MONEY AS A TRANSACTION TECHNOLOGY: A GAME-THEORETIC APPROACH

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ABSTRACT

A barter economy and a monetary economy are modelled using the cooperative game approach. The feature that distinguishes the two economies is the manner in which exchange activities are organized in the face of trasnaction costs. While division of labour or specialization is exploited in the monetary economy's technology of exchange, it is not exploited in that of the barter economy. The presence of a medium of exchange in the monetary economy permits its specialized traders to operate efficiently.

The cooperative game approach admits group rationality along with the usual assumption of individual rationality. Group rationality means that individuals are able to perceive their interdependence. Money is explained as the product of interactions between individual rationality (utility maximizing consumers and profit maximizing traders) and group rationality (the ability to perceive the benefits of monetary exchange versus barter exchange). Consequently, money is viewed not as an object, but as an instituion. Its value reflects the relative superiority of a monetary economy over a barter economy.

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Chapter 1

INTRODUCTION

A. In this thesis I shall attempt to build a sensible model of a monetary economy. The model will include some important features of monetary economics which I believe have not been given enough attention in the literature. To bring out these features, I shall compare my monetary economy with a barter economy. By doing so, I am able to examine the structural differences between monetary and barter exchange. The purpose of this exercise is to understand better how a monetary economy functions.

Neoclassical economic theory does not provide an adequate explanation of the importance of money in modern economics. Persumably, money serves some useful purpose in the Arrow-Debreu models. But these models fail to bring out money's role because they do not describe how goods are exchanged between agents. Recently a number of authors have attempted to prove the usefulness of money ([25, [49] and [64]). While the details of the models differ from one author to another, they all postulate money as something which is inherently useful. In a way, therefore, the conclusion is assumed at the outset. What I propose to do instead is to build models of a monetary economy and a barter economy which are plausible in the light of economic history. Then I shall examine the conditions under which money is in fact useful.

I believe that there is a structural difference between monetary and barter exchange. This structural difference will be modelled rigorously in Chapter 2 and 3. The difference between monetary and barter exchange will be developed by recognizing that real resources are used up when individuals (agents) in an economy exchange goods. The real resources used up when goods are exchanged are called transaction costs. Examples of transaction costs are the cost of moving goods from one agent to another and the legal cost of transferring ownership. There are various ways of organizing the exchange of goods in the presence of transaction costs. The structural difference referred to above is based on the different manner in which goods are exchanged in my monetary and barter economies. I shall argue that the usefulness of money in monetary exchange in contrast to barter exchange is the result of this difference in organizing exchange activities.

Let me be a bit more specific about how transaction costs are handled in my model. I assume each agent is endowed with a certain degree of efficiency in exchanging goods. An agent's ability at performing exchanges is represented by his transaction technology. A transaction technology is similar to the more conventional production technology. While the latter describes feasible outputs for each set of inputs, the former describes all feasible exchanges and their attendant real resource costs. An individual who exchanges the vector y of goods for the vector x of goods will use up real resources as represented by some vector z. The magnitude of the transaction cost vector z depends on the individual's efficiency at exchanging goods. In other words, z depends on an individual's transaction technology. In my model an individual's transaction technology is taken as a primitive. I choose not to investigate the source and nature of transaction costs because it is not necessary for my purposes.

B. In Chapter 2 I shall build a model of an economy which I call a "barter economy." In this economy I require that each individual's exchanges be constrained both by goods in his possession and by his own transaction technology. I do not permit any agent to execute exchanges on behalf of another agent. Each agent in the economy will have some idea about the ratios at which goods are being exchanged. When an agent wants to exchange goods with one or more agents, his desired exchanges will be based on his existing stock of goods, his beliefs about the prevailing exchange ratios between goods, and his transaction technology.

An agent is permitted to engage not only in direct exchange but also in indirect exchanges. He can use some good as an intermediary or link in exchange if it is to his advantage. Furthermore, he is not limited to bilateral exchanges. He can involve himself in multilateral exchange to the extent permitted by his personal credibility. Consequently, my notion of a barter economy is much wider than that which is generally used. More will be said about this in Chapter 2.

Generally, it does not make much sense to postulate the presence of prices in a barter economy. Historically, barter economies were not highly unified, but consisted of a number of isolated markets. While an individual might engage in arbitrage in a local market, the scope of his activities was limited by his information, tastes, initial endowment, and transaction technology. Thus, especially between isolated markets, there would probably have been no high degree of consistency in the ratios at which goods were exchanged. Although my analysis of a barter economy deals with a price vector in connection with existence proofs, my description of barter exchange does not depend on the presence of prices.

The main feature of barter exchange that I want to bring out is the absence of specialization among agents in carrying out exchanges. At each stage in the process of exchanging goods, an individual's planned exchanges are constrained both by goods in his possession and by his transaction technology. The barter economy can be characterized by the statement that division of labour is not exploited in the manner in which goods are exchanged between individuals.

C. In Chapter 3, on the other hand, I shall build a model of an economy which I call a "monetary economy." The crucial difference between this economy and that of Chapter 2 is that I now remove the restriction that each agent must only exchange goods on his own behalf. Here, an individual is permitted to execute exchanges on behalf of others. An agent may act as a trader by buying goods from some individuals and reselling them to others. By acting as a trader, an agent hopes eventually to consume a more desirable bundle of goods than he could have if he only exchanged goods on his own behalf, or, if he permitted some other agent to execute his exchanges. On the other hand, if an agent is not particularly efficient at exchanging goods, it may be to his advantage to have another agent execute his exchanges. Competition between traders will ensure that only the relatively efficient agents act as traders. If the agents with superior transaction technologies are actually performing the exchange tasks, then the manner in which goods are exchanged in this economy is more efficient than that in the barter economy. Thus the crucial structural difference between the two economies is that the monetary economy exploits division of labour in the way goods are exchanged between individuals.

Let me now explain why a medium of exchange has an important role to play in this monetary economy. A trader's act of buying some goods from an individual is separate from his act of selling these same goods to another individual. If the trader's customers always demand spot payment in goods whenever the trader wants to buy goods, the trader may not be able to meet their demands from his inventory. A trader's initial endowment of the goods may have been small, or else, recent trades may have depleted his stock of particular goods. The trader has a problem. He must buy goods in order to sell goods, but this is difficult if individuals always demand spot payment in goods.

In a monetary economy, the presence of a medium of exchange solves the traders problem. As we shall see in the next section, his customers will accept money in exchange for goods. They know that they can buy or order any goods they desire from any trader and pay for them with money. Through the use of a medium of exchange, the trader is no longer constrained at each point in time by goods in his possession. Thus the universal acceptability of money allows the traders in the economy to use their transaction technologies with maximum efficiency.

D. In conclusion, let me describe some of the important features of the monetary economy that emerge from my analysis.

First, I have seen the essential difference between a barter economy and a monetary economy in the manner in which exchange activities are organized in the two economies. To put it briefly, a monetary economy makes use of division of labour or specialization in its technology of exchange while a barter economy does not. On the basis of this structural difference, I have shown that the set of feasible allocations in a monetary economy is larger than that in a barter economy, and hence, that the former is potentially (but not necessarily) more efficient than the latter. The potential benefit of monetary exchange is thus established.

Reliance on specialized traders means separation between the act of sale and the act of purchase of a good both in time and place. For the reasons given below, traders and their customers will accept money in exchange for goods. Consequently, each participant in a transaction is no longer constrained by goods in his possession and/or by his tastes. A medium of exchange cuts the act of sale and the act of purchase loose from the requirements of double coincidence of wants.

Second, in formulating the two economies, I have employed a cooperative game or core theoretic approach. The rationale for this choice lies in the peculiar nature of money. Money has conventionally been introduced into general equilibrium models as an additional good and made to work on the strength of individuals' demand for it. Unlike ordinary goods, however, one cannot legitimately derive an individual's demand for money from his physiological needs. But to ensure the positive exchange value of money, one ends up assuming the usefulness of money. This is essentially what Starr [64], Hahn [25] and others have done.

Given the difficulty of deducing the usefulness of money from individual tastes, one is naturally led to a society-oriented approach in which society and its members perceive the usefulness of money. But it is equally difficult to explain the process of such perception.

Core theory enables us to deal systematically with the two types of economies at both the individual and at the group level. Core theory retains the usual assumption that individuals maximize their utility. It also assumes, however, that individuals are able to perceive their interdependence and that any group of individuals will carry out acts which are of mutual benefit. In other words, core theory recognizes group rationality along with individual rationality. In fact the core is a set of allocations which are rational from the point of view of both individuals and groups.

I stated above that monetary exchange is potentially more efficient than barter exchange and that money permits the traders in a monetary economy to operate effectively. I have explained money as something which emerges as a product of interactions between individual rationality (profit maximizing on the part of traders) and group rationality (ability to perceive benefits of monetary exchange throughout the economy). Thus, in an equilibrium of the monetary economy, any group of individuals is free to break away and use alternative means of exchange. However, no group will do so because there is not any group which can offer all its members greater utility than they can get by remaining in the monetary economy. Thus, an individual must join the monetary economy to exchange goods. To exchange goods, he must use money. For this reason, it can be said that individuals' demand for money is derived both from their chosen environment, namely, the requirements of monetary exchange, and their desire to exchange some of their initial endowments. In short, I have characterized money as something that reflects in its value the relative superiority of a monetary economy over a barter economy.

E. The program of this thesis proceeds as follows. In Chapter 2 I shall present a model of a barter economy. I first model the economy using the cooperative game approach. Then I deduce the structure of prices needed so that competitive behaviour on the part of agents is equivalent to cooperative behaviour. In other words, I derive a competitive barter economy from the cooperative economy.

In Chapter 3 I shall present a model of a monetary economy. I again start from a cooperative economy. Here agents form coalitions for the purpose of exploiting the division of labour in exchange. Because I start with a cooperative economy rather than a competitive economy, I am better able to model the process by which agents are assigned to particular exchange tasks. I again deduce the structure of prices needed so that competitive behaviour is equivalent to cooperative behaviour. I find that a set of buying and a set of selling prices is now needed instead of the one set of prices of the barter economy.

Chapter 4 expands the model of Chapter 3. Here permit agents the choice between monetary and barter exchange. The cooperative approach is particularly suited for this purpose because it allows group rationality. In other words, a group of agents can consider the advantages of breaking away from the monetary economy and of using alternative ways of exchanging goods. In this chapter I present a sufficient condition for monetary exchange to dominate barter exchange. By this I mean that all agents will choose to use monetary exchange rather than barter exchange.

Finally, the appendix contains the proofs which establish the existence of competitive price equilibria in the barter and monetary economies. This is necessary to ensure the logical consistency of my models.

Chapter 2

A "BARTER" ECONOMY

A. Before I present my description of a barter economy, let us first consider how a barter economy is usually described. Jevons [37] has described a barter economy as one where exchange requires the "double coincidence of wants." By this he means that if two agents are to exchange goods, each agent must have something that the other agent wants. In a recent paper, Starr [64] has attempted to model double coincidence of wants. According to Starr, in a barter economy exchange of goods must satisfy two requirements. First, the exchange of goods between a pair of agents must be "price consistent." This means that for any given agent the value of goods supplied to any other agent must equal the value of goods received from him at the current prices. In other words, trades between any two agents must always be cleared between them. A third agent cannot be involved. Thus Starr's "price consistency" assumption restricts trade to bilateral exchange of goods.

Starr's second restriction on exchange is that it be "monotone excess demand diminishing." This restriction on exchange ensures that it is voluntary. By this Starr means that an agent will not give up a good unless he has an excess supply of this good. Conversely, an agent will not accept a good unless he has an excess demand for this good. To put it briefly, the exchange of goods is said to be "monotone excess demand diminishing" if trade between any pair of agents does not increase the excess demand for any good by either agent, or increase the excess supply of any good by either agent. In effect, this restriction prevents the use of an intermediary in exchange.

It is easy to construct simple examples in which Starr's two restrictions on exchange prevents some agent from attaining his desired bundle of goods. This can happen even though the price vector at which goods are being exchanged would be an equilibrium price vector if one of the restrictions was lifted. Starr shows that this difficulty of barter can be circumvented if a good — called money — is appended to the existing list of goods. Starr designates as money that good which is always acceptable in exchange. An agent will accept money even though he does not wish to consume it. The use of money is found to be socially desirable because its use enlarges the set of feasible trades. Allocations of goods which were impossible to achieve through barter exchange can now be achieved because money's universal acceptability overcomes the absence of double coincidence of wants.

B. While Starr's formalization of double coincidence of wants is an important contribution to the theory of money, his paper leads to some serious problems. First, double coincidence of wants certainly is a difficulty of barter exchange. However, I do not believe that it is a historically valid description of barter exchange. In my opinion, double coincidence of wants is too narrow a definition. It restricts trade to bilateral exchange, it precludes all debt contracts, and it does not permit the use of even a limited intermediary in exchange. Historically, there is evidence that multilateral exchange, credit between individuals, and the local use of intermediate goods in exchange occurred in barter economies [20].

Furthermore, Starr does not explain his use of prices in conjunction with his double coincidence of wants requirements. In a barter economy there will be $\ell(\ell-1)/2$ exchange ratios between goods. To be able to reduce these exchange ratios to a set of $\ell-1$ relative prices, someone must be engaged in arbitrage. However, in Starr's economy no agent can use a third good as an intermediate link between two goods. Nor can any agent act as a third party to a transaction between a pair of agents. Clearly arbitrage is ruled out in Starr's economy, and thus double coincidence of wants is inconsistent with Starr's use of a set of ℓ-1 relative prices.

My model of a barter economy is more general than Starr's. It permits agents to engage in arbitrage. Agents are permitted to use indirect exchange, either through a third good or through a third party. However, their arbitrage operations are limited by their initial endowments, tastes, and transaction technologies. Because I consider the structure of trade in an economy along with the presence of a medium of exchange, I am able to retain a meaningful distinction between barter and monetary exchange, in spite of the generality of my barter economy.

C. In Chapter 1 I briefly described my model of a barter economy. The important feature that distinguished it from the monetary economy was that agents were not permitted to execute exchanges for other agents. Each agent's exchanges were constrained both by goods in his possession and by his transaction technology. I shall now be a bit more exact and rigorous in explaining what I mean by this statement.

Let A represent the set of agents in the barter economy. By $a \in A$ we mean that the individual a is a member of the economy. With each agent a we associate the vector $\omega(a)$, agent a's initial endowment of goods. The vector $\omega(a)$ has dimension ℓ, where ℓ is the number of goods in the economy. We write $\omega(a) \in R_+^\ell$, where R_+^ℓ is the non-negative orthant of the Euclidean space R^{ℓ} of dimension ℓ. An agents preferences are represented by $\precsim_a$. The statement $x \precsim_a y$, where x and y are vectors in R^{ℓ}_{+}, means that agent a prefers the bundle of goods y to the bundle x, or else he is indifferent between the two bundles.

In the last chapter we said that an agent's efficiency at exchanging goods is expressed by his transaction technology. An agent's transaction technology is modelled by his transaction set. Suppose agent $a \in A$ wants to exchange the bundle of goods $y(a)$ for the bundle $x(a)$. The vectors $y(a)$ and $x(a)$ are elements of R^{ℓ}_{+}, the commodity or good space. An agent's transaction set will indicate whether the exchange of $y(a)$ for $x(a)$ is technologically feasible. If exchange is feasible, the transaction set will also indicate the real resources required to execute the exchange. Agent a's transaction set is given by $S(a)$, where $S(a)$ is a subset of the Euclidean space of dimension equal to three times the number of goods in the economy or $R^{3\ell}_{+}$. If agent $a \in A$ wants to exchange $y(a) \in R^{\ell}_{+}$ for $x(a) \in R^{\ell}_{+}$, then this exchange is technologically feasible if and only if there exists a vector $z(a) \in R^{\ell}_{+}$ such that

$$\left\{x(a),\ y(a),\ z(a)\right\} \in S(a). \tag{1}$$

The vector $z(a)$ represents the quantities of real resources needed by a to exchange $y(a)$ for $x(a)$.

Even though an exchange of $y(a)$ for $x(a)$ is technologically feasible for agent a, he may not be able to execute the exchange because his initial endowment is insufficient. The exchange $y(a)$ for $x(a)$ with transaction costs $z(a)$ is compatible with a's initial endowment $\omega(a)$ if

$$\omega(a) + x(a) - y(a) - z(a) \geqq 0. \tag{2}$$

This is the material balance condition for agent a.

Relations (1) and (2) express mathematically the crucial restriction that I place on exchange in the barter economy. Each agent's exchanges are constrained both by his transaction technology and by goods in his possession.

I still need one more relation to ensure that material balance is maintained for the entire economy. Suppose that each agent $a \varepsilon A$ wants to exchange some bundle $y(a)$ for $x(a)$ and that there is a vector $z(a)$ such that

$$\{x(a),\ y(a),\ z(a)\} \ \varepsilon\ S(a).$$

Let $f(a) = \omega(a) + x(a) - y(a) - z(a)$ be the disired consumption vector for every $a \varepsilon A$, where $f(a) \geqq 0$. Then the material balance condition for the entire economy is given by

$$\sum_{a \varepsilon A} f(a) = \sum_{a \varepsilon A} \omega(a) - \sum_{a \varepsilon A} z(a).$$

It reads that the total quantity of each good consumed must equal the total initial endowment of this goods minus the total quantity of the good used up in the process of exchanging goods.

D. I have already explained why I use the cooperative approach to modelling. Let me now explain more fully what I mean by the cooperative game approach. The core theoretic or cooperative game approach assumes that economic agents in a social exchange economy will enter into relations with others. Unlike the competitive approach, the cooperative approach assumes that individuals will form coalitions or associations which are of mutual benefit to their members. It is assumed that every possible coalition of agents forms and considers the possibility of redistributing their available goods. An individual will not join a coalition, unless he is offered a more desirable bundle of goods than his initial endowment. Furthermore, he wants to join that coalition which offers him the most desirable bundle of goods. Thus individual rationality is admitted in cooperative economies just as it is in competitive economies.

The main difference is that group rationality is also admitted. Suppose some reallocation of goods throughout the economy is proposed. Group rationality means that this proposed allocation of goods will not be accepted by a coalition if each member of the coalition can get a more desirable bundle of goods from some possible redistribution of the coalition's resources. Rather than exchange goods with the rest of the economy, the members agree to exchange goods only within the coalition.

An allocation of goods to individuals which results from a redistribution of goods among all agents is said to be a core allocation if it does not violate either the individual or group rationality criterion. The set of all core allocations is said to be the *core* of the economy. Clearly, a core allocation is also a Pareto optimal allocation, although the converse need not be true.

The core is an equilibrium concept of cooperative economies which can be compared with the equilibrium price vector of traditional general equilibrium analysis. It has been demonstrated that each competitive allocation, i.e. the distribution of goods among agents after trading at equilibrium prices, belongs to the core. For economies with a finite number of agents, the core is generally larger than the set of competitive allocations. As the number of agents "gets large" the core "shrinks" to the set of competitive allocations [4], [15], [31] and [70]. The equivalence, for large economies, of the core with the set of competitive equilibrium allocations will be exploited throughout this theses.[1]

The cooperative game aspects used in this thesis rely heavily on and borrow freely from the papers by Aumann [4] and [6] and Hildenbrand [31], [32], [33] and [34] with respect to both the definitions of concepts used in models and the techniques used in the proofs of propositions. Other papers of importance to the application of the theory of cooperative games to economics include those by Cornwall [12], Debreu and Scarf [15], Schmeidler [55], Sondermann [62] and Vind [70].

E. The conventional theory of the core assumes, however, that no resource costs are incurred in effecting a redistribution of goods among members of a coalition. By incorporating transaction costs in a cooperative economy, it is possible to consider formally, different ways of organizing an economy's exchange process. Later in Chapter 4 the use of core theory permits me to model the choice between barter and monetary exchange.

We have been considering an economy whose initial state is described by its agents' preferences, endowments and transaction technologies. To give it the flavour of a barter economy, we also added the constraint that each agent must use his own transaction technology in performing exchanges. Let us now set up the economy in its cooperative context.

The set of agents in the economy was given by A. Now let Ω be the set of admissible coalitions of agents. Ω consists of those coalitions which are permitted to form. If the number of agents in the economy is finite, Ω is usually the set of all subsets of A. Formally, Ω is required to be a σ-field, which means that countable unions and finite intersections of its elements are also admissible coalitions. For each coalition $E \in \Omega$ there is defined a real number $\nu(E)$ which represents the fraction of the totality of agents belonging to the coalition E.

An *allocation of commodities*, denoted by f, is a distribution of goods among the agents, where f(a) is the vector assigned to agent a. Coalitions of agents form for the purpose of reallocating initial endowments among their members. To model barter exchange, any reallocation of goods must be in accordance with each agent's transaction set and initial endowment. A given allocation f is said to be attainable for a coalition $E \in \Omega$ using barter exchange, if for each member agent $a \in E$ there exist vectors x(a), y(a) and z(a) in R_+^ℓ such that

i) $(x(a), y(a), z(a)) \in S(a)$,

ii) $f(a) = \omega(a) + x(a) - y(a) - z(a)$, and

iii) $\sum_{a \in E} f(a) = \sum_{a \in E} \omega(a) - \sum_{a \in E} z(a)$.

The first two conditions state that the barter exchange pattern which results in allocation f must be both technologically feasible for each agent and compatible with each agent's initial endowment. The last condition is coalition E's material balance equation.

An allocation which is attainable by the coalition consisting of *all* agents in the economy is said to be a *state of the economy*. A state f of the economy is said to be blocked by the coalition $E \varepsilon \Omega$ if the coalition can redistribute its initial endowments among its members in such a way that the resulting attainable allocation g is perferred to f by all members of the coalition E. The *core* is then defined as a set of state of the economy which cannot be blocked by any admissible coalition.

F. As was mentioned above, the core is an equilibrium concept for cooperative economies which can be compared with the equilibrium price vector of competitive economies. If $p \varepsilon R_+^\ell$ is a vector of prices in the competitive economy, then the budget set for an agent $a \varepsilon A$ can be given by

$$\beta(a,p) = \left\{ (x,y,z) \varepsilon R_+^{3\ell} \;\middle|\; \begin{array}{ll} \text{a)} & (x,y,z) \varepsilon S(a) \\ \text{b)} & \omega(a) + x-y-z \geqq 0, \text{ and} \\ \text{c)} & p \cdot x \leqq p \cdot y \end{array} \right\}$$

The budget set of our "barter" economy can be compared with the usual budget set of an agent in an Arrow-Debreu economy, i.e.

$$\left\{ s \in R^{\ell}_{+} \quad | \ p \cdot s \leqq p \cdot \omega(a) \right\}$$

In this chapter I am interested in discovering under what conditions the allocations of goods resulting from competitive behaviour coincides with the core. I prove that a list of relative prices, one price per good, is sufficient for competitive behaviour to achieve the same result as cooperative behaviour. The competitive version of this economy is just the traditional pure exchange economy with transaction costs. This is in contrast to the model in the next chapter where a set of buying and selling prices and coalition traders are required to achieve the same result.

These statements are established by proving the following propositions.

Proposition 1.

A competitive allocation is also a core allocation.

Proposition 2.

In a "perfectly competitive" economy — where each agent has only a negligible influence on any final allocation of commodities — it is possible to derive from a given core allocation a set of equilibrium relative prices such that the quasi-competitive allocation corresponding to these prices is the given core allocation.

Proposition 3.

Under certain conditions, a quasi-competitive allocation is also a competitive allocation.

Let me explain briefly why it is necessary to work with a quasi-competitive allocation. Below, I shall assign a consumption set X(a) to each agent $a \in A$. For each agent a, X(a) is a subset of R^ℓ_+ and it consists of a's possible consumption vectors. The quasi-competitive equilibrium concept, described by Debreu [14], was introduced because an agent's initial wealth may not be sufficient to ensure him a consumption vector in his consumption set after exchanging goods at a given price vector. When this happens, the demand correspondence used to establish the existence of an equilibrium is discontinuous. If an agent cannot exchange any goods and still remain inside his consumption set, his choice of a consumption vector will be suitable restricted to ensure that the demand correspondence is in fact continuous.

G. In the remainder of this chapter I shall prove in a rigorous manner the propositions made above. But first it is necessary to give precise definition to the concepts introduced.

1. The Measure Space of Agents, (A, Ω, ν)

The economy consists of a measure space of agents (A, Ω, ν) where A is the set of economic agents, Ω is a

σ-field of subsets of A and consists of the admissible set of coalitions and ν is a countably additive function on Ω to R_+. The function ν is called a measure.[2,3]

2. The consumption set correspondence, X

The consumption set correspondence X is a ν-measurable mapping from A to the subsets of R_+^ℓ, minorized by a ν-integrable function. The non-empty, closed convex set X(a) associated with agent a consists of his possible consumption vectors.

3. The set of allocations, L_X

An allocation is a ν-integrable function f from A to R_+^ℓ such that $f(a) \in X(a)$, a.e. a in A. The set of all allocations is denoted by L_X.

4. The initial endowments, ω

The initial distribution of goods among the agents ω is a ν-integrable function from A to R_+^ℓ such that $\omega(a) \in X(a)$, where the initial endowment of agent a is $\omega(a)$.

5. Agent's preferences, $\precsim$

For every $a \in A$ there is defined a quasi-order on X(a) — denoted by $\precsim_a$ and called preference-or-indifference.

This relation is transitive, reflexive and complete. From the relation $\precsim_a$ we also define the relation $\prec_a$ called preference by: $s \prec_a t$ if $s \precsim_a t$ but not $t \precsim_a s$. The two relations have the following properties:

i) $\precsim_a$ is continuous, i.e. if $s \in X(a)$ then the set $\{t \in X(a) | s \precsim_a t\}$ is closed.

ii) $\precsim_a$ exhibits local nonsatiation,[4] i.e. for every $s \in X(a)$ and every open set U containing s, there is a $t \in X(a) \cap U$ such that $s \prec_a t$.

Furthermore, the preference function $\prec$ mapping A into $R^{\ell} \times R^{\ell}$ is ν-measurable (Hildenbrand [31]).

6. Agent transaction technologies, S

The transaction technological correspondence S maps elements of A into subsets of $R_+^{3\ell}$. I assume S has the following properties.

i) $S(a)$ is closed for all $a \in A$.

ii) $(x(a), y(a), z(a)) \in S(a)$ and $x'(a) \leq x(a)$, $y'(a) \leq y(a)$ and $z'(a) \geq z(a)$ then $(x'(a), y'(a), z'(a)) \in S(a)$.

iii) $0 \in S(a)$ for all $a \in A$.

iv) for any $f \varepsilon L_X$ and any $a \varepsilon A$ there exists $x(a)$, $y(a)$, $z(a) \varepsilon R_+^{\ell}$ such that $(x(a), y(a), z(a)) \varepsilon S(a)$ and $\omega(a) + x(a) - y(a) - z(a) > f(a)$.

v) S is a ν-measurable correspondence.

The first three properties need little comment. Condition i) is usually assumed in the literature, condition ii) admits free disposal and condition iii) allows for the possibility of no exchange. Condition iv) is a technological feasibility condition. Because the total resources available in the economy are unbounded from an individuals point of view, this condition implies that given enough resources an individual agent can attain any vector in his consumption set. Property v) is similar to the assumption that the preference function $\precsim$ be ν-measurable.

7. Attainable allocations, K_ω

The allocation $f \varepsilon L_X$ is said to be attainable for coalition $E \varepsilon \Omega$ if and only if there exist ν-integrable functions x, y, z from A to R_+^{ℓ} such that

i) $(x(a), y(a), z(a)) \varepsilon S(a)$

ii) $f(a) = \omega(a) + x(a) - y(a) - z(a)$, and

iii) $\int_E f d\nu = \int_E \omega d\nu - \int_E z d\nu$.

These conditions have already been discussed above. Condition ii) together with condition iii) implies that

iv) $\int_E x(\cdot)d\nu = \int_E y(\cdot)d\nu$.

That is, the total quantity of each commodity received in exchange by all members of the coalition must equal the total quantity of each commodity given up in exchange.[6]

For a given coalition $E \varepsilon \Omega$ the set of all attainable allocations is denoted by $K_\omega(E)$.

8. A *state of the economy* is defined as an allocation which is attainable by the coalition consisting of all agents (a.e. agents) of the economy. $K_\omega(A)$ is the set of all states of the economy.

9. <u>The "barter" economy, Ξ^B</u>

The description of our barter economy is now complete. We denote the economy by

$$\Xi^B = [(A, \Omega, \nu), X, \precsim, S, \omega].$$

10. <u>The core of the economy, $C(\Xi^B)$</u>

A state f of the economy Ξ^B is said to be blocked by the coalition $E \in A$ if there exists an attainable allocation $g \in K_\omega(E)$ such that

i) $f(a) \propto_a g(a)$, a.e. in E

ii) $\nu(E) > 0$.

The core $C(\Xi^B)$ is the set of states of the economy which cannot be blocked by any coalition.

11. <u>Competitive allocations</u>

The price vector of this economy should be interpreted as a list of relative prices, one price per commodity, and is denoted by $p \in R_+^\ell$. Following Hildenbrand [31], three basic states of the economy Ξ^B are defined.

Let $f(= \omega + x - y - z)$ be a state of the economy Ξ^B.

a) <u>Competitive allocation, $W(\Xi^B)$</u>

f is called a competitive allocation or Walras allocation if there exists a price vector $p \in R_+^\ell$, $p \neq 0$ such that

$$p \cdot x(a) \leq p \cdot y(a), \text{ for all } a \in A \text{ and if}$$

$$s(= \omega(a) + x' - y' - z') \in X(a), \ (x', y', z') \in S(a)$$

with $$f(a) \propto_a s,$$

then $$p \cdot x' > p \cdot y'.$$

Let $W(\Xi^B)$ be the set of all competitive allocations.

b) <u>Quasi-competitive allocation, $Q(\Xi^B)$</u>

f is called a quasi-competitive allocation or a quasi-Walras allocation if there exists $p \in R^{\ell}_{+}$, $p \neq 0$ such that:

$$p \cdot x(a) \leq p \cdot y(a), \text{ for all } a \in A \text{ and if}$$

$$s(= \omega(a) + x' - y' - z') \in X(a), \ (x', y', z') \in S(a)$$

with $$f(a) \propto_a s,$$

and if $$\inf_{\substack{(\hat{x},\hat{y},\hat{z}) \in S(a) \\ \omega(a)+\hat{x}-\hat{y}-\hat{z} \in X(a)}} \left\{ p \cdot \hat{x} - p \cdot \hat{y} \right\} < 0$$

then $$p \cdot x' > p \cdot y'.$$

Let $Q(\Xi^B)$ be the set of all quasi-competitive allocation.

c) <u>Expenditure minimizing allocation, $E(\Xi^B)$</u>

f is called an expenditure minimizing allocation or a pseudo-competitive allocation if there exists $p \in R^{\ell}_{+}$, $p \neq 0$ such that

$$p \cdot x(a) \leq p \cdot y(a) \text{ and if}$$

$$s(= \omega(a) + x' - y' - z') \in X(a), \ (x', y', x') \in S(a)$$

with $$f(a) \propto_a s$$

then $$p \cdot x' \geq p \cdot y'.$$

Let $E(\Xi^B)$ be the set of all expenditure minimizing allocations.

From the definitions and comments made above it is clear that

$$W(\Xi^B) \subset Q(\Xi^B) \subset E(\Xi^B).$$

Expenditure minimizing allocations are introduced to facilitate the proofs of the following theorems.

H. The propositions made above will be established by provinggthe following Theorems. Theorem 1 establishes Proposition 1, Theorem 2 and its corollary establish Proposition 2. Finally, an example of an economy in which Proposition 3 is true will be given in Appendix B.

Theorem 1

Every competitive allocation is also a core allocation, i.e. $W(\Xi^B) \subset C(\Xi^B)$.

Theorem 2

If the measure space of agents is non-atomic, then every core allocation is also a pseudo-competitive allocation, i.e. $C(\Xi^B) \subset E(\Xi^B)$.

Corollary

Every core allocation is also a quasi-competitive allocation, i.e. $C(\Xi^B) \subset Q(\Xi^B)$.

K. The following proofs follow closely the proofs by Hildenbrand [31] for coalition production economies.

Proof of Theorem 1. $W(\Xi^B) \subset C(\Xi^B)$.

Let $f \in W(\Xi^B)$ but suppose $f \notin C(\Xi^B)$. Then there exists $E \in \Omega$ with $\nu(E) > 0$ and $h \in K_\omega(E)$ such that $h = \omega + x' - y' - z'$, $(x'(a), y'(a), z'(a)) \in S(a)$ and $f(a) \propto_a h(a)$ for a.e. a in E.

But $f \in W(\Xi^B) \to p \cdot x'(a) > p \cdot y'(a)$, a.e. a in E

$$\to \int_E p \cdot x'(\cdot)d\nu > \int_E p \cdot y'(\cdot)d\nu$$

$$\to \int_E x'(\cdot)d\nu \neq \int_E y'(\cdot)d\nu.$$

But this contradicts the material balance requirement that $h \in K_\omega(E)$. Thus $f \in C(\Xi^B)$.

Proof of Theorem 2. $C(\Xi^B) \subset E(\Xi^B)$.

Let $f \varepsilon C(\Xi^B)$. From the nonsatiation assumption 5.ii) assumption 6.iv) for the transaction technologies, the set

$$\rho(a) = \left\{\left(x(a),\ y(a),\ z(a)\right) \varepsilon\ S(a) \mid f(a) \propto_a \omega(a) + x(a) - y(a) - z(a)\right\}$$

is non-empty for a.e. a in A. Now define the correspondence δ from A to R^ℓ by

$$\delta(a) = \left\{x(a) - y(a) \mid \left(x(a),\ y(a),\ z(a)\right) \varepsilon\ \rho(a)\right\}.$$

Let L_δ be the set of ν-measurable function g from A to R^ℓ such that $g(a) \varepsilon \delta(a)$ for a.e. a in A. Since f is ν-integrable and S is a ν-measurable correspondence $L_\delta \neq \phi$[31, p. 448]. Define the set $U \subset R^\ell$ by

$$U = \bigcup_{\{E\ \varepsilon\ \Omega \mid \nu(E) > 0\}} \left\{\int_E L_\delta d\nu \equiv \left\{\int_E g d\nu \mid g \varepsilon L\right\}\right\}$$

I claim $0 \notin U$. Suppose $0 \varepsilon U$. Then there exists $E \varepsilon \Omega$ with $\nu(E) > 0$ and a function $g : A \to R^\ell$ such that

a) $g \in L_\delta \rightarrow$ letting $x(a) = g(a)^+$, $y(a) =$

$g(a)^-$, where $g^i(a)^+ = \begin{cases} g^i(a) \text{ if } g^i(a) > 0 \\ 0 \text{ otherwise} \end{cases}$

$g^i(a)^- = \begin{cases} -g^i(a) \text{ if } g^i(a) < 0 \\ 0 \text{ otherwise} \end{cases}$

then $g = x - y$ and since δ is ν-measurable, by Theorem B in [33] there exists integrable $z : A \rightarrow R^\ell$ such that $(x(a), y(a), z(a)) \in S(a)$ and $f(a) \propto_a \omega(a) + x(a) - y(a) - z(a) = h(a)$.

b) $$0 = \int_E g d\nu = \int_E (x(\cdot) - y(\cdot))d\nu$$

$$= \int_E x(\cdot)d\nu - \int_E y(\cdot)d\nu.$$

But a) and b) imply $h \in K_\omega(E)$ and h is a blocking allocation for coalition E contradicting $f \in C(\Xi^B)$. Therefore $0 \notin U$.

Because U is the integral of a set correspondence with respect to a non-atomic measure ν, U is convex (see Vind [70]). Using a separating hyperplane theorem, it is possible to show that there exists a vector $p \in R^\ell$, $p \neq 0$ such that $u \in U$ implies $p \cdot u \geq 0$. It is now possible to show that f is a pseudo-competitive allocation for the price vector p. Let $M = \{a \in A | p \cdot x(a) \geq p \cdot y(a)$, for all $(x(a), y(a), z(a)) \in \rho(a)\}$. It is possible to show that $M \in \Omega$. In fact,

$\nu(M) = \nu(A)$. If not, there exists $B \in \Omega$ with $\nu(B) > 0$ such that for every $a \in B$ there exists a point $(x'(a), y'(a), z'(a)) \in \rho(a)$ such that $p \cdot x'(a) < p \cdot y'(a)$. Without loss of generality I can assume that the functions x', y', x' from B to R^{ℓ} are measurable (see Theorem B [33, p. 621]).

But $\quad p \cdot x'(a) < p \cdot y'(a) \quad$ a.e. a in B

$$\rightarrow p \cdot \int_B x'(\cdot)d\nu < p \cdot \int_B y'(\cdot)d$$

$$\rightarrow p \cdot \int_B (x'(\cdot) - y'(\cdot))d\nu < 0.$$

But since $\int_B (x'(\cdot) - y'(\cdot))d\nu \in U$ by construction, we have a contradiction and thus $\nu(M) = \nu(A)$. It just remains to demonstrate that $f = \omega + x - y - z$ satisfies each agents budget constraint. Since $(x(a), y(a), z(a)) \in$ closure of $\rho(a)$ by continuity of preferences we have

$$p \cdot x(a) \geq p \cdot y(a) \quad \text{a.e. } a \text{ in } A.$$

Suppose there exists $C \in \Omega$ with $\nu(C) > 0$ such that

$$p \cdot x(a) > p \cdot y(a), \quad \text{all } a \in C.$$

The last two equations imply that

$$\int_A p \cdot x(\cdot)d\nu > \int_A p \cdot y(\cdot)d\nu$$

or
$$\int_A x(\cdot)d\nu \neq \int_A y(\cdot)d\nu.$$

But this contradicts the material balance constraint that $f \in K_\omega(A)$. Therefore,

$$p \cdot x(a) = p \cdot y(a) \quad \text{a.e. } a \text{ in } A.$$

and thus f is a pseudo-competitive allocation, i.e.

$$f \in E(\Xi^B).$$

Proof of Collary $C(\Xi^B) \subset Q(\Xi^B)$

To prove that f is also a quasi-competitive allocation, it is necessary to show that in the case where

$$\inf_{\substack{(\hat{x},\hat{y},\hat{z}) \in S(a) \\ \omega(a)+\hat{x}-\hat{y}-\hat{z} \in X(a)}} \{p \cdot x - p \cdot y\} < 0$$

then f(a) is a maximal element in the budget set. Since f is a pseudo-competitive allocation, if

$$s \in X(a)$$

where

$$s = \omega(a) + x' - y' - z', \ (x', y', z') \in S(a)$$

and

$$p \cdot x' < p \cdot y'$$

then $$f(a) \propto_a s$$

Let s be in the budget set of $a \in A$, i.e.

$$p \cdot x' \leq p \cdot y'.$$

s can be obtained as a limit of a sequence $\{s_n\}$ where $s_n = \omega(a) + x'_n - y'_n - z'_n$, $(x'_n, y'_n, z'_n) \in S(a)$ with $p \cdot x'_n < p \cdot y'_n$.

Then by continuity of preferences and assumption 6.i) on S(a), we get $f(a) \not\propto_a s$. Thus f(a) is a maximal element in the budget set.

Chapter 3

THE "MONETARY" ECONOMY

A. I stated in Chapter 1 that I want to build a sensible model of a monetary economy. The strategy was to focus on the structure of exchange. While the last chapter dealt with the structure of barter exchange, here I shall examine the structure of monetary exchange. The next chapter will bring together both monetary and barter exchange and will consider the choice between the two methods of exchange.

The barter economy was first modelled by using the cooperative approach. It was assumed that each agent possessed a transaction technology. A coalition of agents formed for the purpose of exchanging goods among its members. However, each individual was restricted to executing only his own exchanges. From a given core allocation, I derived the structure of prices required for competitive behaviour to replicate cooperative behaviour. We discovered that a set of ℓ equilibrium prices, one price per good, was sufficient.

In competitive behaviour, individuals accept these prices as parametric. They attempt to obtain the most desirable consumption bundle in their budget set by exchanging goods at these prices, using their own transaction technology. In Appendix B, I have proven that a set of equilibrium prices does in fact exist. Since each agent is constrained at each stage of the exchange process by goods in his possession and by his transaction technology, a general medium of exchange is not necessary for the barter economy to function efficiently within its given constraints.

B. We will again use the cooperative approach to model the monetary economy. However, in the monetary economy coalitions of agents do not just form for the purpose of exchanging goods. They also form for the purpose of exploiting division of labour in order to reduce transaction costs. I assume that a coalition assigns exchange tasks to its members. Its efficiency at exchanging goods depends on its skill at allocating members to tasks according to their abilities. To capture this idea of division of labour or specialization, I begin by assigning a transaction technology to every coalition. I assume that the transaction technology assigned to a coalition consists of the most efficient subset of its member's transaction technologies. In other words, I assume exchange tasks have been allotted as efficiently as possible. The following chapter will describe a method for obtaining a coalition's transaction technology from its member's transaction technologies.

Furthermore, in this chapter I do not allow individuals to "barter," i.e. use their own transaction technologies to effect exchanges. This assumption will be relaxed in the next chapter when I give agents the choice of barter or monetary exchange. However, here an agent must join a coalition if he wants to obtain more desirable goods. An agent requires access to a coalition's transaction technology in order to trade with its members. His potential trades depend on his environment, i.e. on the coalition to which he belongs. The coalition, on the other hand, wants to admit those individuals whose exchange abilities enhance its technology of exchange and whose initial endowments complement the coalition's initial endowments.

I shall again derive the structure of prices needed for competitive behaviour to replicate cooperative behaviour. I shall show that a list of equilibrium buying and selling prices will do the job. In the competitive version of this model, coalitions act as profit maximizing traders; individuals act as utility maximizing consumers. As a profit maximizing trader, the coalition is willing to buy goods from its members or to sell goods to its members. To cover transaction costs, the trader must establish a differential between his buying and selling prices.

The idea of a coalition trader may seem a bit strange at first. In fact however, the coalition trader consists of a set of individual traders who are individually trying to maximize profits, given their transaction technologies and the parametric list of buying and selling prices. The coalition traders represent the commercial sector of the economy.

In Chapter 1 I discussed the importance of money when some individuals specialize in trade. There I assumed some "institution" was present to provide the needed medium of exchange function. In the cooperative approach that I am using, this "institution" is provided by the coalition. A coalition could set up different, although conceptually equivalent, arrangements to play the role of a medium of exchange. An account could be maintained for each member agent. An agent's account would be credited when he sold goods to one of the coalition traders and debited when he bought goods from a trader. Some clearing arrangements would also be necessary among the traders. An agent's budget constraint would be satisfied if his account was nonnegative at the end of all trading. Alternatively, the coalition could issue fiat money to its consumers and traders. By agreement, the fiat money would always be acceptable in exchange for goods. The precise institution used by a coalition will depend, of course, on the transaction costs incurred in setting up and running the institution. The use of different institutions would be reflected in the coalitions efficiency at executing exchanges, i.e. in its transaction technology. Throughout this thesis I assume that a coalition's transaction technology embodies the optimal configuration of such institutions.

C. Let me be a bit more explicit about the monetary model. For every coalition $E \varepsilon \Omega$, there is specified a subset $T(E)$ of $R^{\ell} \times R^{\ell}$ - the *transaction technological* set. A coalition can only reallocate goods among its members in accordance with the coalition's transaction technology. For a given vector $(x,-y) \varepsilon T(E)$, where $x,y \varepsilon R^{\ell}_{+}$, the vector x denotes the total quantities of goods delivered to member agents by the coalition; the vector y denotes the total quantities of goods obtained from member agents by the coalition. The vector y-x which must belong to R^{ℓ}_{+} represents the real resources used up in effecting the reallocation of goods.

I assume that the transaction set corresponding to each admissible coalition is closed, admits free disposal of resources and allows for the possibility of no exchange within a coalition. These properties need little comment as they are standard assumptions for production sets. I also assume that the correspondence T mapping elements of Ω to subsets of $R^{2\ell}$ is countably additive. This means that for every countable family $\{E_i\}_{i\varepsilon I}$ of pairwise disjoint coalitions in Ω, we have $T(\cup_{i\varepsilon I} E_i) = \Sigma_{i\varepsilon I} T(E_i)$.

The concept of a production set for a coalition of agents in a measure theoretic context was first introduced by Hildenbrand [31]. As I have mentioned, the derivation of T from the transaction abilities of a coalition's members will be considered in the next chapter.

D. An *Allocation of commodities,* denoted by f, is again defined as a distribution of goods among the agents where f(a) - the vector of commodities assigned to agent a - is an element of agent a's consumption set X(a). If the initial endowment is ω, then an allocation f is said to be attainable for coalition E if and only if, letting

$$x(a) = [f(a) - \omega(a)]^+,\ y(a) = [f(a) - \omega(a)]^-$$

$$x = \int_E x(\cdot)d\nu,\ y = \int_E y(\cdot)d\nu,$$

then
$$(x,-y) \varepsilon T(E).$$

The vectors x(a) and y(a) are respectively the quantities of goods received in exchange and the quantities of goods given up in exchange by agent a. The vectors x and y are respectively the total quantities of goods received from the coalition by member agents and the total quantities of goods given up to the coalition by members agents. The definition implies that material balance is maintained for both individual agents and coalitions of agents, since, for every $a \in A$

$$f(a) - \omega(a) = [f(a) - \omega(a)]^{+} - [f(a) - \omega(a)]^{-},$$

whence,

$$f(a) = \omega(a) + x(a) - y(a)\ , \geq 0$$

and

$$\int_E f(\cdot)d\nu = \int_E \omega(\cdot)d\nu + \int_E x(\cdot)d\nu - \int_E y(\cdot)d\nu$$

$$= \int_E \omega(\cdot)d\nu - (y - x)\ .$$

A *state* of the economy is again defined as an allocation which is attainable by the coalition consisting of all agents in the economy. Similarly, the *core* is defined as the set of states of the economy which cannot be blocked by any admissible coalition.

E. In this chapter I am again interested in discovering under what conditions the allocations of goods resulting from competitive behaviour coincide with the core allocations. I establish the fact that a list of equilibrium buying and selling prices are required. The difference between the buying and selling prices reflect the transaction costs incurred by the coalition while transporting goods from one agent to another. The prices represent contracts between the coalition of the entire economy and its member agents regarding the terms of acquiring commodities and the allocation of transport tasks to agents. Because the act of buying and the act of selling a good are separate with respect to both time and place, debt contracts between the coalition as a set of specialized traders and its member consumers are also required to ensure that agent's budget constraints and coalition material balance requirements are satisfied.

These statements are established by proving the following propositions.

Proposition 1:

A competitive allocation is also a core allocation.

Proposition 2:

In a "perfectly competitive" economy — where each agent has only a negligible influence on any final allocation of commodities — it is possible to derive from a given core allocation a set of equilibrium buying and selling prices such that the quasi-competitive allocation corresponding to these prices is the given core allocation.

Proposition 3:

Under certain conditions, a quasi-competitive allocation is also a competitive allocation.

F. These propositions will be proven in a rigorous manner after giving precise definition to some concepts not required in the last chapter.

1. Coalition Transaction Technologies. T[7]

The coalitional transactions correspondence T maps elements of Ω into subsets of $R^{2\ell}$. I assume T has the following properties.

i) $T(E)$ is a closed, convex set for all $E \in \Omega$.

ii) if $(x,-y) \in T(E)$ then $x' \leq x$ and $y' \geq y$ implies $(x',-y') \in T(E)$.

iii) $0 \in T(E)$ for all $E \in \Omega$.

iv) T is dominated by the measure ν, i.e. $E \in \Omega$ with $\nu(E) = 0$ implies $T(E) = \{0\}$.

v) T is countably additive on Ω.

vi) T possesses a Radon-Nikodym derivative [3], [5], [17] and [18]. That is, there exists a correspondence τ mapping A into subsets of $R^{\ell} \times R^{\ell}$ such that for every $E \in \Omega$, $T(E) = \int_E L_\tau d\nu$, where $L_\tau = \{t \mid t$ is a ν-integrable function from A to $R^{\ell} \times R^{\ell}$ such that $t(a) \propto \tau(a)$, a.e. a in A.

vii) T(E) is a compact set for all $E \in \Omega$.

The first three conditions need little comment and are similar to the properties for agent's transaction sets. Condition iv) indicates that only "significant" coalitions are capable of production. Conditions v) and vi) imply constant returns to scale with respect to the nonmarketed factors owned by coalitions, i.e. with respect to member agent's abilities used in the operation of coalitions' transaction technologies. Properties i), iii) - v) plus vii) imply property vi) [31, p. 447].

2. Attainable allocations, K_ω.

The allocation $f \in L_X$ is said to be attainable for coalition $E \in \Omega$ if and only if, letting

$$x(a) = [f(a) - \omega(a)]^+,\ y(a) = [f(a) - \omega(a)]^-,$$

$$x = \int_E x(\cdot)d\nu,\ y = \int_E y(\cdot)d\nu.$$

then $$(x,-y) \varepsilon T(E) .$$

The set of all attainable allocations for coalition E is denoted by $K_\omega(E)$. A state of the economy is an element of $K_\omega(A)$.

3. The "monetary" economy, Ξ^M

The description of the monetary economy in its cooperative game context is now complete and is denoted by

$$\Xi^M = [(A,\ \Omega,\ \nu),\ X,\ \succsim,\ T,\ \omega].$$

4. Prices, Profits and the Radon-Nikodym Derivative

A coalition buys commodities at one set of prices and resells commodities at another set of prices. The differential in the prices pays for the cost of transporting goods from one agent to another. This process could result in a profit or loss for the coalition. The assumption was made that the specialized traders operating the transaction technology of a coalition were profit maximizers. Denote the price vector by $p = (p_b,\ p_s)\ \varepsilon\ R^\ell \times R^\ell$ where p_s and p_b

represent the list of prices the traders respectively pay when buying commodities and receive when selling commodities. The profit function for coalition $E \varepsilon \Omega$ for a price vector p is defined by

$$\Pi(p,E) = \max_{(x,-y) \,\varepsilon\, T(E)} (p_b \cdot x - p_s \cdot y).$$

That is, the coalition's profit function is equal to the maximum difference between the value of commodities sold and the value of commodities purchased, for all feasible combinations of quantities bought and quantities sold.

It is possible to verify that the mapping $\Pi(p, \cdot)$ of Ω into $R \cup \{\infty\}$ has the following properties and therefore is a measure.

i) $\Pi(p,\phi) = 0$,

ii) $\Pi(p,\cdot)$ is countably additive on Ω,

iii) $\Pi(p,\cdot)$ is dominated by the measure ν.

From the Theorem of Randon-Nikodym [60], there exists a function $\pi(p,\cdot)$ of Ω into $R \cup \{\infty\}$, where π is ν-measurable and for every $E \varepsilon \Omega$,

$$\Pi(p,E) = \int_E \pi(p,\cdot)d\nu.$$

The function π evaluated at $a \varepsilon A$, i.e. $\pi(p,a)$ can be interpreted as agent a's share in a coalitions profit (or loss).

An agent's share of profits is independent of the particular coalition he joins because of the countable additivity of the transaction technology correspondence.[8] Furthermore, since T is a compact mapping π is continuous in p.

5. The Core and Competitive Allocations

Let f be a state of the economy Ξ^M and define

$$x(a) = [f(a) - \omega(a)]^+, \quad y(a) = [f(a) - \omega(a)]^-$$

$$x = \int_E x(\cdot)d\nu, \quad y = \int_E y(\cdot)d\nu.$$

a) The core, $C(\Xi^M)$

f is called a core allocation if it cannot be blocked by any coalition $E \in \Omega$.

Let $C(\Xi^M)$ be the set of all core allocations.

b) Competitive allocations, $W(\Xi^M)$

f is called a competitive allocation or Walras allocation if there exists a vector $p = (p_b, p_s) \in R^{2\ell}$, $p \neq 0$ such that

$p_b \cdot x(a) \leq p_s \cdot y(a) + \pi(p, a)$ [illegible]

[illegible] X(a) with [illegible] (a) [illegible] implies

[illegible] $)^+ > p$ [illegible]

[illegible]

i) $p_b \cdot x(a) \leqq p_s \cdot y(a) + \pi(p,a)$
and $s \in X(a)$ with $f(a) \propto_a s$
implies $p_b \cdot [s - \omega(a)]^+ > p_s \cdot$
$[s - \omega(a)]^- + \pi(p,a)$.

ii) $p_b \cdot x - p_s \cdot y = \max_{(\hat{x},-\hat{y}) \in T(a)} \left\{ p_b \cdot \hat{x} - p_s \cdot \hat{y} \right\}$.

Let $W(\Xi^M)$ be the set of all competitive allocations.

c) <u>Quasi-competitive allocations, $Q(\Xi^M)$</u>.

f is called a quasi-competitive allocation or a quasi-Walras allocation if there exists a vector $p = (p_b, p_s) \in R^{2\ell}$, $p \neq 0$ such that

i)) $p_b \cdot x(a) \leqq p_s \cdot y(a) + \pi(p,a)$
and $s \in X(a)$ with $f(a) \propto_a s$ and
$\inf_{r \in X(a)} \{p_b \cdot [r-\omega(a)]^+ - p_s \cdot$
$[r-\omega(a)]^-\} < \pi(p,a)$, implies
$p_b \cdot [s-\omega(a)]^+ > p_s \cdot [s-\omega(a)]^- +$
$\pi(\bar{p},a)$.

ii) same as above.

Let $Q(\Xi^M)$ be the set of all quasi-competitive allocations.

d) <u>Expenditure-minimizing allocations, $E(\Xi^M)$</u>

f is called an expenditure-minimizing allocation or a pseudo-competitive allocation if there exists a vector $p = (p_b, p_s) \in R^{2\ell}$, $p \neq 0$ such that:

i) $\cdot\ x(a) \leqq p_s \cdot y(a) + \pi(p,a)$ and

$s \in X(a)$ with $f(a) \propto_a s$ implies

$p_b \cdot [s - \omega(a)]^+ \geqq p_s \cdot [s - \omega(a)]^- + \pi(p,a)$.

ii) same as above.

Let $E(\Xi^M)$ be the set of all expenditure-minimizing allocations.

The definitions of the competitive allocations correspond to those used by Hildenbrand [31]. As I mentioned the concept of a quasi-equilibrium was first introduced by Debreu [14] to cope with the "basic mathematical difficulty that the demand correspondnence of a consumer may not be upper semi-continuous when his wealth" — at a given price vector — "equals the minimum compatible with his consumption set." From the definitions we again get

$$W(\Xi^M) \subset Q(\Xi^M) \subset E(\Xi^M).$$

G. The propositions made above will be established by proving the following theorems. Theorem 1 establishes Proposition 1, Theorem 2 and its corollary establish Proposition 2. An example of an economy in which Proposition 3 is true will be given in Appendix B.

Theorem 1

Every competitive allocation is also a core allocation, i.e. $W(\Xi^M) \subset C(\Xi^M)$.

Theorem 2

If the measure space of agents is non-atomic, then every core allocation is also a pseudo-competitive allocation, i.e. $C(\Xi^M) \subset E(\Xi^M)$.

Corollary

Every core allocation is also a quasi-competitive allocation, i.e. $C(\Xi^M) \subset Q(\Xi^M)$.

I. The following proofs are again based on those by Hildenbrand [31] for coalition production economies.

Proof of Theorem 1. $W(\Xi^M) \subset C(\Xi^M)$.

Let $f \in W(\Xi^M)$ but suppose $f \notin C(\Xi^M)$.
Then there exists $E \in \Omega$ with $\nu(E) > 0$ and $h \in K_\omega(E)$ such that

$$f(a) \propto_a h(a) \quad \text{a.e. } a \text{ in } E \text{ and}$$

$$\left(\int_E [h(\cdot) - \omega(\cdot)]^+ d\nu, \ - \int_E [h(\cdot) - \omega(\cdot)]^- d\nu\right) \in T(E)$$

But $f \in W(\Xi^M)$ implies

$$p_b \cdot [h(a) - \omega(a)]^+ > p_s \cdot [h(a) - \omega(a)]^- + \pi(p,a) \quad \text{a.e. } a \text{ in } E$$

or

$$p_b \cdot \int_E [h(\cdot) - \omega(\cdot)]^+ d\nu - p_s \cdot \int_E [h(a) - \omega(a)]^- d\nu >$$

$$\int_E \pi(p,\cdot) d\nu = \Pi(p,E) \ .$$

Contradicting the definition of the coalition profit function Π.

Q.E.D.

Proof of Theorem 2. $C(\Xi^M) \subset E(\Xi^M)$.

Let $f \in C(\Xi^M)$. Since $f(a)$ is a nonsatiation consumption vector for almost all agents $a \in A$, the set

$$\psi(a) = \{s \in X(a) \mid f(a) \propto_a s\}$$

is non-empty for a.e. a in A.

Now define the correspondence ρ mapping A into subsets of $R^\ell \times R^\ell$ by

$$\rho(a) = \{([s - \omega(a)]^{+}, - [s - \omega(a)]^{-}) \mid s \in \psi(a)\}.$$

Let L_ρ be the set of measurable functions g from A to $R^\ell \times R^\ell$ such that $g(a) \in \rho(a)$ a.e. a in A.

It is possible to show that $L_\rho \neq \emptyset$ (see Hildenbrand [31]). Define the set $U \subset R^\ell \times R^\ell$ by

$$U = \bigcup_{\{E \in \Omega \mid \nu(E) > 0\}} \left\{\int_E L_\rho d\nu - \int_E L_\tau d\nu\right\}$$

where τ is the Radon-Nikodym derivative of the correspondence T w.r.t. ν. I claim that $0 \notin U$. Suppose $0 \in U$. Then there exists $E \in \Omega$ with $\nu(E) > 0$ and a function $g: A \to R^\ell \times R^\ell$ such that

a) $g \in L_\rho \to$ letting $g(a) = (x(a), -y(a))$
where $x(a), y(a) \in R_+^\ell$, and $\tilde{h}(a) = \omega(a) + x(a) - y(a)$
then $f(a) \propto_a \tilde{h}(a)$, $a \in E$.

b) $\int_E g d\nu \in T(E) = \int_E L_\tau d\nu$.

But, $\int_E g d\nu = \int_E g^{+} d\nu - \int_E g^{-} d\nu$

$$= \int_E (x(\cdot), 0) d\nu - \int_E (0, y(\cdot)) d\nu$$

$$= (x, -y) \in T(E).$$

But a) and b) imply $h \in K_\omega(E)$ and the state f is blocked by coalition E using allocation h. This contradicts $f \in C(\Xi^M)$ and therefore $0 \notin U$.

The set U is convex because it is the union of integrals of a set correspondence w.r.t. the non-atomic measure ν (Vind [70]).

From Minkowski's separating hyperplane theorem there exists a vector $p = (p_b, p_s) \in R^\ell \times R^\ell$ such that $u \in U$ implies

$$p \cdot u \geqq 0$$

Or if $E \in \Omega$, then

$$\left\{p_b \cdot x - p_s \cdot y \mid (x,-y) \in T(E)\right\} \leqq \Pi(p,E)$$

$$\leqq \left\{p_b \cdot \int_E \hat{x}(\cdot)d\nu - p_s \cdot \int_E y(\cdot)d\nu \mid (\hat{x}(a),\right.$$

$$\left. -\hat{y}(a) \in \rho(a),\ a \in E\right\} \tag{1}$$

I now show that f is an expenditure-minimizing allocation for the price-vector $p = (p_b, p_s)$.

Let $M = \{a \in A \mid p_b \cdot x(a) \geqq p_s \cdot y(a) + \pi(p,a),\ (x(a), -y(a) \in \rho(a)\}$. It is possible to show that $M \in \Omega$. In fact $\nu(M) = \nu(A)$. If not, there exists a coalition $B \in \Omega$ with $\nu(B) > 0$ such that, for all $a \in B$ there exists $(\hat{x}(a), -\hat{y}(a)) \in \rho(a)$ such that

$$p_b \cdot \hat{x}(a) < p_s \cdot \hat{y}(a) + \pi(p,a).$$

Without loss of generality I can assume, $\hat{x}$, $\hat{y}$ are measurable functions from B to R^{ℓ}_{+}. Integrating the last equation we get

$$p_b \cdot \int_B \hat{x}(\cdot)d\nu - p_s \cdot \int_B \hat{y}(\cdot)d\nu < \int_B \pi(p,\cdot)d\nu,$$

or

$$p_b \cdot \hat{x} - p_s \cdot y < \Pi(p,B),$$

contradicting relation (1) above.

To show that f(a) belongs to each agent's budget set

let $\quad x(a) = [f(a) - \omega(a)]^{+}, Y(a) = [f(a) - \omega(a)]^{-}.$

Then since $(x(a),-y(a)) \varepsilon$ closure of $\rho(a)$ for a.e. a in A we know from (1) that

$$p_b \cdot x(a) \geqq p_s \cdot y(a) + \pi(p,a), \quad \text{a.e. a in A}.$$

Suppose there exists $C \varepsilon \Omega$, $\nu(C) > 0$ such that

$$p_b \cdot x(a) > p_s \cdot y(a) + \pi(p,a), \quad a \varepsilon C.$$

Then clearly

$$p_b \cdot \int_A x(\cdot)d\nu - p_s \cdot \int_A y(\cdot)d\nu > \int_A (p,\cdot)d\nu$$

or

$$p_b \cdot x - p_s \cdot y > \Pi(p,A)$$

But since $(x,-y) \varepsilon T(A)$ by assumption, we have a contradiction of the definition of Π.

Therefore $\quad p_b \cdot x(a) = p_s \cdot y(a) + \pi(p,a).$

Integrating the last equation we get

$$p_b \cdot x - p_s \cdot y = \Pi(p,A)$$

i.e. f maximizes profits on $T(A)$.

Therefore $f \varepsilon E(\Xi^M)$. Q.E.D.

Proof of Corollary. $C(\Xi^M) \subset Q(\Xi^M)$

To prove that f is also a quasi-competitive allocation, it is necessary to show that in the case $a \varepsilon A$ where

$$\inf_{r \varepsilon X(a)} \left\{p_b \cdot [r - \omega(a)]^+ - p_s \cdot [s - \omega(a)]^-\right\} < \pi(p,a)$$

then f(a) is a maximal element in a's budget set. Since f is an expenditure-minimizing allocation, if

$$s \in X(a) \text{ and } p_b \cdot [s - \omega(a)]^+ < p_s \cdot [s - \omega(a)]^- + \pi(p,a)$$

then $$f(a) \nsucc s.$$

Let s be in the budget set of $a \in A$, i.e.

$$p_b \cdot [s - \omega(a)]^+ \leq p_s \cdot [s - \omega(a)]^- + \pi(p,a).$$

Then s can be obtained as a limit of a sequence $\{s_n\}$ where

$$p_b \cdot [s_n - \omega(a)]^+ < p_s \cdot [s - \omega(a)]^- + \pi(p,a).$$

and thus $$f(a) \nsucc s_n.$$

Then by continuity of preferences we get $f(a) \nsucc s$. Thus f(a) is a maximal element in the budget set, i.e. $f \in Q(\Xi^M)$.

Q.E.D

Chapter 4

THE RELATIVE EFFICIENCY OF A "MONETARY" VERSUS A "BARTER" ECONOMY

A. The presence of division of labour characterized the transaction technology of the monetary economy in Chapter 3; the absence of division of labour characterized that of the barter economy in Chapter 2. In this chapter, I shall be investigating these economies efficiency in the allocation of commodities through competitive trading and I shall develop a procedure for deriving aggregate transaction technologies from the transaction abilities of individual agents.

Recall that an individual in the monetary economy of Chapter 3 had to use a coalition's aggregate transaction technology to obtain more desirable goods. The use of the barter exchange process, which underlies the monetary economy, was not available to agents. In this chapter I shall remove this restriction and I shall allow agents the choice between monetary and barter exchange. It will be said that the monetary economy dominates its underlying barter economy if no group of agents wants to break away from the monetary economy and use barter to allocate goods within the group. I shall establish a sufficient condition, based only on an economy's aggregate monetary and barter transaction technologies, which ensures that a monetary economy dominates its underlying barter economy.

B. It is generally believed that society benefits from the use of money. Recently, some authors have attempted to establish the benefits of monetary exchange by demonstrating that money's presence improves the allocation of resources in an economy. The approach used by these authors is very simple. First, they set up two economies which are identical in all details except that one economy uses "money" while the other does not. Then they show that while competitive exchange with "money" is efficient in allocating commodities, exchange without "money" may fail to be efficient. Money's role in promoting efficient exchange in these models depends on each author's concept of the distinguishing feature of monetary exchange.

In Chapter 2 I discussed in some detail Starr's paper [64], where the presence of a medium of exchange overcomes possible inefficiencies which result from the absence of double coincidence of wants. Ostroy [50], in a similar paper, claims that the presence of "money" in a decentralized economy is capable of improving the efficiency of the trading process. In Ostroy's model, trade occurs as the result of a sequence of simultaneous encounters between pairs of agents. The trading decision of each pair of agents must be based only upon the agents' knowledge of the state of the economy, i.e. onthe prevailing equilibrium prices and the agents' tastes, endowments and trading histories. Ostroy's measure of the efficiency of a trading process is the number of simultaneous bilateral meetings required to move an economy from a state of zero aggregate excess demands to a state of zero individual excess demands.

Every competitive trading process requires some mechanism to ensure that each agent lives within his budget. Budget balance can be ensured under decentralized trade if, for every agent, the value of goods given up equals the value of goods received at each bilateral encounter. However, this rather stringent restriction on trade, called bilateral balance by Ostroy, may conflict with the desire for an efficient trading process.

A more accomodating method of ensuring budget balance involves the use of "money." Suppose an account is maintained for each agent and that all violations of bilateral balance are recorded. Whenever the value of goods that is exchanged by a pair of agents is unequal, one agent's account would be credited while the other's would be debited. At the end of trading, budget balance will have been achieved if each agent's account is nonnegative. Ostroy calls this record keeping device "money" and demonstrates the efficiency of the monetary exchange process.

Other papers which use a similar research strategy, although in a different context, are those by Starett (an asset called "money" permits efficient intertemporal allocation of resources [66]), Feldman (rotating sequences of bilateral trade moves lead to a Pareto optimal allocation if "money" is present [21]) and Ostroy and Starr (the presence of a medium of exchange reduces the information required to coordinate exchange and therefore permits efficient decentralization of the trading process [51]).

C. While these authors succeed in establishing the benefits of monetary exchange, given that money has a role to play, they do not establish the superior efficiency of monetary versus barter exchange. In Chapter 2 and 3 I argued that the mere absence of money was not the proper characterization of a barter economy. I claimed that there is a structural difference in the organization of trade between a barter and a monetary economy, based on the presence of division of labour in the latter's transactions technology. The role of money and the benefits of monetary exchange cannot be established just from an analysis of a monetary economy, but must be established in relation to a barter economy.

Using this test of the superior efficiency of monetary exchange, I shall demonstrate the somewhat startling result that a monetary economy need not be more efficient than its underlying barter economy. The superiority of monetary exchange depends on the proper assignment of agents in the operation of a monetary economy's transactions technology. If tasks are allotted ineffectively to agents, a monetary economy may be less efficient than its associated barter economy. Later in this chapter I shall construct an example to demonstrate this point.

D. While the cited articles deal with the social benefits of "money," they do not provide an adequate explanation of the presence of "money." The emergence of "money" in its role as a dominant medium of exchange has been analyzed by Brunner and Metzler [9] and Nagatani [48]. In both models, some existing commodity achieves the status of a universal intermediary in exchange as the result of unconcerted utility maximizing behaviour on the part of individuals. Because agents do not know the identity of potential trading partners with certainty, direct exchange involves the expenditure of real resources on search behaviour. Indirect exchange may reduce these research costs if there exist goods acceptable as intermediaries in exchange. To an individual agent, the acceptability of a particular good as an intermediary in exchange depends on his information about the goods qualities and properties and about its acceptability to potential trading partners. Through a gradual process of learning by agents, some favourite intermediary in exchange becomes the dominant medium of exchange in these models.

This "individualistic" approach, whose roots lie in the works of Menger [47] and von Mises [67], is in contrast to the "social" or cooperative approach that I am using in this thesis. While these authors' have concentrated on the presence of the object serving as a medium of exchange as the distinguishing feature of a monetary economy, I am concentrating on the structural difference between a monetary and barter economy.

E. The cooperative game approach that I am using in this thesis does not require the usual assumption, that money has positive exchange value, in the existence proof of a monetary equilibrium. Usually, while the demand for any other good is based on the utility agents derive from its consumption, or on its use as a productive agent, the demand for money is based on its objective exchange value (von Mises [67]; Kurz [41]; Nagatani [48]). Agents will use and hold money only if it has positive exchange value, that is, only if they believe that other agents will accept it in exchange for more desirable goods.

Unfortunately, the possibility exists that the equilibrium price of money is not positive (Hahn [24]; Kurz [41]; Starr [65]). When this happens it must be concluded that no trade takes place in the economy, since the demand for money is zero and the use of money is necessary for trade. This problem has been circumvented by Starr [65], who shows that "sufficiently exacting" taxes payable in money will ensure the existence of equilibria with a positive price of money. Starr's approach, which is based on a suggestion by Lerner [45], uses taxes "to create a demand for money independent of its usefulness as a medium of exchange" [65, p. 46].

However, the imposition of taxes upon a pure exchange model, to ensure money's use, appears somewhat ad hoc. Using the cooperative game approach, I have shown that money's usefulness is related to the structure of an economy's technology of exchange. Therefore, to exhibit a monetary equilibrium, I only need to show that an equilibrium exists for the economy of Chapter 3. This is done in Appendix B.

F. The choices that are available to an agent in Starr's model are very limited. An agent must either consume his initial endowment, or else, he must use the monetary exchange process to obtain more desirable goods and/or to obtain money for taxes. Starr uses taxes payable in money to force participation in the monetary economy. He does not give an agent an alternative to monetary exchange.

It is my belief that the use of the monetary exchange process should not be a constraint imposed on the trading behaviour of agents. Rather, it should result from their maximizing behaviour. It is obvious that an agent, who breaks away from the monetary economy by himself, has no choice but to consume his initial endowment of goods because he will have no trading partners.

Therefore, to provide an alternative to monetary exchange, it is necessary to consider the possibility that some groups of agents will break away from the monetary economy and will use an alternative method to exchange goods within the group. This possibility can be analyzed within the framework of this thesis because the cooperative approach admits group rationality along with individual rationality.

In Chapter 3 I assumed that each coalition's transaction technology consisted of the most efficient subset of its members' transaction technologies. Later in this chapter I shall devise a way of constructing this efficient transaction technology from members' transaction technologies. A coalition who is given the choice between barter exchange and the use of this efficient technology will clearly choose the latter. However, in general it is possible that some coalitions are not very skilled at assigning exchange tasks. The next section describes an economy in which agents are not given exchange tasks according to their abilities. Consider for example a traditional society where the eldest son always takes up his father's trade. The point I am trying to make is that for some coalitions, barter exchange might be more efficient than monetary exchange. In other words, the set of feasible allocations attainable through barter is larger than the set of feasible allocations through monetary exchange, at the given assignment of exchange tasks to agents.

However, I shall prove in section I below that for monetary exchange to dominate barter exchange, it is sufficient that for the coalition consisting of the entire economy, the set of feasible allocations attainable through monetary exchange contain the set of feasible allocations attainable through barter. Monetary exchange will dominate barter exchange even though there are smaller coalitions for whom barter exchange is more efficient than monetary exchange.

G. In this section I shall devise an example to illustrate the advantages and disadvantages of monetary exchange and the derivation of aggregate transactions technologies. Consider an economy with a finite number of agents in whcih transaction costs are linear in amounts exchanged and, following Niehans [49], consist simply in a shrinkage by some percentage in the amount of a good that is exchanged. The transaction cost can be interpreted as the cost incurred in transporting the good to or from the market place, with different agents having differing transport abilities.

If in some trading pattern, agent a exchanges the vector y(a) of goods for x(a), the components z(a) of resource costs that are incurred can be given by

$$z_j(a) = \bar{z}_j(a)(x_j(a) + y_j(a)),\ j=1,\cdots,\ell,$$

and agent a's transaction set S(a) is given by

$$S(a) = \left\{ \left[x(a),\ y(a),\ z(a)\right] \middle| x(a),\ y(a) \in R^{\ell}_{+},\ z(a) = \bar{z}(a) \cdot \left[x(a) + y(a)\right] \right\}$$

For any good j either $x_j(a)$ or $y_j(a)$ will be zero, depending on whether good j is received or given up in exchange by agent a.

The vector z of resource costs which are incurred by the entire society in the barter exchange process is the sum of the individual agents' resource costs.

i.e.
$$z = \sum_{a \varepsilon A} z(a),$$
where
$$z_j = \sum_{a \varepsilon A} \bar{z}_j(a)\, x_j(a) + \sum_{a \varepsilon A} \bar{z}_j(a)\, y_j(a).$$

This trading process is a barter exchange process because each agent must transport his own goods.

Suppose that agents are now allowed to specialize in the transportation of certain goods. To capture this idea, for each good j select $\hat{a}(j) \varepsilon A$ such that $\bar{z}_j(\hat{a}(j)) = \min_{a \varepsilon A} \bar{z}_j(a)$. This is, agent $\hat{a}(j)$ is the most efficient of all agents in transporting good j. If for each j, agent $\hat{a}(j)$ is allotted the task of transporting good j, the economy's transactions technology is exploiting the division of labour and therefore it is of the type described in Chapter 3.

The jth component of the total resource cost vector for an arbitrary exchange pattern x, y is given by

$$z_j = \sum_{a \varepsilon A} \bar{z}_j\left(\hat{a}(j)\right)\left(x_j(a) + y_j(a)\right).$$

Clearly, this monetary exchange process is more efficient than the barter process for any trading pattern.

On the other hand, suppose that tasks are allotted by selecting $a'(j) \varepsilon A$ such that $\bar{z}_j(a'(j)) = \max_{a \varepsilon A} \bar{z}_j(a)$.

That is, agent a'(j) is the least efficient of all agents in transporting good j. The economy where agent a'(j) is allotted the task of transporting good j is also a legitimate monetary economy, however, its exchange process is clearly less efficient than the barter exchange process. Therefore, it is possible that society does not benefit from monetary exchange.

H. I now shall consider the problem of deriving transaction technologies from agents' transaction abilities. To begin, consider a barter economy of the type described in Chapter 2 given by

$$\Xi^B = [(A, \Omega, \nu), X, \precsim, S, \omega].$$

Recall that the transactions set S(a) of agent a consists of all feasible combinations of exchanges and attendant resource costs. An exchange of the bundle of goods $\hat{y}(a)$ for the bundle $\hat{x}(a)$ is technologically feasible if there exists a vector $\hat{z}(a)$ of resource costs such that $(\hat{x}(a), \hat{y}(a), \hat{z}(a) \in S(a)$. If $\omega(a)$ represents agent a's initial endowment, the resulting consumption bundle is given by

$$f(a) = \omega(a) + \hat{x}(a) - \hat{y}(a) - \hat{z}(a).$$

It is important to note that $\hat{y}(a)$ is the vector of quantities that is actually given up in exchange to other agents, while $\hat{x}(a)$ is the vector of quantities actually received from other agents. The transaction cost vector $\hat{z}(a)$ consists of goods from agent a's initial endowment and/or goods obtained from others during the process of exchange. Therefore any vector $z(a)$ of resource costs can be decomposed into two components,

$$\hat{z}(a) = \hat{z}_1(a) + \hat{z}_2(a),$$

where $\hat{z}_1(a)$ consists of goods obtained during the process of exchange and $\hat{z}_2(a)$ consists of goods obtained from a's initial endowment.

This distinction was not necessary in the discussion of barter economy because each agent had to bear directly any resource costs which were incurred during the trading process. However, if agent a's transaction abilities are employed in the operation of an aggregate transactions technology, both $\hat{z}_1(a)$ and $\hat{z}_2(a)$ must be obtained from other agents.

Now define

$$x(a) = \hat{x}(a) - \hat{z}_1(a),$$

$$y(a) = \hat{y}(a) + \hat{z}_2(a).$$

Then, if a coalition uses the transaction abilities S(a) of agent a, it must obtain y(a) of goods to deliver x(a) of goods. Therefore, the transaction abilities of agent a as perceived by a coalition can be given by

$$S'(a) = \left\{ \left[x(a), -\ y(a) \right] \middle| x(a) = \hat{x}(a) - \hat{z}_1(a),\ \hat{y}(a) = \hat{y}(a) + \hat{z}_2(a), \right.$$

where

$$\left. \left[\hat{x}(a),\ \hat{y}(a),\ \hat{z}(a) = \hat{z}_1(a) + \hat{z}_2(a) \right] \varepsilon\ S(a) \right\}.$$

From the properties of the correspondence S, it is easy to show that the correspondence S', which maps elements of A into subsets of $R^{\ell} \times R^{\ell}$, has the following properties.

i) S'(a) is closed for all a ε A.

ii) $(x(a), -y(a)) \varepsilon S'(a)$ and $x'(a) \leq x(a)$, $y'(a) \geq y(a)$ then $(x'(a), -\ y'(a)) \varepsilon S'(a)$.

iii) $0 \varepsilon S'(a)$ for all a ε A.

iv) S' is a ν-measurable correspondence.

I shall now use the correspondence S' to construct aggregate transactions technologies for each coalition E ε Ω. Define the correspondence T^*: $\Omega \rightarrow R^{\ell} \times R^{\ell}$ by

$T^*(E) = \{(x,-y) | x,y \in R_+^{\ell}$ and such that there exist ν-integrable functions $x', y': A \to R_+^{\ell}$ with $(x'(a), -y'(a)) \in S'(a)$ for all $a \in E$ and $x = \int_E x'(\cdot)d\nu$; $y = \int_E y'(\cdot)d\nu\}$.

$T^*(E)$ is the integral of the correspondence S' with respect to the measure ν over the set E. Using the notation of Chapter 3, it can also be written as

$$T^*(E) = \int_E D_{S'} d\nu.$$

The set $T^*(E)$ incorporates all possible ways of organizing coalition E's transactions technology by allotting its member agents to various tasks. Since ν is an atomless finite measure, $T^*(E)$ is convex. Furthermore, from the properties of S', it is possible to show that the correspondence T^* satisfies properties i) through v) of Chapter 3 for coalition transactions technologies. If for each $E \in \Omega$, $T^*(E)$ is bounded by the total quantity of resources initially available to coalition E, then T^* is also a compact correspondence and therefore also satisfies property vi).

Thus the economy given by

$$\Xi^{M^*} = [(A, \Omega, \nu), X, \succsim, T^*, \omega]$$

is a legitimate monetary economy whose underlying barter economy is Ξ^B.

Let T be any correspondence from Ω to $R^\ell \times R^\ell$ with $T(E) \subseteq T^*(E)$ for all $E \in \Omega$ and that satisfies the conditions of Chapter 3. Then the economy given by

$$\Xi^M = [(A, \Omega, \nu), X, \precsim, T, \omega]$$

can also be interpreted as a monetary economy whose underlying barter economy is Ξ^B. However, the agents who operate the aggregate transactions technology T are not as efficiently specialized as those that operate T*.

I. Recall that the aggregate transactions set of the monetary economy is a subset of the space $R^{2\ell}$. While the transaction abilities of an individual agent can be represented in the space $R^{2\ell}$, the barter economy's transactions technology cannot. However, it is possible to derive an implicit aggregate transactions set for the barter economy in $R^{2\ell}$. This implicit transactions set can then be compared with the monetary economy's transactions set.

For all $E \in \Omega$, define the correspondence T^B: $\Omega \to R^\ell \times R^\ell$ by

$$T^B(E) = \{(x,-y) | x, \; y \; \varepsilon \; R^\ell_+$$

and such that there exists

$$f \; \varepsilon \; K_\omega(E) \text{ of } \Xi^B$$

with

$$x = \int_E [f-\omega]^+ d\nu; \; y = \int_E [f-\omega]^- d\nu\}.$$

Let $f \; \varepsilon \; K_\omega(E)$ of Ξ^M. Then there exist ν-integrable functions $x', y', z' = z_1' + z_2' : A \rightarrow R^\ell_+$ such that for all $a \; \varepsilon \; A$

$$\left(x'(a), \; y'(a), \; z'(a)\right) \; \varepsilon \; S(a)$$

and

$$f(a) = \omega(a) + x'(a) - y'(a) - z'(a).$$

Substituting $z'(a) = z_1'(a) + z_2'(a)$ as defined above into the last equation and rearranging we get

$$f(a) - \omega(a) = \left(x'(a) - z_1'(a)\right) + \left(y'(a) + z_2'(a)\right)$$

But then

$$[f(a) - \omega(a)]^+ = x'(a) - z_1'(a) \; ,$$
$$[f(a) - \omega(a)]^- = y'(a) + z_2'(a)$$

implies, since $(x'(a) - z_1'(a), \; - (y'(a) + z_2'(a))) \; \varepsilon \; S'(a)$, that

$$T^B(E) \subset T^*(E) \; .$$

Now I shall demonstrate the condition which ensures that a monetary economy will dominate its underlying barter economy.

Theorem: Consider the monetary economy given by

$$\Xi^M = [(A, \Omega, \nu), X, \precsim, T, \omega]$$

and its underlying barter economy Ξ^B whose implicit aggregate transactions technology is given by T^B. If $T^B(A) \subset T(A)$, then the monetary economy Ξ^M dominates the barter economy Ξ^B.

Proof: Let f be a core allocation of Ξ^B. Then since $T^B(A) \subset T(A)$, f is also a state of the economy Ξ^M. Let f* be any core allocation of Ξ^M. I claim that $f(a) \precsim_a f^*(a)$ for a.e. agents in the economy Ξ^M. Otherwise f would be a blocking allocation for some $E \varepsilon \Omega$ contradicting the choice of f*. Thus any agent who is given a choice between Ξ^M and Ξ^B will choose Ξ^M.

Furthermore, it will not be to the advantage of any group of agents to break away from the monetary economy. If g is an attainable allocation for any $E \varepsilon \Omega$ in Ξ^B, with $\nu(E) > 0$, then by definition of f, $g(a) \precsim_a f(a)$ for a.e. agents in E. By the transitivity of preferences, it is also true that $g(a) \precsim_a f^*(a)$ for a.e. agent in E.

Q.E.D.

It is important to note that the condition $T^B(A) \subset T(A)$ applies only to the coalition of the entire economy. No restriction is required on the aggregate transaction sets of smaller coalitions. In other words, the theorem holds even though barter exchange is "more efficient" than monetary exchange for some of the admissible coalitions in the economy. That is, there might be coalitions $E \in \Omega$, $E \neq A$, such that

$$T(E) \subset T^B(E),$$

and monetary exchange will still dominate barter exchange.

FOOTNOTES

[1] I assume that no real resource costs are incurred in the hypothetical formation of a coalition, hypothetical reallocation of goods within the coalition and dissolution of the coalition in the cooperative economy. The analogous assumption in the competitive version of the economy is that no real resource costs are incurred in the determination and dissemination of the equilibrium vector of prices. A more complete analysis would consider the structure of these institutions, the costs incurred in their operation and the efficiency of one institution relative to another. The papers by Feldman [21], Howitt [36] and Ostroy [50] are attempts to analyze the role of money in the operation of these institutions. The purpose of this thesis, however, is to show that the usefulness of money depends on the structure of an economy's transaction technology. To achieve this result, it is sufficient to consider only the transaction costs resulting from the *transportation* of goods from one agent to another.

[2] See Appendix A for definitions of mathematical concepts unusual to economics.

[3] Generally, it is assumed that A is a finite set. For some proofs, however, it is necessary to assume that the measure space is non-atomic. In this case, A must be of the cardinality of the continuum.

[4] The local nonsatistion assumption on $\propto_a$ is weaker than the usual assumption that $\propto_a$ is monotonic, i.e. $s, t, \in X(A)$ with $s < t$ implies $s \propto_a t$. Montonic preferences are assumed in Chapter 5 to ensure that a quasi-competitive price equilibrium is also a competitive price equilibrium.

[5] In Kurz's [41] barter economy, the "market" provides the proof of resources required to effect exchanges. These resources can be contracted by individuals for the purpose of carrying out their exchanges. An agent does not bear directly the resource costs incurred in effecting exchanges, as is the case in my barter economy. Consequently, the barter economy of this chapter is more "primitive" in the degree of commercial development than that of Kurz.

[6] From condition 6ii), y(a) and z(a) can always be chosen so that $x^i(a) > 0$ implies $y^i(a) = 0$. That is, an agent need not buy and sell a good at the same time. Therefore,

$$\begin{aligned} f(a) \in X(a) \subset R_+ &\Rightarrow f(a) = \omega(a) + x(a) - y(a) - z(a) \geqq 0 \\ &\Rightarrow \omega(a) + x(a) \geqq y(a) + z(a) \\ &\Rightarrow \omega(a) \geqq y(a), \text{ and} \\ &\quad\ \omega(a) + x(a) \geqq z(a). \end{aligned}$$

The inequality $\omega(a) \geqq y(a)$ states that an agent cannot sell more than his initial endowment, while the inequality $\omega(a) + x(a) \geqq z(a)$ states that the resource costs incurred by an agent in effecting an exchange cannot exceed his initial endowment plus the quantities of goods acquired in exchange.

[7] The aggregate transaction technology described in this chapter differs from Foley's [22] in that he combines both production and exchange activities. It differs from Kurz's [41] in that he uses separate "buying" and "selling" technologies linked by a medium of exchange. Furthermore, to my knowledge, aggregate transaction technologies have never been studied either in a core theoretic or a measure theoretical context.

[8] Sondermann [62] has obtained "stable" profit distributions in the case certain productive factors show increasing returns to scale for coalition production economies. To incorporate increasing returns to scale in the context of coalition transaction economies, we would have to let T be superadditive on Ω. That is, for every pair of disjoint coalition E_1 and E_2,

$$T(E_1) + T(E_2) \subset T(E_1 \cup E_2).$$

[9] If A is a finite set, then

$$T^*(E) = \sum_{a \varepsilon A} S(a)$$

where Σ indicates the set theoretic sum. To ensure T^* is convex, we must also assume that $S(a)$ is convex for each $a \varepsilon A$.

[10] Kurz [41] and [43] has investigated the existence of an equilibrium in barter and monetary economies under the more reasonable assumption:

$$\int_A \omega(\cdot)d\nu >> 0.$$

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APPENDIX A

MATHEMATICAL CONCEPTS

The following mathematical concepts are unusual to economic theory and so their definitions are gathered here.

1. Measure Space:

A measure space is a triple (A, Ω, ν) where A is a set, Ω is a σ-field of subsets in A, and ν is a countably additive, non-negative set function on Ω with $\nu(A) = 1$.

2. σ-field:

Ω is a σ-field in A if for every countable sequence $\{E_n\}$ of subsets $E_n \in \Omega$,

$$\bigcup_n E_n \in \Omega, \text{ and}$$

$$E_1 - E_2 \in \Omega.$$

Furthermore, $\bigcup_{E \varepsilon \Omega} E = A.$

3. Measurable set:

A set is called measurable if it is an element of Ω.

4. Measure:

A function $\nu : \Omega \to R_+^\ell$ is called a measure if it is countably additive on Ω. That is, for any countable sequence $\{E_n\}$ of disjoint sets in Ω,

$$\nu\left(\bigcup_n E_n\right) = \sum_n \nu\left(E_n\right).$$

5. Almost every $a \varepsilon A$ (a.e. $a \varepsilon A$):

A relation is said to hold for almost every element of A (a.e. $a \varepsilon A$) if the set of those elements for which the relation is not true has measure zero. That is, if $E \varepsilon \Omega$ is the set for which the relation is not true, then

$$\nu(E) = 0$$

6. Measurable function:

A function $f : A \to R$ is called measurable if for every interval $\alpha \subset R$, $f^{-1}[\alpha] \varepsilon \Omega$. f is sometimes called ν-measurable or Ω-measurable.

A vector-valued function $f : A \to R^{\ell}$ is called measurable if each component f^i is measurable.

7. <u>ν-integrable function:</u>

A function $f : A \to R$ is called ν-integrable if it is ν-measurable and if the Lebesque-Stieltjes integral of f with respect to ν over A, denoted by,

$$\int_A f(\cdot)d\nu,$$

exists.

A vector-valued function $f : A \to R^{\ell}$ is called ν-integrable if each component f^i is ν-integrable.

8. <u>Lebesque-Stieltjes integral and properties:</u>

See Sion [60].

9. <u>Correspondence:</u>

A correspondence ψ from A to R^{ℓ} associates with every element a of A a subset $\psi(a)$ of R^{ℓ}. Its graph is

$$G_{\psi} = \left\{(a,r) \in A \times R^{\ell} \mid r \in \psi(a)\right\}.$$

10. Inverse of a correspondence:

The inverse ψ^{-1} of the correspondence is defined as follows: let Γ be a family of subsets of R^ℓ, then

$$\psi^{-1}[\Gamma] = \left\{a \in A \mid \psi(a) \in \Gamma\right\}.$$

11. Strong-inverse of a correspondence:

If X is a subset of R^ℓ, then the strong-inverse ψ^s of the correspondence ψ is given by

$$\psi^s[X] = \left\{a \in A \mid \psi(a) \subset X\right\}$$

12. Upper semicontinuous correspondence:

A correspondence $\psi : A \to R^\ell$ is upper semicontinuous if its graph is closed. That is, for every sequence $\{a_n, r_n\}$ in G_ψ with $\lim\limits_{n\to\infty} (a_n, r_n) = (a,r)$, then $(a,r) \in G_\psi$.

13. Measurable correspondence:

The correspondence $\psi : A \to R^\ell$, where A is part of the measure space (A, Ω, ν), is said to be measurable (ν-measurable or Ω-measurable) if for every open set X in R^ℓ,

$$\psi^s[X] \in \Omega .$$

See Debreu [17] for alternative definitions.

14. Integral of a correspondence:

Consider the correspondence $\psi : A \to R^{\ell}$. Let L_{ψ} be the set of all point-valued $f : A \to R^{\ell}$ such that f is ν-integrable over A and $f(a) \in \psi(a)$ for all $a \in A$. The integral of the correspondence ψ over A is defined by:

$$\int_A \psi(\cdot)d\nu = \left\{\int_A f(\cdot)d\nu \mid f \in L_{\psi}\right\}.$$

The integral $\int_A \psi(\cdot)d\nu$ is also written as $\int_A L_{\psi}\, d\nu$.

See Aumann [5], Debreu [17] and Artstein [3] for properties of the integral of a correspondence.

15. Non-atomic measure space:

The set $E \in \Omega$ is called an atom of the measure space (A, Ω, ν) if $\nu(E) > 0$ and $E \supset F \in \Omega$ implies $\nu(F)$ or $\nu(E-F) = 0$. The measure space is called non-atomic if it has no atoms.

APPENDIX B

THE EXISTENCE OF COMPETITIVE PRICE EQUILIBRIA

A. In this appendix I shall provide the proofs that establish the existence of price equilibria for both the "barter" and "monetary" economies. In view of Theorems 2.1, 2.2, 3.1 and 3.2, I am also establishing the conditions under which the cores of these economies are non-empty. The proofs are based on similar existence proofs by Debreu [13], Aumann [6], Schmeidler [55] and Hildenbrand [33], [34]. The required mathematical tools can be found in Artstein [3], Aumann [5], Debreu [17], Debreu and Schmeidler [18], Schmeidler [56] and Sion [60].

First, I shall demonstrate the existence of a quasi-competitive price equilibrium for the "barter" economy, under the conditions of Chapter 2, in Theorem 1 and for the "monetary" economy, under the conditions of Chapter 3, in Theorem 2. At the end of the appendix, I shall list the additional assumptions required so each quasi-competitive price equilibrium is also a competitive price equilibrium.

B. Consider the "barter" economy

$$\Xi^B = [(A, \Omega, \nu), X, \precsim, S, \omega]$$

as described in Chapter 2. Let

$$\Delta = \left\{ p \in R_+^{\ell} \mid \sum_{i=1}^{\ell} p^i = 1 \right\}$$

be the unit price simplex. Define the budget correspondence $\beta : A \times \Delta \to R_+^{3\ell}$ by

$$\beta(a,p) = \left\{ (x,y,z) \in R_+^{3\ell} \;\middle|\; \begin{array}{ll} \text{a)} & (x,y,z) \in S(a), \\ \text{b)} & \omega(a) + x-y-z \in X(a), \text{ and} \\ \text{c)} & p \cdot x \leqq p \cdot y \end{array} \right\},$$

and the demand correspondence $\phi : A \times \Delta \to R_+^{3\ell}$ by

$$\phi(a,p) = \Big\{ (x,y,z) \in \beta(a,p) \mid \text{for every } (x',y',z') \in \beta(a,p),$$

$$\omega(a) + x'-y'-z' \precsim_a \omega(a) + x-y-z \Big\} .$$

Finally, define the quasi-demand correspondence $\delta : A \times \Delta \to R_+^{3\ell}$ by

$$\delta(a,p) = \begin{cases} \phi(a,p) & \text{if } \inf\limits_{\substack{(\hat{x},\hat{y},\hat{z}) \,\varepsilon\, S(a) \\ \omega(a) + \hat{x}-\hat{y}-\hat{z} \,\varepsilon\, X(a)}} \{p \cdot \hat{x} - p \cdot \hat{y}\} \leqslant 0, \\ \\ \beta(a,p) & \text{otherwise} . \end{cases}$$

Definition I. Quasi-competitive price equilibrium

Let p be a price vector in the unit simplex $\Delta \subset R_+^\ell$ and f be a ν-integrable function from A to R_+^ℓ. The pair (p,f) is called a quasi-competitive price equilibrium of the economy Ξ^B if there exist ν-integrable functions $x,y,z : A \to R_+^\ell$ such that

i) $f(a) = \omega(a) + x(a) - y(a) - z(a)$, a.e. $a \,\varepsilon\, A$,

ii) $\big(x(a),\ y(a),\ z(a)\big) \,\varepsilon\, \delta(a,p)$, $\quad a \,\varepsilon\, A$, and

iii) $\int_A f(\cdot)d\nu \leqq \int_A \omega(\cdot)d\nu - \int_A z(\cdot)d\nu$.

Conditions i) and ii) state that f(a) must be maximal with respect to $\underset{\sim}{\propto}_a$ in agent a's budget set whenever the minimum wealth situation does not occur, while condition iii) is the material balance equation for the entire economy. This last relation can also be written as

iii') $\int_A x(\cdot)d\nu \leqq \int_A y(\cdot)d\nu$.

Definition 2: Competitive price equilibrium

The pair (p,f) is called a competitive price equilibrium of the economy Ξ^B if it forms a quasi-competitive price equilibrium and if the set of agents for whom the minimum wealth situation occurs has measure zero.

i.e. if

$$E^* = \left\{a \in A \mid \inf_{\substack{(\hat{x},\hat{y},\hat{z}) \in S(a) \\ \omega(a)+\hat{x}-\hat{y}-\hat{z} \in X(a)}} \{p \cdot \hat{x} - p \cdot \hat{y}\} \geqq 0\right\}$$

then $\nu(E^*) = 0$.

In other words, for the equilibrium price vector $p \in \Delta$ "most" agents have sufficient wealth to exchange some goods and still remain inside their consumption set.

C. The proof of the following theorem is patterned after the existence proofs by Hildenbrand [33] and [34]. The main difference is that my model portrays a barter economy with transaction costs and individual specific transaction technologies, while in [34] Hildenbrand models a pure exchange economy and in [33] he models a coalition production economy. My economy differs from the barter economy portrayed by Kurz [43] in the specification of the transaction technologies (see footnote 3) and in its measure theoretic context.

In part a) of the proof, to ensure that each agent's budget set is bounded and thus to ensure that his demand set is non-empty, a sequence of "truncated economies" is constructed. In part b) it is shown that each truncated economy has a quasi-competitive price equilibrium by showing its total quasi-demand correspondence satisfies the properties of Debreu's lemma [13, p. 82]. Finally, in part c) it is shown the existence of a sequence of quasi-equilibria for the sequence of truncated economies implies the existence of a quasi-competitive price equilibrium for the original economy.

Theorem 1: If the measure space of agents of the "barter" economy Ξ^B is non-atomic, then there exists a quasi-competitive price equilibrium.

Proof:

Part a) In an economy with a continuum of agents, an agent of measure zero has only an infinitesmal portion of the goods of the entire economy. As Aumann [6] points out, the possibility exists that for a given price vector $p \in \Delta$ the budget set $\beta(a,p)$ for some agent $a \in A$ is unbounded, and hence the demand set $\phi(a,p)$ may be empty. To circumvent this possibility, for every positive integer k consider the truncated consumption set

$$X^k(a) = \left\{s \; \varepsilon \; X(a) \mid s \leqq k\left[\omega(a) + 1\right]\right\}$$

and the truncated transactions set

$$S^k(a) = \left\{(x,y,z) \; \varepsilon \; S(a) \mid (x,y,z) \leqq k\left[\omega(a) + 1, \; \omega(a) + 1, \; \omega(a) + 1\right]\right\}.$$

Define the k-th truncated budget correspondence, β^k, demand correspondence, ϕ^k, and quasi-demand correspondence, δ^k, by replacing X(a) and S(a) with $X^k(a)$ and $S^k(a)$ in the definitions of section B above.

Finally, define the total quasi-demand correspondence $\psi^k : \Delta \to R^\ell$ for the "k-th truncated economy by

$$\psi^k(p) = \Big\{s \; \varepsilon \; R^\ell \mid \text{there exist } \nu\text{-integrable functions } x,y,z : A \to R^\ell_+ \text{ such that } \left[x(a),y(a),z(a)\right] \; \varepsilon \; \delta^k(a,p) \text{ for almost every } a \; \varepsilon \; A \text{ and } s = \int_A \left[x(\cdot) - y(\cdot)\right] d\nu\Big\}.$$

If we define the correspondence $\sigma^k : \Delta \to R^{3\ell}_+$ by

$$\sigma^k(p) = \int_A \delta^k(\cdot,p)d\nu,$$

i.e. $\sigma^k(p)$ is the integral of the quasi-demand correspondence with respect to the measure ν, then ψ^k can also be defined by

$$\psi^k(p) = \left\{(\hat{x}-\hat{y}) \mid (\hat{x},\hat{y},\hat{z}) \in \sigma^k(p)\right\}.$$

<u>Part b)</u>

I claim that the correspondence ψ^k has the following properties:

i) there is a compact set $N \subset R^\ell$ such that $\psi^k(p) \subset N$ for every $p \in \Delta$,

ii) the graph of ψ^k is closed,

iii) for every $p \in \Delta$, $\psi^k(p)$ is non-empty and convex, and

iv) for every $p \in \Delta$, $p \cdot \psi^k(p) \leqq 0$.

To prove property i) let

$$N = \left\{s \in R^\ell \;\middle|\; |s| \leqq k[\int_A (\omega(\cdot) + 1)d\nu]\right\}.$$

Then by construction of ψ^k, for every $p \in \Delta$ and ν-integrable functions $x,y,z : A \to R^\ell_+$ with $(x(a), y(a), z(a)) \in \delta^k(a,p)$ we have $0 \leqq x(a) \leqq k(\omega(a) + 1)$, $0 \leqq y(a) \leqq k(\omega(a) + 1)$ and thus

$$|x(a) - y(a)| \leqq k\left[\omega(a) + 1\right], \text{ a.e. } a \text{ in } A. \qquad (1)$$

Integrating we get

$$\left|\int_A \left[x(\cdot) - y(\cdot)\right]d\nu\right| \leqq \int_A \left|\left[x(\cdot) - y(\cdot)\right]\right|d\nu$$

$$\leqq k\left[\int_A \left[\omega(a) + 1\right]d\nu\right]$$

or $\psi^k(p) \subset N$ for every $p \varepsilon \Delta$.

To prove property ii) let (p_n, s_n) be a sequence in

$$G_{\psi^k} = \left\{(\hat{p},\hat{s}) \ \varepsilon \ \Delta \times R^\ell \mid \hat{s} \ \varepsilon \ \psi^k(\hat{p})\right\},$$

the graph of ψ^k, with $\lim_{n\to\infty}(p_n, s_n) = (p,s)$. That is for every positive integer n there exist ν-integrable functions x_n, y_n, $z_n : A \to R^\ell_+$ such that $(x_n(a), y_n(a), z_n(a)) \ \varepsilon \ \delta^k(a,p)$ and

$$s_n = \int_A \left[x_n(\cdot) - y_n(\cdot)\right]d\nu,$$

$$\lim_{n\to\infty} \int_A \left[x_n(\cdot) - y_n(\cdot)\right]d\nu = \lim_{n\to\infty} s_n = s.$$

From (1) above, the sequence of ν-integrable functions $\{(x_n - y_n)\}$ from $A \to R^\ell$ is bounded pointwise in absolute value by the

ν-integrable function $k(\omega + 1) : A \to R_+^\ell$. Using Theorem E [33, p. 622] (a version of Fatou's lemma), there exists a ν-integrable function $t : A \to R^\ell$ such that

$$\int_A t(\cdot)d\nu = s, \text{ and} \tag{2}$$

$$t(a) \in \text{closure} \left\{ \left[x_n(a) - y_n(a) \right] \right\} \text{ for almost every } a \in A. \tag{3}$$

Since each agent is either a net purchaser or a net supplier of a particular good, but not both, if we let $x(a) = t(a)^+$, $y(a) = t(a)^-$ for all $a \in A$, then $x(a) \in \text{closure } \{x_n(a)\}$ and $y(a) \in \text{closure } \{y_n(a)\}$. Because $S^k(a)$ is compact, for every $a \in A$ there exists $z(a)$ such that $z(a) \in \text{closure } \{z_n(a)\}$ and $(x(a), y(a), z(a)) \in S^k(a)$. From the measurability of the correspondence S^k, $z(a)$ can be chosen for each $a \in A$ such that the function $z : A \to R_+^\ell$ is ν-integrable (Theorem B [33, p. 621].

I also claim that for fixed $\bar{a} \in A$, $\delta^k(\bar{a},\cdot) : \Delta \to R_+^{3\ell}$ is an upper semi-continuous correspondence. Following Schmeidler [55, p. 582], let $\{p_m\}$ and $\{(x_m, y_m, z_m)\}$ be sequences such that $\lim_{m\to\infty} p_m = p$, $\lim_{m\to\infty} (x_m, y_m, z_m) = (x,y,z)$ with $p_m \in \Delta$, $(x_m, y_m, z_m) \in \delta^k(\bar{a},p_m)$. Then we must have

$$p_m \cdot x_m \leqq p_m \cdot y_m \, , \tag{4}$$

$$(x_m, y_m, z_m) \in S^k(\bar{a}), \text{ and} \tag{5}$$

$$\omega(\bar{a}) + x_m - y_m - z_m \in X^k(\bar{a}). \tag{6}$$

Suppose that

$$\inf_{\substack{(\hat{x},\hat{y},\hat{z}) \in S^k(\bar{a}) \\ \omega(\bar{a})+\hat{x}-\hat{y}-\hat{z} \in X^k(\bar{a})}} \left\{p \cdot (\hat{x}-\hat{y})\right\} \geqq 0 .$$

Then since $S^k(\bar{a})$ and $X^k(\bar{a})$ are closed and inequalities are preserved under limits, we get after taking limits on (4), (5) and (6) that

$$(x,y,z) \in \beta^k(\bar{a},p) \tag{7}$$

On the other hand, if

$$\inf_{\substack{(\hat{x},\hat{y},\hat{z}) \in S^k(\bar{a}) \\ \omega(\bar{a}) + \hat{x}-\hat{y}-\hat{z} \in X^k(\bar{a})}} \left\{p \cdot (\hat{x}-\hat{y})\right\} < 0,$$

then since $S^k(\bar{a})$ and $X^k(\bar{a})$ are compact, there exists $(x',y',z') \in \beta^k(\bar{a},p)$ such that

$$p \cdot x' < p \cdot y' . \tag{8}$$

But $\lim_{m\to\infty} p_m = p$ implies there exists a positive integer m_1 such that $m > m_1$

$$\Rightarrow \quad p_m \cdot x' < p_m \cdot y'$$

$$\Rightarrow \quad (x',y',z') \in \beta^k(\bar{a},p_m)$$

$$\Rightarrow \quad \omega(\bar{a}) + x'-y'-z' \underset{\sim}{\propto}_{\bar{a}} \omega(\bar{a}) + x_m-y_m-z_m \ .$$

By continuity of $\underset{\sim}{\propto}_a$ we get after taking limits that

$$\omega(\bar{a}) + x'-y'-z' \underset{\sim}{\propto}_{\bar{a}} \omega(\bar{a}) + x-y-z \ . \tag{9}$$

If we have $(\bar{x},\bar{y},\bar{z}) \in \beta^k(\bar{a},p)$ with

$$p \cdot \bar{x} = p \cdot \bar{y}$$

then $(\bar{x},\bar{y},\bar{z})$ is the limit of points in $\beta^k(\bar{a},p)$ with properties (8) and (9). After taking limits we get

$$\omega(\bar{a}) + \bar{x}-\bar{y}-\bar{z} \underset{\sim}{\propto}_{\bar{a}} \omega(\bar{a}) - x-y-z. \tag{10}$$

But since (7) holds here as well, we have $(x,y,z) \in \psi(\bar{a},p)$. Combining the two cases we get $(x,y,z) \in \delta^k(\bar{a},p)$ and therefore δ^k is upper semi-continuous in p.

Then $\lim_{n\to\infty} p_n = p$, $(x_n(\bar{a}), y_n(\bar{a}), z_n(\bar{a}) \in \delta^k(\bar{a},p_n)$ and $(x(\bar{a}), y(\bar{a}), z(\bar{a})) \in$ closure $\{(x_n(\bar{a}), y_n(\bar{a}), z_n(\bar{a}))\}$ implies by the upper semi-continuity of δ^k in p that $(x(\bar{a}), y(\bar{a}), z(\bar{a})) \in \delta^k(\bar{a},p)$. Therefore,

$$s = \int_A \left[x(\cdot) - y(\cdot)\right] d\nu \in \psi^k(p)$$

and consequently $(p,x) \in G_{\psi^k}$, i.e. the graph of ψ^k is closed.

To show that $\psi^k(p) \neq 0$ for every $p \in \Delta$, it is sufficient to show that $\delta^k(a,p) \neq 0$ for almost every $a \in A$ and that the correspondence $\delta^k(\cdot,p) : A \to R_+^{3\ell}$ is ν-measurable. Since the budget set $\beta^k(a,p)$ is compact, $\omega(a) \in X^k(a)$ for almost every $a \in A$, and $\precsim_a$ is continuous for all $a \in A$, $\delta^k(a,p) \neq 0$ for $p \in \Delta$.

The budget set correspondence can be written as

$$\beta(a,p) = S(a) \cap \rho(a) \cap \mu(a)$$

where

$$\rho(a) = \left\{(x,y,z) \in R_+^{3\ell} \mid \omega(a) + x-y-z \in X^k(a)\right\}, \text{ and}$$

$$\mu(a) = \left\{(x,y,z) \in R^{3\ell} \mid p \cdot x \leqq p \cdot y\right\}.$$

We know that S is a ν-measurable correspondence. The measurability of ρ follows from the measurability of ω and X^k while

$\mu(a)$ is trivially ν-measurable. By Lemma 5.3 of Artstein [3, p. 109], $\beta(\cdot,p)$ is also ν-measurable and thus clearly so is $\beta^k(\cdot,p)$.

By Proposition 4.5 of Debreu [17, p. 360], for fixed $p \varepsilon \Delta$ the set

$$M = \left\{a \varepsilon A \mid \inf_{(x,y,z)\varepsilon S^k(a)\cap\rho(a)} \left\{p \cdot (x-y)\right\} < 0\right\}$$

belongs to $\Omega\mathfrak{M}$ since $S^k \cap \rho$ is a ν-measurable correspondence and since the function $p \cdot (x-y)$ is both continuous on $S^k(a) \cap \rho(a)$ and is also trivially ν-measurable. From Theorem B [33, p. 621], there exists a sequence $\{(x_n, y_n, z_n)\}$ of ν-measurable functions of A into R^ℓ such that $\{x_n(a), y_n(a), z_n(a)\}$ is dense in $\beta^k(a,p)$ for every $a \varepsilon A$.
Define

$$\Theta_n(a) = \left\{(x,y,z) \varepsilon \beta^k(a,p) \mid \omega(a) + x_n(a) - y_n(a) - z_n(a) \underset{\sim}{\propto}_a \omega(a) + x-y-z\right\} \text{ for } a \varepsilon M.$$

Clearly, $\delta^k(a,p) \subset \Theta_n(a)$ for $a \varepsilon M$. Suppose we have $(x,y,z) \varepsilon \bigcap_{n=1}^{\infty} \Theta_n(a)$, but $(x,y,z) \notin \delta^k(a,p)$. That is, there exists $(x',y',z') \varepsilon \beta^k(a,p)$ such that $\omega(a) + x-y-z \propto_a \omega(a) + x'-y'-z'$. Since $\propto_a$ is continuous and $\{(x_n(a), y_n(a), z_n(a))\}$ is dense in $\beta^k(a,p)$, there is an integer $\hat{n}$ such that

$$\omega(a) + x-y-z \propto_a \omega(a) + x_{\hat{n}}(a) - y_{\hat{n}}(a) - z_{\hat{n}}(a)$$

Contradicting the fact that $(x,y,z) \varepsilon \Theta_{\hat{n}}(a)$.
Thus

$$\delta^k(a,p) = \bigcap_{n=1}^{\infty} \Theta_n(a) \text{ for every } a \varepsilon M.$$

But since the correspondence $\beta^k(\cdot,p)$, the function x_n, y_n, z_n and the set M are ν-measurable, Θ_n is ν-measurable for all positive integers n. By Lemma 5.3 of Artstein [3] we again have that $\delta^k(\cdot,p)$ is a ν-measurable correspondence. Using Theorems 1 and 2 of Aumann [5, p. 2], we get that $\delta^k(p)$ is non-empty and convex for every $p \varepsilon \Delta$. Clearly, this implies that $\psi^k(p)$ is non-empty for every $p \varepsilon \Delta$, and it is easy to show that $\psi^k(p)$ is also convex for every $p \varepsilon \Delta$.

Finally, property iv) holds since $s \varepsilon \psi^k(p)$ implies there exist ν-integrable functions $x,y,z : A \to R^{\ell}_{+}$ such that

$$s = \int_A \left(x(\cdot) - y(\cdot)\right) d\nu, \text{ and}$$

$$\left(x(a),\ y(a),\ z(a)\right) \varepsilon \delta^k(a,p) \text{ for a.e. } a \varepsilon A.$$

But since $\delta^k(a,p) \subset \beta^k(a,p)$ we also have

$$p \cdot x(a) \leqq p \cdot y(a).$$

Integrating the last inequality we get

$$\int_A p \cdot x(\cdot)d\nu \leqq \int_A p \cdot y(\cdot)d\nu, \text{ or}$$

$$p \cdot s = \int_A \left[x(\cdot) - y(\cdot)\right]d\nu \leqq 0.$$

Part C

We now can apply Debreu's lemma [13, p. 82], which is based on Kakutani's fixed point theorem, to the correspondence ψ^k. It states that there is a $p \in \Delta$ such that

$$\psi^k(p) \cap R^\ell_- \neq 0.$$

That is, there exist ν-integrable functions $x,y,z : A \to R^\ell_+$ such that

$$\left[x(a),\ y(a),\ z(a)\right] \in \delta^k(a,p), \text{ and} \tag{11}$$

$$\int_A \left[x(\cdot) - y(\cdot)\right]d\nu \leqq 0. \tag{12}$$

But condition (12) is equivalent to

$$\int_A x(\cdot)d\nu \leqq \int_A y(\cdot)d\nu, \tag{13}$$

and condition (11) implies that

$$\left\{x(a),\ y(a),\ z(a)\right\} \ \varepsilon\ S^k(a) \subset S(a).$$

Therefore, if we let $f = \omega + x-y-z$ then the pair (p,f) form a quasi-competitive equilibrium for the "k-th truncated economy."

I have shown that for every positive integer k there is a price vector $p^k\ \varepsilon\ \Delta$ and ν-integrable functions $x^k,\ y^k,\ z^k : A \to R_+^\ell$ such that the pair (p^k, f^k) form a quasi-equilibrium for the "k-th truncated economy" where $f^k = \omega + x^k - y^k - z^k$. Since Δ is compact we can assume without loss of generality that the sequence $\{p^k\}$ converges to the price vector $p^*\ \varepsilon\ \Delta$.

From the material balance requirement we have for each k that

$$0 \leqq \int_A f^k(\cdot)d\nu \leqq \int_A \omega(\cdot)d\nu - \int_A z^k(\cdot)d\nu.$$

Thus we get immediately that

$$0 \leqq \int_A f^k(\cdot)d\nu \leqq \int_A \omega(\cdot)d\nu,$$

and

$$0 \leqq \int_A z^k(\cdot)d\nu \leqq \int_A \omega(\cdot)d\nu.$$

Since an agent cannot sell more than his initial endowment, we must have,

$$0 \leqq y^k(a) \leqq \omega(a), \text{ for every } a \in A$$

and therefore we must have using (13),

$$0 \leqq \int_A x^k(\cdot)d\nu \leqq \int_A y^k(\cdot)d\nu \leqq \int_A \omega(\cdot)d\nu .$$

Hence, $\left\{\int_A x^k(\cdot)d\nu\right\}$, $\left\{\int_A y^k(\cdot)d\nu\right\}$, $\left\{\int_A z^k(\cdot)d\nu\right\}$ and $\left\{\int_A f^k(\cdot)d\nu\right\}$ are all bounded sequences in R^ℓ_+ and by the Bolzano-Weierstrass Theorem each has a convergent subsequence. Without loss of generality, there exist $\hat{x}$, $\hat{y}$, $\hat{z}$, $\hat{f} \in R^\ell_+$ such that

$$\lim_{k\to\infty} \int_A x^k(\cdot)d\nu = \hat{x} \quad , \quad \lim_{k\to\infty} \int_A y^k(\cdot)d\nu = \hat{y} \ ,$$

$$\lim_{k\to\infty} \int_A z^k(\cdot)d\nu = \hat{z} \quad , \quad \lim_{k\to\infty} \int_A f^k(\cdot)d\nu = \hat{f} \ .$$

By Schmeidler's [56] version of Fatou's lemma, there exist ν-integrable functions x^*, y^*, z^*, $f^* : A \to R^\ell_+$ such that

$$\left(x^*(a),\ y^*(a),\ z^*(a),\ f^*(a)\right)$$

is a cluster point of the sequence $\left\{\left(x^k(a),\ y^k(a),\ z^k(a),\ f^k(a)\right)\right\}$ for a.e. $a \in A$, and

$$\int_A x^*(\cdot)d\nu \leqq \hat{x} \quad , \quad \int_A y^*(\cdot)d\nu \leqq \hat{y} \ ,$$

$$\int_A z^*(\cdot)d\nu \leqq \hat{z} \quad , \quad \int_A f^*(\cdot)d\nu \leqq \hat{f} \ . \tag{15}$$

Since inequalities are preserved under limits, the material balance equation must also hold,

i.e. $$\hat{f} \leqq \int_A \omega(\cdot)d\nu - \hat{z} \ .$$

From (15) we get that

$$\int_A f^*(\cdot)d\nu \leqq \int_A \omega(\cdot)d\nu - \int_A z^*(\cdot)d\nu. \tag{16}$$

However, there is a subsequence $\{k'\}$ of the positive integers such that

$$\lim_{k'\to\infty}\left\{x^{k'}(a),y^{k'}(a),z^{k'}(a),f^{k'}(a)\right\} = \left\{x^*(a),y^*(a),z^*(a),f^*(a)\right\}.$$

But for $\hat{k} \varepsilon \{k'\}$ we have

$$f^{\hat{k}}(a) = \omega(a) + x^{\hat{k}}(a) - y^{\hat{k}}(a) - z^{\hat{k}}(a) \text{ a.e. } a\varepsilon A,$$

and taking limits we get

$$f^*(a) = \omega(a) + x^*(a) - y^*(a) - z^*(a) \text{ a.e. } a\varepsilon A. \tag{17}$$

Furthermore,

$$\left\{x^{k'}(a),\ y^{k'}(a),\ z^{k'}(a)\right\} \varepsilon\ \delta^{k'}(a,p^{k'}) \subset \beta^{k'}(a,p^{k'}),$$

S(a) and X(a) closed and the preservation of inequalities under limits implies

$$\left\{x^*(a),\ y^*(a),\ z^*(a)\right\} \varepsilon\ \beta(a,p^*).$$

Thus f* is an attainable allocation.

Suppose for fixed a ε A we have

$$\inf_{\substack{(\hat{x},\hat{y},\hat{z})\varepsilon S(a) \\ \omega(a)+\hat{x}-\hat{y}-\hat{z}\varepsilon X(a)}} \left\{\bar{p}^* \cdot (\hat{x}-\hat{y})\right\} < 0.$$

Then, there exists $(x',y',z')\ \varepsilon\ \beta(a,p^*)$ such that

$$p^* \cdot x' < p^* \cdot y'.$$

But since $\lim_{k'\to\infty} p^{k'} = p^*$, for k ε {k'} large enough, we have

$$p^k \cdot x' < p^k \cdot y' \ ,$$

$$(x',y',z')\ \varepsilon\ S^k(a), \text{ and}$$

$$\omega(a) + x'-y'-z'\ \varepsilon\ X^k(a)\ .$$

But $(x^k(a), y^k(a), z^k(a)) \in \delta^k(a,p^k)$ implies that

$$\omega(a) + x'-y'-z' \underset{\sim}{\propto}_a \omega(a) + x^k(a) - y^k(a) - z^k(a).$$

Taking limits, by the continuity of $\underset{\sim}{\propto}_a$ we get

$$\omega(a) + x'-y'-z' \underset{\sim}{\propto}_a \omega(a) + x^*(a)-y^*(a)-z^*(a) = f^*(a).$$

In the case $(\bar{x},\bar{y},\bar{z}) \in \beta(a,p^*)$ with

$$p^* \cdot \bar{x} = p^* \cdot \bar{y} ,$$

$(\bar{x},\bar{y},\bar{z})$ is the limit of vectors $(x_n,y_n,z_n) \in \beta(a,p^*)$ with

$$p^* \cdot x_n < p^* \cdot y_n ,$$

and

$$\omega(a) + x_n - y_n - z_n \underset{\sim}{\propto}_a \omega(a) + x^*(a) - y^*(a) - z^*(a) .$$

Taking limits again we get

$$\omega(a) - \bar{x}-\bar{y}-\bar{z} \underset{\sim}{\propto}_a \omega(a) + x^*(a) - y^*(a) - z^*(a) .$$

Thus,

$$\left(x^*(a), y^*(a), z^*(a)\right) \in \delta(a,p^*) \tag{18}$$

Equations (16), (17) and (18) imply that (p^*, f^*) is a quasi-competitive price equilibrium for the "barter economy Ξ^B.

Q.E.D.

Corollary. If the measure space of agents of the "barter" economy Ξ^B is non-atomic, or if for every $a \in A$ with $\nu(\{a\}) > 0$, both the preference-or-indifference relation $\succsim_a$ and the transaction set $S(a)$ are convex, then there exists a quasi-competitive price equilibrium.

Proof. The property that ν is non-atomic was required in the theorem to show $\sigma^k(p)$, the integral of the correspondence δ^k with respect to ν, is convex. Since ν is a finite measure, the measure space (A, Ω, ν) has at most a countable number of atoms [33, p. 615]. The set A, therefore, can be decomposed into two subsets $A = A_1 \cup A_2$ where ν is non-atomic on A_1 and A_2 is countable, $a \in A_2 \Rightarrow \nu(a) > 0$.

But $a \in A_2$ implies $\beta(a,p)$ is convex since $S(a)$ and $X(a)$ are convex. Consequently, $\succsim_a$ convex as well implies $\phi(a,p^*)$ is convex or empty and thus $\delta(a,p)$ is convex. Clearly $\sigma^k(p)$ is also convex since

$$\sigma^k(p) = \int_{A_1} \delta^k(\cdot,p)\,d\nu + \sum_{a \in A_2} \delta(a,p).$$

The rest of the theorem goes through as before.

Q.E.D.

D. Consider the monetary economy

$$\Xi^M = [(A, \Omega, \nu), X, \precsim, T, \nu]$$

as described in Chapter 3. Let

$$\Delta = \left\{p = (p_b, p_s) \in R_+^{2\ell} \mid \sum_{i=1}^{\ell} (p_b^i + p_s^i) = 1\right\}$$

be the unit price simplex in $R_+^{2\ell}$. Define the budget correspondence $\beta : A \times \Delta \to R^{2\ell}$ by

$$\beta(a,p) = \Big\{([s - \omega(a)]^+, - [s - \omega(a)]^-) \mid s \in X(a)$$

and

$$p_b \cdot [s - \omega(a)]^+ \leqq p_s \cdot [s - \omega(a)]^- + \pi(p,a)\Big\} ,$$

and the demand correspondence $\phi : A \times \Delta \to R^{2\ell}$ by

$$\phi(a,p) = \Big\{(x,-y) \in \beta(a,p) \mid \text{for every } (x',-y') \in \beta(a,p),$$

$$\omega(a) + x'-y' \precsim_a \omega(a) + x-y\Big\} .$$

Finally, define the quasi-demand correspondence $\delta : A \times \Delta \to R^{2\ell}$ by

$$\delta(a,p) = \begin{cases} \phi(a,p) & \text{if } \inf_{r \varepsilon X(a)} \left\{ p_b \cdot [r-\omega(a)]^+ - p_s \cdot [r-\omega(a)]^- \right\} < \pi(p,a) \\ \beta(a,p) & \text{otherwise} \end{cases}$$

Definition 3. Quasi-competitive price equilibrium

Let p be a price vector in the unit simplex $\Delta \subset R_+^\ell \times R_+^\ell$ and f be a ν-integrable function from A to R_+^ℓ. The pair (p,f) is called a quasi-competitive price equilibrium for the "monetary" economy Ξ^M if

a) $\left([f(a) - \omega(a)]^+, - [f(a) - \omega(a)]^-\right) \varepsilon\ \delta(a,p)$
for almost every $a \varepsilon A$,

b) $\left(\int_A [f(a) - \omega(a)]^+ d\nu\ ,\ - \int_A [f(\cdot) - \omega(\cdot)] d\nu\right) \varepsilon\ T(A)$,
and

c) $$\bar{p}_b \cdot \int_A [f(\cdot) - \omega(\cdot)]^+ d\nu - \bar{p}_s \cdot \int_A [f(\cdot) - \omega(\cdot)]^- d\nu$$

$$= \max_{(x,-y) \varepsilon T(A)} \left\{ p_b \cdot x - p_s \cdot y \right\}.$$

Condition a) implies that f(a) ε X(a) and that f(a) must be maximal with respect to $\succsim_a$ in agent a's budget set whenever the minimum wealth situation does not occur. Condition

b) ensures that f is an attainable allocation, while condition c) is the profit maximizing relation for the coalition traders.

Definition 4. Competitive price equilibrium.

The pair (p,f) is called a competitive price equilibrium for the "monetary" economy Ξ^M if it forms a quasi-competitive price equilibrium and if the set of agents for whom the minimum wealth situation occurs has measure zero.

i.e. if

$$E^* = \left\{a \in A \mid \inf_{r \in X(a)} \left\{p_b \cdot [r-\omega(a)]^+ - p_s \cdot [\hat{r}-\omega(a)]^-\right\} \geqq \pi(p,a)\right\}$$

then $$\nu(E^*) = 0.$$

E. The strategy used in the proof of the next theorem follows closely that of Theorem 1. In part a) the properties of the total quasi-demand correspondence are investigated for a "truncated economy"; in part b) the properties of the supply correspondence for the coalition traders is investigated; in part c) it is shown that the excess quasi-demand correspondence has the properties required by Debreu's lemma. Finally, the existence of a sequence of quasi-equilibria for the sequence of truncated economies is shown in part d) to imply the exitence of a quasi-equilibrium for the original economy.

Theorem 2: If the measure space of agents for the "monetary" economy Ξ^M is non-atomic and if the economy's aggregate transaction set T(A) is compact, then there exists a quasi-competitive price equilibrium.

Proof

Part a

For every positive integer k we again define the truncated consumption set by

$$X^k(a) = \left\{ s \in X(a) \mid s \leqq k(\omega(a) + 1) \right\}$$

Define the k-th truncated budget correspondence, β^k, demand correspondence, ψ^k, and quasi-demand correspondence, δ^k, by replacing X(a) with $X^k(a)$ in the definitions of section D above.

Finally, define the total quasi-demand correspondence $\psi^k : A \to R^{2\ell}$ by

$$\psi^k(p) = \int_A \delta^k(\cdot,p)d\nu$$

$$= \left\{ (\hat{x},-\hat{y}) \mid \hat{x},\hat{y} \in R^\ell_+ \right.$$

and there exist ν-integrable functions

$$x,y : A \to R^\ell_+$$

such that

$$\left[x(a), -y(a)\right] \in \delta^k(a,p), \text{ a.e. } a \in A,$$

and

$$\hat{x} = \int_A x(\cdot)d\nu, \ \hat{y} = \int_A y(\cdot)d\nu\Big\}.$$

I claim that the correspondence ψ^k has the following properties:

i) there is a compact set $N \subset R^{2\ell}$ such that $\psi^k(p) \subset N$ for every $p \in \Delta$,

ii) the graph of ψ^k is closed, and

iii) for every $p \in \Delta$, $\psi^k(p)$ is nonempty and convex.

The proof that ψ^k has these properties follows closely the proof in Theorem 1 above. Therefore we will only sketch parts of it.

To prove property i), let

$$N = \left\{s = (s_1,s_2) \in R^\ell + R^\ell \;\middle|\; |s_i| \leqq k[\int_A(\omega(\cdot) + 1)d\nu]\right\}.$$

If $(\hat{x},-\hat{y}) \in \psi^k(p)$ for $p \in \Delta$ then by construction there exist ν-integrable functions $x,y : A \to R^\ell_+$ such that $(x(a), -y(a)) \in \delta^k(a,p)$ and $\hat{x} = \int_A x(\cdot)d\nu$, $\hat{y} = \int_A y(\cdot)d\nu$. But $(x(a),-y(a)) \in \delta^k(a,p)$ implies

a,p) implies

$$\omega(a) + x(a) - y(a) \in X^k(a)$$

$$\Rightarrow \omega(a) + x(a) - y(a) \leqq k\left[\omega(a) + 1\right]$$

$$\Rightarrow x(a) \leqq (k-1)\omega(a) + k < k\left[\omega(a) + 1\right]$$

Furthermore, since $x^i(a) > 0 \Rightarrow y^i(a) = 0$ and $X^k(a) \subset R_+^\ell$ we have

$$y(a) \leqq \omega(a) < k\left[\omega(a) + 1\right]$$

Therefore

$$\hat{x} = \int_A x(\cdot)d\nu < k \int_A \left[\omega(a) + 1\right]d\nu,$$

$$\hat{y} = \int_A y(\cdot)d\nu < k \int_A \left[\omega(a) + 1\right]d\nu.$$

Hence $\psi^k(p) \subset N$ for every $p \in \Delta$.

To prove property ii), let $(p_n, (\hat{x}_n, -\hat{y}_n))$ be a sequence in G_{ψ^k} with $\lim_{n\to\infty}(p_n, (\hat{x}_n, -\hat{y}_n)) = (p, (\hat{x}, -\hat{y}))$. Then there exist ν-integrable functions $x_n, y_n : A \to R_+^\ell$ such that

$$\hat{x}_n = \int_A x_n(\cdot)d\nu,\ \hat{y}_n = \int_A y_n(\cdot)d\nu, \text{ and}$$

$$\left[x_n(a), -y_n(a)\right] \in \delta^k(a, p_n) \text{ for every } a \in A.$$

However the sequence of ν-integrable functions $\{(x_n, -y_n)\}$ is bounded pointwise in absolute value by the ν-integrable function $k(\omega+1)$. Applying Theorem E [33, p. 622] there exist ν-integrable functions $x,y : A \to R_+^\ell$ such that

$$\int_A \left[x(\cdot), - y(\cdot)\right] d\nu = (\hat{x}, -\hat{y}) , \tag{1}$$

and

$$\left[x(a), - y(a)\right] \in \text{closure} \left\{\left[x_n(a), - y_n(a)\right]\right\}$$

$$\text{for almost every } a \in A. \tag{2}$$

It is possible to show, as before, for fixed $\bar{a} \in A$, that the correspondence $\delta^k(\bar{a},\cdot)$ is upper semi-continuous. Then $\lim_{n\to\infty} p_n = p$, $(x_n(\bar{a}), -y_n(\bar{a})) \in \delta^k(\bar{a},p_n)$ and $(x(\bar{a}), -y(\bar{a}))$ $\in$ closure $\{(x_n(\bar{a}), -y_n(\bar{a})\}$ implies by the upper semi-continiuty of δ^k in p that $(x(\bar{a}), -y(\bar{a})) \in \delta^k(a,p)$. Therefore, $(\hat{x},-\hat{y}) \in G_{\psi^k}$, i.e. the graph of ψ^k is closed.

It is again possible to show that $\psi^k(a,p) \neq 0$ for almost every $a \in A$ and that the correspondence $\psi^k(\cdot,p)$ is ν-measurable. Then since the integral of a correspondence with respect to an atomless measure is convex, $\psi^k(p)$ is convex for every $p \in \Delta$.

Part b.

The supply correspondence for the coalition of traders of the entire economy is defined for $p \in \Delta$ by

$$\eta(p) = \left\{(\hat{x},-\hat{y}) \in T(A) \mid p_b \cdot \hat{x} - p_s \cdot \hat{y} = \Pi(p,A)\right\}.$$

It is easy to see that η has the following properties:

i) for every $p \in \Delta$, $\eta(p)$ is closed and since it is contained in the compact set $T(A)$, it is also compact,

ii) for every $p \in \Delta$, $\eta(p)$ is nonempty and convex, and

iii) the graph of the correspondence η is closed.

By definition,

$$\Pi(p,A) = \max_{(x,-y)\in T(A)} \left[p_b \cdot x - p_s \cdot y\right],$$

The first two properties follow from the continuity and linearity of the function $p_b \cdot x - p_s \cdot y$ in $(x,-y)$ and from the compactness of $T(A)$. Property iii) holds since the function $p_b \cdot x - p_s \cdot y$ is also continuous in (p_b,p_s).

Part c.

Now define the excess quasi-demand correspondence for the k-th truncated economy by

$$\xi^k(p) = \psi^k(p) - \eta(p).$$

The correspondence $\xi^k : \Delta \to R^\ell \times R^\ell$ has the following properties.

i) there is a compact set $N \subset R^{2\ell}$ such that $\xi^k(p) \subset N$ for every $p \in \Delta$,

ii) the graph of ξ^k is closed,

iii) for every $p \in \Delta$, $\xi^k(p)$ is nonempty and convex, and

iv) for every $p \in \Delta$, $p \cdot \xi^k(p) \leqq 0$.

Properties i)-iii) are immediate consequences of the properties of the total quasi-demand and supply correspondences. To establish iv) let $z \in \xi(p) \Rightarrow z = (x'-\hat{x}, -(y'-\hat{y}))$ such that

$$(x',-y') \in \psi^k(p), \; (\hat{x},-\hat{y}) \in \eta(p).$$

But $(x',-y') \in \psi^k(p)$ implies that there exist ν-integrable functions $x,y : A \to R_+^\ell$, such that $(x(a), -y(a)) \varepsilon\ \delta^k(a,p)$ and

$$x' = \int_A x(\cdot)d\nu,\ y' = \int_A y(\cdot)d\nu.$$

But $(x(a), -y(a)) \varepsilon\ \delta^k(a,p) \Rightarrow (x(a), -y(a)) \varepsilon\ \beta^k(a,p)$ and therefore

$$p_b \cdot x(a) - p_s \cdot y(a) \leqq \pi(p,a),\ \text{a.e. in } A.$$

If we integrate the last inequality we get

$$p_b \cdot \int_A x(\cdot)d\nu - p_s \cdot \int_A y(\cdot)d\nu \leqq \int_A \pi(p,a)$$

or

$$p_b \cdot x' - p_s \cdot y' \leqq \Pi(p,A).$$

However, $(\hat{x},-\hat{y})\ \varepsilon\ \eta(p) \Rightarrow p_b \cdot \hat{x} - p_s \cdot \hat{y} = \Pi(p,A)$.

Thus $\qquad p_b \cdot x' - p_s \cdot y' \leqq p_b \cdot \hat{x} - p_s \cdot \hat{y}$,

or

$$p_b \cdot [x'-\hat{x}] - p_s \cdot [y'-\hat{y}] \leqq 0$$

That is, $p \cdot z = (p_b,p_s) \cdot (x'-\hat{x},-(y'-\hat{y})) \leqq 0$.

<u>Part d</u>.

By Debreu's lemma [13, p 82], there is a $p \in \Delta$ such that $\xi^k(p) \cap R_-^{2\ell} \neq \phi$. That is, there exist ν-integrable functions $x,y : A \to R_+^\ell$ such that

$$\left(x(a),\ -y(a)\right) \in \delta^k(a,p),\ \text{a.e. } a \in A, \text{ and} \tag{3}$$

if $$x = \int_A x(\cdot)d\nu \text{ and } y = \int_A y(\cdot)d\nu,$$

then $$(x,-y) \in T(A), \tag{4}$$

and $$p_b \cdot x - p_s \cdot y = \Pi(p,A). \tag{5}$$

Equation (4) follows from property ii) of the transactions correspondence T.

If we define the ν-integrable function $f : A \to R_+^\ell$ by

$$f = \omega + x-y,$$

then f is a state for the economy Ξ^M and the pair (p,f) is a quasi-competitive price equilibrium for the "k-th truncated economy."

Thus for every positive integer k, there is a price vector $p^k \in \Delta$ and ν-integrable functions $x^k,y^k : A \to R_+^\ell$ such that if we let

$$f^k = \omega + x^k - y^k ,$$

then the pair (p^k, f^k) form a quasi-competitive price equilibrium for the "k-th truncated economy."

Since Δ is compact, w.l.o.g. $\lim p^k = p^* = (p_b^*, p_s^*)$. Furthermore, since for every positive integer k,

$$\left(\int_A x^k(\cdot)d\nu, - \int_A y^k(\cdot)d\nu\right) \varepsilon T(A)$$

and since T(A) is compact, each of the sequences $\left\{\int_A x^k(\cdot)d\nu\right\}$ and $\left\{\int_A y^k(\cdot)d\nu\right\}$ has a convergent subsequence. Without loss of generality, there exist $\hat{x}, \hat{y}, \varepsilon R_+^\ell$ such that

$$\lim_{k\to\infty} \int_A x^k(\cdot)d\nu = \hat{x}, \lim_{k\to\infty} \int_A y^k(\cdot)d\nu = \hat{y} .$$

Since no agent can sell more than his initial endowment we also have, for all k, that

$$0 \leqq y^k(a) \leqq \omega(a), \text{ for every } a \ \varepsilon \ A.$$

By applying Schmeidler's [56] version of Fatou's lemma to the first sequence and Schmeidler's corollary to the second sequence, we get that there exist ν-integrable function $x^*, y^* : A \to R_+^\ell$ such that

$$\left(x^*(a), -y^*(a)\right) \text{ is a cluster point of the} \tag{6}$$

$$\text{sequence } \left\{\left(x^k(a),\ -y^k(a)\right)\right\} \text{ for a.e. } a \in A, \text{ and}$$

$$\int_A x^*(\cdot)d\nu \leqq \hat{x},\ \int_A y^*(\cdot)d\nu = \hat{\hat{y}}. \tag{7}$$

But T(A) compact implies that $(\hat{x},-\hat{y}) \in T(A)$. From property ii) on the transactions correspondence T and (7) above we also get

$$\left(\int_A x^*(\cdot)d\nu,\ -\int_A y^*(\cdot)d\nu\right) \varepsilon\varepsilon\ T(A). \tag{8}$$

Since $(x^*(a),-y^*(a))$ is a cluster point of the sequence $\{(x^k(a),\ -y^k(a))\}$, there is a subsequence $\{k'\}$ of the positive integers such that

$$x^*(a) = \lim_{k'\to\infty} x^{k'}(a) \quad , \quad y^*(a) = \lim_{k'\to\infty} y^{k'}(a).$$

Furthermore,

$$\left(x^{k'}(a),\ -y^{k'}(a)\right) \in \delta^{k'}(a,p^{k'}) \subset \beta^{k'}(a,p^{k'})$$

implies that

$$p_b^{k'} \cdot x^{k'}(a) - p_s^{k'} \cdot y^{k'}(a) \leqq \pi(p^{k'},a).$$

Since T(A) is compact, the function $\pi(\cdot,a)$ is continuous on Δ. Taking limits, we get that

$$p_b^* \cdot x^*(a) - p_s^* \cdot y^*(a) \leqq \pi(p^*,a) \tag{9}$$

Since X(a) is closed,

$$\lim_{k'\to\infty} \left\{\omega(a) + x^{k'}(a) - y^{k'}(a)\right\} \varepsilon X(a).$$

Thus, $$\left\{x^*(a), -y^*(a)\right\} \varepsilon \beta(a,p^*).$$

Suppose for a ε A, we have

$$\inf_{r\varepsilon X(a)} \left\{p_b^* \cdot [r-\omega(a)]^+ - p_s^* \cdot [r-\omega(a)]^-\right\} < \pi(p^*,a) \tag{10}$$

and there exists $(\bar{x},\bar{-y}) \varepsilon \beta(a,p^*)$ such that

$$p_b^* \cdot \bar{x} - p_s^* \cdot \bar{y} < \pi(p^*,a).$$

But since $\lim_{k\to\infty} p^k = p^*$, for k ε {k'} large enough

$$p_b^k \cdot \bar{x} - p_s^k \cdot \bar{y} < \pi(p^k, a), \text{ and}$$

$$\omega(a) + \bar{x} - \bar{y} \in X^k(a).$$

But $(x^k(a), -y^k(a)) \in \delta^k(a, p^k)$ implies that

$$\omega(a) + \bar{x} - \bar{y} \succsim_a \omega(a) + x^k(a) - y^k(a).$$

Since $\succsim_a$ is continuous, after taking limits we get

$$\omega(a) + \hat{x} - \hat{y} \succsim_a \omega(a) + x^*(a) - y^*(a).$$

Thus for $r \in X(a)$ satisfying

$$p_b^* \cdot [r - \omega(a)]^+ - p_s^* \cdot [r - \omega(a)]^- < \pi(p^*, a) \tag{11}$$

we have $r \succsim_a f^*(a)$.

Following Hildenbrand [33,p. 620] when (10) holds, for every $s \in X(a)$ with

$$p_b^* \cdot [s - \omega(a)]^+ - p_s^* \cdot [s - \omega(a)]^- = \pi(p^*, a)$$

is the limit of vectors $\hat{r}_n \in X(a)$ with

$$p_b^* \cdot [\hat{}_n - \omega(a)]^+ - p_s^* \cdot [\hat{}_n - \omega(a)]^- < \pi(p^*, a).$$

Thus

$$\left(x^*(a),\ -y^*(a)\right) \in \delta(a, p^*). \tag{12}$$

Finally, I claim that

$$p_b^* \cdot \int_A x^*(\cdot)d\nu - p_s^* \cdot \int_A y^*(\cdot)d\nu = \Pi(p^*, A). \tag{13}$$

We know for $k \in \{k'\}$ that

$$p_b^k \cdot \int_A x^k(\cdot)d\nu - p_s^k \cdot \int_A y^k(\cdot)d\nu = \Pi(p^k, A),$$

and that

$$p_b^k \cdot x^k(a) - p_s^k \cdot y^k(a) \leqq \pi(p^k, a), \text{ for all } a \in A.$$

The last two equations imply there exists $A_1 \in \Omega$ such that $\nu(A_1) = \nu(A)$ and

$$p_b^k \cdot x^k(a) - p_s^k \cdot y^k(a) = \pi(p^k, a), \text{ for all } a \in A_1.$$

Taking limits on the last equation we get

$$p_b^* \cdot x^*(a) - p_s^* \cdot y^*(a) = \pi(p^*, a), \text{ for all } a \in A_1.$$

Integrating the last equation gives (13).

Equations (13), (12) and (8) imply (p^*, f^*), where $f^* = \omega + x^* - y^*$, is a quasi-equilibrium of the "monetary" economy Ξ^M.

Q.E.D.

Corollary: If the measure space of agents for the "monetary" economy Ξ^M is non-atomic, or if for every $a \in A$ with $\nu(\{a\}) > 0$, the preference-or-indifference relation $\succsim_a$ is convex, then there exists a quasi-competitive price equilibrium.

Proof: Same as in the corollary to Theorem 1.

F. The assumptions made in Chapters 2 and 3 were sufficient to prove the existence of a quasi-competitive price equilibrium in both the "barter" and "monetary" economies. To ensure that these equilibria are also competitive price equilibria, additional assumptions must be made. Suppose for both the "barter" and "monetary" economies:

1) $X(a) = R_+^\ell$ for all $a \in A$, i.e. each agent's consumption set is the non-negative orthant of the Euclidean space of dimension ℓ,

2) The preference relation $\propto_a$ is monotonic for all $a \in A$, i.e. $x,y \in R_+^\ell$ with $x < y$ implies $x \propto_a y$,

3) $\omega(a) >> 0$ for all $a \in A$, i.e. each agent possesses positive quantities of every good.

I conjecture that these three additional assumptions are sufficient to ensure that the quasi-competitive price equilibrium in the monetary economy is also a competitive price equilibrium (see Aumann [6], Hildenbrand [33], Kurz [41] and Schmeidler [55]).

To ensure that the quasi-competitive price equilibrium of the "barter" economy is also a competitive price equilibrium.. I conjecture that the assumption:

4) $\mathcal{S}(a)$ is convex for all $a \in A$,

plus the first three assumptions are sufficient. Assumption 4) is necessary, toeensure that an agent will be able to buy positive quantities of all commodities. Otherwise, the transaction cost of starting an exchange plus the amount given up in the exchange may exceed an agent's initial endowment (see Kurz [43]).[10]