

**SINGLY-CONSTRAINED MONOTROPIC NETWORK FLOW PROBLEMS:
A LINEAR TIME TRANSFORMATION TO UNCONSTRAINED
PROBLEMS AND QUALITATIVE SENSITIVITY ANALYSIS**

By

ANTOINE GAUTIER

Ingénieur Diplômé de l'École Centrale des Arts et Manufactures

Paris, France. Promotion 1984

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

in

THE FACULTY OF GRADUATE STUDIES
COMMERCE AND BUSINESS ADMINISTRATION

We accept this thesis as conforming
to the required standard

THE UNIVERSITY OF BRITISH COLUMBIA

November 1990

© ANTOINE GAUTIER, 1990

In presenting this thesis in partial fulfilment of the requirements for an advanced degree at the University of British Columbia, I agree that the Library shall make it freely available for reference and study. I further agree that permission for extensive copying of this thesis for scholarly purposes may be granted by the head of my department or by his or her representatives. It is understood that copying or publication of this thesis for financial gain shall not be allowed without my written permission.

Commerce and Business Administration
The University of British Columbia
6224 Agricultural Road
Vancouver, Canada
V6T 1W5

Date:

Dec 1st 1990

Abstract

This thesis examines several problems related to singly-constrained Monotropic Network Flow Problems. In the first part, a linear time algorithm that reduces the solution of a monotropic network flow problem with an additional linear equality constraint to the solution of lower dimensional subproblems is presented. Of the subproblems, at most one is a singly-constrained monotropic network flow problem while the others are unconstrained. If none of the subproblems is constrained, the algorithm provides a linear-time transformation of constrained to unconstrained monotropic network flow problems. Extensions to nonlinear and inequality constraints are given.

In the second part the qualitative theory of sensitivity analysis for Unconstrained Minimum-Cost Flow Problems presented by Granot and Veinott [GV85] is extended to Minimum-Cost Flow Problems with one additional linear constraint. The departure from the unconstrained network structure is shown to have a profound effect on computational issues. Two natural extensions of the “*less-dependent-on*” partial ordering of the arcs given in [GV85] are presented. One is decidable in linear time while the other yields more information but is NP-complete in general. The Ripple Theorem gives upper bounds on the absolute value of optimal-flow variations as a function of variations in the problem parameter. Moreover, it shows how changes may “ripple down” throughout the network, decreasing in magnitude as one gets “further away” from the arc whose parameter initiated the change. The Theory of Substitutes and Complements presents necessary and sufficient conditions for optimal-flow changes to consistently have the same (or the opposite) sign in two given arcs. The complexity of determining Substitutes and Complements is shown to be NP-complete in general. However, for all intractable problems, families of cases arise from easily recognizable graph structures and can be computed in linear time. The Monotonicity Theory links the changes in the value of the parameters to the change in the optimal arc-flows. Bounds on the rates

of changes are discussed.

We further provide a number of practical situations where our theory may apply. We discuss some Multi-Period Multi-Product Inventory-Production models that can be formulated as nonlinear parametric network flow problems with one additional linear constraint. We then apply our theory to help decision makers understand qualitatively how to respond to changes in the environment such as machine breakdown, strike or variations in inventory carrying costs without additional computation. In a second example, we show how a Cash-Flow Management model can be formulated as a nonlinear parametric network flow problem with one additional linear constraint. The theory is then recommended as a method by which a decision maker could understand qualitatively how to respond to changes in the environment such as variations in interest rates, taxes or asset prices without any additional computation.

Table of Contents

Abstract	ii
List of Tables	vii
List of Figures	viii
Acknowledgement	x
 I Foundations	 xii
1 Introduction	1
1.1 The Singly-Constrained Monotropic Network Flow Problem	1
1.2 Substitutes, complements and Ripples	4
1.3 Applications	5
1.4 Organization	6
2 Notation and Definitions	10
 II Singly-Constrained Network Flow Problems	 14
3 Transformation to an Unconstrained Network Flow Problem	15
3.1 Preliminary remarks	15
3.2 Tucker Representation and Elementary Vectors of $C(\alpha)$	16
3.3 Dependency and Graph Decomposition	20
3.4 Decomposition Into Independent Monotropic Network Flow Problems	43
3.5 Extensions and Remarks	56

4	Ripples, Complements and Substitutes	59
4.1	Introduction	59
4.2	The Ripple Theorem For Constrained Graphs	61
4.2.1	Less-Dependent-On Relation	63
4.2.2	Quantitative Less-Dependent-On Relation	68
4.2.3	The Ripple Theorem	69
4.2.4	Computing the Bounds in the Ripple Theorem	73
4.2.5	Complexity Issues Attached to the Qualitative Less-dependent-on Relation	78
4.3	Complements and Substitutes in Constrained Network Flows	81
4.4	The Monotonicity Theory	90
4.4.1	The Smoothing Theorem	94
4.4.2	Subadditivity of Minimum Cost in the Parameters	97
III	Applications	99
5	A Constrained Inventory-Production Model	100
5.1	Introduction	100
5.2	The Multi-Product Multi-Period Model	101
5.3	Substitutes and Complements in The General Case	105
5.3.1	Pairs of Arcs Corresponding to the Same Product	106
5.3.2	Pairs of Arcs Corresponding to Two Different Products . . .	112
5.3.3	Computational Complexity	114
5.4	Narrow constraints and pseudo-rims	118
5.4.1	Pseudo-rims and their properties	118
5.4.2	Application to Narrow Constraints	122
5.4.3	Recognition of pseudo-rims	127
5.5	First Period Constraints	127

5.5.1	Dependency	128
5.5.2	Positive Arcs, Complements and Substitutes	129
5.5.3	Less-dependent-on Relations	130
5.5.4	Variations in production capacity and inventory holding cost	132
6	A Cash-Flow Asset Allocation Management Model	137
6.1	A Cash-flow Management Model	137
6.2	Properties of the Graph G_{SCF}	141
6.3	Qualitative Sensitivity Analysis	144
A	index	147
	Bibliography	152

List of Tables

5.1	Complements and Substitutes in the Unconstrained	105
5.2	Substitutes and Complements: Computations	115
5.3	Substitutes and Complements, $l = 1$ or 2	116
5.4	Substitutes and Complements: across products 1 and 2	116
5.5	Substitutes and Complements ($l = 1, \dots, p$)	124
5.6	Substitutes and Complements	126
5.7	First Period Example:	130
6.8	Table Signs of Respective Elementary Vector Entries	143
6.9	Substitutes and Complements in the Cash-flow Problem	143
6.10	Coefficients of the Ripple Theorem	144

List of Figures

3.1	Decomposition of the graph G	28
3.2	Rims and Graphs	30
3.3	A one-rim (<i>left</i>) and its expanded version (<i>right</i>)	35
3.4	A palm graph with its branches	38
3.5	The coloring of the palm graph G	39
3.6	Rim Decomposition and Expansion	46
3.7	Rim Expansion	50
3.8	The rim G	54
4.9	Equally dependent pairs of arcs need not disconnect the graph . . .	64
4.10	Construction of the dependent graph G when $G \setminus \{e_k\}$ is a rim	66
4.11	Construction of the dependent graph G when $G \setminus \{e_k\}$ is not bicon- nected	67
4.12	$=_j^d$ does not imply $=_j^q$	69
4.13	Tight bounds in the Ripple Theorem	72
4.14	The extended graph G^*	75
4.15	Only bold arcs verify $\alpha_i \neq 0$	77
4.16	$e_i \leq_j^q e_k$	80
4.17	Substitutes and complements in a constrained graph	82
4.18	Substitutes and complements	87
4.19	The rim G (<i>left</i>) and the constructed graph $\Gamma(G')$ (<i>right</i>)	88
4.20	Arcs with common nodes need not be substitutes or complements .	90
5.21	The basic Inventory-Production Network	103
5.22	A Pseudo-Rim	119

5.23	Projection of elementary vectors in pseudo-rims	121
5.24	A pseudo-rim with four pseudo-components:	123
5.25	The subgraph G^l is a rim with three rim components	128
5.26	Hasse Diagrams, $e_j = x_1^1$	134
5.27	Hasse Diagrams, $e_j = y_4^1$	136
6.28	The graph \mathbf{G}_{GEN}	139
6.29	The graph \mathbf{G}_{SCF}	140

Acknowledgement

The longer one spends researching and writing a doctoral thesis, the more people one is lead to be thankful to at the end. Many are those who helped, one way or another, this long, enjoyable, and above all educational process.

Thank you Frieda for bearing with me during the times of excitement, of discovery, of stalling and of learning. I feel deeply grateful for the patience, the guidance, and the wisdom to recognize where I was going when even I did not know. Grateful too for the many gallons of red ink patiently laid on dozens of successive drafts as, by osmosis, I absorbed some of the craft of academic writing.

A special thanks to the staff here at the faculty of Commerce and Business Administration, to Randy, Barbara who shared her printer and good humour, and Khim who smiled and witnessed my initiation to L^AT_EX. Also to my C290 Students who convinced me I was better suited for teaching than I previously thought, and made the experience fun and enjoyable.

My thanks and love go to many of my friends in Vancouver who helped me see clearly, supported my spirits and helped me through the occasional doubt. For the music, thank you Lenka.

Thanks to my good friends for helping balance the life of the aspiring Ph.D.; and to my surrogate parents, Don, Judy and June, aunts and uncles, who made my Thanksgivings and Christmases real family events.

My recognition also goes to the people of Canada and especially of Vancouver where I have felt more at home than anywhere else in the world, for keeping such a beautiful country and letting me be part of it. A special thanks to the Canadian Government and the World University Service of Canada for making it financially possible.

Finalement, un grand Merci à ma famille, qui j'espère me pardonnera ma longue absence. Nul n'est prophète en son pays, mais est-ce bien une raison pour aller chercher le bonheur

si loin? Merci pour les nombreuses lettres, coups de téléphone, et surtout le support moral qui se fait sentir par delà les mers et montagnes.

Part I

Foundations

Chapter 1

Introduction

1.1 The Singly-Constrained Monotropic Network Flow Problem

A **Monotropic Programming Problem** is an optimization problem in which the objective function is variable-additive and the constraints are of the linear kind. In this work we discuss a special structured Monotropic Programming problem in which one is required to find a flow in a directed network that minimizes an arc-additive objective function, subject to an additional linear constraint. This problem will be referred to as the **Singly-Constrained Monotropic Network Flow Problem (SCMF)**. To state SCMF explicitly, let $G = (N, E)$ be a finite **directed graph** with a set of nodes $N = \{1, 2, \dots, n\}$ and a set of arcs $E = \{e_1, e_2, \dots, e_m\}$ (possibly with parallel arcs). Define the (node-arc) **incidence matrix** $A = (A_{ij})_{\substack{i=1, \dots, n \\ j=1, \dots, m}}$ of G by $A_{ij} = 1$ if node i is the head of arc e_j (that is, $e_j = (\ell, i)$ for some $\ell \in N$), $A_{ij} = -1$ if i is its tail ($e_j = (i, \ell)$ for some $\ell \in N$) and $A_{ij} = 0$ otherwise. A **demand vector** (for G) is a real valued vector $d \in \mathbb{R}^n$ for which $\sum_{i=1}^n d_i = 0$. For a given real-valued row-vector $\alpha \in \mathbb{R}^m$ and a real number $\alpha_0 \in \mathbb{R}$ we define the **additional constraint** $\alpha x = \alpha_0$ for $x \in \mathbb{R}^m$. With each arc $e_i \in E$ we associate a flow $x_i \in \mathbb{R}$, and a cost $b_j(x_j)$ is incurred when the flow in arc e_j is x_j , where $b_j(\cdot)$ is a $+\infty$ or real-valued function on \mathbb{R} . The cost of a flow $x = (x_j) \in \mathbb{R}^m$ is the sum of the cost functions on all the arcs of the graph. SCMF can thus be written as

$$\begin{aligned} P(\alpha, \alpha_0) : \quad & \text{Min} \quad \sum_{j=1}^m b_j(x_j) \\ & \text{s.t.} \quad Ax = d \\ & \quad \quad \alpha x = \alpha_0. \end{aligned}$$

The usual non-negativity and upper/lower bound constraint(s), if any, are naturally incorporated into the cost-function by setting $b_j(x_j)$ equal to $+\infty$ for all forbidden values of x_j . A problem similar to the above, but without the additional constraint, will be referred to as an **Unconstrained Monotropic Network Flow Problem**, or simply an unconstrained problem.

SCMF is an important problem involved in many practical applications. It will occur, for example, within the framework of bicriterion decision analysis [Ba89], where a decision maker may be faced with optimizing a network flow over two criteria, say, for example, cost and project duration, where one of these criteria is a linear function of the arc-flows. In that case, a parametric analysis will optimize one criterion subject to a constraint on the other. Klingman [Kl77] gave an example of a newspaper distribution problem where the average delivery time is constrained, again resulting in a linear side constraint. Other examples include network flow problems with budget or resource constraints. The solution of the Minimum Convex-Cost Dynamic Network Flow Problem [Or82] involves solving a SCMF, which is the computational bottleneck of a strongly polynomial algorithm. The SCMF is also used by Spalti and Liebling [SL89] as a subroutine to solve a location problem for telecommunication satellites.

The special case where no additional constraint is imposed has received considerable attention in the literature. Efficient solution techniques for problems with linear objective functions are too numerous to be surveyed here, while for nonlinear costs one can see for example [Bert76, HK77, DK81, Ro84]. The majority of these algorithms exploit the particular structure of the constraint matrix and the combinatorial properties of network flows and circulations. However, the additional linear constraint partially destroys the attractive features of pure network flow problems. Even though, for linear objective functions, specialized solution methods may be used, see for example [CS77, GK78, BMM88], in the general case of non-linear arc-additive costs, one has to resort to solving a monotropic program (see [Ro84] for references), that is, treat the constraint matrix as not having any particular (combinatorial) properties.

The question of whether the linearly constrained monotropic network flow problem is equivalent to an unconstrained monotropic network flow problem is therefore of interest, both from a practical and from a theoretical point of view. From a practical standpoint, network flow problems are easier to solve and can take advantage of proven efficient optimization codes. From a theoretical one, it is of interest to recognize a family of constrained problems which, although not in an obvious fashion, possess the desirable combinatorial properties of network flow problems. In the same spirit, a number of authors have looked for imbedded network structures in linear programs (see e.g. Bixby and Cunningham [BC80], Cunningham [Cu83] and Truemper [Tr83]). For linear network flow problems with a specific class of constraints, an answer was given by Klingman in [Kl77], where a procedure determines if the additional constraint is equivalent to a partial sum of flows on arcs all directed into or all directed away from a single node. If so, the topology of the network is modified in order to obtain an equivalent problem where one can drop the additional constraint (it will be shown that the procedure presented here covers these cases and is computationally faster). However, in the event of a negative answer, it will be relevant to obtain the “smallest non-transformable” subproblem which, together with other subproblems transformed to unconstrained network flow problems, will solve the original SCMF. Computationally, such a decomposition has the double advantage of both reducing the size of the “hard” (constrained) problem to a minimum, and of breaking down the original problem into more manageable subproblems.

1.2 Substitutes, complements and Ripples

In economics, the study of substitutes and complements has a long history, including works by Hicks and Allen [HA34], Hicks [Hi39], Samuelson [Sa47,Sa74], Lancaster [La62, La65,La66], and Gorman [Go64]. Samuelson [Sa47] poses the problem of whether it is possible to predict the direction of change of certain variables in an economic model given the direction of change in some other variable(s). A first approach to this problem used

the concept of **qualitative determinacy** where only the signs of the effect of one variable on another (Boolean variables) are taken into account. For example, one would use the information that “an increase in paper prices will increase paper recycling activities”, combining it with “an increase in paper recycling activities will create employment” in a fashion reminiscent of artificial intelligence inference. In that framework, Lady [La67] first offered a solution to the problem posed by Samuelson, together with an algorithm based on graph theory to determine qualitative determinacy in economic systems. More results exploiting the structures of *qualitative matrices* of boolean “influence” between variables in an economic system stemmed from the works of Lancaster [La62,La65], Quirk and Ruppert [QR65], Maybee and Quirk [MQ69]. May [Ma73] developed the mathematics of such matrices. Refinements and extensions were provided by Greenberg [Gr81] for matricial forms where the concepts of determinacy and economic correlation were introduced (Greenberg’s concept of *connectivity* is very close to our **dependency**), opening avenues for further theoretical research by Greenberg, Lundgren and Maybee [GLM83,GLM84a,GLM84b]. More recently, the limitations of qualitative matrices which only partially describe inter-variable relation (“by how much does the increase in paper price affect recycling activity?”) were removed as several authors studied linear programming-like systems. Provan [Pr83,Pr86] studies a general linear program and proposes to refine the concepts of substitutes and complements. The richness of the substitute and complement relations between variables is seen therein as a *measure of organization* of a system with interrelated factors. Provan and Kydes [PK80] further study determinacy in network models. Substitutes and complements moved into the area of Network Flows with Shapley [Sh61,Sh62], and further with Veinott [Ve64,Ve65,Ve68] and Gale and Politof [GP81]. A rigorous theory of substitutes and complements for network flows, based partially on Topkis and Veinott’s theory of Lattice Programming [Ve65,TV73,To78,Vefo.], was developed by Granot and Veinott [GV85]. The theory presented by Granot and Veinott explores many facets of sensitivity analysis, both qualitative and quantitative. An extension of that work to multicommodity flow problems was the topic of a doctoral dissertation by Ciurria [Ci89] under the supervision of Prof.

Veinott (see also [CGV89a]). In [CGV89b], sensitivity analysis for multicommodity networks transformable into regular (single-commodity) networks are studied. An application of that theory to multicommodity inventory-production models is given in [CGV89a]. Both the works of Granot and Veinott as well as that of Ciurria, Granot and Veinott rely on the combinatorial nature of (single commodity) network problems. As a byproduct of such structures, combinatorial, networklike algorithms are derived, keeping the complexity of answering most of Samuelson's questions in the class P with complexities which are linear to quadratic in the problem size.

1.3 Applications

In the second part of this thesis, we provide illustrative examples of how our theory may be used to help managers make better decisions. We formulate a Multi-Period Multi-Product Inventory-Production model as a nonlinear parametric network flow problem with one additional linear constraint. We then apply our theory to help decision makers respond to changes in the economic and/or industrial environment such as machine breakdown, strike or variations in inventory carrying costs without additional computation. In a financial application, we show how a Cash-Flow Management model can be formulated as a nonlinear parametric network flow problem with one additional linear constraint. We then examine a number of scenarios and demonstrate how a decision maker could, using our theory, understand qualitatively how to respond to changes in the environment such as variations in interest rates, taxes or asset prices without any additional computation.

1.4 Organization

The following chapters are rather self-contained, each having its own introduction and set of definitions. The notation, as well as basic graph theoretic and network optimization terms are defined in Chapter 2. Further definitions and specific notation are introduced as needed

throughout later chapters. In Chapter 3, we present a decomposition of the SCMF based on a decomposition of the underlying constrained graph¹. The set of optimal solutions of the SCMF is shown to be the cartesian product of the sets of optimal solutions of a number of unconstrained monotropic network flow problems together with, possibly, a unique SCMF. The problem decomposition is based on a partition of the underlying constrained graph. Elementary vectors and their properties [Ro67] are the core of our theory. They behave like “building blocks” that hold the circulation space together, so that two arcs or subgraphs with no common “building block” will be separated by the partitioning process. Although most of the decomposition procedure could be exposed without relying on them, they are introduced for two reasons. First, they provide a rigorous framework and show how the results first emerged. Moreover, they are the basis for a proof of the fact that the decomposition is, in some sense, exhaustive, so that all transformable problems are actually characterized and detected.

Chapter 3 is organized as follows. In section 3.2 the algebraic foundations of the decomposition theory are laid with the study of the circulation space associated with the feasible set, and the characterization of its elementary vectors. Section 3.3 starts with the introduction of the **dependence relation** and of the **dependent components** of a singly-constrained graph. A series of five results is needed in order to establish the Graph Decomposition Theorem (Theorem 6), and a special type of graph structure, the rim, is introduced. The second half of the section provides a theoretical characterization of rims, explores their connections with palm-graph structures, and finally a linear time algorithm that recognizes rims is given (Algorithm I). The implications of the graph decomposition for the SCMF are discussed in section 3.4 where we present a problem decomposition theorem for rims (Theorem 18) which leads to a linear-time algorithm (Algorithm II) that performs the problem decomposition. Some extensions are discussed in Section 3.5.

In Chapter 4 we study the qualitative sensitivity analysis and the theory of substitutes

¹A version of Chapter 3 has appeared as a U.B.C. working paper under the title “A Linear Transformation of a Singly-constrained to an Unconstrained Monotropic Network Flow Problem” [GG89].

and complements² (terms defined in section 4.3). The study of the unconstrained case was the object of [GV85]. Here, however, the additional linear constraint partially destroys the attractive features and some of the results obtained for pure network flow problems. Indeed, even the complexity of solving the problem is considerably increased by the presence of the additional constraint (for the treatment of linear costs, see [CS77,GKKR78,BMM88]).

The cost function is, for most results, required to be convex in the arc-flow, and may reflect real costs as well as upper and/or lower bounds. The sensitivity analysis is performed by modifying the parameter associated with one or several given arcs. Any variation of the cost structure can be represented in this fashion. Our results are concerned with the “reactions” of the optimal flows to such changes. For applications, such parameter variations may, for example, reflect changes in upper and/or lower bounds, additive or multiplicative increases in cost, pricing alternatives, technological advances, discounts, or even simulate the deletion/addition of an arc. In the case of multiple optima, our results are valid for at least one of the possible optimal flows.

The Chapter is organized as follows. Section 4.2 focuses on bounds on the magnitude of changes of the optimal flow in two given arcs, the main results being given in the Ripple Theorem (Theorem 21). To that end, we define two partial orders on the set of arcs, the **less-dependent-on** relation and the **quantitatively-less-dependent-on** relation, as well as a coefficient $M_{j,k}$ which is associated with each pair of arcs (e_j, e_k) . Computational issues associated with the partial orders and the coefficients $M_{j,k}$ are discussed. In section 4.3 we introduce the notions of complements and substitutes pairs of arcs and present a result on the direction of changes in the optimal arc-flow for such pairs of arcs in Lemma 25. Computational complexity is again addressed. The Monotonicity Theory presented in section 4.4 relates and compares the rate of change in the optimal flow of a given arc to that of another arc's parameter when the two arcs are either complements or substitutes, a result stated in the Monotone Optimal-Flow Selection Theorem (Theorem 31). The Smoothing Theorem (Theorem 33) provides an upper bound on the maximum arc-flow change over the network

²A version of Chapter 4 has appeared as a U.B.C. working paper under the title “*Ripples, Complements and Substitutes in Singly-Constrained Monotropic Parametric Network Flow Problems*” [GG90a].

in terms of \mathcal{L}_1 -norm of the corresponding parameter variation. Finally Theorem 34 shows a subadditivity property of the minimum cost.

Chapter 5 deals with an inventory/production problem³. We show how to provide polynomial time solutions to sensitivity analysis questions for the special graphs associated with singly-constrained multi-product inventory-production problems. An extension of the concept of rims is proposed which gives sufficient conditions for the qualitative sensitivity questions to be answerable in linear time for some subsets of arcs. This concept of **pseudo-rim** is applied for a class of singly-constrained multi-product inventory-production problems, and examples are given. Lastly, another class of such problems is studied and illustrates the concept of rims. The qualitative theory developed in Chapter 4 is then fully informative, and comprehensive predictions are made for examples in which variations in production capacity and inventory holding cost are studied.

Chapter 6 illustrates the application of our model to a Cash-Flow Managements problem⁴. Indeed, many cash-flow management models can be represented as network flow problems with arc-gains or losses, see e.g. [M84,GG90b]. In this chapter we show how one such model can be transferred into an equivalent singly-constrained network flow problem. We then apply the qualitative theory in order to help decision makers respond to changes in their environment without having to perform any additional computations. For example, given an existing set of assets, the problem is to decide what would be the “best” distribution of the assets over the next planning period. The objective may be to maximize expected return, to minimize a (quadratic) risk function, or some combination of both. Given the special graph structure that emerges, all sensitivity analysis questions posed in Chapter 4 are answered in time linear in the number of assets, and the theory is fully informative. For example, we are able to give bounds on the rates of change of disinvestment and reinvestment for all

³A version of Chapter 5 has appeared as a U.B.C. working paper under the title “*A Parametric Analysis of a Constrained Nonlinear Inventory-Production Model*” [GG90c].

⁴A version of Chapter 6 has appeared as a U.B.C. working paper under the title “*A Parametric Analysis of a Nonlinear Cash-Flow Asset Allocation Management Model*” [GG90b].

assets and, in some cases, predict directions of changes. Examples are given of increase in tax rates on asset sales and variations in expected rates of return on assets.

□ □ □

Chapter 2

Notation and Definitions

In this chapter, we provide most of the notation and definitions used throughout this thesis. Further definitions, especially those original to this piece of work, will be introduced as required.

Recall from Chapter 1 the problem formulation of the SCMF: Let $G = (N, E)$ be a finite directed graph with a set of nodes $N = \{1, 2, \dots, n\}$ and a set of arcs $E = \{e_1, e_2, \dots, e_m\}$ (possibly with parallel arcs). Define the (node-arc) incidence matrix $A = (A_{ij})_{\substack{i=1, \dots, n \\ j=1, \dots, m}}$ of G by $A_{ij} = 1$ if node i is the head of arc e_j (that is, $e_j = (\ell, i)$ for some $\ell \in N$), $A_{ij} = -1$ if i is its tail ($e_j = (i, \ell)$ for some $\ell \in N$) and $A_{ij} = 0$ otherwise. A demand vector (for G) is a real valued vector $d \in \mathbb{R}^n$ for which $\sum_{i=1}^n d_i = 0$. For a given real-valued row-vector $\alpha \in \mathbb{R}^m$ and a real number $\alpha_0 \in \mathbb{R}$ we define the additional constraint $\alpha x = \alpha_0$ for $x \in \mathbb{R}^m$. With each arc $e_i \in E$ we associate a flow $x_i \in \mathbb{R}$, and a cost $b_j(x_j)$ is incurred when the flow in arc e_j is x_j , where $b_j(\cdot)$ is a $+\infty$ or real-valued function on \mathbb{R} . The cost of a flow $x = (x_j) \in \mathbb{R}^m$ is the sum of the cost functions on all the arcs of the graph. The SCMF can be written as

$$\begin{aligned} P(\alpha, \alpha_0) : \quad & \text{Min} \quad \sum_{j=1}^m b_j(x_j) \\ & \text{s.t.} \quad Ax = d \quad . \\ & \quad \alpha x = \alpha_0. \end{aligned}$$

A flow $x = (x_j) \in \mathbb{R}^m$ is said to be feasible (with respect to a demand d) if $Ax = d$. The set of constrained flows is denoted by $F(\alpha, \alpha_0, d) \equiv \{x \in \mathbb{R}^m : x \geq 0, Ax = d, \alpha x = \alpha_0\}$. The graph $G = (N, E)$ together with the constraint $\alpha x = \alpha_0$ will be referred to as a constrained graph and denoted by (N, E, α) . A real-valued vector $x \in \mathbb{R}^m$ is called a circulation (of G)

if $Ax = 0$, and a **constrained circulation** if it further verifies $\alpha x = 0$. An arc e_i is said to be **neutral** if $\alpha_i = 0$ and **active** if $\alpha_i \neq 0$. We denote by $C(\alpha) \equiv \{x \in \mathbb{R}^m : Ax = 0, \alpha x = 0\}$ space the linear subspace of circulations constrained by the vector α .

The **support** of a given vector $x \in \mathbb{R}^m$, denoted $\text{Supp}(x)$, is the subset of indices $i \in \{1, 2, \dots, m\}$ for which $x_i \neq 0$. Given a linear subspace K of \mathbb{R}^m , an **elementary vector** [Ro67] of K is a vector $x \in K$ whose support is minimal among the supports of nonzero vectors of K . An **elementary support** is the support of an elementary vector. An **elementary circulation** is a constrained circulation whose support is an elementary support, or equivalently an elementary vector of the constrained circulation space $C(\alpha)$. Any vector of K can be expressed as a **conformal sum** (two vectors u and v are conformal if $u_i v_i \geq 0$ for all i) of elementary vectors of K [Ro67, pp. 108–109, Harmonious Superposition Theorem].

A **simple path** p in G is an ordered set of arcs in G for which there exists a set of nodes of G such that every node is shared by a unique pair of successive arcs with respect to the order. If in addition there is a node shared by only the first and the last arc in the ordered set obtained from p , then p is a **simple cycle**.

A vector $c = (c_j) \in \{0, 1, -1\}^m$ is called an **oriented cycle** if it is a circulation and $\text{Supp}(c)$ is a simple cycle on G . If c is an oriented cycle, $-c$ is the **reversed oriented cycle** of c .

If $c_j \neq 0$, then c is said to **contain** arc e_j . If $c_j = 1$ (resp., -1) then e_j is a **forward** (resp., **backward**) arc of c . An oriented cycle c is said to be **neutral** (with respect to the constraint $\alpha x = \alpha_0$) if $\alpha c = 0$ and **active** if $\alpha c \neq 0$. By extension a subgraph of G is **active** (resp., **neutral**) if it contains (resp., does not contain) active cycles. Moreover, an arc is a **positive** (resp., **negative**) arc if and only if every cycle c that contains it as a forward arc verifies $\alpha c \geq 0$ (resp., ≤ 0).

A vector $p \in \{0, 1, -1\}^m$ is called an **oriented path** from node i to node j in G if by adding to E the arc $e = (j, i)$ if it does not exist, the vector $p + e$ possibly in the newly

created graph, is an oriented cycle. Forward (resp., backward) arcs of p are forward (resp., backward) arcs of $p + e$.

A graph $G = (N, E)$ is said to be **connected** if and only if there exists at least one simple path between any two nodes of G , and k -**connected** for $k \geq 2$ if and only if there exists at least k internally node-disjoint paths between any two nodes of G . In particular a graph is **2-connected** or **biconnected** if and only if every pair of distinct arcs is contained in a simple cycle [Wh32, p. 347] and by extension we say that a graph with only one arc is biconnected.

A vector $z \in \{0, 1, -1\}^m$ is a **cut** of G if and only if G can be partitioned into two connected subgraphs $G_1 = (N_1, E_1)$ and $G_2 = (N_2, E_2)$, with $N_1 \cup N_2 = N$ and $N_1 \cap N_2 = \emptyset$, so that $z_j = 1$ (resp., -1) if $e_j = (i_1, i_2)$ (resp., (i_2, i_1)) with $i_1 \in N_1$ and $i_2 \in N_2$, and $z_j = 0$ otherwise. It is a well known result that cuts and simple oriented cycles are orthogonal (see e.g. [Ro84, pp. 115–117]).

A subgraph $G_T = (N, T)$ of G is a **spanning forest** of G if it contains all the nodes of G but no cycles, and a **spanning tree** if, in addition, G_T is connected. If a spanning tree G_T is directed so that one of its nodes, say 1, is not the head of any arc and the unique oriented path $p(i)$ from node 1 to node i in G_T only has forward arcs then G_T is called a **directed spanning tree** and node 1 its **root**. The nodes of $p(i)$ are then the **ancestors** of i and i is their **descendant**. $D(i)$ denotes the set of descendants of node i and we may assume without loss of generality that $i \geq j$ whenever i is a descendant of j . If all the arcs in $E \setminus T$ are oriented from a node to one of its ancestors, then G is called a **palm graph**, T its **underlying tree**, and the arcs in $E \setminus T$ are the **return arcs** while the arcs in T are the **tree arcs**. It was shown by [Ta72] that by using depth-first search any connected undirected graph can be converted into a palm graph by redirecting its arcs in a suitable manner and renumbering the nodes appropriately. This can be done in linear time, see for example [Ta72, AHU74]. In a similar fashion any connected directed constrained graph can be converted into a **constrained palm graph** if by redirecting an arc we mean reversing its orientation and multiplying the corresponding α_i by -1 . We will denote by $f(i), i \neq 1$, the

father of node i , that is, the unique node such that $(f(i), i) \in T$, and $p(i, j) \stackrel{\text{def}}{=} p(j) - p(i)$ is the unique path on T from i to j . We will further associate with each node i a label $\alpha_i^T = \alpha p(i)$.

Let $G_T = (N, T)$ be a spanning forest in G , and e_j an arc in $E \setminus T$. Then there exists a unique oriented cycle in G orienting e_j as a forward arc and whose remaining arcs belong to T . This unique cycle will be denoted by $c(e_j, G_T)$. One can further verify that if G is a palm graph and $e_j = (i, k)$ then $c(e_j, G_T) = p(k, i) + e_j = p(i) - p(k) + e_j$ and thus $\alpha c(e_j, T) = \alpha p(i) - \alpha p(k) + \alpha e_j = \alpha_i^T - \alpha_k^T + \alpha_j$. If $c(e_j, G_T)$ is an active oriented cycle with respect to α , then $G_T \cup \{e_j\} \equiv (N, T \cup \{e_j\})$ is called an α -one-forest (on G). In general, a subgraph S of G is an α -one-forest if and only if it can be constructed in the above manner. We show in Lemma 2 that if α is not a linear combination of rows of A , then there exists at least one α -one-forest.

In the discussions of computational complexity, we will use the term **problem** to designate all possible instances of a given question. A **decision problem** is one whose answer is either “Yes” or “no”, an **evaluation problem** is that of finding the optimal value of an optimization problem, but not necessarily the corresponding optimal decision variables. We will assume that the data is rational and denote by s the **size of an instance**, that is, the number of bits of information required to encode the given instance. Typically, if the data is integer (this is a valid assumption, for rational data can, without loss of generality, be appropriately scaled), the space required to encode $G = (N, E, \alpha)$ together with the demand vector d is

$$s = O(m \log_2 n + \sum_{i=1}^n \log_2(|d_i| + 1) + \sum_{j=1}^m \log_2(|\alpha_j| + 1)).$$

□ □ □

Part II

Singly-Constrained Network Flow Problems

Chapter 3

Transformation to an Unconstrained Network Flow Problem

Chapter Abstract

A linear time algorithm that reduces the solution of a monotropic network flow problem with an additional linear equality constraint to the solution of lower dimensional subproblems is presented. Of the subproblems, at most one is a singly-constrained monotropic network flow problem while the others are unconstrained. If none of the subproblems is constrained, the algorithm provides a linear-time transformation of constrained to unconstrained monotropic network flow problems. Extensions to nonlinear and inequality constraints are given.

□ □ □

3.1 Preliminary remarks

In this chapter, we study the Nonparametric Singly-Constrained Monotropic Network Flow Problem, the SCMF. This problem can be written as

$$\begin{aligned} P(\alpha, \alpha_0): \quad & \text{Min} \quad \sum_{j=1}^m b_j(x_j) \\ & \text{s.t.} \quad Ax = d \\ & \quad \quad \alpha x = \alpha_0. \end{aligned}$$

A problem similar to the above, but without the additional constraint, will be referred to as an **Unconstrained Monotropic Network Flow Problem**, or simply as an **unconstrained problem**. Notice that if x^1 and x^2 are feasible, possibly optimal, for $P(\alpha, \alpha_0)$, then the difference $x^1 - x^2$ belongs to the **circulation space** $C(\alpha) = \{x \in \mathbb{R}^m : Ax = 0, \alpha x = 0\}$. This remark motivates the development of the representation of $C(\alpha)$ in section 3.2.

3.2 Tucker Representation and Elementary Vectors of $C(\alpha)$

In this section we study some characteristics of the subspace $C(\alpha)$ which will then be used in the development of the decomposition algorithm. To that end we will assume without loss of generality that all **leaf arcs** through which no circulation passes were discarded from the outset (this can be done in $O(n)$ operations). Indeed such arcs are of little interest as far as the optimization problem is concerned, for any flow that is feasible for $P(\alpha, \alpha_0)$ has a fixed value on them.

Let $F \in \mathbb{R}^{r \times q}$ be a matrix with r rows and q columns and consider the linear subspace of \mathbb{R}^q defined by $K(F) = \{x \in \mathbb{R}^q : Fx = 0\}$. We will assume without loss of generality that $F = (F^B, F^N)$, where F^B is nonsingular, is of full row rank, so that $(F^B)^{-1}F = (I, \hat{F})$ and $\hat{F} = (\hat{f}_{ij})$, $i = 1, \dots, r$, $j = r+1, \dots, q$. We refer to (I, \hat{F}) as a **Tucker representation** of $K(F)$ [Tuc63, Ro84]. The variables x_1, \dots, x_r are called the **basic variables** while the variables x_{r+1}, \dots, x_q are called the **nonbasic variables** for this particular Tucker representation. Clearly, for any given value x_{r+1}, \dots, x_q of the nonbasic variables, the basic variables x_1, \dots, x_r of a vector $x \in K(F)$ will be uniquely determined by the system of equalities

$$-x_i = \sum_{j=r+1}^q \hat{f}_{ij} x_j \quad \text{for } i = 1, 2, \dots, r.$$

The set B of column vectors of F corresponding to the basic variables is called the **basis for the Tucker representation** of $K(F)$. The next four results provide a characterization of these bases. Note that we may assume in the following that the side constraint $\alpha x = \alpha_0$ is not redundant, that is, that $\alpha \notin K(A)$

LEMMA 1 [Ro84, p. 456]. *Let $F \in \mathbb{R}^{r \times q}$ then a subset B of columns of F is a basis for some Tucker representation of $K(F)$ if and only if it is a basis (in the usual sense of linear algebra) for the subspace spanned by the columns of F , that is for*

$$B(F) \stackrel{\text{def}}{=} \{y \in \mathbb{R}^r : \exists x \in \mathbb{R}^q \text{ with } Fx = y\}.$$

LEMMA 2 *Let $G = (N, E, \alpha)$ be a constrained graph and A its node-arc incidence matrix. If $\alpha \notin K(A)$ then all the α -one-forests of G exist and are the bases for the Tucker representation of $C(\alpha)$.*

PROOF. First, since $\alpha \notin K(A)$, then G contains at least one active cycle. Such an active cycle may be completed into an α -one-forest by adding appropriate arcs. Now, let $G_T = (N, T)$ be a spanning forest in G and e_k an arc in $E \setminus T$ so that $T \cup \{e_k\}$ is an α -one-forest. We will assume without loss of generality that $T = \{e_1, e_2, \dots, e_{n-1}\}$ and that $k = n$. Now, since T is a basis for the system of n linear equations $Ax = 0$, see for example [Ro84], there exists a series of linear operations which, when applied to the rows of the matrix A_α , obtained by replacing the last row of A by α , will result with a matrix $A'_\alpha = \begin{pmatrix} I & A' \\ 0 & \hat{\alpha} \end{pmatrix}$ where I is an $(n-1) \times (n-1)$ identity matrix, $A' = (a_{ij})$ where $a_{ij} = -c(e_j, G_T)_i$, with $c(e_j, G_T)_i$ denoting the i^{th} component of $c(e_j, G_T)$ and $\hat{\alpha}_j = \alpha_j - \sum_{i=1}^{n-1} \alpha_i a_{ij}$.

Note that $\hat{\alpha}_j = \alpha_j + \sum_{i=1}^{n-1} \alpha_i c(e_j, G_T)_i = \alpha c(e_j, G_T)$ for all $j = n, \dots, m$ and, since $G_T \cup \{e_n\}$ is an α -one-forest, $\hat{\alpha}_n \neq 0$. Now by adding to each row i of A'_α row n divided by $\hat{\alpha}_n$ and multiplied by $-a_{in}$ one obtains the matrix $A''_\alpha = (I : \hat{A}_\alpha)$ where I is an $n \times n$ identity matrix, and $\hat{A}_\alpha = (\hat{a}_{ij})$ with

$$\begin{aligned} \hat{a}_{ij} &= a_{ij} - \frac{\hat{\alpha}_j}{\hat{\alpha}_n} a_{in} \quad i = 1, \dots, n-1, \quad j = n+1, \dots, m \\ \hat{a}_{nj} &= \frac{\hat{\alpha}_j}{\hat{\alpha}_n} \quad j = n+1, \dots, m. \end{aligned} \tag{3.1}$$

Now A''_α is a Tucker representation for $C(\alpha)$ whose corresponding basis is the α -one-forest. \square

The origins of Lemma 3 to follow can be traced back to [GKKR78] in which a similar result is used, although not explicitly stated or proved. We include a proof for completeness.

LEMMA 3 *If $\alpha \notin K(A)$, then all bases for the Tucker representation of $C(\alpha)$ are α -one-forests containing an active cycle.*

PROOF. We first show that any basis contains no more than one cycle. Assume, on the contrary that there exists a basis $B \subset E$ for the Tucker representation of $C(\alpha)$ which contains two different simple cycles, say C^1 and C^2 (which differ by more than their orientation). Further let c^1 (resp., c^2) be an oriented cycle corresponding to C^1 (resp., C^2). If c^1 is a neutral cycle, then

$$\begin{bmatrix} A \\ \dots \\ \alpha \end{bmatrix} c^1 = \begin{bmatrix} Ac^1 \\ \dots \\ \alpha c^1 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

is a linear combination of the rows of A_α that are in B and whose coefficients are the entries of c^1 , and thus not all equal to zero. Therefore B is not a basis for the linear subspace

$$B(E, \alpha) \stackrel{\text{def}}{=} \{y \in \mathbb{R}^{n+1} : \exists x \in \mathbb{R}^m \text{ with } y = A_\alpha x\}$$

and by Lemma 1 not a basis for the Tucker representation of $C(\alpha)$. Clearly the same holds if c^2 is a neutral cycle. Let us assume now that both cycles are active. Then similarly

$$\begin{bmatrix} A \\ \dots \\ \alpha \end{bmatrix} [(\alpha c^2)c^1 - (\alpha c^1)c^2] = \begin{bmatrix} (\alpha c^2)Ac^1 - (\alpha c^1)Ac^2 \\ \dots \\ (\alpha c^2)\alpha c^1 - (\alpha c^1)\alpha c^2 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

is a linear combination of rows of A_α that are in B and whose coefficients are the corresponding entries of $(\alpha c^2)c^1 - (\alpha c^1)c^2$, and thus not all equal to zero. A contradiction arises as in the previous case. This proves that any basis contains no neutral cycle and no more than one active cycle.

In Lemma 2 we showed the existence of a basis with cardinality equal to n . (This is always possible since we assume the existence of an α -one-forest). Hence all bases have cardinality equal to n and no more than one cycle, and therefore can be written as $B = G_T \cup \{e_j\}$ where $G_T = (N, T)$ is a spanning forest with $e_j \in E \setminus T$, and B contains a unique cycle $c(e_j, G_T)$ which is active. \square

Combining Lemmas 2 and 3 one obtains

THEOREM 4 *The bases for the Tucker representation of the subspace $C(\alpha)$ are the α -one-forests. \square*

In order to characterize the elementary vectors of the subspace $C(\alpha)$ which play a major role in our algorithm we will describe the General Basis Theorem given by Rockafellar, and its specialization to our content. For further details the reader is referred to [Ro84, p. 457]. To that end, let B be a basis for some Tucker representation of $C(\alpha)$ and let \hat{a}_{ij} , $e_i \in B$, $e_j \in E \setminus B$, be the coefficients in that representation. The General Basis Theorem states that for each $e_j \in E \setminus B$ there exists a unique elementary vector f^j whose support contains e_j positively (that is, $f_j^j = 1$) and otherwise only elements of B . The corresponding elementary vectors f^j form an algebraic basis for $C(\alpha)$ and are determined by

$$f_k^j = \begin{cases} -\hat{a}_{kj} & k \in B \\ 1 & k = j \\ 0 & \text{otherwise.} \end{cases}$$

The subset of $C(\alpha)$ given by $\{f^j : j \in E \setminus B\}$ is called the **fundamental basis associated with B** . Furthermore, any elementary vector is part of at least one such fundamental basis. Hence we may, by the following construction, obtain all of them.

Let $B = G_T \cup \{e_n\}$ be any α -one-forest, and recall from Lemma 2 that (\hat{a}_{ij}) are the entries of the Tucker representation of $C(\alpha)$ corresponding to the basis B and that (a_{ij}) are the entries of A' (recall that $A' = (a_{ij})$ and $a_{ij} = -c(e_j, G_T)_i$). Now, we can use (3.1) and the expression of f_k^j to write

$$\begin{aligned} f^j &= - \sum_{e_k \in B} \hat{a}_{kj} e_k + e_j \\ &= - \sum_{e_k \in G_T} \hat{a}_{kj} e_k - \hat{a}_{nj} e_n + e_j \\ &= - \sum_{e_k \in G_T} a_{kj} e_k + e_j + \sum_{e_k \in G_T} \frac{\hat{\alpha}_j}{\hat{\alpha}_n} a_{kn} e_k - \frac{\hat{\alpha}_j}{\hat{\alpha}_n} e_n \\ &= c(e_j, G_T) - \frac{\hat{\alpha}_j}{\hat{\alpha}_n} c(e_n, G_T). \end{aligned} \tag{3.2}$$

Therefore, f^j is either a neutral oriented cycle (if $\hat{\alpha}_j = 0$), or a weighted sum of two active oriented cycles, in which the weights are determined, up to a multiplicative factor, by the equation $\alpha f^j = 0$. Observe further that any elementary vector thus constructed will fall into one of the following categories.

- (i) $f = c$ and c is a simple neutral cycle satisfying $\hat{\alpha}_j = 0$ in (3.2) for some basis B .
- (ii) f is of the form

$$f = c^1 - (\alpha c^1 / \alpha c^2) c^2 \quad (3.3)$$

with $\alpha c^1 \neq 0$ and $\alpha c^2 \neq 0$. In this situation several cases may occur

- (ii-a) c^1 and c^2 are two disjoint active cycles
- (ii-b) c^1 and c^2 are two simple active cycles intersecting on a single node or on a path.
- (ii-c) c^1 and c^2 are two simple active cycles whose intersection includes more than either a single node or a path. One can see that such a situation would create more than three cycles in $\text{Supp}(f)$ — not a minimal support — and therefore f could not be derived in the fashion described in the General Basis Theorem. Thus this situation does not lead to an elementary vector of $C(\alpha)$.

Furthermore, any vector f belonging to either of the categories (i), (ii-a) or (ii-b) can be constructed in a fashion described above. Thus we have the following theorem.

THEOREM 5 *The family of vectors $H(\alpha)$ for which either (i), (ii-a) or (ii-b) holds is the family of elementary vectors of $C(\alpha)$. \square*

3.3 Dependency and Graph Decomposition

We start this section by defining a dependence relation between arcs in a constrained graph which will lead to a decomposition of the constrained monotropic network flow problem into subproblems of lower dimensions.

Let $G = (N, E, \alpha)$ be a constrained graph. We say that two arcs e_i and e_j in G are **dependent** if and only if there exists an elementary vector $f = (f_k) \in H(\alpha)$ for which $f_i f_j \neq 0$. It can be seen that the dependence relation is an equivalence relation. Indeed, the set of elementary vectors of the circulation space forms a matroid [Ro67]. Therefore, given three arcs e_i , e_j and e_k , if there exists two elementary vectors f^1 and f^2 that respectively contain e_i and e_j , and e_j and e_k , then there exists an elementary vector f^3 that contains e_i and e_k .

A constrained graph $G = (N, E, \alpha)$ is said to be **dependent** if and only if every pair of arcs in E is dependent. A **dependent component** of G is a maximal dependent subgraph of G or, equivalently, a subgraph of G whose set of arcs is an equivalence class for the dependence relation.

We now look into the problem of decomposing a constrained graph $G = (N, E, \alpha)$ into dependent components. To that end, recall that a graph $G = (N, E)$ is **biconnected** if and only if G is connected and either E is a singleton or for any pair of arcs in E there exists a simple cycle containing both arcs. A **biconnected component** of a graph is a maximal biconnected subgraph, and Whitney [Wh32] showed that a graph can be decomposed into biconnected components in a unique fashion. Two biconnected components have at most one node in common (see e.g. [Tut66, pp. 90–92]), and the decomposition of a graph into biconnected components can be done in linear time as described, for example, in [Pa71], [Ta72] and [HT73]. We show how the decomposition procedure into biconnected components can be used to decompose a constrained graph into dependent components.

We begin by decomposing G into, say, ℓ biconnected components $G_r = (N_r, E_r, \alpha_r)$, $r = 1, \dots, \ell$ where α_r is naturally obtained by keeping in α the entries corresponding to arcs in E_r . Then for each biconnected component we test whether G_r contains at least one active cycle. We will assume without loss of generality that for some $1 \leq p_0 \leq \ell$, G_1, \dots, G_{p_0} each contain at least one active cycle, and that G_{p_0+1}, \dots, G_ℓ contain no active cycle.

THEOREM 6 *Let $G_1, \dots, G_{p_0}, G_{p_0+1}, \dots, G_\ell$ be defined as above, and define the graph $G^* = (N^*, E^*) \equiv G_1 \cup G_2 \dots \cup G_{p_0}$. If $p_0 \geq 2$ then $G^*, G_{p_0+1}, \dots, G_\ell$ are the dependent*

components of G . If $p_0 = 1$ there exists a partition of G_1 into $Q \geq 1$ subgraphs $\tilde{G}_q = (N^q, E^q, \alpha^q)$, $q = 1, \dots, Q$, $E_1 = \bigcup_{q=1}^Q E^q$, $E^q \cap E^{q'} = \emptyset$ whenever $q \neq q'$ and $|N^q \cap N^{q'}| = 2$ for $q \neq q'$, such that $\tilde{G}_1, \dots, \tilde{G}_Q, G_{p_0+1} = G_2, \dots, G_t$ are the dependent components of G . Moreover if $p_0 = 1$ and $Q \geq 2$ the dependent components of G_1 are biconnected subgraphs that form a rim (a term which is defined later) and none of them contain any active cycle.

The proof of Théorem 6 requires the following four Lemmas.

LEMMA 7 *Let $G = (N, E, \alpha)$ be a constrained biconnected graph. If there exists an active cycle in G , then for each arc e_i in G there exists an active cycle containing e_i .*

PROOF. Let c^0 be an active cycle in G and e_i an arc in E . If e_i is in c^0 the proof is complete, otherwise let e_j be any arc in c^0 . Since G is biconnected there exists a simple cycle, say c^1 , that contains both e_i and e_j . If $\alpha c^1 \neq 0$ then c^1 is the required active cycle containing e_i and the proof is complete. Otherwise, let the oriented path p which contains e_i as a forward arc be the part of c^1 whose first and last nodes, namely n_1 and n_2 , belong to c^0 , but which has no other nodes in c^0 . Let the oriented path p' (resp., p'') be the part of c^0 that lies between n_1 and n_2 and contains (resp., does not contain) e_j , oriented from n_2 to n_1 . Consider the two cycles $c^2 = p + p'$ and $c^3 = p + p''$, both oriented so that $c_i^2 = c_i^3 = 1$. Clearly $c^2 - c^3 = p' - p'' = \pm c^0$, therefore $\alpha c^2 - \alpha c^3 = \pm \alpha c^0 \neq 0$, which implies $\alpha c^2 \neq \alpha c^3$. Now, since both c^2 and c^3 contain e_i and either $\alpha c^2 \neq 0$ or $\alpha c^3 \neq 0$ the proof follows. \square

LEMMA 8 *Let $G = (N, E, \alpha)$ be a biconnected constrained graph, and suppose that G can be written as the union of two constrained graphs $G_1 = (N^1, E^1, \alpha^1)$ and $G_2 = (N^2, E^2, \alpha^2)$ where $E^1 \cap E^2 = \emptyset$, $E^1 \cup E^2 = E$. Moreover suppose that G_1 is dependent and contains at least one active cycle. Then G_1 and G_2 are dependent.*

PROOF. We start by proving that for any given arc e_i in G_2 there exists an arc in G_1 such that this arc and e_i are dependent. The remainder of the proof follows from the fact that G_1 is dependent and the transitivity of the dependence relation. In order to do this we will exhibit an elementary vector with nonzero entries both on e_i and on some arc in G_1 . Let c^0 be any active cycle of G_1 and e_j be any arc in c^0 . Since G is biconnected there exists a cycle, say c^1 , such that $c_i^1 c_j^1 \neq 0$. If c^1 is a neutral cycle, that is, $\alpha c^1 = 0$, then c^1 is an elementary vector of $H(\alpha)$, e_i and e_j are dependent, completing the proof. Let us assume on the contrary that c^1 is an active cycle and define the simple cycles c^2 and c^3 in a similar manner as it was done in the proof of Lemma 7. If $\alpha c^2 = 0$ (resp., $\alpha c^3 = 0$) then c^2 (resp., c^3) is an elementary vector that contains e_i and at least one arc in G_1 , and the proof follows. Otherwise both $\alpha c^2 \neq 0$ and $\alpha c^3 \neq 0$, and since c^2 and c^3 intersect on the simple path p (defined in the proof of Lemma 7) then $f = c^2 - (\alpha c^2 / \alpha c^3) c^3$ is an elementary vector of $H(\alpha)$ that contains all the arcs of c^0 . Moreover, since $f_i = c_i^2 - (\alpha c^2 / \alpha c^3) c_i^3 = 1 - \alpha c^2 / \alpha c^3 \neq 0$ for we showed in Lemma 7 that $\alpha c^2 \neq \alpha c^3$, f is the required elementary vector and the proof is completed. \square

LEMMA 9 Suppose that a biconnected constrained graph $G = (N, E, \alpha)$ has several dependent components $G_q = (N^q, E^q, \alpha^q)$, $q = 1, \dots, Q$ with $Q \geq 2$. Then no dependent component contains any active cycle, and $|\Delta(q)| = 2$ for any $q = 1, \dots, Q$ where $\Delta(q) = N^q \cap \bigcup_{m \neq q} N^m$.

PROOF. To show the first part of the Lemma, suppose that for some $1 \leq q \leq Q$, G_q contains an active cycle. By applying Lemma 8 to the case where G_1 is G_q and G_2 is $\bigcup_{m \neq q} G_m$ one obtains that G_1 is not independent of the rest of G , yielding a contradiction. To show the second part assume that the result does not hold and that, for example, $|\Delta(1)| \geq 3$ (Clearly $|\Delta(1)| \leq 1$ is impossible for, in a biconnected graph with more than one arc, all node-cuts have at least two elements). Since G is biconnected there exists a simple cycle that contains arcs both in G_1 and in $\tilde{G} = \bigcup_{m \neq 1} G_m$. Moreover since $\Delta(1)$ is the node-cut between G_1 and \tilde{G} , any such cycle contains at least two nodes in $\Delta(1)$.

Let c be such a cycle and assume without loss of generality that the part of c in \tilde{G} is a simple path p that contains no other node of G_1 than its end nodes, say n_1 and n_2 . Indeed if this does not hold let e_j be any arc of G_1 which is in p , n_1 (resp., n_2) the first node of $\Delta(1)$ (possibly an extremity of e_j) encountered on p starting from e_j forward (resp., backward) with respect to the orientation of c , and replace c by the portion of it that lies between n_1 and n_2 in \tilde{G} together with any path joining n_1 to n_2 in G_1 (such a path exists for G_1 is connected).

Since $|\Delta(1)| \geq 3$ there exists a third node common to G_1 and \tilde{G} , say n_3 , and since G is biconnected there exists a simple cycle, say c^0 , that uses two given arcs adjacent to n_3 , one in G_1 and the other in \tilde{G} . Orient c^0 so its first arc after n_3 is in \tilde{G} , denote by n_4 the first node of $\Delta(1)$ encountered on c^0 starting from n_3 and by p' the part of c^0 which lies in \tilde{G} between n_3 and n_4 .

We first look at the case where $n_4 \neq n_1, n_2$ and show that we can assume without loss of generality that p and p' have at most one node in common. Indeed if they do not, let n_5 (resp., n_6) be the first (resp., last) node of p' encountered on p when starting from n_1 (if necessary exchange the names of n_3 and n_4 so that n_5 is encountered on p' before n_6 when starting from n_3), then replace p (resp., p') by the part of p that joins n_1 to n_5 (resp., n_2 to n_6) together with the part of p' that joins n_3 to n_5 (resp., n_4 to n_6) and exchange n_2 and n_3 . Define now two arc-disjoint paths p'' and p''' in G_1 that join respectively n_1 to n_2 and n_3 to n_4 (To prove the existence of p'' and p''' add to G_1 two arcs joining respectively n_1 to n_3 and n_2 to n_4 . Since G_1 is still biconnected there exists a simple cycle of G_1 that contains the two arcs, and removing these two arcs from the cycle results in p'' and p'''). Further define the cycle c^1 (resp., c^2) obtained by joining p and p'' (resp., p' and p'''). Both c^1 and c^2 are active for they have arcs in G_1 and in \tilde{G} , and they meet in at most one node, thus $c^1 - (\alpha c^1 / \alpha c^2) c^2$ is an elementary vector with arcs in G_1 and in \tilde{G} , yielding a contradiction.

We now look at the alternative case where one of n_1 or n_2 , say n_1 , is equal to n_4 . First, we show that we can assume without loss of generality that p and p' meet in at most a single node or a simple path. In order to see this, suppose on the contrary that this situation does

not arise. Let n_5 be the first node of p encountered on p' starting from n_3 (thus $n_5 \neq n_1$) and replace in p' the portion between n_5 and n_1 by the corresponding portion in p , so that the two paths p and p' now possess the above property. Then define two arc-disjoint paths p'' and p''' in G_1 that join respectively n_1 to n_2 and n_1 to n_3 (the proof of their existence is similar to the proof of the existence of p'' and p''' in the previous case, by introducing an arc between n_2 and n_3), and the cycles c^1 and c^2 as above. For similar reasons c^1 and c^2 are active, and now meet in at most a single node or a simple path, thus $c^1 - (\alpha c^1 / \alpha c^2) c^2$ is an elementary vector, yielding a similar contradiction. \square

Lemma 10 to follow is based on the construction we present now.

For a biconnected constrained graph $G = (N, E, \alpha)$ that is comprised of several dependent components, say $G_q = (N^q, E^q, \alpha^q)$ for $q = 1, \dots, Q \geq 2$, we define the cycle graph G^+ induced by G as follows. We will call bolts the nodes of N that are common to at least two dependent components of G and denote by N^+ the set of bolts. Recall from Lemma 9 that each G_q has exactly 2 nodes in common with the other dependent components. For $q = 1, \dots, Q$ we define the arc $e_q^+ = (i, j)$ where i and j are the two nodes common to N^q and $\bigcup_{q' \neq q} N^{q'}$ as described in Lemma 9, and associate with it a scalar $\alpha_q^+ = \alpha p^q$ where p^q is any oriented path in G_q from i to j . Moreover, if $E^+ = \{e_q^+ : q = 1, \dots, Q\}$ and $\alpha^+ = (\alpha_q^+)_{q=1, \dots, Q}$, then the induced cycle graph is the constrained graph $G^+ = (N^+, E^+, \alpha^+)$. We will refer to G as a rim and to its biconnected components $G_q, q = 1, \dots, Q$ as the rim components.

LEMMA 10 *Suppose that $G = (N, E, \alpha)$ is a biconnected constrained graph that is comprised of several dependent components and let $G^+ = (N^+, E^+, \alpha^+)$ be the cycle graph induced by G . Then α^+ is independent of the choice of the paths $(p^q)_{q=1, \dots, Q}$ and G^+ is a simple active cycle.*

PROOF. To see that α_q^+ does not depend on the choice of p^q suppose that there exist two oriented paths in G_q , say, p and p' between the two nodes $i, j \in N^q \cap N^+$ such that $\alpha p \neq \alpha p'$. Clearly $p - p'$ is a circulation on G_q , and therefore a sum of simple cycles [Be73,

pp. 90–91]. Since $\alpha(p - p') \neq 0$, then for at least one of those cycles, say c we have $\alpha c \neq 0$, contradicting the fact that G_q contains no active cycles as was shown in Lemma 9.

Clearly G^+ cannot contain any simple neutral cycle, for otherwise such a cycle would correspond to a simple neutral cycle of G , that is, an elementary vector with arcs in several of the $G_q, q = 1, \dots, Q$, contradicting the fact that the latter are independent. Let us assume now that G^+ contains at least two distinct active cycles, say c^{+1} and c^{+2} . Then the vector $x = (\alpha c^{+2})c^{+1} - (\alpha c^{+1})c^{+2}$ is a constrained circulation on G^+ since it verifies $Ax = 0$ and $\alpha x = 0$, and therefore x is a sum of elementary vectors of G^+ . Let f^+ be any one of those elementary vectors. Since G^+ does not contain neutral cycles f^+ is the weighted sum of two active cycles in G^+ that meet in at most a simple path or a single node, and the same holds for the vector f of G obtained by replacing every arc $e_q^+, q = 1, \dots, Q$, in G^+ by p^q . Therefore f is an elementary vector of G with non-zero components in several dependent components, leading to a contradiction. \square

COROLLARY 11 *If a graph $G = (N, E, \alpha)$ is a rim, then any active cycle in G goes through all the bolts of G and through all of its rim components.*

PROOF. The main consequence of Lemma 10 is a result on the topology of rims. Such graphs are closed “chains” of biconnected subgraphs, each having a single node, a bolt, in common with each one of its two neighbours. Since none of these rim components contains an active cycle, any active cycle has to go through at least two rim components and thus, from the “closed chain” topology of the rim, through all rim components and all the bolts. \square

PROOF OF THEOREM 6. Each subgraph G_{p_0+1}, \dots, G_t is clearly dependent since each is a biconnected component with only neutral cycles, which are also elementary vectors. We first consider the case where $p_0 \geq 2$ and show that, for such values of p_0 , G^* is dependent. To that end let e_i and e_j be two arcs in E^* , for which the following two situations may arise.

(i) e_i and e_j belong to the same biconnected component, say G_1 , of G^* . Then, there exists a simple cycle $c^{i,j}$ containing both e_i and e_j whose arcs are all in G_1 . If $c^{i,j}$ is a neutral

cycle, that is, $\alpha c^{i,j} = 0$, then $c^{i,j}$ is an elementary vector and hence e_i and e_j are dependent. Otherwise $\alpha c^{i,j} \neq 0$, and since $p_0 \geq 2$ there exists an active cycle, say c , in some G_m with $2 \leq m \leq p_0$ such that $f = c^{i,j} - (\alpha c^{i,j} / \alpha c) c$ is an elementary vector of G with $f_i f_j \neq 0$. (ii) e_i and e_j belong to two different biconnected components of G . By Lemma 7 we can find two active cycles c^i and c^j one in each biconnected component with $c^i_i = c^j_j = 1$ that intersect in at most one node. Thus $f = c^i - (\alpha c^i / \alpha c^j) c^j$ is an elementary vector of G for which $f_i f_j \neq 0$, and the two arcs are dependent. We have therefore shown that G^* is dependent.

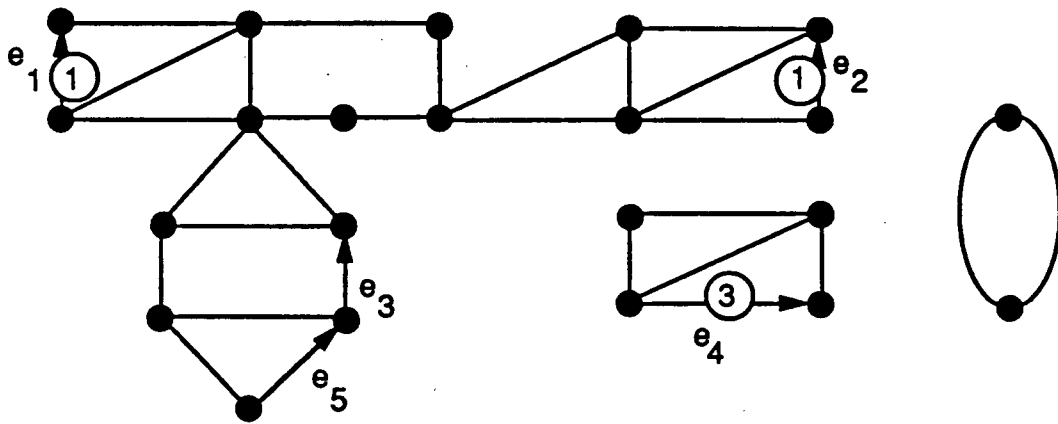
We next show that, when $p_0 \geq 2$, $G^*, G_{p_0+1}, \dots, G_t$ are pairwise independent. Clearly no simple cycle of G contains arcs that belong to more than one of the above subgraphs. Thus, if say G' and $G'' \in \{G^*, G_{p_0+1}, \dots, G_t\}$ are dependent, there must exist an elementary vector f satisfying $f_i f_j \neq 0$ for some $e_i \in G'$ and $e_j \in G''$. Moreover, f is a weighted sum of two active cycles, one contained in G' and the other in G'' . This is impossible since G^* is the only component containing active cycles.

We now look at the case where $p_0 = 1$ and face the decomposition of $G^* = G_1$ into dependent components. From Lemma 9, G_1 either is dependent or can be decomposed into several dependent components, each being a connected subgraph that does not contain any active cycles. Lemma 10, when applied to G_1 in the case it contains several dependent components, completes the proof of Theorem 6 by providing the required rim structure of G_1 . \square

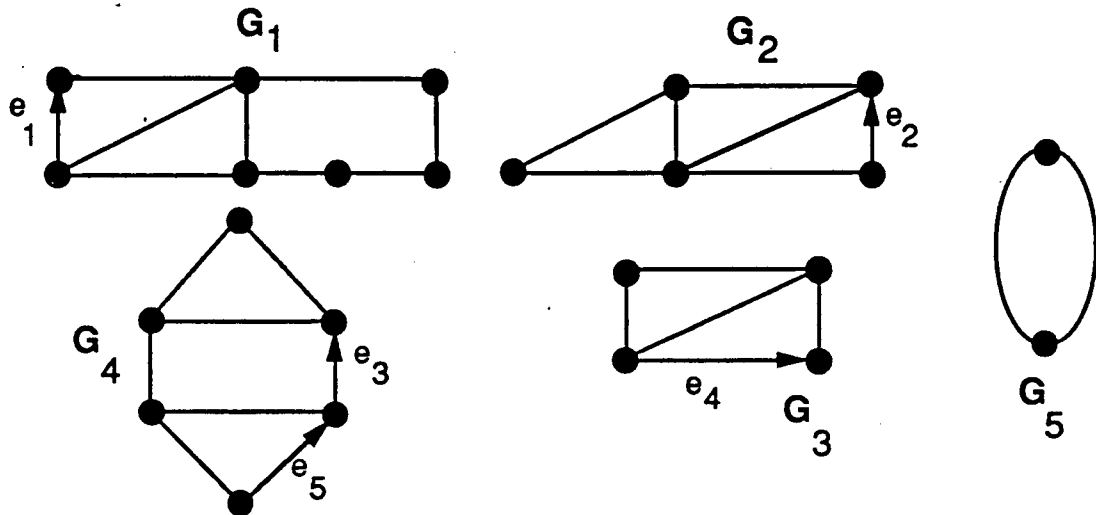
The decomposition of a graph into its dependent components described in Theorem 6 will now be demonstrated in the following example.

EXAMPLE 1 *Figures 3.1-b and 3.1-c illustrate the decomposition procedure applied to the graph G given in Figure 3.1-a (the circled numbers on the arcs designate the nonzero entries of α). As shown in Figure 3.1-b, G has five biconnected components of which G_1, G_2 and G_3 contain active cycles and G_4 and G_5 contain only neutral cycles. \square*

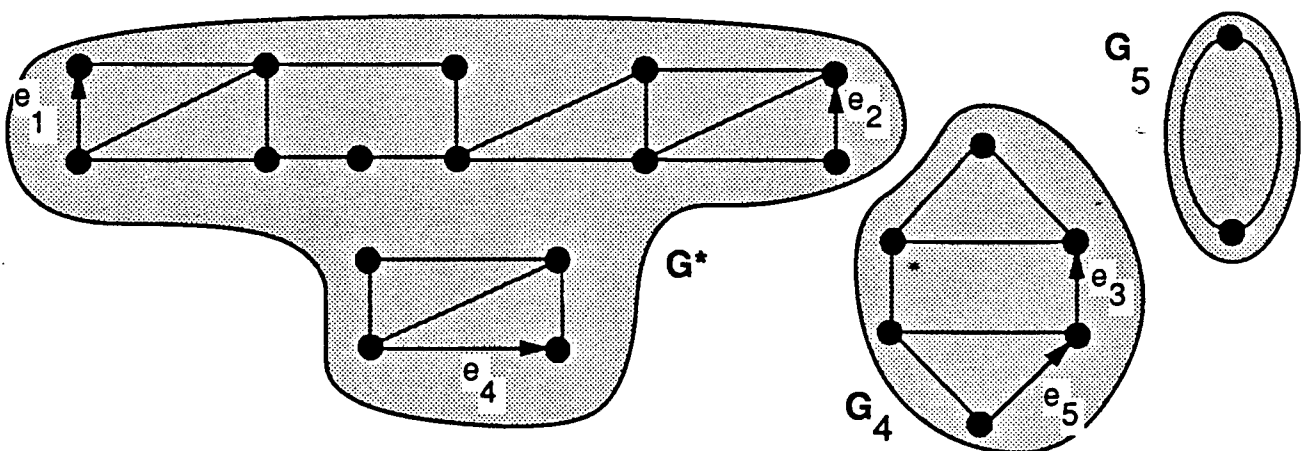
Our last task in this section is to determine whether a biconnected constrained graph (or subgraph) that contains active cycles is a rim, that is, whether $Q \geq 2$. Recall that



a: The Graph G



b: Decomposition into 5 biconnected components



c: Aggregation into 3 dependent components

Figure 3.1: Decomposition of the graph G

a biconnected constrained graph $G = (N, E, \alpha)$ is a rim if and only if it contains at least one active cycle and can be partitioned into $Q \geq 2$ constrained biconnected subgraphs $G_\ell = (N^\ell, E^\ell, \alpha^\ell)$, $\ell = 1, \dots, Q$ such that no subgraph contains an active cycle and, if $Q = 2$, $N^1 \cap N^2 = \{i_1, i_2\}$, whereas if $Q \geq 3$, $N^\ell \cap N^{\ell+1} = \{i_\ell\}$ for $\ell = 1, \dots, Q-1$, $N^Q \cap N^1 = \{i_Q\}$, $N^\ell \cap N^k = \emptyset$ for $k > \ell + 1$.

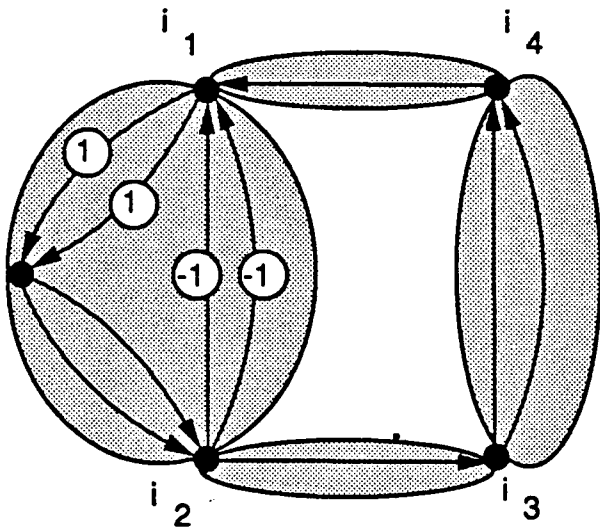
It is important to notice that the fact that a constrained graph is a rim does not depend only on its topology but rather on both the structure of the graph and the nature of the constraint vector α . Example 2 to follow will illustrate this fact.

EXAMPLE 2 In Figures 3.2-a to 3.2-c we present a graph with 5 nodes and 10 arcs. The shaded areas delimit the rim components when the graph is a rim, in which case the bolts are denoted by i_1, i_2 , etc... The reader will notice that, depending on α , the graph may be a rim with various sets of bolts and components (Figures 3.2-a and 3.2-b) or not be a rim at all (Figure 3.2-c).

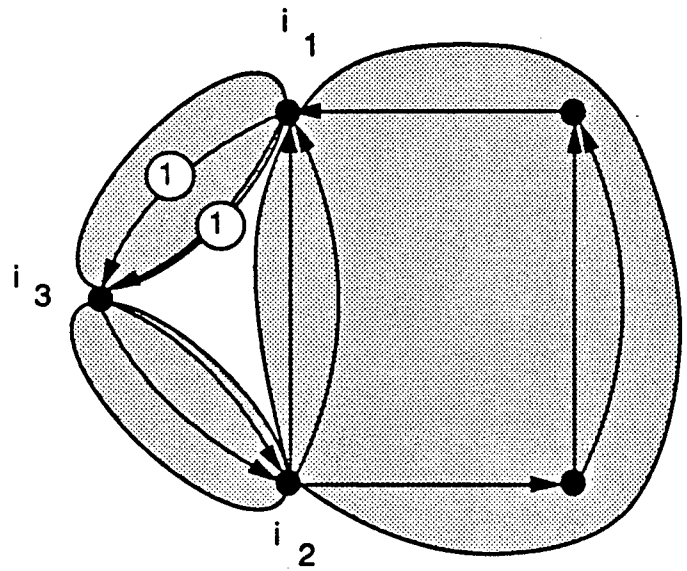
Indeed, it is easy to show that, for the values of α given in Figures 3.2-a and 3.2-b, G is a rim. On the other hand to verify that the constrained graph G shown in Figure 3.2-c is not a rim notice that, if it were one, the only possible bolts would be the nodes n_1 and n_2 , for the cycle $e_1 - e_2$ is active. Therefore, if G were a rim, it would have two components, one of which would contain one of the arcs either e_1 or e_2 and the rest of the graph. However, both subgraphs $G \setminus \{e_1\}$ and $G \setminus \{e_2\}$ contain active cycles and thus do not qualify as rim components, proving that G is not a rim. \square

In the following Theorem we present a characterization of the bolts of a rim.

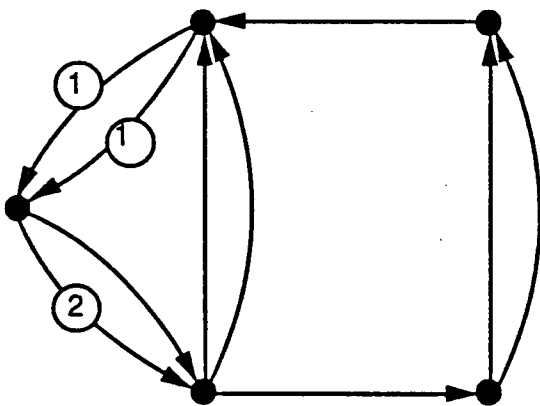
THEOREM 12 Let $G = (N, E, \alpha)$ be a rim and i some node in N . Then i is a bolt of G if and only if the graph $G \setminus \{i\}$ obtained by deleting from G node i and all the arcs adjacent to it does not contain any active cycle.



a: A rim graph with 4 components



b: A rim graph with 3 components



c: A graph which is not a rim

Figure 3.2: Rims and Graphs

PROOF. Observe first that $G \setminus \{i\}$ does not contain any active cycle when i is a bolt, for any active cycle goes through all the bolts as was shown in Corollary 11. We now show that if $G \setminus \{i\}$ does not contain any active cycle then i is a bolt of G . To that end assume on the contrary that all active cycles use node i but that i is not a bolt, that is, it belongs to exactly one component of G , say G_1 (recall from the proof of Corollary 11 that nodes may only belong to either one or two rim components). Let c be any active cycle of G , and write $\pm c$ as $p_1 + p_2 + \dots + p_Q$ where p_ℓ is an oriented path in the rim component G_ℓ from $i_{\ell-1}$ to i_ℓ (with $i_0 = i_Q$). If $G_1 \setminus \{i\}$ contained an oriented path from i_Q to i_1 , say p'_1 , then $p'_1 + p_2 + \dots + p_Q$ would be an active cycle of $G \setminus \{i\}$, as shown in Lemma 10, that does not use node i contradicting the hypothesis. If on the other hand $G_1 \setminus \{i\}$ does not contain any such path then $\{i\}$ is a cut-node of G_1 , which is a biconnected subgraph, again yielding a contradiction and completing the proof. \square

We assume whenever necessary that G is a constrained palm graph with underlying tree T and partition the set of return arcs $E \setminus T$ into two sets R and S as follows. If $c(e_i, T)$ is an active cycle then it is called an **R-cycle** and e_i an **R-arc** ($e_i \in R$) while if $c(e_i, T)$ is a neutral cycle it is called an **S-cycle** and e_i an **S-arc** ($e_i \in S$). Clearly the subgraph H of G whose set of arcs is $T \cup S$ does not contain any active cycles. The biconnected components H_1, \dots, H_h of H will be called the **branches** of G . We further let v_0 be the largest ancestor of all R-arc tails, that is, $v_0 = \max\{l : l \in p(i) \text{ } (i, k) \in R\}$ and w_0 the largest R-arc head, that is, $w_0 = \max\{k : (i, k) \in R\}$. The above definitions are demonstrated in Figure 3.4. In Lemma 13 we introduce a property of v_0 and w_0 that connects these nodes with the R-cycles of G .

LEMMA 13 *Let $G = (N, E, \alpha)$ be a biconnected constrained palm graph, and v_0 and w_0 be defined as above. If $w_0 \leq v_0$, the intersection of all R-cycles is the path $p(w_0, v_0)$, whereas if $v_0 < w_0$ this intersection is empty.*

PROOF. For any R-cycle $c(e_j, T)$, with $e_j = (i, k) \in R$, v_0 is an ancestor of i and $w_0 \geq k$. Thus, if $w_0 \leq v_0$, then $k \leq w_0 \leq v_0 \leq i$, $p(w_0, v_0)$ is a part of $c(e_j, T)$, and hence is common to all R-cycles. Moreover, from the definitions of v_0 and w_0 , no strict ancestor of w_0 is common to all R-cycles, nor does any strict descendant of v_0 . Therefore, the intersection of all R-cycles contains only $p(w_0, v_0)$, and the first part of the Lemma follows. In order to prove the second part, assume $v_0 < w_0$. From the definition of w_0 there exists an R-arc $e_j = (i, w_0) \in R$ where i is a descendant of w_0 , that is, w_0 belongs to $p(i)$. Moreover, from the definition of v_0 , v_0 also belongs to $p(i)$, and thus $v_0 < w_0$ implies that v_0 is a strict ancestor of w_0 . Now, there exists another R-arc, say, $e_{j'} = (i', k) \in R$, such that $p(v_0, i')$ and $p(v_0, w_0)$ have only v_0 in common, for otherwise the immediate descendant of v_0 on $p(v_0, w_0)$ would be an ancestor of all R-arc tails strictly greater than v_0 , contradicting the definition of v_0 . We show that $c(e_{j'}, T)$ has no node in common with $c(e_j, T)$. Indeed, if $k \leq v_0$ we can write $c(e_{j'}, T) = p(k, i') + e_{j'} = p(k, v_0) + p(v_0, i') + e_{j'}$. Now, the nodes of $p(k, v_0)$ are all ancestors of v_0 and thus, since $v_0 < w_0$, $p(k, v_0)$ contains no descendant of w_0 and from the choice of $e_{j'}$ neither does $p(v_0, i')$, whereas $c(e_j, T)$ uses only descendants of w_0 . The same conclusion holds if $k > v_0$, since from the choice of $e_{j'}$, $p(k, i')$ contains no descendant of w_0 . Thus the intersection of all R-cycles is empty and the proof is complete. \square

We now present four results that connect the palm graph and the rim structures that will in turn provide a basis for our rim detection algorithm. In Lemma 10 we showed that if G is a rim then for any active cycle c , $|\alpha c|$ is a constant. We will now show in Lemma 14 that if in addition G is a palm-graph then $\alpha c(e_j, T)$ does not depend on the choice of e_j in R .

LEMMA 14 *Let $G = (N, E, \alpha)$ be a biconnected constrained palm-graph. If G is a rim then $\alpha c(e_j, T)$ does not depend on the choice of e_j in R .*

PROOF. Suppose that the Lemma does not hold and that for some $e_1, e_2 \in R$ we have $\alpha c(e_1, T) \neq \alpha c(e_2, T)$. Since from Lemma 10 we know that $|\alpha c(e_1, T)| = |\alpha c(e_2, T)|$ then

$\alpha c(e_1, T) = -\alpha c(e_2, T)$. By Lemma 13, the intersection of $c(e_1, T)$ and $c(e_2, T)$ is a path, say $p(w, v)$, with $w \leq w_0 \leq v_0 \leq v$ and combining with Corollary 11 we obtain that $p(w_0, v_0)$ contains all the bolts of G , and thus $w_0 < v_0$. Therefore $c = c(e_1, T) - c(e_2, T)$ is an oriented cycle of G , which is active for $\alpha c = \alpha c(e_1, T) - \alpha c(e_2, T) = 2\alpha c(e_1, T) \neq 0$, and thus c also contains all the bolts of G . Now, since v and w are the sole nodes common to c , $c(e_1, T)$ and $c(e_2, T)$ by Corollary 11 they are the only two bolts of G and hence $v = v_0$ and $w = w_0$. From the topological structure of rims exposed in Corollary 11, in a rim with two bolts and two components the part of an active cycle that lies between two bolts belongs entirely to one of the two components. Thus, if $p(w_0, v_0)$ lies in, say, G_1 then $c(e_1, T) - p(w_0, v_0)$ and $c(e_2, T) - p(w_0, v_0)$ lie in G_2 , and therefore so does c . We have obtained an active cycle c inside a rim component, and thus a contradiction to Lemma 9. \square

In the following we will denote by α^r the unique value of $\alpha c(e_j, T)$, $e_j \in R$, whenever G is a rim. Lemma 15 to follow gives a characterization of the bolts of a constrained palm graph that is a rim.

LEMMA 15 *Let G be a biconnected constrained palm graph with underlying tree T and set of S -arcs S . If G is a rim then its set of bolts is equal to the set of cutnodes of H that lie on $p(w_0, v_0)$, together with v_0 (resp., w_0) if it is the tail (resp., head) of all R -arcs.*

PROOF. First recall that H is the subgraph of G whose set of arcs is $T \cup S$ and that a rim contains active cycles, thus $R \neq \emptyset$. We start by showing the first inclusion, namely, that the set of bolts of G is included in the above set. To that end we show that if a node $i_0 \in N$ is a bolt of G it is either a cutnode of H that lies on $p(w_0, v_0)$ or v_0 (resp., w_0) if it is the tail (resp., head) of all R -arcs. Let us first assume that $i_0 = v_0$ (resp., w_0) but that i_0 is neither the tail (resp., head) of all R -arcs nor a cutnode of H . Clearly, i_0 belongs to some branch of G , say H_ℓ , and given any R -arc e_j whose tail (resp., head) is not i_0 (we made the hypothesis that such an arc exists) the R -cycle $c(e_j, T)$ goes through i_0 (since i_0 is a bolt, and by Corollary 11 all active cycles do), and thus through H_ℓ . Let i_1 and i_2 be

respectively the first and last node of $c(e_j, T)$ that belong to the branch H_ℓ and $p(i_1, i_2)$ the portion of $c(e_j, T)$ that lies in H_ℓ . Notice that $c(e_j, T)$ does not lie entirely inside H_ℓ , and thus enters and leaves H_ℓ either through cutnodes of H or, if e_j is incident to some node in H_ℓ , through an extremity of e_j for these are the only nodes common to H_ℓ and the rest of $H \cup \{e_j\}$ (recall that $c(e_j, T)$ is composed of e_j together with arcs of H). Therefore, since i_0 is neither a cutnode of H nor the tail or head of e_j , by the choice of e_j , then i_0, i_1 and i_2 are distinct nodes of H_ℓ . Now, since H_ℓ is biconnected there exists a path in H_ℓ , say p , from i_1 to i_2 that does not go through i_0 . Moreover, from the fact that H_ℓ does not contain any active cycles we can show, as in the proof of Lemma 10, that $\alpha p = \alpha p(i_1, i_2)$ and p , being inside H_ℓ , does not intersect with $c(e_j, T) - p(i_1, i_2)$ which lies entirely outside of H_ℓ . Thus, replacing $p(i_1, i_2)$ in $c(e_j, T)$ by p results in another active cycle $c(e_j, T) - p(i_1, i_2) + p$ of G that does not contain i_0 , contradicting the fact that i_0 is a bolt. The proof for the case where $i_0 \notin \{v_0, w_0\}$ is a bolt but not a cutnode of H on $p(v_0, w_0)$ is identical except that we can take for e_j any R-arc.

In order to prove the reverse inclusion we first show that the cutnodes of H that are on $p(w_0, v_0)$ are bolts of G , and from Theorem 12 it suffices to show that all active cycles of G go through them all. The family of cycles $\{c(e_i, T), e_i \in S \cup R\}$ forms an algebraic basis of the unconstrained circulation space of G (for $S \cup R = E \setminus T$, see e.g. [Ro84, p. 117] and [Ki47] for the introduction of these cycles and the original proof) and any cycle on G can be written as $c = \sum_{e_j \in S \cup R} y_j c(e_j, T)$. Now, let H_ℓ be a branch of G that contains at least one cutnode of H on $p(w_0, v_0)$, say, i_1 . If we regard cycles as flows, then clearly no flow corresponding to an S-cycle leaves H_ℓ through i_1 (or, for that matter, through any other node) for S-cycles lie either entirely inside or outside H_ℓ . However the flow corresponding to any R-cycle $c(e_j, T)$ leaving H_ℓ through i_1 is equal to 1 if the arc $(f(i_1), i_1) \in H_\ell$ and to -1 otherwise, for $c(e_j, T)$ contains the path $p(w_0, v_0)$ which enters or leaves H_ℓ at i_1 ($p(w_0, v_0)$ contains all the bolts of G , and hence i_1). Thus, the flow corresponding to c leaving H_ℓ through i_1 is equal to $\pm \sum_{e_j \in R} y_j$. Now, since the S-cycles are neutral and $\alpha c(e_j, T) = \alpha^r$ for all R-cycles, the fact that c is active implies $\alpha c = \sum_{e_j \in S} y_j \alpha c(e_j, T) + \sum_{e_j \in R} y_j \alpha c(e_j, T) = \alpha^r \sum_{e_j \in R} y_j \neq 0$.

From Lemma 10, $|\alpha c| = |\alpha^r|$, $\sum_{e_j \in R} y_j = \pm 1$, that is, the flow corresponding to c leaving H_t through i_1 is nonzero. Therefore i_1 is a node of c and the proof is complete.

Similarly, if v_0 (resp., w_0) is the tail (resp., head) of all R-arcs then obviously deleting v_0 (resp., w_0) yields a graph with no active cycle and, by Theorem 12, v_0 (resp., w_0) is a bolt. \square

Before giving in Theorem 16 a characterization of palm-graphs that are rims, we introduce a degenerate class of rims which will be referred to as **one-rims**. We call a constrained graph $G = (N, E, \alpha)$ a one-rim if and only if it is biconnected, dependent (and therefore not a rim), and has a unique node $i_1 \in N$, called by extension a bolt, with the following property. There exists a partition of the arcs incident to i_1 into two subsets such that splitting i_1 into two copies of itself, say i'_1 and i''_1 , assigning the arcs of each of these subsets to i'_1 and to i''_1 , respectively, and introducing the **expansion arc** $\bar{e}_1 = (i'_1, i''_1)$ (with $\alpha_{\bar{e}_1} = 0$) results in a rim to be referred to as the **expanded one-rim** (see figure 3.3). One may think of a one-rim as a rim with one component, or as a rim with two components, one of which is degenerate. The appropriate partition of the arcs incident to i_1 will be determined in Algorithm I.

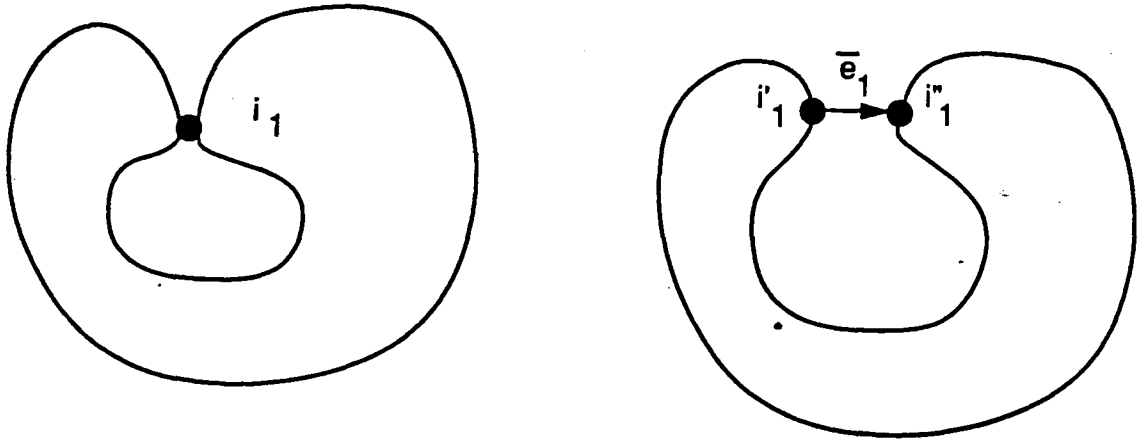


Figure 3.3: A one-rim (left) and its expanded version (right)

To motivate the attention given to one-rims, notice that the set of feasible flows on the original arcs of the expanded one-rim is equal to that of the one-rim, as long as the demands

assigned to i'_1 and i''_1 add up to that of i_1 . Moreover, since the graph obtained by splitting the unique bolt of a one-rim in the fashion described above is a subgraph of its expanded version, and is dependent, it is a rim component of the expanded version, the other component being the singleton $\{\bar{e}_1\}$.

THEOREM 16 *A biconnected constrained palm graph $G = (N, E, \alpha)$ is a rim (resp., a one-rim) if and only if $R \neq \emptyset$, $\{\alpha c(e_i, T), e_i \in R\}$ is a singleton, $w_0 < v_0$ and $p(w_0, v_0)$ contains at least two (resp., exactly one) node(s), each of which is either (i) a cutnode of H , (ii) the tail of all R -arcs or (iii) the head of all R -arcs.*

PROOF. First, let us prove the theorem for rims. We have seen in Lemmas 14 and 15 that these conditions are necessary, and prove now that they are also sufficient.

The part of G that lies (in the order given to N) between two consecutive nodes that verify (i), (ii) or (iii) is a biconnected subgraph that contains no active cycle. Indeed, no R -arc is incident to any node of this subgraph, except possibly for the first, say i_1 , and the last, say i_Q , of these nodes, and thus it is composed of a single branch that possesses the properties required to be a rim component. Furthermore the complement, in G , of the union of the above branches, say G_1 , has by construction exactly two nodes in common with arcs not in G_1 , namely i_1 and i_Q . In order to show that G_1 is a rim component we first show that it contains only neutral cycles. Indeed, the vector $z = \sum_{e_i \in R} e_i$ is a cut of G_1 , for all paths from i_Q to i_1 in G_1 contain a forward R -arc. Any cycle c of G_1 can be written, as in Lemma 15, $c = \sum_{e_i \in S \cup R} c_i c(e_i, T)$, and since cuts and cycles are orthogonal $0 = c^t z = \sum_{e_i \in R} c_i$, thus $\alpha c = \alpha^r \sum_{e_i \in R} c_i = 0$ and c is neutral (recall that α^r is the unique value of $\alpha c(e_j, T)$, $e_j \in R$).

We now claim that G_1 is biconnected, and rule out the trivial case where G_1 is equal to R and is a singleton (a subgraph composed of a single arc is biconnected). Assume on the contrary that G_1 is not biconnected then, since G is biconnected and G_1 has only i_1 and i_Q in common with the rest of G , G_1 is connected. Moreover, the biconnected components of G_1 then form a chain between i_1 and i_Q (for G is biconnected whereas a biconnected component

of G_1 with only one node in common with the rest of G_1 or with the set $\{i_1, i_Q\}$ would also be a biconnected component of G). Thus any active cycle of G goes through all the cutnodes of G_1 for they include a path from i_1 to i_Q in G_1 , and since no biconnected component of G_1 contains active cycles then G is a rim whose set of bolts includes the cutnodes of G_1 , contradicting Theorem 15 and completing our proof for rims.

Now, in order to show that the result is valid for one-rims, it suffices to apply the above to the expanded version of a one-rim (recall that these are rims with two components), and interpret it in terms of the original one-rims (by condensing the expansion arc, thus restoring the original graph). \square

We define in the following a coloring of the nodes and branches of G that, as will be seen, is easily implemented in a depth-first search algorithm, and allows us to determine both the rim components and bolts of G . We color in red all the descendants of v_0 (including v_0) that belong to some R-cycle and in blue all the ancestors of w_0 (including w_0) that belong to some R-cycle. Then, color in red (resp., blue) all the branches of G that contain an arc in T whose extremities are both red (resp., blue) and color in white the remaining nodes and branches. It is clear from Theorem 16 that if G is a rim then $w_0 \leq v_0$, thus no descendant of v_0 is an ancestor of w_0 , and a unique color is assigned to each node. The coloring of the graph G given in Figure 3.4 is demonstrated in Figure 3.5.

The usefulness of the coloring of G is justified by the following theorem.

THEOREM 17 *If G is a rim then the R-arcs and the non-white branches of G form one of its rim components, the others being the white branches.*

PROOF. We start by showing that all the white branches are rim components of G . To that end, first notice that a rim component that lies between two consecutive bolts, say u and v , $u < v$, and which contains the arc $(f(v), v)$, is the unique white branch that contains $(f(v), v)$ (see Figures 3.4 and 3.5). Indeed this white branch is biconnected, contains no active cycles, and since u and v are consecutive bolts on $p(w_0, v_0)$ it contains u . Also recall

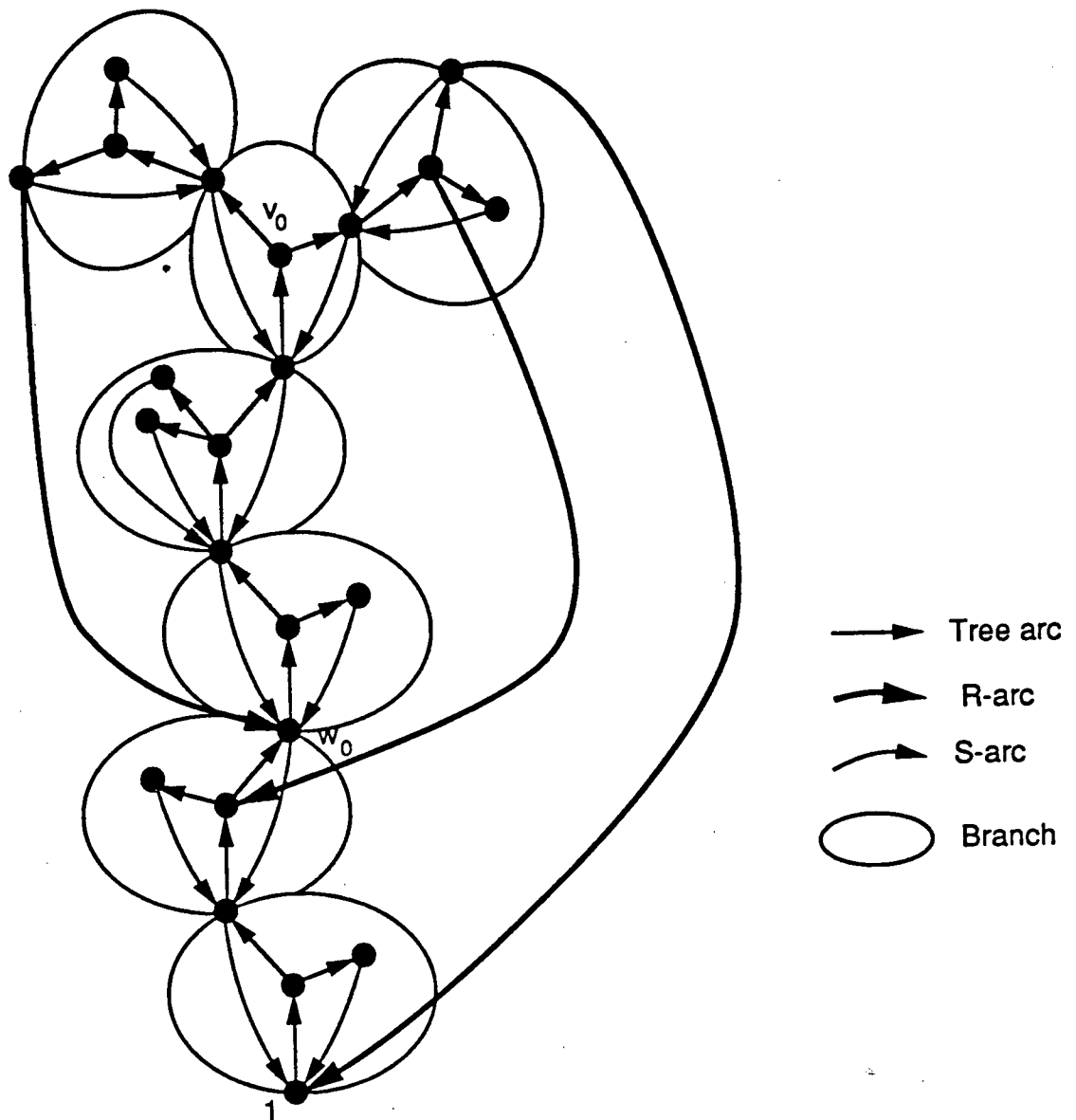


Figure 3.4: A palm graph with its branches

that the branches of G are dependent subgraphs of G (since they are biconnected and contain no active cycles, every pair of arcs in a given branch belongs to some neutral cycle of that branch, that is, to an elementary vector of $H(\alpha)$, and thus is dependent) and since rim components are maximal dependent subgraphs of G , branches either are included in rim components or do not intersect with them. Now, the nodes of the rim component mentioned above are not descendants of v_0 (resp., ancestors of w_0) except possibly for v_0 (resp., w_0) itself, and therefore it does not contain any arc whose extremities are both red (resp., blue) nor, from the above remark, any arc that belongs to a red (resp., blue) branch. Therefore, that rim component is composed of only white branches and, being biconnected, does not contain more than one such branch, proving our claim. Consequently the part of G that lies between the first and last bolt, in the order given to N , and that contains the father of the latter, is a “chain” of branches, each of which has one bolt (a cutnode of H) in common with the next (see Fig 3.5).

We show next that the remaining part of the graph, namely the last rim component G_1 introduced in Theorem 16, is composed entirely of the non-white branches and R-arcs. Let us assume on the contrary that G_1 contains at least one arc of a white branch, say, H_ℓ . From the remark made in the first part of the proof, we obtain that $H_\ell \subset G_1$. Notice further that if a branch H_k is red (resp., blue) then there exists a path on T that goes from v_0 (resp., w_0) to an arc in H_k whose extremities are both red (resp., blue). Therefore we may assume without loss of generality that H_ℓ is a leaf branch of G , that is, a branch that has only one node in common with the rest of H . However, such a leaf branch needs to have a return arc from one of its nodes, other than the cut-node of H , to some other node of $G \setminus H_\ell$, and that return arc is necessarily red (for, since H_ℓ is a leaf of H , no arc of $H \setminus H_\ell$ is incident to other nodes of H_ℓ than the unique cutnode of H that connects H_ℓ to the rest of H). Therefore H_ℓ contains at least one arc that belongs to some R-cycle, and thus two red or two blue nodes, yielding a contradiction and completing the proof. \square

Note that a constrained graph can be transformed into a palm graph in many ways, depending for example on the choice of the root. However if G is a rim its set of bolts is

unique as shown in Theorem 12, and as a consequence of Corollary 11 so is the partition of G into rim components. Therefore the sets of bolts and of rim components, as obtained in Theorems 16 and 17, are independent of the palm graph chosen.

We now present Algorithm I that, given a biconnected constrained graph $G = (N, E, \alpha)$, will determine whether G contains active cycles and, if so, whether G is either a rim or a one-rim. In the latter case, it will either provide its bolts and the rim components or, for a one-rim, its unique bolt and its expanded version. In addition, it will be useful at a later point to have a feasible flow for the unconstrained version of G . Such a flow, denoted by x , can be obtained by sending, for each node $i \in N$, a flow of d_i along the path $p(i)$, so that the flow on each tree arc is equal to the sum of the demands of the descendants of its head (it is a well known result that x is the basic feasible solution, in the usual sense of linear programming, associated with the tree T).

ALGORITHM I

Using any linear-time algorithm based on depth-first search (see e.g. [Ta72], and [HT73]) determine the branches of G (that is, the biconnected components of H) by working on H and ignoring in G the R-arcs (see “Return arc partitioning” below) as they are encountered. Simultaneously, in addition to the operations related to the detection of the branches, perform the following:

Arc redirecting. Whenever an arc $e_i = (u, v)$ is visited as a tree arc and $v < u$ (resp., as a return arc and $u < v$) redirect e_i , that is, replace it by (v, u) and replace α_i by $-\alpha_i$.

Node Labelling. Set $\alpha_1^T = 0$, then, whenever a node v is visited for the first time set $\alpha_v^T = \alpha_{f(v)}^T + \alpha_i$ where $e_i = (f(v), v)$. Recall that $\alpha_v^T = \alpha p(v)$.

Computation of x . Start by setting $x_{(f(i), i)} = d_i$ for all tree arcs $(f(i), i)$, $i \in N$. Then, whenever the search backtracks from a descendant j of i to i , add $x_{(i, j)}$ to $x_{(f(i), i)}$.

Return arc partitioning. If the arc $e_i = (u, v)$ is visited as a return arc and $\alpha_u^T + \alpha_i - \alpha_v^T = 0$ then put e_i in S , otherwise put it in R and color u in red, v in blue. If during the search $\alpha_u^T + \alpha_i - \alpha_v^T (= \alpha c(e_i, T))$ takes on more than one non-zero value end

the search and state that G is not a rim. (Notice that when G does not contain active cycles the algorithm will return $R = \emptyset$ and thus will detect such graphs).

Detection of v_0 and w_0 . When the search visits an R-arc, say (v, w) , for the first time, it colors its tail v , in red, its head w in blue, and continues its exploration from v . At a later point when all the descendants of v have been visited the search backtracks from v and indicates to its ancestors that v is red by coloring them in pink. If during the search a node, say u , is colored in pink twice when backtracking from two of its sons then we conclude that $u \geq v_0$ and we color u in red (notice that it is not indispensable to also color in red all the pink nodes between u and its red ancestors). The last node to be colored in red is v_0 . The first blue node encountered during the backtracking, and thus during the whole search, is w_0 , and all subsequent tree nodes encountered are colored in blue until the last R-arc head is reached. Clearly If $v_0 < w_0$ we conclude that G is not a rim.

Detection of the bolts. In order to keep a record of the cutnodes of H or heads or tails of all R-arcs encountered when backtracking from v_0 , the algorithm keeps a stack of such nodes visited since and including the most recent red node, emptying it completely whenever a new red node is encountered (keeping only the new red node), and stopping the stacking process after the first blue node, namely w_0 , is encountered. If by doing so we obtain an empty stack, then G is neither a rim nor a one-rim, whereas if we obtain a unique (resp., two or more) nodes then G is a one-rim (resp., a rim) and the stack contains its bolt(s).

Rim components (if G is a rim). The Biconnectivity Algorithm we apply to H , as described by [Ta72], keeps a stack of arcs corresponding to each branch, emptying it as cutnodes are encountered. Similarly we keep a stack of rim component starting at the first bolt encountered during the backtracking process and ending with the last bolt, yielding successively G_0, \dots, G_2 , while G_1 is equal to the remainder of the graph.

One-rim split. (if G is a one-rim) replace the bolt i_1 of G by the arc $\bar{e}_1 = (i'_1, i''_1)$, assigning to i'_1 the tree arcs and S-arcs joining i_1 to descendants of it, together with R-arcs whose tails are i_1 , and the remaining arcs to i''_1 .

The proof of the validity of Algorithm I lies in Theorems 16 and 17.

Complexity of Algorithm I

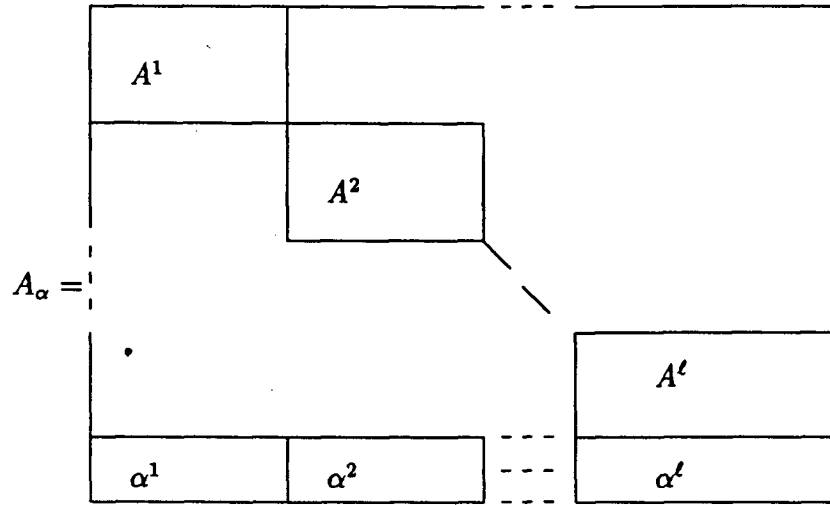
All the operations described above can be performed in a single depth-first search, and the time and memory they require at each step of the search is independent of the size of the graph. Moreover, the space required to maintain the lists corresponding to R , S , $\alpha c(e_j, T)$, $e_j \in R$, the coloring, and the stack of candidate bolts, is at most $O(m + n)$. Therefore, since the depth-first search alone takes $O(m + n)$ (including the operations related to the biconnected components of H , see e.g. [Ta72, HT73]), Algorithm I runs in linear time.

3.4 Decomposition Into Independent Monotropic Network Flow Problems

In this section we develop a decomposition theory to partition the Singly-Constrained Monotropic Network Flow Problem into subproblems where the total number of variables over the subproblems equals the number of variables of the original problem. Of those subproblems at most one is a SCMF, whereas all the others are unconstrained monotropic network flow problems. Our linear time procedure thereby establishes whether $P(\alpha, \alpha_0)$ can be transformed into an unconstrained monotropic network flow problem and if so carries out the transformation.

We will assume without loss of generality that the underlying graph G has been modified in the fashion described by Granot and Veinott [GV85] that is, that the biconnected components of G have been disconnected by creating multiple copies of the nodes common to more than one component (cutnodes of G) and that the demand at the original nodes has been allocated to their copies in the appropriate way. This modification clearly does not affect the values of the α_i 's for there exists a one-to-one correspondence between flows in the original graph and flows in the modified graph that maintains the same flow on each arc.

Now, since the biconnected components G_1, \dots, G_t of G do not have nodes in common, there exists a numbering of E and N for which the constraint matrix A_α of problem $P(\alpha, \alpha_0)$ can be written as



where A^1 (resp., A^2, \dots, A^l) is the incidence matrix corresponding to G_1 (resp., G_2, \dots, G_l) and α^1 (resp., $\alpha^2, \dots, \alpha^l$) the parts of α corresponding to the arcs in G_1 (resp., G_2, \dots, G_l). Following the notations used in Theorem 6 assume without loss of generality that the biconnected components G_1, \dots, G_{p_0} of G contain active cycles, whereas the remaining biconnected components G_{p_0+1}, \dots, G_l do not. Since G_{p_0+1}, \dots, G_l do not contain any active cycle then α^{p_0+1} (resp., $\alpha^{p_0+2}, \dots, \alpha^l$) is a linear combination of rows of A^{p_0+1} (resp., A^{p_0+2}, \dots, A^l) and thus by an appropriate modification of α and α_0 we can assume, without loss of generality, that $\alpha^{p_0+1} = \dots = \alpha^l = 0$. Clearly if $p_0 = 0$, that is, if no biconnected component of G contains active cycles, then $P(\alpha, \alpha_0)$ is feasible if and only if α_0 is also equal to zero, in which case the constraint may be dropped altogether.

If we rearrange the rows of A_α so that the α row follows the last row of A^{p_0} then clearly A_α will have the block-diagonal form. We will denote the north west block containing the rows of A^1, \dots, A^{p_0} together with the α row as $\begin{bmatrix} A^* \\ \alpha \end{bmatrix}$. It is a well known result that any optimization problem with column-additive objective function whose constraint matrix is block-diagonal can be decomposed into independent subproblems, each corresponding to the variables and linear constraints associated with a block of the matrix. We have therefore so far obtained a decomposition of $P(\alpha, \alpha_0)$ into $l - p_0 + 1$ subproblems, only one of which (corresponding to the north-west block) is a SCMF, the other $l - p_0$ being unconstrained

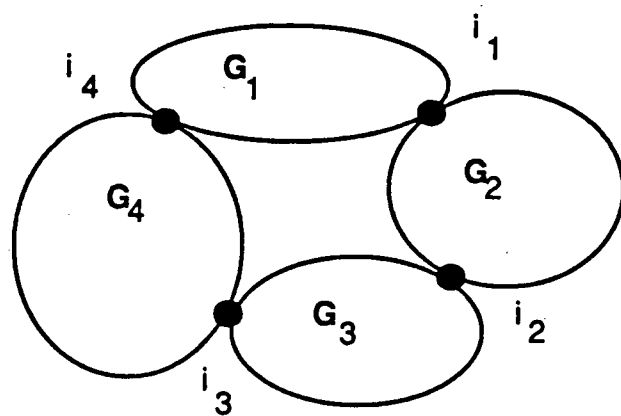
network flow problems. We will show that in some situations we can further transform $[\frac{A^*}{\alpha}]$ to a block-diagonal structure, in which case the problem $P(\alpha, \alpha_0)$ is transformable to an unconstrained network flow problem altogether.

To that end, we return to the study of rims, that is, of biconnected constrained graphs whose number Q of dependent components is at least two (in the sequel we assume without loss of generality that one-rims were replaced by their expanded version as described in section 3.3. Recall that the set of constrained flows on a one-rim is identical to the set of flows on the corresponding rim component of the associated expanded one-rim). We will show in Theorem 18, to follow, how the rim structure is related to the decomposition of the SCMF. We first require the following construction.

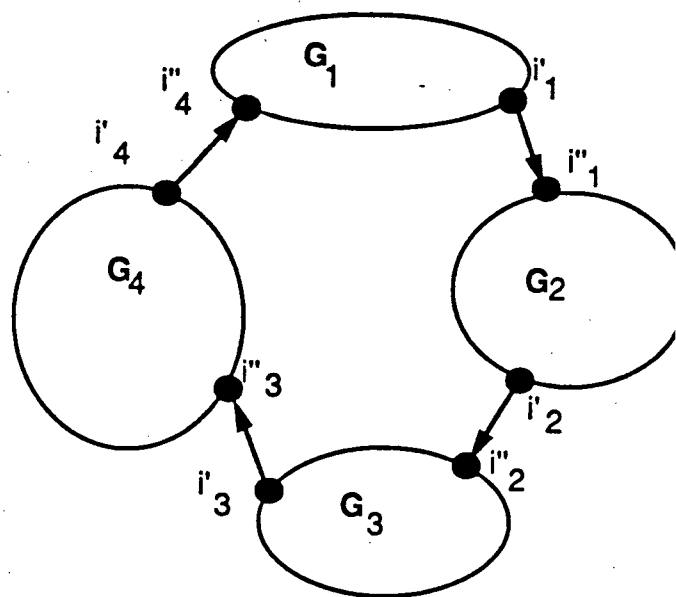
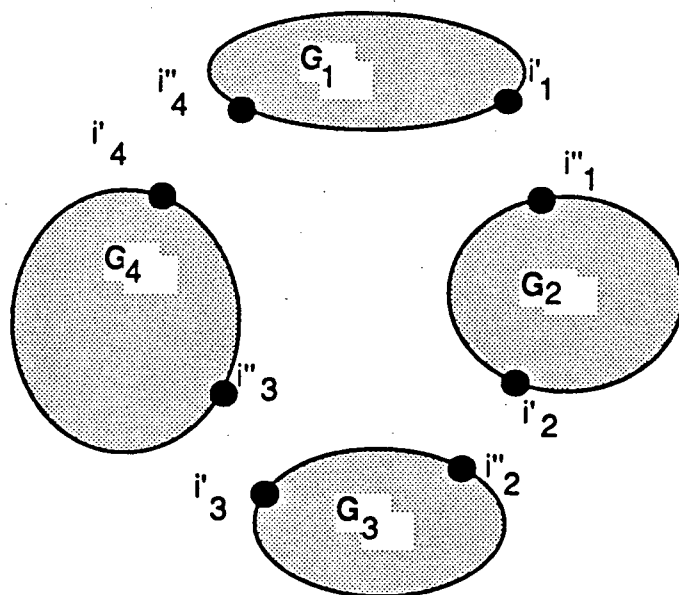
CONSTRUCTION 1 Let $G = (N, E, \alpha)$ be a rim palm graph with rim components $G_q = (N^q, E^q, \alpha^q)$, $q = 1, \dots, Q \geq 2$ and $G_{Q+1} \equiv G_1$. Denote by i_q the bolt that is common to N^q and N^{q+1} (see Figure 3.6-a). Consider now the expanded rim $\bar{G} = (\bar{N}, \bar{E}, \bar{\alpha})$ obtained from G by splitting each bolt i_q into two copies i'_q and i''_q , in such a way that all the arcs of E^q (resp., E^{q+1}) incident to i_q in G are now incident to i'_q (resp., i''_q) in \bar{G} , and by adding to \bar{E} the arcs $\bar{e}_q = (i'_q, i''_q)$, $q = 1, \dots, Q$, with an associated parameter $\bar{\alpha}_{\bar{e}_q} = 0$, the rest of G being unchanged (see Figure 3.6-b). \bar{G} is also a palm graph, and we may define on it $\bar{p}(i)$, $\bar{D}(i)$, $\bar{c}(e_j, T)$ for $i \in \bar{N}$, $e_j \in \bar{E}$ in a similar way as their counterparts $p(i)$, $D(i)$, $c(e_j, T)$, $i \in N$, $e_j \in E$ in G . It will be useful to notice that $\bar{p}(i) = p(i) + \sum_{i \in D(i''_q)} \bar{e}_q \forall i \in \bar{N}$, $\bar{\alpha}\bar{p}(i'_q) = \bar{\alpha}\bar{p}(i''_q) = \alpha p(i_q) = \alpha^T_i$ for all $q = 1, \dots, Q$ and $\bar{\alpha}\bar{p}(i) = \alpha^T_i$ for all other nodes. Moreover $\bar{c}(e_j, T) = c(e_j, T)$ for $e_j \in S$ and $\bar{c}(e_j, T) = c(e_j, T) + \sum_{q=1, \dots, Q} \bar{e}_q$ for $e_j \in R$. Recall that α^w is the unique value of $\alpha c(e_j, T)$, $e_j \in R$, and define on \bar{G} a demand vector \bar{d} as follows

$$\begin{aligned} \bar{d}_{i'_q} &= \sum_{i \in D(i_q)} d_i + \frac{1}{\alpha^r} (\alpha_0 - \sum_{i \neq 1} d_i \alpha^T_i) \\ &\quad \text{for } q = 1, \dots, Q, \\ \bar{d}_{i''_q} &= d_{i_q} - \bar{d}_{i'_q} \end{aligned} \tag{3.4}$$

$$\bar{d}_i = d_i \quad \text{for all other nodes.} \square$$



a: A rim with 4 components

b: The expanded rim \bar{G} 

c: The decomposed rim

Figure 3.6: Rim Decomposition and Expansion

THEOREM 18 Suppose that a biconnected constrained palm graph $G = (N, E, \alpha)$ has several dependent components, say $G_q = (N^q, E^q, \alpha^q)$, $q = 1, \dots, Q \geq 2$. Then the set of constrained flows on G is the Cartesian product of the sets of unconstrained flows on each dependent component when the demands at the nodes shared by two successive dependent components are given by (3.4).

PROOF. First, in the notation of Theorem 6, $p_0 = l = Q \geq 2$, and thus from Theorem 6 G is a rim. Clearly there is an identical one-to-one correspondence between constrained flows on G and the respective entries of constrained flows on \bar{G} , as long as each demand d_{i_q} is shared by i'_q and i''_q in such a way that $\bar{d}_{i'_q} + \bar{d}_{i''_q} = d_{i_q}$. We start by temporarily apportioning the demand at the bolts between their respective copies by setting $\bar{d}_{i'_q} = 0$ and $\bar{d}_{i''_q} = d_{i_q}$ for $q = 1, \dots, Q$, $\bar{d}_i = d_i$ for all other nodes. Then, notice that the flow $\bar{x}^0 = \sum_{i \neq 1} \bar{d}_i \bar{p}(i)$, obtained by sending for each node $i \in \bar{N}$, $i \neq 1$, a flow of \bar{d}_i from node 1 to i , is a feasible unconstrained flow (it is easy to see that it meets the demand requirements, that is, $\bar{A}\bar{x}^0 = \bar{d}$, where \bar{A} is the incidence matrix of \bar{G}). Since this flow may not meet the additional constraint $\alpha x = \alpha_0$, we pick any R-arc $e_k \in R$ and adjust the value of the left-hand-side of the additional constraint by sending the appropriate flow along the active cycle $c(e_k, T)$, defining the constrained flow

$$\bar{x}^1 = \sum_{i \neq 1} \bar{d}_i \bar{p}(i) + \frac{1}{\alpha^r} (\alpha_0 - \sum_{i \neq 1} \bar{d}_i \alpha_i^T) \bar{c}(e_k, T).$$

Indeed, the particular solution \bar{x}^1 also meets the demand requirements for, since $\bar{c}(e_k, T)$ is a cycle, $\bar{A}\bar{c}(e_k, T) = 0$, and $\bar{A}\bar{x}^1 = \bar{A}\bar{x}^0 + 1/\alpha^r (\alpha_0 - \sum_{i \neq 1} \bar{d}_i \alpha_i^T) \bar{A}\bar{c}(e_k, T) = \bar{A}\bar{x}^0 = \bar{d}$, as well as the additional constraint, for

$$\begin{aligned} \bar{\alpha} \bar{x}^1 &= \bar{\alpha} \sum_{i \neq 1} \bar{d}_i \bar{p}(i) + \frac{1}{\alpha^r} (\alpha_0 - \sum_{i \neq 1} \bar{d}_i \alpha_i^T) \bar{\alpha} \bar{c}(e_k, T) \\ &= \sum_{i \neq 1} \bar{d}_i \alpha_i^T + (\alpha_0 - \sum_{i \neq 1} \bar{d}_i \alpha_i^T) = \alpha_0 \end{aligned}$$

(using the fact that, from the choice of $\bar{\alpha}$, $\bar{\alpha}\bar{c}(e_k, T) = \alpha c(e_k, T) = \alpha^r$).

Now, any constrained flow \bar{x} can be expressed as the sum of a particular feasible constrained flow, say \bar{x}^1 , and of some constrained circulation, say, \bar{x}^2 . As in the proof of Lemma 15, \bar{x}^2 can be written $\bar{x}^2 = \sum_{e_j \in S \cup R} y_j \bar{c}(e_j, T)$ where $\alpha \bar{x}^2 = 0$ implies that $\sum_{e_j \in R} y_j = 0$, and thus the general expression of any constrained flow on \bar{G} is given by

$$\bar{x} = \bar{x}^1 + \bar{x}^2 = \sum_{i \neq 1} \bar{d}_i \bar{p}(i) + \frac{1}{\alpha^r} (\alpha_0 - \sum_{i \neq 1} \bar{d}_i \alpha_i^T) \bar{c}(e_k, T) + \sum_{e_j \in S \cup R} y_j \bar{c}(e_j, T). \quad (3.5)$$

Further, notice that the entry of $\bar{p}(i)$ on some arc \bar{e}_q is equal to 1 if $\bar{p}(i)$ uses the arc \bar{e}_q , that is, if i is a descendant of i_q'' in \bar{G} , and to 0 otherwise, whereas the entry of any R-cycle (resp., S-cycle) on that same arc is equal to 1 (resp., 0). Therefore $\bar{x}_{\bar{e}_q} = \bar{x}_{\bar{e}_q}^1 + \bar{x}_{\bar{e}_q}^2$ $= \sum_{i \in D(i_q'')} \bar{d}_i + 1/\alpha^r (\alpha_0 - \sum_{i \neq 1} \bar{d}_i \alpha_i^T)$ for all $q = 1, \dots, Q$ and is independent of the constrained flow \bar{x} . Noticing that, from the choice of $\bar{d}_{i_q'}$ and $\bar{d}_{i_q''}$, the first term $\sum_{i \in D(i_q'')} \bar{d}_i$ is equal to $\sum_{i \in D(i_q')} d_i$, the way i_q' and i_q'' share d_{i_q} can then be adjusted in order to restrict the flow on the arcs \bar{e}_q , $q = 1, \dots, Q$ to zero by adding $\bar{x}_{\bar{e}_q}$ to $\bar{d}_{i_q'}$ and subtracting it from $\bar{d}_{i_q''}$, resulting in the **adjusted version** of \bar{G} where

$$\begin{aligned} \bar{d}_{i_q'} &= \sum_{i \in D(i_q')} d_i + \frac{1}{\alpha^r} (\alpha_0 - \sum_{i \neq 1} d_i \alpha_i^T) \\ &\quad \text{for } q = 1, \dots, Q, \\ \bar{d}_{i_q''} &= d_{i_q} - \bar{d}_{i_q'}. \end{aligned}$$

Indeed, since only the demand at the endpoints of \bar{e}_q , $q = 1, \dots, Q$ are changed and these changes add up to zero for each one of those arcs, the new flows on the adjusted version of \bar{G} are obtained from the old ones by sending along $\bar{e}_q = (i_q', i_q'')$ a flow equal to the change of $\bar{d}_{i_q''}$, resulting in $\bar{x}_{\bar{e}_q} = 0$ for all $q = 1, \dots, Q$. Now, deleting the arcs \bar{e}_q , $q = 1, \dots, Q$ that carry no flow will disconnect the graph into connected components that are the dependent components of the original graph (see the **disconnected version** of \bar{G} in figure 3.6-c) and from the way \bar{x} was constructed in (3.5) the constraint need not be expressed explicitly. The

set of flows on the unconstrained disconnected version of \bar{G} is thus equal to the set of flows on the original version of \bar{G} , and thus to that of G , and the proof follows. \square

Notice that the modified demand vector given by (3.4) can easily be computed as part of Algorithm I, and thus in linear time. Indeed, both $\sum_{i \in D(j)} d_i$ and $\sum_{i \in D(j)} d_i \alpha_i^T$ can be obtained recursively for all nodes j as the depth-first search backtracks to j , the term $\sum_{i \neq 1} d_i \alpha_i^T$ being equal to $\sum_{i \in D(1)} d_i \alpha_i^T - d_1 \alpha_1^T = \sum_{i \in D(1)} d_i \alpha_i^T$ (recall that, by definition, $\alpha_1^T = 0$), and α^r is also obtained in Algorithm I.

The above decomposition is demonstrated on a rim G with $Q = 3$.

EXAMPLE 3 Consider the palm graph G with 7 nodes and 12 arcs given in Figure 3.7-a with the additional constraint $x_{12} - x_{71} + 2x_{23} - 2x_{37} + 3x_{41} + 3x_{51} = 28$ and the demands $d_1 = d_4 = d_5 = d_7 = 0$, and $d_2 = 4$, $d_3 = 1$, $d_6 = -5$

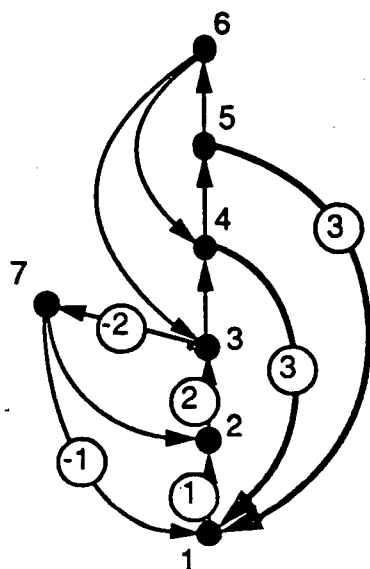
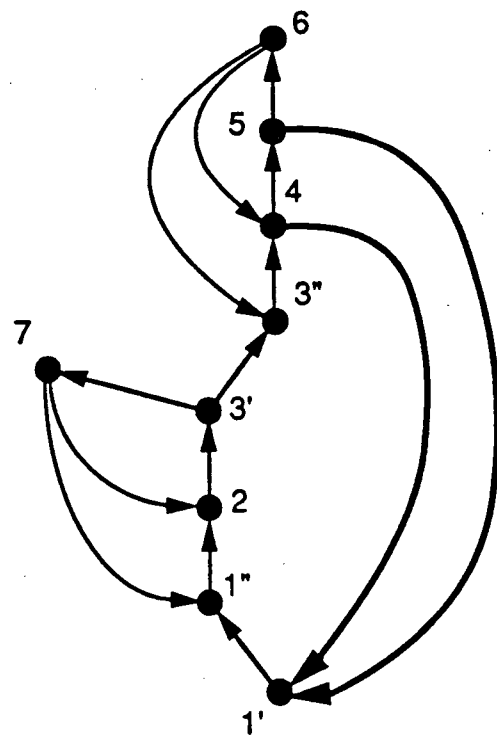
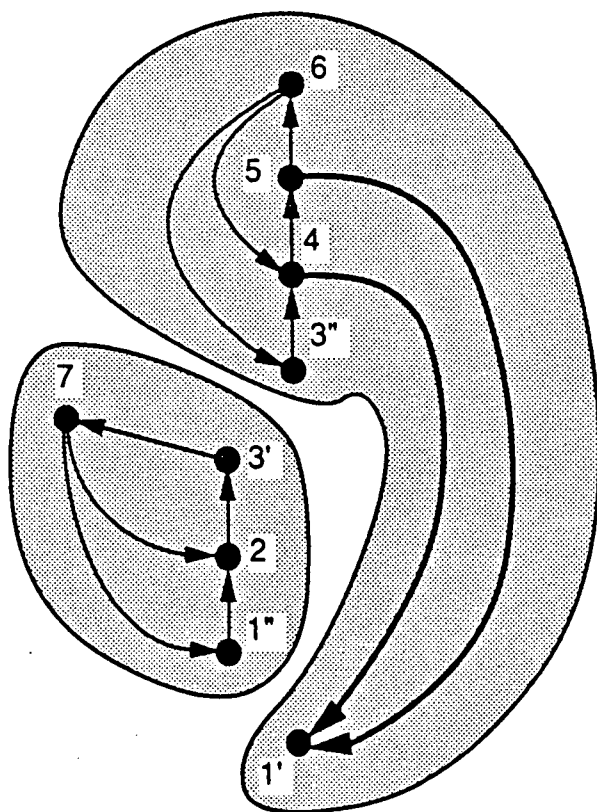
For this palm graph, $T = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (3, 7)\}$, $R = \{(4, 1), (5, 1)\}$, $S = \{(7, 1), (7, 2), (6, 3), (6, 4)\}$, $1 = w_0 < v_0 = 4$. The path $p(w_0, v_0)$ contains node 1 which is the head of all R -arcs and node 3 which is a cutnode of H . $\alpha(e_j, T)$, $e_j \in R$ takes on the unique value $\alpha^r = \alpha((4, 1), T) = \alpha((5, 1), T) = 6$. Thus, from Theorem 16, G is a rim whose two components G_1 and G_2 have for respective node-sets $N_1 = \{3, 4, 5, 6\}$ and $N_2 = \{1, 2, 3, 7\}$, its bolts being the nodes $i_1 = 3$ and $i_2 = 1$. Figure 3.7-b illustrates the expanded rim, \bar{G} in which the bolt i_1 (resp., i_2) is split into nodes $i'_1 = 3'$ and $i''_1 = 3''$ (resp., $i'_2 = 1'$ and $i''_2 = 1''$). Given that $\alpha_1^T = 0$, $\alpha_2^T = 1$, $\alpha_3^T = \alpha_4^T = \alpha_5^T = \alpha_6^T = 3$, $\alpha_7^T = 1$, then from (3.4) we obtain

$$\begin{aligned}\bar{d}_{i'_1} &= \bar{d}_{3'} = \sum_{i \in D(3)} d_i + \frac{1}{\alpha^r} (\alpha_0 - \sum_{i \neq 1} d_i \alpha_i^T) = 2 \\ \bar{d}_{i''_1} &= \bar{d}_{3''} = d_3 - \bar{d}_{3'} = -1\end{aligned}$$

and

$$\begin{aligned}\bar{d}_{i'_2} &= \bar{d}_{1'} = \sum_{i \in D(1)} d_i + \frac{1}{\alpha^r} (\alpha_0 - \sum_{i \neq 1} d_i \alpha_i^T) = 6 \\ \bar{d}_{i''_2} &= \bar{d}_{1''} = d_1 - \bar{d}_{1'} = -6.\end{aligned}$$

The disconnected unconstrained subgraphs are shown in Figure 3.7-c. \square

a: The original graph G b: the expanded rim \bar{G} 

c: The two equivalent subgraphs

Figure 3.7: Rim Expansion

We now return to the original problem $P(\alpha, \alpha_0)$ and use the results obtained above to present Algorithm II which decomposes a SCMF defined on the constrained graph $G = (N, E, \alpha)$ into independent subproblems on the dependent components of G . After a feasibility check (Step 1) Algorithm II isolates the biconnected components of the underlying graph (Step 2) and disconnects them in the manner described by Granot and Veinott [GV85]. Each of the components is then tested for the presence of active cycles and for the rim and one-rim structures (Step 3). Independent unconstrained subproblems are defined on those components that do not contain any active cycle. If all the biconnected components contain no active cycles Algorithm II terminates at Step 4 with an equivalent unconstrained monotropic network flow problem. If several biconnected components contain active cycles or if the unique biconnected component that contains active cycles is not a rim, the algorithm gives the decomposition of the original problem into subproblems, one of which remains a SCMF (Step 5). If, however, a single biconnected component contains active cycles and is either a rim or a one-rim, Algorithm II provides a decomposition of the original problem $P(\alpha, \alpha_0)$ into unconstrained subproblems (step 6).

ALGORITHM II

- Step 1.** Check if the unconstrained network flow problem obtained from $P(\alpha, \alpha_0)$ by removing the constraint $\alpha x = \alpha_0$ is feasible, that is, if the linear system $Ax = d$ is feasible (the total demand on each connected component of G should be zero). If not, $P(\alpha, \alpha_0)$ is infeasible, and terminate.
- Step 2.** Decompose G into say ℓ biconnected components $G_r = (N_r, E_r, \alpha_r)$ $r = 1, \dots, \ell$, and disconnect those that have nodes in common by apportioning the demand at the common nodes between the copies thereof in the fashion described in [GV85].
- Step 3.** Apply Algorithm I to each biconnected component G_r in order to determine if they contain active cycles, and assume without loss of generality that G_r , $1 \leq r \leq p_0$, contain(s) at least one active cycle. If only one of them does, say G_1 , (that is, if $p_0 = 1$), use Algorithm I to determine if it is a rim or a one-rim (in the latter case, replace it

by its expanded version, a rim, and set the cost associated with the expansion arc to zero). Now, if G_1 is a rim expand and disconnect it as described in Theorem 18 and compute the modified demand according to (3.4).

Step 4. For each one of the biconnected components of G that do not contain active cycles create the unconstrained subproblem

$$\begin{aligned} P_k : \quad & \text{Min} \quad \sum_{e_j \in E_k} b_j(x_j) \\ & \text{s.t.} \quad A^k x^k = d^k \quad k = p_0 + 1, \dots, \ell \end{aligned}$$

where $x^k = (x_i)_{i \in E_k}$ and $d^k = (d_i)_{i \in E_k}$, for $k = p_0 + 1, \dots, \ell$. If, further, none of the biconnected components of G contain active cycles that is, if $p_0 = 0$, and if $\alpha \sum_{r=1, \dots, \ell} x(r) = \alpha_0$ (where $x(r)$ is the feasible unconstrained flow for the subgraph G_r , computed as " x " in Algorithm I) then $P(\alpha, \alpha_0)$ is feasible and equivalent to P_k , $k = 1, \dots, \ell$ and thus is transformable to an unconstrained network flow problem. If, However, $p_0 = 0$ and $\alpha \sum_{r=1, \dots, \ell} x(r) \neq \alpha_0$ then $P(\alpha, \alpha_0)$ is infeasible.

Step 5. If G_1 is the unique biconnected component of G that contains active cycles ($p_0 = 1$) but is neither a rim nor a one-rim, or if several biconnected components of G contain active cycles ($p_0 \geq 2$) (recall from Theorem 6 that in those two cases G^* is a dependent component of G) create the subproblem

$$\begin{aligned} P^*(\alpha, \alpha_0) : \quad & \text{Min} \quad \sum_{e_j \in E^*} b_j(x_j) \\ & \text{s.t.} \quad A^* x^* = d^* \\ & \quad \alpha x^* = \alpha_0 \end{aligned}$$

where $x^* = (x_i)_{i \in E^*}$ and $d^* = (d_i)_{i \in E^*}$. $P(\alpha, \alpha_0)$ is equivalent to $P^*(\alpha, \alpha_0)$, together with P_k , $k = p_0 + 1, \dots, \ell$, and thus not transformable to an unconstrained problem.

Step 6. If a single biconnected component of G contains active cycles, that is, $p_0 = 1$ and $G^* = G_1$, and if this biconnected component is a rim (recall that if it was originally a

one-rim, it has been replaced by a rim in Step 3), then for each one of the disconnected rim components of G_1 build the unconstrained subproblem

$$P^q : \text{Min} \sum_{e_j \in E^q} b_j(x_j)$$

$$\text{s.t. } \mathcal{A}^q x^{(q)} = \bar{d}^{(q)} \quad q = 1, \dots, Q$$

where $x^{(q)} = (x_i)_{e_i \in E^q}$, and $\bar{d}^{(q)}, q = 1, \dots, Q$, is the part of the modified demand vector \bar{d} obtained in (3.4) corresponding to N^q . The original problem $P(\alpha, \alpha_0)$ is then equivalent to the $Q + \ell - 1$ independent unconstrained problems $P_k, k = 2, \dots, \ell$, and $P^q, q = 1, \dots, Q$, and thus transformable to an unconstrained problem.

Note that the problems P^q corresponding to the components of the rim G^* that are singletons (i.e., contain only one arc), if any such rim components exist, are trivial one-variable problems of the form $\text{Min}\{b_j(x_j^{(q)}) : x_j^{(q)} = \bar{d}_j^{(q)}\}$ where e_j is the unique arc of G^q (in particular, this is the case of the component $\{\bar{e}_1\}$ of the expanded version of a one-rim). As a result $P(\alpha, \alpha_0)$ really decomposes into only $Q + \ell - 1 - Z$ nontrivial independent unconstrained network flow problems, where Z is the number of rim components of G^* that are singletons.

Proof of the Validity of Algorithm II

The preliminary decomposition into subproblems corresponding to $G^*, G_{p_0+1}, \dots, G_\ell$ is a direct result of the block-diagonal nature of A_α described at the beginning of this section. For the case $p_0 = 0$, it is further necessary to check that the problem $P(\alpha, \alpha_0)$ is feasible (step 4). Notice that this is true if and only if any given feasible unconstrained flow, say x^0 , meets the constraint. Indeed, recall that for any feasible unconstrained flow $x, \alpha x - \alpha x^0 = \alpha(x - x^0) = 0$ for $x - x^0$ is a circulation, and thus a linear combination of simple cycles which are all neutral ($p_0 = 0$). We choose for x^0 the flow $\sum_{r=1, \dots, \ell} x(r)$, where $x(r), r = 1, \dots, \ell$, is a feasible unconstrained flow on the biconnected component G_r of G (obtained in Algorithm I), justifying Step 4. The remainder of the proof lies in Theorems 6 and 18. \square

Complexity of Algorithm II

Step 1 requires the decomposition into connected components, which takes $O(m)$, Step 2 takes $O(m+n)$ [Pa71,Ta72,HT73], Step 3 takes $O(|E_r| + |N_r|)$ on each biconnected component G_r for a total of $O(m+n)$ and Steps 4, 5 and 6 also take linear time, thus Algorithm II runs in $O(m+n)$.

It is interesting to note that in the event where $P(\alpha, \alpha_0)$ is transformable to an unconstrained network flow problem its constraint matrix A_α is not necessarily totally unimodular. The following example shows a constrained network flow problem that is transformable to an unconstrained one but whose matrix is not totally unimodular.

EXAMPLE 4 Consider the graph G with two nodes and two arcs given in Figure 3.8 together with the additional constraint $x_1 - x_2 = 1$. G is clearly a rim whose two components have respectively $\{e_1\}$ and $\{e_2\}$ as their arc sets, and the bolts of G are i_1 and i_2 . Thus any problem $P(\alpha, \alpha_0)$ on G is transformable to an unconstrained network flow problem. However, the constraint matrix of such a problem has $\begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$ as a submatrix, whose determinant is equal to 2, and thus the constraint matrix is not totally unimodular. \square

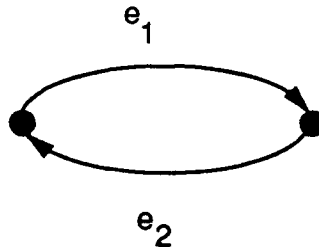


Figure 3.8: The rim G

We now argue that the decomposition presented above is exhaustive in the sense that none of the subproblems obtained can be further decomposed. By “decomposing a problem”, we mean expressing, independently of the cost function, its set of optimal solutions as a cartesian product of optimal solution sets of subproblems obtained by partitioning the underlying graph. First, the subproblems on the biconnected components that do not contain any active cycles, as well as those on the components of a rim, if such a situation occurs,

are unconstrained network flow problems on biconnected graphs and therefore cannot be decomposed (see, for example, [GV85]). We now claim that the constrained subproblem on a dependent subgraph, if any, cannot be decomposed either. In order to see this, suppose on the contrary that, regardless of the cost-function vector $(b_j(x_j))_{e_j \in E^*}$, the subproblem $P^*(\alpha, \alpha_0)$ on the constrained dependent subgraph G^* can be decomposed into two independent subproblems, say, P_1^* and P_2^* , over two subsets E_1^* and E_2^* of the set of arcs E^* . Since G^* is dependent there exists an elementary vector, say f , of the circulation space over G^* that has nonzero components on at least one arc of each of the sets E_1^* and E_2^* . Consider now the problem $P^*(\alpha, \alpha_0)$ obtained by setting $b_j(x_j)$ equal to zero if $x_j = f_j$ or if $e_j \in E_2^*$, and to $+\infty$ otherwise. One of its optimal solutions is $x = f$, and if the subproblem is indeed decomposable then the projection $x^1 = f^1$ obtained by setting to zero all the components of f that correspond to arcs in E_2^* should form a feasible solution for P_1^* . Moreover, the zero vector on E_2^* is clearly optimal for P_2^* , and since the solution set of $P^*(\alpha, \alpha_0)$ is the Cartesian product of those of P_1^* and P_2^* it contains f^1 which is therefore feasible for $P^*(\alpha, \alpha_0)$. Thus f^1 can be written as a sum of conformal elementary vectors of the circulation subspace of G^* whose supports are all included in that of f^1 , which is strictly contained in that of f (for f has nonzero components on at least one arc of E_2^*), contradicting the fact that f is an elementary vector, and proving our claim. A similar proof holds if $P^*(\alpha, \alpha_0)$ decomposes into more than two subproblems, and this trivial extension is omitted. The foremost consequence of the above is that if the original SCMF is transformable to an unconstrained problem, in the sense that its solution set is the cartesian product of the solution sets of unconstrained network flow subproblems on subgraphs of its underlying graph, then Algorithm II will detect it and provide the transformed problem. Conversely, if Algorithm II does not yield such a decomposition, that is, one of the subproblems it yields is a constrained problem, then we may conclude that the original problem does not admit such a transformation.

3.5 Extensions and Remarks

Cost Function We start this section with a remark concerning the cost-function of the problem $P(\alpha, \alpha_0)$. Indeed, the hypothesis that the objective-function be arc-additive can be relaxed in that, given the block-diagonal structure of A_α corresponding to the dependent components of G , it suffices that it be “**dependent component-additive**”. That is, it is sufficient that the objective function can be expressed as a sum of functions whose respective arguments are the sets of variables corresponding to each dependent components of G (the proof is trivial and therefore omitted).

Nonlinear and Inequality Constraints In order to see how our procedure can handle nonlinear and inequality constraints we require the following theorem.

THEOREM 19 *For a biconnected constrained graph $G = (N, E, \alpha)$ the following are equivalent.*

- (a) *The additional constraint is equivalent by a linear transformation to the difference of sum of variables associated with arcs directed into a single node i_1 and a sum of variables associated with arcs directed away from that same node (that is, a sub-sum of the node balance equation for that node)*
- (b) *There exists a way of splitting one node $i_1 \in N$ and introducing an arc, say \bar{e}_1 , between its two copies, such that the additional constraint is equivalent by a linear transformation to $x_{\bar{e}_1} = \delta$, where δ depends on α and the demand vector d .*
- (c) *G is either a one-rim or a rim and i_1 is a bolt.*

PROOF. The fact that (a) and (b) are equivalent is straightforward and only involves combining the node-balance equations for i_1 and its two copies with the constraint. Moreover, it is a direct consequence of theorem 18 that (c) implies (b). In order to show that (b) implies (c), consider the expanded graph \bar{G} obtained in (b) by splitting i_1 into, say, i'_1 and i''_1 and let $\bar{e}_1 = (i'_1, i''_1)$. For the constraint $x_{\bar{e}_1} = \delta$ on \bar{G} , G considered as a subgraph of \bar{G} that

contains no arcs with $\alpha_j \neq 0$, and thus no active cycles. Therefore, G is either a one-rim (if the expanded graph without the expansion arc, $\tilde{G} \setminus \{\bar{e}_1\}$ is biconnected) or a rim (if $\tilde{G} \setminus \{\bar{e}_1\}$ has several biconnected components. Since G is biconnected, these form a "chain" between i_1' and i_1''), i_1 is a bolt, and the proof is complete. \square

A direct consequence of the above is that, for a rim or a one-rim G , the flow on the expansion arc \bar{e}_1 in the expanded graph \tilde{G} defined in Theorem 19 is a linear function of the quantity αx , say, $x_{\bar{e}_1} = \beta \alpha x + \gamma$. Therefore any additional constraint of the form $\Psi(\alpha x) = 0$, where Ψ is any real-valued function, can be incorporated into the cost-function by expanding one of the bolts and setting the cost of the flow on the arc \bar{e}_1 to zero whenever $\Psi((x_{\bar{e}_1} - \gamma)/\beta) = 0$, and to $+\infty$ otherwise, resulting in an unconstrained network flow problem on \tilde{G} . Notice that the above family of constraints includes inequality constraints, for $\alpha x \leq 0$ (resp., ≥ 0) is equivalent to $\Psi(\alpha x) = 0$, where $\Psi(z) = 0$ if $z \leq 0$ (resp., ≥ 0) and $\Psi(z) = +\infty$ otherwise.

Special Types of Constraints In [Kl77], it is shown that the singly-constrained linear network flow problem is equivalent to an unconstrained problem on an expanded graph if the constraint is a sum of flows associated with arcs all directed into or all directed away from a single node. As a consequence of Theorem 19 and the above remark on inequality constraints, the procedure presented here covers all such cases, and carries the transformation a step further by disconnecting the rim components when the additional constraint is of the equality type.

Multiple Constraints If the original problem includes several additional constraints, our procedure can be applied successively to each of them. It is then necessary that the SCMF obtained by keeping only the first constraint be equivalent to an unconstrained problem, which in turn, together with the second constraint, is also equivalent to an unconstrained problem and so on. It is important to remark that the order in which the constraints are

chosen may influence the outcome of the decomposition. We are now studying the multiple-constraint case.

□ □ □

Chapter 4

Ripples, Complements and Substitutes

Chapter Abstract

We extend the qualitative theory of sensitivity analysis for Minimum-Cost Flow Problems presented by Granot and Veinott [GV85] to Minimum-Cost Flow Problems with one additional linear constraint. The departure from the unconstrained network structure is shown to have a profound effect on computational issues. Two natural extensions of the “less-dependent-on” partial ordering of the arcs are presented. One is decidable in linear time while the other yields more information but is NP-complete in general. The Ripple Theorem gives upper bounds on the absolute value of optimal-flow variations as a function of variations in the problem parameter. The Theory of Substitutes and Complements presents necessary and sufficient conditions for optimal-flow changes to consistently have the same (or the opposite) sign in two given arcs. The complexity of determining Substitutes and Complements is shown to be NP-complete in general. However, for all intractable problems, families of cases arise from easily recognizable graph structures and can be computed in linear time. The Monotonicity Theory links the changes in the value of the parameters to the change in the optimal arc-flows, and bounds on the rates of changes are discussed.

□ □ □ •

4.1 Introduction

In this chapter, we turn to the parametric version of the Singly-Constrained Monotropic Network Flow Problem (SCMF) discussed in chapter 3 in order to study sensitivity analysis for such problems. We are interested in the response of optimal solutions and optimal values

to modifications in costs (for instance alternative pricing, varying taxes, profit margin) or in upper/lower bounds (recall that these are incorporated in the objective function). These variations will, in our context, be viewed as changes in cost-function-parameters. To that end, we associate with each arc $e_j \in E$ a **parameter** t_j in a lattice T_j (for background on lattices and lattice programming, see e.g. [CD73,Vefo.]), and a **cost** $b_j(x_j, t_j)$ which incurs when the flow and parameter of arc e_j are respectively x_j and t_j , and $b_j(\cdot, \cdot)$ is a $+\infty$ or real-valued function on \mathfrak{R} . The m -vector $t = (t_j)_{e_j \in E}$ is called the **parameter vector** and lies in the lattice product $T = \times_{e_j \in E} T_j$. The cost of a flow $x = (x_j) \in \mathfrak{R}^m$ is the sum of the cost functions on all the arcs of the graph. The problem to be discussed in this chapter is that of finding a flow with minimum total cost that meets the constraint, for a given parameter vector. Explicitly, the **Parametric Singly-Constrained Monotropic Network Flow Problem (PSCMF)**, also referred to as the **Constrained Problem**, can be written as

$$\begin{aligned} P(\alpha, \alpha_0, t): \quad & \text{Min} \quad \sum_{j=1}^m b_j(x_j, t_j) \\ & \text{s.t.} \quad Ax = d \\ & \quad \quad \alpha x = \alpha_0. \end{aligned}$$

We will denote by $X(t)$ the set of optimal flows of $P(\alpha, \alpha_0, t)$. A problem similar to the above, but without the additional constraint, will be referred to as the **Parametric Unconstrained Monotropic Network Flow Problem**, or simply as the **Unconstrained Problem**. The usual non-negativity and upper/lower bound constraint(s), if any, are naturally incorporated into the cost-function by setting $b_j(x_j, t_j)$ to $+\infty$ for all forbidden values of x_j .

Qualitative sensitivity analysis and the study of substitutes and complements (terms defined in section 4.3) for the unconstrained case was the objective of [GV85]. Here, however, the additional linear constraint partially destroys the attractive features and some of the results obtained for pure network flow problems. The cost function is, for most results, required to be convex in the arc-flow, and may reflect real costs, upper and/or lower bounds. The sensitivity analysis is performed by modifying the parameter associated with one or

several given arcs. Any variation of the cost structure can be represented in this fashion. Our results are concerned with the “reactions” of the optimal flows to such changes. For applications, such parameter variations may for example reflect changes in upper and/or lower bounds, additive or multiplicative increases in cost, pricing alternatives, technological advances, discounts, or even simulate the deletion/addition of an arc. In the case of multiple optima, our results are valid for at least one of the possible optimal flows.

The chapter is organized as follows. Section 4.2 focuses on bounds on the magnitude of change of the optimal flow in two given arcs, the main results being given in the Ripple Theorem (Theorem 21). To that end, we define two partial orders on the set of arcs, the *less-dependent-on* relation and the *quantitatively-less-dependent-on* relation, as well as a coefficient $M_{j,k}$ which is associated with each pair of arcs (e_j, e_k) . Computational issues associated with the partial orders and the coefficients $M_{j,k}$ are discussed. In section 4.3 we introduce the notions of complements and substitutes pairs of arcs and present a result on the direction of changes in the optimal arc-flow for such pairs of arcs in Lemma 25. Computational complexity is again addressed. The Monotonicity Theory presented in section 4.4 relates and compares the rate of change in the optimal flow of a given arc to that of another arc’s parameter when the two arcs are either complements or substitutes, a result stated in the Monotone Optimal-Flow Selection Theorem (Theorem 31). The Smoothing Theorem (Theorem 33) provides an upper bound on the maximum arc-flow change over the network in terms of \mathcal{L}_1 -norm of the corresponding parameter variation. Finally, Theorem 34 shows the subadditivity property of the minimum cost.

4.2 The Ripple Theorem For Constrained Graphs.

We study in this section the effect of a change in the parameter of a given arc, in a constrained network, on the flow throughout the network. We show that the magnitude of the change in any arc depends on how “dependent” (a term to be defined later) the arc is with the arc whose parameter is being changed. First observe that, as a direct result of the problem decomposition presented in Chapter 3, changes in a parameter associated with an

arc in a given dependent component will not affect the optimal flows in arcs belonging to other dependent components. More precisely, if x is an optimal flow for $P(\alpha, \alpha_0, t)$, and if t and t' differ only in their j^{th} entry, then there exists an optimal solution x' to $P(\alpha, \alpha_0, t')$ such that $x_i = x'_i$ for any arc e_i that belongs to a dependent component that does not contain e_j (equivalently, whenever e_i and e_j are independent). In the following, the effects of changes within a given dependent component will be studied.

Recall from Section 3.3 that the dependent components of a constrained graph as obtained from its biconnected components can be classified as follows.

Type (I) Each neutral biconnected component is a dependent component.

Type (II) If the number of active biconnected components is at least 2, then the subgraph of G composed of all of the active biconnected components forms one dependent component.

Type (III) If a unique biconnected component is active and is neither a rim nor a one-rim, then it is also a dependent component in itself.

Type (IV) If a unique biconnected component is active and is a rim (resp., a one-rim) then each of its (resp., its unique) rim component(s) is a dependent component.

Moreover, the components of G thus obtained have no nodes in common.

Now, if the dependent component in question is of type (I) or type (IV), then the associated subproblem is unconstrained, and therefore one can apply the qualitative theory developed by Granot and Veinott [GV85] for Unconstrained Monotropic Network Flow Problems. We will thus assume in the following that G has a single dependent component which is either of type (II) or type (III), that is, where G is composed of either several or of a unique active biconnected component(s). In order to study and compare effects of changes in one arc's parameter over other arcs, we now introduce two partial-orders on the set of arcs E .

4.2.1 Less-Dependent-On Relation

Let $G = (N, E, \alpha)$ be a constrained dependent graph of type (II) or (III) and let e_i, e_j, e_k be arcs in E . We say that e_i is **less dependent on e_j than e_k is** and write $e_i \leq_j^d e_k$ if and only if

$$\text{for all } f \in H(\alpha), f_i f_j \neq 0 \implies f_k \neq 0 \quad (\text{for all } f \in H_j(\alpha), f_i \neq 0 \implies f_k \neq 0).$$

where $H(\alpha)$ is the set of elementary vectors of the circulation space defined in Theorem 5 and $H_j(\alpha)$ is the set of elementary vectors f which contain a given arc e_j . If $e_i \leq_j^d e_k$ and $e_k \leq_j^d e_i$ then e_i and e_k are said to be **d-equally dependent on e_j** and we write $e_i =_j^d e_k$. Clearly, $e_i =_j^d e_k$ if and only if for all elementary vectors $f \in H_j(\alpha)$, either $f_i = f_k = 0$ or $f_i f_k \neq 0$. It is easy to see that the relation \leq_j^d is a partial-order and hence $=_j^d$ is an equivalence relation. The sets \mathcal{E}_j^d of **d-equivalence classes** (equivalence classes induced by the relation $=_j^d$) are partially ordered by \leq_j^d . We will denote by $\mathcal{E}_j^d(k)$ the unique element of \mathcal{E}_j^d containing arc e_k . Note that if two arcs e_k and e_l are in the same d-equivalence class then $\mathcal{E}_j^d(k) = \mathcal{E}_j^d(l)$.

One should observe here that the characterization of the *equally-biconnected-to* equivalence relation $=_j$ given in Granot and Veinott [GV85, Theorem 4] for unconstrained problems is no longer valid for constrained networks. There, $e_i \leq_j e_k$ — e_i is *less-biconnected-to e_j than e_k is* — if and only if every simple cycle which contains e_i and e_j also contains e_k . They state that in a graph G with no additional constraints two arcs e_i and e_k are equally biconnected to a third arc e_j if and only if $e_i = e_k$ or deleting e_i and e_k leaves G disconnected. Example 5 below shows that this characterization is not valid in the constrained case. Theorem 20 to follow provides an operational characterization of the less-dependent-on relation.

EXAMPLE 5 Consider the graph G with 8 arcs and 6 nodes given in Figure 4.9, with $\alpha_1 = 1/2$, $\alpha_6 = 1$, $\alpha_i = 0$ for all $i \neq 1, 6$. Clearly $e_3 =_2^d e_6$, for the only elementary vectors that contain both e_3 and e_2 (resp., e_6 and e_2) are, up to a multiplicative factor, $f = 2(e_1 + e_2 + e_3) - (e_4 + e_5 + e_6) = (2, 2, 2, -1, -1, -1, 0, 0)$ and $g = 2(e_1 + e_2 + e_3) -$

$(e_6 + e_7 + e_8) = (2, 2, 2, 0, 0, -1, -1, -1)$, which also contain e_6 (resp., e_3) and thus $e_3 \leq_2^d e_6$ (resp., $e_3 \geq_2^d e_6$). However, deleting e_3 and e_6 leaves a connected graph. \square

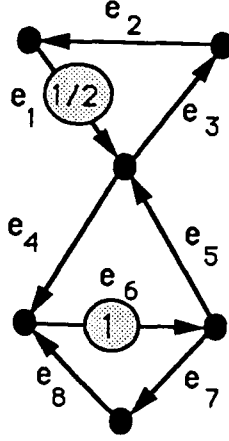


Figure 4.9: Equally dependent pairs of arcs need not disconnect the graph

THEOREM 20 *Let $G = (N, E, \alpha)$ be a constrained dependent graph and let e_i , e_j and e_k be arcs in E . Then $e_i \leq_j^d e_k$ if and only if deleting e_k leaves e_i and e_j independent.*

PROOF. The proof is a straightforward application of the definition of the Less-dependent-on relation, and is added for completeness. Define G' as the constrained graph obtained by deleting e_k and $H'(\alpha)$ to be the set of elementary vectors of the constrained circulation space of G' . It follows trivially from the definition of elementary vectors that $H'(\alpha) = \{f \in H(\alpha) : f_k = 0\}$. Now, $e_i \leq_j^d e_k$ if and only if

$$\forall f \in H(\alpha), \quad f_i f_j \neq 0 \implies f_k \neq 0$$

or equivalently

$$\forall f \in H(\alpha), \quad f_k = 0 \implies f_i f_j = 0$$

which is the same as

$$\forall f \in H'(\alpha), f_i f_j = 0$$

completing the proof. \square

Note that Theorem 20 provides a linear time algorithm to verify whether three arcs e_i , e_j and e_k satisfy $e_i \leq_j^d e_k$, for it suffices to decompose $G \setminus \{e_k\}$ into dependent components in $O(m+n)$ as described in Chapter 3, and check whether or not e_i and e_j are in the same dependent component.

We now apply Theorem 20 to obtain a classification of the cases where $e_i \leq_j^d e_k$ holds. In order to do so, we will start with all the possible configurations of a non-dependent graph $G \setminus \{e_k\}$ and investigate the possible ways of adding an arc $e_k = (n_1, n_2)$ that will result in a dependent graph. For $e_i \leq_j^d e_k$ to hold, it then suffices that the arcs e_i and e_j belong to two different dependent components of $G \setminus \{e_k\}$. The classes of non-dependent graphs that will become dependent when an extra arc e_k is added in an appropriate fashion are described below. We start with the case where $G \setminus \{e_k\}$ is biconnected, and therefore is a rim (e_i and e_j belong to two different rim components).

- If one (resp., two) of the rim components of $G \setminus \{e_k\}$, say G' (resp., G', G''), contains (resp., contain) both n_1 and n_2 , then G is dependent if and only if e_k does not close a neutral cycle in G' (resp., in G' or in G''). For otherwise G would be a rim and $G' \cup \{e_k\}$ (resp., $G' \cup \{e_k\}$ or $G'' \cup \{e_k\}$), one of its rim components, contradicting the fact that G is dependent (notice that, since G' is neutral, one of the cycles that e_k closes in G' is neutral (resp., active) if and only if so are all the cycles that e_k closes in G') (see Figure 4.10-(i)).
- If none of the rim component of $G \setminus \{e_k\}$ contains both n_1 and n_2 , then let c be any active cycle of $G \setminus \{e_k\}$ that contains n_1 and n_2 , and write $c = p_1 - p_2$, where p_1 and p_2 are two arc-disjoint oriented paths from n_2 to n_1 . Notice that $\alpha c = \alpha p_1 - \alpha p_2 \neq 0$, and since $G \setminus \{e_k\}$ is a rim, the set $\{\alpha p_1, \alpha p_2\}$ does not depend on the choice of the active cycle c . Now, since $G \setminus \{e_k\}$ contains active cycles, so does G , and thus G is dependent

if and only if it is not a rim, that is, if both cycles $e_k + p_1$ and $e_k + p_2$ are active or, equivalently, if $\alpha_k \notin \{\alpha_{p_1}, \alpha_{p_2}\}$ (see Figure 4.10-(ii)).

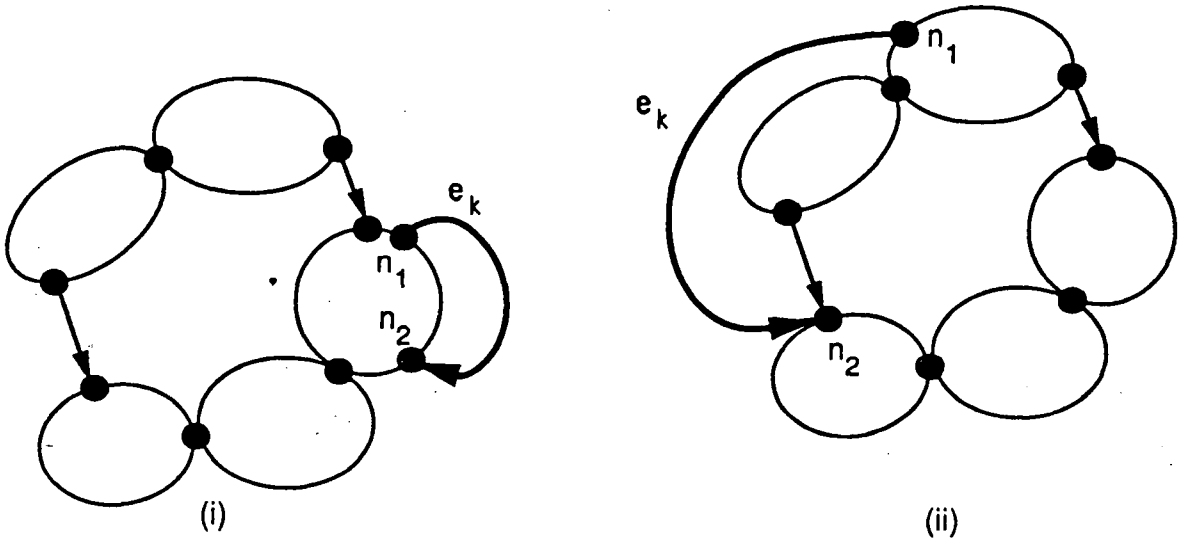


Figure 4.10: Construction of the dependent graph G when $G \setminus \{e_k\}$ is a rim

We now turn to the case where $G \setminus \{e_k\}$ has several biconnected components, say G_1, \dots, G_l , and since $G \setminus \{e_k\}$ is not dependent, then at least one of these, say G_1 , is neutral (e_i and e_j belong to different biconnected components, and at most one of them to an active one). In the following we will use the fact that if an arc e_k is added to a neutral (sub-)graph G' , then $G' \cup \{e_k\}$ is either neutral or one of its biconnected components, which contains e_k , is a rim for all the active cycles go through e_k . Therefore, if G' is neutral, $G' \cup \{e_k\}$ is active if and only if all the cycles that contain e_k are active.

- Let us first assume that one of the biconnected components of $G \setminus \{e_k\}$ contains both n_1 and n_2 . Clearly, it has to be a neutral one, say G_1 , and the cycles created have to be active, for otherwise G_1 would be a neutral biconnected subgraph of G , that is, a dependent component, and G would not be dependent. (From the above remark, $G_1 \cup \{e_k\}$ is then a rim with bolts n_1 and n_2 and rim components G_1 and $\{e_k\}$.) Moreover G_2, \dots, G_l need to be active and thus G_1 is the unique neutral biconnected component of $G \setminus \{e_k\}$ (see Figure 4.11-(i)).

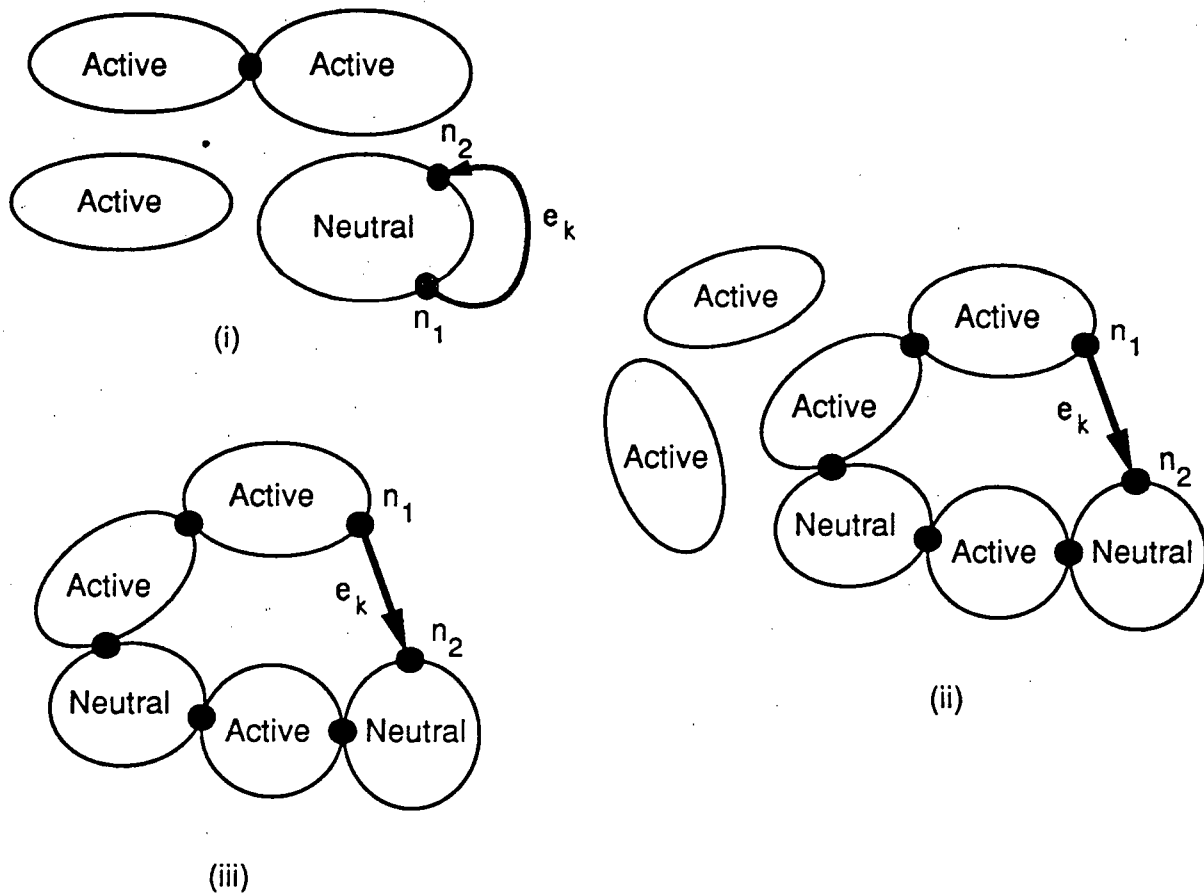


Figure 4.11: Construction of the dependent graph G when $G \setminus \{e_k\}$ is not biconnected

- In the following we assume that none of the biconnected components of $G \setminus \{e_k\}$ contain both n_1 and n_2 . Recall that we assumed G to be of type (II) or of type (III). If G is of type (II), then all the arcs that lie in neutral biconnected components of $G \setminus \{e_k\}$ belong to some active cycle in G , and therefore to some cycle that contains e_k , so that all the neutral biconnected components of $G \setminus \{e_k\}$ are part of a single biconnected component of G . Thus several, but not all, of the biconnected components of $G \setminus \{e_k\}$, including all the neutral ones, form a chain, and n_1 (resp., n_2) belongs exclusively to the first (resp., last) of these subgraphs (see Figure 4.11-(ii)).

If on the contrary G is of type (III), the above remark holds but we require that all the biconnected components of $G \setminus \{e_k\}$ be part of the chain and at least one of them be active, for otherwise G would be a rim (see Figure 4.11-(iii)).

4.2.2 Quantitative Less-Dependent-On Relation

For any three arcs e_i, e_j and e_k in E we say that e_i is (resp., **strictly**) **quantitatively less dependent on e_j than e_k** is and write $e_i \leq_j^q e_k$ (resp., $e_i <_j^q e_k$) if and only if

$$\text{for all } f \in H(\alpha), \quad f_i f_j \neq 0 \implies |f_i| \leq |f_k| \text{ (resp., } |f_i| < |f_k| \text{)}$$

or, equivalently, if

$$|f_i| \leq |f_k| \text{ (resp., } |f_i| < |f_k| \text{) for all } f \in H_j(\alpha).$$

If $e_i \leq_j^q e_k$ and $e_k \leq_j^q e_i$ we say that e_i and e_k are **q-equally dependent on e_j** and write $e_i =_j^q e_k$. Clearly, $e_i =_j^q e_k$ if and only if for all $f \in H_j(\alpha)$, $|f_i| = |f_k|$. One can easily see that the quantitative dependence relation is again a partial-order and hence $=_j^q$ is an equivalence relation. The corresponding sets of **q-equivalence classes \mathcal{E}_j^q** induced by it are partially ordered by \leq_j^q . We will denote by $\mathcal{E}_j^q(k)$ the unique element of \mathcal{E}_j^q that contains arc e_k .

Notice that in the unconstrained case $H(\alpha)$ reduces to the set of simple cycles, and thus both the less-dependent-on and the quantitatively-less-dependent-on relations are equivalent to Granot and Veinott's [GV85] *less-biconnected-to* relation (where $e_i \leq_j e_k$ if and

only if any simple cycle that contains e_i and e_j also contains e_k). In that sense they both constitute genuine extensions of the Granot and Veinott notion. However, it will be shown in example 6 to follow that in the constrained case they are no longer equivalent. Clearly, for e_i, e_j and $e_k \in E$, if $e_i \leq_j^q e_k$ (resp., $e_i =_j^q e_k$) then also $e_i \leq_j^d e_k$ (resp., $e_i =_j^d e_k$). This implies that $\mathcal{E}_j^q(i) \subseteq \mathcal{E}_j^d(i)$ for all $e_i \in E$, but the reverse inclusion is generally not true, as can be seen from the following counter example.

EXAMPLE 6 Consider the graph given in Figure 4.12 with the additional constraint

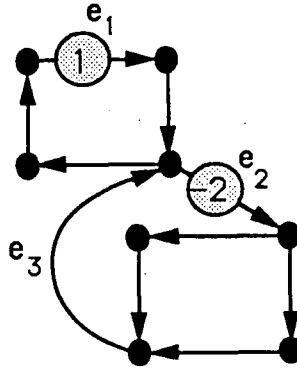


Figure 4.12: $=_j^d$ does not imply $=_j^q$

For any circulation $x \in C(\alpha)$, $x_1 = 2x_2$ by the constraint and $x_2 = x_3$ by the configuration of the graph. Therefore for any elementary vector $f \in H(\alpha)$, $f_1 = 2f_2$ and $f_2 = f_3$. Hence, $e_1 =_2^d e_3$ but $e_1 \neq_2^q e_3$. \square

One should also notice that in contrast with the unconstrained case where the greatest element in the partial order \leq_j always contains arc e_j , in the constrained case one can encounter examples for which there exists an arc e_k such that $e_j <_j^q e_k$. In particular, this may happen if the constraint is of the form $x_j + \alpha_k x_k = \alpha_0$ and $|\alpha_k| < 1$.

4.2.3 The Ripple Theorem

We are ready now to present the Ripple Theorem that will give some characterization of the change in the optimal solution to a Singly-constrained Monotropic Network Flow Problem resulting from a change in the parameter of one arc.

THEOREM 21 The Ripple Theorem. *Let $G = (N, E, \alpha)$ be a dependent constrained graph of type (II) (i.e., several active biconnected components) or type (III) (a unique active biconnected component which is neither a rim nor a one-rim) and suppose that t and t' differ only in one component $e_j \in E$, $b_k(\cdot, t_k)$ is convex for each $e_k \in E \setminus \mathcal{E}_j^d(j)$, x is an element of $X(t)$ and $X(t')$ is nonempty. Then there exists $x' \in X(t')$ such that*

(R.1) $x' - x$ is a sum of conformal elementary circulations, each of whose support contains arc e_j .

(R.2) **D-Ripple Property:** *For any two arcs e_i and e_k in E , if $e_i \leq_j^d e_k$ then*

$$|x'_i - x_i| \leq M_{ki} |x'_k - x_k| \quad (4.6)$$

where

$$M_{ki} = \max\{|f_i/f_k| : f \in H_k(\alpha)\}. \quad (4.7)$$

(R.3) **Q-Ripple Property:** $|x'_k - x_k|$ does not increase the less quantitatively dependent e_k is on e_j .

PROOF. Part (R.1) follows by using Rockafellar's Harmonious Superposition Theorem described in Chapter 2 and the convexity of the cost functions as was done by Granot and Veinott in [GV85]. The proof is included for completeness. Suppose x' is an element of $X(t')$. Then $x' - x$ is a sum of conformal elementary circulations. Let y be the sum of the conformal elementary circulations whose support contains e_j and z the sum of the remaining elementary circulations. Then y and z are conformal and $z_k = 0$ for all $k \in \mathcal{E}_j^d(j)$ since, by definition, any elementary support that does not contain e_j does not contain e_k either whenever $e_k =_j^d e_j$. Therefore, by the convexity hypotheses,

$$b(x + y, t') + b(x + z, t') \leq b(x, t') + b(x', t') \quad (4.8)$$

where $b(x, t) = \sum_{l=1}^m b_l(x_l, t_l)$. Moreover, since the parameter change involves e_j only and $z_j = 0$,

$$b(x + z, t') - b(x, t') = b(x + z, t) - b(x, t). \quad (4.9)$$

Combining (4.8) and (4.9) one obtains

$$b(x + y, t') + b(x + z, t) \leq b(x, t) + b(x', t').$$

Hence, $x + y \in X(t')$ and $x + z \in X(t)$, because $x' \in X(t')$ and $x \in X(t)$. Therefore, $x + y$ is optimal for t' and (R.1) holds. In the remainder of the proof, we will assume that (R.1) holds.

In order to show (R.2) assume that $e_i, e_k \in E$ and that $e_i \leq_j^d e_k$. From (R.1) we can write

$$x' - x = \sum_{\substack{f_a \in H_j(\alpha) \\ \text{conformal}}} z_a f_a \quad \text{where } z_a \geq 0 \text{ for all } a \quad (4.10)$$

and, from the conformality of the sum,

$$|x'_\ell - x_\ell| = \sum_{\substack{f_a \in H_j(\alpha) \\ \text{conformal}}} z_a |f_{a_\ell}| \quad \text{for all } \ell. \quad (4.11)$$

Moreover, $e_i \leq_j^d e_k$ implies that $f_{a_i} = 0$ for all elementary vector $f_a \in H_j(\alpha) \setminus H_k(\alpha)$ and, by the definition of M_{ki} , $|f_{a_i}| \leq M_{ki} |f_{a_k}|$ for all $f_a \in H_k(\alpha)$. Thus,

$$\begin{aligned} |x'_i - x_i| &= \sum_{\substack{f_a \in H_j(\alpha) \\ \text{conformal}}} z_a |f_{a_i}| \\ &= \sum_{\substack{f_a \in H_j(\alpha) \cap H_k(\alpha) \\ \text{conformal}}} z_a |f_{a_i}| + \sum_{\substack{f_a \in H_j(\alpha) \setminus H_k(\alpha) \\ \text{conformal}}} z_a |f_{a_i}| \\ &= \sum_{\substack{f_a \in H_j(\alpha) \cap H_k(\alpha) \\ \text{conformal}}} z_a |f_{a_i}| \\ &\leq \sum_{\substack{f_a \in H_j(\alpha) \cap H_k(\alpha) \\ \text{conformal}}} z_a M_{ki} |f_{a_k}| = \sum_{\substack{f_a \in H_j(\alpha) \\ \text{conformal}}} z_a M_{ki} |f_{a_k}| = M_{ki} |x'_k - x_k| \end{aligned}$$

completing the proof of (R.2).

To prove (R.3) we need to show that for any two arcs $e_i, e_k \in E$ such that $e_i \leq_j^q e_k$ we have $|x'_i - x_i| \leq |x'_k - x_k|$. By hypothesis, $|f_{a_i}| \leq |f_{a_k}|$ for all $f_a \in H_j(\alpha)$, thus (4.11) yields

$$|x'_i - x_i| = \sum_{\substack{f_a \in H_j(\alpha) \\ \text{conformal}}} z_a |f_{a_i}| \leq \sum_{\substack{f_a \in H_j(\alpha) \\ \text{conformal}}} z_a |f_{a_k}| = |x'_k - x_k| \quad (4.12)$$

completing the proof of (R.3). \square

Notice that under the assumptions of the Ripple Theorem, the following special cases are trivial consequences.

- If $x'_k = x_k$ then $x'_i = x_i$.
- Since $e_i \leq_j^d e_j$ for all i , then $|x'_i - x_i| \leq M_{ji}|x'_j - x_j|$ for all i .
- **Constant changes in q-equivalence classes.** In the unconstrained case, Granot and Veinott showed that, under the assumptions of the Ripple Theorem, the magnitude of the change in optimal arc-flow remains constant over all arcs in a given equivalence class for the *less-biconnected-to* relation. This property remains valid for the quantitatively-less-dependent-on relation, since if $e_i =_j^q e_k$, then by definition $|f_{a_i}| = |f_{a_k}|$ for all $f_a \in H_j(\alpha)$, and thus by (4.11), $|x'_i - x_i| = |x'_k - x_k|$. However this is no longer the case for the less-dependent-on relation as demonstrated in example 7 below.

EXAMPLE 7 Non-constant variations in d-equivalence classes. Consider the graph with 7 arcs and 5 nodes given in Figure 4.13, with the added constraint $2x_4 + x_5 = 3$. Let the demand be given by $d_i = 0 \forall i$ and the costs by

$$b_1(x_1, t_1) = (x_1 - t_1)^2, \quad b_9(x_7, t_7) = |x_7 - 1|, \quad b_j(x_j, t_j) = 0 \quad \forall j \neq 1, 7.$$

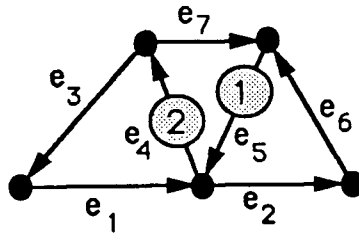


Figure 4.13: Tight bounds in the Ripple Theorem

The assumptions of Theorem 21 are satisfied, and if we further let, for example, $t = (0, 0, \dots, 0)$ and $t' = (1, 0, \dots, 0)$ then it is easy to verify that $X(t) = \{x\}$ and $X(t') = \{x'\}$

where $x = (0, 0, 0, 1, 1, 0, 1)$ and $x' = (1, -2, 1, 2, -1, -2, 1)$. Indeed, zero is a lower bound on the objective value for any value of the parameter, and one can check that both these solutions achieve an objective value of zero, and are unique in that respect. Moreover, e_4 and e_5 are in the same d -equivalence class $\mathcal{E}_1^d(5)$. Indeed, the sole elementary vector $f = (e_1 + e_4 + e_3) - 2(e_2 + e_6 + e_5) = (1, -2, 1, 1, -2, -2, 0)$ that contains e_1 and e_4 (resp., and e_5) also contains e_5 (resp., e_4), and thus $e_4 =_1^d e_5$. However, $|x'_4 - x_4| = 1 \neq 2 = |x'_5 - x_5|$. \square

Before discussing in subsection 4.2.4 the computational complexity involved in determining the bounds $M_{j,k}$ introduced in the Ripple Theorem, we will illustrate in Example 8 that those bounds are indeed the sharpest bounds possible, in the sense that inequality (4.6) in Theorem 21 may be tight.

EXAMPLE 8 **Tight bounds in the Ripple Theorem.** Consider again the graph given in Figure 4.13. The elementary vectors that contain both e_1 and e_2 are

$$f = (e_1 + e_4 + e_3) - 2(e_2 + e_6 + e_5) = (1, -2, 1, 1, -2, -2, 0)$$

$$g = (e_1 + e_2 + e_6 - e_7 + e_3) = (1, 1, 1, 0, 0, 1, -1)$$

and thus $M_{12} = \max\{|2/1|, |1/1|\} = 2$. Now, if we let $t = (0, 0, \dots, 0)$ and $t' = (1, 0, \dots, 0)$ the unique optimal flows for t and t' are respectively

$$x = (0, 0, 0, 1, 1, 0, 1) \text{ and } x' = (1, -2, 1, 2, -1, -2, 1) \text{ (see Example 7).}$$

Thus $|x'_2 - x_2| = M_{12}|x'_1 - x_1|$ and the bound given in equation 4.6 is tight. \square

4.2.4 Computing the Bounds in the Ripple Theorem

In this subsection we discuss the complexity of computing the coefficients $M_{j,k}$ introduced in the Ripple Theorem. This computation in general turns out to be NP-Hard as shown in Theorem 22. However, we provide in the latter part of this subsection special cases in which a polynomial time algorithm can be derived.

THEOREM 22 *The problem [M] of computing the coefficient*

$$M_{jk} = \max\{|f_k/f_j| : f \in H_j(\alpha)\}$$

for any given pair of arcs (e_j, e_k) of a constrained graph $G = (N, E, \alpha)$, is NP-Hard.

PROOF. Assume that there exists an algorithm $Al_{[M]}$ which, given a constrained graph $G = (N, E, \alpha)$ and two arcs $e_j, e_k \in E$, evaluates the coefficient M_{jk} , and runs in at most $P_{[M]}(s)$ elementary operations, where $P_{[M]}(s)$ is some function of s (recall that s is a measure of the size of the input, see Chapter 2). We show in the following that $Al_{[M]}$ can be used to solve an NP-Hard problem, namely the **Maximum Weight Oriented Simple Cycle [MWOSC]** (see [GJ79, p. 213])

$$[\text{MWOSC}] \quad \max\{|\alpha c| : c \text{ simple cycle of } G\}.$$

Consider the following construction. For a given graph $G = (N, E)$, an m -vector $(\alpha_i)_{i=1, \dots, m}$ whose entries will be called **weights**, define the constrained graph $G^* = (N^*, E^*, \alpha^*)$ by

$$N^* = N \cup \{n+1, n+2\}$$

$$E^* = E \cup \{e_{m+1} = (n+1, n+2), e_{m+2} = (n+2, n+1)\}$$

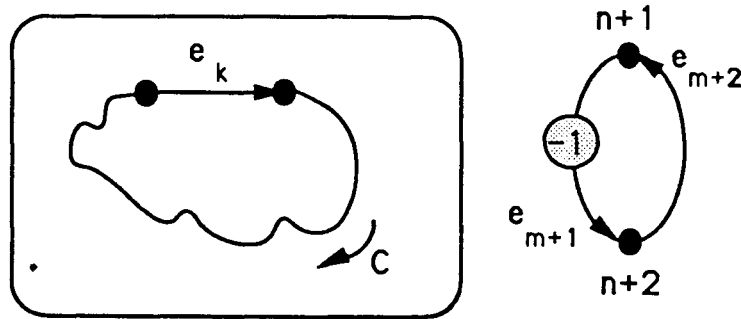
and

$$\alpha_i^* = \begin{cases} \alpha_i & \text{if } i \leq m \\ -1 & \text{if } i = m+1 \\ 0 & \text{if } i = m+2 \end{cases}$$

and let e_k be any given arc in G (See Figure 4.14). Now, since no simple cycle goes through both e_k and e_{m+1} , any elementary vector containing those two arcs has to be a weighted sum of two active disjoint cycles, that is,

$$f = (e_{m+1} + e_{m+2}) + f_k c \quad (4.13)$$

where c is an active cycle of G for which $c_k = 1$ and f_k is determined by (3.3), resulting in

Figure 4.14: The extended graph G^*

$$f_k = -\alpha^*(e_{m+1} + e_{m+2})/\alpha^*c = 1/\alpha^*c = 1/\alpha c. \quad (4.14)$$

Therefore, for the constrained graph G^* , we can write

$$\begin{aligned} M_{k \ m+1} &= \max\{|f_{m+1}/f_k| : f \in H_k^*(\alpha^*)\} \\ &= \max\{|1/f_k| : (e_{m+1} + e_{m+2} + f_k c) \in H_k^*(\alpha^*)\} \\ &= \max\{|\alpha c| : c_k = 1, c \text{ simple cycle of } G\} \end{aligned}$$

where $H_k^*(\alpha^*)$ is the set of elementary vectors of the circulation space of G^* that contain e_k . Thus $Al_{[M]}$ will, when applied to G^* and a particular pair of arcs (e_k, e_{m+1}) , determine the oriented cycle of maximum weight that contains arc e_k . By repeating the above for $k = 1, \dots, m$ and then selecting the largest of the m optimal values, one solves the [MWOSC] in $O(MP_{[M]}(s))$. Since the [MWOSC] is NP-Hard, so is [M], completing the proof. \square

In the following we present five examples where the value of M_{jk} can easily be computed. For convenience we will assume without loss of generality that all elementary vectors f encountered with $f_j \neq 0$ have $f_j = 1$.

Arcs in Rims. In the first part we consider the case where G has several biconnected components, and the biconnected component(s) that contains the arcs e_j and e_k is a rim (are rims). At this point, it is interesting to notice that a biconnected subgraph, say G_1 ,

with a unique arc e_i that verifies $\alpha_i \neq 0$, is a rim whose bolts are the end-nodes of e_i and whose rim components are $\{e_i\}$ and $G_1 \setminus \{e_i\}$. Such subgraphs naturally fall in the category described here.

- If at least one of the biconnected components of G , say G_1 , is a rim, then for every $e_j, e_k \in G_1$, the elementary vectors that contain both e_j and e_k are either simple neutral cycles of G_1 or active cycles of G_1 paired with active cycles not in G_1 . All such elementary vectors verify $|f_j| = |f_k|$, and thus $M_{jk} = 1$.
- If at least two of the biconnected components of G , say G_1 and G_2 , are rims, then for every $e_j \in G_1$ and $e_k \in G_2$, any elementary vector that contains both e_j and e_k is of the form

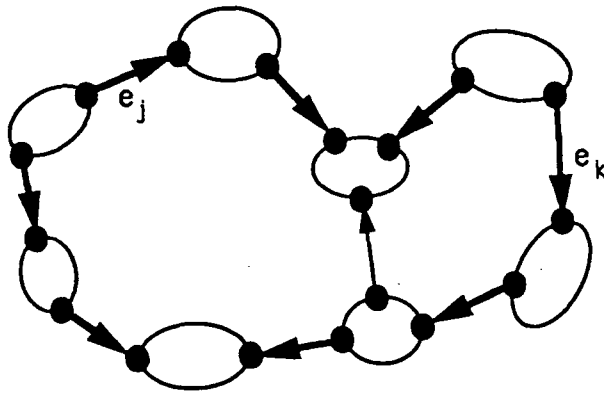
$$f = c^1 + f_k c^2$$

where c^1 and c^2 are respectively active cycles of G_1 and G_2 , $c_j^1 = c_k^2 = 1$ and $f_k = -\alpha c^1 / \alpha c^2$ as in (3.3). Therefore $|f_k / f_j| = |(-\alpha c^1 / \alpha c^2) / 1| = |\alpha c^1 / \alpha c^2|$ is independent of the choice of c^1 and c^2 , and M_{jk} is equal to that ratio.

The support of α on some biconnected component of G is a pair of arcs. Suppose now that $\alpha_i = 0$ for all arcs e_i , $e_i \neq e_j, e_k$ and e_i belongs to the same biconnected component(s) of G as e_j or e_k . Since the above treats the case where the two arcs lie in different biconnected components of G , we may assume that e_j and e_k lie in the same biconnected component of G .

- Let us first assume that G is biconnected but is not a rim (type (III)). Then, any elementary vector which contains both arcs is either a simple neutral cycle (this situation may only occur if $|\alpha_j| = |\alpha_k|$) or a pair of active cycles, say $f = c^1 - (\alpha c^1 / \alpha c^2) c^2$, and we may assume without loss of generality that $c_j^1 = c_k^2 = 1$ and $c_k^1 = c_j^2 = 0$. In either case, $|f_k / f_j| = |\alpha_j / \alpha_k|$, and thus $M_{jk} = |\alpha_j / \alpha_k|$.

- We now suppose that G has several active biconnected components, and that e_j and e_k belong to one of them, say G_1 . Along with the elementary vectors described above, we need to consider any active cycle of G_1 that contains e_j and e_k , together with another active cycle in another biconnected component. For such an elementary vector f , $|f_k/f_j| = 1$, and thus $M_{jk} = \max\{1, |\alpha_j/\alpha_k|\}$.
- Notice that in the above two cases, the assumption on the entries of α may be relaxed by only requiring that, among the arcs that lie in the biconnected component G_1 of G that contains e_j and e_k , only those that are contained in every simple cycle that also contains e_j and those that are contained in every simple cycle that contains e_k ¹ may have nonzero entries in α (see Figure 4.15). Indeed, as far as the study of cycles goes, we may replace α_j (resp., α_k) by α_c , where c is any cycle that verifies $c_j = 1$ (resp., $c_k = 1$), and replace by zero all the other α_i 's, $e_i \in G_1$, $i \neq j, k$.



Only bold arcs have
nonzero alpha's

Figure 4.15: Only bold arcs verify $\alpha_i \neq 0$

¹In a graph G , the set of arcs contained in every simple cycle that contains a given arc e_j is the equivalence class for the \equiv_j relation defined in [GV85] which contains e_j . Alternatively, such an arc, say e_i , is characterized by the fact that $\{e_i\}$ is a biconnected component of $G \setminus \{e_j\}$, and thus can be detected in linear time.

4.2.5 Complexity Issues Attached to the Qualitative Less-dependent-on Relation

In this subsection we present a general statement about the complexity of the problem of determining whether $e_i \leq_j^q e_k$ holds for three given arcs e_i, e_j and e_k . The computation turns out to be NP-complete for the general case, and we provide again special cases where the answer is tractable. We are thankful to Richard Anstee for suggesting improvements to an earlier version of the proof of Theorem 23.

THEOREM 23 *The problem [QLD] of deciding whether three given arcs e_i, e_j and e_k in a constrained network $G = (N, E, \alpha)$ are such that $e_i \leq_j^q e_k$ is NP-complete.*

PROOF. Assume that there exists an algorithm $Al_{[QLD]}$ which, given a constrained graph $G = (N, E, \alpha)$ and a triplet of arcs $e_i, e_j, e_k \in E$, determines whether $e_i \leq_j^q e_k$. We claim that $Al_{[QLD]}$ can be used to solve the decision version of the Maximum Weight Oriented Path [MWOP-D] defined as follows.

Given a vector of weights $(w_i) \in \mathbb{R}^m$, two nodes say n_1 and n_2 , and a real number $\lambda \geq 0$, do all oriented path p from n_1 to n_2 verify $wp \leq \lambda$?

The [MWOP-D] known to be is NP-complete (see [GJ79, p. 213]). Now, add to G the arc $e_0 = (n_2, n_1)$, define the vector α by

$$\begin{aligned} \alpha_0 &= \frac{-2^s}{2^s - \lambda} \\ \alpha_i &= \frac{w_i}{2^s - \lambda} \quad \forall i \geq 1, \end{aligned}$$

and consider again the graph G^* constructed in the proof of Theorem 22. Apply $Al_{[QLD]}$ to the statement " $e_0 \leq_{m+2}^q e_{m+1}$ ". By definition, this statement holds if and only if, for any elementary vector f of G^* which contains both e_0 and e_{m+2} , $|f_0| \leq |f_{m+1}|$. From (4.13) and (4.14), this is equivalent to $|1/\alpha^*c| \leq 1$ for any active cycle c of G^* that contains e_0 , that is, $|\alpha c| \geq 1$ for all active cycles of G that contain e_0 . This is trivially equivalent to $|\alpha c| \geq 1$

for all cycles c of G with $c_0 = 1$. Since such a cycle is of the form $c = e_0 + p$, where p is an oriented path from n_1 to n_2 , we can write that $Al_{[QLD]}$ will return a *YES* if and only if

$$|\frac{1}{2^s - \lambda}(wp - 2^s)| \geq 1$$

for all oriented paths p from n_1 to n_2 . Now, since $|wc| \leq 2^s$,

$$\frac{-1}{2^s - \lambda}(wp - 2^s) \geq 1 \quad \text{and thus} \quad wp \leq \lambda$$

for all path p from n_1 to n_2 , solving the [MWOP-D]. Since the reduction from one problem to the other is clearly linear, the proof follows. \square

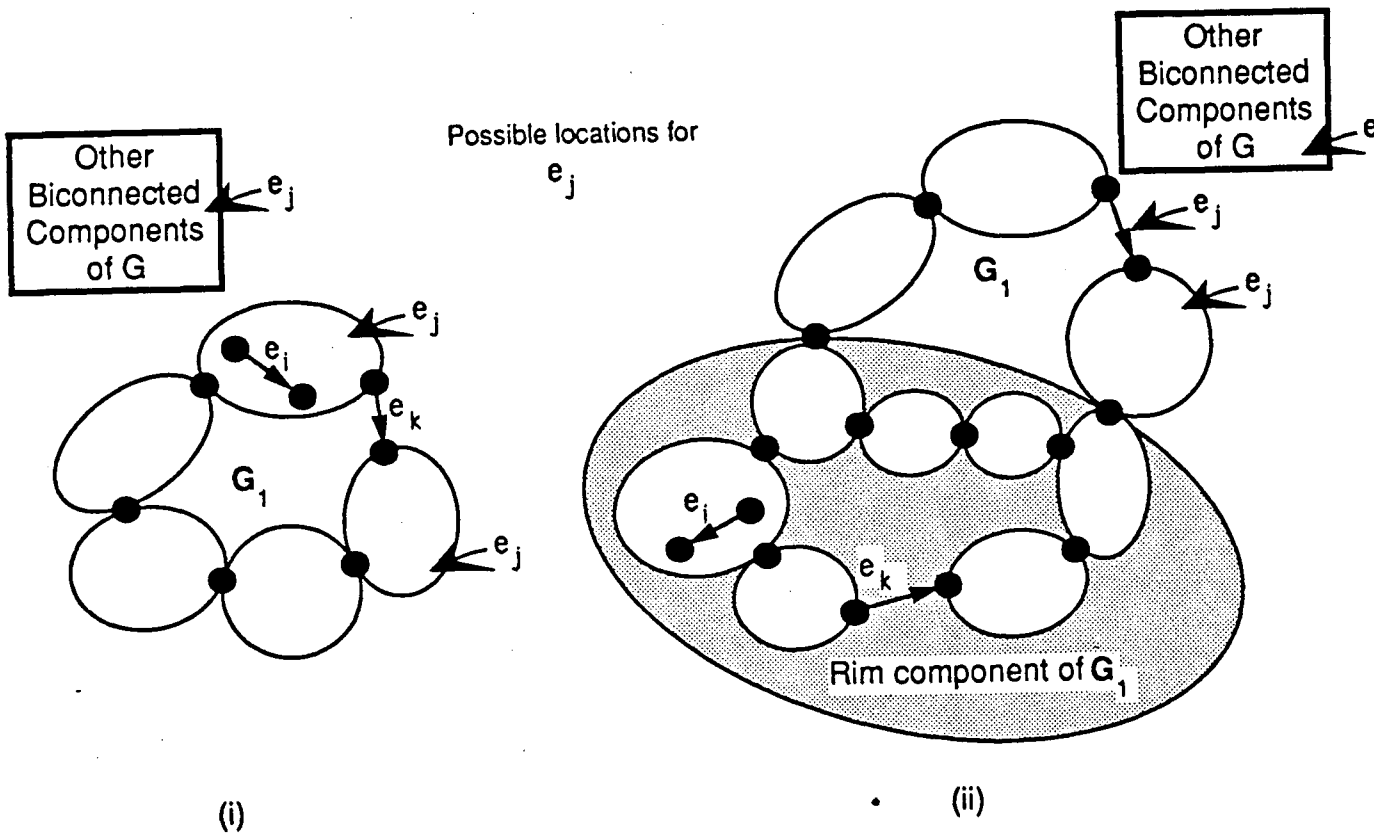
Recall from Subsection 4.2.1 that $e_i \leq_j^q e_k \Rightarrow e_i \leq_j^d e_k$, and thus the list of configurations for which $e_i \leq_j^d e_k$ does not hold (see Subsection 4.2.1) also applies here. We now provide some cases in which one can easily decide whether $e_i \leq_j^q e_k$ does hold. In order to do so, we prove the following lemma.

LEMMA 24 *Let $G = (N, E, \alpha)$ be a constrained graph and e_j, e_k , two arcs in G . Then every active cycle of G which contains e_j also contains e_k if and only if the biconnected component of $G \setminus \{e_k\}$ that contains e_j is neutral.*

PROOF. First, every active cycle of G which contains e_j also contains e_k if and only if e_j does not belong to any active cycle in $G \setminus \{e_k\}$ or, equivalently, if e_j does not belong to any active cycles in the biconnected component of $G \setminus \{e_k\}$ in which it lies. The result follows from Lemma 7 in Chapter 3 which states that a biconnected graph with at least two arcs is active if and only if every arc belongs to an active cycle. \square

- 1• G has at least two biconnected components, one of which, say G_1 , is a rim. Then for any $e_i, e_k \in G_1$ and e_j which does not belong to the rim component of G_1 that contains e_i , any elementary vector that contains both e_i and e_j is of the form given in (3.3) where c^1 is an active cycle of G_1 that contains e_i and c^2 is some active cycle of $G \setminus G_1$ (if $e_j \in G_1$ then c^1 contains e_j , otherwise c^2 does). Conversely, any active

cycle c^1 in G_1 which contains e_j can be completed into an elementary vector that contains e_i as well. Clearly, such an elementary vector contains e_k if and only if c^1 does, and thus $e_i \leq_j^d e_k$ if and only if every active cycle of G_1 which contains e_i also contains e_k . From Lemma 24, this holds if and only if the biconnected component of $G \setminus \{e_k\}$ that contains e_j is neutral. The possible configurations are given in Figure 4.16 where e_j may belong to G_1 (i) or to $G \setminus G_1$ (ii). Since it follows from the above that $|f_i| = |c_i^1| = |f_k| = |c_k^1| = 1$, then $e_i \leq_j^q e_k$ holds whenever $e_i \leq_j^d e_k$ does.


 Figure 4.16: $e_i \leq_j^q e_k$

- 2• G has exactly two biconnected components which are both rims, say G_1 and G_2 . Then for every $e_i \in G_1$, $e_j, e_k \in G_2$, any elementary vector that contains both e_i and e_j is of the form given in (3.3), where c^1 and c^2 are respectively active cycles of G_1 and

G_2 . Therefore $e_i \leq_j^d e_k$ if and only if every active cycle of G_2 which contains e_j also contains e_k . The two possible configurations are given by Lemma 24 (see the subgraph G_1 in Figure 4.16, (i), (ii)). For $e_i \leq_j^q e_k$ to hold, we require the additional condition that $|f_i| \leq |f_k|$ whenever $f_j \neq 0$, which from the above is equivalent to $|\alpha c^2| \leq |\alpha c^1|$.

4.3 Complements and Substitutes in Constrained Network Flows

In this section we define the concepts of complements and substitutes for constrained networks. These notions are crucial to the development of the Monotonicity Theory in section 4.4. The definition of substitutes and complements as given in [GV85] only involved the topology of the graph. Indeed, for the unconstrained network flow problem, two arcs are complements (resp., substitutes) if and only if every simple cycle in the network orients them in the same (resp., in opposite) direction(s). Here, the side constraint $\alpha x = \alpha_0$ comes into play through the elementary vectors, as will be seen in the definition below. Some preliminary results on comparative flow changes between two arcs that are substitutes or complements are given here. However, Theorem 31 in section 4.4 will show the central role played by substitutes and complements in predicting flow changes in a given arc resulting from changes in the parameter, and thus in the cost function, of another arc. We further show here that the problem of determining whether two given arcs are substitutes, complements, or neither is, in general, NP-complete. We then point out particular classes of instances of that problem where a fast solution (where polynomial time algorithms such as Decomposition into Biconnected Components or Rim Recognition suffice) is provided.

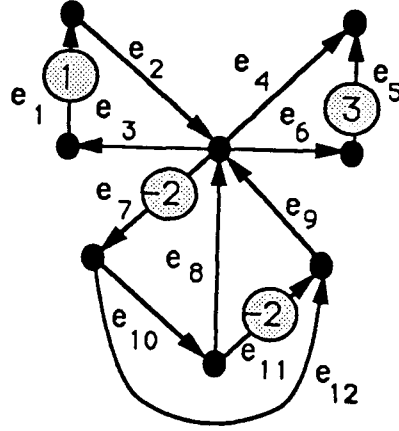
Two arcs e_i and e_j in a constrained graph $G = (N, E, \alpha)$ are said to be **complements** (resp., **substitutes**) if and only if

$$\text{for all elementary vectors } f \in H(\alpha), \quad f_i f_j \geq 0 \quad (\text{resp., } \leq 0). \quad (4.15)$$

(Recall that $H(\alpha)$ is the set of all elementary vectors of the constrained circulation space.)

EXAMPLE 9 In the graph given in Figure 4.17 one can see that e_1 and e_2 are complements, for the elementary vectors that contain them both are

$$\begin{aligned} f^1 &= (e_1 + e_2 + e_3) - 1/3(e_6 + e_5 - e_4) & f^2 &= (e_1 + e_2 + e_3) + 1/2(e_7 + e_{10} + e_8) \\ f^3 &= (e_1 + e_2 + e_3) + 1/4(e_7 + e_{10} + e_{11} + e_9) & f^4 &= (e_1 + e_2 + e_3) + 1/2(-e_8 + e_9 + e_{11}) \\ f^5 &= (e_1 + e_2 + e_3) + 1/2(e_7 + e_{12} + e_9) & f^6 &= (e_1 + e_2 + e_3) + 1/2(e_{10} + e_{11} - e_{12}) \end{aligned}$$



$$x_1 - 2x_7 - 2x_{11} + 3x_5 = 2$$

Figure 4.17: Substitutes and complements in a constrained graph

and one can check that $f_1^i f_2^i > 0$ for $i = 1, \dots, 6$. Similarly (e_1, e_7) and (e_1, e_{11}) , for instance, are other pairs of complements whereas (e_1, e_5) is a pair of substitutes. \square

In order to further motivate the interest in arcs that are complements and substitutes notice that, under the assumptions of the Ripple Theorem, if, in addition, the arc e_j whose parameter is being changed and another arc e_i are complements (resp., substitutes), then

$$(x'_i - x_i)(x'_j - x_j) \geq 0 \quad (\text{resp., } \leq 0).$$

This is a straightforward consequence of (4.10) and the fact that, by the definition of complements (resp., substitutes), $f_a, f_b \geq 0$ (resp., ≤ 0) for all $f_a \in H_j(\alpha)$. Stronger results will be presented in Theorem 31 of section 4.4 and in Lemma 25 to follow. It is also interesting to notice that, although it was shown in Subsection 4.2.3, that optimal arc-flow changes

are not constant across d -equivalence classes, one can predict the comparative direction of changes as shown below.

LEMMA 25 *Variations in pairs of complements and substitutes.* Let $G = (N, E, \alpha)$ be a dependent constrained graph and suppose that t and t' differ only in one component $e_j \in E$, $b_l(\cdot, t_l)$ is convex for each $e_l \in E \setminus \mathcal{E}_j^d(j)$, x is an element of $X(t)$ and $X(t')$ is nonempty. Then there exists $x' \in X(t')$ such that for any $e_k \in \mathcal{E}_j^d(j)$ we have

$$\text{Complementarity Property: } (x'_i - x_i)(x'_k - x_k) \geq 0 \quad (\text{resp., } \leq 0) \quad (4.16)$$

whenever e_i and e_k are complements (resp., substitutes).

PROOF. If $(x'_i - x_i)(x'_k - x_k) = 0$ the proof is complete. Otherwise, notice that $H_j(\alpha) \cap H_i(\alpha) = H_j(\alpha) \cap H_k(\alpha)$ whenever $e_i =_j^d e_k$, and we will denote this set by H_{ijk} . For the optimal flow $x' \in X(t')$ defined in Theorem 21 by (4.10) we obtain

$$\begin{aligned} (x'_i - x_i)(x'_k - x_k) &= \left(\sum_{\substack{f_{a_i} \in H_j(\alpha) \\ \text{conformal}}} z_a f_{a_i} \right) \left(\sum_{\substack{f_{a_k} \in H_j(\alpha) \\ \text{conformal}}} z_a f_{a_k} \right) \\ &= \left(\sum_{\substack{f_{a_i} \in H_{ijk} \\ \text{conformal}}} z_a f_{a_i} \right) \left(\sum_{\substack{f_{a_k} \in H_{ijk} \\ \text{conformal}}} z_a f_{a_k} \right) \end{aligned}$$

where all the terms in both sums are nonzero and $z_a > 0$ for all $f_a \in H_{ijk}$. By conformality, all the terms in the first sum are of the same sign, and so are the terms in the second sum. Now, since e_i and e_j are complements (resp., substitutes), these signs are the same (resp., opposite) and the proof follows. \square

In view of the above two results it is natural to ask whether the property for a pair of arcs to be complements or substitutes is transitive. This is in general not true, as was shown by Granot and Veinott [GV85] for the unconstrained case, and thus for the constrained case as well². Still, in the unconstrained case, every pair of arcs in an equivalence class of the

²Counter examples for the unconstrained case are given in [GV85]. For an example specific to constrained networks, refer to Figure 4.17 where (e_7, e_1) and (e_1, e_{11}) are pairs of complements, but e_7 and e_{11} are neither complements nor substitutes, for the elementary vectors $f^1 = 4(e_1 + e_2 + e_3) + (e_7 + e_{10} + e_{11} + e_9)$ and $f^2 = (e_7 + e_{12} - e_{11} + e_8)$ orient them respectively in the same and in opposite directions.

less-biconnected-to relation defined therein is either complement or substitutes. This result does not hold either for constrained networks, for instance in Figure 4.17 $e_7 =_1^d e_{11}$, $e_7 =_1^q e_{11}$ but e_7 and e_{11} are neither complements nor substitutes. (Indeed, the elementary vectors $f = (e_1 + e_2 + e_3) + 1/4(e_7 + e_{10} + e_{11} + e_9)$ and $g = (e_7 + e_{12} - e_{11} + e_8)$ orient them in opposite directions.) The following result, however, shows that under certain assumptions, some form of transitivity holds, in particular when the arcs involved belong to the same d-equivalence class.

LEMMA 26 *Conditional transitivity of Complements and Substitutes.* *Let e_i , e_j and e_k be three arcs in a constrained dependent graph $G = (N, E, \alpha)$, and suppose that [i] e_i and e_j are complements, [ii] e_i and e_k are complements (resp., substitutes), and [iii] $e_k \leq_j^d e_i^3$. Then e_j and e_k are complements (resp., substitutes)⁴.*

PROOF. Let $f \in H(\alpha)$, and suppose that $f_j f_k \neq 0$. By [iii] $f_i \neq 0$, and thus by [i] and [ii] $f_i f_j > 0$ and $f_i f_k > 0$ (resp., < 0). Therefore $\text{sign}(f_j f_k) = \text{sign}(f_i^2 f_j f_k) = \text{sign}(f_i f_j) \text{sign}(f_i f_k) = 1$ (resp., -1) (Where the real-valued *sign* function is naturally defined by $\text{sign}(z) = 1$ if $z > 0$, -1 if $z < 0$, and $\text{sign}(0) = 0$), completing the proof. \square

THEOREM 27 *The Complexity of Determining Substitutes and Complements.* *The problem of determining whether a given pair of arcs of a constrained graph $G = (N, E, \alpha)$ are substitutes, complements or neither is NP-complete.*

PROOF. We show here that the simpler decision problem of determining whether or not two given arcs are “neither substitutes nor complements”, denoted by [DN], is NP-complete. Assume now that there exists an algorithm, say $Al_{[DN]}$, which when applied to the constrained graph $G = (N, E, \alpha)$ and a pair of arcs (e_i, e_k) answers the above question in, say, $O(P_{[DN]}(s))$ elementary operations. Now, given a constrained graph $G = (N, E, \alpha)$

³In particular, [iii] holds if e_i and e_k are in the same d-equivalence class for $=_j^d$.

⁴If e_i and e_j are substitutes, then trivially a similar result holds, in which the conclusions are reversed.

and a particular arc $e_k \in E$, construct the graph G^* as in the proof of Theorem 22 (See Figure 4.14), and apply $Al_{[DN]}$ to the pair of arcs e_k, e_{m+1} .

Now, from the definition of substitutes and complements, the answer to [DN] is *YES* if and only if there exists a pair of elementary vectors, say f^+ and f^- of $H(\alpha)$ such that $f_{m+1}^+ = f_{m+1}^- = 1$, $f_k^+ > 0$ and $f_k^- < 0$. From equations (4.13) and (4.14) one obtains that

$$f^+ = (e_{m+1} + e_{m+2}) + (1/\alpha c^+)c^+ \text{ and } f^- = (e_{m+1} + e_{m+2}) + (1/\alpha c^-)c^-$$

where c^+ and c^- are active cycles of G with $c_k^+ = c_k^- = 1$. Moreover,

$$f_k^+ = (1/\alpha c^+)c_k^+ > 0, \quad f_k^- = (1/\alpha c^-)c_k^- < 0 \implies \alpha c^+ > 0 \quad \text{and} \quad \alpha c^- < 0.$$

Thus $Al_{[DN]}$ will, in $O(P_{[DN]}(s))$ elementary operations, determine whether the above two cycles exist simultaneously. We now show that $Al_{[DN]}$ can be used to solve the recognition problem associated with the Maximum Weight Oriented Simple Cycle Problem [MWOSC] in time polynomial in $P_{[DN]}(s)$.

In order to do so, first notice that the set $\{\alpha c : c \text{ is a cycle in } G, c_k = 1\}$ has a smallest element, say λ_1 , and a largest element, say λ_2 , and a rough upper bound on their magnitudes is given by $|\lambda_k| \leq 2^s$ for $k = 1, 2$. Moreover, if $Al_{[DN]}$ is applied to the graph obtained by replacing in G^* the quantity α_k by $(\alpha_k - \lambda)$ for some $\lambda \in \mathbb{R}$, then the answer will be *YES* if and only if there exists two cycles c^+ and c^- in G such that $\alpha c^+ > \lambda$ and $\alpha c^- < \lambda$, that is, if and only if $\lambda_1 < \lambda < \lambda_2$. Since an initial element of the interval (λ_1, λ_2) can be obtained in linear time by finding any cycle that contains e_k (complete e_k by any simple path joining its head to its tail using depth-first search), it is possible to use binary search over the interval $[-2^s, 2^s]$ in order to determine both λ_1 and λ_2 , and thus $\lambda(k) = \max\{\alpha c : c \text{ cycle in } G, c_k = \pm 1\} = \max\{|\lambda_1|, |\lambda_2|\}$. The time required to do so is $O(\log_2 2^s P_{[DN]}(s)) = O(s P_{[DN]}(s))$ (see [GJ79, PS82] for the running time of binary search). By repeating the above procedure for $k = 1, \dots, m$, one obtains $\max\{\lambda(k) : k = 1, \dots, m\} = \max\{\alpha c : c \text{ cycle in } G\}$ in $O(ms P_{[DN]}(s))$ elementary operations, proving our claim. Since [MWOSC] is NP-complete, so is [DN], and the proof is complete. \square

There are, however, some cases in which one can determine whether two arcs in a constrained graph $G = (N, E, \alpha)$ are substitutes or complements with little computation. Moreover, the question of whether a given instance of the above problem is one of these special cases can be answered in linear time by well known graph algorithms. The following three examples illustrate this fact.

- a) **“Parallel” arcs.** Suppose that the set of nodes $N \neq \phi$ can be partitioned into two subsets say N_1 and N_2 with $N = N_1 \cup N_2$ and $N_1 \cap N_2 = \phi$ and that there exists a unique pair of arcs say (e_i, e_j) joining N_1 and N_2 with $e_i = (a_i, b_i)$ and $e_j = (a_j, b_j)$ (resp., (b_j, a_j)) with $a_i, a_j \in N_1$ and $b_i, b_j \in N_2$ as is illustrated in Figure 4.18.a. In this case clearly for any circulation x in G we have $x_i = -x_j$ (resp., $x_i = x_j$) regardless of the fact that the network may have some constraints. Therefore, the same will hold for the elementary vectors of the circulation space $C(\alpha)$. Thus, for all $f \in H(\alpha)$, $f_i = -f_j$ (resp., $f_i = f_j$), and the arcs e_i and e_j are substitutes (resp., complements).
- b) **“Series” arcs.** A simple extension of the above result is the case where the set of nodes can be partitioned into say k subsets so that $N = N_1 \cup N_2 \dots \cup N_k$ with $N_\ell \cap N_m = \phi$ for $\ell \neq m \in \{1, \dots, k\}$ and where there exists a unique arc e_ℓ between some node in N_ℓ and another in $N_{\ell+1}$ for all $\ell = 1, \dots, k-1$, a unique arc e_k between some node in N_k and another in N_1 , and no other arc joins any node in N_i with any node in N_j , $i \neq j = 1, \dots, k$. This situation is illustrated on Figure 4.18.b. We can then condense the subsets N_1, \dots, N_k each into a single “supernode” and obtain a simple cycle (see Figure 4.18.c). By the same argument as above two arcs of the subset $\{e_1, \dots, e_k\}$ are complements if they are oriented by this cycle in the same way, and substitutes if they are oriented by it in an opposite way. For any elementary vector of the circulation space $f \in H(\alpha)$, we clearly have, $|f_1| = |f_2| = \dots = |f_k|$ and therefore the arcs e_1, \dots, e_k are all in the same equivalence class for both the dependence and the quantitative dependence relations.
- c) **Arcs with common degree-two nodes.** Suppose now that the arcs e_i and e_j are

in series, i.e., have a node in common, say $a \in N$, and no other arc is incident to a . We can apply a) to this case by considering the subsets $N_1 = \{a\}$ and $N_2 = N_1 \setminus \{a\}$ and assert that e_i and e_j are either substitutes or complements. This is illustrated in Figure 4.18 (d-e).

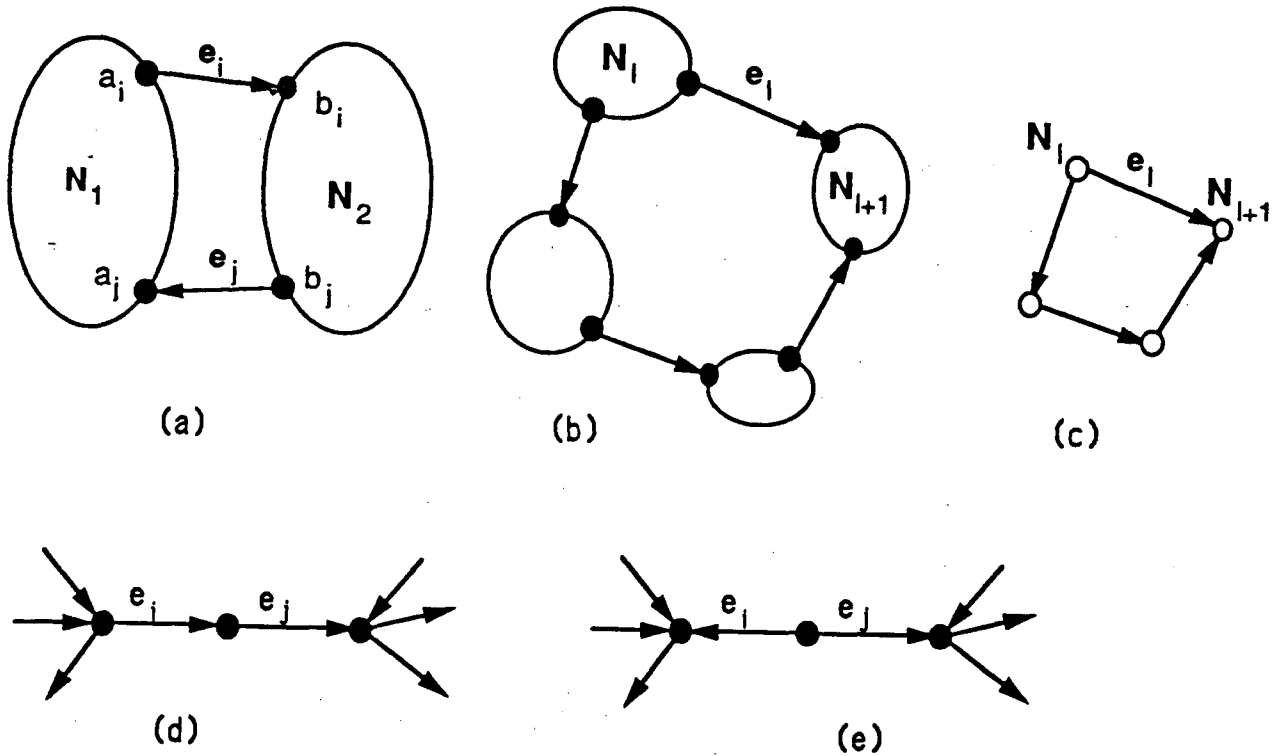


Figure 4.18: Substitutes and complements

For the following cases we require some additional definitions. An arc e_i in a rim G is a **positive** (resp., **negative**) **rim arc** if and only if every positive rim cycle c verifies $c_i \geq 0$ (resp., ≤ 0). Lemma 28 to follow provides a characterization of such arcs using the construction we describe now. Let G be a rim, e_i an arc in one of the rim components of G , say G' , and c any positive rim cycle. Define the unconstrained graph $\Gamma(G') = G' \cup \{e_0\}$ where e_0 joins the two bolts of G' , say n_1 and n_2 , and is oriented from n_1 to n_2 if so is the part of c that lies outside of G' (see Figure 4.19). Note that $\Gamma(G')$ does not depend on the choice of c .

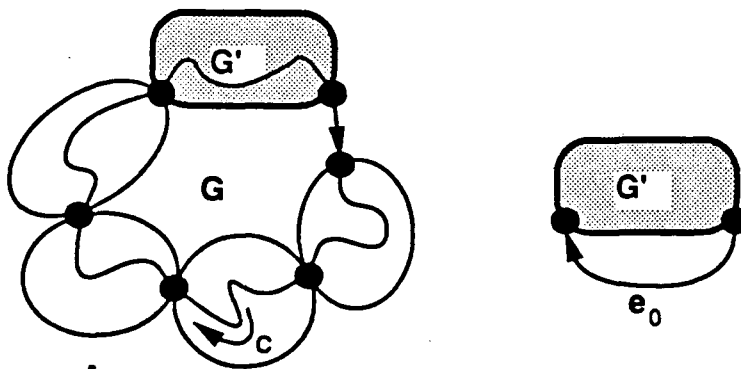


Figure 4.19: The rim G (left) and the constructed graph $\Gamma(G')$ (right)

LEMMA 28 *Characterization of positive rim arcs.* Let $G = (N, E, \alpha)$ be a rim, G' one of its rim components, and e_i an arc in G' . Then e_i is a positive rim arc if and only if e_i and e_0 are complements (resp., substitutes) in $\Gamma(G')$ in the sense defined by Granot and Veinott in [GV85]⁵.

PROOF. First recall that any positive (resp., negative) rim cycle c in G goes through all the bolts and through all the rim components, and thus corresponds to a cycle of $\Gamma(G')$ which contains e_0 as a forward (resp., backward) arc, the correspondence being obtained by replacing the part of c which lies outside of G' by e_0 (resp., $-e_0$). Therefore, every positive rim cycle verifies $c_i \geq 0$ (resp., $c_i \leq 0$) if and only if so does any active cycle $c^{\Gamma(G')}$ in $\Gamma(G')$ which verifies $c_0^{\Gamma(G')} = 1$. Moreover, if $c^{\Gamma(G')}$ is a negative rim cycle then $-c^{\Gamma(G')}$ is a positive rim cycle, the above applies, and the proof follows. \square

The first consequence of Lemma 28 is that one can verify in linear time whether a given arc is a positive rim arc, a negative rim arc, or neither. This observation is now used in the following to identify rim arcs which are substitutes or complements.

- d) **Arcs in a rim.** Suppose that G is of type (II) and that one of its biconnected components, say G_1 , is a rim. Then any elementary vector with arcs in G_1 is either a neutral cycle of G_1 or a pair of active cycles, one in G_1 and the other in $G \setminus G_1$. Therefore

⁵That is, if any oriented cycle in $\Gamma(G')$ orients e_i and e_0 in the same (resp., in opposite) direction.

any two arcs e_i and e_j in G_1 are complements (resp., substitutes) if and only if any cycle c of G_1 verifies $c_i c_j \geq 0$ (resp., ≤ 0), that is, if the two arcs are complements (resp., substitutes) in the sense defined by Granot and Veinott for unconstrained graphs. This can be done in linear time [GV85]. Using similar arguments to those used in the proof of Lemma 28, one can show that two arcs in the same rim component G' of G_1 are substitutes (resp., complements) if and only if they are substitutes (resp., complements) in $\Gamma(G'_1)$. Moreover, if the two arcs lie in different rim components of G_1 , say G' and G'' , then any cycle that contains them both is active, and therefore they are complements (resp., substitutes) if and only if they are both positive or both negative rim arcs (resp., one is a positive and the other a negative rim arc). In particular, it is a simple consequence of the topology of rims that two arcs which belong to different rim components and each are incident to a bolt (not necessarily the same bolt) are conformal.

- e) **Arcs in two rims.** Suppose that G is of type (II), that two of its biconnected components, say G_1 and G_2 , are rims and that $e_i \in G_1$ whereas $e_j \in G_2$. Then any elementary vector that contains both e_i and e_j is of the form $f = c^1 - (\alpha c^1 / \alpha c^2) c^2$ where c^1 and c^2 are respectively positive rim cycles of G_1 and G_2 . Therefore, e_i and e_j are complements (resp., substitutes) if and only if they are both positive or both negative rim arcs (resp., one is a positive and the other a negative rim arc), and the above remark on arcs incident to bolts also holds.

It is important to notice at that point that in contrast with the unconstrained situation [GV85] where it is sufficient for two arcs to be incident to the same node in order to be substitutes or complements, here we also require the common node to be of degree 2 (unless the arcs belong to rims, as shown above). The following example illustrates the fact that the degree requirement is indeed necessary.

EXAMPLE 10 Consider the constrained graph $G = (N, E, \alpha)$ given in Figure 4.20 with the additional constraint $2x_1 + 3x_2 - 10x_6 = 2$. Then

$$f^1 = (e_1 + e_4 - e_3) + 2/5(e_1 + e_2 + e_6) = (7/5, 2/5, -1, 1, 0, 2/5)$$

$$f^2 = (e_1 + e_4 - e_3) - 2/3(e_2 + e_3 + e_5) = (1, -2/3, -5/3, 1, -2/3, 0)$$

are both elementary vectors. Now $f_1^1 f_1^2 > 0$, $f_1^2 f_2^2 < 0$ and thus e_1 and e_2 are neither substitutes nor complements although they share a node. \square

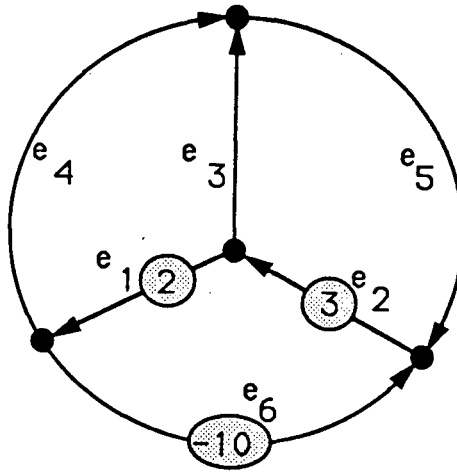


Figure 4.20: Arcs with common nodes need not be substitutes or complements

We now turn to the study of comparative flow changes for substitutes and complements.

4.4 The Monotonicity Theory

The main result of this section provides conditions under which the optimal flow in one arc is monotone in the parameter associated with another arc. We further establish bounds on the rate of change of the flow. Namely, we show how the rate of change of the flow is bounded by a function of the rate of change of the parameter vector t . We start by stating the extensions of two lemmas from Granot and Veinott [GV85] which are needed in the proof of our main result, the Monotone Optimal-Selection Theorem for constrained networks. Consider again the constrained graph $G = (N, E, \alpha)$ and the problem

$$\begin{aligned}
P(\alpha, \alpha_0, t) : \quad & \text{Min} \quad b(x, t) \\
& \text{s.t.} \quad Ax = d \\
& \alpha x = \alpha_0.
\end{aligned}$$

LEMMA 29 **Ascending Optimal-Arc-Flow Multifunction.** *In a dependent constrained graph $G = (N, E, \alpha)$, if T^0 is a chain whose elements differ only in one component $j \in E$, and b_j is subadditive⁶, then the j^{th} projection of $X(t)$, $\pi_j X(t)$, is ascending in t on T^0 .*

PROOF. Our proof is almost identical to the proof of Lemma 11 in [GV85] and is included for completeness. For $t \in T^0$, t_i fixed for all $i \neq j$, the minimum cost $B_j[x_j, t_j]$ over all constrained flows x feasible for $P(\alpha, \alpha_0, t)$ with given flow x_j in arc e_j can be expressed as $B_j[x_j, t_j] = b_j(x_j, t_j) + B'_j[x_j]$, where the second term is the minimum cost of flows in arcs other than e_j . Since $B_j[\cdot, \cdot]$ is subadditive in the variables (x_j, t_j) on $\mathbb{R} \times \pi_j T^0$ and the infimum of B on T^0 is finite, the result follows from the Monotone-Optimal-Selection Theorem of Lattice Programming [To78, Ve65]. \square

Now, if $b(\cdot, t)$ is strictly convex, then $X(t) = \{x(t)\}$ is a singleton and one can interpret the conclusion of Lemma 29 as stating that $x_j(t)$ is nondecreasing in t_j . In order to prove the monotonicity result for the case where not all arc costs are strictly convex we define for a parameter $\varepsilon \in [0, 1]^m$ the following perturbed problem

$$\begin{aligned}
P(\alpha, \alpha_0, t, \varepsilon) : \quad & \text{Min} \quad b(x, t, \varepsilon) \\
& \text{s.t.} \quad Ax = d \\
& \alpha x = \alpha_0.
\end{aligned}$$

Where

$$b(x, t, \varepsilon) = \sum_{e_i \in E} b_i(x_i, t_i, \varepsilon_i) \quad \text{and} \quad b_i(x_i, t_i, \varepsilon_i) = b_i(x_i, t_i) + \varepsilon_i e^{-x_i}. \quad (4.17)$$

⁶Recall that a function f on a lattice T is subadditive if for any two parameters t and t' , $f(t \wedge t') + f(t \vee t') \leq f(t) + f(t')$, where $t \wedge t'$ and $t \vee t'$ are respectively the meet (coordinatewise minimum) and join (coordinatewise maximum) of t and t' — This property is also referred to in the literature as submodularity.

Let $X(t, \varepsilon)$ be the set of optimal solutions of the above perturbed problem, and note that since $b(x, t, 0) = b(x, t)$, then $X(t, 0) = X(t)$.

Furthermore if f is a function from \mathfrak{R}^n into itself, we say that $f(x)$ has an iterated limit g as $x \rightarrow y$ in \mathfrak{R}^n , written $I\lim_{x \rightarrow y} f(x) = g$ if

$$\lim_{x_n \rightarrow y_n} \dots \lim_{x_2 \rightarrow y_2} \lim_{x_1 \rightarrow y_1} f(x) = g.$$

for some order of taking limits (if different orders yield different limits, then the function may have more than one iterated limit).

LEMMA 30 *Iterated Optimal Constrained Flow. In a dependent constrained graph $G = (N, E, \alpha)$, if $t \in T$, $b_j(\cdot, t_j)$ is convex and lower semicontinuous⁷ for each $e_j \in E$, and $X(t)$ is nonempty and bounded, there is a unique $x(t, \varepsilon) \in X(t, \varepsilon)$ for each $\varepsilon \gg 0$ (i.e., $\varepsilon_i > 0$ for all i) and $I\lim_{\varepsilon \downarrow 0} x(t, \varepsilon)$ exists and is in $X(t)$.*

PROOF. The proof is again similar to that of Granot and Veinott [GV85, Lemma 12] except that it uses Theorem 21 (the Ripple Theorem), and is included for completeness. We will show that $X(t, \varepsilon) \neq \emptyset$ and that the graph of the multifunction $X(t, \cdot)$ is compact on $[0, 1]^m$ for any given t . Since $P(\alpha, \alpha_0, t)$ is feasible let $x^0 \in X(t)$, then $b(x^0, t)$ is finite and so is $L \equiv b(x^0, t, 1)$. Moreover, since $b(x, t, \varepsilon)$ is nondecreasing in ε , the level set

$$\underline{X}(t, \varepsilon) \equiv \{x : Ax = d, \alpha x = \alpha_0, b(x, t, \varepsilon) \leq L\}$$

is nonempty, and $\underline{X}(t, \varepsilon) \subseteq \underline{X}(t, 0)$ for all $0 \leq \varepsilon \leq 1$. Now, since $b(\cdot, t)$ is lower semicontinuous and $X(t)$ is nonempty and bounded, then $X(t) = X(t, 0)$ is nonempty and compact. Thus, since the level sets of a lower semicontinuous convex function are all compact if one of them is nonempty and compact [Ro70], $\underline{X}(t, 0)$ is also nonempty and compact. By the above inclusion so is $\underline{X}(t, \varepsilon)$, and thus so is $X(t, \varepsilon)$. Moreover, since $b_j(\cdot, t_j)$ is convex and continuous on the interval on which it is finite, $b(\cdot, t, \cdot)$ is finite and continuous on the compact

⁷Recall that a real-valued convex function f on \mathfrak{R}^m is **lower semicontinuous** if and only if all its level sets $\{x : f(x) \leq a\}$, $a \in \mathfrak{R}$, are closed.

set $\underline{X}(t, 0) \times [0, 1]^m$, and it follows from an extension of [Be63] that for t fixed the graph of the multifunction of $\varepsilon \mapsto X(t, \varepsilon)^8 \subseteq \underline{X}(t, 0)$ is compact on $[0, 1]^m$.

Now, since $b(x, t, \varepsilon)$ is strictly convex on $\underline{X}(t, 0)$ for $\varepsilon \gg 0$, the perturbed problem has a unique solution, say $X(t, \varepsilon) = \{x(t, \varepsilon)\}$ for each ε . By the Ripple Theorem and treating ε as the parameter we have

$$|x_i(t, \varepsilon') - x_i(t, \varepsilon)| \leq M_{ji} |x_j(t, \varepsilon') - x_j(t, \varepsilon)| \quad (4.18)$$

for all arcs e_i and e_j and $0 \ll \varepsilon < \varepsilon' \leq 1$ differing only in the j^{th} component. Moreover, since $b(\cdot, t, \cdot)$ is subadditive then, by Lemma 29, $x_j(t, \varepsilon') - x_j(t, \varepsilon) \geq 0$, and (4.18) yields

$$|x_i(t, \varepsilon') - x_i(t, \varepsilon)| \leq M_{ji} (x_j(t, \varepsilon') - x_j(t, \varepsilon)). \quad (4.19)$$

We now show that $\lim_{\varepsilon \downarrow 0} x(t, \varepsilon)$ exists and is in $X(t, 0) = X(t)$. To this end let $\varepsilon^{(i)}$ be the vector obtained by replacing in ε components 1 through i by zero. We claim that $\lim_{\varepsilon_1 \downarrow 0} x(t, \varepsilon)$ exists and is in $X(t, \varepsilon^{(1)})$. Indeed by the above remark $x_1(t, \varepsilon)$ is nondecreasing in ε_1 , so that $\lim_{\varepsilon_1 \downarrow 0} x_1(t, \varepsilon)$ exists and is finite by the compactness of the graph of $X(t, \cdot)$. Now, by (4.19) and the compactness of the graph of $X(t, \cdot)$, $\lim_{\varepsilon_1 \downarrow 0} x(t, \varepsilon)$ exists and is finite for each $i \geq 2$, so $x(t, \varepsilon^{(1)}) \stackrel{\text{def}}{=} \lim_{\varepsilon_1 \downarrow 0} x(t, \varepsilon)$ exists and is in $X(t, \varepsilon^{(1)})$, proving our claim. By induction, given $x(t, \varepsilon^{(i-1)}) \in X(t, \varepsilon^{(i-1)})$ and the fact that (4.19) also holds if ε is replaced by $\varepsilon^{(i)}$, define $x(t, \varepsilon^{(i)}) \stackrel{\text{def}}{=} \lim_{\varepsilon_i \downarrow 0} x(t, \varepsilon^{(i-1)})$. By a similar argument as above one can show that this limit exists and is in $X(t, \varepsilon^{(i)})$. Lastly, (4.19) also holds if $\varepsilon = 0$, thus $x(t, 0) \stackrel{\text{def}}{=} x(t, \varepsilon^{(m)}) \in X(t, \varepsilon^{(m)}) = X(t, 0) = X(t)$ and the proof is complete. \square

Given an ordering of the arcs, the function $t \mapsto x(t, 0)$ is called an **Iterated Optimal-Constrained-Flow Selection**. Clearly, the selection may depend on the ordering of the arcs, and thus on the order of taking limits, as it does on the type of perturbation applied to the cost function — the perturbation proposed in (4.17) is clearly not the only way to make the costs strictly convex.

⁸That is, the multifunction $X(t, \cdot)$ of ε where t is kept constant

THEOREM 31 Monotone Optimal-Flow Selection. *In a dependent constrained graph $G = (N, E, \alpha)$ suppose that $b_j(\cdot, t_j)$ is convex and lower semicontinuous for each $t_j \in T_j$, $b_j(\cdot, \cdot)$ is subadditive for $e_j \in E$, and $X(t)$ is nonempty and bounded for each $t \in T$. Then there exists an iterated optimal-constrained-flow selection $t \mapsto x(t)$ with the Ripple Property and for which $x_j(t)$ is nondecreasing (resp., nonincreasing) in t_k whenever e_j and e_k are complements (resp., substitutes).*

PROOF. The proof is similar to the proof of Theorem 10 in [GV85]. First consider the case where the cost function $b(x, t)$ is strictly convex in x , and recall that in such a situation $X(t) = \{x(t)\}$ is a singleton, and $x(t, 0) = x(t)$. Now, suppose that t' is obtained from t by increasing t_k to t'_k . By Lemma 29 we obtain that $x_k(t') - x_k(t) \geq 0$, and since e_i and e_j are complements (resp., substitutes) then, by Lemma 25, $x_j(t') - x_j(t) \geq 0$ (resp., ≤ 0), completing the proof. We now look at the case where the arc-costs may not all be strictly convex. Then the perturbed costs $b(x, t, \varepsilon)$ are strictly convex, and thus from Lemma 30 there exists a unique solution $x(t, \varepsilon)$ that is optimal for the perturbed problem for each $t \in T$ and $\varepsilon \gg 0$. Now clearly from the above the result holds for $x(t, \varepsilon)$, $t \in T$. By taking the limit, the result will still hold for $x(t) = x(t, 0) = \text{Ilim}_{\varepsilon \downarrow 0} x(t, \varepsilon)$ which by Lemma 30 will have the desired properties, and the proof is complete⁹. \square

4.4.1 The Smoothing Theorem

The monotone optimal-flow selection theorem shows that changes in arc flows are monotone in parameter changes in certain other arcs. The Smoothing Theorem to follow presents qualitative results regarding the rate of change of the flows compared with changes in the parameters. Following the lines of Granot and Veinott [GV85], we start with two preliminary definitions.

The dual $g^\#$ of a function g of two variables from \Re^2 onto $\Re \cup \{+\infty\}$ is given by the function $g^\#(a, a') = g(a' - a, a')$. The function g is said to be doubly subadditive if both g

⁹Notice that if the costs are strictly convex, then $X(t)$ is the singleton $\{x(t)\}$, $x(t) = x(t, 0)$, and thus the notation is consistent.

and $g^\#$ are subadditive. For examples of doubly subadditive functions the reader is referred to [GV85, p. 489].

Two parameters t and t' of the parameter space T are said to be **monotonically step-connected** if there is a finite sequence $t^0 = t, t^1, \dots, t^k = t'$ such that t^r and t^{r+1} differ at most in one component for all r and t^r is coordinatewise monotone in r . In Lemma 32 we show that, under certain circumstances, a modification of an arc's parameter will not produce an optimal-flow change in this arc larger in magnitude than the variation of the parameter.

LEMMA 32 *In a dependent singly-constrained graph $G = (N, E, \alpha)$, if T^0 is a chain whose elements differ only in their j^{th} entry, $T_j \in \mathbb{R}$, $b_j^\#$ is subadditive and $b_i(\cdot, t_j)$ is convex for all $t \in T^0$ and all $i \neq j$, then the multifunction $t \mapsto t_j - \pi_j X(t)$ is ascending on T^0 .*

PROOF. For $t \in T^0$, let $y_j = t_j - x_j$, then $b_j(x_j, t_j) = b_j^\#(y_j, t_j)$. Therefore $B_j[x_j, t_j] = B_j^\#[y_j, t_j] = b_j^\#(y_j, t_j) + B_j'[t_j - x_j]$ is subadditive on $\mathbb{R} \times \pi_j T^0$ for $b_j^\#$ is subadditive, and $B_j'[\cdot]$ is convex by the Projection Theorem for convex functions. Now, the infimum of B on T^0 is finite, and the proof follows from the Monotone-Optimal-Selection Theorem of Lattice Programming. \square

THEOREM 33 Smoothing Theorem. *In a dependent constrained graph, suppose that $T_i \subseteq \mathbb{R}$, $b_i(\cdot, t)$ is convex and lower semicontinuous for each $t \in T_i$, that b_i is doubly subadditive for all $e_i \in E$, and that $X(t)$ is nonempty and bounded for each $t \in T$. Then there exists an iterated optimal-constrained-flow selection $x(\cdot)$ with the Ripple property and such that $x_i(t)$ and $M_{ji}t_j - x_i(t)$ (resp., $-M_{ji}t_j - x_i(t)$) are nondecreasing (resp., nonincreasing) in t_j whenever e_i and e_j are complements (resp., substitutes). Moreover, $\|x(t') - x(t)\|_\infty \leq M \|t' - t\|_1$ for all monotonically step-connected t and t' in T , where*

$$M = \max\{|f_i/f_j| : e_i, e_j \in E, f \in H_j(\alpha)\}$$

PROOF. The proof is similar to that of Theorem 15 in Granot and Veinott [GV85] and is included for completeness. Suppose that two arcs e_i and e_j are complements and that $t \mapsto x(t)$ is an iterated optimal selection then, by Theorem 31, $x_i(t)$ is nondecreasing in t_j . Moreover, by Lemmas 31 and 32, $t_j - x_j(t)$ is nondecreasing in t_j . Now, let $t \in T$ and let t' be obtained from t by augmenting its j^{th} component. From the Ripple Theorem (Theorem 21) and Lemma 25,

$$\begin{aligned} 0 &\leq x_i(t') - x_i(t) \leq M_{ji}(x_j(t') - x_j(t)) \\ \Rightarrow M_{ji}x_j(t) - x_i(t) &\leq M_{ji}x_j(t') - x_i(t') \end{aligned}$$

and thus, by the choice of t and t' , the quantity $M_{ji}x_j(t) - x_i(t)$ is also nondecreasing in t_j . Consequently

$$M_{ji}t_j - x_i(t) = M_{ji}(t_j - x_j(t)) + (M_{ji}x_j(t) - x_i(t))$$

is the sum of two quantities which are nondecreasing in t_j , and thus is nondecreasing in t_j . Now, if e_i and e_j are substitutes instead of being complements, then by inverting the direction of e_i and replacing $x_i(t)$ by $-x_i(t)$ the arcs become complements, the above conclusions hold, and the proof of the first part of the Theorem follows. To show the second part suppose that t and t' differ in only one component, say j . Then, from the Ripple Theorem (Theorem 21) and the definition of M

$$\begin{aligned} |x_i(t') - x_i(t)| &\leq M_{ji}|x_j(t') - x_j(t)| \leq M|x_j(t') - x_j(t)| \\ &\leq M|t'_j - t_j| \quad \text{by Theorem 29} \end{aligned} \quad (4.20)$$

for all $e_i \in E$. Next, consider the case where t and t' are monotonically step-connected. Then there exists a sequence $t^0 = t, t^1, \dots, t^k = t'$ such that $t^l - t^{l-1}$ has only one non-zero component for each l and t^l is coordinatewise monotone in l , and it follows from (4.20) that

$$\begin{aligned} \|x(t') - x(t)\|_\infty &\leq \sum_{\ell=1}^k \|x(t^\ell) - x(t^{\ell-1})\|_\infty \\ &\leq \sum_{\ell=1}^k M \|t^\ell - t^{\ell-1}\|_1 = M \|t' - t\|_1, \end{aligned}$$

completing the proof. \square

One should observe that in the particular case where e_i and e_j are complements (resp., substitutes)

$$M_{ij} = \text{Max} \quad \{f_j/f_i(\text{resp.}, -f_j/f_i) : f \in H(\alpha), f_i f_j \neq 0\}.$$

4.4.2 Subadditivity of Minimum Cost in the Parameters

In this section we present a result concerning the variations of the minimum cost $\underline{b}(t) = \sum_{j=1}^m b_j(x_j(t), t_j)$ where $x(t)$ is any optimal flow for $P(\alpha, \alpha_0, t)$. For any subset D of E , denote by x_D the sub-vector of x whose entries are indexed by D , and let Y, Z be the partition of E for which $x'_Y \ll x_Y$ and $x'_Z \geq x_Z$.

THEOREM 34 *Let C be a subset of E such that the arcs in C are all pairwise complements, and assume that $\times_{e_j \in E \setminus C} T_j = \{t_{E \setminus C}\}$, $b_j(\cdot, t_j)$ is convex for each $e_j \in E \setminus C$, T is a sublattice, b_C is subadditive and $\underline{b}(\cdot) > -\infty$ on T , then $\underline{b}(\cdot)$ is subadditive on T .*

PROOF. The proof is similar to that of Theorem 17 in Granot and Veinott [GV85] and is included for completeness. First notice that it suffices to show that, for any two feasible (not necessarily optimal) constrained flows x and x' and a pair of parameters (t, t') , $\underline{b}(t \wedge t') + \underline{b}(t \vee t') \leq b(x, t) + b(x', t')$. If either of the last two terms is $+\infty$ then the equality holds trivially, thus in the following we assume that all four terms are finite. Now, $x' - x$ is a sum of conformal elementary circulations, and if y is the sum of the conformal elementary circulations whose support contains an arc in Y and z the sum of the remaining elementary circulations, then y and z are conformal, $y_Y = x'_Y - x_Y$ and $z_Y = 0$. We now claim that $y_Z = 0$, and thus that $z_Z = x'_Z - x_Z$. Indeed, suppose that $y_j \neq 0$ for some $e_j \in Z$, then one of the elementary vectors f that constitute y verifies $f_i f_j \neq 0$ for some $e_i \in Y$. Since the arcs in C are complements, $f_i f_j > 0$, contradicting the fact that f is conformal with $x' - x$. Therefore, $x_C \wedge x'_C = x_C + y_C$ and $x_C \vee x'_C = x_C + z_C$. It now follows from our assumptions on the arc-costs that

$$\underline{b}(t \wedge t') + \underline{b}(t \vee t') \leq b(x + y, t \wedge t') + b(x + z, t \vee t') \leq b(x, t) + b(x', t'). \quad \square$$

The above result can be extended to include substitutes by noticing that two arcs e_i and e_j are substitutes if and only if they are complements in the constrained graph obtained by reversing e_i , replacing α_i by $-\alpha_i$ and $b_i(x_i, t_i)$ by $b_i(-x_i, t_i)$ (one then obtains a result similar to Theorem 19 in [GV85]).

□ □ □

Part III

Applications

Chapter 5

A Constrained Inventory–Production Model

Chapter Abstract

We discuss in this chapter some inventory–production models which can be formulated as nonlinear parametric network flow problems with one additional linear constraint. Although NP-Complete in general, several sensitivity analysis questions can be answered in polynomial time for the class of graphs that arises from this formulation. Special cases where little or no computation is required are studied. We then recommend a method by which a decision maker could understand qualitatively how to respond to changes in the environment such as machine breakdown, strike or variations in inventory carrying costs, and do so without additional computation.

□ □ □

5.1 Introduction

Qualitative sensitivity analysis for inventory–production models with convex costs using the concepts of substitutes and complements were first studied by Veinott [Ve84] for the single commodity case, then by Ciurria [Ci89] and Ciurria, Granot and Veinott [CGV89a] for the multicommodity case. We discuss in this chapter some inventory–production models which can be formulated as nonlinear parametric network flow problems with one additional linear constraint. Several products are manufactured on parallel machines over an horizon of several time periods. The products may be sold or kept as inventory, in quantities which are related by an additional constraint. We will address different questions a manager may be faced with when responding to changes in the environment. These changes can often

be translated into parameter changes in the arc-cost functions. We would like to know the effect(s) of such changes on the decision variables. Where are these effects the strongest? The weakest? Are some variables unaffected? Less affected than others? Do the variables that are affected increase or decrease in value? In order to answer these and other questions, we apply the concepts and definitions from Chapter 4 to the specific constrained graph which underlies the inventory-production model.

The chapter is organized as follows. In Section 5.2 we show how certain inventory models can be stated as (singly-constrained) network flow problems and formulate the Singly-Constrained Multi-Period Multi-Product Inventory-Production Problem as a PSCMF. In Section 5.3 we then study this model in its general form, that is, without making any assumptions on the side constraint, and present some computational aspects of sensitivity analysis, as well as an illustrative example. In Sections 5.4 and 5.5 we present some special classes of constrained subgraphs for which the above computations are unnecessary and sensitivity analysis questions have immediate answers, at least for certain subsets of variables. Examples of such cases are provided.

5.2 The Multi-Product Multi-Period Model

Using a model similar to that of Veinott [Ve84] for the Single-Product Unconstrained case, we formulate in this section the constrained single-item multi-period inventory-production model as a singly-constrained network flow problem. We will adopt the following notation.

n	Number of periods in the planning horizon.
p	Number of Products bought or produced.
x_i^l	Amount of product l produced or ordered in period i (Negative values of x_i^l signify disposal of goods).
y_j^l	Amount of product l stored at the end of period j (Negative values of y_j^l signify backlogged demand).
d_i^l	Market demand to be satisfied by product l in period i .
t_i^l	Parameter associated with the variable x_i^l .
τ_j^l	Parameter associated with the variable y_j^l .
$g_i^l(x_i^l, t_i^l)$	Cost associated with a production level x_i^l of product l in period i .
$h_j^l(y_j^l, \tau_j^l)$	Cost associated with an inventory/backlogging level y_j^l of product l at the end of period j .

where $l = 1, \dots, p$, $i = 1, \dots, n$ and $j = 1, \dots, n - 1$. The constraints of the single-item multi-period inventory-production problem reflect the inventory balance at the end of each period. They can be stated as

$$x_i^l + y_{i-1}^l - y_i^l = d_i^l \quad i = 1, \dots, n \quad (5.21)$$

with the convention that $y_0^l = y_n^l = 0$ (If the situation calls for a positive inventory κ^l at the end of the planning horizon, then replace d_n^l by $d_n^l + \kappa^l$; similarly, starting inventory is subtracted from d_1^l which may then be positive or negative.). Figure 5.21 illustrates the underlying graph in the case where $p = 3$ and $n = 5$. Each production arc joins the **production node** 0 to a **consumption node**, whereas each inventory arc joins a node i to a node $i + 1$. The graph G is composed of p copies of the single product graph (obtained for $p = 1$) called **product subgraphs** and denoted by G^1, \dots, G^p . Note that this formulation will allow for bounds on the variables x_i^l and y_j^l . In order to see this, define the **step-function** μ by

$$\mu(z) = \begin{cases} 0 & \text{if } z \leq 0 \\ +\infty & \text{if } z > 0. \end{cases}$$

If we add now the term $\mu(x_i^l - t_i^l)$ to, say, the cost of production of item l in period i , the model will not allow for a production of more than t_i^l units of that product in the i^{th} period. Similarly, adding to the inventory cost in period i the term $\mu(-y_i^l - \tau_i^l)$ will limit backlogging in period i to τ_i^l (if $\tau_i^l \geq 0$).

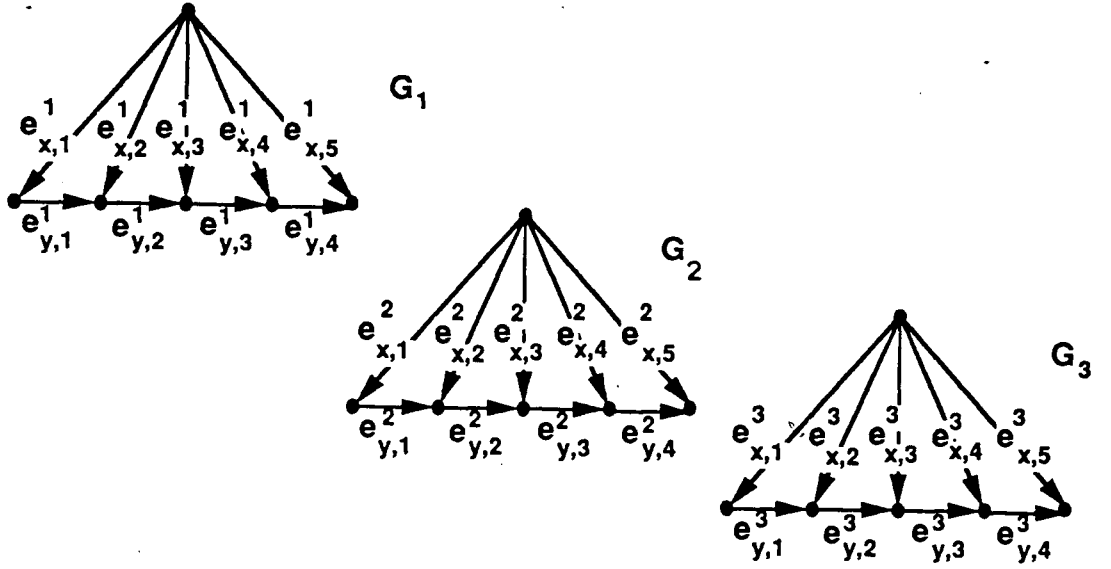


Figure 5.21: The basic Inventory-Production Network
($p = 3$ and $n = 5$)

The Multi-Product Multi-Period Inventory-Production Problem is to find a schedule $(x, y) = (x_1^1, \dots, x_n^p, y_1^1, \dots, y_{n-1}^p)$ which, given a parameter $t = (t_1^1, \dots, t_n^p, \tau_1^1, \dots, \tau_{n-1}^p)$ in a lattice T , minimizes the sum of production and inventory costs. If in addition to the balance equation constraints (5.21) a side constraint which binds the production and inventory levels of some of the p products is added, the problem can be formulated as follows

$$\begin{aligned}
\min \quad & \sum_{l=1}^p \sum_{i=1}^n g_i^l(x_i^l, t_i^l) + \sum_{l=1}^p \sum_{i=1}^{n-1} h_i^l(y_i^l, \tau_i^l) \\
\text{s.t.} \quad & x_i^l + y_{i-1}^l - y_i^l = d_i^l \quad \forall i, l \\
& \sum_{l=1}^p \sum_{i=1}^n \alpha_i^l x_i^l + \sum_{l=1}^p \sum_{i=1}^{n-1} \beta_i^l y_i^l = \alpha_0.
\end{aligned} \tag{5.22}$$

The **side constraint** $\sum_{l=1}^p \sum_{i=1}^n \alpha_i^l x_i^l + \sum_{l=1}^p \sum_{i=1}^{n-1} \beta_i^l y_i^l = \alpha_0$ may express a variety of economic or physical realities. Some examples of possible constraints with their interpretations are

$$\begin{aligned}
\text{Fixed production budget} \quad & \sum_{l=1}^p \sum_{i=1}^n \alpha_i^l x_i^l = \alpha_0 \quad \alpha_i^l > 0 \quad \forall i \\
\text{Fixed total inventory} \quad & \sum_{l=1}^p \sum_{i=1}^{n-1} y_i^l = \alpha_0 \\
\text{Fixed total inventory cost} \quad & \sum_{l=1}^p \sum_{i=1}^{n-1} \beta_i^l y_i^l = \alpha_0 \quad \beta_i^l > 0 \quad \forall i.
\end{aligned}$$

A complete analysis of the **Single-Product Unconstrained Case** where $l = 1$ and the side constraint is absent was given by Veinott [Ve84] where it was shown that every pair of arcs are **conformal**, that is, either substitutes or complements. Indeed, in an unconstrained graph, two arcs are complements (resp., substitutes) if and only if any simple cycle orients them in the same (resp., opposite) direction(s) [GV85]. These results are summarized in Table 5.1 where, for simplicity, we denote the production (resp., inventory) arc for period i by x_i^1 (resp., y_i^1). The reader will notice in Table 5.1 that:

- All production arcs are **substitutes** for they are incident to the same node.
- All inventory arcs are **complements**.
- A Production arc x_i^1 and an inventory arc y_j^1 are **complements** if $i \leq j$ and **substitutes** otherwise.

	x_i^1 $i < j$	x_j^1	x_k^1 $k > j$	y_i^1 $i < j$	y_k^1 $k \geq j$
x_j^1	S	C	S	S	C
y_j^1	C	C	S	C	C

Table 5.1: Complements and Substitutes in the Unconstrained Single-Product Model
C: Complements, S: Substitutes

5.3 Substitutes and Complements in The General Case

In this section we study the Singly-Constrained Multi-Product Multi-Period Production-Inventory model in its general form, that is, without making any assumptions on the side constraint. Recall from Chapter 4 that the definitions of complements and substitutes, the Ripple coefficients, and the partial orders, all depend on the set of elementary vectors of the circulation space. As seen before, the following problems

- (i) Given a pair of arcs, determine whether they are substitutes, complements or neither.
- (ii) Given a pair of arcs, say e_j and e_k , determine the corresponding Ripple coefficient M_{jk} .
- (iii) Given three arcs e_i , e_j and e_k , determine whether or not $e_i \leq_j^q e_k$

are NP-Hard. However, in cases where the number of elementary vectors is bounded by a polynomial in the number of arcs, a polynomial time algorithm may emerge. Here, due to the special structure of the graph underlying the production-inventory problem discussed here, we can enumerate the elementary vectors of the circulation space, and thus answer the above questions, in polynomial time. Recall from Section 3.2, equation(3.3), that the elementary vectors are simple neutral cycles or are of the form $f = c^1 - (\alpha c^1 / \alpha c^2) c^2$, where c^1 and c^2 are simple active cycles that share at most either a single node or a simple path. For convenience we will denote by $c^l(j, k)$, where $1 \leq j < k \leq n$ the unique cycle composed of the arcs $x_j^l, y_j^l, \dots, y_{k-1}^l$ and x_k^l (backward) and define

$$\alpha^l(j, k) \stackrel{\text{def}}{=} \alpha c^l(j, k) = \alpha_j^l - \alpha_k^l + \sum_{i=j}^{k-1} \beta_i^l.$$

Clearly, all simple cycles of G are of that form. The elementary vectors are either simple neutral cycles, that is, $f = c^l(j, k)$ where $\alpha^l(j, k) = 0$, or are of the form

$$f = c^l(j, k) - (\alpha^l(j, k)/\alpha^m(j', k'))c^m(j', k') \quad (5.23)$$

with $\alpha^l(j, k) \neq 0$, $\alpha^m(j', k') \neq 0$, $1 \leq j < k \leq n$, $1 < j' < k' \leq n$ if $l \neq m$, and $1 \leq j < k \leq j' < k' \leq n$ if $l = m$ (if $k = j'$ and $\alpha^l(j, k) = \alpha^l(j', k')$, then $f = c^l(j, k')$ is a neutral cycle.). Also recall from Section 4.3 that two arcs e_i and e_j are complements (resp., substitutes) if and only if for all elementary vectors f , $f_i f_j \geq 0$ (resp., ≤ 0). The above leads to the following results.

5.3.1 Pairs of Arcs Corresponding to the Same Product

In this subsection we deal with pairs of decision variables that correspond to a same product, say l . for the sake of convenience, we drop the product superscript l in the remainder of the subsection.

LEMMA 35 *Substitutes and Complements in the Same Product Subgraph. Suppose that two arcs in the same product subgraph are conformal (that is, either substitutes or complements). Then, they are complements (resp., substitutes) if and only if they are so in the one-product unconstrained model, as described in Table 5.1.*

PROOF. The proof follows from the fact that the set of projections of supports of elementary vectors of the constrained graph on each product subgraph includes the set of simple cycles of that particular subgraph. \square

It remains to determine whether two arcs in the same product subgraph are conformal or not. For example, it follows from Lemma 35 that two inventory arcs, say y_i and y_j , may either not be conformal or be complements. Thus, they will be conformal unless we can find

an elementary vector that orients them in opposite directions. Clearly, the elementary vector in question, if it exists, consists of two active cycles, say c^1 and c^2 , which verify $c_{y_i}^1 = c_{y_j}^2 = 1$ and $(\alpha c^1)(\alpha c^2) > 0$. Assuming without loss of generality that $i < j$, then $c^1 = c(r, s)$ and $c^2 = c(u, v)$ (where $1 \leq r < s \leq u < v \leq n$; the reader may verify that this is the only way to obtain elementary vectors) may be chosen so that s is as small as possible and u as large as possible. If $\alpha c^1 > 0$ and $\alpha c^2 > 0$, then we denote the index of the last inventory arc of c^1 by $L^\oplus(i) = s - 1$ and that of the first inventory arc of c^2 by $U^\oplus(j) = u$, whereas if $\alpha c^1 < 0$ and $\alpha c^2 < 0$, then $L^\ominus(i) = s - 1$ and $U^\ominus(j) = u$ (for a numerical example, see Example 11). Similarly, let $\bar{L}^\oplus(i)$, $\bar{L}^\ominus(i)$ (resp., $\bar{U}^\oplus(i)$, $\bar{U}^\ominus(i)$) be the left-most (resp., right-most) indices of the right (resp., left) production arcs of cycles c^1 (resp., c^2) which verifies $c_{x_i}^1 = 1$ (resp., $c_{x_i}^2 = 1$) and $\alpha c^1 > 0$, $\alpha c^1 < 0$ (resp., $\alpha c^2 > 0$, $\alpha c^2 < 0$). For $j < k$, notice that

$$\alpha c(j, k) = \alpha(j, k) = \sum_{i=j}^{k-1} \gamma_i$$

where $\gamma_i \stackrel{\text{def}}{=} \alpha_i + \beta_i - \alpha_{i+1}$.

We show now how to determine the values $L^\oplus(i)$, $L^\ominus(i)$, $U^\oplus(i)$, $U^\ominus(i)$, $\bar{L}^\oplus(i)$, $\bar{L}^\ominus(i)$, $\bar{U}^\oplus(i)$ and $\bar{U}^\ominus(i)$. To that end, we also define the quantities $l^\oplus(i)$, $l^\ominus(i)$, $u^\oplus(i)$ and $u^\ominus(i)$ as follows.

$$l^\oplus(i) \text{ maximizes } \sum_{k=l^\oplus(i)}^i \gamma_k \text{ over } 1 \leq l^\oplus(i) \leq i.$$

$$l^\ominus(i) \text{ minimizes } \sum_{k=l^\ominus(i)}^i \gamma_k \text{ over } 1 \leq l^\ominus(i) \leq i.$$

$$L^\oplus(i) = \min\{+\infty, m : \sum_{k=l^\oplus(i)}^m \gamma_k > 0, i \leq m \leq n-1\}.$$

$$L^\ominus(i) = \min\{+\infty, m : \sum_{k=l^\ominus(i)}^m \gamma_k < 0, i \leq m \leq n-1\}.$$

$$u^\oplus(i) \text{ maximizes } \sum_{k=i}^{u^\oplus(i)} \gamma_k \text{ over } i \leq u^\oplus(i) \leq n-1.$$

$$u^\ominus(i) \text{ minimizes } \sum_{k=i}^{u^\ominus(i)} \gamma_k \text{ over } i \leq u^\ominus(i) \leq n-1.$$

$$U^\oplus(i) = \max\{-\infty, m : \sum_{k=m}^{u^\oplus(i)} \gamma_k > 0, 0 \leq m \leq i\}.$$

$$U^\ominus(i) = \max\{-\infty, m : \sum_{k=m}^{u^\ominus(i)} \gamma_k < 0, 0 \leq m \leq i\}.$$

$$\bar{L}^\oplus(i) = \begin{cases} i & \text{if } i \geq 2 \text{ and } L^\ominus(i-1) = i-1 \\ \min\{+\infty, m : \sum_{k=i}^{m-1} \gamma_k > 0, i < m \leq n\} & \text{otherwise.} \end{cases}$$

$$\bar{L}^\ominus(i) = \begin{cases} i & \text{if } i \geq 2 \text{ and } L^\oplus(i-1) = i-1 \\ \min\{+\infty, m : \sum_{k=i}^{m-1} \gamma_k < 0, i < m \leq n\} & \text{otherwise.} \end{cases}$$

$$\bar{U}^\oplus(i) = \begin{cases} i & \text{if } i \leq n-1 \text{ and } U^\oplus(i) = i \\ \max\{-\infty, m : \sum_{k=m}^{i-1} \gamma_k < 0, 1 \leq m < i\} & \text{otherwise.} \end{cases}$$

$$\bar{U}^\ominus(i) = \begin{cases} i & \text{if } i \leq n-1 \text{ and } U^\ominus(i) = i \\ \max\{-\infty, m : \sum_{k=m}^{i-1} \gamma_k > 0, 1 \leq m < i\} & \text{otherwise.} \end{cases}$$

With the convention that an empty sum is equal to zero. It is then possible to show the following.

THEOREM 36 Substitutes and Complements in a Product Subgraph.

(i) Two inventory arcs y_i, y_j ($i < j$) are complements if and only if

$$L^\oplus(i) \geq U^\oplus(j) \text{ and } L^\ominus(i) \geq U^\ominus(j).$$

(ii) Two production arcs x_i and x_j ($i < j$) are substitutes unless at least one of the following holds

(ii-a) $\bar{L}^\oplus(i) < \bar{U}^\ominus(j)$ or $\bar{L}^\ominus(i) < \bar{U}^\oplus(j)$

(ii-b) $\bar{L}^\oplus(i) = \bar{U}^\ominus(j) = i_0$ and

- $i < i_0 < j$, or
- $i_0 = i$ and $0 < -\alpha(k, i) < \alpha(i, j)$ for some $k < i$, or
- $i_0 = j$ and $0 < -\alpha(j, k') < \alpha(i, j)$ for some $k' > j$.

(ii-c) $\bar{L}^\ominus(i) = \bar{U}^\oplus(j) = j_0$ and

- $i < j_0 < j$, or
- $j_0 = i$ and $0 < \alpha(k, i) < -\alpha(i, j)$ for some $k < i$, or
- $j_0 = j$ and $0 < \alpha(j, k') < -\alpha(i, j)$ for some $k' > j$.

(iii) A production arc x_i and an inventory arc y_j ($i \leq j$) are complements unless at least one of the following holds.

(iii-a) $\bar{L}^\oplus(i) < U^\oplus(j)$ or $\bar{L}^\ominus(i) < U^\ominus(j)$.

(iii-b) $\bar{L}^\oplus(i) = U^\oplus(j) = i_0$ and

- $i_0 \neq i$, or
- $i_0 = i$ and $0 < \alpha(i, k') < -\alpha(l^\ominus(i-1), i)$ for some $k' > j$.

(iii-c) $\bar{L}^\ominus(i) = U^\ominus(j) = j_0$ and

- $j_0 \neq i$, or
- $j_0 = i$ and $0 < -\alpha(i, k') < \alpha(l^\oplus(i-1), i)$ for some $k' > j$.

(iv) An inventory arc y_i and a production arc x_j ($i < j$) are substitutes unless at least one of the following holds

(iv-a) $L^\oplus(i) + 1 < \bar{U}^\ominus(j)$ or $L^\ominus(i) + 1 < \bar{U}^\oplus(j)$.

(iv-b) $L^\oplus(i) + 1 = \bar{U}^\ominus(j) = i_0$ and

- $i_0 \neq j$, or
- $i_0 = j$ and $0 < -\alpha(j, k') < \alpha(l^\oplus(j-1), j)$ for some $k' > j$.

(iv-c) $L^\ominus(i) + 1 = \bar{U}^\oplus(j) = j_0$ and

- $j_0 \neq j$, or
- $j_0 = j$ and $0 < \alpha(j, k') < -\alpha(l^\ominus(j-1), j)$ for some $k' > j$.

No other pairs of arcs may be substitutes or complements.

PROOF. First, it follows from the above definitions that $U^\oplus(i) \leq i \leq L^\oplus(i)$ and $U^\ominus(i) \leq i \leq L^\ominus(i)$. Moreover, one can check that all cycles c that contain a given inventory arc y_i and verify $\alpha c > 0$ (resp., < 0) also contain both $y_{L^\oplus(i)}$ and $y_{U^\oplus(i)}$ (resp., $y_{L^\ominus(i)}$ and $y_{U^\ominus(i)}$). Similarly, all cycles that contain a given production arc x_i as a forward arc and verify $\alpha c > 0$ (resp., < 0) lie entirely to the left of $x_{L^\oplus(i)}$ (resp., $x_{L^\ominus(i)}$) and to the right of $x_{U^\oplus(i)}$ (resp., $x_{U^\ominus(i)}$). In order to prove (i), recall from Lemma 35 that two inventory arcs y_i, y_j ($i < j$) may only be complements, and will be so if and only if no elementary vector of the form (5.23) orients them in opposite directions. This will hold if and only if there does not exist a pair of active cycles c^1 and c^2 that contain respectively y_i and y_j and such that all inventory arcs of c^1 lie strictly to the left of all inventory arcs of c^2 . From the above remark, such a pair will exist if and only if $L^\oplus(i) < U^\oplus(j)$ or $L^\ominus(i) < U^\ominus(j)$, completing the proof of (i).

In order to show (ii), consider now two production arcs x_i and x_j ($i < j$), and attempt to exhibit an elementary vector f which verifies $f_{x_i} f_{x_j} > 0$. For similar arguments as above, this is possible if $\bar{L}^\oplus(i) < \bar{U}^\ominus(j)$ or if $\bar{L}^\ominus(i) < \bar{U}^\oplus(j)$, and impossible if $\bar{L}^\oplus(i) > \bar{U}^\ominus(j)$ and $\bar{L}^\ominus(i) > \bar{U}^\oplus(j)$, verifying (ii-a). Suppose now that $\bar{L}^\oplus(i) = \bar{U}^\ominus(j) = i_0$, where $i \leq i_0 \leq j$. If f exists, then it will be of the form $f = c^1 - (\alpha c^1 / \alpha c^2) c^2$ where the cycles c^1 and c^2 share the arc x_{i_0} , $c_{x_i}^1 = c_{x_j}^2 = 1$, $\alpha c^1 > 0$ and $\alpha c^2 < 0$. It remains to verify that

$$f_{x_i} f_{x_j} = [1 - (\alpha c^1 / \alpha c^2) c_{x_i}^2] [c_{x_j}^1 - (\alpha c^1 / \alpha c^2)] > 0. \quad (5.24)$$

One is lead to consider the following three cases.

- If $i < i_0 < j$, then $c^1 = c(i, i_0)$, $c^2 = -c(i_0, j)$, $c_{x_i}^2 = c_{x_j}^1 = 0$, thus it follows from (5.24) $f_{x_i} f_{x_j} = -(\alpha c^1 / \alpha c^2) > 0$, and x_i and x_j are not conformal.
- If $i = i_0 < j$, then $c^1 = -c(k, i)$ for some $k < i$, $c^2 = -c(i, j)$, (where $\alpha(k, i) < 0$ and $\alpha(i, j) > 0$) and $c_{x_i}^2 = -1$, $c_{x_j}^1 = 0$. Thus $f_{x_i} f_{x_j} = [1 + \alpha(k, i) / \alpha(i, j)][-\alpha(k, i) / \alpha(i, j)]$ has the sign $\alpha(k, i) + \alpha(i, j)$ and thus may be positive if $0 < -\alpha(k, i) < \alpha(i, j)$ for some $k < i$ (from the definition of $\bar{L}^\oplus(i)$, $i_0 = i \geq 2$).
- If $i < i_0 = j$, then $c^1 = c(i, j)$ and $c^2 = c(j, k')$ for some $k' > j$ (where $\alpha(i, j) > 0$ and $\alpha(j, k') < 0$). Therefore $c_{x_i}^2 = 0$, $c_{x_j}^1 = -1$, and $f_{x_i} f_{x_j} = [-1 - \alpha(i, j) / \alpha(j, k')]$ has the sign of $\alpha(i, j) + \alpha(j, k')$ and thus may be positive if $0 < -\alpha(j, k') < \alpha(i, j)$ for some $k' > j$ (from the definition of $\bar{U}^\oplus(j)$, $i_0 = j \leq n - 1$).

A similar argument holds if $\bar{L}^\ominus(i) = \bar{U}^\oplus(j)$, where α is replaced by $-\alpha$.

To show (iii), observe that since a production arc x_i and an inventory arc y_j ($i \leq j$) may only be complements, we attempt to find an elementary vector f for which $f_{x_i} f_{y_j} < 0$, which will be possible if $\bar{L}^\oplus(i) < U^\oplus(j)$ or if $\bar{L}^\ominus(i) < U^\ominus(j)$, and impossible if $\bar{L}^\oplus(i) > U^\oplus(j)$ and $\bar{L}^\ominus(i) > U^\ominus(j)$. Suppose now that $\bar{L}^\oplus(i) = U^\oplus(j) = i_0$, where $i \leq i_0 \leq j$. If f exists, then $f = c^1 - (\alpha c^1 / \alpha c^2) c^2$ where c^1 and c^2 share the arc x_{i_0} , $c_{x_i}^1 = c_{y_j}^2 = 1$, $\alpha c^1 > 0$ and $\alpha c^2 > 0$. It remains to verify that

$$f_{x_i} f_{y_j} = [1 - (\alpha c^1 / \alpha c^2) c_{x_i}^2][c_{y_j}^1 - (\alpha c^1 / \alpha c^2)] < 0. \quad (5.25)$$

- If $i < i_0 \leq j$, then $c^1 = c(i, i_0)$, $c^2 = c(i_0, k')$ where $k' > j$, $c_{y_j}^1 = c_{x_i}^2 = 0$, thus $f_{x_i} f_{y_j} = -(\alpha c^1 / \alpha c^2) > 0$, and x_i and y_j are not conformal.
- If $i = i_0 \leq j$, then $c^1 = -c(k, i)$ and $c^2 = c(i, k')$ for some $k < i$ and $k' > j$, (where $\alpha(k, i) < 0$ and $\alpha(i, k') > 0$). Therefore $c_{x_i}^2 = 1$ and $c_{y_j}^1 = 0$ and $f_{x_i} f_{y_j} = [1 + \alpha(k, i) / \alpha(i, k')][\alpha(k, i) / \alpha(i, k')]$ has the sign of $-(\alpha(i, k') + \alpha(k, i))$ and

thus may be negative if $0 < \alpha(i, k') < -\alpha(k, i)$ for some $k < i$ and $k' > j$ or, equivalently (from the definition of $\bar{L}^\oplus(i)$, $i_0 = i \geq 2$), if $0 < \alpha(i, k') < -\alpha(l^\oplus(i-1), i)$ for some $k' > j$.

A similar argument holds if $\bar{L}^\oplus(i) = U^\oplus(j)$.

The proof of (iv) follows from similar considerations. Since an inventory arc y_i and a production arc x_j ($i < j$) may only be substitutes, we attempt to find an elementary vector f for which $f_{y_i}, f_{x_j} > 0$, which is possible if $L^\oplus(i) + 1 < \bar{U}^\oplus(j)$ (recall that the head of arc $L^\oplus(i)$ is the node $L^\oplus(i) + 1$) or if $L^\oplus(i) + 1 < \bar{U}^\oplus(j)$, and impossible if $L^\oplus(i) \geq \bar{U}^\oplus(j)$ and $L^\oplus(i) \geq \bar{U}^\oplus(j)$. Suppose now that $L^\oplus(i) + 1 = \bar{U}^\oplus(j) = i_0$, where $i + 1 \leq i_0 \leq j$. If f exists, then $f = c^1 - (\alpha c^1 / \alpha c^2) c^2$ where c^1 and c^2 share the arc x_{i_0} , $c_{y_i}^1 = c_{x_j}^2 = 1$, $\alpha c^1 > 0$ and $\alpha c^2 < 0$. It remains to verify that

$$f_{y_i}, f_{x_j} = [1 - (\alpha c^1 / \alpha c^2) c_{y_i}^2][c_{x_j}^1 - (\alpha c^1 / \alpha c^2)] > 0. \quad (5.26)$$

- If $i + 1 \leq i_0 < j$, then $c^1 = c(k, i_0)$ where $k \leq i$, $c^2 = -c(i_0, j)$, $c_{x_j}^1 = c_{y_i}^2 = 0$, thus $f_{x_i}, f_{y_j} = -(\alpha c^1 / \alpha c^2) > 0$, and y_i and x_j are not conformal.
- If $i + 1 \leq i_0 = j$, then $c^1 = c(k, j)$ and $c^2 = c(j, k')$ for some $k \leq i$ and $k' > j$, (where $\alpha(k, j) > 0$ and $\alpha(j, k') < 0$). Therefore $c_{x_j}^1 = -1$ and $c_{y_i}^2 = 0$ and $f_{x_i}, f_{y_j} = -1 - \alpha(k, j) / \alpha(j, k')$ has the sign $\alpha(k, j) + \alpha(j, k')$ and thus may be positive if $0 < -\alpha(j, k') < \alpha(k, j)$ for some $k \leq i$ and $k' > j$ or, equivalently, if $0 < -\alpha(j, k') < \alpha(l^\oplus(j-1), j)$ for some $k' > j$.

A similar argument holds if $L^\oplus(i) + 1 = \bar{U}^\oplus(j)$. Finally, the last statement is a direct consequence of Lemma 35. \square

5.3.2 Pairs of Arcs Corresponding to Two Different Products

In the following theorem, we provide necessary and sufficient conditions for a given pair of arcs to be substitutes or complements when they correspond to variables in two product

subgraphs. First, recall that an arc is a **positive** (resp., **negative**) arc if and only if every cycle c that contains it as a forward arc verifies $\alpha c \geq 0$ (resp., ≤ 0). In our case,

- A **production arc** x_j^i is a positive (resp., negative) arc if and only if $\alpha^i(i, j) \leq 0$ (resp., ≥ 0) and $\alpha^i(j, k) \geq 0$ (resp., ≤ 0) for all $1 \leq i \leq j \leq k \leq n$.
- An **inventory arc** y_j^i is a positive (resp., negative) arc if and only if $\alpha^i(i, k) \geq 0$ (resp., ≤ 0) for all $1 \leq i \leq j \leq k \leq n$.

THEOREM 37 **Substitutes and Complements Across Two Product Subgraphs.**

Two arcs from two different product subgraphs are complements (resp., substitutes) if and only if one is a positive arc and the other a negative arc (resp., both are positive arcs or both are negative arcs). Moreover, a production arc x_i^j is a positive (resp., negative) arc if and only if $\bar{L}^\ominus(i) = +\infty$ and $\bar{U}^\ominus(i) = -\infty$ (resp., $\bar{L}^\oplus(i) = +\infty$ and $\bar{U}^\oplus(i) = -\infty$), whereas an inventory arc y_i^j is a positive (resp., negative) arc if and only if $L^\ominus(i) = +\infty$ and $U^\ominus(i) = -\infty$ (resp., $L^\oplus(i) = +\infty$ and $U^\oplus(i) = -\infty$).

PROOF. First suppose that some arc e_i is neither a positive nor a negative arc, that is, there exists two active cycles c^1 and c^2 that contain it as a forward arc and such that $\alpha c^1 > 0$ and $\alpha c^2 < 0$. Now, given any other arc e_j in some other product subgraph and any active cycle c that contains that arc as a forward arc (assume without loss of generality that $\alpha c > 0$), one may consider the elementary vectors $f^1 = c^1 - (\alpha c^1 / \alpha c) c$ and $f^2 = c^2 - (\alpha c^2 / \alpha c) c$. Therefore there exist two elementary vectors which verify $f_i^1 f_j^1 < 0$ and $f_i^2 f_j^2 > 0$ and e_i and e_j are neither substitutes nor complements. Moreover, from the definition of substitutes and complements (4.15) and (5.23), a positive and a negative (resp., two positive or two negative) arcs in different product subgraphs will be complements (resp., substitutes). The proof of the second part of the theorem follows directly from the definitions of $\bar{L}^\ominus(i)$, $\bar{U}^\ominus(i)$, $\bar{L}^\oplus(i)$, $\bar{U}^\oplus(i)$, $L^\ominus(i)$, $U^\ominus(i)$, $L^\oplus(i)$ and $U^\oplus(i)$. \square

5.3.3 Computational Complexity

It follows from the above that in order to obtain all pairs of substitutes and complements it suffices to obtain the quantities $l^\oplus(i)$, $l^\ominus(i)$, $L^\oplus(i)$, $L^\ominus(i)$, $u^\oplus(i)$, $u^\ominus(i)$, $U^\oplus(i)$, $U^\ominus(i)$, $\bar{L}^\oplus(i)$, $\bar{L}^\ominus(i)$, $\bar{U}^\oplus(i)$ and $\bar{U}^\ominus(i)$. These may be calculated using the following recursion equations:

$$\begin{aligned}
 l^\oplus(i+1) &= \begin{cases} l^\oplus(i) & \text{if } \sum_{k=l^\oplus(i)}^i \gamma_k > 0 \\ i+1 & \text{otherwise} \end{cases} \\
 l^\ominus(i+1) &= \begin{cases} l^\ominus(i) & \text{if } \sum_{k=l^\ominus(i)}^i \gamma_k < 0 \\ i+1 & \text{otherwise} \end{cases} \\
 u^\oplus(i-1) &= \begin{cases} u^\oplus(i) & \text{if } \sum_{k=i}^{u^\oplus(i)} \gamma_k > 0 \\ i-1 & \text{otherwise} \end{cases} \\
 u^\ominus(i-1) &= \begin{cases} u^\ominus(i) & \text{if } \sum_{k=i}^{u^\ominus(i)} \gamma_k < 0 \\ i-1 & \text{otherwise.} \end{cases}
 \end{aligned}$$

Thus the quantities $l^\oplus(i)$, $l^\ominus(i)$, $L^\oplus(i)$, $L^\ominus(i)$, $u^\oplus(i)$, $u^\ominus(i)$, $i = 1, \dots, n-1$ can be obtained in $O(n)$ for each one of the p product subgraphs. Moreover, if $L^\oplus(i) \geq i+1$ (resp., $L^\ominus(i) \geq i+1$) then $L^\oplus(i+1) = L^\oplus(i)$ (resp., $L^\ominus(i+1) = L^\ominus(i)$), and if $U^\oplus(i) \leq i-1$ (resp., $U^\ominus(i) \leq i-1$) then $U^\oplus(i-1) = U^\oplus(i)$ (resp., $U^\ominus(i-1) = U^\ominus(i)$). One may thus obtain all the pairs of arcs which are either substitutes or complements in at most $O(n^2p)$ elementary operations.

EXAMPLE 11 *In this example, we illustrate the concepts of substitutes and complements for a production-inventory problem where the number of periods is $n = 6$, and the side constraint is of the form*

$$\begin{aligned}
 &2x_1^1 + 3x_2^1 + 5x_3^1 + 7x_4^1 + 11x_5^1 + 13x_6^1 - 9y_1^1 + 14y_2^1 + 7y_3^1 + 0y_4^1 + 17y_5^1 \\
 &+ 2x_1^2 + 3x_2^2 + 5x_3^2 + 7x_4^2 + 11x_5^2 + 13x_6^2 - 9y_1^2 + 14y_2^2 + 7y_3^2 + 0y_4^2 + 17y_5^2 \\
 &\quad + \sum_{l=3}^p \sum_{i=1}^6 \alpha_i^l x_i^l + \sum_{l=3}^p \sum_{i=1}^5 \beta_i^l y_i^l = \alpha_0.
 \end{aligned}$$

Where for products $l = 3, \dots, p$, α_i^l , $i = 1, \dots, 6$, β_i^l , $i = 1, \dots, 5$, and α_0 can be any real numbers. The above quantities for products 1 or 2 as well as pairs of arcs which are substitutes or complements are given in Table 5.2, resulting in the pairs of substitutes and complements given in Tables 5.3 and 5.4.

i	1	2	3	4	5	6
γ_i	-10	12	5	-4	15	
$l^\oplus(i)$	1	2	2	2	2	
$l^\ominus(i)$	1	1	3	4	4	
$L^\oplus(i)$	2	2	3	4	5	
$L^\ominus(i)$	1	$+\infty$	$+\infty$	4	$+\infty$	
$u^\oplus(i)$	5	5	5	5	5	
$u^\ominus(i)$	1	2	4	4	5	
$U^\oplus(i)$	1	2	3	4	5	
$U^\ominus(i)$	1	$-\infty$	$-\infty$	4	$-\infty$	
$L^\oplus(i)$	3	2	4	6	5	$+\infty$
$L^\ominus(i)$	2	$+\infty$	3	4	5	6
$U^\oplus(i)$	1	2	3	4	5	$-\infty$
$U^\ominus(i)$	1	$-\infty$	2	4	3	5

Table 5.2: Substitutes and Complements: Computations

Positive arcs: x_2, y_2, y_3, y_5 Negative arcs: x_6

Now, A manager may be interested in the qualitative effect of variations in, for instance, production cost or inventory capacity on various variables in the model. A reduction in production cost in, say, the first period and the first product, can be modeled by setting

$$g_1^1(x_1^1, K', K'') = K(x_1^1) + K'x_1^1 + K'' \quad (5.27)$$

where $K'x_1^1$ is the variable (linear) production cost of producing product 1 in the first period, K'' may be a setup cost, and $K(x_1^1)$ represents the (convex) remainder of the cost function, possibly including upper and/or lower bounds. The functions $g_1^1(\cdot, \cdot, K'')$ and $g_1^1(\cdot, K', \cdot)$ thus defined are subadditive, and are convex in x_1^1 if so is $K(x_1^1)$. Moreover, it is assumed that all other costs involved are also convex in the flow variables. In this event an increase in K' (or in K'') will have the following effects on the optimal production-inventory schedule.

	x_1^l	x_2^l	x_3^l	x_4^l	x_5^l	x_6^l	y_1^l	y_2^l	y_3^l	y_4^l	y_5^l
x_1^l	C	S					C	C			
x_2^l	S	C					S	C			
x_3^l			C					S	C		
x_4^l				C		S			S		C
x_5^l					C				S	S	C
x_6^l				S		C					S
y_1^l	C	S					C	C			
y_2^l	C	C	S				C	C			
y_3^l			C	S	S				C		
y_4^l					S					C	
y_5^l				C	C	S					C

Table 5.3: Substitutes and Complements, $l = 1$ or 2
C: Complements, S: Substitutes, " ": Neither

	x_1^2	x_2^2	x_3^2	x_4^2	x_5^2	x_6^2	y_1^2	y_2^2	y_3^2	y_4^2	y_5^2
x_1^1											
x_2^1		S				C		S	S		S
x_3^1											
x_4^1											
x_5^1											
x_6^1		C				S		C	C		C
y_1^1											
y_2^1		S				C		S	S		S
y_3^1		S				C		S	S		S
y_4^1											
y_5^1		S				C		S	S		S

Table 5.4: Substitutes and Complements: across products 1 and 2
C: Complements, S: Substitutes, " ": Neither

- (i) A decrease (Monotone Optimal-Flow Selection Theorem) in the production level x_1^1 of product 1 in the first period.
- (ii) An increase (Monotone Optimal-Flow Selection Theorem, x_1^1 and x_2^1 are substitutes) in the production level x_2^1 of product 1 in the second period.
- (iii) A decrease (Monotone Optimal-Flow Selection Theorem, x_1^1 and y_1^1 (resp., y_2^1) are complements) in the inventory levels y_1^1 and y_2^1 of product 1 in the first and second periods

and no other predictions may be made without supplementary data. For similar reasons, an increase in production costs for product 1 in the second period will cause decreases in x_2^1 and y_2^1 and increases in x_1^1 and y_1^1 , as well as decreases in x_6^2 and increases in x_2^2 and y_2^2 , y_3^2 and y_5^2 , whereas an increase in production costs for product 1 in the fourth period will cause decreases in x_4^1 and y_5^1 and increases in x_6^1 and y_3^1 .

Suppose now that a decision maker is interested in the effect of variations in inventory capacity in, say, the first period and the first product. Such a change can be modeled by expressing the inventory cost in the first period for the first product as having the form

$$h_1^1(y_1^1, U) = K'(y_1^1) + \mu(y_1^1 - U),$$

where the function K' includes fixed and variable costs. The function $h_1^1(y_1^1, U)$ thus defined is subadditive, and $h_1^1(\cdot, U)$ is convex if so is K' . Thus a decrease in the inventory capacity U will have the following effects.

- (i) A decrease (Monotone Optimal-Flow Selection Theorem) in the inventory level y_1^1 of product 1 in period 1.
- (ii) A decrease (Monotone Optimal-Flow Selection Theorem, y_1^1 and x_1^1 (resp., y_2^1) are complements) in the production level x_1^1 of product 1 in period 1 and in the inventory level y_2^1 in period 2.

- (iii) *An increase (Monotone Optimal-Flow Selection Theorem, y_1^1 and x_2^1 are substitutes) in the production level x_2^1 of product 1 in period 2.*

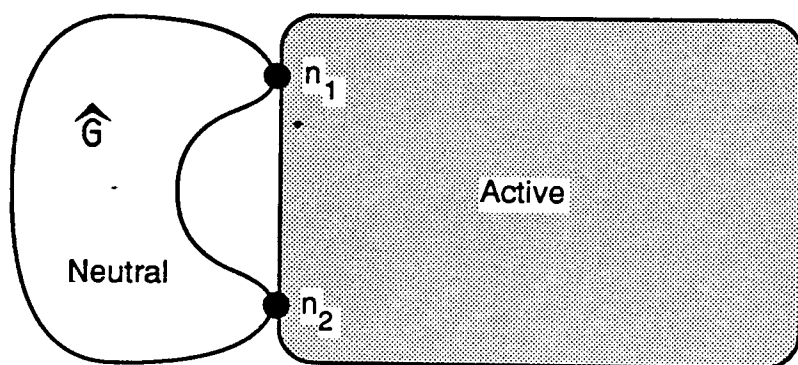
Similarly, a decrease in inventory capacity in the third period for the first product will cause a decrease in x_3^1 and y_3^1 , and an increase in x_4^1 and x_5^1 . \square

5.4 Narrow constraints and pseudo-rims

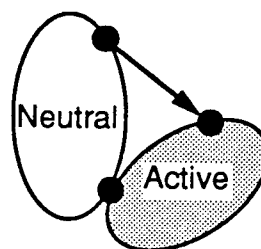
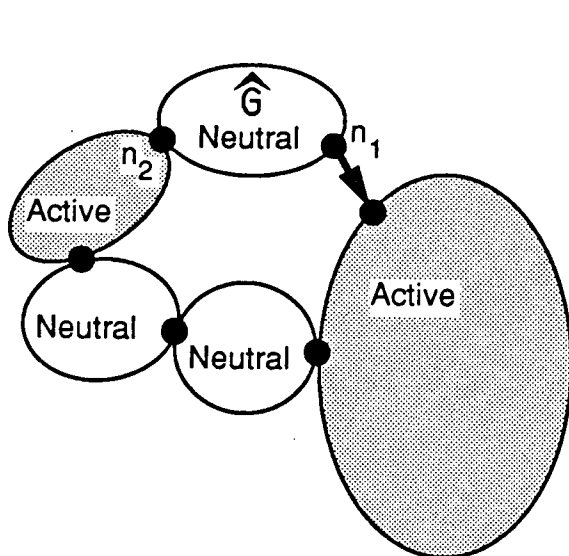
In order to further illustrate the theory of substitutes and complements developed in Chapter 4, we now turn to a special case where the constraint coefficients are concentrated in one part of the graph, that is, when the side constraint involves at most a small number of consecutive periods. In practice, such a situation may happen if the limitations reflected by the constraint are localized in time. It will be seen that, for this special case, it is easy to determine certain pairs of substitutes and complements without computation. We start with definitions of a special class of constrained graphs and the properties thereof, and state two results that will then be applied to this specific case. Note that, even though we develop this theory to apply it to problem (5.22), it also applies to a much larger class of Singly-Constrained Monotropic Network Flow Problems.

5.4.1 Pseudo-rims and their properties

We call a graph G a **pseudo-rim** if and only if it contains a biconnected neutral subgraph, say \hat{G} , called a **pseudo-component**, which only intersects with the rest of G at two nodes, called the **pseudo-bolts** relative to \hat{G} . Note that a pseudo-rim may have one or several pseudo-components, and that these may be attached to one or several of the biconnected components of G . Some of the possible combinations are given in Figure 5.22 (note that a rim is a special kind of pseudo-rim, whose bolts and components are the pseudo-bolts and pseudo-components).



A pseudo-rim with one pseudo-rim component



A pseudo-rim with two biconnected components and five pseudo-rim components

Figure 5.22: A Pseudo-Rim

The interest in pseudo-rims is motivated by the following two results, which follow directly from the definition of pseudo-rims and from the types of elementary vectors in singly-constrained graphs.

LEMMA 38 **Projection of Elementary vectors on pseudo-components.** *The projection of any given elementary vector of a pseudo-rim onto one of its pseudo-components, if nonzero, is either a simple neutral cycle or a simple path that joins the two corresponding pseudo-bolts.*

PROOF. The proof is straightforward, but is included for completeness. Let f be any elementary vector in G that contains at least one arc in \hat{G} . If f is a simple cycle it either lies entirely in the pseudo-component in question, say \hat{G} , in which case no further proof is required, or else, since \hat{G} only shares two nodes with the rest of $G \setminus \hat{G}$, f is composed of two simple paths between the corresponding pseudo-bolts, say n_1 and n_2 , one of which lies entirely inside \hat{G} , and the other, in the remainder of G , and the proof is complete (see Figure 5.23-a). Suppose now that f is of the form $f = c^1 - (\alpha c^1 / \alpha c^2) c^2$, where c^1 and c^2 are simple active cycles that share at most either a single node or a simple path, and assume without loss of generality that c^1 contains at least one arc in the pseudo-component in question. Since c^1 is active, then it may not lie entirely inside \hat{G} which is neutral, and thus we may write c^1 as $p^1 - q^1$, where p^1 (resp., q^1) is a simple path in \hat{G} (resp., in $G \setminus \hat{G}$) that joins pseudo-bolts n_1 to n_2 (see Figure 5.23-b). If c^2 contains no arcs in \hat{G} , then the projection of f on \hat{G} is p^1 , and the proof is complete. Otherwise, we may write in a similar fashion $c^2 = p^2 - q^2$, where p^2 (resp., q^2) is a simple path in \hat{G} (resp., in $G \setminus \hat{G}$) that joins pseudo-bolts n_1 to n_2 . Clearly c^1 and c^2 then share at least one of p^1 or q^1 . In the first case, $p^1 = p^2$ (see Figure 5.23-b), and the proof follows, whereas if $q^1 = q^2$, then

$$\begin{aligned}
f &= c^1 - (\alpha c^1 / \alpha c^2) c^2 \\
&= (p^1 - q^1) - (\alpha(p^1 - q^1) / \alpha(p^2 - q^2))(p^2 - q^2) \\
&= (p^1 - q^1) - (\alpha q^1 / \alpha q^2)(p^2 - q^2) \\
&= (p^1 - q^1) - (p^2 - q^1) = p^1 - p^2
\end{aligned}$$

which lies in \hat{G} . Thus f contains no active cycles, yielding a contradiction and completing the proof. \square

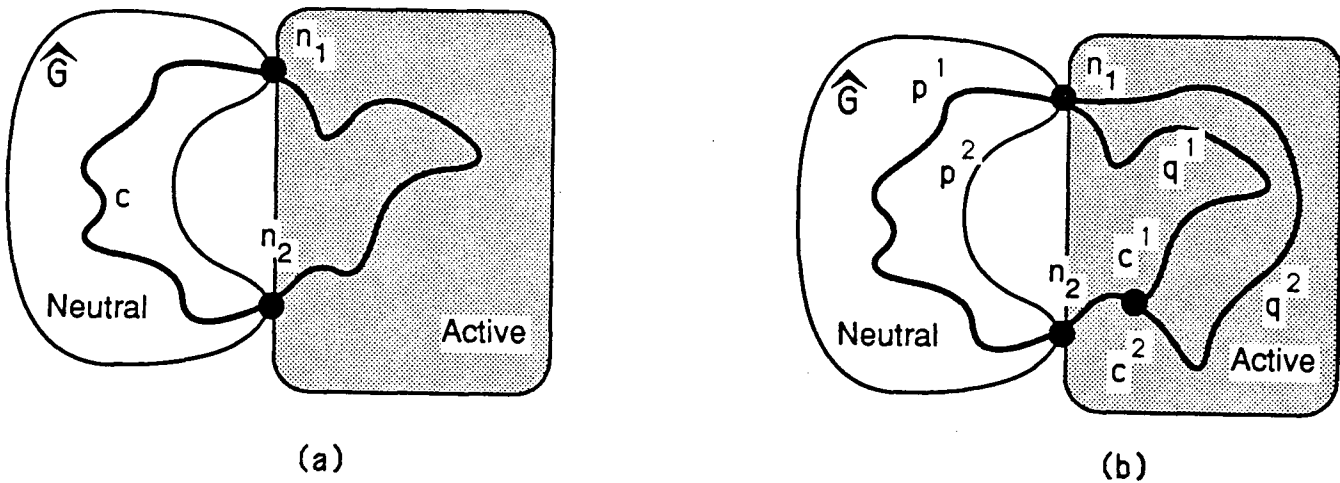


Figure 5.23: Projection of elementary vectors in pseudo-rims

THEOREM 39 Sensitivity Analysis in Pseudo-Components. *Let G be a pseudo-rim, \hat{G} (one of) its pseudo-component(s), and n_1 and n_2 the corresponding pseudo-bolts. Consider the unconstrained graph $G' = \hat{G} \cup \{e_0\}$ where $e_0 = (n_2, n_1)$. Then for any given triplet of arcs e_i , e_j and e_k in \hat{G} , the following holds.*

- (i) e_i and e_j are complements (resp., substitutes) in G if and only if they are complements (resp., substitutes) in G' .
- (ii) $M_{ij} = 1$.

- (iii) $e_i \leq_j^d e_k$ is equivalent to $e_i \leq_j^q e_k$, and holds if and only if a similar relation holds in G' .

PROOF. The proof follows directly from Lemma 38. \square

Thus, as far as arcs that lie inside a pseudo–component are concerned, the theory developed by Granot and Veinott [GV85] holds. Notice however that if the arcs lie in different pseudo–components, these results do not apply.

5.4.2 Application to Narrow Constraints

In this subsection we now apply the above concepts and results to certain classes of multi–period multi–product inventory problems. Pseudo–rims will occur in the subgraph corresponding to product l , $l = 1, \dots, p$, if either one or both of the following hold.

- There exists a period index $k_0 \geq 2$ such that $\alpha_i^l = 0$ for all $i \leq k_0$ and $\beta_i^l = 0$ for all $i \leq k_0 - 1$ (or, more generally, if $\gamma_k^l = 0$ for all $k < k_0$). The two (left) pseudo–components then correspond to the arc sets $\{x_i^l, i \leq k_0, y_j^l, i \leq k_0 - 1\}$ and $\{y_{k_0}^l\}$.
- There exists a period index $k'_0 \leq n - 1$ such that $\alpha_i^l = \beta_i^l = 0$ for all $i \geq k'_0$ (or, more generally, if $\gamma_k^l = 0$ for all $k \geq k'_0$). The two (right) pseudo–components then correspond to the arc sets $\{x_i^l, y_i^l, i \geq k'_0\}$ and $\{y_{k'_0-1}^l\}$ (see Figure 5.24).

From Lemma 38 and Theorem 39, we may conclude that

- (i) inventory arcs in the same pseudo–component are complements.
- (ii) production arcs in the same pseudo–component are substitutes.
- (iii) A production arc x_i^l and an inventory arc y_j^l that lie in the same pseudo–component are complements (resp., substitutes) if $i \leq j$ (resp., $j < i$).

Notice that each of the p product subgraphs may either not be a pseudo–rim or may have one or two pseudo–component(s). We now give two examples of narrow constraints and provide the corresponding tableaux of relations between the arcs and the parameters.

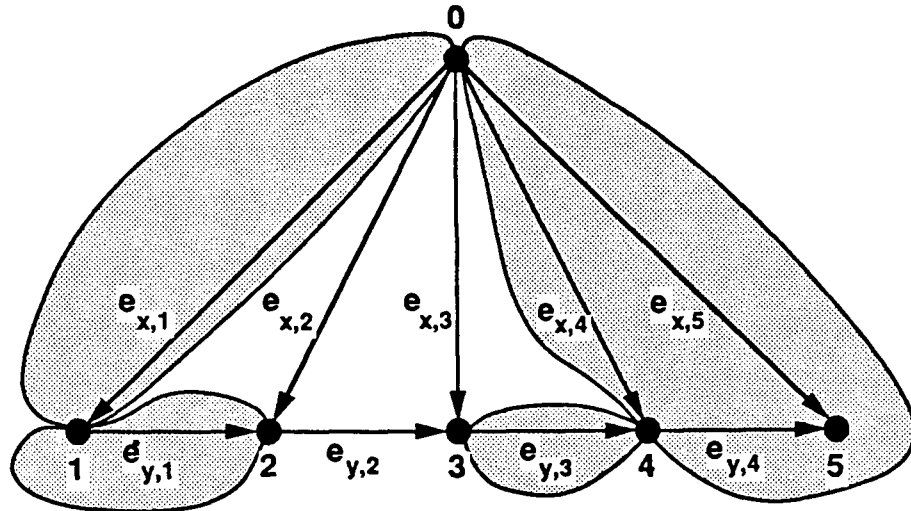


Figure 5.24: A pseudo-rim with four pseudo-components:
The l^{th} product subgraph ($k_0 = 1, k'_0 = 4$)

EXAMPLE 12 First Quarter Budget.

Consider a model with p products and six periods, and suppose that there is a fixed initial operating budget (production and inventory) on the cost of operation in the first three periods. The following side constraint will reflect this fact.

$$\sum_{l=1}^p \sum_{i=1}^3 \alpha_i^l x_i^l + \sum_{l=1}^p \sum_{i=1}^3 \beta_i^l y_i^l = \alpha_0 \quad (5.28)$$

where $\alpha_i^l \geq 0$ is the unit operating cost of producing item l in period i , $\beta_i^l \geq 0$ is the unit inventory operating cost for item l in period i , $i = 1, \dots, 3$, and α_0 is the startup budget. Using Theorem 39 and without further considerations (in particular regardless of the magnitude of the nonzero entries of α), one obtains the table of substitutes and complements given in Table 5.5 (Clearly, more arcs may be substitutes or complements. However, determining these requires the full knowledge of the side constraint and the computations described in Section 5.3.). Moreover, for all pairs of arcs e, e' for which Table 5.5 provides information, the Ripple coefficient $M_{ee'}$ is equal to 1.

Now, suppose that one is interested in the effect of A reduction in production cost in, say, the last (sixth) period and the first product. Such a reduction can be modeled as in

	x_1^l	x_2^l	x_3^l	x_4^l	x_5^l	x_6^l	y_1^l	y_2^l	y_3^l	y_4^l	y_5^l
x_1^l											
x_2^l											
x_3^l											
x_4^l				C	S	S			S	C	C
x_5^l				S	C	S			S	S	C
x_6^l				S	S	C			S	S	S
y_1^l											
y_2^l											
y_3^l				S	S	S			C	C	C
y_4^l				C	S	S			C	C	C
y_5^l				C	C	S			C	C	C

Table 5.5: Substitutes and Complements ($l = 1, \dots, p$)

Example 11, equation (5.27) by setting

$$g_6^1(x_6^1, K', K'') = K(x_6^1) + K'x_6^1 + K'' \quad (5.29)$$

where $K(\cdot)$ and all other costs involved are convex in the flow variables. In this event an increase in K' (or in K'') will have the following effects on the optimal production-inventory schedule.

- (i) A decrease (Monotone Optimal-Flow Selection Theorem) of, say, δ units (where $\delta \geq 0$) in the production level x_6^1 of product 1 in the last period.
- (ii) Increases (Monotone Optimal-Flow Selection Theorem, x_6^1 and x_4^1 (resp., x_5^1) are substitutes) of at most δ units (Ripple property) in the production levels x_4^1 and x_5^1 of product 1 in the fourth and fifth periods. •
- (iii) Increases (Monotone Optimal-Flow Selection Theorem, x_6^1 and y_4^1 (resp., y_5^1) are substitutes) of at most δ units (Ripple property) in the inventory levels y_4^1 and y_5^1 of product 1 in the fourth and fifth periods.

and no further predictions may be made without supplementary data. For similar reasons, an increase in production costs for product 1 in the fifth period will cause a decrease of, say,

δ units (where $\delta \geq 0$) x_3^1 and decreases in y_5^1 and increases in x_4^1 , x_6^1 , and y_4^1 , all of at most δ in magnitude. No further conclusion may be drawn at this point. \square

EXAMPLE 13 Shared Machine.

In this example, suppose that the number of periods is again $n = 6$ and that the same machine is used in the production of item 1 in periods 1 and 2, then it is switched to the production of item 2 in periods 3 and 4, and lastly to that of item 3 in periods 5 and 6. The constraint on total machine time (or capacity) can then be expressed as

$$\alpha_1^1 x_1^1 + \alpha_2^1 x_2^1 + \alpha_3^2 x_3^2 + \alpha_4^2 x_4^2 + \alpha_5^3 x_5^3 + \alpha_6^3 x_6^3 = \alpha_0$$

Using Theorem 39 results in the pairs of substitutes and complements given in Table 5.6.

This time, assume that there is a reduction in production cost for the first product in, the last period. Such a reduction can be modeled as in Example 11, by defining the cost function associated with arc x_6^1 as in (5.29). Under similar convexity assumptions, an increase in K' (or in K'') will have the following effects on the optimal production-inventory schedule.

- (i) A decrease (Monotone Optimal-Flow Selection Theorem) of, say, δ units (where $\delta \geq 0$) in the production level x_6^1 of product 1 in the last period.
- (ii) Increases (Monotone Optimal-Flow Selection Theorem, x_6^1 and x_3^1 (resp., x_4^1 , x_5^1) are substitutes) of at most δ units (Ripple property) in the production levels x_3^1 , x_4^1 and x_5^1 of product 1 in the third, fourth and fifth periods.
- (iii) Increases (Monotone Optimal-Flow Selection Theorem, x_6^1 and y_3^1 (resp., y_4^1 , y_5^1) are substitutes) of at most δ units (Ripple property) in the inventory levels y_2^1 , y_3^1 , y_4^1 and y_5^1 of product 1 in the third, fourth and fifth periods

and no further predictions may be made without supplementary data. \square

First Product											
	x_1^1	x_2^1	x_3^1	x_4^1	x_5^1	x_6^1	y_1^1	y_2^1	y_3^1	y_4^1	y_5^1
x_1^1											
x_2^1											
x_3^1			C	S	S	S		S	C	C	C
x_4^1			S	C	S	S		S	S	C	C
x_5^1			S	S	C	S		S	S	S	C
x_6^1			S	S	S	C		S	S	S	S
y_1^1											
y_2^1			S	S	S	S		C	C	C	C
y_3^1			C	S	S	S		C	C	C	C
y_4^1			C	C	S	S		C	C	C	C
y_5^1			C	C	C	S		C	C	C	C
Second Product											
	x_1^2	x_2^2	y_1^2	y_2^2	x_3^2	x_4^2	y_3^2	x_5^2	x_6^2	y_4^2	y_5^2
x_1^2	C	S	C	C							
x_2^2	S	C	S	C							
y_1^2	C	S	C	C							
y_2^2	C	S	C	C							
x_3^2											
x_4^2											
y_3^2											
x_5^2								C	S	S	C
x_6^2								S	C	S	S
y_4^2								S	S	C	C
y_5^2								C	S	C	C
Third Product											
	x_1^3	x_2^3	x_3^3	x_4^3	x_5^3	x_6^3	y_1^3	y_2^3	y_3^3	y_4^3	y_5^3
x_1^3	C	S	S	S			C	C	C	C	
x_2^3	S	C	S	S			S	C	C	C	
x_3^3	S	S	C	S			S	S	C	C	
x_4^3	S	S	S	C			S	S	S	C	
x_5^3											
x_6^3											
y_1^3	C	S	S	S			C	C	C	C	
y_2^3	C	C	S	S			C	C	C	C	
y_3^3	C	C	C	S			C	C	C	C	
y_4^3	C	C	C	C			C	C	C	C	
y_5^3											

Table 5.6: Substitutes and Complements
 (No further information available without numerical information
 on the side constraint coefficients)

5.4.3 Recognition of pseudo-rims

As was shown in Chapter 3, recognizing whether a given constrained graph is a rim can be achieved in linear time. It is thus natural to ask the same question of pseudo-rims. One way to answer this is by trying all possible pairs of nodes, test whether their removal augments the number of biconnected components of G (by at least one), and whether at least one of the newly created biconnected components is neutral. Since testing for the neutrality of a subgraph can be done in linear time (see Chapter 3), the running time of the whole procedure is a polynomial of degree 3.

5.5 First Period Constraints

We will focus in this section on a another special case where the constraint involves solely quantities related to the first period, that is, $\alpha_i^l = \beta_i^l = 0 \quad \forall i \geq 2$. Such examples will arise if, for instance, resources are limited in the first period. Indeed, a manager may be especially concerned with implementing startup conditions on one or several of the manufacturing chains. For example, a **manpower requirement** in the first period limited to the production of items $1, \dots, p_0$ (where $2 \leq p_0 \leq p$) will be expressed by a side constraint of the form $\sum_{l=1}^{p_0} \alpha_1^l x_1^l = \alpha_0$ where $\alpha_1^l > 0$ is the per-unit work-hour required for the production of item l in the first period and α_0 is the total number of work-hours to be assigned in the first period. In a similar scenario, a plant manager may wish to achieve a \$-value objective on the first period production. Such a constraint can be translated as a side constraint of the form $\sum_{l=1}^p \alpha_1^l x_1^l = \alpha_0$ where $\alpha_1^l \geq 0$ is the \$-value of item l produced (in period 1) and α_0 is the first period objective. In a third example, one may wish to fix the operating expenses (production and inventory) in the first period to meet a limited startup budget, resulting with an additional constraint of the form $\sum_{l=1}^p \alpha_1^l x_1^l + \sum_{l=1}^p \beta_1^l y_1^l = \alpha_0$ where $\alpha_1^l \geq 0$ is the unit operating cost of producing item l in period 1, $\beta_1^l \geq 0$ is the unit inventory operating cost for item l in period 1, and α_0 is the startup budget.

5.5.1 Dependency

It is easy to verify that the examples given above have the common property that each one of the product subgraphs either is neutral or is a rim. Indeed, G^l is a neutral subgraph if and only if $\alpha_i^l + \beta_i^l = 0$. Each such neutral subgraphs is a **dependent component** of G , which is independent of the rest of the graph, and thus the qualitative theory developed by Granot and Veinott [GV85] for unconstrained network flow problems applies. In particular, the coefficients M_{jk} introduced in the Ripple Theorem are identically equal to 1, and the arcs are all either substitutes or complements as described in Table 5.1.

Furthermore, recall that any parametric modification of the cost structure in one dependent component will have no effect on the optimal solution in another dependent component. For that reason in the remainder of this section, we restrict ourselves to product subgraphs which are active and thus are rims, and G is a dependent graph¹. It is easy to verify that the **rim components** of G^l correspond to the arc-sets $\{x_1^l\}$, $\{y_1^l\}$, $\{x_2^l, \dots, x_n^l, y_1^l, \dots, y_{n-1}^l\}$ and its bolts are the nodes 0, 1 and 2 (see Figure 5.25).

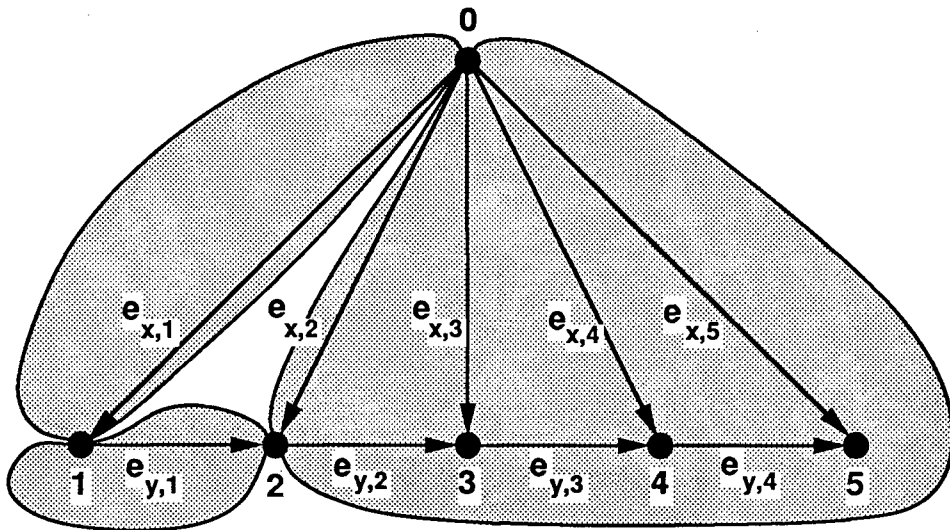


Figure 5.25: The subgraph G^l is a rim with three rim components

¹ It was shown in Chapter 3 that if only one of the product subgraphs is active, and therefore a rim, then the problem is equivalent to an unconstrained one. The theory of Granot and Veinott [GV85] then applies.

5.5.2 Positive Arcs, Complements and Substitutes

Recall from Section 4.3 that two positive or two negative (resp., one positive and one negative) rim arcs in a rim subgraph are complements (resp., substitutes), whereas two positive or two negative (resp., a positive and a negative) rim arcs from two different rim subgraphs are substitutes (resp., complements). Under the above assumptions on the signs of the entries of the vector α , it is obvious that the arcs x_1^l and y_1^l are positive arcs, and any positive rim cycle c in G^l contains both x_1^l and y_1^l as forward arcs and verifies $\alpha c = \alpha_1^l + \beta_1^l > 0$. Incidentally, the elementary vectors are either neutral cycles in a product subgraph, that is simple cycles that do not contain the arcs x_1^l and y_1^l , or are of the form

$$f = c[l] - (\alpha_1^l + \beta_1^l) / (\alpha_1^m + \beta_1^m) c[m] \quad (5.30)$$

where $c[l]$ (resp., $c[m]$) is a simple cycle of G^l (resp., G^m) that contains x_1^l (resp., x_1^m) as a forward arc. A brief look at the structure of the product subgraphs will convince the reader that all inventory arcs are also positive arcs, whereas all but the first production arcs are negative arcs. The following theorem is a direct application of the above and of equation (5.30).

THEOREM 40 Substitutes and Complements in Rims.

- *Two arcs e and e' in a same product subgraph (rim) are complements (resp., substitutes) if and only if they are complements (resp., substitutes) in the unconstrained version of that rim. Moreover, the corresponding Ripple coefficient is given by $M_{ee'} = 1$.*
- *Two arcs e and e' that lie in different product subgraphs are complements (resp., substitutes) if and only if they are substitutes (resp., complements) when placed in the corresponding position but in the same rim, and that rim taken as an unconstrained graph. Moreover, the corresponding coefficient is given by*

$$M_{ee'} = M(l, m) \stackrel{\text{def}}{=} \frac{\alpha_1^l + \beta_1^l}{\alpha_1^m + \beta_1^m}$$

Where e (resp., e') belongs to G^l (resp., G^m). \square

The reader will notice that *every pair of variables is either substitute or complement* (see Table 5.7), and thus the qualitative theory developed in Chapter 4 is most informative. In particular we will show in subsection 5.5.4 that, given an increase in the parameter of one arc, predictions can be made regarding the direction of optimal flow changes in every other variable in the model.

	x_i^1 $i < j$	x_j^1	x_k^1 $k > j$	y_i^1 $i < j$	y_k^1 $k \geq j$
x_j^1	S [1]	C [1]	S [1]	S [1]	C [1]
y_j^1	C [1]	C [1]	S [1]	C [1]	C [1]

	$x_1^{l \neq 1}$	$x_i^{l \neq 1}$ $i \geq 2$	$y_i^{l \neq 1}$
$x_{j=1}^1$	S [$M(1, l)$]	C [$M(1, l)$]	S [$M(1, l)$]
$x_{j \geq 2}^1$	C [$M(1, l)$]	S [$M(1, l)$]	C [$M(1, l)$]
y_j^1	S [$M(1, l)$]	C [$M(1, l)$]	S [$M(1, l)$]

Table 5.7: First Period Example:
Substitutes, Complements and Ripple Coefficients
C: Complements, S: Substitutes, [Ripple Coefficients]

5.5.3 Less-dependent-on Relations

Other examples of questions we would like to address are the following. Given that a parameter is changed, can one be assured that the variation in a given variable will be less than (some constant times) the variation in a second variable? Is it possible to limit the effect of such a change by imposing restrictions on a (small) subset of variables? Is it possible that some variable would vary more than the one whose parameter is being changed? In order to answer the above questions, we will appeal to the two *Less-Dependent-on* relations. Indeed, the D-Ripple property states that if $e_i \leq_j^d e_k$ then the optimal flow variation in arc e_i is at most, in absolute value, M_{ki} times the magnitude of the optimal flow variation in e_k , while the Q-Ripple property, states that the magnitude of the optimal flow change in

a given arc e_k does not increase the less dependent qualitatively on e_j this arc is. For that reason, we investigate in this subsection the cases where one or both of these partial order relations hold. In order to do so, we systematically examine the cases where the three arcs lie in various numbers of product subgraphs and use the definitions of the two relations given in Subsections 4.2.1 and 4.2.2 .

- (a) If the three arcs e_i , e_j and e_k correspond to three different products, then clearly deleting e_k leaves e_i and e_j in the same dependent component, thus $e_i \leq_j^d e_k$ does not hold, and therefore neither does $e_i \leq_j^q e_k$.
- (b) Let us assume that e_i , e_j and e_k lie in two different product subgraphs and consider the following cases.

- (b-1) e_i and e_j belong to the same product subgraph, say G^1 , and e_k belongs to another, say G^2 . In order for the deletion of e_k to cause e_i and e_j to end up in different dependent components, it is necessary and sufficient that (i) $G^2 \setminus e_k$ is not an active subgraph, that is, $e_k = x_1^2$ or y_1^2 , (ii) there are only two active product subgraphs, G^1 and G^2 , and (iii) e_i and e_j lie in different rim components of G^1 (that is, at least one of e_i and e_j belongs to $\{x_1^1, y_1^1\}$).

If the above holds, then it follows from (5.30) that $e_i \leq_j^q e_k$ if and only if

$$|\alpha_1^1 + \beta_1^1| \geq |\alpha_1^2 + \beta_1^2|. \quad (5.31)$$

- (b-2) e_i and e_k belong to the same product graph, say G^1 , and e_j belongs to another, say G^2 . In order for the deletion of e_k to cause e_i and e_j to end up in different dependent components, it is necessary and sufficient that $G^1 \setminus e_k$ is a neutral subgraph, that is, $e_k = x_1^1$ or y_1^1 .

In that event it follows from (5.30) that any elementary vector f that contains e_j verifies $|f_i| = |f_k|$, and thus $e_i \leq_j^q e_k$ also holds.

- (b-3) e_j and e_k belong to the same product graph, say G^1 , and e_i belongs to another, say G^2 . Since e_i and e_j play symmetric roles in $e_i \leq_j^d e_k$, (b-2) also applies here.

If $e_i \leq_j^d e_k$ holds, then (as for (b-1)), $e_i \leq_j^q e_k$ if and only if (5.31) holds.

(c) Lastly we consider the case where the three arcs are in the same product subgraph, say G^1 . From (5.30), the projection of any elementary vector on G^1 either is empty or is a simple cycle, and thus $e_i \leq_j^d e_k$ and $e_i \leq_j^q e_k$ holds if and only if deleting e_k leaves e_i and e_j in two different biconnected components of $G^1 \setminus e_k$. One of the following is necessary.

(c-1) e_k is any inventory arc, and e_i and e_j lie on either side of e_k .

(c-2) $e_k = x_1^1$ and either e_i or e_j is y_1^1 or, by symmetry, $e_k = x_n^1$ and either e_i or e_j is y_{n-1}^1 .

The above information will be best displayed using the **Hasse Diagrams** of both partial orders. Given an arc e_k , the partial order relation \leq_j^d (resp., \leq_j^q) will give rise to a directed graph whose nodes are the equivalence classes for \leq_j^d (resp., \leq_j^q) and where an arc joins a class to another if and only if the arcs in the first class are \leq_j^d (resp., \leq_j^q) than the arcs in the second class, and no class comes in between in the partial order. The resulting graph is a forest of rooted trees² called a **Hasse Diagram**, and the full order can then be inferred from it by applying transitivity. The lower an arc is in the Hasse diagram, the less \leq_j^d (resp., \leq_j^q) it is, and thus, by the D-Ripple property, (resp., Q-Ripple property), the less it will be affected by parameter changes in arc e_j . For that reason, e_j is a root and will usually be chosen as the arc whose parameter change is studied. We will provide examples of Hasse Diagrams in Subsection 5.5.4.

5.5.4 Variations in production capacity and inventory holding cost

In this subsection we give several examples of situations that may occur and answer some questions regarding sensitivity analysis. In the first scenario, we consider the likely effects of a **reduction in production capacity** in, say, the first period and the first product.

² A **rooted tree** is a directed tree such that the unique oriented path from any given node to a specific node, the root, only has forward arcs.

Note that such a reduction may reflect (partial or complete) machine breakdown, manpower reduction or material shortage, or even temporary shutdown of the corresponding production unit. These occurrences can be modeled by setting

$$g_1^1(x_1^1, U) = K(x_1^1) + \mu(x_1^1 - U),$$

where $K(x_1^1)$ represents the (convex) production cost of product 1 in period 1, and the parameter U represents the production capacity for product 1 in the first period.

The function $g_1^1(\cdot, \cdot)$ thus defined is subadditive, and $g_1^1(\cdot, L)$ is convex if so is $K(x_1^1)$. Moreover, it is assumed that all other costs involved are also convex in the flow variables. In this event a decrease in U will have the following effects on the optimal production-inventory schedule.

- (i) A decrease (*Monotone Optimal-Flow Selection Theorem*) of, say, δ units (where $\delta \geq 0$) in the production level x_1^1 and of the inventory level y_1^1 of product 1 in the first period.
- (ii) A decrease (*Monotone Optimal-Flow Selection Theorem*) of at most δ units (*Ripple property*) in the inventory levels of product 1 in periods $2, \dots, n-1$. Moreover, the magnitudes of these variations decrease in time (period index) (*Q-Ripple property*).
- (iii) An increase (*Monotone Optimal-Flow Selection Theorem*) of at most δ units (*Ripple property*) in the production of product 1 in periods $2, \dots, n$. However, these changes may or may not be monotone in the time component.
- (iv) An increase (*Monotone Optimal-Flow Selection Theorem*) of δ^l where $\delta^l \leq M(1, l)\delta = (\alpha_i^1 + \beta_i^1)/(\alpha_i^l + \beta_i^l)\delta$ units (*Ripple property*) in the production level x_1^l and of the inventory level y_1^l of product $l = 2, \dots, p$ in the first period.
- (v) An increase (*Monotone Optimal-Flow Selection Theorem*) of at most δ^l units (*D-Ripple property*) in the inventory levels of product 1 in periods $2, \dots, n-1$. As for product 1, the magnitudes of these variations decrease in time (*Q-Ripple property*).

- (vi) A decrease (*Monotone Optimal-Flow Selection Theorem*) of at most δ^l units (*D-Ripple property*) in the production of product $l = 2, \dots, p$ in periods $2, \dots, n$.

The way the changes ripple down from x_1^1 are reflected in the Hasse Diagrams given in Figure 5.26 for both partial orders $\leq_{x_1^1}^d$ and $\leq_{x_1^1}^q$ (for the sake of convenience, we will assume in these diagrams that $l = 3$, $M(1, 2) \leq 1$ and $M(1, 3) \geq 1$).

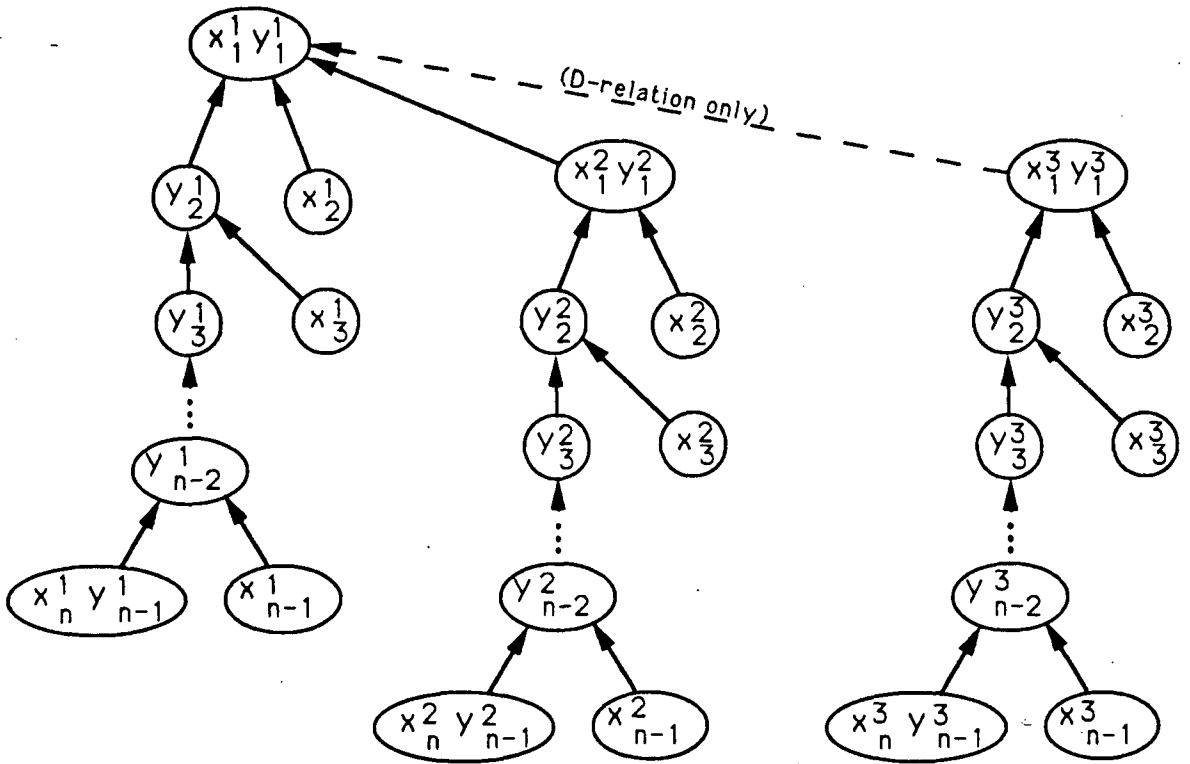


Figure 5.26: Hasse Diagrams, $e_j = x_1^1$

We now turn to a second scenario in which there is a change in the cost associated with the inventory variable y_4^1 (we will assume for the sake of convenience that $n \geq 5$). To that end we may for instance express the inventory cost in the fourth period for the first product as having the form

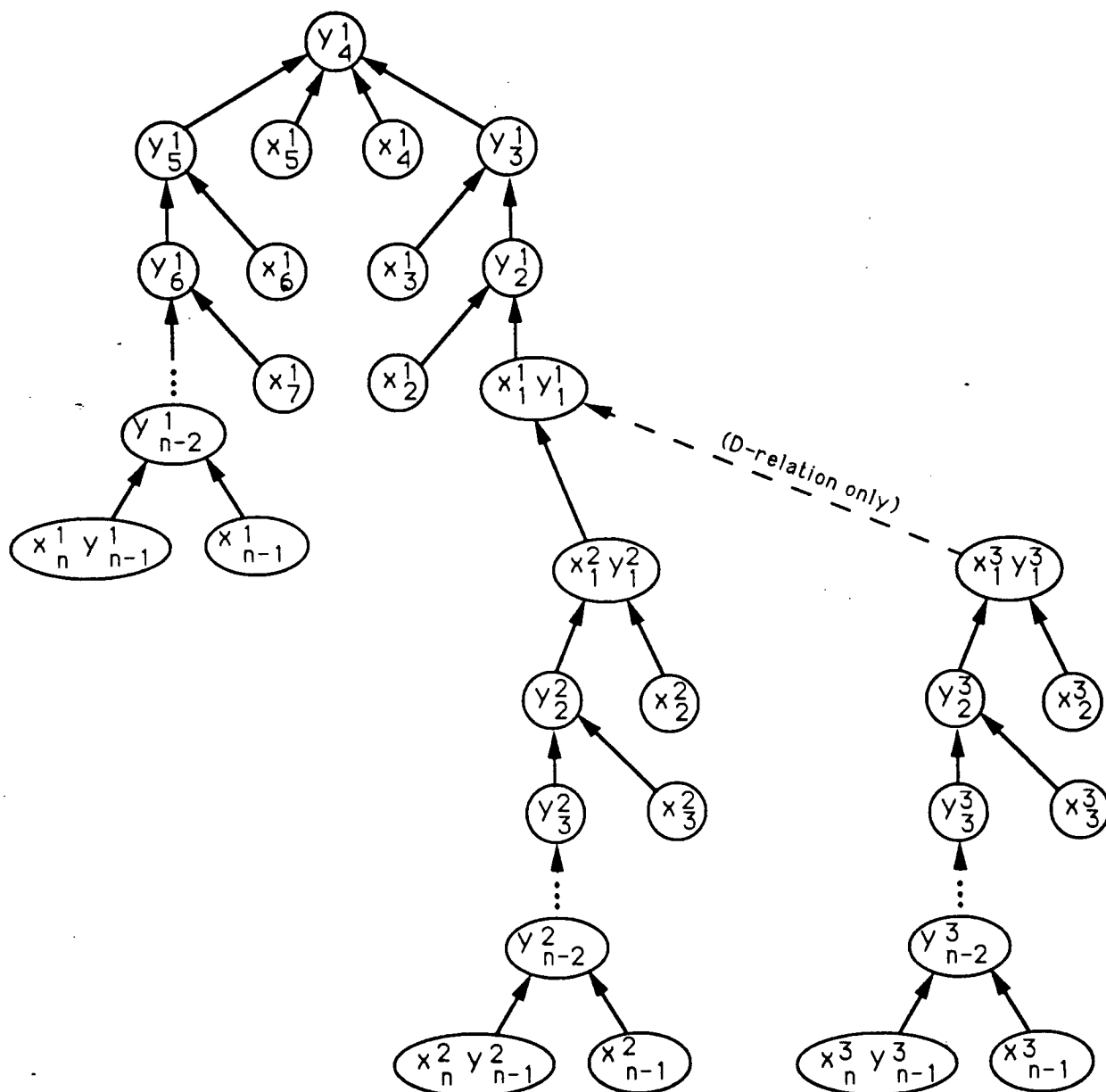
$$h_4^1(y_4^1, H) = K'(y_4^1) + Hy_4^1$$

where the function K' includes upper and lower bounds as well as a fixed cost. The function $h_4^1(y_4^1, H)$ thus defined is subadditive, and $h_4^1(\cdot, H)$ is convex if so is K' . Thus an increase in the marginal inventory cost H will have the following effects.

- (i) A decrease (*Monotone Optimal–Flow Selection Theorem*) of, say, δ units (where $\delta \geq 0$) in the inventory level y_4^1 .
- (ii) A decrease (*Monotone Optimal–Flow Selection Theorem*) of at most δ units (*Ripple property*) in the inventory levels of product 1 in all other periods. Moreover, the magnitudes of these variations decrease as the period index gets further from 4, in the past as well as in the future (*Q–Ripple property*).
- (iii) A decrease (resp., increase) (*Monotone Optimal–Flow Selection*) of at most δ units (*Ripple property*) in the production of product 1 in periods $1, \dots, 4$ (resp., $5, \dots, n$). Let δ^1 ($0 \leq \delta^1 \leq \delta$) be the magnitude of that decrease in the first period production.
- (iv) An Increase (*Monotone Optimal–Flow Selection*) of $\delta^l \leq M(1, l)\delta^1 = (\alpha_i^1 + \beta_i^1)/(\alpha_i^l + \beta_i^l)\delta^1$ units (*D–Ripple property*) in the production level x_1^l and of the inventory level y_1^l of product $l = 2, \dots, p$ in the first period.
- (v) An increase (*Monotone Optimal–Flow Selection*) of at most δ^l units (*D–Ripple property*) in the inventory levels of product 1 in periods $2, \dots, n - 1$. As for product 1, the magnitudes of these variations decrease with time (*Q–Ripple property*) but this time with a maximum variation in the first period.
- (vi) An decrease (*Monotone Optimal–Flow Selection*) of at most δ^l units (*D–Ripple property*) in the production of product $l = 2, \dots, p$ in periods $2, \dots, n$.

The corresponding Hasse Diagrams given in Figure 5.27 for both partial orders $\leq_{y_4^1}^d$ and $\leq_{y_4^1}^q$ (with the same assumption that $l = 3$, $M(1, 2) \leq 1$ and $M(1, 3) \geq 1$).

□ □ □

Figure 5.27: Hasse Diagrams, $e_j = y_4^1$

Chapter 6

A Cash-Flow Asset Allocation Management Model

Chapter Abstract

We show in this chapter how a Cash-Flow Management model can be formulated as a nonlinear parametric network flow problem with one additional linear constraint. We then recommend a method by which a decision maker could understand qualitatively how to respond to changes in the environment such as variations in interest rates, taxes or asset prices without any additional computation.

□ □ □

6.1 A Cash-flow Management Model

Many cash-flow management models can be represented as generalized network flow problems. That is, as network flow problems with arc-gains or losses, see for example [M84]. In this chapter we show how one such model can be transformed into an equivalent singly-constrained network flow problem. One can then apply the qualitative theory developed in Chapter 4 for singly-constrained network flow problems in order to help decision makers respond to changes in their environment without having to perform any additional computations. Given an existing set of assets, the problem is to decide what would be the “best” distribution of the assets over the next planning period. The objective may be to maximize expected return, to minimize a (quadratic) risk function, or some combination of both. We will adopt the notation of Mulvey [M84]. To this end let us denote by

I	Set of asset indices, $ I = n$
b_0	Cash available for investment purposes
b_i	Initial value invested in asset $i \in I$
s_i	Value of asset $i \in I$ sold
p_i	Value of asset $i \in I$ purchased
z_i	Value of asset $i \in I$ unsold
y_i	Additional value invested in asset $i \in I$ in the revised set of assets
x^r	Amount invested in riskless asset in the revised set of assets
y_s	Expected value at the end of the planning horizon of the revised set of assets
$t_{i,1}(t_{i,2})$	Arc multiplier for the buy (sell) transaction
ρ_0	Interest rate on riskless asset
ρ_i	Rate of return on asset $i \in I$
σ_i	Standard deviation of the return on asset i .

Moreover, we associate a convex parametric penalty or carrying cost $\underline{b}(\cdot, \cdot)$ with the quantities s_i, p_i, z_i, y_i , $i \in I$, and x^r . The nonnegativity constraints on the variables will be embedded in this cost as well as upper and/or lower bounds on the amount invested in each asset. For instance, we will denote the total cost associated with the amount x^r invested in the riskless asset by $\underline{b}^r(x^r, t^r)$ where t^r is a parameter associated with this investment. The expression $\sum_{i \in I} \sigma_i^2 y_i^2$ represents the risk factor¹.

Using a goal programming approach, we give a weight $\lambda \geq 0$ to the risk factor in the objective function as was suggested in [M84] for portfolio management and obtain the formulation of the problem as a generalized network flow problem whose objective function is convex. That is

¹Notice that in the portfolio models described for instance in [Ma59,M84] the risk factor is given by $y^t Q y$ where Q is the variance-covariance matrix of the assets (portfolios) returns. In order to apply the Singly-Constrained Network Model described in earlier chapters we further require the limiting assumption that the various assets have **independent rates of return**, thus the non-diagonal elements of Q are zero and $y^t Q y = \sum_{i \in I} \sigma_i^2 y_i^2$. We recognize that the independence assumption does not apply in many real life situations. For that reason, the results presented here only claim to apply in the cases where this assumption can be made.

$$(\text{GEN}) \quad \min \quad -y_s + \lambda \sum_{i \in I} \sigma_i^2 y_i^2 + \sum_{i \in I} b_i^z(z_i, t_i^z) + \sum_{i \in I} b_i^s(s_i, t_i^s)$$

$$+ \sum_{i \in I} b_i^p(p_i, t_i^p) + \sum_{i \in I} b_i^y(y_i, t_i^y) + b^r(x^r, t^r)$$

$$\text{s.t.} \quad b_i - s_i - z_i = 0 \quad \forall i \in I$$

$$t_{i,1}p_i + z_i - y_i = 0 \quad \forall i \in I$$

$$\sum_{i \in I} (1 + \rho_i) y_i + (1 + \rho_0) x^r - y_s = 0$$

$$b_0 + \sum_{i \in I} t_{i,2} s_i - \sum_{i \in I} p_i - x^r = 0.$$

The corresponding graph is given in Figure 6.28 (Triangles on arcs signify arcs with gains).

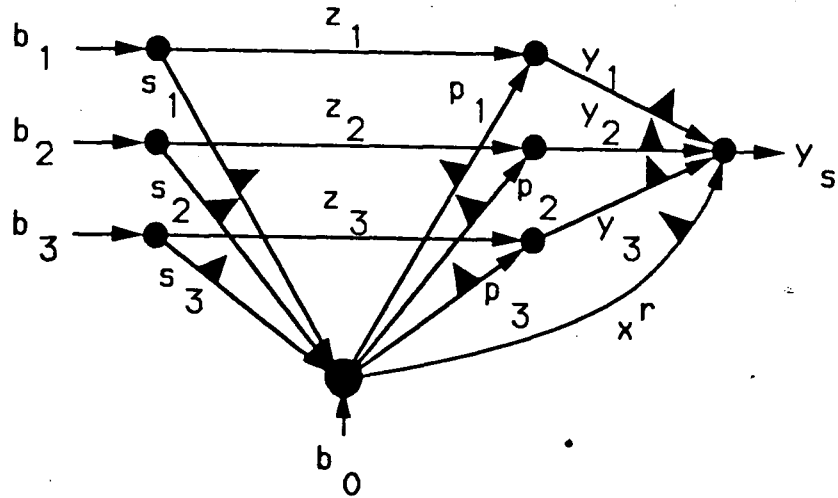


Figure 6.28: The graph G_{GEN}

We now show how this model can be transferred into a Singly-Constrained Monotropic Network Flow problem (SCMF). Indeed, if we let $q_i = t_{i,1}p_i$, we may rewrite (GEN) as

$$(\text{SCF}) \quad \min \quad \sum_{i \in I} b_i^z(z_i, t_i^z, t_i^s) + \sum_{i \in I} b_i^q(q_i, t_i^q) + \sum_{i \in I} b_i^y(y_i, t_i^y) + b^r(x^r, t^r)$$

$$\text{s.t.} \quad q_i + z_i - y_i = 0$$

$$\forall i \in I$$

$$\sum_{i \in I} t_{i,2} z_i + \sum_{i \in I} (1/t_{i,1}) q_i + x^r = b_0 + \sum_{i \in I} t_{i,2} b_i$$

where the costs

$$b_i^z(z_i, t_i^z, t_i^s) = \underline{b}_i^z(z_i, t_i^z) + \underline{b}_i^s(b_i - z_i, t_i^s)$$

$$b_i^q(q_i, t_i^q) = \underline{b}_i^q(q_i/t_{i,1}, t_i^p) \quad (t_i^q \stackrel{\text{def}}{=} t_i^p)$$

$$b_i^y(y_i, t_i^y, \rho_i, \sigma_i) = -(1 + \rho_i)y_i + \underline{b}_i^y(y_i, t_i^y) + \lambda \sigma_i^2 y_i^2$$

$$b^r(x^r, t^r, \rho_0) = -(1 + \rho_0)x^r + \underline{b}^r(x^r, t^r)$$

are also convex. This is a network flow problem on the directed graph given in Figure 6.29 with a side constraint of the form $\alpha x = \alpha_0$ where

$$\alpha_i^z = t_{i,2}, \quad \alpha_i^q = 1/t_{i,1}, \quad \alpha_i^y = 0 \quad \forall i \in I, \quad \alpha^r = 1 \quad \text{and} \quad \alpha_0 = b_0 + \sum_{i \in I} t_{i,2} b_i.$$

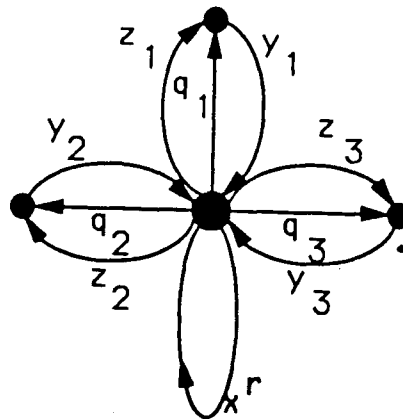


Figure 6.29: The graph G_{SCF}

6.2 Properties of the Graph G_{SCF}

In this section we study various properties of the graph G_{SCF} and verify hypotheses needed in order to apply the theory developed in Chapter 4. Since the elementary vectors play a central role in the theory presented above, we start by obtaining a complete list of them. First notice that, under the assumption that transaction costs are nonzero, G_{SCF} does not contain any neutral cycles. Therefore, all elementary vectors f are weighted sums of simple cycles. Thus, as was shown in Chapter 3, they satisfy

$$f = c_1 - (\alpha c_1 / \alpha c_2) c_2 \quad (6.32)$$

where c_1 and c_2 are two active cycles that either intersect at a single node, on a simple path or do not intersect. Given the simple topology of the graph G_{SCF} , we know that the possible values for c_1 and c_2 are $e_i^z + e_i^y$, $e_i^q + e_i^y$, $e_i^z - e_i^q$ for all $i \in I$, and e^r . We will identify the elementary vectors by their supports, that is, by the variables corresponding to the cycles that form them. Such elementary vectors may, if they involve only arcs in one of the “leaves” of G_{SCF} , be of the form

$$f(z_i, q_i, y_i) = (e_i^z + e_i^y) - (\alpha_i^z / \alpha_i^q)(e_i^q + e_i^y) = e_i^z - t_{i,1} t_{i,2} e_i^q + (1 - t_{i,1} t_{i,2}) e_i^y$$

or, if they involve two leaves of G_{SCF} , say corresponding to $i, j \in I$, they will be of the form

$$f(z_i, y_i, z_j, y_j) = (e_i^z + e_i^y) - (\alpha_i^z/\alpha_j^z)(e_j^z + e_j^y) = e_i^z + e_i^y - (t_{i,2}/t_{j,2})e_j^z - (t_{i,2}/t_{j,2})e_j^y$$

$$f(z_i, y_i, q_j, y_j) = (e_i^z + e_i^y) - (\alpha_i^z/\alpha_j^q)(e_j^q + e_j^y) = e_i^z + e_i^y - (t_{i,2}t_{j,1})e_j^q - (t_{i,2}t_{j,1})e_j^y$$

$$\begin{aligned} f(z_i, y_i, z_j, q_j) &= (e_i^z + e_i^y) - (\alpha_i^z/(\alpha_j^z - \alpha_j^q))(e_j^z - e_j^q) \\ &= e_i^z + e_i^y + (t_{i,2}t_{j,1}/(1 - t_{j,2}t_{j,1}))e_j^z - (t_{i,2}t_{j,1}/(1 - t_{j,2}t_{j,1}))e_j^q \end{aligned}$$

$$f(q_i, y_i, q_j, y_j) = (e_i^q + e_i^y) - (\alpha_i^q/\alpha_j^q)(e_j^q + e_j^y) = e_i^q + e_i^y - (t_{j,1}/t_{i,1})e_j^q - (t_{j,1}/t_{i,1})e_j^y$$

$$\begin{aligned} f(q_i, y_i, z_j, q_j) &= (e_i^q + e_i^y) - \alpha_i^q/(\alpha_j^z - \alpha_j^q)(e_j^z - e_j^q) \\ &= e_i^q + e_i^y + t_{j,1}/(t_{i,1}(1 - t_{j,2}t_{j,1}))e_j^z - t_{j,1}/(t_{i,1}(1 - t_{j,2}t_{j,1}))e_j^q \end{aligned}$$

$$\begin{aligned} f(z_i, q_i, z_j, q_j) &= (e_i^z - e_i^q) - (\alpha_i^z - \alpha_i^q)/(\alpha_j^z - \alpha_j^q)(e_j^z - e_j^q) \\ &= e_i^z - e_i^q - t_{j,1}(1 - t_{i,1}t_{i,2})/(t_{i,1}(1 - t_{j,1}t_{j,2}))e_j^z + t_{j,1}(1 - t_{i,1}t_{i,2})/(t_{i,1}(1 - t_{j,1}t_{j,2}))e_j^q \end{aligned}$$

$$f(z_i, y_i, x^r) = (e_i^z + e_i^y) - (\alpha_i^z/\alpha^r)e^r = e_i^z + e_i^y - t_{i,2}e^r$$

$$f(q_i, y_i, x^r) = (e_i^q + e_i^y) - (\alpha_i^q/\alpha^r)e^r = e_i^q + e_i^y - (1/t_{i,1})e^r$$

$$f(q_i, z_i, x^r) = (e_i^z - e_i^q) - ((\alpha_i^z - \alpha_i^q)/\alpha^r)e^r = e_i^z - e_i^q + (1 - t_{i,2}t_{i,1})/t_{i,1}e^r$$

as well as any combination obtained by reversing i and j . We may assume for practical purposes that $0 < t_{i,1} < 1$, $0 < t_{i,2} < 1$ and thus $0 < 1 - t_{i,1}t_{i,2} < 1$ for all $i \in I^2$.

²In practice, the transaction costs are of the order of 1%. Thus $t_{i,1}$ and $t_{i,2}$ are close to .99.

We may now determine whether a given pair of arcs in G_{SCF} are substitutes, complements, or neither, by simply looking at the signs of the entries of the elementary vectors. For example, we may deduce that e_i^z and e_i^y are complements, for the only elementary vectors that contain them both are $f(z_i, q_i, y_i)$, $f(z_i, y_i, z_j, y_j)$, $f(z_i, y_i, q_j, y_j)$, $f(z_i, y_i, z_j, q_j)$ which all verify $f_i^z f_i^y \geq 0$. Table 6.8 shows the signs of the entries of these elementary vectors ($i \neq j$) and Table 6.9 shows which pairs of arcs are substitutes (S), complements (C) or neither (n).

Support:	e_i^z	e_i^q	e_i^y	e_j^z	e_j^q	e_j^y	e^r
z_i, q_i, y_i	+	-	+				
z_i, y_i, z_j, y_j	+		+	-		-	
z_i, y_i, q_j, y_j	+		+	-		-	
q_i, y_i, z_j, y_j		+	+	-		-	
z_i, y_i, z_j, q_j	+		+	+	-		
z_i, q_i, z_j, q_j	+	-		+		+	
q_i, y_i, q_j, y_j		+	+		-	-	
q_i, y_i, z_j, q_j		+	+	+	-		
z_i, q_i, q_j, y_j	+	-		+	+		
z_i, q_i, z_j, q_j	+	-		-	+		
z_i, y_i, x^r	+		+				-
q_i, y_i, x^r		+	+				-
z_i, q_i, x^r	+	-					+
z_j, y_j, x^r				+		+	-
q_j, y_j, x^r					+	+	-
z_j, q_j, x^r				+	-		+

Table 6.8: Table Signs of Respective Elementary Vector Entries

	z_i	q_i	y_i	z_j	q_j	y_j	x^r
z_i	C	S	C				
q_i	S	C			S	S	.S
y_i	C		C		S	S	S

Table 6.9: Substitutes and Complements in the Cash-flow Problem

Next, we obtain in Table 6.10 the coefficients M_{ij} which appear in (4.7). In order to break ties, we will make the additional assumption that $1/t_{i,1} - t_{i,2} \leq t_{i,2}$ ($1 < 2 t_{i,1} t_{i,2}$) for all $i \in I$ which, for most real situations, is indeed the case.

$x \backslash x'$	e_i^z	e_i^q	e_i^y	e_j^z	e_j^q	e_j^y	e^r
e_i^z	1	1	1	$\frac{t_{i,2}t_{j,1}}{1-t_{i,1}t_{i,2}}$	$\frac{t_{i,2}t_{j,1}}{1-t_{i,1}t_{i,2}}$	$\frac{t_{i,2}}{t_{i,2}}$	$t_{i,2}$
e_i^q	$\frac{1}{t_{i,1}t_{i,2}}$	1	1	$\frac{t_{j,1}}{t_{i,1}(1-t_{j,1}t_{j,2})}$	$\frac{t_{j,1}}{t_{i,1}(1-t_{j,1}t_{j,2})}$	$\frac{1}{t_{i,1}t_{j,2}}$	$\frac{1}{t_{i,1}}$
e_i^y	$\frac{1}{1-t_{i,1}t_{i,2}}$	$\frac{t_{i,1}t_{i,2}}{1-t_{i,1}t_{i,2}}$	1	$\frac{t_{j,1}}{t_{i,1}(1-t_{j,1}t_{j,2})}$	$\frac{t_{j,1}}{t_{i,1}(1-t_{j,1}t_{j,2})}$	$\frac{1}{t_{i,1}t_{j,2}}$	$\frac{1}{t_{i,1}}$
e^r	$\frac{t_{i,1}}{1-t_{i,1}t_{i,2}}$	$\frac{t_{i,1}}{1-t_{i,1}t_{i,2}}$	$\frac{1}{t_{i,2}}$	$\frac{t_{j,1}}{1-t_{i,1}t_{i,2}}$	$\frac{t_{j,1}}{1-t_{i,1}t_{i,2}}$	$\frac{1}{t_{i,2}}$	1

Table 6.10: Coefficients of the Ripple Theorem

6.3 Qualitative Sensitivity Analysis

In the event that the arc costs are convex and, under appropriate parametrization, subadditive, one can apply the results given in Chapter 4 to various scenarios, we provide in the following two such examples. In order to give orders of magnitude, the following statements are respectively given for general values of the transaction costs as functions of $t_{i,1}$ and $t_{i,2}$, for $t_{i,1} = t_{i,2} = .99 \forall i \in I$ and for $t_{i,1} = t_{i,2} = .90 \forall i \in I$.

- **Changes in Sales Taxes.** Assume that a tax is imposed on sales of asset i at rate τ_i .

This will result in an additional cost of $\tau_i s_i = \tau_i(b_i - z_i)$, which may be incorporated in the cost associated with z_i (recall that s_i is not explicitly a part of the SCF formulation). In order to investigate the qualitative effect of changes in the tax rate τ_i on our cash-flow allocation revision, we express the cost associated with z_i as

$$b_i^z(z_i, \tau_i) = \tau_i(b_i - z_i) + \underline{b}_i^z(z_i) + \underline{b}_i^s(b_i - z_i).$$

where $\underline{b}_i^z(z_i)$ and $\underline{b}_i^s(b_i - z_i)$ denote the (non-parametric) costs of selling an amount s_i and keeping an amount z_i of asset i excluding the tax. The function $\underline{b}_i^z(\cdot, \cdot)$ thus defined is subadditive, and $\underline{b}_i^z(\cdot, \tau_i)$ is convex if so is $\underline{b}_i^z(z_i) + \underline{b}_i^s(b_i - z_i)$. In this event an increase in the tax rate τ_i on sales of asset i will have the following effects.

- An increase of, say, $\$ \delta$ (where $\delta \geq 0$) in the amount z_i kept invested in asset i and thus a decrease of $\$ \delta$ in the amount of that asset sold (s_i).

- (ii) A decrease of at most $\$ \delta / t_{i,1}$ (resp., $\$1.01\delta$ if $t_{i,1} = t_{i,2} = .99 \forall i \in I$, and $\$1.11\delta$ if $t_{i,1} = t_{i,2} = .90 \forall i \in I$) in the amount of asset i bought ($p_i = q_i / t_{i,1}$).
 - (iii) An increase of at most $\$ \delta$ in the amount of asset i in the revised set of assets (y_i).
 - (iv) A variation of at most $\$ \delta t_{i,2} t_{j,1} / (1 - t_{j,1} t_{j,2})$ (resp., $\$49.25\delta$, $\$4.26\delta$) in the amount of asset $j \neq i$ (z_j) kept and an opposite variation of the amount of asset j sold (s_j).
 - (v) A variation of at most $\$ \delta t_{i,2} t_{j,1} / (t_{i,1}(1 - t_{j,1} t_{j,2}))$ (resp., $\$49.75\delta$, $\$4.74\delta$) in the amount of asset $j \neq i$ purchased (p_j).
 - (vi) A variation of at most $\$ \delta t_{i,2} / t_{j,2}$ (resp., $\$ \delta$, $\$ \delta$) in the amount of asset $j \neq i$ in the revised set of assets (y_j).
 - (vii) A variation of at most $\$ \delta t_{i,2}$ (resp., $\$.99\delta$, $\$.90\delta$) in the amount invested in riskless asset (x^r).
- **Variations in Expected Rate of Return.** Recall that the cost associated with the level y_i invested in asset i in the revised allocation is given by

$$b_i^y(y_i, t_i^y, \rho_i, \sigma_i) = -(1 + \rho_i)y_i + \underline{b}_i^y(y_i, t_i^y) + \lambda \sigma_i^2 y_i^2.$$

where $\underline{b}_i^y(y_i, t_i)$ denotes the other costs besides asset return and risk component. Suppose now that there is an increase in the rate of return ρ_i and all other parameters remain fixed. We can thus write

$$b_i^y(y_i, \rho_i) = -(1 + \rho_i)y_i + \underline{b}_i^y(y_i) + \lambda \sigma_i^2 y_i^2.$$

The function $b_i^y(y_i, \rho_i)$ thus defined is subadditive, and $b_i^y(\cdot, \rho_i)$ is convex if so is $\underline{b}_i^y(y_i) + \lambda \sigma_i^2 y_i^2$. Thus an increase in the expected rate of return on asset i will have the following effects.

- (i) An increase of, say, $\$ \delta$ (where $\delta \geq 0$) in the amount y_i invested in asset i in the revised set of assets.

- (ii) An increase of at most $\$ \delta / (1 - t_{i,1}t_{i,2})$ (resp., $\$50.25\delta$, $\$5.26\delta$) in the amount of asset i kept in the set of assets ($z_i = q_i/t_{i,1}$).
- (iii) A variation of at most $\$ \delta t_{i,2} / (1 - t_{i,1}t_{i,2})$ (resp., $\$49.75\delta$, $\$4.74\delta$) in the amount purchased of asset i (p_i).
- (iv) A variation of at most $\$ \delta t_{j,1} / (t_{i,1}(1 - t_{j,1}t_{j,2}))$ (resp., $\$50.25\delta$, $\$5.26\delta$) in the amount of asset $j \neq i$ (z_j) kept and an opposite variation of the amount of asset j sold (s_j).
- (v) A decrease of at most $\$ \delta t_{j,1} / (t_{i,1}(1 - t_{j,1}t_{j,2}))$ (resp., $\$50.25\delta$, $\$5.26\delta$) in the amount of asset $j \neq i$ purchased (p_j).
- (vi) A decrease of at most $\$ \delta / t_{i,1}t_{j,2}$ (resp., $\$1.02\delta$, $\$1.23\delta$) in the amount of asset $j \neq i$ in the revised set of assets (y_j).
- (vii) A decrease of at most $\$ \delta / t_{i,1}$ (resp., $\$1.01\delta$, $\$1.11\delta$) in the amount invested in riskless asset (x^r).

Remark. Similar results can be obtained if one considers changes in market prices, interest rates, and more.

- **Subadditivity of the optimal value.** Recall from Subsection 4.4.2 that, under the above assumptions, the optimal value $\underline{b}(\cdot)$ is subadditive in the parameters. In the case where one studies the effect, say, of increasing the tax rates τ_i , $i \in I$, on the sales of more than one asset, the subadditivity reflects the following reality. If increases were to be introduced successively, a given increase will have less impact on the expected return the later it is introduced, all other parameters being equal. Similarly, as far as increases in rates of returns on individual assets are concerned, such an increase will have a more significant effect on the overall objective if it happens alone than if it were to follow a series of similar increases in other assets.

□ □ □

Appendix A

index

2-connected	12
α -one-forest	13
δ	124
$\Delta(q)$	23
[DN]	84
[MWOP-D]	78
[MWOSC]	74
[M]	74
[QLD]	78
$Al_{[DN]}$	84
$Al_{[M]}$	74
$Al_{[QLD]}$	78
$C(\alpha)$ (constrained circulation space)	11
E , set of arcs	10
E , set of arcs	1
$F(\alpha, \alpha_0, d)$, set of constrained flows	10
$H(\alpha)$, set of elementary vectors	20
$H_j(\alpha)$	63
N , set of nodes	1
N , set of nodes	10
$P(\alpha, \alpha_0)$	2
Q	22
s , size of an instance	13
t_j , arc parameter	60
T_j , parameter lattice	60
$X(t)$, set of optimal flows	60
active	11
additional constraint	1, 10
adjusted version	48
ALGORITHM I	41
ALGORITHM II	51
ancestors	12

- arc redirecting 41
- arcs with common degree-two nodes 86
- ascending optimal-arc-flow multifunction 91
- backlogged demand 102
- backward 11
- basic variables 16
- basis for the tucker representation 16
- biconnected component 21
- biconnected 12, 21
- bolts 25
- branches of G 31
- changes in sales taxes 144
- circulation space 15
- circulation 10
- complements 81
- conditional transitivity of complements and substitutes 84
- connected 12
- constrained circulation 11
- constrained graph 10
- constrained palm graph 12
- constrained circulation space 11
- constrained problem 60
- consumption node 102
- contain 11
- cost 10, 1
- cost 60
- cut 12
- cycle graph 25
- d-equally dependent on 63
- d-equivalence classes 63
- D-ripple property 70
- decision problem 13
- demand vector 1, 10
- dependent component 21
- dependent component-additive 56
- dependent 21
- descendant 12
- detection of v_0 and w_0 42
- detection of the bolts 42
- directed graph 1, 10
- directed spanning tree 12
- disconnected version of G 48
- disposal of goods 102

- doubly subadditive 94
- dual 94
- elementary vector 11
- elementary support 11
- elementary circulation 11
- evaluation problem 13
- expanded one-rim 35
- expansion arc 35
- father 13
- feasible 10
- flow 1, 10
- forward 11
- fundamental basis associated with B 19
- hasse diagram 132
- head 1, 10
- incidence matrix 1, 10
- independent rates of return 138
- instances 13
- iterated limit 92
- iterated optimal-constrained-flow selection 93
- iterated optimal constrained flow 92
- join 91
- lattice 60, 103
- leaf branch 40
- leaf arcs 16
- less-dependent-on 61
- lower semicontinuous 92
- manpower requirement 127
- market demand 102
- maximum weight oriented simple cycle 74
- meet 91
- monotone optimal-flow selection 94
- monotonically step-connected 95
- monotropic programming problem 1
- multiple constraints 57
- negative arcs 11, 113
- negative rim arcs 87
- neutral 11
- node labelling 41
- nonbasic variables 16
- nonlinear and inequality constraints 56
- one-rims 35
- one-rim split. (if G is a one-rim) 42

- oriented cycle 11
- oriented path 11
- palm graph 12
- parallel arcs 86
- parameter vector 60
- parameter 60
- parametric singly-constrained monotropic network flow problem (pscmf) 60
- parametric unconstrained monotropic network flow problem 60
- periods 102
- positive arcs 11, 113
- positive rim arcs 87
- problem 13
- product subgraphs 102
- production node 102
- products 102
- pseudo-component 118
- pseudo-bolts 118
- pseudo-rim 118
- q-equally dependent on 68
- q-equivalence classes 68
- Q-ripple property 70
- qualitative determinacy 4
- quantitatively less dependent on 61, 68
- R-cycle 31
- R-arc 31
- reduction in production cost 123
- reduction in production capacity 132
- reduction in production cost 115
- reduction in production cost 125
- return arcs 12
- return arc partitioning 41
- reversed 11
- rim components 25
- rim 25
- risk factor 138
- root 12
- rooted tree 132
- S-cycle 31
- S-arc 31
- schedule 103
- series arcs 86
- shared machine 125
- side constraint 104

simple cycle 11
 simple path 11
 single-product unconstrained case 104
 singly-constrained monotropic network flow problem (scmf) 1
 size of an instance 13
 spanning forest 12
 spanning tree 12
 special types of constraints 57
 step-function 102
 subadditive 91
 submodularity 91
 substitutes 81
 $\text{supp}(x)$ 11
 support 11
 tail 1
 tail 10
 tree arcs 12
 Tucker representation 16
 unconstrained problem 15, 60
 unconstrained monotropic network flow problem 2, 15
 underlying tree 12
 variations in expected rate of return 145
 weights 74

□ □ □

Bibliography

- [AHU74] A. V. Aho, J. E. Hopcroft and J. D. Ullman. *The Design and Analysis of Computer Algorithms*. Addison-Wesley, Reading, Massachusetts, 1974.
- [Ba89] U. Bagchi. *Simultaneous Minimization of Mean and Variation of Flow Time and Waiting Time in Single Machine Systems*. *Opns. Res.* **37**, No. 1, 1989.
- [BMM88] K. Belling-Seib, P. Mevert and Chr. Müller. *Network Flow Problems with One Side Constraint: a Comparison of Three Solution Methods*. *Comput. Opns. Res.* **15** No. 4, 381-394, 1988.
- [Be63] C. Berge. *Topological Spaces*. Macmillan, New York, 115-116, 1963. Translation of *Espaces Topologiques, Fonctions Multivoques*. Dunod, Paris, France.
- [Be73] C. Berge. *Graphs and Hypergraphs*. North-Holland, 1973.
- [Bert76] D. P. Bertsekas. *A New Algorithm for Solution of Restrictive Networks Involving Diodes*. *IEEE Trans. Circuit Sys.* **CAS-23**, 599-608, 1976
- [BC80] R. E. Bixby and W. H. Cunningham. *Converting Linear Programs to Linear Programs*. *Math of Operations Research* **5**, 312-357, 1980.
- [Cu83] W. H. Cunningham. *A Class of Linear Programs Convertible to Network Programs*. *Operations Research* **31** 387-391, 1983.
- [CS77] S. Chen and R. Saigal. *A Primal Algorithm for Solving a Capacitated Network Flow Problem with Additional Linear Constraints*. *Networks*, **7**, 59-79, 1977.
- [Ci89] I. Ciurria. *Substitutes and Complements in Multicommodity Flows and Inventories*. *Doctoral Dissertation, Department of Operation Research, Stanford University*, 1989.
- [CGV89a] I. Ciurria, F. Granot and A. F. Veinott. *Substitutes, Complements and Ripples in Multicommodity Production Planning*. *Working Paper, Faculty of Commerce and Business Administration, The University of British Columbia*, 1989.
- [CGV89b] I. Ciurria, F. Granot and A. F. Veinott. *Substitutes, Complements and Ripples in Multicommodity Flows on Suspension Graphs*. *Working Paper, Faculty of Commerce and Business Administration, The University of British Columbia*, 1989.
- [CD73] P. Crawley and R. P. Dilworth. *Algebraic Theory of Lattices*. Prentice-Hall, 1973.
- [DK81] R. S. Dembo and J. G. Klincewicz. *A Scaled Reduced-gradient Algorithm for Convex Network Flow Problems*. *Math. Prog. Study*, **15**, 125-147, 1981.

- [GP81] D. Gale and T. Politof. *Substitutes and Complements in Network Flow Problems*. *Discrete Applied Math* **3** 175–186, 1981.
- [GJ79] M. R. Garey and D.S. Johnson. *Computers and intractability: A guide to NP-Completeness*. W. H. Freeman and Company, New York, 1979.
- [GG89] A. Gautier and F. Granot. *A Linear Transformation of a Singly-Constrained to an Unconstrained Monotropic Network Flow Problem*. Working paper, Faculty of Commerce and Business Administration, The University of British Columbia. June 1989, revised September 1990.
- [GG90a] A. Gautier and F. Granot. *Ripples, Complements and Substitutes in Singly-Constrained Monotropic Parametric Network Flow Problems*. Working paper, Faculty of Commerce and Business Administration, The University of British Columbia. March 1990.
- [GG90b] A. Gautier and F. Granot. *A Parametric Analysis of a Nonlinear Cash-Flow Asset Allocation Management Model*. Working paper, Faculty of Commerce and Business Administration, The University of British Columbia. May 1990.
- [GG90c] A. Gautier and F. Granot. *A Parametric Analysis of a Constrained Nonlinear Inventory-production Model*. Working paper, Faculty of Commerce and Business Administration, The University of British Columbia. October 1990.
- [GKKR78] F. Glover, D. Karney, D. Klingman and R. Russell. *Solving Singly-constrained Transshipment Problems*. *Transport. Sci.* **12**, 277–297, 1978.
- [Go64] T. Gorman. *More Scope for Qualitative Economics*. *Rev. Econ. St.* **31**, 65–68, 1964.
- [GV85] F. Granot and A. F. Veinott, Jr. *Substitutes, Complements and Ripples in Network Flows*. *Math. Oper. Res.* **10**, 471–497, 1985.
- [Gr81] H. J. Greenberg. *Measuring Complementarity and Qualitative Determinacy in Matricial Forms*. in *Proceedings of the Symposium on Computer Assisted Analysis and Model Simplification*, H. J. Greenberg and J. S. Maybee, Eds. Academic Press, New York, 1981.
- [GLM83] H. J. Greenberg, J. R. Lundgren and J. S. Maybee. *Rectangular Matrices and Signed Graphs*. *SIAM J. Alg. Disc. Meth.* **4** No. 1, 50–61, 1984.
- [GLM84a] H. J. Greenberg, J. R. Lundgren and J. S. Maybee. *Inverting Graphs of Rectangular matrices*. *Discrete Applied Mathematics* **8**, 255–265, 1984.
- [GLM84b] H. J. Greenberg, J. R. Lundgren and J. S. Maybee. *Inverting Signed Graphs*. *SIAM J. Alg. Disc. Meth.* **5** No. 2, 216–222, 1984.
- [HK77] R. V. Helgason and J. L. Kennington. *An Efficient Specialization of the Convex Simplex Method for Nonlinear Network Flow Problems*. Technical report IEOR 99017. Southern Methodist University. 1977.

- [Hi39] J. R. Hicks. *Value and Capital*, 2nd edition, Oxford Press, Clarendon 1939.
- [HA34] J. R. Hicks and R. G. D. Allen. *A Reconsideration of the Theory of Value, Parts I and II*, *Economica*, 1, 52-76, 196-219, 1934.
- [HT73] J. E. Hopcroft and R. E. Tarjan. *Efficient Algorithms for Graph Manipulation*. *Comm. ACM*, 16, 372-378, 1973.
- [Ki47] G. Kirchhoff. *Annalen der Physic und Chemie*, 72, 497, 1847.
- [Kl77] D. Klingman. *Finding Equivalent Network Formulations for Constrained Network Problems*. *Mgmt. Sci.* 23, 737-744, 1977.
- [La67] G. M. Lady. *The Structure of Qualitative Determinate Linear Systems*. Rice University, Houston, Texas, Systems Report No. 19-14, 1967.
- [La62] K. Lancaster. *The scope of Qualitative Economics*. *Rev. Econ. St.* 29, 99-132, 1962.
- [La65] K. Lancaster. *The Theory of Qualitative Linear Systems*. *Econometrica*, 33, 395-408, 1965.
- [La66] K. Lancaster. *The Solution of Qualitative Comparative Static Problems*. *Quart. J. Econ.* 80, 278-295, 1966.
- [Ma59] H. Markowitz. *Portfolio Selection*. John Wiley & Sons, New York, 1959.
- [Ma73] R. M. May. *Stability and Complexity in Model Ecosystems*, Princeton University Press, Princeton, New Jersey, 1973.
- [MQ69] J. Maybee and J. P. Quirk. *Qualitative Problems in Matrix Theory*. *SIAM Review* 11 30-51, 1969.
- [M84] J. M. Mulvey. *Nonlinear Network Models in Finance*. in "Advances in Mathematical Programming and Financial Planning", Vol. 1, pp. 253-271, J.A.I. Press.
- [Or82] J. B. Orlin *Minimum Convex Cost Dynamic Network Flows*. Working Paper, Sloan School of Management, M.I.T. 1982.
- [PS82] C. H. Papadimitriou and K. Steiglitz. *Combinatorial Optimization; Algorithms and Complexity*. Prentice-Hall, Englewood Cliffs, New Jersey, 1982.
- [Pa71] K. Paton. *An Algorithm for the Blocks and Cutnodes of a Graph*. *Comm. ACM*, 14, 468-475, 1971.
- [Pr83] J. S. Provan. *Determinacy in Linear Systems and Networks*. *SIAM J. Alg. Disc. Meth.* 4 262-278, 1983.
- [Pr86] J. S. Provan. *Substitutes and Complements in Constrained Linear Models*. Working paper, Curriculum in Operations Research and Systems Analysis, University of Carolina at Chapel Hill, December 1986.

- [PK80] J. S. Provan and A. S. Kydes. *Correlation and Determinacy in Network Models*. BNL Report No. 51243, Brookhaven National Laboratory, Upton, NY. 1980
- [QR65] J. P. Quirk and R. Ruppert. *Qualitative Economics and the Stability of Equilibrium*, *Review of Economic Studies* **32** 311–326, 1965.
- [Ro67] R. T. Rockafellar. *The Elementary Vectors of a Subspace of R^n* . In *Combinatorial Mathematics and its Applications, Proceedings of the Chapel Hill Conference*, 104–127, R. C. Bose & T. A. Dowling eds., University of North Carolina Press, 1967.
- [Ro70] R. T. Rockafellar. *Convex Analysis*. Princeton University Press, Princeton, N.J., 1970.
- [Ro84] R. T. Rockafellar. *Network Flows and Monotropic Optimization*. John Wiley & Sons, 1984.
- [Sa47] P. A. Samuelson. *Foundations of Economic Analysis*. Athenum, New York, 1971. Originally published by Harvard University Press, 1947.
- [Sa74] P. A. Samuelson. *Complementarity: An Essay on the Fortieth Anniversary of the Hicks–Allen Revolution in Demand Theory*. *J. Econ. Lit.*, 1255–1289, 1974.
- [Sh61] L. S. Shapley. *On Network Flow Functions*. *Naval Res. Logist. Quart.* **8** 151–158, 1961.
- [Sh62] L. S. Shapley. *Complements and Substitutes in the Assignment Problem*. *Naval Res. Logist. Quart.* **9** 45–48, 1962.
- [SL89] S. B. Spalti and T. H. Liebling. *A Special Case of a Network Problem with one Side Constraint*. Working Paper R.O. 890429, Département de Mathématiques, EPF Lausanne, Switzerland, 1989.
- [Ta72] R. E. Tarjan. *Depth-First Search and Linear Algorithms*. *SIAM J. Comput.* **1**, 146–160, 1972.
- [To78] D. M. Topkis. *Minimizing a Submodular Function on a Lattice*. *Oper. Res.* **26**, 305–321, 1978.
- [TV73] D. M. Topkis and A. F. Veinott, Jr. *Monotone Solutions of Extremal problems on Lattices. (abstract)*. *VIIIth Internat. Math. Prog., Abstracts*. Stanford University, 1973.
- [Tr83] K. Truemper. *How to Detect Hidden Networks and Totally Unimodular Section in Linear Programs*. *TIMS/ORSA Bulletin* No. 5, 1978.
- [Tuc63] A. W. Tucker. *Combinatorial Theory Underlying Linear Programs*, In *Recent Advances in Mathematical Programming*, 1–16, R. L. Graves & P. Wolfe eds., MacGraw-Hill, 1963.
- [Tut66] W. T. Tutte. *Connectivity in Graphs*. University of Toronto Press, Toronto, 1966.

- [Ve64] A. F. Veinott. *Production Planning with Convex Costs: a Parametric Study*. *Management Science* **10**, No. 3, 441-460, 1964.
- [Ve65] A. F. Veinott. *Monotone Solutions of Extremal Problems*. (unpublished). 1965.
- [Ve68] A. F. Veinott. *Substitutes and complements in Network Flow Problems*. *Presentation to the TIMS/ORSA meeting, San Francisco, CA*, 1968.
- [Ve84] A. F. Veinott. *Lecture Notes in Inventory Theory*. Department of Operation Research, Stanford University, Stanford, CA. 1984.
- [Vefo.] A. F. Veinott. *Lattice Programming*. Forthcoming.
- [Wh32] H. Whitney. *Non Separable and Planar Graphs*. *Trans. Amer. Math. Soc.*, **34**, 339-362, 1932.

□ □ □