Monetary Policy Analysis in a Small Open Economy: Development and Evaluation of Quantitative Tools

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ABSTRACT

This doctoral thesis consists of four papers, the unifying theme of which is the development and evaluation of quantitative tools for purposes of monetary policy analysis and inflation targeting in a small open economy. These tools consist of alternative macroeconometric models of small open economies which either provide a quantitative description of the monetary transmission mechanism, or yield a mutually consistent set of indicators of inflationary pressure together with confidence intervals, or both. The models vary considerably with regards to theoretical structure, and are estimated with novel Bayesian procedures. In all cases, parameters and trend components are jointly estimated, conditional on prior information concerning the values of parameters or trend components.

The first paper develops and estimates a dynamic stochastic general equilibrium or DSGE model of a small open economy which approximately accounts for the empirical evidence concerning the monetary transmission mechanism, as summarized by impulse response functions derived from an estimated structural vector autoregressive or SVAR model, while dominating that SVAR model in terms of predictive accuracy. The primary contribution of this first paper is the joint modeling of cyclical and trend components as unobserved components while imposing theoretical restrictions derived from the approximate multivariate linear rational expectations representation of a DSGE model.

The second paper develops and estimates an unobserved components model for purposes of monetary policy analysis and inflation targeting in a small open economy. The primary contribution of this second paper is the development of a procedure to estimate a linear state space model conditional on prior information concerning the values of unobserved state variables.

The third paper develops and estimates a DSGE model of a small open economy for purposes of monetary policy analysis and inflation targeting which provides a quantitative description of the monetary transmission mechanism, yields a mutually consistent set of indicators of inflationary pressure together with confidence intervals, and facilitates the generation of relatively accurate forecasts. The primary contribution of this third paper is the development of a Bayesian procedure to estimate the levels of the flexible price and wage equilibrium components of endogenous variables while imposing relatively weak identifying restrictions on their trend components.

The fourth paper evaluates the finite sample properties of the procedure proposed in the third paper for the measurement of the stance of monetary policy in a small open economy with a
Monte Carlo experiment. This Bayesian estimation procedure is found to yield reasonably accurate and precise results in samples of currently available size.
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PREFACE

The last decade has witnessed a revival of academic interest in monetary policy analysis, stimulated by revolutionary developments in theoretical and empirical macroeconomics. From the theoretical perspective, the incorporation of short run nominal price and wage rigidities into dynamic stochastic general equilibrium or DSGE models based on rigorous microeconomic foundations has provided an internally consistent framework for the analysis of the monetary transmission mechanism, which describes the dynamic effects of unsystematic variation in the instrument of monetary policy on indicators and targets, and the optimal conduct of monetary policy. From the empirical perspective, the development of Bayesian procedures to accurately and precisely estimate DSGE models has legitimized their emerging role as quantitative monetary policy analysis tools.

Although the quantitative monetary policy analysis literature is advancing rapidly, significant theoretical and empirical problems remain unsolved. On the theoretical front, DSGE models which yield empirically adequate predictions at all frequencies, as opposed to only cyclical frequencies, remain to be developed. On the empirical front, Bayesian procedures which fully exploit the information content of the levels of observed endogenous variables, while emphasizing the predictions of DSGE models at cyclical frequencies, and deemphasizing them at trend frequencies, are required.

Recent developments in the analysis of monetary policy in open economies have to some extent lagged behind those in the analysis of monetary policy in closed economies, particularly from the empirical perspective. In an open economy, the existence of international trade and financial linkages introduces additional channels through which variation in the instrument of monetary policy affects indicators and targets, complicating the analysis of the monetary transmission mechanism and the optimal conduct of monetary policy. Yet the recent adoption of explicit quantitative inflation targets by the central banks of many economies, particularly those of relatively small and open economies, calls for accurate and precise indicators of inflationary pressure in such economies, together with accurate and precise quantitative descriptions of the monetary transmission mechanism.

This doctoral thesis consists of four papers, the unifying theme of which is the development and evaluation of quantitative tools for purposes of monetary policy analysis and inflation targeting in a small open economy. These tools consist of alternative macroeconometric models of small open economies which either provide a quantitative description of the monetary transmission mechanism, or yield a mutually consistent set of indicators of inflationary pressure.
together with confidence intervals, or both. The models vary considerably with regards to theoretical structure, and are estimated with novel Bayesian procedures. In all cases, parameters and trend components are jointly estimated, conditional on prior information concerning the values of parameters or trend components.

The first paper develops and estimates a DSGE model of a small open economy which approximately accounts for the empirical evidence concerning the monetary transmission mechanism, as summarized by impulse response functions derived from an estimated structural vector autoregressive or SVAR model, while dominating that SVAR model in terms of predictive accuracy. The model features short run nominal price and wage rigidities generated by monopolistic competition and staggered reoptimization in output and labour markets. The resultant inertia in inflation and persistence in output is enhanced with other features such as habit persistence in consumption, adjustment costs in investment, and variable capital utilization. Incomplete exchange rate pass through is generated by short run nominal price rigidities in the import market, with monopolistically competitive importers setting the domestic currency prices of differentiated intermediate import goods subject to randomly arriving reoptimization opportunities. Cyclical components are modeled by linearizing equilibrium conditions around a stationary deterministic steady state equilibrium which abstracts from long run balanced growth, while trend components are modeled as random walks while ensuring the existence of a well defined balanced growth path. Parameters and trend components are jointly estimated with a novel Bayesian procedure, conditional on prior information concerning the values of parameters and trend components.

The primary contribution of this first paper is the joint modeling of cyclical and trend components as unobserved components while imposing theoretical restrictions derived from the approximate multivariate linear rational expectations representation of a DSGE model. This merging of modeling paradigms drawn from the theoretical and empirical macroeconomics literatures confers a number of important benefits. First, the joint estimation of parameters and trend components ensures their mutual consistency, as estimates of parameters appropriately reflect estimates of trend components, and vice versa. It has been shown that decomposing integrated observed endogenous variables into cyclical and trend components with atheoretic deterministic polynomial functions or low pass filters may induce spurious cyclical dynamics, invalidating subsequent estimation, inference and forecasting. Second, basing estimation on the levels as opposed to differences of observed endogenous variables may be expected to yield efficiency gains. A central result of the voluminous cointegration literature is that, if there exist cointegrating relationships, then differencing all integrated observed endogenous variables prior to the conduct of estimation, inference and forecasting results in the loss of information. Third, the proposed unobserved components framework ensures stochastic nonsingularity of the
resulting approximate linear state space representation of the DSGE model, as associated with each observed endogenous variable is at least one exogenous stochastic process. Stochastic nonsingularity requires that the number of observed endogenous variables used to construct the loglikelihood function associated with the approximate linear state space representation of a DSGE model not exceed the number of exogenous stochastic processes, with efficiency losses incurred if this constraint binds. Fourth, the proposed unobserved components framework facilitates the direct generation of forecasts of the levels of endogenous variables as opposed to their cyclical components together with confidence intervals, while ensuring that these forecasts satisfy the stability restrictions associated with balanced growth. These stability restrictions are necessary but not sufficient for full cointegration, as along a balanced growth path, great ratios and trend growth rates are time independent but state dependent, robustifying forecasts to intermittent structural breaks that occur within sample.

The second paper develops and estimates an unobserved components model for purposes of monetary policy analysis and inflation targeting in a small open economy. Cyclical components are modeled as a multivariate linear rational expectations model of the monetary transmission mechanism, while trend components are modeled as random walks while ensuring the existence of a well defined balanced growth path. Although not derived from microeconomic foundations, this unobserved components model of the monetary transmission mechanism in a small open economy arguably provides a closer approximation to the data generating process than existing DSGE models, as fewer cross-coefficient restrictions are imposed. Full information maximum likelihood estimation of this unobserved components model, conditional on prior information concerning the values of trend components, provides a quantitative description of the monetary transmission mechanism in a small open economy, yields a mutually consistent set of indicators of inflationary pressure together with confidence intervals, and facilitates the generation of relatively accurate forecasts.

The primary contribution of this second paper is the development of a procedure to estimate a linear state space model conditional on prior information concerning the values of unobserved state variables. This prior information assumes the form of a set of deterministic or stochastic restrictions on linear combinations of unobserved state variables. In addition to mitigating potential model misspecification and identification problems, exploiting such prior information may be expected to yield efficiency gains in estimation.

The third paper develops and estimates a DSGE model of a small open economy for purposes of monetary policy analysis and inflation targeting. This estimated DSGE model provides a quantitative description of the monetary transmission mechanism in a small open economy, yields a mutually consistent set of indicators of inflationary pressure together with confidence intervals, and facilitates the generation of relatively accurate forecasts. In an extension and
refinement of the DSGE model developed in the first paper, cyclical components are decomposed into subcomponents identified by the presence or absence of short run nominal price and wage rigidities, while investment in housing and investment in capital are separately modeled. In addition to being a necessary step towards providing a more detailed quantitative description of the monetary transmission mechanism in a small open economy, separately modeling investment in housing and investment in capital has implications for the measurement of the stance of monetary policy. Cyclical components are modeled by linearizing equilibrium conditions around a stationary deterministic steady state equilibrium which abstracts from long run balanced growth, while trend components are modeled as random walks while ensuring the existence of a well defined balanced growth path. Parameters and unobserved components are jointly estimated with a novel Bayesian procedure, conditional on prior information concerning the values of parameters and trend components.

The primary contribution of this third paper is the development of a procedure to estimate the levels of the flexible price and wage equilibrium components of endogenous variables while imposing relatively weak, and hence relatively credible, identifying restrictions on their trend components. Based on an extension and refinement of the unobserved components framework proposed in the first paper, this estimation procedure confers a number of benefits of particular importance to the conduct of monetary policy. First, the levels of the flexible price and wage equilibrium components of various observed and unobserved endogenous variables are important inputs into the optimal conduct of monetary policy. In particular, the level of the natural rate of interest, defined as that short term real interest rate consistent with price and wage flexibility, provides a measure of the neutral stance of monetary policy, with deviations of the real interest rate from the natural rate of interest generating inflationary pressure. The proposed unobserved components framework facilitates estimation of the levels as opposed to cyclical components of the flexible price and wage equilibrium components of endogenous variables, while ensuring that they satisfy the stability restrictions associated with balanced growth. Second, given an interest rate smoothing objective derived from a concern with financial market stability, variation in the natural rate of interest caused by shocks having permanent effects may call for larger monetary policy responses than variation caused by shocks having temporary effects. The proposed unobserved components framework yields a decomposition of the levels of the flexible price and wage equilibrium components of endogenous variables into cyclical and trend components, together with confidence intervals which account for uncertainty associated with the detrending procedure. Third, accommodating the existence of intermittent structural breaks requires flexible trend component specifications. However, the joint derivation of empirically adequate cyclical and trend component specifications from microeconomic foundations is a formidable task. The proposed unobserved components framework facilitates estimation of the levels of the flexible
price and wage equilibrium components of endogenous variables while allowing for the possibility that the determinants of their trend components are unknown but persistent.

The fourth paper evaluates the finite sample properties of the procedure proposed in the third paper for the measurement of the stance of monetary policy in a small open economy. In particular, the accuracy and precision of the Bayesian procedure proposed for the estimation of the levels of the flexible price equilibrium components of various observed and unobserved endogenous variables is analyzed with a Monte Carlo experiment, with an emphasis on the levels of the natural rate of interest and natural exchange rate. The data generating process is a calibrated DSGE model of a small open economy featuring long run balanced growth driven by trend inflation, productivity growth, and population growth. Alternative versions of this DSGE model incorporating common deterministic or stochastic trends are considered. Given a large number of artificial data sets generated under these alternative trend component specifications, estimation of the levels of the flexible price equilibrium components of various observed and unobserved endogenous variables is based on a linear state space representation of an approximate unobserved components representation of this DSGE model of a small open economy, in which cyclical components are modeled by linearizing equilibrium conditions around a stationary deterministic steady state equilibrium which abstracts from long run balanced growth, while trend components are modeled as random walks while ensuring the existence of a well defined balanced growth path. Repeated joint estimation of the parameters and unobserved components of this linear state space representation of this approximate unobserved components representation of this DSGE model with the Bayesian procedure under consideration facilitates simulation of the finite sample distributions of estimators of the levels of flexible price equilibrium components, with respect to which accuracy and precision are measured in terms of bias and root mean squared error.

The primary contribution of this fourth paper is the evaluation of the accuracy and precision of the procedure proposed in the third paper for the estimation of the levels of the flexible price equilibrium components of various observed and unobserved endogenous variables, with an emphasis on the levels of the natural rate of interest and natural exchange rate. This Bayesian estimation procedure is found to yield reasonably accurate and precise results in samples of currently available size. In particular, estimates of the levels of the natural rate of interest and natural exchange rate conditional on alternative information sets are approximately unbiased, while root mean squared errors are relatively small, irrespective of whether the data generating process features common deterministic or stochastic trends. Moreover, analytical root mean squared errors appropriately account for uncertainty surrounding estimates of the levels of the natural rate of interest and natural exchange rate.
The remainder of this doctoral thesis consists of four papers. The first paper develops and estimates a DSGE model of a small open economy. The second paper develops and estimates an unobserved components model of the monetary transmission mechanism in a small open economy. The third paper considers the measurement of the stance of monetary policy in a small open economy within a DSGE framework. The fourth paper evaluates the procedure proposed for the measurement of the stance of monetary policy in a small open economy. Closed economy versions of these papers have also been written, and are available at grad.econ.ubc.ca/fvitek.
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CHAPTER 1

Monetary Policy Analysis in a Small Open Economy: A Dynamic Stochastic General Equilibrium Approach

1.1. Introduction

Estimated dynamic stochastic general equilibrium or DSGE models have recently emerged as quantitative monetary policy analysis and inflation targeting tools. As extensions of real business cycle models, DSGE models explicitly specify the objectives and constraints faced by optimizing households and firms, which interact in an uncertain environment to determine equilibrium prices and quantities. The existence of short-run nominal price and wage rigidities generated by monopolistic competition and staggered reoptimization in output and labor markets permits a cyclical stabilization role for monetary policy, which is generally implemented through control of the nominal interest rate according to a monetary policy rule. The persistence of the effects of monetary policy shocks on output and inflation is often enhanced with other features such as habit persistence in consumption, adjustment costs in investment, and variable capital utilization. Early examples of closed economy DSGE models incorporating some of these features include those of Yun (1996), Goodfriend and King (1997), Rotemberg and Woodford (1995, 1997), and McCallum and Nelson (1999), while recent examples of closed economy DSGE models incorporating all of these features include those of Christiano, Eichenbaum and Evans (2005), Altig, Christiano, Eichenbaum and Linde (2005), and Smets and Wouters (2003, 2005).

Open economy DSGE models extend their closed economy counterparts to allow for international trade and financial linkages, implying that the monetary transmission mechanism features both interest rate and exchange rate channels. Building on the seminal work of Obstfeld and Rogoff (1995, 1996), these open economy DSGE models determine trade and current account balances through both intratemporal and intertemporal optimization, while the nominal exchange rate is determined by an uncovered interest parity condition. Existing open economy DSGE models differ primarily with respect to the degree of exchange rate pass through. Models in which exchange rate pass through is complete include those of Benigno and Benigno (2002), McCallum and Nelson (2000), Clarida, Gali and Gertler (2001, 2002), and Gertler, Gilchrist and
Natalucci (2001), while models in which exchange rate pass through is incomplete include those of Adolfson (2001), Betts and Devereux (2000), Kollman (2001), Corsetti and Pesenti (2002), and Monacelli (2005).

In an empirical investigation of the degree of exchange rate pass through among developed economies, Campa and Goldberg (2002) find that short run exchange rate pass through is incomplete, while long run exchange rate pass through is complete. This empirical evidence rejects both local currency pricing, under which the domestic currency prices of imports are invariant to exchange rate fluctuations in the short run, and producer currency pricing, under which the domestic currency prices of imports fully reflect exchange rate fluctuations in the short run. In response to this empirical evidence, Monacelli (2005) incorporates short run import price rigidities into an open economy DSGE model by allowing for monopolistic competition and staggered reoptimization in the import market. These import price rigidities generate incomplete exchange rate pass through in the short run, while exchange rate pass through is complete in the long run.

The economy is complex, and any model of it is necessarily misspecified to some extent. An operational substitute for the concept of a correctly specified model is the concept of an empirically adequate model. A model is empirically adequate if it approximately accounts for the existing empirical evidence in all measurable respects, which as discussed in Clements and Hendry (1998) does not require that it be correctly specified. As argued by Diebold and Mariano (1995), a necessary condition for empirical adequacy is predictive accuracy, which must be measured in relative terms. Quantitative monetary policy analysis and inflation targeting should be based on empirically adequate models of the economy.

Thus far, empirical evaluations of DSGE models have generally focused on unconditional second moment and impulse response properties. While empirically valid unconditional second moment and impulse response properties are necessary conditions for empirical adequacy, they are not sufficient. Moreover, empirical evaluations of unconditional second moment properties are generally conditional on atheoretic estimates of trend components, while empirical evaluations of impulse response properties are generally conditional on controversial identifying restrictions. It follows that the empirical evaluation of predictive accuracy is a necessary precursor to a well informed judgment regarding the extent to which any DSGE model can and should contribute to quantitative monetary policy analysis and inflation targeting.

Existing DSGE models featuring long run balanced growth driven by trend inflation, productivity growth, and population growth generally predict the existence of common deterministic or stochastic trends. Estimated DSGE models incorporating common deterministic trends include those of Ireland (1997) and Smets and Wouters (2005), while estimated DSGE models incorporating common stochastic trends include those of Altig, Christiano, Eichenbaum
and Linde (2005) and Del Negro, Schorfheide, Smets and Wouters (2005). However, as discussed in Clements and Hendry (1999) and Maddala and Kim (1998), intermittent structural breaks render such common deterministic or stochastic trends empirically inadequate representations of low frequency variation in observed macroeconomic variables. For this reason, it is common to remove trend components from observed macroeconomic variables with deterministic polynomial functions or linear filters, such as the difference filter or the low pass filter described in Hodrick and Prescott (1997), prior to the conduct of estimation, inference and forecasting.

Decomposing observed macroeconomic variables into cyclical and trend components prior to the conduct of estimation, inference and forecasting reflects an emphasis on the predictions of DSGE models at business cycle frequencies. Since such decompositions are additive, given observed macroeconomic variables, predictions at business cycle frequencies imply predictions at lower frequencies. As argued by Harvey (1997), the removal of trend components from observed macroeconomic variables with atheoretic deterministic polynomial functions or linear filters ignores these predictions, potentially invalidating subsequent estimation, inference and forecasting. As an alternative, this paper proposes jointly modeling cyclical and trend components as unobserved components while imposing theoretical restrictions derived from the approximate multivariate linear rational expectations representation of a DSGE model.

The development of empirically adequate DSGE models for purposes of quantitative monetary policy analysis and inflation targeting in a small open economy is currently an active area of research. Nevertheless, an estimated DSGE model of a small open economy which approximately accounts for the empirical evidence concerning the monetary transmission mechanism, as summarized by impulse response functions derived from an estimated structural vector autoregressive or SVAR model, while dominating that SVAR model in terms of predictive accuracy, has yet to be developed. This paper develops and estimates a DSGE model of a small open economy which satisfies these impulse response and predictive accuracy criteria. The model features short run nominal price and wage rigidities generated by monopolistic competition and staggered reoptimization in output and labour markets. The resultant inertia in inflation and persistence in output is enhanced with other features such as habit persistence in consumption, adjustment costs in investment, and variable capital utilization. Incomplete exchange rate pass through is generated by short run nominal price rigidities in the import market, with monopolistically competitive importers setting the domestic currency prices of differentiated intermediate import goods subject to randomly arriving reoptimization opportunities. Cyclical components are modeled by linearizing equilibrium conditions around a stationary deterministic steady state equilibrium, while trend components are modeled as random
walks while ensuring the existence of a well defined balanced growth path. Parameters and trend components are jointly estimated with a novel Bayesian procedure.

The organization of this paper is as follows. The next section develops a DSGE model of a small open economy. Estimation, inference and forecasting within the framework of a linear state space representation of an approximate unobserved components representation of this DSGE model are the subjects of section three. Finally, section four offers conclusions and recommendations for further research.

1.2. Model Development

Consider two open economies which are asymmetric in size, but are otherwise identical. The domestic economy is of negligible size relative to the foreign economy.

1.2.1. The Utility Maximization Problem of the Representative Household

There exists a continuum of households indexed by \( i \in [0,1] \). Households supply differentiated intermediate labour services, but are otherwise identical.

1.2.1.1. Consumption and Saving Behaviour

The representative infinitely lived household has preferences defined over consumption \( C_{i,s} \) and labour supply \( L_{i,s} \) represented by intertemporal utility function

\[
U_{i,t} = E_t \sum_{s=1}^{\infty} \beta^{s-t} u(C_{i,s}, L_{i,s}),
\]

where subjective discount factor \( \beta \) satisfies \( 0 < \beta < 1 \). The intratemporal utility function is additively separable and represents external habit formation preferences in consumption,

\[
u_s(C_{i,s}, L_{i,s}) = \nu(c_{i,s} - \alpha C_{i,s-1})^{1-\sigma} \left[ \frac{C_{i,s}}{1-1/\sigma} - \nu^{L_{i,s}} \right],
\]

where \( 0 < \alpha < 1 \). This intratemporal utility function is strictly increasing with respect to consumption if and only if \( \nu^C > 0 \), and given this parameter restriction is strictly decreasing with
respect to labour supply if and only if \( v^L > 0 \). Given these parameter restrictions, this intratemporal utility function is strictly concave if \( \sigma > 0 \) and \( \eta > 0 \).

The representative household enters period \( s \) in possession of previously purchased domestic currency denominated bonds \( B_{i,s}^D \) which yield interest at risk free rate \( i_{s-1} \), and foreign currency denominated bonds \( B_{i,s}^F \) which yield interest at risk free rate \( i_{s-1}^f \). It also holds a diversified portfolio of shares \( \{ x_{i,j,s}^V \} \) in domestic intermediate good firms which pay dividends \( \{ D_{j,s}^V \} \), and a diversified portfolio of shares \( \{ x_{i,k,s}^M \} \) in domestic intermediate good importers which pay dividends \( \{ D_{k,s}^M \} \). The representative household supplies differentiated intermediate labour service \( L_{i,s} \), earning labour income at nominal wage \( W_{i,s} \). Households pool their labour income, and the government levies a tax on pooled labour income at rate \( \tau_s \). These sources of private wealth are summed in household dynamic budget constraint:

\[
B_{i,s+1}^D + \nu B_{i,s+1}^F + \int_0^1 V_{j,s}^Y x_{i,j,s+1}^V dj + \int_0^1 V_{k,s}^M x_{i,k,s+1}^M dk = (1 + i_{s-1})B_{i,s}^D + \nu (1 + i_{s-1}^f)B_{i,s}^F \\
+ \int_0^1 (\Pi_{j,s}^V + V_{j,s}) x_{i,j,s}^V dj + \int_0^1 (\Pi_{k,s}^M + V_{k,s}) x_{i,k,s}^M dk + (1 - \tau_s) \int_0^1 W_{i,s} L_{i,s} dl - P_s^C C_{i,s}.
\]

According to this dynamic budget constraint, at the end of period \( s \), the representative household purchases domestic bonds \( B_{i,s}^D \), and foreign bonds \( B_{i,s}^F \) at price \( \nu \). It also purchases a diversified portfolio of shares \( \{ x_{i,j,s+1}^V \} \) in intermediate good firms at prices \( \{ V_{j,s}^Y \} \), and a diversified portfolio of shares \( \{ x_{i,k,s+1}^M \} \) in intermediate good importers at prices \( \{ V_{k,s}^M \} \). Finally, the representative household purchases final consumption good \( C_{i,s} \) at price \( P_s^C \).

In period \( t \), the representative household chooses state contingent sequences for consumption \( \{ C_{i,s} \} \), domestic bond holdings \( \{ B_{i,s}^D \} \), foreign bond holdings \( \{ B_{i,s}^F \} \), share holdings in intermediate good firms \( \{ x_{i,j,s+1}^V \} \), and share holdings in intermediate good importers \( \{ x_{i,k,s+1}^M \} \) to maximize intertemporal utility function (1) subject to dynamic budget constraint (3) and terminal nonnegativity constraints \( B_{i,T+1}^D \geq 0 \), \( B_{i,T+1}^F \geq 0 \), \( x_{i,j,T+1}^V \geq 0 \) and \( x_{i,k,T+1}^M \geq 0 \) for \( T \to \infty \). In equilibrium, selected necessary first order conditions associated with this utility maximization problem may be stated as

\[
u C_{i,s}(L_{i,s}) = P_s^C \lambda_t,
\]

\[
\lambda_t = \beta (1 + i_t) \nu \lambda_{t+1},
\]

\[
\nu \lambda_t = \beta (1 + i_t^f) \nu \lambda_{t+1}.
\]
\[ V_{j,t}^Y \lambda_t = \beta E_t (\Pi_{j,t+1}^Y + V_{j,t+1}^Y) \lambda_{t+1}, \quad (7) \]
\[ V_{k,t}^M \lambda_t = \beta E_t (\Pi_{k,t+1}^M + V_{k,t+1}^M) \lambda_{t+1}, \quad (8) \]

where \( \lambda_{j,t} \) denotes the Lagrange multiplier associated with the period \( s \) household dynamic budget constraint. In equilibrium, necessary complementary slackness conditions associated with the terminal nonnegativity constraints may be stated as:

\[ \lim_{T \to \infty} \frac{\beta^T \lambda_{t+T} B_{t+T+1}^{p,k}}{\lambda_t} = 0, \quad (9) \]
\[ \lim_{T \to \infty} \frac{\beta^T \lambda_{t+T} E_{t+T} B_{t+T+1}^{p,f}}{\lambda_t} = 0, \quad (10) \]
\[ \lim_{T \to \infty} \frac{\beta^T \lambda_{t+T} V_{j,t+T}^Y X_{j,t+T+1}^Y}{\lambda_t} = 0, \quad (11) \]
\[ \lim_{T \to \infty} \frac{\beta^T \lambda_{t+T} V_{k,t+T}^M X_{k,t+T+1}^M}{\lambda_t} = 0. \quad (12) \]

Provided that the intertemporal utility function is bounded and strictly concave, together with all necessary first order conditions, these transversality conditions are sufficient for the unique utility maximizing state contingent intertemporal household allocation.

Combination of necessary first order conditions (4) and (5) yields intertemporal optimality condition

\[ u_c(C_t, L_t) = \beta E_t (1 + i_t) \frac{P_t^C}{P_{t+1}^C} u_c(C_{t+1}, L_{t+1}), \quad (13) \]

which ensures that at a utility maximum, the representative household cannot benefit from feasible intertemporal consumption reallocations. Finally, combination of necessary first order conditions (4), (5) and (6) yields intratemporal optimality condition

\[ E_t \frac{\beta u_c(C_{t+1}, L_{t+1}) P_{t+1}^C}{u_c(C_t, L_t) P_t^C} (1 + i_t) = E_t \frac{\beta u_c(C_{t+1}, L_{t+1}) P_{t+1}^C}{u_c(C_t, L_t) P_t^C} E_{t+1} (1 + i_{t+1}), \quad (14) \]

which equates the expected present discounted values of the gross real returns on domestic and foreign bonds.
1.2.1.2. Labour Supply and Wage Setting Behaviour

There exist a large number of perfectly competitive firms which combine differentiated intermediate labour services $L_{i,t}$ supplied by households in a monopolistically competitive labour market to produce final labour service $L_t$ according to constant elasticity of substitution production function

$$L_t = \left[ \int \left( \frac{\theta_i^{1-\theta_i}}{\theta_i^{1-\theta_i}} \right)^{\frac{\theta_i}{\theta_i-1}} \right]^{\frac{\theta_i}{\theta_i-1}},$$

(15)

where $\theta_i > 1$. The representative final labour service firm maximizes profits derived from production of the final labour service

$$\Pi_t = W_t L_t - \sum_{i=0}^{1} W_{i,t} L_{i,t},$$

(16)

with respect to inputs of intermediate labour services, subject to production function (15). The necessary first order conditions associated with this profit maximization problem yield intermediate labour service demand functions:

$$L_{i,t} = \left( \frac{W_{i,t}}{W_t} \right)^{-\theta_i} L_t,$$

(17)

Since the production function exhibits constant returns to scale, in competitive equilibrium the representative final labour service firm earns zero profit, implying aggregate wage index:

$$W_t = \left[ \int \left( W_{i,t} \right)^{-\theta_i} di \right]^{-\frac{1}{1-\theta_i}}.$$

(18)

As the wage elasticity of demand for intermediate labour services $\theta_i$ increases, they become closer substitutes, and individual households have less market power.

In an extension of the model of nominal wage rigidity proposed by Erceg, Henderson and Levin (2000) along the lines of Smets and Wouters (2003, 2005), each period a randomly selected fraction $1 - \omega_i$ of households adjust their wage optimally. The remaining fraction $\omega_i$
of households adjust their wage to account for past consumption price inflation according to partial indexation rule

\[ W_{i,t} = \left( \frac{p_{t-1}^c}{p_{t-2}^c} \right)^{\gamma^L} \left( \frac{\bar{p}^c_{t-1}}{\bar{p}^c_{t-2}} \right)^{1-\gamma^L} W_{i,t-1}, \]  

(19)

where \( 0 \leq \gamma^L \leq 1 \). Under this specification, although households adjust their wage every period, they infrequently adjust their wage optimally, and the interval between optimal wage adjustments is a random variable.

If the representative household can adjust its wage optimally in period \( t \), then it does so to maximize intertemporal utility function (1) subject to dynamic budget constraint (3), intermediate labour service demand function (17), and the assumed form of nominal wage rigidity. Since all households that adjust their wage optimally in period \( t \) solve an identical utility maximization problem, in equilibrium they all choose a common wage \( W^*_t \) given by necessary first order condition:

\[
\frac{W^*_t}{W_t} = \frac{E_i \sum \omega^t L_i \left( \frac{\beta^t u_c(C_t, L_{i,t})}{u_c(C_t, L_{i,t})} \right) \theta^t L_i \left( \frac{p_{t-1}^c}{p_{t-2}^c} \right)^{\gamma^L} \left( \frac{\bar{p}^c_{t-1}}{\bar{p}^c_{t-2}} \right)^{1-\gamma^L} W_{i,t-1}}{E_i \sum \omega^t L_i \left( \frac{\beta^t u_c(C_t, L_{i,t})}{u_c(C_t, L_{i,t})} \right) (1 - \theta^t) (1 - \tau^t) \left( \frac{p_{t-1}^c}{p_{t-2}^c} \right)^{\gamma^L} \left( \frac{\bar{p}^c_{t-1}}{\bar{p}^c_{t-2}} \right)^{1-\gamma^L} W_{i,t-1}} L_i.
\]

(20)

This necessary first order condition equates the expected present discounted value of the consumption benefit generated by an additional unit of labour supply to the expected present discounted value of its leisure cost. Aggregate wage index (18) equals an average of the wage set by the fraction \( 1 - \omega^t \) of households that adjust their wage optimally in period \( t \), and the average of the wages set by the remaining fraction \( \omega^t \) of households that adjust their wage according to partial indexation rule (19):

\[
W_t = \left( 1 - \omega^t \right) (W^*_t)^{1-\theta^t} + \omega^t \left[ \left( \frac{p_{t-1}^c}{p_{t-2}^c} \right)^{\gamma^L} \left( \frac{\bar{p}^c_{t-1}}{\bar{p}^c_{t-2}} \right)^{1-\gamma^L} W_{i,t-1} \right]^{\frac{1}{1-\theta^t}}.
\]

(21)

Since those households able to adjust their wage optimally in period \( t \) are selected randomly from among all households, the average wage set by the remaining households equals the value of the aggregate wage index that prevailed during period \( t - 1 \), rescaled to account for past consumption price inflation.
1.2.2. The Value Maximization Problem of the Representative Firm

There exists a continuum of intermediate good firms indexed by \( j \in [0,1] \). Intermediate good firms supply differentiated intermediate output goods, but are otherwise identical. Entry into and exit from the monopolistically competitive intermediate output good sector is prohibited.

1.2.2.1. Employment and Investment Behaviour

The representative intermediate good firm sells shares \( \{s^{V}_{i,j,s+1}\}_{i=0}^{\infty} \) to domestic households at price \( V_{j,s}^{Y} \). Recursive forward substitution for \( V_{j,s}^{Y} \) with \( s > 0 \) in necessary first order condition (7) applying the law of iterated expectations reveals that the post-dividend stock market value of the representative intermediate good firm equals the expected present discounted value of future dividend payments:

\[
V_{j,s}^{Y} = E_t \sum_{s=1}^{\infty} \frac{\beta^{s-1} \lambda_{s}}{\lambda_{t}} \Pi_{j,s}^{Y},
\]

(22)

Acting in the interests of its shareholders, the representative intermediate good firm maximizes its pre-dividend stock market value, equal to the expected present discounted value of current and future dividend payments:

\[
\Pi_{j,s}^{Y} + V_{j,s}^{Y} = E_t \sum_{s=1}^{\infty} \frac{\beta^{s-1} \lambda_{s}}{\lambda_{t}} \Pi_{j,s}^{Y}.
\]

(23)

The derivation of result (22) imposes transversality condition (11), which rules out self-fulfilling speculative asset price bubbles.

Shares entitle households to dividend payments equal to net profits \( \Pi_{j,s}^{Y} \), defined as after tax earnings less investment expenditures:

\[
\Pi_{j,s}^{Y} = (1 - \tau_{s})(P_{j,s}^{Y} Y_{j,s} - W_{s} L_{j,s}) - P_{s} I_{s}.
\]

(24)

Earnings are defined as revenues derived from sales of differentiated intermediate output good \( Y_{j,s} \) at price \( P_{j,s}^{Y} \) less expenditures on final labour service \( L_{j,s} \). The government levies a tax on earnings at rate \( \tau_{s} \), and negative dividend payments are a theoretical possibility.
The representative intermediate good firm utilizes capital $K_s$ at rate $u_{j,s}$ and rents final labour service $L_{j,s}$ given labour augmenting technology coefficient $A_s$ to produce differentiated intermediate output good $Y_{j,s}$ according to constant elasticity of substitution production function

$$F(u_{j,s}K_s,A_sL_{j,s}) = \left[ (\varphi)^{\frac{1}{\theta}} (u_{j,s}K_s)^{\frac{\vartheta-1}{\vartheta}} + (1-\varphi)^{\frac{1}{\theta}} (A_sL_{j,s})^{\frac{\vartheta-1}{\vartheta}} \right]^{\frac{\vartheta}{\vartheta-1}}, \quad (25)$$

where $0 < \varphi < 1$, $\vartheta > 0$ and $A_s > 0$. This constant elasticity of substitution production function exhibits constant returns to scale, and nests the production function proposed by Cobb and Douglas (1928) under constant returns to scale for $\vartheta = 1$.\(^1\)

In utilizing capital to produce output, the representative intermediate good firm incurs a cost $G(u_{j,s},K_s)$ denominated in terms of output:

$$Y_{j,s} = F(u_{j,s}K_s,A_sL_{j,s}) - G(u_{j,s},K_s). \quad (26)$$

Following Christiano, Eichenbaum and Evans (2005), this capital utilization cost is increasing in the rate of capital utilization at an increasing rate,

$$G(u_{j,s},K_s) = \mu \left[ e^{\kappa(u_{j,s}-1)} - 1 \right] K_s, \quad (27)$$

where $\mu > 0$ and $\kappa > 0$. In deterministic steady state equilibrium, the rate of capital utilization is normalized to one, and the cost of utilizing capital equals zero.

Capital is endogenous but not firm-specific, and the representative intermediate good firm enters period $s$ with access to previously accumulated capital stock $K_s$, which subsequently evolves according to accumulation function

$$K_{s+1} = (1 - \delta)K_s + \mathcal{H}(I_s,I_{s-1}). \quad (28)$$

where depreciation rate parameter $\delta$ satisfies $0 \leq \delta \leq 1$. Following Christiano, Eichenbaum and Evans (2005), effective investment function $\mathcal{H}(I_s,I_{s-1})$ incorporates convex adjustment costs,

$$\mathcal{H}(I_s,I_{s-1}) = \nu_s \left[ 1 - \frac{\chi}{2} \left( \frac{I_s - I_{s-1}}{I_{s-1}} \right) \right] I_s, \quad (29)$$

where $\chi > 0$ and $\nu_s > 0$. In deterministic steady state equilibrium, these adjustment costs equal zero, and effective investment equals actual investment.

\(^1\) Invoking L'Hopital's rule yields $\lim_{\delta \to 0} \ln F(u_{j,s}K_s,A_sL_{j,s}) = \varphi \ln(u_{j,s}K_s) + (1-\varphi) \ln(A_sL_{j,s}) - \varphi \ln(1-\varphi) \ln(1-\varphi)\), which implies that $\lim_{\delta \to 0} F(u_{j,s}K_s,A_sL_{j,s}) = \varphi \varphi^{\varphi}(1-\varphi)^{1-\varphi} (u_{j,s}K_s)^{\varphi} (A_sL_{j,s})^{-\varphi}$.\(^{1}\)
In period \( t \), the representative intermediate good firm chooses state contingent sequences for employment \( \{L_{i,t}\}_{s=1}^{\infty} \), capital utilization \( \{u_{j,t}\}_{s=1}^{\infty} \), investment \( \{I_{s,i}\}_{t=1}^{\infty} \), and the capital stock \( \{K_{s+1}\}_{s=1}^{\infty} \) to maximize pre-dividend stock market value (23) subject to net production function (26), capital accumulation function (28), and terminal nonnegativity constraint \( K_{T+1} \geq 0 \) for \( T \rightarrow \infty \). In equilibrium, demand for the final labour service satisfies necessary first order condition

\[
\mathcal{F}_{dL}(u_{j,t}, K_{t}, A_{i,t}) = (1 - \tau) \frac{W_{t}}{P_{t}^{\prime} A_{i,t}} \tag{30}
\]

where \( P_{t}^{\prime} \Phi_{j,t} \) denotes the Lagrange multiplier associated with the period \( s \) production technology constraint. This necessary first order condition equates real marginal cost \( \Phi_{j,t} \) to the ratio of the after tax real wage to the marginal product of labour. In equilibrium, the rate of capital utilization satisfies necessary first order condition

\[
\mathcal{F}_{uK}(u_{j,t}, K_{t}, A_{i,t}) = \frac{\mathcal{G}_{t}(u_{j,t}, K_{t})}{K_{t}} \tag{31}
\]

which equates the marginal product of utilized capital to its marginal cost. In equilibrium, demand for the final investment good satisfies necessary first order condition

\[
Q_{t} \mathcal{H}_{t}(I_{t}, I_{t+1}) + E_{t} \beta_{t+1} \frac{\mathcal{G}_{t+1}(I_{t+1}, I_{t})}{\lambda_{t}} = P_{t}^{\prime} \tag{32}
\]

which equates the expected present discounted value of an additional unit of investment to its price, where \( Q_{j,t} \) denotes the Lagrange multiplier associated with the period \( s \) capital accumulation function. In equilibrium, this shadow price of capital satisfies necessary first order condition

\[
Q_{t} = E_{t} \frac{\beta_{t+1}}{\lambda_{t}} \left\{ p_{t}^{\prime} \Phi_{j,t+1} \left[ u_{j, t+1} \mathcal{F}_{uK}(u_{j, t+1}, K_{t+1}, A_{t+1}, L_{t+1}) - \mathcal{G}_{t+1}(u_{j, t+1}, K_{t+1}) \right] + (1 - \delta) Q_{t+1} \right\} \tag{33}
\]

which equates it to the expected present discounted value of the sum of the future marginal cost of capital, and the future shadow price of capital net of depreciation. In equilibrium, the necessary complementary slackness condition associated with the terminal nonnegativity constraint may be stated as:

\[
\lim_{T \rightarrow \infty} \frac{\beta_{T} \lambda_{T+1}}{\lambda_{t}} Q_{t+1} K_{t+1} = 0. \tag{34}
\]
Provided that the pre-dividend stock market value of the representative intermediate good firm is bounded and strictly concave, together with all necessary first order conditions, this transversality condition is sufficient for the unique value maximizing state contingent intertemporal firm allocation.

1.2.2.2. Output Supply and Price Setting Behaviour

There exist a large number of perfectly competitive firms which combine differentiated intermediate output goods $Y_{jt}$ supplied by intermediate good firms in a monopolistically competitive output market to produce final output good $Y_t$ according to constant elasticity of substitution production function

$$Y_t = \left[ \int_{j=0}^{1} (Y_{jt})^{\theta_t} \frac{\theta_t}{\theta_t - 1} \right]^{\frac{\theta_t - 1}{\theta_t - 1}}, \quad (35)$$

where $\theta_t > 1$. The representative final output good firm maximizes profits derived from production of the final output good

$$\Pi_t^Y = P_t^Y Y_t - \int_{j=0}^{1} P_{jt}^Y Y_{jt} dj, \quad (36)$$

with respect to inputs of intermediate output goods, subject to production function (35). The necessary first order conditions associated with this profit maximization problem yield intermediate output good demand functions:

$$Y_{jt} = \left( \frac{P_{jt}^Y}{P_t^Y} \right)^{\frac{\theta_t}{\theta_t - 1}} Y_t, \quad (37)$$

Since the production function exhibits constant returns to scale, in competitive equilibrium the representative final output good firm earns zero profit, implying aggregate output price index:

$$P_t^Y = \left[ \int_{j=0}^{1} (P_{jt}^Y)^{\frac{\theta_t}{\theta_t - 1}} dj \right]^{\frac{\theta_t - 1}{\theta_t}}. \quad (38)$$
As the price elasticity of demand for intermediate output goods $\theta_Y$ increases, they become closer substitutes, and individual intermediate good firms have less market power.

In an extension of the model of nominal output price rigidity proposed by Calvo (1983) along the lines of Smets and Wouters (2003, 2005), each period a randomly selected fraction $1 - \omega_Y$ of intermediate good firms adjust their price optimally. The remaining fraction $\omega_Y$ of intermediate good firms adjust their price to account for past output price inflation according to partial indexation rule

$$p_{j,t}^Y = \left( \frac{p_{t-1}^Y}{p_{t-2}^Y} \right)^{\gamma_Y} \left( \frac{p_{t-1}^Y}{p_{t-2}^Y} \right)^{1-\gamma_Y} p_{j,t-1}^Y, \quad (39)$$

where $0 \leq \gamma_Y \leq 1$. Under this specification, optimal price adjustment opportunities arrive randomly, and the interval between optimal price adjustments is a random variable.

If the representative intermediate good firm can adjust its price optimally in period $t$, then it does so to maximize to maximize pre-dividend stock market value (23) subject to net production function (26), capital accumulation function (28), intermediate output good demand function (37), and the assumed form of nominal output price rigidity. Since all intermediate good firms that adjust their price optimally in period $t$ solve an identical value maximization problem, in equilibrium they all choose a common price $p_t^{Y,*}$ given by necessary first order condition:

$$P_t^{Y,*} = \frac{\sum_{s=t}^{\infty} (\omega_Y)^{t-1} \frac{\beta^t\lambda_s}{\lambda_s} \theta_s p_{s,t}^Y \Phi_{j,s} \left( \left( \frac{p_{t-1}^Y}{p_{s-1}^Y} \right)^{\gamma_Y} \left( \frac{p_{t-1}^Y}{p_{s-1}^Y} \right)^{1-\gamma_Y} \frac{p_t^Y}{p_s^Y} \right)^{\theta_t^Y} \left( \frac{p_t^{Y,*}}{p_t^Y} \right)^{-\theta_t^Y} p_t^{Y,s}}{\sum_{s=t}^{\infty} (\omega_Y)^{t-1} \frac{\beta^t\lambda_s}{\lambda_s} \theta_s^{t-1}(1-\tau_s) \left( \left( \frac{p_{t-1}^Y}{p_{s-1}^Y} \right)^{\gamma_Y} \left( \frac{p_{t-1}^Y}{p_{s-1}^Y} \right)^{1-\gamma_Y} \frac{p_t^Y}{p_s^Y} \right)^{\theta_t^Y} \left( \frac{p_t^{Y,*}}{p_t^Y} \right)^{-\theta_t^Y} p_t^{Y,s}}. \quad (40)$$

This necessary first order condition equates the expected present discounted value of the after tax revenue benefit generated by an additional unit of output supply to the expected present discounted value of its production cost. Aggregate output price index (38) equals an average of the price set by the fraction $1 - \omega_Y$ of intermediate good firms that adjust their price optimally in period $t$, and the average of the prices set by the remaining fraction $\omega_Y$ of intermediate good firms that adjust their price according to partial indexation rule (39):

$$p_t^Y = \left( 1 - \omega_Y \right) \left( p_t^{Y,*} \right)^{1-\theta_t^Y} + \omega_Y \left[ \left( \frac{p_{t-1}^Y}{p_{t-2}^Y} \right)^{\gamma_Y} \left( \frac{p_{t-1}^Y}{p_{t-2}^Y} \right)^{1-\gamma_Y} p_{t-1}^Y \right]^{\theta_t^Y} \left( \frac{p_t^Y}{p_{t-1}^Y} \right)^{1-\theta_t^Y}. \quad (41)$$
Since those intermediate good firms able to adjust their price optimally in period $t$ are selected randomly from among all intermediate good firms, the average price set by the remaining intermediate good firms equals the value of the aggregate output price index that prevailed during period $t-1$, rescaled to account for past output price inflation.

1.2.3. The Value Maximization Problem of the Representative Importer

There exists a continuum of intermediate good importers indexed by $k \in [0,1]$. Intermediate good importers supply differentiated intermediate import goods, but are otherwise identical. Entry into and exit from the monopolistically competitive intermediate import good sector is prohibited.

1.2.3.1. The Real Exchange Rate and the Terms of Trade

The representative intermediate good importer sells shares $\{x_{s,k,t,s}^M\}$ to domestic households at price $V_{k,t}^M$. Recursive forward substitution for $V_{k,t+s}^M$ with $s > 0$ in necessary first order condition (8) applying the law of iterated expectations reveals that the post-dividend stock market value of the representative intermediate good importer equals the expected present discounted value of future dividend payments:

$$V_{k,t}^M = E_t \sum_{s=t+1}^{\infty} \frac{\beta^{s-t} \lambda_s}{\lambda_t} \Pi_{k,s}^M.$$  \hfill (42)

Acting in the interests of its shareholders, the representative intermediate good importer maximizes its pre-dividend stock market value, equal to the expected present discounted value of current and future dividend payments:

$$\Pi_{k,t}^M + V_{k,t}^M = E_t \sum_{s=t}^{\infty} \frac{\beta^{s-t} \lambda_s}{\lambda_t} \Pi_{k,s}^M.$$  \hfill (43)

The derivation of result (42) imposes transversality condition (12), which rules out self-fulfilling speculative asset price bubbles.

Shares entitle households to dividend payments equal to gross profits $\Pi_{k,s}^M$, defined as earnings less fixed costs:
\[ \Pi_i^M = P_i^M M_{k,s} - \varepsilon_i P_i^{Y,f} M_{k,s} - \Gamma_s, \quad (44) \]

Earnings are defined as revenues derived from sales of differentiated intermediate import good \( M_{k,s} \) at price \( P_i^M \) less expenditures on foreign final output good \( M_{k,s} \). The representative intermediate good importer purchases the foreign final output good at domestic currency price \( \varepsilon_i P_i^{Y,f} \) and differentiates it, generating zero gross profits on average.

The law of one price asserts that arbitrage transactions equalize the domestic currency prices of domestic imports and foreign exports. Define the real exchange rate,

\[ Q_s = \frac{\varepsilon_i P_i^{Y,f}}{P_i^{X}}, \quad (45) \]

which measures the price of foreign output in terms of domestic output. Also define the terms of trade,

\[ T_s = \frac{P_i^M}{P_i^{X}}, \quad (46) \]

which measures the price of imports in terms of exports. Violation of the law of one price drives a wedge \( \Psi_s = \varepsilon_i P_i^{Y,f} / P_i^M \) between the real exchange rate and the terms of trade,

\[ Q_s = \Psi_s T_s, \quad (47) \]

where the domestic currency price of exports satisfies \( P_i^{X} = P_i^{Y} \). Under the law of one price \( \Psi_s = 1 \), and the real exchange rate and terms of trade coincide.

There exist a large number of perfectly competitive firms which combine a domestic intermediate good \( Z_{h,s} \in \{C_{h,s}, I_{h,s}, G_{h,s}\} \) and a foreign intermediate good \( Z_{f,s} \in \{C_{f,s}, I_{f,s}, G_{f,s}\} \) to produce final good \( Z \in \{C_s, I_s, G_s\} \) according to constant elasticity of substitution production function

\[ Z_i = \left[ \frac{1}{\phi_i^Z} (Z_{h,s})^{\psi - 1} + \frac{1}{\phi_i^Z} (Z_{f,s})^{\psi - 1} \right], \quad (48) \]

where \( 0 < \phi_i^Z < 1 \), \( \psi > 1 \) and \( \phi_i^M > 0 \). The representative final good firm maximizes profits derived from production of the final good

\[ \Pi_i^Z = P_i^Z Z_i - P_i^Y Z_{h,s} - P_i^M Z_{f,s}, \quad (49) \]
with respect to inputs of domestic and foreign intermediate goods, subject to production function (48). The necessary first order conditions associated with this profit maximization problem imply intermediate good demand functions:

\[ Z_{h,i} = \phi^2 \left( \frac{P_{Y}^l}{P_{Y}^*} \right)^{-\psi} Z_i, \]  
\[ Z_{f,i} = (1 - \phi^2) \left( \frac{P_{M}^l}{P_{M}^*} \right)^{-\psi} \frac{Z_i}{V_i^M}. \]  

(50)  
(51)

Since the production function exhibits constant returns to scale, in competitive equilibrium the representative final good firm earns zero profit, implying aggregate price index:

\[ P_t^Z = \frac{1}{\phi^2 \left( P_{Y}^l \right)^{-\psi} + (1 - \phi^2) \left( \frac{P_{M}^l}{P_{M}^*} \right)^{-\psi}}. \]  

(52)

Combination of this aggregate price index with intermediate good demand functions (50) and (51) yields:

\[ Z_{h,i} = \phi^2 \left[ \phi^2 + (1 - \phi^2) \left( \frac{T_i}{V_i^M} \right)^{-\psi} \right]^{-\psi} Z_i, \]  
\[ Z_{f,i} = (1 - \phi^2) \left[ (1 - \phi^2) + \phi^2 \left( \frac{T_i}{V_i^M} \right)^{\psi-1} \right]^{-\psi} \frac{Z_i}{V_i^M}. \]  

(53)  
(54)

These demand functions for domestic and foreign intermediate goods are directly proportional to final good demand, with a proportionality coefficient that varies with the terms of trade.

1.2.3.2. Import Supply and Price Setting Behaviour

There exist a large number of perfectly competitive firms which combine differentiated intermediate import goods \( M_{k,i} \) supplied by intermediate good importers in a monopolistically competitive import market to produce final import good \( M_i \) according to constant elasticity of substitution production function.
where \( \theta^M > 1 \). The representative final import good firm maximizes profits derived from production of the final import good

\[
\Pi^M_t = P^M_t M_t - \int_{k=0}^{1} P^M_{k,t} M_{k,t} dk,
\]

(56)

with respect to inputs of intermediate import goods, subject to production function (55). The necessary first order conditions associated with this profit maximization problem yield intermediate import good demand functions:

\[
M_{k,t} = \left( \frac{P_{k,t}^M}{P_t^M} \right)^{-\theta^M} M_t.
\]

(57)

Since the production function exhibits constant returns to scale, in competitive equilibrium the representative final import good firm earns zero profit, implying aggregate import price index:

\[
P_t^M = \left[ \int_{k=0}^{1} (P_{k,t}^M)^{-\theta^M} dk \right]^{1/(1-\theta^M)}.
\]

(58)

As the price elasticity of demand for intermediate import goods \( \theta^M \) increases, they become closer substitutes, and individual intermediate good importers have less market power.

In an extension of the model of nominal import price rigidity proposed by Monacelli (2005) along the lines of Smets and Wouters (2003, 2005), each period a randomly selected fraction \( 1 - \omega^M \) of intermediate good importers adjust their price optimally. The remaining fraction \( \omega^M \) of intermediate good importers adjust their price to account for past import price inflation according to partial indexation rule

\[
P_{k,t}^M = \left( \frac{P_{k,t}^M}{P_{t-1}^M} \right)^{\gamma^M} \left( \frac{P_{k,t}^M}{P_{t-2}^M} \right)^{1-\gamma^M} P_{k,t-1}^M,
\]

(59)

where \( 0 \leq \gamma^M \leq 1 \). Under this specification, the probability that an intermediate good importer has adjusted its price optimally is time dependent but state independent.

If the representative intermediate good importer can adjust its price optimally in period \( t \), then it does so to maximize its pre-dividend stock market value (43) subject to
intermediate import good demand function (57), and the assumed form of nominal import price rigidity. Since all intermediate good importers that adjust their price optimally in period $t$ solve an identical value maximization problem, in equilibrium they all choose a common price $P^{M,*}_t$ given by necessary first order condition

$$P^{M,*}_t = \frac{E_t \sum_{s=1}^\infty (\omega^M)^{s-1} \frac{\beta^{r-1}}{\lambda_i} \Psi_s \left[ \left( \frac{P^M_{t-1}}{P^M_{t-1}} \right)^{\omega^M} \left( \frac{P^M_{t-1}}{P^M_{t-1}} \right)^{1-\omega^M} \frac{P^M_s}{P^M_t} \right] \left( \frac{P^{M,*}_t}{P^M_t} \right)^{-\theta^M}}{E_t \sum_{s=1}^\infty (\omega^M)^{s-1} \frac{\beta^{r-1}}{\lambda_i} (\theta^M_s - 1) \left[ \left( \frac{P^M_{t-1}}{P^M_{t-1}} \right)^{\omega^M} \left( \frac{P^M_{t-1}}{P^M_{t-1}} \right)^{1-\omega^M} \frac{P^M_s}{P^M_t} \right] \left( \frac{P^{M,*}_t}{P^M_t} \right)^{-\theta^M}}$$

(60)

where $\Psi_s = \mathcal{E}_s \omega^{r,f} / P^M_s$ measures real marginal cost. This necessary first order condition equates the expected present discounted value of the revenue benefit generated by an additional unit of import supply to the expected present discounted value of its production cost. Aggregate import price index (58) equals an average of the price set by the fraction $1-\omega^M$ of intermediate good importers that adjust their price optimally in period $t$, and the average of the prices set by the remaining fraction $\omega^M$ of intermediate good importers that adjust their price according to partial indexation rule (59):

$$P^M_t = \left[ (1-\omega^M)(P^{M,*}_t)^{-\theta^M} + \omega^M \left[ \left( \frac{P^M_{t-1}}{P^M_{t-2}} \right)^{\omega^M} \left( \frac{P^M_{t-1}}{P^M_{t-2}} \right)^{1-\omega^M} \frac{P^M_{t-1}}{P^M_{t-1}} \right]^{1-\theta^M} \right]^{1-\theta^M} \right]^{1-\theta^M}$$

(61)

Since those intermediate good importers able to adjust their price optimally in period $t$ are selected randomly from among all intermediate good importers, the average price set by the remaining intermediate good importers equals the value of the aggregate import price index that prevailed during period $t-1$, rescaled to account for past import price inflation.

1.2.4. Monetary and Fiscal Policy

The government consists of a monetary authority and a fiscal authority. The monetary authority implements monetary policy, while the fiscal authority implements fiscal policy.
1.2.4.1. The Monetary Authority

The monetary authority implements monetary policy through control of the nominal interest rate according to monetary policy rule

\[ i_t - \bar{i} = \xi^e (\pi_t^C - \pi_t^C) + \xi^y (\ln Y_t - \ln \bar{Y}_t) + \nu_t^e, \]  

(62)

where \( \xi^e > 1 \) and \( \xi^y > 0 \). As specified, the deviation of the nominal interest rate from its deterministic steady state equilibrium value is a linear increasing function of the contemporaneous deviation of consumption price inflation from its target value, and the contemporaneous proportional deviation of output from its deterministic steady state equilibrium value. Persistent departures from this monetary policy rule are captured by serially correlated monetary policy shock \( \nu_t^e \).

1.2.4.2. The Fiscal Authority

The fiscal authority implements fiscal policy through control of nominal government consumption and the tax rate applicable to the pooled labour income of households and the earnings of intermediate good firms. In equilibrium, this distortionary tax collection framework corresponds to proportional output taxation.

The ratio of nominal government consumption to nominal output satisfies fiscal expenditure rule

\[ \ln \frac{P_t^G G_t}{P_t^r Y_t} - \ln \frac{\bar{P}_t^G G_t}{\bar{P}_t^r \bar{Y}_t} = \zeta^G \left[ \ln \left( \frac{B_{t, t-1}}{P_t^r Y_t} \right) - \ln \left( \frac{\bar{B}_{t, 1}}{\bar{P}_t^r \bar{Y}_t} \right) \right] + \nu_t^G, \]  

(63)

where \( \zeta^G < 0 \). As specified, the proportional deviation of the ratio of nominal government consumption to nominal output from its deterministic steady state equilibrium value is a linear decreasing function of the contemporaneous proportional deviation of the ratio of net foreign debt to nominal output from its target value. This fiscal expenditure rule is well defined only if the net foreign debt is positive. Persistent departures from this fiscal expenditure rule are captured by serially correlated fiscal expenditure shock \( \nu_t^G \).

The tax rate applicable to the pooled labour income of households and the earnings of intermediate good firms satisfies fiscal revenue rule
\begin{equation}
\ln \tau_t - \ln \bar{\tau}_t = \zeta' \ln \left( \frac{-B_{t+1}^G}{P_t^Y Y_t} \right) - \ln \left( \frac{-\bar{B}_{t+1}^G}{\bar{P}_t^Y \bar{Y}_t} \right) + v^t, \tag{64}
\end{equation}

where \( \zeta' > 0 \). As specified, the proportional deviation of the tax rate from its deterministic steady state equilibrium value is a linear increasing function of the contemporaneous proportional deviation of the ratio of net government debt to nominal output from its target value. This fiscal revenue rule is well defined only if the net government debt is positive. Persistent departures from this fiscal revenue rule are captured by serially correlated fiscal revenue shock \( v^t \).

The fiscal authority enters period \( t \) holding previously purchased domestic currency denominated bonds \( B_{t+1}^{G,h} \) which yield interest at risk free rate \( i_{t-1} \), and foreign currency denominated bonds \( B_{t+1}^{G,f} \) which yield interest at risk free rate \( i_{t-1}^f \). It also levies taxes on the pooled labour income of households and the earnings of intermediate good firms at rate \( \tau_t \). These sources of public wealth are summed in government dynamic budget constraint:

\begin{equation}
B_{t+1}^{G,h} + \mathcal{E}_t B_{t+1}^{G,f} = (1 + i_{t-1})B_t^{G,h} + \mathcal{E}_t (1 + i_{t-1}^f)B_t^{G,f} + \tau_t \int_{i=0}^{1} \int_{j=0}^{1} W_{ij} L_{ij} d idi + \tau_t \int_{j=0}^{1} (P_{t}^Y Y_{t,j} - W_{t} L_{t,j}) dj - P_t^G G_t. \tag{65}
\end{equation}

According to this dynamic budget constraint, at the end of period \( t \), the fiscal authority purchases domestic bonds \( B_{t+1}^{G,h} \), and foreign bonds \( B_{t+1}^{G,f} \) at price \( \mathcal{E}_t \). It also purchases final government consumption good \( G_t \) at price \( P_t^G \).

1.2.5. Market Clearing Conditions

A rational expectations equilibrium in this DSGE model of a small open economy consists of state contingent intertemporal allocations for domestic and foreign households and firms which solve their constrained optimization problems given prices and policy, together with state contingent intertemporal allocations for domestic and foreign governments which satisfy their policy rules and constraints given prices, with supporting prices such that all markets clear. Since the domestic economy is of negligible size relative to the foreign economy, in equilibrium \( P_t^f = P_t^c = P_t^f = P_t^g = P_t^x = 0 \) and \( X_t = M_t^c = M_t^f = B_t = 0 \).

Clearing of the final output good market requires that exports \( X_t \), equal production of the domestic final output good less the cumulative demands of domestic households, firms, and the government,
\[ X_t = Y_t - C_{h,t} - I_{h,t} - G_{h,t}, \quad (66) \]

where \( X_t = M'_t \). Clearing of the final import good market requires that imports \( M_t \) satisfy the cumulative demands of domestic households, firms, and the government for the foreign final output good,

\[ M_t = C_{f,t} + I_{f,t} + G_{f,t}, \quad (67) \]

where \( M_t = X'_t \). In equilibrium, combination of these final output and import good market clearing conditions yields aggregate resource constraint:

\[ P_t^Y Y_t = P_t^C C_t + P_t^I I_t + P_t^G G_t + P_t^X X_t - P_t^M M_t. \quad (68) \]

The trade balance equals export revenues less import expenditures, or equivalently nominal output less domestic demand.

Let \( B_{i+1} \) denote the net foreign asset position of the economy, which in equilibrium equals the sum of the domestic currency values of private sector bond holdings \( B_{i+1}^p = B_{i+1}^{p,k} + \mathcal{E}_{i+1} B_{i+1}^{p,l} \) and public sector bond holdings \( B_{i+1}^g = B_{i+1}^{g,k} + \mathcal{E}_{i+1} B_{i+1}^{g,l} \), since domestic bond holdings cancel out when the private and public sectors are consolidated:

\[ B_{i+1} = B_{i+1}^p + B_{i+1}^g. \quad (69) \]

The imposition of equilibrium conditions on household dynamic budget constraint (3) reveals that the expected present discounted value of the net increase in private sector asset holdings equals the expected present discounted value of private saving less domestic investment:

\[ E_{t-1} \frac{\beta \lambda}{\lambda_{t-1}} (B_{i+1}^p - B_t^p) = E_{t-1} \frac{\beta \lambda}{\lambda_{t-1}} \left[ (i_{-1} B_t^p + (1 - \tau_i)P_t^Y Y_t - P_t^C C_t - P_t^I I_t) \right]. \quad (70) \]

The imposition of equilibrium conditions on government dynamic budget constraint (65) reveals that the expected present discounted value of the net increase in public sector asset holdings equals the expected present discounted value of public saving:

\[ E_{t-1} \frac{\beta \lambda}{\lambda_{t-1}} (B_{i+1}^g - B_t^g) = E_{t-1} \frac{\beta \lambda}{\lambda_{t-1}} (i_{-1} B_t^g + \tau_i P_t^Y Y_t - P_t^G G_t). \quad (71) \]

Combination of these household and government dynamic budget constraints with aggregate resource constraint (68) reveals that the expected present discounted value of the net increase in foreign asset holdings equals the expected present discounted value of the sum of net
international investment income and the trade balance, or equivalently the expected present discounted value of national saving less domestic investment:

\[ E_{T_{t-1}} \beta \frac{\lambda_t}{\lambda_{T_{t-1}}} (B_{t+1} - B_t) = E_{T_{t-1}} \beta \frac{\lambda_t}{\lambda_{T_{t-1}}} (i_{t+1}B_t + P_t^X X_t - P_t^M M_t). \]  

(72)

In equilibrium, the current account balance is determined by both intratemporal and intertemporal optimization.

1.2.6. The Approximate Linear Model

Estimation, inference and forecasting are based on a linear state space representation of an approximate unobserved components representation of this DSGE model of a small open economy. Cyclical components are modeled by linearizing equilibrium conditions around a stationary deterministic steady state equilibrium which abstracts from long run balanced growth, while trend components are modeled as random walks while ensuring the existence of a well defined balanced growth path.

In what follows, \( E_{t} x_{t+s} \) denotes the rational expectation of variable \( x_{t+s} \), conditional on information available at time \( t \). Also, \( \hat{x}_t \) denotes the cyclical component of variable \( x_t \), while \( \bar{x}_t \) denotes the trend component of variable \( x_t \). Cyclical and trend components are additively separable, that is \( x_t = \hat{x}_t + \bar{x}_t \).

1.2.6.1. Cyclical Components

The cyclical component of output price inflation depends on a linear combination of past and expected future cyclical components of output price inflation driven by the contemporaneous cyclical components of real marginal cost and the tax rate according to output price Phillips curve

\[ \hat{\pi}_t = \gamma \frac{\beta}{1 + \gamma' \beta} \hat{\pi}_{t+1} + \frac{\beta}{1 + \gamma' \beta} E \hat{\pi}_{t+1} + \frac{(1 - \omega')(1 - \omega' \beta)}{\omega' (1 + \gamma' \beta)} \left[ \ln \Phi + \frac{\tau}{1 - \tau} \ln \hat{r}_t - \frac{1}{\theta' - 1} \ln \hat{\sigma'} \right]. \]  

(73)

where \( \Phi = (1 - \tau) \frac{\theta' - 1}{\theta'} \). The persistence of the cyclical component of output price inflation is increasing in indexation parameter \( \gamma' \), while the sensitivity of the cyclical component of output price inflation to changes in the cyclical components of real marginal cost and the tax rate is
decreasing in nominal rigidity parameter \( \omega^Y \) and indexation parameter \( \gamma^Y \). This output price Phillips curve is subject to output price markup shocks.

The cyclical component of output depends on the contemporaneous cyclical components of utilized capital and effective labour according to approximate linear net production function

\[
\ln \hat{Y}_t = \left(1 - \frac{\theta^Y WL}{\theta^Y - 1 PY}\right) \ln(\hat{u}_t \hat{K}_t) + \frac{\theta^Y WL}{\theta^Y - 1 PY} \ln(\hat{A}_t \hat{L}_t),
\]

(74)

where \( \frac{K}{Y} = \frac{\beta(1-\tau)}{1-\beta(1-\delta)} \left(\frac{\theta^Y - 1}{\theta^Y} - \frac{WL}{PY}\right) \). This approximate linear net production function is subject to output technology shocks.

The cyclical component of the rate of capital utilization depends on the contemporaneous cyclical component of the ratio of capital to effective labour according to approximate linear implicit capital utilization function:

\[
\ln \hat{u}_t = -\frac{\theta^Y WL}{\theta^Y - 1 PY} \left(\kappa^9 + \frac{\theta^Y WL}{\theta^Y - 1 PY}\right) \ln \frac{\hat{K}_t}{\hat{A}_t \hat{L}_t}.
\]

(75)

The sensitivity of the cyclical component of the rate of capital utilization to changes in the cyclical component of the ratio of capital to effective labour is decreasing in capital utilization cost parameter \( \kappa \) and elasticity of substitution parameter \( \theta^Y \). This approximate linear implicit capital utilization function is subject to output technology shocks.

The cyclical component of consumption, investment, or government consumption price inflation depends on a linear combination of past and expected future cyclical components of consumption, investment, or government consumption price inflation driven by the contemporaneous cyclical components of real marginal cost and the tax rate according to Phillips curves:

\[
\hat{\pi}^2_t = \frac{\gamma^Y}{1 + \gamma^Y \beta} \hat{\pi}^2_i + \frac{\beta}{1 + \gamma^Y \beta} E_i \hat{\pi}^2_{i+1} + \frac{(1 - \omega^Y)(1 - \omega^Y \beta)}{\omega^Y (1 + \gamma^Y \beta)} \left[ \ln \hat{\Phi}_i + \frac{\tau}{1 - \tau} \ln \hat{\varepsilon}_i - \frac{1}{\theta^Y - 1} \ln \hat{\theta}_i \right]
\]

\[
- \frac{\gamma^Y (1 - \phi^Z)}{1 + \gamma^Y \beta} \Delta \ln \frac{\hat{\tau}_{i+1}}{\hat{v}^M_{i+1}} + (1 - \phi^Z) \Delta \ln \frac{\hat{\tau}_i}{\hat{v}^M_{i}} - \frac{\beta(1 - \phi^Z)}{1 + \gamma^Y \beta} E_i \Delta \ln \frac{\hat{\tau}_{i+1}}{\hat{v}^M_{i+1}}.
\]

(76)

Reflecting the entry of the price of imports into the aggregate consumption, investment, or government consumption price index, the cyclical component of consumption, investment, or government consumption price inflation also depends on past, contemporaneous, and expected future proportional changes in the cyclical component of the terms of trade. These Phillips curves are subject to output price markup and import technology shocks.
The cyclical component of consumption depends on a linear combination of past and expected future cyclical components of consumption driven by the contemporaneous cyclical component of the real interest rate according to approximate linear consumption Euler equation:

$$\ln \hat{C}_t = \frac{\alpha}{1+\alpha} \ln \hat{C}_{t-1} + \frac{1}{1+\alpha} E_t \ln \hat{C}_{t+1} - \frac{1-\alpha}{1+\alpha} \left[ r_t^c + E_t \ln \frac{\hat{\nu}^c_t}{\hat{\nu}^C_t} \right].$$

(77)

The persistence of the cyclical component of consumption is increasing in habit persistence parameter $\alpha$, while the sensitivity of the cyclical component of consumption to changes in the cyclical component of the real interest rate is increasing in intertemporal elasticity of substitution parameter $\sigma$ and decreasing in habit persistence parameter $\alpha$. This approximate linear consumption Euler equation is subject to preference shocks.

The cyclical component of investment depends on a linear combination of past and expected future cyclical components of investment driven by the contemporaneous cyclical component of the relative shadow price of capital according to approximate linear investment demand function:

$$\ln \hat{I}_t = \frac{1}{1+\beta} \ln \hat{I}_{t-1} + \frac{\beta}{1+\beta} E_t \ln \hat{I}_{t+1} + \frac{1}{\chi(1+\beta)} \ln \left( \hat{\nu}^I_t \frac{\hat{Q}_t}{\hat{P}_t^Y} \right).$$

(78)

The sensitivity of the cyclical component of investment to changes in the cyclical component of the relative shadow price of capital is decreasing in investment adjustment cost parameter $\chi$. This approximate linear investment demand function is subject to investment technology shocks.

The cyclical component of the relative shadow price of capital depends on the expected future cyclical component of the relative shadow price of capital, the contemporaneous cyclical component of the real interest rate, the expected future cyclical component of real marginal cost, and the expected future cyclical component of the marginal product of capital according to approximate linear investment Euler equation:

$$\ln \frac{\hat{O}_t}{\hat{P}_t^Y} = \beta(1-\delta) E_t \ln \frac{\hat{Q}_t}{\hat{P}_t^Y} - \hat{r}_t^Y$$

$$+ \left[ 1 - \beta(1-\delta) \right] E_t \ln \hat{O}_t + \frac{1}{\theta^Y} \frac{W_L}{\theta^Y - 1} E_t \ln \frac{\hat{K}_{t+1}}{A_{t+1} \hat{L}_{t+1}}.$$

(79)

The sensitivity of the cyclical component of the relative shadow price of capital to changes in the cyclical component of the ratio of utilized capital to effective labour is decreasing in elasticity of substitution parameter $\theta$. This approximate linear investment Euler equation is subject to output technology shocks.
The cyclical component of the capital stock depends on the past cyclical component of the capital stock and the contemporaneous cyclical component of investment according to approximate linear capital accumulation function
\[
\ln \hat{K}_{t-1} = (1 - \delta) \ln \hat{K}_t + \delta \ln(\hat{I}_t),
\] (80)
where \(\frac{I}{K} = \delta\). This approximate linear capital accumulation function is subject to investment technology shocks.

The cyclical component of the ratio of nominal government consumption to nominal output depends on the contemporaneous cyclical component of the ratio of net foreign debt to nominal output according to fiscal expenditure rule:
\[
\ln \frac{\hat{P}_t G_t}{\hat{P}_t Y_t} = \zeta^G \ln \left(\frac{-\hat{B}_{t+1}}{\hat{P}_t Y_t}\right) + \hat{\nu}_t^G.
\] (81)
This fiscal expenditure rule ensures convergence of the level of the ratio of net foreign debt to nominal output to its target value in deterministic steady state equilibrium, and is subject to fiscal expenditure shocks.

The cyclical component of the tax rate depends on the contemporaneous cyclical component of the ratio of net government debt to nominal output according to fiscal revenue rule:
\[
\ln \hat{\tau}_t = \zeta^T \ln \left(\frac{-\hat{B}_{t+1}}{\hat{P}_t Y_t}\right) + \hat{\nu}_t^T.
\] (82)
This fiscal revenue rule ensures convergence of the level of the ratio of net government debt to nominal output to its target value in deterministic steady state equilibrium, and is subject to fiscal revenue shocks.

The cyclical component of import price inflation depends on a linear combination of past and expected future cyclical components of import price inflation driven by the contemporaneous cyclical component of the deviation of the domestic currency price of foreign output from the price of imports according to import price Phillips curve:
\[
\hat{\pi}_t^M = \frac{\gamma^M}{1 + \gamma^M} \hat{\tau}_t + \frac{\beta}{1 + \gamma^M} E_t \hat{\pi}_t^M + \frac{(1 - \omega^M)(1 - \omega^M \beta)}{\omega^M (1 + \gamma^M \beta)} \left[ \ln \frac{\hat{P}_t \hat{P}_{t-1}^{Y,t}}{\hat{P}_t M} - \frac{1}{\theta^M - 1} \ln \hat{\sigma}_t^M \right].
\] (83)
The persistence of the cyclical component of import price inflation is increasing in indexation parameter \(\gamma^M\), while the sensitivity of the cyclical component of import price inflation to changes in the cyclical component of real marginal cost is decreasing in nominal rigidity.
parameter \( \omega^M \) and indexation parameter \( \gamma^M \). This import price Phillips curve is subject to import price markup shocks.

The cyclical component of exports depends on the contemporaneous cyclical components of foreign consumption, investment, government consumption, and the terms of trade according to approximate linear export demand function

\[
\frac{X}{Y} \ln \hat{X}_t = (1 - \phi^{C,J}) \frac{C_t'}{Y_t'} \ln \frac{\hat{C}_t'}{\hat{v}_t^{M,J}} + (1 - \phi^{I,J}) \frac{I_t'}{Y_t'} \ln \frac{\hat{I}_t'}{\hat{v}_t^{M,J}} + (1 - \phi^{G,J}) \frac{G_t'}{Y_t'} \ln \frac{\hat{G}_t'}{\hat{v}_t^{M,J}} \\
-\psi \left[ \phi^{C,J} (1 - \phi^{C,J}) \frac{C_t'}{Y_t'} + \phi^{I,J} (1 - \phi^{I,J}) \frac{I_t'}{Y_t'} + \phi^{G,J} (1 - \phi^{G,J}) \frac{G_t'}{Y_t'} \right] \ln \frac{\hat{T}_t'}{\hat{v}_t^{M,J}},
\]

where \( \frac{X}{Y} = 1 - \phi^{C,J} \frac{C}{Y} - \phi^{I,J} \frac{I}{Y} - \phi^{G,J} \frac{G}{Y} \). The sensitivity of the cyclical component of exports to changes in the cyclical component of the foreign terms of trade is increasing in elasticity of substitution parameter \( \psi \). This approximate linear export demand function is subject to foreign import technology shocks.

The cyclical component of imports depends on the contemporaneous cyclical components of consumption, investment, government consumption, and the terms of trade according to approximate linear import demand function

\[
\frac{M}{Y} \ln \hat{M}_t = (1 - \phi^{C,J}) \frac{C_t}{Y_t} \ln \frac{\hat{C}_t}{\hat{v}_t^M} + (1 - \phi^{I,J}) \frac{I_t}{Y_t} \ln \frac{\hat{I}_t}{\hat{v}_t^M} + (1 - \phi^{G,J}) \frac{G_t}{Y_t} \ln \frac{\hat{G}_t}{\hat{v}_t^M} \\
-\psi \left[ \phi^{C,J} (1 - \phi^{C,J}) \frac{C_t}{Y_t} + \phi^{I,J} (1 - \phi^{I,J}) \frac{I_t}{Y_t} + \phi^{G,J} (1 - \phi^{G,J}) \frac{G_t}{Y_t} \right] \ln \frac{\hat{T}_t}{\hat{v}_t^M},
\]

where \( \frac{M}{Y} = (1 - \phi^{C,J}) \frac{C}{Y} + (1 - \phi^{I,J}) \frac{I}{Y} + (1 - \phi^{G,J}) \frac{G}{Y} \). The sensitivity of the cyclical component of imports to changes in the cyclical component of the terms of trade is increasing in elasticity of substitution parameter \( \psi \). This approximate linear import demand function is subject to import technology shocks.

The cyclical component of the real wage depends on a linear combination of past and expected future cyclical components of the real wage driven by the contemporaneous cyclical component of the deviation of the marginal rate of substitution between leisure and consumption from the after tax real wage according to wage Phillips curve:

\[
\ln \frac{\hat{W}_t}{P_t^c} = \frac{1}{1 + \beta} \ln \hat{W}_{t+1} + \beta E_t \frac{\gamma^L}{1 + \beta} \hat{\pi}_t^L + \frac{\gamma^I}{1 + \beta} \hat{\pi}_t^C - \frac{1 + \gamma^L \beta}{1 + \beta} \hat{\pi}_t^C + \frac{\beta}{1 + \beta} E_t \hat{\pi}_t^C \\
+ \frac{(1 - \omega^L)(1 - \omega^I)}{\omega^I (1 + \beta)} \left[ \frac{1}{\eta} \ln \hat{L}_t + \frac{1}{\sigma} \ln \hat{C}_t - \alpha \ln \hat{C}_{t+1} + \frac{\tau}{1 - \tau} \ln \hat{\pi}_t - \ln \frac{\hat{W}_t}{P_t^c} - \frac{1}{\omega^I - 1} \ln \hat{\theta}_t \right].
\]
Reflecting the existence of partial wage indexation, the cyclical component of the real wage also depends on past, contemporaneous, and expected future cyclical components of consumption price inflation. The sensitivity of the cyclical component of the real wage to changes in the cyclical component of consumption price inflation is increasing in indexation parameter \( \gamma \), to changes in the cyclical component of the deviation of the marginal rate of substitution between leisure and consumption from the after tax real wage is decreasing in nominal rigidity parameter \( \omega \), and to changes in the cyclical component of employment is decreasing in elasticity of substitution parameter \( \eta \). This wage Phillips curve is subject to wage markup shocks.

The cyclical component of real marginal cost depends on the contemporaneous cyclical component of the deviation of the after tax real wage from the marginal product of labour according to approximate linear implicit labour demand function:

\[
\ln \phi_i = \ln \frac{\hat{w}_i}{\hat{P}_i - A_i} - \frac{\tau}{1-\tau} \ln \hat{r}_i - \frac{1}{\mathcal{G}} \left( 1 - \frac{\theta}{\theta - 1} \frac{WL}{\hat{P}Y} \right) \ln \frac{\hat{u}_i}{\hat{A}_i \hat{L}_i}.
\] (87)

The sensitivity of the cyclical component of real marginal cost to changes in the cyclical component of the ratio of utilized capital to effective labour is decreasing in elasticity of substitution parameter \( \mathcal{G} \). This approximate linear implicit labour demand function is subject to output technology shocks.

The cyclical component of the nominal interest rate depends on the contemporaneous cyclical components of consumption price inflation and output according to monetary policy rule:

\[
\hat{i}_t = \xi^e \hat{x}^c_t + \xi^f \ln \hat{Y}_t + \nu^f_t.
\] (88)

This monetary policy rule ensures convergence of the level of consumption price inflation to its target value in deterministic steady state equilibrium, and is subject to monetary policy shocks. The cyclical component of the output based real interest rate satisfies \( \hat{r}_t^{\nu} = \hat{i}_t - E_t \hat{x}^c_t \), while the cyclical component of the consumption based real interest rate satisfies \( \hat{r}_t^c = \hat{i}_t - E_t \hat{c}^c_t \).

The cyclical component of the nominal exchange rate depends on the expected future cyclical component of the nominal exchange rate and the contemporaneous cyclical component of the nominal interest rate differential according to approximate linear uncovered interest parity condition:

\[
\ln \hat{e}_t = E_t \ln \hat{e}_{t+1} - (\hat{i}_t - \hat{i}_t^f).
\] (89)
The cyclical component of the real exchange rate satisfies $\ln \hat{Q} = \ln \hat{\xi} + \ln \hat{P}^v$ - $\ln \hat{P}^\xi$, while the cyclical component of the terms of trade satisfies $\ln \hat{T} = \ln \hat{P}^m - \ln \hat{P}^x$, where $\ln \hat{P}^x = \ln \hat{P}^\xi$.

The cyclical component of nominal output depends on the contemporaneous cyclical components of nominal consumption, investment, government consumption, exports, and imports according to approximate linear aggregate resource constraint:

$$\ln(\hat{P}^r \hat{Y}) = \frac{C}{Y} \ln(\hat{P}^c \hat{C}) + \frac{I}{Y} \ln(\hat{P}^i \hat{I}) + \frac{G}{Y} \ln(\hat{P}^g \hat{G}) + \frac{X}{Y} \ln(\hat{P}^x \hat{X}) - \frac{M}{Y} \ln(\hat{P}^m \hat{M}).$$

(90)

In equilibrium, the cyclical component of output is determined by the cumulative demands of domestic and foreign households, firms, and governments.

The cyclical component of the net government debt depends on the past cyclical component of the net government debt, the past cyclical component of the nominal interest rate, the contemporaneous cyclical component of tax revenues, and the contemporaneous cyclical component of nominal government consumption according to approximate linear government dynamic budget constraint

$$E_{t-1} \ln(-\hat{B}^g_{t+1}) = \frac{1}{\beta} [\ln(-\hat{B}^g_t) + \hat{i}_{t+1}] + \left(\frac{B^G}{PY} \right)^{-1} E_{t-1} \left[ \tau \ln(\hat{e}_t \hat{P}^r \hat{Y}) - \frac{G}{Y} \ln(\hat{P}^g \hat{G}) \right].$$

(91)

where $\frac{G^G}{PY} = -\frac{\frac{\tau - G}{Y}}{\frac{1 - \beta}{\frac{1 - \beta}{1 - \beta}}}$. This approximate linear government dynamic budget constraint is well defined only if the level of the net government debt is positive.

The cyclical component of the net foreign debt depends on the past cyclical component of the net foreign debt, the past cyclical component of the nominal interest rate, the contemporaneous cyclical component of export revenues, and the contemporaneous cyclical component of import expenditures according to approximate linear national dynamic budget constraint

$$E_{t-1} \ln(-\hat{B}^f_{t+1}) = \frac{1}{\beta} [\ln(-\hat{B}^f_t) + \hat{i}_{t+1}] + \left(\frac{B}{PY} \right)^{-1} E_{t-1} \left[ \frac{X}{Y} \ln(\hat{P}^x \hat{X}) - \frac{M}{Y} \ln(\hat{P}^m \hat{M}) \right].$$

(92)

where $\frac{X^X}{PY} = -\frac{\frac{X - M}{Y}}{\frac{1 - \beta}{\frac{1 - \beta}{1 - \beta}}}$. This approximate linear national dynamic budget constraint is well defined only if the level of the net foreign debt is positive.

Variation in cyclical components is driven by ten exogenous stochastic processes. The cyclical components of the preference, output technology, investment technology, import technology, output price markup, import price markup, wage markup, monetary policy, fiscal expenditure, and fiscal revenue shocks follow stationary first order autoregressive processes:

$$\ln \hat{\nu}^C_i = \rho_i \ln \hat{\nu}^C_{i-1} + \xi_{i}^{\nu}, \xi_{i}^{\nu} \sim \text{iid } N(0, \sigma^2_i),$$

(93)
\[ \ln \hat{A}_t = \rho^A \ln \hat{A}_{t-1} + \varepsilon^A_t, \varepsilon^A_t \sim \text{iid } \mathcal{N}(0, \sigma^2_A), \]  
(94)

\[ \ln \hat{\nu}^l_t = \rho^\nu \ln \hat{\nu}^l_{t-1} + \varepsilon^\nu_t, \varepsilon^\nu_t \sim \text{iid } \mathcal{N}(0, \sigma^2_\nu), \]  
(95)

\[ \ln \hat{\nu}^M_t = \rho^\nu \ln \hat{\nu}^M_{t-1} + \varepsilon^\nu_t, \varepsilon^\nu_t \sim \text{iid } \mathcal{N}(0, \sigma^2_\nu), \]  
(96)

\[ \ln \hat{\theta}^y_t = \rho^\theta \ln \hat{\theta}^y_{t-1} + \varepsilon^\theta_t, \varepsilon^\theta_t \sim \text{iid } \mathcal{N}(0, \sigma^2_\theta), \]  
(97)

\[ \ln \hat{\theta}^M_t = \rho^\theta \ln \hat{\theta}^M_{t-1} + \varepsilon^\theta_t, \varepsilon^\theta_t \sim \text{iid } \mathcal{N}(0, \sigma^2_\theta), \]  
(98)

\[ \ln \hat{\theta}^L_t = \rho^\theta \ln \hat{\theta}^L_{t-1} + \varepsilon^\theta_t, \varepsilon^\theta_t \sim \text{iid } \mathcal{N}(0, \sigma^2_\theta), \]  
(99)

\[ \hat{\nu}^j_t = \rho^\nu \hat{\nu}^j_{t-1} + \varepsilon^\nu_t, \varepsilon^\nu_t \sim \text{iid } \mathcal{N}(0, \sigma^2_\nu), \]  
(100)

\[ \hat{\nu}^G_t = \rho^\nu \hat{\nu}^G_{t-1} + \varepsilon^\nu_t, \varepsilon^\nu_t \sim \text{iid } \mathcal{N}(0, \sigma^2_\nu), \]  
(101)

\[ \hat{\nu}^I_t = \rho^\nu \hat{\nu}^I_{t-1} + \varepsilon^\nu_t, \varepsilon^\nu_t \sim \text{iid } \mathcal{N}(0, \sigma^2_\nu). \]  
(102)

The innovations driving these exogenous stochastic processes are assumed to be independent, which combined with our distributional assumptions implies multivariate normality. In deterministic steady state equilibrium, \( \nu^c = \nu^l = \nu^M = 1 \) and \( \sigma^2_\nu = \sigma^2_\nu = \sigma^2_\nu = \sigma^2_\nu = \sigma^2_\nu = \sigma^2_\nu = \sigma^2_\nu = \sigma^2_\nu = 0 \).

1.2.6.2. Trend Components

The trend components of the prices of output, consumption, investment, government consumption, and imports follow random walks with time varying drift \( \pi_t \):

\[ \ln \bar{P}^y_t = \pi_t + \ln \bar{P}^y_{t-1} + \varepsilon_{\bar{P}^y_t}, \varepsilon_{\bar{P}^y_t} \sim \text{iid } \mathcal{N}(0, \sigma^2_{\bar{P}^y}), \]  
(103)

\[ \ln \bar{P}^c_t = \pi_t + \ln \bar{P}^c_{t-1} + \varepsilon_{\bar{P}^c_t}, \varepsilon_{\bar{P}^c_t} \sim \text{iid } \mathcal{N}(0, \sigma^2_{\bar{P}^c}), \]  
(104)

\[ \ln \bar{P}^l_t = \pi_t + \ln \bar{P}^l_{t-1} + \varepsilon_{\bar{P}^l_t}, \varepsilon_{\bar{P}^l_t} \sim \text{iid } \mathcal{N}(0, \sigma^2_{\bar{P}^l}), \]  
(105)

\[ \ln \bar{P}^G_t = \pi_t + \ln \bar{P}^G_{t-1} + \varepsilon_{\bar{P}^G_t}, \varepsilon_{\bar{P}^G_t} \sim \text{iid } \mathcal{N}(0, \sigma^2_{\bar{P}^G}), \]  
(106)
\[ \ln \bar{P}^M_t = \pi_t + \ln \bar{P}^M_{t-1} + \epsilon^\mu_t, \epsilon^\mu_t \sim \text{iid } \mathcal{N}(0, \sigma^2_{\bar{P}^M}). \] (107)

It follows that the trend components of the relative prices of consumption, investment, government consumption, and imports follow random walks without drifts. This implies that along a balanced growth path, the levels of these relative prices are time independent but state dependent.

The trend components of output, consumption, investment, government consumption, exports, and imports follow random walks with time varying drifts:

\[ \ln \bar{Y}_t = g_t + \ln \bar{Y}_{t-1} + \epsilon^\bar{Y}_t, \epsilon^\bar{Y}_t \sim \text{iid } \mathcal{N}(0, \sigma^2_{\bar{Y}}), \] (108)

\[ \ln \bar{C}_t = g_t + \ln \bar{C}_{t-1} + \epsilon^\bar{C}_t, \epsilon^\bar{C}_t \sim \text{iid } \mathcal{N}(0, \sigma^2_{\bar{C}}), \] (109)

\[ \ln \bar{I}_t = g_t + \ln \bar{I}_{t-1} + \epsilon^\bar{I}_t, \epsilon^\bar{I}_t \sim \text{iid } \mathcal{N}(0, \sigma^2_{\bar{I}}), \] (110)

\[ \ln \bar{G}_t = g_t + \ln \bar{G}_{t-1} + \epsilon^\bar{G}_t, \epsilon^\bar{G}_t \sim \text{iid } \mathcal{N}(0, \sigma^2_{\bar{G}}), \] (111)

\[ \ln \bar{X}_t = g_t + \ln \bar{X}_{t-1} + \epsilon^\bar{X}_t, \epsilon^\bar{X}_t \sim \text{iid } \mathcal{N}(0, \sigma^2_{\bar{X}}), \] (112)

\[ \ln \bar{M}_t = g_t + \ln \bar{M}_{t-1} + \epsilon^\bar{M}_t, \epsilon^\bar{M}_t \sim \text{iid } \mathcal{N}(0, \sigma^2_{\bar{M}}). \] (113)

It follows that the trend components of the ratios of consumption, investment, government consumption, exports, and imports to output follow random walks without drifts. This implies that along a balanced growth path, the levels of these ratios are time independent but state dependent.

The trend component of the nominal wage follows a random walk with time varying drift \( \pi_t + g_t \), while the trend component of employment follows a random walk with time varying drift \( n_t \):

\[ \ln \bar{W}_t = \pi_t + g_t + \ln \bar{W}_{t-1} + \epsilon^\bar{W}_t, \epsilon^\bar{W}_t \sim \text{iid } \mathcal{N}(0, \sigma^2_{\bar{W}}), \] (114)

\[ \ln \bar{L}_t = n_t + \ln \bar{L}_{t-1} + \epsilon^\bar{L}_t, \epsilon^\bar{L}_t \sim \text{iid } \mathcal{N}(0, \sigma^2_{\bar{L}}). \] (115)

It follows that the trend component of the income share of labour follows a random walk without drift. This implies that along a balanced growth path, the level of the income share of labour is time independent but state dependent. The trend component of real marginal cost satisfies \( \ln \bar{\Phi}_t = \ln \Phi_t \), while the trend component of the rate of capital utilization satisfies \( \ln \bar{u}_t = 0 \). The trend component of the shadow price of capital satisfies \( \ln \bar{Q}_t = \ln \bar{P}^r_t \), while the trend component of the capital stock satisfies \( \ln \bar{K}^{\mu}_t = \ln \bar{K}^{\mu} \).
The trend components of the nominal interest rate, tax rate, and nominal exchange rate follow random walks without drifts:

\[ \tilde{r}_t = \tilde{r}_{t-1} + \varepsilon_t^r, \quad \varepsilon_t^r \sim \text{iid } \mathcal{N}(0, \sigma_r^2), \quad (116) \]

\[ \ln \tilde{\tau}_t = \ln \tilde{\tau}_{t-1} + \varepsilon_t^\tau, \quad \varepsilon_t^\tau \sim \text{iid } \mathcal{N}(0, \sigma_\tau^2), \quad (117) \]

\[ \ln \tilde{\xi}_t = \ln \tilde{\xi}_{t-1} + \varepsilon_t^\xi, \quad \varepsilon_t^\xi \sim \text{iid } \mathcal{N}(0, \sigma_\xi^2). \quad (118) \]

It follows that along a balanced growth path, the levels of the nominal interest rate, tax rate, and nominal exchange rate are time independent but state dependent. The trend component of the output based real interest rate satisfies \( \hat{r}^y_t = \tilde{r} - E_t \tilde{\pi}^y_t \), while the trend component of the consumption based real interest rate satisfies \( \hat{r}^c_t = \tilde{r} - E_t \tilde{\pi}^c_t \). The trend component of the real exchange rate satisfies \( \ln \hat{Q}_t = \ln \tilde{\xi}_t - \ln \tilde{\pi}_t \), while the trend component of the terms of trade satisfies \( \ln \hat{t}_t = \ln \tilde{\pi}_t^y - \ln \tilde{\pi}_t^x \), where \( \ln \tilde{\pi}^y_t = \ln \tilde{\pi}^x_t \). The trend component of the net government debt satisfies \( \ln \hat{\delta}_t = \ln \tilde{\delta}_t - \ln \tilde{\delta}_t \), while the trend component of the net foreign debt satisfies \( \ln \hat{I}_t = \ln \tilde{I}_t - \ln \tilde{I}_t \).

Long run balanced growth is driven by three common stochastic trends. Trend inflation, productivity growth, and population growth follow random walks without drifts:

\[ \pi_t = \pi_{t-1} + \varepsilon_t^\pi, \quad \varepsilon_t^\pi \sim \text{iid } \mathcal{N}(0, \sigma_\pi^2), \quad (119) \]

\[ g_t = g_{t-1} + \varepsilon_t^g, \quad \varepsilon_t^g \sim \text{iid } \mathcal{N}(0, \sigma_g^2), \quad (120) \]

\[ n_t = n_{t-1} + \varepsilon_t^n, \quad \varepsilon_t^n \sim \text{iid } \mathcal{N}(0, \sigma_n^2). \quad (121) \]

It follows that along a balanced growth path, growth rates are time independent but state dependent. All innovations driving variation in trend components are assumed to be independent, which combined with our distributional assumptions implies multivariate normality.

1.3. Estimation, Inference and Forecasting

Unobserved components models feature prominently in the empirical macroeconomics literature, while DSGE models are pervasive in the theoretical macroeconomics literature. The primary contribution of this paper is the joint modeling of cyclical and trend components as
unobserved components while imposing theoretical restrictions derived from the approximate multivariate linear rational expectations representation of a DSGE model.

This merging of modeling paradigms drawn from the theoretical and empirical macroeconomics literatures confers a number of important benefits. First, the joint estimation of parameters and trend components ensures their mutual consistency, as estimates of parameters appropriately reflect estimates of trend components, and vice versa. As shown by Nelson and Kang (1981) and Harvey and Jaeger (1993), decomposing integrated observed endogenous variables into cyclical and trend components with atheoretic deterministic polynomial functions or low pass filters may induce spurious cyclical dynamics, invalidating subsequent estimation, inference and forecasting. Second, basing estimation on the levels as opposed to differences of observed endogenous variables may be expected to yield efficiency gains. A central result of the voluminous cointegration literature surveyed by Maddala and Kim (1998) is that, if there exist cointegrating relationships, then differencing all integrated observed endogenous variables prior to the conduct of estimation, inference and forecasting results in the loss of information. Third, the proposed unobserved components framework ensures stochastic nonsingularity of the resulting approximate linear state space representation of the DSGE model, as associated with each observed endogenous variable is at least one exogenous stochastic process. As discussed in Ruge-Murcia (2003), stochastic nonsingularity requires that the number of observed endogenous variables used to construct the loglikelihood function associated with the approximate linear state space representation of a DSGE model not exceed the number of exogenous stochastic processes, with efficiency losses incurred if this constraint binds. Fourth, the proposed unobserved components framework facilitates the direct generation of forecasts of the levels of endogenous variables as opposed to their cyclical components together with confidence intervals, while ensuring that these forecasts satisfy the stability restrictions associated with balanced growth. These stability restrictions are necessary but not sufficient for full cointegration, as along a balanced growth path, great ratios and trend growth rates are time independent but state dependent, robustifying forecasts to intermittent structural breaks that occur within sample.

1.3.1. Estimation

The traditional econometric interpretation of macroeconomic models regards them as representations of the joint probability distribution of the data. Adopting this traditional econometric interpretation, Bayesian estimation of a linear state space representation of an approximate unobserved components representation of this DSGE model of a small open economy, conditional on prior information concerning the values of parameters and trend
components, facilitates an empirical evaluation of its impulse response and predictive accuracy properties.

1.3.1.1. Estimation Procedure

Let \( x_t \) denote a vector stochastic process consisting of the levels of \( N \) nonpredetermined endogenous variables, of which \( M \) are observed. The cyclical components of this vector stochastic process satisfy second order stochastic linear difference equation

\[
A_0 \hat{x}_t = A_1 \hat{x}_{t-1} + A_2 E_t \hat{x}_{t-1} + A_3 \hat{\nu}_t, \tag{122}
\]

where vector stochastic process \( \hat{\nu}_t \) consists of the cyclical components of \( K \) exogenous variables. This vector stochastic process satisfies stationary first order stochastic linear difference equation

\[
\hat{\nu}_t = B_1 \hat{\nu}_{t-1} + \epsilon_{1,t}, \tag{123}
\]

where \( \epsilon_{1,t} \sim \text{iid } N(0, \Sigma_1) \). The trend components of vector stochastic process \( x_t \) satisfy first order stochastic linear difference equation

\[
C_0 \bar{x}_t = C_1 + C_2 u_t + C_3 \bar{x}_{t-1} + \epsilon_{2,t}, \tag{124}
\]

where \( \epsilon_{2,t} \sim \text{iid } N(0, \Sigma_2) \). Vector stochastic process \( u_t \) consists of the levels of \( L \) common stochastic trends, and satisfies nonstationary first order stochastic linear difference equation

\[
u_{\epsilon} = u_{t-1} + \epsilon_{3,t}, \tag{125}\]

where \( \epsilon_{3,t} \sim \text{iid } N(0, \Sigma_3) \). Cyclical and trend components are additively separable, that is \( x_t = \hat{x}_t + \bar{x}_t \).

If there exists a unique stationary solution to multivariate linear rational expectations model (122), then it may be expressed as:

\[
\hat{x}_t = D_1 \hat{x}_{t-1} + D_2 \hat{\nu}_t. \tag{126}
\]

This unique stationary solution is calculated with the matrix decomposition based algorithm due to Klein (2000).

Let \( y_t \) denote a vector stochastic process consisting of the levels of \( M \) observed nonpredetermined endogenous variables. Also, let \( z_t \) denote a vector stochastic process
consisting of the levels of $N-M$ unobserved nonpredetermined endogenous variables, the cyclical components of $N$ nonpredetermined endogenous variables, the trend components of $N$ nonpredetermined endogenous variables, the cyclical components of $K$ exogenous variables, and the levels of $L$ common stochastic trends. Given unique stationary solution (126), these vector stochastic processes have linear state space representation

$$y_t = F_t^T z_t,$$

$$z_t = G_t + G_t^T z_{t-1} + G_t^T e_{4,t},$$

where $e_{4,t} \sim \text{iid } \mathcal{N}(\mathbf{0}, \Sigma_4)$ and $z_0 \sim \mathcal{N}(z_{00}, P_{00})$. Let $w_t$ denote a vector stochastic process consisting of preliminary estimates of the trend components of $M$ observed nonpredetermined endogenous variables. Suppose that this vector stochastic process satisfies

$$w_t = H_1^T z_t + e_{5,t},$$

where $e_{5,t} \sim \text{iid } \mathcal{N}(\mathbf{0}, \Sigma_5)$. Conditional on known parameter values, this signal equation defines a set of stochastic restrictions on selected unobserved state variables. The signal and state innovation vectors are assumed to be independent, while the initial state vector is assumed to be independent from the signal and state innovation vectors, which combined with our distributional assumptions implies multivariate normality.

Conditional on the parameters associated with these signal and state equations, estimates of unobserved state vector $z_t$ and its mean squared error matrix $P_t$ may be calculated with the filter proposed by Vitek (2006a, 2006b), which adapts the filter due to Kalman (1960) to incorporate prior information. Given initial conditions $z_{00}$ and $P_{00}$, estimates conditional on information available at time $t-1$ satisfy prediction equations:

$$z_{t|t-1} = G_t + G_t^T z_{t-1|t-1},$$

$$P_{t|t-1} = G_t P_{t-1|t-1} G_t^T + G_t \Sigma_4 G_t^T,$$

$$y_{t|t-1} = F_t^T z_{t|t-1},$$

$$Q_{t|t-1} = F_t P_{t|t-1} F_t^T,$$

$$w_{t|t-1} = H_1^T z_{t|t-1},$$

$$R_{t|t-1} = H_1^T P_{t|t-1} H_1^T + \Sigma_5.$$
Given these predictions, under the assumption of multivariate normally distributed signal and state innovation vectors, together with conditionally contemporaneously uncorrelated signal vectors, estimates conditional on information available at time \( t \) satisfy updating equations

\[
\begin{align*}
  z_{i\theta} &= z_{i\theta-1} + K_{y_i}(y_i - y_{i\theta-1}) + K_{w_i}(w_i - w_{i\theta-1}), \\
  P_{i\theta} &= P_{i\theta-1} - K_{y_i}F_{i\theta-1} - K_{w_i}H_iP_{i\theta-1},
\end{align*}
\]

where \( K_{y_i} = P_{i\theta-1}F_{i\theta-1}Q_{y\theta-1}^{-1} \) and \( K_{w_i} = P_{i\theta-1}H_i^TR_{\theta-1}^{-1} \). Given terminal conditions \( z_{T\theta} \) and \( P_{T\theta} \) obtained from the final evaluation of these prediction and updating equations, estimates conditional on information available at time \( T \) satisfy smoothing equations

\[
\begin{align*}
  z_{i\theta} &= z_{i\theta} + J_i(z_{i+1\theta} - z_{i+1\theta}), \\
  P_{i\theta} &= P_{i\theta} + J_i(P_{i+1\theta} - P_{i+1\theta})J_i^T,
\end{align*}
\]

where \( J_i = P_{i\theta}G_i^TP_{i\theta}^{-1} \). Under our distributional assumptions, these estimators of the unobserved state vector are mean squared error optimal.

Let \( \theta \in \Theta \subset \mathbb{R}^J \) denote a \( J \) dimensional vector containing the parameters associated with the signal and state equations of this linear state space model. The Bayesian estimator of this parameter vector has posterior density function

\[
  f(\theta | I_T) \propto f(I_T | \theta) f(\theta),
\]

where \( I_T = \{\{y_i\}_{i=1}^r, \{w_i\}_{i=1}^r\} \). Under the assumption of multivariate normally distributed signal and state innovation vectors, together with conditionally contemporaneously uncorrelated signal vectors, conditional density function \( f(I_T | \theta) \) satisfies:

\[
  f(I_T | \theta) = \prod_{i=1}^T f(y_i | I_{i-1}, \theta) \cdot \prod_{i=1}^T f(w_i | I_{i-1}, \theta).
\]

Under our distributional assumptions, conditional density functions \( f(y_i | I_{i-1}, \theta) \) and \( f(w_i | I_{i-1}, \theta) \) satisfy:

\[
\begin{align*}
  f(y_i | I_{i-1}, \theta) &= (2\pi)^{-\frac{M}{2}} |Q_{y\theta-1}|^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2} (y_i - y_{i\theta-1})^T Q_{y\theta-1}^{-1} (y_i - y_{i\theta-1}) \right\}, \\
  f(w_i | I_{i-1}, \theta) &= (2\pi)^{-\frac{M}{2}} |R_{\theta-1}|^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2} (w_i - w_{i\theta-1})^T R_{\theta-1}^{-1} (w_i - w_{i\theta-1}) \right\}.
\end{align*}
\]
Prior information concerning parameter vector $\theta$ is summarized by a multivariate normal prior distribution having mean vector $\theta_i$ and covariance matrix $\Omega$:

$$f(\theta) = (2\pi)^{-\frac{J}{2}} |\Omega|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2} (\theta - \theta_i)^\top \Omega^{-1} (\theta - \theta_i)\right\}.$$  \hspace{1cm} (144)

Independent priors are represented by a diagonal covariance matrix, under which diffuse priors are represented by infinite variances.

Inference on the parameters is based on an asymptotic normal approximation to the posterior distribution around its mode. Under regularity conditions stated in Geweke (2005), posterior mode $\hat{\theta}_T$ satisfies

$$\sqrt{T}(\hat{\theta}_T - \theta_0) \xrightarrow{d} N(0, -\mathcal{H}_0^{-1}),$$  \hspace{1cm} (145)

where $\theta_0 \in \Theta$ denotes the pseudotrue parameter vector. Following Engle and Watson (1981), Hessian $\mathcal{H}_0$ may be estimated by

$$\mathcal{H}_T = \frac{1}{T} \sum_{t=1}^{T} E_{t-1} \left[ \nabla_\theta \nabla_{\theta}^\top \ln f(y_t | E_{t-1}, \hat{\theta}_T) \right] + \frac{1}{T} \sum_{t=1}^{T} E_{t-1} \left[ \nabla_\theta \nabla_{\theta}^\top \ln f(w_t | E_{t-1}, \hat{\theta}_T) \right]$$

$$+ \frac{1}{T} \nabla_\theta \nabla_{\theta}^\top \ln f(\hat{\theta}_T),$$  \hspace{1cm} (146)

where $E_{t-1} \left[ \nabla_\theta \nabla_{\theta}^\top \ln f(y_t | E_{t-1}, \hat{\theta}_T) \right] = -\nabla_\theta y_{t-1}^\top \Omega_{jt}^{-1} \nabla_\theta y_{t-1} - \frac{1}{2} \nabla_\theta \Omega_{jt}^{-1} (\Omega_{jt}^{-1} \otimes \Omega_{jt}^{-1}) \nabla_\theta \Omega_{jt}^{-1},$

$E_{t-1} \left[ \nabla_\theta \nabla_{\theta}^\top \ln f(w_t | E_{t-1}, \hat{\theta}_T) \right] = -\nabla_\theta w_{t-1}^\top \Omega_{jg}^{-1} \nabla_\theta w_{t-1} - \frac{1}{2} \nabla_\theta \Omega_{jg}^{-1} (\Omega_{jg}^{-1} \otimes \Omega_{jg}^{-1}) \nabla_\theta \Omega_{jg}^{-1},$ and

$\nabla_\theta \nabla_{\theta}^\top \ln f(\hat{\theta}_T) = -\Omega^{-1}.$

1.3.1.2. Estimation Results

The set of parameters associated with this DSGE model of a small open economy is partitioned into two subsets. The first subset is calibrated to approximately match long run averages of functions of observed endogenous variables where possible, and estimates derived from existing microeconometric studies where necessary. The second subset is estimated with the Bayesian procedure described above, conditional on prior information concerning the values of parameters and trend components.

Subjective discount factor $\beta$ is restricted to equal 0.99, implying an annualized deterministic steady state equilibrium real interest rate of approximately 0.04. In deterministic steady state
equilibrium, the output price markup \( \frac{\theta_p}{\theta_{p-1}} \), import price markup \( \frac{\theta^m}{\theta^{m-1}} \), and wage markup \( \frac{\theta^w}{\theta_{w-1}} \) are restricted to equal 1.15. Depreciation rate parameter \( \delta \) is restricted to equal 0.015, implying an annualized deterministic steady state equilibrium depreciation rate of approximately 0.06. In deterministic steady state equilibrium, the consumption import share \( 1 - \phi^c \), investment import share \( 1 - \phi^i \), and government consumption import share \( 1 - \phi^g \) are restricted to equal 0.30. The deterministic steady state equilibrium ratio of consumption to output \( \frac{C}{Y} \) is restricted to equal 0.60, while the deterministic steady state equilibrium ratio of domestic output to foreign output \( \frac{Y}{Y^f} \) is restricted to equal 0.11. In deterministic steady state equilibrium, the foreign consumption import share \( 1 - \phi^{c,f} \), foreign investment import share \( 1 - \phi^{i,f} \), and foreign government consumption import share \( 1 - \phi^{g,f} \) are restricted to equal 0.02. The deterministic steady state equilibrium income share of labour \( \frac{W}{P_Y} \) is restricted to equal 0.50. In deterministic steady state equilibrium, the ratio of government consumption to output \( \frac{G}{Y} \) is restricted to equal 0.20, while the tax rate \( \tau \) is restricted to equal 0.22.

<table>
<thead>
<tr>
<th>Ratio</th>
<th>Value</th>
<th>Ratio</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C/Y )</td>
<td>0.6000</td>
<td>( WL/P_Y )</td>
<td>0.5000</td>
</tr>
<tr>
<td>( I/Y )</td>
<td>0.1723</td>
<td>( K/Y )</td>
<td>2.8710</td>
</tr>
<tr>
<td>( G/Y )</td>
<td>0.2000</td>
<td>( B^c/P_Y )</td>
<td>-0.4950</td>
</tr>
<tr>
<td>( X/Y )</td>
<td>0.3194</td>
<td>( B/P_Y )</td>
<td>-0.6866</td>
</tr>
<tr>
<td>( M/Y )</td>
<td>0.2917</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note: Deterministic steady state equilibrium values are reported at an annual frequency based on calibrated parameter values.

Bayesian estimation of the remaining parameters of this DSGE model of a small open economy is based on the levels of twenty six observed endogenous variables for Canada and the United States described in Appendix 1.A. Those parameters associated with the conditional mean function are estimated subject to cross-economy equality restrictions. Those parameters associated exclusively with the conditional variance function are estimated conditional on diffuse priors. Initial conditions for the cyclical components of exogenous variables are given by their unconditional means and variances, while the initial values of all other state variables are treated as parameters, and are calibrated to match functions of preliminary estimates of trend components calculated with the linear filter described in Hodrick and Prescott (1997). The posterior mode is calculated by numerically maximizing the logarithm of the posterior density kernel with a modified steepest ascent algorithm. Estimation results pertaining to the period 1971Q3 through 2005Q1 are reported in Appendix 1.B. The sufficient condition for the existence of a unique stationary rational expectations equilibrium due to Klein (2000) is satisfied in a neighbourhood around the posterior mode, while the estimator of the Hessian is not nearly
singular at the posterior mode, suggesting that the approximate linear state space representation of this DSGE model of a small open economy is locally identified.

The prior mean of indexation parameter $\gamma^Y$ is 0.75, implying considerable output price inflation inertia, while the prior mean of nominal rigidity parameter $\omega^Y$ implies an average duration of output price contracts of two years. The prior mean of capital utilization cost parameter $\kappa$ is 0.10, while the prior mean of elasticity of substitution parameter $\theta$ is 0.75, implying that utilized capital and effective labour are moderately close complements in production. The prior mean of habit persistence parameter $\alpha$ is 0.95, while the prior mean of intertemporal elasticity of substitution parameter $\sigma$ is 2.75, implying that consumption exhibits considerable persistence and moderate sensitivity to real interest rate changes. The prior mean of investment adjustment cost parameter $\chi$ is 5.75, implying moderate sensitivity of investment to changes in the relative shadow price of capital. The prior mean of indexation parameter $\gamma^M$ is 0.75, implying moderate import price inflation inertia, while the prior mean of nominal rigidity parameter $\omega^M$ implies an average duration of import price contracts of two years. The prior mean of elasticity of substitution parameter $\psi$ is 1.50, implying that domestic and foreign goods are moderately close substitutes in consumption, investment, and government consumption. The prior mean of indexation parameter $\gamma^L$ is 0.75, implying considerable sensitivity of the real wage to changes in consumption price inflation, while the prior mean of nominal rigidity parameter $\omega^L$ implies an average duration of wage contracts of two years. The prior mean of elasticity of substitution parameter $\eta$ is 2.00, implying considerable insensitivity of the real wage to changes in employment. The prior mean of the consumption price inflation response coefficient $\xi^c$ in the monetary policy rule is 1.50, while the prior mean of the output response coefficient $\xi^Y$ is 0.125, ensuring convergence of the level of consumption price inflation to its target value. The prior mean of the net foreign debt response coefficient $\zeta^G$ in the fiscal expenditure rule is −0.10, while the prior mean of the net government debt response coefficient $\zeta^R$ in the fiscal revenue rule is 1.00, ensuring convergence of the levels of the ratios of net foreign debt and net government debt to nominal output to their target values. All autoregressive parameters $\rho$ have prior means of 0.85, implying considerable persistence of shocks driving variation in cyclical components.

The posterior modes of these structural parameters are all close to their prior means, reflecting the imposition of tight independent priors to ensure the existence of a unique stationary rational expectations equilibrium. The estimated variances of shocks driving variation in cyclical components are all well within the range of estimates reported in the existing literature, after accounting for data rescaling. The estimated variances of shocks driving variation in trend components are relatively high, indicating that the majority of variation in the levels of observed endogenous variables is accounted for by variation in their trend components.
Prior information concerning the values of trend components is generated by fitting third order deterministic polynomial functions to the levels of all observed endogenous variables by ordinary least squares. Stochastic restrictions on the trend components of all observed endogenous variables are derived from the fitted values associated with these ordinary least squares regressions, with innovation variances set proportional to estimated prediction variances assuming known parameters. All stochastic restrictions are independent, represented by a diagonal covariance matrix, and are harmonized, represented by a common factor of proportionality. Reflecting little confidence in these preliminary trend component estimates, this common factor of proportionality is set equal to one.

Predicted, filtered and smoothed estimates of the cyclical and trend components of observed endogenous variables are plotted together with confidence intervals in Appendix 1.B. These confidence intervals assume multivariate normally distributed and independent signal and state innovation vectors and known parameters. The predicted estimates are conditional on past information, the filtered estimates are conditional on past and present information, and the smoothed estimates are conditional on past, present and future information. Visual inspection reveals close agreement with the conventional dating of business cycle expansions and recessions.

1.3.2. Inference

Whether this estimated DSGE model approximately accounts for the empirical evidence concerning the monetary transmission mechanism in a small open economy is determined by comparing its impulse responses to domestic and foreign monetary policy shocks with impulse responses derived from an estimated SVAR model.

1.3.2.1. Empirical Impulse Response Analysis

Consider the following SVAR model of the monetary transmission mechanism in a small open economy

\[ A_0 y_t = \mu(t) + \sum_{i=1}^{p} A_i y_{t-i} + B \varepsilon_t, \]  

(147)

where \( \mu(t) \) denotes a third order deterministic polynomial function and \( \varepsilon_t \sim iid \mathcal{N}(0, I) \). Vector stochastic process \( y_t \) consists of domestic output price inflation \( \pi_t^r \), domestic output
ln\(Y_t\), domestic consumption price inflation \(\pi^C_t\), domestic consumption ln\(C_t\), domestic investment price inflation \(\pi^I_t\), domestic investment ln\(I_t\), domestic import price inflation \(\pi^M_t\), domestic exports ln\(X_t\), domestic imports ln\(M_t\), domestic nominal interest rate \(i_t\), nominal exchange rate ln\(E_t\), foreign output price inflation \(\pi^Y_{t'}\), foreign output ln\(Y_{t'}\), foreign consumption ln\(C_{t'}\), foreign investment ln\(I_{t'}\), and foreign nominal interest rate \(i_{t'}\). The diagonal elements of parameter matrix \(A_0\) are normalized to one, while the off diagonal elements of positive definite parameter matrix \(B\) are restricted to equal zero, thus associating with each equation a unique endogenous variable, and with each endogenous variable a unique structural innovation.

This SVAR model is identified by imposing restrictions on the timing of the effects of monetary policy shocks and on the information sets of the monetary authorities, both within and across the domestic and foreign economies. Within the domestic and foreign economies, prices and quantities are restricted to not respond instantaneously to monetary policy shocks, while the monetary authorities can respond instantaneously to changes in these variables. Across the domestic and foreign economies, the domestic monetary authority is restricted to not respond instantaneously to foreign monetary policy shocks, while foreign variables are restricted to not respond to domestic monetary policy shocks.

This SVAR model of the monetary transmission mechanism in a small open economy is estimated by full information maximum likelihood over the period 1971Q3 through 2005Q1. As discussed in Hamilton (1994), in the absence of model misspecification, this full information maximum likelihood estimator is consistent and asymptotically normal, irrespective of the cointegration rank and validity of the conditional multivariate normality assumption. The lag order is selected to minimize multivariate extensions of the model selection criterion functions of Akaike (1974), Schwarz (1978), and Hannan and Quinn (1979) subject to an upper bound equal to the seasonal frequency. These model selection criterion functions generally prefer a lag order of one.

<table>
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<th>(HQ(p))</th>
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<td>1</td>
<td>-110.8778</td>
<td>-102.3386*</td>
<td>-107.4079*</td>
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<tr>
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<td>4</td>
<td>-111.6281*</td>
<td>-89.9197</td>
<td>-102.8068</td>
</tr>
</tbody>
</table>

**Note:** Minimized values of model selection criterion functions are indicated by *.

Since this SVAR model is estimated to provide empirical evidence concerning the monetary transmission mechanism in a small open economy, it is imperative to examine the empirical
validity of its overidentifying restrictions prior to the conduct of impulse response analysis. On the basis of bootstrap likelihood ratio tests, these overidentifying restrictions are not rejected at conventional levels of statistical significance.

<table>
<thead>
<tr>
<th>Test Statistic</th>
<th>P Values</th>
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</tr>
<tr>
<td>278.1389</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

Note: This likelihood ratio test statistic is asymptotically distributed as $\chi^2$. Bootstrap distributions are based on 999 replications.

Theoretical impulse responses to a domestic monetary policy shock are plotted versus empirical impulse responses in Figure 1.1. Following a domestic monetary policy shock, the domestic nominal interest rate exhibits an immediate increase followed by a gradual decline. The domestic currency appreciates, with the nominal exchange rate exhibiting delayed overshooting. These nominal interest rate and nominal exchange rate dynamics induce persistent and generally statistically significant hump shaped negative responses of domestic output price inflation, output, consumption price inflation, consumption, investment price inflation, investment, import price inflation, exports and imports, with peak effects realized after approximately one year. These results are qualitatively consistent with those of SVAR analyses of the monetary transmission mechanism in open economies such as Eichenbaum and Evans (1995), Clarida and Gertler (1997), Kim and Roubini (1995), and Cushman and Zha (1997).
Figure 1.1. Theoretical versus empirical impulse responses to a domestic monetary policy shock

Note: Theoretical impulse responses to a 50 basis point monetary policy shock are represented by black lines, while blue lines depict empirical impulse responses to a 50 basis point monetary policy shock. Asymmetric 95% confidence intervals are calculated with a nonparametric bootstrap simulation with 999 replications.

Theoretical impulse responses to a foreign monetary policy shock are plotted versus empirical impulse responses in Figure 1.2. Following a foreign monetary policy shock, the foreign nominal interest rate exhibits an immediate increase followed by a gradual decline. In response to these nominal interest rate dynamics, there arise persistent and generally statistically significant hump shaped negative responses of foreign output price inflation, output, consumption and investment, with peak effects realized after approximately one to two years. Although domestic output, consumption, investment and imports decline, domestic consumption price inflation, investment price inflation and import price inflation rise due to domestic currency depreciation. These results are qualitatively consistent with those of SVAR analyses of the monetary transmission mechanism in closed economies such as Sims and Zha (1995), Gordon
Figure 1.2. Theoretical versus empirical impulse responses to a foreign monetary policy shock

Note: Theoretical impulse responses to a 50 basis point monetary policy shock are represented by black lines, while blue lines depict empirical impulse responses to a 50 basis point monetary policy shock. Asymmetric 95% confidence intervals are calculated with a nonparametric bootstrap simulation with 999 replications.

Visual inspection reveals that the theoretical impulse responses to domestic and foreign monetary policy shocks generally lie within confidence intervals associated with the corresponding empirical impulse responses, suggesting that this estimated DSGE model approximately accounts for the empirical evidence concerning the monetary transmission mechanism in a small open economy. However, these confidence intervals are rather wide, indicating that considerable uncertainty surrounds this empirical evidence.
1.3.2.2. Theoretical Impulse Response Analysis

In an open economy, exchange rate adjustment contributes to both intratemporal and intertemporal equilibration, while business cycles are generated by interactions among a variety of nominal and real shocks originating both domestically and abroad. Theoretical impulse responses and forecast error variance decompositions to domestic and foreign preference, output technology, investment technology, import technology, output price markup, import price markup, wage markup, monetary policy, fiscal expenditure, and fiscal revenue shocks are plotted in Appendix 1.B.

Following a domestic output technology shock, there arise persistent hump shaped positive responses of domestic output, consumption, investment, and government consumption. Domestic output price inflation, consumption price inflation, investment price inflation, and government consumption price inflation exhibit persistent hump shaped declines in response to a reduction in real marginal cost. The domestic nominal and real interest rates exhibit persistent hump shaped declines in response to a reduction in consumption price inflation, mitigated by an increase in output. The domestic currency appreciates in nominal terms and depreciates in real terms, while the terms of trade deteriorate. Since the increase in nominal output exceeds the increase in domestic demand, the trade balance rises, facilitating an intertemporal resource transfer between the domestic and foreign economies.

Following a domestic monetary policy shock, the domestic nominal and real interest rates exhibit immediate increases followed by gradual declines, inducing persistent hump shaped negative responses of domestic output, consumption, investment, and government consumption. The nominal and real exchange rates overshoot, with immediate appreciations followed by gradual depreciations. Domestic output price inflation, consumption price inflation, investment price inflation, and government consumption price inflation exhibit persistent hump shaped declines in response to a reduction in real marginal cost. These declines in domestic consumption price inflation, investment price inflation, and government consumption price inflation are amplified and accelerated by an improvement in the terms of trade. This reduction in the price of imports in terms of exports induces intratemporal expenditure switching, with a decline in the trade balance reflecting a reduction in nominal output relative to domestic demand.

Following a domestic fiscal expenditure shock, there arise immediate positive responses of domestic output and government consumption, together with persistent hump shaped negative responses of domestic consumption and investment. Domestic output price inflation rises in response to an increase in real marginal cost. The domestic nominal and real interest rates exhibit immediate increases followed by gradual declines, causing the domestic currency to
appreciate in nominal and real terms, while the terms of trade improve. Domestic consumption
price inflation, investment price inflation, and government consumption price inflation rise in
response to an increase in real marginal cost, amplified and accelerated by an improvement in the
terms of trade. Since the increase in nominal output is less than the increase in domestic
demand, the trade balance declines, facilitating an intertemporal resource transfer between the
domestic and foreign economies.

1.3.3. Forecasting

While it is desirable that forecasts be unbiased and efficient, the practical value of any
forecasting model depends on its relative predictive accuracy. In the absence of a well defined
mapping between forecast errors and their costs, relative predictive accuracy is generally
assessed with mean squared prediction error based measures. As discussed in Clements and
Hendry (1998), mean squared prediction error based measures are noninvariant to nonsingular,
scale preserving linear transformations, even though linear models are. It follows that mean
squared prediction error based comparisons may yield conflicting rankings across models,
depending on the variable transformations examined.

To compare the dynamic out of sample forecasting performance of the DSGE and SVAR
models, forty quarters of observations are retained to evaluate forecasts one through eight
quarters ahead, generated conditional on parameters estimated using information available at the
forecast origin. The models are compared on the basis of mean squared prediction errors in
levels, ordinary differences, and seasonal differences. The DSGE model is not recursively
estimated as the forecast origin rolls forward due to the high computational cost of such a
procedure, while the SVAR model is. Presumably, recursively estimating the DSGE model
would improve its predictive accuracy.

Mean squared prediction error differentials are plotted together with confidence intervals
accounting for contemporaneous and serial correlation of forecast errors in Appendix 1.B. If
these mean squared prediction error differentials are negative then the forecasting performance
of the DSGE model dominates that of the SVAR model, while if positive then the DSGE model
is dominated by the SVAR model in terms of predictive accuracy. The null hypothesis of equal
squared prediction errors is rejected by the predictive accuracy test of Diebold and Mariano
(1995) if and only if these confidence intervals exclude zero. The asymptotic variance of the
average loss differential is estimated by a weighted sum of the autocovariances of the loss
differential, employing the weighting function proposed by Newey and West (1987). Visual
inspection reveals that these mean squared prediction error differentials are generally negative,
suggesting that the DSGE model dominates the SVAR model in terms of forecasting performance, in spite of a considerable informational disadvantage. However, these mean squared prediction error differentials are rarely statistically significant at conventional levels, perhaps because the predictive accuracy test due to Diebold and Mariano (1995), which is univariate, typically lacks power to detect dominance in forecasting performance, as evidenced by Monte Carlo evaluations such as Ashley (2003) and McCracken (2000).

Dynamic out of sample forecasts of levels, ordinary differences, and seasonal differences are plotted together with confidence intervals versus realized outcomes in Appendix 1.B. These confidence intervals assume multivariate normally distributed and independent signal and state innovation vectors and known parameters. Visual inspection reveals that the realized outcomes generally lie within their associated confidence intervals, suggesting that forecast failure is absent. However, these confidence intervals are rather wide, indicating that considerable uncertainty surrounds the point forecasts.

1.4. Conclusion

This paper develops and estimates a DSGE model of a small open economy which approximately accounts for the empirical evidence concerning the monetary transmission mechanism, as summarized by impulse response functions derived from an estimated SVAR model, while dominating that SVAR model in terms of predictive accuracy. Cyclical components are modeled by linearizing equilibrium conditions around a stationary deterministic steady state equilibrium which abstracts from long run balanced growth, while trend components are modeled as random walks while ensuring the existence of a well defined balanced growth path. This estimated DSGE model consolidates much existing theoretical and empirical knowledge concerning the monetary transmission mechanism in a small open economy, provides a framework for a progressive research strategy, and suggests partial explanations for its own deficiencies.

Jointly modeling cyclical and trend components as unobserved components while imposing theoretical restrictions derived from the approximate multivariate linear rational expectations representation of a DSGE model confers a number of benefits of particular importance to the conduct of monetary policy. As discussed in Woodford (2003), the levels of the flexible price and wage equilibrium components of various observed and unobserved endogenous variables are important inputs into the optimal conduct of monetary policy, in particular the measurement of the stance of monetary policy. Jointly modeling cyclical and trend components as unobserved components facilitates estimation of the levels of the flexible price and wage equilibrium
components of endogenous variables while imposing relatively weak identifying restrictions on their trend components. The analysis of optimal monetary policy under an inflation targeting regime and the estimation of the levels of flexible price and wage equilibrium components within the framework of an extended and refined version of this DSGE model of a small open economy remains an objective for future research.

Appendix 1.A. Description of the Data Set

The data set consists of quarterly seasonally adjusted observations on twenty six macroeconomic variables for Canada and the United States over the period 1971Q1 through 2005Q1. All aggregate prices and quantities are expenditure based. Model consistent employment is derived from observed nominal labour income and a nominal wage index, while model consistent tax rates are derived from observed nominal output and disposable income. The nominal interest rate is measured by the three month Treasury bill rate expressed as a period average, while the nominal exchange rate is quoted as an end of period value. National accounts data for Canada was retrieved from the CANSIM database maintained by Statistics Canada, national accounts data for the United States was obtained from the FRED database maintained by the Federal Reserve Bank of Saint Louis, and other data was extracted from the IFS database maintained by the International Monetary Fund.
### Table 1.4. Bayesian estimation results

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Prior Distribution</th>
<th>Posterior Distribution</th>
</tr>
</thead>
<tbody>
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<td>Mean</td>
<td>Standard Error</td>
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*Note: All observed endogenous variables are rescaled by a factor of 100.*
Note: Symmetric 95% confidence intervals assume multivariate normally distributed and independent signal and state innovation vectors and known parameters. Shaded regions indicate recessions as dated by the Economic Cycle Research Institute reference cycle.
Figure 1.4. Filtered cyclical components of observed endogenous variables

Note: Symmetric 95% confidence intervals assume multivariate normally distributed and independent signal and state innovation vectors and known parameters. Shaded regions indicate recessions as dated by the Economic Cycle Research Institute reference cycle.
Figure 1.5. Smoothed cyclical components of observed endogenous variables

Note: Symmetric 95% confidence intervals assume multivariate normally distributed and independent signal and state innovation vectors and known parameters. Shaded regions indicate recessions as dated by the Economic Cycle Research Institute reference cycle.
Figure 1.6. Predicted trend components of observed endogenous variables

Note: Observed levels are represented by black lines, while blue lines depict estimated trend components. Symmetric 95% confidence intervals assume multivariate normally distributed and independent signal and state innovation vectors and known parameters. Shaded regions indicate recessions as dated by the Economic Cycle Research Institute reference cycle.
Figure 1.7. Filtered trend components of observed endogenous variables

Note: Observed levels are represented by black lines, while blue lines depict estimated trend components. Symmetric 95% confidence intervals assume multivariate normally distributed and independent signal and state innovation vectors and known parameters. Shaded regions indicate recessions as dated by the Economic Cycle Research Institute reference cycle.
Figure 1.8. Smoothed trend components of observed endogenous variables

Note: Observed levels are represented by black lines, while blue lines depict estimated trend components. Symmetric 95% confidence intervals assume multivariate normally distributed and independent signal and state innovation vectors and known parameters. Shaded regions indicate recessions as dated by the Economic Cycle Research Institute reference cycle.
Figure 1.9. Theoretical impulse responses to a domestic output technology shock

Note: Theoretical impulse responses to a unit standard deviation innovation are represented by blue lines.
Figure 1.10. Theoretical impulse responses to a domestic monetary policy shock

Note: Theoretical impulse responses to a unit standard deviation innovation are represented by blue lines.
Figure 1.11. Theoretical impulse responses to a domestic fiscal expenditure shock

[Diagram of impulse response functions]

Note: Theoretical impulse responses to a unit standard deviation innovation are represented by blue lines.
Figure 1.12. Theoretical impulse responses to a foreign output technology shock

Note: Theoretical impulse responses to a unit standard deviation innovation are represented by blue lines.
Figure 1.13. Theoretical impulse responses to a foreign monetary policy shock

Note: Theoretical impulse responses to a unit standard deviation innovation are represented by blue lines.
Figure 1.14. Theoretical impulse responses to a foreign fiscal expenditure shock

Note: Theoretical impulse responses to a unit standard deviation innovation are represented by blue lines.
Figure 1.16. Mean squared prediction error differentials for levels

Note: Mean squared prediction error differentials are defined as the mean squared prediction error for the DSGE model less that for the SVAR model. Symmetric 95% confidence intervals account for contemporaneous and serial correlation of forecast errors.
Figure 1.17. Mean squared prediction error differentials for ordinary differences

Note: Mean squared prediction error differentials are defined as the mean squared prediction error for the DSGE model less that for the SVAR model. Symmetric 95% confidence intervals account for contemporaneous and serial correlation of forecast errors.
Figure 1.18. Mean squared prediction error differentials for seasonal differences

Note: Mean squared prediction error differentials are defined as the mean squared prediction error for the DSGE model less that for the SVAR model. Symmetric 95% confidence intervals account for contemporaneous and serial correlation of forecast errors.
Figure 1.19. Dynamic forecasts of levels of observed endogenous variables

Note: Realized outcomes are represented by black lines, while blue lines depict point forecasts. Symmetric 95% confidence intervals assume multivariate normally distributed and independent signal and state innovation vectors and known parameters.
Figure 1.20. Dynamic forecasts of ordinary differences of observed endogenous variables

Note: Realized outcomes are represented by black lines, while blue lines depict point forecasts. Symmetric 95% confidence intervals assume multivariate normally distributed and independent signal and state innovation vectors and known parameters.
Figure 1.21. Dynamic forecasts of seasonal differences of observed endogenous variables

Note: Realized outcomes are represented by black lines, while blue lines depict point forecasts. Symmetric 95% confidence intervals assume multivariate normally distributed and independent signal and state innovation vectors and known parameters.
References


Monacelli, T. (2005), Monetary policy in a low pass-through environment, *Journal of Money, Credit, and Banking*, 37, 1047-1066.


CHAPTER 2

An Unobserved Components Model of the Monetary Transmission Mechanism in a Small Open Economy

2.1. Introduction

In recent years, the central banks of many economies have adopted inflation targeting monetary policy regimes. An inflation targeting monetary policy regime is characterized by three primary elements. First, there exists an explicit inflation target, which is typically quite low and is often specified as an interval. Second, achieving an inflation control objective, in the form of minimizing deviations of inflation from its target value, is emphasized relative to achieving an output stabilization objective. Third, the conduct of monetary policy is characterized by a high degree of transparency and accountability.

A stylized qualitative description of the monetary transmission mechanism in a small open economy distinguishes among instruments, indicators, and targets. Under an inflation targeting monetary policy regime, the central bank periodically adjusts a short term nominal interest rate in response to inflationary pressure. Provided that this response is sufficiently large, in the presence of short run nominal rigidities or imperfect information, an increase in the short term nominal interest rate causes an increase in the short term real interest rate, inducing intertemporal reductions in consumption and investment. In an open economy, an increase in the short term nominal interest rate causes a nominal appreciation, while an increase in the short term real interest rate causes a real appreciation. This adjustment of the real exchange rate induces an intratemporal reduction in exports together with an intratemporal increase in imports. In the presence of short run nominal rigidities or imperfect information, the resultant reduction in output is associated with a decline in output price inflation. In an open economy, the resultant reduction in consumption price inflation is amplified and accelerated by the adjustment of the real exchange rate.

Despite the remarkable success of many inflation targeting central banks at achieving low and stable inflation, the development of a mutually consistent set of accurate and precise

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indicators of inflationary pressure remains elusive. Theoretically prominent indicators of inflationary pressure such as the natural rate of interest and natural exchange rate are unobservable. As discussed in Woodford (2003), the natural rate of interest provides a measure of the neutral stance of monetary policy, with deviations of the real interest rate from the natural rate of interest generating inflationary pressure. Within the framework of an unobserved components model of selected elements of the monetary transmission mechanism in a closed economy, Laubach and Williams (2003) find that estimates of the natural rate of interest are relatively imprecise, as evidenced by relatively wide confidence intervals. Jointly estimating this and other indicators of inflationary pressure conditional on a larger information set may be expected to yield efficiency gains.

Definitions of indicators of inflationary pressure such as the natural rate of interest and natural exchange rate vary. Following Laubach and Williams (2003), we define the natural rate of interest as that short term real interest rate consistent with achieving inflation control and output stabilization objectives in the absence of shocks having temporary effects. In this long run equilibrium, there does not exist a cyclical stabilization role for monetary policy generated by nominal rigidities or imperfect information. In contrast, Woodford (2003) defines the natural rate of interest as that short term real interest rate consistent with achieving inflation control and output stabilization objectives in the absence of nominal rigidities. In this short run equilibrium, although there does not exist a cyclical stabilization role for monetary policy, the natural rate of interest varies in response to shocks having both temporary and permanent effects. Given an interest rate smoothing objective derived from a concern with financial market stability, it may be optimal for a central bank to adjust the short term nominal interest rate primarily in response to variation in the natural rate of interest caused by shocks having permanent effects.

Within the framework of a linear state space model, prior information concerning the values of unobserved state variables is often available in the form of deterministic or stochastic restrictions. Within the framework of an unobserved components model, prior information concerning the values of unobserved components is often available from alternative estimators. The primary methodological contribution of this paper is the development of a procedure to estimate a linear state space model conditional on prior information concerning the values of unobserved state variables. This prior information assumes the form of a set of deterministic or stochastic restrictions on linear combinations of unobserved state variables. In addition to mitigating potential model misspecification and identification problems, exploiting such prior information may be expected to yield efficiency gains in estimation.

This paper develops and estimates an unobserved components model for purposes of monetary policy analysis and inflation targeting in a small open economy. In an extension of the empirical framework developed by Laubach and Williams (2003), cyclical components are
modeled as a multivariate linear rational expectations model of the monetary transmission mechanism, while trend components are modeled as random walks while ensuring the existence of a well defined balanced growth path. Although not derived from microeconomic foundations, this unobserved components model of the monetary transmission mechanism in a small open economy arguably provides a closer approximation to the data generating process than existing dynamic stochastic general equilibrium models, as fewer cross-coefficient restrictions are imposed. Full information maximum likelihood estimation of this unobserved components model, conditional on prior information concerning the values of trend components, provides a quantitative description of the monetary transmission mechanism in a small open economy, yields a mutually consistent set of indicators of inflationary pressure together with confidence intervals, and facilitates the generation of relatively accurate forecasts.

The organization of this paper is as follows. The next section develops an unobserved components model of the monetary transmission mechanism in a small open economy. In section three, unrestricted and restricted estimators of unobserved state variables are derived within the framework of a linear state space model. Estimation, inference and forecasting within the framework of a linear state space representation of our unobserved components model are the subjects of section four. Finally, section five offers conclusions and recommendations for further research.

2.2. The Unobserved Components Model

Consider two structurally isomorphic economies which are asymmetric in size. The domestic economy is of negligible size relative to the foreign economy, and hence takes the rational expectations equilibrium of the foreign economy as exogenous. The central banks of the domestic and foreign economies pursue inflation control and output stabilization objectives. Cyclical components are modeled as a multivariate linear rational expectations model of the monetary transmission mechanism, while trend components are modeled as random walks while ensuring the existence of a well defined balanced growth path.

In what follows, \( E_t x_{it} \) denotes the rational expectation of variable \( x_{it} \), conditional on information available at time \( t \). Also, \( \hat{x}_t \) denotes the cyclical component of variable \( x_t \), while \( \bar{x}_t \) denotes the trend component of variable \( x_t \). Cyclical and trend components are additively separable, that is \( x_t = \hat{x}_t + \bar{x}_t \).
2.2.1. Cyclical Components

The cyclical component of output price inflation $\tilde{\pi}_t^Y$ depends on a linear combination of past and expected future cyclical components of output price inflation driven by the contemporaneous cyclical component of output according to output price Phillips curve

$$\tilde{\pi}_t^Y = \phi_{1,1}\tilde{\pi}_{t-1}^Y + \phi_{1,2}E_t\tilde{\pi}_{t+1}^Y + \theta_{1,1}\ln\hat{Y}_t + \varepsilon_t^{\rho^Y}, \varepsilon_t^{\rho^Y} \sim \mathcal{N}(0,\sigma_{\rho^Y}^2),$$

(1)

where the level of output price inflation satisfies $\Delta\ln P_t^Y$. The sensitivity of the cyclical component of output price inflation to changes in the cyclical component of output is increasing in $\theta_{1,1} > 0$.

The cyclical component of consumption price inflation $\tilde{\pi}_t^C$ depends on a linear combination of past and expected future cyclical components of consumption price inflation driven by the contemporaneous cyclical component of output according to consumption price Phillips curve

$$\tilde{\pi}_t^C = \phi_{2,1}\tilde{\pi}_{t-1}^C + \phi_{2,2}E_t\tilde{\pi}_{t+1}^C + \theta_{2,1}\ln\hat{Y}_t$$

$$-\phi_{2,1}\theta_{2,2}\Delta\ln\hat{Q}_{t-1} + \theta_{2,2}\Delta\ln\hat{Q}_t - \phi_{2,2}\theta_{2,2}\Delta\ln\hat{Q}_{t+1} + \varepsilon_t^{\rho^C}, \varepsilon_t^{\rho^C} \sim \mathcal{N}(0,\sigma_{\rho^C}^2),$$

(2)

where the level of consumption price inflation satisfies $\Delta\ln P_t^C$. The cyclical component of consumption price inflation also depends on past, contemporaneous, and expected future proportional changes in the cyclical component of the real exchange rate. The sensitivity of the cyclical component of consumption price inflation to changes in the cyclical component of output is increasing in $\theta_{2,1} > 0$, and to changes in the cyclical component of the real exchange rate is increasing in $0 < \theta_{2,2} < 1$.

The cyclical component of output $\ln\hat{Y}_t$ follows a stationary second order autoregressive process driven by the contemporaneous cyclical component of foreign output and a linear combination of the past cyclical components of the real interest rate and real exchange rate

$$\ln\hat{Y}_t = \phi_{3,1}\ln\hat{Y}_{t-1} + \phi_{3,2}\ln\hat{Y}_{t-2} + \theta_{3,1}\ln\hat{Y}_t + \theta_{3,3}\hat{r}_{t-1} + \theta_{3,3}\ln\hat{Q}_{t+1} + \varepsilon_t^{\gamma}, \varepsilon_t^{\gamma} \sim \mathcal{N}(0,\sigma_{\gamma}^2),$$

(3)

where the level of the real interest rate satisfies $r_t = i_t - E_t\pi_t^C$, while the level of the real exchange rate satisfies $\ln\hat{Q}_t = \ln S_t + \ln P_t^R - \ln P_t^V$. The sensitivity of the cyclical component of output to changes in the cyclical component of foreign output is increasing in $\theta_{3,1} > 0$, to changes in the cyclical component of the real interest rate is decreasing in $\theta_{3,3} < 0$, and to changes in the cyclical component of the real exchange rate is increasing in $\theta_{3,3} > 0$.

The cyclical component of consumption $\ln\hat{C}_t$ follows a stationary second order autoregressive process driven by the past cyclical component of the real interest rate:
\[ \ln \hat{C}_t = \phi_{4,1} \ln \hat{C}_{t-1} + \phi_{4,2} \ln \hat{C}_{t-2} + \theta_{4,1} \hat{r}_{t-1} + \varepsilon_t^C, \ \varepsilon_t^C \sim \text{iid } \mathcal{N}(0, \sigma^2_C). \] (4)

The sensitivity of the cyclical component of consumption to changes in the cyclical component of the real interest rate is decreasing in \( \theta_{4,1} < 0 \).

The cyclical component of investment \( \ln \hat{I}_t \) follows a stationary second order autoregressive process driven by the contemporaneous cyclical component of output:

\[ \ln \hat{I}_t = \phi_{5,1} \ln \hat{I}_{t-1} + \phi_{5,2} \ln \hat{I}_{t-2} + \theta_{5,1} \ln \hat{Y}_t + \theta_{5,2} \ln \hat{Y}_{t-2} + \varepsilon_t^I, \ \varepsilon_t^I \sim \text{iid } \mathcal{N}(0, \sigma^2_I). \] (5)

The sensitivity of the cyclical component of investment to changes in the cyclical component of output is increasing in \( \theta_{5,1} > 0 \).

The cyclical component of exports \( \ln \hat{X}_t \) follows a stationary second order autoregressive process driven by the contemporaneous cyclical component of foreign output and the past cyclical component of the real exchange rate:

\[ \ln \hat{X}_t = \phi_{6,1} \ln \hat{X}_{t-1} + \phi_{6,2} \ln \hat{X}_{t-2} + \theta_{6,1} \ln \hat{Y}_t + \theta_{6,2} \ln \hat{Q}_{t-1} + \varepsilon_t^X, \ \varepsilon_t^X \sim \text{iid } \mathcal{N}(0, \sigma^2_X). \] (6)

The sensitivity of the cyclical component of exports to changes in the cyclical component of foreign output is increasing in \( \theta_{6,1} > 0 \), and to changes in the cyclical component of the real exchange rate is increasing in \( \theta_{6,2} > 0 \).

The cyclical component of imports \( \ln \hat{M}_t \) follows a stationary second order autoregressive process driven by the contemporaneous cyclical component of output and the past cyclical component of the real exchange rate:

\[ \ln \hat{M}_t = \phi_{7,1} \ln \hat{M}_{t-1} + \phi_{7,2} \ln \hat{M}_{t-2} + \theta_{7,1} \ln \hat{Y}_t + \theta_{7,2} \ln \hat{Q}_{t-1} + \varepsilon_t^M, \ \varepsilon_t^M \sim \text{iid } \mathcal{N}(0, \sigma^2_M). \] (7)

The sensitivity of the cyclical component of imports to changes in the cyclical component of output is increasing in \( \theta_{7,1} > 0 \), and to changes in the cyclical component of the real exchange rate is decreasing in \( \theta_{7,2} < 0 \).

The cyclical component of wage inflation \( \hat{\pi}_t^w \) depends on a linear combination of past and expected future cyclical components of wage inflation driven by the contemporaneous cyclical component of the unemployment rate according to wage Phillips curve

\[ \hat{\pi}_t^w = \phi_{8,1} \hat{\pi}_{t-1}^w + \phi_{8,2} \mathbb{E}_t \hat{\pi}_{t+1}^w + \theta_{8,1} \hat{\mu}_t \\
- \phi_{8,1} \theta_{8,2} \hat{\pi}_{t-1}^c + \phi_{8,2} \mathbb{E}_t \hat{\pi}_{t+1}^c + \varepsilon_t^w, \ \varepsilon_t^w \sim \text{iid } \mathcal{N}(0, \sigma^2_w). \] (8)
where the level of wage inflation satisfies \( \pi^w_t = \Delta \ln W_t \). The cyclical component of wage inflation also depends on past, contemporaneous, and expected future cyclical components of consumption price inflation. The sensitivity of the cyclical component of wage inflation to changes in the cyclical component of the unemployment rate is decreasing in \( \theta_{8,1} < 0 \), and its to changes in the cyclical component of consumption price inflation is increasing in \( 0 < \theta_{8,2} < 1 \).

The cyclical component of employment \( \ln \hat{L}_t \) follows a stationary second order autoregressive process driven by the contemporaneous cyclical component of output:

\[
\ln \hat{L}_t = \phi_{1,1} \ln \hat{L}_{t-1} + \phi_{1,2} \ln \hat{L}_{t-2} + \theta_{3,1} \ln \hat{Y}_t + \varepsilon^i_t, \quad \varepsilon^i_t \sim \text{iid } \mathcal{N}(0, \sigma^2_i).
\]  

(9)

The sensitivity of the cyclical component of employment to changes in the cyclical component of output is increasing in \( \theta_{3,1} > 0 \).

The cyclical component of the unemployment rate \( \hat{u}_t \) follows a stationary second order autoregressive process driven by the contemporaneous cyclical component of output:

\[
\hat{u}_t = \phi_{10,1} \hat{u}_{t-1} + \phi_{10,2} \hat{u}_{t-2} + \theta_{10,1} \ln \hat{Y}_t + \varepsilon^u_t, \quad \varepsilon^u_t \sim \text{iid } \mathcal{N}(0, \sigma^2_u).
\]  

(10)

The sensitivity of the cyclical component of the unemployment rate to changes in the cyclical component of output is decreasing in \( \theta_{10,1} < 0 \).

The cyclical component of the nominal interest rate \( \hat{i}_t \) follows a stationary first order autoregressive process driven by the contemporaneous cyclical components of consumption price inflation and output:

\[
\hat{i}_t = \phi_{11,1} \hat{i}_{t-1} + \theta_{11,2} \ln \hat{Y}_t + \varepsilon^i_t, \quad \varepsilon^i_t \sim \text{iid } \mathcal{N}(0, \sigma^2_i).
\]  

(11)

The sensitivity of the cyclical component of the nominal interest rate to changes in the cyclical component of consumption price inflation is increasing in \( \theta_{11,2} > 0 \), and to changes in the cyclical component of output is increasing in \( \theta_{11,2} > 0 \).

The cyclical component of the nominal exchange rate \( \ln \hat{S}_t \) depends on a linear combination of past and expected future cyclical components of the nominal exchange rate driven by the contemporaneous cyclical component of the nominal interest rate differential:

\[
\ln \hat{S}_t = \phi_{12,1} \ln \hat{S}_{t-1} + \phi_{12,2} E_t \ln \hat{S}_{t+1} + \theta_{12,1} (\hat{i}_t - \hat{i}_{t-1}) + \varepsilon^\delta, \quad \varepsilon^\delta \sim \text{iid } \mathcal{N}(0, \sigma^2_\delta).
\]  

(12)

The sensitivity of the cyclical component of the nominal exchange rate to changes in the cyclical component of the nominal interest rate differential is decreasing in \( \theta_{12,1} < 0 \).
2.2.2. Trend Components

The trend components of the prices of output \( \ln P_t^Y \) and consumption \( \ln P_t^C \) follow random walks with time varying drift \( \pi_t \):

\[
\ln P_t^Y = \pi_t + \ln P_{t-1}^Y + \varepsilon_t^Y, \quad \varepsilon_t^Y \sim \text{iid } N(0, \sigma_{\pi_t}^2),
\]

\[
\ln P_t^C = \pi_t + \ln P_{t-1}^C + \varepsilon_t^C, \quad \varepsilon_t^C \sim \text{iid } N(0, \sigma_{\pi_t}^2).
\]

(13) (14)

It follows that the trend component of the relative price of consumption follows a random walk without drift. This implies that along a balanced growth path, the level of this relative price is time independent but state dependent.

The trend components of output \( \ln Y_t \), consumption \( \ln C_t \), investment \( \ln I_t \), exports \( \ln X_t \), and imports \( \ln M_t \) follow random walks with time varying drift \( g_t + n_t \):

\[
\ln Y_t = g_t + n_t + \ln Y_{t-1} + \varepsilon_t^Y, \quad \varepsilon_t^Y \sim \text{iid } N(0, \sigma_{g_t}^2),
\]

\[
\ln C_t = g_t + n_t + \ln C_{t-1} + \varepsilon_t^C, \quad \varepsilon_t^C \sim \text{iid } N(0, \sigma_{g_t}^2),
\]

\[
\ln I_t = g_t + n_t + \ln I_{t-1} + \varepsilon_t^I, \quad \varepsilon_t^I \sim \text{iid } N(0, \sigma_{g_t}^2),
\]

\[
\ln X_t = g_t + n_t + \ln X_{t-1} + \varepsilon_t^X, \quad \varepsilon_t^X \sim \text{iid } N(0, \sigma_{g_t}^2),
\]

\[
\ln M_t = g_t + n_t + \ln M_{t-1} + \varepsilon_t^M, \quad \varepsilon_t^M \sim \text{iid } N(0, \sigma_{g_t}^2).
\]

(15) (16) (17) (18) (19)

It follows that the trend components of the ratios of consumption, investment, exports, and imports to output follow random walks without drifts. This implies that along a balanced growth path, the levels of these ratios are time independent but state dependent.

The trend component of the nominal wage \( \ln W_t \) follows a random walk with time varying drift \( \pi_t + g_t \), while the trend component of employment \( \ln L_t \) follows a random walk with time varying drift \( n_t \):

\[
\ln W_t = \pi_t + g_t + \ln W_{t-1} + \varepsilon_t^W, \quad \varepsilon_t^W \sim \text{iid } N(0, \sigma_{\pi_t}^2),
\]

\[
\ln L_t = n_t + \ln L_{t-1} + \varepsilon_t^L, \quad \varepsilon_t^L \sim \text{iid } N(0, \sigma_{n_t}^2).
\]

(20) (21)

It follows that the trend component of the income share of labour follows a random walk without drift. This implies that along a balanced growth path, the level of the income share of labour is time independent but state dependent.
The trend components of the unemployment rate $\bar{u}$, nominal interest rate $\bar{i}$, and nominal exchange rate $\ln S$ follow random walks without drifts:

$$\bar{u}_t = \bar{u}_{t-1} + \epsilon_{\bar{u}}^u, \quad \epsilon_{\bar{u}}^u \sim \text{iid } N(0, \sigma_{\bar{u}}^2), \quad (22)$$

$$\bar{i}_t = \bar{i}_{t-1} + \epsilon_{\bar{i}}^i, \quad \epsilon_{\bar{i}}^i \sim \text{iid } N(0, \sigma_{\bar{i}}^2), \quad (23)$$

$$\ln S_t = \ln S_{t-1} + \epsilon_{\ln S}^\delta, \quad \epsilon_{\ln S}^\delta \sim \text{iid } N(0, \sigma_{\ln S}^2). \quad (24)$$

It follows that along a balanced growth path, the levels of the unemployment rate, nominal interest rate, and nominal exchange rate are time independent but state dependent.

Long run balanced growth is driven by three common stochastic trends. Trend inflation $\pi_t$, productivity growth $g_t$, and population growth $n_t$ follow random walks without drifts:

$$\pi_t = \pi_{t-1} + \epsilon_{\pi}^\pi, \quad \epsilon_{\pi}^\pi \sim \text{iid } N(0, \sigma_{\pi}^2), \quad (25)$$

$$g_t = g_{t-1} + \epsilon_{g}^g, \quad \epsilon_{g}^g \sim \text{iid } N(0, \sigma_{g}^2), \quad (26)$$

$$n_t = n_{t-1} + \epsilon_{n}^n, \quad \epsilon_{n}^n \sim \text{iid } N(0, \sigma_{n}^2). \quad (27)$$

It follows that along a balanced growth path, growth rates are time independent but state dependent. As an identifying restriction, all innovations are assumed to be independent, which combined with our distributional assumptions implies multivariate normality.

### 2.3. Estimation of Unobserved State Variables

State space models consist of signal and state equations. The signal equation specifies a static relationship between a vector of observed signal variables and a vector of unobserved state variables, while the state equation specifies a dynamic relationship governing the evolution of this vector of unobserved state variables. The objective of state space analysis is to estimate the state vector, given the signal vector.

Within the framework of a linear state space model, if the signal and state innovation vectors are multivariate normally distributed and independent, then conditional on the parameters associated with the signal and state equations, mean squared error optimal estimates of the state vector may be calculated with the filter due to Kalman (1960). If the signal and state innovation vectors are not multivariate normally distributed, then these state vector estimates retain minimum mean squared error status within the class of linear estimators. Estimation, inference
and forecasting within the framework of a linear state space model is discussed in Hamilton

Within the framework of a linear state space model, prior information concerning the values
of state variables is often available in the form of deterministic or stochastic restrictions. This
section derives unrestricted and restricted estimators of state variables within the framework of a
linear state space model. The former approach is standard, while the latter is a contribution of
this paper. In addition to mitigating potential model misspecification and identification
problems, exploiting prior information concerning the values of state variables may be expected
to yield efficiency gains in estimation.

2.3.1. Unrestricted Estimation of Unobserved State Variables

Let \( y_t \) denote a vector stochastic process consisting of \( N \) observed nonpredetermined
endogenous variables, let \( x_t \) denote a vector stochastic process consisting of \( M \) observed
exogenous or predetermined endogenous variables, and let \( z_t \) denote a vector stochastic process
consisting of \( K \) unobserved state variables. Suppose that these vector stochastic processes have
linear state space representation

\[
y_t = A_1 x_t + A_2 z_t + A_3 \varepsilon_{1,t}, \tag{28}
\]

\[
z_t = B_1 x_t + B_2 z_{t-1} + B_3 \varepsilon_{2,t}, \tag{29}
\]

where \( \varepsilon_{1,t} \sim \text{iid } \mathcal{N}(0, \Sigma_1) \), \( \varepsilon_{2,t} \sim \text{iid } \mathcal{N}(0, \Sigma_2) \) and \( z_0 \sim \mathcal{N}(z_0, P_0) \). The signal and state
innovation vectors are assumed to be independent, while the initial state vector is assumed to be
independent from the signal and state innovation vectors, which combined with our distributional
assumptions implies multivariate normality.

Within the framework of this linear state space model, define \( z_{t|t-1} = \text{E}(z_t | \mathcal{I}_{t-1}) \),
\( P_{t|t-1} = \text{Var}(z_t | \mathcal{I}_{t-1}) \), \( y_{t|t-1} = \text{E}(y_t | \mathcal{I}_{t-1}) \) and \( Q_{t|t-1} = \text{Var}(y_t | \mathcal{I}_{t-1}) \), where \( \mathcal{I}_{t-1} = \{ \{ y_s \}_{s=1}^{t-1}, \{ x_s \}_{s=1}^{t-1} \} \).

Conditional on the parameters associated with the signal and state equations, these conditional
means and variances satisfy prediction equations:

\[
z_{t|t-1} = B_1 x_t + B_2 z_{t-1}, \tag{30}
\]

\[
P_{t|t-1} = B_2 P_{t-1} B_2^T + B_3 \Sigma_2 B_3^T, \tag{31}
\]

\[
y_{t|t-1} = A_1 x_t + A_2 z_{t|t-1}, \tag{32}
\]
These predicted estimates of the means and variances of the signal and state vectors are conditional on past information.

Given these predicted estimates, estimates of the state vector conditional on past and present information may be derived with Bayesian updating. Define \( \mathbf{z}_{|t} \) as that argument which maximizes posterior density function:

\[
f(\mathbf{z}_{|t} | \mathbf{y}, \mathcal{I}_{t-1}) = \frac{f(\mathbf{y}_t | \mathbf{z}_{|t}, \mathcal{I}_{t-1}) f(\mathbf{z}_{|t} | \mathcal{I}_{t-1})}{f(\mathbf{y}_t | \mathcal{I}_{t-1})}
\]

Under the assumption of multivariate normally distributed signal and state innovation vectors, \( \mathbf{z}_{|t} \) minimizes objective function

\[
S(\mathbf{z}_t) = (\mathbf{z}_t - \mathbf{z}_{|t-1})^T \mathbf{P}_{|t-1}^{-1} (\mathbf{z}_t - \mathbf{z}_{|t-1}) - (\mathbf{y}_t - \mathbf{y}_{|t-1})^T \mathbf{Q}_{|t-1}^{-1} (\mathbf{y}_t - \mathbf{y}_{|t-1}),
\]

subject to signal equation (28). The necessary first order condition associated with the implied unconstrained minimization problem yields

\[
\mathbf{z}_{|t} = \mathbf{z}_{|t-1} + \mathbf{K}_t (\mathbf{y}_t - \mathbf{y}_{|t-1}),
\]

where \( \mathbf{K}_t = \mathbf{P}_{|t-1}^{-1} \mathbf{A}_2^T \mathbf{Q}_{|t-1}^{-1}. \) This necessary first order condition is sufficient if \( \mathbf{P}_{|t-1}^{-1} - \mathbf{A}_2^T \mathbf{Q}_{|t-1}^{-1} \mathbf{A}_2 \) is positive definite. Define \( \mathbf{P}_{|t} \) as the mean squared error of \( \mathbf{z}_{|t} \), conditional on \( \mathcal{I}_{t-1} \). Within the framework of this linear state space model, this mean squared error matrix satisfies:

\[
\mathbf{P}_{|t} = \mathbf{P}_{|t-1} - \mathbf{K}_t \mathbf{A}_2 \mathbf{P}_{|t-1}.
\]

Under our distributional assumptions, \( \mathbf{z}_{|t} \) equals the mean of posterior density function \( f(\mathbf{z}_t | \mathbf{y}_t, \mathcal{I}_{t-1}) \), and is therefore mean squared error optimal. Given initial conditions \( \mathbf{z}_{|0} \) and \( \mathbf{P}_{|0} \), recursive evaluation of equations (30), (31), (32), (33), (36) and (37) yields predicted and filtered estimates of the state vector.

Given these predicted and filtered estimates, estimates of the state vector conditional on past, present and future information may be derived with Bayesian updating. Define \( \mathbf{z}_{|\tau} \) as that argument which maximizes posterior density function:

\[
f(\mathbf{z}_{|\tau} | \mathbf{z}_{|\tau+1}, \mathcal{I}_t) = \frac{f(\mathbf{z}_{|\tau+1} | \mathbf{z}_{|\tau}, \mathcal{I}_t) f(\mathbf{z}_{|\tau} | \mathcal{I}_t)}{f(\mathbf{z}_{|\tau+1} | \mathcal{I}_t)}.
\]

Under the assumption of a multivariate normally distributed state innovation vector, \( \mathbf{z}_{|\tau} \) minimizes objective function

\[
Q_{|\tau-1} = A_2 \mathbf{P}_{|\tau-1} \mathbf{A}_2^T + A_3 \mathbf{\Sigma}_I A_3^T.
\]
subject to state equation (29). The necessary first order condition associated with the implied unconstrained minimization problem yields

\[ S(z_t) = (z_t - z_{\text{opt}}) \mathbf{P}^{-1}_{t|t-1} (z_t - z_{\text{opt}}) - (z_{t+1} - z_{t+1|t}) \mathbf{P}^{-1}_{t+1|t} (z_{t+1} - z_{t+1|t}), \]  

(39)

where \( J_t = \mathbf{P}_{t|t-1} \mathbf{J}^\top \mathbf{P}_{t+1|t-1} \). This necessary first order condition is sufficient if \( \mathbf{P}_{t|t-1} - \mathbf{B}_2^\top \mathbf{P}_{t+1|t-1} \mathbf{B}_2 \) is positive definite. Define \( \mathbf{P}_{t|t} \) as the mean squared error of \( z_{t|t} \), conditional on \( I_t \). Within the framework of this linear state space model, this mean squared error matrix satisfies:

\[ \mathbf{P}_{t|t} = \mathbf{P}_{t|t-1} + J_t (\mathbf{P}_{t+1|t-1} - \mathbf{P}_{t+1|t}) J_t^\top. \]  

(41)

Under our distributional assumptions, \( z_{t|t} \) equals the mean of posterior density function \( f(z_t | z_{t+1}, I_t) \), and is therefore mean squared error optimal. Given terminal conditions \( z_{t|T} \) and \( \mathbf{P}_{t|T} \) obtained from the final evaluation of the prediction and updating equations, recursive evaluation of equations (40) and (41) yields smoothed estimates of the state vector.

### 2.3.2. Restricted Estimation of Unobserved State Variables

Let \( \mathbf{y}_t \) denote a vector stochastic process consisting of \( N \) observed nonpredetermined endogenous variables, let \( \mathbf{x}_t \) denote a vector stochastic process consisting of \( M \) observed exogenous or predetermined endogenous variables, and let \( \mathbf{z}_t \) denote a vector stochastic process consisting of \( K \) unobserved state variables. Suppose that these vector stochastic processes have linear state space representation

\[ \mathbf{y}_t = \mathbf{A}_1 \mathbf{x}_t + \mathbf{A}_2 \mathbf{z}_t + \mathbf{A}_3 \mathbf{e}_{1,t}, \]  

(42)

\[ \mathbf{z}_t = \mathbf{B}_1 \mathbf{x}_t + \mathbf{B}_2 \mathbf{z}_{t-1} + \mathbf{B}_3 \mathbf{e}_{2,t}, \]  

(43)

where \( \mathbf{e}_{1,t} \sim \text{iid } \mathcal{N}(0, \Sigma_1), \; \mathbf{e}_{2,t} \sim \text{iid } \mathcal{N}(0, \Sigma_2) \) and \( \mathbf{z}_0 \sim \mathcal{N}(\mathbf{z}_{0|0}, \mathbf{P}_{0|0}) \). Let \( \mathbf{w}_t \) denote a vector stochastic process consisting of \( J \) observed synthetic variables. Suppose that this vector stochastic process satisfies

\[ \mathbf{w}_t = \mathbf{C}_1 \mathbf{z}_t + \mathbf{C}_2 \mathbf{e}_{3,t}, \]  

(44)
where $\varepsilon_{t} \sim \text{iid } \mathcal{N}(0, \Sigma_{t})$. Conditional on known parameter values, this signal equation defines a set of deterministic or stochastic restrictions on linear combinations of unobserved state variables. The signal and state innovation vectors are assumed to be independent, while the initial state vector is assumed to be independent from the signal and state innovation vectors, which combined with our distributional assumptions implies multivariate normality.

Within the framework of this linear state space model, define $z_{t_{0}} = E(z_{t} \mid I_{t_{0}})$, $P_{t_{0}} = \text{Var}(z_{t} \mid I_{t_{0}})$, $y_{t_{0}} = E(y_{t} \mid I_{t_{0}})$, $Q_{t_{0}} = \text{Var}(y_{t} \mid I_{t_{0}})$, $w_{t_{0}} = E(w_{t} \mid I_{t_{0}})$ and $R_{t_{0}} = \text{Var}(w_{t} \mid I_{t_{0}})$, where $I_{t_{0}} = \{y_{t_{0}}^{-1}, \ldots, y_{t_{0}}, w_{t_{0}}^{-1}, \ldots, w_{t_{0}} \}$. Conditional on the parameters associated with the signal and state equations, these conditional means and variances satisfy prediction equations:

\begin{align*}
    z_{t_{0}} &= B_{t_{0}} x_{t_{0}} + B_{z} z_{t_{0}-1}, \\
    P_{t_{0}} &= B_{z} P_{t_{0}-1} B_{z}^{T} + B_{z} \Sigma_{z} B_{z}^{T}, \\
    y_{t_{0}} &= A_{t} x_{t_{0}} + A_{z} z_{t_{0}-1}, \\
    Q_{t_{0}} &= A_{z} P_{t_{0}-1} A_{z}^{T} + A_{z} \Sigma_{z} A_{z}^{T}, \\
    w_{t_{0}} &= C_{z} z_{t_{0}-1}, \\
    R_{t_{0}} &= C_{z} P_{t_{0}-1} C_{z}^{T} + C_{z} \Sigma_{z} C_{z}^{T}.
\end{align*}

These predicted estimates of the means and variances of the signal and state vectors are conditional on past information.

Given these predicted estimates, estimates of the state vector conditional on past and present information may be derived with Bayesian updating. Define $z_{t_{0}}$ as that argument which maximizes posterior density function:

\begin{equation}
    f(z_{t} \mid y_{t}, w_{t}, I_{t_{0}}) = \frac{f(y_{t} \mid z_{t}, w_{t}, I_{t_{0}}) f(w_{t} \mid z_{t}, I_{t_{0}}) f(z_{t} \mid I_{t_{0}})}{f(y_{t} \mid w_{t}, I_{t_{0}}) f(w_{t} \mid I_{t_{0}})}.
\end{equation}

Under the assumption of multivariate normally distributed signal and state innovation vectors, together with conditionally contemporaneously uncorrelated signal vectors, $z_{t_{0}}$ minimizes objective function

\begin{align*}
    S(z_{t}) &= (z_{t} - z_{t-1})^{T} P_{t-1}^{-1} (z_{t} - z_{t-1}) \\
    &
    - (y_{t} - y_{t-1})^{T} Q_{t-1}^{-1} (y_{t} - y_{t-1}) - (w_{t} - w_{t-1})^{T} R_{t-1}^{-1} (w_{t} - w_{t-1}),
\end{align*}
subject to signal equations (42) and (44). The necessary first order condition associated with the implied unconstrained minimization problem yields

$$z_{t|t} = z_{t|t-1} + K_y (y_t - y_{t|t-1}) + K_w (w_t - w_{t|t-1}),$$

(53)

where $K_y = P_{y|t-1} A^T_2 Q^{-1}_{y|t-1}$ and $K_w = P_{w|t-1} C^T_1 R^{-1}_{w|t-1}$. This necessary first order condition is sufficient if $P_{y|t-1} - A^T_2 Q^{-1}_{y|t-1} A_2 - C^T_1 R^{-1}_{w|t-1} C_1$ is positive definite. Define $P_{t|t}$ as the mean squared error of $z_{t|t}$, conditional on $I_{t-1}$. Within the framework of this linear state space model, this mean squared error matrix satisfies:

$$P_{t|t} = P_{y|t-1} - K_y A_2 P_{y|t-1} - K_w C_1 P_{t|t-1},$$

(54)

Under our distributional assumptions, $z_{t|t}$ equals the mean of posterior density function $f(z_t | y_t, w_t, I_{t-1})$, and is therefore mean squared error optimal. Given initial conditions $z_{0|0}$ and $P_{0|0}$, recursive evaluation of equations (45), (46), (47), (48), (49), (50), (53) and (54) yields predicted and filtered estimates of the state vector.

Given these predicted and filtered estimates, estimates of the state vector conditional on past, present and future information may be derived with Bayesian updating. Define $z_{i|T}$ as that argument which maximizes posterior density function:

$$f(z_i | z_{i-1}, I_i) = \frac{f(z_{i+1} | z_{i}, I_{i}) f(z_i | I_i)}{f(z_{i+1} | I_i)}.$$  

(55)

Under the assumption of a multivariate normally distributed state innovation vector, $z_{i|T}$ minimizes objective function

$$S(z_t) = (z_t - z_{i|T})^T P_{t|t}^{-1} (z_t - z_{i|T}) - (z_{t+1} - z_{i|T})^T P_{t+1|t}^{-1} (z_{t+1} - z_{i|T}),$$  

(56)

subject to state equation (43). The necessary first order condition associated with the implied unconstrained minimization problem yields

$$z_{i|T} = z_{i|t} + J_i (z_{i+1|T} - z_{i+1|t}),$$  

(57)

where $J_i = P_{t|t} B_2^T P_{t+1|t}^{-1}$. This necessary first order condition is sufficient if $P_{t|t}^{-1} - B_2^T P_{t+1|t}^{-1} B_2$ is positive definite. Define $P_{t|t}$ as the mean squared error of $z_{i|T}$, conditional on $I_i$. Within the framework of this linear state space model, this mean squared error matrix satisfies:

$$P_{t|t} = P_{t|t} + J_i (P_{t+1|T} - P_{t+1|t}) J_i^T.$$  

(58)
Under our distributional assumptions, $z_{t\mid \pi}$ equals the mean of posterior density function $f(z_t | \mathbf{z}_t, I_t)$, and is therefore mean squared error optimal. Given terminal conditions $z_{t\mid \pi}$ and $P_{t\mid \pi}$ obtained from the final evaluation of the prediction and updating equations, recursive evaluation of equations (57) and (58) yields smoothed estimates of the state vector.

### 2.4. Estimation, Inference and Forecasting

Although unobserved components models feature prominently in the empirical macroeconomics literature, an unobserved components model of the monetary transmission mechanism has yet to be developed and estimated. Given that the monetary transmission mechanism is a cyclical phenomenon, it seems natural to model it within the framework of an unobserved components model.

#### 2.4.1. Estimation

The traditional econometric interpretation of macroeconometric models regards them as representations of the joint probability distribution of the data. Adopting this traditional econometric interpretation, the parameters and trend components of our unobserved components model of the monetary transmission mechanism in a small open economy are jointly estimated by full information maximum likelihood, conditional on prior information concerning the values of trend components.

#### 2.4.1.1. Estimation Procedure

Let $\mathbf{x}_t$ denote a vector stochastic process consisting of the levels of $N$ nonpredetermined endogenous variables, of which $M$ are observed. The cyclical components of this vector stochastic process satisfy third order stochastic linear difference equation

$$A_0 \hat{x}_t = A_1 \hat{x}_{t-1} + A_2 \hat{x}_{t-2} + A_3 \epsilon_{t-1} + \epsilon_{t,1},$$

where $\epsilon_{t,1} \sim \text{iid } N(0, \Sigma_t)$. If there exists a unique stationary solution to this multivariate linear rational expectations model, then it may be expressed as:

$$\hat{x}_t = B_1 \hat{x}_{t-1} + B_2 \hat{x}_{t-2} + B_3 \epsilon_{t,1},$$

(60)
This unique stationary solution is calculated with the matrix decomposition based algorithm due to Klein (2000).

The trend components of vector stochastic process $x_t$ satisfy first order stochastic linear difference equation

$$C_0\tilde{x}_t = C_1v_t + C_2\tilde{x}_{t-1} + \varepsilon_{2,t},$$

(61)

where $\varepsilon_{2,t} \sim \text{iid } \mathcal{N}(0, \Sigma_2)$. Vector stochastic process $v_t$ consists of the levels of $L$ common stochastic trends, and satisfies first order stochastic linear difference equation

$$v_t = v_{t-1} + \varepsilon_{3,t},$$

(62)

where $\varepsilon_{3,t} \sim \text{iid } \mathcal{N}(0, \Sigma_3)$. Cyclical and trend components are additively separable, that is $x_t = \tilde{x}_t + \tilde{\tilde{x}}_t$.

Let $y_t$ denote a vector stochastic process consisting of the levels of $M$ observed nonpredetermined endogenous variables. Also, let $z_t$ denote a vector stochastic process consisting of the contemporaneous levels of $N - M$ unobserved nonpredetermined endogenous variables, the contemporaneous and lagged cyclical components of $N$ nonpredetermined endogenous variables, the contemporaneous trend components of $N$ nonpredetermined endogenous variables, and the levels of $L$ common stochastic trends. Given unique stationary solution (60), these vector stochastic processes have linear state space representation

$$y_t = Fz_t,$$  

(63)

$$z_t = G_1z_{t-1} + G_2\varepsilon_{4,t},$$  

(64)

where $\varepsilon_{4,t} \sim \text{iid } \mathcal{N}(0, \Sigma_4)$ and $z_0 \sim \mathcal{N}(z_{00}, P_{00})$. Let $w_t$ denote a vector stochastic process consisting of preliminary estimates of the trend components of $M$ observed nonpredetermined endogenous variables. Suppose that this vector stochastic process satisfies

$$w_t = H_1z_t + \varepsilon_{5,t},$$  

(65)

where $\varepsilon_{5,t} \sim \text{iid } \mathcal{N}(0, \Sigma_5)$. Conditional on known parameter values, this signal equation defines a set of stochastic restrictions on selected unobserved state variables. The signal and state innovation vectors are assumed to be independent, while the initial state vector is assumed to be independent from the signal and state innovation vectors, which combined with our distributional assumptions implies multivariate normality.
Conditional on the parameters associated with these signal and state equations, estimates of unobserved state vector $z_t$ and its mean squared error matrix $P_t$ may be calculated with the filter derived previously. Given initial conditions $z_{0|0}$ and $P_{0|0}$, estimates conditional on information available at time $t-1$ satisfy prediction equations:

\begin{align}
    z_{t|t-1} &= G_t z_{t-1|t-1}, \\
    P_{t|t-1} &= G_t P_{t-1|t-1} G_t^T + G_t \Sigma_z G_t^T, \\
    y_{t|t-1} &= F_t z_{t|t-1}, \\
    Q_{t|t-1} &= F_t P_{t|t-1} F_t^T, \\
    w_{t|t-1} &= H_t z_{t|t-1}, \\
    R_{t|t-1} &= H_t P_{t|t-1} H_t^T + \Sigma_z.
\end{align}

Given these predictions, under the assumption of multivariate normally distributed signal and state innovation vectors, together with conditionally contemporaneously uncorrelated signal vectors, estimates conditional on information available at time $t$ satisfy updating equations:

\begin{align}
    z_{t|t} &= z_{t|t-1} + K_y (y_t - y_{t|t-1}) + K_w (w_t - w_{t|t-1}), \\
    P_{t|t} &= P_{t|t-1} - K_y F_t P_{t|t-1} - K_w H_t P_{t|t-1},
\end{align}

where $K_y = P_{t|t-1} F_t^T Q_{t|t}^{-1}$ and $K_w = P_{t|t-1} H_t^T R_{t|t}^{-1}$. Given terminal conditions $z_{T|T}$ and $P_{T|T}$ obtained from the final evaluation of these prediction and updating equations, estimates conditional on information available at time $T$ satisfy smoothing equations:

\begin{align}
    z_{t|T} &= z_{t|t} + J_t (z_{t+1|T} - z_{t+1|t}), \\
    P_{t|T} &= P_{t|t} + J_t (P_{t+1|T} - P_{t+1|t}) J_t^T,
\end{align}

where $J_t = P_{t|t} G_t^T P_{t+1|t}^{-1}$. Under our distributional assumptions, these estimators of the unobserved state vector are mean squared error optimal.

Let $\theta \in \Theta \subset \mathbb{R}^K$ denote a $K$ dimensional vector containing the parameters associated with the signal and state equations of this linear state space model. The full information maximum likelihood estimator $\hat{\theta}_T$ of this parameter vector maximizes conditional loglikelihood function:
Under the assumption of multivariate normally distributed signal and state innovation vectors, together with conditionally contemporaneously uncorrelated signal vectors, the contributions to this conditional loglikelihood function satisfy 

\[ \ell_{y_0}(\theta) = \ell_{y_0}(\theta) + \ell_{w_0}(\theta), \]

where:

\[ \ell_{y_0}(\theta) = -\frac{M}{2} \ln(2\pi) - \frac{1}{2} \ln |Q_{0, y_{t-1}}| - \frac{1}{2} (y_t - y_{0_{t-1}})^T Q_{0, y_{t-1}}^{-1} (y_t - y_{0_{t-1}}), \]

\[ \ell_{w_0}(\theta) = -\frac{M}{2} \ln(2\pi) - \frac{1}{2} \ln |R_{0, w_{t-1}}| - \frac{1}{2} (w_t - w_{0_{t-1}})^T R_{0, w_{t-1}}^{-1} (w_t - w_{0_{t-1}}). \]

Under regularity conditions stated in Watson (1989), full information maximum likelihood estimator \( \hat{\theta}_T \) is consistent and asymptotically normal,

\[ \sqrt{T}(\hat{\theta}_T - \theta_0) \xrightarrow{d} \mathcal{N}(0, \mathcal{A}_0^{-1}\mathcal{B}_0\mathcal{A}_0^{-1}), \]

where \( \theta_0 \in \Theta \) denotes the true parameter vector. Following Engle and Watson (1981), consistent estimators of \( \mathcal{A}_0 \) and \( \mathcal{B}_0 \) are given by

\[ \hat{\mathcal{A}}_T = \frac{1}{T} \sum_{t=1}^{T} a_t(\hat{\theta}_T), \]

\[ \hat{\mathcal{B}}_T = \frac{1}{T} \sum_{t=1}^{T} b_t(\hat{\theta}_T)b_t(\hat{\theta}_T)^T, \]

where \( a_t(\hat{\theta}_T) = a_{y_t}(\hat{\theta}_T) + a_{w_t}(\hat{\theta}_T) \) and \( b_t(\hat{\theta}_T) = b_{y_t}(\hat{\theta}_T) + b_{w_t}(\hat{\theta}_T). \)

Under our distributional assumptions,

\[ a_{y_t}(\hat{\theta}_T) = \nabla_{\theta} y_{t}^T Q_{0, y_{t-1}}^{-1} \nabla_{\theta} y_{t-1} + \frac{1}{2} \nabla_{\theta} Q_{0, y_{t-1}}^T (Q_{0, y_{t-1}}^{-1} \otimes Q_{0, y_{t-1}}^{-1}) \nabla_{\theta} Q_{0, y_{t-1}}, \]

\[ a_{w_t}(\hat{\theta}_T) = \nabla_{\theta} w_{t}^T R_{0, w_{t-1}}^{-1} \nabla_{\theta} w_{t-1} + \frac{1}{2} \nabla_{\theta} R_{0, w_{t-1}}^T (R_{0, w_{t-1}}^{-1} \otimes R_{0, w_{t-1}}^{-1}) \nabla_{\theta} R_{0, w_{t-1}}, \]

\[ b_{y_t}(\hat{\theta}_T) = \nabla_{\theta} \ell_{y_t}(\hat{\theta}_T) \] and \( b_{w_t}(\hat{\theta}_T) = \nabla_{\theta} \ell_{w_t}(\hat{\theta}_T). \) If the signal and state innovation vectors are multivariate normally distributed, then the conditional information matrix equality holds, and \( \mathcal{A}_0 = \mathcal{B}_0. \)
2.4.1.2. Estimation Results

Our unobserved components model of the monetary transmission mechanism in a small open economy is estimated by full information maximum likelihood, conditional on prior information concerning the values of trend components. The data set consists of the levels of twenty observed endogenous variables for Canada and the United States described in Appendix 2.A. The initial values of state variables are treated as parameters, and are calibrated to match functions of preliminary estimates of trend components calculated with the linear filter described in Hodrick and Prescott (1997). The conditional loglikelihood function is maximized numerically with a modified steepest ascent algorithm. Estimation results pertaining to the period 1972Q1 through 2005Q1 appear in Appendix 2.B, with robust $t$ ratios reported in parentheses. The sufficient condition for the existence of a unique stationary rational expectations equilibrium due to Klein (2000) is satisfied in a neighbourhood around the full information maximum likelihood estimate, while the outer product of the gradient estimator of the information matrix is not nearly singular at the full information maximum likelihood estimate, suggesting that the linear state space representation of this unobserved components model is locally identified.

Prior information concerning the values of trend components is generated by fitting third order deterministic polynomial functions to the levels of all observed endogenous variables by ordinary least squares. Stochastic restrictions on the trend components of all observed endogenous variables are derived from the fitted values associated with these ordinary least squares regressions, with innovation variances set proportional to estimated prediction variances assuming known parameters. All stochastic restrictions are independent, represented by a diagonal covariance matrix, and are harmonized, represented by a common factor of proportionality. Reflecting little confidence in these preliminary trend component estimates, this common factor of proportionality is set equal to one.

The signs of all parameter estimates are consistent with our priors, while most are statistically significant at conventional levels. Estimates of the variances of innovations associated with both cyclical and trend components are often statistically significant at conventional levels, suggesting that the levels of the observed endogenous variables under consideration are subject to shocks having both temporary and permanent effects.

Predicted, filtered and smoothed estimates of the cyclical and trend components of observed endogenous variables are plotted together with confidence intervals in Appendix 2.B. These confidence intervals assume multivariate normally distributed and independent signal and state innovation vectors and known parameters. The predicted estimates are conditional on past information, the filtered estimates are conditional on past and present information, and the
smoothed estimates are conditional on past, present and future information. Visual inspection reveals close agreement with the conventional dating of business cycle expansions and recessions.

In order to examine whether our unobserved components model of the monetary transmission mechanism in a small open economy is dynamically complete in mean and variance, we subject the levels and squares of the predicted standardized residuals to the autocorrelation test of Ljung and Box (1978). We also examine whether there exist significant departures from conditional normality with the test of Jarque and Bera (1980). The predicted standardized residual vector \( \xi_{t|t-1} \) is related to the predicted ordinary residual vector \( \tilde{\xi}_{t|t-1} \) by \( \xi_{t|t-1} = Q_{t|t-1}^{1/2} \tilde{\xi}_{t|t-1} \), where \( \xi_{t|t-1} = y_t - y_{t|t-1} \). The inverse square root of predicted conditional covariance matrix \( Q_{t|t-1}^{-1/2} \) is calculated with a spectral decomposition as \( Q_{t|t-1}^{-1/2} = X_{t|t-1} A_{t|t-1}^{-1/2} X_{t|t-1}^T \), where \( X_{t|t-1} \) denotes a square matrix containing distinct orthonormal eigenvectors, while \( A_{t|t-1} \) denotes a diagonal matrix containing the corresponding positive eigenvalues.

We find moderate evidence of autocorrelation in the predicted standardized residuals, suggesting that the conditional mean function is dynamically incomplete. Furthermore, we find strong evidence of autoregressive conditional heteroskedasticity in the predicted standardized residuals, suggesting that the conditional variance function is dynamically incomplete. Finally, we find strong evidence of departures from normality in the predicted standardized residuals, in part attributable to the existence of excess kurtosis. These residual diagnostic test results suggest that our full information maximum likelihood estimation results are consistent and asymptotically normal, but are asymptotically inefficient.

2.4.2. Inference

Achieving low and stable inflation calls for accurate and precise indicators of inflationary pressure, together with an accurate and precise quantitative description of the monetary transmission mechanism. Our unobserved components model of the monetary transmission mechanism in a small open economy addresses both of these challenges within a unified empirical framework.

2.4.2.1. Quantifying Inflationary Pressure

Theoretically prominent indicators of inflationary pressure such as the natural rate of interest and natural exchange rate are unobservable. As discussed in Woodford (2003), the natural rate
of interest provides a measure of the neutral stance of monetary policy, with deviations of the real interest rate from the natural rate of interest generating inflationary pressure. It follows that the key to achieving low and stable inflation is the conduct of a monetary policy under which the short term nominal interest rate tracks variation in the natural rate of interest as closely as possible.

Predicted, filtered and smoothed estimates of the natural rate of interest are plotted together with confidence intervals versus corresponding estimates of the real interest rate in Figure 2.1. This concept of the natural rate of interest represents that short term real interest rate consistent with achieving inflation control and output stabilization objectives in the absence of shocks having temporary effects. Visual inspection reveals that our estimates of the natural rate of interest exhibit persistent low frequency variation and are relatively precise, as evidenced by relatively narrow confidence intervals. Periods during which the estimated real interest rate exceeds the estimated natural rate of interest are closely aligned with the conventional dating of recessions, suggesting that tight monetary policy was to varying degrees a contributing factor.

Predicted, filtered and smoothed estimates of the natural exchange rate are plotted together with confidence intervals versus the observed real exchange rate in Figure 2.2. This concept of the natural exchange rate represents that real exchange rate consistent with achieving inflation control and output stabilization objectives in the absence of shocks having temporary effects. Visual inspection reveals that our estimates of the natural exchange rate exhibit persistent low frequency variation and are relatively precise, as evidenced by relatively narrow confidence intervals.
Figure 2.2. Predicted, filtered and smoothed estimates of the natural exchange rate

LREXCH_P  LREXCH_F  LREXCH_S

Note: Observed levels are represented by black lines, while blue lines depict estimated trend components. Symmetric 95% confidence intervals assume multivariate normally distributed and independent signal and state innovation vectors and known parameters. Shaded regions indicate recessions as dated by the Economic Cycle Research Institute reference cycle.

2.4.2.2. Quantifying the Monetary Transmission Mechanism

The monetary transmission mechanism describes the dynamic effects of unsystematic variation in the instrument of monetary policy on indicators and targets. In a small open economy, the monetary transmission mechanism features both interest rate and exchange rate channels, while an inflation targeting central bank must react to shocks originating both domestically and abroad. Estimated impulse responses to domestic and foreign monetary policy shocks are plotted in Figure 2.3 and Figure 2.4, providing a quantitative description of the monetary transmission mechanism in a small open economy.

In response to a domestic monetary policy shock, the domestic nominal and real interest rates exhibit immediate increases followed by gradual declines. The domestic currency appreciates in nominal and real terms, with the nominal exchange rate exhibiting delayed overshooting. These real interest rate and real exchange rate dynamics induce persistent hump shaped reductions in domestic output, consumption, investment, exports and imports, together with persistent hump shaped declines in domestic output price inflation and consumption price inflation, with peak effects realized after one to two years. These output dynamics are associated with a persistent hump shaped reduction in domestic employment, together with a persistent hump shaped increase in the domestic unemployment rate, inducing a persistent hump shaped decline in domestic wage inflation, with peak effects realized after one to two years. These results are qualitatively consistent with those of structural vector autoregressive analyses of the monetary transmission mechanism in open economies such as Eichenbaum and Evans (1995), Clarida and Gertler (1997), Kim and Roubini (1995), and Cushman and Zha (1997).
In response to a foreign monetary policy shock, the foreign nominal and real interest rates exhibit immediate increases followed by gradual declines. These real interest rate dynamics induce persistent hump shaped reductions in foreign output, consumption and investment, together with a persistent hump shaped decline in foreign inflation, with peak effects realized after one to two years. These output dynamics are associated with a persistent hump shaped reduction in foreign employment, together with a persistent hump shaped increase in the foreign unemployment rate, inducing a persistent hump shaped decline in foreign wage inflation, with
peak effects realized after one to two years. The domestic currency depreciates in nominal and real terms, with the nominal exchange rate exhibiting delayed overshooting. These real interest rate and real exchange rate dynamics induce persistent hump shaped reductions in domestic output, exports and imports, together with persistent hump shaped declines in domestic output price inflation and consumption price inflation. These output dynamics are associated with a persistent hump shaped reduction in domestic employment, together with a persistent hump shaped increase in the domestic unemployment rate, inducing a persistent hump shaped decline in domestic wage inflation. These results are qualitatively consistent with those of structural vector autoregressive analyses of the monetary transmission mechanism in closed economies such as Sims and Zha (1995), Gordon and Leeper (1994), Leeper, Sims and Zha (1996), and Christiano, Eichenbaum and Evans (1998, 2005).
2.4.3. Forecasting

While it is desirable that forecasts be unbiased and efficient, the practical value of any forecasting model depends on its relative predictive accuracy. As a benchmark against which to evaluate the predictive accuracy of our unobserved components model of the monetary transmission mechanism in a small open economy, we consider the autoregressive integrated
moving average or ARIMA class of models. In particular, we consider ARIMA models for the levels of observed endogenous variables \( y_{it} \) of the form

\[
\Delta^d y_{it} = \mu_i + \sum_{j=1}^{p_i} \phi_{ij} \Delta^d y_{i,t-j} + \epsilon_{it} + \sum_{k=1}^{q_i} \theta_{ik} \epsilon_{i,t-k},
\]

where \( \epsilon_{it} \sim \text{iid } \mathcal{N}(0,\sigma^2) \). Theoretical support for this univariate forecasting framework is provided by the decomposition theorem due to Wold (1938), which states that any covariance stationary purely linearly indeterministic scalar stochastic process has an infinite order moving average representation. As discussed in Clements and Hendry (1998), any infinite order moving average process can be approximated to any required degree of accuracy by an autoregressive moving average process, with the required autoregressive and moving average orders typically being relatively low.

The ARIMA models are estimated by maximum likelihood over the period 1972Q3 through 2005Q1. The autoregressive, ordinary difference, and moving average orders are jointly selected to minimize the model selection criterion function proposed by Schwarz (1978).\(^1\) Those ARIMA model specifications deemed optimal are employed throughout our forecast performance evaluation exercise.

In the absence of a well defined mapping between forecast errors and their costs, relative predictive accuracy is generally assessed with mean squared prediction error based measures. As discussed in Clements and Hendry (1998), mean squared prediction error based measures are noninvariant to nonsingular, scale preserving linear transformations, even though linear models are. It follows that mean squared prediction error based comparisons may yield conflicting rankings across models, depending on the variable transformations examined.

To evaluate the dynamic out of sample forecasting performance of our unobserved components model of the monetary transmission mechanism in a small open economy, we retain forty quarters of observations to evaluate forecasts one through eight quarters ahead, generated conditional on parameters estimated using information available at the forecast origin. The models are compared on the basis of mean squared prediction errors in levels, ordinary differences, and seasonal differences. The unobserved components model is not recursively estimated as the forecast origin rolls forward due to the high computational cost of such a procedure, while the ARIMA models are. Presumably, recursively estimating the unobserved components model would improve its predictive accuracy.

---

\(^1\) The autoregressive order \( p \), ordinary difference order \( d \), and moving average order \( q \), are jointly selected subject to upper bounds of four, two and two, respectively.
Mean squared prediction error differentials are plotted together with confidence intervals accounting for contemporaneous and serial correlation of forecast errors in Appendix 2.B. If these mean squared prediction error differentials are negative then the forecasting performance of the unobserved components model dominates that of the ARIMA models, while if positive then the unobserved components model is dominated by the ARIMA models in terms of predictive accuracy. The null hypothesis of equal squared prediction errors is rejected by the predictive accuracy test of Diebold and Mariano (1995) if and only if these confidence intervals exclude zero. The asymptotic variance of the average loss differential is estimated by a weighted sum of the autocovariances of the loss differential, employing the weighting function proposed by Newey and West (1987). Visual inspection reveals that these mean squared prediction error differentials are of variable sign, suggesting that the unobserved components model matches the ARIMA models in terms of forecasting performance, in spite of a considerable informational disadvantage. However, these mean squared prediction error differentials are rarely statistically significant at conventional levels, perhaps because the predictive accuracy test due to Diebold and Mariano (1995), which is univariate, typically lacks power to detect dominance in forecasting performance, as evidenced by Monte Carlo evaluations such as Ashley (2003) and McCracken (2000).

Dynamic out of sample forecasts of levels, ordinary differences, and seasonal differences are plotted together with confidence intervals versus realized outcomes in Appendix 2.B. These confidence intervals assume multivariate normally distributed and independent signal and state innovation vectors and known parameters. Visual inspection reveals that the realized outcomes generally lie within their associated confidence intervals, suggesting that forecast failure is absent. However, these confidence intervals are rather wide, indicating that considerable uncertainty surrounds the point forecasts.

2.5. Conclusion

This paper develops and estimates an unobserved components model of the monetary transmission mechanism in a small open economy for purposes of monetary policy analysis and inflation targeting. This estimated unobserved components model provides a quantitative description of the monetary transmission mechanism in a small open economy, yields a mutually consistent set of indicators of inflationary pressure together with confidence intervals, and facilitates the generation of relatively accurate forecasts.

Definitions of indicators of inflationary pressure such as the natural rate of interest and natural exchange rate vary, while estimates are typically sensitive to identifying restrictions. It
follows that combinations of estimates of indicators of inflationary pressure derived under alternative definitions from dissimilar models may be more useful for purposes of monetary policy analysis and inflation targeting in a small open economy than any of the constituents: An examination of the inflation control and output stabilization benefits conferred by combining alternative estimates remains an objective for future research.

Appendix 2.A. Description of the Data Set

The data set consists of quarterly seasonally adjusted observations on twenty macroeconomic variables for Canada and the United States over the period 1971Q1 through 2005Q1. All aggregate prices and quantities are expenditure based. Employment is derived from observed nominal labour income and a nominal wage index, while the unemployment rate is expressed as a period average. The nominal interest rate is measured by the three month Treasury bill rate expressed as a period average, while the nominal exchange rate is quoted as an end of period value. National accounts data for Canada was retrieved from the CANSIM database maintained by Statistics Canada, national accounts data for the United States was obtained from the FRED database maintained by the Federal Reserve Bank of Saint Louis, and other data was extracted from the IFS database maintained by the International Monetary Fund.
## Appendix 2.B. Tables and Figures

### Table 2.1. Full information maximum likelihood estimation results, domestic economy

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Note: Rejection of the null hypothesis at the 1%, 5% and 10% levels is indicated by ***, ** and *, respectively.
Table 2.2. Full information maximum likelihood estimation results, foreign economy

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$\mathcal{L}(\hat{\theta}_j) = -6605.462$

Note: Rejection of the null hypothesis at the 1%, 5% and 10% levels is indicated by ***, ** and *, respectively.
Figure 2.5. Predicted cyclical components of observed endogenous variables

Note: Symmetric 95% confidence intervals assume multivariate normally distributed and independent signal and state innovation vectors and known parameters. Shaded regions indicate recessions as dated by the Economic Cycle Research Institute reference cycle.
Figure 2.6. Filtered cyclical components of observed endogenous variables

Note: Symmetric 95% confidence intervals assume multivariate normally distributed and independent signal and state innovation vectors and known parameters. Shaded regions indicate recessions as dated by the Economic Cycle Research Institute reference cycle.
Figure 2.7. Smoothed cyclical components of observed endogenous variables

Note: Symmetric 95% confidence intervals assume multivariate normally distributed and independent signal and state innovation vectors and known parameters. Shaded regions indicate recessions as dated by the Economic Cycle Research Institute reference cycle.
Figure 2.8. Predicted trend components of observed endogenous variables

Note: Observed levels are represented by black lines, while blue lines depict estimated trend components. Symmetric 95% confidence intervals assume multivariate normally distributed and independent signal and state innovation vectors and known parameters. Shaded regions indicate recessions as dated by the Economic Cycle Research Institute reference cycle.
Figure 2.9. Filtered trend components of observed endogenous variables

Note: Observed levels are represented by black lines, while blue lines depict estimated trend components. Symmetric 95% confidence intervals assume multivariate normally distributed and independent signal and state innovation vectors and known parameters. Shaded regions indicate recessions as dated by the Economic Cycle Research Institute reference cycle.
Figure 2.10. Smoothed trend components of observed endogenous variables

Note: Observed levels are represented by black lines, while blue lines depict estimated trend components. Symmetric 95% confidence intervals assume multivariate normally distributed and independent signal and state innovation vectors and known parameters. Shaded regions indicate recessions as dated by the Economic Cycle Research Institute reference cycle.
Figure 2.11. Mean squared prediction error differentials for levels

Note: Mean squared prediction error differentials are defined as the mean squared prediction error for the unobserved components model less that for the ARIMA model. Symmetric 95% confidence intervals account for contemporaneous and serial correlation of forecast errors.
Figure 2.12. Mean squared prediction error differentials for ordinary differences

Note: Mean squared prediction error differentials are defined as the mean squared prediction error for the unobserved components model less that for the ARIMA model. Symmetric 95% confidence intervals account for contemporaneous and serial correlation of forecast errors.
Figure 2.13. Mean squared prediction error differentials for seasonal differences

Note: Mean squared prediction error differentials are defined as the mean squared prediction error for the unobserved components model less that for the ARIMA model. Symmetric 95% confidence intervals account for contemporaneous and serial correlation of forecast errors.
Figure 2.14. Dynamic forecasts of levels of observed endogenous variables

Note: Realized outcomes are represented by black lines, while blue lines depict point forecasts. Symmetric 95% confidence intervals assume multivariate normally distributed and independent signal and state innovation vectors and known parameters.
Figure 2.15. Dynamic forecasts of ordinary differences of observed endogenous variables

Note: Realized outcomes are represented by black lines, while blue lines depict point forecasts. Symmetric 95% confidence intervals assume multivariate normally distributed and independent signal and state innovation vectors and known parameters.
Figure 2.16. Dynamic forecasts of seasonal differences of observed endogenous variables

Note: Realized outcomes are represented by black lines, while blue lines depict point forecasts. Symmetric 95% confidence intervals assume multivariate normally distributed and independent signal and state innovation vectors and known parameters.
References


Wold, H. (1938), A Study in the Analysis of Stationary Time Series, Almqvist and Wiksell.
CHAPTER 3

Measuring the Stance of Monetary Policy in a Small Open Economy: A Dynamic Stochastic General Equilibrium Approach

3.1. Introduction

Estimated dynamic stochastic general equilibrium or DSGE models have recently emerged as quantitative monetary policy analysis and inflation targeting tools. As extensions of real business cycle models, DSGE models explicitly specify the objectives and constraints faced by optimizing households and firms, which interact in an uncertain environment to determine equilibrium prices and quantities. The existence of short run nominal price and wage rigidities generated by monopolistic competition and staggered reoptimization in output and labour markets permits a cyclical stabilization role for monetary policy, which is generally implemented through control of the short term nominal interest rate according to a monetary policy rule. The persistence of the effects of monetary policy shocks on output and inflation is often enhanced with other features such as habit persistence in consumption, adjustment costs in investment, and variable capital utilization. Early examples of closed economy DSGE models incorporating some of these features include those of Yun (1996), Goodfriend and King (1997), Rotemberg and Woodford (1995, 1997), and McCallum and Nelson (1999), while recent examples of closed economy DSGE models incorporating all of these features include those of Christiano, Eichenbaum and Evans (2005), Altig, Christiano, Eichenbaum and Linde (2005), Smets and Wouters (2003, 2005), and Vitek (2006c).

Open economy DSGE models extend their closed economy counterparts to allow for international trade and financial linkages, implying that the monetary transmission mechanism features both interest rate and exchange rate channels. Building on the seminal work of Obstfeld and Rogoff (1995, 1996), these open economy DSGE models determine trade and current account balances through both intratemporal and intertemporal optimization, while the nominal

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exchange rate is determined by an uncovered interest parity condition. Existing open economy DSGE models differ primarily with respect to the degree of exchange rate pass through. Models in which exchange rate pass through is complete include those of Benigno and Benigno (2002), McCallum and Nelson (2000), Clarida, Gali and Gertler (2001, 2002), and Gertler, Gilchrist and Natalucci (2001), while models in which exchange rate pass through is incomplete include those of Adolfson (2001), Betts and Devereux (2000), Kollman (2001), Corsetti and Pesenti (2002), Monacelli (2005), and Vitek (2006d).

Recent research has emphasized the implications of developments in the housing market for the conduct of monetary policy. Existing DSGE models incorporating a housing market include those of Aoki, Proudman and Vlieghe (2004) and Iacoviello (2005), both of which focus on the implications of financial market frictions for the monetary transmission mechanism. In addition to abstracting from open economy elements of the monetary transmission mechanism, these papers do not consider the implications of developments in the housing market for the measurement of the stance of monetary policy.

Existing DSGE models featuring long run balanced growth driven by trend inflation, productivity growth, and population growth generally predict the existence of common deterministic or stochastic trends. Estimated DSGE models incorporating common deterministic trends include those of Ireland (1997) and Smets and Wouters (2005), while estimated DSGE models incorporating common stochastic trends include those of Altig, Christiano, Eichenbaum and Linde (2005) and Del Negro, Schorfheide, Smets and Wouters (2005). However, as discussed in Clements and Hendry (1999) and Maddala and Kim (1998), intermittent structural breaks render such common deterministic or stochastic trends empirically inadequate representations of low frequency variation in observed macroeconomic variables. For this reason, it is common to remove trend components from observed macroeconomic variables with deterministic polynomial functions or linear filters, such as the difference filter or the low pass filter described in Hodrick and Prescott (1997), prior to the conduct of estimation, inference and forecasting. As an alternative, Vitek (2006c, 2006d) proposes jointly modeling cyclical and trend components as unobserved components while imposing theoretical restrictions derived from the approximate multivariate linear rational expectations representation of a DSGE model.

This merging of modeling paradigms drawn from the theoretical and empirical macroeconomics literatures confers a number of important benefits. First, the joint estimation of parameters and trend components ensures their mutual consistency, as estimates of parameters appropriately reflect estimates of trend components, and vice versa. As shown by Nelson and Kang (1981) and Harvey and Jaeger (1993), decomposing integrated observed endogenous variables into cyclical and trend components with atheoretic deterministic polynomial functions or low pass filters may induce spurious cyclical dynamics, invalidating subsequent estimation,
inference and forecasting. Second, basing estimation on the levels as opposed to differences of observed endogenous variables may be expected to yield efficiency gains. A central result of the voluminous cointegration literature surveyed by Maddala and Kim (1998) is that, if there exist cointegrating relationships, then differencing all integrated observed endogenous variables prior to the conduct of estimation, inference and forecasting results in the loss of information. Third, the proposed unobserved components framework ensures stochastic nonsingularity of the resulting approximate linear state space representation of the DSGE model, as associated with each observed endogenous variable is at least one exogenous stochastic process. As discussed in Ruge-Murcia (2003), stochastic nonsingularity requires that the number of observed endogenous variables used to construct the loglikelihood function associated with the approximate linear state space representation of a DSGE model not exceed the number of exogenous stochastic processes, with efficiency losses incurred if this constraint binds. Fourth, the proposed unobserved components framework facilitates the direct generation of forecasts of the levels of endogenous variables as opposed to their cyclical components together with confidence intervals, while ensuring that these forecasts satisfy the stability restrictions associated with balanced growth. These stability restrictions are necessary but not sufficient for full cointegration, as along a balanced growth path, great ratios and trend growth rates are time independent but state dependent, robustifying forecasts to intermittent structural breaks that occur within sample.

The primary contribution of this paper is the development of a procedure to estimate the levels of the flexible price and wage equilibrium components of endogenous variables while imposing relatively weak, and hence relatively credible, identifying restrictions on their trend components. Based on an extension and refinement of the unobserved components framework proposed by Vitek (2006c, 2006d), this estimation procedure confers a number of benefits of particular importance to the conduct of monetary policy. First, as discussed in Woodford (2003), the levels of the flexible price and wage equilibrium components of various observed and unobserved endogenous variables are important inputs into the optimal conduct of monetary policy. In particular, the level of the natural rate of interest, defined as that short term real interest rate consistent with price and wage flexibility, provides a measure of the neutral stance of monetary policy, with deviations of the real interest rate from the natural rate of interest generating inflationary pressure. The proposed unobserved components framework facilitates estimation of the levels as opposed to cyclical components of the flexible price and wage equilibrium components of endogenous variables, while ensuring that they satisfy the stability restrictions associated with balanced growth. Second, given an interest rate smoothing objective derived from a concern with financial market stability, variation in the natural rate of interest caused by shocks having permanent effects may call for larger monetary policy responses than variation caused by shocks having temporary effects. The proposed unobserved components
framework yields a decomposition of the levels of the flexible price and wage equilibrium components of endogenous variables into cyclical and trend components, together with confidence intervals which account for uncertainty associated with the detrending procedure. Third, as discussed in Clements and Hendry (1999) and Maddala and Kim (1998), accommodating the existence of intermittent structural breaks requires flexible trend component specifications. However, the joint derivation of empirically adequate cyclical and trend component specifications from microeconomic foundations is a formidable task. The proposed unobserved components framework facilitates estimation of the levels of the flexible price and wage equilibrium components of endogenous variables while allowing for the possibility that the determinants of their trend components are unknown but persistent.

The secondary contribution of this paper is the estimation of the levels of the flexible price and wage equilibrium components of various observed and unobserved endogenous variables while imposing relatively weak identifying restrictions on their trend components, with an emphasis on the levels of the natural rate of interest and natural exchange rate. Definitions of indicators of inflationary pressure such as the natural rate of interest and natural exchange rate vary, and numerous alternative procedures for estimating the natural rate of interest have been proposed, several of which are discussed in a survey paper by Giammarioli and Valla (2004). Within the framework of a calibrated DSGE model of a closed economy, Neiss and Nelson (2003) find that estimates of the deviation of the real interest rate from the natural rate of interest exhibit economically significant high frequency variation. Within the framework of an estimated DSGE model of a closed economy, Smets and Wouters (2003) find that estimates of the deviation of the real interest rate from the natural rate of interest exhibit economically significant high frequency variation and are relatively imprecise, as evidenced by relatively wide confidence intervals. In addition to abstracting from open economy elements of the monetary transmission mechanism, these papers abstract from the trend component of the natural rate of interest, as they employ estimation procedures which involve the preliminary removal of trend components from observed macroeconomic variables with atheoretic deterministic polynomial functions.

This paper develops and estimates a DSGE model of a small open economy for purposes of monetary policy analysis and inflation targeting. This estimated DSGE model provides a quantitative description of the monetary transmission mechanism in a small open economy, yields a mutually consistent set of indicators of inflationary pressure together with confidence intervals, and facilitates the generation of relatively accurate forecasts. The model features short run nominal price and wage rigidities generated by monopolistic competition and staggered reoptimization in output and labour markets. The resultant inertia in inflation and persistence in output is enhanced with other features such as habit persistence in consumption and labour supply, adjustment costs in housing and capital investment, and variable capital utilization.
Incomplete exchange rate pass through is generated by short run nominal price rigidities in the import market, with monopolistically competitive importers setting the domestic currency prices of differentiated intermediate import goods subject to randomly arriving reoptimization opportunities. Cyclical components are modeled by linearizing equilibrium conditions around a stationary deterministic steady state equilibrium which abstracts from long run balanced growth, while trend components are modeled as random walks while ensuring the existence of a well defined balanced growth path. Parameters and unobserved components are jointly estimated with a novel Bayesian procedure, conditional on prior information concerning the values of parameters and trend components.

The organization of this paper is as follows. The next section develops a DSGE model of a small open economy. Estimation, inference and forecasting within the framework of a linear state space representation of an approximate unobserved components representation of this DSGE model are the subjects of section three. Finally, section four offers conclusions and recommendations for further research.

3.2. Model Development

Consider two open economies which are asymmetric in size, but are otherwise identical. The domestic economy is of negligible size relative to the foreign economy.

3.2.1. The Utility Maximization Problem of the Representative Household

There exists a continuum of households indexed by $i \in [0, 1]$. Households supply differentiated intermediate labour services, but are otherwise identical.

3.2.1.1. Consumption, Saving and Investment Behaviour

The representative infinitely lived household has preferences defined over consumption $C_{i,t}$, housing $H_{i,t}$, and labour supply $L_{i,t}$ represented by intertemporal utility function

$$U_{i,t} = E_t \sum_{s=1}^{\infty} \beta^{s-t} u(C_{i,s}, H_{i,s}, L_{i,s}),$$  \hspace{1cm} (1)
where subjective discount factor $\beta$ satisfies $0 < \beta < 1$. The intratemporal utility function is additively separable and represents external habit formation preferences in consumption, housing, and labour supply,

$$u(C_{i,s}, H_{i,s}, L_{i,s}) = \nu^C \left[ \frac{(C_{i,s} - \alpha^C C_{i,s-1})^{1-\alpha \sigma}}{1-\alpha \sigma} + \nu^H \frac{(H_{i,s} - \alpha^H H_{i,s-1})^{1-\alpha \eta}}{1-\alpha \eta} - \nu^L \frac{(L_{i,s} - \alpha^L L_{i,s-1})^{1+\eta}}{1+\eta} \right].$$

(2)

where $0 \leq \alpha^C < 1$, $0 \leq \alpha^H < 1$ and $0 \leq \alpha^L < 1$. This intratemporal utility function is strictly increasing with respect to consumption if and only if $\nu^C > 0$, and given this parameter restriction is strictly increasing with respect to housing if and only if $\nu^H > 0$, and is strictly decreasing with respect to labour supply if and only if $\nu^L > 0$. Given these parameter restrictions, this intratemporal utility function is strictly concave if $\sigma > 0$ and $\eta > 0$.

The representative household enters period $s$ in possession of previously purchased domestic currency denominated bonds $B^{P,h}_{i,s}$ which yield interest at risk free rate $i^{h}_{i,s}$, and foreign currency denominated bonds $B^{P,f}_{i,s}$ which yield interest at risk free rate $i^{f}_{i,s}$. It also holds a diversified portfolio of shares $\{x_{i,s+j}^Y\}_{j=0}^1$ in domestic intermediate good firms which pay dividends $\{I^Y_{i,s+j}\}_{j=0}^1$, and a diversified portfolio of shares $\{x_{i,s+k}^M\}_{k=0}^1$ in domestic intermediate good importers which pay dividends $\{I^M_{i,s+k}\}_{k=0}^1$. The representative household supplies differentiated intermediate labour service $L_{i,s}$, earning labour income at nominal wage $W_{i,s}$. Households pool their labour income, and the government levies a tax on pooled labour income at rate $\tau_s$. These sources of private wealth are summed in household dynamic budget constraint:

$$B^{P,h}_{i,s+1} + \varepsilon_s B^{P,f}_{i,s+1} + \int_{j=0}^{1} V^{Y}_{i,s+j} x^Y_{i,s+j} dj + \int_{k=0}^{1} V^{M}_{i,s+k} x^M_{i,s+k} dk = (1 + i^{h}_{i,s}) B^{P,h}_{i,s} + \varepsilon_s (1 + i^{f}_{i,s}) B^{P,f}_{i,s} + \int_{j=0}^{1} (I^Y_{i,s+j} + V^{Y}_{i,s+j}) x^Y_{i,s+j} dj + \int_{k=0}^{1} (I^M_{i,s+k} + V^{M}_{i,s+k}) x^M_{i,s+k} dk + (1 - \tau_s) \int_{j=0}^{1} W_{i,s} L_{i,s} dj - P^C_s C_{i,s} - P^H_s I^H_{i,s}$$

(3)

According to this dynamic budget constraint, at the end of period $s$, the representative household purchases domestic bonds $B^{P,h}_{i,s+1}$, and foreign bonds $B^{P,f}_{i,s+1}$ at price $\varepsilon_s$. It also purchases a diversified portfolio of shares $\{x^Y_{i,s+j}\}_{j=0}^1$ in intermediate good firms at prices $\{V^{Y}_{i,s+j}\}_{j=0}^1$, and a diversified portfolio of shares $\{x^M_{i,s+k}\}_{k=0}^1$ in intermediate good importers at prices $\{V^{M}_{i,s+k}\}_{k=0}^1$. Finally, the representative household purchases final consumption good $C_{i,s}$ at price $P^C_s$, and final housing investment good $I^H_{i,s}$ at price $P^H_s$.

The representative household enters period $s$ in possession of previously accumulated housing stock $H_{i,s}$, which subsequently evolves according to accumulation function

$$H_{i,s+1} = (1 - \delta^H_s) H_{i,s} + \mathcal{H}^H (I^H_{i,s}, I^H_{i,s-1}).$$

(4)
where depreciation rate parameter $\delta^H$ satisfies $0 \leq \delta^H \leq 1$. Effective housing investment function $\mathcal{H}^H(I_{i,s}^H, I_{i,s-1}^H)$ incorporates convex adjustment costs,

$$\mathcal{H}^H(I_{i,s}^H, I_{i,s-1}^H) = \psi_i^{ii} \left[ 1 - \frac{2 \delta^H}{2} \left( \frac{I_{i,s}^H - I_{i,s-1}^H}{I_{i,s-1}^H} \right)^2 \right] I_{i,s}^H,$$

where $\chi^H > 0$ and $\psi_i^{ii} > 0$. In deterministic steady state equilibrium, these adjustment costs equal zero, and effective investment equals actual investment.

In period $t$, the representative household chooses state contingent sequences for consumption $\{C_{i,s}\}_{s=0}^\infty$, investment in housing $\{I_{i,s}^H\}_{s=0}^\infty$, the stock of housing $\{H_{i,s+1}\}_{s=0}^\infty$, domestic bond holdings $\{B_{i,s+1}^B\}_{s=0}^\infty$, foreign bond holdings $\{B_{i,s+1}^F\}_{s=0}^\infty$, share holdings in intermediate good firms $\{x_{i,s+1}^k\}_{s=0}^\infty$, and share holdings in intermediate good importers $\{x_{i,s+1}^M\}_{s=0}^\infty$ to maximize intertemporal utility function (1) subject to dynamic budget constraint (3), housing accumulation function (4), and terminal nonnegativity constraints $H_{i,T+1} \geq 0$, $B_{i,T+1}^B \geq 0$, $B_{i,T+1}^F \geq 0$, $x_{i,j,T+1}^V \geq 0$ and $x_{i,k,T+1}^M \geq 0$ for $T \to \infty$. In equilibrium, selected necessary first order conditions associated with this utility maximization problem may be stated as

$$u_c(C_{i,s}, H_{i,s}, L_{i,s}) = P_i^C \lambda_i,$$

$$Q^H \mathcal{H}^H(I_{i,s}^H, I_{i,s-1}^H) + E_i \frac{\beta \lambda_{t+1}}{\lambda_t} Q_{t+1}^H \mathcal{H}^H(I_{t+1}^H, I_{t}^H) = P_t^H,$$

$$Q_t^H = E_i \frac{\beta \lambda_{t+1}}{\lambda_t} \left[ \frac{u_i(H_{i,s+1}, L_{i,s+1})}{\lambda_{t+1}} + (1 - \delta^H) Q_{t+1}^H \right],$$

$$\lambda_t = \beta (1 + i_t) E_i \lambda_{t+1},$$

$$\varepsilon_i \lambda_t = \beta (1 + i_t') E_i \varepsilon_{i,t} \lambda_{t+1},$$

$$V_{j,t}^Y \lambda_t = \beta E_i (TV_{j,t+1}^Y + V_{j,t+1}^Y) \lambda_{t+1},$$

$$V_{k,t}^M \lambda_t = \beta E_i (TV_{k,t+1}^M + V_{k,t+1}^M) \lambda_{t+1},$$
Provided that the intertemporal utility function is bounded and strictly concave, together with all
necessary first order conditions, these transversality conditions are sufficient for the unique
utility maximizing state contingent intertemporal household allocation.

Combination of necessary first order conditions (6) and (9) yields intertemporal optimality
condition

\[
Q_t^H = \frac{\beta u_C(C_{t+1}, H_{t+1}, L_{t+1})}{u_C(C_t, H_t, L_t)} P_{t+1}^C u_C(C_{t+1}, H_{t+1}, L_{t+1}) + (1 - \delta^H) Q_t^H,
\]

which equates the expected present discounted value of an additional unit of investment in
housing to its price. Combination of necessary first order conditions (6) and (8) yields
intertemporal optimality condition

\[
Q_t^H = \frac{\beta u_C(C_{t+1}, H_{t+1}, L_{t+1})}{u_C(C_t, H_t, L_t)} P_{t+1}^C u_C(C_{t+1}, H_{t+1}, L_{t+1}) + (1 - \delta^H) Q_{t+1}^H,
\]

where 

\[
\lim_{T \to +\infty} \frac{\beta^T \lambda_{t,T}}{\lambda_t} Q_{t+1}^H H_{t+1} H_{t+1} = 0,
\]

\[
\lim_{T \to +\infty} \frac{\beta^T \lambda_{t,T}}{\lambda_t} B_{t,T+1}^p = 0,
\]

\[
\lim_{T \to +\infty} \frac{\beta^T \lambda_{t,T}}{\lambda_t} E_{t+1} B_{t,T+1}^p = 0,
\]

\[
\lim_{T \to +\infty} \frac{\beta^T \lambda_{t,T}}{\lambda_t} v_{j+1}^T x_{j+1}^T = 0,
\]

\[
\lim_{T \to +\infty} \frac{\beta^T \lambda_{t,T}}{\lambda_t} v_{k+1}^T x_{k+1}^T = 0.
\]
which equates the shadow price of housing to the expected present discounted value of the sum of the future marginal cost of housing, and the future shadow price of housing net of depreciation. Finally, combination of necessary first order conditions (6), (9) and (10) yields intratemporal optimality condition

\[ E_t \frac{\beta u c(C_t, H, L)}{u c(C_t, H, L)} \frac{P_t^c}{P_{t-1}^c} (1+i) = E_t \frac{\beta u c(C_{t-1}, H_{t-1}, L_{t-1})}{u c(C_{t-1}, H_{t-1}, L_{t-1})} \frac{P_{t-1}^c}{P_{t-1}^c} \frac{E_{t+1}}{E_t} (1+i'), \]

which equates the expected present discounted values of the gross real returns on domestic and foreign bonds.

3.2.1.2. Labour Supply and Wage Setting Behaviour

There exist a large number of perfectly competitive firms which combine differentiated intermediate labour services \( L_{i,t} \) supplied by households in a monopolistically competitive labour market to produce final labour service \( L_t \) according to constant elasticity of substitution production function

\[ L_t = \left[ \int_{L_{i,t}}^{\theta^e} \frac{\theta^e - 1}{\theta^{e-1}} \right]^{\theta^e}, \]

where \( \theta^e > 1 \). The representative final labour service firm maximizes profits derived from production of the final labour service

\[ \Pi_t = W_t L_t - \int_{i=0}^{1} W_{it} L_{i,t} di, \]

with respect to inputs of intermediate labour services, subject to production function (22). The necessary first order conditions associated with this profit maximization problem yield intermediate labour service demand functions:

\[ L_{i,t} = \left( \frac{W_{it}}{W_t} \right)^{-\theta^e} L_t. \]

Since the production function exhibits constant returns to scale, in competitive equilibrium the representative final labour service firm earns zero profit, implying aggregate wage index:
As the wage elasticity of demand for intermediate labour services $\theta^L_t$ increases, they become closer substitutes, and individual households have less market power.

In an extension of the model of nominal wage rigidity proposed by Erceg, Henderson and Levin (2000) along the lines of Smets and Wouters (2003, 2005), each period a randomly selected fraction $1 - \omega^L$ of households adjust their wage optimally. The remaining fraction $\omega^L$ of households adjust their wage to account for past consumption price inflation according to partial indexation rule

$$ W_{t,i} = \left( \frac{p^C_{t-1}}{p^C_{t-2}} \right)^{\gamma^L} \left( \frac{p^C_{t-1}}{p^C_{t-2}} \right)^{1-\gamma^L} W_{t,i-1}, \quad (26) $$

where $0 \leq \gamma^L \leq 1$. Under this specification, although households adjust their wage every period, they infrequently adjust their wage optimally, and the interval between optimal wage adjustments is a random variable.

If the representative household can adjust its wage optimally in period $t$, then it does so to maximize intertemporal utility function (1) subject to dynamic budget constraint (3), housing accumulation function (4), intermediate labour service demand function (24), and the assumed form of nominal wage rigidity. Since all households that adjust their wage optimally in period $t$ solve an identical utility maximization problem, in equilibrium they all choose a common wage $W^*_t$ given by necessary first order condition:

$$ \frac{W^*_t}{W_t} = - \frac{E_i \sum_{j=1}^{\infty} (\omega^L)^{j-1} \beta^{-j} u_c(C_t, H_t, L_t) \theta^L E u_c(C_t, H_t, L_t) \left[ \left( \frac{p^C_{t-1}}{p^C_{t-2}} \right)^{\gamma^L} \left( \frac{p^C_{t-1}}{p^C_{t-2}} \right)^{1-\gamma^L} \frac{W^*_t}{W_t} \right]^{\gamma^L} \left( \frac{W^*_t}{W_t} \right)^{1-\gamma^L} L_t}{E_i \sum_{j=1}^{\infty} (\omega^L)^{j-1} \beta^{-j} u_c(C_t, H_t, L_t) (\theta^L - 1)(1 - \tau_j) \frac{W_t}{P^e_t} \left[ \left( \frac{p^C_{t-1}}{p^C_{t-2}} \right)^{\gamma^L} \left( \frac{p^C_{t-1}}{p^C_{t-2}} \right)^{1-\gamma^L} \frac{W_t}{W_t} \right]^{\gamma^L} \left( \frac{W_t}{W_t} \right)^{1-\gamma^L} L_t} \quad (27) $$

This necessary first order condition equates the expected present discounted value of the consumption benefit generated by an additional unit of labour supply to the expected present discounted value of its leisure cost. Aggregate wage index (25) equals an average of the wage set by the fraction $1 - \omega^L$ of households that adjust their wage optimally in period $t$, and the average of the wages set by the remaining fraction $\omega^L$ of households that adjust their wage according to partial indexation rule (26):
\[ W_t = \left\{ (1 - \omega^t)(W_t^*)^{-\theta^t} + \omega^t \left[ \left( \frac{P_{t-1}}{P_{t-2}} \right)^{\gamma^t} \left( \frac{P_{t-1}}{P_{t-2}} \right)^{1-\gamma^t} W_{t-1} \right]^{-\theta^t} \right\}^{1/1-\theta^t}. \]  

(28)

Since those households able to adjust their wage optimally in period \( t \) are selected randomly from among all households, the average wage set by the remaining households equals the value of the aggregate wage index that prevailed during period \( t-1 \), rescaled to account for past consumption price inflation.

If all households were able to adjust their wage optimally every period, then \( \omega^t = 0 \) and necessary first order condition (27) would reduce to:

\[ (1 - \xi_t) \frac{\bar{W}_t}{P_t^C} = -\frac{\theta^t}{\theta^t - 1} u_c(C_t, H_t, L_t). \]

(29)

In flexible price and wage equilibrium, each household sets its after tax real wage equal to a time varying markup over the marginal rate of substitution between leisure and consumption, and labour supply is inefficiently low.

3.2.2. The Value Maximization Problem of the Representative Firm

There exists a continuum of intermediate good firms indexed by \( j \in [0,1] \). Intermediate good firms supply differentiated intermediate output goods, but are otherwise identical. Entry into and exit from the monopolistically competitive intermediate output good sector is prohibited.

3.2.2.1. Employment and Investment Behaviour

The representative intermediate good firm sells shares \( \{x_{i,j,t+1}^Y\}_{i=0}^{1} \) to domestic households at price \( V_{j,t}^Y \). Recursive forward substitution for \( V_{j,t,s}^Y \) with \( s > 0 \) in necessary first order condition (11) applying the law of iterated expectations reveals that the post-dividend stock market value of the representative intermediate good firm equals the expected present discounted value of future dividend payments:

\[ V_{j,t}^Y = E_t \sum_{s=t+1}^{s} \frac{P_{t+1}^{s-1} \lambda_s}{\lambda_t} \Pi_{j,s}^Y. \]

(30)
Acting in the interests of its shareholders, the representative intermediate good firm maximizes its pre-dividend stock market value, equal to the expected present discounted value of current and future dividend payments:

$$\Pi^Y_{j,t} + V^Y_{j,t} = E \sum_{t=0}^{\infty} B^t \frac{\lambda_t}{\lambda_s} \Pi^Y_{j,t}.$$  \hspace{1cm} (31)

The derivation of result (30) imposes transversality condition (16), which rules out self-fulfilling speculative asset price bubbles.

Shares entitle households to dividend payments equal to net profits $$\Pi^Y_{j,t}$$, defined as after tax earnings less expenditures on investment in capital:

$$\Pi^Y_{j,t} = (1 - \tau_s)(P^s_{j,t}Y^s_{j,t} - W^s_{j,t}L^s_{j,t}) - P^s_{i,t}I^s_{i,t}.$$  \hspace{1cm} (32)

Earnings are defined as revenues derived from sales of differentiated intermediate output good $$Y^s_{j,t}$$ at price $$P^s_{j,t}$$ less expenditures on final labour service $$L^s_{j,t}$$. The government levies a tax on earnings at rate $$\tau_s$$, and negative dividend payments are a theoretical possibility.

The representative intermediate good firm utilizes capital $$K^s_{j,t}$$ at rate $$u^s_{j,t}$$ and rents final labour service $$L^s_{j,t}$$ given labour augmenting technology coefficient $$A^s_s$$ to produce differentiated intermediate output good $$Y^s_{j,t}$$ according to constant elasticity of substitution production function

$$\mathcal{F}(u^s_{j,t}, K^s_{j,t}, A^s_s L^s_{j,t}) = \left[ (\varphi)^{\frac{\vartheta - 1}{\vartheta}} (u^s_{j,t} K^s_{j,t})^{\vartheta - 1} + (1 - \varphi)^{\frac{1}{\vartheta}} (A^s_s L^s_{j,t})^{\vartheta - 1} \right]^\frac{1}{\vartheta},$$  \hspace{1cm} (33)

where $$0 < \varphi < 1$$, $$\vartheta > 0$$ and $$A^s_s > 0$$. This constant elasticity of substitution production function exhibits constant returns to scale, and nests the production function proposed by Cobb and Douglas (1928) under constant returns to scale for $$\vartheta = 1$$.\(^1\)

In utilizing capital to produce output, the representative intermediate good firm incurs a cost $$\mathcal{G}(u^s_{j,t}, K^s_{j,t})$$ denominated in terms of output:

$$Y^s_{j,t} = \mathcal{F}(u^s_{j,t}, K^s_{j,t}, A^s_s L^s_{j,t}) - \mathcal{G}(u^s_{j,t}, K^s_{j,t}).$$  \hspace{1cm} (34)

Following Christiano, Eichenbaum and Evans (2005), this capital utilization cost is increasing in the rate of capital utilization at an increasing rate,

\(^1\) Invoking L'Hospital's rule yields $$\lim_{t \to \infty} \ln \mathcal{F}(u^s_{j,t}, K^s_{j,t}, A^s_s L^s_{j,t}) = \varphi \ln(u^s_{j,t} K^s_{j,t}) + (1 - \varphi) \ln(A^s_s L^s_{j,t}) - \varphi \ln \varphi - (1 - \varphi) \ln(1 - \varphi)$$, which implies that $$\lim_{t \to \infty} \mathcal{F}(u^s_{j,t}, K^s_{j,t}, A^s_s L^s_{j,t}) = \varphi^\varphi ((1 - \varphi)^{-1})^\varphi (u^s_{j,t} K^s_{j,t})^\varphi (A^s_s L^s_{j,t})^{1 - \varphi}.$$
where \( \mu > 0 \) and \( \kappa > 0 \). In deterministic steady state equilibrium, the rate of capital utilization is normalized to one, and the cost of utilizing capital equals zero.

Capital is endogenous but not firm-specific, and the representative intermediate good firm enters period \( s \) with access to previously accumulated capital stock \( K_s \), which subsequently evolves according to accumulation function

\[
K_{s+1} = (1 - \delta^K) K_s + \mathcal{H}^K(I_s^K, I_{s+1}^K),
\]

where depreciation rate parameter \( \delta^K \) satisfies \( 0 \leq \delta^K \leq 1 \). Following Christiano, Eichenbaum and Evans (2005), effective capital investment function \( \mathcal{H}^K(I_s^K, I_{s+1}^K) \) incorporates convex adjustment costs,

\[
\mathcal{H}^K(I_s^K, I_{s+1}^K) = \nu_s^K \left[ 1 - \frac{\chi^K}{2} \left( \frac{I_s^K - I_{s+1}^K}{I_{s+1}^K} \right)^2 \right] I_s^K,
\]

where \( \chi^K > 0 \) and \( \nu_s^K > 0 \). In deterministic steady state equilibrium, these adjustment costs equal zero, and effective investment equals actual investment.

In period \( s \), the representative intermediate good firm chooses state contingent sequences for employment \( \{L_{j,s}\}_{s=1}^\infty \), capital utilization \( \{u_{j,s}\}_{s=1}^\infty \), investment in capital \( \{I_s^K\}_{s=1}^\infty \), and the capital stock \( \{K_{s+1}\}_{s=1}^\infty \) to maximize pre-dividend stock market value (31) subject to net production function (34), capital accumulation function (36), and terminal nonnegativity constraint \( K_{T+1} \geq 0 \) for \( T \to \infty \). In equilibrium, demand for the final labour service satisfies necessary first order condition

\[
\mathcal{F}_{AL}(u_{j,s}, A_j, L_{j,s}) \Phi_{j,s} = (1 - \tau_s) \frac{W_s}{P_t^r A_t},
\]

where \( P_t^r \Phi_{j,s} \) denotes the Lagrange multiplier associated with the period \( s \) production technology constraint. This necessary first order condition equates real marginal cost \( \Phi_{j,s} \) to the ratio of the after tax real wage to the marginal product of labour. In equilibrium, the rate of capital utilization satisfies necessary first order condition

\[
\mathcal{F}_{uk}(u_{j,s}, K_t, A_j, L_{j,s}) = \frac{G_{uk}(u_{j,s}, K_t)}{K_t},
\]
which equates the marginal product of utilized capital to its marginal cost. In equilibrium, demand for the final capital investment good satisfies necessary first order condition

\[ Q^K_t \mathcal{H}_K^K (I^{K*}, I^{K}) + E_t \frac{\beta \lambda_t}{\lambda_t} Q^K_{t+1} \mathcal{H}_K^K (I^{K*}, I^{K}) = P_t^F, \]  

(40)

which equates the expected present discounted value of an additional unit of investment in capital to its price, where \( Q^K_s \) denotes the Lagrange multiplier associated with the period \( s \) capital accumulation function. In equilibrium, this shadow price of capital satisfies necessary first order condition

\[ Q^K_t = E_t \frac{\beta \lambda_t}{\lambda_t} \left\{ P^F t \Phi (u_{t+1}, L_{t+1}, A_{t+1}) - G_k (u_{t+1}, K_{t+1}) \right\} + (1 - \delta^K)Q^K_{t+1}, \]  

(41)

which equates it to the expected present discounted value of the sum of the future marginal cost of capital, and the future shadow price of capital net of depreciation. In equilibrium, the necessary complementary slackness condition associated with the terminal nonnegativity constraint may be stated as:

\[ \lim_{t \to \infty} \frac{\beta^T \lambda_{t+1}}{\lambda_t} Q^K_{t+1} K_{t+1} = 0. \]  

(42)

Provided that the pre-dividend stock market value of the representative intermediate good firm is bounded and strictly concave, together with all necessary first order conditions, this transversality condition is sufficient for the unique value maximizing state contingent intertemporal firm allocation.

3.2.2.2. Output Supply and Price Setting Behaviour

There exist a large number of perfectly competitive firms which combine differentiated intermediate output goods \( Y_{j,t} \) supplied by intermediate good firms in a monopolistically competitive output market to produce final output good \( Y_t \) according to constant elasticity of substitution production function

\[ Y_t = \left[ \int_{j=0}^{1} (Y_{j,t}) \frac{dY_{j,t}}{dY_{j,t}} \right]^{\theta F / \theta F - 1}, \]  

(43)
where $\theta^Y > 1$. The representative final output good firm maximizes profits derived from production of the final output good

$$\Pi^Y_t = P^Y_t Y_t - \int_{j=0}^1 P^Y_{j,t} Y_{j,t} dj,$$

(44)

with respect to inputs of intermediate output goods, subject to production function (43). The necessary first order conditions associated with this profit maximization problem yield intermediate output good demand functions:

$$Y_{j,t} = \left( \frac{P^Y_{j,t}}{P^Y_t} \right)^{-\theta^Y} Y_t.$$  

(45)

Since the production function exhibits constant returns to scale, in competitive equilibrium the representative final output good firm earns zero profit, implying aggregate output price index:

$$P^Y_t = \left[ \int_{j=0}^1 (P^Y_{j,t})^{1-\theta^Y} dj \right]^{-\frac{1}{1-\theta^Y}}.$$  

(46)

As the price elasticity of demand for intermediate output goods $\theta^Y$ increases, they become closer substitutes, and individual intermediate good firms have less market power.

In an extension of the model of nominal output price rigidity proposed by Calvo (1983) along the lines of Smets and Wouters (2003, 2005), each period a randomly selected fraction $1 - \omega^Y$ of intermediate good firms adjust their price optimally. The remaining fraction $\omega^Y$ of intermediate good firms adjust their price to account for past output price inflation according to partial indexation rule

$$P^Y_{j,t} = \left( \frac{P^Y_{j,t-1}}{P^Y_{t-2}} \right)^{\gamma^Y} \left( \frac{P^Y_{t-1}}{P^Y_{t-2}} \right)^{1-\gamma^Y} P^Y_{j,t-1},$$

(47)

where $0 \leq \gamma^Y \leq 1$. Under this specification, optimal price adjustment opportunities arrive randomly, and the interval between optimal price adjustments is a random variable.

If the representative intermediate good firm can adjust its price optimally in period $t$, then it does so to maximize pre-dividend stock market value (31) subject to net production function (34), capital accumulation function (36), intermediate output good demand function (45), and the assumed form of nominal output price rigidity. Since all intermediate good firms
that adjust their price optimally in period $t$ solve an identical value maximization problem, in equilibrium they all choose a common price $P_{j,t}^{*,*}$ given by necessary first order condition:

$$P_{j,t}^{*,*} = \frac{\mathbb{E}_{s \in S} \left( \omega^Y \right)^{s-t} \beta^Y \lambda_s \theta^Y_s \Phi_{j,s} \left[ \left( \frac{P_{s,1}^Y}{P_{s,0}^Y} \right)^{Y} \left( \frac{P_{s,1}^Y}{P_{s,0}^Y} \right)^{-Y} \frac{P_{s,1}^Y}{P_{s,0}^Y} \right]^{p^Y_s} \left( \frac{P_{s,1}^Y}{P_{s,0}^Y} \right)^{-d^Y_s}}{\mathbb{E}_{s \in S} \left( \omega^Y \right)^{s-t} \beta^Y \lambda_s \theta^Y_s \Phi_{j,s} \left( \omega^Y - 1 \right)(1 - r_i) \left( \frac{P_{s,1}^Y}{P_{s,0}^Y} \right)^{Y} \left( \frac{P_{s,1}^Y}{P_{s,0}^Y} \right)^{-Y} \frac{P_{s,1}^Y}{P_{s,0}^Y} \right]^{p^Y_s} \left( \frac{P_{s,1}^Y}{P_{s,0}^Y} \right)^{-d^Y_s} P_{s,Y_s}} \tag{48}$$

This necessary first order condition equates the expected present discounted value of the after tax revenue benefit generated by an additional unit of output supply to the expected present discounted value of its production cost. Aggregate output price index (46) equals an average of the price set by the fraction $1 - \omega^Y$ of intermediate good firms that adjust their price optimally in period $t$, and the average of the prices set by the remaining fraction $\omega^Y$ of intermediate good firms that adjust their price according to partial indexation rule (47):

$$P_t^Y = \left[ (1 - \omega^Y) \left( \frac{P_{t-1}^Y}{P_{t-1}^Y} \right)^{-d^Y} + \omega^Y \left[ \left( \frac{P_{t-1}^Y}{P_{t-1}^Y} \right)^{Y} \left( \frac{P_{t-1}^Y}{P_{t-1}^Y} \right)^{-Y} \frac{P_{t-1}^Y}{P_{t-1}^Y} \right]^{p^Y} \right]^{1 - d^Y}$$

Since those intermediate good firms able to adjust their price optimally in period $t$ are selected randomly from among all intermediate good firms, the average price set by the remaining intermediate good firms equals the value of the aggregate output price index that prevailed during period $t-1$, rescaled to account for past output price inflation.

If all intermediate good firms were able to adjust their price optimally every period, then $\omega^Y = 0$ and necessary first order condition (48) would reduce to

$$(1 - \bar{d}_t) \bar{P}_t^Y = \frac{\theta^Y_t}{\theta^Y_{t-1}} \bar{P}_t^Y \bar{\Phi}_t,$$

where $\bar{P}_t^Y = \bar{P}_t^Y$. In flexible price and wage equilibrium, each intermediate good firm sets its after tax price equal to a time varying markup over nominal marginal cost, and output supply is inefficiently low.
3.2.3. The Value Maximization Problem of the Representative Importer

There exists a continuum of intermediate good importers indexed by $k \in [0, 1]$. Intermediate good importers supply differentiated intermediate import goods, but are otherwise identical. Entry into and exit from the monopolistically competitive intermediate import good sector is prohibited.

3.2.3.1. The Real Exchange Rate and the Terms of Trade

The representative intermediate good importer sells shares $\{x_{i(k,s)}\}_{i=0}^{1}$ to domestic households at price $V_{k,s}^M$. Recursive forward substitution for $V_{k,s+1}^M$ with $s > 0$ in necessary first order condition (12) applying the law of iterated expectations reveals that the post-dividend stock market value of the representative intermediate good importer equals the expected present discounted value of future dividend payments:

$$V_{k,s}^M = \mathbb{E}_t \sum_{s=1}^{\infty} \frac{B^{s-t}}{\lambda_s} \Pi_{k,s}^M.$$  \hspace{1cm} (51)

Acting in the interests of its shareholders, the representative intermediate good importer maximizes its pre-dividend stock market value, equal to the expected present discounted value of current and future dividend payments:

$$\Pi_{k,s}^M + V_{k,s}^M = \mathbb{E}_t \sum_{s=1}^{\infty} \frac{B^{s-t}}{\lambda_s} \Pi_{k,s}^M.$$  \hspace{1cm} (52)

The derivation of result (51) imposes transversality condition (17), which rules out self-fulfilling speculative asset price bubbles.

Shares entitle households to dividend payments equal to gross profits $\Pi_{k,s}^M$, defined as earnings less fixed costs:

$$\Pi_{k,s}^M = P_{k,s}^M M_{k,s} - E_s P_{k,s}^f M_{k,s} - \Gamma_s.$$  \hspace{1cm} (53)

Earnings are defined as revenues derived from sales of differentiated intermediate import good $M_{k,s}$ at price $P_{k,s}^M$ less expenditures on foreign final output good $M_{k,s}$. The representative intermediate good importer purchases the foreign final output good at domestic currency price $E_s P_{k,s}^f$ and differentiates it, generating zero gross profits on average.
The law of one price asserts that arbitrage transactions equalize the domestic currency prices of domestic imports and foreign exports. Define the real exchange rate,

\[ Q_s = \frac{\mathcal{E} p_x^f}{p_x}, \tag{54} \]

which measures the price of foreign output in terms of domestic output. Also define the terms of trade,

\[ T_s = \frac{p_M}{p_x}, \tag{55} \]

which measures the price of imports in terms of exports. Violation of the law of one price drives a wedge \( \Psi_s = \mathcal{E} p_x^f / p_M \) between the real exchange rate and the terms of trade,

\[ Q_s = \Psi_s T_s, \tag{56} \]

where the domestic currency price of exports satisfies \( p_x^f = p_x^f \). Under the law of one price \( \Psi_s = 1 \), and the real exchange rate and terms of trade coincide.

There exist a large number of perfectly competitive firms which combine a domestic intermediate good \( Z_{h,i} \in \{C_{h,i}, I_{h,i}^H, I_{h,i}^E, G_{h,i}\} \) and a foreign intermediate good \( Z_{f,i} \in \{C_{f,i}, I_{f,i}^H, I_{f,i}^E, G_{f,i}\} \) to produce final good \( Z_t \in \{C_t, I_t^H, I_t^E, G_t\} \) according to constant elasticity of substitution production function

\[ Z_t = \left( \frac{1}{\phi^2} (Z_{h,i})^\psi + (1 - \phi^2) (Z_{f,i})^\psi \right)^{\frac{\psi}{\psi - 1}}, \tag{57} \]

where \( 0 < \phi^2 < 1 \), \( \psi > 1 \) and \( \nu_t^M > 0 \). The representative final good firm maximizes profits derived from production of the final good

\[ \Pi_t^Z = P_t^c Z_t - P_t^i Z_{h,i} - P_t^m Z_{f,i}, \tag{58} \]

with respect to inputs of domestic and foreign intermediate goods, subject to production function (57). The necessary first order conditions associated with this profit maximization problem imply intermediate good demand functions:

\[ Z_{h,i} = \phi^2 \left( \frac{P_t^i}{P_t^c} \right)^\psi Z_t, \tag{59} \]
Since the production function exhibits constant returns to scale, in competitive equilibrium the representative final good firm earns zero profit, implying aggregate price index:

\[ P_i^Z = \left[ \phi^Z \left( P_i^V \right)^{1-\psi} + \left( 1 - \phi^Z \right)^{1-\psi} \left( \frac{P_i^M}{V_i^M} \right)^{1-\psi} \right]^{\frac{1}{1-\psi}}. \] (61)

Combination of this aggregate price index with intermediate good demand functions (59) and (60) yields:

\[ Z_{h,i} = \phi^Z \left[ \phi^Z + \left( 1 - \phi^Z \right)^{1-\psi} \left( \frac{T_i^M}{V_i^M} \right)^{1-\psi} \right]^{\frac{1}{1-\psi}} Z_i, \] (62)

\[ Z_{f,i} = (1 - \phi^Z) \left[ (1 - \phi^Z) + \phi^Z \left( \frac{T_i^M}{V_i^M} \right)^{1-\psi} \right]^{\frac{1}{1-\psi}} Z_i. \] (63)

These demand functions for domestic and foreign intermediate goods are directly proportional to final good demand, with a proportionality coefficient that varies with the terms of trade.

3.2.3.2. Import Supply and Price Setting Behaviour

There exist a large number of perfectly competitive firms which combine differentiated intermediate import goods \( M_{k,i} \) supplied by intermediate good importers in a monopolistically competitive import market to produce final import good \( M_i \) according to constant elasticity of substitution production function

\[ M_i = \left[ \frac{1}{\theta_i^{M-1}} \int_{k=0}^{\theta_i^M} (M_{k,i}) \overline{\theta_i^M}^M dk \right] \overline{\theta_i^M}^{M-1}, \] (64)

where \( \theta_i^M > 1 \). The representative final import good firm maximizes profits derived from production of the final import good.
\[ \Pi^M_t = P^M_t M_t - \sum_{k=0}^{1} P^M_{k,t} M_{k,t} \text{dk}, \]  

(65)

with respect to inputs of intermediate import goods, subject to production function (64). The necessary first order conditions associated with this profit maximization problem yield intermediate import good demand functions:

\[ M_{k,t} = \left( \frac{P^M_{k,t}}{P^M_t} \right)^{-\theta^M} M_t. \]  

(66)

Since the production function exhibits constant returns to scale, in competitive equilibrium the representative final import good firm earns zero profit, implying aggregate import price index:

\[ P^M_t = \left[ \sum_{k=0}^{1} (P^M_{k,t})^{-\theta^M} \text{dk} \right]^{1/(1-\theta^M)}. \]  

(67)

As the price elasticity of demand for intermediate import goods \( \theta^M \) increases, they become closer substitutes, and individual intermediate good importers have less market power.

In an extension of the model of nominal import price rigidity proposed by Monacelli (2005) along the lines of Smets and Wouters (2003, 2005), each period a randomly selected fraction \( 1 - \omega^M \) of intermediate good importers adjust their price optimally. The remaining fraction \( \omega^M \) of intermediate good importers adjust their price to account for past import price inflation according to partial indexation rule

\[ P^M_{k,t} = \left( \frac{P^M_{t-1}}{P^M_{t-2}} \right)^{\gamma^M} \left( \frac{P^M_{t-1}}{P^M_{t-2}} \right)^{1-\gamma^M} P^M_{k,t-1}, \]  

(68)

where \( 0 \leq \gamma^M \leq 1 \). Under this specification, the probability that an intermediate good importer has adjusted its price optimally is time dependent but state independent.

If the representative intermediate good importer can adjust its price optimally in period \( t \), then it does so to maximize to maximize pre-dividend stock market value (52) subject to intermediate import good demand function (66), and the assumed form of nominal import price rigidity. Since all intermediate good importers that adjust their price optimally in period \( t \) solve an identical value maximization problem, in equilibrium they all choose a common price \( P^M_t^* \) given by necessary first order condition.
where \( \Psi_s = \frac{E_s P_r^{s, f}}{P_s^M} \) measures real marginal cost. This necessary first order condition equates the expected present discounted value of the revenue benefit generated by an additional unit of import supply to the expected present discounted value of its production cost. Aggregate import price index (67) equals an average of the price set by the fraction \( 1 - \omega^M \) of intermediate good importers that adjust their price optimally in period \( t \), and the average of the prices set by the remaining fraction \( \omega^M \) of intermediate good importers that adjust their price according to partial indexation rule (59):

\[
P_t^M = \left\{ (1 - \omega^M)(P_t^{M,*})^{-\phi^M} + \omega^M \left[ \left( \frac{P_t^{M,*}}{P_t^{s,-1}} \right)^{1-\phi^M} \left( \frac{P_t^{M,*}}{P_t^{s,-1}} \right)^{1-\phi^M} \right]^{1-\phi^M} \right\}^{-\phi^M}. \tag{70}
\]

Since those intermediate good importers able to adjust their price optimally in period \( t \) are selected randomly from among all intermediate good importers, the average price set by the remaining intermediate good importers equals the value of the aggregate import price index that prevailed during period \( t - 1 \), rescaled to account for past import price inflation.

If all intermediate good importers were able to adjust their price optimally every period, then \( \omega^M = 0 \) and necessary first order condition (69) would reduce to

\[
\tilde{P}_t^{M,*} = \frac{\theta_t^M}{\theta_t^M - 1} \tilde{P}_t^{M} \tilde{p}_t^r, \tag{71}
\]

where \( \tilde{P}_t^{M,*} = \tilde{P}_t^{M} \). In flexible price and wage equilibrium, each intermediate good importer sets its price equal to a time varying markup over nominal marginal cost, and import supply is inefficiently low.
3.2.4. Monetary and Fiscal Policy

The government consists of a monetary authority and a fiscal authority. The monetary authority implements monetary policy, while the fiscal authority implements fiscal policy.

3.2.4.1. The Monetary Authority

The monetary authority implements monetary policy through control of the short term nominal interest rate according to monetary policy rule

\[ i_t - i^* = \xi^s (\pi_t^c - \bar{\pi}_t^c) + \xi^r (\ln Y_t - \ln \bar{Y}_t) + \nu_t^r, \]  

(72)

where \( \xi^s > 1 \) and \( \xi^r > 0 \). As specified, the deviation of the nominal interest rate from its flexible price and wage equilibrium value is a linear increasing function of the contemporaneous deviation of consumption price inflation from its target value \( \pi_t^c = \bar{\pi}_t^c \), and the contemporaneous proportional deviation of output from its flexible price and wage equilibrium value. Persistent departures from this monetary policy rule are captured by serially correlated monetary policy shock \( \nu_t^r \).

3.2.4.2. The Fiscal Authority

The fiscal authority implements fiscal policy through control of nominal government consumption and the tax rate applicable to the pooled labour income of households and the earnings of intermediate good firms. In equilibrium, this distortionary tax collection framework corresponds to proportional output taxation.

The ratio of nominal government consumption to nominal output satisfies fiscal expenditure rule

\[ \ln \frac{P_t^G G_t}{P_t^Y Y_t} - \ln \frac{\bar{P}_t^G G_t}{\bar{P}_t^Y Y_t} = \zeta^G \left[ \ln \left( \frac{B^G_{st}}{P^G_{st} Y_t} \right) - \ln \left( \frac{\bar{B}^G_{st}}{\bar{P}^G_{st} \bar{Y}_t} \right) \right] + \nu_t^G, \]  

(73)

where \( \zeta^G < 0 \). As specified, the proportional deviation of the ratio of nominal government consumption to nominal output from its deterministic steady state equilibrium value is a linear decreasing function of the contemporaneous proportional deviation of the ratio of net foreign debt to nominal output from its target value. This fiscal expenditure rule is well defined only if
the net foreign debt is positive. Persistent departures from this fiscal expenditure rule are captured by serially correlated fiscal expenditure shock $v_f^e$.

The tax rate applicable to the pooled labour income of households and the earnings of intermediate good firms satisfies fiscal revenue rule

$$\ln r - \ln \bar{r} = \zeta^r \left[ \ln \left( \frac{B_{t+1}^G}{P^f Y_t} \right) - \ln \left( \frac{\bar{B}_{t+1}^G}{P^f \bar{Y}_t} \right) \right] + v_r^r,$$

where $\zeta^r > 0$. As specified, the proportional deviation of the tax rate from its deterministic steady state equilibrium value is a linear increasing function of the contemporaneous proportional deviation of the ratio of net government debt to nominal output from its target value. This fiscal revenue rule is well defined only if the net government debt is positive. Persistent departures from this fiscal revenue rule are captured by serially correlated fiscal revenue shock $v_r^r$.

The fiscal authority enters period $t$ holding previously purchased domestic currency denominated bonds $B_{t+1}^{G,h}$ which yield interest at risk free rate $i_{t-1}$, and foreign currency denominated bonds $B_{t+1}^{G,f}$ which yield interest at risk free rate $i_{t-1}^f$. It also levies taxes on the pooled labour income of households and the earnings of intermediate good firms at rate $r_t$. These sources of public wealth are summed in government dynamic budget constraint:

$$B_{t+1}^{G,h} + \varepsilon_e B_{t+1}^{G,f} = (1 + i_{t-1})B_t^{G,h} + \varepsilon_e (1 + i_{t-1}^f)B_t^{G,f} + \tau_t \int_{j=0}^{1} \int_{l=0}^{1} W_{ij} L_{ij} dldi + \tau_t \int_{j=0}^{1} (P_{j}^f Y_{jt} - W_{ij} L_{ij}) dj - P^G_t G_t.$$  \hfill (75)

According to this dynamic budget constraint, at the end of period $t$, the fiscal authority purchases domestic bonds $B_{t+1}^{G,h}$, and foreign bonds $B_{t+1}^{G,f}$ at price $\varepsilon_e$. It also purchases final government consumption good $G_t$ at price $P^G_t$.

### 3.2.5. Market Clearing Conditions

A rational expectations equilibrium in this DSGE model of a small open economy consists of state contingent intertemporal allocations for domestic and foreign households and firms which solve their constrained optimization problems given prices and policy, together with state contingent intertemporal allocations for domestic and foreign governments which satisfy their policy rules and constraints given prices, with supporting prices such that all markets clear.
Since the domestic economy is of negligible size relative to the foreign economy, in equilibrium
\[ P^{r,f} = P^{c,f} = P^{x,f} = P^{g,f} = P^{k,f} \] and \[ X_t^f = M_t^f = B_{t+1}^p = B_{t+1}^g = 0. \]

Clearing of the final output good market requires that exports \( X_t \) equal production of the domestic final output good less the cumulative demands of domestic households, firms, and the government,
\[ X_t = Y_t - C_{t,h} - I_{t,h} - I_{t,k} - G_{t,h}, \] (76)
where \( X_t = M_t^f \). Clearing of the final import good market requires that imports \( M_t \) satisfy the cumulative demands of domestic households, firms, and the government for the foreign final output good,
\[ M_t = C_{t,f} + I_{t,f} + I_{t,k} + G_{t,f}, \] (77)
where \( M_t = X_t^f \). In equilibrium, combination of these final output and import good market clearing conditions yields aggregate resource constraint:
\[ P^r Y_t = P^c C_t + P^{m} I_{t}^{m} + P^{x} I_{t}^{x} + P^{k} G_t + P^i X_t - P^m M_t. \] (78)
The trade balance equals export revenues less import expenditures, or equivalently nominal output less domestic demand.

Let \( B_{t+1} \) denote the net foreign asset position of the economy, which in equilibrium equals the sum of the domestic currency values of private sector bond holdings \( B_t^p = B_{t+1}^p + \mathcal{E}_t B_{t+1}^{p,f} \) and public sector bond holdings \( B_t^G = B_{t+1}^G + \mathcal{E}_t B_{t+1}^{G,f} \), since domestic bond holdings cancel out when the private and public sectors are consolidated:
\[ B_{t+1} = B_{t+1}^p + B_{t+1}^G. \] (79)
The imposition of equilibrium conditions on household dynamic budget constraint (3) reveals that the expected present discounted value of the net increase in private sector asset holdings equals the expected present discounted value of private saving less domestic investment:
\[ E_{t-1} \frac{\beta_t}{\lambda_{t-1}} (B_{t+1}^p - B_t^p) = E_{t-1} \frac{\beta_t}{\lambda_{t-1}} \left[ i_{t-1} B_t^p + (1 - \tau_t) P^r Y_t - P^c C_t - P^{m} I_{t}^{m} - P^{x} I_{t}^{x} \right] . \] (80)
The imposition of equilibrium conditions on government dynamic budget constraint (75) reveals that the expected present discounted value of the net increase in public sector asset holdings equals the expected present discounted value of public saving:
Combination of these household and government dynamic budget constraints with aggregate resource constraint (78) reveals that the expected present discounted value of the net increase in foreign asset holdings equals the expected present discounted value of the sum of net international investment income and the trade balance, or equivalently the expected present discounted value of national saving less domestic investment:

\[ E_{t-1} \frac{\beta \lambda}{\lambda_{t-1}^r} (B_{t+1}^G - B_t^G) = E_{t-1} \frac{\beta \lambda}{\lambda_{t-1}^r} (i_{t-1} B_t^g + \tau_t p'_t Y_t - p_t^G G_t). \] 

(81)

In equilibrium, the current account balance is determined by both intratemporal and intertemporal optimization.

3.2.6. The Approximate Linear Model

Estimation, inference and forecasting are based on a linear state space representation of an approximate unobserved components representation of this DSGE model of a small open economy. Cyclical components are modeled by linearizing equilibrium conditions around a stationary deterministic steady state equilibrium which abstracts from long run balanced growth, while trend components are modeled as random walks while ensuring the existence of a well defined balanced growth path.

In what follows, \( E_t x_{t+s} \) denotes the rational expectation of variable \( x_{t+s} \), conditional on information available at time \( t \). Also, \( \hat{x}_t \) denotes the cyclical component of variable \( x_t \), \( \tilde{x}_t \) denotes the flexible price and wage equilibrium component of variable \( x_t \), and \( \overline{x}_t \) denotes the trend component of variable \( x_t \). Cyclical and trend components are additively separable, which implies that \( x_t = \hat{x}_t + \tilde{x}_t \) and \( \overline{x}_t = \hat{x}_t + \overline{x}_t \), where \( \overline{x}_t = \overline{x}_t \).

3.2.6.1. Cyclical Components

The cyclical component of output price inflation depends on a linear combination of past and expected future cyclical components of output price inflation driven by the contemporaneous cyclical components of real marginal cost and the tax rate according to output price Phillips curve
$$\hat{\pi}_t^y = \frac{\gamma^y}{1 + \gamma^y \beta} \hat{\pi}^y_{t-1} + \frac{\beta}{1 + \gamma^y \beta} E_t \hat{\pi}_t^y + \frac{(1 - \omega^y)(1 - \omega^y \beta)}{\omega^y (1 + \gamma^y \beta)} \left[ \ln \dot{\phi}_t + \frac{\tau}{1 - \tau} \ln \hat{\pi}_t - \frac{1}{\theta^y - 1} \ln \hat{\theta}_t^y \right], \quad (83)$$

where $\Phi = (1 - \tau)\frac{\theta^y - 1}{\theta^y - 1 - \theta^y (1 - \delta)}$. The persistence of the cyclical component of output price inflation is increasing in indexation parameter $\gamma^y$, while the sensitivity of the cyclical component of output price inflation to changes in the cyclical components of real marginal cost and the tax rate is decreasing in nominal rigidity parameter $\omega^y$ and indexation parameter $\gamma^y$. This output price Phillips curve is subject to output price markup shocks.

The cyclical component of output depends on the contemporaneous cyclical components of utilized capital and effective labour according to approximate linear net production function

$$\ln \hat{Y}_t = \left( 1 - \frac{\theta^y}{\theta^y - 1} \frac{WL}{PY} \right) \ln (\hat{u}_t, \hat{K}_t) + \frac{\theta^y}{\theta^y - 1} \frac{WL}{PY} \ln (\hat{A}_t, \hat{L}_t), \quad (84)$$

where $\hat{K}_t = \frac{\beta (1 - \tau)}{1 - \beta (1 - \delta)} \left( \frac{\theta^y - 1}{\theta^y - \frac{WL}{PY}} \right)$. This approximate linear net production function is subject to output technology shocks.

The cyclical component of the rate of capital utilization depends on the contemporaneous cyclical component of the ratio of capital to effective labour according to approximate linear implicit capital utilization function:

$$\ln \hat{u}_t = -\frac{\theta^y}{\theta^y - 1} \frac{WL}{PY} \left( \kappa + \frac{\theta^y}{\theta^y - 1} \frac{WL}{PY} \right)^{-1} \ln \hat{\frac{K}{L}_t}. \quad (85)$$

The sensitivity of the cyclical component of the rate of capital utilization to changes in the cyclical component of the ratio of capital to effective labour is decreasing in capital utilization cost parameter $\kappa$ and elasticity of substitution parameter $\theta$. This approximate linear implicit capital utilization function is subject to output technology shocks.

The cyclical component of consumption, housing investment, capital investment or government consumption price inflation depends on a linear combination of past and expected future cyclical components of consumption, housing investment, capital investment or government consumption price inflation driven by the contemporaneous cyclical components of real marginal cost and the tax rate according to Phillips curves:

$$\hat{\pi}_t^y = \frac{\gamma^y}{1 + \gamma^y \beta} \hat{\pi}^y_{t-1} + \frac{\beta}{1 + \gamma^y \beta} E_t \hat{\pi}_t^y + \frac{(1 - \omega^y)(1 - \omega^y \beta)}{\omega^y (1 + \gamma^y \beta)} \left[ \ln \dot{\phi}_t + \frac{\tau}{1 - \tau} \ln \hat{\pi}_t - \frac{1}{\theta^y - 1} \ln \hat{\theta}_t^y \right]
- \frac{\gamma^y (1 - \phi^y)}{1 + \gamma^y \beta} \Delta \ln \frac{\hat{\pi}_t^y}{\Pi_{t-1}^y} + (1 - \phi^y) \Delta \ln \frac{\hat{\pi}_t^y}{\Pi_{t-1}^y} \frac{\beta}{1 + \gamma^y \beta} E_t \Delta \ln \frac{\hat{\pi}_t^y}{\Pi_{t-1}^y}. \quad (86)$$
Reflecting the entry of the price of imports into the aggregate consumption, housing investment, capital investment or government consumption price index, the cyclical component of consumption, housing investment, capital investment or government consumption price inflation also depends on past, contemporaneous, and expected future proportional changes in the cyclical component of the terms of trade. These Phillips curves are subject to output price markup and import technology shocks.

The cyclical component of consumption depends on a linear combination of past and expected future cyclical components of consumption driven by the contemporaneous cyclical component of the consumption based real interest rate according to approximate linear consumption Euler equation:

\[ \ln \hat{C}_t = \frac{\alpha^c}{1 + \alpha^c} \ln \hat{C}_{t-1} + \frac{1}{1 + \alpha^c} E_t \ln \hat{C}_{t+1} - \sigma \frac{1 - \alpha^c}{1 + \alpha^c} \left[ \hat{r}^c_t + E_t \ln \hat{\nu}^c_t \right]. \]  

(87)

The persistence of the cyclical component of consumption is increasing in habit persistence parameter \( \alpha^c \), while the sensitivity of the cyclical component of consumption to changes in the cyclical component of the consumption based real interest rate is increasing in intertemporal elasticity of substitution parameter \( \sigma \) and decreasing in habit persistence parameter \( \alpha^c \). This approximate linear consumption Euler equation is subject to preference shocks.

The cyclical component of investment in housing depends on a linear combination of past and expected future cyclical components of investment in housing driven by the contemporaneous cyclical component of the relative shadow price of housing according to approximate linear housing investment demand function:

\[ \ln \hat{i}_t^H = \frac{1}{1 + \beta} \ln \hat{i}_{t-1}^H + \frac{\beta}{1 + \beta} E_t \ln \hat{i}_{t+1}^H + \frac{1}{\chi^H (1 + \beta)} \ln \left( \frac{\hat{Q}_t^H \hat{Q}_t^H}{\hat{\rho}_t^H} \right). \]  

(88)

The sensitivity of the cyclical component of investment in housing to changes in the cyclical component of the relative shadow price of housing is decreasing in housing investment adjustment cost parameter \( \chi^H \). This approximate linear housing investment demand function is subject to housing investment technology shocks.

The cyclical component of the relative shadow price of housing depends on the expected future cyclical component of the relative shadow price of housing, the contemporaneous cyclical component of the consumption based real interest rate, and the expected future cyclical component of the marginal rate of substitution between housing and consumption according to approximate linear housing investment Euler equation:
\[
\ln \frac{\hat{Q}_t^H}{P_t^C} = \beta (1-\delta^H) \ln \frac{\hat{Q}^H_{t+1}}{P^C_{t+1}} - \frac{1-\beta(1-\delta^H)}{\sigma} \left[ \ln \frac{\hat{H}_{t+1} - \alpha^H \ln \hat{H}_t}{1 - \alpha^H} - \ln \frac{\hat{C}_{t+1} - \alpha^C \ln \hat{C}_t}{1 - \alpha^C} \right].
\]  
(89)

The sensitivity of the cyclical component of the relative shadow price of housing to changes in the cyclical component of the ratio of adjusted housing to adjusted consumption is decreasing in intertemporal elasticity of substitution parameter \(\sigma\).

The cyclical component of the stock of housing depends on the past cyclical component of the stock of housing and the contemporaneous cyclical component of investment in housing according to approximate linear housing accumulation function

\[
\ln \hat{H}_{t+1} = (1-\delta^H) \ln \hat{H}_t + \delta^H \ln (\hat{v}_t^H \hat{i}_t^H),
\]  
(90)

where \(\hat{v}_t^H = \delta^H\). This approximate linear housing accumulation function is subject to housing investment technology shocks.

The cyclical component of investment in capital depends on a linear combination of past and expected future cyclical components of investment in capital driven by the contemporaneous cyclical component of the relative shadow price of capital according to approximate linear capital investment demand function:

\[
\ln \hat{i}_t^K = \frac{1}{1+\beta} \ln \hat{i}_{t-1}^K + \frac{\beta}{1+\beta} \ln \hat{i}_{t+1}^K + \frac{1}{\chi^K (1+\beta)} \ln \left( \hat{v}_t^K \frac{\hat{Q}_t^K}{P_t^K} \right).
\]  
(91)

The sensitivity of the cyclical component of investment in capital to changes in the cyclical component of the relative shadow price of capital is decreasing in capital investment adjustment cost parameter \(\chi^K\). This approximate linear capital investment demand function is subject to capital investment technology shocks.

The cyclical component of the relative shadow price of capital depends on the expected future cyclical component of the relative shadow price of capital, the contemporaneous cyclical component of the output based real interest rate, the expected future cyclical component of real marginal cost, and the expected future cyclical component of the marginal product of capital according to approximate linear capital investment Euler equation:
The sensitivity of the cyclical component of the relative shadow price of capital to changes in the cyclical component of the ratio of utilized capital to effective labor is decreasing in elasticity of substitution parameter $\vartheta$. This approximate linear capital investment Euler equation is subject to output technology shocks.

The cyclical component of the stock of capital depends on the past cyclical component of the stock of capital and the contemporaneous cyclical component of investment in capital according to approximate linear capital accumulation function

$$\ln K_{t+1} = (1 - \delta^K) \ln K_t + \delta^K \ln (\hat{I}_t^* / I_{t+1}^*),$$

where $\hat{I}_t^*/I_{t+1}^* = \delta^K$. This approximate linear capital accumulation function is subject to capital investment technology shocks.

The cyclical component of the ratio of nominal government consumption to nominal output depends on the contemporaneous cyclical component of the ratio of net foreign debt to nominal output according to fiscal expenditure rule:

$$\ln \frac{\hat{P}_t^G \hat{G}_t}{\hat{P}_t^Y \hat{Y}_t} = \zeta^G \ln \left( -\frac{\hat{B}_{t+1}^G}{\hat{P}_t^Y \hat{Y}_t} \right) + \hat{\nu}_t^G.$$  (94)

This fiscal expenditure rule ensures convergence of the level of the ratio of net foreign debt to nominal output to its target value in deterministic steady state equilibrium, and is subject to fiscal expenditure shocks.

The cyclical component of the tax rate depends on the contemporaneous cyclical component of the ratio of net government debt to nominal output according to fiscal revenue rule:

$$\ln \hat{\tau}_t = \zeta^\tau \ln \left( -\frac{\hat{B}_{t+1}^G}{\hat{P}_t^Y \hat{Y}_t} \right) + \hat{\nu}_t^G.$$  (95)

This fiscal revenue rule ensures convergence of the level of the ratio of net government debt to nominal output to its target value in deterministic steady state equilibrium, and is subject to fiscal revenue shocks.

The cyclical component of import price inflation depends on a linear combination of past and expected future cyclical components of import price inflation driven by the contemporaneous
cyclical component of the deviation of the domestic currency price of foreign output from the price of imports according to import price Phillips curve:

$$\hat{\pi}_t^M = \frac{\gamma^M}{1 + \gamma^M} \hat{\pi}_{t-1}^M + \frac{\beta}{1 + \gamma^M} E \hat{\pi}_{t+1}^M + \frac{(1 - \omega^M)(1 - \omega^M \beta)}{\omega^M (1 + \gamma^M \beta)} \left[ \ln \frac{\hat{\xi}^{\pi \pi} - 1}{\hat{\pi}_t^M} - \frac{1}{\theta^M - 1} \ln \hat{\theta}_t^M \right]. \quad (96)$$

The persistence of the cyclical component of import price inflation is increasing in indexation parameter $\gamma^M$, while the sensitivity of the cyclical component of import price inflation to changes in the cyclical component of real marginal cost is decreasing in nominal rigidity parameter $\omega^M$ and indexation parameter $\gamma^M$. This import price Phillips curve is subject to import price markup shocks.

The cyclical component of exports depends on the contemporaneous cyclical components of foreign consumption, housing investment, capital investment, government consumption, and the terms of trade according to approximate linear export demand function

$$\frac{X}{Y} \ln \hat{X}_t = (1 - \phi^C) \frac{C}{Y} \ln \hat{C} + (1 - \phi^H) \frac{I^H}{Y} \ln \hat{I}^H + (1 - \phi^K) \frac{I^K}{Y} \ln \hat{I}^K + (1 - \phi^G) \frac{G}{Y} \ln \hat{G} \left[ \phi^C (1 - \phi^C) \frac{C}{Y} + \phi^H (1 - \phi^H) \frac{I^H}{Y} + \phi^K (1 - \phi^K) \frac{I^K}{Y} + \phi^G (1 - \phi^G) \frac{G}{Y} \right] \ln \hat{\hat{\xi}}^{\xi^e} \quad (97)$$

where $\frac{X}{Y} = 1 - (1 - \phi^C) \frac{C}{Y} - (1 - \phi^H) \frac{I^H}{Y} - (1 - \phi^K) \frac{I^K}{Y} - (1 - \phi^G) \frac{G}{Y}$. The sensitivity of the cyclical component of exports to changes in the cyclical component of the foreign terms of trade is increasing in elasticity of substitution parameter $\psi$. This approximate linear export demand function is subject to foreign import technology shocks.

The cyclical component of imports depends on the contemporaneous cyclical components of consumption, housing investment, capital investment, government consumption, and the terms of trade according to approximate linear import demand function

$$\frac{M}{Y} \ln \hat{M}_t = (1 - \phi^C) \frac{C}{Y} \ln \hat{C} + (1 - \phi^H) \frac{I^H}{Y} \ln \hat{I}^H + (1 - \phi^K) \frac{I^K}{Y} \ln \hat{I}^K + (1 - \phi^G) \frac{G}{Y} \ln \hat{G} \left[ \phi^C (1 - \phi^C) \frac{C}{Y} + \phi^H (1 - \phi^H) \frac{I^H}{Y} + \phi^K (1 - \phi^K) \frac{I^K}{Y} + \phi^G (1 - \phi^G) \frac{G}{Y} \right] \ln \hat{\hat{\xi}}^{\xi^i} \quad (98)$$

where $\frac{M}{Y} = (1 - \phi^C) \frac{C}{Y} + (1 - \phi^H) \frac{I^H}{Y} + (1 - \phi^K) \frac{I^K}{Y} + (1 - \phi^G) \frac{G}{Y}$. The sensitivity of the cyclical component of imports to changes in the cyclical component of the terms of trade is increasing in elasticity of substitution parameter $\psi$. This approximate linear import demand function is subject to import technology shocks.
The cyclical component of the real wage depends on a linear combination of past and expected future cyclical components of the real wage driven by the contemporaneous cyclical component of the deviation of the marginal rate of substitution between leisure and consumption from the after tax real wage according to wage Phillips curve:

\[
\ln \frac{\dot{W}_t}{\hat{P}_t} = \frac{1}{1 + \beta} \ln \frac{\dot{W}_{t-1}}{\hat{P}_{t-1}} + \frac{\beta}{1 + \beta} \ln \frac{\dot{W}_{t-1}}{\hat{P}_{t-1}^e} + \frac{\gamma}{1 + \beta} \dot{\hat{R}}_{t-1} - \frac{1 + \gamma \beta}{1 + \beta} \dot{\hat{R}}_t + \frac{\beta}{1 + \beta} E_t \dot{\hat{R}}_{t+1} \\
+ \frac{(1 - \omega^t)(1 - \omega^t \beta)}{\omega^t (1 + \beta)} \left[ \frac{1}{\eta} \ln \frac{\dot{L}_t - \alpha \dot{L}}{1 - \alpha} + \frac{1}{\sigma} \ln \frac{\dot{C}_t - \alpha \dot{C}}{1 - \alpha} + \frac{\tau}{1 - \tau} \ln \frac{\dot{\hat{Y}}_t - \ln \frac{\dot{W}_t}{\hat{P}_t^e}}{\theta^t - 1} \ln \hat{\delta}_t^t \right].
\]

Reflecting the existence of partial wage indexation, the cyclical component of the real wage also depends on past, contemporaneous, and expected future cyclical components of consumption price inflation. The sensitivity of the cyclical component of the real wage to changes in the cyclical component of consumption price inflation is increasing in indexation parameter \( \gamma^t \), to changes in the cyclical component of the deviation of the marginal rate of substitution between leisure and consumption from the after tax real wage is decreasing in nominal rigidity parameter \( \omega^t \), and to changes in the cyclical component of adjusted employment is decreasing in elasticity of substitution parameter \( \eta \). This wage Phillips curve is subject to wage markup shocks.

The cyclical component of real marginal cost depends on the contemporaneous cyclical component of the deviation of the after tax real wage from the marginal product of labour according to approximate linear implicit labour demand function:

\[
\ln \phi_t = \ln \frac{\dot{W}_t}{\hat{P}_t \hat{A}} - \frac{\tau}{1 - \tau} \ln \hat{\delta}_t - \frac{1}{\theta^t - 1} \ln \frac{\dot{W}_t}{\hat{P}_t \hat{A}} \ln \frac{\dot{K}_t}{\hat{A} \hat{L}_t}.
\]

The sensitivity of the cyclical component of real marginal cost to changes in the cyclical component of the ratio of utilized capital to effective labour is decreasing in elasticity of substitution parameter \( \theta^t \). This approximate linear implicit labour demand function is subject to output technology shocks.

The adjusted cyclical component of the nominal interest rate depends on the contemporaneous adjusted cyclical components of consumption price inflation and output according to monetary policy rule:

\[
\dot{i}_t - \dot{\hat{i}}_t = \xi^\tau (\dot{\hat{R}}_t - \dot{\hat{R}}_t^e) + \xi^\tau (\ln \dot{\hat{Y}}_t - \ln \dot{\hat{Y}}_t) + \dot{\nu}_t.
\]

This monetary policy rule ensures convergence of the level of consumption price inflation to its target value in flexible price and wage equilibrium, and is subject to monetary policy shocks.
The cyclical component of the output based real interest rate satisfies $\hat{r}_t^r = \hat{i}_t - E_t \hat{\pi}_t^{cy}$, while the cyclical component of the consumption based real interest rate satisfies $\hat{r}_t^c = \hat{i}_t - E_t \hat{\pi}_t^{cy}$.

The cyclical component of the nominal exchange rate depends on the expected future cyclical component of the nominal exchange rate and the contemporaneous cyclical component of the nominal interest rate differential according to approximate linear uncovered interest parity condition:

$$\ln \hat{\delta}_t = E_t \ln \hat{\delta}_{t+1} - (\hat{i}_t - \hat{i}_t^r).$$  \hfill (102)

The cyclical component of the real exchange rate satisfies $\ln \hat{\delta}_t = \ln \hat{\delta}_{t+1} + \ln \hat{p}_t^{r, f} - \ln \hat{p}_t^{r, x}$, while the cyclical component of the terms of trade satisfies $\ln \hat{T}_t = \ln \hat{p}_t^{r, M} - \ln \hat{p}_t^{r, X}$, where $\ln \hat{p}_t^{r, X} = \ln \hat{p}_t^{r, Y}$.

The cyclical component of nominal output depends on the contemporaneous cyclical components of nominal consumption, housing investment, capital investment, government consumption, exports, and imports according to approximate linear aggregate resource constraint:

$$\ln(\hat{P}_t^{r, X} \hat{Y}_t) = \frac{C}{Y} \ln(\hat{P}_t^{r, C} \hat{C}_t) + \frac{I_t^{II}}{Y} \ln(\hat{P}_t^{r, C} \hat{I}_t^{II}) + \frac{I_t^{IX}}{Y} \ln(\hat{P}_t^{r, C} \hat{I}_t^{IX}) + \frac{G}{Y} \ln(\hat{P}_t^{r, G} \hat{G}_t)$$

$$+ \frac{X}{Y} \ln(\hat{p}_t^{r, X} \hat{X}_t) - \frac{M}{Y} \ln(\hat{p}_t^{r, M} \hat{M}_t).$$  \hfill (103)

In equilibrium, the cyclical component of output is determined by the cumulative demands of domestic and foreign households, firms, and governments.

The cyclical component of the net government debt depends on the past cyclical component of the net government debt, the past cyclical component of the nominal interest rate, the contemporaneous cyclical component of tax revenues, and the contemporaneous cyclical component of nominal government consumption according to approximate linear government dynamic budget constraint

$$E_t \ln(\hat{B}_t^{G, i}) = \frac{1}{\beta} \left[ \ln(-\hat{B}_t^{G, c}) + \hat{i}_{t-1} \right] + \left( \frac{B_t^{G}}{P_t Y} \right)^{-1} \left[ \tau \ln(\hat{\delta}_t^{r, X} \hat{Y}_t) - \frac{G}{Y} \ln(\hat{p}_t^{r, G} \hat{G}_t) \right].$$  \hfill (104)

where $\frac{B_t^{G}}{P_t Y} = - \frac{\beta}{1 - \beta} \left( \tau \frac{G}{Y} \right)$. This approximate linear government dynamic budget constraint is well defined only if the level of the net government debt is positive.

The cyclical component of the net foreign debt depends on the past cyclical component of the net foreign debt, the past cyclical component of the nominal interest rate, the contemporaneous cyclical component of export revenues, and the contemporaneous cyclical component of import expenditures according to approximate linear national dynamic budget constraint.
where \( \frac{B}{PY} = -\frac{\beta}{1-\beta} \left( \frac{X}{Y} - \frac{M}{Y} \right) \). This approximate linear national dynamic budget constraint is well defined only if the level of the net foreign debt is positive.

Variation in cyclical components is driven by eleven exogenous stochastic processes. The cyclical components of the preference, output technology, housing investment technology, capital investment technology, import technology, output price markup, import price markup, wage markup, monetary policy, fiscal expenditure, and fiscal revenue shocks follow stationary first order autoregressive processes:

\[
\begin{align*}
\ln\hat{\nu}_t^C &= \rho_{\nu}^C \ln \hat{\nu}_{t-1}^C + \epsilon_{\nu t}^C, \epsilon_{\nu t}^C \sim \mathcal{N}(0, \sigma_{\nu C}^2), \\
\ln\hat{A}_t &= \rho_A \ln \hat{A}_{t-1} + \epsilon_A^t, \epsilon_A^t \sim \mathcal{N}(0, \sigma_A^2), \\
\ln\hat{\nu}_t^{Kt} &= \rho_{\nu}^{Kt} \ln \hat{\nu}_{t-1}^{Kt} + \epsilon_{\nu t}^{Kt}, \epsilon_{\nu t}^{Kt} \sim \mathcal{N}(0, \sigma_{\nu K}^2), \\
\ln\hat{\nu}_t^{Mt} &= \rho_{\nu}^{Mt} \ln \hat{\nu}_{t-1}^{Mt} + \epsilon_{\nu t}^{Mt}, \epsilon_{\nu t}^{Mt} \sim \mathcal{N}(0, \sigma_{\nu M}^2), \\
\ln\hat{\theta}_t^{\sigma^\nu} &= \rho_{\sigma^\nu} \ln \hat{\theta}_{t-1}^{\sigma^\nu} + \epsilon_{\sigma^\nu t}, \epsilon_{\sigma^\nu t} \sim \mathcal{N}(0, \sigma_{\sigma^\nu}^2), \\
\ln\hat{\theta}_t^{\sigma^\rho^\nu} &= \rho_{\sigma^\rho^\nu} \ln \hat{\theta}_{t-1}^{\sigma^\rho^\nu} + \epsilon_{\sigma^\rho^\nu t}, \epsilon_{\sigma^\rho^\nu t} \sim \mathcal{N}(0, \sigma_{\sigma^\rho^\nu}^2), \\
\hat{\nu}_t^\nu &= \rho_{\nu} \hat{\nu}_{t-1} + \epsilon_{\nu t}^\nu, \epsilon_{\nu t}^\nu \sim \mathcal{N}(0, \sigma_{\nu}^2), \\
\hat{\nu}_t^\sigma &= \rho_{\nu} \hat{\nu}_{t-1} + \epsilon_{\nu t}^\sigma, \epsilon_{\nu t}^\sigma \sim \mathcal{N}(0, \sigma_{\nu}^2), \\
\hat{\nu}_t^\nu &= \rho_{\nu} \hat{\nu}_{t-1} + \epsilon_{\nu t}^\nu, \epsilon_{\nu t}^\nu \sim \mathcal{N}(0, \sigma_{\nu}^2).
\end{align*}
\]

The innovations driving these exogenous stochastic processes are assumed to be independent, which combined with our distributional assumptions implies multivariate normality. In flexible price and wage equilibrium, \( \omega^V = \omega^M = \omega^L = 0 \) and \( \sigma_{\nu}^2 = 0 \). In deterministic steady state
equilibrium, \( v^c = v' = v^M = 1 \) and \( \sigma_{v^c}^2 = \sigma_{v'}^2 = \sigma_{v^M}^2 = \sigma_{v^c}^2 = \sigma_{v'}^2 = \sigma_{v^M}^2 = \sigma_{v^c}^2 = \sigma_{v'}^2 = \sigma_{v^M}^2 = 0 \).

### 3.2.6.2. Trend Components

The trend components of the prices of output, consumption, housing investment, capital investment, government consumption, and imports follow random walks with time varying drift \( \pi_t \):

\[
\ln \widetilde{P}_t^Y = \pi_t + \ln \widetilde{P}_{t-1}^Y + \epsilon_t^{\pi T} \sim iid \mathcal{N}(0, \sigma_{\pi T}^2),
\]

\[
\ln \widetilde{P}_t^C = \pi_t + \ln \widetilde{P}_{t-1}^C + \epsilon_t^{\pi T} \sim iid \mathcal{N}(0, \sigma_{\pi T}^2);
\]

\[
\ln \widetilde{P}_t^H = \pi_t + \ln \widetilde{P}_{t-1}^H + \epsilon_t^{\pi T} \sim iid \mathcal{N}(0, \sigma_{\pi T}^2),
\]

\[
\ln \widetilde{P}_t^K = \pi_t + \ln \widetilde{P}_{t-1}^K + \epsilon_t^{\pi T} \sim iid \mathcal{N}(0, \sigma_{\pi T}^2),
\]

\[
\ln \widetilde{P}_t^G = \pi_t + \ln \widetilde{P}_{t-1}^G + \epsilon_t^{\pi T} \sim iid \mathcal{N}(0, \sigma_{\pi T}^2),
\]

\[
\ln \widetilde{P}_t^M = \pi_t + \ln \widetilde{P}_{t-1}^M + \epsilon_t^{\pi T} \sim iid \mathcal{N}(0, \sigma_{\pi T}^2).
\]

It follows that the trend components of the relative prices of consumption, housing investment, capital investment, government consumption, and imports follow random walks without drifts. This implies that along a balanced growth path, the levels of these relative prices are time independent but state dependent.

The trend components of output, consumption, housing investment, capital investment, government consumption, exports, and imports follow random walks with time varying drift \( g_t + n_t \):

\[
\ln \bar{Y}_t = g_t + n_t + \ln \bar{Y}_{t-1} + \epsilon_t^Y, \epsilon_t^Y \sim iid \mathcal{N}(0, \sigma_Y^2),
\]

\[
\ln \bar{C}_t = g_t + n_t + \ln \bar{C}_{t-1} + \epsilon_t^C, \epsilon_t^C \sim iid \mathcal{N}(0, \sigma_C^2),
\]

\[
\ln \bar{I}_t^H = g_t + n_t + \ln \bar{I}_{t-1}^H + \epsilon_t^{\pi T}, \epsilon_t^{\pi T} \sim iid \mathcal{N}(0, \sigma_{\pi T}^2),
\]

\[
\ln \bar{I}_t^K = g_t + n_t + \ln \bar{I}_{t-1}^K + \epsilon_t^{\pi T}, \epsilon_t^{\pi T} \sim iid \mathcal{N}(0, \sigma_{\pi T}^2),
\]
\[
\ln \tilde{G}_t = g_t + n_t + \ln \tilde{G}_{t-1} + \varepsilon_t^G, \quad \varepsilon_t^G \sim \text{iid } \mathcal{N}(0, \sigma^2_G), \quad (127)
\]

\[
\ln \tilde{X}_t = g_t + n_t + \ln \tilde{X}_{t-1} + \varepsilon_t^X, \quad \varepsilon_t^X \sim \text{iid } \mathcal{N}(0, \sigma^2_X), \quad (128)
\]

\[
\ln \tilde{M}_t = g_t + n_t + \ln \tilde{M}_{t-1} + \varepsilon_t^M, \quad \varepsilon_t^M \sim \text{iid } \mathcal{N}(0, \sigma^2_M). \quad (129)
\]

It follows that the trend components of the ratios of consumption, housing investment, capital investment, government consumption, exports, and imports to output follow random walks without drifts. This implies that along a balanced growth path, the levels of these great ratios are time independent but state dependent. The trend component of the shadow price of housing satisfies \( \ln \tilde{Q}_t^H = \ln \tilde{P}_t^H \), while the trend component of the housing stock satisfies \( \ln \tilde{H}_{t+1} = \ln \tilde{H}_t \).

The trend component of the nominal wage follows a random walk with time varying drift \( \pi_t + g_t \), while the trend component of employment follows a random walk with time varying drift \( n_t \):

\[
\ln \tilde{W}_t = \pi_t + g_t + \ln \tilde{W}_{t-1} + \varepsilon_t^W, \quad \varepsilon_t^W \sim \text{iid } \mathcal{N}(0, \sigma^2_W), \quad (130)
\]

\[
\ln \tilde{L}_t = n_t + \ln \tilde{L}_{t-1} + \varepsilon_t^L, \quad \varepsilon_t^L \sim \text{iid } \mathcal{N}(0, \sigma^2_L). \quad (131)
\]

It follows that the trend component of the income share of labour follows a random walk without drift. This implies that along a balanced growth path, the level of the income share of labour is time independent but state dependent. The trend component of real marginal cost satisfies \( \ln \tilde{\Phi}_t = \ln \tilde{\Phi} \), while the trend component of the rate of capital utilization satisfies \( \ln \tilde{u}_t = 0 \). The trend component of the shadow price of capital satisfies \( \ln \tilde{Q}_t^K = \ln \tilde{P}_t^K \), while the trend component of the capital stock satisfies \( \ln \tilde{K}_{t+1} = \ln \tilde{K}_t \).

The trend components of the nominal interest rate, tax rate, and nominal exchange rate follow random walks without drifts:

\[
\tilde{r}_t = \tilde{r}_{t-1} + \varepsilon_t^r, \quad \varepsilon_t^r \sim \text{iid } \mathcal{N}(0, \sigma^2_r), \quad (132)
\]

\[
\ln \tilde{r}_t = \ln \tilde{r}_{t-1} + \varepsilon_t^\tilde{r}, \quad \varepsilon_t^\tilde{r} \sim \text{iid } \mathcal{N}(0, \sigma^2_r). \quad (133)
\]

\[
\ln \tilde{E}_t = \ln \tilde{E}_{t-1} + \varepsilon_t^\tilde{E}, \quad \varepsilon_t^\tilde{E} \sim \text{iid } \mathcal{N}(0, \sigma^2_E). \quad (134)
\]

It follows that along a balanced growth path, the levels of the nominal interest rate, tax rate, and nominal exchange rate are time independent but state dependent. The trend component of the output based real interest rate satisfies \( \tilde{r}_t^Y = \tilde{r}_t - E_t \tilde{r}_{t+1}^Y \), while the trend component of the consumption based real interest rate satisfies \( \tilde{r}_t^C = \tilde{r}_t - E_t \tilde{r}_{t+1}^C \). The trend component of the real exchange rate satisfies \( \ln \tilde{Q}_t = \ln \tilde{E}_t + \ln \tilde{P}_t^J - \ln \tilde{P}_t^Y \), while the trend component of the terms of
trade satisfies $\ln \bar{T}_i = \ln \bar{P}_t^m - \ln \bar{P}_t^y$, where $\ln \bar{P}_t^x = \ln \bar{P}_t^y$. The trend component of the net government debt satisfies $\ln \left( \frac{\bar{B}_t^n}{\bar{P}_t^y} \right) = \ln \left( \frac{\bar{B}_t^c}{\bar{P}_t^y} \right)$, while the trend component of the net foreign debt satisfies $\ln \left( \frac{\bar{B}_t^n}{\bar{P}_t^y} \right) = \ln \left( \frac{\bar{B}_t^c}{\bar{P}_t^y} \right)$.

Long run balanced growth is driven by three common stochastic trends. Trend inflation, productivity growth, and population growth follow random walks without drifts:

$$\pi_t = \pi_{t-1} + \epsilon_t^\pi, \quad \epsilon_t^\pi \sim \text{iid } \mathcal{N}(0, \sigma_\pi^2), \quad (135)$$

$$g_t = g_{t-1} + \epsilon_t^g, \quad \epsilon_t^g \sim \text{iid } \mathcal{N}(0, \sigma_g^2), \quad (136)$$

$$n_t = n_{t-1} + \epsilon_t^n, \quad \epsilon_t^n \sim \text{iid } \mathcal{N}(0, \sigma_n^2). \quad (137)$$

It follows that along a balanced growth path, growth rates are time independent but state dependent. All innovations driving variation in trend components are assumed to be independent, which combined with our distributional assumptions implies multivariate normality.

### 3.3. Estimation, Inference and Forecasting

Quantitative monetary policy analysis and inflation targeting should be based on empirically adequate models of the economy, ones which approximately account for the existing empirical evidence in all measurable respects, at all frequencies. The monetary transmission mechanism is a cyclical phenomenon, involving dynamic interrelationships among deviations of the levels of various observed and unobserved endogenous variables from the levels of their flexible price and wage equilibrium components. Measurement of the stance of monetary policy involves estimation of the levels of the flexible price and wage equilibrium components of particular unobserved endogenous variables, while inflation targeting involves the generation of forecasts of the levels of particular observed endogenous variables.

Within a DSGE framework, a first best approach to the conduct of quantitative monetary policy analysis and inflation targeting entails the joint derivation of empirically adequate cyclical and trend component specifications from microeconomic foundations. This approach, which should promote invariance to monetary policy regime shifts for reasons identified by Lucas (1976), is complicated by the existence of intermittent structural breaks, accounting for which requires flexible trend component specifications, as discussed in Clements and Hendry (1999) and Maddala and Kim (1998). Within a DSGE framework, a second best approach to the conduct of quantitative monetary policy analysis and inflation targeting entails the derivation of
empirically adequate cyclical component specifications from microeconomic foundations, augmented with flexible trend component specifications. This approach, proposed by Vitek (2006c, 2006d), is based on the presumption that the determinants of trend components are unknown but persistent, and is extended and refined in this paper.

3.3.1. Estimation

The traditional econometric interpretation of macroeconometric models regards them as representations of the joint probability distribution of the data. Adopting this traditional econometric interpretation, the parameters and unobserved components of a linear state space representation of an approximate unobserved components representation of this DSGE model of a small open economy are jointly estimated with a Bayesian procedure, conditional on prior information concerning the values of parameters and trend components.

3.3.1.1. Estimation Procedure

Let \( x_t \) denote a vector stochastic process consisting of the levels of \( N \) nonpredetermined endogenous variables, of which \( M \) are observed. The cyclical components of this vector stochastic process satisfy second order stochastic linear difference equation

\[
A_0 \hat{x}_t = A_1 \hat{x}_{t-1} + A_2 E_t \hat{x}_{t+1} + A_3 \hat{x}_t + A_4 \hat{v}_t,
\] (138)

where vector stochastic process \( \hat{x}_t \) consists of the flexible price and wage equilibrium components of \( N \) nonpredetermined endogenous variables. The cyclical components of this vector stochastic process satisfy second order stochastic linear difference equation

\[
B_0 \hat{v}_t = B_1 \hat{v}_{t-1} + B_2 E_t \hat{x}_{t+1} + B_3 \hat{v}_t,
\] (139)

where vector stochastic process \( \hat{v}_t \) consists of the cyclical components of \( K \) exogenous variables. This vector stochastic process satisfies stationary first order stochastic linear difference equation

\[
\hat{v}_t = C_1 \hat{v}_{t-1} + \varepsilon_{1,t},
\] (140)

where \( \varepsilon_{1,t} \sim \text{iid } N(0, \Sigma_1) \). The trend components of vector stochastic process \( x_t \) satisfy first order stochastic linear difference equation
\[ D_0 \bar{x}_t = D_1 + D_2 u_t + D_3 \bar{x}_{t-1} + \epsilon_{2,t}, \]  

where \( \epsilon_{2,t} \sim \text{iid } \mathcal{N}(0, \Sigma_2) \). Vector stochastic process \( u_t \) consists of the levels of \( L \) common stochastic trends, and satisfies nonstationary first order stochastic linear difference equation

\[ u_t = u_{t-1} + \epsilon_{3,t}, \]  

where \( \epsilon_{3,t} \sim \text{iid } \mathcal{N}(0, \Sigma_3) \). Cyclical and trend components are additively separable, which implies that \( x_t = \hat{x}_t + \bar{x}_t \) and \( \bar{x}_t = \hat{x}_t + \bar{x}_t \), where \( \bar{x}_t = \bar{x}_t \).

If there exists a unique stationary solution to multivariate linear rational expectations model (138), then it may be expressed as:

\[ \hat{x}_t = S_1 \hat{x}_{t-1} + S_2 \bar{x}_{t-1} + S_3 \hat{\nu}_t. \]  

If there exists a unique stationary solution to multivariate linear rational expectations model (139), then it may be expressed as:

\[ \hat{x}_t = T_1 \hat{x}_{t-1} + T_2 \hat{\nu}_t. \]  

These solutions are calculated simultaneously with the matrix decomposition based algorithm due to Klein (2000).

Let \( y_t \) denote a vector stochastic process consisting of the levels of \( M \) observed nonpredetermined endogenous variables. Also, let \( z_t \) denote a vector stochastic process consisting of the levels of \( N-M \) unobserved nonpredetermined endogenous variables, the cyclical components of \( N \) nonpredetermined endogenous variables, the cyclical components of the flexible price and wage equilibrium components of \( N \) nonpredetermined endogenous variables, the trend components of \( N \) nonpredetermined endogenous variables, the cyclical components of \( K \) exogenous variables, and the levels of \( L \) common stochastic trends. Given unique stationary solutions (143) and (144), these vector stochastic processes have linear state space representation

\[ y_t = F_t z_t, \]  

\[ z_t = G_1 + G_2 z_{t-1} + G_3 \epsilon_{4,t}, \]  

where \( \epsilon_{4,t} \sim \text{iid } \mathcal{N}(0, \Sigma_4) \) and \( z_0 \sim \mathcal{N}(z_{00}, P_{00}) \). Let \( w_t \) denote a vector stochastic process consisting of preliminary estimates of the trend components of \( M \) observed nonpredetermined endogenous variables. Suppose that this vector stochastic process satisfies
\[ w_t = H_t z_t + \varepsilon_{5,t}, \]  
(147)

where \( \varepsilon_{5,t} \sim \text{iid } \mathcal{N}(0, \Sigma_5). \) Conditional on known parameter values, this signal equation defines a set of stochastic restrictions on selected unobserved state variables. The signal and state innovation vectors are assumed to be independent, while the initial state vector is assumed to be independent from the signal and state innovation vectors, which combined with our distributional assumptions implies multivariate normality.

Conditional on the parameters associated with these signal and state equations, estimates of unobserved state vector \( z_t \) and its mean squared error matrix \( P_t \) may be calculated with the filter proposed by Vitek (2006a, 2006b), which adapts the filter due to Kalman (1960) to incorporate prior information. Given initial conditions \( z_{0|0} \) and \( P_{0|0} \), estimates conditional on information available at time \( t - 1 \) satisfy prediction equations:

\[
\begin{align*}
  z_{t|t-1} &= G_t z_{t-1|t-1}, \\
  P_{t|t-1} &= G_t P_{t-1|t-1} G_t^T + G_t \Sigma_4 G_t^T, \\
  y_{t|t-1} &= F_t z_{t|t-1}, \\
  Q_{t|t-1} &= F_t P_{t|t-1} F_t^T, \\
  w_{t|t-1} &= H_t z_{t|t-1}, \\
  R_{t|t-1} &= H_t P_{t|t-1} H_t^T + \Sigma_5.
\end{align*}
\]

Given these predictions, under the assumption of multivariate normally distributed signal and state innovation vectors, together with conditionally contemporaneously uncorrelated signal vectors, estimates conditional on information available at time \( t \) satisfy updating equations:

\[
\begin{align*}
  z_{t|t} &= z_{t|t-1} + K_y (y_t - y_{t|t-1}) + K_w (w_t - w_{t|t-1}), \\
  P_{t|t} &= P_{t|t-1} - K_y F_t P_{t|t-1} - K_w H_t P_{t|t-1},
\end{align*}
\]

where \( K_y = P_{t|t-1} F_t^T Q_{t|t-1}^{-1} \) and \( K_w = P_{t|t-1} H_t^T R_{t|t-1}^{-1}. \) Given terminal conditions \( z_{T|T} \) and \( P_{T|T} \) obtained from the final evaluation of these prediction and updating equations, estimates conditional on information available at time \( T \) satisfy smoothing equations:

\[
\begin{align*}
  z_{t|T} &= z_{t|t} + J_t (z_{t+1|T} - z_{t+1|t}),
\end{align*}
\]

(156)
\[ \mathbf{P}_{\theta|Y} = \mathbf{P}_{\theta} + J_{t}(\mathbf{P}_{t+1|Y} - \mathbf{P}_{t+1|Y})J_{t}^T, \] (157)

where \( J_{t} = \mathbf{P}_{t|Y} \mathbf{P}_{t+1|Y}^{-1}. \) Under our distributional assumptions, these estimators of the unobserved state vector are mean squared error optimal.

Let \( \mathbf{\theta} \in \Theta \subseteq \mathbb{R}^J \) denote a \( J \) dimensional vector containing the parameters associated with the signal and state equations of this linear state space model. The Bayesian estimator of this parameter vector has posterior density function

\[ f(\mathbf{\theta} | Y) \propto f(Y | \theta)f(\mathbf{\theta}), \] (158)

where \( Y = \{ \{ y_s \}_{s=1}^T, \{ w_s \}_{s=1}^T \}. \) Under the assumption of multivariate normally distributed signal and state innovation vectors, together with conditionally contemporaneously uncorrelated signal vectors, conditional density function \( f(Y | \theta) \) satisfies:

\[ f(Y | \theta) = \prod_{t=1}^T f(y_t | Y_{t-1}, \theta) \prod_{t=1}^T f(w_t | Y_{t-1}, \theta). \] (159)

Under our distributional assumptions, conditional density functions \( f(y_t | Y_{t-1}, \theta) \) and \( f(w_t | Y_{t-1}, \theta) \) satisfy:

\[ f(y_t | Y_{t-1}, \theta) = (2\pi)^{-\frac{M}{2}} | \mathbf{Q}_{y_{t-1}} |^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2} (y_t - y_{y_{t-1}})^T \mathbf{Q}_{y_{t-1}}^{-1} (y_t - y_{y_{t-1}}) \right\}, \] (160)

\[ f(w_t | Y_{t-1}, \theta) = (2\pi)^{-\frac{M}{2}} | \mathbf{R}_{w_{t-1}} |^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2} (w_t - w_{w_{t-1}})^T \mathbf{R}_{w_{t-1}}^{-1} (w_t - w_{w_{t-1}}) \right\}. \] (161)

Prior information concerning parameter vector \( \mathbf{\theta} \) is summarized by a multivariate normal prior distribution having mean vector \( \mathbf{\theta}_0 \) and covariance matrix \( \Omega \):

\[ f(\mathbf{\theta}) = (2\pi)^{-\frac{D}{2}} | \Omega |^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2} (\mathbf{\theta} - \mathbf{\theta}_0)^T \Omega^{-1} (\mathbf{\theta} - \mathbf{\theta}_0) \right\}. \] (162)

Independent priors are represented by a diagonal covariance matrix, under which diffuse priors are represented by infinite variances.

Inference on the parameters is based on an asymptotic normal approximation to the posterior distribution around its mode. Under regularity conditions stated in Geweke (2005), posterior mode \( \hat{\mathbf{\theta}}_T \) satisfies
where $\theta_0 \in \Theta$ denotes the pseudotrue parameter vector. Following Engle and Watson (1981), Hessian $\mathcal{H}_0$ may be estimated by

$$
\sqrt{T}(\hat{\theta}_T - \theta_0) \xrightarrow{d} \mathcal{N}(0, -\mathcal{H}_0^{-1}),
$$

(163)

where $\theta_0 \in \Theta$ denotes the pseudotrue parameter vector. Following Engle and Watson (1981), Hessian $\mathcal{H}_0$ may be estimated by

$$
\mathcal{H}_T = \frac{1}{T} \sum_{t=1}^{T} \left[ \nabla_\theta \nabla_\theta ^T \ln f(y_t | I_{t-1}, \hat{\theta}_T) \right] + \frac{1}{T} \sum_{t=1}^{T} E_{t-1} \left[ \nabla_\theta \nabla_\theta ^T \ln f(w_t | I_{t-1}, \hat{\theta}_T) \right]
$$

(164)

$$
+ \frac{1}{T} \nabla_\theta \nabla_\theta ^T \ln f(\hat{\theta}_T),
$$

where

$$
E_{t-1} \left[ \nabla_\theta \nabla_\theta ^T \ln f(y_t | I_{t-1}, \hat{\theta}_T) \right] = -\nabla_\theta y_{\hat{\theta}_{t-1}} \mathcal{Q}_y^{-1} \nabla_\theta y_{\hat{\theta}_{t-1}} - \frac{1}{2} \nabla_\theta \mathcal{Q}_y^{-1} \mathcal{Q}_y^{-1} \mathcal{Q}_y^{-1} \nabla_\theta \mathcal{Q}_y^{-1},
$$

$$
E_{t-1} \left[ \nabla_\theta \nabla_\theta ^T \ln f(w_t | I_{t-1}, \hat{\theta}_T) \right] = -\nabla_\theta w_{\hat{\theta}_{t-1}} \mathcal{R}_w^{-1} \nabla_\theta w_{\hat{\theta}_{t-1}} - \frac{1}{2} \nabla_\theta \mathcal{R}_w^{-1} \mathcal{R}_w^{-1} \mathcal{R}_w^{-1} \nabla_\theta \mathcal{R}_w^{-1},
$$

and

$$
\nabla_\theta \nabla_\theta ^T \ln f(\hat{\theta}_T) = -\Omega^{-1}.
$$

### 3.3.1.2. Estimation Results

The set of parameters associated with this DSGE model of a small open economy is partitioned into two subsets. The first subset is calibrated to approximately match long run averages of functions of observed endogenous variables where possible, and estimates derived from existing microeconometric studies where necessary. The second subset is estimated with the Bayesian procedure described above, conditional on prior information concerning the values of parameters and trend components.

Subjective discount factor $\beta$ is restricted to equal 0.99, implying an annualized deterministic steady state equilibrium real interest rate of approximately 0.04. In deterministic steady state equilibrium, the output price markup $\phi^o$, import price markup $\phi^m$, and wage markup $\phi^w$ are restricted to equal 1.15. Depreciation rate parameter $\delta^H$ is restricted to equal 0.01, implying an annualized deterministic steady state equilibrium depreciation rate of approximately 0.04, while depreciation rate parameter $\delta^K$ is restricted to equal 0.02, implying an annualized deterministic steady state equilibrium depreciation rate of approximately 0.08. In deterministic steady state equilibrium, the consumption import share $1 - \phi^C$, housing investment import share $1 - \phi^H$, capital investment import share $1 - \phi^K$, and government consumption import share $1 - \phi^G$ are restricted to equal 0.30. The deterministic steady state equilibrium ratio of consumption to output $\frac{C}{Y}$ is restricted to equal 0.60, while the deterministic steady state equilibrium ratio of domestic output to foreign output $\frac{Y}{Y^*}$ is restricted to equal 0.11. In deterministic steady state equilibrium, the foreign consumption import share $1 - \phi^C$, foreign housing investment import share $1 - \phi^H$, foreign capital investment import share $1 - \phi^K$, and
foreign government consumption import share $1 - \phi^G x^F$ are restricted to equal 0.02. The deterministic steady state equilibrium income share of labour $w_L / p_Y$ is restricted to equal 0.65, while the deterministic steady state equilibrium ratio of housing to output $h / y$ is restricted to equal 6.00. In deterministic steady state equilibrium, the ratio of government consumption to output $g / y$ is restricted to equal 0.20, while the tax rate $\tau$ is restricted to equal 0.22.

Table 3.1. Deterministic steady state equilibrium values of great ratios

<table>
<thead>
<tr>
<th>Ratio</th>
<th>Value</th>
<th>Ratio</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>C / Y</td>
<td>0.6000</td>
<td>W / L / P Y</td>
<td>0.6500</td>
</tr>
<tr>
<td>I'' / Y</td>
<td>0.0600</td>
<td>H / Y</td>
<td>1.5000</td>
</tr>
<tr>
<td>I' / Y</td>
<td>0.1138</td>
<td>K / Y</td>
<td>1.4224</td>
</tr>
<tr>
<td>G / Y</td>
<td>0.2000</td>
<td>B / P Y</td>
<td>-0.4950</td>
</tr>
<tr>
<td>X / Y</td>
<td>0.3183</td>
<td>B / P Y</td>
<td>-0.6487</td>
</tr>
<tr>
<td>M / Y</td>
<td>0.2921</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note: Deterministic steady state equilibrium values are reported at an annual frequency based on calibrated parameter values.

Bayesian estimation of the remaining parameters of this DSGE model of a small open economy is based on the levels of twenty nine observed endogenous variables for Canada and the United States described in Appendix 3.A. Those parameters associated with the conditional mean function are estimated subject to cross-economy equality restrictions. Those parameters associated exclusively with the conditional variance function are estimated conditional on diffuse priors. Initial conditions for the cyclical components of exogenous variables are given by their unconditional means and variances, while the initial values of all other state variables are treated as parameters, and are calibrated to match functions of preliminary estimates of trend components calculated with the linear filter described in Hodrick and Prescott (1997). The posterior mode is calculated by numerically maximizing the logarithm of the posterior density kernel with a modified steepest ascent algorithm. Estimation results pertaining to the period 1971Q3 through 2005Q3 are reported in Appendix 3.B. The sufficient condition for the existence of a unique stationary rational expectations equilibrium due to Klein (2000) is satisfied in a neighbourhood around the posterior mode, while the estimator of the Hessian is not nearly singular at the posterior mode, suggesting that the approximate linear state space representation of this DSGE model of a small open economy is locally identified.

The prior mean of indexation parameter $\gamma^y$ is 0.75, implying considerable output price inflation inertia, while the prior mean of nominal rigidity parameter $\omega^y$ implies an average duration of output price contracts of two years. The prior mean of capital utilization cost parameter $\kappa$ is 0.10, while the prior mean of elasticity of substitution parameter $\theta$ is 0.75, implying that utilized capital and effective labour are moderately close complements in
production. The prior mean of habit persistence parameter $\alpha^C$ is 0.95, while the prior mean of intertemporal elasticity of substitution parameter $\sigma$ is 2.75, implying that consumption exhibits considerable persistence and moderate sensitivity to real interest rate changes. The prior mean of habit persistence parameter $\alpha^H$ is 0.95, while the prior mean of housing investment adjustment cost parameter $\chi^H$ is 1.25, implying considerable sensitivity of housing investment to changes in the relative shadow price of housing. The prior mean of capital investment adjustment cost parameter $\chi^K$ is 5.75, implying moderate sensitivity of capital investment to changes in the relative shadow price of capital. The prior mean of indexation parameter $y^M$ is 0.75, implying moderate import price inflation inertia, while the prior mean of nominal rigidity parameter $\omega^M$ implies an average duration of import price contracts of two years. The prior mean of elasticity of substitution parameter $\psi$ is 1.50, implying that domestic and foreign goods are moderately close substitutes in consumption, housing investment, capital investment, and government consumption. The prior mean of indexation parameter $y^L$ is 0.75, implying considerable sensitivity of the real wage to changes in consumption price inflation, while the prior mean of nominal rigidity parameter $\omega^L$ implies an average duration of wage contracts of two years. The prior mean of habit persistence parameter $\alpha^L$ is 0.95, while the prior mean of elasticity of substitution parameter $\eta$ is 0.75, implying considerable insensitivity of the real wage to changes in employment. The prior mean of the consumption price inflation response coefficient $\xi^C$ in the monetary policy rule is 1.50, while the prior mean of the output response coefficient $\xi^Y$ is 0.125, ensuring convergence of the level of consumption price inflation to its target value. The prior mean of the net foreign debt response coefficient $\zeta^G$ in the fiscal expenditure rule is -0.10, while the prior mean of the net government debt response coefficient $\zeta^T$ in the fiscal revenue rule is 1.00, ensuring convergence of the levels of the ratios of net foreign debt and net government debt to nominal output to their target values. All autoregressive parameters $\rho$ have prior means of 0.85, implying considerable persistence of shocks driving variation in cyclical components.

The posterior modes of these structural parameters are all close to their prior means, reflecting the imposition of tight independent priors to ensure the existence of a unique stationary rational expectations equilibrium. The estimated variances of shocks driving variation in cyclical components are all well within the range of estimates reported in the existing literature, after accounting for data rescaling. The estimated variances of shocks driving variation in trend components are relatively high, indicating that the majority of variation in the levels of observed endogenous variables is accounted for by variation in their trend components.

Prior information concerning the values of trend components is generated by fitting third order deterministic polynomial functions to the levels of all observed endogenous variables by ordinary least squares. Stochastic restrictions on the trend components of all observed
endogenous variables are derived from the fitted values associated with these ordinary least squares regressions, with innovation variances set proportional to estimated prediction variances assuming known parameters. All stochastic restrictions are independent, represented by a diagonal covariance matrix, and are harmonized, represented by a common factor of proportionality. Reflecting little confidence in these preliminary trend component estimates, this common factor of proportionality is set equal to one.

Predicted, filtered and smoothed estimates of the cyclical and trend components of observed endogenous variables are plotted together with confidence intervals in Appendix 3.B. These confidence intervals assume multivariate normally distributed and independent signal and state innovation vectors and known parameters. The predicted estimates are conditional on past information, the filtered estimates are conditional on past and present information, and the smoothed estimates are conditional on past, present and future information. Visual inspection reveals close agreement with the conventional dating of business cycle expansions and recessions.

Predicted, filtered and smoothed estimates of deviations of the levels of observed endogenous variables from their flexible price and wage equilibrium components, in addition to the levels of these flexible price and wage equilibrium components, are plotted together with confidence intervals in Appendix 3.B. Visual inspection reveals that a relatively low proportion of variation in the cyclical components of observed endogenous variables is accounted for by variation in the cyclical components of their flexible price and wage equilibrium components. This result suggests that a relatively high proportion of business cycle variation is accounted for by short run nominal price and wage rigidities, which amplify and propagate the effects of a variety of nominal and real shocks having temporary effects.

3.3.2. Inference

Achieving low and stable inflation calls for accurate and precise indicators of inflationary pressure, together with an accurate and precise quantitative description of the monetary transmission mechanism. This estimated DSGE model of a small open economy addresses both of these challenges within a unified framework.
3.3.2.1. Quantifying the Stance of Monetary Policy

Theoretically prominent indicators of inflationary pressure such as the natural rate of interest and natural exchange rate are unobservable. As discussed in Woodford (2003), the level of the natural rate of interest provides a measure of the neutral stance of monetary policy, with deviations of the real interest rate from the natural rate of interest generating inflationary pressure. It follows that the key to achieving low and stable inflation is the conduct of a monetary policy under which the short term nominal interest rate tracks variation in the level of the natural rate of interest as closely as possible, although also achieving an interest rate smoothing objective derived from a concern with financial market stability may call for larger monetary policy responses to variation in the natural rate of interest caused by shocks having permanent effects than to variation caused by shocks having temporary effects.

Definitions of indicators of inflationary pressure such as the natural rate of interest and natural exchange rate vary. Following Neiss and Nelson (2003), we define the natural rate of interest as that short term real interest rate consistent with past, present and future price and wage flexibility. Under this definition, the natural rate of interest is a function only of exogenous variables. In contrast, Woodford (2003) defines the natural rate of interest as that short term real interest rate consistent with current and future price and wage flexibility, conditional on the state of the economy. Under this definition, the natural rate of interest is a function of both exogenous and predetermined endogenous variables. As argued by Neiss and Nelson (2003), it seems odd to define the natural rate of interest such that it depends on predetermined endogenous variables, and by implication past monetary policy shocks given short run nominal price and wage rigidities.

Predicted, filtered and smoothed estimates of the level and trend component of the consumption based natural rate of interest are plotted together with confidence intervals versus corresponding estimates of the consumption based real interest rate in Figure 3.1. Visual inspection reveals that predicted estimates of the level of the natural rate of interest exhibit economically significant low frequency variation and are relatively imprecise, as evidenced by relatively wide confidence intervals, while filtered and smoothed estimates exhibit economically and statistically significant high frequency variation and are relatively precise, as evidenced by relatively narrow confidence intervals. Visual inspection also reveals that predicted, filtered and smoothed estimates of the trend component of the natural rate of interest exhibit economically and statistically significant low frequency variation and are relatively precise, as evidenced by relatively narrow confidence intervals. Given delays in data availability, these results suggest that accurate and precise measurement of the neutral stance of monetary policy on the basis of the level of the natural rate of interest can occur only retrospectively in practice, while inaccurate
but precise measurement of the neutral stance of monetary policy on the basis of the trend component of the natural rate of interest can take place contemporaneously in practice. This is problematic, as periods during which the estimated real interest rate exceeds the estimated natural rate of interest are closely aligned with the conventional dating of recessions, suggesting that tight monetary policy was to varying degrees a contributing factor.

Figure 3.1. Predicted, filtered and smoothed estimates of the natural rate of interest

![Graphs showing predicted, filtered, and smoothed estimates of the natural rate of interest.](image)

**Note:** Estimated levels are represented by black lines, while blue and red lines depict estimated flexible price and wage equilibrium components and trend components, respectively. Symmetric 95% confidence intervals assume multivariate normally distributed and independent signal and state innovation vectors and known parameters. Shaded regions indicate recessions as dated by the Economic Cycle Research Institute reference cycle.

In an open economy, the level of the consumption based natural rate of interest should fluctuate in response to a variety of shocks having both temporary and permanent effects, originating both domestically and abroad. In particular, the cyclical component of the natural rate of interest should fluctuate in response to a variety of real shocks having temporary effects, while the trend component of the natural rate of interest should fluctuate in response to a variety of nominal and real shocks having permanent effects. As noted by Woodford (2003), it is not obvious that the level of the natural rate of interest should be expected to evolve smoothly, given its dependence on such a diverse set of shocks.

The dynamic effects of a variety of real shocks having temporary effects on the level of the consumption based natural rate of interest, and the relative contributions of these real shocks to variation in its cyclical component, may be analyzed with theoretical impulse responses and forecast error variance decompositions. Visual inspection of theoretical impulse responses plotted in Appendix 3.B reveals that the level of the natural rate of interest declines in response to a temporary foreign output technology shock, and rises in response to a temporary foreign fiscal expenditure shock. Visual inspection of theoretical forecast error variance decompositions plotted in Appendix 3.B reveals that approximately 89% of variation in the cyclical component of the natural rate of interest is accounted for by the foreign output technology shock at all
horizons, while approximately 7% of this variation is accounted for by the foreign fiscal expenditure shock at all horizons.

Predicted, filtered and smoothed estimates of the level and trend component of the natural exchange rate are plotted together with confidence intervals versus the observed real exchange rate in Figure 3.2. This concept of the natural exchange rate represents that real exchange rate consistent with past, present and future price and wage flexibility. Visual inspection reveals that predicted, filtered and smoothed estimates of both the level and trend component of the natural exchange rate exhibit economically and statistically significant high frequency variation and are relatively precise, as evidenced by relatively narrow confidence intervals. Visual inspection also reveals that a relatively high proportion of variation in the observed real exchange rate is accounted for by variation in the level of the natural exchange rate, while a relatively high proportion of variation in the level of the natural exchange rate is accounted for by variation in its trend component. These results suggest that a relatively high proportion of variation in the observed real exchange rate is accounted for by nominal and real shocks having permanent effects, while a relatively high proportion of variation in the cyclical component of the real exchange rate is accounted for by real shocks having temporary effects. It follows that a relatively low proportion of cyclical real exchange rate variation is accounted for by short run nominal price and wage rigidities.

Figure 3.2. Predicted, filtered and smoothed estimates of the natural exchange rate

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Note: Observed levels are represented by black lines, while blue and red lines depict estimated flexible price and wage equilibrium components and trend components, respectively. Symmetric 95% confidence intervals assume multivariate normally distributed and independent signal and state innovation vectors and known parameters. Shaded regions indicate recessions as dated by the Economic Cycle Research Institute reference cycle.
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The dynamic effects of a variety of real shocks having temporary effects on the level of the natural exchange rate, and the relative contributions of these real shocks to variation in its cyclical component, may be analyzed with theoretical impulse responses and forecast error variance decompositions. Visual inspection of theoretical impulse responses plotted in Appendix 3.B reveals that the level of the natural exchange rate rises in response to a temporary domestic
output technology shock, corresponding to a real depreciation, and declines in response to a temporary foreign output technology shock, corresponding to a real appreciation. Visual inspection of theoretical forecast error variance decompositions plotted in Appendix 3.B reveals that approximately 53% of variation in the cyclical component of the natural exchange rate is accounted for by the domestic output technology shock at all horizons, while approximately 21% of this variation is accounted for by the foreign output technology shock at all horizons.

The finite sample properties of the estimation procedure proposed in this paper are evaluated with a Monte Carlo experiment in Vitek (2006e), with an emphasis on the levels of the natural rate of interest and natural exchange rate. Joint estimation of the parameters and unobserved components of a linear state space representation of an approximate unobserved components representation of a relatively parsimonious DSGE model of a small open economy with this Bayesian procedure is found to yield reasonably accurate and precise results in samples of approximately the size considered in this paper. In particular, estimates of the levels of the natural rate of interest and natural exchange rate conditional on alternative information sets are approximately unbiased, while analytical root mean squared errors appropriately account for uncertainty surrounding them, irrespective of whether the data generating process features common deterministic or stochastic trends.

3.3.2.2. Quantifying the Monetary Transmission Mechanism

Whether this estimated DSGE model provides an accurate quantitative description of the monetary transmission mechanism in a small open economy is determined by comparing its impulse responses to domestic and foreign monetary policy shocks with impulse responses derived from an estimated structural vector autoregressive or SVAR model.

Consider the following SVAR model of the monetary transmission mechanism in a small open economy

$$A_0 y_t = \mu(t) + \sum_{i=1}^{n} A_i y_{t-i} + B\varepsilon_t,$$

where $\mu(t)$ denotes a third order deterministic polynomial function and $\varepsilon_t \sim iid \mathcal{N}(0, \Sigma)$. Vector stochastic process $y_t$ consists of domestic output price inflation $\pi_Y^i$, domestic output $\ln Y_t$, domestic consumption price inflation $\pi_C^i$, domestic consumption $\ln C_t$, domestic housing investment price inflation $\pi_I^{hi}$, domestic housing investment $\ln I_t^{hi}$, domestic capital investment price inflation $\pi_I^{k}$, domestic capital investment $\ln I_t^{k}$, domestic import price inflation $\pi_M^i$, domestic imports $\ln M_t$, domestic nominal interest rate $i_t$, nominal
exchange rate $\ln E$, foreign output price inflation $x_{t}^{y,f}$, foreign output $\ln Y_{t}^{f}$, foreign consumption $\ln C_{t}^{f}$, foreign housing investment $\ln I_{t}^{h,f}$, foreign capital investment $\ln I_{t}^{k,f}$, and foreign nominal interest rate $i_{t}^{f}$. The diagonal elements of parameter matrix $A_{0}$ are normalized to one, while the off diagonal elements of positive definite parameter matrix $B$ are restricted to equal zero, thus associating with each equation a unique endogenous variable, and with each endogenous variable a unique structural innovation.

This SVAR model is identified by imposing restrictions on the timing of the effects of monetary policy shocks and on the information sets of the monetary authorities, both within and across the domestic and foreign economies. Within the domestic and foreign economies, prices and quantities are restricted to not respond instantaneously to monetary policy shocks, while the monetary authorities can respond instantaneously to changes in these variables. Across the domestic and foreign economies, the domestic monetary authority is restricted to not respond instantaneously to foreign monetary policy shocks, while foreign variables are restricted to not respond to domestic monetary policy shocks.

This SVAR model of the monetary transmission mechanism in a small open economy is estimated by full information maximum likelihood over the period 1971Q3 through 2005Q3. As discussed in Hamilton (1994), in the absence of model misspecification, this full information maximum likelihood estimator is consistent and asymptotically normal, irrespective of the cointegration rank and validity of the conditional multivariate normality assumption. The lag order is selected to minimize multivariate extensions of the model selection criterion functions of Akaike (1974), Schwarz (1978), and Hannan and Quinn (1979) subject to an upper bound equal to the seasonal frequency. These model selection criterion functions generally prefer a lag order of one.

Table 3.2. Model selection criterion function values

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Note: Minimized values of model selection criterion functions are indicated by *.

Since this SVAR model is estimated to provide empirical evidence concerning the monetary transmission mechanism in a small open economy, it is imperative to examine the empirical validity of its overidentifying restrictions prior to the conduct of impulse response analysis. On the basis of bootstrap likelihood ratio tests, these overidentifying restrictions are not rejected at conventional levels of statistical significance.
Table 3.3. Results of tests of overidentifying restrictions

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Note: This likelihood ratio test statistic is asymptotically distributed as $\chi^2_{4}$. Bootstrap distributions are based on 999 replications.

Theoretical impulse responses to a domestic monetary policy shock are plotted versus empirical impulse responses in Figure 3.3. Following a domestic monetary policy shock, the domestic nominal interest rate exhibits an immediate increase followed by a gradual decline. The domestic currency appreciates, with the nominal exchange rate exhibiting delayed overshooting. These nominal interest rate and nominal exchange rate dynamics induce persistent and generally statistically significant hump shaped negative responses of domestic output price inflation, output, consumption price inflation, consumption, housing investment price inflation, housing investment, capital investment price inflation, capital investment, import price inflation, exports and imports, with peak effects realized after approximately one year. These results are qualitatively consistent with those of SVAR analyses of the monetary transmission mechanism in open economies such as Eichenbaum and Evans (1995), Clarida and Gertler (1997), Kim and Roubini (1995), Cushman and Zha (1997), and Vitek (2006d).
Figure 3.3. Theoretical versus empirical impulse responses to a domestic monetary policy shock

Note: Theoretical impulse responses to a 50 basis point monetary policy shock are represented by black lines, while blue lines depict empirical impulse responses to a 50 basis point monetary policy shock. Asymmetric 95% confidence intervals are calculated with a nonparametric bootstrap simulation with 999 replications.

Theoretical impulse responses to a foreign monetary policy shock are plotted versus empirical impulse responses in Figure 3.4. Following a foreign monetary policy shock, the foreign nominal interest rate exhibits an immediate increase followed by a gradual decline. In response to these nominal interest rate dynamics, there arise persistent and generally statistically significant hump shaped negative responses of foreign output price inflation, output,
consumption, housing investment and capital investment, with peak effects realized after approximately one to two years. Although domestic output, consumption, housing investment, capital investment and imports decline, domestic consumption price inflation, housing investment price inflation, capital investment price inflation and import price inflation rise due to domestic currency depreciation. These results are qualitatively consistent with those of SVAR analyses of the monetary transmission mechanism in closed economies such as Sims and Zha (1995), Gordon and Leeper (1994), Leeper, Sims and Zha (1996), Christiano, Eichenbaum and Evans (1998, 2005), and Vitek (2006c, 2006d).
Figure 3.4. Theoretical versus empirical impulse responses to a foreign monetary policy shock

Note: Theoretical impulse responses to a 50 basis point monetary policy shock are represented by black lines, while blue lines depict empirical impulse responses to a 50 basis point monetary policy shock. Asymmetric 95% confidence intervals are calculated with a nonparametric bootstrap simulation with 999 replications.

Visual inspection reveals that the theoretical impulse responses to domestic and foreign monetary policy shocks generally lie within confidence intervals associated with the corresponding empirical impulse responses, suggesting that this estimated DSGE model provides an accurate quantitative description of the monetary transmission mechanism in a small open
economy. However, these confidence intervals are rather wide, indicating that considerable uncertainty surrounds this empirical evidence.

3.3.3. Forecasting

While it is desirable that forecasts be unbiased and efficient, the practical value of any forecasting model depends on its relative predictive accuracy. To compare the dynamic out of sample forecasting performance of the DSGE and SVAR models, forty quarters of observations are retained to evaluate forecasts one through eight quarters ahead, generated conditional on parameters estimated using information available at the forecast origin. The models are compared on the basis of mean squared prediction errors in levels, ordinary differences, and seasonal differences. The DSGE model is not recursively estimated as the forecast origin rolls forward due to the high computational cost of such a procedure, while the SVAR model is. Presumably, recursively estimating the DSGE model would improve its predictive accuracy.

Mean squared prediction error differentials are plotted together with confidence intervals accounting for contemporaneous and serial correlation of forecast errors in Appendix 3.B. If these mean squared prediction error differentials are negative then the forecasting performance of the DSGE model dominates that of the SVAR model, while if positive then the DSGE model is dominated by the SVAR model in terms of predictive accuracy. The null hypothesis of equal squared prediction errors is rejected by the predictive accuracy test of Diebold and Mariano (1995) if and only if these confidence intervals exclude zero. The asymptotic variance of the average loss differential is estimated by a weighted sum of the autocovariances of the loss differential, employing the weighting function proposed by Newey and West (1987). Visual inspection reveals that these mean squared prediction error differentials are generally negative, suggesting that the DSGE model dominates the SVAR model in terms of forecasting performance, in spite of a considerable informational disadvantage. However, these mean squared prediction error differentials are rarely statistically significant at conventional levels, perhaps because the predictive accuracy test due to Diebold and Mariano (1995), which is univariate, typically lacks power to detect dominance in forecasting performance, as evidenced by Monte Carlo evaluations such as Ashley (2003) and McCracken (2000).

Dynamic out of sample forecasts of levels, ordinary differences, and seasonal differences are plotted together with confidence intervals versus realized outcomes in Appendix 3.B. These confidence intervals assume multivariate normally distributed and independent signal and state innovation vectors and known parameters. Visual inspection reveals that the realized outcomes generally lie within their associated confidence intervals, suggesting that forecast failure is
absent. However, these confidence intervals are rather wide, indicating that considerable uncertainty surrounds the point forecasts.

### 3.4. Conclusion

This paper develops and estimates a DSGE model of a small open economy for purposes of monetary policy analysis and inflation targeting which provides a quantitative description of the monetary transmission mechanism, yields a mutually consistent set of indicators of inflationary pressure together with confidence intervals, and facilitates the generation of relatively accurate forecasts. Cyclical components are modeled by linearizing equilibrium conditions around a stationary deterministic steady state equilibrium which abstracts from long run balanced growth, while trend components are modeled as random walks while ensuring the existence of a well defined balanced growth path. Parameters and unobserved components are jointly estimated with a Bayesian procedure, conditional on prior information concerning the values of parameters and trend components.

Definitions of indicators of inflationary pressure such as the natural rate of interest and natural exchange rate vary, while estimates are typically sensitive to identifying restrictions. It follows that combinations of estimates of indicators of inflationary pressure derived under alternative definitions from dissimilar models may be more useful for purposes of monetary policy analysis and inflation targeting in a small open economy than any of the constituents. An examination of the inflation control and output stabilization benefits conferred by combining alternative estimates remains an objective for future research.

### Appendix 3.A. Description of the Data Set

The data set consists of quarterly seasonally adjusted observations on twenty nine macroeconomic variables for Canada and the United States over the period 1971Q1 through 2005Q3. All aggregate prices and quantities are expenditure based. Model consistent employment is derived from observed nominal labour income and a nominal wage index, while model consistent tax rates are derived from observed nominal output and disposable income. The nominal interest rate is measured by the three month Treasury bill rate expressed as a period average, while the nominal exchange rate is quoted as an end of period value. National accounts data for Canada was retrieved from the CANSIM database maintained by Statistics Canada, national accounts data for the United States was obtained from the FRED database maintained by
the Federal Reserve Bank of Saint Louis, and other data was extracted from the IFS database maintained by the International Monetary Fund.
### Appendix 3.B. Tables and Figures

Table 3.4. Bayesian estimation results

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*Note:* All observed endogenous variables are rescaled by a factor of 100.
Figure 3.5. Predicted cyclical components of observed endogenous variables

Note: Estimated cyclical components are represented by blue lines, while red lines depict estimated deviations from flexible price and wage equilibrium components. Symmetric 95% confidence intervals assume multivariate normally distributed and independent signal and state innovation vectors and known parameters. Shaded regions indicate recessions as dated by the Economic Cycle Research Institute reference cycle.
Figure 3.6. Filtered cyclical components of observed endogenous variables

Note: Estimated cyclical components are represented by blue lines, while red lines depict estimated deviations from flexible price and wage equilibrium components. Symmetric 95% confidence intervals assume multivariate normally distributed and independent signal and state innovation vectors and known parameters. Shaded regions indicate recessions as dated by the Economic Cycle Research Institute reference cycle.
Figure 3.7. Smoothed cyclical components of observed endogenous variables

Note: Estimated cyclical components are represented by blue lines, while red lines depict estimated deviations from flexible price and wage equilibrium components. Symmetric 95% confidence intervals assume multivariate normally distributed and independent signal and state innovation vectors and known parameters. Shaded regions indicate recessions as dated by the Economic Cycle Research Institute reference cycle.
Figure 3.8. Predicted trend components of observed endogenous variables

Note: Observed levels are represented by black lines, while blue and red lines depict estimated flexible price and wage equilibrium components and trend components, respectively. Symmetric 95% confidence intervals assume multivariate normally distributed and independent signal and state innovation vectors and known parameters. Shaded regions indicate recessions as dated by the Economic Cycle Research Institute reference cycle.
Figure 3.9. Filtered trend components of observed endogenous variables

Note: Observed levels are represented by black lines, while blue and red lines depict estimated flexible price and wage equilibrium components and trend components, respectively. Symmetric 95% confidence intervals assume multivariate normally distributed and independent signal and state innovation vectors and known parameters. Shaded regions indicate recessions as dated by the Economic Cycle Research Institute reference cycle.
Figure 3.10. Smoothed trend components of observed endogenous variables

Note: Observed levels are represented by black lines, while blue and red lines depict estimated flexible price and wage equilibrium components and trend components, respectively. Symmetric 95% confidence intervals assume multivariate normally distributed and independent signal and state innovation vectors and known parameters. Shaded regions indicate recessions as dated by the Economic Cycle Research Institute reference cycle.
Figure 3.11. Theoretical impulse responses to a domestic output technology shock

Note: Theoretical impulse responses to a unit standard deviation innovation under sticky price and wage equilibrium are represented by blue lines, while green lines depict theoretical impulse responses to a unit standard deviation innovation under flexible price and wage equilibrium.
Figure 3.12. Theoretical impulse responses to a domestic monetary policy shock

Note: Theoretical impulse responses to a unit standard deviation innovation under sticky price and wage equilibrium are represented by blue lines, while green lines depict theoretical impulse responses to a unit standard deviation innovation under flexible price and wage equilibrium.
Figure 3.13. Theoretical impulse responses to a domestic fiscal expenditure shock

Note: Theoretical impulse responses to a unit standard deviation innovation under sticky price and wage equilibrium are represented by blue lines, while green lines depict theoretical impulse responses to a unit standard deviation innovation under flexible price and wage equilibrium.
Figure 3.14. Theoretical impulse responses to a foreign output technology shock

Note: Theoretical impulse responses to a unit standard deviation innovation under sticky price and wage equilibrium are represented by blue lines, while green lines depict theoretical impulse responses to a unit standard deviation innovation under flexible price and wage equilibrium.
Figure 3.15. Theoretical impulse responses to a foreign monetary policy shock

Note: Theoretical impulse responses to a unit standard deviation innovation under sticky price and wage equilibrium are represented by blue lines, while green lines depict theoretical impulse responses to a unit standard deviation innovation under flexible price and wage equilibrium.
Figure 3.16. Theoretical impulse responses to a foreign fiscal expenditure shock

Note: Theoretical impulse responses to a unit standard deviation innovation under sticky price and wage equilibrium are represented by blue lines, while green lines depict theoretical impulse responses to a unit standard deviation innovation under flexible price and wage equilibrium.
Figure 3.17. Theoretical forecast error variance decompositions under sticky price and wage equilibrium.
**Figure 3.19. Mean squared prediction error differentials for levels**

**Note:** Mean squared prediction error differentials are defined as the mean squared prediction error for the DSGE model less that for the SVAR model. Symmetric 95% confidence intervals account for contemporaneous and serial correlation of forecast errors.
Figure 3.20. Mean squared prediction error differentials for ordinary differences

Note: Mean squared prediction error differentials are defined as the mean squared prediction error for the DSGE model less that for the SVAR model. Symmetric 95% confidence intervals account for contemporaneous and serial correlation of forecast errors.
Figure 3.21. Mean squared prediction error differentials for seasonal differences

Note: Mean squared prediction error differentials are defined as the mean squared prediction error for the DSGE model less that for the SVAR model. Symmetric 95% confidence intervals account for contemporaneous and serial correlation of forecast errors.
Figure 3.22. Dynamic forecasts of levels of observed endogenous variables

Note: Realized outcomes are represented by black lines, while blue lines depict point forecasts. Symmetric 95% confidence intervals assume multivariate normally distributed and independent signal and state innovation vectors and known parameters.
Figure 3.23. Dynamic forecasts of ordinary differences of observed endogenous variables

Note: Realized outcomes are represented by black lines, while blue lines depict point forecasts. Symmetric 95% confidence intervals assume multivariate normally distributed and independent signal and state innovation vectors and known parameters.
Figure 3.24. Dynamic forecasts of seasonal differences of observed endogenous variables

Note: Realized outcomes are represented by black lines, while blue lines depict point forecasts. Symmetric 95% confidence intervals assume multivariate normally distributed and independent signal and state innovation vectors and known parameters.
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CHAPTER 4

Measuring the Stance of Monetary Policy in a Small Open Economy: A Monte Carlo Evaluation

4.1. Introduction

The real business cycle or RBC class of models introduced by Kydland and Prescott (1982) and Long and Plosser (1983) was originally intended to provide a unified theoretical and empirical framework for the joint analysis of business cycle dynamics and long run growth. As extensions of the neoclassical growth model due to Ramsey (1928) and Solow (1956), in the absence of shocks which generate business cycle fluctuations, RBC models generally converge to well defined balanced growth paths along which great ratios and trend growth rates are time and state independent. However, evaluations of the business cycle predictions of this class of models based on comparisons between theoretical and empirical unconditional second moments typically abstract from their predictions for long run growth, as the comparisons are conditional on atheoretic decompositions of the levels of endogenous variables into cyclical and trend components.

The dynamic stochastic general equilibrium or DSGE class of models has recently emerged as the dominant theoretical and empirical framework for the analysis of the monetary transmission mechanism and the optimal conduct of monetary policy. As extensions of RBC models, the subclass of DSGE models generally employed in contemporary monetary policy analyses features short run nominal price rigidities generated by monopolistic competition and staggered reoptimization in output markets. Early examples of closed economy DSGE models within this subclass include those of Yun (1996), Goodfriend and King (1997), Rotemberg and Woodford (1995, 1997), and McCallum and Nelson (1999), while early examples of open economy DSGE models within this subclass include those of McCallum and Nelson (2000), Clarida, Galí and Gertler (2001, 2002), and Gertler, Gilchrist and Natalucci (2001). In parallel with the RBC methodology, evaluations of the predictions of these DSGE models with regards to the monetary transmission mechanism based on comparisons between theoretical and empirical impulse response functions typically abstract from their predictions for long run growth, as do measurements of the stance of monetary policy based on flexible price equilibrium concepts,
again being conditional on atheoretic decompositions of the levels of endogenous variables into cyclical and trend components.

The existence of a well defined balanced growth path along which great ratios and trend growth rates are time independent is desirable, as it ensures the mutual stability of long horizon forecasts of the levels of endogenous variables. However, within a DSGE framework, ensuring the existence of a well defined balanced growth path requires the imposition of restrictions which potentially limit the empirical adequacy of cyclical and trend component specifications. As discussed in King, Plosser and Rebelo (1988), ensuring the existence of a well defined balanced growth path restricts the classes of functions representing preferences and technologies which may be considered, potentially limiting the empirical adequacy of cyclical component specifications. Ensuring the existence of a well defined balanced growth path also restricts the types of exogenous stochastic processes responsible for driving both business cycle dynamics and long run growth which may be considered, potentially limiting the empirical adequacy of trend component specifications. As discussed in Canova, Finn and Pagan (1994), the prediction of DSGE models featuring long run balanced growth driven by trend inflation, productivity growth, and population growth that the levels of observed endogenous variables should fluctuate around common deterministic or stochastic trends is often rejected empirically.

A central theme of the voluminous cointegration literature surveyed by Maddala and Kim (1998) is that, while empirical support for the existence of cointegrating relationships of a form consistent with the existence of a well defined balanced growth path often arises, it typically does so conditional on intermittent structural breaks. As discussed in Clements and Hendry (1999), failure to allow for the existence of such intermittent structural breaks is a dominant source of forecast failure in macroeconometric models. These observations suggest that allowing the balanced growth paths towards which DSGE models converge in the absence of shocks to be state dependent should robustify estimation, inference and forecasting to intermittent structural breaks that occur within sample.

Due to the curse of dimensionality, DSGE models are generally solved with perturbation methods, which require the existence of a stationary deterministic steady state equilibrium around which to approximate equilibrium conditions. In cases where DSGE models feature long run balanced growth driven by trend inflation, productivity growth and population growth, a stationary deterministic steady state equilibrium may be obtained by appropriately deflating endogenous variables by common deterministic or stochastic trends. However, if the balanced growth paths towards which DSGE models converge in the absence of shocks exhibit a flexible form of state dependence, then existing perturbation methods are not applicable.

Quantitative monetary policy analysis and inflation targeting should be based on empirically adequate models of the economy, ones which approximately account for the existing empirical
evidence in all measurable respects, at all frequencies. As discussed in Woodford (2003), the monetary transmission mechanism is a cyclical phenomenon, involving dynamic interrelationships among deviations of the levels of various observed and unobserved endogenous variables from the levels of their flexible price equilibrium components. Measurement of the stance of monetary policy involves estimation of the level of the natural rate of interest, defined as that short term real interest rate consistent with flexible prices, while inflation targeting involves the generation of forecasts of the levels of particular observed endogenous variables.

Within a DSGE framework, a first best approach to the conduct of quantitative monetary policy analysis and inflation targeting entails the joint derivation of empirically adequate cyclical and trend component specifications from microeconomic foundations. This approach, which should promote invariance to monetary policy regime shifts for reasons identified by Lucas (1976), is complicated by the existence of intermittent structural breaks, accounting for which requires flexible trend component specifications, as discussed in Clements and Hendry (1999) and Maddala and Kim (1998). Within a DSGE framework, a second best approach to the conduct of quantitative monetary policy analysis and inflation targeting entails the derivation of empirically adequate cyclical component specifications from microeconomic foundations, augmented with flexible trend component specifications. This approach, proposed by Vitek (2006c, 2006d), is based on the presumption that the determinants of trend components are unknown but persistent, and is extended and refined in Vitek (2006e, 2006f).

The primary objective of this paper is to evaluate the finite sample properties of the procedure proposed by Vitek (2006f) for the measurement of the stance of monetary policy in a small open economy under alternative trend component specifications. Towards this end, the accuracy and precision of the Bayesian procedure proposed for the estimation of the levels of the flexible price equilibrium components of various observed and unobserved endogenous variables is analyzed with a Monte Carlo experiment, with an emphasis on the levels of the natural rate of interest and natural exchange rate. The secondary objective of this paper is to describe in a pedagogical manner the application of this procedure to the estimation of a simple but economically interesting DSGE model of a small open economy. Joint estimation of the parameters and unobserved components of a linear state space representation of an approximate unobserved components representation of this DSGE model with this Bayesian procedure, conditional on prior information concerning the values of parameters and trend components, is found to yield reasonably accurate and precise results in samples of currently available size. In particular, estimates of the levels of the natural rate of interest and natural exchange rate conditional on alternative information sets are approximately unbiased, while root mean squared errors are relatively small, irrespective of whether the data generating process features common
deterministic or stochastic trends. Moreover, analytical root mean squared errors appropriately account for uncertainty surrounding estimates of the levels of the natural rate of interest and natural exchange rate.

The organization of this paper is as follows. The next section develops a DSGE model of a small open economy. Alternative approximate unobserved components representations of this DSGE model are described in section three. The design and results of a Monte Carlo experiment for analyzing the accuracy and precision of the procedure proposed for the measurement of the stance of monetary policy in a small open economy are discussed in section four. Finally, section five offers conclusions and recommendations for further research.

4.2. Model Development

Consider two open economies which are asymmetric in size, but are otherwise identical. The domestic economy is of negligible size relative to the foreign economy.

4.2.1. The Utility Maximization Problem of the Representative Household

There exists a continuum of identical households indexed by \( i \in [0,1] \). The representative infinitely lived household has preferences defined over consumption \( C_{i,s} \) and labour supply \( L_{i,s} \), represented by intertemporal utility function

\[
U_{i,s} = E_t \sum_{i=1}^{\infty} \beta^{t-i} u(C_{i,s}, L_{i,s}),
\]

where subjective discount factor \( \beta \) satisfies \( 0 < \beta < 1 \). The representative household consists of \( N_s \) identical members, and has intratemporal utility function:

\[
u(C_{i,s}, L_{i,s}) = N_s v \left( \frac{C_{i,s}}{N_s}, \frac{L_{i,s}}{N_s} \right).
\]

The intratemporal utility function of the representative household member is multiplicatively separable:

\[
\nu \left( \frac{C_{i,s}}{N_s}, \frac{L_{i,s}}{N_s} \right) = \frac{1}{1-1/\sigma} \left( \frac{C_{i,s}}{N_s} \right)^{1-1/\sigma} \exp \left[ -\frac{1-1/\sigma}{1+1/\eta} \left( \frac{L_{i,s}}{N_s} \right)^{1+1/\eta} \right].
\]
In order to ensure the existence of a well defined balanced growth path, the marginal utility of consumption is homogeneous in consumption, while the marginal utility of leisure is homogenous of one higher degree in consumption.

The representative household enters period $s$ in possession of a previously purchased diversified portfolio of internationally traded domestic currency denominated bonds $B_{i,s}$ which completely spans all relevant uncertainty. It also holds a diversified portfolio of shares $\{x_{i,s}\}_{j=0}^1$ in domestic intermediate good firms which pay dividends $\{\Pi_{j,s}\}_{j=0}^1$. The representative household supplies final labour service $L_{i,s}$, earning labour income at nominal wage $W_s$. These sources of wealth are summed in household dynamic budget constraint:

$$E_s Q_{s,s+1} B_{i,s+1} + \int_{j=0}^1 V_{j,s} x_{i,j,s+1} dj = B_{i,s} + \int_{j=0}^1 (\Pi_{j,s} + V_{j,s}) x_{i,j,s} dj + W_s L_{i,s} - P^C C_{i,s}. \quad (4)$$

According to this dynamic budget constraint, at the end of period $s$, the representative household purchases a diversified portfolio of state contingent bonds $B_{i,s+1}$, where $Q_{s,s+1}$ denotes the price of a bond which pays one unit of the domestic currency in a particular state in the following period, divided by the conditional probability of occurrence of that state. It also purchases a diversified portfolio of shares $\{x_{i,j,s+1}\}_{j=0}^1$ at prices $\{V_{j,s}\}_{j=0}^1$. Finally, the representative household purchases final consumption good $C_{i,s}$ at price $P^C_s$.

In period $t$, the representative household chooses state contingent sequences for consumption $\{C_{i,s}\}_{s=t}^\infty$, labour supply $\{L_{i,s}\}_{s=t}^\infty$, bond holdings $\{B_{i,s+1}\}_{s=t}^\infty$, and share holdings $\{\{x_{i,j,s+1}\}_{j=0}^1\}_{s=t}^\infty$ to maximize intertemporal utility function (1) subject to dynamic budget constraint (4) and terminal nonnegativity constraints $B_{i,T+1} \geq 0$ and $x_{i,j,T+1} \geq 0$ for $T \to \infty$. In equilibrium, selected necessary first order conditions associated with this utility maximization problem may be stated as

$$u_C(C_t, L_t) = P^C_t \lambda_t, \quad (5)$$

$$-u_L(C_t, L_t) = W_t \lambda_t, \quad (6)$$

$$Q_{s,s+1} \lambda_t = \beta \lambda_{s+1}, \quad (7)$$

$$V_{j,s} \lambda_t = \beta E_t((\Pi_{j,s+1} + V_{j,s+1}) \lambda_{s+1}), \quad (8)$$

where $\lambda_{s,t}$ denotes the Lagrange multiplier associated with the period $s$ household dynamic budget constraint. In equilibrium, necessary complementary slackness conditions associated with the terminal nonnegativity constraints may be stated as:
Provided that the intertemporal utility function is bounded and strictly concave, together with all necessary first order conditions, these transversality conditions are sufficient for the unique utility maximizing state contingent intertemporal household allocation.

The absence of arbitrage opportunities requires that short term nominal interest rate \( i \) satisfy
\[
\frac{1}{1+i} = E_t Q_t^{l+1}.
\]
Combination of this equilibrium asset pricing relationship with necessary first order conditions (5) and (7) yields intertemporal optimality condition
\[
u_c(C_t, L_t) = \beta E_t (1+i) \frac{P_c}{P_{t+1}} u_c(C_{t+1}, L_{t+1}),
\]
which ensures that at a utility maximum, the representative household cannot benefit from feasible intertemporal consumption reallocations. Finally, combination of necessary first order conditions (5) and (6) yields intratemporal optimality condition
\[
\frac{u_c(C_t, L_t)}{u_c(C_t, L_t)} = \frac{W_t}{P_t^c},
\]
which equates the marginal rate of substitution between leisure and consumption to the real wage.

4.2.2. The Value Maximization Problem of the Representative Firm

There exists a continuum of intermediate good firms indexed by \( j \in [0,1] \). Intermediate good firms supply differentiated intermediate output goods, but are otherwise identical. Entry into and exit from the monopolistically competitive intermediate output good sector is prohibited.
4.2.2.1. Employment Behaviour

The representative intermediate good firm sells shares \( \{x_{i,t}\}_{i=0}^{1} \) to domestic households at price \( V_{j,t} \). Recursive forward substitution for \( V_{j,t+s} \) with \( s > 0 \) in necessary first order condition (8) applying the law of iterated expectations reveals that the post-dividend stock market value of the representative intermediate good firm equals the expected present discounted value of future dividend payments:

\[
V_{j,t} = E_{t} \sum_{s=1}^{\infty} \frac{\beta^{s-t}}{\lambda_{t}} \Pi_{j,s}. 
\]  

Acting in the interests of its shareholders, the representative intermediate good firm maximizes its pre-dividend stock market value, equal to the expected present discounted value of current and future dividend payments:

\[
\Pi_{j,s} + V_{j,s} = E_{t} \sum_{s=1}^{\infty} \frac{\beta^{s-t}}{\lambda_{t}} \Pi_{j,s}. 
\]  

The derivation of result (13) imposes transversality condition (10), which rules out self-fulfilling speculative asset price bubbles.

Shares entitle households to dividend payments equal to profits \( \Pi_{j,s} \), defined as revenues derived from sales of differentiated intermediate output good \( Y_{j,s} \) at price \( P_{j,s}^{Y} \) less expenditures on final labour service \( L_{j,s} \):

\[
\Pi_{j,s} = P_{j,s}^{Y} Y_{j,s} - W_{j,s} L_{j,s}. 
\]  

The representative intermediate good firm rents final labour service \( L_{j,s} \) given labour augmenting productivity coefficient \( A_{s} \) to produce differentiated intermediate output good \( Y_{j,s} \) according to production function

\[
Y_{j,s} = A_{s} L_{j,s}, 
\]  

where \( A_{s} > 0 \). In order to ensure the existence of a well defined balanced growth path, this production function is homogeneous of degree one.

In period \( t \), the representative intermediate good firm chooses a state contingent sequence for employment \( \{L_{j,s}\}_{s=1}^{m} \) to maximize pre-dividend stock market value (14) subject to production function (16). In equilibrium, demand for the final labour service satisfies necessary first order condition
where \( P^Y_{i,s} \) denotes the Lagrange multiplier associated with the period \( s \) production technology constraint. This necessary first order condition equates real marginal cost \( \Phi_i \) to the ratio of the real wage to the marginal product of labour.

4.2.2.2. Output Supply and Price Setting Behaviour

There exist a large number of perfectly competitive firms which combine differentiated intermediate output goods \( Y_{j,t} \) supplied by intermediate good firms in a monopolistically competitive output market to produce final output good \( Y_t \) according to constant elasticity of substitution production function

\[
Y_t = \left[ \frac{\theta}{\theta - 1} \int_{j=0}^{1} (Y_{j,t})^{\frac{\theta}{\theta - 1}} dj \right]^{\frac{\theta - 1}{\theta}},
\]

(18)

where \( \theta > 1 \). The representative final output good firm maximizes profits derived from production of the final output good

\[
\Pi_t^Y = P_t^Y Y_t - \int_{j=0}^{1} P^Y_{j,t} Y_{j,t} dj,
\]

(19)

with respect to inputs of intermediate output goods, subject to production function (18). The necessary first order conditions associated with this profit maximization problem yield intermediate output good demand functions:

\[
Y_{j,t} = \left( \frac{P^Y_{j,t}}{P^Y_t} \right)^{-\theta} Y_t.
\]

(20)

Since the production function exhibits constant returns to scale, in competitive equilibrium the representative final output good firm earns zero profit, implying aggregate output price index:

\[
P^Y_t = \left[ \frac{1}{\int_{j=0}^{1} (P^Y_{j,t})^{1-\theta} dj} \right]^{1-\theta}.
\]

(21)
As the price elasticity of demand for intermediate output goods \( \theta \) increases, they become closer substitutes, and individual intermediate good firms have less market power.

In an adaptation of the model of nominal output price rigidity proposed by Calvo (1983), each period a randomly selected fraction \( 1 - \omega \) of intermediate good firms adjust their price optimally. The remaining fraction \( \omega \) of intermediate good firms adjust their price to account for past steady state output price inflation according to indexation rule:

\[
P^Y_{j,t} = \frac{P^Y_{j,t-1}}{P^Y_{t-1}} P^Y_{j,t-1}.
\]  

(22)

Under this specification, optimal price adjustment opportunities arrive randomly, and the interval between optimal price adjustments is a random variable.

If the representative intermediate good firm can adjust its price optimally in period \( t \), then it does so to maximize to maximize pre-dividend stock market value (14) subject to production function (16), intermediate output good demand function (20), and the assumed form of nominal price rigidity. Since all intermediate good firms that adjust their price optimally in period \( t \) solve an identical value maximization problem, in equilibrium they all choose a common price \( P^{Y,*}_t \) given by necessary first order condition:

\[
\frac{P^{Y,*}_t}{P^Y_t} = \left( \frac{\theta}{\theta - 1} \right) \frac{E_t \sum_{s=t}^{\infty} \omega^{s-t} \frac{\beta^{s-t} \lambda_s}{\lambda_t} \Phi_s \left( \frac{P^Y_t}{P^Y_{t-1}} \right)^{\theta} P^Y_s Y_s}{E_t \sum_{s=t}^{\infty} \omega^{s-t} \frac{\beta^{s-t} \lambda_s}{\lambda_t} \left( \frac{P^Y_s}{P^Y_{t-1}} \right)^{\theta} P^Y_s Y_s}.
\]  

(23)

This necessary first order condition equates the expected present discounted value of the revenue benefit generated by an additional unit of output supply to the expected present discounted value of its production cost. Aggregate output price index (21) equals an average of the price set by the fraction \( 1 - \omega \) of intermediate good firms that adjust their price optimally in period \( t \), and the average of the prices set by the remaining fraction \( \omega \) of intermediate good firms that adjust their price according to indexation rule (22):

\[
P^Y_t = \left[ (1 - \omega)(P^{Y,*}_t)^{-\theta} + \omega \left( \frac{P^Y_{t-1}}{P^Y_{t-2}} \right)^{\theta} \right]^{1/\theta}.
\]  

(24)

Since those intermediate good firms able to adjust their price optimally in period \( t \) are selected randomly from among all intermediate good firms, the average price set by the remaining
intermediate good firms equals the value of the aggregate output price index that prevailed during period $t - 1$, rescaled to account for past steady state output price inflation.

If all intermediate good firms were able to adjust their price optimally every period, then $\omega = 0$ and necessary first order condition (23) would reduce to

$$\tilde{p}_t^x = \frac{\theta}{\theta - 1} \tilde{p}_t^v \tilde{P}_t,$$

(25)

where $\tilde{p}_t^x = \tilde{P}_t^v$. In flexible price equilibrium, each intermediate good firm sets its price equal to a constant markup over nominal marginal cost, and output supply is inefficiently low.

4.2.3. International Trade and Financial Linkages

In an open economy, exchange rate adjustment contributes to both intratemporal and intertemporal equilibration, while business cycles are generated by interactions among a variety of nominal and real shocks originating both domestically and abroad.

4.2.3.1. International Trade Linkages

The law of one price asserts that arbitrage transactions equalize the domestic currency prices of domestic imports and foreign exports. Let $\mathcal{E}_s$ denote the nominal exchange rate, which measures the price of foreign currency in terms of domestic currency, and define the real exchange rate,

$$Q_s = \frac{\mathcal{E}_s P_v^\prime}{P_t^v},$$

(26)

which measures the price of foreign output in terms of domestic output. Under the law of one price, the real exchange rate coincides with the terms of trade, which measures the price of imports in terms of exports.

There exist a large number of perfectly competitive firms which combine a domestic intermediate consumption good $C_h$, and a foreign intermediate consumption good $C_f$, to produce final consumption good $C$ according to constant elasticity of substitution production function.
where $0 < \phi < 1$ and $\psi > 1$. The representative final consumption good firm maximizes profits derived from production of the final consumption good

$$\Pi_i^C = P_i^C C_i - P_i^T C_{h,i} - C_i P_i^{x,f} C_{f,i},$$

with respect to inputs of domestic and foreign intermediate consumption goods, subject to production function (27). The necessary first order conditions associated with this profit maximization problem imply intermediate consumption good demand functions:

$$C_{h,i} = \phi \left( \frac{P_i^T}{P_i^C} \right)^\psi C_i,$$

$$C_{f,i} = (1 - \phi) \left( \frac{C_i P_i^{x,f}}{P_i^C} \right)^\psi C_i.$$

Since the production function exhibits constant returns to scale, in competitive equilibrium the representative final consumption good firm earns zero profit, implying aggregate consumption price index:

$$P_i^C = \left[ \phi (P_i^T)^{1-\psi} + (1 - \phi) (C_i P_i^{x,f})^{1-\psi} \right]^{1/\psi}.$$  

Combination of this aggregate consumption price index with intermediate consumption good demand functions (29) and (30) yields:

$$C_{h,i} = \phi \left[ \phi + (1 - \phi) (C_i)^{1-\psi} \right]^{\psi} C_i,$$

$$C_{f,i} = (1 - \phi) \left[ (1 - \phi) + \phi (C_i)^{\psi-1} \right]^{\psi} C_i.$$  

These demand functions for domestic and foreign intermediate consumption goods are directly proportional to final consumption good demand, with a proportionality coefficient that varies with the real exchange rate.
4.2.3.2. International Financial Linkages

Under the assumption of complete international financial markets, utility maximization by domestic and foreign households implies intertemporal optimality conditions

\[
Q_{t,t+1} = \frac{\beta u_c(C_{t,t+1}, L_{t,t+1})}{u_c(C_t, L_t)} \frac{P_t^c}{P_{t+1}^c},
\]

\[
Q_{t,t+1} = \frac{\beta u_c(C_{t,t+1}, L_{t,t+1})}{u_c(C_t, L_t)} \frac{P_{t+1}^y}{P_t^y} \frac{\varepsilon_t}{\varepsilon_{t+1}},
\]

respectively. Combination of these intertemporal optimality conditions with real exchange rate definition (26) yields international risk sharing condition:

\[
Q_t \propto \frac{u_c(C_f^t, L_f^t)}{u_c(C_t, L_t)} \frac{P_t^c}{P_t^y}.
\]

Under the assumption that the domestic economy is of negligible size relative to the foreign economy, this international risk sharing condition induces stationarity of consumption per unit of effective labour, and of the real net foreign asset position per unit of effective labour, which equals zero in deterministic steady state equilibrium.

4.2.4. Monetary Policy

The government consists of a monetary authority which implements monetary policy through control of the short term nominal interest rate according to monetary policy rule

\[
i_t - \tilde{y}_t = \xi (\pi_t^c - \tilde{\pi}_t^c) + \zeta (\ln Y_t - \ln \tilde{Y}_t) + v_t,
\]

where \(\xi > 1\) and \(\zeta > 0\). As specified, the deviation of the nominal interest rate from its flexible price equilibrium value is a linear increasing function of the contemporaneous deviation of consumption price inflation from its target value \(\tilde{\pi}_t^c = \bar{\pi}_t^c\), and the contemporaneous proportional deviation of output from its flexible price equilibrium value. Persistent departures from this monetary policy rule are captured by serially correlated monetary policy shock \(v_t\).
4.2.5. Market Clearing Conditions

A rational expectations equilibrium in this DSGE model of a small open economy consists of state contingent intertemporal allocations for domestic and foreign households and firms which solve their constrained optimization problems given prices and policy, together with state contingent intertemporal allocations for domestic and foreign governments which satisfy their policy rules, with supporting prices such that all markets clear.

Clearing of the final output good market requires that production of the final output good equal the cumulative demands of domestic and foreign households:

\[ Y_t = C_{h,t} + C_{f,t} \] (38)

The assumption that the domestic economy is of negligible size relative to the foreign economy is represented by parameter restriction \( \phi' = 1 \), under which \( P^t, f = P^c, f \) in equilibrium.

4.3. The Approximate Linear Model

Estimation and inference are based on a linear state space representation of an approximate unobserved components representation of this DSGE model of a small open economy. A first best approximation is considered in which cyclical and trend component specifications are jointly derived from microeconomic foundations. Under this approach, along the balanced growth path towards which the economy converges in the absence of shocks, great ratios and trend growth rates are time and state independent. A second best approximation is also considered in which cyclical component specifications are derived from microeconomic foundations, and are combined with more flexible trend component specifications. Under this approach, along the balanced growth path towards which the economy converges in the absence of shocks, great ratios and trend growth rates are time independent but state dependent.

In what follows, \( E_t, x_{t+s} \) denotes the rational expectation of variable \( x_{t+s} \), conditional on information available at time \( t \). Also, \( \tilde{x}_t \) denotes the cyclical component of variable \( x_t \), \( \hat{x}_t \) denotes the flexible price equilibrium component of variable \( x_t \), and \( \bar{x}_t \) denotes the trend component of variable \( x_t \). Cyclical and trend components are additively separable, which implies that \( x_t = \hat{x}_t + \tilde{x}_t \) and \( \bar{x}_t = \tilde{x}_t + \bar{x}_t \), where \( \bar{x}_t = \bar{x}_t \).
4.3.1. First Best Approximation

Cyclical components are modeled by applying stationarity inducing transformations consistent with the existence of a well defined balanced growth path along which all variables are constant or grow at constant rates to equilibrium conditions, then linearizing them around the resultant stationary deterministic steady state equilibrium, while trend components are modeled by imposing the cointegrating relationships implied by this balanced growth path.

4.3.1.1. Cyclical Components

The cyclical component of output price inflation depends on the expected future cyclical component of output price inflation and the contemporaneous cyclical component of real marginal cost according to output price Phillips curve

\[ \hat{\pi}_t^y = \beta'E_{t+1} \hat{\pi}_{t+1}^y + \frac{(1-\omega)(1-\omega\beta')}{\omega} \left[ \left( \frac{1}{\phi + \eta} \right) \ln \frac{\hat{y}_i}{A_i N_i} - \frac{1-\theta}{\phi} \left( \ln \frac{\hat{y}_i}{A_i N_i} + (\omega(1+\phi) - \phi) \ln \hat{Q}_i \right) \right], \]

where \( \beta' = \beta(1+n)(1+g)^{-1/\sigma} \). Reflecting the existence of international trade linkages, the cyclical component of real marginal cost depends not only on the contemporaneous cyclical component of domestic output per unit of effective labour, but also on the contemporaneous cyclical components of foreign output per unit of effective labour and the real exchange rate.

The cyclical component of consumption price inflation depends on the expected future cyclical component of consumption price inflation and the contemporaneous cyclical component of real marginal cost according to consumption price Phillips curve:

\[ \hat{\pi}_t^c = \beta'E_{t+1} \hat{\pi}_{t+1}^c + \frac{(1-\omega)(1-\omega\beta')}{\omega} \left[ \left( \frac{1}{\phi + \eta} \right) \ln \frac{\hat{y}_i}{A_i N_i} - \frac{1-\theta}{\phi} \left( \ln \frac{\hat{y}_i}{A_i N_i} + (\omega(1+\phi) - \phi) \ln \hat{Q}_i \right) \right] + (1-\phi) \ln \frac{\hat{Q}_i}{\hat{Q}_{t+1}} - \beta'(1 - \phi)E_i \ln \frac{\hat{Q}_{t+1}}{\hat{Q}_i}. \]

Reflecting the entry of the price of imports into the aggregate consumption price index, the cyclical component of consumption price inflation also depends on contemporaneous and expected future proportional changes in the cyclical component of the real exchange rate.

The cyclical component of output per unit of effective labour depends on the expected future cyclical component of output per unit of effective labour and the contemporaneous cyclical component of the real interest rate according to approximate linear consumption Euler equation:
Reflecting the existence of international trade linkages, the cyclical component of output per unit of effective labour also depends on expected future proportional changes in the cyclical components of foreign output per unit of effective labour and the real exchange rate.

The adjusted cyclical component of the nominal interest rate depends on the contemporaneous adjusted cyclical components of consumption price inflation and output according to monetary policy rule:

$$i_t = i_t^* + \xi (\hat{\pi}_t^C - \hat{\pi}_t^F) + \zeta (\ln \hat{Y}_t - \ln \hat{Y}_t^F) + \hat{v}_t.$$  (42)

This monetary policy rule ensures convergence of the level of consumption price inflation to its target value in flexible price equilibrium.

The cyclical component of the real exchange rate depends on the contemporaneous cyclical components of the domestic and foreign marginal utilities of consumption according to approximate linear international risk sharing condition:

$$\ln \hat{Q}_t = \frac{1}{\phi^2 \sigma + \psi (1 + \phi)(1 - \phi)} \left[ \left( \ln \frac{\hat{Y}_t}{\hat{A}_t} - \ln \frac{\hat{Y}_t^F}{\hat{A}_t^F} \right) + \ln \frac{\hat{A}_t}{\hat{A}_t^F} + (1 - \phi) \ln \frac{\hat{N}_t}{\hat{N}_t^F} \right].$$  (43)

The cyclical component of the real interest rate satisfies $\hat{r}_t = \bar{i}_t - E_t \hat{\pi}_t^C$, while the cyclical component of the real exchange rate satisfies $\ln \hat{Q}_t = \ln \hat{E}_t + \ln \hat{P}_t^Y - \ln \hat{P}_t^F$.

Variation in cyclical components is driven by three exogenous stochastic processes. The cyclical components of the productivity, population, and monetary policy shocks follow stationary first order autoregressive processes:

$$\ln \hat{A}_t = \rho_A \ln \hat{A}_{t-1} + \epsilon_A^\Delta, \quad \epsilon_A^\Delta \sim \text{iid } \mathcal{N}(0, \sigma_A^2),$$  (44)

$$\ln \hat{N}_t = \rho_N \ln \hat{N}_{t-1} + \epsilon_N^\hat{N}, \quad \epsilon_N^\hat{N} \sim \text{iid } \mathcal{N}(0, \sigma_N^2),$$  (45)

$$\hat{v}_t = \rho_v \hat{v}_{t-1} + \epsilon_v^\hat{v}, \quad \epsilon_v^\hat{v} \sim \text{iid } \mathcal{N}(0, \sigma_v^2).$$  (46)
The innovations driving these exogenous stochastic processes are assumed to be independent, which combined with our distributional assumptions implies multivariate normality. In flexible price equilibrium, \( \omega = 0 \) and \( \sigma^2 = 0 \).

### 4.3.1.2. Trend Components

The trend components of the prices of output \( \ln \bar{P}^y \) and consumption \( \ln \bar{P}^c \) are driven by common deterministic or stochastic trend \( \ln \bar{P}_t \), while the trend component of output \( \ln \bar{Y}_t \) is driven by common deterministic or stochastic trends \( \ln A_t \) and \( \ln \bar{N}_t \):

\[
\ln \bar{P}^y = \pi + \ln \bar{P}^y_{t-1} + \epsilon^y_t, \tag{47}
\]

\[
\ln \bar{P}^c = \pi + \ln \bar{P}^c_{t-1} + \epsilon^c_t, \tag{48}
\]

\[
\ln \bar{Y}_t = g + n + \ln \bar{Y}_{t-1} + \epsilon^A_t + \epsilon^N_t. \tag{49}
\]

It follows that along a balanced growth path, the level of the relative price of consumption is time and state independent.

The trend components of the nominal interest rate \( \bar{i}_t \) and nominal exchange rate \( \ln \bar{E}_t \) are time and state independent:

\[
\bar{i}_t = \bar{i}_{t-1}, \tag{50}
\]

\[
\ln \bar{E}_t = \ln \bar{E}_{t-1}. \tag{51}
\]

The trend component of the real interest rate satisfies \( \bar{r}_t = \bar{i}_t - E_t \bar{P}^c_{t+1} \), while the trend component of the real exchange rate satisfies \( \ln \bar{Q}_t = \ln \bar{E} + \ln \bar{P}^r_{t+1} - \ln \bar{P}^r_t \).

Long run balanced growth is driven by three common deterministic or stochastic trends. The trend components of the price level \( \ln \bar{P}_t \), productivity \( \ln \bar{A}_t \), and population \( \ln \bar{N}_t \) follow random walks with constant drifts:

\[
\ln \bar{P}_t = \pi + \ln \bar{P}_{t-1} + \epsilon^p_t, \epsilon^p_t \sim \text{iid } \mathcal{N}(0, \sigma^2_p), \tag{52}
\]

\[
\ln \bar{A}_t = g + \ln \bar{A}_{t-1} + \epsilon^A_t, \epsilon^A_t \sim \text{iid } \mathcal{N}(0, \sigma^2_A), \tag{53}
\]

\[
\ln \bar{N}_t = n + \ln \bar{N}_{t-1} + \epsilon^N_t, \epsilon^N_t \sim \text{iid } \mathcal{N}(0, \sigma^2_N). \tag{54}
\]
If $\sigma_p^2 = \sigma_A^2 = \sigma_b^2 = 0$ then these common trends are deterministic, and are otherwise stochastic. As an identifying restriction, all innovations are assumed to be independent, which combined with our distributional assumptions implies multivariate normality.

4.3.2. Second Best Approximation

Cyclical components are modeled by linearizing equilibrium conditions around a stationary deterministic steady state equilibrium which abstracts from long run balanced growth, while trend components are modeled as random walks while ensuring the existence of a well defined balanced growth path.

4.3.2.1. Cyclical Components

The cyclical component of output price inflation depends on the expected future cyclical component of output price inflation and the contemporaneous cyclical component of real marginal cost according to output price Phillips curve:

$$\hat{\pi}_t^y = \beta E_t \hat{\pi}_{t+1}^y + \frac{(1-\omega)(1-\omega\beta)}{\omega} \left[ \left( \frac{1}{\phi} + \frac{1}{\eta} \right) \ln \frac{\hat{Y}_t}{A_N} - \frac{1-\phi}{\phi} \left( \ln \frac{\hat{Y}_t^f}{A_N} + (\psi(1+\phi) - \phi) \ln \hat{Q}_t \right) \right].$$

Reflecting the existence of international trade linkages, the cyclical component of real marginal cost depends not only on the contemporaneous cyclical component of domestic output per unit of effective labour, but also on the contemporaneous cyclical components of foreign output per unit of effective labour and the real exchange rate.

The cyclical component of consumption price inflation depends on the expected future cyclical component of consumption price inflation and the contemporaneous cyclical component of real marginal cost according to consumption price Phillips curve:

$$\hat{\pi}_t^c = \beta E_t \hat{\pi}_{t+1}^c + \frac{(1-\omega)(1-\omega\beta)}{\omega} \left[ \left( \frac{1}{\phi} + \frac{1}{\eta} \right) \ln \frac{\hat{Y}_t}{A_N} - \frac{1-\phi}{\phi} \left( \ln \frac{\hat{Y}_t^f}{A_N} + (\psi(1+\phi) - \phi) \ln \hat{Q}_t \right) \right] + (1-\phi) \ln \frac{\hat{Q}_t}{\hat{Q}_{t-1}} - \beta (1-\phi) E_t \ln \frac{\hat{Q}_{t+1}}{\hat{Q}_t}.$$
Reflecting the entry of the price of imports into the aggregate consumption price index, the cyclical component of consumption price inflation also depends on contemporaneous and expected future proportional changes in the cyclical component of the real exchange rate.

The cyclical component of output per unit of effective labour depends on the expected future cyclical component of output per unit of effective labour and the contemporaneous cyclical component of the real interest rate according to approximate linear consumption Euler equation:

\[
\ln \frac{\dot{Y}}{A_iN_t} = E_t \ln \frac{\dot{Y}_{t+1}}{A_{i+1}N_{t+1}} - \frac{\phi}{1+\phi(\sigma-1)} \left[ \sigma \dot{r}_t - E_t \ln \frac{\hat{A}_{t+1}}{A_t} \right] \\
\quad - \frac{1-\phi}{1+\phi(\sigma-1)} \left[ E_t \Delta \ln \frac{\dot{Y}_{t+1}}{A_{i+1}N_{t+1}} + \psi(1+\phi)E_t \ln \frac{\hat{Q}_{t+1}}{Q_t} - \dot{\nu}_t \right].
\]  

(57)

Reflecting the existence of international trade linkages, the cyclical component of output per unit of effective labour also depends on expected future proportional changes in the cyclical components of foreign output per unit of effective labour and the real exchange rate.

The adjusted cyclical component of the nominal interest rate depends on the contemporaneous adjusted cyclical components of consumption price inflation and output according to monetary policy rule:

\[
\dot{i}_t - \bar{i}_t = \xi (\hat{x}^c_t - \hat{x}^c_{t+1}) + \zeta (\ln \dot{Y}_t - \ln \dot{Y}_{t+1}) + \nu_t.
\]  

(58)

This monetary policy rule ensures convergence of the level of consumption price inflation to its target value in flexible price equilibrium.

The cyclical component of the real exchange rate depends on the contemporaneous cyclical components of the domestic and foreign marginal utilities of consumption according to approximate linear international risk sharing condition:

\[
\ln \hat{Q}_t = \frac{1}{\phi \sigma + \psi(1+\phi)(1-\phi)} \left[ (1+\phi(\sigma-1)) \left( \ln \frac{\dot{Y}}{A_iN_t} - \ln \frac{\dot{Y}^f}{A^f_iN^f_t} \right) + \ln \frac{\hat{A}}{A^f} + (1-\phi) \ln \frac{\hat{N}_t}{N^f_t} \right].
\]  

(59)

The cyclical component of the real interest rate satisfies \( \dot{r}_t = \ddot{i}_t - E_t \hat{x}^c_{t+1} \), while the cyclical component of the real exchange rate satisfies \( \ln \hat{Q}_t = \ln \hat{E}_t + \ln \hat{P}^y_t - \ln \hat{P}^y_{t+1} \).

Variation in cyclical components is driven by three exogenous stochastic processes. The cyclical components of the productivity, population, and monetary policy shocks follow stationary first order autoregressive processes:
The innovations driving these exogenous stochastic processes are assumed to be independent, which combined with our distributional assumptions implies multivariate normality. In flexible price equilibrium, \( \omega = 0 \) and \( \sigma^2_v = 0 \).

4.3.2.2. Trend Components

The trend components of the prices of output \( \ln \bar{P}_t^v \) and consumption \( \ln \bar{P}_t^c \) follow random walks with time varying drift \( \pi_t \), while the trend component of output \( \ln \bar{Y}_t \) follows a random walk with time varying drift \( g_t + n_t \):

\[
\ln \bar{P}_t^v = \pi_t + \ln \bar{P}_{t-1}^v + \epsilon^v_t, \quad \epsilon^v_t \sim \text{iid } \mathcal{N}(0, \sigma^2_v),
\]

(63)

\[
\ln \bar{P}_t^c = \pi_t + \ln \bar{P}_{t-1}^c + \epsilon^c_t, \quad \epsilon^c_t \sim \text{iid } \mathcal{N}(0, \sigma^2_c),
\]

(64)

\[
\ln \bar{Y}_t = g_t + n_t + \ln \bar{Y}_{t-1} + \epsilon^v_t, \quad \epsilon^v_t \sim \text{iid } \mathcal{N}(0, \sigma^2_v).
\]

(65)

It follows that the trend component of the relative price of consumption follows a random walk without drift. This implies that along a balanced growth path, the level of this relative price is time independent but state dependent.

The trend components of the nominal interest rate \( \bar{i}_t \) and nominal exchange rate \( \ln \bar{E}_t \) follow random walks without drifts:

\[
\bar{i}_t = \bar{i}_{t-1} + \epsilon_i^v, \quad \epsilon_i^v \sim \text{iid } \mathcal{N}(0, \sigma_i^2),
\]

(66)

\[
\ln \bar{E}_t = \ln \bar{E}_{t-1} + \epsilon_i^e, \quad \epsilon_i^e \sim \text{iid } \mathcal{N}(0, \sigma_i^2).
\]

(67)

It follows that along a balanced growth path, the levels of the nominal interest rate and nominal exchange rate are time independent but state dependent. The trend component of the real interest rate satisfies \( \bar{r}_t = \bar{i}_t - \ln P_{t-1}^c \), while the trend component of the real exchange rate satisfies \( \ln \bar{Q}_t = \ln \bar{E}_t + \ln \bar{P}_{t-1}^v - \ln \bar{P}_t^v \).
Long run balanced growth is driven by three common stochastic trends. Trend inflation $\pi_t$, productivity growth $g_t$, and population growth $n_t$ follow random walks without drifts:

$$\pi_t = \pi_{t-1} + \varepsilon_t^\pi, \quad \varepsilon_t^\pi \sim \text{iid } N(0, \sigma_\pi^2),$$  \tag{68}

$$g_t = g_{t-1} + \varepsilon_t^g, \quad \varepsilon_t^g \sim \text{iid } N(0, \sigma_g^2),$$  \tag{69}

$$n_t = n_{t-1} + \varepsilon_t^n, \quad \varepsilon_t^n \sim \text{iid } N(0, \sigma_n^2).$$  \tag{70}

It follows that along a balanced growth path, growth rates are time independent but state dependent. As an identifying restriction, all innovations are assumed to be independent, which combined with our distributional assumptions implies multivariate normality.

**4.4. Estimation and Inference**

The finite sample properties of the procedure proposed by Vitek (2006f) for the measurement of the stance of monetary policy in a small open economy are analyzed within the framework of these alternative approximate unobserved components representations of this DSGE model with a Monte Carlo experiment. Each replication of this Monte Carlo experiment consists of two steps. In the first step, a linear state space representation of the first best approximation to this DSGE model of a small open economy is simulated conditional on calibrated parameter values and initial conditions. In generating artificial data sets, both deterministic and stochastic trend component specifications are employed. In the second step, the parameters and unobserved components of a linear state space representation of the second best approximation to this DSGE model are jointly estimated with a Bayesian procedure, conditional on prior information concerning the values of parameters and trend components. Averaging the differences and squared differences between estimated and simulated levels of the flexible price equilibrium components of various observed and unobserved endogenous variables across replications of this Monte Carlo experiment, with an emphasis on the levels of the natural rate of interest and natural exchange rate, facilitates measurement of accuracy and precision in terms of bias and root mean squared error.

The linear state space representation of the second best approximation to this DSGE model of a small open economy approximately nests the linear state space representation of the first best approximation, irrespective of whether common trends are deterministic or stochastic. To elaborate, the cyclical component specifications differ only with respect to the discount factor entering into the coefficients of the Phillips curves, while the trend component specifications are
fully nested under restrictions on variance parameters if common trends are deterministic, and are approximately nested under restrictions on variance parameters if common trends are stochastic. Furthermore, under the first best approximation to this DSGE model, the exogenous stochastic processes associated with the stochastic trend component specification nest commonly employed types of exogenous stochastic processes under parameter restrictions. It follows that the estimated model associated with this Monte Carlo experiment is approximately correctly specified for the data generating process, in the sense that it approximately nests the data generating process, while the exogenous stochastic processes associated with the data generating process nest commonly employed types of exogenous stochastic processes.

4.4.1. Estimation

Let \( x_t \) denote a vector stochastic process consisting of the levels of \( N \) nonpredetermined endogenous variables, of which \( M \) are observed. The cyclical components of this vector stochastic process satisfy second order stochastic linear difference equation

\[
A_0 \hat{x}_t = A_1 \hat{x}_{t-1} + A_2 E_t \hat{x}_{t+1} + A_3 \hat{x}_t + A_4 \hat{v}_t,
\]

where vector stochastic process \( \hat{x}_t \) consists of the flexible price equilibrium components of \( N \) nonpredicted endogenous variables. The cyclical components of this vector stochastic process satisfy second order stochastic linear difference equation

\[
B_0 \hat{x}_t = B_1 \hat{x}_{t-1} + B_2 E_t \hat{x}_{t+1} + B_3 \hat{v}_t,
\]

where vector stochastic process \( \hat{v}_t \) consists of the cyclical components of \( K \) exogenous variables. This vector stochastic process satisfies stationary first order stochastic linear difference equation

\[
\hat{v}_t = C_t \hat{v}_{t-1} + e_{1,t},
\]

\(^1\) Under the first best approximation with common deterministic trends \( \sigma_{\alpha_1} = \sigma_{\alpha_2} = \sigma_{\alpha_3} = \sigma_{\alpha_4} = \sigma_{\alpha_5} = \sigma_{\alpha_6} = 0 \) and \( \sigma_{\alpha_7} = \sigma_{\alpha_8} = \sigma_{\alpha_9} = \sigma_{\alpha_10} = \sigma_{\alpha_11} = \sigma_{\alpha_12} = 0 \). Under the first best approximation with common stochastic trends \( \sigma_{\beta_1} = \sigma_{\beta_2} = \sigma_{\beta_3} = \sigma_{\beta_4} = \sigma_{\beta_5} = \sigma_{\beta_6} = \sigma_{\beta_7} = \sigma_{\beta_8} = \sigma_{\beta_9} = \sigma_{\beta_{10}} = \sigma_{\beta_{11}} = \sigma_{\beta_{12}} = 0 \) and \( \sigma_{\beta_{13}} = \sigma_{\beta_{14}} = \sigma_{\beta_{15}} = \sigma_{\beta_{16}} = \sigma_{\beta_{17}} = \sigma_{\beta_{18}} = \sigma_{\beta_{19}} = \sigma_{\beta_{20}} = \sigma_{\beta_{21}} = \sigma_{\beta_{22}} = \sigma_{\beta_{23}} = \sigma_{\beta_{24}} = 0 \).

\(^2\) Consider the exogenous stochastic process governing the evolution of the level of productivity. Under the first best approximation, this exogenous stochastic process has structural form representation \( \ln \alpha_t = \ln \alpha_t + \ln \hat{\alpha}_t \), where \( \ln \hat{\alpha}_t = \rho_{\alpha_1} \ln \hat{\alpha}_{t-1} + \epsilon_{\alpha_1} \) with \( \epsilon_{\alpha_1} \sim \text{iid} \mathcal{N}(0, \sigma_{\alpha_1}^2) \), and \( \ln \hat{\alpha}_t = g + \ln \hat{\alpha}_t + \epsilon_{\alpha_1} \) with \( \epsilon_{\alpha_1} \sim \text{iid} \mathcal{N}(0, \sigma_{\alpha_2}^2) \). Under the assumption that \( \epsilon_{\alpha_1} \) and \( \epsilon_{\alpha_2} \) are independent, it has reduced form representation \( \Delta \ln \alpha_t = (1 - \rho_{\alpha_1}) g + \rho_{\alpha_1} \Delta \ln \hat{\alpha}_{t-1} + \epsilon_{\alpha_1} + \theta_1 \epsilon_{\alpha_2} + \theta_2 \epsilon_{\alpha_2} \) with \( \epsilon_{\alpha_1} \sim \text{iid} \mathcal{N}(0, \sigma_{\alpha_2}^2) \), where \( \theta_1 \) and \( \sigma_{\alpha_2} \) are functions of \( \rho_{\alpha_1}, \sigma_{\alpha_2}^2 \) and \( \sigma_{\alpha_3} \).
where $e_{ij} \sim \text{iid } \mathcal{N}(0, \Sigma_i)$). The parameter matrices differ depending on whether the first best or second best approximation is employed.

The trend components of vector stochastic process $x_t$ satisfy first order stochastic linear difference equation

$$D_0 \ddot{x}_t = D_1 + D_2 u_t + D_3 \ddot{x}_{t-1} + \varepsilon_{2,t},$$  

where $\varepsilon_{2,t} \sim \text{iid } \mathcal{N}(0, \Sigma_2)$. Vector stochastic process $u_t$ consists of the levels of $L$ common stochastic trends, and satisfies nonstationary first order stochastic linear difference equation

$$u_t = u_{t-1} + \varepsilon_{3,t},$$  

where $\varepsilon_{3,t} \sim \text{iid } \mathcal{N}(0, \Sigma_3)$. The parameter matrices differ depending on whether the first best or second best approximation is employed, in addition to whether common trends are deterministic or stochastic.\(^3\) Cyclical and trend components are additively separable, which implies that $x_t = \hat{x}_t + \bar{x}_t$ and $\ddot{x}_t = \hat{x}_t + \ddot{x}_t$, where $\dddot{x}_t = \dddot{x}_t$.

If there exists a unique stationary solution to multivariate linear rational expectations model (71), then it may be expressed as:

$$\hat{x}_t = S_1 \hat{x}_{t-1} + S_2 \hat{x}_{t-1} + S_3 \hat{v}_t,$$

If there exists a unique stationary solution to multivariate linear rational expectations model (72), then it may be expressed as:

$$\hat{x}_t = T_1 \hat{x}_{t-1} + T_2 \hat{v}_t.$$

These solutions are calculated simultaneously with the matrix decomposition based algorithm due to Klein (2000).

Let $y_t$ denote a vector stochastic process consisting of the levels of $M$ observed nonpredetermined endogenous variables. Also, let $z_t$ denote a vector stochastic process consisting of the levels of $N - M$ unobserved nonpredetermined endogenous variables, the cyclical components of $N$ nonpredetermined endogenous variables, the cyclical components of the flexible price equilibrium components of $N$ nonpredetermined endogenous variables, the trend components of $N$ nonpredetermined endogenous variables, the cyclical components of $K$ exogenous variables, and the levels of $L$ common stochastic trends. Given unique stationary solutions (76) and (77), these vector stochastic processes have linear state space representation

\(^3\) Under the first best approximation with common deterministic trends $D_2 = 0$, $\Sigma_2 = 0$ and $\Sigma_3 = 0$. Under the first best approximation with common stochastic trends $D_2 = 0$ and $\Sigma_3 = 0$. 
\[ y_t = F_t z_t, \]  
\[ z_t = G_1 + G_2 z_{t-1} + G_3 e_{4,t}, \]

where \( e_{4,t} \sim \text{iid } \mathcal{N}(0, \Sigma_4) \) and \( z_0 \sim \mathcal{N}(z_{00}, P_{00}) \). Let \( w_t \) denote a vector stochastic process consisting of preliminary estimates of the trend components of \( M \) observed nonpredetermined endogenous variables. Suppose that this vector stochastic process satisfies

\[ w_t = H_t z_t + e_{5,t}, \]

where \( e_{5,t} \sim \text{iid } \mathcal{N}(0, \Sigma_5) \). Conditional on known parameter values, this signal equation defines a set of stochastic restrictions on selected unobserved state variables. The signal and state innovation vectors are assumed to be independent, while the initial state vector is assumed to be independent from the signal and state innovation vectors, which combined with our distributional assumptions implies multivariate normality.

Conditional on the parameters associated with these signal and state equations, estimates of unobserved state vector \( z_t \) and its mean squared error matrix \( P_t \) may be calculated with the filter proposed by Vitek (2006a, 2006b), which adapts the filter due to Kalman (1960) to incorporate prior information. Given initial conditions \( z_{00} \) and \( P_{00}, \) estimates conditional on information available at time \( t-1 \) satisfy prediction equations:

\[ z_{t|t-1} = G_1 + G_2 z_{t-1|t-1}, \]  
\[ P_{t|t-1} = G_2 P_{t-1|t-1} G_2^\top + G_3 \Sigma_4 G_3^\top, \]
\[ y_{t|t-1} = F_t z_{t|t-1}, \]  
\[ Q_{t|t-1} = F_t P_{t|t-1} F_t^\top, \]
\[ w_{t|t-1} = H_t z_{t|t-1}, \]  
\[ R_{t|t-1} = H_t P_{t|t-1} H_t^\top + \Sigma_5. \]

Given these predictions, under the assumption of multivariate normally distributed signal and state innovation vectors, together with conditionally contemporaneously uncorrelated signal vectors, estimates conditional on information available at time \( t \) satisfy updating equations

\[ z_t = z_{t|t-1} + K_y (y_t - y_{t|t-1}) + K_w (w_t - w_{t|t-1}), \]
\[ P_{jT} = P_{jT-1} - K_y H_j P_{jT-1}, \]  

(88)

where \( K_y = P_{jT-1} F_j^T Q_{jT}^{-1} \) and \( K_w = P_{jT-1} H_j^T R_{jT}^{-1} \). Given terminal conditions \( z_{jT} \) and \( P_{jT} \), obtained from the final evaluation of these prediction and updating equations, estimates conditional on information available at time \( T \) satisfy smoothing equations

\[ z_{jT} = z_{jT} + J_i (z_{T+1} - z_{T+1}), \]

(89)

\[ P_{jT} = P_{jT} + J_i (P_{T+1} - P_{T+1}) J_i^T, \]

(90)

where \( J_i = P_{jT} \Phi_j^T P_{jT} \). Under our distributional assumptions, these estimators of the unobserved state vector are mean squared error optimal.

Let \( \theta \in \Theta \subset \mathbb{R}^J \) denote a \( J \) dimensional vector containing the parameters associated with the signal and state equations of this linear state space model. The Bayesian estimator of this parameter vector has posterior density function

\[ f(\theta | I_T) = f(I_T | \theta) f(\theta), \]

(91)

where \( I_T = \{ \{ y_s \}_{s=1}^T, \{ w_s \}_{s=1}^T \} \). Under the assumption of multivariate normally distributed signal and state innovation vectors, together with conditionally contemporaneously uncorrected signal vectors, conditional density function \( f(I_T | \theta) \) satisfies:

\[ f(I_T | \theta) = \prod_{t=1}^T f(y_t | I_{t-1}, \theta) \cdot \prod_{t=1}^T f(w_t | I_{t-1}, \theta). \]

(92)

Under our distributional assumptions, conditional density functions \( f(y_t | I_{t-1}, \theta) \) and \( f(w_t | I_{t-1}, \theta) \) satisfy:

\[ f(y_t | I_{t-1}, \theta) = (2\pi)^{-\frac{M}{2}} |Q_{\theta_{t-1}}^{-1}|^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2} (y_t - y_{\theta_{t-1}})^T Q_{\theta_{t-1}}^{-1} (y_t - y_{\theta_{t-1}}) \right\}, \]

(93)

\[ f(w_t | I_{t-1}, \theta) = (2\pi)^{-\frac{M}{2}} |R_{\theta_{t-1}}^{-1}|^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2} (w_t - w_{\theta_{t-1}})^T R_{\theta_{t-1}}^{-1} (w_t - w_{\theta_{t-1}}) \right\}. \]

(94)

Prior information concerning parameter vector \( \theta \) is summarized by a multivariate normal prior distribution having mean vector \( \theta_i \) and covariance matrix \( \Omega \):

\[ f(\theta) = (2\pi)^{-\frac{J}{2}} |\Omega|^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2} (\theta - \theta_i)^T \Omega^{-1} (\theta - \theta_i) \right\}. \]

(95)
Independent priors are represented by a diagonal covariance matrix, under which diffuse priors are represented by infinite variances.

Inference on the parameters is based on an asymptotic normal approximation to the posterior distribution around its mode. Under regularity conditions stated in Geweke (2005), posterior mode $\hat{\theta}_r$ satisfies

$$\sqrt{T}(\hat{\theta}_r - \theta_0) \overset{d}{\to} N(0, -H_0^{-1}),$$

where $\theta_0 \in \Theta$ denotes the pseudottrue parameter vector. Following Engle and Watson (1981), Hessian $H_0$ may be estimated by

$$H_r = \frac{1}{T} \sum_{t=1}^{T} E_{t-1} \left[ \nabla_\theta \nabla_\theta^T \ln f(y_t | \mathcal{I}_{t-1}, \hat{\theta}_r) \right] + \frac{1}{T} \sum_{t=1}^{T} E_{t-1} \left[ \nabla_\theta \nabla_\theta^T \ln f(w_t | \mathcal{I}_{t-1}, \hat{\theta}_r) \right]$$

$$+ \frac{1}{T} \nabla_\theta \nabla_\theta^T \ln f(\hat{\theta}_r),$$

where

$$E_{t-1} \left[ \nabla_\theta \nabla_\theta^T \ln f(y_t | \mathcal{I}_{t-1}, \hat{\theta}_r) \right] = -\nabla_\theta y_{t|\delta_{\theta}} Q_{\delta_{\theta}-1}^{-1} \nabla_\theta y_{\delta_{\theta}-1} - \frac{1}{2} \nabla_\theta Q_{\delta_{\theta}-1}^{-1} \left( \frac{Q_{\delta_{\theta}-1}^{-1} \otimes Q_{\delta_{\theta}-1}^{-1}}{2} \right) \nabla_\theta Q_{\delta_{\theta}-1},$$

$$E_{t-1} \left[ \nabla_\theta \nabla_\theta^T \ln f(w_t | \mathcal{I}_{t-1}, \hat{\theta}_r) \right] = -\nabla_\theta w_{t|\delta_{\theta}} R_{\delta_{\theta}-1}^{-1} \nabla_\theta w_{\delta_{\theta}-1} - \frac{1}{2} \nabla_\theta R_{\delta_{\theta}-1}^{-1} \left( R_{\delta_{\theta}-1}^{-1} \otimes R_{\delta_{\theta}-1}^{-1} \right) \nabla_\theta R_{\delta_{\theta}-1},$$

and

$$\nabla_\theta \nabla_\theta^T \ln f(\hat{\theta}_r) = -\Omega^{-1}.$$

### 4.4.2. Inference

The design of this Monte Carlo experiment is realistic in the sense that the true parameter values are all well within the range of estimates reported in the existing literature, after accounting for data rescaling, while the sample size is consistent with the span and frequency of real data sets typically employed in the estimation of DSGE models. The true values of parameters are reported in Table 4.1. Under both deterministic and stochastic trend component specifications, artificial data sets consist of 200 simulated observations on the levels of eight observed endogenous variables, namely domestic and foreign price levels, outputs, nominal interest rates, and the nominal exchange rate. This sample size corresponds to 50 years of quarterly observations.
Table 4.1. True values of parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>0.9900</td>
<td>$\rho_1$</td>
<td>0.5000</td>
<td>$\pi$</td>
<td>0.5000</td>
</tr>
<tr>
<td>$\eta$</td>
<td>1.0000</td>
<td>$\rho_2$</td>
<td>0.5000</td>
<td>$g$</td>
<td>0.2500</td>
</tr>
<tr>
<td>$\omega$</td>
<td>0.8000</td>
<td>$\rho_3$</td>
<td>0.5000</td>
<td>$\eta$</td>
<td>0.5000</td>
</tr>
<tr>
<td>$\phi$</td>
<td>0.7000</td>
<td>$\sigma_1^2$</td>
<td>0.2500</td>
<td>$\sigma_2^2$</td>
<td>0.2500</td>
</tr>
<tr>
<td>$\psi$</td>
<td>1.5000</td>
<td>$\sigma_3^2$</td>
<td>0.2500</td>
<td>$\sigma_3^2$</td>
<td>0.1250</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>1.0000</td>
<td>$\sigma_4^2$</td>
<td>0.2500</td>
<td>$\sigma_4^2$</td>
<td>0.1250</td>
</tr>
<tr>
<td>$\theta$</td>
<td>7.6667</td>
<td>$\sigma_5^2$</td>
<td>0.2500</td>
<td>$\sigma_5^2$</td>
<td>0.2500</td>
</tr>
<tr>
<td>$\xi$</td>
<td>1.5000</td>
<td>$\sigma_6^2$</td>
<td>0.2500</td>
<td>$\sigma_6^2$</td>
<td>0.1250</td>
</tr>
<tr>
<td>$\zeta$</td>
<td>0.1250</td>
<td>$\sigma_7^2$</td>
<td>0.2500</td>
<td>$\sigma_7^2$</td>
<td>0.1250</td>
</tr>
</tbody>
</table>

Note: The data generating process is calibrated at a quarterly frequency under the assumption that all observed endogenous variables are rescaled by a factor of 100.

The set of parameters associated with this DSGE model of a small open economy is partitioned into two subsets. The first subset is calibrated to equal true values, while the second subset is estimated with the Bayesian procedure described above, conditional on prior information concerning the values of parameters and trend components. Those parameters associated with the conditional mean function are estimated conditional on cross-economy equality restrictions and informative independent priors, while those parameters associated exclusively with the conditional variance function are estimated conditional on diffuse priors. The means of informative marginal prior distributions equal true values, penalizing deviations from them. Initial conditions for the cyclical components of exogenous variables are given by their unconditional means and variances, while the initial values of all other state variables are treated as parameters, and are calibrated to equal true values. The posterior mode is calculated by numerically maximizing the logarithm of the posterior density kernel with a modified steepest ascent algorithm.

Prior information concerning the values of trend components is generated by fitting first order deterministic polynomial functions to the levels of all observed endogenous variables by ordinary least squares. Stochastic restrictions on the trend components of all observed endogenous variables are derived from the fitted values associated with these ordinary least squares regressions, with innovation variances set proportional to estimated prediction variances assuming known parameters. All stochastic restrictions are independent, represented by a diagonal covariance matrix, and are harmonized, represented by a common factor of proportionality. Reflecting little confidence in these preliminary trend component estimates, this common factor of proportionality is set equal to one.

This Monte Carlo experiment indicates that joint estimation of parameters and unobserved components with the Bayesian procedure under consideration yields reasonably accurate and precise results. Parameter estimation results under deterministic and stochastic trend component
specifications are reported in Table 4.2 and Table 4.3, respectively. Examination of these results reveals that, under both deterministic and stochastic trend component specifications, the modes of the marginal posterior distributions of the parameters exhibit statistically insignificant differences from true values at conventional levels, while posterior standard errors are relatively small. However, posterior standard errors based on asymptotic distribution theory tend to overstate uncertainty surrounding estimates of the parameters, implying that inference on them based on an asymptotic normal approximation to the posterior distribution around its mode tends to be conservative. That estimates of those parameters associated with the conditional mean function are approximately unbiased is in part attributable to the design of this Monte Carlo experiment, under which the means of informative marginal prior distributions equal true values, penalizing deviations from them. Nevertheless, the data remain informative with respect to these parameters, as prior standard errors are larger than posterior standard errors. Examination of these results also reveals that the modes of the marginal posterior distributions of parameters tend to exhibit smaller deviations from true values under a deterministic trend component specification than under a stochastic trend component specification, while posterior standard errors are generally smaller. These results are to be expected, as prior information concerning the values of trend components represents the belief that common trends are deterministic as opposed to stochastic.
Table 4.2. Experimental results under deterministic trend specification, parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>True Value</th>
<th>Prior Distribution</th>
<th>Posterior Distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>SE</td>
<td>Mode</td>
</tr>
<tr>
<td>η</td>
<td>1.000000</td>
<td>1.000000</td>
<td>1.001700</td>
</tr>
<tr>
<td>ω</td>
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<td>0.800000</td>
<td>0.817700</td>
</tr>
<tr>
<td>ψ</td>
<td>1.500000</td>
<td>1.500000</td>
<td>1.529300</td>
</tr>
<tr>
<td>σ</td>
<td>1.000000</td>
<td>1.000000</td>
<td>1.055300</td>
</tr>
<tr>
<td>ξ</td>
<td>1.500000</td>
<td>1.500000</td>
<td>1.486400</td>
</tr>
<tr>
<td>ζ</td>
<td>0.125000</td>
<td>0.125000</td>
<td>0.125280</td>
</tr>
<tr>
<td>ρ₁</td>
<td>0.500000</td>
<td>0.500000</td>
<td>0.515920</td>
</tr>
<tr>
<td>ρ₂</td>
<td>0.500000</td>
<td>0.500000</td>
<td>0.493610</td>
</tr>
<tr>
<td>ρ₃</td>
<td>0.500000</td>
<td>0.500000</td>
<td>0.501490</td>
</tr>
<tr>
<td>σ₁</td>
<td>0.250000</td>
<td>-</td>
<td>0.247010</td>
</tr>
<tr>
<td>σ₂</td>
<td>0.250000</td>
<td>-</td>
<td>0.248150</td>
</tr>
<tr>
<td>σ₃</td>
<td>0.250000</td>
<td>-</td>
<td>0.249110</td>
</tr>
<tr>
<td>σ₄</td>
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<td>-</td>
<td>0.245890</td>
</tr>
<tr>
<td>σ₅</td>
<td>0.250000</td>
<td>-</td>
<td>0.247250</td>
</tr>
<tr>
<td>σ₆</td>
<td>0.250000</td>
<td>-</td>
<td>0.249250</td>
</tr>
<tr>
<td>σ₇</td>
<td>0.000000</td>
<td>-</td>
<td>0.009145</td>
</tr>
<tr>
<td>σ₈</td>
<td>0.000000</td>
<td>-</td>
<td>0.009100</td>
</tr>
<tr>
<td>σ₉</td>
<td>0.000000</td>
<td>-</td>
<td>0.009897</td>
</tr>
<tr>
<td>σ₁₀</td>
<td>0.000000</td>
<td>-</td>
<td>0.009766</td>
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<tr>
<td>σ₁₁</td>
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<td>-</td>
<td>0.009765</td>
</tr>
<tr>
<td>σ₁₂</td>
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<td>0.009404</td>
</tr>
<tr>
<td>σ₁₃</td>
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<td>0.009900</td>
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<tr>
<td>σ₁₄</td>
<td>0.000000</td>
<td>-</td>
<td>0.009650</td>
</tr>
<tr>
<td>σ₁₅</td>
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<td>-</td>
<td>1.00×10⁻⁵</td>
</tr>
<tr>
<td>σ₁₆</td>
<td>0.000000</td>
<td>-</td>
<td>5.00×10⁻⁷</td>
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<tr>
<td>σ₁₇</td>
<td>0.000000</td>
<td>-</td>
<td>5.00×10⁻⁷</td>
</tr>
<tr>
<td>σ₁₈</td>
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<td>1.00×10⁻⁴</td>
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<tr>
<td>σ₁₉</td>
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<td>-</td>
<td>5.00×10⁻⁷</td>
</tr>
<tr>
<td>σ₂₀</td>
<td>0.000000</td>
<td>-</td>
<td>5.00×10⁻⁷</td>
</tr>
</tbody>
</table>

Note: The ensemble modes, standard errors, and asymptotic standard errors of the marginal posterior distributions of parameters are calculated by averaging posterior modes, squared deviations of posterior modes from true values, and asymptotic standard errors across 100 replications, respectively. The parameters are estimated subject to identifying restrictions $σ_i^2 = σ_j^2$ and $σ_i^2 = σ_j^2$. 
### Table 4.3. Experimental results under stochastic trend specification, parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>True Value</th>
<th>Prior Distribution</th>
<th>Posterior Distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Mean</td>
<td>SE</td>
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<td>1.000000</td>
<td>0.100000</td>
</tr>
<tr>
<td>( \omega )</td>
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<td>0.800000</td>
<td>0.080000</td>
</tr>
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<td>( \psi )</td>
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<td>1.500000</td>
<td>0.150000</td>
</tr>
<tr>
<td>( \sigma )</td>
<td>1.000000</td>
<td>1.000000</td>
<td>0.100000</td>
</tr>
<tr>
<td>( \xi )</td>
<td>1.500000</td>
<td>1.500000</td>
<td>0.150000</td>
</tr>
<tr>
<td>( \zeta )</td>
<td>0.125000</td>
<td>0.125000</td>
<td>0.012500</td>
</tr>
<tr>
<td>( \rho_1 )</td>
<td>0.500000</td>
<td>0.500000</td>
<td>0.050000</td>
</tr>
<tr>
<td>( \rho_2 )</td>
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<td>0.500000</td>
<td>0.050000</td>
</tr>
<tr>
<td>( \rho_3 )</td>
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<td>0.500000</td>
<td>0.050000</td>
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<tr>
<td>( \sigma_1 )</td>
<td>0.250000</td>
<td>-</td>
<td>( \infty )</td>
</tr>
<tr>
<td>( \sigma_2 )</td>
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<td>-</td>
<td>( \infty )</td>
</tr>
<tr>
<td>( \sigma_3 )</td>
<td>0.250000</td>
<td>-</td>
<td>( \infty )</td>
</tr>
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<td>( \sigma_4 )</td>
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<td>-</td>
<td>( \infty )</td>
</tr>
<tr>
<td>( \sigma_5 )</td>
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<td>-</td>
<td>( \infty )</td>
</tr>
<tr>
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<td>-</td>
<td>( \infty )</td>
</tr>
<tr>
<td>( \sigma_7 )</td>
<td>0.250000</td>
<td>-</td>
<td>( \infty )</td>
</tr>
<tr>
<td>( \sigma_8 )</td>
<td>0.250000</td>
<td>-</td>
<td>( \infty )</td>
</tr>
<tr>
<td>( \sigma_9 )</td>
<td>0.250000</td>
<td>-</td>
<td>( \infty )</td>
</tr>
<tr>
<td>( \sigma_{10} )</td>
<td>0.000000</td>
<td>-</td>
<td>( \infty )</td>
</tr>
<tr>
<td>( \sigma_{11} )</td>
<td>0.000000</td>
<td>-</td>
<td>( \infty )</td>
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<tr>
<td>( \sigma_{12} )</td>
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<td>-</td>
<td>( \infty )</td>
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<td>( \sigma_{13} )</td>
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<td>( \sigma_{14} )</td>
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<td>( \sigma_{15} )</td>
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<td>-</td>
<td>( \infty )</td>
</tr>
<tr>
<td>( \sigma_{16} )</td>
<td>0.000000</td>
<td>-</td>
<td>( \infty )</td>
</tr>
<tr>
<td>( \sigma_{17} )</td>
<td>0.000000</td>
<td>-</td>
<td>( \infty )</td>
</tr>
<tr>
<td>( \sigma_{18} )</td>
<td>0.000000</td>
<td>-</td>
<td>( \infty )</td>
</tr>
<tr>
<td>( \sigma_{19} )</td>
<td>0.000000</td>
<td>-</td>
<td>( \infty )</td>
</tr>
<tr>
<td>( \sigma_{20} )</td>
<td>0.000000</td>
<td>-</td>
<td>( \infty )</td>
</tr>
<tr>
<td>( \sigma_{21} )</td>
<td>0.000000</td>
<td>-</td>
<td>( \infty )</td>
</tr>
<tr>
<td>( \sigma_{22} )</td>
<td>0.000000</td>
<td>-</td>
<td>( \infty )</td>
</tr>
</tbody>
</table>

**Note:** The ensemble modes, standard errors, and asymptotic standard errors of the marginal posterior distributions of parameters are calculated by averaging posterior modes, squared deviations of posterior modes from true values, and asymptotic standard errors across 100 replications, respectively. The parameters are estimated subject to identifying restrictions \( \sigma_2 = \sigma_2 \) and \( \sigma_3 = \sigma_3 \).

Theoretically prominent indicators of inflationary pressure such as the natural rate of interest and natural exchange rate are unobservable. As discussed in Woodford (2003), the level of the natural rate of interest provides a measure of the neutral stance of monetary policy, with deviations of the real interest rate from the natural rate of interest generating inflationary pressure. It follows that the key to achieving low and stable inflation is the conduct of a monetary policy under which the short term nominal interest rate tracks variation in the level of the natural rate of interest as closely as possible.

This Monte Carlo experiment indicates that joint estimation of parameters and unobserved components with the Bayesian estimation procedure under consideration yields reasonably accurate and precise estimates of the level, cyclical component and trend component of the natural rate of interest conditional on alternative information sets, irrespective of whether the...
data generating process features common deterministic or stochastic trends. The results of estimating the level, cyclical component and trend component of the natural rate of interest under deterministic and stochastic trend component specifications are reported in Table 4.4 and Table 4.5, respectively. This concept of the natural rate of interest represents that real interest rate consistent with past, present and future price flexibility. The predicted estimates are conditional on past information, the filtered estimates are conditional on past and present information, and the smoothed estimates are conditional on past, present and future information. Examination of these results reveals that, under both deterministic and stochastic trend component specifications, estimates of the level, cyclical component and trend component of the natural rate of interest conditional on alternative information sets are approximately unbiased, while root mean squared errors are relatively small. That estimates of the natural rate of interest are generally more accurate and precise under a deterministic trend component specification than under a stochastic trend component specification, as evidenced by smaller biases and root mean squared errors, reflects the design of this Monte Carlo experiment, under which prior information concerning the values of trend components represents the belief that common trends are deterministic as opposed to stochastic. Examination of these results also reveals that analytical root mean squared errors appropriately account for uncertainty surrounding estimates of the natural rate of interest, with size distortions being small under both deterministic and stochastic trend component specifications. These size distortions may be partially attributed to the fact that analytical root mean squared errors do not account for parameter uncertainty, whereas simulated root mean squared errors do, inflating them to some extent.

Table 4.4. Experimental results under deterministic trend specification, natural rate of interest

<table>
<thead>
<tr>
<th>Estimate</th>
<th>Level</th>
<th>Cyclical Component</th>
<th>Trend Component</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Bias</td>
<td>RMSE</td>
<td>ARMSE</td>
</tr>
<tr>
<td>Predicted</td>
<td>0.002016</td>
<td>1.978920</td>
<td>1.977000</td>
</tr>
<tr>
<td>Filtered</td>
<td>0.001975</td>
<td>0.967920</td>
<td>1.177400</td>
</tr>
<tr>
<td>Smoothed</td>
<td>0.001120</td>
<td>0.867760</td>
<td>1.008440</td>
</tr>
</tbody>
</table>

Note: The ensemble biases, root mean squared errors, and analytical root mean squared errors of the marginal posterior distributions of state variables are calculated by averaging deviations of posterior means from simulated values, squared deviations of posterior means from simulated values, and analytical mean squared errors across 100 replications, respectively. Under the data generating process, the unconditional mean of the natural rate of interest is 4%, expressed at an annual percentage rate. All results are reported at an annual percentage rate.
Table 4.5. Experimental results under stochastic trend specification, natural rate of interest

<table>
<thead>
<tr>
<th>Estimate</th>
<th>Level</th>
<th>Cyclical Component</th>
<th>Trend Component</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Bias</td>
<td>RMSE</td>
<td>ARMSE</td>
</tr>
<tr>
<td>Predicted</td>
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<td>2.033000</td>
</tr>
<tr>
<td>Filtered</td>
<td>0.006227</td>
<td>1.362800</td>
<td>1.333400</td>
</tr>
<tr>
<td>Smoothed</td>
<td>0.016802</td>
<td>1.396280</td>
<td>1.197800</td>
</tr>
</tbody>
</table>

Note: The ensemble biases, root mean squared errors, and analytical root mean squared errors of the marginal posterior distributions of state variables are calculated by averaging deviations of posterior means from simulated values, squared deviations of posterior means from simulated values, and analytical mean squared errors across 100 replications, respectively. Under the data generating process, the unconditional mean of the natural rate of interest is 4%, expressed at an annual percentage rate. All results are reported at an annual percentage rate.

This Monte Carlo experiment indicates that joint estimation of parameters and unobserved components with the Bayesian procedure under consideration also yields reasonably accurate and precise estimates of the level, cyclical component and trend component of the natural exchange rate conditional on alternative information sets, irrespective of whether the data generating process features common deterministic or stochastic trends. The results of estimating the level, cyclical component and trend component of the natural exchange rate under deterministic and stochastic trend component specifications are reported in Table 4.6 and Table 4.7, respectively. This concept of the natural exchange rate represents that real exchange rate consistent with past, present and future price flexibility. Examination of these results reveals numerous parallels with the results of estimating the natural rate of interest. Under both deterministic and stochastic trend component specifications, estimates of the level, cyclical component and trend component of the natural exchange rate conditional on alternative information sets are approximately unbiased, while root mean squared errors are relatively small. Furthermore, estimates of the natural exchange rate are generally more accurate and precise under a deterministic trend component specification than under a stochastic trend component specification, as evidenced by smaller biases and root mean squared errors. Finally, analytical root mean squared errors appropriately account for uncertainty surrounding estimates of the natural exchange rate, with size distortions being small under both deterministic and stochastic trend component specifications.
Table 4.6. Experimental results under deterministic trend specification, natural exchange rate

<table>
<thead>
<tr>
<th>Estimate</th>
<th>Level</th>
<th>Cyclical Component</th>
<th>Trend Component</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Bias</td>
<td>RMSE</td>
<td>ARMSE</td>
</tr>
<tr>
<td>Predicted</td>
<td>-0.001269</td>
<td>0.633730</td>
<td>0.656290</td>
</tr>
<tr>
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<td>0.319740</td>
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<td>0.237530</td>
<td>0.293560</td>
</tr>
</tbody>
</table>

Note: The ensemble biases, root mean squared errors, and analytical root mean squared errors of the marginal posterior distributions of state variables are calculated by averaging deviations of posterior means from simulated values, squared deviations of posterior means from simulated values, and analytical mean squared errors across 100 replications, respectively. Under the data generating process, the unconditional mean of the natural exchange rate is normalized to one.

Table 4.7. Experimental results under stochastic trend specification, natural exchange rate

<table>
<thead>
<tr>
<th>Estimate</th>
<th>Level</th>
<th>Cyclical Component</th>
<th>Trend Component</th>
</tr>
</thead>
<tbody>
<tr>
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<td>Bias</td>
<td>RMSE</td>
<td>ARMSE</td>
</tr>
<tr>
<td>Predicted</td>
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<td>0.943340</td>
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<td>0.305880</td>
</tr>
<tr>
<td>Smoothed</td>
<td>0.004117</td>
<td>0.396570</td>
<td>0.263190</td>
</tr>
</tbody>
</table>

Note: The ensemble biases, root mean squared errors, and analytical root mean squared errors of the marginal posterior distributions of state variables are calculated by averaging deviations of posterior means from simulated values, squared deviations of posterior means from simulated values, and analytical mean squared errors across 100 replications, respectively. Under the data generating process, the unconditional mean of the natural exchange rate is normalized to one.

4.5. Conclusion

This paper evaluates the finite sample properties of a novel procedure proposed by Vitek (2006f) for the measurement of the stance of monetary policy in a small open economy with a Monte Carlo experiment. Joint estimation of the parameters and unobserved components of a linear state space representation of an approximate unobserved components representation of a DSGE model of a small open economy with this Bayesian procedure, conditional on prior information concerning the values of parameters and trend components, is found to yield reasonably accurate and precise results in samples of currently available size. In particular, estimates of the levels of the natural rate of interest and natural exchange rate conditional on alternative information sets are approximately unbiased, while root mean squared errors are relatively small, irrespective of whether the data generating process features common deterministic or stochastic trends. Moreover, analytical root mean squared errors appropriately account for uncertainty surrounding estimates of the levels of the natural rate of interest and natural exchange rate.

The design of this Monte Carlo experiment could be extended or refined along numerous dimensions. The data generating process could be estimated rather than calibrated, potentially enhancing its empirical realism. However, the trend component specification of the data
generating process under consideration is too restrictive to accommodate the existence of intermittent structural breaks in real data sets spanning a reasonably long period, irrespective of whether common trends are deterministic or stochastic. In order to analyze the robustness of the finite sample properties of the estimation procedure under consideration to forms of model misspecification not associated with approximation error, alternative data generating processes could be considered, perhaps driven by different types of exogenous stochastic processes. However, the set of potential forms of such model misspecification is large, and the computational cost of evaluating the implications of individual forms of model misspecification is high, while the exogenous stochastic processes associated with the estimated model nest commonly employed types of exogenous stochastic processes.

References


