MAXIMUM LIKELIHOOD IDENTIFICATION OF LINEAR DISCRETE-TIME SYSTEMS

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ABSTRACT / RESUME

Abstract

The theoretical properties of the Maximum Likelihood estimator, for both single input-single output and multivariable systems, are considered. New results relative to convergence properties of some identification methods of single input-single output systems are obtained. A unified approach to the Maximum Likelihood identification method of multivariable systems is proposed. Numerical tests on a computer are performed.

Résumé

Nous considérons les propriétés théoriques de l'estimateur du Maximum de Vraisemblance dans le cas de systèmes monovariables et de systèmes multivariables. Nous étudions des méthodes d'identification de systèmes monovariables, et de nouveaux résultats relatifs à la convergence de ces méthodes sont obtenus. Nous proposons une approche globale de l'identification des systèmes multivariables au sens du Maximum de Vraisemblance. Cette procédure est illustrée par des exemples numériques.

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INTRODUCTION

Preliminaries

During the past decade, increasing attention has been devoted to the different aspects of system identification. However, although these topics have been discussed in a multitude of papers and a rather large number of survey papers [3,8,9,25] have been published, the field of identification does not appear as a unified subject. It is then of importance to have in mind the basic concepts characterizing an identification problem.

One of the basic ingredients in the formulation of the identification problem is the choice of the class of models [21] (parametric, non parametric) and of the model structure (linear-non linear, continuous-discrete time,...). Such choices cannot be done systematically and depend on the a priori knowledge of the case treated and on the purpose of the identification.

Due to the fact that, in most realistic situations, the measurements made on the system considered are corrupted by random disturbances, one may investigate the identification problem through statistical methods, leading to an estimation problem. The estimation problem can be formulated as the choice of an estimator (Least-Squares, Markov, Bayes, Maximum Likelihood). Such a formulation makes it possible to derive mathematical properties of the estimates and provides a rigorous framework to the field of identification. It must be noticed that the probabilistic interpretation can be eluded by taking into account the effects of the disturbances from an empirical point of view.

In any situation (probabilistic or deterministic), the identification problem can conceptually be stated as the finding of the model--subjected to the same input signals as the system--that is optimal in some sense. The optimality has to be defined using a criterion with respect to the output signals, which are related to the parameters values through a functional relationship. The criterion is generally defined by means of a loss function, leading to an optimization problem. The choice of an optimization method (derivative-type methods, search methods) is, once again, strongly related to the problem under study.

All these aspects are equally important steps of the identification problem which then cannot be considered as a simple estimation of a set of parameters or as a simple optimization problem.

Aim of Thesis

The present work is concerned with identification of linear discrete-time systems.

As outlined above, the choice of an estimator is a crucial step of the identification problem. A very popular choice is that of the Least-Squares (LS) estimator. However, it is well known that the

LS estimator is generally biased. In order to overcome the problem of correlated noise, one may postulate a system with correlated noise, by introducing a noise model. Depending on the a priori knowledge of the system and of the noise, the following estimators may be chosen:

- the Markov estimator for which the knowledge of the covariance matrix of the noise is needed
- the Bayes' estimator for which the knowledge of the probability density function of the noise and of the parameters values is needed
- the Maximum Likelihood estimator for which the probability density function of the noise is needed.

The assumption of known covariance matrix of the noise or of known probability density function of the parameters values severely limits the practical applicability of the two first estimators. The aim of Chapter 1 is to analyze, for single input-single output systems, the mathematical properties of the Maximum Likelihood estimator. Two new results will be obtained:

- in case of a Moving Average noise model (Åström [2]), it will be shown that the Maximum Likelihood estimates may converge to wrong values. Counterexamples to general convergence will be given (Section 3).
- in case of an Autoregressive noise model, it will be established that - under suitable assumptions - the Maximum Likelihood estimates converge to the

true values of the parameters (Section 4). This result provides a convergence proof of the well-know Generalized-Least-Squares (GLS) method, proposed by Clarke [6].

In Chapter 2, multivariable systems will be considered. The identification of multivariable systems cannot be viewed as a simple generalization of the single input-single output case and, before formulating the identification problem, the following considerations must be taken into account : the choice generally made for the class of the models (state-space representation) implies the determination of a suitable canonical form and the derivation of a canonical set of input-output relations.

The problem of finding state-space canonical forms have been investigated by Luenberger[15], Gopinath [12] and Mayne [16]. It can be shown that the use of a selector matrix or of a set of indexes allows one to describe the structure of a canonical model. In Section 2, the results relative to this problem will be reported.

Although the problem of deriving a set of canonical input-output relations has been studied by many authors, no general approach is yet available in literature. The methods proposed by Gopinath[15], Ackerman [1], Zuercher [28] or

Guidorzi[13] are subject to strong restrictions. In Section 3, the solution to this problem in the most general case is proposed and comparison with the previous methods is established. The basic differences between the approach presented in this Section and the previous ones lie in the fact that it allows a discrimination of the input-output data and that it is extended to the noisy case.

In Section 4, using the results of Chapter 1, a Maximum Likelihood estimator is defined and the consistency of the estimates, in case of correlated noise, is proved.

In Section 5, the optimization problem is discussed. The various types of minimization algorithms are described and compared.

Finally, in Chapter 3, experimental results are presented. The theoretical results of Chapter 1 and Chapter 2 are tested on numerical examples.

Chapter 1: SINGLE INPUT - SINGLE OUTPUT SYSTEMS

1 - INTRODUCTION

As pointed out in the introductory Chapter, a type of identification problem may be obtained by embedding it in a probabilistic framework, leading to an estimation problem. Such a point of view allows a rigorous approach to the field of identification and constitutes one of its main aspects.

The aim of this Chapter is to investigate the mathematical properties of the Maximum Likelihood estimation method for various types of noise model structures.

(i) In Section 3, the Moving Average noise model will be considered.

(ii) In Section 4, the Autoregressive noise model will be studied.

In both cases, two classes of Maximum Likelihood estimators will be investigated and new results relative to the convergence properties of some existing methods will be established.

2. PRELIMINARIES

The class of linear discrete-time single input - single output systems considered is represented by the difference equation

$$A(z^{-1})\ y(k) = B(z^{-1})\ u(k) + n(k) \tag{1.1}$$

where

$\{y(k),\ k=1,\dots,N\}$ is the system output

$\{u(k),\ k=1,\dots,N\}$ is the system input

$\{n(k),\ k=1,\dots,N\}$ is an additive zero-mean noise

and

$$A(z^{-1}) = 1 + \sum_{i=1}^{n_a} a_i z^{-i} \tag{1.2}$$

$$B(z^{-1}) = \sum_{i=0}^{n_b} b_i z^{-i} \tag{1.3}$$

It is assumed that the noise process can be expressed as a process driven by a white noise e(k)

$$n(k) = H(z^{-1})\ e(k) \tag{1.4}$$

where

$$H(z^{-1}) = \frac{C(z^{-1})}{D(z^{-1})} \tag{1.5}$$

$$C(z^{-1}) = 1 + \sum_{i=1}^{n_c} c_i z^{-i} \tag{1.6}$$

$$D(z^{-1}) = 1 + \sum_{i=1}^{n_d} d_i z^{-i} \tag{1.7}$$

The Likelihood function L can then be defined as the probability density function of $\epsilon(k)$, where the numbers $\epsilon(k)$ are the so-called residuals defined by

$$H(z^{-1})\, \epsilon(k) = A(z^{-1})\, y(k) - B(z^{-1})\, u(k) \tag{1.8}$$

Assuming that e is $(0, \sigma_e^2)$ gaussian, the residual $\epsilon(k)$ is a sequence of independent and gaussian $(0, \sigma_e^2)$ variables and the log-Likelihood function Λ takes the form

$$\Lambda = - \frac{1}{2\sigma_e^2} \sum_{k=1}^{N} \epsilon^2(k) - N \log \sigma_e - N \log 2\pi \tag{1.9}$$

The Maximum Likelihood estimation procedure can then be interpreted as the finding of a model

$$\hat{A}(z^{-1})\, y(k) = \hat{B}(z^{-1})\, u(k) + \hat{H}(z^{-1})\, \epsilon(k) \tag{1.10}$$

in such a way that the log-Likelihood function Λ is maximized.

From (1.9), it is clear that maximizing Λ is equivalent to minimizing the loss function

$$V_N(\underline{a},\underline{b},\underline{c},\underline{d}) = \frac{1}{2N}\sum_{k=1}^{N} \epsilon^2(k) \tag{1.11}$$

where $\epsilon(k)$ is obtained through the equation of error

$$\epsilon(k) = \hat{H}^{-1}(z^{-1})\ \hat{A}(z^{-1})\ y(k) - \hat{H}^{-1}(z^{-1})\ \hat{B}(z^{-1})\ u(k) \tag{1.12}$$

In the following, there is an advantage in putting

$$\underline{p} = \left[\underline{a}^T\ \underline{b}^T\ \underline{c}^T\ \underline{d}^T\right]^T \tag{1.13}$$

where $\underline{a}^T = \left[a_1\ a_2 \ldots a_{n_a}\right]$, $\underline{b}^T = \left[b_0\ b_1 \ldots b_{n_b}\right]$, $\underline{c}^T = \left[c_1\ c_2 \ldots c_{n_c}\right]$ and $\underline{d}^T = \left[d_1\ d_2 \ldots d_{n_d}\right]$.

The Maximum Likelihood estimator can thus be defined as follows : the Maximum Likelihood estimate(s) of $\underline{p}$, say $\hat{\underline{p}}$, is(are) the absolute minimum point(s) of $V_N(\hat{\underline{p}})$

$$E_1 : \hat{\underline{p}} \in S_1 = \left\{ \hat{\underline{p}}_N^M : V_N(\hat{\underline{p}}_N^M) = \min_{\hat{p}} V_N(\hat{\underline{p}}) \right\} \tag{1.14}$$

However, from equation (1.12), it is clear that ϵ is non linear in $\hat{\underline{c}}$ and $\hat{\underline{d}}$. Consequently, the finding of $\hat{p}_N^M$ has to be done iteratively using a search routine. It is then of importance to know if $V_N(\hat{\underline{p}})$ has local extremum points $\hat{\underline{p}}_N^m$. This leads us to redefine the Maximum Likelihood estimator in the following way

$$E_2 : \hat{p} \in S_2 = \left\{ \hat{\underline{p}}_N^m : \left. \frac{\partial V(\hat{\underline{p}})}{\partial \hat{p}} \right|_{\hat{\underline{p}}=\hat{\underline{p}}_N^m} = 0 \right\} \tag{1.15}$$

The estimator E_1 (resp. E_2) will be consistent if and only if

$$S_1 (\text{resp. } S_2) = \{ \underline{p} \} \tag{1.16}$$

Clearly $S_1 \subseteq S_2$ and E_2 consistent implies E_1 consistent.

From a practical point of view, it is highly desirable that E_2 is consistent. E_2 inconsistent means that, even if the global minimum point of the loss function coincides with the true value of the parameter vector $\underline{p}$ (E_1 consistent), the Maximum Likelihood estimation procedure may not converge into S_1 and may then give a wrong estimate.

The equation (1.4) can be rewritten as

$$D(z^{-1})\ n(k) = C(z^{-1})\ e(k) \tag{1.17}$$

which implies that the noise process is modeled as an autoregressive moving average (ARMA) process. In the next Sections, the following cases will be investigated

a) MA noise model (D=1). This case has been treated by Åström [2], Åström and Söderström [4] and Söderström [24].

b) AR noise model (C=1). This choice of noise structure is the basic ingredient in the formulation of the Generalized Least-Squares algorithm proposed by Clarke [6].

In both cases, the statistical properties of the estimates will be considered. It will be shown that

- for a MA model (Section 3), the estimator E_1 is consistent while E_2 is inconsistent.
- for a AR noise model (Section 4), the estimators E_1 and E_2 are consistent.

The concept of persistently exciting signals will be used in the following. A signal u(k) is said to be persistently exciting of order n [2] if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} u(k)\, u(k+j) = R_u(j) \qquad (1.18a)$$

exists and

$$A_n = \left[R_u(i-j)\right] \begin{matrix} i=1,\dots,n \\ j=1,\dots,n \end{matrix} \qquad (1.18b)$$

is positive definite. If u(k) is persistently exciting of

order n
$$\sum_{i=0}^{n} h_i \, u(k-i) = 0 \tag{1.19a}$$

implies

$$h_i = 0 \quad i=0,\ldots,n \tag{1.19b}$$

3. MOVING AVERAGE NOISE MODEL

3.1. Introduction

If a Moving Average (MA) noise model is used, the output data are governed by the difference equation

$$A(z^{-1})\, y(k) = B(z^{-1})\, u(k) + C(z^{-1})\, e(k) \tag{1.20}$$

The model structure is then given by

$$\hat{A}(z^{-1})\, y(k) = \hat{B}(z^{-1})\, u(k) + \hat{C}(z^{-1})\, \epsilon(k) \tag{1.21}$$

To apply Maximum Likelihood (ML) method, it is assumed that

- A1. All the processes are ergodic

- A2. The polynomials $A(z^{-1})$ and $C(z^{-1})$ have all their zeros inside the unit circle [this condition implies that the system (1.20) is stable]

- A3. The polynomials $A(z^{-1})$, $B(z^{-1})$ and $C(z^{-1})$ are relatively prime [this condition implies that the system (1.20) is controllable either from u or from e $\big]$.

With these assumptions, the ML estimate of $\underline{p} = \left[\underline{a}^T\ \underline{b}^T\ \underline{c}^T\right]^T$, say $\hat{\underline{p}} = \left[\hat{\underline{a}}^T\ \hat{\underline{b}}^T\ \hat{\underline{c}}^T\right]^T$, is obtained by minimization of the loss function

$$V_N(\hat{\underline{p}}) = \frac{1}{2N} \sum_{k=1}^{N} \epsilon^2(k) \tag{1.22}$$

where the residual $\epsilon(k)$ depends on $\hat{\underline{p}}$ via (1.21)

$$\epsilon(k) = \frac{\hat{A}(z^{-1})}{\hat{C}(z^{-1})}\, y(k) - \frac{\hat{B}(z^{-1})}{\hat{C}(z^{-1})}\, u(k) \tag{1.23}$$

As a first result, the global minimum points of the loss function will be considered and conditions will be given for the estimator E_1 to be consistent. This will be the object of Section 3.2.

As mentioned above, it is of importance to know if there is a unique local minimum point of $V_N(\hat{\underline{p}})$. In practice, there are cases for which the loss function can have more than one local minimum. The reasons for such difficulties are the following

- the model structure may not be appropriate
- the number of data, N, may be too small
- the model structure may have the inherent property that $V_N(\hat{\underline{p}})$ has several local minima.

In order to avoid the first two pitfalls, the following additional assumptions will be made

- A4. It is assumed that (1.20) really holds
- A5. The asymptotic loss function

$$V(\hat{\underline{p}}) = \lim_{N\to\infty} V_N(\hat{\underline{p}}) = E\left[\epsilon^2\right] \qquad (1.24)$$

will be considered instead of $V_N(\hat{\underline{p}})$.

The case covered by the third explanation will be the main object of Section 3.3., where it will be shown that the model structure considered implies that, in general, a unique local minimum cannot be obtained and consequently that E_2 is inconsistent.

3.2. Global minimum points

Let us define

- $\hat{\underline{p}}_N^M$ the global minimum point(s) of $V_N(\hat{\underline{p}})$:

$$V_N(\hat{\underline{p}}_N^M) = \min_{\hat{p}} V_N(\hat{\underline{p}}) \qquad (1.25)$$

- $\hat{\underline{p}}^M$ the global minimum point(s) of $V(\hat{\underline{p}})$

$$V(\hat{\underline{p}}^M) = \min_{\hat{p}} V(\hat{\underline{p}}) \qquad (1.26)$$

Åström has shown [2] that

$$\hat{\underline{p}}^M = \underline{p} \qquad (1.27)$$

$$\lim_{N \to \infty} \hat{\underline{p}}_N^M = \hat{\underline{p}}^M \text{ with probability 1} \tag{1.28}$$

provided u is persistently exciting of order $n_a + n_b$. These two relations establish that the global minimum point(s) of the loss function $V_N(\hat{\underline{p}})$ converge(s) as $N \to \infty$ to the unique global minimum point of the asymptotic loss function $V(\hat{\underline{p}})$, which coincides with the true value of the parameters, $\underline{p}$.

This means that, under the assumptions A1 - A5, $S_1 = \{\underline{p}\}$ and that

Theorem 1.1

The ML estimator E_1 is asymptotically consistent.

The results of this Section do not give any information about local extremum points and thus do not prove that $\hat{\underline{p}}$ converges to $\underline{p}$. The convergence properties of the ML method crucially depends on the existence of multiple local extrema of the loss function.

It must be noticed that, although the ML method has been extensively applied to systems whose model is given by (1.20), no general result relative to local extremum points is available in literature - except in the case of a pure ARMA process ($B \equiv 0$) [4] . In the next Section a class of counter examples to a unique local minimum is given : this implies the in-consistency of E_2 and the non-convergence of the method described above, which is one the most commonly used in system identification.

3.3. Local extremum points

Combining (1.20) and (1.21) and dropping the arguments z^{-1} and k, for convenience, yields

$$\epsilon = \frac{\hat{A}B - A\hat{B}}{A\hat{C}} u + \frac{\hat{A}C}{A\hat{C}} e \tag{1.29}$$

Since u and e are assumed to be uncorrelated, the asymptotic loss function takes then the form

$$2V = E\left[\epsilon^2\right] = E\left[\frac{\hat{A}B - A\hat{B}}{A\hat{C}} u\right]^2 + E\left[\frac{\hat{A}C}{A\hat{C}} e\right]^2 \tag{1.30}$$

which can be rewritten, using Parseval's theorem, as

$$2V = \frac{\sigma_e^2}{2\pi i} \oint \frac{\hat{A}C}{A\hat{C}}(z) \frac{\hat{A}C}{A\hat{C}}(z^{-1}) \frac{dz}{z}$$

$$+ \frac{1}{2\pi i}\oint \frac{\hat{A}B - A\hat{B}}{A\hat{C}}(z) \frac{\hat{A}B - A\hat{B}}{A\hat{C}}(z^{-1}) \Phi_u(z) \frac{dz}{z} \tag{1.31}$$

where the integration path is the unit circle and $\Phi_u(z)$ is the discrete power spectrum of u.

We can now consider the stationary points of V, i.e.

the points which are solutions of

$$\frac{\partial V}{\partial \underline{\hat{p}}} = \underline{0} \tag{1.32}$$

Let us first consider the solutions of $\partial V/\partial \hat{c}_j = 0$, $j = 1,..,n_c$. After simple calculations, it can be written

$$\frac{\partial V}{\partial \hat{c}_j} = -\frac{\sigma_e^2}{2\pi i}\oint \frac{\hat{A}C}{A\hat{C}\hat{C}}(z)\ \frac{\hat{A}C}{A\hat{C}}(z^{-1})\ z^{j-1}\ dz$$

$$-\frac{1}{2\pi i}\oint \frac{\hat{A}B-A\hat{B}}{A\hat{C}\hat{C}}(z)\ \frac{\hat{A}B-A\hat{B}}{A\hat{C}}(z^{-1})\ \Phi_u(z)\ z^{j-1}\ dz \quad j=1,...,n_c$$

(1.33a)

or

$$\frac{\partial V}{\partial \hat{c}_j} = \oint \frac{f(z)}{A^*\hat{C}^*}\ \Phi_u(z)\ z^{j-1}\ dz \qquad j=1,...,n_c \tag{1.33b}$$

where

$$f(z) = -\sigma_e^2\ \frac{\hat{A}C}{A\hat{C}\hat{C}}\ \hat{A}^*C^* - \frac{\hat{A}B-A\hat{B}}{A\hat{C}\hat{C}}\ (\hat{A}^*B^*-A^*\hat{B}^*)\ \Phi_u \tag{1.34a}$$

$$A^*(z) = z^{n_a}\ A(z^{-1}) \tag{1.34b}$$

$$B^*(z) = z^{n_b}\ B(z^{-1}) \tag{1.34c}$$

$$C^*(z) = z^{n_c}\ C(z^{-1}) \tag{1.34d}$$

Since the integrand of $\partial V/\partial \hat{c}_j$ has n_a+n_c poles ($f(z)$ is analytic), the equation $\partial V/\partial \underline{\hat{c}} = \underline{0}$ has several solutions in $\underline{\hat{c}}$.

Let us now consider $\partial V/\partial \underline{\hat{a}}$ and $\partial V/\partial \underline{\hat{b}}$. Using (1.30), one may write

$$\frac{\partial V}{\partial \hat{a}_j} = E\left[\frac{B}{A\hat{C}} z^{-j}u\right]\left[\frac{\hat{A}B-A\hat{B}}{A\hat{C}} u\right] + E\left[\frac{C}{A\hat{C}} z^{-j}e\right]\left[\frac{\hat{A}C}{A\hat{C}} e\right] \qquad j=1,\ldots,n_a$$

$$\frac{\partial V}{\partial \hat{b}_j} = -E\left[\frac{1}{\hat{C}} z^{-j}u\right]\left[\frac{\hat{A}B-A\hat{B}}{A\hat{C}} u\right] \qquad j=0,\ldots,n_b \tag{1.35}$$

or

$$\frac{\partial V}{\partial \hat{a}_j} = \sum_{i=1}^{n_a} \hat{a}_i \; E\left[\frac{B}{A\hat{C}} z^{-j}u\right]\left[\frac{B}{A\hat{C}} z^{-i}u\right] + E\left[\frac{C}{A\hat{C}} z^{-j}e\right]\left[\frac{C}{A\hat{C}} z^{-i}e\right]$$
$$- \sum_{i=0}^{n_b} \hat{b}_i \; E\left[\frac{B}{A\hat{C}} z^{-j}u\right]\left[\frac{1}{\hat{C}} z^{-i}u\right] + \frac{\partial V}{\partial \hat{a}_j}(0) \tag{1.36a}$$

$$\frac{\partial V}{\partial \hat{b}_j} = -\sum_{i=1}^{n_a} \hat{a}_i \; E\left[\frac{1}{\hat{C}} z^{-j}u\right]\left[\frac{B}{A\hat{C}} z^{-i}u\right]$$
$$+ \sum_{i=0}^{n_b} \hat{b}_i \; E\left[\frac{1}{\hat{C}} z^{-j}u\right]\left[\frac{1}{\hat{C}} z^{-i}u\right] + \frac{\partial V}{\partial \hat{b}_j}(0) \tag{1.36b}$$

The equation

$$\frac{\partial V}{\partial \underline{\hat{a}}} = \underline{0} \qquad \frac{\partial V}{\partial \underline{\hat{b}}} = \underline{0} \tag{1.37a}$$

can then be rewritten as

$$\begin{bmatrix} M & -N \\ -N^T & P \end{bmatrix} \begin{bmatrix} \underline{\hat{a}} \\ \underline{\hat{b}} \end{bmatrix} = \begin{bmatrix} \underline{g} \\ \underline{r} \end{bmatrix} \tag{1.37b}$$

where M, N, P, $\underline{q}$ and $\underline{r}$ are defined by

$$M_{ij} = E\left[\frac{B}{A\hat{C}}\, z^{-i}u\right]\left[\frac{B}{A\hat{C}}\, z^{-j}u\right] + E\left[\frac{C}{A\hat{C}}\, z^{-i}e\right]\left[\frac{C}{A\hat{C}}\, z^{-j}e\right] \quad (1.39a)$$

$$N_{ij} = E\left[\frac{B}{A\hat{C}}\, z^{-i}u\right]\left[\frac{1}{\hat{C}}\, z^{-j}u\right] \quad (1.39b)$$

$$P_{ij} = E\left[\frac{1}{\hat{C}}\, z^{-i}u\right]\left[\frac{1}{\hat{C}}\, z^{-j}u\right] \quad (1.39c)$$

$$\underline{q} = -\frac{\partial V}{\partial \underline{\hat{a}}}(0) \quad (1.40a)$$

$$\underline{r} = -\frac{\partial V}{\partial \underline{\hat{b}}}(0) \quad (1.40b)$$

For any $\hat{C}$, the matrices M and P are positive definite. The equation (1.38) has then, at least, one solution in $\underline{\hat{a}}$ and $\underline{\hat{b}}$, for any $\underline{\hat{c}}$. Since (1.33) has several solutions in $\underline{\hat{c}}$, the gradients of V may vanish for several values of $\underline{\hat{p}}$.

The points $\underline{\hat{p}}$ satisfying (1.33) and (1.38) belong to S_2. This means that E_2 is inconsistent.

Remark

The above analysis is valid for any (A,B,C). The non-uniqueness of the estimates is thus an inherent property of the model considered.

We have thus shown the existence of counter-examples to a unique local minimum and consequently to general convergence of the ML method in the case of a MA noise model. This drawback is an inherent property of the model structure considered.

In the next section, a similar analysis for the Autoregressive noise model case is proposed.

4. AUTOREGRESSIVE NOISE MODEL

4.1. Introduction

In the case of a Autoregressive (AR) noise model, the equations (1.8) and (1.10) take the form

$$A(z^{-1})\ y(k) = B(z^{-1})\ u(k) + \frac{1}{C(z^{-1})} e(k) \tag{1.41}$$

$$\hat{A}(z^{-1})\ y(k) = \hat{B}(z^{-1})\ u(k) + \frac{1}{\hat{C}(z^{-1})} \epsilon(k) \tag{1.42}$$

Similarly with the previous case, the ML estimate of $\hat{\underline{p}} = \left[\hat{\underline{a}}^T\ \hat{\underline{b}}^T\ \hat{\underline{c}}^T\right]^T$, is obtained by minimizing the loss function

$$V_N(\hat{\underline{p}}) = \frac{1}{2N} \sum_{k=1}^{N} \epsilon^2(k) \qquad (1.43)$$

where the residual $\epsilon(k)$ is given by

$$\epsilon(k) = \hat{C}(z^{-1})\ \hat{A}(z^{-1})\ y(k) - \hat{C}(z^{-1})\ \hat{B}(z^{-1})\ u(k) \qquad (1.44)$$

Assuming the conditions A1-A5 of Section 3 still hold, we shall analyze first, the global minimum points (Section 4.2) and second, the local extremum points (Section 4.3) of the asymptotic loss function $V(\hat{\underline{p}})$. The asymptotic consistency of the estimators E_1 and E_2 will be proved.

4.2. - Global minimum points

The purpose of this section is to prove the asymptotic consistency of the ML estimator E_1. To do so, we shall proceed as follows :

- First, we shall prove that the global minimum point $\hat{\underline{p}}^M$ of the asymptotic loss function $V(\hat{\underline{p}})$ coincides with the true value of the parameters vector, $\underline{p}$
- Second, we shall show that the global minimum point(s) of $V_N(\hat{p})$, $\hat{\underline{p}}_N^M$ converge(s) to $\hat{\underline{p}}^M$ as $N \rightarrow \infty$.

Combining (1.41) and (1.42) allows us to rewrite the equation (1.44) and to obtain the equation of error

$$\epsilon = \hat{C}\ \frac{\hat{A}B - A\hat{B}}{A}\ u + \frac{\hat{A}\hat{C}}{AC}\ e \qquad (1.45)$$

It then follows

$$2V = E\left[\epsilon^2\right] = E\left[\hat{C}\,\frac{\hat{A}B - A\hat{B}}{A}\,u\right]^2 + E\left[\frac{\hat{A}\hat{C}}{AC}\,e\right]^2 \tag{1.46}$$

and

$$2V \geqslant E\left[\frac{\hat{A}\hat{C}}{AC}\,e\right]^2 = \frac{1}{2\pi i}\oint \frac{\hat{A}\hat{C}}{AC}(z)\,\frac{\hat{A}\hat{C}}{AC}(z^{-1})\,\frac{dz}{z} = \bar{V} \tag{1.47}$$

where the integration path is the unit circle $|z| = 1$.

We can now postulate the

<u>Lemma 1.1</u>

$$2V \geqslant \sigma_e^2 \tag{1.48}$$

The equality holds if and only if $\hat{\underline{p}} = \underline{p}$ $(\hat{A}=A,\ \hat{B}=B$ and $\hat{C}=C)$.

<u>Proof</u> :

Let us consider the integral

$$I = \frac{\sigma_e^2}{2\pi i}\oint\left[\frac{\hat{A}\hat{C}}{AC}(z) - 1\right]\left[\frac{\hat{A}\hat{C}}{AC}(z^{-1}) - 1\right]\frac{dz}{z} \geqslant 0 \tag{1.49}$$

Then

$$I = \bar{V} - 2\,\frac{\sigma_e^2}{2\pi i}\oint \frac{\hat{A}\hat{C}}{AC}(z)\,\frac{dz}{z} + \frac{\sigma_e^2}{2\pi i}\oint \frac{dz}{z} \tag{1.50}$$

Clearly, since $A(z^{-1})$ and $C(z^{-1})$ have all their zeros inside the unit circle, (assumption A2) and $A(0) = C(0) = 1$, it follows that

$$\frac{1}{2\pi i}\oint \frac{\hat{A}\hat{C}}{AC}(z)\,\frac{dz}{z} = \frac{\hat{A}(0)\ \hat{C}(0)}{A(0)\ C(0)} = 1 \tag{1.51}$$

$$\frac{1}{2\pi i}\oint \frac{dz}{z} = 1$$

and

$$I = \overline{V} - \sigma_e^2 \geqslant 0 \tag{1.52}$$

Hence

$$2V \geqslant \sigma_e^2 \tag{1.53}$$

The equality holds if and only if

(i) $I = 0$ which implies

$$\hat{A}\hat{C} = AC \tag{1.54}$$

(ii) $E\left[\hat{C}\,\frac{\hat{A}B - A\hat{B}}{A}\,u\right] = 0$, which is equivalent, provided u is persistently exciting of order n_b+n_c, to

$$\hat{A}B = A\hat{B} \tag{1.55}$$

Combining (1.54) and (1.55) and using Assumption A3 yields

$$\hat{A} = A,\ \hat{B} = B,\ \hat{C} = C \tag{1.56}$$

Q.E.D.

For all N, $V_N(\hat{\underline{p}})$ and $V(\hat{\underline{p}})$ are polynomials. Then, the assertion of convergence of V_N to V as $N \to \infty$ implies that $V_N(\hat{p})$

converges uniformly to $V(\hat{\underline{p}})$ on every compact set. Hence, considering the estimates in a compact set G allows us to establish the

Lemma 1.2

$$\lim_{N\to\infty} \hat{\underline{p}}_N^M = \hat{\underline{p}}^M \text{ with probability 1} \tag{1.57}$$

Proof :

Let α be an arbitrary small positive real number and G_α be the set

$$G_\alpha = \left\{ \hat{\underline{p}} : \left\| \hat{\underline{p}} - \hat{\underline{p}}^M \right\| < \alpha \right\} \tag{1.58}$$

Since V is continuous, there exists $\beta > 0$ such that

$$\min_{G-G_\alpha} V(\hat{\underline{p}}) \geqslant V(\hat{\underline{p}}^M) + \beta \tag{1.59}$$

Since V_N converges to V uniformly, there exists an integer M such that if $N > M$ then

$$\left| V_N(\hat{\underline{p}}) - V(\hat{\underline{p}}) \right| \leqslant \beta \tag{1.60}$$

for all $\hat{p} \in G$.

Thus

$$\min V_N(\hat{\underline{p}}) \geqslant \min_{G-G_\alpha} V(\hat{\underline{p}}) - \beta \geqslant V(\hat{\underline{p}}^M) + 2\beta \tag{1.61}$$

This means that $\hat{\underline{p}}_N^M$ is in G_α. Q.E.D.

Finally, the lemmas 1.1 and 1.2 yields the

Theorem 1.2 :

The ML estimator E_1 is asymptotically consistent.

4.3 - Local extremum points

The goal of this Section is to show that the estimator E_2 is asymptotically consistent, i.e. that $S_2 = \{\underline{p}\}$. To do so, we shall prove that the gradients of $V(\hat{\underline{p}})$ with respect to $\hat{\underline{p}}$ vanish only at $\hat{\underline{p}} = \underline{p}$.

In a recent paper [23], Söderström has shown the existence of counter-examples to a unique local minimum of V, for "small" signal-to-noise ratios : this is a major drawback to the implementation of this method. However, it will be shown that the introduction of one additional assumption allows us to overcome this problem.

We shall thus investigate the conditions for the gradients of V to vanish i.e.

$$\frac{\partial V}{\partial \hat{p}} = \underline{0} \qquad (1.62)$$

Defining the polynomials

$$\hat{F} = \hat{A}\hat{C} = 1 + \sum_{k=1}^{n} \hat{f}_k \; z^{-k} \qquad (1.63)$$

$$\hat{H} = \hat{B}\hat{C} = \sum_{k=0}^{m} \hat{h}_k \; z^{-k} \qquad (1.64)$$

where

$$n = n_a + n_b \tag{1.65a}$$

$$m = n_b + n_c \tag{1.65b}$$

confers to V the form

$$2V = E\left[\frac{\hat{F}}{AC}e\right]^2 + E\left[\frac{\hat{F}B - \hat{H}A}{A}u\right]^2 \tag{1.66}$$

and allows us to rewrite (1.62) as

$$\frac{\partial V}{\partial \hat{a}_j} = \sum_{k=1}^{n} \hat{c}_{k-j} \frac{\partial V}{\partial \hat{f}_k} = 0 \qquad j=1,\ldots,n_a \tag{1.67a}$$

$$\frac{\partial V}{\partial \hat{b}_j} = \sum_{k=0}^{m} \hat{c}_{k-j} \frac{\partial V}{\partial \hat{h}_k} = 0 \qquad j=0,\ldots,n_b \tag{1.67b}$$

$$\frac{\partial V}{\partial \hat{c}_j} = \sum_{k=1}^{n} \hat{a}_{k-j} \frac{\partial V}{\partial \hat{f}_k} = \sum_{k=0}^{m} \hat{b}_{k-j} \frac{\partial V}{\partial \hat{h}_k} = 0 \qquad j=1,\ldots,n_c \tag{1.67c}$$

In the following, it will be assumed that

<u>A6</u> The polynomials $\hat{A}$ and $\hat{C}$, and $\hat{B}$ and $\hat{C}$ are relatively prime.

With this assumption, (1.67) implies

$$\frac{\partial V}{\partial \hat{f}_j} = 0 \qquad j=1,\ldots,n \tag{1.68a}$$

$$\frac{\partial V}{\partial \hat{h}_j} = 0 \qquad j=0,\ldots,m \tag{1.68b}$$

Using (1.66), (1.68) can be rewritten as

$$E\left[\frac{\hat{F}}{AC}\,e\right]\left[\frac{1}{AC}\,z^{-j}e\right] + E\left[\frac{\hat{F}B-\hat{H}A}{A}\,u\right]\left[\frac{B}{A}\,z^{-j}u\right] = 0 \tag{1.69a}$$

$$E\left[\frac{\hat{F}B-\hat{H}A}{A}u\right]\left[z^{-j}u\right] = 0 \tag{1.69b}$$

or

$$\sum_{i=1}^{n} \hat{f}_i\; E\left[\frac{z^{-i}}{AC}e\right]\left[\frac{z^{-j}}{AC}e\right] + \sum_{i=1}^{n} \hat{f}_i\; E\left[\frac{B}{A}\,z^{-i}u\right]\left[\frac{B}{A}\,z^{-j}u\right]$$

$$-\sum_{i=0}^{m} \hat{h}_i\; E\left[z^{-i}u\right]\left[\frac{B}{A}\,z^{-j}u\right] + \frac{\partial V}{\partial \hat{f}_j}(0) = 0 \tag{1.70a}$$

$$\sum_{i=1}^{n} \hat{f}_i\; E\left[\frac{B}{A}\,z^{-i}u\right]\left[z^{-j}u\right] - \sum_{i=0}^{m} \hat{h}_i\; E\left[z^{-i}u\right]\left[z^{-j}u\right] + \frac{\partial V}{\partial \hat{h}_j}(0) = 0 \tag{1.70b}$$

Let us define the matrices

$$M_1 = \left[E\left[\frac{z^{-i}}{AC}e\right]\left[\frac{z^{-j}}{AC}e\right]\right]_{\substack{i=1,\dots,n\\ j=1,\dots,n}} \tag{1.71}$$

$$M_2 = \left[E\left[\frac{B}{A}\,z^{-i}u\right]\left[\frac{B}{A}\,z^{-j}u\right]\right]_{\substack{i=1,\dots,n\\ j=1,\dots,n}} \tag{1.72}$$

$$N = \left[E\left[z^{-i}u\right]\left[\frac{B}{A}\,z^{-j}u\right]\right]_{\substack{i=1,\dots,n\\ j=1,\dots,m}} \tag{1.73}$$

$$P = \left[E\left[z^{-i}u\right]\left[z^{-j}u\right]\right]_{\substack{i=1,\dots,m\\ j=1,\dots,m}} \tag{1.74}$$

and the vectors

$$\hat{\underline{f}}^T = \left[\hat{f}_1 \ \hat{f}_2 \ldots \ \hat{f}_n\right] \quad (1.75)$$

$$\hat{\underline{h}}^T = \left[\hat{h}_1 \ \hat{h}_2 \ldots \ \hat{h}_m\right] \quad (1.76)$$

$$\underline{q}^T = \frac{\partial V}{\partial \hat{\underline{f}}}(0) \quad (1.77)$$

$$\underline{r}^T = \frac{\partial V}{\partial \hat{\underline{h}}}(0) \quad (1.78)$$

The equation (1.70) takes then the form

$$(M_1 + M_2)\ \hat{\underline{f}} - N\ \hat{\underline{h}} + \underline{q} = 0 \quad (1.79a)$$

$$N^T\ \hat{\underline{f}} - P\ \hat{\underline{h}} + \underline{r} = 0 \quad (1.79b)$$

or equivalently

$$(M_1 + M_2 - N\ P^{-1}\ N^T)\ \hat{\underline{f}} = N\ P^{-1}\ \underline{r} - \underline{q} \quad (1.80a)$$

$$\hat{\underline{h}} = P^{-1}\ N^T\ \hat{\underline{f}} + P^{-1}\ \underline{r} \quad (1.80b)$$

Defining

$$v = \frac{B}{A}\ u \quad (1.81)$$

yields

$$M_2 = \left[R_v(i-j)\right] \quad (1.82a)$$

$$N = \left[R_{uv}(i-j)\right] \quad (1.82b)$$

$$N^T = \left[R_{vu}(i-j)\right] \quad (1.82c)$$

$$P = \left[R_u(i-j)\right] \quad (1.82d)$$

Clearly, the matrix $M_2 - N\ P^{-1}\ N^T$ is positive definite. Since M_1 is positive definite, the matrix $M_1 + M_2 - N\ P^{-1}\ N^T$, as a sum of two symmetric positive definite matrices, is positive definite and then non-singular.

The equation (1.80) has thus a unique solution, given by

$$\hat{\underline{f}} = (M_1+M_2-NP^{-1}N^T)^{-1}(NP^{-1}\underline{r}-q) \qquad (1.83a)$$

$$\hat{\underline{h}} = P^{-1}N^T(M_1+M_2-NP^{-1}N^T)^{-1}(NP^{-1}\underline{r}-q) + P^{-1}\underline{r} \qquad (1.83b)$$

This proves that the gradients of V vanish for one and only one value of $\hat{\underline{p}}$. From Theorem 1.2, this value coincides with $\underline{p}$. We have thus established that under assumption A1-A6, $S_2 = \left\{ \underline{p} \right\}$ and that

Theorem 1.3

The M.L. estimator E_2 is asymptotically consistent.

A direct consequence of the above study is that - provided the assumptions A1-A6 are verified - the ML method, in case of an AR noise model, is convergent.

It must be noticed that the Generalized Least-Squares (GLS) identification algorithm, proposed by Clarke [6]is nothing else than a ML method for which the loss function is minimized via a gradient algorithm. The theorems 1.2 and 1.3 thus establish the convergence proof of the GLS method.

Before concluding we shall proceed to make some comments about the various assumptions which have been made.

4.4. Comments

The reason why a finite N can cause difficulties is that the values of $V_N(\hat{\underline{p}})$, for fixed $\hat{\underline{p}}$, are stochastic variables while $V(\hat{\underline{p}})$, for fixed $\hat{\underline{p}}$, is a deterministic function.

The assumption A6 is a restrictive assumption since no minimization algorithm can guarantee that this condition is fulfilled at each stage of the procedure. It is however easy, when a stationary point is reached, to test if it is so or not.

In the whole section, it was assumed that the noise process may be modelled as an AR process

$$C(z^{-1})\ n(k) = e(k) \tag{1.84}$$

This means that there exists a noise whitening filter which can be represented by its truncated impulse response $\{c_i, i=0,\ldots,n_c\}$ This assumption has the drawback that there are no systematic rules for choosing the order n_c of the autoregression such that the c_i's, for $i > n_c$, can be neglected. However, we can reasonably argue the following points:

- an analysis of the noise can facilitate the choice of the order of the autoregression
- n_c can be determined iteratively by increasing it until the reduction of the loss function is no longer significant.

5 - CONCLUSION

In this Chapter, the ML estimator, for linear single input-single output systems, has been investigated. The statistical properties of the ML estimator have been established: it has been shown to depend on the noise model structure considered. For this purpose, two new results have been obtained

- for a MA noise model, the ML method may not converge. Counterexamples to general convergence can be found.
- for an AR noise model, the ML method is convergent.

In the next Chapter, the above study will be extended to the multivariable case where it will be shown that, once a canonical structure is defined, the extension is straightforward.

Chapter 2: MULTIVARIABLE SYSTEMS

1 - INTRODUCTION

As outlined in the previous chapter, the general definition of systems identification allows many degrees of freedom in the formulation of the identification problem, leading to take into account the following points :

- The choice of a class of models : impulse response - transfer function - state-space representation.
- The determination of a class of input signals
- The choice of the estimator
- The choice of the optimization algorithm.

When formulating and solving such problems, it is important to have the particular type of the system treated and the final goal of the identification procedure in mind. This cannot then be done from a purely mathematical point of view. It is, however, highly desirable to achieve a unified approach of these problems and, by embedding them in an abstract framework, obtain general results.

(i) The choice made for the class of the models of multivariable systems is generally that of state-space representation. Such a choice implies

- the determination of the system order and of a suitable canonical form

- the derivation of a set of input-output relations.

The use of a selector matrix S, as suggested by Gopinath [12], or of a set N of indexes, as shown by Mayne [16] allows one to describe the structure of a canonical model. These approaches have the drawback that there is no method for the choice of S or N. However, since such a method should be applied to input-output sequences, this problem cannot be performed in realistic cases - i.e. when no precise a priori knowledge of the noise characteristics is available. The various aspects of the determination of a canonical structure will be treated in Section 2.

Although the problem of deriving a set of input-output relations from the canonical state-space representation has been investigated by many authors, no unified approach can be found in literature. The methods proposed by Gopinath [12], Ackerman [1], Zuercher [28] or Guidorzi [13] are subject to strong restrictions. In Section 3, it is shown how to work out this problem in the most general case, permitting the decomposition of the system into subsystems. Moreover this approach is extended to the noisy case in which a suitable noise model is introduced in such a fashion as to preserve the advantages of the decomposition - even in the case of correlated noise.

(ii) In Section 4, a Maximum Likelihood estimator is described and the statistical properties of the Maximum Likelihood estimates are investigated.

(iii) Finally, in Section 5, the computational aspects are discussed. Two classes of optimization algorithms are described and compared.

2 - CANONICAL STRUCTURES

2.1. Introduction

The class of linear discrete-time multivariable systems considered is represented by the state equations

$$\underline{z}(k+1) = F\underline{z}(k) + G\underline{u}(k) \tag{2.1a}$$

$$\underline{y}(k) = H\underline{z}(k) \tag{2.1b}$$

where $\underline{z}(k)\epsilon R^n$ (state vector), $\underline{u}(k)\epsilon R^m$ (input vector), $\underline{y}(k)\epsilon R^r$ (output vector), $F\epsilon R^n \times R^n$ (state matrix), $G \epsilon R^n \times R^m$ (control matrix) and $H \epsilon R^r \times R^m$ (observation matrix).

It is well known (see e.g. [15]) that (F_1, G_1, H_1) and (F_2, G_2, H_2) are similar - i.e. they lead to the same transfer function $K(z) = H(zI - H)^{-1}G$ - if there exists a non-singular matrix $T \epsilon R^n \times R^n$ such that

$$F_2 = TF_1T^{-1} \tag{2.2a}$$

$$G_2 = TG_1 \tag{2.2b}$$

$$H_2 = H_1T^{-1} \tag{2.2c}$$

The non-uniqueness of (F, G, H) results in a degree of arbitrariness in the realization algorithm $K \longrightarrow (F, G, H)$, which cannot be removed by simple normalization.

The next sections show that each representation (F, G, H) is characterized by a set of integers $(n_1, n_2, \ldots, n_s)$ and that, if $(n_1, n_2, \ldots, n_s)$ is known, the representation (F, G, H) of (2.1) is unique.

2.2. Preliminaries

Let

$$\Gamma = \left[H^T \; (HF)^T \ldots \; (HF^{n-1})^T\right]^T \quad \Delta = \left[G \; FG \ldots \; F^{n-1}G\right] \qquad (2.3)$$

be, respectively, the observability and controllability matrices. (F, G) and (F, H) are assumed to be, respectively, controllable and observable, which is equivalent to

$$\text{rank}(\Gamma) = \text{rank}(\Delta) = n \qquad (2.4)$$

Let $\underline{h}_i^T$ denote the i-th row of H and let I and J denote the sets

$$I = \{1, 2, \ldots, r\} \qquad (2.5)$$

$$J = \{j_1, j_2, \ldots, j_s\} \qquad (2.6)$$

where $1 \leqslant j_k \leqslant r$ for all $k = 1, 2, \ldots s$, $s \leqslant r$. J is a permutation of any subspace of I.

Let T be the non-singular (n x n) matrix defined by

$$T = \begin{bmatrix} T_{j_1} \\ T_{j_2} \\ \vdots \\ T_{j_s} \end{bmatrix} \qquad\qquad T_i = \begin{bmatrix} \underline{h}_i^T \\ \underline{h}_i^T F \\ \vdots \\ \underline{h}_i^T F^{n_i} \end{bmatrix} \quad i \in J \tag{2.7}$$

Since the pair (F,H) is observable, T non-singular implies

$$\sum_{i \in J} (n_i+1) = s \tag{2.8}$$

$$T = P \tag{2.9}$$

where P is a permutation matrix.

From (2.7) and (2.9) it is clear that the transformation matrix T is completely defined by either the set $\{n_{j_1}, n_{j_2}, \ldots, n_{j_s}\}$ or the matrix P. This matrix is nothing else than the selector matrix, which is the basic concept of the approach suggested by Gopinath [12] . Although these two definitions of T are equivalent, the set of indexes $\{n_{j_1}, n_{j_2}, \ldots, n_{j_s}\}$, rather than P, will be used in the following.

2.3. Canonical forms

Defining the new state vector $\underline{x}(k) = T\underline{z}(k)$ yields the state-space equations

$$\underline{x}(k+1) = A\underline{x}(k) + B\underline{u}(k) \tag{2.10a}$$

$$\underline{y}(k) = C\underline{x}(k) \tag{2.10b}$$

where (A, B, C) is obtained from (F, G, H) through (2.2).

We can now establish the

<u>Lemma 2.1</u> :

Let

- $\underline{a}_i^T$ be the i-th row of A
- $\underline{b}_i^T$ be the i-th row of B
- $\underline{c}_i^T$ be the i-th row of C
- $\underline{e}_i^T$ be the i-th unit vector
- ν_i, i = 1,2,...s, the indexes

$$\nu_1 = 1 \qquad \nu_i = \sum_{k=1}^{i-1} n_{j_k} + i$$

Then

$$\text{(i)} \quad \underline{a}_i^T = \underline{e}_{i+1}^T \qquad i \in \{1,2,\ldots,n\} - \{\nu_1, \nu_2, \ldots, \nu_s\} \tag{2.11a}$$

$$\underline{a}_i^T = \underline{a}_i^T \qquad i \in \{\nu_1, \nu_2, \ldots, \nu_s\} \tag{2.11b}$$

$$\text{(ii)} \quad \underline{c}_{j_i}^T = \underline{e}_{\nu_i}^T \qquad i = 1,2,\ldots,s \tag{2.12a}$$

$$\underline{c}_i^T = \underline{c}_i^T \qquad i \in \{1,2,\ldots,r\} - \{j_1, \ldots j_s\} \tag{2.12b}$$

Before investigating these results further, it is necessary to prove the

<u>Lemma 2.2</u> :

The triplet (A, B, C), defined by (2.11) - (2.12) is unique.

<u>Proof</u> :

Let us assume there exist two representations (A_1, B_1, C_1) and (A_2, B_2, C_2) of (2.10)

Let Γ_1 and Γ_2 be the corresponding observability matrices. Since the representations are similar,

$$\Gamma_1 = \Gamma_2 T \tag{2.13}$$

and there exists a non singular permutation matrix P such that

$$P\Gamma_1 = \begin{bmatrix} I_n \\ \hline M_1 \end{bmatrix} \qquad P\Gamma_2 = \begin{bmatrix} I_n \\ \hline M_2 \end{bmatrix} \tag{2.14}$$

where I_n is the n x n unit matrix.

Combining (2.13) and (2.14) yields

$$P\Gamma_1 = P\Gamma_2 T = \begin{bmatrix} T \\ \hline M_2 T \end{bmatrix} \tag{2.15}$$

and $T = I_n$. QED

The lemma 2.1 and 2.2 establish that the knowledge of a set of indexes $\{n_1, n_2, \ldots n_s\}$ implies the uniqueness of the triplet (A, B, C) characterizing the state representation of a multivariable system.

In the following, we can assume, without loss of generality, that

$$j_i = i \qquad i = 1, \ldots, s \tag{2.16}$$

i.e.

$$J = \{1, 2, \ldots s\} \quad 1 \leqslant s \leqslant r \tag{2.17}$$

Hence, the canonical form of the triplet (A, B, C) , given in (2.11) and (2.12) can be rewritten. The matrices A, B and C exhibit the following structure

(i) A matrix

$$A = \begin{array}{c} \left[\begin{array}{c} A_1 \\ \hdashline A_2 \\ \hdashline \vdots \\ \hdashline A_s \end{array}\right] \begin{array}{l} n_1+1 \\ \\ n_2+1 \\ \\ \\ \\ n_s+1 \end{array} \\ \longleftarrow n \longrightarrow \end{array} \qquad \sum_{1}^{s} (n_i+1) = n \tag{2.18}$$

where $A_i, i=1,2,\ldots,s,$ is the $\left[(n_i+1) \times n\right]$ matrix

$$A_i = \begin{array}{c} \longleftarrow n \longrightarrow \\ \longleftarrow \nu_i \longrightarrow \qquad\qquad \\ \left[\begin{array}{cccccccc} 0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \multicolumn{8}{c}{\text{———————— } \underline{a}_i^T \text{ ————————}} \end{array}\right] \end{array} \; n_i+1 \tag{2.19}$$

(ii) B matrix

$$B = \begin{array}{c} \left[\begin{array}{c} B_1 \\ \hdashline B_2 \\ \hdashline \vdots \\ \hdashline B_s \end{array}\right] \begin{array}{l} n_1+1 \\ \\ n_2+1 \\ \\ \\ \\ n_s+1 \end{array} \; n \\ \longleftarrow m \longrightarrow \end{array} \tag{2.20}$$

where B_i, $i = 1,2,\ldots,s$, is the $(n_i+1) \times m$ matrix

$$B_i = \underbrace{\begin{bmatrix} \underline{b}_{\nu_i}^T \\ \underline{b}_{\nu_i+1}^T \\ \vdots \\ \underline{b}_{\nu_i+n_i}^T \end{bmatrix}}_{m} \; \updownarrow n_i+1 \tag{2.21}$$

(iii) <u>C matrix</u>

$$C = \underbrace{\left[\begin{array}{c} \underline{e}_{\nu_1}^T \\ \vdots \\ \underline{e}_{\nu_s}^T \\ \hdashline \underline{c}_1^T \\ \vdots \\ \underline{c}_{r-s}^T \end{array}\right]}_{n} \; \begin{array}{c} \updownarrow s \\ \\ \\ \updownarrow r-s \end{array} \tag{2.22}$$

The state equations take then the form

$$\left[\begin{array}{c} x_{\nu_1}(k+1) \\ \vdots \\ x_{\nu_1+n_1}(k+1) \\ \hline x_{\nu_2}(k+1) \\ \vdots \\ x_{\nu_2+n_2}(k+1) \\ \hline \vdots \\ \hline x_{\nu_s}(k+1) \\ \vdots \\ x_{\nu_s+n_s}(k+1) \end{array}\right] = \left[\begin{array}{c} \\ A_1 \\ \\ \hline \\ A_2 \\ \\ \hline \\ \hline \\ A_s \\ \\ \end{array}\right] \left[\begin{array}{c} x_{\nu_1}(k) \\ \vdots \\ x_{\nu_1+n_1}(k) \\ \hline x_{\nu_2}(k) \\ \vdots \\ x_{\nu_2+n_2}(k) \\ \hline \vdots \\ \hline x_{\nu_s}(k) \\ \vdots \\ x_{\nu_s+n_s}(k) \end{array}\right] + \left[\begin{array}{c} \underline{b}_{\nu_1}^T \\ \vdots \\ \underline{b}_{\nu_1+n_1}^T \\ \hline \underline{b}_{\nu_2}^T \\ \vdots \\ \underline{b}_{\nu_2+n_2}^T \\ \hline \vdots \\ \hline \underline{b}_{\nu_s}^T \\ \vdots \\ \underline{b}_{\nu_s+n_s}^T \end{array}\right] \underline{u}(k)$$

$$(2.23a)$$

$$\left[\begin{array}{c} \underline{y}_1(k) \\ \hline \underline{y}_2(k) \end{array}\right] = \left[\begin{array}{c} \left[\underline{e}_{\nu_i}^T\right]_{i=1,\ldots,s} \\ \hline \bar{C} \end{array}\right] \underline{x}(k) \qquad (2.23b)$$

where

$$\underline{y}_1(k) = [y_1(k)\ y_2(k)\dots y_s(k)]^T \qquad (2.24a)$$

$$\underline{y}_2(k) = [y_{s+1}(k)\ y_{s+2}(k)\dots y_r(k)]^T \qquad (2.24b)$$

Having obtained a canonical structure for the matrices A, B and C, a set of input-output equations can now be deduced from (A, B, C). Remarkable features of this formulation are that it exhibits the structure of the system and that it shows the link between the internal (state-space) representation and the external (input-output) equations. This extends the results of Gopinath [12], Zuercher [28] and Guidorzi[13].

3 - CANONICAL INPUT-OUTPUT EQUATIONS

3.1 Introduction

A direct consequence of the structure of the matrices A and C, derived in the above Section, is that the original system has been decomposed into s interconnected subsystems. The main goal of this Section is to derive an input-output representation of the system such that the subsystems can be separately identified.

Section 3.2 will be concerned with the noise-free case while the noisy case will be treated in Section 3.3.

3.2. Noise-free version

According to (2.23) the output vector $\underline{y}(k)$ can be divided into two components $\underline{y}_1(k)$ and $\underline{y}_2(k)$, $\underline{y}_1(k)$ [resp. $\underline{y}_2(k)$] being related to $\underline{x}(k)$ through a relation whose parameters are known [resp. unknown]. The aim of this section is the derivation of an input-output equation $\underline{u} \longrightarrow \underline{y}_1$, involving the parameters of the matrices A and B and then an input-output equation $\underline{u} \longrightarrow \underline{y}_2$ containing the unknown parameters of C.

3.2.1. Parameters of A and B

From (2.19) and (2.21), it follows that the i-th subsystem, i = 1,2,...,s, is described by the set of equations

$$
\begin{aligned}
x_{\nu_i}(k) &= x_{\nu_i+1}(k) + \underline{b}_{\nu_i}^T \underline{u}(k) \\
x_{\nu_i+1}(k) &= x_{\nu_i+2}(k) + \underline{b}_{\nu_i+1}^T \underline{u}(k) \\
&\cdots\cdots\cdots\cdots\cdots \\
x_{\nu_i+n_i-1}(k) &= x_{\nu_i+n_i}(k) + \underline{b}_{\nu_i+n_i-1} \underline{u}(k) \\
x_{\nu_i+n_i}(k) &= \underline{a}_i^T \underline{x}(k) + \underline{b}_{\nu_i+n_i}^T \underline{u}(k)
\end{aligned}
\qquad (2.25)
$$

Combining (2.22) and (2.25) leads to

$$
\begin{aligned}
x_{\nu_i}(k) &= y_i(k) \\
x_{\nu_i+1}(k) &= zy_i(k) - \underline{b}_{\nu_i}^T \underline{u}(k) \\
x_{\nu_i+2}(k) &= z^2 y_i(k) - \underline{b}_{\nu_i+1}^T \underline{u}(k) - \underline{b}_{\nu_i}^T z\underline{u}(k) \\
&\cdots\cdots\cdots\cdots\cdots \\
x_{\nu_i+n_i}(k) &= z^{n_i} y_i(k) - \underline{b}_{\nu_i+n_i-1}^T u(k) - \ldots - \underline{b}_{\nu_i}^T z^{n_i-1} \underline{u}(k)
\end{aligned}
\tag{2.26}
$$

Remark

It must be emphasized that z^{-1} is the time delay operator, defined by $z^{-j} f(k) = f(k-j)$.

Then, since

$$
\underline{x}(k) = \left[x_{\nu_1}(k) \ \ldots \ x_{\nu_1+n_1}(k) \ \middle| \ \ldots \ \middle| \ x_{\nu_s}(k) \ \ldots \ x_{\nu_s+n_s}(k) \right]^T \tag{2.27}
$$

the whole state vector can be expressed as

$$
\underline{x}(k) = M(z)\,\underline{y}_1(k) - N(z)\,\underline{u}(k) \tag{2.28}
$$

where

$$M(z) = \left[\begin{array}{c:c:c:c} \begin{matrix}1\\z\\ \cdot\\ \cdot\\ z^{n_1}\end{matrix} & \underline{0} & \cdots & \underline{0} \\ \hdashline \underline{0} & \begin{matrix}1\\z\\ \cdot\\ \cdot\\ z^{n_2}\end{matrix} & & \underline{0} \\ \hdashline \\ \hdashline \underline{0} & \underline{0} & & \begin{matrix}1\\z\\ \cdot\\ \cdot\\ z^{n_s}\end{matrix} \end{array}\right] \quad (n \times s) \tag{2.29}$$

and

$$N(z) = \left[\begin{array}{c} 0 \\ \underline{b}_{\nu_1}^T \\ \cdot \\ \cdot \\ \underline{b}_{\nu_1+n_1-1}^T + z\underline{b}_{\nu_1+n_1-2}^T + \cdots + z^{n_1-1}\underline{b}_{\nu_1}^T \\ \hline 0 \\ \underline{b}_{\nu_2}^T \\ \cdot \\ \cdot \\ \underline{b}_{\nu_2+n_2-1}^T + z\underline{b}_{\nu_2+n_2-2}^T + \cdots + z^{n_2-1}\underline{b}_{\nu_2}^T \\ \hline \vdots \\ \hline 0 \\ \underline{b}_{\nu_s}^T \\ \cdot \\ \cdot \\ \underline{b}_{\nu_s+n_s-1}^T + z\underline{b}_{\nu_s+n_s-2}^T + \cdots + z^{n_s-1}\underline{b}_{\nu_s}^T \end{array}\right] \quad (n \times m)$$

(2.30)

The substitution of (2.28) into (2.10a) leads to the input-output relation

$$\Big[(zI - A)\, M\Big]\, \underline{y}_1(k) = \Big[(zI - A)\, N + B\Big]\, \underline{u}(k) \qquad (2.31)$$

In (2.31), however, only the $(\nu_2 - 1)$th, $(\nu_3 - 1)$th, ..., $(\nu_s - 1)$th, n-th equations are significant, the remaining ones being identities. By removing these identities, (2.31) takes the form

$$P(z)\ \underline{y}_1(k) = Q(z)\ \underline{u}(k) \tag{2.32}$$

where P and Q are polynomial matrices in z

$$P(z) = \left[P^*_{ij}(z)\right] \quad \begin{matrix} i = 1,..,s \\ j = 1,..,s \end{matrix} \tag{2.33a}$$

$$Q(z) = \left[Q^*_{ij}(z)\right] \quad \begin{matrix} i = 1,..,s \\ j = 1,..,m \end{matrix} \tag{2.33b}$$

The polynomials $P^*_{ij}(z)$ and $Q^*_{ij}(z)$ are obtained by simple inspection of (2.31). It follows

$$P^*_{ij}(z) = \delta_{ij} z^{n_j+1} - a_{i,\nu_j+n_j} z^{n_j} - \dots - a_{i,\nu_j+1} z - a_{i,\nu_j}$$

$$i = 1,\dots,s \qquad j = 1,\dots,s \tag{2.34a}$$

$$Q^*_{ij}(z) = \beta_{\nu_i+n_i,j} z^{n_i} + \dots + \beta_{\nu_i+1,j} z + \beta_{\nu_i,j}$$

$$i = 1,\dots,s \qquad j = 1,\dots,m \tag{2.34b}$$

where the β's are related to the a's and b's through the equation

$$\begin{bmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1m} \\ \beta_{21} & & & \beta_{2m} \\ \vdots & & & \\ \beta_{n1} & \cdots & \cdots & \beta_{nm} \end{bmatrix} = L \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & & & b_{2m} \\ \vdots & & & \\ b_{n1} & \cdots & \cdots & b_{nm} \end{bmatrix} \tag{2.35}$$

with

$$L = \begin{bmatrix} L_{ij} \end{bmatrix} \begin{matrix} i = 1,..,s \\ j = 1,..,s \end{matrix} \tag{2.36}$$

$$L_{ii} = \begin{bmatrix} -a_{i,\nu_i+1} & -a_{i,\nu_i+2} & \cdots & -a_{i,\nu_i+n_i} & 1 \\ -a_{i,\nu_i+2} & & & 1 & \\ \vdots & & \cdot & & \\ \vdots & \cdot & & & \\ -a_{i,\nu_i+n_i} & & & 0 & \\ 1 & & & & \end{bmatrix} \tag{2.37a}$$

$$L_{ij} = \begin{bmatrix} -a_{i,\nu_j+1} & -a_{i,\nu_j+2} & \cdots & -a_{i,\nu_j+n_j} & 0 \\ -a_{i,\nu_j+2} & & & 0 & \\ \vdots & & \cdot & & \\ \vdots & \cdot & & & \\ -a_{i,\nu_j+n_j} & & & 0 & \\ 0 & & & & \\ 0 & & & & \end{bmatrix} \tag{2.37b}$$

the matrices L_{ij} being $\left[(n_i + 1) \times (n_j + 1)\right]$.

Remark

The matrix L is always non singular : det (L) = 1, since det $(L_{ij}) = \delta_{ij}$.

We then obtain s subsystems whose input-output description is

$$P^{*}_{i1}(z)\, y_1(k) + \dots + P^{*}_{is}(z)\, y_s(k) =$$

$$Q^{*}_{i1}(z)\, u_1(k) + \dots + Q^{*}_{im}(z)\, u_m(k) \qquad i=1,..,s \tag{2.38}$$

Defining the integers

$$\bar{n}_i = \max\,(n_1, n_2, \dots, n_{i-1}, n_i+1, n_{i+1}, \dots, n_s) \qquad i=1,\dots,s \tag{2.39}$$

and the parameters

$$p^{i}_{jk} = -\, a_{i,\nu_j+n_j-k+1} \qquad j=1,..,s \qquad k=1,..,n_j+1$$

$$q^{i}_{jk} = \beta_{\nu_i+n_i-k+1,\,j} \qquad j=1,..,s \qquad k=1,..,n_i+1$$

$$i=1,..,s \tag{2.40}$$

allows us to define the reciprocal polynomials

$$P_{ij}(z^{-1}) = z^{-\bar{n}_i}\, P^{*}_{ij}(z)$$

$$= \delta_{ij} z^{n_j-\bar{n}_i+1} + p^{i}_{j1} z^{n_j-\bar{n}_i} + \dots + p^{i}_{j,n_j+1} z^{-\bar{n}_i}$$

$$= \delta_{ij} z^{n_j-\bar{n}_i+1} + \sum_{k=1}^{n_j+1} p^{i}_{jk} z^{n_j-\bar{n}_i-k+1}$$

$$i=1,..,s \quad j=1,..,s \tag{2.41}$$

$$Q_{ij}(z^{-1}) = z^{-\bar{n}_i} Q^*_{ij}(z)$$

$$= q^i_{j1} z^{n_i-\bar{n}_i} + q^i_{j2} z^{n_i-\bar{n}_i+1} + \dots + q^i_{j,n_i+1} z^{-\bar{n}_i}$$

$$= \sum_{k=1}^{n_j+1} q^i_{jk} z^{n_i-\bar{n}_i-k+1}$$

$$i=1,\dots,s \quad j=1,\dots,m \tag{2.42}$$

and to rewrite (2.38) in the form

$$P_{i1}(z^{-1})\, y_1(k) + \dots + P_{is}(z^{-1})\, y_s(k) =$$

$$Q_{i1}(z^{-1})\, u_1(k) + \dots + Q_{im}(z^{-1})\, u_m(k) \qquad i=1,..,s \tag{2.43}$$

In summary, a set of s input-output equations of the form

$$\sum_{j=1}^{s} P_{ij}(z^{-1})\, y_j(k) = \sum_{j=1}^{m} Q_{ij}(z^{-1})\, u_j(k) \tag{2.44a}$$

$$P_{ij}(z^{-1}) = \delta_{ij} z^{n_j-\bar{n}_i+1} + \sum_{k=1}^{n_j+1} p^i_{jk} z^{n_j-\bar{n}_i-k+1} \qquad i=1,..,s \tag{2.44b}$$

$$Q_{ij}(z^{-1}) = \sum_{k=1}^{n_i+1} q^i_{jk} z^{n_i-\bar{n}_i-k+1} \qquad j=1,..,s \tag{2.44c}$$

$$i=1,\dots,s$$

has been obtained.

Recalling (2.40), (2.44) can be used in order to estimate the parameters of A and B.

3.2.2. Parameters of C

In equations (2.44), which allow us to identify the system dynamics, only the information contained in $\underline{u}$ and $\underline{y}_1$ is used. In order to identify the matrix C, we shall now derive a set of equations making use of the information provided by $\underline{y}_2$.

According to (2.23), it follows

$$\underline{y}_2(k) = \overline{C}\underline{x}(k) \qquad (2.45)$$

where

$$\overline{C} = \begin{bmatrix} \underline{c}_1^T \\ \underline{c}_2^T \\ \vdots \\ \underline{c}_{r-s}^T \end{bmatrix} \qquad (2.46)$$

Then, combining (2.46) and (2.28) yields

$$\underline{y}_2(k) = \overline{C}M(z)\,\underline{y}_1(k) - \overline{C}N(z)\,\underline{u}(k) \qquad (2.47)$$

or, equivalently

$$P'(z)\,\underline{y}_1(k) - \underline{y}_2(k) = Q'(z)\,\underline{u}(k) \qquad (2.48)$$

where $P'(z)$ and $Q'(z)$ are the matrices

$$P'(z) = \overline{C}M(z) = \left[P'^{*}_{ij}(z) \right]_{\substack{i=1,\ldots,r-s \\ j=1,\ldots,s}} \qquad (2.49a)$$

$$Q'(z) = \overline{C}N(z) = \left[Q'^{*}_{ij}(z) \right]_{\substack{i=1,\ldots,r-s \\ j=1,\ldots,m}} \qquad (2.49b)$$

whose constitutive element P'^{*}_{ij} and Q'^{*}_{ij} are obtained from simple calculations

$$P'^{*}_{ij}(z) = c_{i,\nu_j+n_j} z^{n_j} + c_{i,\nu_j+n_j-1} z^{n_j-1} + \ldots + c_{i,\nu_j+1} z + c_{i,\nu_j}$$

$$i = 1,\ldots,r-s \qquad j = 1,\ldots,s \qquad (2.50a)$$

$$Q'^{*}_{ij}(z) = \gamma_{j\tilde{n},i} z^{\tilde{n}-1} + \gamma_{j\tilde{n}-1,i} z^{\tilde{n}-2} + \ldots + \gamma_{(j-1)\tilde{n},i} z + \gamma_{(j-1)\tilde{n}+1,i}$$

$$i = 1,\ldots,r-s \qquad j = 1,\ldots,m \qquad (2.50b)$$

in which

$$\tilde{n} = \max(n_i) \qquad (2.51)$$

$$\gamma_{(j-1)\tilde{n}+k,i} = \sum_{u=1}^{s} \sum_{v=1}^{n_u-k+1} c_{i,\nu_u+v+k-1}\, b_{\nu_u+v-1,j} \qquad (2.52)$$

We then obtain r-s subsystems whose input-output description is

$$P_{i1}^{'*}(z)\, y_1(k) + \dots + P_{is}^{'*}\, y_s(k) - y_{s+i}(k) =$$

$$Q_{i1}^{'*}(z)\, u_1(k) + \dots + Q_{im}^{'*}(z)\, u_m(k) \qquad i=1,\dots,r-s \qquad (2.53)$$

Recalling (2.51) and introducing the parameters

$$p_{jk}^{i} = c_{i,\nu_j+n_j-k+1} \qquad j=1,..,s \quad k=1,..,n_j+1$$

$$q_{jk}^{i} = \gamma_{jn-k+1,i} \qquad j=1,\dots,m \quad k=1,..,\tilde{n}$$

$$i=1,\dots,r-s \qquad (2.54)$$

allows us to define the reciprocal polynomials

$$\begin{aligned} P_{ij}^{'}(z^{-1}) &= z^{-\tilde{n}}\, P_{ij}^{'*}(z) \\ &= p_{j1}^{i} z^{n_j-\tilde{n}} + p_{j2}^{i} z^{n_j-\tilde{n}+1} + \dots + p_{j,n_j+1} z^{-\tilde{n}} \\ &= \sum_{k=1}^{n_j+1} p_{jk}^{i} z^{n_j-\tilde{n}-k+1} \qquad i=1,..,r-s \quad j=1,..,s \qquad (2.55) \end{aligned}$$

$$\begin{aligned} Q_{ij}^{'}(z^{-1}) &= z^{-\tilde{n}}\, Q_{ij}^{'*}(z) \\ &= q_{j1}^{i} z^{-1} + q_{j2}^{i} z^{-2} + \dots + q_{j,n}^{i} z^{-\tilde{n}} \\ &= \sum_{k=1}^{\tilde{n}} q_{jk}^{i} z^{-k} \qquad i=1,..,r-s \quad j=1,..,m \qquad (2.56) \end{aligned}$$

and to rewrite (2.53) in the form

$$P'_{i1}(z^{-1})\ y_1(k) + \ldots + P'_{is}(z^{-1})\ y_s(k) - y_{s+i}(k) =$$

$$Q'_{i1}(z^{-1})\ u_1(k) + \ldots + Q'_{im}(z^{-1})\ u_m(k) \qquad i=1,\ldots,r-s \qquad (2.57)$$

Remark

Using (2.54) and (2.52), the q^i_{jk} 's can be expressed in term of the p^i_{jk} 's

$$q^i_{jk} = \sum_{u=1}^{s} \sum_{v=1}^{n_u-\tilde{n}+k} p^i_{u,n_u-v-\tilde{n}+k+1}\ b_{\nu_u+v-1,j} \qquad (2.58)$$

so that the number of unknown parameters in (2.53) is n.

In summary, a set of r-s input-output equations of the form

$$\sum_{j=1}^{s} P'_{ij}(z^{-1})\ y_j(k) - y_{s+i}(k) = \sum_{j=1}^{m} Q'_{ij}(z^{-1})\ u_j(k) \qquad (2.59a)$$

$$P'_{ij}(z^{-1}) = \sum_{k=1}^{n_j+1} p^i_{jk} z^{n_j-\tilde{n}-k+1} \qquad j=1,\ldots,s \qquad (2.59b)$$

$$Q'_{ij}(z^{-1}) = \sum_{k=1}^{n} q^i_{jk} z^{-k} \qquad j=1,\ldots,m \qquad (2.59c)$$

$i=1,\ldots,r-s$

has been obtained.

Recalling (2.54), (2.59) allows us to estimate the parameters of the matrix C.

Equations (2.44) and (2.59) constitute the canonical input-output description of (2.10). The equivalence between the class of canonical state-space representations (2.10) and the input-output description (2.44)-(2.59) has thus been obtained.

In the next Section, the results obtained above are extended to the case of data corrupted by noise.

3.3. Noisy version

If we assume that the input and output sequences are corrupted by an additive noise

$$y_i(k) = \tilde{y}_i(k) + v_i(k) \qquad i=1,\ldots,r \tag{2.60}$$

$$u_i(k) = \tilde{u}_i(k) + w_i(k) \qquad i=1,\ldots,m \tag{2.61}$$

the equations (2.44) and (2.59) take the form

$$P\,\underline{y}_1 = Q\,\underline{u} + P\,\underline{v}_1 + Q\,\underline{w} \tag{2.62}$$

$$P'\underline{y}_1 - \underline{y}_2 = Q'\underline{u} + P'\underline{v}_1 - \underline{v}_2 + Q'\underline{w} \tag{2.63}$$

where

$$\underline{v}_1(k) = \left[v_1(k) \quad v_2(k) \ \ldots \ v_s(k)\right]^T \tag{2.64}$$

$$\underline{v}_2(k) = \left[v_{s+1}(k) \quad v_{s+2}(k) \ \ldots \ v_r(k)\right]^T \tag{2.65}$$

$$\underline{v}(k) = \left[\underline{v}_1^T(k) \quad \underline{v}_2^T(k)\right]^T \tag{2.66}$$

$$\underline{w}(k) = \left[w_1(k) \quad w_2(k) \ \ldots \ w_m(k)\right]^T \tag{2.67}$$

The i-th component $n_i(k)$ of the noise vector

$$\underline{n}(k) = P\ \underline{v}_1 + Q\ \underline{w} \tag{2.68}$$

defined by

$$n_i(k) = \sum_{j=1}^{s} P_{ij}(z^{-1})\ v_i(k) + \sum_{j=1}^{m} Q_{ij}(z^{-1})\ w_i(k) \tag{2.69}$$

has a power-density spectrum given by

$$\begin{aligned}\Phi_{n_i}(\xi) = &\sum_{j=1}^{s} P_{ij}(\xi)\ P_{ij}(\xi^{-1})\ \Phi_{v_i}(\xi) \\ &+ \sum_{j=1}^{m} Q_{ij}(\xi)\ Q_{ij}(\xi^{-1})\ \Phi_{w_i}(\xi)\end{aligned} \tag{2.70}$$

where Φ_{v_i} and Φ_{w_i} are the power-density spectra of v_i and w_i.

Since $\Phi_{n_i}(\xi)$ can be expressed as the product

$$\Phi_{n_i}(\xi) = \Psi_{n_i}(\xi)\ \Psi_{n_i}(\xi^{-1}) \tag{2.71}$$

then

$$H_i(\xi^{-1}) = \frac{\sigma_i}{\Psi_{n_i}(\xi^{-1})} \tag{2.73}$$

σ_i constant

is the transfer function of a realizable filter.

Since, in addition, (2.73) is equivalent to

$$\Phi_{n_i}(\xi)\ H_i(\xi)\ H_i(\xi^{-1}) = \sigma_i^2 \tag{2.74}$$

$H_i(\xi^{-1})$ is the transfer function of a whitening filter.

It follows immediately that

$$n_i(k) = \frac{1}{H_i(z^{-1})} e_i(k) \tag{2.75}$$

where $e_i(k)$ is a sequence of independant random variables

$$E[e_i] = 0 \tag{2.76a}$$

$$E[e_i(j)\ e_i(k)] = \sigma_i^2\ \delta_{jk} \tag{2.76b}$$

If the whitening filter is represented by its impulse response, the process (2.75) takes the form of an auto-regressive process

$$n_i(k) = \frac{1}{D_i(z^{-1})} e_i(k) \tag{2.77}$$

where

$$D_i(z^{-1}) = 1 + \sum_{j=1}^{n_d^i} d_j^i z^{-j} \tag{2.78}$$

The equation (2.62) can, then, be rewritten in the form

$$P\ \underline{y}_1 = Q\ \underline{u} + D\ \underline{e} \tag{2.79}$$

where

$$\underline{e}(k) = \left[e_1(k)\ e_2(k)\ \dots\ e_s(k)\right]^T \tag{2.80}$$

$$D(z^{-1}) = \mathrm{diag}\left[\frac{1}{D_1(z^{-1})}\ \dots\ \frac{1}{D_s(z^{-1})}\right] \tag{2.81}$$

The equation (2.63), in the same manner, becomes

$$P'\underline{y}_1 - \underline{y}_2 = Q\,\underline{u} + D'\underline{e}' \tag{2.82}$$

where

$$\underline{e}'(k) = \left[e_1'(k)\ e_2'(k)\ \dots\ e_{r-s}'(k)\right]^T \tag{2.83}$$

$$D'(z^{-1}) = \mathrm{diag}\left[\frac{1}{D_1'(z^{-1})}\ \cdots\ \frac{1}{D_{r-s}'(z^{-1})}\right] \tag{2.84}$$

We then obtain the noisy version of the equations (2.44) and (2.59)

$$\sum_{j=1}^{s} P_{ij}(z^{-1})\, y_j(k) = \sum_{j=1}^{m} Q_{ij}(z^{-1})\, u_j(k) + \frac{1}{D_i(z^{-1})}\, e_i(k)$$

$$i=1,\dots,s \tag{2.85a}$$

$$\sum_{j=1}^{s} P_{ij}'(z^{-1})\, y_j(k) - y_{s+i}(k) = \sum_{j=1}^{m} Q_{ij}'(z^{-1})\, u_j(k) + \frac{1}{D_i'(z^{-1})}\, e_i'(k)$$

$$i=1,\dots,r-s \tag{2.85b}$$

These results generalize those obtained in Section 3.2. The equivalence between the state-space representation and the input-output description of a multivariable system has thus been established in the case of data corrupted by noise.

It must be pointed out that, as revealed by (2.85), the advantages of the decomposition of the system into subsystems has been preserved — in the sense that these subsystems can be separately identified - although no restrictive assumption on noise characteristics has been made.

It is now a simple matter to formulate the identification problem. However, before stating and solving this problem, we shall proceed to some comments.

3.4. Comments

The structure of the matrices A, B and C, given in Section 2, and, consequently, the structure of the input-output equations, derived in Section 3, result from an arbitrary choice of the indexes n_i. It would be of importance to propose an efficient method for the finding of the set $\{n_1, n_2, \ldots, n_s\}$, optimal with respect to a given criterion.

However, whatever criterion we choose, such a method should be based on the measured input-output sequences and would lead to the following problem : how can one take into account the noise whose statistics are unknown ? For example, the minimal representation case leads us to consider input-output data matrices : how can we assert that these matrices are singular or not ? (Is a matrix whose determinant is 10^{-10} singular ?). It then seems that this problem cannot be solved in realistic cases.

Nevertheless, the choice for the n_i's can be guided by some physical considerations :

- The structure may be assigned by the physical meaning of the state variables.
- If the j-th output of the system is markedly more free from noise than the remaining ones, a maximization of the corresponding n_j could be advantageous.
- The number of unknown parameters is

  in (2.85a) : $n+m(n_i+1)+n_d^i$ $\qquad i=1,\ldots,s$

  in (2.85b) : $n+n_d^i$ $\qquad i=1,\ldots,r-s$

  If n is a multiple of s, it can be judicious, in order to minimize the computing requirements, to choose $n_i = n/s$, since the subsystems are separately identified.

Another problem arises with regard to the noise model : introducing a noise filter increases the number of unknown parameters which affects the identification procedure, since the efficiency of any optimization algorithm decreases when the number of parameters increases. However, the noise model allows us

- to reduce the effect of the choice of the n_i's on the identification results since this effect is directly related to the amount of noise corrupting the different outputs.
- to improve the efficiency of some simple minimization algorithms, which are sensitive to noise.

Moreover such an approach provides a probabilistic framework which permits us to establish the mathematical properties of the estimates.

4 - MAXIMUM LIKELIHOOD ESTIMATOR

4.1. Introduction

The object of this Section is to present a Maximum Likelihood estimator. The equation of error and the likelihood function will be derived (Section 4.2) and the consistency of the Maximum Likelihood estimates will be proved (Section 4.3).

4.2. Equation of error

Recalling (2.85) leads us to choose the model

$$\sum_{j=1}^{s} \hat{P}_{ij}(z^{-1})\; y_j(t) = \sum_{j=1}^{m} \hat{Q}_{ij}(z^{-1})\; u_j(t) + \frac{1}{\hat{D}_i(z^{-1})}\; \epsilon_i(t)$$

$i=1,\ldots,s$ (2.86a)

$$\sum_{j=1}^{s} \hat{P}'_{ij}(z^{-1})\; y_j(t) - y_{s+i}(t) = \sum_{j=1}^{m} \hat{Q}'_{ij}(z^{-1})\; u_j(t) + \frac{1}{\hat{D}'_i(z^{-1})}\; \epsilon'_i(t)$$

$i=1,\ldots,r-s$ (2.86b)

where

$$\hat{P}_{ij}(z^{-1}) = \delta_{ij} z^{n_j - \bar{n}_i + 1} + \sum_{k=1}^{n_j+1} \hat{p}^{i}_{jk} z^{n_j - \bar{n}_i - k + 1} \qquad (2.87)$$

$$\hat{Q}_{ij}(z^{-1}) = \sum_{k=1}^{n_i+1} \hat{q}^{i}_{jk} z^{n_i - \bar{n}_i - k + 1} \qquad (2.88)$$

$$\hat{P}'_{ij}(z^{-1}) = \sum_{k=1}^{n_j+1} \hat{p}^{i\prime}_{jk} z^{n_j - \tilde{n} - k + 1} \qquad (2.89)$$

$$\hat{Q}'_{ij}(z^{-1}) = \sum_{k=1}^{\tilde{n}} \hat{q}^{i\prime}_{jk} z^{-k} \qquad (2.90)$$

$$\hat{D}_i(z^{-1}) = 1 + \sum_{j=1}^{n_d^i} \hat{d}^{i}_{j} z^{-j} \qquad (2.91)$$

$$\hat{D}'_i(z^{-1}) = 1 + \sum_{j=1}^{n_d^i} \hat{d}^{i\prime}_{j} z^{-j} \qquad (2.92)$$

The residuals $\epsilon_i(t)$ $\left[\epsilon'_i(t)\right]$ are independant and Gaussian $(0, \hat{\sigma}_i)$ $\left[(0, \hat{\sigma}'_i)\right]$.

Rewriting (2.86) in the form

$$\epsilon_i(t) = \hat{D}_i \left[\sum_{j=1}^{s} \hat{P}_{ij}\ y_j(t) - \sum_{j=1}^{m} \hat{Q}_{ij}\ u_j(t) \right]$$

$$i=1,\ldots,s \tag{2.93a}$$

$$\epsilon_i'(t) = \hat{D}_i' \left[\sum_{j=1}^{s} \hat{P}_{ij}'\ y_j(t) - y_{s+i}(t) - \sum_{j=1}^{m} \hat{Q}_{ij}'\ u_j(t) \right]$$

$$i=1,\ldots,r-s \tag{2.93b}$$

yields the so-called equations of error

$$\epsilon_i(t) = \sum_{h=0}^{n_d^i} \hat{d}_h^i \left[\sum_{j=1}^{s} \sum_{k=1}^{n_j+1} \hat{p}_{jk}^i\ y_j(t+n_j-\bar{n}_i-k-h+1) \right.$$

$$\left. + y_i(t+n_i-\bar{n}_i-h+1) - \sum_{j=1}^{m} \sum_{k=1}^{n_i+1} \hat{q}_{jk}^i\ u_j(t+n_i-\bar{n}_i-k-h+1) \right]$$

$$i=1,\ldots,s \tag{2.94a}$$

$$\epsilon_i'(t) = \sum_{h=0}^{n_d^i} \hat{d}_h^{i\prime} \left[\sum_{j=1}^{s} \sum_{k=1}^{n_j+1} \hat{p}_{jk}^{i\prime}\ y_j(t+n_j-\tilde{n}-k-h+1) \right.$$

$$\left. - y_{s+i}(t-h) - \sum_{j=1}^{m} \sum_{k=1}^{\tilde{n}} \hat{q}_{jk}^{i\prime}\ u_j(t-k-h) \right]$$

$$i=1,\ldots,r-s \tag{2.94b}$$

The Gaussianess of ϵ_i and ϵ_i' allows us to define the log-Likelihood functions

$$L_i = -\frac{1}{2\hat{\sigma}_i^2} \sum_{t=1}^{N} \epsilon_i^2(t) + N\log\hat{\sigma}_i + N\log 2\pi \quad i=1,..,s \qquad (2.95a)$$

$$L_i' = -\frac{1}{2\hat{\sigma}_i'^2} \sum_{t=1}^{N} \epsilon_i'^2(t) + N\log\hat{\sigma}_i' + N\log 2\pi \quad i=1,..,r-s \qquad (2.95b)$$

where N is the number of available data.

The Maximum Likelihood estimates of $\hat{P}_{ij}, \hat{Q}_{ij}, \hat{P}_{ij}'$ and $\hat{Q}_{ij}'$ are obtained by maximization of L_i and L_i' or equivalently by minimization of the loss functions

$$W_i = \frac{1}{2N} \sum_{t=1}^{N} \epsilon_i^2(t) \qquad i=1,\ldots,s \qquad (2.96a)$$

$$W_i' = \frac{1}{2N} \sum_{t=1}^{N} \epsilon_i'^2(t) \qquad i=1,\ldots,r-s \qquad (2.96b)$$

Having defined the Maximum Likelihood estimation problem as the finding of the global minima of W_i and W_i', we can determine the statistical properties of the estimates by examining these minima.

4.3. Statistical properties of the estimates

In the whole Section, the following assumptions are made :

- A1 . To ensure that the model structure is appropriate, it is assumed that (2.85) really holds.
- A2 . For all (i,j), the polynomials $P_{ij}(z^{-1})$, $P'_{ij}(z^{-1})$, $D_i(z^{-1})$ and $D'_i(z^{-1})$ are assumed to have all their zeros inside the unit circle. This implies that the equations (2.85) are stable.
- A3 . For all i=1,...,s and j=1,...,min(m,s) [i=1,...,r-s and j=1,...,min(m,r-s)] the polynomials $P_{ij}(z^{-1})$ and $Q_{ij}(z^{-1})$ [$P'_{ij}(z^{-1})$ and $Q'_{ij}(z^{-1})$] are relatively prime. These conditions are fulfilled provided the system is controllable from $\underline{u}$.
- A4 . All stochastic processes are ergodic.

Moreover

- Since, for a finite number of data, N, the properties of W_i and W'_i may change drastically from one experiment to another, the asymptotic loss functions

$$V_i = \lim_{N\to\infty} W_i \qquad i=1,\dots,s \tag{2.97a}$$

$$V'_i = \lim_{N\to\infty} W'_i \qquad i=1,\dots,r-s \tag{2.97b}$$

will be considered in the following.

- For convenience, $E\left[f(t)\right]$ will denote

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} f(t) \tag{2.98}$$

provided the limit exists. If f(t) is an ergodic stochastic process, $E\left[f(t)\right]$ is the expected value of f(t).

The output data are governed by

$$P\,\underline{y}_1 = Q\,\underline{u} + D\,\underline{e} \tag{2.99a}$$

$$P'\underline{y}_1 - \underline{y}_2 = Q'\underline{u} + D'\underline{e}' \tag{2.99b}$$

while the model is defined by

$$\hat{P}\,\underline{y}_1 = \hat{Q}\,\underline{u} + \hat{D}\,\underline{\epsilon} \tag{2.100a}$$

$$\hat{P}'\underline{y}_1 - \underline{y}_2 = \hat{Q}'\underline{u} + \hat{D}'\underline{\epsilon}' \tag{2.100b}$$

Let us, first, consider (2.99a) and (2.100a). Combining these two equations yields

$$\underline{\epsilon} = \left[\hat{D}^{-1}\,\hat{P}\,P^{-1}\,D\right]\underline{e} + \left[\hat{D}^{-1}\,\hat{P}\,P^{-1}\,Q - \hat{D}^{-1}\,\hat{Q}\right]\underline{u} \tag{2.101}$$

Then, assuming that $\underline{u}$ and $\underline{e}$ are uncorrelated and setting

$$F = \hat{D}^{-1}\,\hat{P}\,P^{-1}\,D \tag{2.102}$$

$$G = \hat{D}^{-1}\,\hat{P}\,P^{-1}\,Q - \hat{D}^{-1}\,\hat{Q} \tag{2.103}$$

allows us to write

$$E\left[\underline{\epsilon}\ \underline{\epsilon}^T\right] = E\left[F\ \underline{e}\ \underline{e}^T\ F^T\right] + E\left[G\ \underline{u}\ \underline{u}^T\ G^T\right] \tag{2.104}$$

which clearly implies that

$$E\left[\underline{\epsilon}\ \underline{\epsilon}^T\right] \geqslant E\left[F\ \underline{e}\ \underline{e}^T\ F^T\right] = \overline{V} \tag{2.105}$$

where $A > B$ means that A-B is positive definite.

Applying Parseval's theorem, $\overline{V}$ becomes (Assumption A4)

$$\overline{V} = \frac{1}{2\pi i} \oint_{|\zeta|=1} F(\zeta)\ R\ F^T(\zeta^{-1})\ \frac{d\zeta}{\zeta} \tag{2.106}$$

where

$$R = E\left[\underline{e}\ \underline{e}^T\right] \tag{2.107}$$

Let us consider the integral

$$J = \frac{1}{2\pi i} \oint_{|\zeta|=1} \left[F(\zeta) - I\right]\ R \left[F^T(\zeta^{-1}) - I\right] \frac{d\zeta}{\zeta} \geqslant 0 \tag{2.108}$$

Then

$$J = \overline{V} + J_1 - 2J_2 \tag{2.109}$$

where

$$J_1 = \oint_{|\xi|=1} R \frac{d\xi}{\xi} = R \tag{2.110}$$

$$J_2 = \oint_{|\xi|=1} F(\xi)\, R \frac{d\xi}{\xi} = R \tag{2.111}$$

The latter equality is due to the fact that the elements of $F(\xi)$ have their zeros outside the unit circle (Assumption A2) and that $F(0) = \hat{D}^{-1}(0)\ \hat{P}(0)\ P^{-1}(0)\ D(0) = I$ since $D_i(0) = \hat{D}_i(0) = 1$ and $P_{ij}(0) = \hat{P}_{ij}(0) = \delta_{ij}$.

Hence

$$J = \overline{V} - R \geqslant 0 \tag{2.112}$$

and, recalling (2.105) and (2.107)

$$E\left[\underline{\epsilon}\ \underline{\epsilon}^T\right] - E\left[\underline{e}\ \underline{e}^T\right] \geqslant 0 \tag{2.113}$$

Applying Sylvester's theorem implies that the diagonal elements must be positive

$$V_i = E\left[\epsilon_i^2\right] \geqslant E\left[e_i^2\right] = \sigma_i^2 \qquad i=1,\ldots,s \tag{2.114}$$

The equality, in (2.114) occurs if and only if

(i) $J = \overline{V} - R = 0$ i.e.

$$F = \hat{D}^{-1}\ \hat{P}\ P^{-1}\ D = I \tag{2.115}$$

(ii) the second term of the right-hand side of (2.104) vanishes i.e. - provided $\underline{u}$ is persistently exciting of order $2n + \bar{n}_d + 1$ (see chapter 1) - if

$$G := \hat{D}^{-1} \hat{P} P^{-1} Q - \hat{D}^{-1} \hat{Q} = 0 \tag{2.116}$$

Combining (2.115) and (2.116) yields

$$\hat{D} P = D \hat{P} \tag{2.117}$$

$$\hat{D} Q = D \hat{Q} \tag{2.118}$$

which implies (Assumption A3)

$$\hat{P} = P \tag{2.119}$$

$$\hat{Q} = Q \tag{2.120}$$

$$\hat{D} = D \tag{2.121}$$

We have then established the

<u>Lemma 2.3.</u>

$$V_i \geqslant \sigma_i^2 \qquad i=1,\dots,s \tag{2.122}$$

The equalities are obtained if and only if $\hat{P} = P$, $\hat{Q} = Q$ and $\hat{D} = D$.

The lemma 2.3 states that the estimates of the parameters of the matrices A and B converge to the true values as $N \longrightarrow \infty$.

Let us now consider (2.99b) and (2.100b). Remarking, from (2.100a), that $\underline{y}_1$ is defined by

$$\underline{y}_1 = P^{-1} Q \underline{u} + P^{-1} D \underline{e} \qquad (2.123)$$

the equations (2.99b) and (2.100b) can be rewritten as

$$\underline{y}_2 = (P' P^{-1} Q - Q') \underline{u} + P' P^{-1} D \underline{e} - D' e' \qquad (2.114a)$$

$$\underline{y}_2 = (\hat{P}' P^{-1} Q - \hat{Q}') \underline{u} + \hat{P}' P^{-1} D \underline{e} - \hat{D}' \underline{\epsilon}' \qquad (2.114b)$$

from which we can deduce

$$\underline{\epsilon}' = \left[\hat{D}'^{-1} (\hat{P}' - P') P^{-1} Q - \hat{D}'^{-1} (\hat{Q}' - Q')\right] \underline{u} + \left[\hat{D}'^{-1} (\hat{P}' - P') P^{-1} D\right] \underline{e} + \left[\hat{D}'^{-1} D'\right] \underline{e}' \qquad (2.115)$$

Proceeding as above lead us to the

Lemma 2.4.

$$V_i' \geqslant \sigma_i'^2 \qquad i=1,\ldots,r-s \qquad (2.116)$$

The equalities are obtained if and only if $\hat{P}' = P'$, $\hat{Q}' = Q'$ and $\hat{D}' = D'$.

The lemma 2.4 proves that the estimates of the parameters of the matrix C converge to the true values as $N \to \infty$.

We then obtain the

Theorem 2.1

The Maximum Likelihood estimate of the triplet (A, B, C) is asymptotically consistent.

The remaining problem is the minimization of the loss functions. Inspecting (2.96) and (2.94) shows that minimizing W_i and W_i' is a non-linear optimization problem. The object of the next section is to present the various classes of non-linear optimization algorithms.

5 - MINIMIZING THE LOSS FUNCTIONS

5.1. Introduction

The nonlinear minimization problem can be formally stated as

$$\text{Minimize } V(\underline{p}) \qquad \underline{p} \in E^n \tag{2.127}$$

A large number of methods have been proposed to solve the general nonlinear minimization problem [20,14]. These methods can be divided in two classes :

- The methods that use derivatives
- The methods that do not use derivatives

In the following, the most commonly used methods are briefly described

5.2. Minimization procedures using derivatives

We first consider how to solve problem (2.127) by algorithms that make use of first and, possibly, second derivatives of $V(\underline{p})$.

5.2.1. Gradient methods [14]

At the k-th stage, the transition from a point $\underline{p}^{(k)}$ to another point $\underline{p}^{(k+1)}$ is given by

$$\underline{p}^{(k+1)} = \underline{p}^{(k)} - \lambda_k V_{\underline{p}}(p^{(k)}) \qquad (2.128)$$

The negative of the gradient gives the direction for optimization but not the magnitude of the step, so that various steepest descent algorithms are possible, depending on the choice of λ. Many methods of selecting λ are available. It can be shown that, under suitable conditions, the method converges as $k \longrightarrow \infty$. However this theoretical result is of little interest in practice because the rate of convergence can be intolerably slow.

5.2.2. Newton's methods [19]

The second-derivate methods, among which Newton's method is the best known, originate from the quadratic approximation of $V(\underline{p})$. The transition from $\underline{p}^{(k)}$ to $\underline{p}^{(k+1)}$ is

$$\underline{p}^{(k+1)} = \underline{p}^{(k)} - \lambda_k \left[V_{\underline{pp}}(p^{(k)}) \right]^{-1} V_{\underline{p}}(\underline{p}^{(k)}) \qquad (2.129)$$

The convergence is guaranteed if the inverse of the Hessian matrix of V is positive definite. This is a major drawback to this method since for functions which are not strictly convex, Newton's method can diverge.

5.2.3. Quasi-Newton methods [14]

Quasi-Newton methods approximate the Hessian matrix or its inverse but use information from only first-order derivatives. At stage (k+1), $p^{(k+1)}$ is computed from $\underline{p}^{(k)}$ through

$$\underline{p}^{(k+1)} = \underline{p}^{(k)} - \lambda_k \; A(p^{(k)}) \; V_{\underline{p}}(p^{(k)}) \qquad (2.130)$$

where A is an approximation of the inverse of the Hessian. The problem of approximating the Hessian has been investigated by Pearson[17], Davidon - Fletcher - Powell[14] and Fletcher [10].

5.3. Minimization procedures without using derivatives

This Section is concerned with derivative-free type of methods (search method). As a general rule, first-derivative and second-derivative methods converge faster than search methods. However, in practice, the derivative-type methods have two main drawbacks to their implementation. First, it can be laborious or impossible to provide analytical functions for the derivatives. Second, the derivative-type methods require a large amount of problem preparation as compared with search methods.

Two of the many existing search algorithms are briefly described. A complete description of these two methods is given in [14].

5.3.1. Rosenbrock's method [22]

Rosenbrock's method locates $\underline{p}^{(k+1)}$ by successive unidimensional searches from an initial point $\underline{p}^{(k)}$ along a set of orthonormal directions $v_1^{(k)}, \ldots, v_n^{(k)}$ generated by Gram - Schmidt procedure.

5.3.2. Powell's method [17]

Powell's method locates the minimum of V by successive unidimensional searches from an initial point along a set of conjugate directions.

Choosing an algorithm is not a simple matter since no one method appears to be far superior to all the others. The choice of a particular algorithm rests on the formulation of the problem and the experience of the practitioners.

For the identification of multivariable systems, search methods seem more appropriate than derivative-type methods because of the generally large number of parameters. However, if the computational (storage) requirements are considered, a quasi-Newton method can be chosen. In Chapter 3, the various aspects of these problems will be treated on some examples.

6 - CONCLUSION

In this chapter a unified approach to the identification of multivariable systems has been presented.

Although the problem of deriving the input-output description of a multivariable system from its state-space representation has been deeply investigated by many authors, no general result is available in literature. The results of Section 3 bridge this gap. Compared with the previous methods, the present one has the following advantages.

- This is the most general approach, since all the previous methods are special cases of the above method.

a) If the number of subsystems is equal to the number of outputs ($s = r$), one obtains the input-output representation derived by Gopinath [12] and Zuercher [28].

b) If the n_i's are chosen as large as possible, one obtains the so-called minimization realization derived by Ackerman [1].

c) If the two above conditions can be satisfied simultaneously, the input-output representation, suggested by Guidorzi [13], is obtained.

- The identification of the system dynamics (matrices A and B) is based on the information provided by $\underline{u}$ and $\underline{y}_1$. This allows a discrimination of the input-output data which is of interest if some outputs of the system are markedly more free from noise than the other ones.

- Equations (2.28) and (2.25) provide a very simple state estimator. Having estimated the parameters of P and Q (i.e. A and B), equation (2.99a) gives an estimate of the output vector $\underline{y}_1$, say $\tilde{\underline{y}}$, from which the noise has been removed

$$\hat{P}\tilde{\underline{y}}_1 = \hat{Q}\underline{u} \tag{2.131}$$

Then, provided $\underline{u}$ is noise-free, the state estimator

$$\hat{\underline{x}}(k) = M\tilde{\underline{y}}_1(k) - \hat{N}\underline{u}(k) \tag{2.132}$$

gives an unbiased estimate of $\underline{x}(k)$. This formulation leads to an alternative approach to the identification of the parameters of the observation matrix: the parameters of C can be estimated through equation (2.45).

The Maximum Likelihood estimator, introduced in Section 4, allows us to establish the consistency of the estimates in case of correlated noise.

All these results constitute a global approach to the identification of linear multivariable systems.

The whole procedure described in the previous Sections can be extended, in an entirely obvious way, to systems whose input-output structure is known a priori (transfer functions, impulse response).

Due to the recursive structure of the computation scheme, the procedure can be extended, with only minor modifications, to an on-line identification procedure.

Chapter 3 : EXPERIMENTS

1 - INTRODUCTION

The aim of this Chapter is to present some examples of Maximum Likelihood indentification of multivariable systems.

For this purpose, tests will be made on simulated data

- for different minimizations algorithms
- for different noise powers

2 - EXPERIMENT 1

We consider the following third-order system

$$\underline{x}(k+1) = \begin{bmatrix} 0 & 1 & 0 \\ a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \underline{x}(k) + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} \underline{u}(k) \qquad (3.1)$$

$$\underline{y}(k) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underline{x}(k) \qquad (3.2)$$

with 2 inputs and 2 outputs.

This system has been simulated on an IBM 370 computer. The inputs and the outputs were sequences of length $N = 250$.

The loss function was minimized via the Fletcher algorithm [10]. Tables 3.1. and 3.2. give the results for different values of the signal-to-noise ratio,

$$\frac{S}{N} = \frac{1}{s} \sum_{i=1}^{s} \frac{\sum_{k=1}^{N} \left[y_i(k)-\bar{y}_i\right]^2}{\sum_{k=1}^{N} \left[n_i(k)-\bar{n}_i\right]^2}$$

3. EXPERIMENT 2

The following system has been simulated on an IBM 370 computer:

$$\underline{x}(k+1) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & 0 & 0 & 1 \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix} \underline{x}(k) + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ b_{41} & b_{42} \end{bmatrix} \underline{u}(k) \qquad (3.3)$$

$$\underline{y}(k) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ c_{11} & c_{12} & c_{13} & c_{14} \end{bmatrix} \underline{x}(k) \qquad (3.4)$$

The identification of this system has been achieved by using the Rosenbrock minimization algorithm for a set $N = 250$ input-output data. The results are reported in Tables 3.3 and 3.4.

The results, given in Tables 3.1 - 3.4 show the usefulness, in case of small signal-to-noise-ratios, of the noise model which allows one to eliminate the bias on the parameters. This confirms the theoretical results of the previous chapters.

parameters	true values	estimated values
a_{11}	- 0.5	- 0.5000
a_{12}	- 0.25	- 0.2499
a_{13}	0.25	0.2500
a_{21}	1.375	1.3749
a_{22}	1.5	1.5000
a_{23}	- 0.75	- 0.7500
b_{11}	0.2	0.1999
b_{12}	0.3	0.2999
b_{21}	0.1	0.0999
b_{22}	- 0.275	- 0.2750
b_{31}	0.85	0.8499
b_{32}	0.95	0.9500

Table 3.1 : $\frac{S}{N} = \infty$

parameters	true values	estimated values $n_d = 0$	estimated values $n_d = 3$
a_{11}	- 0.5	0.083	- 0.489
a_{12}	- 0.25	- 0.236	- 0.252
a_{13}	0.25	0.018	0.249
a_{21}	1.375	1.005	1.328
a_{22}	1.5	1.485	1.506
a_{23}	- 0.75	- 0.595	- 0.751
b_{11}	0.2	0.181	0.210
b_{12}	0.3	0.329	0.296
b_{21}	0.1	0.084	0.097
b_{22}	- 0.275	- 0.237	- 0.259
b_{31}	0.85	0.426	0.879
b_{32}	0.95	0.874	0.898

Table 3.2 : $\frac{S}{N} = 10$

(n_d : noise model order)

parameters	true values	estimated values $n_d = 0$	estimated values $n_d = 3$
a_{11}	- 1	- 1.274	- 1.061
a_{12}	0.5	0.067	0.506
a_{13}	2	2.186	2.003
a_{14}	1.5	1.324	1.491
a_{21}	1	0.645	0.979
a_{22}	2.5	2.484	2.485
a_{23}	0.25	- 0.016	0.261
a_{24}	1	0.715	1.083
b_{11}	0.5	0.674	0.501
b_{12}	0.5	0.323	0.518
b_{21}	2	2.711	2.215
b_{22}	- 1	- 1.617	- 1.184
b_{31}	4	3.077	3.899
b_{32}	3.2	1.006	2.994
b_{41}	2.7	- 0.199	2.645
b_{42}	- 0.2	- 1.157	- 0.247
c_{11}	1	0.874	1.012
c_{12}	- 1	- 1.015	- 1.024
c_{13}	- 1	- 1.637	- 1.127
c_{14}	1	0.421	0.919

<u>Table 3.3</u> : $\frac{S}{N} = 10$

(n_d : noise model order)

parameters	true values	estimated values $n_d = 0$	estimated values $n_d = 3$
a_{11}	- 1	1.429	- 1.064
a_{12}	0.5	- 0.126	0.512
a_{13}	2	2.114	2.004
a_{14}	1.5	1.677	1.494
a_{21}	1	0.429	0.875
a_{22}	2.5	2.734	2.515
a_{23}	0.25	- 0.126	0.299
a_{24}	1	0.834	1.077
b_{11}	0.5	0.722	0.494
b_{12}	0.5	0.299	0.577
b_{21}	2	2.612	2.129
b_{22}	- 1	-1.739	- 1.227
b_{31}	4	2.984	3.725
b_{32}	3.2	1.008	2.999
b_{41}	2.7	- 0.237	2.825
b_{42}	- 0.2	- 0.975	- 0.251
c_{11}	1	0.827	1.108
c_{12}	- 1	- 1.115	- 1.009
c_{13}	- 1	- 1.725	- 1.118
c_{14}	1	0.622	0.905

Table 3.4. : $\frac{S}{N} = 3$

(n_d : noise model order)

CONCLUSIONS

The purpose of this thesis is to investigate the Maximum Likelihood identification of linear discrete-time systems.

The statistical properties of the Maximum Likelihood estimator, for single input-single output systems, are analyzed. Two different types of noise model structure are considered and two new results relative to the convergence properties of identification methods are obtained.

A general method for deriving the input-output description of a multivariable system from its state-space representation is proposed. It is shown that this approach is superior, from different points of view, to those proposed in literature.

Based on the results obtained in the single input-single output case, a Maximum Likelihood estimation procedure for multivariable systems is described and convergence properties are derived.

It is shown that the results can be generalized in various directions.

Finally, numerical results of Maximum Likelihood identification are obtained.

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