ \ \forall\, t=T_1+2,\ldots,\ T_2 \\
Z_n\cdot\theta_n - \beta_n s_n^2 + \beta_n\left(\epsilon_{T_1+1} - \bar{\epsilon}_{T_1+1}\right)\cdot V_n^{T_1+1} = \beta_n B_n^2\ ,\ \ \forall\, n\in N_{T_2} \\
Z_n\cdot\theta_n \geq 0,\ s_n^1\geq 0\ ,\ \forall\, n\in\ N_{T_1} \\
Z_n\cdot\theta_n \geq 0,\ s_n^2\geq 0\ ,\ \forall\, n\in N_{T_2} \\
\epsilon_0\geq\mathbf{0},\ \bar{\epsilon}_0\geq\mathbf{0},\ \epsilon_{T_1}\geq\mathbf{0},\ \bar{\epsilon}_{T_1}\geq\mathbf{0}\ ,
\end{array}
\tag{2.33}
$$

where

$V_{t,n}\equiv\left[C_{t,n},P_{t,n}\right]$: Purchase prices in scenario n, $n\in N_t$.
$\bar{V}_{t,n}\equiv\left[\bar{C}_{t,n},\bar{P}_{t,n}\right]$: Sale prices in scenario n, $n\in N_t$.
$V_n^t\equiv\left[C_n^t,P_n^t\right]$: Payoff in the event n, $n\in N_t$, of an option purchased at time t.
$\epsilon_{\mathbf{t}}\equiv\left[\epsilon_t^C,\epsilon_t^P\right]$: Amount of options purchased at time t, where $t=0,T_1+1$.
$\bar{\epsilon}_{\mathbf{t}}\equiv\left[\bar{\epsilon}_t^C,\bar{\epsilon}_t^P\right]$: Amount of options sold at time t, where $t=T_1,...,T_2-1$.

The dual is

$$
\begin{aligned}
&Min_{\ y_0,\mathbf{y},\mathbf{x}}\ \ y_0^1\beta_0 W_0 - \textstyle\sum_{n\in N_{T_1}} x_n^1\beta_n B_n^1 - \sum_{n\in N_{T_2}} x_n^2\beta_n B_n^2 \\
&+\textstyle\sum_{n\in N_{T_1}} \left(U_1\left([f+\alpha s_n^1]\beta_n\right) - \alpha s_n^1 U_1'\left([f+\alpha s_n^1]\beta_n\right)\right) p_n \\
&+\textstyle\sum_{n\in N_{T_2}} \left(U_2\left([f+\alpha s_n^2]\beta_n\right) - \alpha s_n^2 U_2'\left([f+\alpha s_n^2]\beta_n\right)\right) p_n \\
&s.t. \\
&\alpha p_n U_1'\left([f+\alpha s_n^1]\beta_n\right) - x_n^1 - q y_n^2 \le 0,\ \forall\ n\in N_{T_1} \\
&x_n^1 - y_n^1 + y_n^2 \le 0,\ \forall\ n\in\ N_{T_1} \\
&(y_n^1 Z_n - \textstyle\sum_{m\epsilon C(n)} y_m^1 Z_m) = 0,\ \forall\ n\in N_t,\ \forall\ t=0,\ldots,T_1-1 \\
&\textstyle\sum_{n\in N_{T_1}} x_n^1\beta_n V_n^0 - \beta_0 V_0 y_0^1 + \sum_{n\in N_{T_1}} y_n^2\beta_n V_n^0 \le 0 \\
&\textstyle\sum_{n\in N_{T_2}} x_n^2\beta_n V_n^{T_1+1} - \sum_{n\in N_{T_1+1}} y_n^2\beta_n V_{T_1+1,n} \le 0 \\
&-\textstyle\sum_{n\in N_{T_1}} x_n^1\beta_n V_n^0 + \beta_0\bar{V}_0 y_0^1 - \sum_{n\in N_{T_1}} y_n^2\beta_n V_n^0 \le 0 \\
&-\textstyle\sum_{n\in N_{T_2}} x_n^2\beta_n V_n^{T_1+1} + \sum_{n\in N_{T_1+1}} y_n^2\beta_n \bar{V}_{T_1+1,n} \le 0 \\
&\alpha p_n U_2'\left([f+\alpha s_n^2]\beta_n\right) - x_n^2 \le 0,\ \forall\ n\in N_{T_2} \\
&x_n^2 - y_n^2 \le 0,\ \forall\ n\in N_{T_2} \\
&(y_n^2 Z_n - \textstyle\sum_{m\in C(n)} y_m^2 Z_m) = 0,\ \forall\ n\in N_t,\ \forall\ t=T_1,\ldots,T_2-1\ ,
\end{aligned}
\tag{2.34}
$$

where $y_n^2 \equiv \sum_{m\in C(n)} y_m^2,\ \forall\ n\in N_{T_1}$.

The dual yields necessary conditions under which it is optimal to invest in options at $t=0$ and $t=T_1+1$.

Proposition 2.6.1 (Optimal Option Investment at $t=0$)

(i) If $\epsilon_0^{C,*} > 0$ $\left(\epsilon_0^{P,*} > 0\right)$, *then* $\beta_0 C_0 \le E^{Q_1^*}\left[\beta_{T_1} C_{T_1}^0\right]$ $\left(\beta_0 P_0 \le E^{Q_1^*}\left[\beta_{T_1} P_{T_1}^0\right]\right)$.

(ii) If $\bar{\epsilon}_0^{C,*} > 0$ $\left(\bar{\epsilon}_0^{P,*} > 0\right)$, *then* $\beta_0 \bar{C}_0 \le E^{Q_1^*}\left[\beta_{T_1} C_{T_1}^0\right]$ $\left(\beta_0 \bar{P}_0 \le E^{Q_1^*}\left[\beta_{T_1} P_{T_1}^0\right]\right)$.

where $Q_1^* \equiv \left\{\frac{(y_n^1)^*}{(y_0^1)^*}\right\}_{n\in N_{T_1}}$ *is a martingale measure for the process* $\{Z_t\}_{t=0,\ldots,T_1}$, *and* $E^{Q_1^*}[\,\cdot\,]$ *denotes the expected value operator under* Q_1^*, *and based (conditional) on the information at* $t=0$.

Proof: (i) Assume $\epsilon_0^{C,*} > 0$. Then, by complementarity,

$$\beta_0 C_0 \left(y_0^1\right)^* = \sum_{n \in N_{T_1}} \left(\left(x_n^1\right)^* + \left(y_n^2\right)^*\right) \beta_n C_n^0 \, .$$

From the first two sets of dual restrictions

$$y_n^1 \geq x_n^1 + y_n^2 \geq x_n^1 + q y_n^2 \geq \alpha p_n U_1^{'} \left(f + \alpha s_n^1\right) > 0 \ , \ \forall \, n \in N_{T_1} \ ,$$

where the strict inequality follows from the fact that $U(\cdot)$ is strictly increasing and differentiable. From the third set of restrictions,

$$y_0^1 = \sum_{n \in N_{T_1}} y_n^1 \, .$$

Hence, $Q_1^* \equiv \left\{ \frac{\left(y_n^1\right)^*}{\left(y_0^1\right)^*} \right\}_{n \in N_{T_1}}$ is a probability measure. Therefore, from the fourth set of dual restrictions, we obtain

$$\beta_0 C_0 \leq \sum_{n \in N_{T_1}} \left(\frac{\left(y_n^1\right)^*}{\left(y_0^1\right)^*} \right) \beta_n C_n^0 = E^{Q_1^*} \left[\beta_{T_1} C_{T_1}^0\right] \ .$$

The martingale property of Q_1^* comes from the third set of restrictions in (2.34).

(ii) Assume $\bar{\epsilon}_0^{C,*} > 0$. By complementarity

$$\beta_0 \bar{C}_0 \left(y_0^1\right)^* = \sum_{n \in N_{T_1}} \left(\left(x_n^1\right)^* + \left(y_n^2\right)^*\right) \beta_n C_n^0$$

Therefore, by the properties of y_0^1 and y_n^1 discussed in (i),

$$\beta_0 \bar{C}_0 \leq \sum_{n \in N_{T_1}} \left(\frac{(y_n^1)^*}{(y_0^1)^*} \right) \beta_n C_n^0 = E^{Q_1^*} \left[\beta_{T_1} C_{T_1}^0 \right] \ .$$

Q.E.D.

Proposition 2.6.2 (Optimal Option Investment at $t = T_1 + 1$)

(i) If $\epsilon_{T_1+1}^{C,*} > 0$ $\left(\epsilon_{T_1+1}^{P,*} > 0 \right)$, *then*

$$E^{Q_2^*} \left[\beta_{T_2} C_{T_2}^{T_1+1} \right] \geq E^{Q_2^*} \left[\beta_{T_1+1} C_{T_1+1} \right] \ , \ \left(E^{Q_2^*} \left[\beta_{T_2} P_{T_2}^{T_1+1} \right] \geq E^{Q_2^*} \left[\beta_{T_1+1} P_{T_1+1} \right] \right) \ ,$$

and, if (ii) $\bar{\epsilon}_{T_1+1}^{C,*} > 0$ $\left(\bar{\epsilon}_{T_1+1}^{P,*} > 0 \right)$, *then*

$$E^{Q_2^*} \left[\beta_{T_2} C_{T_2}^{T_1+1} \right] \geq E^{Q_2^*} \left[\beta_{T_1+1} \bar{C}_{T_1+1} \right] \ , \ \left(E^{Q_2^*} \left[\beta_{T_2} P_{T_2}^{T_1+1} \right] \geq E^{Q_2^*} \left[\beta_{T_1+1} \bar{P}_{T_1+1} \right] \right) \ ,$$

where $Q_2^* \equiv \left\{ \frac{(y_n^2)^*}{(y_0^2)^*} \right\}_{n \in N_{T_2}}$ *is a martingale measure for the process* $\{Z_t\}_{t=T_1+1,\dots,T_2}$, *and* y_0^2 *is defined as*

$$y_0^2 \equiv \sum_{n \in N_{T_1+1}} y_n^2 \ .$$

$E^{Q_2^*}[\ \cdot\ ]$ *denotes the expected value operator under* Q_2^* *and based (conditional) on the information at* $t = 0$.

<u>Proof:</u> (i) Assume $\epsilon_{T_1+1}^{C,*} > 0$. Then, by the application of the complementarity property to the fifth set of dual restrictions

$$\sum_{n\in N_{T_2}} \left(x_n^2\right)^* \beta_n C_n^{T_1+1} = \sum_{n\in N_{T_1+1}} \left(y_n^2\right)^* \beta_n C_{T_1+1,n} \ .$$

From second and third to last set of dual restrictions

$$y_n^2 \geq x_n^2 \geq \alpha p_n U_2' \left(f + \alpha s_n^2\right) > 0 \ , \ \forall \ n \in N_{T_2} \ ,$$

where the strict inequality follows from the assumption that U is strictly increasing and differentiable. Therefore,

$$\sum_{n\in N_{T_2}} \left(y_n^2\right)^* \beta_n C_n^{T_1+1} \geq \sum_{n\in N_{T_1+1}} \left(y_n^2\right)^* \beta_n C_{T_1+1,n} \ . \tag{2.35}$$

and $Q_2^* \equiv \left\{ \frac{\left(y_n^2\right)^*}{\left(y_0^2\right)^*} \right\}_{n\in N_{T_2}}$ defines a probability measure. Dividing both sides of (2.35) by $\left(y_0^2\right)^*$ yields

$$E^{Q_2^*}\left[\beta_{T_2} C_{T_2}^{T_1+1}\right] \geq E^{Q_2^*}\left[\beta_{T_1+1} C_{T_1+1}\right] \ .$$

The martingale property of Q_2^* comes from the last set of restrictions in (2.34).

(ii) The proof is analogous to (i).

Q.E.D.

Propositions 2.6.1 and 2.6.2 state that the purchase and sale of options at $t = 0$ and $t = T_1 + 1$ is optimal if these options are not overpriced under Q_1^* and Q_2^*, respectively. The conditions of Proposition 2.6.2 are a sort of forward-looking conditions.

In addition to the optimal investment conditions in terms of the embedded martingale measures Q_1^* and Q_2^*, the dual problem (2.34) provides us with other insights. For instance, the inequalities

$$y_n^1 \geq x_n^1 + y_n^2 \geq x_n^1 + qy_n^2 \geq \alpha p_n U_1' \left([f + \alpha s_n^1] \beta_n \right) \ , \ \forall \ n \in N_{T_1} \ , \tag{2.36}$$

and

$$y_n^2 \geq x_n^2 \geq \alpha p_n U_2' \left([f + \alpha s_n^2] \beta_n \right) \ \forall \ n \in N_{T_2} \ , \tag{2.37}$$

imply that

$$\begin{array}{rcccl} y_0^1 q_n^1 & = & y_n^1 & \geq & \alpha p_n U_1' \left([f + \alpha s_n^1] \beta_n \right) \ , \ \forall \ n \in N_{T_1} \\ y_0^2 q_n^2 & = & y_n^2 & \geq & \alpha p_n U_2' \left([f + \alpha s_n^2] \beta_n \right) \ , \ \forall \ n \in N_{T_2} \end{array} \tag{2.38}$$

This set of inequalities establishes (scaled) lower bounds for the martingale measures Q_1^* and Q_2^* which are of the same form as those in Section 2.4 for the single-period monitoring case. The probability value that the martingale measure Q_i^* assigns to each scenario depends on the natural or given probability and the marginal utility under the utility function U_i, $(i = 1, 2)$, of the profit over the benchmark for such scenario. In particular, observe that the Q_1^* probability values at T_1 do not depend on then the utility gained over the fee at monitoring date T_2. Therefore, the Q_1^* probabilities at T_1 are, in a certain sense, independent of the future scenarios at T_2. This does not occur for *high water marks.*

High Water Marks

At the beginning of the time horizon $(t = 0)$ there is a preestablished benchmark B^1 for the first monitoring date T_1. For the second monitoring date T_2, the corresponding benchmark,

B^2, is determined through the policy of *high water marks.* Hence, B^2 depend upon B^1 and the (positive) profit obtained with respect to the benchmark, if any. Therefore, we add

$$\begin{aligned} -\beta_n B_n^2 + \beta_n B_n^1 + (1-\alpha)\beta_n s_n^1 &= 0, \ \forall\, n \in N_{T_1} \\ \beta_n B_n^2 - \beta_{a(n)} B_{a(n)}^2 &= 0, \ \forall\, n \in N_t \ , \ \forall\, t = T_1 + 1, \ldots, \ T_2 \ , \end{aligned} \tag{2.39}$$

to formulation (2.33) to model such a benchmark determination policy. The first set of restrictions in (2.39) models the benchmark update at time T_1 by adding the adjusted profit $(1-\alpha)s^1$ (after subtracting the incentive fee paid to the hedge fund manager) to the benchmark B^1. The second set of restrictions *rolls over* the updated benchmark value at T_1 until time T_2 where the hedge fund portfolios's value is measured against its benchmark again. Then, (2.33) and (2.39) implies:

- the definition of additional dual variables (l_n) that satisfy

$$l_n - \sum_{m \in C(n)} l_m = 0 \ , \ \forall\, n \in N_t \ , \quad t = T_1, \ldots, T_2 - 1 \ , \tag{2.40}$$

$$-x_n^2 + l_n = 0 \ , \quad \forall\, n \in N_{T_2} \ , \tag{2.41}$$

- the replacement of the first set of restrictions in (2.34)

$$\alpha p_n U_1' \left([f + \alpha s_n^1] \beta_n \right) - x_n^1 - q y_n^2 \leq 0 \ , \ \forall\, n \in N_{T_1}$$

by

$$\alpha p_n U_1' \left([f + \alpha s_n^1] \beta_n \right) - x_n^1 - q y_n^2 + (1-\alpha) l_n \leq 0 \ , \ \forall\, n \in N_{T_1} \ , \tag{2.42}$$

- and the addition of the term $\sum_{n\in N_{T_2}} l_n\beta_n B_n^1$ to the dual objective function of (2.34).

Restriction (2.42) expresses the optimal investment conditions in terms of implicit martingale measures and defining (scaled) lower bounds for these measures. So

$$\alpha p_n U_1'([f+\alpha s_n^1]\beta_n) + (1-\alpha)l_n \leq x_n^1 + q y_n^2 \ .$$

But, $x_n^1 + qy_n^2 \leq x_n^1 + y_n^2 \leq y_n^1$, $\forall\ n \in N_{T_1}$. The fact that $U_1'(\cdot)$ is strictly increasing, together with (2.40), (2.41), and $x_n^2 \geq \alpha p_n U_2'([f+\alpha s_n^2]\beta_n)$ yields

$$0 < \alpha p_n U_1'([f+\alpha s_n^1]\beta_n) + (1-\alpha)l_n \ , \ \forall\ n \in N_{T_1} \ .$$

Therefore,

$$0 < \alpha p_n U_1'([f+\alpha s_n^1]\beta_n) + (1-\alpha)l_n \leq x_n^1 + q y_n^2 \leq y_n^1 \ . \tag{2.43}$$

Hence, $Q_1 \equiv \left\{\frac{y_n^1}{y_0^1}\right\}_{n\in N_{T_1}}$ defines a probability measure.

Propositions 2.6.1 and 2.6.2 hold for the *high water marks* policy, although with different embedded martingale measures Q_1^* and Q_2^*; Q_1^* has different lower bounds than the corresponding ones for the case of fixed benchmarks (see equation (2.38)). To see this, observe that if we apply Equation (2.40) repeatedly

$$l_n = \sum_{m\in D^{T_2}(n)} l_m \ ,$$

where $D^{T_2}(n)$ is the set of *descents* of node n by time T_2. Then, by equations (2.41) and (2.37),

$$\sum_{m \in D^{T_2}(n)} l_m = \sum_{m \in D^{T_2}(n)} x_m^2 \geq \sum_{m \in D^{T_2}(n)} \alpha p_m U_2'([f + \alpha s_m^2]\beta_m) .$$

Therefore, $l_n \geq \sum_{m \in D^{T_2}(n)} \alpha p_m U_2'([f + \alpha s_m^2]\beta_m)$. Hence, using (2.43), Q_1^* is bounded from below by

$$\left\{ \alpha p_n + (1-\alpha) \sum_{m \in D^{T_2}(n)} \alpha p_m U_2'([f + \alpha s_m^2]\beta_m) \right\}_{n \in N_{T_1}} , \tag{2.44}$$

i.e. ,

$$y_n^{1,*} = y_0^1 q_n^{1,*} \geq \alpha p_n U_1'\left([f + \alpha s_n^1]\beta_n\right) + (1-\alpha) \left(\sum_{m \in D^{T_2}(n)} \alpha p_m U_2'([f + \alpha s_m^2]\beta_m) \right) , \ \forall\, n \in N_{T_1} .$$

Therefore, the value of the embedded (martingale) probability measure Q_1^* for each scenario $n \in N_{T_1}$ depends on the combination of natural probabilities and marginal utilities in n at T_1, and *all* its descendant scenarios by monitoring date T_2.

2.6.2 Multiple Monitoring Dates

We now extend the results to the multi-period monitoring case. Assume that there are m monitoring dates $T_1, \ldots, T_m$ ($m \geq 2$) where the portfolio's value is measured against its benchmark. As in the two-period case, we assume that the hedge fund manager follows self-financing strategies between monitoring dates and that a proportion q (with $q \leq 100\ \%$) of the profit over the benchmark is subtracted from the hedge fund portfolio's value at each monitoring date T_j, except for the last one (i. e., $j = 1, \ldots, m-1$). The subtracted amount should cover *at least* the incentive fee of the hedge fund manager (i.e., $q \geq \alpha$). As in the two-period case, the re-balance of the investment positions does not occur at T_j but a period later at $T_j + 1$. The utility function over the entire horizon is assumed to be time-additive and each of its utility components belongs to the class of utility functions $\mathcal{U}$.

Theorem 2.6.1 *(i) If* $\epsilon^{C,*}_{T_j+1} > 0$, *then* $E^{Q^*_{j+1}}\left[\beta_{T_{j+1}} C^{T_j+1}_{T_{j+1}}\right] \geq E^{Q^*_{j+1}}\left[\beta_{T_j+1} C_{T_j+1}\right]$,

and (ii) if $\bar{\epsilon}^{C,*}_{T_j+1} > 0$, *then* $E^{Q^*_{j+1}}\left[\beta_{T_{j+1}} C^{T_j+1}_{T_{j+1}}\right] \geq E^{Q^*_{j+1}}\left[\beta_{T_j+1} \bar{C}_{T_j+1}\right]$,

where $Q^*_{j+1} = \left\{ \frac{\left(y^{j+1}_n\right)^*}{\left(y^*_0\right)^*} \right\}_{n \in N_{T_{j+1}}}$ *is a martingale measure, and* $E^{Q^*_j}[\,\cdot\,]$ *is the expected value operator under* Q^*_j *and based (conditionally) on the information at* $t = 0$.

Proof: Only (i) is proved since (ii) is completely analogous.

Fixed or Stock Index Based Benchmarks

The corresponding dual restrictions are

$$\alpha p_n U'_{j+1}\left([f + \alpha s^{j+1}_n]\beta_n\right) - x^{j+1}_n - q y^{j+2}_n \leq 0 \;\forall\; n \in N_{T_{j+1}}\,, \tag{2.45}$$

$$x^{j+1}_n + y^{j+2}_n - y^{j+1}_n \leq 0\,,\; \forall\; n \in N_{T_{j+1}}\,, \tag{2.46}$$

$$\sum_{n \in N_{T_{j+1}}} \left(x^{j+1}_n + y^{j+2}_n\right) \beta_n C^{T_j+1}_n - \sum_{n \in N_{T_j}} y^{j+1}_n \beta_n C_{T_j+1,n} \leq 0 \tag{2.47}$$

for $j = 0, \ldots, m-2$, and

$$\alpha p_n U_m^{'}\left([f+\alpha s_n^m]\beta_n\right) - x_n^m \leq 0 \ , \ \forall\ n \in N_{T_m} \ , \tag{2.48}$$

$$x_n^m - y_n^m \leq 0 \ , \ \forall\ n \in \ N_{T_m} \ , \tag{2.49}$$

$$\sum_{n \in N_{T_m}} \left(x_n^m\right) \beta_n C_n^{T_{m-1}+1} - \sum_{n \in N_{T_{m-1}}} y_n^m \beta_n C_{T_{m-1}+1,n} \leq 0 \tag{2.50}$$

for T_{m-1}. Assume $\epsilon_{T_j+1}^{C,*} > 0$. Then,

<u>Case 1:</u> $j = 0, \ldots, m-2$. Applying the complementarity property to (2.47) yields

$$\sum_{n \in N_{T_{j+1}}} \left(\left(x_n^{j+1}\right)^* + \left(y_n^{j+2}\right)^*\right) \beta_n C_n^{T_j+1} = \sum_{n \in N_{T_j}} \left(y_n^{j+1}\right)^* \beta_n C_{T_j+1,n} \ .$$

From (2.45) and (2.46) $0 < \alpha p_n U_{j+1}^{'}\left([f+\alpha s_n^{j+1}]\beta_n\right) \leq x_n^{j+1} + q y_n^{j+2} \leq x_n^{j+1} + y_n^{j+2} \leq y_n^{j+1}$.

Hence, $\sum_{n \epsilon N_{T_{j+1}}} y_n^{j+1} \beta_n C_n^{T_j+1} \geq \sum_{n \epsilon N_{T_j}} y_n^{j+1} \beta_n C_{T_j+1,n}$, and, dividing by $y_0^{j+1} \equiv \sum_{n \epsilon N_{T_j}} y_n^{j+1}$,

$$E^{Q_{j+1}^*}\left[\beta_{T_{j+1}} C_{T_{j+1}}^{T_j}\right] \geq E^{Q_{j+1}^*}\left[\beta_{T_j} C_{T_j}\right] \ .$$

<u>Case 2:</u> The proof is analogous to Case 1.

The case of put options is proved in an analogous manner.

High Water Marks

The dual restrictions for high water marks are the same as the two cases treated for fixed or stock index benchmarks except that (2.45) and (2.48) are replaced respectively by

$$\alpha p_n U'_{j+1}\left([f+\alpha s_n^{j+1}]\beta_n\right) - x_n^{j+1} - q y_n^{j+2} + (1-\alpha) l_n^{j+1} \le 0 \ , \ \forall\, n \in N_{T_{j+1}} \ , \quad j = 0, \ldots, m-2 \ ,$$

and $\alpha p_n U'_m\left([f+\alpha s_n^m]\beta_n\right) - x_n^m + (1-\alpha) l_n^m \le 0$, $\forall\, n \in N_{T_m}$, where l_n^{j+1} satisfies

$$\begin{array}{rcl} l_n^{j+1} - \sum_{m \in C(n)} l_m^{j+1} & = & 0 \ , \ \forall\, n \in N_t \ \ t = T_{j+1}, \ldots, T_{j+2}-1, \ j = 0, \ldots, m-2 \ , \\ -x_n^{j+1} + l_n^{j+1} & = & 0 \ , \ \forall\, n \in N_{T_{j+1}} \ , \ j = 1, \ldots, m-1. \end{array}$$

The proof for this benchmark policy follows the same structure used for the case of fixed or stock index benchmarks.

Q.E.D.

2.7 Advanced Models

The models given here incorporate risk management features and other considerations that are not so far. These models include measures of underperformance (with respect to a benchmark), risk management of options, and other risky factors such as short selling constraints. We focus on the single monitoring case and depart from the following basic set of constraints

$$\begin{array}{l} Z_0 \cdot \theta_0 + \beta_0 \left(\epsilon_0 \cdot V_0 - \bar{\epsilon}_0 \cdot \bar{V}_0\right) = \beta_0 W_0 \ , \\ Z_n \cdot (\theta_n - \theta_{a(n)}) = 0 \ , \ \forall\, n \in N_t \ , \ \forall\, t = 1, \ldots, \ T \ , \\ Z_n \cdot \theta_n - \beta_n s_n + \beta_n \left(\epsilon_0 - \bar{\epsilon}_0\right) \cdot V_n = \beta_n B_n \ , \ \forall\, n \in N_T \\ Z_n \cdot \theta_n \ge 0, \ s_n \ge 0 \ , \ \forall\, n \in N_T \ , \\ \epsilon_0 \ge \mathbf{0}, \ \bar{\epsilon}_0 \ge \mathbf{0} \ . \end{array} \tag{2.51}$$

2.7.1 Underperformance Risk Management

The risk of underperforming the benchmark is a major issue for any hedge fund manager, whose performance and compensation fee is based on it. Therefore, appropriate risk measures of underperformance should be included in the objective function of the hedge fund manager. An important class of measures in that sense are so called *downside-risk aversion measures* (Fishburn (1977)) that penalize the shortfall of the portfolio relative to a given benchmark B_T. Within this class of downside-risk aversion measures, there is a subclass of measures that is widely applied and which is known as the class of *lower partial moment measures* (Bawa and Lindenberg (1977)). In this subclass, the penalization is done through partial (statistical) moments of the shortfall. That is, underperformance measures are of the form

$$R_\gamma(B_T) \equiv E\left[Max\left(B_T - W_T, 0\right)^\gamma\right] \ ,$$

where $\gamma \geq 0$ and W_T is the value of the portfolio at time T. The values of γ determine the weight that the investor gives to small or large deviations. The larger the γ, the more the investor cares about larger deviations and vice versa. Popular values of γ are $\gamma = 0$, which defines the so called *shortfall probability*; $\gamma = 1$, which is simply the *expected shortfall*; and $\gamma = 2$, which defines the *downside variance.*

Underperformance measures of the form of $R_\gamma(B_T)$ may be combined with the goal of maximizing the hedge fund manager's compensation fee, using the objective function

$$Max_{\ \theta,\epsilon,\mathbf{s}} \quad E\left[\left(f + \alpha s_T\right)\beta_T\right] - A\ R_\gamma\left(B_T \beta_T\right) \ , \tag{2.52}$$

where A is a nonnegative constant that determines the *risk aversion* of the hedge fund manager (towards underperformance) or equivalently, a *tradeoff parameter* between expected profit and underperformance risk. The objective function (2.52) is

$$Max_{\ \theta,\epsilon,s} \quad \sum_{n\in N_T}[f+\alpha s_n]\beta_n p_n - A\sum_{n\in N_T}(SF_n\beta_n)^{\gamma}\, p_n \ , \tag{2.53}$$

where SF_n is nonnegative and represents the shortfall relative to the benchmark B_n in scenario $n \in \ N_T$, and must satisfy the following set of (modified) constraints

$$Z_n \cdot \theta_n - \beta_n s_n + \beta_n\left(\epsilon_0 - \bar{\epsilon}_0\right)\cdot V_n^0 + \beta_n SF_n = \beta_n B_n \ , \ \forall\ n \in N_T \ , \tag{2.54}$$

plus nonnegativity restrictions, where $A > \alpha$ to assure feasibility. Consider the case of the expected shortfall as the measure of underperformance, i. e., $\gamma = 1$. Then,

$$SF_n^* = Max\left(\beta_n B_n - \left[Z_n \cdot \theta_n + \beta_n\left(\epsilon_0 - \bar{\epsilon}_0\right)V_n^0\right], 0\right) \ .$$

since the objective function can be re-expressed as

$$\sum_{n\in N_T} f\beta_n p_n + \sum_{n\in N_T}\left[\alpha\left(Z_n\cdot\theta_n^* + \beta_n\left(\epsilon_0^* - \bar{\epsilon}_0^*\right)V_n^0 - \beta_n B_n\right)p_m + (\alpha - A)SF_n^* p_n\right] \ .$$

Hence, if $Z_n \cdot \theta_n^* + \beta_n\left(\epsilon_0^* - \bar{\epsilon}_0^*\right)V_n^0 > \beta_n B_n$ for some $n \in N_T$, then, given that $(\alpha - A) < 0$, then $SF_n^* = 0$. Otherwise, if $Z_n \cdot \theta_n^* + \beta_n\left(\epsilon_0^* - \bar{\epsilon}_0^*\right)V_n^0 < \beta_n B_n$, for some $n \in N_T$, the objective function is

$$\sum_{n\in N_T} f\beta_n p_n + \sum_{n\in N_T}\left[A\left(Z_n\cdot\theta_n^* + \beta_n\left(\epsilon_0^* - \bar{\epsilon}_0^*\right)V_n^0 - \beta_n B_n\right)p_m + (\alpha - A)s_n^* p_n\right] \ .$$

Thus, $(\alpha - A) < 0$, implies that $s_n^* = 0$. Hence, by feasibility, SF_n^* must satisfy that $SF_n^* = \beta_n B_n - Z_n \cdot \theta_n^* - \beta_n (\epsilon_0^* - \bar{\epsilon}_0^*) V_n^0$.

The inclusion of the set of variables $(SF_n)_{n \in N_T}$ extends the feasible region defined by (2.51) and adds the restrictions

$$x_n \leq Ap_n \ , \ \forall \ n \in N_T \tag{2.55}$$

to the dual associated to (2.51) (see (2.28) in Section 2.4).

The model formed by the objective function (2.53) subject to the constraints (2.51) (with the modified restriction (2.54)) possesses the same optimality conditions for option investment stated in Proposition 2.4.1. The addition of the dual restriction (2.55) gives us more structure; and, hence, more insights about the optimal solutions. For instance, if $SF_n^* > 0$, then, by complementarity, $x_n^* = Ap_n$. Analogously, $s_n^* > 0$ implies $x_n^* = \alpha p_n$. Furthermore, from our previous analysis, $s_n^* SF_n^* = 0$. Therefore,

$$x_n^* = \begin{cases} \alpha p_n & \text{if } s_n^* > 0 \\ Ap_n & \text{if } SF_n^* > 0 \end{cases} \quad \forall \ n \in N_T \ .$$

That is, the dual optimal variable x_n^* plays the role of an indicator function of profits and shortfalls for each scenario.

2.7.2 Option Risk Management

Hedge fund managers should consider appropriate risk management for any trading strategy that is implemented in their pursuit of superior returns. We propose some risk measures for the case of strategies involving options and the way these measures could be included in our framework. We also study the implications of the inclusion of such measures on the nature of the optimal strategies.

Buying options is less risky than selling options since the maximum loss is bounded. At worst, the premium paid for the option is lost if the option expires out of the money. Therefore, an appropriate risk measure for the purchase of an option is

$$\psi(V_0\beta_0)\ , \tag{2.56}$$

where ψ is the probability that the option expires out of the money (ψ can be computed once the scenarios have been set up and therefore it is known at the moment of solving the model), and V_0 is the option's price (call or put). Hence, assuming homogeneity of the risk measure (i.e., a double position have double risk, see Artzner, et al. (1999)), a model that incorporates a tradeoff between profit and risk of purchased options in our framework is

$$Max\ _{\theta,\epsilon_0,\mathbf{s}} \quad \sum_{n\in N_T}[f+\alpha s_n]\beta_n p_n - \epsilon_0[\psi(V_0\beta_0)] \tag{2.57}$$

subject to (2.51). The inclusion of $\epsilon_0[\psi(V_0\beta_0)]$ yields the following proposition:

Proposition 2.7.1 *If* $\epsilon_0^{C,*} > 0$, *then* $\beta_0 C_0 = \sum_{n\in N_T}\left(\frac{x_n^*}{\psi+y_0^*}\right)\beta_n C_n < E^{Q^*}[\beta_T C_T]$, *where* $Q^* = \left\{\frac{y_n^*}{y_0^*}\right\}_{n\in N_T}$. *Analogously, if* $\epsilon_0^{P,*} > 0$ *then,*

$$\beta_0 P_0 = \sum_{n\in N_T}\left(\frac{x_n^*}{\psi+y_0^*}\right)\beta_n P_n < E^{Q^*}[\beta_T P_T]\,.$$

Proof: The proof is analogous to that of Proposition 2.3.1. The key is that the corresponding dual problem differs only in the last two constraints from (2.2). The last two constraints are

$$\begin{aligned}&\textstyle\sum_{n\in N_T} x_n\beta_n C_n - (y_0+\psi)\,\beta_0 C_0(1+tc^C)+\eta_C = 0\ ,\\ &\textstyle\sum_{n\in N_T} x_n\beta_n P_n - (y_0+\psi)\,\beta_0 P_0(1+tc^P)+\eta_P = 0\ ,\end{aligned}$$

Q.E.D.

i.e., buying options in the current model is optimal if these are underpriced (under Q^*).

Selling options implies taking a position where losses are, in principle, unbounded and, hence, risk management is crucial. Hedging the risk of sold options is usually carried out through either the implementation of a portfolio of assets that replicates the payout of the option or the purchase of an option of the same characteristics (i.e., same expiry date and exercise price). The latter is partially implicit in the basic formulation (2.51) since it allows for buying and selling options of the same type although the model does not enforce the purchase of options if stock index options are sold. Perfect replication of the option's payoff is not always possible since our framework allows for incomplete markets. Therefore, it is imperative to include risk measures into the basic formulation. We addressed this problem for the basic problem in Section 2.3.2 by including

$$\delta\left[\sum_{n\in N_T}\beta_n\left(\epsilon_0\cdot V_n\right)p_n\right]\ .$$

We now address the risk management for sold options in a broader sense. For instance, we consider a class of risk measures that satisfies two properties: homogeneity and *a priori* determination (i.e., once the scenarios have been set up, the risk measure can be determined). Two members of this class of measures are Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR) (see Mina and Yi Xiao (2001)). VaR is the *maximum* expected loss under a certain confidence level and CVaR is defined, given a certain level of confidence, as the expected loss given that the loss has surpassed the VaR of the corresponding confidence level. Although, VaR is the most widely used risk measure in the financial context, CVaR has more *appealing* properties. For instance, Artzner, et al. (1999) prove that CVaR is a coherent risk measure (while VaR is not), Rockafellar and Uryasev (2000 and 2000a) prove that CVaR is convex and continuous, and Rockafellar, et al. (2000b) prove that CVaR is an expectation-bounded risk measure (while VaR is not). Other convex risk measures, which place more weight on losses, are discussed in Rockafellar and Ziemba (2000) and Cariño and Ziemba (1998). If VaR is computed at the confidence level of $(1-\eta)\%$, then a tradeoff between profit and risk over sold options can be modelled via

$$Max_{\ \theta,\epsilon_0,\mathbf{s}} \quad \sum_{n\in N_T}[f+\alpha s_n]\beta_n p_n - \bar{\epsilon}_0(\eta VaR) \tag{2.58}$$

subject to (2.51) or

$$Max_{\ \theta,\epsilon_0,\mathbf{s}} \quad \sum_{n\in N_T}[f+\alpha s_n]\beta_n p_n - \bar{\epsilon}_0\,(\eta CVaR) \tag{2.59}$$

subject to (2.51) if CVaR is used. Application of duality leads to

Proposition 2.7.2 *If* $\bar{\epsilon}_0^{C,*}>0$ $\left(\bar{\epsilon}_0^{P,*}>0\right)$, *then*

*(i) under model (*2.58*),*

$$\beta_0\bar{C}_0=\sum_{n\in N_T}\left(\frac{x_n^*}{y_0^*}\right)\beta_n C_n+\left(\frac{1}{y_0^*}\right)(\eta VaR)\ ,\ \left(\beta_0\bar{P}_0=\sum_{n\in N_T}\left(\frac{x_n^*}{y_0^*}\right)\beta_n P_n+\left(\frac{1}{y_0^*}\right)(\eta VaR)\right)\ ,$$

*(ii) under model (*2.59*),*

$$\beta_0\bar{C}_0 = \sum_{n\in N_T}\left(\frac{x_n^*}{y_0^*}\right)\beta_n C_n+\left(\frac{1}{y_0^*}\right)(\eta\ CVaR)\ ,\left(\beta_0\bar{P}_0 = \sum_{n\in N_T}\left(\frac{x_n^*}{y_0^*}\right)\beta_n P_n + \left(\frac{1}{y_0^*}\right)(\eta\ CVaR)\right)$$

Proof: Only (i) is proved since (ii) is completely analogous. Assume $\bar{\epsilon}_0^{C,*} > 0$. Then, by complementarity, $y_0^*\beta_0\bar{C}_0 = \sum_{n\in N_T} x_n^*\beta_n C_n + \eta VaR$. Therefore,

$$\beta_0\bar{C}_0 = \sum_{n\in N_T}\left(\frac{x_n^*}{y_0^*}\right)\beta_n C_n + \left(\frac{1}{y_0^*}\right)(\eta VaR)\ .$$

Q.E.D.

2.7.3 Other Extensions

Restrictions on the positions of the portfolio

None of the models studied before impose any restrictions on the asset positions within the portfolio. Nevertheless, such restrictions often appear in practice. For instance, liquidity and short-selling restrictions are common investment constraints that investors impose on the management of their portfolios. Such restrictions can be included into the set of constraints (2.51) through the addition of the constraints

$$\theta_n \geq b_n\ ,\ \forall\ n \in N_t\ , t = 0,\ldots,T,$$

where $b_n \in \Re$. Therefore, the dual restriction, $y_n Z_n - \sum_{m\in C(n)} y_m Z_m = 0$, for all $n \in N_t$, and $t = 0,\ldots,T-1$, is replaced by

$$y_n Z_n - \sum_{m\in C(n)} y_m Z_m - \lambda_n = 0\ ,\ \forall\ n \in N_t,\ t = 0,\ldots,T-1\ ;$$

$$x_n - y_n + \lambda_n^0 \leq 0 \ , \ \forall \ n \in N_T \ ;$$

$$\lambda_n \geq \mathbf{0}, \ \forall \ n \in N_t, \ t = 0, \ldots, T-1 \ ;$$

and the term $-\sum_{n \in N_T} \lambda_n b_n$ is added to the dual function.

Although $y_n \geq 0$, $\forall \ n \in \ N_t$, $\left\{ q_n \equiv \frac{y_n}{y_0} \right\}_{n \in N_T}$ is no longer a probability measure on (Ω, F_T).

Further Extensions

The model (2.51) assumes that the hedge fund manager buys and sells options with the same payoff. A natural extension is to allow for buying and selling options with different payoffs and/or different times to maturity. For instance, the budget constraint is still of the form

$$Z_0 \cdot \theta_0 + \beta_0 \left(\epsilon_0 \cdot V_0 - \bar{\epsilon}_0 \cdot \bar{V}_0 \right) = \beta_0 W_0 \ ,$$

but now $\epsilon_0 = \left(\epsilon_0^1, \ldots, \epsilon_0^N \right)$ and $\bar{\epsilon}_0 = \left(\bar{\epsilon}_0^1, \ldots, \epsilon_0^M \right)$, where ϵ_0^j $(\bar{\epsilon}_0^j)$ is the purchase (sale) price of the j^{th} option (European calls or puts) for $j = 1, \ldots, N$ $(j = 1, \ldots, M)$. Self-financing restrictions are modelled using

$$Z_n \cdot \left(\theta_n - \theta_{a(n)} \right) - \beta_n \left(\epsilon_0^t \cdot V_n^t - \bar{\epsilon}_0^t \cdot \bar{V}_n^t \right) = 0, \ \forall \ n \in N_t \ ,$$

where ϵ_0^t $(\bar{\epsilon}_0^t)$ is the vector of positions of the option purchased (sold) at time 0 and which matures at time t, and V_n^t $(\bar{V}_n^t)$ is the vector of corresponding payoffs. The Benchmark restriction is

$$Z_n \cdot \theta_n - \beta_n s_n + \beta_n \left(\epsilon_0^T \cdot V_n^T - \bar{\epsilon}_0^T \cdot \bar{V}_n^T \right) = \beta_n B_n, \ \forall\, n \in N_T \ .$$

Duality can be applied to obtain insights on the structure of optimal investments as in this chapter.

2.8 Conclusions

This chapter develops a duality framework for a variety of models of the problem faced by a hedge fund manager whose goal is to maximize his expected compensation fee using option investment strategies. This framework leads to explicit characterizations of the optimal strategies that contrast with many stochastic programming models used in finance that only provide numerical results.

The application of the duality framework for the hedge fund manager's problem leads to the general, and expected, conclusion that the purchase or sale of Stock Index Options is optimal if these options are not overpriced or not underpriced respectively. However, our approach derives the *pricing threshold* that determines the overpricing or underpricing of such options, and the relation of this threshold with the parameters that characterize his compensation scheme. We obtain explicit relations between the predetermined percentage that the manager obtains from the profits over a specified benchmark and the *embedded* probability measure that defines the pricing threshold. Moreover, we generalize our results to a broad class of utility functions, risk management considerations, short selling restrictions, multiple monitoring dates, and different policies for determining the benchmark, such as fixed and high water marks schemes.

The framework has several advantages: First, it allows for a broad class of distributions including the commonly observed *fat-tailed* return price distributions. Second, it is applicable to both, incomplete and complete markets. Third, the mathematics used to construct this framework are simpler, and hence more accessible, than those in the continuous-time framework. Fourth, the implementation of this framework to establish optimality conditions is relatively simple.

Our duality framework can be extended to consider more complex or general problems such as more general convex compensation schemes that include problems such as the compensation of a corporate manager that controls firm leverage or the compensation of a trader at a security firm.

Chapter 3

Incentives and Design of Option-Like Compensation Schemes

3.1 Introduction

Two of the most significant changes in corporate compensation practices in the last fifteen years have been, on the one hand, the escalation and recent decline of the average executive compensation amount and, on the other hand, the replacement of base salaries by stock options as the single largest component of executive compensation (Hall and Murphy (2003)). For instance, since the early 1990s stock options have replaced base salaries as the single largest component of executive compensation in almost all industry sectors and, during the same period of time, firms related to the technology sector (computers, software, internet, telecommunications and networking) have had a pronounced increase in employee stock options (Hall and Liebman (1998), Murphy (1999)).

The explosive growth in the use of options as a compensation can be roughly explained by two main factors. First, the belief that option-like compensation schemes are capable of aligning interests between executives, employees and shareholders while, at the same time, they can attract, motivate and retain high qualified executives and employees (e.g. DeFusco et al (1990)). Second, practical reasons for companies related with the conservation of cash and the reduction of reporting accounting expenses (e.g. Oyer (2004)).

Regardless of its causes, the use of option-like compensation schemes has generated concerns about their design and the behavior they induce, as well as about the valuation of the options embedded in these payment schemes from the executives' and shareholders' point of view.

The design and induced behavior of these option-like compensation schemes are intimately related. The compensation scheme design determines the induced behavior of shareholders, executives, and employees. Conversely, a desired induced behavior characterizes the required compensation schemes.

Since the early nineties, there has been an intensive research on the issue of induced behavior of option-like compensation schemes (Murphy (1999)). Most of these studies support the idea that these compensation mechanisms can be effectively used to align interests and induce desired behaviors, such as the implementation of riskier strategies by an investment manager. However, recent studies by Carpenter (2000), Lewellen (2006), Ross (2004) and Braido and Ferreira (2006) conclude that this type of payment schemes do not necessarily imply the *expected* or *desired* behaviors, unless some assumptions on the executives' and employees' preferences, or on the specific form of the compensation scheme, are made. For instance, Carpenter considers a risk averse manager compensated with a call option on the assets he controls. Under certain assumptions, Carpenter obtains the manager's explicit optimal policy and finds that this type of option compensation does not necessarily lead to greater risk-taking, as it might be intuitively *expected* [1]. Ross (2004) corroborates Carpenter's theoretical findings in a simpler static framework and proves that, among the class of convex compensation schemes, such as vanilla call options, there is no single scheme that would make all strictly-concave utility functions uniformly display lower (absolute) risk aversion in their domains. However, very recently, Braido and Ferreira (2006) find, in a static setting too, that for the particular case of a call option, there exist conditions on the stock price distribution -not considered by Ross (2004)- under which all strictly-concave managers prefer riskier projects to safer ones.

Many of the ideas developed recently about induced risk incentives by option-like compensation schemes (e.g., Carpenter (2000), Ross (2004), Braido and Ferreira (2006)) have been applied to other important corporate issues such as optimal leverage decisions. For instance, Cadenillas, et al. (2004) and Carlson and Lazrak (2005) are two important, and complementary, studies that have analyzed the optimal leverage choice from the shareholder's

[1] From option pricing theory, it is well known that the higher the volatility, the higher the option value. Therefore, it is intuitive to think that a manager who holds a stock option as part of his compensation, will be encouraged to take higher levels of volatility in order to increase his expected compensation.

and the manager's point of view.

Research on corporate compensation design has been importantly influenced by agency theory (Ross (1973)). This theory considers the problem of a person, the principal, who wants to induce another person, the agent, to take some action which is costly to the agent. The principal's problem consists in designing an incentive payment from the principal to the agent that induces the agent to take the best action from the principal's point of view. In the context of executive compensation the shareholders play the role of the principal, while the executives represent the agents. The majority of executive compensation design studies that are based on agency theory, visualize executive compensation as a *remedy* to the agency problem (e.g. Murphy (1999) and Core, et al. (2003) for surveys). However, there exist some recent studies that consider executive compensation not only as a remedy, but also as part of an agency problem itself (Bebchuk and Fried (2003)), arguing that managers usually have a certain degree of power or influence over the people (e.g. board of directors) that set their compensation payments.

For the shareholders, writing compensation options represents an opportunity cost, while for the executives or employees these options are an expected benefit. However, the expected benefits usually do not equate the opportunity costs of issuing these options. In fact, these costs are often greater than the expected benefits, since compensation options typically have trading and hedging restrictions that an outside investor would not have. The magnitude of the difference between the expected benefits and the opportunity costs derived from these options is directly related to their induced behavior and, thereafter, their design. For instance, the smaller this magnitude is, the stronger the induced incentives are, and vice versa. Therefore, the design of these options is connected to the control of this magnitude.

Determining the company's opportunity cost from issuing compensation options is less challenging than estimating their value for executives or employees, since they have trading and hedging constraints than an outside investor would not have. In fact, the outside investor's *freedom* usually *fits* into the model assumptions made by the existing frameworks to value options and therefore, opportunity costs can be usually estimated by using an *appropriate* option-valuation model.

Hedging constraints on compensation options imply that executives' and employees' valuations depend on their particular preferences. Therefore, research efforts on the valuation of these options have mostly relied on utility-based frameworks with two types of approaches: a certainty equivalent approach, which determines the riskless cash compensation that the executive or employee would exchange for the option (e.g. Hall and Murphy (2000,2002) and Lambert, et al. (1991)), and an optimal strategy approach in which, given an objective function, the optimal exercising policy is obtained and used to value the option (e.g. Hudart (1994) and Marcus and Kulatilaka (1994)). Nevertheless, there are some studies that have addressed the valuation problem without considering an explicit preference-based model. For example, Carpenter (1998) shows that a simple extension of the American option (binomial) model is reasonable to value, in practice, these compensation options.

Characterizing the optimal exercise policy of a compensation option is not only important for obtaining the subjective value of compensation options, but also for determining the hedging cost for the shareholders. This is because even though the outside investor has, in principle, no trading restrictions, the person who actually exercises the option is the executive or employee. Theory of optimal exercise of compensation options is still under development. Most part of the efforts on this direction are based on the intuition that, given the payoff risk and the undiversified condition of the executive or employee, the compensation option holder chooses an option exercise policy as a consequence of a greater utility maximization problem that includes other decisions, such as portfolio and consumption choice and managerial strategy. The majority of the papers under this perspective, make some kind of exogenous assumptions about how non-option wealth is invested (e.g. Huddart (1994), Marcus and Kulatilaka (1994) and Carpenter (1998)) or about the exercise policy (e.g. Jennergren and Naslund (1993), Cvitanic, et al. (2004), and Hull and White (2004)).

This chapter focuses on the design and induced behavior of option-like compensation schemes for a fund manager. It considers payment schemes composed by cash and two call options with the same underlying -the fund value- but different strike prices. There are two main goals: First, studying the induced incentives of this type of compensation package -fund value and risk incentives- and analyzing if these compensation schemes have the same unexpected theoretical features that have been documented for the single-option case. Second, finding criterions to determine appropriate parameter values of the considered compensation package to induce, as much as possible, specific behaviors.

In order to achieve these goals, we develop our analysis in a continuous-time framework similar to the one used by Carpenter (2000). This framework presents two main advantages for our study. First, it assumes that the fund value is governed by a geometric Brownian motion that allow us to derive explicit formulas that facilitate our analysis. Second, it is easily adaptable to general convex compensations schemes, as the one treated in this chapter.

We obtain the explicit optimal risk-taking strategy for an undiversified risk-averse manager who maximizes the expected utility value of his compensation package. We find that including two distinct options in the compensation scheme does not eliminate the unexpected risk-taking behavior observed for the single-option case, but it does provide us with a significant higher control on the manager's implied actions. In particular, we obtain that the sensitivity of optimal risk-taking choice with respect to fund value is roughly proportional to a *stability factor*, which involves only the compensation parameters. Therefore, the smaller the value of this factor, the less sensitive is the risk-taking decision to changes in the fund value. This result can be used, for example, by investors to decrease the incentives to rebalance the fund's portfolio by appropriately choosing the compensation parameters. For instance, we consider in our analysis a situation in which investors desire to induce an strategy that leads to a range of fund values of the form $\{0\} \cup (\underline{V}, \infty)$, where $\underline{V}$ is a minimal target value. This type of strategy is typically implied by a single-option compensation scheme (Carpenter (2000)) and can be induced by the double-option payment scheme considered in this chapter. However, this strategy can involve, in both cases, drastic changes of optimal risk-taking decisions given small variations of the fund value, specially when the options are out-of-the money. This is costly in terms of transaction costs or investment irreversibility or lumpiness. Therefore, it is imperative to be able to control the possibility of such drastic changes. Although there exists a stability factor for the single-option case that could be used to control stability, this is hardly possible in practice since there are *typical* values for the corresponding compensation parameters. However, for the double-option case, the stability factor involves two extra parameters, namely the second-option exercise price and the relative weight of the second-option payoff in the compensation scheme, for which there are <u>not</u> such *typical* values. Hence, adding a second option provide to investors with two degrees of freedom for controlling stability. We propose specific criterions to determine optimal values for these extra parameters. Therefore, this stability factor, together with other results obtained in this chapter, provide incentives for investors to grant the manager additional options.

Nevertheless, we also discover that the risk-taking optimal profile can change abruptly for a certain second option's threshold exercise price. This means that two managers with very similar option-like compensation packages can have different risk-taking profiles. In other words, a marginal change in the compensation structure may imply drastic changes in the risk-taking incentives and hence a policy implication of our work suggest that investors should pay attention to global compensation scheme of their managers. To our knowledge, this is one of the few studies on how exercise price decisions may affect managerial risk-taking behavior (Hjortshøj (2006)).

This chapter is structured as follows: Section two describes the model. Section three analyzes the utility function derived from the manager's preferences over his compensation package. Section four develops closed-form expressions for the manager's optimal choice of terminal fund value and volatility. Section five illustrates different types of fund value and risk incentives that the proposed option-like compensation scheme is, in theory, able to induce. Section six proposes simple criterions, based on the theoretical results and analysis of the previous sections, to determine appropriate compensation parameter values in order to induce specific fund value or risk profiles. Section seven presents the conclusions.

3.2 The Model

We consider the problem of an undiversified risk-averse manager who maximizes the expected utility value of his compensation package. We assume that the manager makes decisions regarding the risk level of the fund, given a compensation package. To address this problem we consider a continuous-time framework that is described in the following subsections.

3.2.1 The Fund Value

We depart from the standard assumption that the value of a fund is governed by a geometric Brownian motion (Black and Scholes (1973), Merton (1974)) and consider the extended model

$$dV_t = \mu V_t dt + \alpha \nu_t V_t dt + \nu_t V_t dz_t, \tag{3.1}$$

where μ and α are fixed parameters, $(\nu_t)_{t\geq 0}$ is an adapted process and $(z_t)_{t\geq 0}$ is a standard Brownian motion. Model (3.1) is equivalent to the one used in Carpenter (2000) and a particular case of the one used in Cadenillas, et al. (2004). The parameter μ is exogenous and represents the expected return due to the fund's momentum which cannot be affected by the manager. In an equilibrium setting, μ would be related to the prevailing interest free rate in the economy. The process $(\nu_t)_{t\geq 0}$ represents the volatility of the fund's value and it is assumed that it can be controlled by the manager and that it has an impact on the expected return of the fund. This impact is captured by the parameter α and so it can be interpreted as the expected return per unit of risk undertaken by the manager, as measured by the volatility.

3.2.2 The Manager Problem

The manager is risk averse with utility function[2]

$$U(x) = \frac{x^{1-\gamma}}{1-\gamma}, \quad \gamma > 0,$$

where x is the compensation that the manager receives and γ represents the manager's risk aversion, and who receives a compensation package formed by a fixed salary X_0, a proportion p of a call option payoff on the fund's value with a a strike value B, that represents a benchmark payoff for the manager, and a proportion q of the payoff of another call option on the fund's value with a strike price $B + K$, where $K > 0$. Therefore, the total value of the compensation package is

$$\phi(V) = X_0 + p(V - B)^+ + q(V - B - K)^+$$

[2]This is the power utility function, the most popular member of the class of CRRA utility functions.

where $(x)^+ \equiv max(x, 0)$. We assume that the values B and K are preestablished by the investors. If we were in a firm value context, the parameter B might represent the leverage level of the firm, then the compensation package is composed by a fixed salary X_0, a proportion p of the *shares* of the firm, and a proportion q of the payoff of a call option on the *levered value* of the firm.

The goal of the manager is to maximize the expected derived utility of his compensation package at the end of the horizon time $[0, T]$, by choosing the level of risk for the fund. That is, the objective of the manager is

$$\sup_{(\nu_t)_{t\in[0,T]}} E\left[\frac{(\phi(V_T))^{1-\gamma}}{1-\gamma}\right], \tag{3.2}$$

where the volatility process ν is well behaved and is such that

$$V_t \geq 0, \quad 0 \leq t \leq T.$$

It can be shown that the solution to Problem (3.2) is uniquely characterized (e.g. Karatzas et al (1987)) as the solution of the *static problem*

$$\sup_{V_T} E\left[\frac{(\phi(V_T))^{1-\gamma}}{1-\gamma}\right], \tag{3.3}$$

under the positivity constraint $V_T \geq 0$, i.e. under the assumption that the fund's value is always nonnegative, and the constraint

$$E\left[\xi_T V_T\right] \leq V_0,$$

where

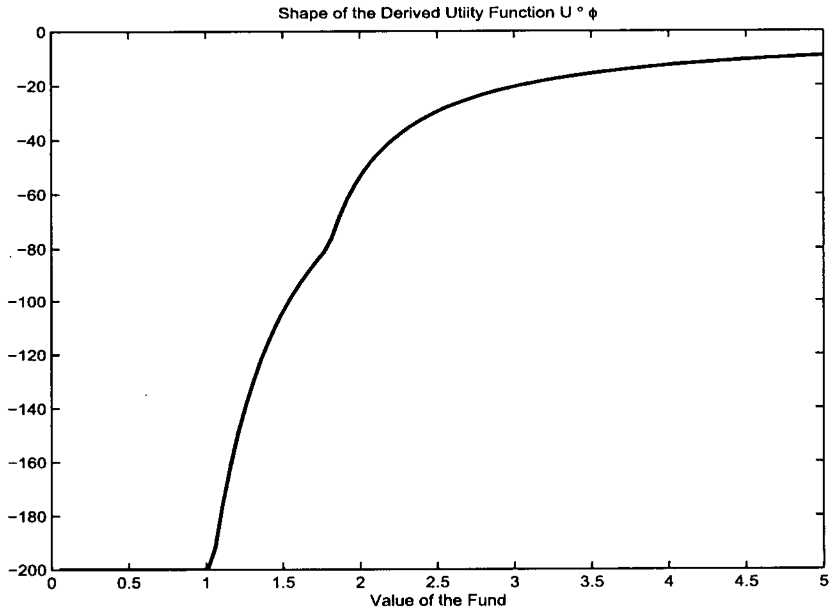

Figure 3.1: **Shape of $U \circ \phi$.** This is the graph of $U \circ \phi$ when the following parameters are considered: $X_0 = 0.005$, $p = 0.01$, $q = 0.02$, $B = 1.04$, $K = 0.76$ and $\gamma = 2$. Note that $(U \circ \phi)(0) = (-1)(0.005^{-1}) = -200$.

$$\xi_t = \exp\left\{-\mu t - \frac{\alpha^2}{2}t - \alpha z_t\right\}, \text{ for } t \in [0, T], \tag{3.4}$$

and which, in the case that μ represents the (instantaneous) interest free rate, represents a *state price density* or a *pricing kernel.*

3.3 Derived Utility Function Analysis

Problem (3.3) cannot be solved directly using standard theory since the derived utility function $U \circ \phi = \phi^{1-\gamma}/(1-\gamma)$ is not concave overall its domain (See Figure 3.1). This lack of concavity is expected up to some extent since the convex compensation schedule ϕ is designed with the intend of inducing the manager to be less risk-averse and therefore with the goal of making its utility function *less concave.*

In order to solve Problem (3.3) we construct an equivalent problem where $U\circ\phi$ is replaced by an *appropriate* concave function which we refer to as the *concavification function* of $U\circ\phi$. This section is devoted to the related analysis of the derived utility function $U\circ\phi$ in order to construct its concavification function.

3.3.1 Motivation

Concavity of the concavification function of $U\circ\phi$ is required in order to apply standard methods while its *appropriateness* refers roughly to the property that its solution is also optimal for the original problem. Intuitively, one would expect that the concavification function of $U\circ\phi$, say $\tilde{U}$, is *closed* to it since $\tilde{U}$ should reflect the essential set of preferences represented by the derived utility function $U\circ\phi$, at least over a neighborhood of the optimal solution. Thus, in such sense, the construction of $\tilde{U}$ entails to find the *closest* concave function to $U\circ\phi$. This *closeness* to $U\circ\phi$ must be ruled by the referred *appropriateness* of $\tilde{U}$. That is, it must be guided by the premise of assuring that the optimal solution under $\tilde{U}$ is also a solution of Problem (3.3). In other words, the optimal solution must not change if $U\circ\phi$ is replaced by $\tilde{U}$. One way of satisfying such premise is to restrict our search of $\tilde{U}$ to the set of all the concave functions that *dominate* $U\circ\phi$, i.e. to consider the set of functions

$$G=\left\{g:\Re^{+}\rightarrow\Re\mid g\text{ is concave and }g(x)\geq(U\circ\phi)(x)\text{ for all }x\geq0\right\}.$$

The reason to focus on this class of functions is the following: suppose that it exists $g\in G$ such that

$$E\left[g(X^{*})\right]=E\left[(U\circ\phi)(X^{*})\right] \tag{3.5}$$

where X^* is the optimal solution of Problem (3.3) when $U \circ \phi$ is replaced by g. Then,

$$E\left[(U \circ \phi)(X^*)\right] = E\left[g(X^*)\right] \geq E\left[g(X)\right] \geq E\left[(U \circ \phi)(X)\right]$$

for all X. Thus, X^* must be an optimal solution of Problem (3.3). Therefore, the construction of the concavification function $\tilde{U}$ can be formulated as determining a concave function that dominates $U \circ \phi$ and that satisfies condition (3.5). This condition is precisely what we mean by the *closest* concave function to $U \circ \phi$.

3.3.2 Concavification Function Construction

As previously motivated, the concavification function of $U \circ \phi$ is roughly the *closest* concave function that dominates $U \circ \phi$. Using this idea, we provide a geometrical motivation for the construction of the concavification function $\tilde{U}$, derived from conditions established in terms of the absolute risk aversion induced by the compensation schedule ϕ. Then, we state formal concavification results for the particular compensation scheme ϕ that we consider.

Geometrical Motivation

To motivate the geometrical construction of the concavification function $\tilde{U}$, we assume that it is of the form $(U \circ \phi) + h$, where h is a function, and analyze the conditions that should be imposed on h to assure that $\tilde{U}$ is the *closest* concave function that dominates $U \circ \phi$. Dominance of $\tilde{U}$ over $U \circ \phi$ holds by simply imposing the restriction that $h(x) \geq 0$ for all $x \geq 0$. Concavity and *optimal closeness* of $\tilde{U}$ require a deeper analysis. To perform such analysis we use the concept of absolute risk aversion (Pratt (1964)). We start off by assuming that h is twice differentiable over all its domain and that its first derivative does not vanish over the interval $[0, B)$. Therefore, the induced absolute risk aversion of $(U \circ \phi) + h$ can be decomposed as follows:

$$\begin{aligned} A_{(U\circ\phi)+h} &\equiv -\tfrac{(U\circ\phi)''+h''}{(U\circ\phi)'+h'} \\ &= \begin{cases} -\tfrac{h''}{h'} & \text{if } x < B. \\ \left(\tfrac{(U\circ\phi)'}{(U\circ\phi)'+h'}\right)\left(-\tfrac{(U\circ\phi)''}{(U\circ\phi)'} - \tfrac{h''}{(U\circ\phi)'}\right) & \text{if } x \in (B, B+K) \cup (B+K, \infty). \end{cases} \end{aligned}$$

Therefore, in the case that $x < B$, the induced absolute risk aversion of $\tilde{U}$, and thereafter its concavity, depends on the term

$$A_h \equiv -\frac{h''(x)}{h'(x)} \ .$$

So, $\tilde{U}$ is concave for $x < B$ if and only if $A_h \geq 0$. On the other hand, in the case that $x \in (B, B+K) \cup (B+K, \infty)$ observe that $A_{(U\circ\phi)+h}$ can be rewritten as

$$F\left\{A_\phi + A_U(\phi)\left[\phi' - 1\right] + \left[A_U(\phi) - \frac{h''}{(U\circ\phi)'}\right]\right\}$$

where $F = (U\circ\phi)' / \left[(U\circ\phi)' + h'\right]$. Hence, if we assume that the concavification function $\tilde{U}$ is within the class of (strictly) monotone utility functions, then $F(x) \geq 0$ for all $x \geq 0$. Therefore, the induced absolute risk aversion of $\tilde{U}$, and thereafter its concavity, depends on the expression

$$A_\phi + A_U(\phi)\left[\phi' - 1\right] + \left[A_U(\phi) - \frac{h''}{(U\circ\phi)'}\right] \ .$$

Thus, for the case $x \in (B, B+K) \cup (B+K, \infty)$, the absolute risk aversion of $\tilde{U}$, and its concavity, depend on the interaction of three effects (Ross (2004)): a *convexity* effect, represented by A_ϕ; a *magnification* effect, represented by $A_U(\phi)\left[\phi' - 1\right]$; and a *translation* effect, represented by $\left[A_U(\phi) - \frac{h''}{(U\circ\phi)'}\right]$. Hence, $\tilde{U}$ is concave in this case if and only if the *translation* effect is positive and it offsets both, the *convexity* and the *magnification* effects. That is, concavity of $\tilde{U}$ requires that

$$\left[A_U(\phi) - \frac{h''}{(U\circ\phi)'}\right] \geq -\left(A_\phi + A_U(\phi)\left[\phi' - 1\right]\right) ,$$

or equivalently

$$\begin{aligned} h'' &\leq (U\circ\phi)'\left\{A_\phi + A_U(\phi)\left[\phi' - 1\right] + A_U(\phi)\right\} \\ &= -\left(U\circ\phi\right)'' \end{aligned}$$

In summary, if the concavification function $\tilde{U}$ is of the form $(U\circ\phi) + h$, then h must satisfy the following conditions:

Dominance (D):

- $h(x) \geq 0$, for all $x \geq 0$.

Monotonicity (M):

- $h'(x) > -\left(U\circ\phi\right)'(x)$, for all $x > B$.

Concavity (C):

1. $A_h(x) = -\frac{h''(x)}{h'(x)} \geq 0$, for all $x < B$.
2. $h''(x) \leq -\left(U\circ\phi\right)''(x)$, for all $x \in (B, B+K) \cup (B+K, \infty)$.

Therefore, intuitively, the function $(U \circ \phi) + h$ is the *closest* concave function that dominates $U \circ \phi$ if h is the *smallest* function that satisfies the **D**, **M** and **C** conditions.

If the **D** and **M** conditions were the only ones considered, h would simply be the *zero* function (i.e. $h(x) = 0$ for all x). However, the *zero* function does not satisfy the **C** conditions. For instance, under the assumption that h is twice differentiable, the *zero* function clearly does not satisfy the **C.2** condition. Therefore, the **C** conditions are the key ones. We focus first on **C.2** $-h'' \geq (U \circ \phi)''$, and then analyze conditions **C.1**, **D**, and **M**. In particular, we study the *minimal* conditions under which $-h''$ satisfies **C.2**. That is, we analyze the condition

$$-h'' = (U \circ \phi)'' \ . \tag{3.6}$$

Although we are interested on the fulfillment of (3.6) for $x > B$, we omit for the moment this range restriction and analyze the consequences of this condition. Equation (3.6) implies that h must be of the form

$$h(x) = b + mx - (U \circ \phi)(x) \ .$$

That is, we need to *add* to $U \circ \phi$ what it is necessary to obtain a linear function, at least over the range of values in which $U \circ \phi$ is not concave. This makes sense considering that linear functions are *minimal* concave functions, since any linear function is both, concave and convex. We now analyze the range of values for b and m that would lead h to satisfy the **D** and **M** conditions. If h satisfies the **D** condition, then

$$b + mx \geq (U \circ \phi)(x) \tag{3.7}$$

i.e., the linear function $b+mx$ must dominate the derived utility function $(U \circ \phi)(x)$. It turns out that h must satisfy $h(0)=0$ in order to be *optimal*[3]. This implies $b=(U \circ \phi)(0)$, which means that this linear function must be *anchored* to the point $(0,(U \circ \phi)(0))$.

The inequality (3.7) cannot be satisfied in a strict manner for all $x \geq 0$ since that would imply the existence of a concave function that is *closer* to $U \circ \phi$. Indeed, it exists $\tilde{x} > B$ such that

$$b+m\tilde{x}=(U \circ \phi)(0)+m\tilde{x}=(U \circ \phi)(\tilde{x}) \ .$$

Therefore, $b+mx$ must be the line that is tangent to the graph of $U \circ \phi$ at $\tilde{x}$ that passes through the point $(0,(U \circ \phi)(0))$. Thus, $\tilde{x}$ satisfies

$$(U \circ \phi)(0)=(U \circ \phi)(\tilde{x})+(U \circ \phi)^{'}(\tilde{x})(0-\tilde{x}) \ .$$

and $m=(U \circ \phi)^{'}(\tilde{x}) \geq 0$. Hence, note that

$$\begin{aligned} h(x) &= b+mx-(U \circ \phi)(x) \\ &= (U \circ \phi)(0)+\left[(U \circ \phi)^{'}(\tilde{x})\right]x-(U \circ \phi)(x) \end{aligned}$$

satisfies

$$h^{'}(x)=(U \circ \phi)^{'}(\tilde{x})-(U \circ \phi)^{'}(x) \geq -(U \circ \phi)^{'}(x)$$

i.e., h satisfies the **M** condition. Moreover, note that

[3]If $h(0)>0$, it is always possible to construct a function $\tilde{h}$ such that $\tilde{h}(0)=0$, $(U \circ \phi)+\tilde{h}$ concave, and $0 \leq \tilde{h}(x) \leq h(x)$ for all $x \in [0,B)$, given that $U \circ \phi$ is flat over this interval.

$$
\begin{aligned}
A_h(x) &= -\frac{h''(x)}{h'(x)} \\
&= -\frac{(U\circ\phi)''(x)}{(U\circ\phi)'(\tilde{x})-(U\circ\phi)'(x)} \\
&= 0
\end{aligned}
$$

for $x > B$. Therefore, $h(x) = (U \circ \phi)(0) + \left[(U \circ \phi)'(\tilde{x})\right] x - (U \circ \phi)(x)$ satisfies the **D**, **M**, and **C** conditions. However, note that for $x > \tilde{x}$, it is enough, and thus optimal, to set $h(x) = 0$ in order to satisfy the **D**, **M**, and the **C** conditions.

So,

$$
h(x) = \begin{cases} (U \circ \phi)(0) + \left[(U \circ \phi)'(\tilde{x})\right] x - (U \circ \phi)(x) & \text{for } x \leq \tilde{x}, \\ 0 & \text{otherwise} \end{cases}
$$

That is, the geometrical construction of the concavification function entails *drawing* a line that is tangent to the graph of $U \circ \phi$ that passes through the point $(0, (U \circ \phi)(0))$. The concavification function will be formed by this tangent line, for values less than or equal to the tangency point $\tilde{x}$, and by the derived utility function otherwise.

We now formalize the construction of the concavification function for the type of compensation scheme ϕ that is considered in this chapter.

Concavification Results

The current analysis of the concavification function of $U \circ \phi$ begins from its conceptualization as the *closest concave function that dominates* $U \circ \phi$. In the motivational part, the idea of closeness is *mapped* into a set of conditions that characterize the *offsetting forces* of the convexity effects induced by the compensation scheme ϕ. Therefore, *closest* within this context means to satisfy *minimal conditions* to offset these convexity effects, which lead to the described geometry of the concavification function construction.

Although, the previous motivational subsection provides us with a good understanding of the construction and shape of the concavification function, its concept has not been fully formalized yet. To formalize the concept of concavification function, we use the characterization of a concave function f in terms of its hypograph, defined by $\{(x, y)|y \leq f(x)\}$. For instance, a function f is concave if and only if its hypograph is a convex set (Zangwill (1969)). Therefore, using this characterization, the concavification function of f can be formally described as the function $\tilde{f}$ whose hypograph is the smallest convex set that contains the hypograph of f. The smallest convex set that contains a given set is called the *convex hull* of such set. Hence, the definition of the concavification of f can be rephrased in the following manner:

Definition 3.3.1 (Concavification) *Let f be a real valued function on $\Re$ and let*

$$H_f = \{(x, y)|y \leq f(x)\}$$

be the hypograph of f. Let H_f^* be the convex hull of H_f. Then, if it exists a function $\tilde{f}$ such that its hypograph coincides with the convex hull of H_f, i.e. if $H_{\tilde{f}} = H_f^*$, then $\tilde{f}$ is said to be the concavification of f.

Armed with the above definition, we formally characterize the concavification function for particular cases derived from the consideration of $U \circ \phi$ over special subsets of its domain and then combine *appropriately* such cases to define the concavification function of $U \circ \phi$ overall its domain. To follow this strategy, we need to establish first the following preliminary results:

Lemma 3.3.1 *Let $\gamma > 0$. Then, it exists $V_1 > B$ such that*

$$\begin{array}{ll} \frac{X_0^{1-\gamma}}{1-\gamma} = \frac{(X_0+p(V_1-B))^{1-\gamma}}{1-\gamma} - pV_1 \left(X_0 + p\left(V_1 - B\right)\right)^{-\gamma} & , \textit{if } \gamma \neq 1, \textit{ or} \\ \log(X_0) = \log\left(\left(X_0 + p\left(V_1 - B\right)\right) - \frac{pV_1}{X_0+p(V_1-B)}\right. & , \textit{if } \gamma = 1. \end{array} \tag{3.8}$$

<u>Proof</u>: See Appendix 3.

Lemma 3.3.2 *Let $\gamma > 0$. Then, it exists $V_4 > B + K$ such that*

$$\begin{array}{l} \frac{X_0^{1-\gamma}}{1-\gamma} = \frac{(X_0+p(V_4-B)+q(V_4-B-K))^{1-\gamma}}{1-\gamma} - (p+q)\, V_4 \left[X_0 + p\left(V_1 - B\right) + q\left(V_4 - B - K\right)\right]^{-\gamma} , \\ \textit{if } \gamma \neq 1, \textit{ or} \\ \log(X_0) = \log\left(\left(X_0 + p\left(V_1 - B\right)\right) - \frac{(p+q)V_4}{X_0+p(V_1-B)+q(V_4-B-K)}\right. , \\ \textit{if } \gamma = 1. \end{array} \tag{3.9}$$

<u>Proof</u>: See Appendix 3.

Lemma 3.3.3 *Let $\gamma > 0$. Then, there exist $V_2, V_3 \in \Re$ such that satisfy the following system of equations:*

$$\begin{aligned}(1-\gamma)(X_0 + p(V_2 - B))^{1-\gamma} &= (1-\gamma)(X_0 + p(V_3 - B) + q(V_3 - B - K))^{1-\gamma} \\ &+ (p+q)(X_0 + p(V_3 - B) + q(V_3 - B - K))^{-\gamma}(V_2 - V_3)\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}(1-\gamma)(X_0 + p(V_3 - B) + q(V_3 - B - K))^{1-\gamma} &= (1-\gamma)(X_0 + p(V_2 - B))^{1-\gamma} \\ &+ p(X_0 + p(V_2 - B))^{-\gamma}(V_3 - V_2)\end{aligned} \tag{3.11}$$

if $\gamma \neq 1$, or

$$\begin{aligned}\log(X_0 + p(V_2 - B)) &= \log(X_0 + p(V_3 - B) + q(V_3 - B - K)) \\ &+ (p+q)(X_0 + p(V_3 - B) + q(V_3 - B - K))^{-1}(V_2 - V_3)\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}\log(X_0 + p(V_3 - B) + q(V_3 - B - K)) &= \log(X_0 + p(V_2 - B)) \\ &+ p(X_0 + p(V_2 - B))^{-1}(V_3 - V_2)\end{aligned} \tag{3.13}$$

if $\gamma = 1$.

Proof: See Appendix 3.

The previous results establish, under certain assumptions, the existence of four *tangency* points over the graph of $U \circ \phi$. For instance, it is established the existence of

- the points $(V_1, (U \circ \phi)(V_1))$ and $(V_4, (U \circ \phi)(V_4))$ on the graph of $U \circ \phi$ for which the corresponding tangent lines have the property of crossing through the point $(0, (U \circ \phi)(0))$, and

- the points $(V_2, (U \circ \phi)(V_2))$ and $(V_3, (U \circ \phi)(V_3))$ for which the line that passes through them is tangent to the graph of $U \circ \phi$ at both points.

These tangency points are crucial for the construction of the concavification function of $U \circ \phi$.

Next proposition proposes and characterizes formally, under certain assumptions, the concavification function of $U \circ \phi$ when its domain is restricted to the interval $[0, B+K]$.

Proposition 3.3.1 *Assume $\gamma > 0$, let $V_1 > B$ such that it satisfies equation (3.8), and define the function*

$$\tilde{U}_1(v) = \begin{cases} (U \circ \phi)(0) + v\,(U \circ \phi)'(V_1) & \text{for } v \in [0, V_1] \\ (U \circ \phi)(v) & \text{for } v > V_1 \end{cases}.$$

If $V_1 < B+K$, then $\tilde{U}_1$ is the concavification function of the $U \circ \phi$ restricted to the domain $[0, B+K]$.

<u>Proof</u>: See Appendix 3.

The following proposition characterizes the concavification function of $U \circ \phi$ over the interval $[B, \infty)$.

Proposition 3.3.2 *Let $\gamma > 0$ and assume that V_2 and V_3 satisfy equations (3.10) and (3.11), if $\gamma \neq 1$, or equations (3.12) and (3.13) otherwise. Define the following function:*

$$\tilde{U}_2(v) = \begin{cases} (U \circ \phi)(V_2) + (v - V_2)(U \circ \phi)'(V_2) & \textit{for } v \in (V_2, V_3] \\ (U \circ \phi)(v) & \textit{otherwise.} \end{cases}$$

Then, $\tilde{U}_2$ is the concavification function of $U \circ \phi$ over the interval $[B, \infty)$.

Proof: See Appendix 3.

So far, it has been proved that $\tilde{U}_1$ and $\tilde{U}_2$ are the concavification functions of $U \circ \phi$ restricted to $[0, B + K]$ and $[B, \infty)$, respectively. We use these two functions to define a *candidate* function for being the concavification of $U \circ \phi$ overall its domain. The construction of this candidate function is based on the idea of combining $\tilde{U}_1$ and $\tilde{U}_2$ in such way that its combination possess, at least, the property of dominating the derived utility $U \circ \phi$. Therefore, a *natural* candidate in such sense is the following function:

$$\tilde{U}_3(v) = Max\left\{\tilde{U}_1(v), \tilde{U}_2(v)\right\} \text{ for all } v \geq 0 \ .$$

It is immediate that

$$\tilde{U}_3(v) \geq (U \circ \phi)(v) \text{ for all } v \geq 0$$

given that both, $\tilde{U}_1$ and $\tilde{U}_2$, dominate $U \circ \phi$. However, $\tilde{U}_3$ is in general not a concave function. Nevertheless, it turns out that if $\tilde{U}_3$ is concave then $\tilde{U}_3$ must be the concavification function of $U \circ \phi$. This fact is proved in the next proposition.

Proposition 3.3.3 *If $\tilde{U}_3$ is concave, then $\tilde{U}_3$ is the concavification of $U \circ \phi$.*

Proof: See Appendix 3.

There are two natural issues that follow from the previous result:

1. Finding conditions under which $\tilde{U}_3$ is concave, and
2. Proposing another candidate function in case that $\tilde{U}_3$ is not concave.

The first issue is implicitly, although partially, addressed in the proof of Proposition 3.3.3 (See Appendix 3) where the condition $V_2 \geq V_1$ is proved to be necessary from the assumption of concavity of $\tilde{U}_3$. The next proposition proves that this condition is also sufficient to assure that $\tilde{U}_3$ is concave.

Proposition 3.3.4 *$\tilde{U}_3$ is concave if and only if $V_2 \geq V_1$.*

Proof: See Appendix 3.

If $\tilde{U}_3$ is not concave, we need to look for another *candidate* function to be the concavification function of $U \circ \phi$. Note that $\tilde{U}_3$ dominates $U \circ \phi$ and it is at least *by pieces* its concavification function. Therefore, the search for an alternative candidate function should depart in some way from $\tilde{U}_3$. A *natural* guess is to propose the concavification function of $\tilde{U}_3$ as a candidate function. It results that this is the right guess. This is proved in the next two propositions.

Proposition 3.3.5 *Assume that $\tilde{U}_3$ is not concave. Then, its concavification function is the function $\tilde{U}$ defined as*

$$\tilde{U}(v) = \begin{cases} (U \circ \phi)(V_4) + (U \circ \phi)'(V_4)(v - V_4) & \text{for } v \in [0, V_4] \\ (U \circ \phi)(v) & \text{for } v > V_4 \end{cases} \tag{3.14}$$

where V_4 satisfies the equation

$$
\begin{aligned}
(1-\gamma)\left(X_0^{1-\gamma}\right) &= (1-\gamma)\left(X_0+p\left(V_4-B\right)+q\left(V_4-B-K\right)\right)^{1-\gamma} \\
&- V_4(p+q)\left(X_0+p\left(V_4-B\right)+q\left(V_4-B-K\right)\right)^{-\gamma}
\end{aligned}
\tag{3.15}
$$

if $\gamma \neq 1$, *and*

$$
\begin{aligned}
\log(X_0) &= \log\left(X_0+p\left(V_4-B\right)+q\left(V_4-B-K\right)\right) \\
&- \tfrac{(p+q)V_4}{X_0+p(V_4-B)+q(V_4-B-K)}
\end{aligned}
\tag{3.16}
$$

if $\gamma = 1$.

Proof: See Appendix 3.

Proposition 3.3.6 *$\tilde{U}$ is the concavification of $U \circ \phi$.*

Proof: See Appendix 3.

Shape of the concavification of $U \circ \phi$

The concavification function of $U \circ \phi$ has two possible shapes. It has a *four-piece* shape if $\tilde{U}_3$ is concave, and hence characterized by $\tilde{U}_3$ itself (See Figure 3.3). Otherwise, the concavification function has a *two-piece* shape characterized by the function $\tilde{U}$ defined in (3.14) (See Figure 3.2). Therefore, the shape type of the concavification function of $U \circ \phi$ depends on the concavity of $\tilde{U}_3$ and hence, it is characterized by Proposition 3.3.4.

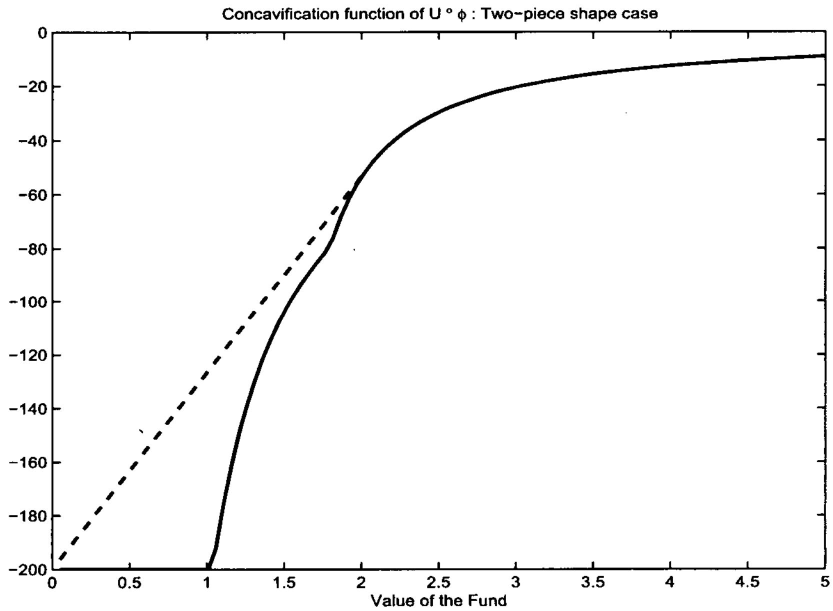

Figure 3.2: **Two-piece shape Concavification of** $U \circ \phi$. This is the graph of $U \circ \phi$ and its concavification for a case in which the concavification function has a *two-piece* shape. The parameters are: $X_0 = 0.005$, $p = 0.01$, $q = 0.02$, $B = 1.04$, $K = 0.76$ and $\gamma = 2$. $U \circ \phi$ and its concavification function coincide, for this particular example, over the interval $[2.0544, \infty)$. The concavification function displayed in this graph corresponds to the derived utility function of Figure 3.1.

3.4 Optimal Solution

The main reason for the construction of the concavification function of $U \circ \phi$ is the definition of an alternative problem to (3.3), by replacing $U \circ \phi$ by its concavification function, for which standard techniques can be applied and whose solution is also a solution of (3.3). In this section we solve this alternative problem and demonstrate that its solution is optimal for (3.3) too [4].

[4] Refer to Corollary 3.4.1.

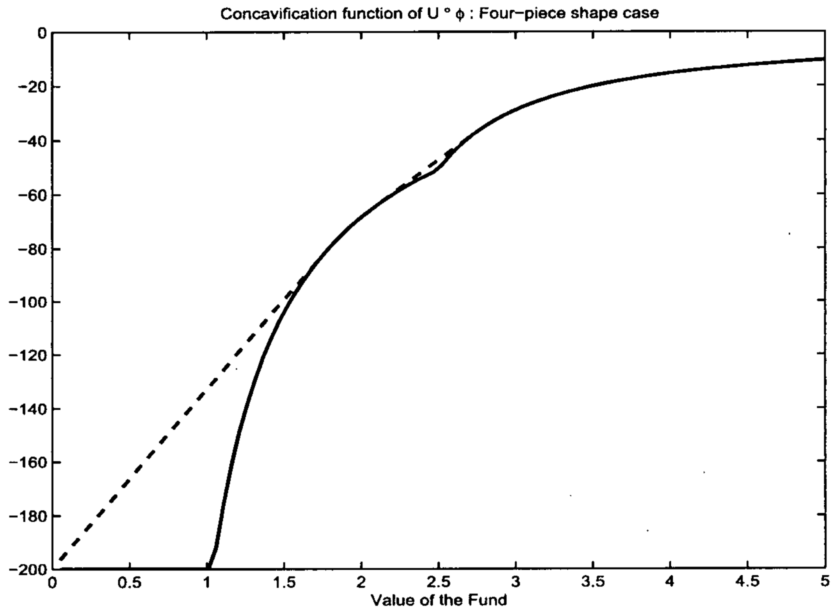

Figure 3.3: **Four-piece shape Concavification of** $U \circ \phi$. This is the graph of $U \circ \phi$ and its concavification for a case in which the concavification function has a *four-piece* shape. The parameters are: $X_0 = 0.005$, $p = 0.01$, $q = 0.02$, $B = 1.04$, $K = 1.46$ and $\gamma = 2$. $U \circ \phi$ and its concavification function coincide, for this particular example, over the intervals $[1.7611, 2.0858]$ and $[2.7391, \infty)$.

3.4.1 Optimal Fund Value

Consider the alternative problem

$$\begin{aligned} &\sup_{V_T} \quad E\left[\tilde{U}(V_T)\right] \\ &s.t. \\ &\qquad\quad E\left[\xi_T V_T\right] \leq V_0 \\ &\qquad\quad V_T \geq 0 \end{aligned} \tag{3.17}$$

where $\tilde{U}$ is the concavification function of $U \circ \phi$, and where $(\xi_t)_{t\in[0,T]}$ is the pricing kernel as defined in (3.4). The following proposition states explicitly the optimal terminal fund value of Problem (3.17).

Proposition 3.4.1 *Let $\gamma > 0$ and V_1 and V_2 as defined in Lemmas 3.3.1 and 3.3.3, respectively. Then, if $V_2 \geq V_1$ and*

$$g_{4p}(\lambda) \equiv E\left[\xi_T h_1(\lambda\xi_T)\, 1_{\{V_2 \geq h_1(\lambda\xi_T) > V_1\}}\right] + E\left[\xi_T h_2(\lambda\xi_T)\, 1_{\{h_1(\lambda\xi_T) > V_2\}}\right] < \infty \tag{3.18}$$

for all $\lambda \geq 0$, where

$$\begin{aligned} h_1(y) &= \frac{\left(\frac{y}{p}\right)^{-1/\gamma} - X_0}{p} + B\ , \\ h_2(y) &= \frac{\left(\frac{y}{p+q}\right)^{-1/\gamma} - X_0 + qK}{p+q} + B\ , \end{aligned}$$

the optimal terminal fund value of Problem (3.17) is

$$\begin{aligned} V_T^* &= \left[B + \tfrac{1}{p}\left[\left(\tfrac{\lambda}{p}\right)^{-\frac{1}{\gamma}} \xi_T^{-\frac{1}{\gamma}} - X_0\right]\right] 1_{\left\{p[X_0 + p(V_2 - B)]^{-\gamma} \,\leq\, \lambda\xi_T \,<\, p[X_0 + p(V_1 - B)]^{-\gamma}\right\}} \\ &+ \left[B + \left(\tfrac{q}{p+q}\right) K + \left(\tfrac{1}{p+q}\right)\left[\left(\tfrac{\lambda}{p+q}\right)^{-\frac{1}{\gamma}} \xi_T^{-\frac{1}{\gamma}} - X_0\right]\right] 1_{\left\{\lambda\xi_T \,<\, p[X_0 + p(V_2 - B)]^{-\gamma}\right\}} \end{aligned} \tag{3.19}$$

where λ is a nonnegative scalar that satisfies

$$E\left[\xi_T V_T^*\right] = V_0 \tag{3.20}$$

where V_0 is the initial fund value. Otherwise, if

$$g_{2p}(\lambda) \equiv E\left[\xi_T h_2(\lambda\xi_T)\, 1_{\{h_2(\lambda\xi_T) > V_4\}}\right] < \infty \tag{3.21}$$

for all $\lambda \geq 0$, where V_4 is defined as in Lemma 3.3.2, then the optimal terminal fund value is given by

$$V_T^* = \left[B + \left(\frac{q}{p+q}\right)K + \left(\frac{1}{p+q}\right)\left[\left(\frac{\lambda}{p+q}\right)^{-\frac{1}{\gamma}} \xi_T^{-\frac{1}{\gamma}} - X_0\right]\right] 1_{\{\lambda\xi_T \,<\, (p+q)[X_0+p(V_4-B)+q(V_4-B-K)]^{-\gamma}\}} \tag{3.22}$$

where λ is a nonnegative scalar that satisfies condition (3.20).

Proof: Assumption (3.18) implies that g_{4p} is continuous and strictly decreasing and hence, it assures that Equation (3.20) has a solution for some $\lambda \geq 0$. Suppose that $V_2 \geq V_1$. Therefore, the concavification function of $U \circ \phi$ is

$$\tilde{U}(v) = \begin{cases} (U \circ \phi)(V_1) + (U \circ \phi)'(V_1)(v - V_1) & \text{, if } v < V_1. \\ (U \circ \phi)(v) & \text{, if } v \in [V_1, V_2] \cup (V_3, \infty). \\ (U \circ \phi)(V_2) + (U \circ \phi)'(V_2)(v - V_2) & \text{, if } v \in (V_2, V_3]. \end{cases}$$

By standard theory (Karatzas et al (1987) and Cox and Huang (1989)),

$$\tilde{U}'(V_T^*) = \lambda\xi_T \ a.s. \tag{3.23}$$

where λ is a nonnegative scalar that satisfies Equation (3.20). Therefore, condition (3.23) and definition of $\tilde{U}$ implies

$$\tilde{U}'(V_T^*(\omega)) = \begin{cases} p\left[X_0 + p\left(V_1 - B\right)\right]^{-\gamma} & \text{, if } \lambda\xi_T(\omega) \in C_1 \\ (p+q)\left[X_0 + p\left(V_T^*(\omega) - B\right) + q\left(V_T^*(\omega) - B - K\right)\right]^{-\gamma} & \text{, if } \lambda\xi_T(\omega) \in C_2 \\ p\left[X_0 + p\left(V_T^*(\omega) - B\right)\right]^{-\gamma} & \text{, otherwise} \end{cases}$$

for all $\omega \in \Omega$ for which equation (3.20) has a a solution, and where

$$\begin{aligned} C_1 &= \left\{x | x = p\left[X_0 + p\left(V_1 - B\right)\right]^{-\gamma}\right\} \\ C_2 &= \left\{x | x < p\left[X_0 + p\left(V_2 - B\right)\right]^{-\gamma}\right\} \end{aligned}$$

Therefore, if $\lambda\xi_T(\omega) < p\left[X_0 + p\left(V_1 - B\right)\right]^{-\gamma}$, for some $\omega \in \Omega$, then

$$
\begin{aligned}
V_T^*(\omega) &= \left(\tilde{U}'\right)^{-1}\left(\lambda\xi_T(\omega)\right) \\
&= \begin{cases}
\left[B + \left(\frac{q}{p+q}\right)K + \left(\frac{1}{p+q}\right)\left[\left(\frac{\lambda}{p+q}\right)^{-\frac{1}{\gamma}}\xi_T(\omega)^{-\frac{1}{\gamma}} - X_0\right]\right] & \text{, if } \lambda\xi_T(\omega) \in C_2 \\
\left[B + \frac{1}{p}\left[\left(\frac{\lambda}{p}\right)^{-\frac{1}{\gamma}}\xi_T(\omega)^{-\frac{1}{\gamma}} - X_0\right]\right] & \text{, if } \lambda\xi_T(\omega) \in \left(C_1 \cup C_2\right)'
\end{cases}
\end{aligned}
$$

Note that if $\lambda\xi_T(\omega) \in C_1$ then $V_T^*(\omega) \in (0, V_1]$ ($\tilde{U}'$ is not defined at $v = 0$), while if $\lambda\xi_T(\omega) > p\left[X_0 + p\left(V_1 - B\right)\right]^{-\gamma}$, then equation (3.20) does not have a solution for $\lambda \geq 0$. We extend the definition of $\tilde{U}'$ to consider the latter two cases using the following *mapping* [5]:

$$
\tilde{u}'\left(V_T(\omega)\right) = \begin{cases}
p\left[X_0 + p\left(V_1 - B\right)\right]^{-\gamma} & \text{, if } \lambda\xi_T(\omega) \in C_1 \\
(p+q)\left[X_0 + p\left(v - B\right) + q\left(v - B - K\right)\right]^{-\gamma} & \text{, if } \lambda\xi_T(\omega) \in C_2 \\
\left[p\left[X_0 + p\left(V_1 - B\right)\right], \infty\right) & \text{, if } \lambda\xi_T(\omega) \in C_3 \\
p\left[X_0 + p\left(v - B\right)\right]^{-\gamma} & \text{, otherwise}
\end{cases}
$$

where $C_3 = \left\{x | x > p\left[X_0 + p\left(V_1 - B\right)\right]^{-\gamma}\right\}$. Hence, condition (3.23) generalizes to

$$
\tilde{u}'\left(V_T^*\right) = \lambda\xi_T \ a.s.
$$

from which condition $\lambda\xi_T(\omega) \geq p\left[X_0 + p\left(V_1 - B\right)\right]^{-\gamma}$ implies $V_T^*(\omega) \in [0, V_1]$. Therefore, the optimal terminal fund value can be expressed as

$$
\begin{aligned}
V_T^* &= \left[B + \frac{1}{p}\left[\left(\frac{\lambda}{p}\right)^{-\frac{1}{\gamma}}\xi_T^{-\frac{1}{\gamma}} - X_0\right]\right] 1_{\left\{p\left[X_0 + p\left(V_2 - B\right)\right]^{-\gamma} \leq \lambda\xi_T < p\left[X_0 + p\left(V_1 - B\right)\right]^{-\gamma}\right\}} \\
&+ \left[B + \left(\frac{q}{p+q}\right)K + \left(\frac{1}{p+q}\right)\left[\left(\frac{\lambda}{p+q}\right)^{-\frac{1}{\gamma}}\xi_T^{-\frac{1}{\gamma}} - X_0\right]\right] 1_{\left\{\lambda\xi_T < p\left[X_0 + p\left(V_2 - B\right)\right]^{-\gamma}\right\}}
\end{aligned}
$$

[5]Formally speaking, $\tilde{u}'$ is a *subdifferential* of $\tilde{U}'$ (e.g. Rockafellar (1970)).

The proof of expression (3.22) is similar.

Q.E.D.

Corollary 3.4.1 *The optimal solution of Problem* (3.17) *is also optimal for Problem* (3.3).

Proof: By construction of the concavification function $\tilde{U}$ it holds the condition

$$E\left[\tilde{U}(V_T)\right] \geq E\left[U(V_T)\right]$$

for V_T such that

$$\begin{aligned} &E\left[\xi_T V_T\right] \leq V_0 \\ &V_T \geq 0 \end{aligned}$$

From Proposition 3.4.1 it is clear that

$$E\left[\tilde{U}(V_T^*)\right] = E\left[U(V_T^*)\right]$$

where V_T^* is the optimal solution of Problem (3.17). Therefore, it follows that V_T^* must be an optimal solution of Problem (3.3).

Q.E.D.

3.4.2 Optimal Risk Taking

We now explicitly characterize the manager's optimal risk-taking strategy.

Proposition 3.4.2 *Let $\gamma > 0$ and V_1 and V_2 as defined in Lemmas* 3.3.1 *and* 3.3.3, *respectively. If $V_2 \geq V_1$, then the manager's optimal volatility choice for any $t \in [0, T)$ is given by*

$$
\begin{aligned}
\nu_t^* &= \alpha(1-\gamma^*) \\
&+ \left[\left(B - \tfrac{X_0 - qK}{p+q}\right) e^{-\mu(T-t)}\right] \left(\tfrac{1}{V_t^*}\right) \left[\tfrac{n(d_{T-t}(m_2,1))}{\sqrt{T-t}} - \alpha(1-\gamma^*)\mathcal{N}\left(d_{T-t}(m_2,1)\right)\right] \\
&+ \left[\left(B - \tfrac{X_0}{p}\right) e^{-\mu(T-t)}\right] \left(\tfrac{1}{V_t^*}\right) \left[\tfrac{n(d_{T-t}(m_1,1)) - n(d_{T-t}(m_2,1))}{\sqrt{T-t}}\right] \\
&- \left[\left(B - \tfrac{X_0}{p}\right) e^{-\mu(T-t)}\right] \alpha(1-\gamma^*) \left[\mathcal{N}\left(d_{T-t}(m_1,1)\right) - \mathcal{N}\left(d_{T-t}(m_2,1)\right)\right] \left(\tfrac{1}{V_t^*}\right) \\
&+ \left[V_t^* - \left(B - \tfrac{X_0 - qK}{p+q}\right) e^{-\mu(T-t)} \mathcal{N}\left(d_{T-t}(m_2,1)\right)\right] \left(\tfrac{1}{V_t^*}\right) G(X_0, p, B, q, K, \gamma) \\
&- \left[\left(B - \tfrac{X_0}{p}\right) e^{-\mu(T-t)}\right] \left[\mathcal{N}\left(d_{T-t}(m_1,1)\right) - \mathcal{N}\left(d_{T-t}(m_2,1)\right)\right] \left(\tfrac{1}{V_t^*}\right) G(X_0, p, B, q, K, \gamma)
\end{aligned}
$$

where $\gamma^* = 1 - 1/\gamma$,

$$
G(X_0, p, B, q, K, \gamma) = \frac{\frac{(p+q)^{-\gamma^*}}{\sqrt{T-t}} n\left(d_{T-t}(m_2,\gamma^*)\right) + \frac{p^{-\gamma^*}}{\sqrt{T-t}} \left[n\left(d_{T-t}(m_1,\gamma^*)\right) - n\left(d_{T-t}(m_2,\gamma^*)\right)\right]}{(p+q)^{-\gamma^*} \mathcal{N}\left(d_{T-t}(m_2,\gamma^*)\right) + p^{-\gamma^*} \left[\mathcal{N}\left(d_{T-t}(m_1,\gamma^*)\right) - \mathcal{N}\left(d_{T-t}(m_2,\gamma^*)\right)\right]}
$$

and $d_u(\cdot,\cdot)$ is the function defined by

$$
d_u(m,\beta) \equiv \left[\log m - \left(-\mu u + \alpha^2 u \left(\beta - \frac{1}{2}\right)\right)\right] / \left(\alpha\sqrt{u}\right) \ , \tag{3.24}
$$

$\mathcal{N}(\cdot)$ *and* $n(\cdot)$ *are the standard normal cumulative distribution and density,* $\gamma^* = 1 - 1/\gamma$, *and* λ *is the nonnegative scalar that satisfies Equation (3.47). Otherwise, if* $V_2 < V_1$, *the manager's optimal volatility choice for any* $t \in [0, T)$ *is*

$$
\begin{aligned}
\nu_t^* &= \alpha(1-\gamma^*) \\
&+ \left[\left(B - \tfrac{X_0 - qK}{p+q}\right) e^{-\mu(T-t)}\right] \left(\tfrac{1}{V_t^*}\right) \left[\tfrac{n(d_{T-t}(m_4,1))}{\sqrt{T-t}} - \alpha(1-\gamma^*)\mathcal{N}\left(d_{T-t}(m_4,1)\right)\right] \\
&+ \left[V_t^* - \left(B - \tfrac{X_0 - qK}{p+q}\right) e^{-\mu(T-t)} \mathcal{N}\left(d_{T-t}(m_4,1)\right)\right] \left(\tfrac{1}{V_t^*}\right) \left[\tfrac{n(d_{T-t}(m_4,\gamma^*))}{\mathcal{N}(d_{T-t}(m_4,\gamma^*))\sqrt{T-t}}\right] .
\end{aligned}
$$

Proof: Assume that $V_2 \geq V_1$. Therefore, the optimal fund value for any time $t \in [0, T]$ is given by (See Proposition 3.7.1 in the Appendix 3)

$$
\begin{aligned}
V_t^* &= \left[B - \tfrac{X_0 - qK}{p+q}\right] e^{-\mu(T-t)} \mathcal{N}\left(d_{T-t}(m_2,1)\right) \\
&+ \left[(p+q)^{-\gamma^*} \lambda^{\gamma^*-1} \xi_t^{\gamma^*-1}\right] e^{-\left(\mu + \frac{\alpha^2(1-\gamma^*)}{2}\right)\gamma^*(T-t)} \mathcal{N}\left(d_{T-t}\left(m_2,\gamma^*\right)\right) \\
&+ \left[B - \tfrac{X_0}{p}\right] e^{-\mu(T-t)} \left[\mathcal{N}\left(d_{T-t}(m_1,1)\right) - \mathcal{N}\left(d_{T-t}(m_2,1)\right)\right] \\
&+ \left[p^{-\gamma^*} \lambda^{\gamma^*-1} \xi_t^{\gamma^*-1}\right] e^{-\left(\mu + \frac{\alpha^2(1-\gamma^*)}{2}\right)\gamma^*(T-t)} \left[\mathcal{N}\left(d_{T-t}(m_1,\gamma^*)\right) - \mathcal{N}\left(d_{T-t}(m_2,\gamma^*)\right)\right]
\end{aligned}
$$

where

$$
m_i = \left[\frac{p}{\lambda \xi_t}\right] \left[X_0 + p\left(V_i - B\right)\right]^{-\gamma} ,
$$

for $i = 1, 2$. Therefore,

$$
\begin{aligned}
dV_t^* &= \left[B - \tfrac{X_0 - qK}{p+q}\right] e^{-\mu(T-t)} d\mathcal{N}\left(d_{T-t}(m_2,1)\right) \\
&+ \left[B - \tfrac{X_0 - qK}{p+q}\right] \mathcal{N}\left(d_{T-t}(m_2,1)\right) e^{-\mu(T-t)} \mu dt \\
&+ \left[B - \tfrac{X_0}{p}\right] e^{-\mu(T-t)} d\left[\mathcal{N}\left(d_{T-t}(m_1,1)\right) - \mathcal{N}\left(d_{T-t}(m_2,1)\right)\right] \\
&+ \left[B - \tfrac{X_0}{p}\right] \left[\mathcal{N}\left(d_{T-t}(m_1,1)\right) - \mathcal{N}\left(d_{T-t}(m_2,1)\right)\right] e^{-\mu(T-t)} \mu dt \\
&+ \left[(p+q)^{-\gamma^*} \lambda^{\gamma^*-1} \xi_t^{\gamma^*-1}\right] d\left(e^{-\left(\mu + \frac{\alpha^2(1-\gamma^*)}{2}\right)\gamma^*(T-t)} \mathcal{N}\left(d_{T-t}\left(m_2,\gamma^*\right)\right)\right) \\
&+ \left[p^{-\gamma^*} \lambda^{\gamma^*-1} \xi_t^{\gamma^*-1}\right] d\left(e^{-\left(\mu + \frac{\alpha^2(1-\gamma^*)}{2}\right)\gamma^*(T-t)} \left[\mathcal{N}\left(d_{T-t}(m_1,\gamma^*)\right) - \mathcal{N}\left(d_{T-t}(m_2,\gamma^*)\right)\right]\right)
\end{aligned}
$$

By Itô's lemma,

$$
\begin{aligned}
d\left(\mathcal{N}\left(d_{T-t}(m_i,\beta)\right)\right) &= \left[\tfrac{1}{2} n'\left(d_{T-t}(m_i,\beta)\right)\right] dt + n\left(d_{T-t}(m_i,\beta)\right) d\left(d_{T-t}(m_i,\beta)\right) \quad \text{,for } i = 1,2, \\
d\left(\xi_t^{\gamma^*-1}\right) &= \left[\frac{\mu + \frac{\alpha^2}{2}}{\gamma} + \frac{\alpha^2}{2\gamma^2}\right] \xi_t^{\gamma^*-1} dt + \left(\frac{\alpha}{\gamma}\right) \xi_t^{\gamma^*-1} dz_t
\end{aligned}
$$

and,

$$
\begin{aligned}
d\left(d_{T-t}(m_i,\beta)\right) &= d\left(\frac{\log\left(\left[\frac{p}{\lambda\xi_t}\right]\left[X_0 + p(V_i - B)\right]^{-\gamma}\right)}{\alpha\sqrt{T-t}}\right) \\
&= \frac{\log\left(\left[\frac{p}{\lambda\xi_t}\right]\left[X_0 + p(V_i - B)\right]^{-\gamma}\right)}{\alpha(T-t)} dt + \frac{1}{\alpha\sqrt{T-t}} d\left(\log\left(\left[\frac{p}{\lambda\xi_t}\right]\left[X_0 + p\left(V_i - B\right)\right]^{-\gamma}\right)\right) \\
&= \frac{1}{\alpha\sqrt{T-t}}\left[\mu + \frac{\alpha^2}{2} + \log\left(\left[\frac{p}{\lambda\xi_t}\right]\left[X_0 + p\left(V_i - B\right)\right]^{-\gamma}\right)\right] dt + \frac{1}{\sqrt{T-t}} dz_t
\end{aligned}
$$

for $\beta = 1, \gamma^* = 1 - 1/\gamma$. Thus, the *stochastic part* of V_t^* is given by

$$
\begin{aligned}
& \left[B - \tfrac{X_0 - qK}{p+q}\right] e^{-\mu(T-t)} \tfrac{n(d_{T-t}(m_2,1))}{\sqrt{T-t}} \\
+ \quad & \left[B - \tfrac{X_0}{p}\right] e^{-\mu(T-t)} \left[\tfrac{n(d_{T-t}(m_1,1))}{\sqrt{T-t}} - \tfrac{n(d_{T-t}(m_2,1))}{\sqrt{T-t}}\right] \\
+ \quad & \left[(p+q)^{-\gamma^*} \lambda^{\gamma^*-1} \xi_t^{\gamma^*-1}\right] e^{-\left(\mu + \frac{\alpha^2(1-\gamma^*)}{2}\right)\gamma^*(T-t)} \left[\mathcal{N}\left(d_{T-t}\left(m_2, \gamma^*\right)\right)\left(\tfrac{\alpha}{\gamma}\right) + \tfrac{n(d_{T-t}(m_2,\gamma^*))}{\sqrt{T-t}}\right] \\
+ \quad & \left[p^{-\gamma^*} \lambda^{\gamma^*-1} \xi_t^{\gamma^*-1}\right] e^{-\left(\mu + \frac{\alpha^2(1-\gamma^*)}{2}\right)\gamma^*(T-t)} \left[\mathcal{N}\left(d_{T-t}(m_1, \gamma^*)\right)\left(\tfrac{\alpha}{\gamma}\right) + \tfrac{n(d_{T-t}(m_1,\gamma^*))}{\sqrt{T-t}}\right] \\
- \quad & \left[p^{-\gamma^*} \lambda^{\gamma^*-1} \xi_t^{\gamma^*-1}\right] e^{-\left(\mu + \frac{\alpha^2(1-\gamma^*)}{2}\right)\gamma^*(T-t)} \left[\mathcal{N}\left(d_{T-t}(m_2, \gamma^*)\right)\left(\tfrac{\alpha}{\gamma}\right) + \tfrac{n(d_{T-t}(m_2,\gamma^*))}{\sqrt{T-t}}\right]
\end{aligned}
\tag{3.25}
$$

On the other hand, V_t^* must satisfy

$$
dV_t^* = \left[\mu V_t^* + \alpha \nu_t^* V_t^*\right] dt + \nu_t^* V_t^* dz_t \ .
$$

Therefore, $\nu_t^* V_t^*$ must equate expression (3.25), from which the result follows.

Q.E.D.

3.5 Induced Incentives

Based on the theoretical results of the previous section, we analyze and illustrate in this section the different fund value and risk incentive profiles that can be induced by the compensation scheme ϕ.

3.5.1 Fund Value Incentives

The optimal choice of terminal fund value implies a sort of *gambling behavior* of the manager. In the two-piece shape case (See Equation (3.22)), the solution implies an all-or-nothing strategy. Either the manager is far out ($V_T^* = 0$) or well in the money ($V_T^* > V_4 > B+K > B$) for the two embedded options of the manager's compensation package. In the four-piece shape case (See Equation (3.19)), the optimal solution implies that the manager is either far out ($V_T^* = 0$) or well in the money for *at least* one of the two embedded options of his compensation package. That is, either $V_T^* > V_3 > B + K > B$ or $V_2 \geq V_T^* > V_1 > B$. This *gambling behavior* of the manager is up to some extent expected. The manager will get the same compensation, and thus the same utility, either if $V_T = 0$ or $0 < V_T < B$, although he will have to use part of his budget in the latter case. Hence, if the manager chooses to be out of the money, he will be as far out of the money as possible. On the other hand, if the manager decides to be in the money, he will target for firm values that will provide him a sufficiently small marginal utility. For instance, in the two-piece shape case, the manager aims for fund values with smaller marginal utility than the one provided by a fund value equal to V_4, and characterized in terms of the pricing kernel ξ_T (See Figure 3.4). The intuition behind the threshold value of V_4 relies in the fact

$$\zeta \left(U \circ \phi\right)(0) + (1-\zeta)\left(U \circ \phi\right)(V_4) \geq \left(U \circ \phi\right)\left(\zeta \cdot 0 + (1-\zeta) V_4\right) \ , \forall\, \zeta \in (0,1) \ .$$

That is, the weighted average utility of the zero and V_4 payoffs, surpasses the utility of the weighted average of those payoffs. Therefore, any strategy that implies to target for fund values between zero and V_4 could be dominated, in the sense of expected utility, by a strategy that takes either the value of zero, with an appropriate chosen probability $\mathcal{P}$, or V_4 with probability $1 - \mathcal{P}$ (See Appendix 3).

For the four-piece shape case, the manager targets for fund values with a marginal utility necessarily smaller than that associated to the fund value V_1, but without aiming for fund values between V_2 and V_3. The reason for excluding $[V_2, V_3)$ is that the utility of any fund

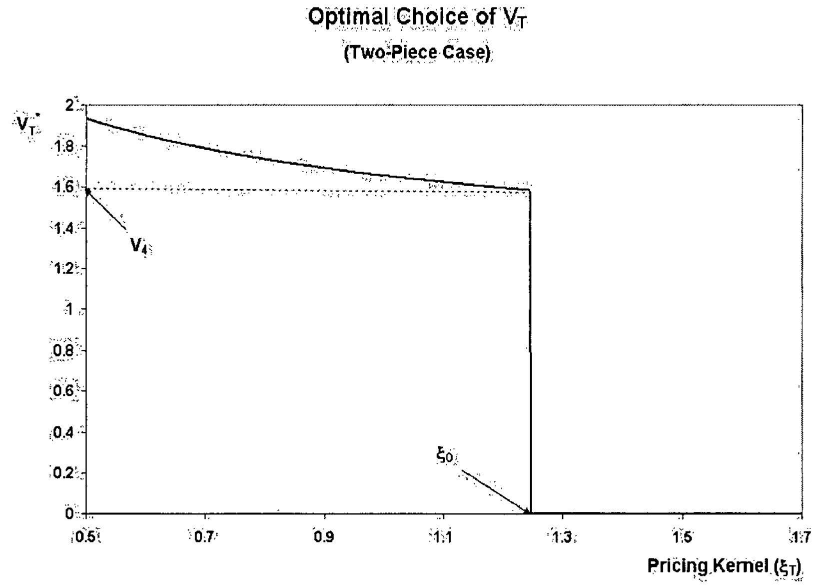

Figure 3.4: **Optimal Choice of V_T: Two-piece shape case.** This is a graph that shows the optimal terminal value as a function of the pricing kernel ξ_T for a particular example. The parameters are: $X_0 = 0.005$, $p = 0.01$, $q = 0.02$, $B = 1.04$, $K = 0.16$, $\gamma = 2$, $V_0 = 1$, $\mu = 0.04$, $\alpha = 0.4$, $T = 1$. The relevant tangency point in this case is $V_4 = 1.5838$, which corresponds to the threshold value $\xi_0 = (1/\lambda)\,(U \circ \phi)'\,(V_4) = 1.2462$.

value between V_2 and V_3 is dominated by the corresponding weighted average of the utility values of V_2 and V_3. Therefore, in this case, the manager targets for fund values in the range $(V_1, V_2]$ or the range (V_3, ∞). The election between these two ranges can be characterized in terms of the pricing kernel ξ_T (See Figure 3.5).

3.5.2 Risk Incentives

One of our primary goals is to explore the risk incentives induced by the compensation scheme ϕ and analyze if these incentives present the same unexpected features documented by Carpenter (2000). In particular, we are interested in studying if ϕ can induce the manager to take lower volatility levels than investors would like. In order to do such analysis, we

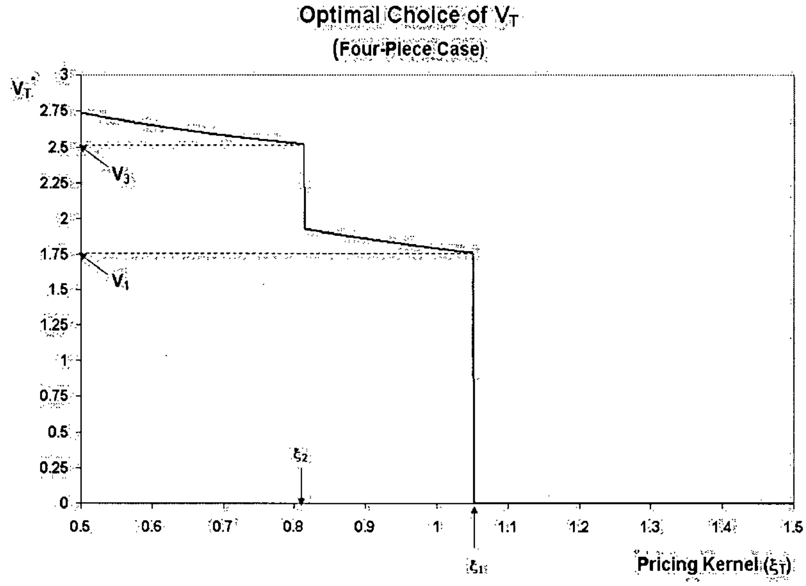

Figure 3.5: **Optimal Choice of V_T: Four-piece shape case**. This is the graph of the optimal terminal value as a function of the pricing kernel ξ_T. The parameters are: $X_0 = 0.005$, $p = 0.01$, $q = 0.02$, $B = 1.04$, $K = 1.26$, $\gamma = 2$, $V_0 = 1$, $\mu = 0.04$, $\alpha = 0.4$, $T = 1$. The relevant tangency points are in this case $V_1 = 1.7611$ and $V_3 = 2.5147$, with the corresponding threshold values $\xi_1 = 1.0524$ and $\xi_2 = 0.8144$.

consider first an example of the single option case (i.e. $q = 0$ %) for which, contrary to what it is supposed to be designed for, the compensation scheme induces less risk (volatility) taking of the manager than if he were trading his own account (characterized by Merton's constant (0.2 in this case), see Figure 3.6). Then, we study the impact of adding the second option payoff into the compensation scheme. This analysis is divided in three parts. First, we analyze the effect of parameter K for a fixed $q \neq 0$. Second, we study the impact of q, for a fixed K. Third, we analyze the combined effect of K and q.

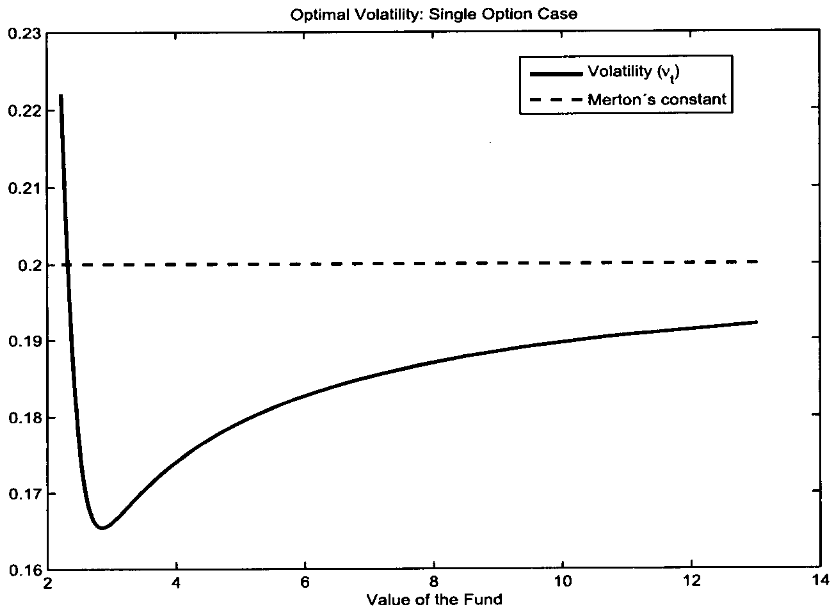

Figure 3.6: **Optimal Volatility: Single-Option Case**. This graph shows the optimal volatility strategy for the case of a single option (i.e. $q = 0\%$) with the following parameters: $X_0 = 0.005$, $p = 0.01$, $B = 1.04$, $\gamma = 2$, $V_0 = 1$, $\mu = 0.04$, $\alpha = 0.4$, $T = 1$.

K Effect

We computed the optimal volatility curve, as a function of the fund value, for different values of K using the same parameters of the single-option example considered in Figure 3.6, except that we now set q equal to 2 %. We choose $q = 2\%$ to make our compensation scheme ϕ *more convex* than with $q = 0$ % (See Figure 3.7), for fund values greater or equal than $B + K$. This should, intuitively, make more attractive for the manager to target for fund values greater or equal than $B + K$. Figures 3.8 and 3.9 show these volatility curves.

A primer observation that we can derived from Figures 3.8 and 3.9 is that the compensation scheme ϕ shows, for $q \neq 0$, the same unexpected features observed for the single-option case studied by Carpenter (2000). For instance, all volatility curves lie below Merton's

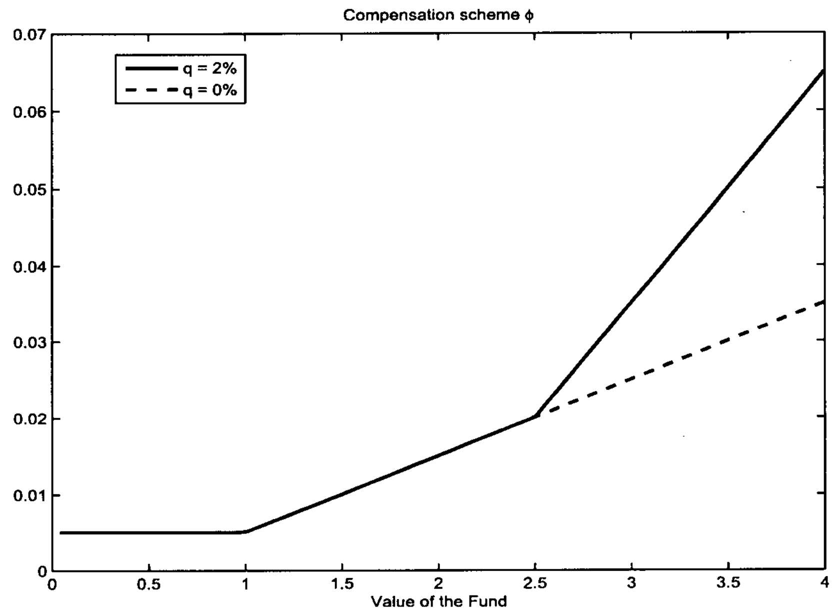

Figure 3.7: **Compensation Scheme Shape**. This is a graph shows the shape of the compensation scheme for the single option case ($q = 0\%$), and the double option case ($q = 2\%$).

constant for most of the range of fund values considered.

Other observations that can be made from Figures 3.8 and 3.9 are the following:

1. The minimal value of the optimal volatility curve for the case $q = 0$ % (around 0.16) is greater or equal than the minimal value associated with each of the volatility curves plotted. That is, for the particular example and values of K considered, the effect of the second option is inducing less volatility taking than with a single option ($q = 0$ %).

2. For values of K associated with the same concavification function shape (Figure 3.8 corresponds to a two-piece shape, while Figure 3.9 is associated with a four-piece shape), we observe that the larger the value of K,

   (a) the smaller the minimal value of the optimal volatility curve, and

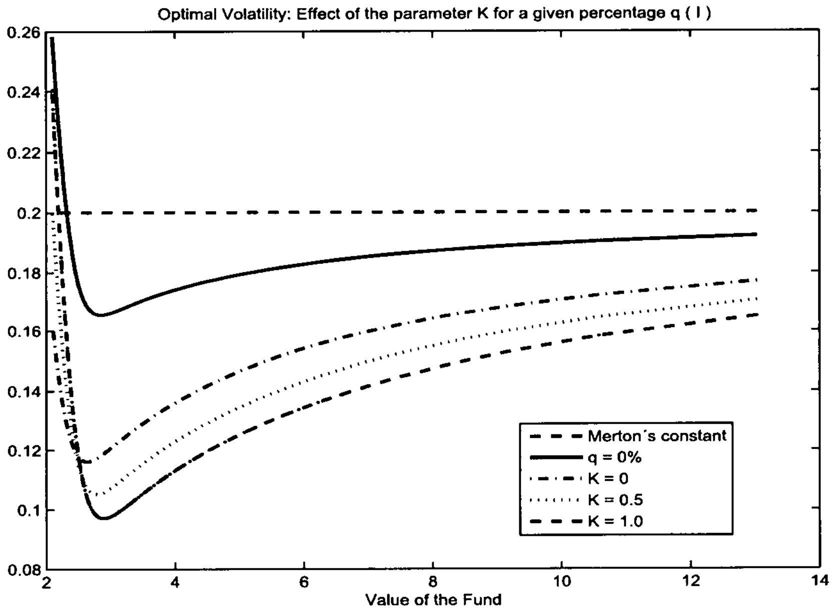

Figure 3.8: K **Effect (I)**. This graph shows the effect of parameter K on the optimal volatility decisions, for a fixed q ($= 2$ %). The parameters used are: $X_0 = 0.005$, $p = 0.01$, $B = 1.04$, $\gamma = 2$, $V_0 = 1$, $\mu = 0.04$, $\alpha = 0.4$, and $T = 1$. The values of K considered are 0, 0.5, and 1.0. The concavification function associated with these values has a two-piece shape.

(b) the steeper the volatility curve for values smaller than the fund value at which the minimum volatility is reached, which we denote by $\overline{V}(K)$ (e.g. $\overline{V}(1) \approx 2.6$). For fund values larger than $\overline{V}(K)$, the volatility curve has essentially the same type of curvature for all the values of K considered.

3. The minimal values of the optimal volatility curves that are shown in Figure 3.9, which correspond to values of K *larger than one*, are smaller than the corresponding minimal values showed in Figure 3.8, which corresponds to values of K *smaller than one* (see also Figure 3.10). That is, for the particular parameters considered, the minimal volatility value decreases as K increases, regardless of the shape of the concavification function. Unfortunately, this is not always true as it is shown in Figure 3.11.

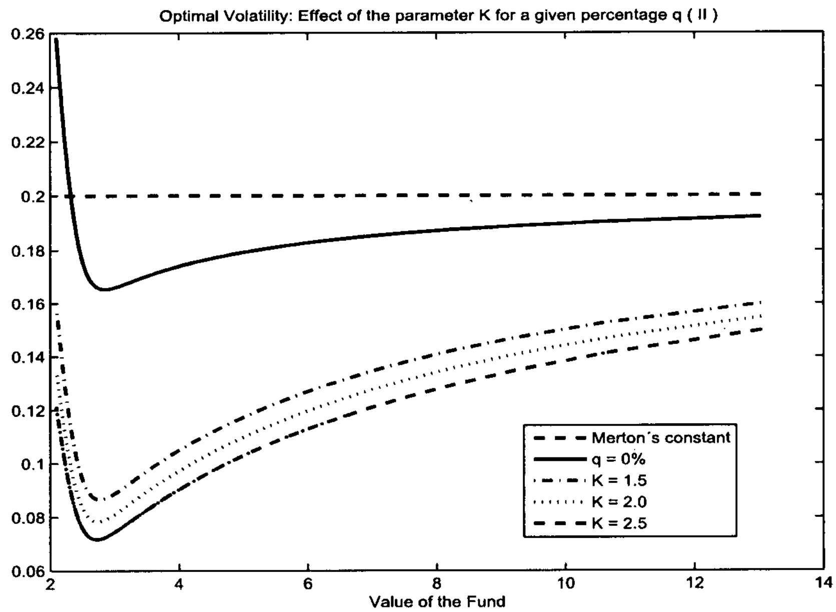

Figure 3.9: K **Effect (II)**. This graph exemplifies the effect of parameter K on the optimal volatility decisions, for a fixed q ($= 2$ %). The parameters used are: $X_0 = 0.005$, $p = 0.01$, $B = 1.04$, $\gamma = 2$, $V_0 = 1$, $\mu = 0.04$, $\alpha = 0.4$, and $T = 1$. The values of K considered are 1.5, 2.0, and 2.5. The concavification function associated with these values has a four-piece shape.

Observation 2 can be explained in the following manner: the higher the second option exercise price, the smaller the manager's incentive to pursue higher risk-taking once he is in the money, but the higher the risk incentives if the manager is out of the money. Nevertheless, Observation 2 does not hold always. For instance, if q is relatively large with respect to p then, for sufficiently large values of K, it holds that the larger the value of K, the larger the minimal value of the optimal volatility curve (See Figure 3.11). This fact can be reconcile with Observation 2 by arguing that the manager will pursue higher risk-taking if the relative weight of the second option payoff with respect to the compensation scheme is sufficiently large, even in the case that both options are in the money.

In addition to the piecewise monotonic behavior of the minimal optimal volatility as

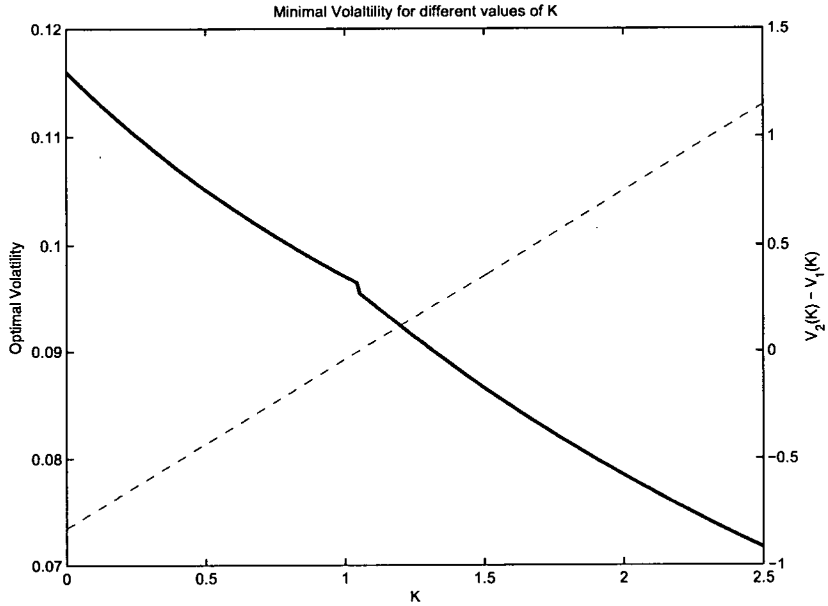

Figure 3.10: **Minimal Optimal Volatility: K Effect (I)**. This graph exemplifies the effect of the parameter K on the minimal volatility decision, for a fixed q ($= 2$ %). The solid line represents the minimal volatility decision, while the dashed lined corresponds to the difference between the tangent points V_2 and V_1, for each value of K considered. The concavification function has a four-piece shape if and only if $V_2 - V_1 > 0$ (Proposition 3.3.4). The parameters used are: $X_0 = 0.005$, $p = 0.01$, $B = 1.04$, $\gamma = 2$, $V_0 = 1$, $\mu = 0.04$, $\alpha = 0.4$, and $T = 1$.

a function of K, Figures 3.10 and 3.11 show an abrupt change of the minimal volatility decision at $K \approx 1.05$ and $K \approx 1.38$, respectively. Coincidentally, these values of K correspond, respectively, to the threshold values, *ceteris paribus*, between a two-piece and a four-piece shape concavification function. The size and direction of this discontinuity (*upward* or *downward*) depend on the parameter q. This phenomena is relevant because of its policy implications: it means that it is possible that managers with similar compensation schemes behave in a different way.

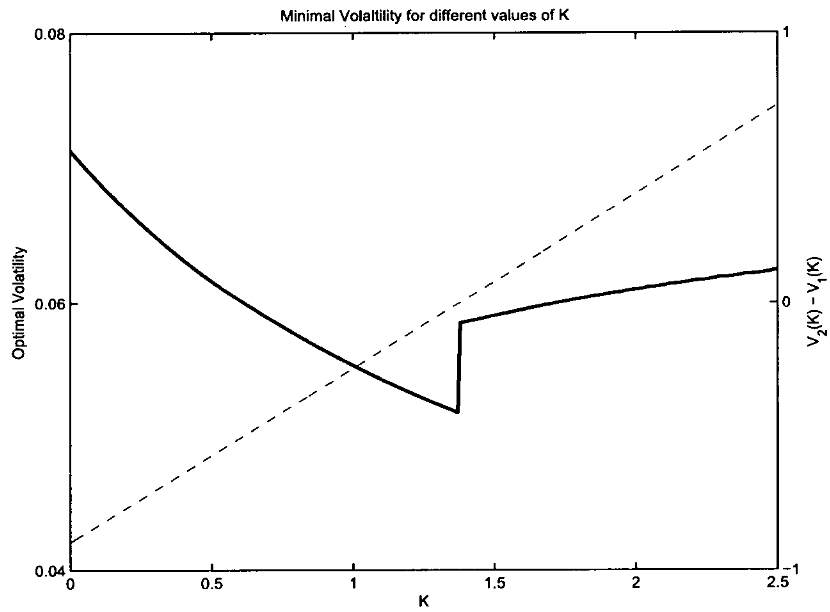

Figure 3.11: **Minimal Optimal Volatility: K Effect (II)**. This graph exemplifies the effect of the parameter K on the minimal volatility decision, for a fixed q ($= 10$ %). The solid line represents the minimal volatility decision, while the dashed lined corresponds to the difference between the tangent points V_2 and V_1, for each value of K considered. The concavification function has a four-piece shape if and only if $V_2 - V_1 > 0$ (Proposition 3.3.4). The parameters used are: $X_0 = 0.005$, $p = 0.01$, $B = 1.04$, $\gamma = 2$, $V_0 = 1$, $\mu = 0.04$, $\alpha = 0.4$, and $T = 1$.

q Effect

In order to get a sense of the q effect, we computed the optimal volatility, as a function of the firm value, for different values of q. To make this numerical exercise comparable with those done for the analysis of the K effect, we use the same parameters of the single-option example considered in Figure 3.6, except that we now set $K = 0.5$. We plotted these optimal curves in Figure 3.12.

For the particular case and values of q considered, the effects are analogous to those observed in the analysis of the K effect. For instance,

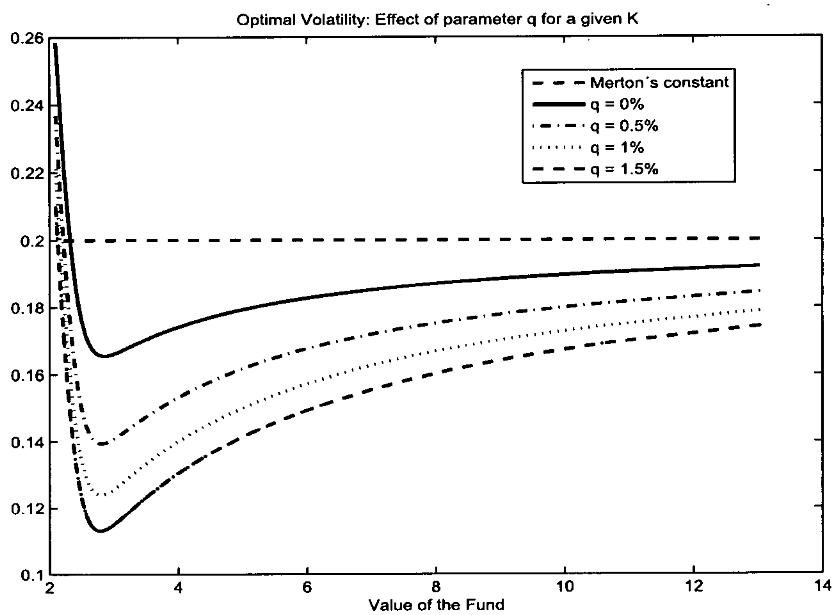

Figure 3.12: q **Effect**. This graph illustrates the effect of parameter q on the optimal volatility decision, for a fixed K ($= 0.5$). The parameters used are: $X_0 = 0.005$, $p = 0.01$, $B = 1.04$, $\gamma = 2$, $V_0 = 1$, $\mu = 0.04$, $\alpha = 0.4$, and $T = 1$. The values of q considered are 0.5%, 1%, and 1.5%.

- it is induced less risk-taking than with a single option ($q = 0\%$), and
- minimal volatility decisions decrease as q increases, although, contrary to what it is observed in the K effect case, such monotonic behavior is continuous over all the range of q values considered (see Figure 3.13).

Combined Effect of K and q

From the numerical analysis done previously, we conclude that the inclusion of a second option payoff in the compensation scheme does not eliminate the unexpected volatility levels that may arise in the single option-case. However, this analysis considers separately the effect of the two parameters that determine the second option, K and q. Therefore, a *natural*

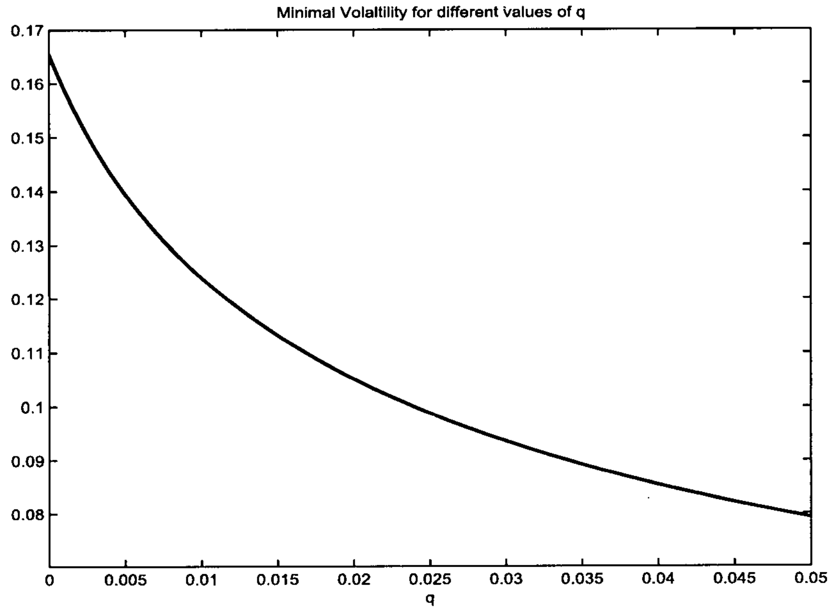

Figure 3.13: **Minimal Optimal Volatility: q Effect**. This graph illustrates the effect of parameter q on the optimal volatility decision, for a fixed K ($= 0.5$). The parameters used are: $X_0 = 0.005$, $p = 0.01$, $B = 1.04$, $\gamma = 2$, $V_0 = 1$, $\mu = 0.04$, $\alpha = 0.4$, and $T = 1$.

question is if K and q can be chosen simultaneously in such way that the unexpected risk profiles -when investors want to induce higher risk-taking- can be mitigated, at least in some degree. This is indeed possible as it is exemplified in Figure 3.14. This figure shows that for $K = 2.5$ and relatively small values of q (approximately less than or equal to 0.15%), the minimal volatility value is higher than that associated to the single-option case (q = 0 %).

Optimal volatility decisions can be more complex than those exemplified previously. For instance, if q is relatively small and K is relatively large the shape of the optimal volatility curve can take the form shown in Figure 3.15. In this case, the rationale of the manager is the following:

- If *both options are out of the money*, i.e. $V < B$, the manager will take higher risk-

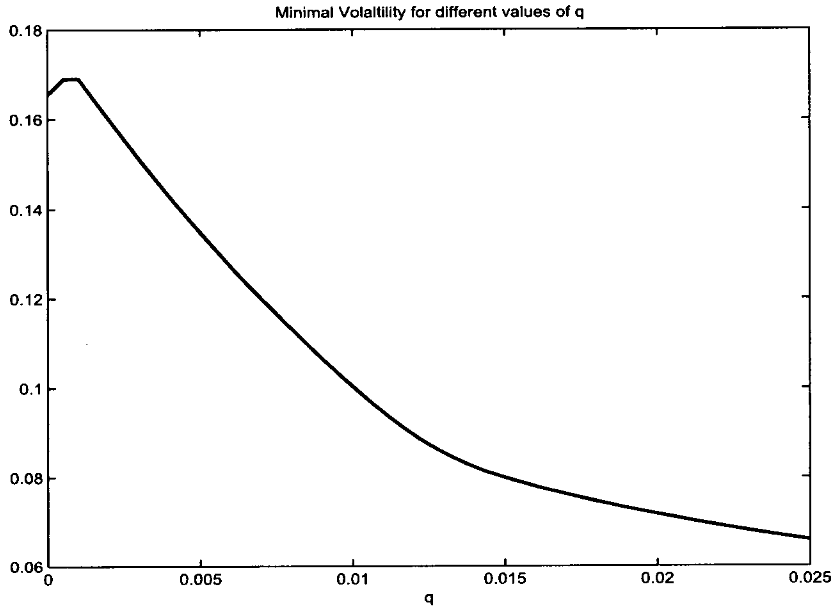

Figure 3.14: **Mitigation of Unexpected Risk Profiles.** This graph plots the maximal minimal volatility level for a given q. It shows that the unexpected risk profile observed for the single-option case illustrated in Figure 3.6 can be mitigated. The parameters used are: $X_0 = 0.005$, $p = 0.01$, $B = 1.04$, $K = 2.5$, $\gamma = 2$, $V_0 = 1$, $\mu = 0.04$, $\alpha = 0.4$, $T = 1$, and $T - t = 0.5$.

taking than if he were trading his own account and, the farther the option is from being in the money, the higher the risk the manager will take in order to increase the possibility that the first option, with payoff $(V - B)^+$, ends up in the money at time T.

- If *only the first option is in the money* then,
  - if the fund value V is <u>not</u> *sufficiently* above the benchmark B ($B < V \leq V^*$ in Figure 3.15), then the higher the fund value, the lesser the risk the manager will take. The manager acts in this way to decrease the chance that the first option, which is already in the money, ends out of the money at time T.
  - if the fund value V is *sufficiently* above the benchmark B but still significantly

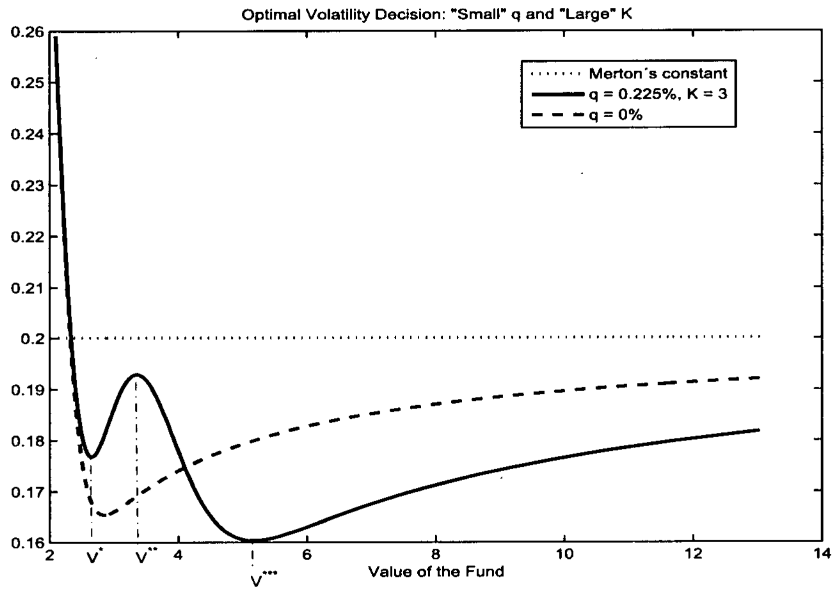

Figure 3.15: **Optimal Volatility Decision: Small q and Large K.** This graph shows the optimal volatility decisions for a relatively small q (= 0.225 %) and a relatively large K (= 3), and the corresponding single-option optimal volatility curve (q = 0%). The parameters used are: $X_0 = 0.005$, $p = 0.01$, $B = 1.04$, $\gamma = 2$, $V_0 = 1$, $\mu = 0.04$, $\alpha = 0.4$, and $T = 1$.

below the other benchmark value $B + K$ ($V^* < V \leq V^{**}$ in Figure 3.15) then, the higher the fund value, the higher the risk the manager will take. That is, the manager will increase risk-taking as the fund value increases, once the first option is *sufficiently* deep in the money. The manager behaves this way to increase both, the expected first option payoff and the possibility that the second option, with payoff $(V - B - K)^+$, ends in the money at time T. Note in Figure 3.15 that the difference between the volatility curves associated, respectively, with ($q = 0.225\%, K = 3$) and $q = 0\%$, captures the manager's risk appetite induced to increase the chance that the second option ends in the money.

– if the fund value V is *sufficiently* closed to the benchmark value $B + K$

($V^{**} < V < B + K$ in Figure 3.15) then, the higher the fund value, the lesser the risk the manager will take. In this situation, the second option is relatively near to be in the money and therefore, the closer is the fund value to $B + K$, the higher the incentive of the manager to guarantee that the second option ends up in the money at time T.

- If *both options are in the money* then
  - if the fund value V is <u>not</u> *sufficiently* above the benchmark value $B + K$ ($B + K < V \leq V^{***}$ in Figure 3.15), then the higher the fund value, the lesser the risk the manager will take. In this case, given that the second option is not *sufficiently* deep in the money, the manager tries to reduce the possibility that the second option ends out of the money at time T.
  - if the fund value V is *sufficiently* above the benchmark $B + K$ ($V > V^{***}$), then the higher the fund value, the higher the risk the manager will take. That is, once both options are *sufficiently* deep in the money, the manager will increase risk-taking as fund value grows in order to increase both expected payoffs.

The observations made in this section suggest that an appropriate selection of the second option parameters (q, K) allow investors to have a *significant* higher control on the manager's risk profile they want to incentive than with a single option. Next section analyzes this issue further.

3.6 Incentives Design

This section is devoted to the design of terminal fund value and risk profiles. For this purpose, analytical results related to the sensitivity and stability of the induced profiles with respect to the compensation parameters are obtained. Based on these results, specific criterions are proposed in order to induce particular fund value and risk profiles.

3.6.1 Terminal Fund Value Design

The range of (optimal) terminal fund values depends on the shape of the concavification function of $U \circ \phi$. In the two-piece shape case, the range is of the form $\{0\} \cup (V_4, \infty)$. Otherwise, in the four-piece shape case, the range is of the form $\{0\} \cup (V_1, V_2] \cup (V_3, \infty)$. That is, in the two-piece shape case the manager follows an all-or-nothing strategy while in the four-piece shape case the manager targets for a less disperse range of fund values than in the two-piece case. Therefore, it is crucial for the investors, or whoever establishes the manager's compensation scheme, to understand the way in which the compensation parameters determine the shape of the concavification function. To do so, recall that this shape can be characterized in terms of the sign of the difference $V_2 - V_1$ (See Proposition 3.3.4). If $V_2 - V_1$ is negative we have a two-piece shape concavification function. Otherwise, we have four-piece shape concavification function.

Four-piece shape concavification function

In the case that investors want to induce a terminal fund value range of the form

$$\{0\} \cup (V_1, V_2] \cup (V_3, \infty) \ ,$$

it is important to characterize the set of compensation parameters values that guarantees condition $V_2 - V_1 \geq 0$, and their impact on V_1, V_2 and V_3. To accomplish these two goals, we proceed in the following manner: First, we analyze the effect on V_1 from the set of parameters that determines it: X_0, p, and B. Second, we study the impact of q and K on V_2, *ceteris* *paribus* (X_0, p, B). Third, we do analysis on the simultaneous effect of (X_0, p, B, q, K) on[6] V_1 and V_2.

Proposition 3.6.1 (Sensitivity of V_1 with respect to X_0, B, and p) *Let* $\gamma > 0$, $p \in (0,1)$, *and* $X_0 > 0$. *Then,*

1. $\frac{\partial V_1}{\partial B} > 0$.

2. $\frac{\partial V_1}{\partial p} < 0$ *if and only if*

$$V_1 > B\left[1 + \left\{(p\gamma V_1)^{-1}(X_0 + p(V_1 - B))\right\}\right] \tag{3.26}$$

3. $\frac{\partial V_1}{\partial X_0} > 0$ *if and only if*

$$1 + \frac{p\gamma V_1}{(X_0 + p(V_1 - B))} < \frac{X_0^{-\gamma}}{(X_0 + p(V_1 - B))^{-\gamma}}$$

Proof: See Appendix 3.

Proposition 3.6.1 states on one hand, that V_1 increases as B increases. On the other hand, changes in the fixed compensation X_0 or the proportion p could either derive in increments or decrements of V_1 (See Figure 3.16). These results are useful to determine the impact, both qualitative and quantitative, of the decisions made over the compensation parameters X_0, p, and B.

In the following proposition we analyze the sensitivity of V_2 with respect to the second option compensation parameters q and K.

[6] V_3 is a linear transformation of V_2 (see Equation (3.40)). Therefore V_2 determines V_3 and viceversa.

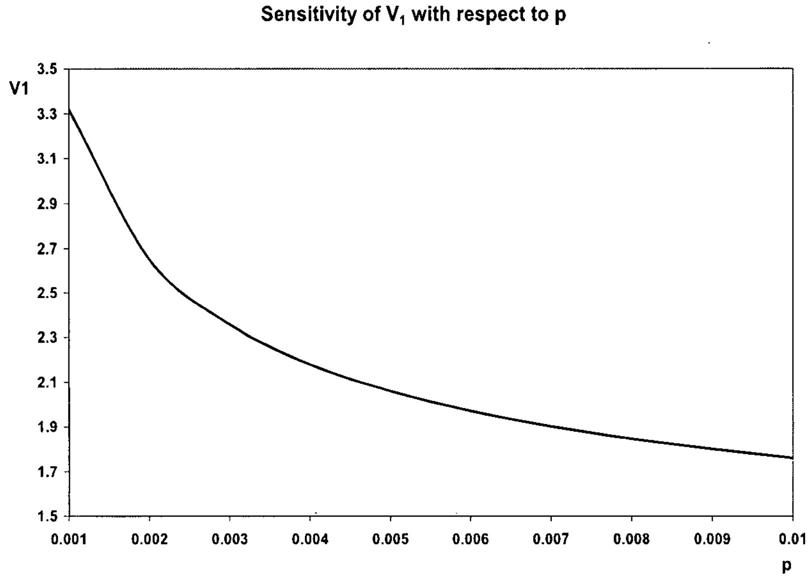

Figure 3.16: **Sensitivity of V_1 with respect to p.** This graph exemplifies the way V_1 changes as p changes. The parameters used are: $X_0 = 0.005$, $B = 1.04$, and $\gamma = 2$. For these parameters, and the range of values considered for p, condition (3.26) of Proposition 3.6.1 is satisfied and therefore $\frac{\partial V_1}{\partial p} < 0$.

Proposition 3.6.2 (Sensitivity of V_2 with respect to q and K) *Let $\gamma > 0$ and $p \in (0,1)$. Then,*

1. $\frac{\partial V_2}{\partial K} < 0$ *if and only if* $\gamma < 1$.

2. $\frac{\partial V_2}{\partial q} < 0$ *if and only if*

$$\left(\frac{1}{1-\gamma}\right)\left[\left(1-\frac{1}{\beta}\right)\frac{\partial \alpha}{\partial q} + \left(\frac{\alpha}{\beta^2}\right)\frac{\partial \beta}{\partial q}\right] < 0$$

where α and β are defined in (3.39) in Appendix 3.

Proof: See Appendix 3.

Proposition 3.6.2 states that V_2 increases or decreases (proportionally) as K increases, depending upon if the risk aversion parameter γ is, respectively, strictly greater or smaller than one. The sensitivity of V_2 with respect to q is more complex. It depends on the risk aversion parameter γ and the sign of $\frac{\partial \alpha}{\partial q}$. For instance, assume that $\gamma < 1$. Then, $\frac{\partial \alpha}{\partial q} < 0$ implies that V_2 increases as q increases. However, $\frac{\partial \alpha}{\partial q} > 0$ does not imply that V_2 decreases as q increases.

A practical consequence of Propositions 3.6.1 and 3.6.2 for the design of the compensation scheme is the following: suppose manager's risk aversion parameter is $\gamma > 1$ and that investors' first priority is to establish the threshold value V_1. Thereafter, they select (X_0, p, B) accordingly, and for which Proposition 3.6.1 is helpful. In order to determine q and K, and given result one of Proposition 3.6.2 ($\frac{\partial V_2}{\partial K} > 0$ iff $\gamma > 1$), it is useful to obtain, for each value of q, the minimal value of K that assures $V_2 - V_1 \geq 0$, which we denote by $K^*(q)$. Let Q be the set of values of q considered by the investors. Hence, the set of values $\{B + K^*(q) | \; q \in Q\}$ specifies the lower bound, and thereafter the range of values, of the second option exercise price that induces a four-piece shape concavification function for each value q considered (see Figure 3.17). Hence, if investors want to induce a four-piece shape concavification function, they choose q associated with the range of values of K they are *more comfortable with.* That is, they select the interval $[K^*(q), \infty)$ they prefer the most. Finally, they choose $K \in [K^*(q), \infty)$ using some criterion.

This *two-phase procedure* for establishing the compensation parameters is not only practical but also useful to give a characterization[7] of the set of compensation parameter values that guarantees $V_2 - V_1 \geq 0$. For instance,

[7]There are other ways to characterize condition $V_2 - V_1 \geq 0$. For example, given (X_0, p, B), we could determine a threshold value of q for each value of K, say $q^*(K)$, in the same spirit that $K^*(q)$ was defined. However, the determination of feasible intervals will not be as straight forward as in the case of $K^*(q)$ because the sign of $\frac{\partial V_2}{\partial q}$ depends on the risk aversion parameter γ and the $\frac{\partial \alpha}{\partial q}$ (Result 2 in Proposition 3.6.2).

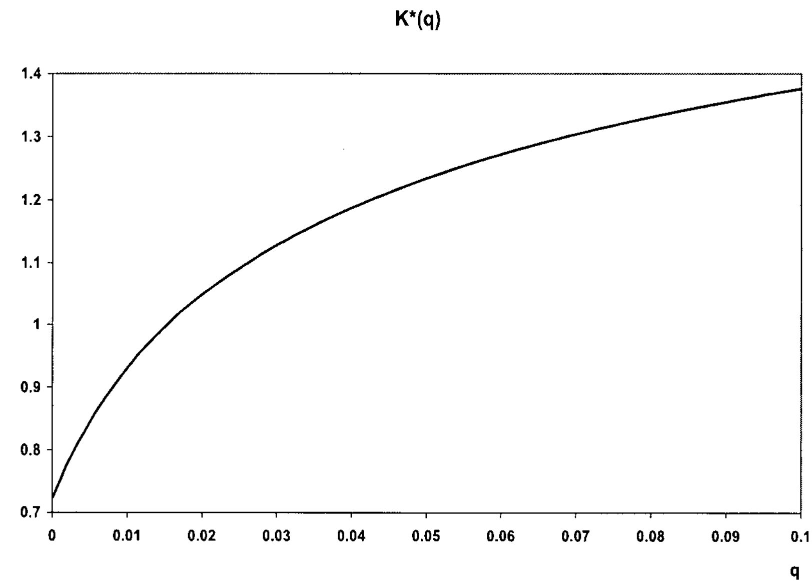

Figure 3.17: $K^*(q)$: This graph shows the minimal value of K required to assure that condition $V_2 - V_1 \geq 0$ is satisfied for each value of q, and given a set of parameters (X_0, p, B). The parameters used are: $X_0 = 0.005$, $B = 1.04$, and $\gamma = 2$. The range of values of q considered is $Q \equiv [0.001, 0.1]$.

$$\{(X_0, p, B, q, K) \mid V_2 - V_1 \geq 0\} = \bigcup_{(X_0, p, B) \in \Re_+^3} \{(X_0, p, B, q, K) \mid K \geq K^*(q), q \in \Re_+\}$$

Compensation parameters can also be chosen simultaneously to induce a specific terminal fund value profile. For example, suppose investors want to incentive a fund value range of the form $\{0\} \cup (V_1, V_2] \cup (V_3, \infty)$ and that they prefer the strategies that lead to a final fund value within the range $(V_1, V_2]$ rather than those strategies that lead to the range (V_3, ∞). Therefore, they would like to increase the length of the interval $(V_1, V_2]$ and the value of V_3. In order to determine the required compensation parameters to induce this specific fund value incentive profile, investors can solve the following optimization problem:

$$
\begin{aligned}
& Max_{\ X_0,p,B,q,K} \quad \delta V_3 + (1-\delta)(V_2 - V_1) \\
& s.t. \\
& \qquad\qquad V_2 - V_1 \geq 0 \\
& \qquad\qquad (X_0, p, B, q, K) \in A \subseteq \Re^5_+
\end{aligned}
\tag{3.27}
$$

where δ and $(1-\delta)$, for $\delta \in [0,1]$, represent, respectively, the relative weight that investors assign to strategies that lead to fund values within the interval (V_3, ∞) and $(V_1, V_2]$, where the set A specifies other restrictions that might apply to the compensation parameters, including nonnegativity conditions. Section 3.6.3 develops a numerical example for this criterion.

Two-piece shape concavification function

In the case that investors want to induce a terminal fund value of the form $\{0\} \cup (V_4, \infty)$, there are two crucial issues at hand: the characterization of the set of compensation parameter values that implies $V_2 - V_1 < 0$ and their impact on V_4. The following proposition establishes the sensitivity of V_4 with respect to each of the compensation parameters.

Proposition 3.6.3 (Sensitivity of V_4) *Let $\gamma > 0$ and $p \in (0,1)$. Then,*

1. $\frac{\partial V_4}{\partial B} > 0$.
2. $\frac{\partial V_4}{\partial p} < 0$ *if and only if*

$$V_4 > B\left[1 + \left\{((p+q)\gamma V_4)^{-1}(X_0 + p(V_4 - B) + q(V_4 - B - K))\right\}\right]$$

3. $\frac{\partial V_4}{\partial q} < 0$ *if and only if*

$$V_4 > (B+K)\left[1 + \left\{((p+q)\gamma V_4)^{-1}(X_0 + p(V_4 - B) + q(V_4 - B - K))\right\}\right]$$

4. $\frac{\partial V_4}{\partial K} > 0$.

5. $\frac{\partial V_4}{\partial X_0} > 0$ *if and only if*

$$\frac{X_0^{-\gamma}}{\left(X_0 + p\left(V_4 - B\right) + q\left(V_4 - B - K\right)\right)^{-\gamma}} > 1 + \frac{\left(p+q\right)\gamma V_4}{\left(X_0 + p\left(V_4 - B\right) + q\left(V_4 - B - K\right)\right)}$$

<u>Proof:</u> The proof of all the results uses the same methodology in Proposition 3.6.1.

Q.E.D.

The sensitivity of V_4 with respect to the type of compensation parameter (proportions and exercise prices) is identical to that of V_1: V_4 decreases, under certain conditions, as the value of the proportions p and q increase; V_4 increases as the option exercise prices B and $B + K$ increase; and the sensitivity of V_4 with respect to the fixed salary X_0 depends on a certain condition that involves, among other quantities, the risk aversion parameter γ.

3.6.2 Risk Profile Design

Two important features of manager's risk profile for the investors are the range of the volatility levels that the manager can take and the sensitivity of the risk-taking decisions with respect to the firm values and the market conditions.

Range of Volatility Values

The range of volatility values is of the form $(\underline{\nu}, \infty)$, where

$$\underline{\nu}\left(\mathbf{P}\right) = Min_{\;V\,\geq\,0}\;\; \nu^*\left(V, \mathbf{P}\right)\;,$$

V represents the fund value, and $\mathbf{P} \equiv (X_0, p, B, q, K)$ is the vector of compensation parameters. The analysis done in 3.5.2 illustrates that regardless the convexity of the compensation scheme ϕ, $\underline{\nu}$ can be below the volatility level that the manager will take if he were trading his own account. However, it is also observed in 3.5.2 that if the compensation parameters are chosen appropriately, this undesired feature -for the investors that want to induce higher risk-taking- can be mitigated and even eliminated in some cases. For instance, given the compensation parameters (X_0, p, B, q), K can be chosen equal to $K^*(q)$, where $K^*(q)$ is defined as

$$K^*(q) = ArgMax\ _K\ Min\ _V\ \nu^*\left(q, K, V\right)\ ,$$

where $\nu^*\left(q, K, V\right)$ denotes the optimal volatility choice given the parameter values $(q, K)$and the firm value V, in order to mitigate as much as possible the undesired *low volatility effect* for any firm value.

In general, investors can set a minimum level ML and choose a vector of compensation parameters $\mathbf{P}^*$ that solves the following optimization problem

$$Min\ _{\mathbf{P}\in\mathcal{P}}\ \left[\underline{\nu}\left(\mathbf{P}\right) - ML\right]^2$$

where the set $\mathcal{P}$ specifies particular restrictions that the vector of compensation parameters $\mathbf{P}$ must satisfy.

Stability

Drastic changes of optimal risk-taking decisions given small variations of the fund value and the market conditions are costly in terms of transactions costs or investment irreversibility or lumpiness. Therefore, it is crucial for investors to induce *stable* risk-taking decisions. Stability could be indirectly achieved by controlling the range of volatility values. However, under the current model we only have control over the lower bound of the range of volatility values $\underline{\nu}$. Hence, it is necessary to study directly the sensitivity of the optimal volatility with respect to changes of the firm value V. In order to accomplish such analysis, we uss the following result.

Lemma 3.6.1 (Sensitivity of volatility with respect to the fund value) *Let $\gamma > 0$ and V_1 and V_2 as defined in Lemmas 3.3.1 and 3.3.3, respectively. If $V_2 \geq V_1$, then*

$$\begin{aligned}
\frac{\partial \nu_t^*}{\partial V_t} &= \left(\tfrac{X_0 - qK}{p+q}\right) e^{-\mu(T-t)} \left(\tfrac{1}{V_t^2}\right) \left[\tfrac{n(d_{T-t}(m_2,1))}{\sqrt{T-t}} - \mathcal{N}\left(d_{T-t}(m_2,1)\right)\left\{\alpha\left(1-\gamma^*\right) + G(X_0,p,B,q,K,\gamma)\right\}\right] \\
&- \left(\tfrac{X_0}{p}\right) e^{-\mu(T-t)} \left(\tfrac{1}{V_t^2}\right) \left[\tfrac{n(d_{T-t}(m_2,1))}{\sqrt{T-t}} - \mathcal{N}\left(d_{T-t}(m_2,1)\right)\left\{\alpha\left(1-\gamma^*\right) + G(X_0,p,B,q,K,\gamma)\right\}\right] \\
&- \left(B - \tfrac{X_0}{p}\right) e^{-\mu(T-t)} \left(\tfrac{1}{V_t^2}\right) \left[\tfrac{n(d_{T-t}(m_1,1))}{\sqrt{T-t}} - \mathcal{N}\left(d_{T-t}(m_1,1)\right)\left\{\alpha\left(1-\gamma^*\right) + G(X_0,p,B,q,K,\gamma)\right\}\right]
\end{aligned}$$

where $d_u(\cdot,\cdot)$ is the function defined by

$$d_u(m,\beta) \equiv \left[\log m - \left(-\mu u + \alpha^2 u \left(\beta - \frac{1}{2}\right)\right)\right] / \left(\alpha\sqrt{u}\right) \ ,$$

$\mathcal{N}(\cdot)$ and $n(\cdot)$ are the standard normal cumulative distribution and density, $\gamma^* = 1 - 1/\gamma$, and λ is the nonnegative scalar that satisfies Equation (3.47). Otherwise, if $V_2 < V_1$, then

$$\begin{aligned}
\frac{\partial \nu_t^*}{\partial V_t} &= \frac{\left(B - \frac{X_0 - qK}{p+q}\right) e^{-\mu(T-t)}}{\sqrt{T-t}} \left[\tfrac{\mathcal{N}(d_{T-t}(m_4,1))}{\mathcal{N}(d_{T-t}(m_4,\gamma^*))} n\left(d_{T-t}\left(m_4,\gamma^*\right)\right) - n\left(d_{T-t}\left(m_4,1\right)\right)\right] \left(\tfrac{1}{V_t^2}\right) \\
&+ \left(B - \tfrac{X_0 - qK}{p+q}\right) e^{-\mu(T-t)} \left[\alpha(1-\gamma^*)\mathcal{N}\left(d_{T-t}\left(m_4,1\right)\right)\right] \left(\tfrac{1}{V_t^2}\right)
\end{aligned}$$

Proof: Differentiate with respect to V_t the optimal volatility expressions obtained in Proposition 3.4.2.

Q.E.D.

The explicit expressions obtained in the previous lemma show a direct, and useful, link between the compensation parameters and the sensitivity of the volatility with respect to the fund value. For instance, for the two-piece shape case (i.e., $V_2 < V_1$), observe that $\frac{\partial \nu_t^*}{\partial V_t}$ can be factorized in terms of the expression

$$\left(B - \frac{X_0 - qK}{p+q} \right) , \tag{3.28}$$

which involves <u>all</u> the parameters that determine the compensation scheme ϕ, without considering the risk aversion parameter γ. In other words, expression (3.28) is a *scaling* factor of $\frac{\partial \nu_t^*}{\partial V_t}$. Therefore, we can control the stability of ν_t^* with respect to V by adjusting the absolute value of this factor. The smaller the absolute value of this factor, the more stable is ν_t^* with respect to V.

Factor (3.28) can be used to establish conditions to maintain a stability level. For example, suppose that investors want to keep a stability level[8] of ϵ. Hence, we can infer the value of any compensation parameter to maintain the stability level ϵ. For instance, suppose that X_0, p, B, and q are given. Thus,

$$K = -\left(1 + \frac{p}{q}\right)(B - \epsilon) + \left(\frac{X_0}{q}\right) \tag{3.29}$$

assures

[8]The value of ϵ could be chosen to guarantee, for example, that

$$\frac{\left|\frac{\partial \nu^*}{\partial V}\right|}{V} \leq \beta \,\%$$

where β is a predetermined percentage and $V \in \Re_+$.

$$\left(B - \frac{X_0 - qK}{p+q}\right) = \epsilon \tag{3.30}$$

Moreover, we can deduce the required increment or reduction of a parameter value. For instance, from (3.29) we obtain

$$\frac{\partial K}{\partial q} = \left(\frac{1}{q^2}\right)\left[p\,(B-\epsilon) - X_0\right]$$

So, $\frac{\partial K}{\partial q} \geq 0$ if and only if $pB \geq X_0 + p\epsilon$. That is, if the manager's fixed salary is *small enough* (i.e. $X_0 \leq pB - p\epsilon$) then a larger value of q will require a larger value of K in order to preserve the same stability level ϵ. This is consistent with intuition. If the manager's fixed salary is relatively small (i.e. $X_0 \leq pB - p\epsilon$) and q is increased, the manager will be encouraged to take higher risks to get a better expected compensation. To mitigate an excessive increase in the volatility level, the exercise price of the second option, $B+K$, must be increased.

In the four-piece shape case, there are two factors -instead of one- controlling the size of $\frac{\partial \nu^*}{\partial V}$, that involve only compensation parameters. These factors are:

$$\frac{X_0 - qK}{p+q} - \frac{X_0}{p}$$

and

$$B - \frac{X_0}{p}\,.$$

Stability can be controlled by setting the value of the factors equal to some predetermined constants[9], say ϵ_1 and ϵ_2, and therefore establishing conditions and implicit relationships that compensation parameters must satisfy, as it is illustrated previously for the two-piece shape case. A particular and relevant case occurs when the constants ϵ_1 and ϵ_2 are equal. This assumption implies that $\frac{\partial \nu^*}{\partial V}$ is proportional to the single factor

$$\frac{X_0 - qK}{p+q} - \frac{X_0}{p} = B - \frac{X_0}{p}$$

from which we derive the implicit relation

$$B - \frac{X_0 - qK}{p+q} = 0 \ .$$

Therefore, factor (3.28) is relevant for the stability of $\frac{\partial \nu^*}{\partial V}$ in any concavification case. For instance, condition (3.30), for ϵ *small*, allows to control stability through a single factor for both cases.

3.6.3 An Example

The goal of this section is to exemplify some of the theoretical results that are developed in the previous sections. We focus on the selection of the second option parameters q and K, given that in practice there exists a *typical* set of values for the first option parameters and the fixed salary. That is, we assume that the compensation parameters X_0, p, and B are given, the risk aversion parameter γ is known, and propose specific criterions to choose q and K.

For instance, suppose that the initial or current fund value is $V_0 = 1$, that the coefficient

[9]Using some criterion as that outlined in the precedent footnote to the previous one.

of risk aversion is $\gamma = 2$, and the following compensation parameters $X_0 = 0.005$, $p = 0.01$, and $B = 1$. That is, investors compensate the manager with 1% of the profits above the current firm's value and a fixed salary equivalent to 0.5% of the actual firm's value. We assume that investors decide first on the range type of terminal firm values and then on the stability of the optimal volatility decisions.

To determine the type of terminal fund range, we need the reference value V_1, in this case equal[10] to 1.7071, and select q and K such that

- $V_2 \geq V_1$, if investors want to induce a terminal fund value range of the form $\{0\} \cup (V_1, V_2] \cup (V_3, \infty)$,
- or $V_2 < V_1$, if investors want to induce a range of the form $\{0\} \cup (V_4, \infty)$

We suppose, for simplicity, that q and K are not chosen simultaneously.

Choosing q first

Assume that investors set $q = 2\%$ and want to determine K. The election of K depends primarily on the type of terminal fund value range they want to induce. Given that $\gamma > 1$, we know that V_2 is an increasing function of K (Refer to Proposition 3.6.2). Therefore, it must exist a threshold value $K^*(q)$ such that $K \leq K^*(q)$ implies a terminal fund value of the form $\{0\} \cup (V_4, \infty)$. Otherwise, if $K > K^*(q)$, the terminal fund value range is of the form $\{0\} \cup (V_1, V_2] \cup (V_3, \infty)$. Under the given assumptions, $K^*(q) = 1.0305$ (See Figure 3.18).

Suppose that investors want to induce a range of the form $\{0\} \cup (V_4, \infty)$. How should they select $K \in [0, K^*(q)]$? There are certainly many ways to do such selection, depending upon what extra criterions investors would like to consider. To illustrate how this extra information could be used to determine a specific K, we consider the following situation:

[10]Refer to Lemma 3.3.1 for the characterization of V_1.

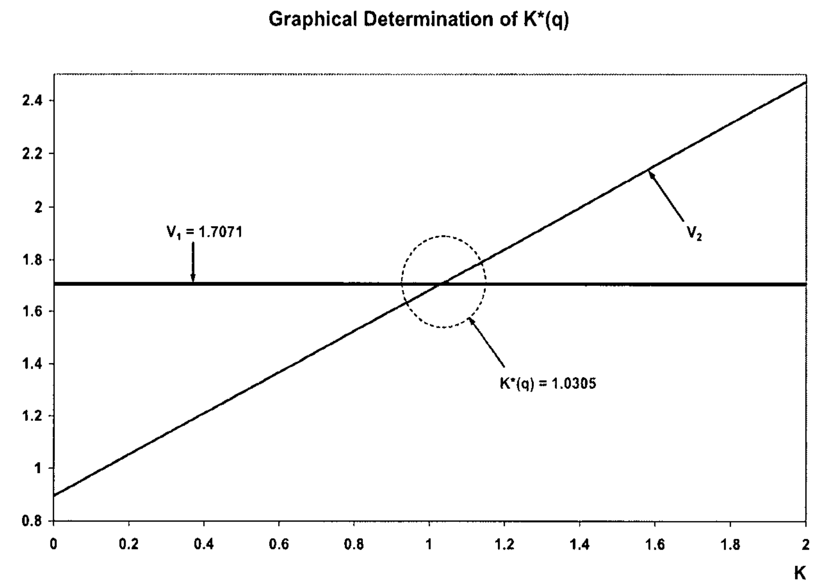

Figure 3.18: **Determination of $K^*(q)$:** This graph shows V_2 as function of K, given risk aversion parameter $\gamma = 2$ and the parameter values $X_0 = 0.005$, $p = 0.01$, $B = 1$, and $q = 0.02$. $K^*(q)$ is determined as the value at which $V_2\,(K^*(q))$ equates V_1, which in this case is equal to 1.7071. The terminal fund value is of the form $\{0\} \cup (V_4, \infty)$ if and only if $K \in [0, K^*(q)]$.

Assume that investors want to increase fund's value but without inducing the manager to follow a relatively unstable risk-taking policy. To determine the *appropriate* value of K, investors can solve the following optimization problem

$$Max_{\ K\in[0,K^*(q)]}\ (1-\delta)\,V_4 - \delta\left(B - \frac{X_0 - qK}{p+q}\right)^2 \tag{3.31}$$

where $1-\delta$ and δ, for $\delta \in [0,1]$, are the relative weights that investors assess, respectively, to a minimum target fund value, if the manager decides to be in the money, and stability. Recall that it was shown in 3.6.2 that the term $B - \frac{X_0 - qK}{p+q}$ is a key factor for controlling the stability of optimal risk-taking decisions. By solving Problem (3.31), investors try to choose

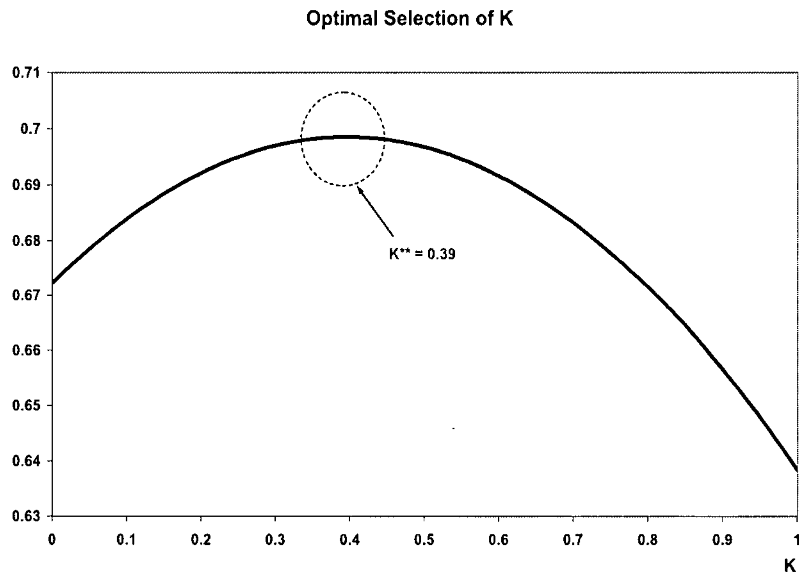

Figure 3.19: **Optimal Selection of K:** This graph shows the objective function of Problem (3.31) for $K \in [0, 1]$. The optimal value is reached at $K^{**} = 0.39$. The optimal exercise price for the second option is then $B + K^{**} = 1.39$.

the value of K that leads to the *best tradeoff* between the minimum target value $\dot{V}_4$ and stability. The form of the objective function in (3.31) emulates the classical mean-variance approach of portfolio theory (Markowitz (1952)). The squared of the factor $B - \frac{X_0 - qK}{p+q}$ can be directly related to the *variability* of the portfolio's return [11].

We solved Problem (3.31) for $\delta = 35\%$. We obtained that the value of K that leads to

[11]Let $V : \Omega \rightarrow \Re$ a random variable and $G : \Re \rightarrow \Re$ a real valued function. Therefore, by Taylor's Theorem

$$G(V(\omega)) \approx G(V_0) + (V(\omega) - V_0)G'(V_0)$$

for $\omega \in \Omega$, and $V_0 \in \Re$. Hence, the variance of $G(V)$, which we denote by $VAR(G(V))$, can be approximated in the following manner:

$$VAR(G(V)) \approx \left[G'(V_0)\right]^2 VAR(V)$$

If V represents the fund's value and $G(V)$ the optimal volatility decision ν^* then, by the analysis developed in 3.6.2, $\left[G'(V_0)\right]^2$ is proportional to $\left[B - \frac{X_0 - qK}{p+q}\right]^2$.

the best tradeoff is $K^{**} = 0.39$ (See Figure 3.19). Hence, investors should set a percentage of $q = 2\%$ and an exercise price of $B + K^{**} = 1.39$ for the second option included in the compensation scheme. In other words, investors should set a second benchmark, 39% higher than the current firm's value ($V_0 = 1$), if the manager is compensated with 2% of the profits over such benchmark.

<u>Choosing K first</u>

Assume now that investors set $K = 0.8$ and want to determine q. The election of q depends primarily on the type of terminal fund value range they want to induce. Under the given parameters, V_2 is a decreasing function of q (Refer to Proposition 3.6.2). Therefore, it exists a threshold value $q^*(K)$ such that if $q \leq q^*(K)$ then the fund value range is of the form $\{0\} \cup (V_1, V_2] \cup (V_3, \infty)$. Otherwise, if $q > q^*(K)$, the fund value range is of the form $\{0\}\,(V_4, \infty)$. In this case, $q^*(K) = 0.0036$ (See Figure 3.20).

Suppose that investors want to induce a terminal fund value range of the form $\{0\} \cup (V_1, V_2] \cup (V_3, \infty)$. Thus, investors must select q from $[0, q^*(K)]$.

Furthermore, assume that investors want to amplify the range $(V_1, V_2]$ as much as possible and, at the same time, to maximize the minimum target value V_3. To define the value of q, investors can solve the following optimization problem:

$$Max_{\ q\in[0,q^*(K)]}\ \tilde{\delta} V_3 + \left(1 - \tilde{\delta}\right)(V_2 - V_1) \tag{3.32}$$

where $\tilde{\delta}$ and $\left(1 - \tilde{\delta}\right)$, for $\tilde{\delta} \in [0, 1]$, are the relative weights that investors assess, respectively, to a minimum target value V_3 and the length of the range of fund values $(V_1, V_2]$. Note that Problem (3.32) is a particular case of Problem (3.27) outline in 3.6.1.

We solved Problem (3.32) for $\delta = 55\%$. We obtained that the value of q that leads to

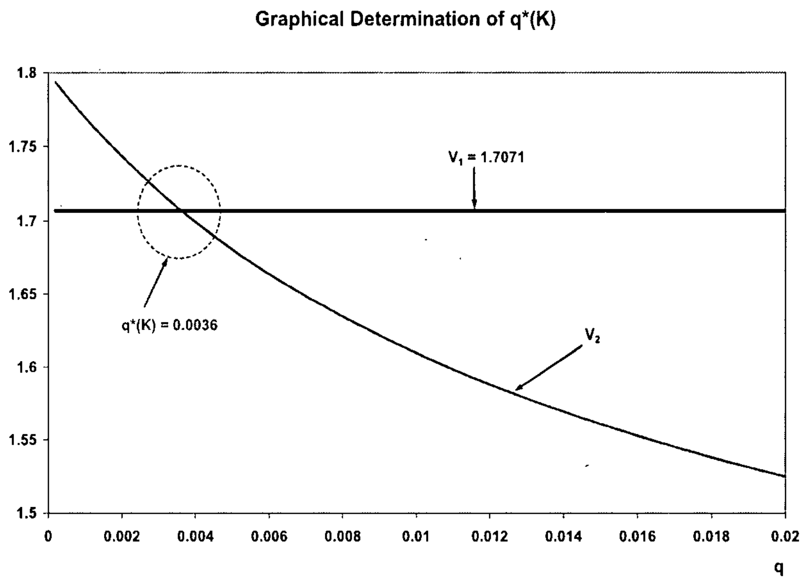

Figure 3.20: **Determination of $q^*(K)$:** This graph shows V_2 as function of q, given risk aversion parameter $\gamma = 2$ and the parameter values $X_0 = 0.005$, $p = 0.01$, $B = 1$, and $K = 0.80$. The point at which the function $V_2(q)$ crosses the value of $V_1 = 1.7071$ is $q^*(K) = 0.0036$. The terminal fund value is of the form $\{0\} \cup (V_1, V_2] \cup (V_4, \infty)$ if and only if $q \in [0, q^*(K)]$.

the best *tradeoff* between strategies that lead to terminal firm values in $(V_1, V_2] \cup (V_3, \infty)$ is $q^{**} = 0.18\%$ (See Figure 3.21). That is, investors should compensate the manager with 0.18% of the profits over a benchmark set 80% above the current firm's value.

3.7 Conclusions

The main motivation of this research is to contribute on the understanding of the induction and design of risk incentives derived from option-like compensation schemes. In particular, this study focus on the intriguing finding that option-like compensation schemes might generate unexpected risk-taking policies. For instance, single-option compensation schemes,

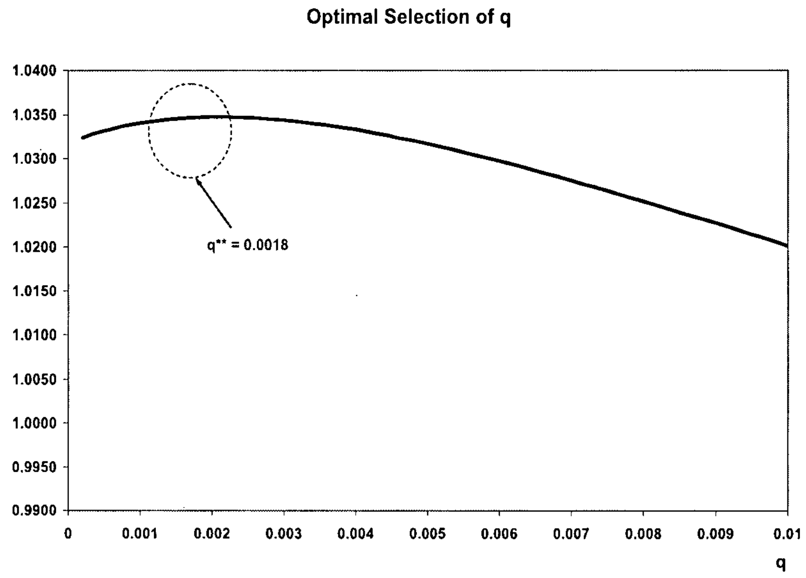

Figure 3.21: **Optimal Selection of** q**:** This graph shows the objective function of Problem (3.32) for $q \in [0, 0.02]$. The optimal value is reached at $q^{**} = 0.0096$.

which are in principle established to induce higher risk-taking, can imply that the manager actually takes less risk than if he were trading his own account. We studied the case of a compensation scheme formed by two options instead of one, as it is a common case in practice. Our expectation was to find out that two-option compensation schemes could alleviate, in some degree, the unexpected features of the single-option case given the extra *degrees of freedom* (exercise price and the proportion that the second option represents from the whole compensation package). We corroborated our expectation by obtaining the explicit optimal risk-taking policy and making a detailed analysis on the induction and design of such optimal policy.

The theoretical results obtained and the criterions proposed here on relation to the design of incentives for the type of option-like compensation schemes considered, should be useful in practice and a starting point for adapting these schemes to particular needs or proposing others.

Final Remarks

The main motivation of this Ph.D. thesis is understanding the behavior of investors that have to operate under risk tolerance constraints or managers who are compensated through *standard* and relatively more sophisticated option-like compensation schemes. Our analysis is developed in two different model frameworks: a discrete-time framework, that can easily consider incomplete markets and *fat tailed* distributions, and under which we study investment under risk constraints and standard option-like compensations; and a continuous-time framework, in which we explore the optimal strategies and induced incentives of slightly more complex option-like compensation schemes than the *typical* single-option case observed in practice.

The discrete-time analysis framework derives many important and useful insights about the nature of the optimal manager's strategies followed by the type of investors and managers considered. For instance, in the case of an investor subject to risk tolerance constraints, we uncover implicit probability measures that are used to assess the risk of implementing optimal strategies that involve to invest at the limit of the tolerated risk. For a manager or executive paid with a *typical* option-like compensation scheme, the optimal strategy leads to an expected gain that is proportional to the difference between the initial capital and the expected market value of replicating the benchmark. Moreover, if options are considered in the investment universe, we establish explicit optimal investment conditions for such instruments.

The study realized in the continuous-time framework reveals that although compensation schemes formed by two different options might present the same counterintuitive behavior that the *standard* single-option case, these more complex compensation packages can be designed to mitigate this kind of behavior and to induce richer terminal firm value and risk taking profiles. We obtain several explicit theoretical results on the sensitivity of the terminal firm value and the stability of the risk taking decisions with respect to the compensation parameters. Furthermore, we propose and illustrate specific criterions that combine some of these theoretical results to determine concrete firm value and risk profiles.

The analysis done in this Ph.D. thesis should be useful to administration boards, that are responsible for establishing compensation packages and risk limits for investment managers, and to researchers who are interested in the induced behavior and the design of option-like compensation packages. The results can be extended to consider other monotone risk-averse utility functions and different convex risk measures. The discrete-time framework developed in this thesis should be particularly appealing, because of its relative simplicity and generality, to explore the induced behavior under other investment restrictions, while the *concavification* technique developed in the continuous-time framework, can be applied to more sophisticated option-like compensation schemes than the one studied in this research work.

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Appendix 1

Lemma 3.7.1 *Let $f : \Re \rightarrow \Re$ be a real-valued function that is monotone nondecreasing. Assume that $\lim_{x \to a^+} f(x)$ (limit of f as x approaches to a from the right) exists for some $a \in \Re$ and is equal to L. Then, $f(x) \geq L$ for $x > a$.*

Proof: The proof is by contradiction. Assume that there exists $\bar{x} > a$ such that $f(\bar{x}) < L$. Let $\epsilon = \frac{L - f(\bar{x})}{2} > 0$. Therefore, there must exist $\delta > 0$ such that if $0 < x - a < \delta$ then

$$|f(x) - L| < \epsilon = \frac{L - f(\bar{x})}{2} \tag{3.33}$$

Let $\tilde{\delta} = min\,(\delta, \bar{x} - a)$. Hence, $0 < x - a < \tilde{\delta}$ implies $x < \bar{x}$ and (3.33). Therefore,

$$\frac{f(\bar{x}) - L}{2} < f(x) - L$$

then $f(\bar{x}) - L < f(x) - L - \left(\frac{L - f(\bar{x})}{2}\right)$, which implies

$$f(x) > f(\bar{x}) + \left(\frac{L - f(\bar{x})}{2}\right) > f(\bar{x})$$

thus leading to a contradiction, since f is monotone nondecreasing and $x < \bar{x}$. Therefore, $f(x) \geq L$ for all $x > a$.

Q.E.D.

Appendix 2

1. **Derivation of the Dual Problem (2.2)**

The Lagrangian of (2.1) is

$$\begin{aligned}
&L(\theta, \epsilon_0^C, \epsilon_0^P, s, y_0, y, x, \lambda, \mu, \eta_C, \eta_P) = \\
&\alpha \left[\textstyle\sum_{n\in N_T} \beta_n s_n p_n\right] \\
&-y_0(Z_0 \cdot \theta_0 + \epsilon_0^C \beta_0 C_0(1+tc^C) + \epsilon_0^P \beta_0 P_0(1+tc^P) - \beta_0 W_0) \\
&-\textstyle\sum_{t=1}^{T} \sum_{n\in N_t} y_n[Z_n \cdot (\theta_n - \theta_{a(n)})] \\
&+\textstyle\sum_{n\in N_T} x_n(Z_n \cdot \theta_n - \beta_n s_n + \epsilon_0^C \beta_n C_n + \epsilon_0^P \beta_n P_n - \beta_n B_n) \\
&+\textstyle\sum_{n\in N_T} \lambda_n(Z_n \cdot \theta_n) + \sum_{n\in N_T} \mu_n \beta_n s_n + \eta_C \epsilon_0^C + \eta_P \epsilon_0^P \ ,
\end{aligned} \tag{3.34}$$

where

$$\lambda_n \geq 0,\ \mu_n \geq 0,\ \forall\ n \in N_T\ ,$$
$$\eta_C \geq 0,\ \eta_P \geq 0,$$

and $-y_0$, $-y_n$ and x_n are free variables since they correspond to equality constraints. The signs have been chosen to give later on a contextual interpretation. The Lagrangian can be rewritten as (see Item 3 of this Appendix)

$$\begin{aligned}
&L(\theta, \epsilon_0^C, \epsilon_0^P, s, y_0, y, x, \lambda, \mu, \eta_C, \eta_P) = \\
&y_0 \beta_0 W_0 - \textstyle\sum_{n\in N_T} x_n \beta_n B_n \\
&+\textstyle\sum_{n\in N_T} (\alpha p_n + \mu_n - x_n)\beta_n s_n + \sum_{n\in N_T} (\lambda_n + x_n - y_n)\,(Z_n \cdot \theta_n) \\
&-\textstyle\sum_{t=0}^{T-1} \sum_{n\in N_t} (y_n Z_n - \sum_{m\in C(n)} y_m Z_m) \cdot \theta_n \\
&+\epsilon_0^C[\textstyle\sum_{n\in N_T} x_n \beta_n C_n - \beta_0 C_0(1+tc^C)(y_0) + \eta_C] \\
&+\epsilon_0^P[\textstyle\sum_{n\in N_T} x_n \beta_n P_n - \beta_0 P_0(1+tc^P)(y_0) + \eta_P]
\end{aligned} \tag{3.35}$$

The dual constraints require that the factors of the primal variables are zero, and the dual objective function comes from all the Lagrangian terms that do not involve primal variables. So, the dual is

$$
\begin{aligned}
&Min_{\ y_0,\mathbf{y},\mathbf{x},\lambda,\mu,\eta_C,\eta_P} \ \ y_0\beta_0 W_0 - \textstyle\sum_{n\in N_T} x_n\beta_n B_n \\
&s.t. \\
&\alpha p_n + \mu_n - x_n = 0, \ \forall \ n \in N_T \\
&\lambda_n + x_n - y_n = 0, \ \forall \ n \in N_T \\
&(y_n Z_n - \textstyle\sum_{m\in C(n)} y_m Z_m) = 0, \ \forall \ n \in N_t, \ \forall \ t = 0,\ldots,T-1 \\
&\textstyle\sum_{n\in N_T} x_n\beta_n C_n - \beta_0 C_0(1+tc^C)(y_0) + \eta_C = 0 \\
&\textstyle\sum_{n\in N_T} x_n\beta_n P_n - \beta_0 P_0(1+tc^P)(y_0) + \eta_P = 0 \\
&\lambda_n \geq 0, \ \mu_n \geq 0, \ \forall \ n \in N_T \\
&\eta_C \geq 0, \ \eta_P \geq 0 \ .
\end{aligned}
$$

Q.E.D.

2. **Derivation of the Dual Problem** (2.14)

The Lagrangian of (2.14) can be expressed as (see (3.34) and (3.35))

$$
\begin{aligned}
&L(\theta, \bar{\epsilon}_0^C, \bar{\epsilon}_0^P, s, y_0, y, x, \lambda, \mu, \eta_C, \eta_P) = \\
&y_0\beta_0 W_0 - \textstyle\sum_{n\in N_T} x_n\beta_n B_n \\
&+\textstyle\sum_{n\in N_T}(\alpha p_n + \mu_n - x_n)\beta_n s_n + \textstyle\sum_{n\in N_T}(\lambda_n + x_n - y_n) Z_n \cdot \theta_n \\
&-\textstyle\sum_{t=0}^{T-1}\textstyle\sum_{n\in N_t}(y_n Z_n - \textstyle\sum_{m\in C(n)} y_m Z_m)\theta_n \\
&+\bar{\epsilon}_0^C[y_0\beta_0\bar{C}_0(1+tc^C) - \textstyle\sum_{n\in N_T} x_n\beta_n C_n - \textstyle\sum_{n\in N_T}(\beta_n C_n)p_n + \eta_C] \\
&+\bar{\epsilon}_0^P[y_0\beta_0\bar{P}_0(1+tc^P) - \textstyle\sum_{n\in N_T} x_n\beta_n P_n - \textstyle\sum_{n\in N_T}(\beta_n P_n)p_n + \eta_P] \ .
\end{aligned}
\tag{3.36}
$$

where

$$
\lambda_n \geq 0, \ \mu_n \geq 0, \ \forall \ n \in N_T \ ,
$$
$$
\eta_C \geq 0, \ \eta_P \geq 0,
$$

and $-y_0$, $-y_n$ and x_n are free variables since they correspond to equality constraints.

The dual constraints require that the factors of the primal variables are zero; and the dual objective function comes from the Lagrangian terms that do not involve primal variables. The dual is

$$
\begin{aligned}
& Min_{\ y_0,\mathbf{y},\mathbf{x},\lambda,\mu,\eta_C,\eta_P} \quad y_0\beta_0 W_0 - \textstyle\sum_{n\in N_T} x_n\beta_n B_n \\
& s.t. \\
& \alpha p_n + \mu_n - x_n = 0, \ \forall\ n \in N_T\ , \\
& \lambda_n + x_n - y_n = 0, \ \forall\ n \in N_T\ , \\
& (y_n Z_n - \textstyle\sum_{m\in C(n)} y_m Z_m) = 0, \ \forall\ n \in N_t, \ \forall\ t = 0,\ldots,T-1\ , \\
& y_0\beta_0\bar{C}_0(1+tc^C) - \textstyle\sum_{n\in N_T} x_n\beta_n C_n - \sum_{n\in N_T}(\beta_n C_n)p_n + \eta_C = 0\ , \\
& y_0\beta_0\bar{P}_0(1+tc^P) - \textstyle\sum_{n\in N_T} x_n\beta_n P_n - \sum_{n\in N_T}(\beta_n P_n)p_n + \eta_P = 0\ , \\
& \lambda_n \geq 0, \ \mu_n \geq 0, \ \forall\ n \in N_T\ , \\
& \eta_{Call} \geq 0, \ \eta_P \geq 0\ .
\end{aligned}
$$

Q.E.D.

3. **Equivalence between equations (3.34) and (3.35)**

The equivalence between equations (3.34) and (3.35) follows from showing that

$$
y_0\,(Z_0\cdot\theta_0) + \textstyle\sum_{t=1}^{T}\sum_{n\in N_t} y_n\left[Z_n\cdot\left(\theta_n - \theta_{a(n)}\right)\right] \ =
$$

$$
\textstyle\sum_{n\in N_T} y_n\,(Z_n\cdot\theta_n) + \sum_{t=0}^{T-1}\sum_{n\in N_t}\left(y_n Z_n - \sum_{m\in C(n)} y_m Z_m\right)\cdot\theta_n
$$

Proof:

$$y_0\left(Z_0\cdot\theta_0\right)+\sum_{t=1}^{T}\sum_{n\in N_t} y_n\left[Z_n\cdot\left(\theta_n-\theta_{a(n)}\right)\right] =$$

$$\sum_{t=0}^{T}\sum_{n\in N_t} y_n\left(Z_n\cdot\theta_n\right)-\sum_{t=1}^{T}\sum_{n\in N_t} y_n Z_n\cdot\theta_{a(n)} =$$

$$\sum_{t=0}^{T-1}\left[\sum_{n\in N_t} y_n\left(Z_n\cdot\theta_n\right)-\sum_{n\in N_{t+1}} y_n\left(Z_n\cdot\theta_{a(n)}\right)\right]+\sum_{n\in N_T} y_n\left(Z_n\cdot\theta_n\right)$$

.

Observe that

$$\sum_{n\in N_{t+1}} y_n\left(Z_n\cdot\theta_{a(n)}\right)=\sum_{n\in N_t}\left(\sum_{m\in C(n)} y_m Z_m\right)\cdot\theta_n\ .$$

Hence,

$$y_0\left(Z_0\cdot\theta_0\right)+\sum_{t=1}^{T}\sum_{n\in N_t} y_n\left[Z_n\cdot\left(\theta_n-\theta_{a(n)}\right)\right] =$$

$$\sum_{t=0}^{T-1}\left[\sum_{n\in N_t} y_n\left(Z_n\cdot\theta_n\right)-\sum_{n\in N_t}\left(\sum_{m\in C(n)} y_m Z_m\right)\cdot\theta_n\right]+\sum_{n\in N_T} y_n\left(Z_n\cdot\theta_n\right) =$$

$$\sum_{t=0}^{T-1}\sum_{n\in N_t}\left(y_n Z_n-\sum_{m\in C(n)} y_m Z_m\right)\cdot\theta_n+\sum_{n\in N_T} y_n\left(Z_n\cdot\theta_n\right)$$

.

Q.E.D.

4. Proof of Equation (2.10)

If $Z_n\cdot\theta_n$ is unrestricted for $n\in N_T$, then the term

$$\sum_{n\in N_T}\lambda_n(Z_n\cdot\theta_n)$$

vanishes from the Lagrangian function of Problem (2.1) (see (3.34) in Item 1 of this Appendix); hence, $\sum_{n \in N_T} (\lambda_n + x_n - y_n) (Z_n \cdot \theta_n)$ is replaced by

$$\sum_{n \in N_T} (x_n - y_n) Z_n \cdot \theta_n$$

in the Lagrangian equation (3.35), and $x_n = y_n$ for all $n \in N_T$. Thus, the dual objective function is

$$y_0 \left(\beta_0 W_0 - \sum_{n \in N_T} \left(\frac{y_n}{y_0} \right) \beta_n B_n \right) = y_0 \left(\beta_0 W_0 - E^Q \left[\beta_T B_T \right] \right) .$$

Q.E.D.

Appendix 3

<u>Proof of Lemma 3.3.1</u>

Consider the function $g : [B, \infty) \to \Re$ defined in the following manner:

$$g(x) = \begin{cases} \frac{(X_0+p(x-B))^{1-\gamma}}{1-\gamma} - px\,(X_0 + p\,(x-B))^{-\gamma} - \frac{X_0^{1-\gamma}}{1-\gamma} & \text{if } \gamma \neq 1. \\ \log\,(X_0 + p\,(x-B)) - \frac{px}{X_0+p(x-B)} - \log(X_0) & \text{if } \gamma = 1. \end{cases}$$

Clearly, this function is differentiable over its domain and hence it is also continuous. Therefore, it is enough to show that there exist $\underline{x}$ and $\bar{x}$ such that $g(\underline{x}) < 0 < g(\bar{x})$ and then apply the Intermediate Value Theorem to claim that there must exist V_1 in the interior of the domain of g such that $g(V_1) = 0$, from which we can conclude (3.8). Note that

$$g(B) = -pBX_0^{-\gamma} < 0\ .$$

To prove that it exists $\bar{x}$ such that $g(\bar{x}) > 0$, observe that the function g can be expressed, for the case of $\gamma \neq 1$, in the following manner:

$$g(x) = -\frac{X_0^{1-\gamma}}{1-\gamma} + (X_0 + p\,(x-B))^{-\gamma}\left[\left(\frac{X_0}{1-\gamma} - pB\right) + \left(\frac{\gamma}{1-\gamma}\right)p\,(x-B)\right]\ .$$

Thus, by applying L' Hôpital's Theorem, it follows that

$$\lim_{x\to\infty}(X_0 + p\,(x-B))^{-\gamma}\left[\left(\frac{X_0}{1-\gamma} - pB\right) + \left(\frac{\gamma}{1-\gamma}\right)p\,(x-B)\right] = \begin{cases} \infty & \text{if } \gamma < 1. \\ 0 & \text{if } \gamma > 1. \end{cases}$$

Thus,

$$\lim_{x\to\infty} g(x) = \begin{cases} \infty & \text{if } \gamma < 1. \\ -\frac{X_0^{1-\gamma}}{1-\gamma} & \text{if } \gamma > 1. \end{cases}$$

Note that $-\frac{X_0^{1-\gamma}}{1-\gamma} > 0$ for $\gamma > 1$. For the case of $\gamma = 1$, observe that

$$\lim_{x\to\infty} \left(\frac{px}{X_0 + p\,(x - B)} \right) = 1 \ .$$

Hence,

$$\lim_{x\to\infty} g(x) = \infty$$

for $\gamma = 1$. Therefore, for any $\gamma > 0$, it must exist $\bar{x}$ such that $g(\bar{x}) > 0$. Hence, by the Intermediate Value Theorem, it must exists V_1 in the interior of the interval $[B, \bar{x}]$, so $V_1 > B$, such that $g(V_1) = 0$ and thus it follows that V_1 must satisfy equation (3.8).

Q.E.D.

<u>**Proof of Lemma** 3.3.2</u>

The proof is similar to the proof of Lemma 3.3.1.

Q.E.D.

<u>**Proof of Lemma** 3.3.3</u>

Let $\gamma > 0$ and $\gamma \neq 1$. Then, the system formed by equations (3.10) and (3.11) imply that

$$p\,(X_0 + p\,(V_2 - B))^{-\gamma} = (p+q)\,(X_0 + p\,(V_3 - B) + q\,(V_3 - B - K))^{-\gamma} \qquad (3.37)$$

or equivalently,

$$X_0 + p(V_2 - B) = \left(1 + \frac{q}{p}\right)^{-\frac{1}{\gamma}} (X_0 + p(V_3 - B) + q(V_3 - B - K)) \tag{3.38}$$

Solving for V_2 from equation (3.38) leads to

$$V_2 = \beta V_3 + \alpha$$

where:

$$\begin{aligned} \beta &= \left(1 + \tfrac{q}{p}\right)^{1-\frac{1}{\gamma}} \\ \alpha &= \frac{X_0\left[\left(1+\frac{q}{p}\right)^{-\frac{1}{\gamma}} - 1\right]}{p} + (1-\beta)B - K\left(\tfrac{q}{p}\right)\left(1 + \tfrac{q}{p}\right)^{-\frac{1}{\gamma}} \end{aligned} \tag{3.39}$$

Thus,

$$V_3 = \left(\frac{1}{\beta}\right) V_2 - \left(\frac{\alpha}{\beta}\right) \tag{3.40}$$

Also, from equation (3.38), we deduce that

$$\frac{(X_0 + p(V_2 - B))^{1-\gamma}}{1-\gamma} = \frac{\left(1 + \frac{q}{p}\right)^{1-\frac{1}{\gamma}}}{1-\gamma} (X_0 + p(V_3 - B) + q(V_3 - B - K))^{1-\gamma}$$

i.e.,

$$\frac{(X_0 + p\,(V_2 - K))^{1-\gamma}}{1-\gamma} = \beta \frac{(X_0 + p\,(V_3 - B) + q\,(V_3 - B - K))^{1-\gamma}}{1-\gamma}\ . \tag{3.41}$$

Therefore, using (3.37), (3.40) and (3.41), we can rewrite equation (3.10) in the following manner:

$$\frac{(X_0 + p(V_2 - B))^{1-\gamma}}{1-\gamma} = \left[\frac{1}{\beta}\right]\left[\frac{(X_0 + p(V_2 - B))^{1-\gamma}}{1-\gamma}\right] + p(X_0 + p(V_2 - B))^{-\gamma}\left(V_2 - \left(\frac{1}{\beta}V_2 - \frac{\alpha}{\beta}\right)\right)$$

So,

$$\frac{(X_0 + p(V_2 - B))^{1-\gamma}}{1-\gamma}\left[1 - \frac{1}{\beta}\right] = p(X_0 + p(V_2 - B))^{-\gamma}\left[V_2\left(1 - \frac{1}{\beta}\right) + \frac{\alpha}{\beta}\right] .$$

Hence,

$$\frac{(X_0 + p(V_2 - B))}{1-\gamma}\left[1 - \frac{1}{\beta}\right] = p\left[V_2\left(1 - \frac{1}{\beta}\right) + \frac{\alpha}{\beta}\right] .$$

Therefore,

$$V_2\left[p\left(1 - \frac{1}{\beta}\right)\left(\frac{\gamma}{1-\gamma}\right)\right] = p\left(\frac{\alpha}{\beta}\right) - \left(\frac{1 - \frac{1}{\beta}}{1-\gamma}\right)(X_0 - pB)$$

from which

$$V_2 = \frac{p\left(\frac{\alpha}{\beta}\right) - \left(\frac{1-\frac{1}{\beta}}{1-\gamma}\right)(X_0 - pB)}{p\left(1 - \frac{1}{\beta}\right)\left(\frac{\gamma}{1-\gamma}\right)} \tag{3.42}$$

Q.E.D.

Proof of Proposition 3.3.1

V_1 exits by Lemma 3.3.1 and hence the function $\tilde{U}_1$ is well defined. Denote the derived utility function $U \circ \phi$ restricted to the domain $[0, B+K]$ by $(U \circ \phi)\,|_{[0,B+K]}$. We need to show that the hypograph of $\tilde{U}_1$, restricted to $[0, B+K]$, is the convex hull of the hypograph of $(U \circ \phi)\,|_{[0,B+K]}$. Let $H_{\tilde{U}_1}$ and $H_{U\circ\phi}$ be, respectively, the hypographs of $\tilde{U}_1$ and $U \circ \phi$ restricted to $[0, B+K]$. Thus, we have to prove the following conditions:

1. $H_{\tilde{U}_1}$ is a convex set.

2. $H_{\tilde{U}_1} \supseteq H_{U\circ\phi}$.

3. If $A \supseteq H_{U\circ\phi}$ and A is convex then $A \supseteq H_{\tilde{U}_1}$.

1. By the construction of $\tilde{U}_1$, $\tilde{U}_1''$ exists over the interval $[0, B+K]$ and it is defined as

$$\tilde{U}_1''(v) = \begin{cases} 0 & \text{for } v \in [0, V_1]. \\ (U \circ \phi)''(v) & \text{otherwise.} \end{cases}$$

where

$$(U \circ \phi)''(v) = U'(\phi(v))\,\phi''(v) + U''(\phi(v))\left(\phi'(v)\right)^2 . \tag{3.43}$$

Note that the assumptions that U is concave and differentiable, together with the observation that the compensation scheme satisfies the condition $\phi''(v) = 0$ for all $v \geq 0$, imply that $(U \circ \phi)''(v) \leq 0$ for all $v > V_1$. Therefore, $\tilde{U}_1''(v) \leq 0$ for all $v \geq 0$, and thus it follows that $\tilde{U}_1$ must be concave over all its domain. Hence, by the characterization of a concave function in terms of its hypograph, $H_{\tilde{U}_1}$ is a convex set.

2. By construction of $\tilde{U}_1$, $\tilde{U}_1$ dominates $U \circ \phi$ over the interval $[0, B + K]$, that is $\tilde{U}_1(v) \geq (U \circ \phi)(v)$ for all $v \in [0, B + K]$. Therefore, using the definition of the hypograph of a function, it is clear that $H_{\tilde{U}_1} \supseteq H_{U \circ \phi}$.

3. Let A be a convex set such that $A \supseteq H_{U \circ \phi}$. We have to show that $A \supseteq H_{\tilde{U}_1}$. To do so, we partition A, $H_{\tilde{U}_1}$, and $H_{U \circ \phi}$ in the following manner:

$$
\begin{array}{rclcrcl}
A & = & A_1 \bigcup (A \setminus A_1) & , \text{ with } & A_1 & = & \{(v, y) \in A | v \in [0, V_1]\} \\
H_{\tilde{U}_1} & = & H^1_{\tilde{U}_1} \bigcup \left(H_{\tilde{U}_1} \setminus H^1_{\tilde{U}_1}\right) & , \text{ with } & H^1_{\tilde{U}_1} & = & \{(v, y) \in H_{\tilde{U}_1} | v \in [0, V_1]\} \\
H_{U \circ \phi} & = & H^1_{U \circ \phi} \bigcup \left(H_{U \circ \phi} \setminus H^1_{U \circ \phi}\right) & , \text{ with } & H^1_{U \circ \phi} & = & \{(v, y) \in H_{U \circ \phi} | v \in [0, V_1]\}
\end{array}
$$

and where $C \setminus B$ denotes the set formed by all the elements of C that are not contained in B. That is, we partition each of these sets with respect to the *tangency point* V_1. The rationale of this partition is to prove $A \supseteq H_{\tilde{U}_1}$ by pieces. First, we prove that $(A \setminus A_1) \supseteq \left(H_{\tilde{U}_1} \setminus H^1_{\tilde{U}_1}\right)$ and then we prove that $A_1 \supseteq H^1_{\tilde{U}_1}$.

- $(A \setminus A_1) \supseteq \left(H_{\tilde{U}_1} \setminus H^1_{\tilde{U}_1}\right)$:

  By the design of $\tilde{U}_1$, $\tilde{U}_1$ and $U \circ \phi$ coincide over the interval (V_1, ∞). In particular, $\tilde{U}_1(v) = (U \circ \phi)(v)$ for $v \in [V_1, B + K]$. Hence, it must be satisfied that $H_{\tilde{U}_1} \setminus H^1_{\tilde{U}_1} = H_{U \circ \phi} \setminus H^1_{U \circ \phi}$. Thus, it is clear that the assumption $A \supseteq H_{U \circ \phi}$ implies $(A \setminus A_1) \supseteq \left(H_{\tilde{U}_1} \setminus H^1_{\tilde{U}_1}\right)$.

- $A_1 \supseteq H^1_{\tilde{U}_1}$:

  Observe first that the assumption $A \supseteq H_{U \circ \phi}$ assures that $A_1 \supseteq H^1_{U \circ \phi}$. Then, by construction of $\tilde{U}_1$, $\tilde{U}_1$ dominates $U \circ \phi$ over the interval $[0, V_1]$ and thus, it is satisfied that $H^1_{\tilde{U}_1} \supseteq H^1_{U \circ \phi}$. Therefore, to prove $A_1 \supseteq H^1_{\tilde{U}_1}$ it is enough to show that $A_1 \supseteq \left(H^1_{\tilde{U}_1} \setminus H^1_{U \circ \phi}\right)$. To accomplish this task, we use the following strategy: choose an arbitrary point in $\left(H^1_{\tilde{U}_1} \setminus H^1_{U\circ\phi}\right)$ and show that this point can be expressed as a convex combination of two elements of A_1. Then, we prove that A_1 is a convex set and thus conclude that the arbitrary point chosen must belong to A_1, by the convexity of A_1.

Let $(v_0, y_0) \in \left(H^1_{\tilde{U}_1} \setminus H^1_{U\circ\phi}\right)$ and consider the line that passes through the points $(0, (U \circ \phi)(0))$ and (v_0, y_0). This line is described by the equation

$$l(y) = (U \circ \phi)(0) + \left(\frac{y_0 - (U \circ \phi)(0)}{v_0}\right)(v - v_0)$$

where its slope, by construction of $\tilde{U}_1$, satisfies

$$\left(\frac{y_0 - (U \circ \phi)(0)}{v_0}\right) \leq (U \circ \phi)'(V_1) .$$

Recall that V_1 satisfies the following equation:

$$(U \circ \phi)(V_1) = (U \circ \phi)(0) + V_1 (U \circ \phi)'(V_1) .$$

Therefore,

$$\begin{aligned}
(U \circ \phi)(V_1) &= (U \circ \phi)(0) + V_1 (U \circ \phi)'(V_1) \\
&\geq (U \circ \phi)(0) + v_0 (U \circ \phi)'(V_1) \\
&\geq (U \circ \phi)(0) + v_0 \left(\tfrac{y_0 - (U \circ \phi)(0)}{v_0}\right) \\
&= l(v_0) \\
&= y_0
\end{aligned}$$

Thus, we deduce that l must cross the graph of $U \circ \phi$ at some value $v^* \leq V_1$. That is, it exists (v^*, y^*) such that $v^* \leq V_1$ and $l(v^*) = (U \circ \phi)(v^*)$. Clearly, $(v^*, y^*) \in H_{U\circ\phi}$. Therefore, the point (v_0, y_0) belongs to the line segment between the points $(0, (U \circ \phi)(0))$ and (v^*, y^*). Hence, it must exist $\lambda \in [0, 1]$ such that

$$(v_0, y_0) = \lambda(0, (U \circ \phi)(0)) + (1 - \lambda)(v^*, y^*) \ .$$

Now, we claim that A_1 is a convex set. To prove such claim let $(\bar{v}, \bar{y}), (\tilde{v}, \tilde{y}) \in A_1$ and $\eta \in (0, 1)$. Observe that it must be satisfied the condition

$$\eta\bar{v} + (1 - \eta)\tilde{v} \leq V_1$$

since $\bar{v}$ and $\tilde{v}$ are, by definition of A_1, less than or equal to V_1. On other side, $A_1 \subset A$ and A convex implies that

$$\eta\,(\bar{v}, \bar{y}) + (1 - \eta)\,(\tilde{v}, \tilde{y}) \in A \ .$$

Therefore, by definition of A_1,

$$\eta\,(\bar{v}, \bar{y}) + (1 - \eta)\,(\tilde{v}, \tilde{y}) \in A_1$$

and so, A_1 is a convex set.

To finish the proof of (3), recall that $(0, (U \circ \phi)(0)), (v^*, y^*) \in H_{U \circ \phi} \subseteq A_1$ and use the convexity of A_1 to conclude that $(v_0, y_0) \in A_1$ and thus, since (v_0, y_0) is arbitrary, $A_1 \supseteq H_{\tilde{U}_1}$.

Therefore, from conditions (1),(2), and (3), we conclude that $H_{\tilde{U}_1}$ is the smallest convex set that contains to $H_{U \circ \phi}$. Hence, by definition 3.3.1, $\tilde{U}_1$ must be the convavification of $U \circ \phi$ over the interval $[0, B + K]$.

Q.E.D.

Proof of Proposition 3.3.2

By Lemma 3.3.3, V_2 and V_3 exist and therefore, $\tilde{U}_2$ is well defined. Denote the derived utility function $U \circ \phi$ restricted to $[B, \infty)$ by $(U \circ \phi)|_{[B,\infty)}$. The goal is to show that the hypograph of $\tilde{U}_2$, restricted to $[B, \infty)$, is the convex hull of the hypograph of $(U \circ \phi)|_{[B,\infty)}$. Let $H_{\tilde{U}_2}$ and $H_{U\circ\phi}$ be, respectively, the hypographs of $\tilde{U}_2|_{[B,\infty)}$ and $(U \circ \phi)|_{[B,\infty)}$. Thus, we need to prove the following conditions:

1. $H_{\tilde{U}_2}$ is a convex set.

2. $H_{\tilde{U}_2} \supseteq H_{U\circ\phi}$.

3. If $A \supseteq H_{U\circ\phi}$ and A is a convex set then $A \supseteq H_{\tilde{U}_2}$.

1. By the construction of $\tilde{U}_2$, $\tilde{U}_2''$ exists over the interval $[B, \infty)$ and it is defined as

$$\tilde{U}_2''(v) = \begin{cases} 0 & \text{for } v \in (V_2, V_3]. \\ (U \circ \phi)''(v) & \text{for all } v > 0 \text{ such that } v \notin \{(V_2, V_3] \cup \{0\} \cup \{B\}\} \end{cases}$$

where

$$(U \circ \phi)''(v) = U'(\phi(v))\,\phi''(v) + U''(\phi(v))\left(\phi'(v)\right)^2 \leq 0$$

The assumptions that U is concave and differentiable, together with the observation that the compensation scheme satisfies condition $\phi''(v) = 0$ for all $v \geq 0$, imply that $(U \circ \phi)''(v) \leq 0$ for all $v > 0$ such that $v \notin \{(V_2, V_3] \cup \{0\} \cup \{B\}\}$. Therefore, $(U \circ \phi)''(v) \leq 0$ for all $v > 0$ such that $v \neq B$. Thus, $\tilde{U}_2$ must be a concave function. Hence, by the characterization of a concave function in terms of its hypograph, $H_{\tilde{U}_2}$ must be a convex set.

2. By the construction of $\tilde{U}_2$, $\tilde{U}_2$ dominates $U \circ \phi$ over the interval $[B, \infty)$, that is $\tilde{U}_2(v) \geq (U \circ \phi)(v)$ for all $v \in [B, \infty)$. Therefore, using the definition of hypograph of a function, we obtain that $H_{\tilde{U}_2} \supseteq H_{U\circ\phi}$.

3. Let A be a convex set such that $A \supseteq H_{U\circ\phi}$. We need to show that $A \supseteq H_{\tilde{U}_2}$. To accomplish that, partition A, $H_{\tilde{U}_2}$, and $H_{U\circ\phi}$ in the following manner:

$$
\begin{array}{rclcrcl}
A & = & A_1 \bigcup (A \setminus A_1) & , \text{ with} & A_1 & = & \{(v, y) \in A | v \in (V_2, V_3]\} \\
H_{\tilde{U}_2} & = & H^1_{\tilde{U}_2} \bigcup \left(H_{\tilde{U}_2} \setminus H^1_{\tilde{U}_2}\right) & , \text{ with} & H^1_{\tilde{U}_2} & = & \left\{(v, y) \in H_{\tilde{U}_2} | v \in (V_2, V_3]\right\} \\
H_{U\circ\phi} & = & H^1_{U\circ\phi} \bigcup \left(H_{U\circ\phi} \setminus H^1_{U\circ\phi}\right) & , \text{ with} & H^1_{U\circ\phi} & = & \{(v, y) \in H_{U\circ\phi} | v \in (V_2, V_3]\}
\end{array}
$$

and where $C \setminus B$ denotes the set formed by all the elements of C that are not contained in B. That is, we partition each of these sets with respect to the *tangency points* V_2 and V_3. The rationale of this partition is to prove $A \supseteq H_{\tilde{U}_2}$ by pieces. First, to prove $(A \setminus A_1) \supseteq \left(H_{\tilde{U}_2} \setminus H_{\tilde{U}^1_2}\right)$, and then, to prove that $A_1 \supseteq H_{\tilde{U}^1_2}$.

- $(A \setminus A_1) \supseteq \left(H_{\tilde{U}_2} \setminus H^1_{\tilde{U}_2}\right)$:

  By design of $\tilde{U}_2$, $\tilde{U}_2$ coincides with $U \circ \phi$ over the set $[0, V_2) \cup (V_3, \infty)$. That is, $(U \circ \phi)(v) = \tilde{U}_2(v)$ for all $v \in [0, V_2) \cup (V_3, \infty)$. Hence, it must be satisfied that $H_{\tilde{U}_2} \setminus H_{\tilde{U}^1_2} = H_{U\circ\phi} \setminus H^1_{U\circ\phi}$. Thus, it is clear that the assumption $A \supseteq H_{U\circ\phi}$ implies $(A \setminus A_1) \supseteq \left(H_{\tilde{U}_2} \setminus H_{\tilde{U}^1_2}\right)$.

- $A_1 \supseteq H^1_{\tilde{U}_2}$:

  The assumption $A \supseteq H_{U\circ\phi}$ assures that $A_1 \supseteq H^1_{U\circ\phi}$. Then, by construction of $\tilde{U}_2$, $\tilde{U}_2$ dominates $U \circ \phi$ over all its domain. Hence, it is satisfied that $H_{\tilde{U}^1_2} \supseteq H^1_{U\circ\phi}$. Therefore, to prove $A_1 \supseteq H^1_{U\circ\phi}$ it is enough to prove that $A_1 \supseteq \left(H^1_{\tilde{U}_2} \setminus H^1_{U\circ\phi}\right)$. To do so, we follow the next strategy: choose an arbitrary point in $\left(H^1_{\tilde{U}_2} \setminus H^1_{U\circ\phi}\right)$ and prove that this point can be expresses as a convex combination of two elements of the set A_1. Then, prove that A_1 is a convex set and thus conclude that the arbitrary point chosen must belong to A_1, by the convexity of A_1.

Let $(v_0, y_0) \in \left(H^1_{\tilde{U}_2} \setminus H^1_{U\circ\phi}\right)$ and consider the line that passes through the points $(V_2, (U \circ \phi)(V_2))$ and (v_0, y_0). This line is described by the equation

$$l(y) = (U \circ \phi)(V_2) + \left(\frac{y_0 - (U \circ \phi)(V_2)}{v_0 - V_2}\right)(v - V_2)$$

where its slope, by construction of $\tilde{U}_2$, satisfies

$$\left(\frac{y_0 - (U \circ \phi)(V_2)}{v_0 - V_2}\right) \leq (U \circ \phi)'(V_3) .$$

Recall that V_3 satisfies the equation

$$(U \circ \phi)(V_3) = (U \circ \phi)(V_2) + (V_3 - V_2)(U \circ \phi)'(V_3)$$

Therefore,

$$\begin{aligned}
(U \circ \phi)(V_3) &= (U \circ \phi)(V_2) + (V_3 - V_2)(U \circ \phi)'(V_3) \\
&\geq (U \circ \phi)(V_2) + (v_0 - V_2)(U \circ \phi)'(V_3) \\
&\geq (U \circ \phi)(V_2) + (v_0 - V_2)\left(\tfrac{y_0 - (U\circ\phi)(V_2)}{v_0 - V_2}\right) \\
&= l(v_0) \\
&= y_0
\end{aligned}$$

Thus, we deduce that l must cross the graph of $U \circ \phi$ at some value $v^* \leq V_3$. That is, it exists (v^*, y^*) such that $v^* \leq V_3$ and $l(v^*) = (U \circ \phi)(v^*)$. Clearly, $(v^*, y^*) \in$

$H_{U\circ\phi}$. Therefore, the point (v_0, y_0) belongs to the line segment between the points $(V_2, (U \circ \phi)(V_2))$ and (v^*, y^*). Hence, it must exists $\lambda \in [0,1]$ such that

$$(v_0, y_0) = \lambda(V_2, (U \circ \phi)(V_2)) + (1-\lambda)(v^*, y^*) \ .$$

We claim that A_1 is a convex set. To prove this claim, let $(\bar{v}, \bar{y}), (\tilde{v}, \tilde{y}) \in A_1$ and $\eta \in (0,1)$. By definition of A_1, $\bar{v}$ and $\tilde{v}$ are less than or equal to V_3, and greater than V_2. Hence,

$$V_2 < \eta\bar{v} + (1-\eta)\tilde{v} \leq V_3 \ .$$

On the other hand, $A_1 \subset A$ and A convex implies that

$$\eta(\bar{v}, \bar{y}) + (1-\eta)(\tilde{v}, \tilde{y}) \in A \ .$$

Therefore, by definition of A_1, $\eta(\bar{v}, \bar{y}) + (1-\eta)(\tilde{v}, \tilde{y}) \in A_1$, and so, A_1 is convex.

To end the proof of (3), recall that $(V_2, (U \circ \phi)(V_2)), (v^*, y^*) \in H_{U\circ\phi} \subseteq A_1$ and use the convexity of A_1 to conclude that $(v_0, y_0) \in A_1$. Thus, since (v_0, y_0) is arbitrary, we deduce that $A_1 \supseteq H_{\tilde{U}_2}$.

So, from conditions (1),(2), and (3) we conclude that $H_{\tilde{U}_2}$ is the smallest convex set that contains to $H_{U\circ\phi}$. Hence, by definition 3.3.1, $\tilde{U}_2$ must be the concavification of $U \circ \phi$ over the interval $[B, \infty)$.

Q.E.D.

Proof of Proposition 3.3.3

Let $H_{\tilde{U}_3}$ and $H_{U\circ\phi}$ be, respectively, the hypographs of $\tilde{U}_3$ and $U \circ \phi$. We need to show that $H_{\tilde{U}_3}$ is the convex hull of $H_{U\circ\phi}$.

By construction of $\tilde{U}_3$, $\tilde{U}_3(v) \geq (U \circ \phi)(v)$ for all $v \geq 0$ and hence, $H_{\tilde{U}_3} \supseteq H_{U\circ\phi}$. On the other hand, concavity of $\tilde{U}_3$ implies that $H_{\tilde{U}_3}$ is a convex set. Therefore, the only issue that remains to be proved is that $H_{\tilde{U}_3}$ is the smallest set that satisfies the latter two properties. To prove this, the strategy is the following: show that the assumption of concavity of $\tilde{U}_3$ implies that $V_2 \geq V_1$ and hence, deduce that $\tilde{U}_3$ coincides with $\tilde{U}_1$ over the interval $[0, V_2]$, and with $\tilde{U}_2$ over the interval (V_2, ∞). Therefore,

$$H_{\tilde{U}_3} = H_{\tilde{U}_1|_{[0,V_2]}} \cup H_{\tilde{U}_2|_{(V_2,\infty)}} \text{ and } H_{\tilde{U}_1|_{[0,V_2]}} \cap H_{\tilde{U}_2|_{(V_2,\infty)}} = \emptyset$$

where $H_{\tilde{U}_1|_{[0,V_2]}}$ is the hypograph of $\tilde{U}_1$ restricted to $[0, V_2]$, and $H_{\tilde{U}_2|_{(V_2,\infty)}}$ represents the hypograph of $\tilde{U}_2$ restricted to (V_2, ∞). Finally, use the fact that $\tilde{U}_1|_{[0,V_2]}$ must be the concavification function of $U \circ \phi$ restricted to $[0, V_2]$, while $\tilde{U}_2|_{(V_2,\infty)}$ is the concavification function of $U \circ \phi$ restricted to (V_2, ∞) to deduce that any convex set A such that $A \supseteq H_{U\circ\phi}$ must satisfy that $A \supseteq H_{\tilde{U}_3}$. We now prove this strategy in detail:

By construction of $\tilde{U}_3$, the following conditions are satisfied:

- $V_3 \geq V_1$.
- $$\begin{aligned} \tilde{U}_3(V_3) &= (U \circ \phi)(V_3) \\ \tilde{U}_3(V_1) &= (U \circ \phi)(V_1) \end{aligned} .$$
- $\tilde{U}_3$ is twice differentiable and,

  $$-\ (U \circ \phi)'(V_2) = (U \circ \phi)'(V_3) \ ,$$

– under the assumption of concavity, $\tilde{U}_3''(v) \leq 0$ for all $v \geq 0$.

Thus,

$$(U \circ \phi)'(V_2) = (U \circ \phi)'(V_3) = \tilde{U}_3'(V_3) \leq \tilde{U}_3'(V_1) = (U \circ \phi)'(V_1) .$$

Therefore,

$$(U \circ \phi)'(V_2) \leq (U \circ \phi)'(V_1)$$

which, by the concavity of $U \circ \phi$ over the interval $[B, B+K]$, implies that $V_2 \geq V_1$. Hence, by the definition of $\tilde{U}_1$ and $\tilde{U}_2$,

$$\begin{array}{rcll} \tilde{U}_1(v) & = & \tilde{U}_2(v) & , \quad \text{for } v \in [V_1, V_2], \\ \tilde{U}_1(v) & \geq & \tilde{U}_2(v) & , \quad \text{for } v < V_1 \\ \tilde{U}_2(v) & \geq & \tilde{U}_1(v) & , \quad \text{for } v > V_2 \end{array}$$

Therefore, by the construction of $\tilde{U}_3$,

$$\tilde{U}_3 = \begin{cases} \tilde{U}_1(v) & \text{for } v \leq V_2 \\ \tilde{U}_2(v) & \text{for } v > V_2 \end{cases} \tag{3.44}$$

So,

$$H_{\tilde{U}_3} = H_{\tilde{U}_1|_{[0,V_2]}} \cup H_{\tilde{U}_2|_{(V_2,\infty)}} \text{ and } H_{\tilde{U}_1|_{[0,V_2]}} \cap H_{\tilde{U}_2|_{(V_2,\infty)}} = \emptyset \tag{3.45}$$

From Propositions 3.3.1 and 3.3.2 we know that $\tilde{U}_1$ and $\tilde{U}_2$ are, respectively, the concavification functions of $U \circ \phi$ restricted to $[0, B+K]$ and $[B, \infty)$. Thus, given that $V_2 \geq V_1$, it is clear that $\tilde{U}_1|_{[0,V_2]}$ must be the concavification function of $U \circ \phi$ restricted to $[0, V_2]$, while $\tilde{U}_2|_{(V_2,\infty)}$ is the concavification function of $U \circ \phi$ restricted to (V_2, ∞).

Let A be a convex set such that $A \supseteq H_{U\circ\phi}$ and partition A and $H_{U\circ\phi}$ in the following manner:

$$\begin{aligned} A &= A_1 \bigcup (A \setminus A_1) \quad &, \quad A_1 &= \{(v,y) \in A | v \in [0, V_2]\} \\ H_{U\circ\phi} &= H^1_{U\circ\phi} \bigcup \left(H_{U\circ\phi} \setminus H^1_{U\circ\phi}\right) \quad &, \quad H^1_{U\circ\phi} &= \{(v,y) \in H_{U\circ\phi} | v \in [0, V_2]\} \end{aligned}$$

where $C \setminus B$ denotes the set of all the elements of C that ar not in B. Thus, by definition of this partition,

$$A_1 \supseteq H^1_{U\circ\phi} \text{ and } (A \setminus A_1) \supseteq \left(H_{U\circ\phi} \setminus H^1_{U\circ\phi}\right)$$

Hence, since $\tilde{U}_1|_{[0,V_2]}$ and $\tilde{U}_2|_{[V_2,\infty)}$ are, respectively, the concavification functions of $U \circ \phi$ over $[0, V_2]$ and (V_2, ∞), it must be satisfied then that

$$A_1 \supseteq H_{\tilde{U}_1|_{[0,V_2]}} \text{ and } (A \setminus A_1) \supseteq H_{\tilde{U}_2|_{(V_2,\infty)}}$$

Therefore, by (3.45), $A \supseteq H_{\tilde{U}_3}$.

Q.E.D.

<u>Proof of Proposition 3.3.4</u>

The necessity of condition $V_2 \geq V_1$ is proved in the first part of the proof of the previous proposition.

We now prove the sufficiency of condition $V_2 \geq V_1$. This condition implies, by the definition of $\tilde{U}_1$ and $\tilde{U}_2$, that

$$\begin{aligned} \tilde{U}_1(v) &= \tilde{U}_2(v) \quad , \quad \text{for } v \in [V_1, V_2], \\ \tilde{U}_1(v) &\geq \tilde{U}_2(v) \quad , \quad \text{for } v < V_1, \\ \tilde{U}_2(v) &\geq \tilde{U}_1(v) \quad , \quad \text{for } v > V_2. \end{aligned}$$

Therefore, by construction of $\tilde{U}_3$,

$$\tilde{U}_3 = \begin{cases} \tilde{U}_1(v) & \text{for } v \leq V_2 \\ \tilde{U}_2(v) & \text{for } v > V_2 \end{cases} .$$

Hence, since $\tilde{U}_1$ and $\tilde{U}_2$ are concave, twice differentiable and $\tilde{U}_1'(V_2) = \tilde{U}_2'(V_2)$, then $\tilde{U}_3''(v) \leq 0$ for all $v \geq 0$, and so we conclude that $\tilde{U}_3$ must be concave.

Q.E.D.

Proof of Proposition 3.3.5

The proof is completely analogous to the proof in Proposition 3.3.1 except that in the current case V_4, instead of V_1, plays the role of the *pivot* to partition the domain of $\tilde{U}_3$. Conditions (3.15) and (3.16) are simply the explicit statement that the point $(0, (U \circ \phi)(0))$ belongs to the tangent line of the graph of $(U \circ \phi)|_{[B+K,\infty)}$ at $(V_4, (U \circ \phi)(V_4))$. That is, V_4 must satisfy

$$(U \circ \phi)(0) = (U \circ \phi)(V_4) + (0 - V_4)(U \circ \phi)'(V_4)$$

Q.E.D.

Proof of Proposition 3.3.6

If $\tilde{U}_3$ is concave then, by Definition 3.3.1, $\tilde{U}(v) = \tilde{U}_3(v)$ for all $v \geq 0$, and hence, by Proposition 3.3.3, $\tilde{U}$ is the concavification of $U \circ \phi$. Otherwise, assume that $\tilde{U}_3$ is not concave and let $H_{\tilde{U}}$ be the hypograph of $\tilde{U}$. By construction, $\tilde{U}$ is concave and satisfies $\tilde{U}(v) \geq \tilde{U}_3(v) \geq (U \circ \phi)(v)$ for all $v \geq 0$. Therefore, $H_{\tilde{U}}$ is a convex set that contains to $H_{U\circ\phi}$. Hence, what remains to be shown is that $H_{\tilde{U}}$ is the smallest set with the latter two properties. Let A be a convex set such that $A \supseteq H_{U\circ\phi}$. We need to prove that $A \supseteq H_{\tilde{U}}$. It is enough to show that $A \supseteq H_{\tilde{U}_3}$ given that $\tilde{U}$ is the concavification of $\tilde{U}_3$. To prove $A \supseteq H_{\tilde{U}_3}$, we decompose A and $H_{U\circ\phi}$ into

$$\begin{aligned} A &= A_1 \cup A_2 \\ H_{U\circ\phi} &= H^1_{U\circ\phi} \cup H^2_{U\circ\phi} \end{aligned}$$

where:

$$\begin{aligned} A_1 &= \{(v,y) \in A | v \in [0, B+K]\} \\ A_2 &= \{(v,y) \in A | v \in [B, \infty)\} \\ H^1_{U\circ\phi} &= \{(v,y) \in H_{U\circ\phi} | v \in [0, B+K]\} \\ H^2_{U\circ\phi} &= \{(v,y) \in H_{U\circ\phi} | v \in [B, \infty)\} \end{aligned}$$

Thus, $A \supseteq H_{U\circ\phi}$ implies $A_1 \supseteq H^1_{U\circ\phi}$ and $A_2 \supseteq H^2_{U\circ\phi}$. Therefore, given that $\tilde{U}_1$ and $\tilde{U}_2$ are, respectively, the concavification functions of $U \circ \phi$ restricted to $[0, B+K]$ and $[B, \infty)$, it must be satisfied that $A_1 \supseteq H^1_{\tilde{U}_1}$ and $A_2 \supseteq H^2_{\tilde{U}_2}$, where

$$\begin{aligned} H^1_{\tilde{U}_1} &= \left\{(v,y) \in H_{\tilde{U}_1} | v \in [0, B+K]\right\} \\ H^2_{\tilde{U}_2} &= \left\{(v,y) \in H_{\tilde{U}_2} | v \in [B, \infty)\right\} \end{aligned}$$

Hence, $A_1 \cup A_2 \supseteq H^1_{\tilde{U}_1} \cup H^2_{\tilde{U}_2}$. Moreover, by construction of $\tilde{U}_1$ and $\tilde{U}_2$ and the definition of $\tilde{U}_3$,

$$H^1_{\tilde{U}_1} \cup H^2_{\tilde{U}_2} = H_{\tilde{U}_1} \cup H_{\tilde{U}_2} = H_{\tilde{U}_3} \ .$$

Thus,

$$A = A_1 \cup A_2 \supseteq H_{\tilde{U}_3} \ .$$

Therefore, since $\tilde{U}$ is the concavification function of $\tilde{U}_3$, $A \supseteq H_{\tilde{U}}$ must be satisfied.

Q.E.D.

<u>**Proof of Proposition** 3.6.1:</u>

By construction, V_1 satisfies $V_1 \geq B$ and (see Lemma 3.3.1)

$$0 = (U \circ \phi)(V_1) - (U \circ \phi)(0) - V_1 (U \circ \phi)'(V_1) \ . \tag{3.46}$$

The results are obtained by deriving the previous equation with respect to B, p, and X_0 and then solving of the corresponding partial derivative, as it is shown next:

1. Deriving (3.46) with respect to B leads to

$$\begin{aligned} 0 &= (X_0 + p(V_1 - B))^{-\gamma} p \left[\tfrac{\partial V_1}{\partial B} - 1\right] + p^2 \gamma V_1 (X_0 + p(V_1 - B))^{-\gamma-1} \left[\tfrac{\partial V_1}{\partial B} - 1\right] \\ &- p (X_0 + p(V_1 - B))^{-\gamma} \left[\tfrac{\partial V_1}{\partial B}\right] \end{aligned}$$

or equivalently,

$$0 = p\left[\frac{\partial V_1}{\partial B} - 1\right] + p^2\gamma V_1\left(X_0 + p\left(V_1 - B\right)\right)^{-1}\left[\frac{\partial V_1}{\partial B} - 1\right] - p\left[\frac{\partial V_1}{\partial B}\right]$$

from which

$$\frac{\partial V_1}{\partial B} = 1 + \left(X_0 + p\left(V_1 - B\right)\right)\left(p\gamma V_1\right)^{-1} > 0 \,.$$

2. Deriving (3.46) with respect to p leads to

$$\begin{aligned} 0 &= \left(X_0 + p\left(V_1 - B\right)\right)^{-\gamma}\left[p\tfrac{\partial V_1}{\partial p} + \left(V_1 - B\right)\right] \\ &+ p\gamma V_1\left(X_0 + p\left(V_1 - B\right)\right)^{-\gamma-1}\left[p\tfrac{\partial V_1}{\partial p} + \left(V_1 - B\right)\right] \\ &- V_1\left(X_0 + p\left(V_1 - B\right)\right)^{-\gamma} \\ &- p\left(X_0 + p\left(V_1 - B\right)\right)^{-\gamma}\left[\tfrac{\partial V_1}{\partial p}\right] \end{aligned}$$

or equivalently,

$$0 = -B + p\gamma V_1\left(X_0 + p\left(V_1 - B\right)\right)^{-1}\left[p\frac{\partial V_1}{\partial p} + \left(V_1 - B\right)\right] \,.$$

Hence,

$$\frac{\partial V_1}{\partial p} = \frac{B\left[1 + \left\{\left(p\gamma V_1\right)^{-1}\left(X_0 + p\left(V_1 - B\right)\right)\right\}\right] - V_1}{p}$$

from which the result follows.

3. Deriving (3.46) with respect to X_0 leads to

$$\begin{aligned} 0 &= \left(X_0 + p\left(V_1 - B\right)\right)^{-\gamma}\left[1 + p\tfrac{\partial V_1}{\partial X_0}\right] - X_0^{-\gamma} \\ &+ p\gamma V_1\left(X_0 + p\left(V_1 - B\right)\right)^{-\gamma-1}\left[1 + p\tfrac{\partial V_1}{\partial X_0}\right] - p\left(X_0 + p\left(V_1 - B\right)\right)^{-\gamma}\left[\tfrac{\partial V_1}{\partial X_0}\right] \end{aligned}$$

or equivalently,

$$\begin{aligned} 0 &= \left(p^2\gamma V_1\right)\left(X_0+p\left(V_1-B\right)\right)^{-1}\left[\tfrac{\partial V_1}{\partial X_0}\right]+1-\tfrac{X_0^{-\gamma}}{\left(X_0+p\left(V_1-B\right)\right)^{-\gamma}} \\ &+ p\gamma V_1\left(X_0+p\left(V_1-B\right)\right)^{-1} . \end{aligned}$$

Thus,

$$\frac{\partial V_1}{\partial X_0}=\left(p^2\gamma V_1\right)^{-1}\left(X_0+p\left(V_1-B\right)\right)\left[\frac{X_0^{-\gamma}}{\left(X_0+p\left(V_1-B\right)\right)^{-\gamma}}-1-p\gamma V_1\left(X_0+p\left(V_1-B\right)\right)^{-1}\right]$$

from which result (3) follows.

Q.E.D.

Proof of Proposition 3.6.2:

1. Deriving (3.42) with respect to K leads to

$$\frac{\partial V_2}{\partial K}=\left(\frac{p^2\gamma}{(1-\gamma)\beta}\right)\left[\frac{\left(1-\frac{1}{\beta}\right)\frac{\partial\alpha}{\partial K}+\left(\frac{\alpha}{\beta^2}\right)\frac{\partial\beta}{\partial K}}{p^2\left(1-\frac{1}{\beta}\right)^2\left(\frac{\gamma}{1-\gamma}\right)^2}\right]$$

where α and β are defined in (3.39). Therefore, the sign of $\frac{\partial V_2}{\partial K}$ depends on the expression

$$\left(\frac{p^2\gamma}{(1-\gamma)\beta}\right)\left[\left(1-\frac{1}{\beta}\right)\frac{\partial\alpha}{\partial K}+\left(\frac{\alpha}{\beta^2}\right)\frac{\partial\beta}{\partial K}\right] .$$

From (3.39)

$$\begin{aligned} \tfrac{\partial\alpha}{\partial K} &= \left(\tfrac{q}{p}\right)\left(1+\tfrac{q}{p}\right)^{-\frac{1}{\gamma}} \\ \tfrac{\partial\beta}{\partial K} &= 0 \end{aligned} .$$

Therefore,

$$\frac{\partial V_2}{\partial K} = \left(\frac{p^2\gamma}{(1-\gamma)\beta}\right)\left(1-\frac{1}{\beta}\right)\left(\frac{q}{p}\right)\left(1+\frac{q}{p}\right)^{-\frac{1}{\gamma}}$$

which does not depend on K. Thus, V_2 is a linear function with respect to K. Moreover, from (3.39), $(1-1/\beta) < 0$ if and only if $\gamma < 1$. Hence, $\frac{\partial V_2}{\partial K} < 0$ if and only if $\gamma < 1$.

2. Deriving (3.42) with respect to q leads to

$$\frac{\partial V_2}{\partial q} = \left(\frac{p^2\gamma}{(1-\gamma)\beta}\right)\left[\frac{\left(1-\frac{1}{\beta}\right)\frac{\partial\alpha}{\partial q} + \left(\frac{\alpha}{\beta^2}\right)\frac{\partial\beta}{\partial q}}{p^2\left(1-\frac{1}{\beta}\right)^2\left(\frac{\gamma}{1-\gamma}\right)^2}\right]$$

where α and β are defined in (3.39). The result follows from the above equation.

Q.E.D.

Lemma 3.7.2 *Let $X \sim N(\mu, \sigma^2)$ be defined in a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and let $A \subseteq \Omega$. Then,*

$$E\left[e^{\beta X}\mathbf{1}_{\{X\in A\}}\right] = e^{\mu\beta+\frac{1}{2}\sigma^2\beta^2} P\left(\hat{X} \in A\right)$$

where $\hat{X} \sim N(\mu + \beta\sigma^2, \sigma^2)$.

Proof: Just note that

$$\begin{aligned}
\beta x - \tfrac{1}{2\sigma^2}(x-\mu)^2 &= \tfrac{2\sigma^2\beta x-(x^2-2x\mu+\mu^2)}{2\sigma^2} \\
&= -\tfrac{1}{2\sigma^2}\left(x^2 - 2x(\mu+\beta\sigma^2) + \mu^2\right) \\
&= -\tfrac{1}{2\sigma^2}\left((x-(\mu+\beta\sigma^2))^2 - \beta\sigma^2(2\mu+\beta\sigma^2)\right) \\
&= -\tfrac{1}{2\sigma^2}\left(x-(\mu+\beta\sigma^2)\right)^2 + \mu\beta + \tfrac{1}{2}\sigma^2\beta^2 \ .
\end{aligned}$$

Hence,

$$\begin{aligned} E\left[e^{\beta X}\mathbf{1}_A\right] &= e^{\mu\beta+\frac{1}{2}\sigma^2\beta^2}\int_A \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{\left(x-(\mu+\beta\sigma^2)\right)^2}{2\sigma^2}} \\ &= e^{\mu\beta+\frac{1}{2}\sigma^2\beta^2} P\left(\hat{X}\in A\right) \end{aligned}$$

where $\hat{X} \sim N\left(\mu+\beta\sigma^2, \sigma^2\right)$.

Q.E.D.

Lemma 3.7.3 (Characterization of λ) *If $V_2 \geq V_1$, the parameter λ must satisfy the equation*

$$\begin{aligned} V_0 &= \left[B-\tfrac{X_0}{p}\right] e^{-\mu T}\left[\mathcal{N}(d_T(m_1,1))-\mathcal{N}(d_T(m_2,1))\right] \\ &+ \left(\tfrac{1}{p}\right)\left(\tfrac{\lambda}{p}\right)^{\gamma^*-1} e^{-\left(\mu+\frac{\alpha^2(1-\gamma^*)}{2}\right)\gamma^* T}\left[\mathcal{N}(d_T(m_1,\gamma^*))-\mathcal{N}(d_T(m_2,\gamma^*))\right] \\ &+ \left[B-\tfrac{X_0-qK}{p+q}\right] e^{-\mu T}\mathcal{N}\left(d_T(m_2,1)\right) \\ &+ \left(\tfrac{1}{p+q}\right)\left(\tfrac{\lambda}{p+q}\right)^{(\gamma^*-1)} e^{-\left(\mu+\frac{\alpha^2(1-\gamma^*)}{2}\right)\gamma^* T}\mathcal{N}(d_T(m_2,\gamma^*)) \end{aligned} \tag{3.47}$$

where $d_u(\cdot,\cdot)$ is the function defined by

$$d_u(m,\beta) \equiv \left[\log m - \left(-\mu u + \alpha^2 u\left(\beta-\frac{1}{2}\right)\right)\right] / \left(\alpha\sqrt{u}\right) \ , \tag{3.48}$$

$\mathcal{N}(\cdot)$ is the standard normal cumulative distribution, $\gamma^* = 1-1/\gamma$, and

$$m_i = \left[\frac{p}{\lambda\xi_t}\right]\left[X_0 + p\left(V_i - B\right)\right]^{-\gamma} \ , \ for\ i = 1,2,$$

and where V_1 and V_2 are defined in Lemmas 3.3.1 *and* 3.3.3*, respectively. Otherwise,* λ *must satisfy the equation*

$$\begin{aligned} V_0 &= \left[B - \tfrac{X_0 - qK}{p+q}\right] e^{-\mu T} \mathcal{N}\left(d_T(m_4, 1)\right) \\ &+ \left(\tfrac{1}{p+q}\right)\left(\tfrac{\lambda}{p+q}\right)^{(\gamma^*-1)} e^{-\left(\mu + \frac{\alpha^2(1-\gamma^*)}{2}\right)\gamma^* T} \mathcal{N}(d_T(m_4, \gamma^*)) \end{aligned} \quad . \tag{3.49}$$

where

$$m_4 = \left[\frac{p+q}{\lambda \xi_t}\right] \left[X_0 + p\left(V_4 - B\right) + q\left(V_4 - B - K\right)\right]^{-\gamma} ,$$

and V_4 is defined in Lemma 3.3.2.

Proof: Assume that $V_2 \geq V_1$. Therefore, condition (3.20) is equivalent to

$$\begin{aligned} V_0 &= \left[B - \tfrac{X_0}{p}\right] E\left[\xi_T 1_{\{p[X_0 + p(V_2 - B)]^{-\gamma} \leq \lambda \xi_T < p[X_0 + p(V_1 - B)]^{-\gamma}\}}\right] \\ &+ \left(\tfrac{1}{p}\right)\left(\tfrac{\lambda}{p}\right)^{\gamma^*-1} E\left[\xi_T^{\gamma^*} 1_{\{p[X_0 + p(V_2 - B)]^{-\gamma} \leq \lambda \xi_T < p[X_0 + p(V_1 - B)]^{-\gamma}\}}\right] \\ &+ \left[B - \tfrac{X_0 - qK}{p+q}\right] E\left[\xi_T 1_{\{\lambda \xi_T < p[X_0 + p(V_2 - B)]^{-\gamma}\}}\right] \\ &+ \left(\tfrac{1}{p+q}\right)\left(\tfrac{\lambda}{p+q}\right)^{(\gamma^*-1)} E\left[\xi_T^{\gamma^*} 1_{\{p[X_0 + p(V_2 - B)]^{-\gamma} \leq \lambda \xi_T < p[X_0 + p(V_1 - B)]^{-\gamma}\}}\right] \end{aligned} \tag{3.50}$$

By Lemma 3.7.2

$$E\left[\xi_T^{\beta} \mathbf{1}_{\{\log \xi_T \in A\}}\right] = e^{-\left[\mu + \left(\frac{\alpha^2}{2}\right)(1-\beta)\right]\beta T} P\left(\hat{X} \in A\right) \tag{3.51}$$

where $\hat{X} \sim N\left(-\left[\mu + \left(\frac{\alpha^2}{2}\right)(1-\beta)\right] T, \alpha^2 T\right)$. The result follows from applying (3.51) to (3.50).

The proof of condition (3.49) is similar.

Q.E.D.

Proposition 3.7.1 (Optimal Firm Value for any time $t \in (0,T]$) *Let $\gamma > 0$ and V_1 and V_2 as defined in Lemmas 3.3.1 and 3.3.3, respectively. If $V_2 \geq V_1$, then the optimal fund value for any $t \in [0,T)$ is*

$$\begin{aligned} V_t^* &= \left[B - \tfrac{X_0 - qK}{p+q}\right] e^{-\mu(T-t)} \mathcal{N}\left(d_{T-t}(m_2, 1)\right) \\ &+ \left[(p+q)^{-\gamma^*} \lambda^{\gamma^*-1} \xi_t^{\gamma^*-1}\right] e^{-\left(\mu + \frac{\alpha^2(1-\gamma^*)}{2}\right)\gamma^*(T-t)} \mathcal{N}\left(d_{T-t}\left(m_2, \gamma^*\right)\right) \\ &+ \left[B - \tfrac{X_0}{p}\right] e^{-\mu(T-t)} \left[\mathcal{N}\left(d_{T-t}(m_1, 1)\right) - \mathcal{N}\left(d_{T-t}(m_2, 1)\right)\right] \\ &+ \left[p^{-\gamma^*} \lambda^{\gamma^*-1} \xi_t^{\gamma^*-1}\right] e^{-\left(\mu + \frac{\alpha^2(1-\gamma^*)}{2}\right)\gamma^*(T-t)} \left[\mathcal{N}\left(d_{T-t}(m_1, \gamma^*)\right) - \mathcal{N}\left(d_{T-t}(m_2, \gamma^*)\right)\right] \end{aligned}$$

where $d_u(\cdot,\cdot)$ is defined by expression (3.48),

$$m_i = \left[\frac{p}{\lambda \xi_t}\right] \left[X_0 + p\left(V_i - B\right)\right]^{-\gamma} \text{ , } \textit{for } i = 1, 2,$$

$\mathcal{N}(\cdot)$ is the standard normal cumulative distribution, $\gamma^* = 1 - 1/\gamma$, λ is a nonnegative scalar that satisfies Equation (3.47). Otherwise, the optimal fund value for any $t \in [0,T)$ is given by

$$\begin{aligned} V_t^* &= \left[B - \tfrac{X_0 - qK}{p+q}\right] e^{-\mu(T-t)} \mathcal{N}\left(d_{T-t}(m_4, 1)\right) \\ &+ \left[(p+q)^{-\gamma^*} \lambda^{\gamma^*-1} \xi_t^{\gamma^*-1}\right] e^{-\left(\mu + \frac{\alpha^2(1-\gamma^*)}{2}\right)\gamma^*(T-t)} \mathcal{N}\left(d_{T-t}\left(m_4, \gamma^*\right)\right) \end{aligned}$$

where

$$m_4 = \left[\frac{p+q}{\lambda \xi_t}\right] [X_0 + p(V_4 - B) + q(V_4 - B - K)]^{-\gamma} \ ,$$

V_4 is defined as in Lemma 3.3.2, and λ is a nonnegative scalar that satisfies Equation (3.49).

Proof: Assume that $V_2 \geq V_1$. Note that $(\xi_t^* V_t^*)_{t \geq 0}$ is a martingale. Therefore,

$$\xi_t V_t^* = E_t\left[\xi_T V_T^*\right]$$

for all $t \in [0, T]$ and where $E_t[\ \cdot\ ]$ is the expected value operator with information up to time t. Hence, the above expression and (3.19) lead to

$$\begin{aligned}
V_t^* &= \tfrac{1}{\xi_t^*} E_t\left[\xi_T V_T^*\right] \\
&= \left[B - \tfrac{X_0}{p}\right] E\left[\left(\tfrac{\xi_T}{\xi_t}\right) 1_{\{p[X_0+p(V_2-B)]^{-\gamma} \leq \lambda\xi_T < p[X_0+p(V_1-B)]^{-\gamma}\}}\right] \\
&+ \left(\tfrac{1}{p}\right)\left(\tfrac{\lambda}{p}\right)^{\gamma^*-1} \xi_t^{\gamma^*-1} E\left[\left(\tfrac{\xi_T}{\xi_t}\right)^{\gamma^*} 1_{\{p[X_0+p(V_2-B)]^{-\gamma} \leq \lambda\xi_T < p[X_0+p(V_1-B)]^{-\gamma}\}}\right] \\
&+ \left[B - \tfrac{X_0-qK}{p+q}\right] E\left[\left(\tfrac{\xi_T}{\xi_t}\right) 1_{\{\lambda\xi_T < p[X_0+p(V_2-B)]^{-\gamma}\}}\right] \\
&+ \left(\tfrac{1}{p+q}\right)\left(\tfrac{\lambda}{p+q}\right)^{(\gamma^*-1)} \xi_t^{\gamma^*-1} E\left[\left(\tfrac{\xi_T}{\xi_t}\right)^{\gamma^*} 1_{\{\lambda\xi_T < p[X_0+p(V_2-B)]^{-\gamma}\}}\right]
\end{aligned} \tag{3.52}$$

By Lemma 3.7.2

$$E\left[\xi_T^{\beta} \mathbf{1}_{\{\log \xi_T \in A\}}\right] = e^{-\left[\mu + \left(\frac{\alpha^2}{2}\right)(1-\beta)\right]\beta T} P\left(\hat{X} \in A\right) \tag{3.53}$$

where $\hat{X} \sim N\left(-\left[\mu + \left(\frac{\alpha^2}{2}\right)(1-\beta)\right] T, \alpha^2 T\right)$. The result follows from applying (3.53) to (3.52). The proof for the case $V_2 < V_1$ is similar.

Q.E.D.

Observation made in 3.5.1

Any strategy that leads to fund values between 0 and V_4 can be characterized in terms of a density function $g(x)$, for $x \in (0, V_4)$, that specifies the likelihood of choosing the intermediate value x. By construction,

$$(U \circ \phi)(x) \leq \lambda(x)(U \circ \phi)(0) + (1 - \lambda(x))(U \circ \phi)(V_4)$$

where $\lambda(x) \in (0, 1)$ and it satisfies $x = \lambda(x) \cdot 0 + (1 - \lambda(x)) V_4$. Therefore,

$$\begin{aligned} \int_0^{V_4} (U \circ \phi)(x) dg(x) &\leq \left[\int_0^{V_4} \lambda(x) dg(x)\right] (U \circ \phi)(0) + \left[1 - \int_0^{V_4} \lambda(x) dg(x)\right] (U \circ \phi)(V_4) \\ &= \mathcal{P}(U \circ \phi)(0) + [1 - \mathcal{P}](U \circ \phi)(V_4) \end{aligned}$$

where $\mathcal{P} = \int_0^{V_4} \lambda(x) dg(x)$. Clearly, $\mathcal{P} \in [0, 1]$. Hence, following a strategy that takes on the payoff of zero with probability $\mathcal{P}$ and the payoff of V_4, with probability $1 - \mathcal{P}$ is better, in the sense of expected utility, than any strategy that takes on intermediate values between zero and V_4.