GRADE 7 STUDENTS' UNDERSTANDINGS OF DIVISION: A CLASSROOM CASE STUDY

By

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ABSTRACT

This study is concerned with Grade 7 students' conceptual and procedural understandings of division. Although division is formally introduced in Grade 3 or 4, late intermediate students frequently have difficulty understanding both the concepts and the procedures associated with division.

The classroom case study was chosen as the method of investigation for this study. Because the researcher was also the enrolling teacher of the group of 22 Grade 7 students, the conditions of the study were as similar as is possible to regular classroom instruction. The investigation followed a unit of study in division of whole numbers and decimal fractions from the pretest, through instruction, to the posttest. The researcher elicited students' understandings of division in computational and problem-solving situations in a variety of ways. Students wrote a pencil-and-paper pretest which was designed to reveal understandings. Areas of interest identified by the pretest were then investigated through small group and whole class discussions. Instruction was based on eliciting and confronting students' beliefs regarding division, and on strengthening conceptual understanding of both division and decimal fractions.

Students viewed division procedurally, attaching little meaning to the processes associated with the division algorithm. Approximately one fourth of the students were uncertain about the meaning of the two forms of notation, and most read "b $\overline{)a}$ " as "b goes into a." When asked to use manipulative materials to reflect a division question, some students were unable to do so independently.

It was found that students relied heavily on the partitive model of division. Although some students demonstrated an understanding of quotitive division, these students also tended to rely on partition and turned to quotition only when it became apparent that partition was not appropriate. This reliance on partition influenced the students' ability to solve story problems requiring division. Students were able to solve story problems which fit the partitive model: the divisor is a whole number and is less than the whole number dividend. In situations where this was not true, students had difficulty. In these cases, students reversed the terms of the question or chose an operation other than division.

These results led to an investigation of students' beliefs about division. The belief that "division always makes smaller" was common. This belief stems from partition with whole numbers where it is true.

A related belief held by students is that the divisor must be smaller than the dividend. An exception is the case where this would necessitate a divisor less than one. In this case, students preferred a larger whole number as the divisor. Division by a number less than one was seen as illegitimate.

Division involving decimal fractions was generally difficult for students. Weak place value concepts, coupled with a belief that whole numbers and decimal fractions were two separate and unrelated number systems, contributed to difficulty when solving problems. Students had few representations for decimal fractions which compounded their difficulty. The dominance of partition and the tendency to overgeneralize whole number rules appear to be partly responsible for this.

When solving problems students showed little evidence of planning or looking back. Generally they found the numbers in the problem and performed the operation that seemed appropriate. Decisions about operations were often driven by the relative size of the numbers in the problem and by the beliefs mentioned earlier. Because they omitted the looking back phase of problem solving, students rarely accounted for remainders and did not recognize when an answer was unreasonable.

Implications for instruction resulting from this study centre on the assessment of students' understanding of division. This can be accomplished in the regular classroom setting through pencil-and-paper tests, small group work, whole class discussions, and individual interviews. Beliefs which may interfere with learning must be revealed and confronted. Asking students to defend and justify their thinking is part of this process.

Students' reliance on partition and their procedural view of division suggest changes in the way in which division is introduced in the early intermediate years. Delaying the formal introduction of the division algorithm to Grade 5 would allow more students time to develop their conceptual understanding of partition and quotition. Students should focus on estimation and reasonableness of responses. Introduction of division involving decimal fractions, including numbers less than one, could be accomplished by using manipulative materials and calculators. Contexts in which the divisor is greater than the dividend should also be introduced in the early intermediate years. Procedures, when finally introduced, should be linked to the concepts.

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Chapter 1

THE PROBLEM

With the approach of the year 2000, society has begun to question the focus of education and reconsider the characteristics of an educated citizen. In British Columbia, the Sullivan Royal Commission on Education (Province of British Columbia,1988b) made recommendations which called for the revision of curricula for students in the primary, intermediate, and secondary grades. Although these revised programs are in various stages of development, common themes can be found. Emphasis is being placed on the learner as a constructor of knowledge. Problem solving and critical thinking figure prominently. Emphasis on learning "what" is being replaced by learning "how" and "why." Interpreted from the perspective of mathematics education, one might view this philosophy as a welcome reinforcement of existing good practice rather than a change in direction.

As early as 1980 the National Council of Teachers of Mathematics published a report entitled "An Agenda For Action - Recommendations for School Mathematics of the 1980s" which outlined a set of recommendations for mathematics curricula for the coming decades. At the core of the report was the recommendation that problem solving should be the focus of school mathematics in the 1980s. In British Columbia evidence that this call has been heeded exists in the form of a revised mathematics curriculum (Province of British Columbia, 1988a) supported by new authorized textbook series, all of which look upon problem solving as the core of mathematics.

Much of the mathematics education research conducted since the mid-seventies has centered on problem solving (Lester, 1983) which leads one to hope that a trickle-down effect might have occurred, improving instruction and learning. Despite the emphasis this was not the case; problem solving remained relatively low in standings as determined by results of mathematics assessments such as the 1983 National Assessment of Education Progress (Lindquist, Carpenter, Silver, & Matthews, 1983) and the 1985 British Columbia Mathematics

Assessment (Robitaille & O'Shea, 1985). In the results of the fourth NAEP assessment (Kouba, Brown, Carpenter, Lindquist, Silver, & Swafford, 1988) it was reported that seventh-grade students were more successful on one-step problems than on two-step and nonroutine problems. Between 78 and 95% of students responding could correctly solve one-step addition translation problems. One-step division translation problems were more difficult. Between 28 and 74% of students responding experienced success. Sowder (1988) cautions that even students who are correctly solving one-step story problems may be relying on naive strategies such as trying all the operations and choosing the most reasonable answer.

The lack of understanding which underlies students' difficulties in problem solving can be traced to a poor conceptual base and the failure to connect concepts with the many procedures and processes from which we select when solving problems (Hiebert & Wearne, 1986; Carpenter, 1986; Silver, 1986; Riley, Greeno, & Heller, 1983). Teaching practice which focuses only on transmitting a repertoire of such skills, ignoring the development and interconnections of related concepts, will not help students become expert problem solvers. When connections are not made between conceptual and procedural knowledge, inadequate conceptions and ineffective invented algorithms can result (Resnick, 1983b; Silver, 1986; Case, 1978).

These incomplete and alternate conceptions are highly stable in nature, resisting change unless confronted by a conflict-creating situation (Hewson & Hewson, 1984). Studies conducted with pre-service teachers indicate that conceptions such as "the quotient must be less than the dividend in a division problem" and "the divisor must be a whole number" were common (Tirosh & Graeber, 1990). Illustrations such as these lead to strong arguments for the development of instructional strategies that tie procedures to concepts. If this element is missing in the lesson planned by the teacher, the student will make an individual attempt to tie the new knowledge to an existing structure. For this reason it is essential that students are presented with ways to assimilate new knowledge with old so that they do not create misunderstandings which may

endure through adulthood. What is called for, then, is a critical examination of current teaching practices and the adoption of more conceptually-based approaches to instruction.

The Problem Area

Problem solving is generally viewed as being composed of four phases as first described by George Polya in 1945: understanding the problem, forming a plan, carrying out the plan, and checking back (Polya, 1973). Many authors identify the problem representation phase of understanding the problem as the main site of problem-solving difficulty (Greeno, 1980; Mayer, 1982; Riley et al., 1983; Hiebert & Wearne, 1986; Silver, 1986). Results from an NAEP assessment confirm that students virtually ignore all phases of the problem-solving process except carrying out whatever mathematical operation seems appropriate (Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1980).

Problem solving refers to a broad range of problem types and to the strategies and procedures used in the solution of these problems. The type of problem used in this study is the translation problem. In a study of students' ability to choose operations in solving routine story problems, Sowder (1988) refers to Polya who stressed the importance of students being able to "translate a real situation into mathematical terms" (p. 149). It is in this sense that the term "translation problem" is used: students read a story problem, determine the operation or opertions required, carry out the calculations, and interpret the answer in terms of the problem text.

Attempts to explain the lack of instructional focus on understanding in problem solving have been made in the literature. The failure to emphasize understanding can perhaps be attributed to the fact that teaching procedures and algorithms does not require as much time and effort as teaching with meaning (Skemp, 1976; Baroody & Ginsburg, 1986). This promotion of procedural knowledge over conceptual knowledge leads to many of the difficulties experienced by students when attempting to solve problems. When children enter school, conceptual and procedural knowledge are closely related and are often indistinguishable. In 1983, Carpenter

and Moser found that young children could analyze and solve simple word problems before receiving formal instruction by inventing informal modeling and counting strategies. After instruction in addition and subtraction children tended to rely on the algorithm with little attention to the modeling they had displayed previously (Carpenter, 1986). Introducing procedural knowledge without corresponding ties to conceptual knowledge results in ineffective, unstable procedural knowledge which is soon forgotten, does not transfer, and is applied without the ability to check the reasonableness of a response (Hiebert & Lefevre, 1986).

Evidence of student inability to connect meaning with procedure can be found in many sources. Silver (1988) reports that in the 1982 NAEP assessment only 24% of 13-year-olds were able to correctly solve this translation problem: An army bus holds 36 soldiers. If 1,128 soldiers are being bused to their training site, how many buses are needed? Students often chose answers that contained common or decimal fractions, reflecting their reliance on procedure. Bell, Fischbein, and Greer (1984) conducted a study on choice of operation in problem-solving settings with 12- and 13-year-olds. Of the 30 students interviewed, only 14 were able to correctly respond to this problem: A rowing crew covers a 3 km course in 7.2 minutes. How far did they row in 1 minute? Nine of the remaining students reversed the numbers (7.2 ÷ 3), yielding an answer of 2.4 km per minute; upon checking we would see 7 minutes multiplied by at least 2 km per minute would make the race course at least 14 km long. The authors attribute, in part, the difficulty with problems of this nature to beliefs students hold regarding the appropriateness of dividing one number by a larger number.

Based on a series of interviews with students in the middle grades, Sowder (1988) categorized student problem-solving strategies into four groups: coping strategies, computation-driven strategies, slightly less immature strategies, and desired strategies. Sowder presented the following problem to one student: A bag of snack food has 4 vitamins and weighs 228 grams. How many grams of snack food are there in 6 bags? The student suggested she would probably divide to find the solution because there was a "big number and a little number." Later, she changed her mind and chose multiplication because her answer seemed too

small. Her advice for other students was: "If you see a big number and a little number, go for the division. If that doesn't work, then you can try the other ones..." (Sowder, 1988, p. 4). These examples clearly demonstrate how students approach problems procedurally, without regard to meaning.

Most students must be taught how to make the links between prior knowledge and the concepts presented in the current learning situation. If this is not done, students will be unable to represent problems and choose appropriate routes to problem solutions. Failure to make these necessary connections can result in inadequate conceptions and ineffective invented algorithms (Case, 1978; Resnick, 1983b; Silver, 1986).

Despite the number of studies which call for links between conceptual and procedural knowledge, there is little mention of practical suggestions which could be implemented in the classroom to nurture these links. The general purpose of this study was to use and document an instructional approach to division of whole numbers and decimal fractions which attempted to begin from the conceptions held by students and move towards linking their concepts to problem representation. Teaching techniques which helped to uncover student thinking and which assisted students in making connections between concepts and procedures were described.

Division was chosen as the focus for instruction because it is the operation which causes the most difficulty at the Grade 7 level (Grossnickle & Perry, 1985). There are generally considered to be two models of division to which elementary students are exposed: partitive division and quotitive division. Partitive division involves the sharing of an object or collection of objects into a number of equal parts or groups. In this type of division, the dividend is larger than the divisor, the divisor is a whole number, and the quotient is smaller than the dividend (Fischbein, Deri, Nello, & Marino, 1985). Quotitive division is sometimes termed measurement division (Irons, 1981), in which one seeks to determine how many times one quantity is contained in another. When the quotient is a whole number, the model can be viewed as repeated subtraction (Fischbein et al., 1985). Few students have complete conceptions of

both the partitive and quotitive meanings of division and it is rare that a student can model or explain the algorithm in concrete terms.

When a division question involves decimal fractions, student achievement drops dramatically (Carpenter, Matthews, Lindquist, & Silver, 1984). A possible explanation for this drop in achievement centres on students' understanding of division. Because their experience is generally limited to one model, partition, many students have misunderstandings related to division. Division always results in a smaller number, and division must be of a larger number by a smaller number are two examples of such conceptions (Bell et al., 1984). The difficulty introduced by the inclusion of decimal fractions is not limited to problem-solving situations. In the fourth NAEP results (Kouba et al., 1988), seventh grade students were more successful with addition and multiplication of decimal fractions than in subtraction and division. Other factors such as students' understanding of decimal fractions themselves may also affect their performance.

Research Questions

1. What beliefs and understandings do Grade 7 students hold regarding division of whole numbers and of decimal fractions?
2. How can teachers elicit students' understandings in the classroom setting?
3. What kinds of teaching strategies can teachers employ that assist students in making connections between concepts and procedures?
4. What is the impact of this instructional approach that focuses on linking students' conceptions to procedures with respect to conceptual and procedural understanding, and students' ability to represent translation problems involving division?

Rationale for the Study

Techniques for revealing student thinking about division with whole numbers and decimal fractions, and the development of teaching strategies to deal with incomplete conceptions

would be of interest to the educational community. By identifying the entry level concepts students hold concerning division, documenting instructional interaction, and modifying instruction based on this interaction, a number of practical suggestions can be made regarding the linking of conceptual knowledge with problem representation.

The study is, in essence, an example of the practical application of the theories regarding the interrelation of conceptual and procedural knowledge. Of most immediate importance is a reordering of the emphasis on teaching procedures to assessing conceptual understanding which can then be used as an instructional base. This attention to the entry level concepts of students and its impact on the instructional program in the classroom aligns the study with current theories of constructivism. As students construct knowledge, so do teachers (Magoon, 1977) and the dynamic interplay of these two elements is at the core of the study.

Limitations of the Study

The classroom case study is a technique of investigation which provides the researcher with a great deal of information about the students in a given class. The students' and researcher's personalities, the classroom dynamics, and the instructional focus all contribute to student learning. Unlike other methods of investigation, the classroom case study allows for these many facets of learning because it is built on interaction within and observation of an established class of students. In this particular classroom case study the researcher was also the regular classroom teacher. This arrangement resulted in several benefits. The researcher's familiarity with the students and their educational profiles led to informed interpretation of students' responses, and the level of trust already established with the class permitted open and frank discussions from the beginning of the study.

The study took place within the confines of one specific class. Traditional generalizations to all Grade 7 students, even within the school district, cannot be claimed. Researchers and

classroom teachers may, however, see patterns of behaviour which are applicable to other groups of students. Replications of the study with other Grade 7 students would be of interest.

The study centered on division using whole numbers and decimal fractions. The results, therefore, may not apply to conceptual understanding of division with common fractions or negative integers. The study would have to be replicated to provide information about these areas.

The students involved in the study were the members of a class of French immersion Grade 7 students which had essentially been intact since Kindergarten; some of the students, however, were in split classes for a period of two years. The group of students was largely homogeneous with respect to their educational background and experience.

Chapter 2

REVIEW OF THE RELATED LITERATURE

This chapter contains a review of the literature related to the teaching experiment as a vehicle of investigation, as well as the literature related to students' understanding of mathematical problem solving involving division of whole numbers and decimal fractions. The chapter is divided into five major sections:

1. Literature which discusses the teaching experiment in both the Soviet Union and in North America.
2. Literature which discusses conceptual and procedural knowledge and the effects of these types of knowledge on problem solving.
3. Literature which discusses research on children's understanding of division.
4. Literature which discusses research on children's understanding of decimal fractions.
5. Literature which discusses research on problem solving requiring division.

Literature of Relevance to the Teaching Experiment

Much of the research on the teaching and learning of mathematics that has taken place in the past two decades has had little impact on changing classroom practices (Romberg & Carpenter, 1986). Possible reasons for this vary from a lack of unified theories of teaching and learning (Romberg & Carpenter, 1986; Resnick, 1983a; Skemp, 1976) to the failure to involve teachers in participatory roles in research (Kilpatrick, 1981). The focus of mathematics education research has changed and is continuing to change.

Prior to 1970, most North American research in mathematics education was based on quantitative investigations carried out in controlled, experimental settings (Carpenter, 1980). This was not necessarily true on the international scene, however. In the Soviet Union, for example, educational psychologists used the teaching experiment to investigate such topics as

the development of mental operations, the formation of mathematical concepts, and the nature and development of thought. Unlike North American research, the Soviet findings had a direct impact both on the curriculum and on the methods of teaching mathematics (Kilpatrick & Wirszup, 1969).

Soviet Teaching Experiments

The Soviet research in question was carried out in the classrooms of state schools from 1946 to the early 1960s. Often referred to as a "teaching experiment," the Soviet model involved studying development in the changing conditions of instruction, varying the conditions, and demonstrating how the nature of the child's development changed in the process (Kilpatrick & Wirszup, 1969). Groups of students at varying levels were often asked to reason aloud as they worked through problems and further information was gathered through interviews and introspection upon problem completion (Menchinskaya, 1969; Yaroshchuk, 1969; Doblaev, 1969).

The theory base for the research relied upon a belief that scientific and everyday concepts are formed under the influence of adult teaching, a view contrary to the Piagetian approach. Also of note is the identification of a child's experience as an important factor in learning. Menchinskaya (1969) commented as early as 1950 that personal experience can sometimes distort rather than clarify concepts, and the resulting naive conceptions are resistant to change even after instruction. Another concern centered on presenting instruction in varying ways so that concepts were not solely associated with one presentation and could therefore be used in numerous applications (Menchinskaya, 1969; Zykova, 1969). The issue of concept acquisition was between spontaneous concepts, child-generated on the basis of concrete experience and the child's own mental effort; and scientific concepts, the product of direct instruction or interaction with adults. According to Vygotskii, the interplay between these two types of concepts led to development because the formal structure of the scientific concepts helped organize the child's own spontaneous concepts (Menchinskaya, 1969).

Another influential Soviet researcher, Krutetskii, investigated individual mathematical problem-solving abilities. In studies conducted over many years Krutetskii grouped students who had some incapacity in mathematics into six categories consisting of varying degrees of combinations of verbal and visual ability. Krutetskii followed this study with a twelve-year investigation in an attempt to categorize all students. As a result, he concluded that there appear to be three types of students: analytic, geometric, and harmonic which is a combination of the other two. Perhaps the most important feature of Krutetskii's long-term study is the relationship he uncovered between problem-solving ability and perceptions of problem structure. Using clinical interviews and nonexperimental procedures, Krutetskii found good and poor problem solvers differed as to what they perceived to be the important elements of a problem. Good problem solvers had the ability to distiguish relevant from irrelevant information, to see quickly and accurately the mathematical structure of a problem, to generalize across a wide range of similar problems, and to remember a problem's structure for a long time (Lester, 1980).

While the Soviet literature was not solely responsible for the change in focus of North American research, it had a considerable influence on the call for integration of cognitive psychology and educational research (Lester, 1980; Carpenter, 1980; Confrey, 1981; Kantowski, 1977; Silver, 1981; Sowder, 1980).

North American Research

Perhaps one of the earliest North American studies to focus on cognitive processes in problem solving was the work of Kilpatrick. In his study, Kilpatrick attempted to develop a technique for analyzing behaviour during the solution of a mathematical problem, and to demonstrate how the technique could be used to delineate an individual's approach to a broad class of problems using a clinical method (Kilpatrick, 1967). In brief, Kilpatrick found his coding system revealed students belonged to one of four groups, one using an algebraic approach to problems and three using varying degrees of trial and error. Kilpatrick's recommendations

included encouraging teachers to study the problem-solving behaviour of their students and classify the processes students used in order to improve teaching, calling for further research into how students use trial and error in solving problems, and encouraging behavioural analyses of mathematical problem solving to balance research on oddity problems and anagrams.

Kantowski (1977), influenced by the Soviet work and by Kilpatrick's coding scheme, undertook a study on the processes involved in solving complex, nonroutine problems. Kantowski used a teaching experiment model involving a group of high-ability Grade 9 students and found that the use of heuristics resulted in an improvement in the number of correct solutions. Of greater significance were her recommendations for further research. Among these recommendations were the following: repeat the investigation with students of lower ability in line with Krutetskii's belief that students of differing ability solve problems with different processes, and at different levels with varied content; investigate if and how processes are affected by instruction in heuristics; determine if there is an optimal amount of content needed before working on a process; investigate if there is a relationship between looking back and problem-solving success; and study the effect of making students aware of their problem-solving processes. Much of the research that was to follow in the 1980s centered on topics of this nature.

Further research on the characteristics of good and poor problem solvers was conducted by Silver in 1981 when he replicated Krutetskii's study on problem structure recall. Silver's findings supported Krutetskii's main hypothesis that high ability students recall structure and low ability recall detail, but he also found that good problem solvers recalled details and poor problem solvers recalled the question and the content. Neither group said they made use of the information from the target or model problems to solve the related research problems. The inability of poor problem solvers to recall structure was not linked to faulty memory as detail recall was unaffected. Silver suggested further research was in order to determine whether difficulties were related to mathematical or information-processing deficits.

The late 1970s and early 1980s brought a call from educational psychologists and mathematics researchers alike to form a more concerted, theory-bound research base that would tie together the two fields (Skemp, 1976; Larkin, 1980; Lester, 1980; Resnick, 1983a; Confrey, 1981; Mayer, 1982). The quantitative, product-oriented studies of the past were seen as insufficient; they could not help researchers meet their goals of improving teaching and learning. There were perhaps two central reasons for this failure: a lack of theory behind the research and an over-emphasis on procedures and algorithms in instruction. A lack of theory was noted by Skemp (1976) when he defined two types of understanding, relational and instrumental, and explained that much of our school time is devoted to instrumental understanding in mathematics. This focus on instrumental understanding, or understanding of procedures, was due primarily to a lack of a theory which unified thoughts and goals.

Deficiencies in understanding were signalled by studies which revealed a lack of conceptual understanding in spite of an apparent mastery of procedures. An example of such a study would be Erlwanger's investigation into children's conceptions of mathematics (1973) which revealed that a child named Benny, who had seemingly mastered concepts in a programmed learning unit, had in fact developed a number of buggy algorithms to compensate for his inadequate understanding.

Resnick (1983b), in summarizing findings from cognitive research in mathematics and science, stated that students who did well on textbook problems often could not apply the laws and formulas to practical problems. As part of her recommendations, Resnick urged researchers and teachers to focus on qualitative analysis of learning and to help children make sense of procedures and formulas. Mayer (1982) agreed. He suggested that the poor performance of students on story problems pointed to the need for a better understanding of how to provide mathematical instruction.

The Teaching Experiment. In an article which examined the impact of research in mathematics education, Kilpatrick (1981) suggested a number of possible explanations for the apparent ineffectiveness of studies conducted in the recent past. Kilpatrick acknowledged many

factors ranging from insufficient funds to insufficient knowledge, but underlined the lack of teachers in a participatory research role as a partial explanation for the failure of research to influence practice. He suggested that as researchers move out into the classroom and work in close contact with teachers, teaching, learning, and research will all benefit.

Some of the most exciting studies in recent years are those in which the researcher has attempted to understand children's mathematical thinking. Cobb and Steffe (1983) believe there is no substitute for experiencing the intimate interaction involved in teaching children when exploring their construction of mathematical knowledge. As advocates of the teaching experiment methodology, they outlined three main reasons for researchers acting as teachers. First, researchers cannot rely solely on theoretical analysis to understand children's mathematical realities. Second, experiences children gain through their interactions with adults in a teaching situation greatly influence their construction of mathematical knowledge, whereas in a clinical interview situation, they do not. What a child does and says in a clincial interview may not mirror his or her actions in the classroom setting. Third, the context within which the child constructs the knowledge is crucial to understanding this knowledge (Cobb & Steffe, 1983).

The teaching experiment consists of a number of aspects, the first of which is modeling. Models are the explanations formulated in research. They are based on the observation of children's behaviour, and on the researcher's interpretation of the meaning that children attribute to their own behaviour. Teaching episodes form the second aspect of this method of investigation. The teacher tests the limits of a model he has of the child's knowledge, investigates how the model may change, and interprets the child's behaviour. The course of the teaching episode is determined by the child and the teacher, and spontaneously designed tasks may be used. Teaching episodes are often audio-video taped and the record may be used to plan subsequent instruction. Individual interviews form the third aspect of the teaching experiment. These interviews focus on the child rather than the task and assist the researcher in formulating models of what may go on in the head of the child. A series of teaching episodes and

interviews take place over an extended period of time -- anywhere from six weeks to two years. They are often carried out with small numbers of children in laboratory-type conditions (Steffe, 1983).

When we begin delving into students' mathematical thinking, many interesting and surprising outcomes result. For example, in an investigation into problem-solving strategies used by elementary and junior high students, Sowder (1989) found six immature strategies that were commonly used. He was particularly surprised when he interviewed a seventh-grader in a school's gifted program. When presented with a story problem, this student used the immature, computation-driven strategy of trying all operations on the numbers, then choosing the most reasonable answer. Even for students who have been identified as having superior mathematical ability, this reliance on computation fails them when presented with multi-step and nonroutine problems.

Burns (1986) also found that students who are successful at computation frequently are not able to reason through their own responses. When asked why there was a zero in the second row of a two-digit by two-digit multiplication question, children responded: "Nobody in our group can remember why we put the zero in the second line. We told you before that it is a rule to put the zero in the second line" (Burns, 1986, p. 35). These students could not explain their work and did not see the relationship between the computation and their prior place value knowledge.

Weiland (1985) suggested that teachers begin with the child's own thinking regarding a particular mathematical concept when planning instruction of standard algorithms. Through the use of interviews, Weiland believes teachers can find out how children describe the actions they used to solve word problems involving operations such as division. Teachers can then match their instruction to children's mathematical realities when introducing standard algorithms.

Examples such as these highlight the value of the teaching experiment. Involving teachers, researchers, and students together in the learning situation enables a closer, more meaningful investigation of the teaching and learning processes. Children do not learn by being

"told" about mathematics; they are active participants in the construction of their own knowledge. Similarly, we as teachers cannot understand solely by being told about research; we must construct personal knowledge about children's understandings through our own inquiries. As researchers in our own classrooms, we are in excellent positions to determine children's understandings of mathematical concepts.

A Constructivist Perspective of Learning and Teaching

In recent years one branch of educational research has focused on learners as constructors of knowledge rather than recipients of knowledge. From this viewpoint, Resnick (1983) believes learners look for meaning and try to find regularity and order in the events of the world. This learning depends on the prior knowledge of the individual because it is the framework through which the learner interprets new material. Osborne and Wittrock (1983) agree. In order to learn with understanding, the learner must actively construct meaning. Each of us must invent a model or explanation for new material that organizes the information in a way that makes sense to us, and fits our logic, our real world experiences, or both. This view of learning and the learner has an impact on classroom instruction.

One implication for teaching from a constructivist perspective is that the learner should be involved in a number of ways. The importance of considering the conceptions of the learners must be acknowledged. Conceptions students bring with them to the learning environment must be elicited, and the learning experiences planned by the teacher should be adapted to the needs and perspectives of the participants. In some cases the learners' conceptions will have to be exposed to conflict situations to enable the students to reject inadequate conceptions (Driver, 1985). Because these conceptions are resistant to change (Hewson & Hewson, 1984), eliciting and assessing students' conceptions must be an ongoing process, and requires a reflexive curriculum that adapts to the needs of the students (Driver, 1985).

It is important for teachers to be aware of the nature of the conceptions their students may hold prior to instruction, and the need to plan learning activities that assist students in the

development of a complete and stable conceptual base. In a recent article, Nesher (1987) discussed the origin of students' alternate conceptions. She stated that errors do not occur randomly, but often originate in a conceptual framework based on earlier acquired knowledge derived from previous instruction. Rather than errors, she advocates the use of the term "misconceptions" since they are outgrowths of an already acquired system of concepts and beliefs wrongly applied to an extended domain. These conceptions, such as "multiplication makes bigger" and "division makes smaller," are the result of a fragmented or incomplete conceptual base.

The term "misconception" is somewhat negative, and implies the student has little understanding of the concept in question. Preferred terms would be "incomplete" or "alternate" conceptions. These terms indicate the student has some understanding upon which the teacher and student may build.

Literature of Relevance to Conceptual and Procedural Understanding

Conceptual and procedural knowledge are two terms used to refer to types of knowledge frequently discussed in literature on learning. In the next section, these types of knowledge will be defined, and the roles they play in mathematical problem solving will be discussed.

Conceptual and Procedural Knowledge

The outcome of the call for the establishment of a theory base for mathematics education research has been an examination by cognitive psychologists and mathematics researchers of the types of knowledge involved in mathematics, and inquiries into how these types of knowledge are acquired. Over the years, it has generally come to be accepted that there are two main types of knowledge, conceptual and procedural. From Scheffler's 1965 labelling of knowledge types as "knowing that" and "knowing how to" (Hiebert & Lefevre, 1986), to Carpenter's "organismic" and "mechanistic" knowledge (1980), the models differ to varying degrees in terminology if not in definition.

One of the most concise reviews of the conceptual and procedural knowledge issue was completed by Hiebert and Lefevre in 1986. Summarizing the debate which has taken place over the past century, the authors noted the central issue has been the distinction between skill and understanding, and which should receive greater emphasis in instruction. The debate has moved from Thorndike's argument for skill learning in the 1920s and Brownell's emphasis on understanding in the 1930s, through to Bruner's case for understanding in the 1960s and Gagne's return to skill learning in the 1970s. A sample of current investigations into this area reveals a variety of terms which differ from those of the past.

One group of researchers, typified by Resnick (1982), refers to the two knowledge types as semantics and syntax. Semantics, as in language, involves the meanings associated with mathematical symbols. Syntax is associated with the rules governing the movement and association of the symbols. Riley et al. (1983) proposed three schemata which correspond to conceptual and procedural knowledge. Problem schemata involve knowledge for understanding semantic relationships (conceptual knowledge) and action schemata involve knowledge about the actions involved in problems (procedural knowledge). The third schemata, strategic knowledge, are used for planning solutions to problems and can be viewed as an overlap of the two knowledge types. Gelman and Gallistel referred to the two knowledge types as principles and skills, VanLehn used the terms schematic and teleologic, Tulving called the two semantic and episodic memory, and Anderson used the terms declarative and procedural knowledge (Hiebert & Lefevre, 1986). In 1976, Skemp wrote about relational and instrumental understanding in mathematics. By relational understanding, Skemp means knowing both what to do and why. Instrumental understanding refers to rules without reasons or the learning of fixed plans. Baroody and Ginsburg (1986) prefer the terms meaningful and mechanical knowledge, but their groupings are essentially the same. Meaningful knowledge refers to semantics or the implicit and explicit knowledge of concepts and principles, and mechanical knowledge refers to the knowledge of facts and procedures.

Carpenter (1980), in an overview of research in cognitive development, outlined the distinctions between organismic and mechanistic understanding as proposed by Reese and Overton in 1970. The organismic model is concerned with the development of complex cognitive systems and how the child processes or operates on information. This model is associated with the work of Piaget and his followers and attempts to obtain a measure of the underlying structures that children apply to a variety of problem situations. The mechanistic model, based on the behavioristic theories of Gagne, is concerned with the development of discrete, chainlike associations. From an organismic point of view the learner is an active participant in the acquisition of knowledge, whereas the mechanistic model views the learner as reactive (Carpenter, 1980).

In 1984, James Hiebert wrote at length about form and understanding. Form included knowledge about symbols such as numerals, operations and relations, and phrases but also included the rules, procedures, and algorithms for manipulating the symbols. Understanding centered on ideas about how mathematics works, both in personal experience and school instruction. By 1986, Hiebert, in collaboration with Lefevre, had refined their definitions and had adopted the names conceptual and procedural knowledge for the two categories.

Conceptual Knowledge. The emphasis on relationships and interconnectedness of information stored in memory is what separates conceptual from procedural knowledge. The linking of information can take place in two ways and on two levels.

> Conceptual knowledge is characterized most clearly as knowledge that is rich in relationships. It can be thought of as a connected web of knowledge, a network in which the linking of relationships are as prominent as the discrete pieces of information. Relationships pervade the individual facts and propositions so that all pieces of information are linked to some network (Hiebert & Lefevre, 1986, pp. 3-4).

Two pieces of information previously stored in memory can be linked at a later date, and an existing piece of information in memory can be linked to something newly learned. On a primary level the relationship is no more abstract than the information being presented. On a reflective level relationships become free of the context in which the information was presented and can be more abstract (Hiebert & Lefevre, 1986).

Procedural Knowledge. Procedural knowledge can be separated into two parts: the formal language, or symbol representation system, and the algorithms or rules for completing mathematical tasks (Hiebert & Lefevre, 1986). The formal language system involves a familiarity with the symbols and rules used to write in standard form. The second part, rules or algorithms, consists of the sequential procedures used to complete tasks. Within this second part are yet again two divisions or kinds of procedures. The first involves the procedures that operate on standard symbols (e.g., 5, =, +) and the second is the problem-solving strategies or actions that operate on nonstandard mathematical symbols such as concrete objects or visual diagrams.

A distinction which Hiebert and Lefevre draw between conceptual and procedural knowledge is that conceptual knowledge must be learned meaningfully whereas procedural knowledge may or may not be. However, the authors advocate the meaningful learning of procedures because this links them to conceptual knowledge. This brings into focus the next part of the debate: the relationship between conceptual and procedural knowledge and which, if either, of the two should come first in instruction.

Hiebert and Lefevre (1986), after defining the two knowledge types, note that not all learning fits into one category or the other; some knowledge seems to overlap both and some knowledge seems to be neither conceptual nor procedural. The critical element is not the identification of types of knowledge as either conceptual or procedural, but the creation of links between the two. Only when a student is competent in conceptual and procedural knowledge and connects the two, can he or she solve problems with understanding.

Problem Solving and Conceptual and Procedural Knowledge

Mathematical problem solving has generally been viewed as consisting of four stages: understanding, planning, carrying out the plan, and looking back (Polya, 1973). For some researchers the four steps can be condensed to three by grouping understanding with planning (Hiebert & Wearne, 1986), but the framework remains the same. Within this basic

framework lie the elements of the problem-solving process. Expectedly, how researchers view the roles of conceptual and procedural knowledge in the different stages of the process varies to some extent.

Hiebert and Wearne (1986) believe links between concepts and procedures are necessary for students to fully develop their problem-solving ability. They identified three sites in the process where links can be made. Site 1 is the initial point in the problem-solving process and is where symbols are given meaning and a representation of the problem is formed. Site 2 is where the execution of procedures takes place and Site 3 corresponds to the looking back stage where the reasonableness of answers is checked.

At Site 1, Hiebert and Wearne found that some students overgeneralize syntactic conventions and apply rules without full understanding. Ensuring that connections between procedural and conceptual knowledge are made would alleviate errors that may be due to an incomplete or a flawed concept base. At Site 2, procedural execution, students tend to focus on the syntactic features of a problem when selecting procedures, ignoring the meaning behind the manipulation of symbols. While the connection of conceptual knowledge with procedural is not critical at this site if the students are accurate in their recall of rules, linking the concrete to the symbolic would contribute to accuracy. The evaluation of solutions, Site 3, requires a linking of conceptual with procedural knowledge that is not presently evident in most of our students. Research shows students ignore the looking back stage of problem solving (Hiebert & Wearne, 1986; Carpenter et al., 1980; Kantowski, 1977); Hiebert and Wearne (1986) suggest this is due to the absence of an underlying conceptual base in procedural knowledge. Whereas in Site 2 a lack of conceptual knowledge may not preclude procedural accuracy, at Site 3 such a lack prevents evaluation of the reasonableness of an answer.

In Skemp's model of understanding (1976), relational and instrumental mathematics correspond roughly to conceptual and procedural knowledge. He says relational mathematics consists of the construction of a conceptual structure, or schema, which students theoretically can use to produce unlimited plans for problem solving. In contrast, instrumental mathematics

is the learning of context-bound, fixed plans which enable students to begin at one starting point and progress to one end point with no awareness of the relationship between successive stages and the final goal.

In 1982, Skemp introduced the notion of symbolic understanding which he defined as "...mutual assimilation between a symbol system and a conceptual structure, dominated by the conceptual structure" (p. 61). This appears to be the linking of conceptual with procedural knowledge, or in Skemp's words, the combining of two schemata: the symbol system and the structure of mathematical concepts. Skemp accounts for problem-solving difficulty by explaining these two schemata compete for input, and that the symbol system often dominates. While Skemp says this is natural because all communication is in symbolic form, to be understood relationally it must be attracted to a conceptual structure and there must be connectors from one system to the other. Skemp does not detail the specific locations in the problem-solving process this should take place, but it is clear that without a clear conceptual structure and interconnections, most input will be assimilated into the symbol system.

Riley, Greeno, and Heller (1983) examined the relationship between conceptual and procedural knowledge in the performance and development of problem-solving ability. These authors believe improvement in performance results mainly from improved understanding of certain conceptual relationships. While formal arithmetic may play a role in the acquisition of conceptual structures, children are unable to conceptualize formal arithmetic in a way that makes it useful for problem solving. The authors distinguish three kinds of knowledge involved in problem solving: problem schemata for understanding semantic relationships, action schemata for representing knowledge about actions used in problem solving, and strategic knowledge which is used for planning solutions to problems. The conceptual base for problem schemata is clear, as is the procedural base for action schemata. Strategic knowledge can be seen as a overlapping of the two areas, integrating understanding with action.

Riley et al. (1983) identify three main ways conceptual and procedural knowledge interact during problem solving. First is the role of schemata in the selection of actions, second

involves the use of schemata to monitor the effects of selected actions on a problem, and third, conceptual knowledge can influence which actions are selected. The authors believe the acquisition of schemata for understanding the problem in a way that relates it to available action schemata is the main locus for improvement of problem-solving skill. Children who are more skilled problem solvers have acquired schemata that act as principles for organizing the information in a problem.

Mayer (1982) described four types of knowledge that relate to problem solving. These are linguistic or factual knowledge, schema knowledge which concerns relations among problem types, algorithmic knowledge, and strategic knowledge. Mayer groups linguistic or factual knowledge and schema knowledge together under the umbrella of representation of the problem, and algorithmic and strategic knowledge as solution of the problem. Mayer noted difficulties in problem solving frequently result from inadequate representation of the problem, and rarely from faulty solution procedures. He advocates a focus on process, ensuring procedures are related to conceptual knowledge. This type of learning leads to superior transfer and long-term retention (Mayer, 1982).

It is the relationships among, and not the distinctions between, the elements of conceptual and procedural knowledge that interest Silver (1986). While some authors argue in favour of a concept base upon which procedural knowledge can be built and others argue the reverse, Silver suggests it may be more productive to examine the links between the two. Silver identified four factors central to successful problem solving: the problem task environment, long-term memory, working memory, and mental representation (Silver, 1987). The facts, concepts, and the interrelations among them form the task environment. The function of this environment is two-fold: to allow access to external information, and to provide external memory for information generated by the problem solver. Long-term memory is the store of basic facts, processes, problem types, heuristics and algorithms that goes into problem solving. It also involves beliefs about mathematics, knowledge about the real world, and metacognitive knowledge. Working memory is where information from the problem task

environment and knowledge from long-term memory function together to form a mental representation of a problem. Silver (1987) believes it is the quality of the problem representation that is central to the problem-solving process because inaccurate or incomplete representations may make a problem difficult or impossible.

Perhaps one of the most common and unfortunate reasons for the lack of a focus on understanding in problem solving is that procedures and algorithms are much easier to teach and to assess. Teaching with meaning requires a great deal of time and effort whereas rote learning and practice do not (Skemp, 1976; Baroody & Ginsburg, 1986).

The promotion of procedural knowledge over conceptual knowledge, while understandable in practical terms, leads to many of the difficulties experienced by students when attempting to solve problems. By far the most frequently cited reason for the lack of problem-solving success is the failure to link conceptual and procedural knowledge (Carpenter, 1986, Hiebert & Lefevre, 1986; Silver, 1986; Hiebert & Wearne, 1986; Greeno, 1980; Gelman & Meck, 1986; Riley et al., 1983).

Riley et al. (1983) identify the acquisition of schemata for understanding the problem in a way that relates it to available action schemata as the main locus for improvement. The authors give examples of addition and subtraction problems involving change, combine, and compare procedures and found that difficulty often related to an inability to form a correct initial representation of the problem. This was particularly evident when students attempted to solve change or compare problems. The researchers discovered students had weak part-whole understandings which prevented them from forming a meaningful representation of the problem. When this understanding was missing, students often resorted to the application of incorrect, alternate procedures.

Hiebert and Lefevre (1986) found that conceptual and procedural knowledge were closely related and often indistinguishable in preschool children. As children progress through school, the relatedness of concepts and procedures diminishes as the emphasis on procedural skill increases. Conventional instructional methods fail to draw connection between the two.

This failure results in an over-reliance on symbol manipulation rules which often exceed students' conceptual knowledge. Hiebert and Lefevre say that without ties to conceptual knowledge, procedural knowledge lacks effectiveness and stability, is soon forgotten, does not transfer, and is applied inappropriately without the ability to check the reasonableness of a response (Hiebert & Lefevre, 1986).

In an example from their work on decimal and common fractions, Hiebert and Wearne (1986) found students had trouble selecting the features of whole numbers that could be generalized to decimals. Students either under- or overgeneralized understandings relating to the magnitude of a number or the role of zero, resulting in errors. Hiebert and Wearne believe students must link written symbols with their conceptual understandings if they are to establish appropriate meanings for notation.

Semantics in problem solving refers to the underlying conceptual meaning behind mathematical symbols. Because this term is frequently equated with conceptual knowledge it is not surprising some researchers identify weak semantics as an area in need of improvement in problem solving. Carpenter has found young children are relatively successful at analyzing and solving simple word problems before receiving instruction (Carpenter, 1986). The analyses they apply are based on accurate representations of the problem and are formed from the semantics of the problem. Later, regardless of the approach used, children justified their solutions by referring to these same semantic features. As students progress through school, the increasing focus on procedures, or syntax, results in a loss of semantic understanding. Often the syntactic features of the problem, rather than the semantic features, provide the basis for selection of procedures and for checking the appropriateness of responses (Hiebert & Wearne, 1986).

Related to connections between conceptual and procedural knowledge is the notion of transfer of knowledge from one set of experiences to another. In his replication of Krutetskii's study on recall of mathematical problem information when solving related problems, Silver (1981) found that students frequently recalled problem details, but did not make use of

structural similarities. Lester (1982) also reported that the mere fact that problems are similar structurally does not imply they are of equal difficulty or that transfer will occur. Just as students do not always see similarities in problems, neither do they see connections between mathematical knowledge and skills they possess and problems they are asked to solve (Carpenter et al., 1980). Transfer from one type of knowledge or problem schema to another is not automatic; connections must be taught and experienced.

When connections are not made between conceptual and procedural knowledge, incomplete or alternate conceptions and ineffective invented algorithms can result (Resnick, 1983b; Silver, 1986; Case, 1978). These conceptions are highly stable in nature, resisting change unless confronted by a conflict-creating situation (Hewson & Hewson, 1984). This point in itself is one of the strongest arguments for the development of instructional strategies that tie procedures to concepts. If this element is missing in the lesson planned by the teacher, the student will make an individual attempt to tie the new knowledge to an existing structure; this is the essense of the constructivist view of learning which dominates the current research in mathematics and science education. For this reason it is essential that students be presented with ways to assimilate new knowledge with old. The fact that current instructional practices are not based on informal student strategies and do not take account of individual entry points aids in explaining some of the difficulty students have in becoming expert problem solvers.

An example of the types of strategies students invent can be found in a 1988 study by Sowder. He reported only rare students were able to solve problems by choosing strategies which reflected understanding. He categorized the majority of strategies students used into several groupings of "naive" strategies such as finding all the numbers in a problem and performing an operation upon them, looking at the numbers and deciding which operation is indicated by the relative size of the numbers, and looking for key words or phrases such as "all together" and assigning an operation based on a fixed meaning of the phrase.

A final area which may be responsible for some of the difficulty students experience when solving problems is the beliefs students hold about mathematics and the types of problems

they are frequently asked to solve. Students tend to work toward precise answers for specific problems (Lesh, Landau, & Hamilton, 1983) and spend much of their time on routine knowledge rather than on the application of mathematics to problematic situations (Lindquist et al., 1983). The NAEP results from 1985 found high school students believe mathematics is mostly memorization, there is only one way to solve a problem, and problems should be solved in a few minutes or less (Silver, 1987). Students cannot be blamed for their beliefs as these beliefs are based on their experiences. If mathematics instruction simply teaches routines and algorithmic steps for solving narrow problems (Schoenfeld, 1982), then students' conceptions of what mathematics entails will not change.

Improving Connections Between Conceptual and Procedural Knowledge

The location of areas for improving connections between conceptual and procedural knowledge in the problem-solving process have been identified and specific weaknesses have been outlined. How can these changes be realized? Suggestions for improvement fall into two general categories: those which are global or theoretical in nature, and those which are of a more specific, practical nature.

The global recommendations centre primarily around theories underlying research and instruction. At the core of much of the research outlined in this paper has been the constructivist nature of learning. In this vein, Jere Confrey (1981) devised a seven part plan for researchers and instructors to follow:

1. Identify relevant concepts to be taught;
2. Determine the students' alternative, private conceptions of the concepts, perhaps through their responses to a problem;
3. Identify terminology and symbols attached to those public concepts, and to those private conceptions;
4. Propose possible routes from private to public concepts through a series of developmental stages, both conceptual and linguistic;
5. Apply a theory of conceptual change (conflict);
6. Devote attention to processes which help form concepts such as generalization, prediction, abstraction, curtailment, etc.;

7. Assess students on problems that involve flexible and original instances of the concept, and that require problem solving strategies as well as recall of previous instances (Confrey, 1981, p. 12).

By following a plan such as this when designing research or instruction, Confrey believes we will bring concepts and procedures closer together. Perhaps the greatest strength of her plan is the fact that it is based on moving from conceptions held by students entering instruction, and not on the concept to be attained.

Silver (1987) and Case (1982) echo the need to base instruction on a constructivist model, with Silver adding that this can be done in part by selecting a controlled number of excellent examples and counterexamples of the concept. Case encourages us to determine the stage of development the students are at and then design instruction to bring them as efficiently as possible from their current level to the desired level. A related issue is the need for the development of reliable concept tests to assess the status of student understanding (Resnick, 1983b; Lester, 1983; Sowder, Threadgill-Sowder, Moyer, & Moyer, 1986). Such tests will assist educators in planning effective instruction that takes into account the various entry points of students.

Another area for instructional focus is metacognition. Metacognition is what one knows about one's own cognitive performance and how one regulates cognitive actions during performance (Garofalo, 1987). Schoenfeld found that expert problem solvers use metacognition to monitor and assess the strategies they use and that they possess intuitions against which the progress of a solution is gauged. Metacognition can also include the conscious and unconscious belief systems that may determine the approaches people take to certain problems (Schoenfeld, 1982). Silver believes a focus should be placed on metalevel processes such as planning, monitoring, and evaluation. Garofalo (1987) advocates assignments which require students to carefully reflect on their mathematics knowledge and behaviour. Teachers should ensure that students are aware that some problems have more than one answer, take differing amounts of time to solve, and that not all information in a problem is necessary for its solution.

The next set of recommendations focuses on improving problem solving in a variety of ways. Because many researchers identified problem representation as the most critical part of the problem-solving process, the first set of recommendations deals with improvement in that area. The remaining recommendations are general in nature, and are aimed at all three sites in the problem-solving process.

Hiebert and Wearne (1986) believe problem representation at Site 1 can be improved by helping students create meaning for symbols by establishing a rich store of conceptual knowledge. Additional instructional time is required for this and could be taken from Site 2, execution of procedures. To check for understanding of operation symbols at Site 1, Hiebert and Wearne propose the use of estimation, or asking students to relate a computation expression to a story problem. In addition to enriching the concept base, Hiebert and Lefevre (1986) note that instruction can be limited to procedures that can be related to the existing conceptual knowledge of students. Lester, in reviewing how contemporary cognitive psychologists view the role of problem respresentation, summarized by stating that understanding processes are very complex and influence all other aspects of problem-solving behaviour. He recommends stressing the understanding phase of problem solving by directing attention at training students how to gain understanding, by representing problems in different ways, and by recognizing that the representation chosen will affect problem difficulty (Lester, 1982). Other researchers such as Riley et al. (1983) state that representation of the problem is critical, and can identify qualities that good problem solvers have, but do not detail specific ways in which improvement can be made. They state that children who are more skilled have acquired schemata that act as principles for organizing the information in a problem and then call for further research into the analysis of the process of understanding. This is not unusual, however, as almost all researchers call for further work in the field.

The remainder of the recommendations from the literature consists of a number of specific suggestions for focusing attention in problem solving. A central theme is the creation of links between conceptual and procedural knowledge and the problem-solving process. These

suggestions are intended for use in one or more of the three sites in the problem-solving process.

1. The NAEP recommendations of 1983 were as follows:
   a) higher-level cognitive activity must be addressed instructionally;
   b) students must be given opportunity to engage in real problem solving, not memorization and repetition;
   c) supplemental material must be used to extend the text if the text focuses on lower-level objectives;
   d) students must be asked to defend their reasoning, justify answers, and explain why results are reasonable (Lindquist et al., 1983).
2. The issue of teacher modeling is raised by Alan Schoenfeld (1983). He suggests teachers follow this advice:
   a) teachers act as role models by going through the process even when the answer is known, by solving problems with students and using their ideas, and by solving new problems presented by students;
   b) teachers can act as a coach;
   c) several solutions to problems can be investigated;
   d) teachers should be realistic about the amount of material they hope to cover;
   e) emphasis should be placed on what students are to get out of a lesson, for example, looking back will not happen just because it is done in the example.
3. Sowder et al. (1986) make a number of specific recommendations in the close of their article on assessing conceptual understanding. These are:
   a) check pupil understanding of operations before working on problems;
   b) provide more opportunity to work on concepts by reviewing with the class the types of situations the different operations fit;
   c) emphasize the language used in mathematical problem solving;
   d) use concrete materials in a variety of ways;

e) have students make up questions for written or pictured data;

f) give students beginning story sentences and have them produce the question;

g) present problems which involve all operations;

h) have students make up problems for a given operation.

4. A number of authors encourage teachers to familiarize students with two types of strategies or heuristics: global and specific. Global strategies are needed for obtaining an overall view of a problem, and specific strategies are needed for application within the global framework (Suydam,1980; Larkin,1980).

5. Students should be presented with realistic word problems taken from real-life settings according to Lesh et al. (1983). This will contribute to establishing an accurate representation of the problem and will ultimately have more meaning for the student.

6. Skemp (1982) encourages us to emphasize the language behind mathematics, as a failure to establish concept structures will result in a domination by the symbol system. To accomplish this, we should:

   a) sequence material so new material is presented in a way that enables its assimilation into a conceptual structure;

   b) begin instruction with physical embodiments of mathematical concepts so that sensory input goes first to the conceptual structure and is then connected with the symbolic representation;

   c) stay with language much longer, especially in the early years;

   d) use notations as the need for them arises; and

   e) use transitional, informal notations as bridges to formal notations since the informal notations are already attached to the child's conceptual structure.

7. Allow children sufficient time to solve problems (Suydam, 1980). Do fewer problems, but do them well. Removing the focus from the completion of a set of problems and placing it on the investigation of a set of problems will help them understand your priorities.

8. Use problems with extraneous information (Suydam, 1980). This forces students to attempt a representation rather than opting for what Sowder (1988) calls a "naive" strategy.
9. Ensure there is a good deal of pre- and post-solution discussion, both teacher-students and students-students. This is the time when conceptions, accurate and inaccurate, are exposed and observable. Make use of strategies such as cooperative learning small groups to help students to broaden their experience bases, practise using mathematical language, see multiple perspectives, involve themselves in more higher-level thinking and dissuade each other from using inappropriate representations and strategies (Gilbert-Macmillan & Leitz, 1986).
10. Make use of manipulatives in modeling problems for students and when having students solve problems (Suydam, 1980; Skemp, 1982; Hiebert, 1984; Carpenter, 1986).
11. Have students create word problems from pictures, diagrams (Sowder et al., 1986) and number sentences (Silver, 1986; Bell et al., 1986; Sowder et al., 1986). This exposes their understanding of operation and forces them to look for connections within their conceptual framework.
12. Ask students to defend their reasoning, justify their answers, and explain why their results are reasonable (Lindquist et al.,1983). This encourages students to look back from their solution to the initial problem representation to check the fit.

Literature of Relevance to Children's Understanding of Division

Of the four arithmetic operations, division has long been considered the most difficult for elementary school students (Grossnickle & Perry, 1985). Results of the 1983 NAEP indicate that although 13-year-olds were able to perform most routine calculations, their achievement was significantly lower in division exercises which involved two-digit divisors (Carpenter et al., 1984). In British Columbia, the 1985 Mathematics Assessment also found Grade 7 students weak in division with two-digit divisors. Only 67% of students were able to

select the correct answer to $45 \overline{)1232}$. The authors found this result discouraging because students had been introduced to the long division algorithm as early as Grade 4 (Robitaille & O'Shea, 1985).

Researchers who have conducted investigations centering specifically on student achievement in division have uncovered similar findings. Laing and Meyer (1982) tested a total of 331 seventh, eighth, and ninth grade students on their proficiency with one- and two-digit divisors. Approximately 21% of all students were unable to correctly answer the question $8 \overline{)972}$; just over 58% of these students were unable to calculate the answer to the question $27 \overline{)26460}$. That students find the division algorithm difficult is an acknowledged fact. The queston of why it poses such difficulty is the focus of another area of investigation.

When compared to the other algorithms students work with in elementary school, one might initially conclude that division is simply more complex; it requires multiplication and subtraction, and the number of steps involved is neither constant nor predictable. While errors in multiplication and subtraction do occur, the majority of difficulties with division questions cannot be attributed to a lack of students' proficiency with the subtraction or multiplication algorithms (Laing & Meyer, 1982). Various forms of the algorithm, such as the Greenwood and the Pyramid algorithms, have been taught to students to determine if one form is simpler and more effective than the others, but student achievement has not improved (Slesnick, 1982; Laing & Meyer, 1982). Bell et al. (1984) found that elementary students have impoverished conceptions of division which contribute to the errors they make. Students generally work with one-digit whole number divisors, and are exposed to partitive division much more frequently than quotitive. Bell, Swan, and Taylor (1981) reported difficulties associated with the division symbols. Students were confused about the direction of the $a \div b$ symbolism, confusing it with $a \overline{)b}$. Slesnick (1982) found that the division algorithm requires higher cognitive processes than the understanding of fundamental division concepts, leading her to suggest that an early introduction to concepts and a delay in the introduction of the algorithm may result in better student understanding.

Because of these difficulties with the division algorithm, students often develop conceptions which further impede progress. One of the most common beliefs students hold about division is "the smaller number must always be divided into the larger number," and "division makes smaller" (Bell et al., 1981). Hart (1981) reported that 51% of a group of 11- and 12-year-olds believed that it was impossible to divide one whole number by a larger whole number. When solving story problems, other researchers have found that the numbers and their relative size often determine which operation a student will choose (Fischbein et al., 1985; Sowder, 1988). In a discussion with a Grade 6 student, Sowder (1988) reported the following conversation:

> Interviewer: "Now, what made you think of this (subtraction)?"
> "Well, nothing else would work. Adding wouldn't work. Multiplying wouldn't work. So, and dividing wouldn't work, so right away that left only one thing..."
> Interviewer: "Why didn't you think of divide?"
> "That's usually what I think of when I see a big number and a one-digit number. I just try to divide" (p. 7).

The student failed to represent the problem in a qualitative way, choosing instead a computation-driven strategy. In this case, division was ruled out as the appropriate operation because the numbers involved did not fit the student's conception of division. Examples such as this serve to highlight the narrow scope of children's conceptions of division.

Remainders add another dimension to difficulty with division. As students progress through elementary school they gain experience expressing remainders in three symbolic formats. By the time they reach Grade 7, students will be able to express $10 \div 4$ as 2 r2, $2\frac{1}{2}$, or 2.5. Silver (1988) reported students who had little difficulty with these forms of expression for remainders in computational settings often experienced difficulty in problem-solving situations. The situational context of the problem sometimes requires incrementing the remainder to find a reasonable solution. The following problem illustrates this need: Mary has 100 brownies which she will put into containers that hold exactly 40 brownies each. How many containers will she use for all the brownies? Silver described the difference between this

problem and the problems students often solve in elementary school as the need for semantic processing. In his view, the successful solution of a problem depends on mapping between and among a least three referential systems: the story text, the story situation, and the mathematical model. Failure to map the calculation back on to the story text or implied story situation results in an inappropriate solution.

Literature of Relevance to Children's Understanding of Decimal Fractions

Most students believe that decimal fractions are a separate number system from that of whole numbers (Hiebert, 1987; Bell et al., 1981; Hart, 1981). This belief, and the complications which accompany it, result in poor achievement in both computation and problem solving involving decimal fractions. Results from the second NAEP assessment (Carpenter et al., 1981) indicated that middle school students had difficulty on concept and computation exercises with decimals. More than 40% of students were unable to add .70 + .40 + .20, and 63% of students could not divide one decimal fraction by another. At the time of the fourth NAEP assessment, little improvement had been noted. About 40 to 60% of Grade 7 students were successful with decimal computations; subtraction and division appeared to be more difficult than addition and multiplication. Although few problem-solving items required calculations with decimals, students performed best on one-step problems involving whole numbers (Kouba et al., 1988).

Why do students fail to see decimal fractions as an extension of whole numbers? A lack of conceptual development would seem to be responsible (Kouba et al., 1988). Hart (1981) found students had difficulty supplying referents for decimal fractions when asked to write a story to match the addition sentence: 6.4 + 2.3 + 8.7. Bell et al. (1984) had similar results when they asked students to write story problems to correspond to open sentences. One student supplied the following story for 8.7 ÷ 59.1: There are 8.7 boxes and 59.1 wheels. How many wheels would you get in a box? In a series of interviews with 12- to 16-year-olds Bell et al. (1981) reported that students read 11.9 miles per hour as "11 miles 9 minutes per hour" and

thought that 0.8 was equivalent to one eighth. Sometimes students simply ignored the decimal point in a number and interpreted decimal fractions such as 0.47 as 47 (Bell et al., 1984). If students are not able to tie the symbols associated with decimal notation to concrete referents within their experience, it is not surprising they have difficulty with calculation and problem solving.

Although the rules associated with operations involving decimals are relatively simple, students' reliance on procedure tends to take the place of understanding (Grossnickle & Perry, 1985). This procedural focus results not only in the misapplication of rules (Hart, 1981), but also in the creation of inappropriate rules as students struggle to make their own conceptual connections (Hiebert, 1987). In their efforts to make sense of mathematics, students sometimes overgeneralize features of whole numbers to decimal fractions (Hiebert & Wearne, 1986). An example of this can be found in Hart's (1981) study. When 12-year-olds were asked to divide 24 by 20, 8% responded with 1.4. In an effort to accomodate the remainder, it was placed after the decimal point in the quotient as if it represented four tenths. Apparently, students saw no conflict in equating $\frac{4}{20}$, with 0.4. Another rule commonly borrowed from whole numbers and applied to decimal fractions is "more digits means a bigger number." A student who holds this belief will indicate that 0.1814 is greater than 0.385. Hiebert and Wearne (1986) found that approximately 44% of Grade 7 students responded in this way and other researchers (Greer, 1987; Hart, 1981) had similar findings. To illustrate how students can create or modify rules it is useful to consider another example provided by Hiebert and Wearne (1986). Eighty-one Grade 7 students were asked to calculate the answer to $3 \div 0.6$. Fifty-one percent responded with the answer 0.2. Students ignored the decimal point, divided the 6 by the 3, and inserted the decimal point in front of the 2.

When students' lack of understanding of decimal fractions is combined with their concepts and beliefs concerning division, further confusion results. The beliefs "multiplication makes larger" and "division makes smaller" influence students to choose the inappropriate

operation in exercises and when solving problems. When asked which would result in the bigger answer, 8 x 0.4 or 8 ÷ 0.4, 50% of 12-year-olds in one study chose multiplication (Hart, 1981). In an investigation of choice of operations in verbal problems, Bell, Fischbein, and Greer (1984) found that students chose multiplication over division to solve this problem: A convict is digging his way out of a prison cell. After the first day he has only dug a tunnel of length 0.174 miles. At this rate how long will it take him to reach a forest 3 miles away? Students perceived correctly that their answer had to be larger, but did not believe that it could be obtained through division. The reverse of this situation also seems to be true.

When multiplication is the required operation in a problem but one or more of the numbers is a decimal fraction, students seem to choose division. In the following problem presented to 30 subjects, three chose multiplication and 16 chose division (Bell et al., 1984): To fit a picture of a dress onto a page of a magazine, the picture has to be reduced to 0.14 of its original size. In the original picture the length of the dress was 2 m. What will be its length in the magazine? A quick mental estimate would tell students that 0.5 of its original size would be 1 m, so 0.25 of its original size would be 0.5 m, and 0.125 would be 0.25 m. Dividing 0.14 by 2 would result in an answer that was far too small when compared with 0.25, but these students did not make that mental check; their belief in "division makes smaller" was the overriding factor in their decision making.

Other conceptions which influence achievement include the belief that decimal fractions less than one cannot divide whole numbers. In a problem which required the division of 900 by 0.75, only 25% of Grade 7 students responded correctly; the remaining students multiplied or subtracted (Fischbein et al., 1985). A curious inconsistency with students' whole number beliefs arises when students are asked to divide a small decimal fraction by a whole number. One would expect the students to reverse the number and divide, but this is not the case. When presented with a question such as 0.75 ÷ 5, students ignore the decimal point and carry out the division. Division of any number, whether greater than or less than the divisor, whole number or decimal fraction, was more difficult with decimal fractions less than one than with those

greater than one (Fischbein et al., 1985). Student uncertainty when dividing decimal fractions is not limited to situations where the divisor is greater than the dividend. Problems also occur when the items have unlike terms, such as 4.32 ÷ 2.4 (Carpenter et al., 1981). The "move the decimal over" rule fails students because of the lack of meaning associated with it. In a situation such as this students will divide 432 by 24, failing to see the place value ramifications of their error.

All of these difficulties indicate a need for instruction which focuses on conceptual development in a number of areas. Representations for decimal fractions must be concrete and meaningful for students. The introduction of arithmetic operations such as division must be done through the use of manipulatives, and the use of the standard algorithm in isolation should be delayed until students can demonstrate understanding with concrete materials and in pictorial form. To help develop an intuitive sense of decimal fractions and to develop students' metacognitive abilities, estimation should be a key part of instruction. Finally, teachers must be aware of the existence of student beliefs and conceptions which may interfere with instruction, and must plan lessons which attempt to reveal and confront these beliefs.

Literature of Relevance to Problem Solving and Division

The research on division and problem solving tends to centre on the factors influencing the student's choice of operation when solving routine translation problems. Students' limited understanding of the division operation is often cited as the source of problem solving difficulty (Hendrickson, 1986; Fischbein et al., 1985; Bell et al., 1984). Translation problems involving decimal fractions pose further difficulty for students. Weak place value understanding (Hart, 1981), a lack of intuitive size of decimal fractions (Bell et al., 1981), and a failure to integrate decimal fractions with whole numbers (Greer, 1987, Hiebert, 1987) all contribute to the poor success rate when solving problems of this nature. A student's procedural focus when solving problems also effects outcomes. The use of naive problem-solving

strategies and the failure of students to accomodate remainders when solving division problems are two of the results. Each of these areas will now be examined in greater detail.

Students have difficulty conceptualizing division beyond certain restricted classes of numbers (Greer, 1987). The presence of large whole numbers or decimal fractions in a problem makes it difficult for students to recognize the operation required for solution (Bell et al., 1981; Hart, 1981). This inability to translate the story text into symbolic form has been related to the accessibility of an appropriate model of division. Fischbein et al. (1985) believe that the fundamental operations of arithmetic are linked to implicit, unconscious, intuitive models which mediate when students solve problems. For division the dominant model is partition. When presented with problems which require the division of one whole number by a smaller whole number, students are able to identify the required operation and carry out the calculation. Translation problems become difficult when they involve large numbers or decimal fractions, when the action conflicts with the required operation, or when beliefs such as "division makes smaller" interfere. The same students who are able to solve whole number problems are unable to solve problems with the same structures when the numbers are changed. In their investigation of division and problem solving, Bell et al. (1984) reported that 12- and 13-year-olds found the easiest problems were partition problems in which the divisor and the dividend were whole numbers, and the divisor was less than the dividend. Students were also successful with quotition problems which involved whole number divisors that were less than the decimal fraction dividends. However, when problems involved a divisor less than one, or a decimal fraction divisor greater than the dividend the success rate dropped dramatically. In cases such as these students were likely to reverse the terms of a division or resort to another operation such as multiplication.

When students do not have the appropriate models for operations available to them when they are attempting to solve problems they often resort to immature problem-solving strategies based on rules and procedures. Sowder (1988) found that students use strategies such as "try all the operations and choose the most reasonable answer," and "the size of the

numbers tells you which operation to use" because they generally work with one-step whole number problems. When common or decimal fractions are involved, or the problems require more than one step, these strategies fail. Sowder advocates an instructional focus on conceptual knowledge, and on experience linking conceptual understanding with procedural understanding.

Silver (1988) reports that difficulty with word problems involving division with remainders can be linked to students' failure to link the story text with the story situation and the mathematical model. Results from various NAEP assessments support this statement: students often err by choosing answers that contain common or decimal fractions. Silver speculated this was because students lacked a procedure for interpreting remainders. In a study based on helping students engage in referential mapping, Silver found enhanced performance was possible. Increased sensitivity and attention to relevant semantic and referential mappings was encouraged through asking several related questions based on one calculation.

A number of researchers have asked students to write word problems for open sentences (Bell et al., 1984; Hart, 1981; Ekenstam & Greger, 1983). Results from these investigations confirmed the reliance of students on the partitive model of division and illustrated the students' unwillingness to divide one number by a larger one. In addition these investigations have revealed uncertainty in a number of other areas. Students are unsure about the meaning of division symbolism (Bell et al., 1981). The familiar notation $a \overline{)b}$ is confused with $a \div b$. Students interpret the two forms as being equivalent, unaware of the importance of placement of numbers within each. Students have unstable place value concepts with respect to decimal fractions (Bell et al., 1981). A number such as 0.8 was interpreted as one eighth, and 0.45 hours was interpreted as 45 minutes. A lack of meaningful representations for decimal fractions has also been uncovered through having students write word problems (Ekenstam & Greger, 1983). In their study on the problem-solving ability of 12- and 13-year-olds, Ekenstam and Greger (1983) reported that students used decimal fractions to refer to objects normally associated with whole numbers. For example, students referred to "2.5 people" and "3.7 cows." Bell et al. (1984) had similar results. Students referred to "59.1 pieces," "0.47

of a mouse," and "8.7 boxes." These examples reflect a lack of experience with decimal fractions themselves, the failure to view decimal fractions as an extension of the whole number system, and an unfamiliarity with appropriate structures for word problems containing decimal fractions.

Implications of the Literature

Recent trends in the type of research being conducted in mathematics education centre on studies which are qualitative in nature. The trend towards methodologies such as the teaching experiment (Steffe, 1983; von Glasersfeld, 1987) is encouraging because of the emphasis on the ongoing interaction between the teacher and the student. Exploration of how students construct knowledge from the interplay between past experience and new material is the focus of such research. The role of reflection by the student and the teacher is an important component in studies of this nature.

An area which would be suitable for investigation through the use of a teaching experiment is conceptual and procedural knowledge and their respective roles in learning. The most frequently cited reason for the lack of problem-solving success is the failure to link conceptual with procedural knowledge (Carpenter, 1986; Hiebert & Lefevre, 1986; Silver, 1986). Young children who enter school are relatively successful at analyzing and solving simple word problems before instruction. They base their analyses on accurate representations of the problem and can justify their solutions by referring to the semantics of the problem (Carpenter, 1986). After instruction their focus tends to be on procedures, and results in a loss of meaning. Strategies which will help students build links between conceptual and procedural knowledge in problem solving are required.

Students also experience a separation between concepts and procedures when performing operations such as division. Division has long been considered the most difficult of the four arithmetic operations (Grossnickle & Perry, 1985). Although it is a complex algorithm,

students' difficulties with division result from incomplete conceptions rather than a lack of procedural efficiency. Elementary students generally work with one-digit whole number divisors which are less than the dividend. This contributes to a reliance on the partitive model of division. From this experience, students develop limited models of division which lead to beliefs such as "division always makes smaller" and "the smaller number must always be divided into the larger number" (Bell et al., 1981). The ability of students to solve problems is negatively affected by these beliefs. The development of strategies to broaden students' understanding of division and problem solving is required.

Similarly, a lack of conceptual development with respect to decimal fractions also influences students' ability to solve problems. Students have difficulty supplying referents for decimal fractions in addition (Hart, 1981) and division contexts (Bell et al., 1984). This lack of understanding translates once again into a reliance on procedures (Grossnickle & Perry, 1985). The overgeneralization of whole number rules to decimal fractions is another result of lack of understanding (Hiebert & Wearne, 1981). The whole number rule "more digits means a larger number" misleads students in a decimal fraction setting, as does the "division always makes smaller" rule. Conceptions such as these are stable and interfere with the solution of problems. It is clear from the research literature that students need more experience with models for decimal fractions, and opportunities to use decimal fractions in meaningful contexts.

From a review of the literature it appears many students view mathematics learning as a series of unrelated areas. This perspective sees mathematics as a subject separate from real-life experiences. It also sees few relationships among various areas within mathematics. What is required is an approach to instruction and learning which attempts to acknowledge, reveal, and build upon the students' understanding. Teaching strategies must be developed which help students link prior knowledge with new knowledge, and learning in school with learning in the real world. Relationships among the concepts and procedures learned in mathematics, and their application in meaningful settings must be encouraged.

Chapter 3

METHODS AND PROCEDURES

In mathematics assessments such as the 1983 NAEP (Carpenter et al., 1984) it has been found that, although students are often able to perform isolated calculations accurately, they are frequently unable to solve problems which call for the application of similar calculations. In the Fourth NAEP Assessment of Mathematics (Kouba et al., 1988), it was also reported that despite successful skill learning in many areas, students exhibit serious gaps in their knowledge of basic underlying concepts, and are frequently unable to apply the skills they have learned.

In an attempt to examine students' conceptions of the division algorithm and its application, the researcher planned an instructional program to reveal and build upon students' conceptions of division of whole numbers and decimal fractions. The classroom case study was chosen as the vehicle for this investigation. A type of teaching experiment, the classroom case study differs from Steffe's model of the teaching experiment (1983) in three ways. It involves instruction of a class of students rather than individuals or small groups, the researcher acts as both teacher and observer, and the research takes place in the classroom setting rather than under laboratory-type conditions. In describing the difference between the teaching experiment and a clinical interview, von Glasersfeld (1987) suggests the interview aims at establishing where the child is, and the experiment aims at ways and means of getting the child on. In the researcher's interpretation, the main difference between the teaching experiment and the classroom case study is the number of children involved. The depth of understanding the researcher can develop when working with one student cannot be compared to the understanding developed when working with a class, but such is the dilemma of the classroom teacher.

This classroom case study centered on regular mathematics instruction in a class enrolled by the researcher. An instructional sequence of 16 lessons was planned before the study began. These intended lessons were adapted and changed to reflect the needs of the students

as the study progressed. To provide accurate records of instruction and interaction, lessons were video-taped and copies of student work from the lessons were retained by the researcher.

This chapter is divided into five areas of discussion:

1. The subjects
2. The researcher
3. The overview of the instructional sequence
4. The instructional program
   a) the pretest and posttest
   b) the lessons
5. The method of analysis.

The Subjects

The subjects of the study were 22 Grade 7 students enrolled in early French immersion in an elementary school in an urban school district in British Columbia. These students were part of a larger group of 38 that generally received instruction together for English language arts and mathematics. For French instruction the students were divided into two groups; one of 22 students and one of 16 students. The groups were of unequal size due to the lack of adequate instructional space. During the study the researcher continued to instruct the larger group of 38 for language arts, but instructed only the group of 22 for mathematics. To facilitate video-taping and movement in the classroom, the group of 16 was instructed by the principal for the duration of the study. Of the 22 students 12 were female and 10 were male. Pseudonyms have been used in reporting the results.

The school involved in the study was a dual track elementary school, offering instruction in French and English from Kindergarten through Grade 7. English students came from families living in the immediate catchment area of the school which consisted primarily of rental accomodation and older homes. These families ranged from middle to lower income, and many were single-parent families. French immersion students came from various catchment

areas across the school district, most from families who owned their own homes and were in the middle to upper income bracket. Generally, immersion students lived with both parents, and one or both were professionals. The students in the study reflected a cross-section of the immersion population of the school.

With three exceptions, all students had been in the same school from Kindergarten through Grade 7. Most had similar if not identical experiences in intermediate mathematics instruction, as only a few students had formed part of split classes from Grades 4 through 6. None of the teachers who had instructed the classes was a mathematics specialist. Students had received mathematics instruction in French for Grades 1 through 3, and in English for Grades 4 through 7.

The Researcher

In a descriptive study such as this it is important to understand the perspective of the researcher, who in this case acted as both investigator and reporter. The researcher began a teaching career as a generalist elementary teacher with a four-year Bachelor of Education degree. Prior to the study, the researcher commenced the coursework necessary to complete a fifth year in the department of Mathematics and Science Education at the University of British Columbia. Of the courses taken, three in particular provided the researcher with opportunities to investigate student learning in mathematics. These were Diagnosis and Remediation in Elementary School Mathematics, Mathematics Education (Elementary), and Mathematics Teaching: Problem Solving. This coursework and the experience teaching intermediate mathematics prompted the researcher to pursue a Master's degree in this area.

The notion of constructivism was introduced to the researcher in a Mathematics and Science Education course early in her studies, and it served as the basis for reflection on teaching and learning. An investigation of the roles of conceptual and procedural knowledge, as well as studies on representation in problem solving, followed. Through coursework, classroom practice, and involvement on the school district's mathematics committee, the researcher has

focused on the teaching and learning of mathematics prior to the study. During this time the researcher presented numerous workshops on the use of manipulatives, mathematics program planning, problem solving, data analysis, and geometry.

Overview of the Instructional Sequence

Based on a survey of the literature related to achievement in division and on the researcher's teaching experience, it was expected that a variety of student conceptions about division with whole numbers and decimal fractions would be uncovered. In anticipation of uncovering these conceptions the researcher planned a set of lessons beginning with modeling the concrete division of whole numbers moving though division of decimals in a problem-solving setting. What follows is a description of the anticipated difficulties and the proposed instructional programme.

The researcher believed that while students could accurately perform calculations involving division of whole numbers, they would not generally be able to model division with manipulatives nor would they be able to represent open division sentences by writing story problems. Students were expected to have a limited understanding of the meanings of division. Of the two most common meanings for division, partition and quotition, partition was expected to be the most developed. Problem solving involving division of whole numbers was also expected to be a source of difficulty because of the students' weak conceptual understanding of division. It was anticipated that students would have a limited view of the role of the remainder in problem solving situations due to their strong procedural base. The researcher believed division of decimal fractions would pose more difficulties for students. Some students were known to have weak place value understanding of decimal fractions, and others had experienced difficulty in a previous unit involving multiplication of decimal fractions. Students had previously shown little independent use of estimation skills and were expected to rely on procedural knowledge when approaching calculations and problem solving involving division of decimal fractions.

The researcher used a variety of teaching strategies which were designed elicit and develop students' thinking about division, decimal fractions, and translation problems requiring division. A major focus of instruction was on minimizing the number of questions or problems presented in a lesson. The researcher explored questions from a number of perspectives and focused on depth of understanding. Working on these activities in pairs or small groups, students had increased opportunities to express and justify their thinking. Whole class discussions centered on revealing beliefs, explaining reasoning, justifying responses, and challenging others. The use of manipulative materials and other aids was also central to instruction. Students used base 10 blocks, 10-by-10 grids, and diagrams on a daily basis. Another focus of instruction was on the language associated with division and with decimal fractions. By acting as a model the researcher attempted to tie meaning to symbols and procedures. Students were encouraged to follow this example. The use of familiar problem contexts and the creation of story problems to accompany division open sentences were important strategies for a number of reasons. They helped students develop a meaningful framework for division and they revealed students' thinking about models of division. Beliefs such as "the divisor must be less than the dividend" were exposed in this way. Students' understanding of representations for decimal fractions were also revealed using such strategies. Posing questions which would create conflict for students was another strategy which the researcher used to elicit understanding and challenge beliefs. These questions and problems forced students to confront beliefs such as decimal fractions less than one cannot be divisors, and provided the researcher with opportunities to assess development.

The Instructional Program

A series of 16 one-hour classes were planned for a four-week period during the months of November and December of 1988. Students received mathematics instruction from the researcher in their own classroom at the regularly scheduled times. Because the researcher

enrolled this class for instruction in mathematics, this study represented little change in the daily pattern of the students.

Each lesson was video-taped so that data could be transcribed. On Mondays and Wednesdays a fellow graduate student conducted the taping of whole class and small group work. On Tuesdays and Thursdays the camera was generally stationary and focused on the class as a whole. When appropriate, the researcher trained the camera on one group after the introduction of the lesson.

Pretest and Posttest

To assist the researcher in planning for the unit, a pretest (Appendix A) was administered to the subjects in the week before the instructional unit began. This test consisted of seven sections covering five areas of interest to the researcher.

Sections A and B each contained five division questions, the former with whole numbers (e.g. $16 \overline{)219}$) and the latter with decimal fractions (e.g. $0.7 \overline{)2.28}$). By the time students reach Grade 7 they have been working with the algorithmic division of whole numbers for three years and decimal fractions for one to two years. An understanding of the group's entry computational ability provided the researcher with information useful for initial instruction.

Section C consisted of four translation problems (e.g. A necklace is 58.5 cm long. It is made from 13 hinged pieces of gold. How long is each piece of gold?), two of which required consideration of the role of the remainder (#1 and #3). In all of these problems the dividend was greater than the divisor. The researcher was interested in how the students would account for the remainder and in how they would deal with the division of a decimal fraction by a whole number in a problem-solving setting.

Section D was composed of 10 multiple choice questions where the students had to read a translation problem and select the correct open sentence from a choice of six (e.g. 2.45 kg of jellybeans are packed 0.35 kg to a bag. How many bags can be filled?). Two of these problems involved two steps (#1 and #2), two had divisors which were greater than the dividends (#9

and #10), and six contained decimal fractions (#3, #4, #5, #6, #9, #10). Although all questions could be solved using division, the distractors for each included open sentences with all operations and one which had reversed terms in the division open sentence. These questions were included to determine if the students could detect the need for division in the problems and to check for the tendency to reverse the terms in a division open sentence.

Sections E and F consisted of three questions designed to determine the students' understanding of the effect of relative size of the dividend and the divisor on the quotient when working with decimal fractions (e.g. Consider these numbers: 6.25 9.4 0.73 5.78 12.6 1.291. Choose the two numbers which will result in the largest possible quotient. Write the division question.)

Section G required students to write story problems for six division open sentences (e.g. 11.4 ÷ 8). Four of the six questions involved a decimal fraction as at least one of the terms (#a, #b, #e, #f), and two questions (#c and #e) had divisors greater than their dividends. All open sentences were written in the form a ÷ b to avoid confusion with the b)a notation. There were three reasons for including questions of this nature. Firstly, the researcher believed that students' ability to represent division open sentences through story problems reflected, in part, their conceptual understanding of division. Secondly, the researcher was interested in exploring the tendency of students to reverse the terms of a division open sentence whose divisor was greater than its dividend. Finally, the researcher wanted to investigate students' representations of decimal fractions in story problems.

Following the instructional program a posttest (Appendix B) was administered to the subjects. It was similar to the pretest so that comparisons could be drawn. Problem structures and contexts were of the same type, and in some cases the identical questions were asked (e.g. Section F). In addition to the seven sections of the pretest, the posttest included an eighth section consisting of five true or false statements related to some of the beliefs regarding division and decimals that the students had expressed throughout the unit (e.g. 4.5 ÷ 3.1 is the

same question as $4.5 \overline{)3.1}$). The researcher wanted to determine both the students' ability to detect erroneous statements and their ability to provide examples disproving such statements.

Overview of the Lessons

A series of 16 lesson plans was written in advance of the study, based on expectations held by the researcher. However, due to information provided by the pretest and insights gained through the course of instruction, these lessons were modified as the study progressed. This section contains a brief overview of each of 19 lessons that took place in the study.

Week 1.

Lesson 1: Students used base 10 blocks in triads to reveal thinking and language associated with the division algorithm.

Lesson 2: Introduction of partitive and quotitive division through modeling with base 10 blocks and use of story problems. Discussion of what makes division easy or difficult.

Lesson 3: Further exploration of partitive and quotitive division through student-composed story problems. Role of the remainder.

Lesson 4: Role of the remainder: Can the remainder always be broken into smaller pieces? Is the sharing of the remainder related to partitive and quotitive division? Groups discuss how to solve multi-step story problems.

Week 2.

Lesson 5: Discussion of story problems from Lesson 4, justified thinking and processes chosen to solve. Introduction of representation and division of decimal fractions using base 10 blocks.

Lesson 6: Review of using base 10 blocks to model division of decimal fractions. Use of base 10 blocks to model and solve story problems.

Lesson 7 : Transition from division of decimal fractions by one-digit whole numbers to two-digit whole numbers emphasizing estimation and using overhead to model. Renaming tenths to hundredths. Creating story problems for division questions.

Lesson 8: Beliefs and conceptions regarding the division of one decimal fraction by another were uncovered and explored. Modeling of such questions with base 10 blocks and 10-by-10 grids. Renaming to equivalent questions using estimation and checking with calculators.

Week 3.

Lesson 9: Using 10-by-10 grids, review of division of one decimal fraction by another. Creating story problems for questions. Practice questions in small groups.

Lesson 10 : Review of questions assigned in Lesson 9. Student demonstration of division using grids. Set of story problems involving division of decimal fractions was discussed in small groups and assigned for completion.

Lesson 11: Writing of story problems based on a division question. Three question review on division of one decimal fraction by another. Class discussion of review. Writing out plans for a problem set after discussion in small groups.

Lesson 12 : Lengthy class discussion of the opening warm-up activity involving student's story problems that required the division of a decimal fraction by a larger whole number. Multiple representations of the problem. Discussion regarding why students were reluctant to divide a decimal fraction by a larger whole number. True or false oral quiz on beliefs regarding division and decimal fractions. Comparison of plans for solution of problems assigned in Lesson 11.

Week 4.

Lesson 13: Solutions for problem set as the focus of class discussion. Exploration of division of decimal fractions in which one term contained hundredths and the other, tenths. Discussion of seven students' story problems chosen from those written in Lesson 11. Student evaluation of the problems to determine if each contained division, was logical, and could be answered.

Lesson 14: Exploration of troublesome division questions: decimal fraction divisor and whole number dividend, divisor containing tenths and dividend containing

hundredths, and decimal fraction divisor in the thousandths. Further exploration of student belief that no decimals should remain in the dividend. Five question review on types of questions discussed at the beginning of the lesson. Discussion of student story problems assigned in Lesson 13.

Lesson 15: Small groups working on a problem set while researcher works with five students who had difficulty with the review.

Lesson 16: Focus on determining correct placement of numbers in a division open sentence based on story problems. Introduction of a strategy ("word frame") to assist students in representing problems. Application of strategy to student-written story problems and to a new set of story problems.

Week 5.

Lesson 17: Review lesson centering on writing story problems for open division sentences. Problem set from Lesson 16 as focus for discussion. Assignment of a new problem set. Students work in groups.

Lesson 18: Review of representing division questions through story problems using the word frame strategy. Solutions to problem set from Lesson 17 presented by students. Discussion on justification and reasoning behind solutions.

Lesson 19: Review for test including partitive and quotitive division, role of remainder, division of whole number and decimal fractions, examination of students beliefs, and writing problems for division open sentences.

Lesson 20: Posttest.

Description of Lessons

This section describes in greater detail the lessons which took place. The student practice sheets referred to in the lessons are contained in Appendix C. These practice sheets were developed as the study progressed using feedback from the lessons and the best information available to the researcher.

Lesson 1

To expose the language and actions students associated with division of whole numbers the first lesson centered on having students act as teachers in groups of three. One was the student, one the teacher, and one an observer whose job it was to record the language used and the actions of their group members. Using base 10 blocks, "teachers" were asked to instruct "students" how to divide questions such as 68 ÷ 12. The researcher circulated and noted the language and actions used, looking specifically for meaningful use of manipulatives. Following the small group work, observers presented their observations to the class and the language and actions were recorded at the board. Part of the discussion centered on the use of story problems to convey meaning. Students struggled to create problems which involved accounting for a remainder. The class was asked to write a brief statement describing how they felt when division was difficult and when it was easy.

Lesson 2

The class began with a discussion of what made division difficult or easy. To introduce partitive and quotitive division, the researcher told the class they would be working with two kinds of division in the lesson, and asked for volunteers to assist in describing how division could differ. Taking the question 15 ÷ 5, Stan volunteered to demonstrate. Other students were asked to describe Stan's actions as he modeled partitive and quotitive division using base 10 blocks. Some students could not see the difference between the two. The researcher then repeated Stan's modeling, with emphasis placed on language, drawing on the students' comments. Phrases like "shared among" and "separated into groups of" were used in place of the students' phrase "goes into." The names partitive and quotitive were assigned to the two methods. Word problems were used to illustrate when the two types might be used. For example: A group of 94 students are going ice skating. A bus will hold 45 students. How many buses will be required? The class was asked to think about the actions involved; either "sharing" out students onto an

unknown number of buses, or taking out a group of students that would fill a bus. Following the discussion of the two types of division, students modeled each in groups at their desks. Then in pairs they created story problems to reflect the actions of partitive and quotitive division. Four story problems were assigned for homework (Practice Page #1 - Appendix C). Students were to decide if each was partitive or quotitive, and were to solve each showing their work.

Lesson 3

The four problems assigned for homework were discussed. Students were asked to tell whether they were partitive or quotitive and explain why. Much discussion focused on the difference between "dealing" things out one at a time or in small groups (partitive), and taking out a group of a specific size (quotitive). Students were asked how remainders would be accounted for in each problem. Students then shared the problems they had written on the previous day and each group decided if the question was partitive or quotitive (Practice Page #2 - Appendix C). Decisions had to be justified. The researcher asked pairs of students to create one partitive and one quotitive problem with free choice of numbers so that their understanding could be checked.

Lesson 4

From the previous lessons it had become clear that students struggled with the role of the remainder. This lesson focused on the types of things that could be shared into fractional parts and those that could not and would therefore imply some other form of accomodation. Students were asked to generate two lists of objects: one that could be shared into parts, and one that could not. Once completed, students were asked to think about partitive and quotitive division and how the two types might relate to the lists they had just made. The researcher's intent was to encourage further discussion of the two types of division, and to stimulate thinking about how various objects might be divided. Students worked to make one column fit the quotitive model, and one the partitive. It seemed they assumed it could be done because the instructor had directed them to think about it. After deciding there was no simple rule for

dealing with remainders, students were assigned a set of problems, some of which involved remainders (Practice Page #3 - Appendix C). Their task was to discuss each problem in their group to ensure understanding, and then individuals completed the work. The researcher travelled from group to group listening to discussion and challenging thinking. At the end of the class, students were asked which questions caused their group the most difficulty. Possible reasons for the difficulty were discussed.

Lesson 5

The problems from the previous day were reviewed, with much of the discussion focusing on the problem which involved the prediction of the time it might require for a sixth chick to emerge from its shell. Two problems surfaced: an unclear conception of averages, and the representation of time in decimal form. From there, the researcher went on to the renaming of base 10 blocks as decimal fractions so that division of decimal fractions could be modeled concretely. Once the students were able to rename decimal fractions ranging from 1.7 to 0.04 with the blocks, the researcher asked students to think about how a decimal fraction like 1.09 might be shared among 3. Using a combination of researcher and student modeling, students worked to complete questions involving whole number divisors and decimal fraction dividends in both exercise and story problem formats.

Lesson 6

After reviewing naming and dividing decimal fractions (e.g. $4\overline{)3.4}$) with base 10 blocks, the students worked in small groups with the manipulatives while solving story problems (Practice Page #4 - Appendix C). Students were encouraged to record pictures of their solutions. The researcher circulated, asking questions of the students regarding their representations of the division problems. The task became difficult when some problems required renaming large numbers and we began to run short of sufficient rods to complete the renaming. Because the task was becoming unnecessarily frustrating, the lesson was curtailed.

Lesson 7

Using the base 10 blocks side-by-side with the algorithm, students discussed and solved questions such as $3\overline{)0.18}$ in a whole class setting. Students modeled questions at their desks and on the overhead projector. Problem contexts were created for each question in an attempt to reveal the meaning students attached to the decimal fractions. The same procedure was followed for 2-digit divisors, except that the use of estimation was introduced. Much time was spent on linking the algorithmic renaming of one whole to tenths, tenths to hundredths, and hundredths to thousandths with the actions using base 10 blocks. Students were dissuaded from referring to "adding a zero" when renaming. Following this, the problem set assigned in the previous class was reviewed.

Lesson 8

The researcher began by reminding students of the type of questions they had worked with on the previous day, questions like $7\overline{)2.8}$, and of how well they had seemed to understand them. The students were then presented with the question, $0.7\overline{)2.8}$, and were asked how they might approach such a question. The researcher's thinking was that students would see the similarity between the questions, with the divisor being smaller by a factor of 10. Many students offered a variety of methods with which the question could be approached, and justified their thinking. The bulk of the lesson centered on the exploration of each student's suggestion, followed by reactions from the class. Students formed into three groups, each defending an answer: 4.0, 0.4, or 0.04. Through this discussion many interesting beliefs concerning decimal fractions and the division algorithm were uncovered and explored. Eventually, the question was modeled with base 10 blocks and shown on the overhead using 10-by-10 grids. Exercises on renaming questions by multiplying or dividing by 10, 100 and 1000 were completed together, with an emphasis on estimation and checking the solutions to renamed questions with calculators. A portion of a practice sheet (Practice Page #5 - Appendix C) was assigned.

Lesson 9

The focus of this lesson was the reinforcement of the use of 10-by-10 grids to show division of decimal fractions. In a whole class setting, students were asked to visualize the grids to form estimates for division questions such as 1.6 ÷ 0.2 and 1.4 ÷ 0.4. Estimates were then checked by shading the grids into the appropriate size "chunks." Because both the divisor and the dividend both contained tenths, some students held with the belief that "you just forget the decimals and divide." One student, Kate, expressed her approach more meaningfully; she said to rename each to tenths and divide. A question like 1.6 ÷ 0.2 became 16 tenths ÷ 2 tenths, not unlike multiplying both terms by 10 but with more meaning. Each of these approaches was followed through with a pairing of the algorithm and the actions and language used in each. The practice sheet (Practice Page #5 - Appendix C) assigned on the previous day was discussed. Students had been unable to provide estimates for some questions, especially those with numbers containing digits in the hundredths or thousandths (e.g. 6.8 ÷ 0.25). Using Kate's approach of renaming, an estimate was found for 680 hundredths ÷ 25 hundredths. Students were asked to go back over their sheets and use either renaming or tying questions to story problems as strategies for determining estimates. Finding suitable representations for decimal fractions became the focus at this point, as students were content with suggestions such as 2.3 workmen or 2.3 containers. One student tried to accomodate 0.97 ÷ 2.3 as a comparison of two different amounts of kilograms of beans. Because of these difficulties, questions involving the division of one decimal fraction by a larger one were left to another day for further investigation. Students were assigned five questions to solve using the 10-by-10 grids and were encouraged to estimate (Practice Page #6 - Appendix C).

Lesson 10

Students demonstrated their understanding of the homework from the previous day by shading the 10-by-10 grids to indicate the answers to the questions on the practice sheet. The class was comfortable with the work and had completed it successfully. Some students were able

to provide other forms of reasoning for defending their responses. For example when asked how they might know 2.7 ÷ 0.9 is 3 without completing the shading, one student responded that we were using 3 squares, and 9 (tenths) is 1 (tenth) less than a whole, so the answer is 3. At this point the students seemed ready for further exploration so a set of story problems involving decimal fractions was presented (Practice Page #7 - Appendix C). The initial discussion centered on this problem: A stack of quarters is 6.3 cm high. If each quarter is 0.15 cm thick, how much money is in the stack? Students were asked to explain aloud how they would approach the problem. While the researcher presented this problem because of the parallel between it and the grid questions completed earlier, the students saw no such connections independently. One student suggested multiplication, another suggested division - either 6.3 ÷ 0.15 or 0.15 ÷ 6.3. Finally a suggestion was made to draw a picture, and while doing so the researcher asked how the problem was similar to the grid problems. After further discussion, the day's problems were read aloud. Students worked on them in their groups, discussing the approaches to be taken. While the groups worked, the researcher visited two groups to check students' ability to divide questions such as 3.9 ÷ 0.6, questions similar to those presented in the grid form a few days earlier (Practice Page #8 - Appendix C). Surprisingly, only one student in each group had the correct response. There was no evidence of any estimation, and the task was generally approached procedurally with reliance on the algorithm. After reminding the students of the grids and asking them to visualize their responses on the grids, the students returned to the review sheet. While they worked, the rest of the class was surveyed as to the difficulty of the story problems. "Difficult" questions ranged from those requiring the most time to those that had decimal fractions in the thousandths.

Lesson 11

This lesson began with presenting one of the questions from the review sheet (Practice Page #8 - Appendix C) given to the two groups on the previous day. Students were given the question, 3.9 ÷ 0.6, and their peers' answers: 650, 65, 6.5, and 0.65. The researcher

commented on how surprised she had been with the answers because everyone had done so well with the grid problems, and the asked for possible reasons for the continued difficulty. One student suggested it was because the ones on grid paper had been exact and didn't have "remainders" (were whole numbers). Three questions (7.2 ÷ 0.8, 6.8 ÷ 0.8, 6.0 ÷ 1.5) were placed on the overhead to serve as a review, then were discussed as a class. Students commented they would rename each as tenths or multiply both terms by 10. Very few students made errors, so the follow up discussion centered on how one might check the reasonableness of an answer. A set of problems unrelated to division of decimal fractions was presented (Practice Page #9 - Appendix C). Students were directed to read each carefully, discuss them in their groups, and write out their plans for solution. Answers were not required.

Lesson 12

This lesson was to focus around a true or false quiz regarding beliefs associated with division of decimal fractions, to be followed with the presentation of plans for solution for the story problems assigned on the previous day. The quiz was used to close the lesson because a more interesting discussion arose from the opening activity. As a warm-up on the previous day, students were asked to write a story problem for any two of the numbers which had been placed on the board. They had to ensure the problems required division for solution, and included at least one decimal fraction. Joani had written the following problem which the researcher chose to present for discussion because it required the division of one number by a larger number: You have 22 pairs of jeans and after you shorten each pair, there's 4.5 metres of cloth left over. About how much did each pair lose? After deciding that division was the operation to use to solve this problem, students were asked how they would proceed. Geoffrey, who was considered to have good conceptual understanding of most concepts presented in Grade 7, said he would divide 22 by 4.5, obtaining an answer of about 4 or 5. He received general agreement from the class. This provided the researcher with the opportunity to use conceptual conflict to encourage the students to question their own thinking. When asked what the 4 or 5

represented, students quickly scrambled from metres to decimetres to centimetres, emphasizing the commitment to their answer. The only reflection that took place was on the type of unit to be used, not on the way in which the answer had been achieved. Students were then asked to visualize what was being shared. The researcher drew a picture on the board and directed the students to think about the action of the problem, to think about what was being shared and to try to relate the problem to the grid paper. The drawing on the board was partitioned into 22 strips and the "story" it told, 4.5 ÷ 22, was compared to the original suggestion of 22÷ 4.5. A discussion followed, centering on what the students thought the reasons might be for reversing the numbers. The discussion of the belief that the larger number is always divided by the smaller number led into an oral true or false quiz that had been originally planned. After reviewing the quiz, a short time was spent on the plans for the solution of the problems assigned the previous day (Practice Page #9 - Appendix C). Although these problems were not related specifically to division there was a tendency to rely on computation to solve them. The use of diagrams to represent the actions in some of the problems enabled some students to change their solution plans.

Lesson 13

From previous discussions with the students and from error analysis of their work, the most difficult division questions involved dividing a number by a larger one, had a decimal fraction in the thousandths as one term, or had terms with digits of differing place values (e.g. 4.2 ÷ 3.34). The researcher gave the class a practice sheet (Practice Page #10 - Appendix C) containing questions of these types and asked for estimates and calculations using pencil and paper. While the students worked, the researcher went from group to group checking for difficulty. Problems such as multiplying one term by 10 and the other by 100 were dealt with on a group basis. Students were asked to explain why an error such as this was wrong, and how it would affect the answer. Connections were again made between renaming to tenths or hundredths and multiplying by 10 or 100. A second component of the lesson was a practice sheet containing a selection of the problems written by the students in Lesson 11 (Practice Page

#11 - Appendix C). Some of the student problems lacked information or contained illogical representations of decimal fractions. To encourage reflection upon their thinking and the thinking of their peers, students were asked to determine if each problem required division for its solution and if the problem was logical. Estimates were requested and students were directed to use calculators to carry out calculations.

Lesson 14

To inititate discussion, the following questions were written on the board: 0.03 $\overline{)142}$, 0.075 $\overline{)26.1}$, 3.2 $\overline{)4.36}$. These questions represented a sample of those causing the most difficulty in the previous lesson and in the unit as a whole. Students were asked to consider why the researcher had chosen these questions to put on the board and how they might be approached. Generally, the perception was that they had been chosen because they all had to have one or more zeros "added" to them, including the last one. This confirmed the researcher's suspicion that students believed all decimals had to be removed, including those in the dividends, before a question could be attempted. By asking for estimates for three parallel questions (3.2 $\overline{)4.36}$, 32 $\overline{)43.6}$, and 320 $\overline{)436}$), students were convinced that the removal of decimals from the dividend was unnecessary although many expressed the opinion that it was easier. Again, a conflict situation was required to strengthen the researcher's position. Students were asked to consider the question 3.2 $\overline{)4.3567129}$. Using their preference for removing the decimal in the dividend, this would result in having to divide by 32 000 000. After some discussion of the types of problems encountered when dividing small decimal fractions, a review of similar questions was administered to the class (Practice Page #8 - Appendix C). The researcher observed students as they worked, specifically looking for estimates and for questions which posed difficulties. Following the review, the assignment from the previous day which involved assessing other students' story problems was discussed (Practice Page #11 - Appendix C). Students identified which of the problems could or could not be solved and explained what the

difficulties were. Errors they found included missing information and illogical representations of decimal fractions such as 4.5 A's on a report card and filling a car with 0.35 kg of gasoline.

Lesson 15

Because of an early dismissal on this day, the principal was unable to meet with his group of 16 students so the entire grouping of 38 was together for mathematics instruction. The researcher decided to assign the remainder of the problem set from Lesson 12 to the class as a whole, and work with the few students who demonstrated continuing difficulty with questions from the review (Practice Page #8 - Appendix C). For these students rounding and estimating was a problem. Two students had difficulty with place value concepts and did not see the connection between multiplying by 10 or 100 and where the digits were placed relative to the decimal. Further work with base 10 blocks and renaming of ones to tenths to hundredths and vice versa was necessary.

Lesson 16

At this point in the unit the researcher was confident that most students were aware that one number could be divided by a larger number and that students were capable of solving division questions involving decimal fractions using estimation and paper and pencil. What remained difficult was constructing the appropriate equations for simple translation problems involving division of decimal fractions. In this lesson the researcher introduced a strategy to help students correctly represent this type of story problem. The following problems were placed on the board to initiate discussion:

1. Four gophers are digging a tunnel that will be 5.2 km long. If they dig 0.35 km in one day, how many days will it take them to finish?
2. Karen is making ornaments. She has 0.85 g of gold dust to be distributed evenly over 7 ornaments. How much gold dust will each ornament get?

On the board the researcher drew a blank division frame like the one below and asked the students to think of what was being divided, excluding the numbers.

#3

#2 _____) #1 _____

Students initially responded with the units or with the actual numbers. Upon prompting they were able to suggest phrases from the problems which could be inserted into the frame.

#3 number of days required

#2 how much per day) #1 how far in total

Once the problem had been translated into this form the corresponding numbers could be substituted, an estimate made, and the calculation completed. When the second problem was approached in the same fashion only two students reversed the digits, reluctant to divide 0.85 by 7. When Stan was prepared to argue his position for 7 ÷ 0.85, he was encouraged to explain his reasoning and answer questions from his classmates. They convinced him that it was the gold dust, rather than the ornaments, that was being shared. This division frame strategy was then used to examine the solutions to the students' problems from Lesson 11. Students who had made reversals were able to detect their own errors. Towards the end of the class another problem set containing some mulit-step problems was assigned (Practice Page #12 - Appendix C). Students were directed to use the frame if appropriate and to record their thinking beside their work.

Lesson 17

Because the unit was coming to a close, the researcher decided to review the use of the "word frame," have students write problems for the types of division questions which had posed difficulties, and ask students to justify why they believed their answers were correct. The lesson began with students writing story problems for two division questions on the board: 6 ÷ 18 and 2.7 ÷ 9. After the researcher collected the cards, a few students presented their problems using the word frame. All but two students had interpreted 6 ÷ 18 as 18 $\overline{)6}$. With help from their peers they were able to see why their problems did not fit the question, but were unable to think of ones that did in that setting. With the second question, 2.7 ÷ 9, the representations for 2.7 were logical and their story problems reflected understanding. The

remainder of the lesson was spent working through the multi-step problems assigned in Lesson 16. Again, the word frame was used to assist in translating the problems and to help students visualize the actions. A final set of story problems requiring division was assigned (Practice Page #13 - Appendix C).

Lesson 18

In this lesson the class looked at creating a story problem for a difficult division question and reviewed the solutions to the problems assigned in the previous lesson. The researcher encouraged the students to justify their answers and ask questions of each other. In their attempts to create a problem to fit 3.7 ÷ 4.5 students struggled with representations for 4.5. Initially some wanted to consider people, but after some thought were able to suggest kilograms of gumdrops shared among dozens of cookies and litres of cottage cheese shared among canteloupes. In reviewing the problems assigned in Lesson 17, the researcher was as interested in the students' reasons for why they believed their answers were correct as in the answers themselves. The first question involved determining the number of times a screw would have to be turned to move a certain distance into a piece of wood. Justifications for answers included using the word frame, stating that the answer seemed realistic, and drawing scale diagrams.

Lesson 19

In this final review lesson the researcher presented questions from across the unit of study (partitive and quotitive division, role of the remainder, writing story problems for equations, dividing a number by a larger number, etc.), but students were given the opportunity to make and present questions to their peers in preparation for the test. Their questions included true and false statements and division questions with decimal fractions in the hundredths or less. Their interpretations of what was difficult provided the researcher with further insight into their thinking.

Lesson 20

During this class, a final unit test was administered (Appendix B). It paralleled the pretest with the exception of a final page of true or false questions aimed at the specific beliefs confronted in the unit.

The Method of Analysis

Data records for the study were transcriptions of the video-tapes, notes made by the researcher, and student artifacts such as records of daily work and copies of pre- and posttests. The analysis should show (a) the conceptions and beliefs Grade 7 students hold regarding division of whole numbers and decimal fractions, (b) the types of instructional techniques which assist in revealing student thinking, and (c) the results of a variety of instructional strategies which were aimed at helping students develop stronger conceptual understanding of division.

As the study progressed, the researcher took daily notes to assist both in planning for future instruction and in later analysis of data. Following the study, the video-tapes were transcribed and examined together with data collected from the written artifacts of the students to determine student understandings in three content areas: students' understandings of division, students' understandings of decimal fractions, and students' understandings of problem solving requiring division. The four specific research questions were investigated within each of these content areas. Question 1 centered on the beliefs and understandings Grade 7 students hold regarding division. This area was explored through an analysis of the pretest results and through classroom investigation. Question 2 focused on how teachers might elicit students' understandings in the classroom setting. Classroom investigation into this area was carried out through a variety of strategies employed in whole class and small group settings. Question 3 involved an inquiry into and assessment of the types of teaching strategies that might help students make connections between concepts and procedures. Question 4 concerned the impact of such an instructional approach on linking conceptual and procedural knowledge, and on

representations of division translation problems. A combination of classroom and posttest results provided the researcher with information regarding this question.

The three content areas are subdivided into two or three subareas. Each subarea was examined in terms of pretest results, classroom investigation, instructional decisions, posttest results, and classroom results. The pretest results describe the students' responses to various sections of the pretest as they relate to each subsection. Generally, trends in responses were used to illustrate the kinds of student understanding. Classroom investigation involved further exploration into students' understanding. Examples from classroom interaction were reviewed. Samples of students' work, collected from both individuals and small groups, were examined. Teacher strategies used to elicit student thinking were evaluated. Instructional decisions examined the success of strategies and activities employed by the researcher to build upon students' understanding. The posttest results were examined with respect to differences from the pretest results. Classroom results consisted of the researcher's interpretation of the students' development of understanding. Results were based partially on perceptions developed by the researcher during discussions with individuals, small groups, and the class as a whole. Evidence gathered from student practice pages was also used to assess students' initial understandings and their development.

Chapter 4

THE RESULTS OF THE INVESTIGATION

In this chapter the results of the investigation are separated into three content areas. These areas are:

1. Students' Understandings of Division
2. Students' Understandings of Decimal Fractions
3. Students' Understandings of Problem Solving Requiring Division

Each of the content areas is divided into two or three subareas. Students' understandings of division is separated into three subareas concerning interpretations of division notation, the actions associated with division, and the roles of the divisor and dividend. Students' understandings of decimal fractions is separated into subareas describing place value and decimal fractions, and students' representations of decimal fractions. Students' understanding of problem solving requiring division is explored in the final section: students' approaches to translation problems and their ability to write story problems are the subareas discussed. Each of these is in turn separated into five subsections: Pretest Results, Classroom Investigation, Instructional Decisions, Posttest Results, and Classroom Results.

Based on previous experience and a review of the literature, the researcher had identified a number of areas related to division of whole numbers and decimal fractions which might pose difficulty for students in Grade 7. At the onset of the study the researcher prepared and administered a pretest (Appendix A) which provided initial information on the students' understanding. Throughout the instructional program further data were collected through large and small group discussions and from the written work completed in class. The sections which follow detail students' initial understandings, how these understandings were influenced, and their understandings at the end of the instructional program.

Students' Understandings of Division

The Grade 7 students involved in this study had incomplete conceptions of division. In the first subarea, students' interpretations of division notation are discussed. Student's models of division and the actions associated with those models are described in the second subarea. Finally, students' interpretations of the divisor and the dividend are detailed.

Interpretations of Division Notation

Does $a \div b$ mean the same as $a\overline{)b}$? A number of the students in the study were unsure about the answer to this question. Students had difficulty both in choosing the correct division open sentence for translation problems and in knowing which open sentences would result in quotients greater than one. When asked to write story problems for open division sentences some students reversed the terms of the question even when the divisor was less than the dividend.

Pretest Results. Section D of the pretest contained 10 questions which required matching a story problem with the correct open sentence. An example of this type of question is given below:

> Karen is working on a macrame project that requires pieces of cord that are 0.85 m long. If she has 15 m of cord to cut into pieces, how much cord will she have left over when she is finished cutting?

a. $15 + 0.85$	c. $15\overline{)0.85}$	e. $15 - 0.85$
b. $15 \div 0.85$	d. 0.85×15	f. 15×0.85

Table 1 details the results for each of the questions in Section D of the pretest and posttest. Student responses have been separated into three groups: correct responses, reversal of the dividend and divisor, and other operations (addition, subtraction, or multiplication).

With one exception, the most commonly chosen distractor was the alternate division open sentence. Question #3 involved area and most students chose a distractor containing multiplication. The researcher expected reversals in two of these questions because the correct

solutions involved divisors which were greater than the dividends. However, students sometimes chose to reverse the terms even when the divisor should have been less than the dividend, leading the researcher to suspect confusion with notation. Many students were able to correctly choose the answer for question #8 in which a baker planned to use 4200 g of flour to make 12 cakes. Four of the remaining students, however, opted for the alternate division open sentence. The correct response was written as 4200 ÷ 12, whereas the distractor was written 4200)12. In a similar question the distractor was written with the a ÷ b notaton and it also drew responses, indicating that confusion was not limited to one type of notation.

Table 1
Student Responses To Section D: Choosing An Open Sentence

	Correct		Reversal		Other Operation	
Question	Pretest	Posttest	Pretest	Posttest	Pretest	Posttest
1	13	17	6	5	3	0
2	14	16	n/a	n/a	8	6
3	7	12	3	1	12	9
4	17	18	3	4	2	0
5	16	18	3	1	3	3*
6	14	18	7	3	1	1
7	20	19	1	2	1	1
8	16	16	4	5	2	1
9	3	9	18	12	1	1
10	5	16	10	4	7	2

*One student did not respond.

In Section F students were asked to choose the division questions which would result in a quotient greater than one. Two types of division notation were included and students were able to make more than one choice. Table 2 contains the responses chosen by students in the pretest and posttest. Only 11 students correctly chose $4.7 \overline{)6.01}$ and 3.2 ÷ 1.25. Three students thought

$5.1 \overline{)4.75}$ would result in a quotient greater than 1, and five students thought 7.6 ÷ 8.3 would yield similar results. The role of decimal fractions cannot be discounted, but was more likely a factor in $5.1 \overline{)4.75}$ than in 7.6 ÷ 8.3 because of the inclusion of hundredths in the former question.

Table 2
Section F Responses: Which quotients are greater than one?

Question	Pretest	Posttest
$5.1 \overline{)4.75}$	3	0
3.2 ÷ 1.25	11	19
0.995 ÷ 1.1	7	2
$4.7 \overline{)6.01}$	11	19
7.6 ÷ 8.3	5	3

Although writing problems for open sentences was expected to be a difficult task for students, it was not expected that they would reverse the terms in questions in which the divisors were less than the dividends. In Section G, students were asked to write story problems for these division questions: 12.5 ÷ 9, 15.2 ÷ 3.5, 4 ÷ 20, 19 ÷ 3, 1.6 ÷ 2.4, 3 ÷ 0.6. Four of the six questions had dividends which were greater than their divisors, yet reversals of terms occurred in all questions except 12.5 ÷ 9. As few as three and as many as nine students reversed the terms in the questions 19 ÷ 3, 15.2 ÷ 3.5, and 3 ÷ 0.6.

Classroom Investigation. To determine if problems of this nature could be attributed, wholly or in part, to confusion with division notation, the researcher had students write the symbols for examples given in words. Using the a ÷ b notation, students were asked to write the symbols for the words "twenty-seven divided by six." Students were then asked to write a similar problem using the $b \overline{)a}$ notation. All students were able to use the first type of notation correctly, but three students reversed the terms in the second example. In a class discussion, the researcher encouraged the students to reflect on why this type of error might happen. It

was revealed that all students wrote the $b\overline{)a}$ notation from left to right which does not parallel the actions implied.

Instructional Decisions. The researcher decided to approach students' confusion with division notation by having them write the symbols in the order which does parallel the action: the dividend first, followed by the division symbol, and then the divisor. Students were asked to first write what was being divided, then write the "$)\overline{}$" symbol while thinking "divided by" or "shared among." Finally, they were to finish with the number that determined the size of the chunk (quotitive situation) or the number of groups (partitive situation). This method was modeled consistently by the researcher throughout the instructional program and was adopted by students to differing degrees; some used it exclusively, others used it only when the numbers involved were decimal fractions less than one. In addition to the short-term specific focus on notation, the researcher stressed the language of division in each lesson. Partitive situations were commonly described as "sharing" or "dealing out" and quotitive as "groups of," or "chunks of" in measurement settings.

Posttest Results. Interpretation of division notation was checked specifically in only one question in the posttest. Twenty-one of twenty-two students were able to state that $4.5 \div 3.1$ does not equal $4.5\overline{)3.1}$. Fewer students chose the alternate division open sentences for story problems, and a greater number of students were able to identify division questions which would result in a quotient greater than one. It is difficult, however, to attribute these results solely to a clearer understanding of division notation. Other factors such as representation of decimal fractions and beliefs about the relative size of the dividend and divisor may have had an influence on the outcome.

Classroom Results. Based on student responses in class discussions the researcher was confident in most students' ability to consistently interpret correctly the two forms of the division notation, $a \div b$ and $a\overline{)b}$. They were able to take division situations presented orally and represent them using the two forms of notation. Students were also able to "catch" the researcher when the terms in a question were deliberately reversed. Two students continued to

have difficulty with interpretion of notation, and received additional instruction and practice in this area. In the section which follows, students' understandings of the models and actions associated with division are explored.

Interpretations of the Actions Associated with Division

Two intuitive models are associated with division, the partitive model and the quotitive model. Grade 7 students in this study had varying degrees of familiarity with these models. In the section which follows, students' ability to use manipulatives to demonstrate the actions associated with these models, and their interpretation of these models are examined.

Pretest Results. Students' interpretations of the actions associated with division can be inferred from two sections of the pretest. In Section D students were asked to choose the open sentence which would help them solve a story problem. Each set of distractors included all operations and some involved two steps. The number of correct answers for each of the 10 questions in this section ranged from a low of 3 to a high of 20 as detailed in Table 1. While the most commonly chosen distractor was the inverse division open sentence, students frequently chose distractors containing multiplication.

When asked to write story problems for division open sentences in Section G, students had extreme difficulty with five of the six questions (See Table 4). The number of correct answers per question in this section ranged from a low of 1 to a high of 15. There was a total of 41 correct responses out of a possible 132 (22 students x 6 questions). Generally, students were neither able to describe the actions implied in a story problem as an open sentence, nor create scenarios which reflected the division actions described by open sentences. They did not appear to have the the ability to move from the symbolic to the concrete and vice versa.

Classroom Investigation. To further investigate understanding of division in the instructional program, students were asked to use base 10 blocks to model the actions implied by two questions: 68 ÷ 12 and 82 ÷ 13. Although students clearly understood which number was

the divisor and which was the dividend, none was able to model the question solely by using manipulatives. Student interpretation focused on the procedures associated with the algorithm. These examples were taken from the discussion following the modeling.

Dorothy: I rounded 12 to 10 and 68 to 70. Then Darin said it would be about 7. I worked out 7 times 12. Then I used blocks. I took the 12 and separated 68 into groups of 12.

Sue: I asked Geoffrey, "How many of these (12 blocks) goes into 70 of these (blocks)?"

While many of the students reported that they used blocks to model the questions, like Sue, they often made one term with blocks, made the other and solved the problem using pencil and paper. They did not independently choose the number of blocks needed to represent the dividend and use quotitive or partitive actions to determine the quotient. Sue's use of the term "goes into" was representative of the language used by most students. This also signalled the tendency of students to rely on procedure, and on a left-to-right interpretation of division. The apparent lack of ability to see the actions associated with division was supported when students were asked to create story problems which would match the questions. A selection of student examples follows.

Amy: We said there were 82 apples and 13 people. We took out groups of 13.

Stan: A man went to the market and he had $68.00. He wanted to buy 12 pies. How much money did he have left?

Annie: A man had $68.00 and he wanted to buy 12 packs of potatoes. How much was each pack of potatoes and how much did he have left over?

Amy created a partitive question, then talked about solving it using quotition. She did not appear to be able to visualize the actions involved and describe them. Stan and Annie created questions which focused on the fact that there would be a remainder in the division, but paid no attention to the purpose of the division, and both of their problems required additional information. It seems as if their responses were motivated by a desire to meet the researcher's request by

simply creating a problem which contained the necessary numbers rather than by their understanding of division.

Instructional Decisions. The researcher used a number of strategies which proved helpful in assisting students to develop a more complete understanding of division. These consisted of teaching students directly about the partitive and quotitive models of division; consistently relating questions to story problems; having students write story problems for division questions, using models, pictures and diagrams wherever possible; and using a "word frame" to translate problems into open sentences.

After introducing partitive and quotitive division through modeling division of whole numbers with base ten blocks and having students demonstrate their understanding of the actions involved, students worked in pairs to classify story problems according to type and later wrote story problems to reflect the two types of division. The researcher stressed the language associated with each type. Partitive was sharing between or among, and often was referred to as the "dealing out" of objects into groups. Quotitive was taking out groups of a predetermined size. Student understanding of the partitive model was strong; all pairs were able to generate a story problem which reflected sharing. Quotitive division was more difficult. Only three pairs of students were able to write whole number story problems which were based on taking out groups of a specified size. The failure of the remaining students to write quotitive problems, even after modeling, practice, and discussion, underscored their reliance on the partitive model. Increasing the group size to four students and directing them to reach consensus on problems resulted in greater opportunities for students to challenge each other and justify their thinking. All groups were able to create quotitive story problems in this setting. The researcher's choice of the term "dealing out" to represent the action in partitive questions proved to be somewhat of a hindrance. Students focused heavily on the one-at-a-time aspect of card dealing. In any problem which could have been solved by assigning more than one object to a number of people or things, students tended to believe it was a quotitive problem. In a partitive problem which

had two Grade 7 students giving their hockey cards to a number of Grade 3 students, students explained why they believed the problem to be quotitive.

Annie: I think it's quotitive. If you put the amount of cards together you have a large number, then give the cards out, like five to each person. Then see how much you have left, then keep on going.

In a quotitive problem which concerned the number of cartons necessary to pack 336 eggs, one student reasoned through his answer this way:

Stan: I put partitive. You can't pick up 12 eggs at once. You have to fill it one at a time.

Students adopted the familiar notion of card dealing and interpreted it to mean that it could only involve groups of one. It was necessary for the researcher to attempt to correct this problem by returning to the card model and discuss how cards might be dealt out more than one at a time. The base 10 blocks were again used to show how in partitive division it is the number of groups that is predetermined, and that in quotitive division it is the size of the group or chunk that is predetermined.

Another strategy which was used to help students build their conceptions of division was the use of diagrams to represent division questions and story problems. After a number of lessons in which base 10 blocks were used for showing division with whole numbers and decimal fractions, students were introduced to the use of 10-by-10 grids to represent division of decimal fractions by whole numbers and by decimal fractions. Questions such as $2.8 \div 4$ and $2.8 \div 0.4$ were demonstrated for the class by the researcher and the students. Sections of the grids were shaded to show the number of sections of a specified size. This visual aid was particularly useful when the divisor was a decimal fraction because students had difficulty estimating with numbers less than one. Before students were introduced to the algorithm for division they worked exclusively with pictorial representations until their estimation skills enabled them to predict a reasonable solution. When the algorithm was introduced, it was presented side-by-side with the grid model.

Some students had difficulty understanding division questions not set in a context. This was especially true when decimal fractions were involved in at least one term. When asked to provide estimates for questions such as 15.5 ÷ 9, a small number of students were unable to do so until the question was rephrased as a story problem. Helping students represent the question by creating a context for them seemed to enable them to visualize the question and formulate an estimate. For this reason students rarely worked on division exercises in isolation. When they did, part of the assignment generally included the creation of problem settings to fit the exercises. The problems written by students were often discussed in small groups or with the class as a whole. Students offered advice on improvement, pointing out aspects of story problems which did not make sense. Selected student problems were often used as the warm-up activity, as the basis for discussion, or were part of the day's assignment. The use of student generated problems made their work more motivating and meaningful because they had a sense of ownership and were familiar with the problem contexts.

A final strategy which was used to assist students determine the actions involved in story problems was the use of a word frame. This strategy involved substituting words or phrases from the problems for numbers in the $a\overline{)b}$ notation. Students had to visualize the problem rather than focus on the numbers involved, and this proved difficult at first. The following excerpt from video-tape transcriptions illustrates how the frame was used.

Researcher: Four gophers are digging a tunnel that will be 5.2 km long. If they dig 0.35 km in one day, how many days will it take them to finish? Without any numbers, what do you have to divide? Let's put some words in. I want three words or phrases to go in these blanks.

#3

#2 _____) #1 _____

What do we want to divide?

Justin: Metres by metres.

Researcher: Maybe I'm not being clear. I don't want the units, I want the thing that's being divided. Like the number of cases of pop by the number of bottles.

Annie: 0.35.

Researcher: Sorry, I don't want the number.

Dave: How long they have to dig by how much they dig per day.

Researcher: How far in total by how much per day? (writes it in the frame)

#3

#2 how much per day) #1 how far in total

What will the answer give us in words, then?

Mathew: The number of days.

Researcher: (Adds "number of days" to the frame.) Okay, now let's put the numbers in.

Elaine: 0.35) 5.2.

Researcher: Anyone with it reversed? (No hands were raised.) Approximate answer? 0.35 is close to one half. How many halves in 5?

Gary: About 10.

The word frame assisted students in accurately representing division story problems as division open sentences. Not only did it help the students determine the action of the dividend and the divisor, it assisted them in thinking about the function of the quotient because it required them to use words rather than numbers. When students completed the question, they had a contextual and numerical answer and were better equipped to consider the reasonableness of their response.

Posttest Results. On the posttest, students were better able to choose the appropriate open sentence for division story problems. Correct answers for the 10 questions ranged from a low of 9 to a high of 19 as detailed in Table 1. Again the most commonly chosen distractor was the alternate division open sentence, but this distractor was chosen less frequently . Distractors containing multiplication were chosen by only two students. In Section G, writing story problems, students improved in their ability to create contexts for open sentences. There

was a total of 63 correct answers out of a possible 110 (22 students x 5 questions). In the posttest students were less likely to write problems which required an operation other than division and reversed the terms of the division question less often (pretest reversals: 33 in 6 questions, posttest reversals: 22 in 5 questions).

Classroom Results. In the classroom setting, students were much more confident in their ability to both model division questions with manipulatives and represent story problems in pictorial and symbolic form. Students could identify and model partitive and quotitive division story problems. Students became more adept at writing story problems to reflect open division questions. Creating a partitive situation for a division open sentence was easier than creating a quotitive situation, and questions in which the divisor was less than the dividend were easier than questions in which it was greater. When the divisor was greater than the dividend, some students wrote problems which required subtraction or multiplication rather than division. Further investigations into the students' interpretations of the divisor and the dividend are detailed in the next section.

Interpretations of the Divisor and the Dividend

In surveying the literature on division with decimal fractions and whole numbers, two widely-held student beliefs surface: division makes smaller, and the divisor must be less than the dividend (Bell, Swan, & Taylor, 1981; Hart, 1981). It is not surprising that students' understandings of division centre on these two beliefs; they are grounded in the early experiences of division with whole numbers where these beliefs hold true. It is not until the introduction of numbers less than one in the intermediate grades that reliance upon these whole number understandings causes difficulty for students. At that time students begin to modify and distort questions and procedures to fit their framework for division. What follows is a description of the Grade 7 students' understandings of the roles of the divisor, the dividend and the quotient in both whole number and decimal fraction settings.

Pretest Results. In the pretest results, evidence can be found to show that students believe the divisor should be less than the dividend. Section G required the students to write story problems for division open sentences. Two of the six questions had divisors which were greater than the dividends: 4 ÷ 20 and 1.6 ÷ 2.4. In the first example, 4 ÷ 20, 15 of 22 students wrote story problems which called for a solution by reversing the numbers. In the second example, 1.6 ÷ 2.4, only three students were able to write a story problem which reflected the question. A variety of other attempts to accomodate the problem were made, including reversing the numbers; using addition, subtraction or multiplication; and distorting the information. Two students made no attempt at the problem. The examples below illustrate a sampling of student responses.

4 ÷ 20: If you had 20 dollars and you wanted to divide evenly between four people, how much would each person get?

1.6 ÷ 2.4: 2.4 yards of string must be divided into 1.6 yard lengths. How many lengths will there be?

1.6 ÷ 2.4: If you had $2.40 and you wanted 1.6 grams of flour at 50¢ a gram, how much is left over?

In Section E students were first asked to chose two numbers from a selection of five which would result in the largest possible quotient when divided. They were then asked to choose two numbers which would result in the smallest possible quotient. Table 3 shows the open sentences students chose in the pretest and posttest. Ten students gave the correct response for the first question (12.6 ÷ 0.73). The remaining students wrote a variety of open sentences which had divisors less than the dividends. In the second question, only three students gave the correct response (0.73 ÷ 12.6) and the remaining 19 students wrote open sentences that had one commonality: all were structured so that the divisor was less than the dividend.

Table 3

Section E: Open sentences resulting in the largest and smallest possible quotients

Open Sentence	Pretest	Posttest
Largest Possible Quotient		
12.6 ÷ 0.73	10	20
12.6 ÷ 1.291	1	1
12.6 ÷ 5.78	2	0
1.291 ÷ 0.73	3	0
12.6 ÷ 9.4	4	0
9.4 ÷ 6.25	1	0
1.291 ÷ 6.25	1	0
0.73 ÷ 12.6	0	1
Smallest Possible Quotient		
0.73 ÷ 12.6	3	19
1.291 ÷ 12.6	1	1
0.73 ÷ 1.291	1	0
9.4 ÷ 12.6	0	1
6.25 ÷ 5.78	1	1
12.6 ÷ 9.4	3	0
1.291 ÷ 0.73	7	0
12.6 ÷ 6.25	1	0
5.78 ÷ 0.73	1	0
12.6 ÷ 1.291	1	0
9.4 ÷ 0.73	1	0
12.6 ÷ 0.73	2	0

When presented with questions where the divisor was greater than the dividend, students reacted in a variety of ways. In a computation question in Section B, students grappled with the question $1.5 \overline{)0.45}$. Some ignored the decimal and divided. Three students made no attempt at all. One student thought the answer would be -1.05, and another responded with 0.00 r45. When examining the story problem written for 1.6 ÷ 2.4 in Section G, a similar response pattern emerges. Only 3 of 22 students were able to correctly represent the problem. Some chose an alternate operation, a few reversed the terms, and the remainder either distorted the problem information or gave no response at all.

Students were asked to choose the correct open sentence for division story problems in Section D. In two of these questions, #9 and #10, a considerable number of students chose distractors which were reversals of the correct division open sentence (see Table 1). The two problems, "Paint" and "Race," called for the division of one decimal fraction by a larger decimal fraction.

Paint: An artist wants to use some very expensive paint on a series of sculptures she has made. If she had 0.27 of a litre of paint and 30 sculptures, how much paint can she use on each?

Race: A cross-country race was run over a course that was 8.4 km long. If the first runner completed the race with an average speed of 11.2 km per hour, how long did it take him to finish the race?

In "Paint," 18 of 22 students chose the distractor $0.27 \overline{)30}$. In "Race," 10 students chose $8.4 \overline{)11.2}$. Although there is some question of students' understanding of the division notation as reported earlier, the large number of students choosing the reversed division distractor suggets there is more than confusion with notation influencing responses. When this evidence is combined with evidence from the other sections of the test, it appears students were not comfortable dividing either whole number or decimal fraction dividends by divisors greater than those dividends.

It might be surprising, then, to find that in some instances not only were students prepared to divide by a divisor greater than the dividend, they reversed the terms of a division question in order to do so. In Section D, choosing an open sentence for division story problems, the problems "Tunnel" and "Cord" implied the division of a whole number by a decimal fraction less than one.

Tunnel: A new tunnel is being dug at the Strike-It-Rich gold mine. In one day the miners are able to tunnel 0.275 km. At this rate, how long will it take them to complete a 2 km tunnel?

Cord: Karen is working on a macrame project that requires pieces of cord that are 0.85 m long. If she has 15 m of cord to cut into pieces, how much cord will she have left over when she is finished cutting?

In "Tunnel" 3 of 22 students chose the distractor 0.275 ÷ 2. In "Cord" 7 of 22 students chose $15\overline{)0.85}$. In each of these problems the divisor should have been less than the dividend, the divisors being decimal fractions less than one and the dividends being whole numbers. At least two possible explanations exist for this apparent change in student thinking: these reversals could have been influenced by the students' desire to divide by a whole number, or by the belief that the decimal fractions in these cases were in fact "larger" than the whole numbers because they appeared larger when the decimal point was ignored. Students' preference for whole number divisors and their tendency to ignore decimal points reflects their whole number, partition model for division.

A third possible explanation for reversing terms in division questions such as these is that some students believe division by a decimal fraction less than one is impossible. Section B of the pretest contained division exercises involving decimal fractions. In the one question where students were asked to divide by a decimal fraction less than one (2.282 ÷ 0.7), three students made no attempt. In Section G, writing story problems, the one question which involved division by a decimal fraction less than one caused students the most difficulty. Only 1 of the 22 students

was able to create a story problem for 3 ÷ 0.6; a number resorted to other operations, some reversed the terms, and one student was unable to make any response.

Classroom Investigation. Analysis of the pretest results indicated students had difficulty to differing degrees with the role of the divisor, dividend, and the quotient in division of whole numbers and decimal fractions. Although possible reasons for these difficulties could be hypothesized from evidence in the literature, it was during the course of classroom investigation that the thinking and reasoning of this group of Grade 7s became clearer to the researcher.

In the study, the students' first exposure to a situation in which the divisor was greater than the dividend came when the researcher introduced division of a decimal fraction by a whole number using base 10 blocks. The use of physical models enabled students to see that a division such as 1.09 ÷ 3 was possible. When using the blocks to complete exercises or to solve story problems involving division by a whole number, most students did not reverse terms or resist in any way the division of one number by a larger one. In a limited way, they were able to relate story problems to exercises such as 0.18 ÷ 3:

Anthony: You have 3 cities and 0.18 megatonnes of sludge. How much sludge does each city get?

Annie: There are 3 kids and they wanted to share out 0.18 of something. How much does each get?

Geoffrey: There are 0.18 million people in America. The Russians invade and put them in concentration camps. How many in each camp?

Jaylene: There are 3 children. One looked in his pocket and found 18¢. How much would each get?

Anthony created a problem in which 0.18 could be turned into something large enough to share among three. Annie could see a potential setting for the problem, but could not relate 0.18 to anything in her experience. Geoffrey realized the inaccuracy of his figures, but created

a problem which matched the division. Jaylene related 0.18 to money, one of the most common representations for decimal fractions amongst the group.

Difficulties became evident when the divisor was a decimal fraction or when problem solving was involved. When students were presented with a computation question in which the divisor was greater than the dividend and no concrete or pictorial aids were available, they made attempts at solution but were not generally successful.

When presented with the question $0.7 \overline{)2.8}$, which was similar to one students had done on the previous day ($7 \overline{)2.8}$) many fascinating beliefs emerged. The understanding which had seemed evident with the base 10 blocks vanished. Students began to rely solely on procedure, drawing on the rules they associated with division of decimal fractions. In the following excerpt from the video-tape transcription, four students, Geoffrey, Justin, Darin, and Anthony, explain their thinking regarding the division of 2.8 by 0.7. Other students comment and question the thinking of their peers.

Geoffrey: You can take any number in the world, it can have a decimal or not, once you times it by 10, you move the decimal point over 1. Say you have 32 and times it by 10, you get 320, right? So you move the decimal over one.

Researcher: So you're saying you multiply that (32) by 10, you get 320. You move the decimal points here (32.0 x 10 = 320.0)

Geoffrey: Okay. So in 7 ...

Researcher: So with 0.7, all you're doing is...

Joani: Shifting it over one to the other side.

Researcher: (Writes on the board.) 0.7 x 10 = (0)7.0

Joani: Yeah, but Geoffrey, if you times that by 10 at the start of the equation, what do you do at the end? Do you divide the answer by 10 at the end of the question?

Geoffrey: I think you divide by 100.

Researcher: Do you understand Joani's question? She says if you multiply these numbers by 10 at the beginning, then isn't the answer that you get going to be too large? Reaction?

Darin: No. Well at the end you just put the decimal points back in.

Students: But where do you put them?

Researcher: Do you want to follow that through?

Darin: Well, after you times it ($7\overline{)28}$) by 10 and you've done the question, you put the decimal points back in and you move the decimal point up.

Researcher: Tell me exactly where I should put the decimal points.

Darin: In between the 2 and the 8. And then back in front of the 7. And then you move that decimal point up in front of the 4.

Researcher: So this is what I'd want here? $7\overline{)28} = .7\overline{)2.8}$ (.4)

Gary: You got it!

Geoffrey: No.

Researcher: Have a look at it. Think about it. Then I'd like to hear some reactions. That's an interesting way of thinking it through.

Gary: Oh no! That's wrong.

Justin: Isn't there two numbers after the decimal points?

Researcher: Yes, in the original question.

Justin: Then the answer is 0.04.

Researcher: Instead of the 0.4, you think we should be seeing 0.04?

Justin: Yeah.

Researcher: Why?

Justin: There is one number behind the decimal point here (.7) and one there (2.8) ...

Researcher: So in the quotient there should also be two numbers behind the decimal? That's

interesting. Okay. So we've seen three people's ways of walking through that and we've got three different answers, don't we? One thing we're all sure of - there's a four in there someplace!

Anthony: I agree with Justin because with the whole 7 dividing 2.8, the answer is 0.4, so that's 0.7 dividing 2.8 so you can't get the same answer.

Researcher: So 2.8 ÷ 0.7 couldn't give you the same answer as 28 ÷ 7? Because they're different numbers?

Anthony: And they're not whole numbers.

Researcher: How many people think Anthony's logic sounds good there? (About 12 hands) So we've got about 12 people who agreed 28 ÷ 7 couldn't give you the same answer as 2.8 ÷ .7. Okay, thanks.

Marian: I agree with Geoffrey 'cause .7 is less than one so if you multiply that 4 whole, you would get about 2.8.

Geoffrey: Yeah, how do you ...

Researcher: Would you like to go over that again?

Marian: You can check it by multiplying 4 wholes by 0.7.

Researcher: The number 4, which has an invisible decimal there (4.0), by 0.7?

Joani: That's wrong!

Kate: That's right!

Researcher: If you want to multiply that through?

Marian: 4 x 7 is 28 and there are two numbers behind the decimal place (4.0 x 0.7) so it's 2.8.

Geoffrey: Ta da!

Researcher: Ta da? So 4 wholes times 0.7, a decimal fraction, can give you a decimal fraction?

(General agreement from the students.)

Researcher: What does that do to what Anthony just said a minute ago?

Geoffrey: Destroys it!

Marian: Blows it away!

Researcher: You think it destroys it, but what about the people who agreed with him?

Annie: That's just another way to do it.

Students: No .... What?

Joani: You can't get more than one answer.

Researcher: Well, I think a lot of people think in math there is only one answer, and it's not necessarily true. Annie, what do you think here? It's just a different way to do it?

Annie: It's just a different way to do it and get the same answer.

Researcher: Do you get the same answer? Darin's is .4, or 4 tenths, Geoffrey's is 4, or 4 ones. Is it all right, Annie, to get those two answers?

Annie: But Geoffrey made a mistake. There are two decimal points and he has to move them over. You can't ignore them.

Researcher: (Reviews how Geoffrey multiplied each side by 10, moving them over.)

Annie: I still agree with Anthony.

Researcher: New voice? Amy.

Amy: You moved the decimal points over by times 10, but the answer, you didn't move it back over.

Students: Yeah!

Researcher: Where do you think I should move it?

Amy: In front of the 4.

Researcher: 0.4 Because I've multiplied this (divisor) by 10 and this (dividend) by 10, I should move this (decimal in the quotient) back over?

Students: Yeah!

Amy: 'Cause it's not 28, it's 2.8 so you've got to move it back.

Researcher: We multiply these two by 10, so we have to divide the answer by 10?

Amy: Move it back.

Researcher: That's what moving it back is, isn't it? Divide by 10.

Dorothy: I agree with Amy's logic. Because he did 28 ÷ 7, not 2.8 ÷ .7, so I would say it's 4.

Gary: I think Geoffrey's logic is good, 'cause when you multiply it, that will be your answer. That's what we were taught a long time ago.

Researcher: When you check a division by multiplying?

Mathew: I agree with Darin.

Researcher: Take them out of there, do the dividing, put them back in, bump them up.

Mathew: Yeah.

Geoffrey seems to have a good understanding of the procedures involved in moving the decimal points when dividing decimal fractions, and can explain how this is related to multiplying by a power of 10. When Joani questions him about the need to then divide the quotient by 10, Geoffrey becomes unsure about his response and cannot reason through that there is no need to alter the quotient. He does not see that the 10 in the divisor and the 10 in the dividend will also be divided. In fact, he sees the effect of these tens as having increased the quotient by a factor of 100. For Darin, the situation is very simple. With total disregard to place value and the meaning behind the division, he first ignores the decimal point when dividing, then positions it as one might when adding or subtracting decimal fractions. His logic is simple, and appeals to some of his fellow students. Justin applies the rules he associates with multiplication of decimal fractions to division. Again, the place value of the numbers involved is lost through the application of a procedure for which the student has little understanding. Anthony agrees with Justin, then takes the reasoning one step further. He sees that 2.8 ÷ 7 cannot equal 2.8 ÷ 0.7, but thinks the answer to the second question should be smaller than 0.4 because the divisor is smaller. When asked if 2.8 ÷ 0.7 could yield the same answer as 28 ÷ 7, Anthony replied that it could not because the two sets of numbers were different, and the first set was not whole numbers. With just over half of the class agreeing with Anthony, it was clear

that many students could not see the relationship between the two sets of numbers, and that they perhaps viewed whole numbers and decimal fractions as being two separate and unrelated systems.

What is most striking about this exchange of ideas amongst the students is the procedural base of the discussion. At no point did a student refer back to base 10 blocks or any concrete situation which might have described the example. From Geoffrey's multiplying by 10, to Darin's disappearing and reappearing decimal point, students talked about the question only in abstract terms. Whether they were unable to relate the question to a concrete context, or whether they did not see the relevance of doing so was not clear at the time.

In problem-solving situations, students sometimes seemed to have a clear understanding of the roles of the divisor and the dividend. At other times, however, they resorted to reversing the terms. In an example from Lesson 10, students explain how they would approach this problem: A stack of quarters is 6.3 cm high. If each quarter is 0.15 cm thick, how much money is in the stack?

Joani: I'm going to write out my equation then divide. Either 6.3 ÷ 0.15 or 0.15 ÷ 6.3.

Researcher: So you think it's a divide question?

Joani: Yeah.

Researcher: Are you sure?

Joani: Yeah.

Marian: It's two steps. First you divide, then multiply.

Joani: First I'd write out my equation.

Researcher: Equation's important?

Joani: Well, you got to know what you're doing before you can do it.

Researcher: Tell me what to do.

Joani: 6.3 ÷ 0.15

Researcher: Are you sure it goes that way? A minute ago you said you had to decide if it went this way or that way.

Joani: Is this a trick question?

Researcher: No.

Joani: You need to see how many times 0.15 can divide 6.3.

Stewart: You can draw a picture. (He describes a stack, cut up.)

Researcher: What am I doing with this, like we did with the grid paper? How is this problem similar?

Kathryn: Dividing it into groups of 0.15 cm.

Researcher: When I do that, what will I find out?

Joani: How many quarters you have.

Researcher: And then we'll be finished?

Students: No.

Researcher: Tell me why.

Darin: You'd know how many quarters, but not how much money.

The students appeared to have a clear understanding of the problem, could explain a way to approach its solution, and could justify their thinking. At first Joani seemed to be unsure of the positioning of the two terms, although she was committed to division as a course of solution. The contributions of other students added to the scenario, clarifying the action and the purpose of the action. A few days later during Lesson 12, the students were discussing how to solve a problem written by Joani. It read:

> You have 22 pairs of jeans and after you shorten each pair, there's 4.5 metres of cloth left over. About how much did each pair lose?

The understanding demonstrated earlier no longer seemed to be evident in the students' thinking concerning this problem.

Researcher: How would you find the answer to this problem?

Geoffrey: I divided 22 by 4.5.

Researcher: And what would be your answer?

Geoffrey: About 4 or 5.

Researcher: 4 or 5 what?

Geoffrey: Metres.

Researcher: Oh?

Geoffrey: No, not really.

Researcher: Comments on Geoffrey's answer.

Marian: 4 or 5 cm.

Researcher: 4 or 5 cm because it doesn't make sense to cut off 4 or 5 metres?

Sara: 4 or 5 tenths?

Researcher: Interesting! Can you visualize what you are doing here? What's the quantity that you have to be shared out?

Kate: 4.5

Researcher: 4.5 metres of cloth. (Draws a diagram of a piece of cloth 4.5 m long.)

Joani: Like you sewed it together to make a quilt.

Researcher: What's the task in the question?

Kathryn: I'd move the decimal and ...

Researcher: I want you to forget the question ($4.5 \overline{)22}$) and focus on the drawing. We have 4.5 metres of cloth... what do we want to know?

Stan: You want to find out how many cm or dm go into, or were taken off each pair.

Joani: I thought that problem was easy!

Researcher: How come we're having so much difficulty?

Students: It's the numbers.

Reearcher: There's a problem here, folks. A huge problem. I don't think anybody's seeing it. You want to take this 4.5 m and find out how much of that came off one pair of jeans. Think about the grid paper. Now I want you to look at the story Geoffrey was telling me. This ($22 \div 4.5$) says what?

Students: 22 pieces divided among 4.5.

Researcher: Is that the same story as this story (the picture)?

Students: No.

Researcher: What's the big problem here, folks? (About half of the hands go up.)

Gary: It should be $22 \overline{)4.5}$

Researcher: If we read the story this way: 4.5 metres of cloth ÷ by 22 pairs of jeans, we'd be able to find our answer. What have you got that you are trying to divide up?

Students: 4.5 metres.

Researcher: How many chunks are you trying to divide it into?

Students: 22.

All students in the class were convinced the solution to this problem required dividing 22 by 4.5, including the problem's author. When the answer of about 4 or 5 was obtained, students struggled to make it fit by changing the units from metres to centimetres or decimetres. Prompting through the drawing of a diagram did not alter the thinking; students were committed to the division of the larger number by the smaller number and were only able to see the reversal once it was pointed out to them.

Students' whole number framework for division resulted in another type of confusion. In the computation question, $0.04 \overline{)6.0}$, students were generally able to calculate the answer but often could not give a reasonable estimate. The majority of estimates for this question were between 100 and 200, but included responses such as 2 and 24. Three students were unable to give any estimate at all. It appeared as if some estimates had been changed on the student worksheets after the calculations had been completed. Although students experience discomfort with this type of division in a computation setting, it is when students are free to make decisions in the problem-solving setting that this difficulty has the most impact.

When presented with a problem which required the division of a whole number by a decimal fraction less than one, sometimes students reversed the terms to enable them to divide by the whole number. In one lesson, students were given this problem: A sheet of paper is 0.075 cm thick. About how many sheets would it take to make a stack 5 cm thick? Rather than divide a whole number by a decimal fraction less than one, some students reversed the terms

obtaining an answer of 0.015 and then attempted to interpret this. Others avoided the division by guessing and checking through multiplication. Their unwillingness to divide a whole number by a decimal fraction less than one led them to either choose an alternate form of solution, or to try to accomodate an incorrect answer. It should be noted that students most often made this mistake when the digits in the decimal fraction appeared to be a multiple of the whole number, as they are in the example above. Students were asked to rate the problem as easy, just right, or difficult. None rated it easy, six rated it just right, and 16 rated it difficult.

Instructional Decisions. The transition from division of whole numbers to decimal fractions seemed to be most logically approached by maintaining a whole number divisor and introducing a decimal fraction dividend. This allowed students to use their familiar whole number strategies while exploring the use of base 10 blocks to represent decimal fractions. Division of decimal fractions less than one seemed logical and feasible to students because they could separate tenths, hundredths and thousandths into groups. Paralleling the actions with the algorithm made a smooth transition from concrete to abstract. Students were clearly able to divide decimal fractions by whole numbers, in both concrete and symbolic situations. Whole number and partitive rules for division hold true in this case, and it was upon this knowledge that students built their framework for dividing all decimal fractions.

The researcher chose to then introduce division by a decimal fraction less than one using a question almost identical to one used previously with the exception of the divisor: instead of the whole number 7, the decimal fraction 0.7 was used. Using place value and logic, the researcher intended to guide students through the process of estimating an answer, then modeling it with materials. When the topic of how to approach such a question was opened up to the students, their responses were entirely procedural in nature and reflected nothing of their experience with concrete materials. Although the researcher had expected some difficulty when dividing one decimal fraction by another, the complexity of the confusion and the extent to which it was shared by a large majority of students was surprising. Students required a common language through which they could relate to questions of this nature so, using quotition, division

by decimal fracions less than one was shown through base 10 blocks, 10-by-10 grids, and measurement situations. Students worked primarily in small groups, explaining and justifying to a partner or to the class in demonstrations. These hands-on experiences served to give students a concrete base for their division work, and proved to them that division by a number less than one was in fact possible.

Although students may have been convinced that division by a very small decimal fraction was possible, they continued to resist doing so. Relating problems to a similar one already solved by students seemed to be of assistance. In a first attempt at this problem, students neither reversed the terms nor multiplied; they had few suggestions about how it might be solved:

A laser beam removes 0.004 mm of the surface of a piece of metal on each pass of the beam. The metal is 1.2 mm thick. How many passes would it take to have no metal left?

When a comparison was drawn between this problem and the "stack of quarters" problem described earlier, students were able to see the similarity in the actions of the problem and the numbers involved and could then justify dividing by 0.004. Their ability to represent problem situations seemed to be fleeting in nature, and they often required prompting or assistance when the numbers involved were decimal fractions, or the divisor was greater than the dividend.

Overcoming the tendency to want to divide the larger number by the smaller number required much work. Although only a few students initially believed it impossible to divide one number by a larger one, students subconsciously resisted doing so. Introducing decimal fractions, especially those less than one, compounded the difficulty. Division of a number less than one had no meaning for some students. Encouraging the use of models, diagrams and pictures helped provide a context for such divisions. Writing story problems to reflect division questions of this nature, in conjunction with the use of the word frame to analyze similar story problems provided by the researcher, enabled students to begin to develop a set of experiences they could draw upon. This excerpt from the lesson transcriptions illustrates how students used the word frame to help analyze problems which required division of a decimal fraction less than one by a larger number.

Researcher: Okay. Now the gold dust. There's 0.85 grams and 7 ornaments. If it's to be distributed evenly, how much will each ornament get? What are you dividing, and what are you dividing it by? Let's put those words inside.

#3
#2_____) #1_____

I'll give you a few minutes to try it in your book. Words only, no numbers. (A few minutes later.) What have we got that we're dividing?

Dorothy: How many grams of gold dust.

Researcher: What are we dividing that by?

Jaylene: Number of ornaments.

#3
#2 ornaments) #1 gold dust

Researcher: Okay, so what? What do we know?

Darin: How much gold dust will be on each ornament.

Researcher: Great. Now let's translate that into numbers.

Kathryn: 0.85

Researcher: Divided by?

Geoffrey: 7.

In group discussion situations when thinking was closely monitored, students were able to correctly interpret division situations such as this one. Their attention was focused on the objects in the problem rather than on the numbers involved. In an example mentioned previously, the jeans problem which required the division of 4.5 by 22, students did not independently employ the kind of thinking exhibited above. This served to remind the researcher of the lasting nature of students' conceptions, and signalled the need to return frequently to similar situations to help strengthen students' ties to this new experience.

Posttest Results. Change in students' interpretations of the role of the divisor can be inferred from a number questions in the posttest. In Section G, writing problems to match division open sentences, there were two questions with divisors greater than dividends: $6 \div 36$

and 1.6 ÷ 2.4. The results from this section of the pretest and posttest can be found in Table 4. In the whole number question, over half of the students were able to create story problems. A number reversed the terms or distorted the problem. Although not ideal, this was a considerable improvement over the pretest scores. In the decimal fraction question, eight students created corresponding story questions, and the remainder of the students reversed the terms, had difficulty with decimal representation, or distorted the numbers. Again, this was an improvement over the pretest scores, especially with respect to the number of reversals. All students attempted the problem, and the majority of difficulties centered on the students' inability to represent the decimal fractions. The examples below illustrate the four types of responses:

1. Correct

   If you have 1.6 ml of food colouring and you want to share it among 2.4 kg of batter, how much food colouring does each kilogram of batter get?

2. Decimal Representation

   There are 2.4 pepper shakers that are full and in them there is 1.6 g of pepper. How much does each one hold?

3. Reversal

   2.4 yards of string must be divided into 1.6 yard lengths. How many lengths will this be?

4. Distortion

   Sandy wants to make 2 puppets. She has 2.4 metres of fabric and 1.6 meters of paper to decorate. How much fabric would she need for each puppet, and how much paper?

If the students who had difficulty with decimal representation are included with those who were able to write a story problem for the question, then 18 of the 22 students could be considered to have accurately reflected the action of the division.

In Section E, choosing two numbers which would result in the smallest possible quotient, 19 students correctly chose 12.6 ÷ 0.73 (See Table 3). Two of the three remaining students

wrote questions which had divisors greater than dividend. The third student had the correct numbers but reversed the terms.

One computation question in Section B required the division of a decimal fraction less than one by a larger decimal fraction. Fifteen students were able to correctly divide $1.5\overline{).48}$; three students made multiplication or subtraction errors, and four made mistakes relating to place value.

Table 4
Students' Responses to Section G: Writing Story Problems

Question	Correct	Reversal	Operation	Decimal	Distortion	No Response
Type 1						
Pretest: 19÷3	15	2	2	-	1	0
Posttest: n/a	-	-	-	-	-	-
Type 2						
Pretest: 4÷20	3	15	4	0	0	0
Posttest: 6÷36	14	7	0	0	0	0
Type 3						
Pretest: 12.5÷9	11	0	8	2	1	0
Posttest:11.4 ÷8	22	0	0	0	0	0
Type 4						
Pretest: 15.2÷3.5	9	3	5	5	0	1
Posttest: 15.2÷3.5	14	2	1	5	0	0
Type 5						
Pretest: 1.6÷2.4	3	3	6	5	3	2
Posttest: 1.6÷2.4	9	2	1	10	0	0
Type 6						
Pretest: 3÷0.6	1	9	4	2	6	0
Posttest: 8÷0.7	8	11	0	3	0	0

Section D, choosing the correct open sentence for a story problem, two questions required the students to select answers with divisors greater than dividends:

Wax: A ski team wants to try using some very expensive wax on their skis just before an important race. If they have 0.97 of a litre of the wax and 20 pairs of skis, how much wax can be used on each pair of skis?

Swim: A swim race took place on a course that was 2.4 km long. If the first swimmer completed the race with an average speed of 3.9 km per hour, how long did it take him to finish the race?

The pretest and posttest results for these problems are detailed in Table 1. "Wax" is similar to "Paint" in the pretest. Although there was a dramatic increase in the number of students who chose the correct response in the posttest, twelve students preferred the distractor with the reversed terms. "Swim" compares with "Race" in the pretest. Again, more students were able to choose the correct option in the posttest. The relative success of students on "Swim" when compared with "Wax" may be attributable to the division of a decimal fraction less than one by a significantly larger whole number. It appears that students are not so reluctant to divide one number by a larger one if the two numbers are relatively close in size.

Two of the questions in this section of the posttest called for the division of a whole number by a decimal fraction less than one. "Highway" is similar to the pretest question "Tunnel," and "Fabric" is similar to "Cord":

Highway: A new extension is being made to a highway. In one day the crew is able to construct 0.575 km of the highway. At this rate, how long will it take them to complete the 8 km extension?

Fabric: Belinda is working on an art project that requires pieces of fabric that are 0.75 m long. If she has 13 m of fabric to cut into pieces, how much fabric will she have left over when she is finished cutting?

On the pretest, students were fairly successful with these two questions. Sixteen of 22 and 14 of 22 were able to solve Tunnel and Cord respectively. The majority of those who made errors

chose distractors with reversed terms, preferring to divide by the whole number. In the posttest, 18 of 22 had both Highway and Fabric correct. Highway had one reversal, two multiplications and one no answer. Fabric had three reversals and one multiplication. Most students were less reluctant to divide whole numbers by decimal fractions less than one, but this tendency remained for some.

Computation questions from Section B which involved division by decimal fractions less than one were handled well. Of the 22 students, 18 correctly calculated $0.003 \overline{)2.7}$ and 17 found the solution to $0.7 \overline{)2.28}$. All students attempted the questions with some making errors in multiplication and subtraction. Place value errors, the most common in the pretest, were much reduced. In Section G, writing problems for an open sentence with a decimal fraction divisor less than one was still difficult. Only six students were able to write a story problem for $8 \div 0.7$; 11 reversed the terms and the rest had difficulty with decimal representation. The examples below provide illustration.

1. Correct

   There is 8 litres of water in the gauge and I want to pour it into .7 litre containers. How many containers will be full?

2. Reversal

   There was .7 of a m^2 of bark mulch to be spread between 8 m. How much did each m get?

3. Decimal Representation

   Stewart had 8 g of gold and he wanted to split it evenly among .7 gold holders. How much does each gold holder get?

Students appeared to be willing to divide by a decimal fraction less than one but were not always able to create an appropriate problem-solving setting. Of the 11 students who reversed the terms, 10 created situations where the decimal fraction referred to a partitive situation where a quantity of goods was to be shared among eight objects or people, for example: 0.7 kg of gumballs and 8 paper bags. Those who were successful viewed the problem from a quotitive perspective, for example: 0.7 m pieces of fabric from 8 metres of fabric.

In the true or false items in the posttest, all 22 students recognized that it was possible to divide a number by a larger number. When asked if they should always divide the larger number by the smaller one, all 22 recognized that this was false.

Classroom Results. From their work in the classroom, students demonstrated growth over the instructional period. Initially they were not sure of the meaning behind division of decimal fractions, and had a limited understanding of division of whole numbers when the divisor was greater than the dividend. Although there were many different entry points, students consciously or subconsciously believed the divisor should be less than the dividend at the beginning of the study. Working with concrete materials, developing problem contexts and attempting to solve conflict-creating problems served to enable students to identify situations in which the divisor could be greater than the dividend and gave meaning to such situations.

Students developed their ability to both estimate and calculate division of whole numbers and decimal fractions by decimal fractions. Most students were able to round decimal fractions greater than one to the nearest whole number, and those less than one to either one half (0.5) or one tenth (0.1). Estimates could then be made, although somewhat inaccurately. Some students were not able to estimate by giving a specific number, but chose rather to say "closer to zero than to one." Computation was developed through the multiplication of both terms by a power of 10, and students quickly became comfortable with this procedure. Once students had demonstrated they were able to complete calculations up to the thousandths, calculators were used in conjunction with estimation. As in the posttest, students were very successful on computation items in the classroom setting.

Throughout the study students were asked to write story problems for open sentences. Here are a few of their responses to a question posed near the end of the study:

$6 \div 18$: There were 18 people and they went out for a glass of milk. There were 6 litres of milk. How much did each person get?

6 ÷ 18: Suzy was picking out things from her attic. She found a board 6 m long and she wanted to divide it up into 18 pieces because her project was to make a bird cage with 18 sides. How long would each board be?

However, when students were not consciously made aware of a division setting which required the divisor to be greater than the dividend they often subconsciously reverted to their previous belief and were then unlikely to be able to identify their error independently. An example of this type of behaviour is found in the problem presented earlier where students were attempting to solve Joani's story problem about the amount of material cut from 22 pairs of jeans. Until it was pointed out to the class that there was an error in the way they had set up the solution, none of the students noticed the reversal of the terms. Instead they tried to find ways to make their answer fit the question by varying the units of measure. By the end of the study, however, students were able to discuss their thinking and reflect on why they might have made the errors they did. In the excerpt below, students explain why they subconsciously reversed the terms in the jeans problem:

Researcher: This (4.5 $\overline{)22}$) was giving us completely distorted information. But how many people recognized right away that we had the numbers reversed? (No hands.) Why do you think hat happened?

Stan: Sometimes a word problem, you get it switched around, depending on the way you hear it. You kind of ignore what's being divided.

Researcher: If you ignore what's being divided, what do you focus on?

Stan: The other number. You tend to focus on the other number and you put it into what's being divided.

Researcher: Joani, why do you think this happened? I don't think you saw Geoffrey's problem.

Joani: I guess I just wasn't thinking. Some people tend to divide the bigger number by the smaller number, instead of the small number by the big number. You just reverse them because you think the big number should be divided by the small number.

Researcher: It feels better to do this? 4.5 $\overline{)22}$.

Students: It looks better.

Researcher: This looks weird? 22 $\overline{)4.5}$.

Students: Yeah.

Stan: Yeah, cause you'll get a decimal (less than one).

Although none of the students immediately saw that the terms had been reversed, they could in retrospect reflect on why it might have happened and could articulate their thinking to others. This public discussion of the reasoning behind student responses gave it value and contributed to the students' development of their metacognitive abilities.

Students clearly were committed to the belief that the divisor must be less than the dividend. This belief probably evolved from the students' experience with division of whole numbers but was generalized to include decimal fraction situations. The next section centres on students' understandings of decimal fractions, and the impact that understanding has on students' ability to solve problems.

Students' Understandings of Decimal Fractions

If division is the most difficult of the four basic operations (Bell, Swan, & Taylor, 1981), then it becomes even more complex when it involves decimal fractions. In this study, students who appeared to have clear understandings of the concept of division with whole numbers made errors which were inconsistent with this understanding when faced with division of decimal fractions. Assessing students' understandings of decimal fractions, and finding ways to help them build upon those understandings became an important component of the instructional program. Students' difficulties with decimal fractions fell under two main headings: place value and representation of decimal fractions. Representation refers to the concrete or real life examples students connect to decimal fractions.

Place Value and Decimal Fractions

From a review of the literature on decimal fractions it appears students do not always see decimal fractions and whole numbers as belonging to the same number system (Hart, 1981; Greer, 1987). Because they do not see decimal fractions as an extension of place value of whole numbers (Hiebert & Wearne,1986) students apply separate rules to each system and remain unaware of the conflict this presents. In assessing the understanding of the Grade 7 students in this study, a number of results indicated that although they were able to name the columns from tenths to millionths some students were not familiar with the relationship between decimal fraction columns and whole number columns.

Pretest Results. Because the pretest was a pencil and paper test, student thinking about place value had to be inferred from a limited number of written responses. An indication that students might view whole numbers and decimal fractions as separate systems can be found in Section B, computation involving decimal fractions. One question called for the remainder of a whole number division to be recorded as a decimal fraction. Only 6 of 22 students were able to provide a decimal fraction remainder for 29 ÷ 4. Thirteen of the students noted the remainder as "r 1" and the balance made other mistakes which prevented them from dealing with a remainder.

Clearer evidence that students fail to see the two systems as related comes from the overall results of Sections A and B, division with whole numbers and division with decimal fractions. A total of two place value errors were made on the five division of whole number questions. In Section B, dividing decimal fractions, a total of 24 place value errors were made in the same number of questions. All questions involved with one- or two- digit divisors and two- to four- digit dividends.

When asked to select the questions which would result in quotients greater than one in Section F, seven students chose the distractor 0.995 ÷ 1.1. Although there was some question as to their understanding of the notation and the position of the terms, it appeared some students

disregarded or did not understand the role of the decimal point and saw 0.995 as being greater than 1.1.

Classroom Investigation. It was clear from the pretest and from discussions with the students prior to the study that the group as a whole had limited experience with decimal fractions. They felt confident with addition and subtraction, and had learned the procedures associated with the multiplication algorithm. Division of decimal fractions, the students freely admitted, was the hardest of all four operations. One day the class had been asked to think about what made division easy or difficult, and report back in the next lesson. Here is a sampling of their responses.

Dave: It's easy as long as I have it taught to me, like not just a quick, "You're supposed to do this, now go to it," but something that explains it.

Researcher: Every time you do something with division?

Dave: Something new in division.

Researcher: Like?

Dave: Decimal in both parts.

Joani: Division is easy, but problem solving is sometimes kind of hard to figure out what the proper way is.

Mathew: Division is hard.

Researcher: Always?

Mathew: Small numbers are okay, just big numbers and problem solving is hard.

Gary: Mathew said division with small numbers is easy. Do you mean you find decimals easy?

Jaylene: I think some is hard and some is easy. Decimals are harder and the really high numbers, to divide into another really high number is harder.

Geoffrey: Division is harder than adding, subtracting, and multiplying.

Researcher: It's harder that the other three?

Geoffrey: Yeah, but it's still pretty easy except when you put in the fractions and the decimals.

Kate: Most of it's easy. I only get stuck on the whole number divided by decimals.

Annie: I have problems doing the equation when it's like 75 ÷ 13, like division by two numbers.

Researcher: It sounds like decimals are harder than whole numbers, and the larger the number of digits in the divisor, the harder it is. For some, dividing is okay, it's just when it's in problem solving that it is trouble.

Students knew that division involving decimal fractions was the most difficult of the four basic operations, but they were not aware of the reasons for this. Lack of experience with the algorithm was certainly one factor, but as the study progressed it became clear that their minimal understandings of place value coupled with their inability to represent decimal fractions in story problems was of greater concern.

When the Grade 7 students were asked to read decimal fractions, without exception they read the decimal point as "point" and named the digits that followed. The decimal fraction 4.5 was read as "four point five"; 21.07 as "twenty-one point oh seven." With practice they could read each correctly, but tended not to do so unless reminded. This interfered with their ability to estimate responses to questions such as 0.47 ÷ 0.09 because they focused on the procedural aspect of moving the decimal point rather than on separating 47 hundredths into groups of nine hundredths.

While it would be difficult to say that students viewed whole numbers and decimal fractions as two completely different systems, some students did not believe there should be crossover between the two. In discussing the question 2.8 ÷ 0.7, Anthony remarked that it could not result in the whole number 4 because 28 ÷ 7 was 4. 2.8 ÷ 0.7 had to have a smaller answer because the numbers were smaller. Certainly Anthony and the half of the class that agreed with him did not see that it was the relationship between the respective dividends and divisors that was the key to the solution, rather than the magnitude of the individual numbers

involved. In discussing this type of response, students could not defend why a whole number division should not have a decimal fraction answer and vice versa, but remarked that it "looked better."

This lack of ability to see the connections between the various place value columns was highlighted once again when students worked through division of a decimal fraction by a whole number using materials. With the base 10 blocks students were to model a division question and record the answer. In approaching the question $3 \overline{)5.3}$ students were able to create the number 5.3 with blocks and separate it into 3 piles of 1 and 7 tenths. At this point students became confused and most commonly finished the question by responding 1.7 r 2, not seeing the conflict presented by combining two forms of recording the remainder. Similarly, some students simply recorded the remainder in a whole number division as a decimal. When asked to divide 768 by 5, some students performed the following calculation:

```
      153.3
5 ) 768
    500
     268
     250
      18
      15
       3
```

With at least half of the class responding in this way it was evident they had little understanding of the relationship between the divisor and the remainder, and the conflict in meaning of moving the 3 from the ones column to the tenths column. This emphasized the need to focus on the actions of division and avoid the algorithm until students were able to think through the meaning of the placement of the digits.

Later, when working with division of two decimal fractions, students again demonstrated their lack of understanding of place value. In attempting the division $3.2 \overline{)4.25}$, students multiplied the divisor by 10 and the dividend by 100. Their motivation centered on making both numbers whole numbers, failing to see how this would distort the quotient. In fact, the students rarely referred to multiplying both terms of a question by a power of ten; they were

more apt to refer to "moving the decimal." In situations where the divisor was a decimal fraction less than one and the dividend was a whole number, students tended to multiply the decimal fraction by a power of ten and leave the whole number untouched. Students would transform a question like $0.03 \overline{)142}$ to $3 \overline{)142}$ and fail to see the answer would be 100 times too small. Although students were quite capable of rounding decimal fractions greater than one to the nearest whole number, they tended to use the same approach with decimal fractions in the hundredths or less. Estimating answers to questions such as the one mentioned above was difficult because students tended to round any decimal fraction less than one to the whole number one, producing an estimate which would match the answer. In Lesson 14 some students continued to have difficulty both estimating and calculating such questions. Elaine estimated $0.04 \overline{)6.0}$ as 2, ignoring the place value of the digits. Annie estimated $4.6 \overline{)3.45}$ as close to 0 and was content with an answer of 75; it is uncertain whether she saw the estimate and the answer as unrelated or was unable to use place value skills to have the digits 7 and 5 reflect a number close to 0.

This lack of place value understanding and a reliance on algorithmic-based thinking contributed to invented and distorted algorithms. Justin, when trying to explain how to divide 2.8 by 0.7, focused on the number of digits behind the decimal points to produce an answer of 0.04. A number of students shared Amy's reasoning; she believed that the answer should be 0.4 because 2.8 and 0.7 had to be multiplied by 10, so the answer of 4.0 would have to be divided by 10. Darin believed the decimal points should be removed from the question and then reinserted when the computation was complete, "bumping" them up into the quotient. Although other factors were involved in the formation of these beliefs, students who had a clear and extensive knowledge of place value would have been less likely to make errors of this type.

Instructional Decisions. Part of the difficulty with place value of decimal fractions was that students had few concrete ties for these numbers. Before introducing division of decimal fractions, students used base 10 blocks to name numbers both greater than and less than one. The researcher was careful to model the correct language, reading a number such as 3.04 as

"three and four hundredths." Although the researcher did not insist on this type of response from the students, they were encouraged to use the correct verbal structure and were often asked if there was another way to say the number in question.

Writing numbers based on models was a useful way to help students build their place value understanding. Comparing three flats to three rods and to three unit cubes enabled the class to see the difference between a 3 in the ones', the tenths' and the hundredths' places. Once students seemed to be sure of naming various decimal fractions less than and greater than one, the researcher carried on to introduce division of decimal fractions.

It was at this point it became clear that although students had a good superficial knowledge of the place value of whole numbers and decimal fractions, they were unaware of the relationships between the two and within each. A large place value chart from the thousands to the thousandths was placed on the board and the digit 3 was placed in the ones column. By multiplying or dividing by 10, students moved the 3 into adjacent columns. By multiplying or dividing by 100, the students were entitled to move the 3 two columns. In isolation, this was a relatively simple task which students appeared to understand well. After this initial instruction, discussion of place value was integrated with the lesson at hand. In the following excerpt from the transcripts, students talk through the division of 78.4 by 32:

Researcher: Let's go through this. 7 tens. Can you share that?

Students: No.

Researcher: Rename them as ones. 78 ones to share. How many each?

Sara: 2.

Researcher: How many will that use up?

Students: 64.

Researcher: How many left to share?

Students: 14.

Researcher: 14 ones left to share among 32. Can I do that?

Students: No.

Researcher: I'll have to rename the 14 ones. What's smaller than ones?

Students: Tenths.

Researcher: How many tenths will I get for all those ones?

Students: 140.

Researcher: But I already have 4, so there are...

Students: 144.

Researcher: We've got 144 tenths to share out among 32 people. We've passed from whole numbers to decimal numbers so I'll put that decimal point in there so I don't get confused. 144 shared among 32. How many do I have enough for each?

Students: 4. 2.

Researcher: Enough for 1?

Students: Yes.

Researcher: Enough for 2?

Students: Yes.

Researcher: Enough for 3? 4?

Students: 4.

Researcher: How many would that use up?

Students: 128.

Researcher: How many would that leave me with?

Students: 16.

Researcher: 16 tenths left to share. Can't do it with 32.

Annie: So you add a zero.

Researcher: I'm not adding. I could change the tenths for hundredths.

Dorothy: Isn't that hundredths already?

Researcher: (Walked through it again, using the base 10 blocks to show the change from ones to tenths to hundredths.) Now we can share out 160 hundredths. How many each?

Students: 5.

Researcher: So we used up?

Students: 160.

Researcher: Right. So our answer would be 2.45.

This same procedure was followed when dividing decimal fractions by single-digit whole numbers. Again, students were comfortable with this type of renaming, and could model the actions with base 10 blocks. The introduction of a decimal fraction divisor caused some difficulty, however. At this point students began to focus on the procedures and paid little attention to place value. They could not see how three different questions, $0.7\overline{)2.8}$, $7\overline{)28}$, and $70\overline{)280}$ could all result in the same answer. They believed the factor of 10 or 100 should be reflected in the quotient. Using patterns, the researcher had students work though a series of related division questions. Beginning with questions that could be solved mentally, such as $10 \div 5$, $100 \div 50$, $1000 \div 500$, students moved to using a calculator to solve those involving decimal fractions. In this way they were able to see that $1 \div 0.5$ and $0.1 \div 0.05$ were equal both to each other and to the whole number questions. For a small number of students this was not sufficient proof, and they had to return to the base 10 blocks to see the physical evidence.

Posttest Results. As was the case in the pretest, no items in the posttest were specifically targetted as place value items. However, a number of items yield some information about student growth over the length of the study. In Section B of the pretest and posttest, students were asked to divide one whole number by another and record the remainder in decimal form. Twenty of twenty-two students were able to do so in the posttest, whereas only six had demonstrated this ability in the pretest. In general, students had greater success with dividing whole numbers and decimal fractions in the posttest. No errors in Section A were due to place value problems. A total of eight place value errors were made in the six questions in Section B, compared to the pretest where there were a total of 24 errors in five questions.

In Section F students were asked to choose the questions which would result in a quotient greater than one:

Circle the questions below which will result in a quotient that is greater than 1.

a. $5.1 \overline{)4.75}$ b. $3.2 \div 1.25$ c. $0.995 \div 1.1$

d. $4.7 \overline{)6.01}$ e. $7.6 \div 8.3$

Pretest and posttest results for this question are detailed in Table 2. It is interesting to note that only two students chose the distractor $0.995 \div 1.1$ compared to seven in the pretest. Nineteen students correctly identified $4.7 \,)6.01$ and $3.2 \div 1.25$ as open sentences which would result in quotients greater than one. Again it is difficult to separate place value and division notation understanding, but it seems clear that students were aware dividing 0.995 by 1.1 would not result in a number greater than one.

Classroom Results. From the researcher's perspective, the students improved their place value understanding with respect to decimal fractions over the period of the study. Students could reason through division questions such as $4 \overline{)2.6}$ and $1.5 \overline{)0.75}$. They became more adept at estimating answers. In early classroom work, estimates for the division of one decimal fraction by a larger one had been 0 or -1. By the end of the investigation students were able to provide more reasonable estimates. For example, Anthony estimated $5.376 \div 8.4$ as "less than one." Division by decimal fraction divisors in the hundredths or thousandths, or those which students could not easily estimate, remained difficult. Students like Joani were able to estimate $13.95 \div .09$ as 1300, but this was not common. In this case, students were more likely to produce the correct calculation than a reasonable estimate.

Students continued to refer to "moving the decimal points" and "getting rid of the decimal" rather than multiplying or dividing divisors and dividends by a power of ten. This reliance on procedural language reflected a lack of understanding which the researcher believes limited the students' understanding of the actions involved. In the following excerpt, Stan talks about why questions like $3.2 \overline{)4.36}$ might have been introduced for discussion:

Stan: I think you put them on the board cause when you multiply by 10, you'll have to add a zero.

Researcher: For example, in the last one, where will I have to write a zero?

Stan: Multiply by 10 and it's 32 $\overline{)43.6}$. To take off the decimals, you have to add a zero to 32.

Stan is part way there. He realizes that the "adding" of zeros is related to multiplication by 10, but is still so fixed on removing the decimal points that he intends to multiply by 10 again so that none remain in the dividend. When asked what multiplying by 10 twice was the same as, Anthony replied "1000." Almost all of the students prefered to multiply by a power of 10 until both numbers became whole numbers. This signalled a continuing lack of place value sense, and a reliance on an overgeneralized procedure. When confronted with a conflict-creating situation, dividing 4.3628756 by 3.2, students realized that extending the divisor by an additional six digits made the question unnecessarily awkward but tended to continue in the pattern they had established.

In the last days of the study, students were working on a problem which required a divisor of 0.004. Typically, this was the most challenging type of divisor for students because it was difficult to estimate with and was often a much smaller number than the dividend. Darin, who was identified as having weak place value concepts, demonstrated his growth in this area:

Researcher: Let's look at the laser beam one. A laser removes 0.004 mm from a piece of metal with each pass it makes. Get a picture in your head: each time the beam passes over the top...How many passes before the metal is gone? It's simple when you think of what's happening, but it can be hard on paper. Why?

Sara: All the zeros.

Researcher: What could you do to make it seem easier?

Darin: Multiply by 1000.

Researcher: Why 1000?

Darin: That would give you 4.

Researcher: Why?

Darin: 'Cause 4 thousandths times 1000 is 4.

By the end of the study, students' place value concepts were more uniform and tended to work for students rather than against them. About half of the class could clearly articulate the value of digits to the thousandths' place and could defend "moving the decimal" in terms of multiplication and division by powers of 10. The remainder of the class could best be described as having evolving place value concepts. Although their accuracy improved, their reasoning was inconsistent and did not always reflect understanding. In addition to difficulty understanding place value of decimal fractions, students often were unable to represent decimal fractions in a meaningful way. The following sections deals with the investigation into this area.

Students' Representations of Decimal Fractions

Most students in Grade 7 are introduced to decimal fractions in Grades 3 or 4. By the time they reach Grade 7 one might assume students would have had a variety of experiences using decimal fractions and would have developed an assortment of representations for these numbers. Unfortunately, this does not seem to be the case. Students in this Grade 7 class had very limited representations for decimal fractions, and thus interfered with their ability to complete calculations and solve problems. Students relied heavily on money, which tended to limit the range of decimal fractions they could work with to hundredths. Decimal fractions less than one posed the most difficulty for students. Because they had little intuitive sense of the size of these numbers, they could not estimate answers or check the reasonableness of solutions.

Pretest Results. The pretest contained only one section which involved free responses from students: Section G, writing story problems. From this one section, however, it was possible to learn a great deal about students' entry points with respect to representation of decimal fractions. Table 4 details the results for this section, however examples of student problems provide an insight into the errors made. Although all six questions involved division of whole numbers or decimal fractions to tenths only, over half of the students used money in

their problems. Two students used money exclusively, in one case disregarding the place value of digits.

3 ÷ 0.6: If you had $3.00 and you wanted some gum at ¢0.6 a piece, how many pieces could you buy?

A question like 1.6 ÷ 2.4 was expected to be difficult for students, but when they limited themselves to using money to represent the decimal fractions it became even more complex. In this problem the student is unable to create a problem that involves division.

1.6 ÷ 2.4: If you had $1.60 and next month after getting your allowance you had $2.40, how much would you get?

For the student to chose subtraction, consciously or subconsciously, is not surprising because story problems which contain two or more items involving money would most likely be addition or subtraction problems. The student could not see a way of representing the two numbers as a partitioning of money. The concept of ratio would have been useful here, but instruction in that topic had not yet taken place.

Also common to students was the tendency to assign decimal fractions to items that could not meaningfully be broken into parts. In this problem, both representations for the decimal fractions are illogical:

1.6 ÷ 2.4: There is 1.6 boxes of apples. 2.4 trucks come along. How many does each truck get?

While this example is perhaps the most obvious of the group, many students created representations like "3.5 groups," "2.4 equal parts," or "2.4 people." Certainly the students' dominant partitive model for division contributed to their choices, but they did not seem to have any internal checking mechanism for recognizing the conflict they had created.

Another difficulty which surfaced was the use of inappropriate amounts. In some cases the amounts of items to be divided were too large, and in other cases, too small. The following problems provide illustration:

Kate: If Clara went to the store to buy 12.5 lbs of chocolate chips for a cake for 9 people, how many chocolate chips would each person get?

Annie: There was 0.6 cm of rope and 3 men wanted to share it equally. How much would they each get?

Although the students were able to create representations for the quantities in these problems, the units of measure they chose reflected a lack of understanding of quantity as affected by the numbers assigned to those measures.

The use of decimal fractions with time appeared in a number of students' questions. The fact that the hour is based on 60 did not interfere with students' willingness to use a base 10 system to represent it.

15.2 ÷ 3.5: The kid ran 3.5 km in 15.2 minutes. How long did each km take?

1.6 ÷ 2.4: You've been listening to your cassettes for 1.6 hours and you have only listened to 2.4 tapes. How long did each tape last?

Both of these problems are structurally correct and the numbers assigned to the units are appropriate. These students may have been able to justify and explain their answers had they been asked to work them out, however it is doubtful they would have been able to explain what 15.2 minutes meant in terms of minutes and seconds. The second question, dividing 1.6 by 2.4, would result in 0.6 repeating. Interpretation of this decimal fraction in terms of hours or minutes would also have been challenging.

Finally, some students were unable to represent the decimal fractions at all. Two students did not respond to the questions 1.6 ÷ 2.4 and 3 ÷ 0.6. Another two students responded with story problems of this nature:

15.2 ÷ 3.5: The school children had to solve a problem, 15.2 ÷ 3.5. Help the children find the answer.

A few students changed the decimal fractions into numbers with which they felt comfortable. For example, 0.6 was changed to .6th of a pie in one case, and to $6.00 in another. One student ignored the decimal point altogether and treated the number as if it were six.

The information gleaned from the pretest indicated a number of areas for investigation in the study: representations other than money for decimal fractions was one area of concern; students' beliefs about the partitioning of various objects required further investigation; and student's understanding of the appropriateness of assigning decimal fractions to time required assessment.

Classroom Investigation. Before formally beginning work with decimal fractions, students explored division of whole numbers in a problem-solving setting. In an investigation of the role of the remainder in story problems the issue of continuing to divide the remainder into parts emerged. Although the original discussion had centered on the need to reflect on rounding a quotient up or down depending on the setting, the direction of the discussion changed when one student suggested an answer of 4.2 kids. In the next lesson, students described what they believed could be divided into parts and what could not.

Researcher: Yesterday we ran into a problem where we had 4.2 of a kid. It's difficult to take 0.2 of a kid! What would you take?

Students: The hands! The feet!

Researcher: Today I'd like you to help me generate two lists: something that you can keep sharing in a division situation, and something that you can't. Yesterday Geoffrey brought up renaming the blocks when we were sharing 67 ÷ 5 and sharing out to tenths. We talked about cookies. When you have 2 left over, you can keep sharing.

Stan: You can share out tenths of cookies.

Researcher: Perhaps. But you can at least share halves. Some things you can share, some you can't.

What can you keep on sharing? Anything you've come across in word problems?

Dave: You can share some inanimate objects.

Researcher: Such as?

Dave: Food.

Stan: You can't share animate objects like people.

Researcher: Would that include food?

Stan: Okay. Just people.

Researcher: That's safer.

Annie: You can share out numbers.

Researcher: Like 10 ÷ 3? You'd have one left over. You could break that into smaller parts.

Jaylene: A box of pencils.

Researcher: You can give out so many boxes to people and break up the remaining box.

Geoffrey: A $10.00 bill.

Researcher: Money? Except when...?

Geoffrey: You get down to pennies.

Gary: Shapes.

Researcher: Can you give me some examples?

Gary: A circle.

Researcher: Are you thinking of fractions?

Gary: Yeah, well, you can keep cutting it up.

Dorothy: Cannot. Marbles. If you got down to where there's 2 left over and there's 5 people, you can't break them in half.

Researcher: What makes the difference between when you're sharing out marbles or cookies? If you share out 17 cookies among 4 people, they get 4 each and one left over. Then what?

Joani: Cut the last cookie into 4 pieces.

Researcher: If we share out 17 marbles among 4 people, they get 4 each and there is one left over. Then what?

Students: You can't cut it up.

Researcher: Why not?

Stan: Well you could if you had a sharp enough knife.

Researcher: Would you want to even if you could?

Geoffrey: Marbles are useless when they're cut up.

Researcher: So what makes the difference between the marbles and the cookies? Why can you continue to break up one and not the other?

Gary: You can break up the cookies because if you cut them up you can still use them. But if you cut up a marble, it's useless.

Researcher: So the difference is it loses its value. Let's keep that in mind. It might help you later on.

Students exhibited a good understanding of what could be meaningfully divided and could give examples of objects that could not be divided and retain their value. For the most part, their examples centered on partition of whole numbers and came from their experience. When we reached division of decimal fractions later in the instructional unit the same type of understanding was not evident.

The base 10 blocks were used to introduce the students to decimal fractions. Once the they had begun to establish an understanding of the relationships between the whole numbers, the tenths, the hundredths and the thousandths students were asked to provide representations for the decimal fractions they were dividing. A question like 0.18 ÷ 3 became 3 cities sharing 0.18 megatonnes of sludge, or 0.18 million people in 3 concentration camps. Students tended to represent decimal fractions as portions of large quantities.

Writing story problems for questions served a dual purpose. In addition to helping students visualize the actions of division it gave them the opportunity to represent decimal fractions both less than and greater than one. Generally, students were successful when dividing a decimal fraction greater than one by a whole number. Dividing a decimal fraction less than one by a whole number was more difficult, but the most difficult settings were when two decimal fractions were involved. Students attempted to create a problem for 0.97 ÷ 2.3:

Marian: 0.97 of a litre of milk and 2.3 containers.

Researcher: 2.3 containers? It would work if it was two containers.

Annie: 0.97 kg of beans. Someone else has 2.3. How much more does one have than the other?

Joani: That's subtraction.

Sara: There's 0.97 grams of something. He wants to divide it out among 2.3 of his workmen.

Researcher: 2.3? Just from the knees down?

Geoffrey: Two 1 litre containers and one 0.3 litre container.

At this point the students had determined that a measurement of some type would be the best representation for the decimal fraction dividend, but they were having difficulty with the divisor. Still working with a partitive model, the students were creating problems modeled after those with whole number divisors. In many cases the divisor was represented by people who would share out a quantity of some item. Sara's problem above is representative of this type of problem.

The question of assigning decimal fractions to time came up early in the study when the students were asked to solve the following problem about the time it required for the chicks they were studying in science to emerge from their shells once the shells had been cracked:

> Of the 12 eggs incubated, only 5 have hatched so far. The chicks took varying amounts of time to hatch: 2 h 38 min, 3 h 15 min, 2 h 47 min, 3 h 05 min, 2 h 52 min. Suppose a 6th chick hatched. How long would you expect it to take? Why?

Of the students who attempted a solution, most approached the problem by writing each time as a decimal fraction and adding the times together. Amy solved the problem this way:

```
                    2.87
2.38            5 ) 14.37
3.15                10
2.47                43
3.05                40
2.52                37
14.37               35
                     2
```

The last chick would take 3 h and 17 min.

There is an interesting mixture of understanding demonstrated here. On the one hand the student ignored the element of base 60 in time by directly translating each time into a decimal fraction. However, when she saw the answer had exceeded 60 minutes she regrouped to form another hour. What Amy did not reflect upon at this point was the fact that the average she obtained, 3 h 17 minutes, was greater than each of the times she had added together. This approach was common to that of other students, with some students dividing the total by 6 instead of 5. These students obtained an answer of 2.39 h which they translated into 2 h 39 minutes. Again, none of the students noticed this time did not reflect the average of the times they had added together. In the discussion which followed this problem, students began to notice their errors quickly as they began to justify their approach.

Researcher: How did you approach it?

Joani: We added up the hours and the minutes and we got 14.37 and then we divided by 6.

Researcher: Why 6?

Joani: Oh, because there's 6 chicks. If there's another one that hatched, there was 5 plus 1 is 6. You divide by 6 and it comes out to 2 hours and 30 minutes.

Researcher: OK, so you divide by the number of chicks whose times you have there plus the one whose time you don't have yet?

Joani: Yeah.

Researcher: Any reactions to that?

Stan: I think you have the idea but I think you divided it wrong. It should only be divided by 5, because if you add all these (the times the chicks took to hatch) you'll only have 5 numbers and then you're dividing by 6. So you'll get a much less number, like if you had 2 hours and 37 minutes, well according to this question there's only one chick that had that.

Geoffrey: If you add up all the numbers you have, then divide, that gives you the average, right? But that doesn't work on this.

Researcher: We're having trouble with the time aspect of this one. That's what has made this a harder question than it is. Let's look at how to find an average with time. (List the times, keep the hours and minutes separate, add them up and get 14 hours and 37 minutes.)

Joani: The way Stan and I did it would be very wrong. I took 14.37 and divided it by 5. You should take the 14 and divide it by 5, then take 37 and divide it by 5.

Researcher: Let's look at that for a second. Can you take 14 hours and 37 minutes and say that's the same as the decimal number 14.37?

Students: No...

Researcher: Why not?

Students: It's time.

Researcher: So?

Students: Time's not based on 100.

Researcher: So?

Students: So it's not a decimal. It's two different things.

Researcher: If we do this (14.37 = 14 hrs 37 min), what are we saying then? If 14.37 could be translated to time, what would it be? How many minutes would there be in an hour, then?

Students: 100.

Researcher: Are there?

Students: No.

Researcher: That's why it can't be expressed as a decimal, then. There's another way you could do this so it would be less painful.

Joani: You could change the hours to minutes and then divide.

Once the addition of the times had been modeled with the hours and minutes kept separate, students could see that the total was much different than the one they had obtained by treating

each time as a decimal fraction. From there the group could recognize the need to treat the time in minutes to facilitate the division.

Ignoring the decimal point and treating the decimal fraction as a whole number had surfaced as an area for investigation in the pretest. A few students had avoided representing decimal fractions in this way. This rarely happened in the classroom setting, perhaps because the students had been reminded of the place value ramifications of doing so. Three students, Mathew, Annie and Elaine, did occasionally ignore a decimal point and represent the number as a whole number. They had difficulty especially with decimal fraction divisors less than one, often choosing subtraction or multiplication in place of division. Their difficulties were not limited to representing decimal fractions. They shared a need to further investigate place value with whole numbers and had unstable concepts of both multiplication and division.

From the classroom investigation and the evidence in the pretest, it appeared students required work in a number of areas. The meaningfulness of partitioning various types of items or objects was clearly an area of concern. Students were in need of developing their understanding of decimal fractions less than one through the development of a repertoire of concrete examples that could be tied to these numbers. Finally, and perhaps most importantly, students had to develop an awareness of the need to reflect on their answers to determine the appropriateness of their responses.

Instructional Decisions. Students had demonstrated some understanding of the appropriateness of partitioning various objects through their discussion as described in the previous section. They realized that partitioning, or breaking up, some objects like marbles was illogical because the object lost its value. However, in assignments following this discussion students were often unable to give appropriate examples for decimal fractions indicating a need for further development of their experience base. To accomplish this, three main approaches were used. First, the researcher acted as a model providing students with appropriate representations for decimal fractions. Because the students' had demonstrated a reliance on money, the researcher deliberately avoided using examples of this type. Although

very familiar with money, students tended not to see the place value relationships between dollars and cents. Measurement in litres, meters, and kilograms was most useful because of students' familiarity with the units and their understanding of the place value aspects of the smaller units associated with each. Examples such as "five thousandths of a metre" had meaning for students. Students could picture the examples, and could translate them into concrete examples in the classroom. Second, students spent a significant amount of time creating story problems for division sentences. Whether orally or in writing, students were asked to tie examples to decimal fraction division questions on a regular basis. These questions provided the basis for discussion, were used to check understanding, and often formed part of a successive lesson. Third, students almost always worked in pairs or small groups when generating examples and evaluating ther examples of others. The need for students to develop reflection was evident; using small groups enabled more immediate feedback and gave students a comfortable setting in which they could challenge each other's thinking. Because story problems were often written in pairs, students shared the responsibility for their answers and had their partner's support. In the excerpt from the transcript that follows, students explain why some of the story problems their peers wrote had either inappropriate representations of decimal fractions or did not meet the condition of requiring division:

Researcher: Let's look at the problems I gave you yesterday. The ones your classmates wrote. How many of the seven could you actually figure out?

Marian: Five.

Kate: Six.

Researcher: Most had five or six. Which ones didn't work out?

Dave: The second part of number one. How much more ribbon do you need to have 10 cm per gift. That isn't division.

Researcher: Okay. Any that just didn't work out?

Fiona: Number five.

Researcher: How many agree? Twenty-two report cards, 4.5 A's to be shared out. How many per person?

Justin: Can't share out 4.5 A's.

Researcher: Another one?

Dorothy: The last one. Gas. It's in kilograms. It should be litres.

Researcher: Would it be okay if it was litres?

Marian: No, there's not much: 0.35 of a litre.

Researcher: You could do it, but should you? Does it make sense to share out 0.35 litres of gas among 22 trucks?

Marian: No.

By reproducing students' work on a practice sheet with no names attached to the various problems, students were free to critique each other's problems in a non-threatening way. Students who had made errors were able to benefit from the feedback of their peers, rather than solely from the instructor. Not only did students point out the weaknesses of story problems such as these, but they also suggested how the problem might be improved.

Posttest Results. Section G of the posttest involved writing story problems for open division sentences. The results for this section are detailed in Table 4. As in the pretest, all six questions involved division of whole numbers or decimal fractions to tenths only. In general, the students demonstrated growth in their ability to represent open division sentences as story problems. Specifically, students were better able to assign meaningful examples to the decimal fractions in the questions. In the pretest almost half of the students used money in at least one of their problems. In the posttest the use of money was limited to one student who relied upon it for every question. Although students continued to have some difficulty representing numbers like 1.6, more students were able to relate decimal fractions to measures of distance, mass, and volume than they had in the pretest. The dominant model for division continued to be partition, but some students used quotition when two decimal fractions were involved. Some students

introduced ratio into their problems, a topic which had not been formally introduced. The growth in ablility to represent decimal fractions can be seen in the samples below.

Dave: I have 1.6 kg of sugar. My mom told me to put it into 2.4 kg of batter for cookies. How much sugar per kg did the batter get?

Kathryn: If you had 15.2 litres of milk and some containers and each could hold 3.5 litres, how many containers would you need?

The most difficult question to write a problem for was 1.6 ÷ 2.4. Two decimal fractions and a divisor greater than the dividend combined to make this a confusing question. In each of the other questions involving decimal fractions, between one and three students included items such as "15.2 containers." For the question 1.6 ÷ 2.4, eight students referred to items of this type in the posttest. Although this might seem discouraging at first, only three students were able to adequately represent the same problem in the pretest.

Students did not choose time as a representation for decimal fractions in the posttest, nor did they change the numbers by omitting or moving the decimal points. One student did, however, refer to "0.7 of a dozen" demonstrating that there was still some confusion with mixing decimal fractions with quantities based on numbers other than ten.

Classroom Results. In the classroom students were more able, both in groups and as individuals, to represent decimal fractions meaningfully. Students were able to tie concrete examples to single decimal fractions; creating story problems for division questions which involved one decimal fraction and one whole number did not pose much difficulty. Students drew on their knowledge of metric measurement and generally relied on partition.

Sue: One day Sue went to the market to buy 2.7 kg of chocolate chips. She wanted to make 9 dozen cookies. How many grams of chocolate chips does each dozen have?

The most difficult task continued to be writing story problems for the division of one decimal fraction by another. Students seemed to avoid using quotition, preferring instead to find a way to fit division of two decimal fractions into a partitive situation. This is best shown in the two story problems below.

Gary: Tom had 4.5 canteloupes and 3.7 litres of cottage cheese. How much cottage cheese would there be per canteloupe?

Marian: Joani had 3.7 kg of gumdrops. She wanted to make 4.5 dozen cookies. There are 12 cookies in a dozen. How many grams of gumdrops does each dozen get?

Most students were able to recognize inappropriate use of amounts in story problems, and could suggest ways to improve them. When Jaylene wrote a problem which required sharing 10.7 grams of jellybeans among 22 children, her group suggested changing the grams to kilograms so the problem would be more realistic. Similarly, Geoffrey's group objected to his use of grams in this problem: Your diet says you can have 0.35 grams of cereal a day. How many days would it take to finish 4.5 grams? Geoffrey, reluctant to change his problem, defended it by saying it was a very strict diet. The ability to explain decisions and defend choices had developed in most of the students. Working in groups had provided the students with a comfortable setting in which to present their thoughts and test their responses.

The preceding sections have discussed the students' understandings of division and decimal fractions. How these two areas relate in terms of problem solving is the topic of the final content area.

Students' Understandings of Problem Solving Requiring Division

The investigation into students' understandings of problem solving requiring division centered on two areas. The first was their ability to translate story problems into the appropriate division open sentence and interpret the answer in a meaningful way. The second was their ability to write story problems for division open sentences. Central to both of these areas is the student's intuitive model for division. The dominance of the partitive model over quotitive model was evident with both whole numbers and decimal fractions, however problem settings which involved decimal fractions proved to be more difficult.

Translation Problems

Students' understanding of translation problems requiring division is examined in the following section. Of particular interest to the researcher was the students' ability to translate story problems into open sentences and the degree to which students reflected on responses.

Pretest Results. Because of the length of the pretest, students were asked to solve only four free response translation problems. Two of these problems involved division of whole numbers (#1 and #3) and two required division of a decimal fraction by a whole number (#2 and #4). The two whole number problems also required the accommodation of a remainder. These problems are listed below. Table 5 shows the students' results on these problems.

1. Pencils are sold in packages of 12. Mr. Wilson needs 185 pencils. How many packages must he buy?
2. A metal chain is 31.68 cm long. It is made from 9 links. How long is each link?
3. Mrs. Wood's class collected items for the food bank. The students were able to pack 241 items into 26 bags. If the students made sure the food was distributed equally, how many items were in each bag?
4. After 8 laps around the training track, the odometer on Sharon's bicycle showed 10.8 km. How long was each lap?

From these problems the researcher obtained information about four areas of interest: the recognition of division as the required operation for solution, division with decimal fractions in a problem-solving situation, the accommodation of a remainder, and evidence of planning and checking back.

Almost all students recognized that division was the operation required to solve these translation problems. In question #1, two students used multiplication to "guess and check" a solution. One student could see division was required in the first three questions, but could not carry out the computation. In the fourth question he chose to add 8 and 1.8, obtaining 2.6. Questions #2 and #4 required division of a decimal fraction by a whole number. Of the 22

students, 18 correctly solved #2 and nine correctly solved #4. On a similar computation item from Section B, 6) 115.5, only seven students obtained the correct answer. The accommodation of a remainder was necessary in questions #1 and #3. Question #1 required students to take an answer of 15.41 and increment it to 16. Four students were able to do this, and another four rounded their answers down to 15. The remaining students made no attempt to deal with the remainder, or made procedural errors. Question #3 required the interpretation of 9.26 as 9 or 10 items. Seven students responded correctly, and 12 students left the answer as 9.26 or 9 r 7. The remaining students made procedural errors.

Table 5
Translation Problems: Pretest and Posttest Results

	Correct		Remainder		Multiplication		Other	
Question	Pretest	Posttest	Pretest	Posttest	Pretest	Posttest	Pretest.	Posttest
1	4	13	11	4	2	0	3	5
2	18	19	n/a	n/a	0	0	4	3
3	7	9	12	9	0	0	3	4
4	9	19	n/a	n/a	0	0	13	3

Other than the division calculations there was no planning evident for the four problems. Although the problems were not difficult, each could have been represented by a picture. There was little evidence of checking back. Two students used multiplication to check their division computation, but neither of these students recognized the need to increment the answer for question #1. A number of students made place value errors in question #4, obtaining an answer of 13.5 km rather than 1.35 km. Again, there was no evidence of checking back to determine the reasonableness of the solution.

Section D in the pretest was composed of 10 division translation problems for which students had to choose from amongst six open sentences containing the four basic operations. One distractor in each question had reversed terms in the division. An example is given below.

A new tunnel is being dug at the Strike-It-Rich gold mine. In one day the miners are able to tunnel 0.275 km. At this rate, how long will it take them to complete a 2 km tunnel?

a. 0.275 x 2	c. 0.275 ÷ 2	*e. 0.275)2
b. 2 + 0.275	d. 2 - 0.275	f. 0.275 - 2

The intent of these questions was to determine if students could identify division as the operation required for solution, and if they could determine which numbers represented the dividend and the divisor. Results from these questions can be found in Table 1. Generally, students were able to identify division as the method of solution in the 10 problems. Only on question #3 did more students choose an operation other than division. This may be due to the fact that the question involved area, with which students frequently associate multiplication. Questions #2 and #10 also drew a large number of responses containing other operations. Question #2 involved an average, and two steps for solution. Question #10 involved rate which may explain why so many students opted for multiplication. In every question some students chose distractors with reversed terms in the division open sentence. From a low of one student to a high of 18, students demonstrated a lack of ability to translate division story problems into open sentences which would result in reasonable solutions.

Classroom Investigation. Results from the classroom investigation matched those from the pretest. Students generally did not make self-directed efforts to accommodate remainders, nor did they independently plan and look back when solving problems. When presented with translation problems which involved decimal fractions or when the divisor was greater than the dividend, students often reversed the terms in division open sentences. Problems with familiar settings and structures were generally easier than those with settings which were new to students or contained unfamiliar structures. Problem-solving choices were sometimes driven by factors such as the relative size of numbers and a reliance on partitive division.

Students responded to problems which required accommodation of the remainder in a predictable fashion. When asked how many 45 passenger buses would be required to transport

94 people, most students responded 2 r 4. All students, when challenged to explain this answer, could revise it to include another bus or find an alternate form of transportation for the remaining people. Through the presentation of a number of similar problems, it appeared students required prompting to interpret remainders. When solving a problem which asked how much one chick would weigh if five weighed a total of 768 grams, Sara responded, "On day two one of them would weigh 153 grams with a remainder of 3."

Although it may be argued that little planning is required for translation problems, students tended to neither plan nor reflect upon their problem solutions. When using calculators most students were content with writing the result of their calculation as the entire solution. When using paper and pencil, students could refer to their calculations but often could not recall why they had chosen the numbers involved in these calcualtions. This approach handicapped students because they were unable to discuss their solutions and explain their reasoning in subsequent lessons. Students generally justified their solutions only when asked, and interpreted confirmation of their calculations to be "looking back." Students did not estimate before calculating, and had nothing with which to compare their answers.

When presented with story problems which required the division of one number by a larger number, some students reversed the terms in the division question to reflect their understanding of division. Students approached the following problem in a number of ways: A swinging metronome makes 15 clicks in 6.3 seconds. At this rate, how long will it take it to make 24 clicks? Some students decided to divide 15 by 6.3 because "you can't do it the other way." Others chose to divide 6.3 by 15 because "63 doesn't go into 15." And there were those who wanted to know how many seconds there were per click. In the first example students were driven by dividing the larger of the two numbers by the smaller. In the second, students ignored the decimal point for the purpose of division and viewed 6.3 as the larger number, 63. Some may have been influenced by partition, and chose to divide by 15 because it was the whole number. Only the students in the third group focused on the meaning of the problem and were able to explain their choices without referring to procedures.

Translation problems which contained familiar structures and settings were the easiest type of problems for students to solve. These problems were of the type often found in textbook series, usually requiring the student to find the cost per item or the number received per person in a sharing situation. These problems tend to be partitive in nature and often involve a whole number divisor. When translation problems stray from this model, they become more difficult for students. In the following problem, many students reversed the dividend and the divisor: The thickness of one sheet of paper is 0.075 cm. How many sheets of paper are in a stack 5 cm high? Because the quotitive model of division was not firmly in place for the students, they tended to view it as a partitive situation and tried to divide 0.075 by 5. Also influencing students was the relationship between the digits; students ignored the decimal point and could see 75 ÷ 5 as a more appropriate division than 5 ÷ 0.075.

Instructional Decisions. Although most students could determine when division was required to solve a story problem, their reliance on procedure and their beliefs about division often led to incorrect responses. Because students did not check the reasonableness of their answers their errors went unnoticed. The major thrusts of instruction were to help students develop an awareness for more thoughtful analysis of problems, to present strategies which would assist students with their analysis, and to encourage and provide the opportunity for reflection.

One suggestion in the literature on problem solving is to present fewer problems to students and spend more time on each (Suydam, 1980; Schonfeld, 1983). The use of small group work is also encouraged as it enables more students to express their views and hear multiple perspectives on a problem (Gilbert-Macmillan & Leitz, 1986). These approaches were two of those used in the instructional program. After problems were presented to the class as a whole, students worked in small groups to read and clarify the tasks. While the groups were working, the researcher visited with each checking understanding and asking questions about their plans for solution. Often the groups would report their plans back to the class before working through to a solution. In this way students were able to hear many different

perspectives on a problem and had the opportunity to clarify their thinking before proceding any further. In some groups this process worked very effectively. In others, some students were more concerned with who was "right" than they were with ensuring all groups members understood the problems.

In the following excerpts two groups discuss this problem: The combined weight of the five chicks will be about 1475 g on day 7. If all 12 chicks had hatched what would their combined mass be on day 7? The first group argues about the answer to a problem without discussing their plans for solution.

Stan: It should add up to 176.

Stewart: No, no, no!

Annie: Yes, you have to...

Stewart: What would the combined mass be on day 7? It is 1475 on day 7.

Stan: This is day 2 already.

Stewart: Day 7, day 7. It's 1475 ÷ 5 and then you would add 295 to that, times 2.

Mathew: Where, where, where, where?

Annie: Well, I got 1675.2.

Researcher: You are still focusing on differences in the exact answer. What I'm concerned about is are you approaching it in the same way, or are you approaching it in different ways? If you're approaching it in one way and getting different answers, that's one set of difficulties. If you're approaching it in different ways and getting different answers, that's another set of difficulties. So which camp are we in?

Annie: We all used different ways and got different answers.

Researcher: Well, that's logical, then. If I do something a different way than you do, I might get a different answer. So now you've got to focus on how you got there, not on exact answers. That's really what you were supposed to talk about before you began working.

The next group discusses the process involved, rather than focusing on the solution.

Jaylene: If all the chickens hatched, what would their combined weight be on day 7?

Fiona: Oh, god!

Jaylene: We have to divide this. You divide, then times this.

Matt: What's the point of divide then times it? Well it says if all the chicks hatched, what is their combined weight.

Jaylene: Now, Matt. Look. You divide into 1475 because you have to find out how much each weighs on day 7, so you get 295. Then it's 295 times 12 if all of them hatched. So you divide 1475 by 5 and get 295.

Anthony: 295...

Jaylene: No, that's how much one of them weighed on day 7 and that's why you do 295 times 12. I got 3540.

Matt: Why would it be 5 into?

Jaylene: Because. That's how may of ours (chicks) hatched.

Matt: It should be 12, not 5. You want to find out what all of them weigh.

Researcher: The combined weight is talking about how many chicks?

Matt: 5.

Researcher: And the question asks you about what? If all of them had hatched. How many things do you have to do in this problem?

Jaylene: Two.

Teacher: What do you need to know?

Jaylene: How much one chick weighs. And it's 295. An then do 12 times 295.

Researcher: Would that be an exact answer? Would all the chicks weigh that exactly?

Jaylene: No, one might be fatter.

Researcher: How could you accomodate that in your answer?

Anthony: About?

Although much of the discussion in each group was directed by one group member the outcomes of the discussions were quite different. In the second group the students related their

calculations back to the problem, whereas in the first group the focus was solely on procedures and answers. Working in this setting on a daily basis enabled students to develop their thinking and communication skills. With encouragement and reinforcement, students focused more on processes and less on outcomes over the course of the study.

Students required assistance with representation of translation problems, especially when they were unfamiliar with the problem structure or the numbers in the problem influenced the students' decision making. Visualizing the action in problems and drawing pictures or diagrams helped some students determine which number was the dividend and which was the divisor. Although this strategy was modeled by the researcher, students were reluctant to adopt it. Some said it took too much time and others believed it was childish. A strategy which was adopted by the students was the word frame. As described in an earlier section, the word frame acted as an aid for translation of the problem from words into symbols. Students focused on the content of the problem to decide what was being partitioned or separated. This word or phrase was then placed in the division word frame in the #1 position. Students placed the phrase that represented what was doing the sharing in partitive situations, or word that referred to the size of the piece in quotitive situations, in the #2 position. The phrase representing the outcome of the problem was written in the #3 position.

#2 _____ #3 _____
) #1 _____

A problem which might be best approached using the word frame would be one like the following in which the divisor is greater than the dividend: The thickness of one sheet of paper is 0.075 cm. How many sheets are there in a stack 5 cm high? Once in the word frame, the problem would look like this.

#2 thickness of one sheet) #3 number of sheets in the stack
#1 height of the stack of paper

With the analysis of the problem focused on the content, students then substituted the quantities from the problem for the phrases and carried out the calculations. This strategy helped students represent problems and avoid reversing the dividend and divisor. The tendency

to rely on the partitive model was another obstacle that the word frame strategy helped students overcome. By concentrating on the words ignoring the numbers, students were less likely to automatically divide by the whole number.

Reflection on problem solutions was encouraged in the same way as problem analysis. Students worked in pairs to groups to determine if their answer was reasonable. Interpreting remainders was a beginning focus for reflection. After class discussions on the usefulness of breaking various objects into smaller parts, students reviewed rounding numbers. Once they became aware that the principles of rounding were not always appropriate when applied to problem solving, students worked together to determine when incrementing an answer was more appropriate than rounding.

Encouraging the development of metacognitive behaviour was part of every lesson. Whether through class discussion, talking in small groups or with individuals, or recording thoughts in writing students were asked each day to reflect, react, or justify. Students were asked to rate problems according to difficulty and explain why they believed their answers were correct. This was difficult at first, as students had little experience expressing their thoughts.

Any student could challenge another's answer or thinking provided he or she could justify the challenge. The researcher's role was to ask for justification and explanation and to sometimes provide conflict-creating situations. In this excerpt the researcher challenges a student's reasoning by providing an example which causes conflict:

Geoffrey: I got another answer. I came up with two hours and 56.5 minutes.

Researcher: Okay.

Geoffrey: I took the difference in minutes between the earliest time and the latest time and it turned out to be 37 minutes. And then I divided it in half and I added that to the earliest number. And that turned out to be two hours and a half.

Stan: So that's half way between all of the them.

Researcher: Half way between all of them, or half way between the earliest one and the latest one?

Geoffrey: Half way between the earliest and the latest which would be the average.

Researcher: Would that be the average? All right. Here's an example for you. Let's say there were five chicks that hatched. The first one took 2 minutes 0 seconds, the second one took 2 minutes 5 seconds, the third took 2 minutes 10 seconds, the fourth took 2 minutes 15 seconds, and the fifth took 10 minutes. The first 4 were all around 2 minutes and the last one took 10 minutes. If a sixth hatched, how long would you expect it to take?

Geoffrey: I don't like you!

Researcher: You don't like me? But if you go by what you just said, take the earliest and the latest, then what happens to your logic?

Geoffrey: But two hours 56 minutes sounds good. Just look at the numbers.

Through exchanges such as this one students were encouraged to look beneath the surface of their answers and test their logic. Although students were uncomfortable with this at first because they viewed the resulting work as an addition to the assignment, they came to see how they might reduce the time invested in solving problems by asking themselves questions like this during the process of solution.

Posttest Results. Section C of the posttest was similar to the same section in the pretest. Students were asked to solve four translation problems, two which involved decimal fractions and two which required accommodation of the remainder. All students used division to solve the problems, and results were considerably better than in the pretest (see Table 5). The problem settings and structures are similar to those in the pretest.

1. Cookies are sold in packages of 36. Mr. Black needs 450 cookies for a meeting. How many packages must he buy?
2. A necklace is 58.5 cm long. It is made from 13 hinged pieces of gold. How long is each piece of gold?

3. Tom works in a chocolate factory. His job is to decide how many chocolates should be packed into each box. If there are 526 chocolates and 36 boxes, how many should be put in each box?

4. After 8 circuits around the track, the train in Bobbi's new train set had travelled 21.6 metres. How long was each circuit?

Questions #2 and #4 produced higher success rates than question #1 and #3 which required the accommodation of remainders. Students generally made two types of errors: they failed to consider the role of the remainder, or they made a computation error when carrying out their calculations. Although the remainders were not dealt with by all students, those who did accommodate the remainder in their answers demonstrated growth. In question #1 the answer of 12.5 had to be incremented to 13. A sampling of student responses follow:

Gary: 12.5 packages, but you'll need another package so it's 13.

Justin: He will need 13 packages with 18 cookies left over.

Dorothy: You can't buy a half, so 13.

In question #3 students had to take an answer of 14.6 and reduce it to 14. Two students explain their decisions.

Jaylene: Tom would put 14 in each box and would have some for himself.

Sara: 14 in each box and Tom would need some more boxes.

Students who did not explain the remainder showed no evidence of looking back to check the reasonableness of their solutions. There was some evidence of reflection on a number of the students' tests. A few students checked their division calculations by multiplying, eleven made estimates before carrying out their solutions, three students drew diagrams, and three used the word frame.

In Section D students were required to choose the correct open sentence for each of 10 translation problems. Each problem was followed by six open sentences using one of the four basic operations. One distractor for each question used the correct operation, division, but had the divisor and the dividend reversed. The purpose of the questions was to determine if students

could identify division as the required operation, and if they could isolate the correct division open sentence. More students chose the correct answer in all but one question. Fewer students chose the distractor with reversed terms in all but two questions, and fewer students chose other operations to solve the problems.

Classroom Results. Students generally performed better in the classroom setting on translation problems than they did in the posttest because of the small group structure. When students were able to discuss problems with others they made fewer errors when choosing solution processes and tended to reflect more on their solutions. Two problems given to students near the end of the study were similar to those in Section C of the posttest:

1. The thickness of a dollar bill is 0.1 mm. How much money is in a 50 mm stack of dollar bills?
2. How many 15¢ stamps can you buy for $10.00?

As they worked in their groups the researcher was able to observe students helping each other.

Problem #1

Joani: You've got to divide the 50 mm into...

Kate: No, you don't because then you get less than what you're supposed to. It's the other way.

Problem #2

Dorothy: $10.00 divided by 15¢. That's 66.6.

Darin: 66.6 pieces of paper. You can't get .6 pieces of paper so take off the 6 and it's 66.

Dorothy: It's 66 stamps.

Twenty-one of twenty-two students were successful on all of these questions, and accounted for the remainder in the second question. Working in groups facilitated the reflection that might not have occurred had students been working independently.

The use of the word frame reduced the number of reversals when students were interpreting translation problems, but did not eliminate them. When students estimated answers they had another method of checking the reasonableness of their responses. However students

found it difficult to estimate when one or both of the numbers in the question were decimal fractions in the hundredths or thousandths. In general, students had improved their ability to interpret and solve division translation problems but continued to be influenced by their beliefs about the division. Students were more likely to check answers and reflect upon solutions when they worked with at least one other person.

Writing Story Problems

Writing story problems to reflect division open sentences is the topic of the final section. This task afforded students with the opportunity to reflect their understanding of the models of division, and to explore their understanding of the roles of the divisor and dividend in problem-solving settings.

Pretest Results. Writing story problems for division open sentences was a difficult task for students. The open sentences included decimal fractions and situations in which the divisor was greater than the dividend. Students reversed the terms, ignored decimal points, and used other operations. Several students distorted the numbers to reflect numbers with which they were more comfortable and others introduced additional numbers. In Table 4, the results from the pretest show that students had the most success with open sentences where the divisor was less than the dividend and was a whole number.

These results are not surprising. Students are most familiar with partitive story problems involving whole numbers. By using such a model to interpret these questions students had difficulty representing decimal fractions as the divisor and tended to reverse the terms in situations where the divisor was greater than the dividend. The following examples provide illustration:

$4 \div 20$: While the man was at the market he saw a series of books at $4.00 each. Altogether the whole series cost $20.00. How many books were there?

15.2 ÷ 3.5: Betty had 15.2 stickers but then she split them up into 3.5 groups. How many did she have in each group?

3 ÷ 0.6: You want to make 3 cakes and you've got 0.6 L of milk. How much does each cake get?

In the final example the terms were reversed even though the divisor was less than the dividend. In this case the student was influenced by the partitive model. A number of factors were involved in the students' lack of ability to complete this task: beliefs about the divisor, dominance of partition, and representation of decimal fractions. Given these factors, the one open sentence which should have been easy for students was 19 ÷ 3. Only 15 of the 22 students were able to write a translation problem to reflect this division. It appeared students were not able to write translation problems to reflect division open sentences at the onset of the instructional program.

Classroom Investigation. Students were asked to write problems for open sentences early in the study. The actions associated with partitive and quotitive division were modeled for and practised by students. After identifying partitive and quotitive problems students worked in groups to write their own. One group provided these examples:

Partitive: George has 67 marbles. He wants to divide them between 5 kids. How much would each kid get?

Quotitive: The baseball coach had 5 packs of gum and he wanted to share it between 67 people.

In another group the understanding of quotitive division was more visible, but their partitive question required more information:

Partitive: The girl went to the market and she wanted to buy 5 pies and she had 67 dollars. How many pies could she buy?

Quotitive: Two people were in a yo-yo contest for 67 hours. How many 5 hours were there?

When the focus was removed from partitive and quotitive division, students continued to have difficulty writing problems:

Annie: A man had $68.00 and he wanted to buy 12 packs of potatoes. How much was each pack and how much did he have left over?

Students did not reflect on their problems to determine if their question could be answered. Their goal seemed to be writing a problem which contained the numbers, rather than one which represented the division.

Further difficulty was experienced when decimal fractions were introduced. Presented with the question 0.84 ÷ 21, students were unable to create a translation problem. They could not connect 0.84 with any quantity that could be shared among 21. Story problems for open sentences which contained two decimal fractions were even more difficult. Without a whole number divisor, students struggled to find representations for the numbers and mesh them in a problem:

0.97 ÷ 2.3: There's 0.97 L of milk and 2.3 containers. How much in each?

As in the pretest, students made one of four types of errors when faced with a difficult problem. They reversed the divisor and dividend, used an inappropriate respresentation for a decimal fraction, changed the operation, or distorted the numbers. The most common errors were reversals and inappropriate decimal representations.

Instructional Decisions. Helping students develop their ability to write story problems was an integral part of the instructional program. Because proficiency in writing problems is linked with the students' understanding of division and decimal fractions, writing and interpreting problems was a part of most lessons.

Students began by building models of division with manipulatives and then paired their actions with story problems. This work was done in group or partner settings to encourage development of the language associated with partitive and quotitive division. Completed problems were often critiqued in groups and then discussed with the class. Sometimes these problems formed the basis of ensuing lessons, or were used as warm-up activities. Once students had identified a deficiency in a problem they were encouraged to find a way to correct it. In the following excerpt, students discuss a problem written by one of their peers:

22 trucks came to a gas station. All of the trucks get 0.35 kg of gas altogether. How much did one truck get?

Dorothy: The gas. It's in kilograms. It should be litres.

Researcher: Would it be okay if it was in litres?

Marian: No, there's not much. 0.35 of a litre.

Researcher: Could it be improved?

Stewart: Not unless there's something bigger than litres.

Writing, reflection, and discussion provided the students with opportunities to expand their experience base. Using their problems in the lessons lent credibility to the task and encouraged reflection.

Posttest Results. Table 4 shows growth in the students' ability to write story problems. All 22 students were able to write a story problem for a decimal fraction divided by a whole number, whereas only 11 had been able to do so in the pretest. Students were less prone to reverse the terms when writing problems to relfect the division of one whole number by a larger whole number. Thirteen students were able to write a problem for this type of question in the posttest compared to two students in the pretest. Results for problems involving the division of two decimal fractions, and the divison of a whole number by a decimal fraction show increases but indicate students continued to have difficulty in these areas. The most common type of error remained reversals. A considerable number of students reversed the divisor and dividend when writing problems. The final question, $8 \div 0.7$, had 11 reversals; students were still influenced by the partitive model and chose to divide by the whole number. Difficulties representing decimal fractions was also a factor, especially with the question $1.6 \div 2.4$. Two students reversed the terms, but 10 could not represent the decimal fractions appropriately. One student used an operation other than division and one or two students distorted the numbers or added numbers in each question.

Classroom Results. In the classroom, students were generally successful at writing story problems for division open sentences. Partner and group work, coupled with class

discussions provided students with opportunities to revise and edit their problems. Students began to develop their ability to reflect upon their own work and the work of others. They became aware of various types of structural flaws to watch for, and made fewer of these errors as the study progressed. Students became more comfortable dividing whole numbers by decimal fractions, but tended to avoid writing problems which had a divisor greater than the dividend. Given the choice of five numbers on the board, students wrote the following problems:

22 ÷ 4.5: If you had 22 kg of jellybeans and you wanted to fit 4.5 kg in each bag, how many bags do you need?

10.7 ÷ 0.35: There are three kids going to the market. They had to walk 10.7 miles and they walked 0.35 miles per hour. How long did it take to get to the market?

Without prompting, these students had created quotitive situations to match their division open sentences. This was uncommon, however. Most students chose to divide by 22, the only whole number included in the set on the board.

By the end of the study the students had developed their ability to write story problems. Some were able to write problems for any division open sentence presented to them and others were able only to write problems if the question could be interpreted in a partitive sense. A small number of students still appeared to believe the divisor must be less than the dividend and were unable to write problems for situations in which this was not true. The representation of decimal fractions continued to pose difficulty for some students. Generally these were students who had strong partitive models for division, and so could not find logical situations for dividing by a decimal fraction.

Discussion of the Results

The Grade 7 students in this study had varying degrees of understanding with respect to division, decimal fractions, and translation problems requiring division. These understandings were revealed through pretest results and classroom investigations in small group and whole class settings.

Approximately one fourth of the students had difficulty understanding division notation. This difficulty centered on equating "a ÷ b" with " a $\overline{)b}$." Most students referred to "a $\overline{)b}$ " as "a goes into b." The failure to attach meaning to division notation reflected the students' general lack of development of division models. Although all students understood partitive division and relied on it heavily in problem solving, few students had a similar grasp of quotitive division. Initially most students were unable to model either partitive or quotitive division with manipulative materials. The students' interpretations of the divisor and the dividend also mirrored this reliance on partition. Most students believed the divisor must always be less than the dividend, and that division always makes smaller. Because of this, students tended to reverse terms or use another operation when the divisor was greater than the dividend, or when it was a decimal fraction less than one.

Division involving decimal fractions introduced another element of difficulty. Students were unsure of the relationship between whole numbers and decimal fractions, and had weak place value concepts of number to thousandths. Few students were able to meaningfully represent decimal fractions because of their reliance on partition and their lack of experience with decimal fractions. Students required extensive work with materials and with formulating appropriate representations for decimal fractions. Students treated division of decimal fractions in a procedural fashion. They exhibited little understanding when "moving decimals" and saw no conflict in "moving the decimal" one position in the divisor and two positions in the dividend. Their lack of understanding of division and decimal fractions had a visible impact on their ability to interpret division translation problems.

Few students could recognize division as the required operation when the divisor was greater than the dividend or was a decimal fraction less than one. In these cases, students tended to reverse the terms of the division or chose another operation. Most students chose operations based on the relative size of the numbers involved. Few students made attempts to interpret remainders in problems, and a very small number of students showed any evidence of either the planning or looking back stages of problem solving. When asked to write story problems to

reflect division open sentences, students could accurately reflect a question in which the divisor and dividend were whole numbers, and the divisor was less than the dividend. In cases where decimal fractions were involved or the divisor was greater than the dividend, students had difficulty with the appropriate model of division and meaningful representations for decimal fractions.

Chapter 5

CONCLUSIONS

This chapter deals with a summary of the study that has been undertaken, discusses the results, presents implications for instruction, and makes recommendations for further research. The results of the study are organized around the research questions.

Summary

The review of the related literature documented the need for the exploration of teaching strategies which would help students become more thoughtful when solving problems requiring division. Traditionally, classroom instruction has focused on procedures rather than on concepts (Skemp, 1976; Baroody & Ginsburg, 1986), and little has been done to connect concepts and procedures (Hiebert & Wearne, 1986). This approach to instruction neither recognizes nor builds upon the understandings students bring with them to the learning situation.

The inability of students to solve translation problems is a reflection of this emphasis on mathematical procedures. Results from NAEP assessments show students virtually ignore all phases of the problem-solving process except carrying out whatever mathematical operation seems appropriate (Carpenter et al., 1980). Another outcome of the failure to connect concepts with procedures is the development of incomplete conceptions (Resnick, 1983b). One example comes from a study investigating students' choice of operation when solving problems. Researchers found that students had limited conceptions of division which influenced their ability to solve problems. Students' early experiences with division resulted in the beliefs that division always results in a smaller number, and that a larger number must be divided by a smaller number (Bell et al., 1984).

The results from investigations into students' understandings in the areas of division, decimal fractions, and problem solving prompted the researcher to pursue a study which

examined students' understandings in these areas, and how they might be elicited and influenced. The investigation consisted of the administration of a pretest followed by a series of 19 lessons. The pretest revealed a number of areas relating to division and decimal fractions which required further investigation. The lessons provided the researcher with the opportunities to investigate students' understandings in greater depth, and to attempt to influence student thinking. Small group and whole class discussions were used extensively. Students were encouraged to explain and justify their thinking. Following the instructional unit, a posttest was administered. This posttest contained items similar to those in the pretest. An additional section on students' beliefs concerning division was included.

The data collected and analyzed are discussed with reference to the following research questions:

1. What beliefs and understandings do Grade 7 students hold regarding division of whole numbers and of decimal fractions?
2. How can teachers elicit students' understandings in the classroom setting?
3. What kinds of teaching strategies can teachers employ that assist students in making connections between concepts and procedures?
4. What impact does this instructional approach that focuses on linking students' conceptions to procedures have on student understanding of representations of problems involving division and decimal fractions?

Conclusions

Conclusions regarding the students' understandings of division, decimal fractions, and problem solving are drawn in the following section. They have been organized around the four research questions. These conclusions are then discussed in relation to previous research findings.

Question 1

What beliefs and understandings do Grade 7 students hold regarding division of whole numbers and of decimal fractions?

The Grade 7 students in this study demonstrated understandings of division which were consistent with the previous research literature. The dominant model for division was partition. The students' beliefs regarding division and their understanding of the relationship between whole numbers and decimal fractions interfered with their ability to solve problems.

Approximately one fourth of the students were unsure about the relationship between the two forms of division notation. These students read both "a $\overline{)b}$" and "a ÷ b" as "a divided by b." This belief appeared to be due to the left-to-right method of both reading and writing most notation. A smaller group of these students believed "a $\overline{)b}$" and "b $\overline{)a}$" were equivalent, and the remaining students read a $\overline{)b}$ as "a goes into b." Through the use of meaningful language and encouraging students to write the "a $\overline{)b}$" notation in the order which reflects the division of the dividend by the divisor, almost all students were able to overcome their confusion with the two types of notation.

Students relied on the partitive model in their interpretation of division. When asked to model a division question with base 10 blocks, students frequently relied on their knowledge of the division algorithm to first make the two terms of the division and then try to parallel the algorithm with the blocks. They tended to equate the concept of division with the division algorithm. Students who could model division relied heavily on the partitive model. Students had difficulty identifying division as the required operation for solving translation problems when the divisior was greater than the dividend or was a decimal fraction less than one. When they wrote story problems for division questions of this type, the problems often reflected the reverse of the given question. Student understanding of the quotitive model was incomplete at best. Many students were initially unable to choose the correct open sentence for story problems involving quotition. They chose distractors which had the terms of the division reversed, or opted for other operations such as multiplication. With instruction on quotition

and a focus on suitable contexts for division questions which implied quotition, the majority of students developed the ability to model, solve, and write quotitive story problems.

The reliance on the partitive model contributed to beliefs such as the divisor must be less than the dividend. This was especially true when both terms were decimal fractions or when one term was a decimal fraction and one a whole number. In the first case, students would divide by the smaller of the two numbers, and in the second case they would divide by the whole number. Through instruction in quotition, experiences with manipulatives, and creating story problems to reflect open sentences which did not match the partitive model, students became more receptive of divisors which did not match their initial conceptions.

Another misunderstanding common to many of the students in this study was the belief that division always makes smaller. Because many whole number questions involve the division of one whole number by a smaller whole number, some students did not believe the quotient could be larger than the dividend when dividing by decimal fractions less than one. This belief is a result of the overgeneralization of whole number rules, and it negatively influenced success when problems involved decimal fractions. It led to students using operations other than division when solving problems involving decimal fractions less than one. Some students ignored the decimal point and reversed the terms of the division. By providing students with opportunities to confront this belief by using base 10 blocks and 10-by-10 grids, and by writing story problems to reflect open sentences which had divisors less than one, students demonstrated growth in this area. Writing story problems or discussing solutions with others tended to increase their ability to notice reversals and incorrect solution paths.

Students had weak place value understanding of decimal fractions. They were unable to see the relationship between and among whole numbers, tenths, hundredths, and thousandths. Students interpreted decimal fractions such as 0.995 as being greater than one, failing to see that 995 thousandths is less than one. Some students recognized that the decimal fraction represented a portion of the whole, but did not always relate it to base 10. They made many place value errors in computation questions involving decimal fractions but very few in those

with whole numbers. By using models to develop connections between whole numbers and decimal fractions, and using patterns to show how multiplication or division by powers of 10 affected place value, students strengthened their understanding of the relationship between whole numbers and decimal fractions. Students made fewer place value errors in computation situations and most students could defend "moving the decimal" in terms of multiplication or division by a power of 10.

When discussing division of decimal fractions, students in this study talked of "moving the decimal" with little understanding of the reasoning behind the action. This led to errors, demonstrating the students' lack of understanding of the concepts behind the procedures. Even when students in this study recognized the relationship between moving the decimal and multiplying by a power of 10, they did not see the need to maintain equivalence with the initial question. Multiplying the divisor by one power of ten and the dividend by another was common. The choice was driven by the procedural focus on the number of digits behind the decimal rather than by the concept of multiplying both terms by the same number. When working through the division algorithm with decimal fractions, students saw no conflict in transferring a remainder in the ones column to the tenths column without actually dividing. This underlined their lack of understanding of the relationship between the divisor and the remainder, and its effect on place value. Students' computational ability had improved by the end of the study. On the posttest, few students made place value errors when dividing decimal fractions and students had dramatically increased their ability to identify division questions which would result in quotients greater than one.

Initially, the Grade 7 students in this study did not view whole numbers and decimal fractions as belonging to the same number system. They believed that the division of two whole numbers could not result in a decimal fraction, and that the division of two decimal fractions could not result in a whole number. When presented with two similar translation problems, students could determine the required operation for the one containing only whole numbers but could not always do so for the problem containing decimal fractions.

The belief that decimal fractions and whole numbers are not related, together with students' reliance on the partitive model, resulted in the belief that numbers less than one should not be divisors. When a question involved a decimal fraction divisor less than one, students tended to ignore the decimal, reverse the terms, make place value errors, use other operations, or omit the question. By the end of the study, students had made considerable progress in this area. In the classroom setting they were able to write story problems for open sentences of this nature, and showed improvement in their ability to choose the correct open sentence for translation problems requiring division by a number less than one.

Students had difficulty representing decimal fractions in a meaningful way. Decimal fractions to hundredths could be correctly represented with money, but students had difficulty with other representations. They tended to assign values such as 3.6 to objects normally referred to by a whole number. Students were often unable to represent decimal fractions such as 0.006 in any way at all. Students did not always represent decimal fractions incorrectly, but the quantities they referred to were often unrealistic. Some were far too small, like 0.6 cm of rope shared among three people, and others were far too large. Occasionally students would ignore the decimal point altogether and represent the decimal fraction as a whole number. Creating story problems individually and with a partner helped students develop their ability to represent decimal fractions. Students related decimal fractions to measures of distance, mass, and volume. By the end of the study only a few students continued to assign decimal fractions to objects such as containers.

Estimation with decimal fractions which could not be meaningfully rounded to a whole number was problematic for students. Students used rules for rounding without reflection. When asked to provide an estimate for a question such as $0.078 \div 0.036$, students would round the first number to one and the second number to zero and produce an estimate of zero. This thinking reflected a lack of place value understanding and resulted in the inability to check reasonableness of responses. Students' conceptions of dividing by zero were also revealed by these estimates. By focusing on estimating when completing calculations and solving problems,

students built upon their experience, and developed their intuitive sense of the relative size of decimal fractions and its impact on the division. By the end of the study students rounded decimal fractions over one to nearest whole number, and those under one to the nearest half or nearest tenth. Some students were able to estimate accurately to hundredths. Others gave estimates of "less than one" or "close to zero."

One discovery which has not been reported in the research literature involves estimates for division of decimal fractions less than or close to one. A number of students gave estimates of negative one for questions such as 0.04 ÷ 1.2. This suggests students believe that such divisions are somewhat illegitimate, and will result in a number so small it will be less than zero. It also reflects the belief that division makes smaller. The work on development of representations for decimal fractions and the focus on estimation helped to change this belief. This type of response did not occur again in either the posttest or classroom results.

Question 2

How can teachers elicit students' understandings in the classroom setting?

Several teaching strategies used in the instructional program proved to be useful for revealing student thinking. By focusing on a few questions or problems in each class the researcher and the students were able to direct their efforts to the processes involved. Valuing the thinking behind the solutions and allowing time for this thinking to be expressed encouraged students to express themselves publicly.

Cooperative partner and small group work provided students with opportunity to express, explore, and reflect upon their thinking. When teamed with whole class discussions in which the students were asked to explain and justify their responses, these strategies revealed beliefs about many areas including notation, models of division, and the role of the divisor and dividend.

Writing story problems to reflect division open sentences proved to be a useful strategy for revealing beliefs about division, and about decimal fractions. Initially students wrote problems which only reflected the partitive model of division. Their problems called for

situations in which the divisor was less than the dividend, and was often a whole number. Problems which involved decimal fractions revealed students' lack of representations for these numbers, and their beliefs about the role that could be played by a decimal fraction in a division open sentence.

Question 3

What kinds of teaching strategies can teachers employ that assist students in making connections between concepts and procedures?

Small group and partner work, combined with teacher and student demonstrations using manipulatives, formed the basis for developing students' conceptions of division. After reviewing the actions for partitive and quotitive division, students were introduced to writing story problems to develop understanding. Working in small groups, both writing problems and reflecting on problems, was extremely useful. Students began to visualize the actions in the problem and reflect on their work and the work of others.

The use of concrete models and pictorial representations of these models was another important strategy employed by the researcher. When discussing how one might go about solving $2.8 \div 0.7$, students referred solely to procedure, making no connection to models, manipulatives, place value, or possible contexts. Using base 10 blocks followed by shading 10-by-10 grids enabled students to visualize the numbers in a decimal fraction division. When using models, students did not reverse the terms of a division question.

Creating contexts for division questions was a strategy which enabled students to build upon their models of division. Using students' problems in ensuing lessons served to motivate students and encourage reflection. In familiar contexts students were more likely to be able to determine the dividend and divisor regardless of the relative size of the two numbers. For this reason, problems set in an unfamiliar context were often related back to a previously solved and similar problem. This was an effective strategy for some students, but others did not see the relationship between the problems. Similarly, substituting whole numbers for decimal

fractions enabled part of the group to better visualize the problem. Others could not see the parallels between the problems and required other strategies.

Focusing more attention on fewer problems was used to develop more thoughtful analysis of problems. Completing fewer problems allowed for more time to be spent on each. Clarifying meaning, either in a whole class or small group setting, could be accomplished through rephrasing of the problem, drawing pictures, or relating the problem to one previously solved. There was also time for justifying reasoning. Because the focus had been removed from completing a set of problems, students were encouraged to explain and defend their decisions. During these discussions students often found flaws in their thinking and modified their plans for solution.

Using examples which provoked conflict resulted in meaningful discussion and tested students' commitment to their beliefs. Presenting conflict-creating situations proved useful when working to disspell such beliefs as the divisor must be less than the dividend. Once the students had established a level of comfort with division of decimal fractions and had become used to justifying their thinking in both small group and whole class settings, posing questions designed to create conflict and test commitment to beliefs became a part of each lesson. Students attempted to solve the problem, and had to respond to researcher's questions such as "How could he have only a part of a turn?" and "Can the average be less than all of the numbers in the set?" Students then had to re-examine their thinking and either justify their response or change their approach.

In addition to discussion of story problem contexts and the actions which took place within them, students required a bridging strategy to help them translate the problems into number sentences. This was especially true for problems which required the division of one number by a larger number, those which contained two decimal fractions, and those which required the division of a whole number by a decimal fraction less than one. Students placed words or phrases from a problem in a division frame, then substituted the corresponding numbers. This helped to neutralize interfering beliefs such as the divisor must be less than the

dividend because it removed the focus from the numbers in a problem. An example of a problem analyzed by using the "word frame" is given below.

> Four gophers are digging a tunnel that will be 5.2 km long. If they dig 0.35 km in one day, how many days will it take them to finish?

how long it will take

how far in one day) how far all together

$$0.35 \overline{)\ 5.2}\ = \text{about15 days}$$

By using the word frame students avoided reversing the dividend and divisor, and were less likely to automatically divide a decimal fraction by a whole number. This strategy helped students accomodate remainders because the quotient was also represented by a phrase. Students did not respond with answers such as "4.02 sheets of paper" when using the word frame. Although the primary focus was to provide students with a strategy for analyzing problems, the word frame also helped with interpretation of the quotient because it tied the numbers in the question to phrases from the problem. Students were more likely to interpret remainders correctly when using the word frame.

The importance of the language used to refer to partitive and quotitive division became evident when students adopted a term used by the researcher. The term "dealing out" was used to refer to a partitive situation because it provided a clear image of sharing among a group. It was an unfortunate choice in terms because students became fixed on the notion of dealing out cards one at a time and assumed that if more than one was dealt at one time, the problem was quotitive. This experience demonstrated how easily students can be influenced through the instructor's choice of language and models. Referring to partitive situations as "sharing among" and quotitive situations as "separated into chunks of" generally helped students visualize the actions.

Question 4

What impact does this instructional approach that focuses on linking students' conceptions to procedures have on student understanding of representations of problems involving division and decimal fractions?

Students were initially able to identify division as the required operation in partitive situations which involved no more than one decimal fraction. When problems involved whole numbers and the dividend was greater than the divisor, students could accurately translate the problem into a division open sentence. This was not true if two decimal fractions were involved, or if a whole number was to be divided by a decimal fraction. In these cases beliefs such as the divisor must be less than the dividend affected students' choices. The most common error was reversing the terms in the division open sentence. A small number of students chose other operations, usually multiplication.

Students did not interpret or accomodate the remainder in a problem without prompting. When working with whole numbers, calculations were completed and remainders were noted. When working with decimal fractions, students interpreted that a calculation to a few decimal places sufficed as an answer. Students could generally interpret remainders when prompted. Drawing attention to the problem context cued students to the need to accomodate the remainder. Some students then interpreted the remainder meaningfully, but others focused on rounding the number according to a rule rather than using the problem context.

Students did not record any evidence of planning or looking back independently. Generally the written record for a problem was the calculation or calculations carried out to solve the problem. Few students represented problems through drawings, and the only evidence of looking back was related to checking calculations. Very few of the students in this study drew diagrams to clarify the problems. Students did not estimate before performing calculations and often could not explain why they had chosen the numbers in their calculations.

Writing story problems for division open sentences was a difficult task for students. Other than the division of one whole number or decimal fraction by a smaller whole number,

few students were able to adequately represent open sentences with story problems. Students reversed the divisor and dividend, distorted the numbers in the question, used other operations, or forced the numbers to fit their limited models of division. Students' reliance on the partitive model of division was the main cause of their lack of ability to represent some open sentences with story problems. Because the partitive model implies a whole number divisor and a dividend greater than that divisor, students were unable to represent problems which did not fall into this category. Some students also had difficulty representing open sentences in which one whole number was to be divided by a smaller whole number.

By the end of the study, the students had developed the ability to identify division as the required operation in translation problems. The only problems which remained difficult were those which involved the division of a very small number by a much larger one. In these cases, students were prone to reverse the dividend and the divisor. The strong partitive influence continued to lead students to divide by the smaller number. Strategies such as drawing diagrams, using the word frame, and working with a partner helped students represent and interpret these types of translation problems.

By focusing on placing division in problem-solving contexts, students developed their ability to write translation problems to match division open sentences. Students continued to rely on the partitive model, especially when the numbers involved were whole numbers. When the dividend was a decimal fraction less than one, students tended to assign a large unit of measure to it to enable them to view it in a partitive fashion. All students were able to write story problems for division open sentences which could be interpreted in a partitive fashion. Students' ability to write quotitive problems had increased throughout the study. Their experience with manipulatives and their exposure to correct models of quotitive problems contributed to this growth. The most difficult questions for which students had to write story problems were those involving the division of one decimal fraction by a larger decimal fraction.

Individually students did not perform as well as they did in groups. Through observation and examination of work samples it appeared that students' metacognitive behaviour was

increased in group settings. Students were more likely to reflect on the contextual requirements of a problem when they discussed problems with their peers. These results emphasize the success of cooperative problem solving.

Teaching strategies employed in the instructional program positively influenced the students' ability to represent and solve division translation problems. The entry beliefs held by students relating to both division and decimal fractions were strong and resistant to change. Continued emphasis on instruction which focuses on understanding would be required to enable all students to achieve a desirable level of understanding.

Discussion

The discussion which follows is organized under four headings: students' understanding of division, students' understanding of decimal fractions, students' understanding of problem solving requiring division, and teaching strategies. The results of the study are related to the previous research literature. Reflections of the researcher are noted where appropriate.

Students' Understanding of Division

Findings regarding the students' understandings of division notation are consistent with those reported in the research. Bell et al. (1984) found confusion with notation in their study of 12- and 13-year-olds. Students read $a \div b$, $a \overline{)b}$ and $b\overline{)a}$ all as "a divided by b." Bell et al. (1981) also noted problems with understanding the various symbolisms for division. Students in this study were also unaware that the order of appearance of numbers was significant, and confused $a \div b$ with $a \overline{)b}$.

The exploration of this group of Grade 7 students' understanding of division echoes the research in this area. Fischbein et al. (1985) found that tacit models associated with the arithmetical operations mediate when students solve problems. For division, these models are partition (sharing) and quotition (repeated subtraction). The authors reported that students in Grade 5 had only one intuitive model for division - partition - and by Grade 7, were in a

transitional stage. Some students in the present study had acquired the quotitive model but it was not yet stable or influential. Bell et al. (1984) found that students favoured partition when they were asked to write story problems for division open sentences. Students in this study were successful when solving problems that had a familiar context. These problems often involved money, or were partitive in nature. Substituting whole numbers for decimal fractions enabled some students to solve a problem they had previously been unable to attempt. Not all students found this strategy useful, however. They did not see that the method of solution was invariable when the numbers were changed. Bell et al. (1981) had similar results. They found that students could see the operation required when using familiar numbers such as 2, 6 and 10, but when problems with similar contexts contained decimal fractions, students could not identify the operation.

The Grade 7 students in this study generally believed the divisor must be less than the dividend. In a study conducted by Bell et al. (1984), 12- and 13-year-olds were asked to solve various story problems requiring division. Students in their study who made errors either reversed the terms, used multiplication or subtraction, or did not respond. Students in the present study made the same types of errors. When writing story problems to match division open sentences, their students reversed terms, forced a question which would have been best represented by a quotitive problem into a partitive situation, or altered the numbers to ease embodiment. These are the types of responses given by Grade 7 students in this study.

The belief that the divisor must be less than the dividend was common. The dominance of the partitive model of division and the misinterpretation of decimal fractions less than one are partly responsible for this type of thinking. These results are also found in the research. Bell et al. (1984) and Fischbein et al. (1985) reported that the strongest factor in determining the placement of terms in a division sentence was the belief that the divisor must be less than the dividend. However, they reported that in cases where the dividend was a decimal fraction less than the whole number divisor, students did not operate according to this belief. Instead, the

bias towards partition dominated resulting in students choosing to divide by a whole number--a result common to this study.

In computation situations students in the present study sometimes ignored the decimal point and divided. Numbers such as 0.75 were interpreted as 75. Fischbein et al. (1985) suggest students' reliance on the partitive model leads them to drop the decimal point and treat the decimal fraction as a whole number. The mediation of the implicit model, in this case partition, makes a division like 0.75 ÷ 5 intuitively feasible whereas 5 ÷ 0.75 is not. Bell et al. (1984) also found students ignored the decimal point and treated decimal fractions as whole numbers when writing story problems. Some students in the present study believed division involving numbers less than one was impossible. In her study, Hart (1981) discovered a general reluctance to divide one number by a larger number. When asked to divide 16 by 20, 51% of 12-year-olds and 47% of 13-year-olds responded that there was no answer. Hart attributed this to a reliance on the partitive model of division.

Students in this study held held the belief that "division always makes smaller." This belief led to two types of errors when solving problems. Students chose operations other than division or reversed the terms of the question. Some researchers have reported related findings. Bell et al. (1984) noted that division by decimal fractions less than one proved difficult. In situations which required such a division, students chose multiplication responses, reversed the terms and ignored decimal points, or gave no response. Fischbein et al. (1985) found that 62% of Grade 7 students in their study were able to correctly solve a story problem which involved the division of 15 by 3.25. Only 38% of the same students were able to solve a problem which required dividing 3 by 0.15.

Another curious belief related to "division makes smaller" is worth noting. When estimating answers to questions which involved the division of two small decimal fractions, some students in this study thought the result would be so small that it would result in a negative number. This finding was of interest to the researcher because the students had not received formal instruction in negative integers in either their Grade 6 or Grade 7 mathematics

programs. There is no mention in the previous research of the division of two decimal fractions resulting in a negative number.

Students' Understandings of Decimal Fractions

Students in this study treated whole numbers and decimal fractions as separate number systems. They believed that the answer when dividing two decimal fractions could not be a whole number, and when dividing two whole numbers could not be a decimal fraction. Hart reported similar findings. In her study, students indicated that division of one whole number by another should not result in a decimal fraction (Hart, 1981).

Initially few seventh graders in this study could relate the fractional portion of a decimal fraction to tenths, hundredths, or thousandths. This lack of place value understanding is consistent with research findings. Bell et al. (1981) reported that students interpreted 11.9 miles per hour as 11 miles 9 minutes per hour, 1.07 lb as 1 pound, 7 ounces, 0.8 as an eighth, and 0.45 hours as 45 minutes. In a study conducted by Hart, one student explained that 0.75 was larger than 0.8 because 0.75 was "nothing before and seventy-five; this (0.8) is nothing before and just eight" (Hart, 1981, p. 52). The student clearly saw no relationship between the zero and the other digits, nor did she see the significance of the place value of the digits. This lack of recognition of place value was underlined when students read decimal fractions aloud. Grade 7 students in this study read decimal fractions in a meaningless way. A number such as 4.23 was read "four point two three." Hart had similar findings when she asked students to read 0.29. Students in her study said "point" and then "two nine" or "twenty-nine" rather then naming the hundredths.

When discussing division of decimal fractions, students in this study talked of "moving the decimal" with little understanding of the reasoning behind the action. Grossnickle and Perry (1985) reported that one of the most common ways students treated decimal fractions in division questions was to move the decimal point by using carets. This led to errors, demonstrating the students' lack of understanding of the concepts behind the procedures.

Students had very limited representations for decimal fractions. Hart (1981) found approximately 35% of 12- and 13-year-olds were able to correctly represent three decimal fractions in a division question. The others assigned objects such as apples, candies, and books to the numbers. Even when asked to represent decimal fractions in an addition question, the percentage of students able to do so was similar. Results reported by other researchers echo these findings. Ekenstam and Greger (1983) found students were often unable to represent decimal fractions when writing story problems. Representations such as these were common: "Patrik has 3.7 cows and 2.8 pigs," and "A person wanted to find the result of 15 ÷ 2.5. Find the result for him." Bell et al. (1984) also asked students to write story problems to correspond to open sentences. One student supplied the following story for 8.7 ÷ 59.1: There are 8.7 boxes and 59.1 wheels. How many wheels would you get in a box? Students in the present study made similar errors. One student referred to sharing 1.6 boxes of apples among 2.4 trucks. They also used decimal fractions to refer to time and quantities such as dozens. Occasionally students would ignore the decimal point altogether and represent the decimal fraction as a whole number, a finding common to the research (Bell et al., 1984). Seventh graders in this study clearly had similar difficulty representing fractional parts of a whole, and could not see the conflicts in logic and place value presented by their representations.

This inability to represent decimal fractions led to further difficulty. Students were unable to check the reasonableness of their responses because they had little intuitive sense of the size of the numbers. Because of this, they tended to rely on procedures and whole number generalizations when solving problems. Hiebert and Lefevre (1986) believe that if students' representations have rich associations, these associations guide the selection of procedures when solving problems. Conversely, the opposite appears to be true. The absence of a rich network of representations leads to the inability to adequately interpret and solve story problems.

Students' Understanding of Problem Solving Requiring Division

Fischbein et al. (1985) found an intervening intuitive model explained the difficulties students experience when attempting to solve most translation problems. The dominant model for division at the Grade 7 level is partition, a finding common to this study. The authors note that although implicit models may sometimes facilitate problem solving, they often divert or block the solution process when contradictions emerge between the model and the solution algorithm. By using this model students are able to solve partitive problems in which the divisor is a whole number and is less than the dividend, but the model fails in situations where the divisor is a decimal fraction or is greater than the dividend.

Students in this study were able to identify division as the required operation in partitive situations which involved no more than one decimal fraction. Beliefs such as the divisor must be less than the dividend influenced their ability to represent problems. The most common error was reversing the terms in the division open sentence. These findings mirror those in the literature. Ekenstam and Greger (1983) reported that students who could solve a one-step translation problem involving whole numbers could not solve the same problem when decimal fractions were substituted. They sometimes chose another operation or simply had no idea of the approach to take. Greer (1987) reported that when students were asked to write the calculations for two similar rate problems, 57% of students were able to write the correct calculation when the divisor was less than the dividend whereas only 20% could do so when the divisor was greater than the dividend. Students opted instead for reversed division, or multiplication. Bell et al. (1984) also found that translation problems requiring the division of a smaller number by a larger one elicited reversals. Students chose multiplication over division if small decimal fractions were involved, except in cases where both numbers were less than one, or when one number would evenly divide the other if the decimal points were ignored (eg. 2.4, 0.48). In these cases, students reversed the terms of the division question. In a study conducted by Bell et al. in 1981 students had difficulty seeing what operation was

required in division problems. The authors believe this was partially due to a lack of intuitive sense of the size of decimal numbers. Other factors which influenced students' ability to solve translation problems were a misunderstanding of the symbolisms for division, an unawareness of the importance of order of appearance of numbers, conceptual misunderstandings related to place value, and beliefs such as "division makes smaller."

Writing story problems for given division open sentences was a difficult task for students in this study. A number of researchers have reported similar findings. Ekenstam and Greger (1983) asked students to write problems to match arithmetical statements. When these statements involved decimal fractions the students had difficulty. Many said they could not write problems if they were not able to use money. Some students in their study used decimal fractions inappropriately, assigning objects normally measured by whole numbers to decimal fractions. In a division example, 15 ÷ 2.5, some students viewed the problem from a partitive perspective by sharing 15 kroner among 2.5 people. Others created computational problems which directed the reader to help a person find the answer to 15 ÷ 2.5.

In this study, the students' reliance on the partitive model of division was the main cause of their lack of ability to write story problems for some open sentences. Because the partitive model implies a whole number divisor and a dividend greater than that divisor, students were unable to write problems for questions which did not fall into this category. Some students reversed the terms and others used a different operations or wrote unrelated problems. Bell et al. (1984) asked students to write division story problems for a variety of open sentences. Students found questions with whole number divisors easier than those with decimal fraction divisors, a finding which is similar to results from the present study. Only about 15% of students in the study conducted by Bell et al. could write problems which reflected the division of one decimal fraction by a larger one. Fewer than 15% of seventh graders in this study could do so.

Grade 7 students in this study did not interpret or accomodate the remainder in a problem without prompting. They were content with responses calculated to a few decimal

places, or written in the form "10 r 2." Students did not map their answers back to the problem context. The 1983 NAEP results on problem solving also suggest students do not adequately identify the unknown in a problem. Although 70% of students correctly performed the calculation in a problem concerning the number of buses required to transport 1128 soldiers, only 23% of students accounted for the remainder in solving the problem (Lindquist et al., 1983). Silver (1988) found that students could express remainders in routine computational settings, but experienced difficulty when the computation is embedded in a problem situation. Silver attributed this difficulty to the fact that the answer must be constructed from the data supplied by the computation and the story situation. Students in Silver's study tended to map only from the story text to the mathematical model and failed to return to the story text.

Students did not record any evidence of planning or looking back independently. This is a common finding in the research. In their investigation of checking back, Ekenstam and Greger (1983) interviewed children and presented them with solved problems. The students were to decide if the problems had been solved correctly. Nearly all children checked only the numerical compuations. Many students had poor estimation skills, especially with respect to decimal fractions. In the Second NAEP results (Carpenter et al., 1980), looking back was one of the stages least attended to by students. It was reported that students had more difficulty estimating whether an answer was reasonable than they did performing calcuations. This difficulty inhibited students' ability to check an answer by comparing it to an estimate.

Teaching Strategies

The most successful strategies used in this study included having students explain and justify their thinking in small group and whole class settings, writing story problems for open sentence, and using the word frame to solve translation problems. These strategies encouraged students to think about the meaning of division and helped them develop their understanding of decimal fractions.

The importance of the language became evident when students adopted a term used by the researcher when explaining partitive and quotitive division. The term "dealing out" was used to refer to a partitive situation because it provided a clear image of sharing among the members of a group. It was an unfortunate choice in terms because students became fixed on the notion of dealing out cards one at a time and assumed that if more than one was dealt at one time, the problem was quotitive. This experience demonstrated how easily students can be influenced through the instructor's choice of language and models. Careful thought and reflection on the choice of language is required.

Sequencing instruction is another area which requires reflection when planning. Thinking to build on whole number understanding, the researcher chose to introduce decimal fraction division by first dividing decimal fractions by whole numbers. Division of one decimal fraction by a smaller one was then introduced. The decision to parallel a question like $2.8 \div 7$ with $2.8 \div 0.7$ seemed logical. It was effective in that it revealed student beliefs, but it may not have been the best instructional decision. Although these questions were easier for students than dividing one decimal fraction by a larger one, this approach reinforced the students' partitive model of division. In retrospect, it would have been better to introduce questions of all types concurrently. For example, a question such as $2.8 \div 0.7$ could have been introduced first, then followed immediately by $2.8 \div 7$, and a question such as $0.4 \div 8$ could have been followed by $0.4 \div 0.08$.

Implications for Instruction

The assessment of students' understanding of division should form the core of instruction. Among those things that need to be assessed are:

a) students' models of division and their understanding of division notation;

b) beliefs such as "the divisor must be less than the dividend";

c) the relationship between whole numbers and decimal fractions;

d) the effect of decimal fractions on students' beliefs involving division.

These beliefs and understandings can be elicited through individual interviews, small group work and observation, and whole class discussions in which students are asked to explain and defend their thinking. All of these strategies can be employed in the normal course of instruction, and can be carried out by the regular classroom teacher.

The students' reliance on procedures when dividing suggests a lack of conceptual development. Students should work with division in contextual situations using manipulatives for a longer period of time before the algorithm is introduced. Delaying the formal introduction of the division algorithm to Grade 5 would allow students more time to develop models for both quotitive and partive division. Translation problems involving both whole numbers and decimal fractions could be solved by using materials and the strategy "Act It Out." Procedures, when finally introduced, must be linked to concepts in the course of instruction. Symbols must be paired with physical embodiments, and efforts should be made to tie new knowledge to students' existing structures. Bridging strategies such as the word frame will have to be developed to enable students to move from contextual division situations to symbolic representations.

The development of models for partitive and quotitive division should become a focus of instruction earlier in a student's schooling. Decimal fractions are introduced in Grade 3 in the British Columbia mathematics curriculum; teachers can begin the exploration of quotition at the point in time. By presenting division in contexts requiring both forms of division and modeling the actions for each, teachers should help students develop a more complete conceptual framework for division. Particular emphasis should be placed on quotition because it does not appear to occur as frequently as partition does in the student's every day life. The appropriate language associated with the two types of division should be stressed when working with manipulatives and when solving problems. Creating story problems to reflect division, both with manipulatives and symbols, should be encouraged because it enables students to test and check and refine their models of division.

The introduction of decimal fractions should focus on models. Manipulatives such as base 10 blocks and the many forms of metric measurement provide rich opportunities for students to

explore part-whole relationships with decimal notation. The appropriate language for referring to decimal fractions should also be stressed. Story problems involving decimal fractions should be presented by the teacher, and created by students. Students must be given opportunities to test their representations of decimal fractions in division contexts.

Beliefs such as "division makes smaller" and "the divisor must be less than the dividend" probably cannot be prevented. However, through earlier experience with decimal fraction divisors less than one, and experiences requiring the division of one whole number by a larger one, students will be able to confront these beliefs earlier in their career and make the appropriate accommodations. These questions must be introduced through meaningful contexts and should be explored with manipulatives and calculators. When beliefs such as these develop, teachers must use conflict-creating situations which force students to examine their commitment to their beliefs.

Calculation should be de-emphasized in both computation and problem solving. Students should focus on estimation, and should perform operations with calculators. Differences between estimates and calculations should be explored. Students must be taught how to estimate; they require more sophisticated strategies than "rounding up" or "rounding down." Greater emphasis should be placed on the planning and looking back stages of problem solving. Formulating estimates is one component of this, but students must also be encouraged to defend their reasoning and justify their answers.

Teachers should make use of partner and small group work in addition to whole class activities. The cooperative nature of small group work de-emphasizes the focus on answers and encourages discussion on the solution process. Because students have more opportunities to express their thinking, incomplete conceptions and misleading solution paths can be exposed and corrected in a non-threatening setting. The development of mathematical language is facilitated and encouraged through increased opportunity and exposure to the language of others. Metacognitive skills develop as students listen to peers and attempt to relate what they hear to

their own thinking, as they explain their reasoning to others, and as they evaluate the comments of others.

Recommendations for Further Research

The classroom case study is essentially an exploratory study in a limited environment. For this reason, results cannot be generalized to a specific population. It is clear, however, that from findings in this study and those reported in the research, several topics bear further investigation.

The development of students' models for division could be explored. Questions which might be asked include:

1. Why do students rely of the partitive model of division?
   a) What role does the textbook have in creating this reliance?
   b) What role does real-life experience have in creating this reliance?
   c) What role do other experiences in school mathematics have in creating this reliance?
2. Can quotitive division be successfully introduced to students at an early age? If so, what strategies faciliate this introduction?

A conceptual focus on division instruction should result in more complete conceptions of division, and should prevent the misapplication and distortion of rules and procedures. Questions which might be asked include:

1. Does a conceptual focus on instruction of division result in better understanding of division procedures?
2. Do students who have been introduced to division using a conceptual focus develop alternate conceptions or beliefs such as "division makes smaller"?
3. What instructional timelines are required to reverse erroneous beliefs based on incomplete or alternate conceptions?
4. How can conceptual understanding of division be adequately assessed?

Students' representations for decimal fractions require further investigation. Questions which might be asked include:

1. What strategies can be employed to help students develop varied and meaningful models for decimal fractions?
2. What is the optimum grade level for successful introduction of decimal fraction concepts?
3. Does the use of the calculator facilitate development of decimal fraction concepts?

Various teaching and assessment strategies employed in this study require further investigation. Questions which might be asked include:

1. Exploration of conceptual understanding thorough discussion requires a great deal of time. Can written inventories be developed which assess students' conceptual understanding of division and other operations?
2. Does writing story problems for division open sentences contribute to the development of models of division, or does it simply reflect models of division?
3. Does writing story problems from an early age contribute to the development of models of division?
4. Do students who solve problems in pairs or small groups increase their ability to solve problems individually?
5. What strategies can be used to alter erroneous beliefs or alternate conceptions?
6. How can the development of metacognition be encouraged in students?

Finally, the classroom case study itself bears further investigation. Involving teachers in researching their own classrooms should contribute to a greater depth of understanding of students' thinking, and should result in more reflective teaching. Investigations into the use of this model linked to student achievement and to self-evaluation by teachers would be worthwhile.

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APPENDIX A

THE PRETEST

Pretest **Name** _____

A. Solve. Write remainders in the following format: $5\overline{)12}$ 2 R 2

1. $6\overline{)119}$
2. $15\overline{)345}$
3. $21\overline{)2205}$
4. $27\overline{)184}$
5. $18\overline{)306}$

B. Solve. Give answers to 2 decimal places.

1. $4\overline{)29}$
2. $6\overline{)115.5}$
3. $.7\overline{)2.28}$
4. $2.1\overline{)7.35}$
5. $1.5\overline{)0.45}$

C. Solve. Show all of your work. Circle the answer.

1. Pencils are sold in packages of 12. Mr. Wilson needs 185 pencils. How many packages must he buy?

2. A metal chain is 31.68 cm long. It is made from 9 links. How long is each link?

Name _____

3. Mrs. Wood's class collected items for the food bank. The students were able to pack 241 items into 26 bags. If the students made sure the food was distributed equally, how many items were in each bag?

4. After 8 laps around the training track, the odometer on Sharon's bicycle showed 10.8 km. How long was each lap?

D. For each of the following questions, **do not** calculate the answer. Choose one of the multiple choice options to indicate which operation or combination of operations should be performed in order to find the solution to the problem.

1. How long will you have to wait in line to ride a roller coaster if there are 75 people in front of you? Each ride lasts 5 minutes and can hold 25 people.

   a. (75 - 25) x 5
   b. (75 + 25) ÷ 5
   c. 75 $\overline{)25}$ x 5
   d. (75 ÷ 25) x 5
   e. (5 x 25) - 75
   f. (5 x 75) - 25

2. In two 8 hour working days, Mr. Andrews painted 5 rooms. What was the average amount of time spent painting each room?

   a. (2 + 8) ÷ 5
   b. (8 x 2) ÷ 5
   c. 8 $\overline{)2}$ x 5
   d. (8 x 2) x 5
   e. (8 ÷ 2) x 5
   f. 5 $\overline{)8}$ ÷ 2

3. The area of a room is 24.75 m. The length of one side of the room is 4.5 m and the height of the room is 3 m. What is the width of the room?

   a. 24.75 ÷ 4.5
   b. 4.5 x 3
   c. 4.5 x 24.75
   d. 24.75 - (4.5 x 3)
   e. 24.75 - 4.5
   f. 4.5 ÷ 24.75

4. 2.45 kg of raisins are packed 0.35 kg to a bag. How many bags can be filled?

   a. 2.45 + 0.35
   b. 2.45 - 0.35
   c. 2.45 $\overline{)0.35}$
   d. 0.35 x 2.45
   e. 0.35 ÷ 2.45
   f. 0.35 $\overline{)2.45}$

Name _____

5. A new tunnel is being dug at the Strike-It-Rich gold mine. In one day the miners are able to tunnel 0.275 km. At this rate, how long will it take them to complete a 2 km tunnel?

a. 0.275 x 2

b. 2 + 0.275

c. 0.275 ÷ 2

d 2 - 0.275

e. 0.275 $\overline{)2}$

f. 0.275 - 2

6. Karen is working on a macrame project that requires pieces of cord that are 0.85 m long. If she has 15 m of cord to cut into pieces, how much cord will she have left over when she is finished cutting?

a. 15 + 0.85

b. 15 ÷ 0.85

c. 15 $\overline{)0.85}$

d. 0.85 x 15

e. 15 - 0.85

f. 15 x 0.85

7. A cassette carrying case can hold 24 cassettes. I have twice as many cassettes as my sister. If there are 300 cassettes in my collection, how many carrying cases will I need to purchase?

a. (2 x 300) ÷ 24

b. 300 - (2 x 24)

c. 24 ÷ 300

d. 24 $\overline{)300}$

e. 300 x 24

f. (300 ÷ 2) ÷24

8. A baker has 4200 g of flour which will be used to make 12 cakes. How much flour will be used for each cake?

a. 4200 ÷ 12

b. 12 + 4200

c. 4200 x 12

d. 12 x 4200

e. 4200 $\overline{)12}$

f. 4200 - 12

9. An artist wants to use some very expensive paint on a series of sculptures she has made. If she has 0.27 of a litre of the paint and 30 sculptures, how much paint can she use on each?

a. 0.27 x 30

b. 30 - 0.27

c. 0.27 ÷ 30

d. 0.27 $\overline{)30}$

e. 0.27 + 30

f. 30 x 0.27

Name _____

10. A cross-country race was run over a course that was 8.4 km long. If the first runner completed the race with an average speed of 11.2 km per hour,how long did it take him to finish the race?

a. 8.4 + 11.2

b. 11.2 $\overline{)8.4}$

c. 11.2 x 8.4

d. 11.2 - 8.4

e. 8.4 x 11.2

f. 8.4 $\overline{)11.2}$

E. For the next two problems consider these numbers:

6.25 9.4 0.73 5.78 12.6 1.291

1. Choose the two numbers which will result in the **largest** possible quotient. Write the division question.

_____

2. Choose the two numbers which will result in the **smallest** possible quotient. Write the division question.

_____

F. Circle the questions below which will result in a quotient that is greater than 1.

a. 5.1 $\overline{)4.75}$

b. 3.2 ÷ 1.25

c. .995 ÷ 1.1

d. 4.7 $\overline{)6.01}$

e. 7.6 ÷ 8.3

Name _____

G. Write a story problem for each of the questions below. Do **not** solve your problems.

a. $12.5 \div 9$

_____

_____

b. $15.2 \div 3.5$

_____

_____

c. $4 \div 20$

_____

_____

d. $19 \div 3$

_____

_____

e. $1.6 \div 2.4$

_____

_____

f. $3 \div 0.6$

_____

_____

APPENDIX B

THE POSTTEST

Posttest **Name**_____

A. Solve. Write remainders in the following format: $\begin{array}{r} 2\ R\ 2 \\ 5\overline{)12} \end{array}$

1. $16\overline{)219}$

2. $24\overline{)225}$

B. Solve. Give answers to 2 decimal places.

1. $8\overline{)44}$

2. $6\overline{)11.5}$

3. $0.003\overline{)2.7}$

4. $0.7\overline{)2.28}$

5. $2.1\overline{)7.35}$

6. $1.5\overline{)0.48}$

C. Solve. Show all of your work. Check to see that the answer is reasonable. Circle the answer.

1. Cookies are sold in packages of 36. Mr. Black needs 450 cookies for a meeting. How many packages must he buy?

2. A necklace is 58.5 cm long. It is made from 13 hinged pieces of gold. How long is each piece of gold?

3. Tom works in a chocolate factory. His job is to decide how many chocolates should be packed into each box. If there are 526 chocolates and 36 boxes, how many should be put in each box?

4. After 8 circuits around the track, the train in Bobbi's new train set had travelled 21.6 metres. How long was each circuit?

Name _____

D. For each of the following questions, **do not** calculate the answer. Choose one of the multiple choice options to indicate which operation or combination of operations should be performed in order to find the solution to the problem.

1. How long will you have to wait in line to watch a film at a small theatre if there are 105 people in front of you? Each showing of the film lasts 8 minutes and the theatre can hold 25 people.

   a. (105 - 25) x 8
   b. (105 + 25) ÷ 8
   c. 105 $\overline{)25}$ x 8
   d. (105 ÷ 25) x 8
   e. (8 x 25) - 105
   f. (8 x 105) - 25

2. In three 7 hour working days, Mrs. Danson wrote 5 computer programs. What was the average amount of time spent writing each program?

   a. (3 + 7) ÷ 5
   b. (7 x 3) ÷ 5
   c. 7 $\overline{)3}$ x 5
   d. (7 x 3) x 5
   e. (7 ÷ 3) x 5
   f. 5 $\overline{)7}$ ÷ 3

3. The area of the bottom of a box is 22.5 cm^2. The length of one side of the box is 4.5 cm and the height of the box is 3 cm. What is the width of the box?

   a. 22.5 ÷ 4.5
   b. 4.5 x 3
   c. 4.5 x 22.5
   d. 22.5 - (4.5 x 3)
   e. 22.5 - 4.5
   f. 4.5 ÷ 22.5

4. 2.45 kg of jellybeans are packed 0.35 kg to a bag. How many bags can be filled?

   a. 2.45 + 0.35
   b. 2.45 - 0.35
   c. 2.45 $\overline{)0.35}$
   d. 0.35 x 2.45
   e. 0.35 ÷ 2.45
   f. 0.35 $\overline{)2.45}$

5. A new extension is being made to a highway. In one day the crew is able to construct 0.575 km of the highway. At this rate, how long will it take them to complete the 8 km extension?

   a. 0.575 x 8
   b. 8 + 0.575
   c. 0.575 ÷ 8
   d. 8 - 0.575
   e. 0.575 $\overline{)8}$
   f. 0.575 - 8

Name _____

6. Belinda is working on an art project that requires pieces of fabric that are 0.75 m long. If she has.13 m of fabric to cut into pieces, how much fabric will she have left over when she is finished cutting?

a. 13 + 0.75

b. 13 ÷ 0.75

c. 13 $\overline{)0.75}$

d. 0.75 x 13

e. 13 - 0.75

f. 13 x 0.75

7. A large packing box can hold 24 cases of pop. A case of pop sells for $4.25. If there are 300 cases of pop that must be shipped, how many packing boxes will I need?

a. ($4.25 x 300) ÷ 24

b. 300 - ($4.25 x 24)

c. 24 ÷ 300

d. 24 $\overline{)300}$

e. 300 x 24

f. (300 ÷ $4.25) - 24

8. A chemist has 6200 g of copper nitrate which will be used in 12 experiments. How much copper nitrate will be used for each experiment?

a. 6200 ÷ 12

b. 12 + 6200

c. 6200 x 12

d. 12 x 6200

e. 6200 $\overline{)12}$

f. 6200 - 12

9. A ski team wants to try using some very expensive wax on their skis just before an important race. If they have 0.97 of a litre of the wax and 20 pairs of skis, how much wax can be used on each pair of skis?

a. 0.97 x 20

b. 20 - 0.97

c. 0.97 ÷ 20

d. 0.97 $\overline{)20}$

e. 0.97 + 20

f. 20 x 0.97

10. A swim race took place on a course that was 2.4 km long. If the first swimmer completed the race with an average speed of 3.9 km per hour, how long did it take him to finish the race?

a. 2.4 + 3.9

b. 3.9 $\overline{)2.4}$

c. 3.9 x 2.4

d. 3.9 - 2.4

e. 2.4 x 3.9

f. 2.4 $\overline{)3.9}$

Name _____

E. For the next two problems consider these numbers:

6.25 9.4 0.73 5.78 12.6 1.291

1. Choose the two numbers which will result in the **largest** possible quotient. Write the division question. _____

2. Choose the two numbers which will result in the **smallest** possible quotient. Write the division question. _____

F. Circle the questions below which will result in a quotient that is greater than 1.

a. $5.1 \overline{)4.75}$ b. $3.2 \div 1.25$ c. $.995 \div 1.1$

d. $4.7 \overline{)6.01}$ e. $7.6 \div 8.3$

G. Write a story problem for each of the questions below. Do **not** solve your problems.

a. $11.4 \div 8$

_____

_____

b. $15.2 \div 3.5$

_____

_____

c. $6 \div 36$

_____

_____

d. $1.6 \div 2.4$

_____

_____

e. $8 \div 0.7$

_____

_____

Name_____

H. For each of the following questions answer **true** or **false**. If you answer **false** give a reason or an example to explain your thinking.

1. In a word problem which contained the numbers 80 and 3 you would probably divide these two numbers.

true **false**

2. It is possible to divide a small number by a large number. **true** **false**

3. $4.5 \div 3.1$ is the same question as $4.5\overline{)3.1}$ **true** **false**

4. When you have two numbers, you always divide the larger one by the smaller one.

true **false**

5. Whenever you divide two numbers, the quotient (answer) is always smaller than the dividend (the number you divided).

true **false**

APPENDIX C

STUDENT PRACTICE PAGES

Practice Page # 1

Division and Problem Solving - Partitive and Quotitive Division Name_____

Date_____

For each of the following problems, identify whether the partitive (sharing) or the quotitive (taking out groups) method of division implied. Solve the problem. Show all of your work.

1. Amy and Sara have been collecting hockey cards for many years. They've decided, however, that hockey cards are decidedly "uncool" for Grade 7s so they are going to give them to the 27 students in Grade 3. How many cards will each student get if Amy has 218 cards and Sara has 242 cards?

2. Stan wanted to make sure he had enough Hallowe'en candy to last him 24 days. To do this, he separated his haul of 410 items into 24 brown bags. How many munchies did he have per day?

3. Dave got a job on a chicken farm. His job was to fill egg cartons and pack them into larger boxes. If each carton holds 12 eggs, how many can he fill with 336 eggs? If the larger boxes hold 6 dozen eggs, how many of these can Dave fill?

4. Anthony and Gary entered a yo-yo marathon contest. After 1558 hours of continuous yo-yoing, they finally collapsed. How many days did the two boys last?

Practice Page # 2

Partitive, Quotitive: Which is Which?

Name_____

Date _____

- On a banana tree there were 67 bananas and 5 hungry gorillas. How many bananas would each gorilla get?

- We have 67 students and 5 classrooms. How many students are in each class?

- A girl went to the market and wanted to buy 5 pies. She had 67 dollars. How many pies could she buy?

- Two people were in a yo-yo contest for 67 hours. How many 5 hours were there?

- The chocolate factory has 67 pieces of chocolate and fits 5 in each box. How many boxes will they need?

- The ice cream store has 67 ice cream cones and 5 parties to deliver to equally. How much does each party get?

- The basketball coach had 5 packs of gum and he wanted to share it between 67 people?

- George has 67 marbles. He wants to divide them between 5 kids. How much would each kid get?
- 67 credit cards, 5 lines. How many credit cards does each line get?

- 67 people, 5 flights everyday to Los Angeles. How many people equally can go on each flight?

Practice Page # 3

Problem Solving Name_____

Date _____

- The students in Mme Fortin's class have been hatching chicks as part of a science unit. Of the 12 eggs incubated, so far only 5 have hatched.
- The chicks are ready to begin eating chicken feed. Mme Fortin has bought 570 g of feed that is intended to last the 7 days we will have the chicks.
- The combined mass of the chicks is 768 g on day 2. Their combined mass will be approximately 1475 on day 7.
- The chicks took varying amounts of time to hatch:
  2 h 38 min., 3 h 15 min., 2 h 47 min., 3 h 05 min., 2 h 52 min.

Consider the information above carefully. When you answer each problem remember to show your thinking and your work. Comment on any remainders.

1. If we incubated 60 eggs, how many would you expect to hatch? Why?

2. How much does 1 chick weigh on day 2?

3. If all chicks had hatched, what would their combined mass be on day 7?

4. What is the average weight gain for 1 chick from day 2 to day 7?

5. How much food did Mme Fortin think each chick would eat in 1 day?

6. Suppose a 6th chick hatched. How long would you expect it to take? Why?

Practice Page # 4

Problem Solving With Decimals

Name _____

Date _____

When solving these problems **do not** try to work them out with pencil and paper. Use the base 10 blocks. Draw sketches of what you did to solve the problems.

1. A swinging metronome makes 15 clicks in 6.3 seconds. At this rate, how long will it take for the metronome to make 24 clicks?

2. Jessie scored a total of 57 points in the basketball season. If there were 12 games in the season including the final game, what was her average number of points scored per game?

3. Glen paid $6.25 for 125 g of a specialty cheese. How much would 1 g cost? How much would one kg cost?

Practice Page # 5

Dividing a Decimal Fraction
By Another Decimal Fraction

Name_____

Date_____

For each question below first give an estimate. Round decimals to the nearest whole number and divide mentally. Put your estimate in the ☁.

Then, rewrite each question in the form of an equivalent division question that would be easier to solve.

Do not work the answers out with pencil and paper. Check to see that each form of the question yields the same answer by using a calculator. Record the answer on the _____ .

1. $0.4 \overline{)7.2}$ ☁ _____

$\overline{)}$

2. $1.5 \overline{)9.1}$ ☁ _____

$\overline{)}$

3. $0.5 \overline{)0.84}$ ☁ _____

$\overline{)}$

4. $0.25 \overline{)6.8}$ ☁ _____

$\overline{)}$

5. $0.07 \overline{)14.9}$ ☁ _____

$\overline{)}$

6. $2.3 \overline{)0.97}$ ☁ _____

$\overline{)}$

7. $7.9 \overline{)64.5}$ ☁ _____

$\overline{)}$

8. $0.99 \overline{)3.98}$ ☁ _____

$\overline{)}$

9. $4.5 \overline{)0.85}$ ☁ _____

$\overline{)}$

10. $1.8 \overline{)17.6}$ ☁ _____

$\overline{)}$

Are your estimates and your answers close? Can you account for any discrepancies?

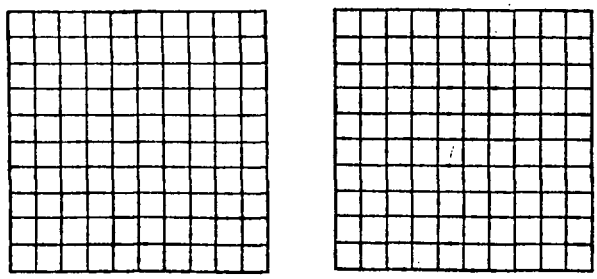
Practice Page # 6

Show by shading:

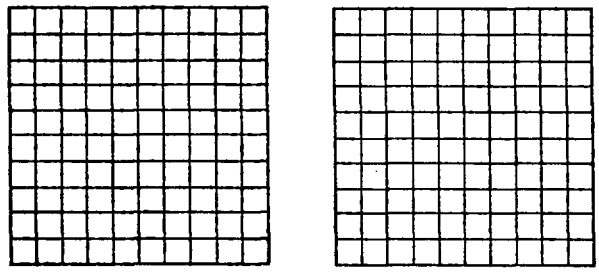
Name_____

Date_____

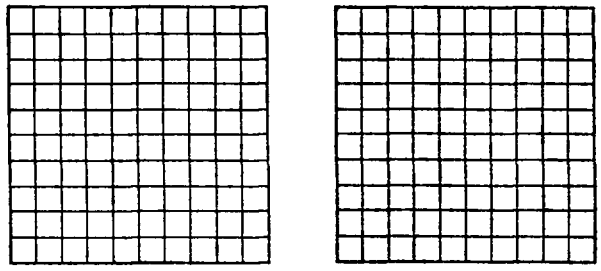
1. 1.6 ÷ 0.4



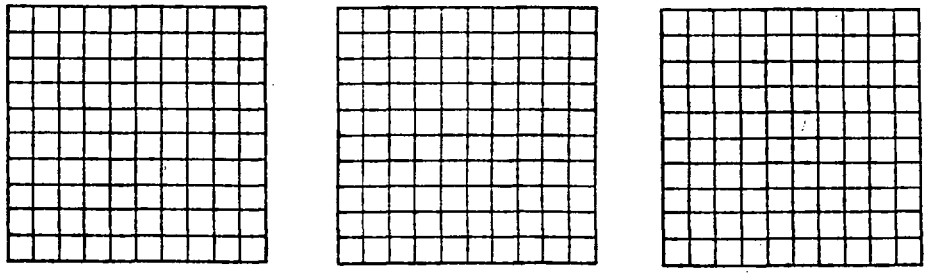
2. 1.2 ÷ 0.4



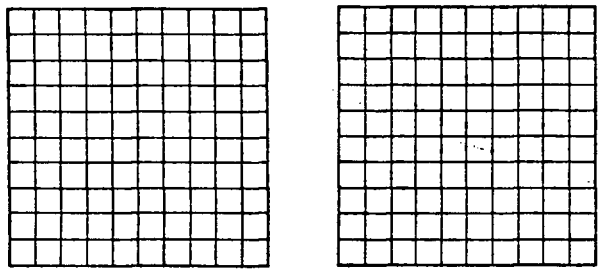
3. 2.0 ÷ 0.5



4. 2.7 ÷ 0.9



5. 1.8 ÷ 0.1



Practice Page # 7

Problem Solving

Name_____

Date _____

1. The thickness of a dollar bill is 0.1 millimetre. How much money is in a 50 millimetre stack of dollar bills?

° easy ° just right ° difficult

2. A sheet of paper is 0.075 cm thick. About how many sheets would it take to make a stack 5 cm thick?

° easy ° just right ° difficult

3. How many 15 ¢ stamps can you buy for $10.00?

° easy ° just right ° difficult

4. Four students were going on a camping trip. They agreed to share the weight of supplies evenly, but could not agree what fair loads would be. Here are the supplies. Can you determine 4 fair loads?

tent - 7.0 kg	stove - 2.75 kg	charcoal - 4.0 kg
guitar - 5.3 kg	4 sleeping bags - 1.75 kg each	hammock - 3.5 kg
axe - 2.75 kg	utensils - 2.2 kg	food - 4.75 kg

° easy ° just right ° difficult

Practice Page # 8

Dividing With Decimal Fractions

Name_____

Date_____

1. $0.6 \overline{)3.9}$

2. $3.4 \overline{)22.1}$

3. $4.6 \overline{)3.45}$

4. $0.04 \overline{)6.0}$

5. $1.1 \overline{)5.5}$ is the same as:

____ $\overline{)55.0}$

$0.11 \overline{)\_\_\_\_}$

$110 \overline{)\_\_\_\_}$

$0.011 \overline{)\_\_\_\_}$

____ $\overline{)0.55}$

How do you know? Be prepared to defend your answer. (Know <u>what</u> and <u>why</u>!)

Practice Page # 9

Problem Solving Review

Name_____

Date_____

1. A jar containing 40 marbles has a mass of 155 g. The same jar with 20 marbles in it has a mass of 95 g. What is the mass of the jar?

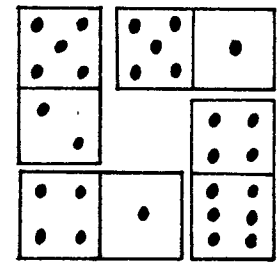
2. A square garden has a perimeter of 12 m.
   a. What is the area of the garden?
   b. Fence posts are placed 1 m apart around the outside. How many posts are needed?

3. Two trains leave the station at the same time and travel in opposite directions. One train travels at 76 km/h and the other travels at 67 km/h. How far apart will they be in 3 hours?

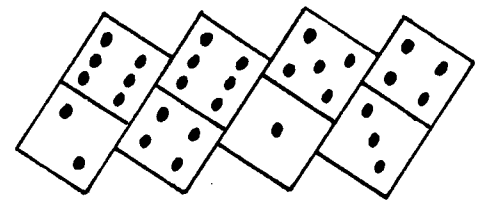
4. Six boys fill 6 notebooks in 6 weeks, and 4 girls fill 4 notebooks in 4 weeks. How many notebooks will 12 boys and 12 girls fill in 12 weeks?

5. All of my pets are dogs except two, all are cats except two, and all are budgies except two. How many pets do I have?

6. The sum of each of the sides is the same in the square.



Arrange the four dominoes below so that the sum of each of the sides is the same.



7. Which detergent is the best buy?

8. A carpenter can saw a board at the rate of 1.5 min./cut. How long will it take to cut a 4 m board into 11 pieces of the same size?

Practice Page # 10

Division Review

Name_____

Date_____

Write your estimates in the cloud.

1. $1.5 \overline{)34.5}$

2. $6.3 \overline{)34.65}$

3. $0.09 \overline{)13.95}$

4. $8.4 \overline{)5.376}$

Practice Page # 11

Problem Solving - Division 11's Creations Name_____

Date_____

Read each problem carefully. Decide if it contains division. Does it make sense? If it does not make sense, identify the problem. If it **can** and **should** be solved, estimate an answer and use a calculator to solve it.

1. If you have 10.7 cm of ribbon and you want to have enough ribbon for 22 gifts. How many cm of ribbon would each gift get? How much more ribbon do you need to have 10 cm per gift?

2. The woman went to the market. That day her kids had asked her to buy some jelly beans for them. Being the good mommy she is she bought 10.7 g of jelly beans. She had 22 children and wanted to split them evenly. How much would each kid get?

3. Your diet says you can have 0.35 g of cereal a day. How many days would it take to finish 4.5 g?

4. There are 22 gumballs in a jar. It weighs 10.7 g. If there were 23 gumballs, how much would it weigh?

5. There were 22 students getting report cards. There were 4.5 A's to be shared out evenly. How many A's to each person?

6. There are three kids going to the market. They had to walk 10.7 km and they walked 0.35 km per hour. How long did it take to get to the market?

7. 22 trucks came to a gas station. All of the trucks get 0.35 kg of gas all together. How much gas did one truck get?

Practice Page # 12

Problem Solving

Name_____

Date_____

1. Diana is ordering graph paper. She can buy a 12-package box will 300 sheets per package for $72.00, or a 6-package box with 200 sheets per package for $30.00. Which would be the better buy?

Work Thinking

2. Books are being packed and shipped to a new school. Each book weighs 1.3 kg. There are 24 books to a carton and there are 15 cartons. The local shipping company is charging $70.20 to move them. What is the price per kilogram for shipping the books?

Work Thinking

3. The total prize money in a golf tournament was $83 950.00. Eagle Jones won first place and collected $23 725.00. What was the average share won by the 15 other players?

Work Thinking

Practice Page # 13

Problem Solving

Name_____

Date_____

Read each problem carefully. Get a picture in your mind of what is happening. Draw a sketch - it really helps. If the problem involves division, write it out this way to help you get the number in the right places: # of marbles per kid) marbles to be shared / # of kids sharing **Why do you think you are right?**

1. One turn of a screw moves the screw into some wood 0.16 of a cm. How many turns are needed to move a screw 2.4 cm into a wooden block?

   I think I'm right because_____

2. Karen weighs 54.7 kg. Through fitness testing she knows 21.88kg of her weight is muscle. She says she can predict the muscle mass of her friend Cindy who weighs 48.6 kg. How could she do this? What number did she predict?

   I think I'm right because_____

3. To make a fire extinguisher, you can mix baking soda and vinegar. Susan had 650.5 ml of vinegar and 514.75 g of baking soda. These quantities are enough to make 5 extinguishers. How much of each ingredient would she use for 1 extinguisher?

   I think I'm right because_____

4. Bob wants to buy a mountain bike on sale for $128.00. His parents agree to pay half of the cost. He plans to pay for his share in 5 equal payments. How much will each payment be?

   I think I'm right because_____

5. A laser beam removes 0.004 mm of the surface of a piece of metal on each pass of the beam. The metal is 1.2 mm thick. How many passes would it take to have no metal left?

   I think I'm right because_____