DRAWING AS PROBLEM-SOLVING: 
YOUNG CHILDREN'S MATHEMATICAL REASONING THROUGH 
THE ACT OF DRAWING 

by 

CAROLE SAUNDRY 

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE 
REQUIREMENTS FOR THE DEGREE OF 

MASTER OF ARTS 

in 

THE FACULTY OF GRADUATE STUDIES 

(Mathematics Education) 

THE UNIVERSITY OF BRITISH COLUMBIA 

August 2006 

© Carole Saundry, 2006
Abstract

This project investigates how young children employ mathematical reasoning through the act of drawing while solving mathematical problems. Included is an examination of how grade 2 children responded when presented with two mathematical problems to solve, the kinds of pictures or representations they drew spontaneously – and the purpose for those drawing – what students were thinking while they drew, and how (or if) the act of drawing or the drawings themselves were helping students to reason through the problems.

Considered together, video footage of students using drawing as a problem-solving strategy, children’s completed representational work and student interview data informed the creation of a framework for examining students' drawing as problem-solving strategies and behaviours. Four categories of student behaviour emerged from this analysis, namely: the use of virtual manipulatives, the application of a system, the use of imagery and an examination of the sophistication of students' representations. These categories form a framework that provides a lens for understanding children's mathematical reasoning as expressed through and derived from the act of drawing.

Questions regarding problem structure and complexity and their impact on student reasoning through pictures are also addressed. Implications of the study are elaborated, and involve a re-conceptualizing of drawing in mathematical problem-solving to include a more performance-based approach, observing thinking in the act of thinking, reasoning while drawing. Recommendations for classroom practice are made regarding how teachers can support children in developing these active and metacognitive skills. Further research directions are suggested in the discussion of the results.
# Table of Contents

Abstract .................................................................................................................. ii  
Table of Contents ................................................................................................. iii  
List of Tables .......................................................................................................... vi  
List of Figures .......................................................................................................... vii  
List of Video Clips .................................................................................................. ix  
Acknowledgements ................................................................................................. x  
Dedication ................................................................................................................ xi  

## CHAPTER 1 – INTRODUCTION ........................................................................... 1

**Drawing As Problem-solving:**  
**Young Children’s Mathematical Reasoning Through the Act of Drawing** .......... 1  
  
- Background and rationale .................................................................................. 1  
- Conceptualizing the study: developing the research question ......................... 7  
- Related studies .................................................................................................. 8  
- Methods ............................................................................................................ 10  
- Significance of the research problem ............................................................... 11  
- Structure of this document .............................................................................. 12  

## CHAPTER 2 – LITERATURE REVIEW ............................................................. 14

**Introduction** .................................................................................................. 14  

**Interpreting the research** ................................................................................ 15  
  
- Imagery and mathematical problem-solving .................................................. 15  
- Dynamic imagery and mathematics achievement:  
  Comparative studies .......................................................................................... 19  
- Visualization and diagrammatic capacity ......................................................... 21  
- Assessing image-making; the complexity of language .................................. 22  
- Spatial-visualizing capacity and number understanding ............................... 23  
- Visualization and symbolism in mathematics ................................................. 25  
- Drawings as dynamic representations ............................................................. 26  
- Drawings as manipulatives .............................................................................. 27  
- Explicit instruction in drawing to problem-solve ........................................ 28  
- Reform and representation:  
  Valuing drawing as mathematical reasoning .................................................. 31  

**Summary** ........................................................................................................ 33  

Contribution to a current body of research ........................................................ 35
What is drawing as problem-solving? ................................................................. 110
   Creating a definition ....................................................................................... 110
   New understandings given video observations
   From "drawing the solution" to "drawing thinking" ....................................... 113

The Drawing as Problem-solving Framework .................................................. 115
   Some questions and reflections ..................................................................... 115
   Virtual manipulatives ..................................................................................... 115
   From physical models to mental images ....................................................... 116
   Sophistication of the drawing ....................................................................... 117

What makes for successful drawing as problem-solving? .............................. 123
   The active use of virtual manipulatives ......................................................... 123
   The flexible use of strategies ....................................................................... 126
   The power of a system .................................................................................. 130
   What about children who don't draw? ......................................................... 135
   What about children for whom drawing is not helpful? .............................. 139

Dispositions and mathematical sense-making
How they influenced drawing as problem-solving behaviours ...................... 140
   Asking permission to draw .......................................................................... 141
   But what's the division answer going to be?
   Classroom culture and student expectations ............................................ 142

Understanding how drawing helped: Insights from the interview results ........ 144
   What the visualizers reported ..................................................................... 147

Implications for practice .................................................................................. 150
   Types of problems ....................................................................................... 150
   Types of learners .......................................................................................... 151
   Addressing and supporting metacognition:
   Questions and prompts for classroom teachers ......................................... 152
   Classroom strategies ..................................................................................... 153

Summary and further research directions ......................................................... 154

References ........................................................................................................ 157

Appendix A: Interview data ............................................................................ 161
Appendix B: Consent forms ............................................................................ 167
Appendix C: Copy of the UBC Research Ethics Board's
   Certificates of Approval ................................................................................ 177
List Of Tables

Table 1  Video Data Recording Form
Excerpt from Group 1A................................................. 46
Table 2  Revised Video Data Recording Form
Excerpt from Group 4A................................................. 49
Table 3  Summary of strategies by group
Cookie Problem.......................................................... 53
Table 4  Summary of strategies by group
Wheels Problem.......................................................... 54
Table 5  Summary of type of response against
drawing or not.............................................................. 59
Table 6  Summary of strategies by group for the
Cookie Problem.......................................................... 72
Table 7  Summary of strategies by group for the
Wheels Problem.......................................................... 73
Table 8  Summary of Virtual Manipulative use for the
Cookie and Wheels Problems (all groups)........................ 75
Table 9  Summary of Systems use for the Cookie and
Wheels Problems (all groups)........................................... 81
Table 10 Summary of Imagery use for the Cookie and
Wheels Problems (all groups).......................................... 87
Table 11 Summary of Sophistication of the Representation
for the Cookie and Wheels problems (all groups)...... 93
Table 12 Frequency of Drawing as Problem-solving
Behaviour and Characteristics by Cluster of Themes. 100
Table 13 Interview data:
Type of support and frequency...................................... 103
# List Of Figures

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Figure 1</td>
<td>Jason's cookies</td>
<td>6</td>
</tr>
<tr>
<td>Figure 2</td>
<td>Overhead and Handheld Camera Views</td>
<td>41</td>
</tr>
<tr>
<td>Figure 3</td>
<td>Anthony's lines</td>
<td>59</td>
</tr>
<tr>
<td>Figure 4</td>
<td>Melissa's sets</td>
<td>59</td>
</tr>
<tr>
<td>Figure 5</td>
<td>Orson works through the Cookie Problem</td>
<td>62-63</td>
</tr>
<tr>
<td>Figure 6</td>
<td>Lisa's line up</td>
<td>66</td>
</tr>
<tr>
<td>Figure 7</td>
<td>Ruby's chart</td>
<td>66</td>
</tr>
<tr>
<td>Figure 8</td>
<td>Orson works through the Wheels Problem</td>
<td>68-69</td>
</tr>
<tr>
<td>Figure 9</td>
<td>Sara's sets</td>
<td>77</td>
</tr>
<tr>
<td>Figure 10</td>
<td>Shauna's sets</td>
<td>77</td>
</tr>
<tr>
<td>Figure 11</td>
<td>David's line</td>
<td>78</td>
</tr>
<tr>
<td>Figure 12</td>
<td>Anthony distributes</td>
<td>79</td>
</tr>
<tr>
<td>Figure 13</td>
<td>Larry distributes</td>
<td>79</td>
</tr>
<tr>
<td>Figure 14</td>
<td>Pricilla's counters</td>
<td>80</td>
</tr>
<tr>
<td>Figure 15</td>
<td>Jason keeps track with tallies</td>
<td>80</td>
</tr>
<tr>
<td>Figure 16</td>
<td>Cathy's numbers</td>
<td>82</td>
</tr>
<tr>
<td>Figure 17</td>
<td>Susanne's box</td>
<td>83</td>
</tr>
<tr>
<td>Figure 18</td>
<td>Susanne's box 2</td>
<td>83</td>
</tr>
<tr>
<td>Figure 19</td>
<td>Pricilla's wheels</td>
<td>84</td>
</tr>
<tr>
<td>Figure 20</td>
<td>Jessie checks and rechecks</td>
<td>84</td>
</tr>
<tr>
<td>Figure 21</td>
<td>Sara eliminates</td>
<td>85</td>
</tr>
<tr>
<td>Figure 22</td>
<td>Norman's work</td>
<td>86</td>
</tr>
<tr>
<td>Figure 23</td>
<td>Eyes up</td>
<td>88</td>
</tr>
<tr>
<td>Figure 24</td>
<td>Iris's drawing</td>
<td>89</td>
</tr>
<tr>
<td>Figure 25</td>
<td>Trent's recording</td>
<td>91</td>
</tr>
<tr>
<td>Figure 26</td>
<td>Lisa's people</td>
<td>94</td>
</tr>
<tr>
<td>Figure 27</td>
<td>Lisa's people 2</td>
<td>94</td>
</tr>
</tbody>
</table>
### List Of Video Clips

<table>
<thead>
<tr>
<th>Video Clip</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Video Clip 1</td>
<td>Orson’s flexible thinking</td>
<td>64</td>
</tr>
<tr>
<td>Video Clip 2</td>
<td>Jason keeps track with tallies</td>
<td>80</td>
</tr>
<tr>
<td>Video Clip 3</td>
<td>Sara eliminates</td>
<td>85</td>
</tr>
<tr>
<td>Video Clip 4</td>
<td>Grant talks tallies</td>
<td>98</td>
</tr>
<tr>
<td>Video Clip 5</td>
<td>Sara’s flexibility</td>
<td>128</td>
</tr>
<tr>
<td>Video Clip 6</td>
<td>Norman’s system</td>
<td>133</td>
</tr>
<tr>
<td>Video Clip 7</td>
<td>Mark’s use of imagery</td>
<td>137</td>
</tr>
</tbody>
</table>
Acknowledgements

Although only one name appears on the front of this document, there were dozens who contributed to its creation. I would like to then acknowledge the support of the people who made this possible...

To Dr. Cynthia Nicol of UBC, whose friendship and guidance made this journey a rewarding and worthwhile learning experience; to the teachers who opened up their classrooms in the name of professional growth and mathematical understanding; and to my family who helped to make space for me to think, ponder and cogitate in while I pursued this investigation.

But mostly, I owe a debt of gratitude to the children involved in this study, who shared exactly who they are, how they think, and what’s important to remember when you do math. They taught me well.
Dedication

For Cameron, whose visual approach to mathematical problem-solving inspires me to learn all I can...
Chapter 1 – Introduction

Drawing As Problem-solving:

Young Children's Mathematical Reasoning Through the Act of Drawing

Young children approach mathematical problem-solving in a range of creative ways, applying their learning strengths while they work through problems. This study examined how a group of grade 2 students approached and solved mathematical tasks using their visual-spatial strengths; specifically, how young children used drawing as problem-solving strategies in the solution of mathematical problems. Working in small groups, students solved 2 problems, and the behaviours they exhibited while reasoning through the problems were videotaped. These reasoning behaviours were then analyzed and collated into a framework describing drawing as problem-solving. From this data and the results of interview questions, three broad areas were considered: representational complexity, the flexible and systematic use of drawing as problem-solving strategies, and the ability to access imagery. This study suggests links between these areas and problem-solving capacity in young children, and proposes instructional strategies for supporting reasoning through the act of drawing.

Background and rationale

Years of my professional career have been spent observing young children. In my role as curriculum coordinator for the Richmond School District, I have been charged with the implementation of a district-wide assessment of kindergarten learners, a play-based performance assessment aimed at identifying learner strengths across multiple intelligences and problem-solving. The assessment, entitled DISCOVER: Learner Strengths is adapted from the work of Dr. June C. Maker of the
University of Arizona. (2001) The results of this assessment on a district-wide level indicate that very young children entering school in Richmond have strengths in 2 areas predominantly – logical-mathematical intelligence and visual-spatial intelligence. From the assessment itself, a judgment of logical-mathematical capacity would be made on the existence of 3 of 4 possible sets of indicators: basic mathematical concepts of more than and less than, ability to sort and classify using objects and language, the systematic approach to tangram puzzle problems and making constructions that included notions of measurement and ordering. Visual-spatial capacity in the assessment was not restricted to a measure of artistic ability in terms of drawing, but extended to a child’s attention to pattern and design on the pieces used, the ability to use translations to advantage in the solution of tangram puzzle problems, and the ability to see abstract objects as something in the environment.

The predominance of these strengths in learners of various ethnic, cultural and linguistic backgrounds led me to wonder if there was a connection between the existence of visual-spatial and logical-mathematical intelligence – if being visually able supported mathematical capacity in some way. Likewise, supporting teachers to use these results to inform their practice meant encouraging them to strengthen these strengths in their young children. As a result, I began to design tasks for use in the early primary classroom explicitly connecting mathematical and visual aspects. I wondered at the existence of a causal relationship between spatial ability and mathematical capacity; in particular, if a child’s strength for visualizing, transforming images and construction could be channeled into mathematical problem-solving ability. In a community where English as a second language is a major educational concern, having options for supporting thinking that include the whole child –
engaging the whole brain – was appealing, both to me and the teachers with whom I work.

Involvement with the writing team for the B.C. Early Numeracy Project allowed me an opportunity to study some of the early research into the area of imagery and mathematical problem-solving. The team was charged with creating a numeracy assessment for children in early primary – ultimately focusing on kindergarten and grade one students – to assess, identify and then support these students at risk in their early years of school. My role in the project was to examine and summarize research in spatial sense, imagery and visualization, to determine both its significance for the project and to establish a way in which imagery and visual thinking might be built into the assessment and instructional components. The work of Grayson Wheatley, Erma Yackel and Paul Cobb (Wheatley & Cobb, 1990; Yackel & Wheatley, 1990) had implications for classroom practice, in particular a tangram-like task that ultimately became part of the Early Numeracy Project assessment. More importantly, though, the process of research helped to connect me with researchers who had begun to ask some of the same questions as I had regarding the connection between imagery and mathematical capacity.

The research I accessed (Wheatley & Cobb, 1990; Reynolds & Wheatley, 1997; Thomas, Mulligan, and Goldin, 2002; Gray, Pitta, and Tall, 2000, Owens, Mitchelmore, Outhred & Pegg, 1996) suggested there was a connection between imagery and problem-solving capacity but did not explicitly state how these strengths might transfer, or whether teaching a child to visualize would make him or her a better problem-solver. It did however open my eyes to the signs of visualization in my work in classrooms. As a coordinator I am invited into classrooms to co-plan, co-
teach and co-assess student work. Having access to hundreds of students of varying ages allowed me a window through which to observe and notice children thinking and processing mathematical ideas using imagery and visualization. Certain problems in particular seemed to elicit an image-making response, causing groups of students (and adults in workshop situations) to gaze upwards, to gesture and finger point while reasoning through the task. What exactly were they "looking" at? And how did their image help them to reason through the problem? I began to focus in on these students and ask them this very question, explaining that their upwards glances made me wonder what they were seeing in their heads. Some could answer that they were looking at an object, a picture or image from the problem; still others reported they were moving or manipulating the image in some way. Young children on the other hand, although they exhibited the same outward signs (up-cast eyes, gesturing) were less able to describe their mental processes, responding instead that they "just knew" the answer.

Questions then arose about language and the ability to access metacognition to describe thinking processes. If young children were accessing imagery to reason through a problem, they seemed less able to describe it; and in our community of learners of English, communication was even more complicated. How could I access young students' thought processes, their image-making activity without requiring language? My work with a group of kindergarten children gave me some possible direction. As part of a video project, I set up an experiment involving a group of 3 boys in May of their kindergarten year. The question I was investigating at that time was to determine the extent to which sharing solutions to a mathematical problem impacted the behaviour of others working through similar problems. I asked the boys to
solve a division problem – 5 cookies shared by 2 children – and gave them access to whatever they needed to solve the problem, including manipulatives, paper and crayons. I hoped that sharing their solutions after they had worked through the problem would expand the strategies available to and used by the children while they solved progressively more difficult problems on subsequent days. What happened surprised me.

In the first problem-solving interview with this group of children, I posed the problem aloud. While two of the boys began chatting about the problem, one child, Jason, sat quietly staring at the ceiling, then put his hands together palm to palm and made chopping motions 5 times over. He quickly picked up a paper and marker and drew 5 circles, then drew vertical lines through each one, muttering “cut in half, cut in half...” with each line drawn. Next, he drew 2 people under each cookie, numbered the people and explained that everyone would get “5 cookies. 5 cookies cut in half.” His drawing follows (Figure 1 – Jason’s cookies) and even as an artifact gathered after problems solving took place, it is clear that Jason’s solution suggests 5 halves be allotted to each child. At the time I marveled at his thinking, noticing the product more than the process. So too did the other boys in the group. They left off the manipulatives – never once using them through the three problems presented – and attempted instead to draw their solutions like Jason had.

As the kindergarten problem-solving study progressed, two other more complex problems were presented. In each case the children opted only to draw, using Jason’s representations as a model. Lorne borrowed Jason’s structure for the representation of the solution to the second problem (7+2), adding a knife to facilitate the cutting of the pieces and
talking about the actions suggested by his picture. One boy, Charlie, was less able to communicate his solutions with pencil and paper due to fine motor issues. In the second and third problems (7+2 and 15+2) he did not produce a drawing to show his solution but rather solved the problems through the act of drawing (creating a partial drawing and then stating his answer) or by visualizing the solution without drawing at all. When I probed him for more information, he said simply, “I thought it in my brain.”

Through repeated viewings of this footage in workshops situations with teachers, I began to notice different things – evidence of student image-making, separate from the products they created. Each of the boys demonstrated visualizing behaviours through upraised eyes and gesturing, both while they were thinking about the problem and again afterwards when they were telling how they had solved the problem. And yet afterwards only 2 of the boys had produced a paper-based artifact for the 3 problems. As a classroom teacher, this situation was a very familiar
one. Each of the boys had solved the problem, performed the cognitive work, each had accessed imagery to support them in reasoning through the problems, but only 2 had created a product.

More questions emerged. How had drawing helped each of these children? To what extent did it provide a scaffold for thinking? At what point was drawing as a problem-solving strategy no longer helpful? And when was visualization enough?

Conceptualizing the study: developing the research question
Several years passed. Jason, Lome and Charlie had moved on to grade 2. I remained curious as to whether these drawing as problem-solving behaviours persisted in the later grades, and how these particular children, now in separate classrooms, might solve complex mathematical problems if given the opportunity to interact with them. Thinking even more broadly, though, I began to wonder how other children of this age group might engage with mathematical problems in this way. Ultimately I set up my masters research study to involve the entire grade 2 cohort at one school - 33 children in all - in an attempt to address my questions regarding student imagery and how it might support students in mathematical problem-solving. If Jason and Lome and Charlie could access imagery and drawing as problem-solving as early as kindergarten, then who else was using these visual strategies? And if they were, how were these strategies helping?

My background and experience with young children and the study of kindergarten students allowed me to hone in on a more specific set of questions for this study. Imagery and the role of representation in supporting student thinking or drawing as problem-solving and how it
mediates student reasoning became the focus for this investigation. Specifically, I articulated the following research questions:

- How do young children access visualization and imagery in their resolution of mathematical tasks?
- How does drawing as problem-solving support students in making sense of and reasoning through a mathematical problem?

Related studies

Of course, related studies have been enacted. Previous research in mathematical representation has examined children’s capacity to represent their thinking on paper (Diezmann, 2000, 2006; Deizmann and English, 2001; Goldin and Kaput, 1996; Owens and Clement, 1998; Smith, 2003; Woleck, 2001). In particular, this work sought to assess student thinking as shown in the pictures they drew, or the diagrams they were able to create in response to a problem situation. Research into representation examined student samples to gain insight into student thinking and development. Smith and Woleck, both teacher researchers, spoke to the rich assessment data available to teachers in, as Woleck terms it, “listening to their pictures”. Smith and Woleck saw student drawings of mathematical thinking as a powerful assessment tool.

Research into representation, however, attends to and seeks to interpret student thinking after the act of thinking is complete. That is, student work in the form of diagrams or pictures was analyzed after the child had finished working on the problem. In reading this research on representation, I had to wonder about the valid mathematical thinking that never made it to the page. How was that thinking being accounted for? Characterized by Charlie, who in kindergarten just “thought it in [his] brain”, my experience of working with young mathematicians has been
mainly an oral and visual one, where scant or no printed artifact is produced. And yet these children are thinking, and thinking deeply. By their own description, while solving mathematical problems, they are creating and manipulating pictures in their heads.

My research journey then lead me to learn more about visualization and its role in problem-solving. I looked to Owens, Mitchelmore, Outhred and Pegg (1996), Owens and Clement (1998), Wheatley (1990), Reynolds (1997), Yackel (1990), Thomas et al (2002), and Gray et al (2000) for some clarification on how students use imagery to solve problems. Each of these researchers credited visual imagery or visualization with supporting students in their problem-solving efforts. Some even suggested that gifted mathematicians use imagery more often and more efficiently than other children (Thomas et al, 2002; and Gray et al, 2000). My experience working with gifted children in both segregated and regular classroom settings certainly allowed me to witness this. Still I was not able to satisfy my curiosity about how these visual processes are accessed by young children and applied to the solution of mathematical problems. In considering the results of our district-wide DISCOVER: Learner Strengths assessment, many children in kindergarten tended to have logical-mathematical and visual-spatial strengths operating in a kind of partnership. The literature, however, did not seem to shed light on how these processes might be developing concurrently in young children.

In designing this study, I wanted to set up a situation – a performance task – in which students' thinking would be assessed during the act of thinking. Acknowledging the rich assessment data to be gained from student representation, I wanted to collect and analyze children's pictures; but more so, I wanted to observe students while they were working to
represent their thinking processes. I hoped to observe and describe students' reasoning behaviours while they drew in response to a mathematical problem. *Drawing as a problem-solving* is the term I use to describe this mathematical reasoning through the act of drawing. In addition, for those children who might access imagery to support their representational activity, I wanted to create a research situation that would promote student use of and talk about visualization. All this in an effort to address the main research questions:

- How do young children access visualization and imagery in their resolution of mathematical tasks?
- How does drawing as problem-solving support students in making sense of and reasoning through a mathematical problem?

The structure of the study through which I attempted to answer these questions is described below.

**Methods**

The participants for this study were 33 students from a suburban school district. Two problems were posed of the grade 2 students who worked in small groups (3-4 children). Students were video-taped while they worked through the tasks, both from above and using a hand-held camera to capture both the talk and the act of drawing as it happened. The problems were both routine and non-routine in nature. Students were asked to solve a sharing problem (There are 18 cookies and 12 children. How many will each child get?) as well as a combination problem (18 wheels, what toys could there be?). I read the problems aloud to the students, and they were presented with a piece of paper and a pencil to use.

During the course of the child's problem-solving, I posed the following
metacognitive questions: “Can you tell me what you’re thinking? What does this part of your drawing mean?” After the problems were completed, I asked questions around the act of drawing, including: “What strategy did you use to solve the problem? How did it help you?” Students’ comments - the questions they asked, their meta-cognitive talk, and their interactions with other children - were transcribed verbatim. A further language sample was recorded when children were asked how drawing helped them solve the problem. These answers helped to clarify what students were thinking while they drew and the degree to which drawing was seen as helpful. These language samples proved helpful in elaborating on students’ reasoning through drawing – how they used the pictures in their heads and the marks on the page to solve problems.

**Significance of the research problem**

This study will attempt to bring together my background experience with the DISCOVER: Learner Strengths assessment, my classroom based observations of visual-spatial thinking across multiple grade levels, my related research on imagery for the Early Numeracy Project and my informal video study of problem-solving strategies of Kindergarten learners. The project itself grew out of both structured and informal observations of children; as such, the research has its roots in early primary classrooms. This study is significant in that it attempts to connect research with practice, in creating a framework for observing drawing as problem-solving behaviours and a set of related instructional tasks. Significant too is this study’s divergence from previous work. While other researchers have focused on students’ drawings and what can be learned about thinking after the event, this study will focus on the act of drawing and students’ reasoning behaviours as they doodle towards a solution. Featuring two performance-based tasks, this study presents an opportunity to access
student thinking as it happens; while video footage of students drawing and clips featuring students explaining their ideas provide insight into their reasoning. Students' use of language – a developmental cornerstone in early primary – has a critical role to play in this study, both as a way to access thinking and as a means to expose children's dispositions.

Unlike many other studies featured in the related research, this one highlights young learners and their particular approach to problem-solving – an approach that can be both mathematically structured and visually chaotic. An examination of how young children access these seemingly complementary logical-mathematical and visual-spatial strengths to solve mathematical problems may shed light on student capacity and provide a way to support students who need it.

**Structure of this document**

This introductory chapter has outlined my particular professional background and how the various pieces of my work have combined to focus in on the area of visualization and representation as they influence and support young children's mathematical problem-solving. Chapter 2 will expand on the research related to these areas and describe the extent to which my own research questions are only partially explained through existing research. Research methods used during the course of this study will be described in Chapter 3 – including how I set up the problem-solving interviews, how I analyzed and collated video footage and how I sorted interview data.

Chapter 4 presents an analysis of this data to determine how it is that students reason through the act of drawing, and the extent to which imagery impacts the process. A framework describing characteristics of
these drawing as problem-solving activities and featuring student samples is also included in Chapter 4. In Chapter 5, I discuss the results, consider the range of student strategies, and elaborate on how imagery and systematic thinking impact student success with drawing as problem-solving. Implications for classroom practice and further research directions are presented in this final chapter as well.
Chapter 2 – Literature Review

Introduction

I turned to the research in considering how young children access visualization and imagery in their resolution of mathematical tasks, and in designing a research study that would access it. My research question was largely concerned with visual-spatial imagery; as such I looked to giants in the field – Owens, Thomas, Mulligan, Yackel and Wheatley – all of whom have studied the visualizing efforts of student mathematicians. It became clear that to some degree, visual thinking was supporting student problem-solving. Comparative studies confirmed this.

But in addition to imagery, I wanted to determine how drawing as problem-solving supports students in making sense of and reasoning through a mathematical problem. Research in this area proved harder to find. Although a broad movement in the area of student representation had yielded a large body of work, I struggled to find studies that looked at young children, or studies that assessed student thinking in the act of thinking. I attended the Psychology of Mathematics Education conference in Bergen, Norway in 2004 and went to every session I could find on representation, diagrams, the use of pictures, but found that these studies did not answer my questions. Instead, they addressed issues related to how students might interpret pictures in solving problems, or the notion of explicit instruction to develop diagrammatic capacity. Largely these studies on representation seemed focused on assessing mathematical understanding after problem-solving had taken place.

It appeared as though I had a question worth researching – connected to previous studies but unique in its performance-based emphasis and in the
age of its participants. I wanted to understand what led a child to pick up a pencil and begin to draw in response to a problem, what visual processes took place before and concurrent with drawing. I wanted to explore how visualization was connected to drawing – the observable indicator of image-making – to understand what Jason, the kindergarten student, was seeing in his head when he chopped the cookies, what Charlie thought in his brain.

**Interpretation of the research**

My interpretation of these other research projects follows. I have attempted to organize the other research into sets or topics, adding a heading to describe the work where appropriate. Generally, the work outlined below flows from visualization and dynamic imagery and their use in mathematical problem-solving situations, to the analysis and assessment of students’ mathematical understanding through their representations, both dynamic and static in nature. Studies highlighting instructional implications in representation and the stance presented by the National Council of Teachers of Mathematics wrap up the chapter. Understanding of my own research question grew from reading and reflecting on these works. Other questions that were raised for me are threaded throughout; some were answered in part during the course of my study, still others warrant further investigation.

**Imagery and mathematical problem-solving**

Grayson Wheatley and others (Wheatley & Cobb, 1990; Reynolds & Wheatley, 1997; Thomas, Mulligan, and Goldin, 2002; Gray, Pitta, and Tall, 2000) have researched spatial sense and mental imagery, and have posited a connection between a child’s capacity to make a mental image and his or her ability to solve mathematical problems. Wheatley
asserts that spatial sense (or imagery) is useful in numerical as well as geometric settings, and that it is foundational to mathematical reasoning (Reynolds & Wheatley, 1997; Wheatley, 1990; Wheatley and Cobb, 1990). Wheatley and Cobb (1990) describe mathematics as the activity of creating relationships, and assert that these relationships have their roots in visual imagery. Their analysis of young children's spatial constructions outlines the processes children employ in the solution of spatial problems, and points to dynamic imagery as a powerful tool. Dynamic imagery, the ability to make and manipulate a mental image, is seen to be key in problem-solving, leading a child to think more flexibly in response to a problem. Wheatley and Cobb state that:

Mathematical problem solving is often a matter of reasoning analytically, constructing an image, using the image to support additional conceptual reasoning... a process of building from images to analysis and analysis to images [that] may continue through many cycles. (1990, p. 161)

Wheatley developed a series of activities designed to improve mental imagery, and proposed that this in turn would increase student success when approaching mathematical problem-solving tasks (Yackel & Wheatley, 1990). These tasks, included in an instructional program called "Image-Maker", involve children of all ages. One of a series of simple line drawings is presented to children for a few seconds, and children are then asked to "draw what they saw". The representations students make are then shown and the images students create "in the mind's eye" are shared aloud. Specific language related to geometry and transformations emerge. Through this activity, it is hoped that students will develop the capacity to create and re-present an image and develop the language to describe mental images (Yackel & Wheatley, 1990). I have used these Image-Maker tasks with children and adults alike, and
have seen them published in teacher resource materials (Buschman, 2003) where they are recommended for promoting language and visual-spatial thinking in students in the primary grades. They provide a powerful connection point for children who have visual-spatial strengths – since being a good Image-Maker does not require that a student be artistic, but visually adept in terms of constructing mental images.

Wheatley and Reynolds (1997) also suggest that if teachers can support students in the use of dynamic imagery, children will be more powerful mathematical problem-solvers; that cognitive structures are enhanced, and the development of mathematical reasoning progresses more fluidly. The Image-Maker program is designed to improve students’ visual imagery (Yackel & Wheatley, 1990), but in their research, a correlation between the practice of image making and the development of mathematical reasoning were not made explicit. Understanding how this kind of thinking develops required a more theoretical frame. I turned to the work of Kay Owens for that framework.

Kay D. Owens has written and co-authored volumes of work on visualization and spatio-mathematical problem solving as they relate to primary-aged children. Her research has been extensive, long-term and involves large samples of children; her findings are supported by both qualitative and quantitative data. She has developed and refined a framework for assessing and describing 5 different kinds of imagistic processing, and offers tasks to assess a child’s level of functioning. She emphasizes the role of visual imagery in “establishing the meaning of the problem, in channeling the problem-solving approaches of the students, and in influencing the students' cognitive constructions” (Owens & Clement, 1998, p.216). This article was rich with compiled work,
summarizing a wealth of research and a theoretical framework for assessing and understanding students' capacity to solve spatial problems using imagistic processing.

Owens' work with young students has focused on students' construction of understanding through language and constructivist experiences. In 1996 she and Outhred investigated how older students construct an understanding of area through unit grids; in her 1998 study of children's experiences of tiling areas, she and Clements observed students covering shapes by comparing the lengths of sides and different angles by overlaying card-cutout representations. In these two studies, Owens makes connections between students' movement of objects to find area, and tessellation work – the latter requiring more dynamic use of spatial problem-solving strategies.

In a later article, (1999) Owens, Leberne and Harrison elaborate on this notion of dynamic imagery as a strategy for spatial problem-solving. This article proved extremely helpful in beginning to define and compare types of imagery – namely static and dynamic imagery. Owens et al, in speaking of the young student's development of geometric understandings, defines three important aspects that contribute to this development: orientation and motion, part-whole recognition and classification and language (1999, p. 29). In particular, the notion of orientation and motion were interesting to me in considering how children develop effective structures for solving problems using imagery. Owens and her colleagues distinguish between static pictorial imagery – in which the pictures in the head do not change – and dynamic imagery in which multiple examples exist and are changeable. The way a young child acquires dynamic imagery, says Owens, Leberne and Harrison (1999), is
through active investigation – the construction of understanding through exploration. "Through active investigation, imagery is more likely to be dynamic or representative of a pattern, relationship or rule." (1999, p. 27). This seemed to suggest that for young children, the way to develop dynamic imagery that supports pattern-based mathematical thinking is through active exploration, rather than through direct teaching. Diezmann and English (2001) offer a different perspective later in this chapter.

**Dynamic imagery and mathematics achievement: Comparative studies**

So what does imagery look like in children? How does it impact student performance? If, according to Owens et al (1999) dynamic imagery is more flexible and changeable than static imagery, and students who possess dynamic imagery tend to create images that are more representative of a rule or pattern-based understanding, do students who use dynamic imagery out-perform those with static pictorial imagery? The following studies describe comparative projects to this effect.

Thomas, Mulligan, and Goldin (2002) examined the developmental sequence through which elementary-aged children progress in the internalization of the number system (the counting sequence from 1-100) and posited that this development could be witnessed in young children’s external representations of the number system. The authors sought to investigate the specific role of imagery in representations and the construction of relational understanding in mathematics. High ability children (classified gifted) were interviewed and compared to those of average ability; distinctions were made between the types of imagery (the internal representations of number) used by both groups. High ability children tended to use dynamic, fluid and changing imagery to describe
number, demonstrating flexibility in their understanding. Average ability students, by contrast, displayed more static visualizations of number – their images were concrete, and less mobile. The authors state that varied experiences of the number system and an array of contexts for imaging and visualizing number relationships significantly deepen number concepts and add "movement" to the images created by children (p.129); that powerful problem-solvers utilize dynamic imagery in solution finding, by manipulating internal representations of the base ten system in flexible ways.

Gray, Pitta, and Tall also compared two groups of students – low and high achievers – in their study (2000). Children aged 8-12 were given a series of mental arithmetic problems to solve and were asked to describe “what came to mind” as they worked through the problems, describing their strategies and images. In solving the arithmetic problems, low achievers constructed images that were story-based, episodic and rich in detail, even colour. Children in this group carried out “procedures in the mind” and described the numbers as “spinning” and “in a jumble” (p.14). By contrast, high achievers reported images that were more abstract, and featured relational understandings (p.12). They described symbolic images and numbers “flashing” in their heads – as though they were being presented with the answer. For these children, images of number and number relationships were generative and meaningful.

Gray, Pitta, and Tall (2000) and Thomas, Mulligan, and Goldin (2002) agree that spatial patterns and visual imagery play an important role in establishing relationships between numbers, and help to solidify part-whole thinking. They concur with other researchers that imagery can be static or dynamic in nature (Gray, Pitta, & Tall, 2000; Owens & Clement,
1998; Owens, Mitchelmore, Outhred, & Pegg, 1996; Owens, 1999; Wheatley & Cobb, 1990). Gray et al (2000) and Thomas et al (2002) move beyond Owens’ 1999 study to suggest that imagery – a child’s ability to build, re-present and transform images – can be applied to other problems, even those that are not geometric in nature. Higher achieving mathematical problem solvers, say Gray and Thomas, tend to use dynamic and flexible imagery in their solution of problems, while those who use static or unchanging images tend to rely on and over-apply procedural understandings (Gray, Pitta & Tall, 2000; Thomas, Mulligan & Goldin, 2002).

Despite the difference in the ages of the students involved in these studies (Gray, Pitta & Tall, 2000; Thomas, Mulligan & Goldin, 2002; Owens, Leberne & Harrison, 1999), there was consistency seen in the results. These researchers seem to agree within the realm of their research projects, that dynamic imagery is a powerful mathematical thinking tool. This made me reconsider my thinking about conceptual and procedural knowledge in mathematics. Perhaps conceptual understanding is derived from dynamic, changing images, while procedural knowledge comes from static and inflexible ones. I began to wonder then if there was a way to add movement to these otherwise static images – to make dynamic that which was unchanging and support students who struggle. Diezmann brings up the issue of struggling students in her article, “Making sense with diagrams: Students’ difficulties with feature-similar problems” (2000).

**Visualization and diagrammatic capacity**

Diezmann (2000) makes connections between diagrammatic representations and spatial and number sense. She proposes that students who struggle to represent their thinking diagrammatically may
also struggle with either (or both) spatial sense or number sense; that the inability to visualize a mathematical situation affects a student's ability to represent that situation in a diagram. Diezmann writes:

Students' difficulties and errors in generating accurate and effective diagrams are generally associated with students' lack of expertise in diagrammatic representation. However the results of this cross-study comparison suggests that effective diagrammatic representation also depends on a sound mathematical knowledge base, which includes sense-making in mathematical situations. (p. 233)

If not all students can accurately generate effective diagrams, as suggested in Diezmann's study, what are the implications for drawing as problem-solving capacity? Are there students for whom drawing as problem-solving is too difficult or not accessible by virtue of a shaky mathematical foundation?

Assessing image-making; the complexity of language
Diezmann (2000) states that representations function as both a student response as well as an indicator of emerging mathematical understandings; that is, a child can represent an answer through a diagram (a representation or picture) but also that the child's mathematical understandings can be inferred from these artifacts. In contrast, this study proposes that the act of drawing itself presents a rich opportunity to observe and assess student thinking in the act of thinking. While research into representation highlights the many ways students can show what they know and can do, the study of drawing as problem-solving provides performance assessment data – a window into the mathematical activity in progress.
Measuring mental imagery is difficult, since it relies on a student’s ability to use language to describe the image and process being undertaken mentally. These “internal representational capacities” must be inferred (Goldin, 1996), as they produce no formal product to be examined and described. Rather, it is the talk about these images and the internal processes that is the focus of study. A young child’s capacity to describe the pictures in his or her head – indeed to be metacognitive at all – poses a particular challenge for researchers looking to examine mental imagery as a factor in mathematical thinking, particularly in very young learners. Without a formal product or language to describe the process of thinking, evidence of mathematical reasoning through imagery must be generated through careful, informed observation on the part of researchers. And without language to describe, explain and reflect on thinking, does student learning coalesce? Sue Gifford (2005) discusses the relationships that exist between language, visual image-making and numeracy and makes recommendations for students who struggle to mathematically.

Spatial-visualizing capacity and number understanding
Language is featured in the work of Sue Gifford (2005). In her review of research related to dyscalculia (the impaired ability to solve mathematical problems) in young children, Gifford speaks to the interrelatedness of language, visualizing and the development of number sense.

However, number representation also depends on language in a complex way, and on the development of symbolising. ... A complex picture emerges, whereby visualising, finger use, language and symbolic representations of numbers develop interdependently, suggesting that various areas of the brain work together to develop number understanding. (p. 38)
She highlights the interdependent processes of the developing brain, and proposes a multi-modal intervention strategy including support with visual reasoning for students who are diagnosed with dyscalculia. Specifically, she recommends creating networks of images and vivid associations to support memory and meaning-making (p. 50). She proposes a focus on teaching to students' strengths, in particular visual capacity, which emerges early in young children: "It seems that young children first solve number problems by visual images, and so a spatial visualising capacity is indicated." (p. 37)

I appreciated two things about Gifford's research. First, she promotes teaching to a student's area of strength. This is consistent with the philosophy behind the DISCOVER: Learner Strengths assessment and its instructional component, in which addressing areas of need is accomplished through a child's area of greatest strength. In our district, for example, language development is promoted by engaging young learners in visual-spatial, logical mathematical tasks – areas of relative strengths for them. Gifford suggests a similar route for supporting children who struggle, by accessing visual capacity to address mathematical difficulty.

Second, Gifford speaks to the creation of networks of images and vivid associations to support meaning-making. Her means of supporting children who struggle takes the form of a broad-based, multi-modal approach in which visual images and language are key. This rings of Owens' work (Owens, Leberne and Hamilton, 1999), in which dynamic imagery is defined as flexible, moving and concerned with relationships and patterns. It seems as though Gifford's approach to supporting
students with difficulty advocates for the same flexible, sense-making
network as Owens suggests exists for those students who develop
dynamic imagery. So how do these images – and more importantly,
these systems of images – develop and link to concepts? David Tall
outlines one theory.

Visualization and Symbolism in Mathematics
In his 1994 address to the Commission Internationale pour l'Étude et
l'Amélioration de l'Enseignement des Mathématiques in Toulouse, David
Tall extended Bruner’s theory of representation to consider students
visualization and conceptualizing of symbols. Tall found that children use
number (the symbol for the numeral) to represent the process of counting
and also to symbolize the concept of that number; that operationally
children overlap concept and process within symbolic representation. Tall
and Gray coined the term “procept” to describe the students’
encapsulation of the concept and process into one symbolic
representation. Although he speaks here about formal symbols (numerals,
signs for operations, algebraic symbols and Euclidean formulae), I
wondered at the applicability of Tall and Gray’s notion of procept to
drawing as problem-solving representations.

The symbol can be spoken, heard, seen and read and the
combination of these sensory perceptions and actions gives it a
cognitive existence as a mathematical object. But it is more powerful
than that – the process can be used to do mathematics and the
object can be used to think about it. (p. 2)

In terms of drawing as problem-solving, the representation itself – the tally,
the stick-man, the cookie – stands for both the object and the action
being performed on it. The duality of the representation as both a way to
do the math and a way to think about the math poses an interesting
connection to my masters research project, in which students’ visual
images are converted to representations and then operated on by way of reasoning through the problems presented.

**Drawings as dynamic representations**

Operating on representations to reason through a problem suggests that drawings – the symbol – are a way to do and think about the math, and not simply a way to record a response. When children draw in response to a mathematical problem, there is more going on than simply representing a solution; Tall maintains that a child’s interaction with the symbol on the page is far more active, more dynamic than that. The creation of these representations on the page appears to have a dynamic, cognitively involved component.

Woleck (2001) and Smith (2003) present a similar perspective on drawing in math class. They maintain that drawing supports children in modeling a problem and therefore in arriving at a solution for it (Smith, 2003, Woleck, 2001). Kristine Reed Woleck writes from a classroom-teacher/researcher perspective in her article “Listen to their Pictures: An investigation of Children’s Mathematical Drawings” (2001, p. 215-227). She describes her grade one students’ drawings in response to math problems and explains that these dynamic representations signify an essential cognitive bridge between the concrete and the abstract in mathematics. As children strip away the unnecessary features of a problem, removing all but the essential elements of the problem’s structure, they become generalizable to more than the single context in which they are being used.

Rather than drawing people, clothed and seated in bus seats to represent how many buses are needed to accommodate the class for a field trip, a child using a more abstract representation may opt for stick figures or
encircled tally marks to indicate groups of children and bus seats. These latter forms of representation allow for more flexibility of use – the essential feature of divisive thinking is maintained, and less attention is paid to the context of the problem. I wonder though, what is it that leads a child to strip away the unnecessary features of a problem over time, and represent it in a less idiosyncratic way? Or can pictorial, idiosyncratic representations support learning beyond early childhood?

Although Woleck acknowledges that children may use drawings as scaffold for thinking, and mentions specifically the use of pictures as a place-holder or pictorial manipulatives to be counted during problem-solving, the focus of her action-research is on the use of drawings to communicate mathematical ideas after problem-solving. My research question is more concerned with Tall’s notion (1994) of doing and thinking about the math in the generative sense. That is, how do children create symbolic representations that allow them to actively engage with the images in the mind’s eye as they are being translated to the page? What does this process look like, and how does it develop?

**Drawings as manipulatives**

Woleck (2001) and Smith (2003) did provide some insight into how young children interpret and use their representations in a mathematical sense. In Stephen Smith’s study of third graders’ response to “the candy bar problem”, he noted that each of the children involved in the study drew – both to manipulate the objects in the problem (drawing as problem-solving) and to represent their thinking afterwards (drawing of problem-solving). He noted that the children’s use of drawings differed in terms of their idiosyncratic nature – two of the children used and relied on the context of the problem in their communication of both their
mathematical processes and the final problem product, while two others tended to strip away context and represent their thinking in less idiosyncratic ways. Smith emphasizes that the artifacts children produce in solving mathematical problems – including language, drawings and constructions – cannot be considered separate from the students' mathematical reasoning.

Here, and in Woleck's 2001 study, these researchers suggest that young children use drawings in the same way as a manipulative – that the drawing serves as a perfectly designed mathematical tool to represent whatever is being manipulated in the problem. Unlike traditional manipulatives (like Cuisinaire Rods) that have been constructed by those who have a firm grasp of a concept to represent that concept, a drawing is uniquely matched to the problem and the problem context. The degree of abstraction from the actual object is quite low, and as such, these drawings serve the purpose of taking children from concrete representations to more abstract ones, such as structured diagrams or equations.

In terms of my study, the notion of drawings as a manipulative proved to be a helpful way of considering young children's meaning-making efforts, a purpose for their interaction with the representation they created. Virtual manipulatives, a term coined later in my project, was based on Woleck (2001) and Smith's (2003) work, with a more dynamic spin suggested in the qualifier "virtual", calling to mind the dynamic imagery and meaning-making work of Owens (Owens and Clements, 1998; Owens, Leberne and Hamilton, 1999), Wheatley (1990) and Tall (1994).
**Explicit instruction in drawing to problem-solve**

For Smith (2003) and Woleck (2001), pictures are seen as one component of mathematical representation in general, which includes oral language, written words and numbers. These drawings are generated by children, represent understandings and are purposeful. Other researchers consider that idiosyncratic drawing serves little purpose, clouds the structure of a problem and detracts from the big mathematical ideas. Like manipulatives, they suggest that these drawings be discarded as soon as possible and an efficient and standardized structure for representing and organizing information be adopted. In Cypress, for example, this is precisely what happens at 4th grade, when children are taught structured diagrams to match the operations and are discouraged from using any other form of representation (Pantziara, Gagatsis, & Pitta-Pantazi, 2004). Current Cyprian research in the use of visual images to support mathematical understanding focuses on the reading of pictures – the interpretation of an image and the degree to which the image supports or detracts from a child’s capacity to problem-solve (Elia & Philippou, 2004). Focused largely on intermediate aged-children’s interpretations of images, the research task does not ask children to solve mathematical problems by drawing, but rather by reading information presented pictorially. In terms of image production, the research reported that intermediate-aged children were taught to use very specific schema for representing mathematical operations; however, the research does not address a child’s spontaneous and constructed representations or the development of these images over time and with experience.

van Garderen and Montague’s study of grade 6 students of varying abilities (students with learning disabilities, average achievers and gifted learners) describes students’ visual-spatial representations during
mathematical problem-solving (2003). The researchers found that children with learning disabilities tended to draw primarily pictorial representations that illustrated persons, places, or things described in the problem, while gifted students tended to create representations that illustrated relationships between elements of the problem. One wonders whether this dependence on pictorial representations is based on students' lack of sense-making of the mathematical situation (Diezmann, 2000), or whether this strategy, like counting all for very young children, was the only one available to and accessed by these students. Like counting all, the use of pictorial representations — although less efficient and with greater potential for inaccuracy — can still generate a solution.

It is this sort of result, however, that leads researchers to seek to support students in moving beyond their constructed idiosyncratic or pictorial drawings to more structured and formal representations. Novick, Hurley and Francis (1999) have defined 4 general-purpose diagrams that can be used to match the structure of any given problem; Diezmann and English (2001) and Diezmann (2006) describe ways that teachers can support their students in understanding the relationship between a problem and the diagram needed (2001, p. 86-87, 2006, p. 2-439). Diezmann and English promote diagram literacy — “the ability to understand and use and to think and learn in terms of images” in their study of 10 year old mathematics students (p. 77). The authors' desire to connect image-making and mathematical understanding is consistent with the work of Wheatley and Cobb (1990), Yackel and Wheatley (1990) and Owens and Clements (1998) among others.

An important distinction is the application of Diezmann's research findings. While Diezmann and English highlight the importance of talk and
modeling when working with diagrams, they also recommend that students be explicitly taught these four specific diagrams and when to apply them, and should practice them within structured problem sets. They maintain that students should be presented with sets of problems that can be represented with a particular diagram and that the relationship between a problem and its corresponding diagram should be explicitly taught. I believe that Owens, Leberne and Harrison (1999) would disagree with this approach. Their research supports the development of dynamic imagery, flexible networks of spatial understanding through exploration and investigation – particularly for young students. Although the participants in the Diezmann study (Diezmann and English, 2001) are 10 years old and approaching a developmental stage at which these relationships could be introduced and discussed, caution must be exercised when accelerating this developmental process for younger students.

The desire to support diagram literacy (Nickerson, 1994) in elementary school-aged students should not supercede the natural development and constructed meaning-making for young children promoted by the NCTM and researchers like Woleck (2001) and Smith (2003). Smith (2003) acknowledges that children need to bridge from idiosyncratic to mathematical representations in order to understand and communicate mathematics (p.273) and makes the case for sharing, discussing and analyzing peers' solutions as a developmentally appropriate way to bridge from one to the other.

Reform and representation: Valuing drawing as mathematical reasoning
Acknowledging the need to develop language to describe mathematical processes and thinking, proponents of the reform
movement in mathematics education have highlighted the significance of representation and communication in mathematics. The NCTM Principles and Standards (2000) make this explicit. Teachers are directed to support children's use of representations to model and solve problems, and to promote the use of representations to organize, record and communicate mathematical ideas. The concern here is this: although the NCTM encourages teachers to have their children represent their thinking with manipulatives and with drawings, these drawings may be seen in traditional math classrooms to be superfluous pieces that detract from the purpose of problem-solving rather than supporting it. Like the use of manipulatives, pictures can be interpreted as a temporary crutch, to be discarded as quickly as possible in favour of more standard representations and structures. In my experience as Curriculum Coordinator, more traditional math classrooms seem to emphasize the use of abstract and general representations over the idiosyncratic ones employed by young children.

The NCTM Principles and Standards go on to say that a constructivist stance should be adopted when considering students' drawings in math class, since a young child's capacity to represent mathematical ideas develops over time. With experience and an educational program that includes sharing and talk, a child's representations of mathematical ideas will develop and become less idiosyncratic and more generalizable (Smith, 2003). Most educational researchers embrace this developmental notion; how it is interpreted and implemented, however, varies.
Summary

My research journey allowed me to refine my study, both in terms of its intent and the nature of the questions addressed within it. Wondering how young children access visualization and imagery in their resolution of mathematical tasks lead me to the work of Owens and Outhred (1996), Owens and Clements (1998) and Owens, Leberne and Hamilton (1999), Wheatley and Cobb (1990), Reynolds & Wheatley (1997) and Thomas, Mulligan and Goldin, (2002). Their studies confirmed that imagery is accessed in the solution of mathematical problems – both those that are geometric and numerical in nature. In fact, dynamic imagery contributes to the development of number sense, in allowing for flexibility in thinking about constructions of number (Thomas, Mulligan and Goldin, 2002). Flexibility and movement in imagery are key to success with problem-solving, as observed in the performance of students of various abilities (Gray, Pitta & Tall, 2000; Thomas, Mulligan & Goldin, 2002). Higher achieving mathematical problem solvers tend to use dynamic and flexible imagery in their solution of problems, while those who use static or unchanging images tend also to rely on procedural knowledge (Gray, Pitta & Tall, 2000; Thomas, Mulligan & Goldin, 2002). Gifford’s work (2005) provided a summary of ways to address the needs of students with dyscalculia, recommending, like Gray et al (2000), Thomas et al (2002), that support with visual thinking and the development of dynamic imagery would influence mathematical understanding. This provided confirmation for the direction of my study; if I could observe and recognize dynamic imagery in action – acknowledged as a powerful contributor to mathematical reasoning and number sense – then perhaps I could begin to answer the question of how young children access visualization and imagery in their resolution of mathematical tasks.
My second research question asked: "How does drawing as problem-solving support students in making sense of and reasoning through a mathematical problem?". For support in developing my own understanding and in designing a study to begin to answer this question, I looked to research on representation. Much of this research concerns itself with the assessment of student thinking after problem-solving had taken place and so the notion of reasoning through a problem was not clarified for me through these works. Nonetheless, I did see connections through the notion of drawings as manipulatives (Smith, 2003; Woleck, 2001) and the fascinating concept of the procept (the encapsulation of the symbol as a way to both do and think about the math) suggested by Tall and Gray (1994). These three studies presented a way to consider representations as more active – more dynamic in their use in problem-solving.

Further in my survey of the literature on representation, I encountered a way to describe and distinguish between the types of students' representations. Students' representation show different stages: idiosyncratic or pictorial, in which the level of abstraction between the picture on the page and the context of the problem is quite low, and abstract representations that are more generalizable. (Woleck, 2001; Smith, 2003; van Garderen and Montague, 2003). In my exploration, I felt concern regarding the instructional directions suggested by certain researchers (Diezmann and English, 2001; Novick, Hurley, Francis, 1999; Nickerson, 1994). In an effort to support children in developing the most efficient representational strategies possible, some suggest direct instruction and practice, while others pose a more constructivist approach (Gifford, 2005; Owens, Leberne and Hamilton, 1999, NCTM, 2000). More than the abstraction of the marks on the page, attention needs to be
paid to how students are using their representations, doing and thinking about the math. Although research into representation provides some clues into how students solve problems and communicate mathematical understanding, this study proposes paying attention to the processes that precede, or take place concurrent with the production of these representations.

**Contribution to a current body of research**

Research in the area of young children's drawing as problem solving is limited. While there is acknowledgement that drawings assist children in representing their mathematical thinking and that this form of representation has merit (Smith, 2003), there is little research focused on how representational capacity develops in young children, or what imagistic thinking transpires while drawing. In my project, I hope to link research on imagery – particularly dynamic imagery – and research on representation, to consider how students' visual images are converted to representations and then (or concurrently) operated on by way of reasoning through the problems presented.

The hope is that this study might contribute to a larger body of research related to the representation of mathematical thinking, and to fill a gap in considering how the process of drawing – not simply its product – has a central role to play in fostering children's mathematical reasoning. Kyriakides (2006) concurs. In his article entitled “Modeling Fractions with Area: the Salience of Vertical Partitioning”, he argues that we as mathematics researchers and educators should “...draw our attention not so much on the images (either mental or concrete) but through the images” in order to understand the reasoning children do in the act of problem solving. Quoting John Mason, he affirms that mathematics
images 'are not mere marks on paper but indicate or speak to entities that are almost palpable, almost substantial' (p. 4-17).

By addressing key questions related to this topic and by identifying observable indicators of mathematical reasoning while children are engaged in drawing, I aspire to contribute to the field of imagery and mathematical representation, and in turn support classroom teachers in implementing strategies for strengthening student learning; helping them to focus instructional decision making not on the images but through them.
Chapter 3 - Methods

BEFORE THE STUDY

Who was involved
This study involved 33 children in second grade, 16 boys and 17 girls. The children were all from the same school, but drawn from 4 different classrooms. Children were invited to participate; parents gave consent for participation in the study and videotaping of students while they worked (see Appendix B – Consent forms). Problem-solving sessions took place in groups of 3 or 4, with students from the same classes grouped together. Each problem-solving session lasted approximately 45 minutes, allowing for the completion of 2 problems. Problem-solving sessions took place in late May 2005. Data from the children’s responses to the problems was collected and analyzed to determine a baseline or framework for describing drawing-as-problem-solving themes, including student behaviours and characteristics of the drawings they made.

The school in question is a suburban school with approximately 350 students in grades K-7. This is a multi-cultural school, in which more than 22 different nationalities are represented. English as a second language is an area of ongoing support at this school. Selecting children from the same school and from the same grade level was done intentionally. In creating a framework, the hope was to work with students who had had a similar background and early school experience.
The process

The problem-solving tasks

The problem solving session was intended to access students' mathematical reasoning through drawing as well as to provide a context for observing drawing as problem-solving behaviours. As such, question structures were chosen and the complexity of those questions constructed to elicit as much as possible from the children. Having worked with some of these children when they were in kindergarten, I had a sense of their capacity and areas of strength. In kindergarten, I had engaged a small group of children in solving a series of division by sharing problems with varied results in terms of strategies, ways of dealing with remainders and complexity and range of representations. Although I had placed manipulatives on the table at that point, these very young children all picked up markers and paper to work through the problem. I was surprised to see them choosing this mode of problem-solving. More so, I was intrigued about the image-making taking place for these children. I became increasingly curious about the impact of visualization on problem-solving; in particular, how young children's capacity to make a picture in their heads supports them in their mathematical reasoning. For these children, visualization played an important role in finding the answers – eyes were raised, hands "chopped" and children spoke aloud about the action they were orchestrating in their minds eye.

For this current study, I hoped to interact with these same children in researching my new question: How does the act of drawing support children in reasoning through a mathematical problem?
The Cookie Problem

In order to address this research question, I needed to design a problem that was sufficiently complex that it could not be solved without mental effort, but still provide children with access to the math no matter what their capacity. The division as sharing context was the starting point, and the numbers carefully chosen. The first question I asked of children was as follows:

There are 18 cookies on a plate and 12 children who want to share them. How can the children share the cookies? How much will everyone get?

This question required children to divide a whole number by another whole number and deal with a remainder. Although this kind of division problem is not part of the grade 2 curriculum, the real-world context and the range of correct answers accepted (1 and a half, 3 halves, one and some left over) made this question both sufficiently challenging and accessible. During the study itself, I did adapt this question for children who answered it effortlessly (asking: “If you had 9 cookies and 6 children who wanted to share, how many cookies would everyone get?”); or for those children about whom I sought more information in terms of mathematical reasoning through drawing (“Answer this similar question – but without drawing this time.”).

The Wheels Problem

The wheels problem presented a different kind of challenge. This was a part-part-whole problem in which the parts were variable. This problem featured more than one right answer and more than one method for solving it. In fact, I had used this question with another group of grade 1
and 2 children in another school district with interesting results. The open-ended structure of this problem was intentional – my hope was that children would use drawing as a problem-solving strategy in order to solve it. Likewise, the problem was too difficult to be solved mentally, and so the thought was that drawing would be employed as a system for recording and managing the elements of the problem. The problem read:

Cam got 18 wheels for Christmas. What toys could he have gotten?

**Videotaping**

Two video cameras were used for this study. The first was mounted on a tripod and recorded overhead footage of the group as they worked. This camcorder was directed at the entire group in an effort to capture interactions between the children (looking at one another's work, talking, comparing ideas), the point at which the children put pencil to paper (to monitor persistence in terms of time on task) and the frequency of visualization activity as observed through upraised eyes.

The handheld camcorder was used to capture close-ups of the children's hands, their pencils in action and any comments or self-talk they offered. In focusing in closely on the child while working on the task, I hoped to film the "act" of drawing as problem-solving, including any processing, self-correction or the application of new strategies while problem-solving. The two cameras in concert provided 2 important vantage points and a means of cross-referencing the video data; for those moments when I was focused in on one particular child, the overhead camera was able to capture video data from the other 2-3 children.
Figure 2 – Overhead and Handheld Camera Views

<table>
<thead>
<tr>
<th>Overhead Camera</th>
<th>Handheld Camera</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Image" /></td>
<td><img src="image2" alt="Image" /></td>
</tr>
</tbody>
</table>

One group works on the cookies problem, as filmed on the overhead camcorder. I am getting a close up of Orson... ...using the handheld camcorder, while he works on the problem.
DURING THE STUDY

My role as observer

Children were taken from their classrooms to work in small groups of 3 or 4 for a brief 45-minute problem-solving session. During this 45-minute block, both problems were presented and the interview question was asked of the children. Working in groups with classmates was intended to provide children with a familiar, school like situation. We worked in a small resource room around a round table. On the table were pencils and erasers and nothing else. I began by setting the context with the children. I told them that I was interested in knowing more about how children think when they solve math problems, and that I was going to be filming them while they worked on a series of problems. (A letter to this effect had been read to all children before they elected to participate in the study.) I turned on the video cameras and had them introduce themselves on film, then I read the cookie problem out loud to the children, and answered any questions that arose to be sure everyone understood the problem. I turned on the handheld camera and began to film children as they worked, zooming in on their fingers and their emerging drawings.

While children worked, I tried to be non-invasive with my questioning. When I did want some more information, I was invitational in my prompts, asking questions about children's thinking:

- What are you thinking about?
- What's your idea?
- What are you doing?

I also asked about the child's drawing or parts of it, to determine the purpose:

- What do these lines mean?
• What does this part of your drawing mean?
• How does that help you?

How the students responded
The problems were posed of the children orally, and a blank paper with only the problem written on it was presented to the students. The initial question asked how 12 children could share 18 cookies. There were no manipulatives on the table, and no directives given by me for solving the problem. All of the children attempted the problems, and each child tried, when asked, to describe his or her thinking. All students worked on the problem for the allotted 20 minutes without prompting from me, indicating their perseverance and engagement with the problems. Working in small groups with their classmates provided the children with a familiar learning situation, and allowed them to interact with one another. Occasionally children would encourage one another to stick with the task or even gave each other support with problem-solving strategies ("Just draw faces – just do it like mine..."). Some children spontaneously compared their results at the end of the problem-solving session (eg. Jason and Grant, Norman and Lorne), the content of which is discussed later in this paper.

How my observational practice shifted
What I attended to
While working with the second group of 3 children, I began to watch for repetitions of the first group’s strategies. I watched for the creation of sets, the distribution of items, and paid attention to the movement of the child’s hands while processing the problem. I began to attend to children’s strategic, systematic approach. Systems like elimination (crossing off items) and keeping track behaviours (counting and recounting, drawing boxes, etc) were observed carefully. The
sophistication of the drawing, although interesting, would remain after the children had finished working; I recognized that the paper artifact could be analyzed afterwards. I moved my attention elsewhere, focusing instead on fingers and pencil tips, on postures and upraised eyes – the performance-based indicators that would not remain after the children had left, except in the form of video footage.

Adding in the Interview question

Still, the children’s work was curious to me. Their behaviours were conforming to a pattern somewhat, and yet because the children tended to work independently, there was not a lot of talk to eavesdrop on. Likewise, I was hesitant to ask questions while children worked for fear of distracting them or redirecting them inadvertently. Clearly the act of drawing (or the drawings themselves) were helping, but I could not be sure how. I decided to ask a specific question between the two tasks to try to elicit some thinking from the children. So, between the Cookie Problem and the Wheels Problem, I asked the following question:

What strategy did you use to solve the problem?
How did it help you?

I constructed this question carefully, not wanting to suggest to children that drawing would help, so asked instead what had helped them, and then inquired as to how. The children answered that drawing had helped them; their descriptions of how varied. Of those able to articulate an answer, analogous thinking emerged (“Drawing is like...” “Drawing is...”). Student thinking is classified and discussed later in this paper. During the course of the study itself however, students’ description of both their drawings and how the act of drawing had helped allowed me to observe their drawing-as-problem-solving in a different way, and to anticipate actions I had not up to that point.
AFTER THE STUDY

Analysis of the information - How I described and recorded events

After the study I gathered student work and made several copies, classifying some according to strategy used or sophistication. I soon realized however that the task of assessing the students' product was not going to provide me with information on drawing-as-problem-solving behaviours. In order to assess those behaviours I had to watch and analyze the videos.

I watched each of the handheld camcorder videos and timed and annotated events by child. Each event (shift in drawing, a new phase of problem solving) was described with a running commentary using anecdotal language according to characteristics. Student commentary was recorded verbatim and key clips were cropped and saved for later viewing. As I watched more video I established a code and a more sophisticated system for recording the behaviours that resurfaced. My own comments or questions were added in a separate column as a way of keeping track of emerging questions or themes.

Below is a sample from early video analysis notes from Group 1A. It shows data from the Cookie Problem and how the 4 children responded within a particular time period. Student quotes and what they did in response to the task are noted; my questions or reflections are contained within the final column. Note the theme column where I attempted to assign a label to the observed behaviours.
Table 1 – Video Data Recording Form – Excerpt from Group 1A

The Cookie Problem: What students did, what they said

<table>
<thead>
<tr>
<th>Time</th>
<th>Observations</th>
<th>Behaviour</th>
<th>Questions, comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>CLIP2</td>
<td>Mark – eyes raised, finger counting – “Trying to count by like stuff – 3’s and 2’s and ones” Did any of those things help? “Not really.”</td>
<td>SPATIAL IMAGERY</td>
<td>More spatially oriented than others? Drawing did not help him, but was using a system in his head</td>
</tr>
<tr>
<td>2:30</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>CLIP2</td>
<td>Joanna – drew two sets – one group of cookies, one group of children circled groups of cookies in 2’s, split some. Asked “what age are the children?”</td>
<td>SET CREATION</td>
<td>Looking for proportional relationship – small kids, less cookies Fairness an issue</td>
</tr>
<tr>
<td>3:28</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>CLIP2</td>
<td>Melissa: “This is going to take a while.” What? “Drawing these pictures!”</td>
<td>PICTORIAL USE OF DRAWING</td>
<td>Drawing not simplified – still pictorial</td>
</tr>
<tr>
<td>1:47</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>CLIP2</td>
<td>Sara drawing faces on people</td>
<td>PICTORIAL USE OF DRAWING</td>
<td>How much was her solution affected by other children? Eyes on Melissa, Joanna, but then abandoned their solution ideas for her own</td>
</tr>
<tr>
<td>4:00</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>CLIP2</td>
<td>Melissa – numbered children 1-12. Then numbering cookies on front of page to match/represent distribution. “They all get one cookie but then there’s extras. I haven’t thought about that yet. I’ll think about that after I do that.”</td>
<td>DISTRIBUTION KEEPING TRACK</td>
<td></td>
</tr>
<tr>
<td>4:06</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
The descriptors and their origins

The preceding format was labour intensive. It split up the children’s responses by time and made reflecting on a single child’s behaviours challenging. As behaviours surfaced multiple times, the list of "behaviours" became overwhelming and distracting. In all, I had observed and identified 14 different behaviours, listing them in the order of their occurrence within the first group’s session. They were:

- Creating sets
- Distribution
- Keeping track
- Elimination
- Check and recheck
- Counters
- Solution representation
- Spatial imagery
- Pictorial use of drawing
- Set creation
- Dynamic imagery
- Visualization
- Eyes up
- Iconic use of pictures

This list of behaviours grew out of the children’s performance in response to the problems. Like assessing any performance task, I could not describe student performance until I had seen it. And, as each new behaviour was observed, it was added to the growing list above.

An attempt was made to organize the list and to establish commonalities within it. Upon reflection, there arose 4 categories or themes under which each of these behaviours fell. Duplicate behaviours were deleted and amalgamated under one theme heading.
### Theme and behaviour

<table>
<thead>
<tr>
<th>Virtual manips</th>
<th>Imagery</th>
<th>Sophistication of representation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Creating sets</td>
<td>1. Dynamic imagery</td>
<td>1. Pictorial use of drawing</td>
</tr>
<tr>
<td>2. Distribution</td>
<td>2. Visualization</td>
<td>2. Iconic use of pictures</td>
</tr>
<tr>
<td>3. Counters</td>
<td>3. Eyes up</td>
<td>3. Abstract thinking</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Systems</th>
<th>Represented</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Keeping track</td>
<td></td>
</tr>
<tr>
<td>2. Elimination</td>
<td></td>
</tr>
<tr>
<td>3. Check and recheck</td>
<td></td>
</tr>
</tbody>
</table>

The table on the following page (Table 2 – Revised Video Data Recording Form – Excerpt from Group 4A) shows annotated video data from Group 4A using these new themes and behaviours. It outlines the Cookie Problem, Trent’s response to it and my questions about his response. Note the numbers in each theme column, which correspond to the behaviour observed.
### Table 2 – Revised Video Data Recording Form – Excerpt from Group 4A

#### The Cookie Problem - What students did, what they said

<table>
<thead>
<tr>
<th>CLIP SAVED AS</th>
<th>Time</th>
<th>Behaviour noted and Comment</th>
<th>VIRTUAL MANIPS</th>
<th>IMAGERY</th>
<th>SOPHISTICATION</th>
<th>Questions, comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Trent’s image</td>
<td></td>
<td>Trent – very faint finger counting movements, looking up... quiet... “I don’t know what to write.” Audio only – camera pointed at the floor… Trent took me off camera to say: “I counted up to 12 in my head and then I counted up to 18 - 13, 14, 15, 16, 17, 18...” (showing me on his fingers how he counted) “then I counted the cookies split in 2 – 2, 4, 6, 8, 10, 12.” That works, doesn’t it? “whispering... - “What should I write??” Can you just write what you said? He reads his solution aloud...</td>
<td>1, 2, 3</td>
<td></td>
<td></td>
<td>Not putting anything on paper... risk taking? He motions me away from the group to ask if his idea works – does NOT want to be on camera and wrong... Clearly Trent has solved the problem BEFORE he begins to write it down. This solution is very sophisticated and completely visualized. Dynamic imagery.</td>
</tr>
<tr>
<td>CLIP2</td>
<td>0:45</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Trent explains and re-thinks</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Influence of other research

Two of these sets of descriptors were derived from other research. The terms iconic and pictorial refer to the type of drawing that children created in response to the problem situation. These terms speak to the artistic detail and sophistication of the drawing. Pictorial as a descriptor was borrowed from the work of Diezmann (2000), Smith (2003) and Woleck (2001). Pictorial representations are detailed and idiosyncratic; that is, they are closely if not specifically matched to the context of the problem. In contrast, iconic representations are more abstract, generalizable, and feature few context-specific details. This terminology and definition was drawn from the research of Tall (1999), Gray, Pitta and Tall (2000) and Thomas, Mulligan and Goldin (2002).

Imagery, including dynamic imagery and visualization, references the work of Owens et al (1996, 1999). Owen describes imagery as a mental process that includes "...the recognition of shapes and parts of shapes, the transformations of images in the mind, the visual analysis of shapes and mentally viewing shapes from other perspectives" (Owens, Mitchelmore, Outhred & Pegg, 1996). Normally associated with geometric understandings, imagery has a role to play in the resolution of mathematical problems. Wheatley and Cobb state that:

Mathematical problem solving is often a matter of reasoning analytically, ... a process of building from images to analysis and analysis to images [that] may continue through many cycles. (1990, p. 161).

In describing this state of change, of cycling through problem solving, Wheatley and Cobb refer to the dynamic nature of image-making and its potential for supporting reasoning through mathematical problems.
Mark's performance was a good example of what Wheatley calls dynamic imagery. His upraised eyes and finger counting, as well as his oral description of the process he used all support the label of dynamic imagery. While one cannot assume that raised eyes is proof of visualization, it occurred often enough to warrant noting, even while looking solely through the handheld camera. The overhead camera allowed for more observation of this behaviour while Mark and other students solved the problems.

Another behaviour worth noting in students’ response to these problems was their use of gestures. Their gesturing – head bobbing, finger counting, pointing, and chopping – likewise provided evidence of the internal construction and manipulation of mental images. In Presmeg’s earliest research on teachers’ use of gestures to transmit visual information to their pupils, (1985, quoted in Presmeg, 2006) she stated that “teachers’ use of gesture was one of the surest indicators that they had a mental image that they were intentionally or inadvertently conveying to their students”. I would suggest that in this study, children’s use of gestures (whether conscious or unconsciously performed) was an indication that students were operating on a mental image in a dynamic way. Presmeg describes a gesture as a sign vehicle indicating an object in someone’s cognition. (2006) The inter-relation between gestures and mental objects – and the complementary way in which gestures support visualization and mathematical reasoning – made student gestures worth noticing and annotating in the course of this study.

Questions generated through analysis
While examining student work and recording the type and frequency of their responses to the task, I began to reflect on my own thinking, my
preconceptions and what I had expected to see in the students' performance. In particular, I began to wonder about the appropriateness of the task for certain children, those who were not able to use drawing-as-problem-solving because they had not created a stable image of the problem. I wondered as well about those visual spatial students who did NOT draw and the mental processes these students were using. This resurfaced my initial inquiry and renewed my curiosity about visual spatial thinkers and their capacity in terms of mathematical problem-solving.

Two further questions arose from this. How can children be supported in their efforts to problem-solve? And how can teachers be supported in recognizing drawing-as-problem-solving in all its forms? These issues will be addressed in the discussion chapter.

**Sorting and charting behaviours by group, by task, by behaviour**

Once video analysis was complete for all 11 groups of children, data was collated into one larger document in order to gain a sense of the frequency of the behaviours within and between groups. This document resulted in a consolidated list of the number of behaviours by type, by task and by group. Each of these was totaled to get a sense of which behaviours surfaced the most often, by which group(s) and how they impacted the relative success of the children. Some groups were more visual spatially oriented (Group 4B); some groups employed systems more readily (Group 3A); some groups drew pictorially (Group 3B). Examining these behaviours by task also highlighted how the nature of the task (the cookie problem as opposed to the wheels problem) invited different strategies. See Tables 3 and 4, Summary of Strategies by Group – Cookie Problem and Summary of Strategies by Group – Wheels Problem. The results of these tables will be discussed in Chapter 4.
Note: The numeral in each cell refers to the total number of occurrences of the behaviour or strategy used by any member of the group.

<table>
<thead>
<tr>
<th></th>
<th>3</th>
<th>2</th>
<th>2</th>
<th>3</th>
<th>5</th>
<th>4</th>
<th>1</th>
<th>3</th>
<th>2</th>
<th>2</th>
<th>2</th>
<th>2</th>
<th>3</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>4</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>4</td>
<td>3</td>
<td>1</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>4</td>
<td>3</td>
<td>1</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>4</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>3</td>
<td>4</td>
<td>1</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>4</td>
<td>3</td>
<td>1</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>4</td>
<td>3</td>
<td>1</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>4</td>
<td>3</td>
<td>1</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>4</td>
<td>3</td>
<td>1</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>4</td>
<td>3</td>
<td>1</td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Creating Sets</th>
<th>Distribution</th>
<th>Counters</th>
<th>Keeping Track</th>
<th>Elimination</th>
<th>Check &amp; Recheck</th>
<th>Dynamic Imagery</th>
<th>Visualization</th>
<th>Eyes Up</th>
<th>Pictorial</th>
<th>Iconic</th>
<th>Abstract Thinking Represented</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>4</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>4</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>4</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>4</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>4</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>4</td>
<td>3</td>
<td>1</td>
</tr>
</tbody>
</table>

Students' names

Stephanie, Shauna, Jorge, Iris
Hannah, Zarah, Susie, Jessie
Trent, David, Charlie
Ruby, Mona, Rebecca, Karl
Norman, Lorne, Cathy
Anthony, Avril, Orson
Priscilla, Martin, Larry, John
Jason, Lisa, Grant, Susanne
Mark, Joanna, Melissa, Sara

Table 3 - Summary of strategies by group - Cookie Problem
**Table 4** - Summary of strategies by group - Wheels Problem

Note: The numeral in each cell refers to the total number of occurrences of the behaviour or strategy used by any member of the group.

<table>
<thead>
<tr>
<th></th>
<th>Creating Sets</th>
<th>Virtual Manips</th>
<th>Systems</th>
<th>Imagery</th>
<th>Sophistication</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

**Students' names**

- Hannah, Zarah, Suile, Jessie
- Trent, David, Charlie
- Ruby, Mona, Rebecca, Karl
- Norman, Lorne, Cathy
- Anthony, Avril, Orson
- Priscilla, Martin, Larry, John
- Jason, Lisa, Grant, Susanne
- Mark, Joanne, Melissa, Sara
Determining contributors to success
From these more organized charts it was possible to do a somewhat more quantitative analysis. An assessment of the successful completion of the questions by task and by group followed, later compared to the type and sophistication of the strategies used. This was done in an attempt to correlate the type of drawing as problem-solving strategy and the sophistication of the drawings with a child’s capacity to arrive at a solution that approached 1 and a half for the Cookie Problem. Success was then more generally defined to include distributing all the cookies in some way (one each now, give the remaining 6 away).

“Success” could not be matched directly to a single strategy used or even the sophistication of the drawing, however. Rather, success seemed affected by a variety of interacting factors. The predisposition to use visualization and image-making in concert with drawing, and the application of a system to the solution of the problem had a significant effect on students’ capacity to resolve the problem. This will be explored in more detail in the discussion chapter.

Analysis of Interview data
Again, the students’ videotaped responses to the interview question (“What strategy did you use? How did it help you?”) were annotated, transcribed and analyzed. There emerged 2 sets of interview data, describing how much and why drawing was helpful to students. The first categorized students’ statements regarding the degree to which drawing had helped them solve the cookie problem. The second looked more closely at students’ statements to classify the way in which drawing had helped. Themes emerged in both sets of data; cross-referencing these two data sets helped to clarify how students who did not find drawing
helpful were using their drawings as opposed to those who were aided by their drawings or the act of drawing itself.
Chapter 4 - Results

Overview

Students completed 2 problems during the small group interviews, the Cookie Problem and the Wheels Problem. Each of the 11 groups of students and their responses to the problems are described below, with trends in student approach to the individual tasks. Following the “Observations by Task” section for each of the Cookie and Wheels Problems is an overview of the drawing as problem-solving behaviours noted during all of the problem-solving interviews. These behaviours have been collated and grouped into a drawing as problem-solving framework, with elaborated descriptions of each cluster of behaviours and exemplars to illustrate each one. This second section is entitled “Observations by Strategy” as it focuses on the particular drawing as problem-solving strategies used by the students across both of the problem-solving tasks.

Collation and analysis of this data helps to elaborate a response to the main research question: How do young children access visualization and imagery in their resolution of mathematical tasks? How does drawing as problem-solving support students in making sense of and reason through a mathematical problem?
OBSERVATIONS BY TASK
How children drew and how it helped them

The Cookie Problem – General Themes

Trends in the solution (product)

The first question posed of the children asked them to share 18 cookies among 12 children. Children employed several different methods to solve the problem, and twenty of the 33 children were able to arrive at “one and a half cookies each” as an answer. Of the thirteen who did not, dividing the 6 remaining whole cookies into fractional parts proved challenging. Rather than giving up, students proposed a range of alternate solutions for the problem. Some attempted to determine a whole number solution to the problem to avoid breaking up the cookies. Karl* thought that every second child should have 2 cookies, and Joanna asked which children were older so she could distribute more to the “bigger kids”. Other solutions dealt with the remainder by excluding or subtracting it from the whole. Martin suggested giving away the remaining 6 cookies to “some other children”; while Jessie decided that the 6 leftover cookies could be “for tomorrow”. Three children did not demonstrate an understanding of the concept of division as sharing, and as a result, were not successful in arriving at a solution that made sense. Their responses were very different from those described above as “alternate solutions” to the problem. Mona, for instance, drew two sets of pictures on her sheet of paper – 18 cookies and 12 children – and then added the two sets together to arrive at 30 as a response. When asked, “What does 30 mean?” Mona responded, “It’s what the answer is for the whole thing.” Implications for these three children will be addressed in the discussion chapter.

* All names in this document have been changed.
Table 5 – Summary of type of response against drawing or not

<table>
<thead>
<tr>
<th>Response type</th>
<th>Drew to solve</th>
<th>Did not draw</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5 cookies each</td>
<td>16</td>
<td>4</td>
</tr>
<tr>
<td>Alternate response</td>
<td>10</td>
<td>0</td>
</tr>
<tr>
<td>Mistaken response</td>
<td>3</td>
<td>0</td>
</tr>
</tbody>
</table>

Trends in the drawing (process)

Twenty of the 33 children interviewed arrived at the answer, most of them through drawing or some kind of manipulation of an image on the page. For those who drew to solve the problem, the approaches used followed particular patterns; most began by drawing sets of cookies and/or children and then operating on these images in some way.

Some students drew cookies and children then connected one to the other with lines (eg. Anthony – Figure 3); some children drew circles, counted out, labeled or partitioned off 12 of them, then split the remaining cookies in 2 parts (eg. Melissa – Figure 4), while still others drew
18 cookies, cut them all in half and distributed 3 halves to each “person” (eg. John). John explains his thinking below:
(Note: My comments to the students are in italics.)

You break all the cookies in half and then you have 36 cookies. Then you could give 36 to the kids.

You give 36 to the kids? Tell me about that part.

Cuz you break all the cookies in half in the middle and it gives you 36 halves. Then you give the 36 halves to the children until there’s no more. ...they get a whole and a half.

Children’s successful attempts at solving the cookie problem indicated that they understood the notion of sharing, and that they were concerned both with using up all the cookies as well as making sure each of the children got a fair share. All but four (Mark, Trent, Stephanie and Charlie) used drawing in processing the problem in some way, to represent the sets of children and cookies, to separate the dedicated whole cookies from the remaining six and in some cases to cancel out those cookies that had been “eaten”. Each of these drawing as problem-solving strategies is described in more detail in the “After Problem-solving” section of this chapter; one child’s use of several of these strategies is outlined in the next section.
One child's processing of the Cookie Problem

Orson uses drawing as a problem-solving strategy

Orson is a child who made several attempts at this problem. His problem-solving attempts were fairly typical of children in the cohort of 33, and so his performance on both the Cookie Problem and the Wheels Problem is described below. Following Orson through his particular process of thinking helps to elaborate a number of strategies and make the framework more clear. Field notes from video observation are included below to highlight key elements and to trace his processing of the problem.

Orson began to solve the cookie problem by drawing 18 cookies in a circle at the bottom corner of his page. "I'm drawing them on a plate," he told his friend. He counted out the cookies and put a small mark on each one, doubling up on three of the 18 cookies. "Are they like taking 12 away?" he asked me, explaining, "It says 12 there." (Figure 5a) At this point, Orson stopped and re-read the question. "This is hard..." he commented, (Figure 5b) then asked for a new sheet of paper, "to write lines."
Orson's first attempt seems to indicate he thinks this is a subtraction problem.

Orson has shifted his thinking somehow. Drawing lines is a way to distribute cookies – to assign them to an owner. What clicked for him in re-reading the problem?

Orson's strategy now involves counting, creating sets of 4 and distributing these sets by connecting lines to "children".

Orson's use of lines. Orson realizes he has run out of cookies to continue with this solution. Note the lines dividing cookies into sets of 4, and a line joining these sets to a child.

On his new sheet of paper, Orson drew 12 children (stick men) and sketched 18 cookies underneath. Beginning on the right side of the page, he counted and partitioned off 4 cookies with a line. He drew a line
connecting this set of 4 cookies to an individual child. (Figure 5c) He persisted in this way, creating 4 sets of 4 cookies and then he stopped, realizing he had run out of cookies. (Figure 5d) Embarrassed, Orson flipped his paper over.

I asked him to explain his thinking.

I drew 12 kids and 18 cookies and I made lines of 4 and then made a line to one person and then to the other persons.

Why did you choose 4?
Because I thought it would maybe be 4.

How are you going to change your idea now?
Make it maybe like 2.

<table>
<thead>
<tr>
<th>Figure 5e</th>
<th>Figure 5f</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="Image" /></td>
<td><img src="image2.png" alt="Image" /></td>
</tr>
</tbody>
</table>

Orson uses circles to enclose sets of one child and 2 cookies, starting with 2 cookies each, as he indicated he would try next.

Orson's self corrects and erases. "Only 9 kids can have cookies, so, I'll just erase one each [erasing]...that didn't work either!"

This time, Orson tried 2 cookies per child, but did not use the partitioning system he had employed before. Instead, he drew circles to enclose sets
of cookies and children. (Figure 5e) He used self-talk as a running commentary to guide his thinking: “Only 9 kids can have cookies, so, I’ll just erase one each. [erasing]...that didn’t work either!” (see Figure 5f) In the end, Orson settled on one cookie for each child and 4 left over for the next day as his attempts to erase extra cookies left him with only 16 cookies rather than 18, and, it appears, 14 children as well.

To view Orson working through the Cookie Problem, please watch Video Clip 1, entitled: “Orson’s flexible thinking”

In each one of Orson’s attempts to solve the problem, he used drawing as a problem-solving strategy to make sense of the problem; that is, Orson processed and tested out possible groupings (4’s then 2’s then 1’s) by drawing and redrawing his emerging ideas. It was through his drawings that Orson manipulated the elements of the problem. He created and then counted the items in each set, he made estimates, he distributed sets of cookies to children, he self-corrected and adjusted his thinking. In the end, Orson persisted for more than 15 minutes on this problem, drawing and thinking aloud, and presented three different drawing as problem-solving strategies.
OBSERVATIONS BY TASK

How children drew and how it helped them

The Wheels Problem – General Themes

Trends in the solution (product)
The wheels problem asked children to determine which vehicles could be represented by 18 wheels. Once again, the children solved the problem in a variety of ways. Drawing was used by most of the children to support them in keeping track of sets of wheels, and how many wheels they had accounted for in solving the problem. Children who used drawing as a problem-solving strategy for this task had to consider the multiple parts of the whole of 18 in assigning wheels to different vehicles. Children produced detailed drawings that closely resembled the actual object in solving this problem (ex. drawings of scooters, rollerblades, toy cars and trains).

Trends in the drawing (process)
Students used drawing as a means of keeping track of the number of wheels that had already been assigned to vehicles. David, for example, drew 2 cars, paused, then made a truck with 10 wheels to complete his solution. For David, drawing provided him with a way to record a cumulative total. This was confirmed when he recorded numbers at the top of the page to complement his pictures: 10, 14, 18. Lisa drew 18 wheels in a long line, then circled wheels and added vehicles above them, stopping when the 18 wheels were used up. (Figure 6 – Lisa’s line up)
Ruby paired drawing and strategies incorporated from other disciplines to solve the problem. She wrote the word for the vehicle she wanted to use, sketched the appropriate number of wheels beside it and then recorded a running total of the wheels used so far. (Figure 7 – Ruby’s chart) I asked her why there were no vehicles on her page. She replied, “I drew wheels. Just the wheels.” Did this help? How? “Cuz it was a shortcut. If I drewed the plane it would take too long, if I drewed the motorcycle, the truck and everything it would take too long for me to do it.”
Ruby's solution showed sophistication in thinking – and an acknowledgement that sets of circles can "stand for" a plane or car. Her systematic approach and logical thinking in approaching the problem were evident here.
One child's processing of the Wheels Problem

Orson uses drawing as a problem-solving strategy

Orson began to solve this new problem by asking: "Can we make a list? Can I make a web?" He generated a brainstormed list of possible vehicles (Figure 8a) and then chose carefully from the printed list, commenting: "I need 4 things with 4 wheels... 4+4+4+4 is 18." When I asked him to "prove it", Orson suggested 4 vehicles from his list and counted aloud using his fingers from 8 to 16 – "4, 8, 9, 10, 11, 12, 13, 14, 15, 16..." He paused, looked up and exclaimed, "It's 4 plus 4 plus 4 plus 4 plus 4 plus 2! A scooter!" then added the word "scooter" to his list. (Figure 8b)

Figure 8 – Orson works through the Wheels Problem

<table>
<thead>
<tr>
<th>Figure 8a</th>
<th>Figure 8b</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="Orson's list" /></td>
<td><img src="image2.png" alt="Orson's list" /></td>
</tr>
</tbody>
</table>
| Orson's list, in which he brings in his language-based strength to support him in problem-solving. Orson visualizes the wheels on each vehicle and counts aloud: "I need 4 things with 4 wheels... 4+4+4+4 is 18." | "It's 4 plus 4 plus 4 plus 4 plus 4 plus 2! A scooter!"

To this point, Orson had been using visualization as a problem-solving strategy (imagining the wheels attached to the vehicles), pairing his
mental image with his list. Orson's list allowed him to keep track of the possible vehicles so that he could then visualize the wheels and record his solution in a systematic way.

Next Orson drew the solution to the problem, making a picture of each of the vehicles, and counting the 4 wheels on each vehicle as he added each one. (Figure 8c) Lastly he added numbers and “plus” signs to the diagram. (Figure 8d)

Figure 8c

Orson draws pictures to prove his thinking. His drawings allow him to count to check; his list supports him in being systematic.

Figure 8d

Cam got 18 wheels for Christmas. What toys could he have gotten?

Orson’s completed page – note the addition statements.

Orson’s solution once again demonstrates a range of drawing as problem-solving strategies and flexibility in the application of those
strategies. Within this context, Orson uses strategies that were different from the ones he employed in the cookie problem: in particular, his use of visualization paired with the scaffold of the list and then pictures used to represent the solution rather than acting as the vehicle for thinking through the problem.
Summary - Children's use of drawing as problem-solving

As I observed the children while they solved these problems, I noticed several commonalities both in terms of the products they created and in the processes they used to solve the problems. Although the problems had different structures, children used drawing as a problem solving strategy in both cases to construct meaning for the task, manipulate the elements of the problems and arrive at a solution. The next section of this chapter outlines these drawing as problem-solving strategies and behaviours and the commonalities noted across both the cookies and wheels problems.

A table summarizing the occurrence of these strategies and the frequency of their occurrence follows on the next 2 pages.
Note: The numeral in each cell refers to the total number of occurrences of the behaviour or strategy used by any member of the group.

<table>
<thead>
<tr>
<th></th>
<th>CREATING SETS</th>
<th>DISTRIBUTION</th>
<th>VIRTUAL MANIPS</th>
<th>COUNTERS</th>
<th>KEEPING TRACK</th>
<th>ELIMINATION</th>
<th>CHECK &amp; RECHECK</th>
<th>DYNAMIC IMAGERY</th>
<th>VISUALIZATION</th>
<th>EYES UP</th>
<th>PICTORIAL</th>
<th>ICONIC</th>
<th>ABSRACT THINKING REPRESENTED</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td>0</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>0</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>1</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Stephanie, Shauna, Jorge, Hannah, Zarah, Susie, Trent, David, Charlie, Ruby, Mona, Cathy, Norman, Lorne, Cathy, Anthony, Avril, Orion, Priscilla, Melissa, Larry, Jason, Lisa, Grant, Susanne, Karl, Andrew, Sarah, John, Mark, Joanna, Melissa, Sara, Mary, Philip, Melissa.
Note: The numeral in each cell refers to the total number of occurrences of the behaviour or strategy used by any member of the group.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>9</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 7 - Summary of strategies by group for the Wheels Problem
OBSERVATIONS BY STRATEGY

Creating the framework
Children’s drawing as problem-solving strategies and their frequency will be examined in some detail below. Through examination of the video data, 4 clusters of drawing as problem-solving themes including student behaviours and drawing characteristics were defined: virtual manipulatives, systems, imagery and sophistication. What follows is a description of the cluster of indicators by theme, and then examples – print and video samples – of each type of strategy within the cluster.

Within each section is a chart indicating the cluster of indicators by problem, matched to the groups of children who demonstrated these behaviours. Together, these 4 clusters make up a framework of drawing as problem-solving behaviours and characteristics.
Virtual manipulatives

Within this set of behaviours, the most commonly occurring cluster of behaviours, children used their pictures, sketches or representations as manipulatives – like counters to be moved or acted upon. Table 8 – Summary of Virtual Manipulative use for the Cookie and Wheels problems (all groups) – provides a summary of these actions for the groups, organized by question. Note that the numeral in each cell refers to the total number of occurrences of the behaviour or strategy used by any member of the group. For example, within the first group, only one of the children used their pictures like counters to solve the Wheels Problem, while in the Cookie Problem, there were 4 occurrences observed within the group.

Table 8 – Summary of Virtual Manipulative use for the Cookie and Wheels problems (all groups)

<table>
<thead>
<tr>
<th>VIRTUAL MANIPULATIVES</th>
<th>COOKIES PROBLEM</th>
<th>WHEELS PROBLEM</th>
<th>Students' names</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>CREATING SETS</td>
<td>DISTRIBUTION</td>
<td>COUNTERS</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>
Through observation during problem solving I established that children were using their drawings like physical objects. The notion of action is key here; children’s solution-finding and reasoning was supported by modeling the action in the problem. As a virtual manipulative, pictures, tallies, circles and cookies could all be transformed. Children who demonstrated these behaviours drew both in iconographic and pictorial style – that is, the sophistication of their drawings did not preclude their use of virtual manipulative strategies. Lisa, for example, drew very elaborate faces for the cookie problem and used those pictorial representations as counters in her solution-finding, whereas John drew simple circles to represent both cookies and children and “moved” them with lines. Both processes involved using pictures like manipulatives, but the level of abstraction of those pictures differed.

Creating sets
Children who created sets organized their representations into distinct groups. Sometimes these groups were circled or boxed; occasionally the sets were organized in rows or columns, or drawn on one side of divided page or another. The point of creating sets was to organize the elements of the problem in a visual way so that an action could take place on them. Circles or boxes helped children keep track or separate the elements of the problem. The following examples show clearly the elements of the problem in 2 separate groupings; one represents a developed solution (Figure 9 – Sara’s sets) the other shows Shauna’s groups and final solution (Figure 10 – Shauna’s sets).
Another form of creating or delineating sets is shown in David's work. David drew 18 cookies in an initial set, then counted out 12 of them, marking each one with a dot, and then drew a dividing line to create a set of eaten and a set of uneaten cookies. When asked to tell about this last mark on the page, he responded: “The line is stopping it there.” He then wrote numbers to the left and the right of the line (12 and 6) to show how many whole cookies were on each side. (Figure 11 – David’s Line) Drawing was important here because it allowed David to delineate between the parts of his solution; in this case to mark how many of the whole cookies had been allotted to children and which ones would have to be treated differently. This is an example of drawing-as-problem-solving; it shows the purposeful use of a line, which David himself describes as “stopping it”. “It” might be interpreted as the number of children in possession of whole cookies. David later confirmed that the strategy that helped him the most in solving the problem was: “When I drew the line.”
There are 18 cookies on a plate and 12 children who want to share them. How can the children share the cookies? How much will everyone get?

David draws a set of 18 cookies—the 12 children are understood. The vertical line, says David, divides the whole cookies from the half ones: "The line is stopping it there."

**Distribution**

Distribution is an action, characterized by the "moving" of the elements of a problem. Sometimes the movement was accomplished by tracing lines from one set of objects to the items in another set; thereby distributing or sharing out each of the items. Distribution applied to both whole and partial cookies as a way of allocating the set. In order to establish the response to the problem, each of the lines connecting the portion amount to the receiver then had to be counted, and the shared portion (the quotient) verified for fairness (see Figure 12—Anthony distributes). This drawing as problem-solving strategy lent itself well to division problems, where the action of sharing was replicated by the physical distribution of items. Larry's work gives a clear indication of this. (Figure 13—Larry distributes) During the process of solving this problem, Larry lost track of the cookies he had distributed and used finger counting to verify his idea. "This is tricky," he commented. "Oops! I forgot to join this."
Pricilla’s thinking

It’s lines, so I can help. So I can think. (counting and then boxing groups of lines)
What are the lines for?
So I can count them.
So when you put those lines there, Pricilla, what are you counting?
How many kids will have cookies.
So each of those lines is a cookie?
Ya.

Video of Jason shows a range of drawing as problem-solving strategies, including the use of drawing as counters. In the clip indicated below, Jason counts concurrently with his finger and his pencil, matching cookies to children (Figure 15 – Jason keeps track with tallies). The marks on the page (simple lines for people) are not part of his solution but rather are the tools for arriving at that solution. To watch Jason as he keeps track, please view Video Clip 2, entitled: “Jason keeps track with tallies”
Systems

The following strategies describe ways in which children approached the task systematically. Each of these strategies was developed and applied spontaneously by the child. The drawings employed in each case were either the object of the system (they were “operated on” in a systematic way) or the drawings or symbols were the system itself (tallies, etc). Even children who struggled with the problems initially had more success once they adopted a systematic approach than those who attempted other strategies. Table 9 outlines the different systems used by the children and their frequency. As before, the numeral in each column indicates the number times each strategy was used by members of the group. For example, one of the members of the first group used an elimination strategy in the resolution of the Cookies Problem, but no one in the second group used the check and recheck strategy to solve the Wheels Problem.

Table 9 – Summary of Systems use for the Cookie and Wheels problems (all groups)

<table>
<thead>
<tr>
<th>SYSTEMS</th>
<th>COOKIES PROBLEM</th>
<th>WHEELS PROBLEM</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>KEEPING TRACK</td>
<td>KEEPING TRACK</td>
</tr>
<tr>
<td></td>
<td>ELIMINATION</td>
<td>ELIMINATION</td>
</tr>
<tr>
<td></td>
<td>CHECK &amp; RECHECK</td>
<td>CHECK &amp; RECHECK</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>3</th>
<th>1</th>
<th>3</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mark</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Joanna</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Melissa</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sara</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Jason</td>
<td></td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lisa</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Grant</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Susanne</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pricilla</td>
<td>1</td>
<td>4</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>Martin</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Larry</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>John</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Anthony</td>
<td>3</td>
<td>4</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Avril</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Orson</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Norman</td>
<td>4</td>
<td>3</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Lorne</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cathy</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ruby</td>
<td>1</td>
<td>0</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>Mona</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Rebecca</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Karl</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Trent</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>David</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Charlie</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hannah</td>
<td>5</td>
<td>1</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>Zarah</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Susie</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Jessie</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Stephanie</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Shauna</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Jorge</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Iris</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Students' names

Mark, Joanna, Melissa, Sara
Jason, Lisa, Grant, Susanne
Pricilla, Martin, Larry, John
Anthony, Avril, Orson
Norman, Lorne, Cathy
Ruby, Mona, Rebecca, Karl
Trent, David, Charlie
Hannah, Zarah, Susie, Jessie
Stephanie, Shauna, Jorge, Iris
Keeping track

Children operated on their drawings in this case. Keeping track behaviours occurred when children labeled or numbered their drawings, to keep track of which ones had been operated on, distributed or used up. Sometimes these labels or numbers were added initially along with the pictures. Other times, they were added after the initial set was drawn as a way of verifying or recording a partially completed mental strategy.

<table>
<thead>
<tr>
<th>Figure 16 – Cathy’s numbers</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Cookies and Kids</strong></td>
</tr>
<tr>
<td>There are 18 cookies on a plate and 12 children who want to share them. How can the children share the cookies? How much will everyone get?</td>
</tr>
<tr>
<td><img src="image" alt="Cathy's numbers" /></td>
</tr>
</tbody>
</table>

Here, Cathy has numbered her cookies in wholes and parts as a way of keeping track of which child is allotted which cookie or portion of it. In her solution, the numbers serve 2 purposes – one as a strategy for keeping track and the other as a way of labeling the children, with the numbers from 1 to 12. In each case, the number corresponds to the child who will eat the cookies.

Circling or boxing objects was another indication that children were using drawing to keep track. The drawing as problem-solving observed in this case was fairly abstract (ie – in the form of the box or the circle), but it served the purpose of separating those objects that had been operated on from others. (Figure 14 – Pricilla’s counters, above) Below, Susanne’s drawing (Figure 17 – Susanne’s box) shows how she thought about the problem and used drawing to keep track of cookies – and to show the cookies she has “taken away” to the right of the boxed cookies. Later, her box serves to separate whole cookies (a single cookie allotted to a
single child) from partial ones (split in half and distributed to children by an identifying number). (Figure 18 – Susanne’s box 2)

<table>
<thead>
<tr>
<th>Figure 17 – Susanne’s box</th>
<th>Susanne’s thinking</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>“I drew 18 cookies and I counted them – to 12 – and then I took these away...”</td>
</tr>
<tr>
<td></td>
<td>Susanne’s box indicates a set of cookies (12) given as wholes to the 12 children – not pictured, but suggested by the drawing.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Figure 18 – Susanne’s box 2</th>
<th>In the second part of her solution, Susanne explains:</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>“I cut them in half, like counting by 2’s, - 2,4,6,8,10,12 ...12 of the half cookies.”</td>
</tr>
</tbody>
</table>

Still more abstract was the use of tallies to keep track of objects moved, operated on, or assigned to a place. For some, these marks were central to the solution, or represented the solution-finding process in its entirety (Figure 29 – Grant’s tallies, below). Other times the use of a tally was a supplemental system, used to support the allotment of items, as in Pricilla’s work. (Figure 19 – Pricilla’s wheels)
Check and recheck
Check and recheck as a system required children to return to their original set of drawings over and over, to count and re-count the items to verify or disprove a partial solution. This system was akin to “guess and check” in drawing form, but allowed children the scaffold of going back over their thinking during problem-solving. In some cases, check and recheck behaviours allowed children to self-correct. Check and recheck as a drawing as problem-solving strategy supported children in navigating through their own processing of the information in the problem. Here, (Figure 20) Jessie circles pairs of cookies to distribute to the children. He checks his work partway through, then stops.
**Elimination**

As a drawing as problem-solving strategy, elimination was a fairly sophisticated system. Elimination required the use of 2 separate but concurrent actions – the movement of one object and the canceling of another. Children developed a range of strategies for tracking which items had been “used up”, including erasing, crossing out, stroking off or colouring in objects. To observe Sara using her system of elimination, please view Video Clip 3, entitled, “Sara eliminates”.

<table>
<thead>
<tr>
<th>Figure 21 – Sara eliminates</th>
<th>Sara’s thinking</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image" alt="Sara eliminating" /></td>
<td><em>Tell me about these x’s here.</em></td>
</tr>
<tr>
<td></td>
<td><em>That means people have those cookies.</em></td>
</tr>
</tbody>
</table>
Of all the systems used, this one was the most supportive of children's solution-finding efforts; that is, when children used the elimination system they were more likely to arrive at a correct answer, even after several false starts. Norman's work (Figure 22) is below. He began by drawing sets of cookies and then children. He worked doggedly for 11 minutes and employed a range of strategies, but only solved the problem once he used elimination to help him (colouring in each of the 6 remaining cookies while re-writing them as a fraction above the receiving child's head).

<table>
<thead>
<tr>
<th>Figure 22 – Norman’s work</th>
<th>Norman’s thinking</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image" alt="Figure 22" /></td>
<td>I draw 18 cookies and I accidentally draw 22 so I crossed it out, and I draw 12 kids and I put every number in the cookies and then I put the same amount of cookies and when I got to 12, ... and then I crack two and I got one half... and I shade it in like this...</td>
</tr>
</tbody>
</table>

*How did the shading in help you?*

*It doesn’t make me ‘lost’ count...*
Imagery

Imagery or visualization behaviours were indicated through the children’s oral descriptions of what they were thinking (I can see it..., you just break the cookies..., you put these ones and those ones together...), in student hand signals or gestures (chopping, slicing, separating groups, motioning “over there”), or wordlessly through upraised eyes, or murmuring and counting on fingers. Table 10 outlines student use of these observable indicators of image-making and instances of visualization and dynamic imagery use as reported by the children.

Table 10 – Summary of Imagery use for the Cookie and Wheels problems (all groups)

<table>
<thead>
<tr>
<th>IMAGERY</th>
<th>COOKIES PROBLEM</th>
<th>WHEELS PROBLEM</th>
<th>Students’ names</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>VISUALIZATION</td>
<td>DYNAMIC IMAGERY</td>
<td>EYES UP</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>
Children who used these strategies did so to support solution finding. The timing of the use of imagery varied – some children visualized initially to make sense of the problem (Jason, Shauna), some stopped to visualize when they became stumped, (Iris, Grant) others used only visualization to solve the problem. This last group of children did not draw anything on their paper to help them – no tally marks, no sets of counters – instead they recorded only their solution to the problem. (Charlie, Trent, Mark, Stephanie)

**Eyes up**

Any time a child was seen to be “thinking”, with eyes raised upwards and to the right, this was counted as an occurrence of “eyes up” behaviour. “Eyes up” was the observable indicator of image-making or visualization. (Figure 23 – Eyes up) Although many children raised their eyes to “think” while problem-solving, students were not always able to describe what they were doing or what they were “looking at” when raising their eyes (Iris, Cathy, Lorne).

Figure 23 – Eyes up

| Grant | Lorne | Jason |
Visualization

This behaviour was observed when children described “seeing” sets of objects in their heads and then imagining an action, or an operation being performed on them. Children who used visualization as a problem-solving strategy did not always draw the image they created and used mentally. Rather, they could “see the answer” or the process to follow in their heads.

Some children used mental strategies in concert with drawing to sort out how many cookies each child should get (Iris, Jason). Although these children struggled to explain their process, their intensity, their raised eyes, gestures and head-bobbing indicated they were working hard to manipulate the elements of the problem. Iris used her picture and visualizing strategies to solve the problem (Figure 24) and then talked about what she did:

Iris’s Thinking

There are 12 kids so I kinda count 12 and the other ones I leaved them there and... there’s 6 left... and 6 plus 6 equals 12... so if you cut them in half there’s 2 of each one so I made each one have half of it so all of them would have one and a half.
Four children were successful in their attempt to solve the cookie problem, but did not draw pictures or representations to support their thinking. Rather, these children solved the problem visually using mental images and then recorded their answers afterwards. (Mark, Trent, Charlie, Stephanie). When I observed Mark with his eyes raised to the ceiling, counting on his fingers, I asked what he was thinking. He responded: “I’m trying to count by like stuff – 3’s and 2’s and ones...” He returned to the task, gazing upwards, counting on the fingers of his other hand. Stephanie too spent time looking up at the ceiling, once she had drawn 18 cookies on her sheet. I asked her what she saw in her head when solving this problem. “Just see like cookies and kids and like and see like, how can you share them in your head.”

Trent’s response to the cookie problem is a good example of “seeing things in your head”. When asked to solve the problem, Trent grew quiet, and looked up at the ceiling. He wrote nothing on his page for a few minutes and then called me over to him – he motioned for me to follow him away from the table and away from the overhead camera. I kept the handheld camera rolling to record his voice, but pointed it at the floor. Whispering, Trent told me that everyone would get 1 and a half cookies – and then explained how he had solved the problem:
Trent's thinking:

I counted up to 12 in my head and then I counted up to 18 - 13, 14, 15, 16, 17, 18...

(showing me on his fingers how he counted)

Then I counted the cookies split in two - 2, 4, 6, 8, 10, 12

Trent's recording of the solution (Figure 25) is a printed version of the process he followed in his head to solve the problem, and indicates his use of visualization as a problem-solving strategy.

**Dynamic imagery**

When students employed dynamic imagery in the solution of these problems, movement or change in the elements of the problem were key features. However, unlike the distribution behaviour (as described in the virtual manipulatives section), dynamic imagery was a mental process that did not result in a drawing. The manipulation and division of the cookies, the design of the vehicles was done through pictures in the students' heads – pictures some children described as "movies". The students' hands and fingers moved to mirror the action taking place in their heads. When asked, students could describe the movement of the objects and what was happening to them.
Mark arrived at solutions to both problems through dynamic imagery. For the cookie problem he sat without writing or drawing for over 11 minutes, although his fingers moved as did his lips while he murmured. He explained that each child would get 3 halves.

If you cut six cookies in half everyone gets half a cookie, but if you cut six cookies three times – cuz 6 plus 6 plus 6 is 18 – so you cut six cookies in half and you cut 6 cookies in half and you cut 6 cookies in half so everybody gets 3 halves of a cookie.

Where did you get that idea from?
It just came in my head.

For Mark, the image did more than just “come into his head”. The image of the cookies and children was pictured in his visual working memory, where he manipulated, cut and distributed the cookies in 3 sets of 6 and allotted each child 3/2 cookies. An in-depth look at Mark’s problem-solving process and links to video evidence follows in the discussion chapter.
Sophistication of representation

Indicators grouped under this heading speak to the degree of complexity of the drawing – the detail included and the sophistication of the thinking involved. In some cases, drawings were little more than marks on the page (tallies, lines, circles) while in other cases the representations were very realistic, full of detail and design. In this section, artistic drawings are contrasted with more abstract representations. Table 11 summarizes these characteristics of student representations.

Table 11 – Summary of Sophistication of the Representation for the Cookie and Wheels problems (all groups)

<table>
<thead>
<tr>
<th>SOPHISTICATION OF REPRESENTATION</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>PICTORIAL</strong></td>
</tr>
<tr>
<td>COOKIES PROBLEM</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>4</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>3</td>
</tr>
</tbody>
</table>
Pictorial use of drawing

Pictorial drawings closely resembled reality and contained elaborate detail. Children's work was "pretty" – and included artistic renderings or cartoon-like pictures. (see Figures 26 and 27) Contextually, these drawings were very closely linked, and idiosyncratic in their production.

Figure 26 – Lisa's people

Figure 27 – Lisa's people 2

Pictorial drawings took more time and effort to draw than more iconic representations – for Lisa, drawing her elaborate people took more than 7 minutes, and in her final drawing (Figure 27), we see that Lisa miscalculates, dividing the 18th cookie into 6 pieces. Melissa noted herself that pictorial representations were time consuming: "This is going to take a while, ... drawing these pictures!" For some, the production of these pictorial representations was a way of drawing the solution rather than processing the information in the problem; that is, the image on the page was intended by the child to be the product of the problem-solving event.
rather than process-related. For others, there was a greater degree of processing, thinking through the images produced – although the production of these images did indeed take more time.

**Iconic representations**

Iconic representations were less detailed than pictorial ones. In this kind of drawing, images were more abstract and generalizable beyond this problem – people were represented as circles or sticks or numbers, vehicles were shown by lines or circles. (See Figure 28 – Martin’s vehicles, and Figure 29 – Shauna’s wheels) Iconic representations are simple, but they demonstrate sophistication in their application; these simple images stand for another more complex one. The iconic representations used by the children were short-term memory aids, a temporary place holder or marker to support solution finding. Their simplicity and lack of detail allowed children to process information without becoming distracted by detail.

<table>
<thead>
<tr>
<th>Figure 28 – Martin’s vehicles</th>
<th>Figure 29 – Shauna’s wheels</th>
</tr>
</thead>
<tbody>
<tr>
<td>Martin has produced only wheels and distributes them into “vehicles”. Next, he names the vehicles.</td>
<td>Shauna begins with words to describe her vehicles, and, like Ruby, draws wheels as placeholders for her to count.</td>
</tr>
</tbody>
</table>
Abstract thinking represented
Sophisticated mathematical thinking was evident in the students' processing of their task, at times displaying an unexpected level of abstraction. For example, there were children who represented the solution to the cookie problem as a single unit; that is, children showed the answer to the problem as a single thing – a plate with one and a half cookies on it. (Sara, Jorge) Other children used a short cut to simplify the amount of recording necessary or to keep track of multiple objects with a single image or set of characters. Like an “x” in an algebraic expression, this image stood for multiple other sets of objects. Children who used a short cut demonstrated a level of sophistication in terms of their mathematical thinking; their drawing was not necessarily sophisticated, but their thinking certainly was.

In the Cookie Problem, Grant asked questions about the quality of the drawings. “Do the pictures have to be like real nice and all that?” and then encouraged his peers to draw simply: “Just draw happy faces,” he told Lisa. In the wheels task, Grant developed a system for representing his vehicles that did not involve drawing – a system that actually matched his thinking about the problem. (Figure 30 – Grant’s tallies) Grant asked if he could “use tallies for cars”. He wrote the word cars on his paper and then recorded tallies, counting 4, 8, 12, 16 as he drew each one. He drew a fifth tally stroke through his first four, then paused and erased the line, realizing he had made too many and needed something with just 2 wheels. Instead, he wrote “dirt bikes” and a single tally to complete his solution. I asked him what he had done.
I wrote "4 wheels for cars" so I drew 4 tallies. It ends up with 16 wheels.

*Every one of those tallies means what?*

Four, like instead of drawing 4 wheels...

*So if I was going to read this...*

(Grant, touching tallies) So that's eight – these two - and that's 12, that's 16.

And a dirt bike has 2 wheels so after 16 plus 2 is 18.

*And this little line here?*

It's a tally.

*It's a tally for just one dirt bike?*

*Ya.*

Grant's process shows a different type of thinking; more abstract than that of his peers who drew each vehicle and all the wheels on them. Grant's tallies represent a single vehicle and its wheels. His solution is multiplicative in its representation as opposed to additive like David's cumulative total or subtractive like Lisa's line of wheels; he counts by 4's and then adds 2 to find the solution. Grant's thinking about the problem demonstrated level of sophistication both in its process and in the abstraction of his drawings.
To listen to Grant's explanation of his thinking, please view Video Clip 4, entitled "Grant talks tallies".
The Drawing as Problem-solving Framework

Summary of the findings

There were four clusters of drawing-as-problem-solving behaviours observed in the children’s work that contributed to the creation of the framework. While the sophistication of representation cluster speaks more specifically to the abstract nature of the product, the virtual manipulatives and systems clusters describe how children used their drawings and the act of drawing to process the elements of the problems. Imagery described the visual thinking that students engaged in while solving these mathematical problems. A discussion of these categories and implications for classroom practice follow in the final chapter.

These four categories or clusters of drawing-as-problem-solving behaviours and characteristics were observed across both the Cookie Problem and the Wheels Problem. In both cases, the behaviours or characteristics of the drawings assisted children in processing the information in the problems. The frequency of their application differed, however. In Table 12 – Frequency of Drawing as Problem-solving Behaviour and Characteristics by Cluster of Themes – these trends were noted. Numerals in the columns are an indication of how many times children in the cohort of 33 were observed performing these behaviors (e.g. Children were observed 9 times with eyes upraised in the Cookies Problem).
Table 12– Frequency of Drawing as Problem-solving Behaviour and Characteristics by Cluster of Themes

<table>
<thead>
<tr>
<th>CLUSTER OF THEMES</th>
<th>FREQUENCY</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>COOKIES</td>
</tr>
<tr>
<td>VIRTUAL MANIPULATIVES</td>
<td>66</td>
</tr>
<tr>
<td>▪ Creating sets</td>
<td>25</td>
</tr>
<tr>
<td>▪ Distribution</td>
<td>16</td>
</tr>
<tr>
<td>▪ Counters</td>
<td>25</td>
</tr>
<tr>
<td>SYSTEMS</td>
<td>56</td>
</tr>
<tr>
<td>▪ Keeping track</td>
<td>20</td>
</tr>
<tr>
<td>▪ Elimination</td>
<td>13</td>
</tr>
<tr>
<td>▪ Check and recheck</td>
<td>23</td>
</tr>
<tr>
<td>IMAGERY</td>
<td>24</td>
</tr>
<tr>
<td>▪ Dynamic Imagery</td>
<td>3</td>
</tr>
<tr>
<td>▪ Visualizing</td>
<td>12</td>
</tr>
<tr>
<td>▪ Eyes Up</td>
<td>9</td>
</tr>
<tr>
<td>SOPHISTICATION</td>
<td>37</td>
</tr>
<tr>
<td>▪ Pictorial Representation</td>
<td>16</td>
</tr>
<tr>
<td>▪ Iconic Representation</td>
<td>15</td>
</tr>
<tr>
<td>▪ Abstract Thinking Represented</td>
<td>6</td>
</tr>
</tbody>
</table>

The Cookie Problem, the more complex problem of the two, required significantly more use of pictures as virtual manipulatives for creating sets, counting and distribution. Both problems required the use of systems; however in the Wheels Problem these systems were largely made up of keeping track and check and recheck behaviours (34/39 examples). In the Cookie Problem, children accessed elimination as a strategy 8 times more often.

The type of problem impacted the frequency of both visualization and pictorial representations in the case of the Wheels Problem. That is, children visualized the vehicles and drew them in detail for the wheels problem, whereas for the Cookie Problem they were equally as likely to
use iconic or pictorial representations in their work. Abstract thinking was more often seen with the more complex Cookie Problem than in the Wheels Problem that required a more straightforward response.

Visualization took the place of virtual manipulatives in the Wheels Problem situation, while in the Cookie Problem students applied both virtual manipulative strategies and visualization in concert to solve the problem.

The complexity of the problem and the nature of the problem structure influenced the ways in which children interacted with it. More distribution was seen in the division as sharing problem, while more “eyes up” visualizing behaviours and pictorial representations were noted in the open-ended Wheels Problem.
Interview data

In the interview data, there were four themes evident in students’ responses.

The first category included children who did not answer the question “How did drawing help?” These children described their process for solving the problem rather than talking about drawing as a problem-solving strategy; they were not able to reflect metacognitively about the question. The second and third groups included children who indicated that drawing helped them but could not say how, and then students who said that drawing as a problem-solving strategy had helped them to keep track of the parts of the problem and manipulate the pieces. A fourth group of students responded that drawing had not helped them at all as they had used “the pictures in their heads” to solve the problem instead.

Within children’s responses, there were likewise patterns in the reasons they cited for the helpfulness of drawing as a problem-solving strategy. Students described the ways in which they had used their drawings like manipulatives, to count, move, distribute or otherwise act out the problem. Children spoke about how drawing helped them to keep track, to test out an idea or how children were able to apply a system to their solution-finding through drawing.

A summary of students’ responses and the reasons for their responses follows. The table, (Table 13 – Interview data: Type of support and frequency) gives a numerical overview of the interview data for children who responded that drawing had helped them. This is followed by examples of each type of support, or how drawing had helped, as described by the children (Figure 30 – Sample interview responses by
category). A complete list of children's interview statements is contained within Appendix 1 – Interview Data.

Table 13 – Interview data: Type of support and frequency

<table>
<thead>
<tr>
<th>TYPE OF SUPPORT (HOW DRAWING HELPED)</th>
<th>NUMBER OF REFERENCES TO EACH TYPE OF SUPPORT BY STUDENT</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subject of an action</td>
<td>7</td>
</tr>
<tr>
<td>Distribution</td>
<td>5</td>
</tr>
<tr>
<td>Memory aid</td>
<td>7</td>
</tr>
<tr>
<td>Elimination system</td>
<td>3</td>
</tr>
<tr>
<td>Pictures as iconic representations</td>
<td>1</td>
</tr>
</tbody>
</table>

Examples of student's explanations of how drawing as problem-solving helped

Figure 31 – Sample interview responses by category

Drawings were the subject of an action; drawing is a mathematical act

Ruby says:
I drewed one cookie for each children then I saw the 6 leftover and I knew that I had to cut them into pieces...

Grant says:
Cuz you can draw faces and draw cookies to the faces... instead of just keeping it in your head.

Anthony says:
[Drawing is] ...like acting what's happening. Drawing's like acting because... the kids take cookies.

Orson says:
It's like acting because it's like what people do. Did you have any acting in yours, Orson?
Yes, like taking away, like giving people.

Jessie says:
...the drawing is sort of like doing the math. Like drawing the math. What part of the math were you drawing, Jessie?
It's like we're doing take-aways. Like I did one, there's 1, 2, 3 and 1, 2, 3, and it makes six. And it makes 12.
Drawing as an Elimination System

Norman says:

...and I shade it in like this.

How did the shading in help you?

It doesn’t make me lost count...

Drawing allowed for trial and error

Shauna says:

It helps because I can try different things, like 3’s or ones if that works.

Cathy says:

It helps me draw so like, uh, in case I do like four in half and it doesn’t work then like sometimes it can help me, so if I so if I have to cut it in half or just leave it...

Drawing as a Memory aid:

Zarah says:

It’s like you could look at the picture and do it with the picture instead of doing it in your brain and making it explode kinda.

Pricilla says:

It helped instead of in your head and get mixed up.

What did the drawing do to make it easier?

By looking at it and thinking, how much I can cross off?

Lisa says:

So when we don’t have to do it in our heads and doing it in our heads is harder.

Pictures as iconic representations

Lome says:

I could like draw math and stuff, like circles, except...they were cookies. Like if there were 15 houses..., I could probably do that, only like I wouldn’t draw full houses with so much detail, I’d use up all my time just using one house... I actually just draw circles for like everything.
Interview data: Summary of the findings

Children were inclined to draw in responding to each of these problems. This was an indication that students could see that using pictures would be a purposeful activity that would support them in meaning-making. Their capacity to describe the usefulness of the pictures or the act of drawing itself varied with the child’s ability to access language to describe these relationships. Several were able to explain not only how their pictures helped (in allowing them something to act on or an object to distribute, as a memory support) but also how drawing itself made problem-solving more manageable (using a system of elimination).
Chapter 5 – Discussion

Overview

This chapter provides the forum to examine the findings and to discuss the main research questions, namely: "How do young children access visualization and imagery in their resolution of mathematical tasks? How does drawing as problem-solving support students in making sense of and reasoning through a mathematical problem?" Results from the study reported in the previous chapter, as well as the children’s interview responses and the application of the framework for assessing and describing drawing as problem solving will be discussed here. An examination of the conditions contributing to successful drawing as problem-solving will also be considered, and implications for classroom practice will follow.

Reflecting on the problems as illustrative of drawing as problem-solving: A comparison of the Cookie Problem and the Wheels Problem

One question I sought to answer involved the degree of complexity of the problems presented for solution through drawing. Specifically, how complex did a problem have to be in order for a child to pick up a pencil and draw to reason towards a solution? Questions related to the complexity of the problem and the problem structure (What word(s), situations or actions in a mathematical situation lead a child to pick up a pencil and sketch that action?) are addressed below. In an attempt to clarify these issues, the following presents an examination of the Cookie and Wheels Problems, both in terms of their structure and difficulty level, and students’ responses to those problems.
The Cookie Problem, by virtue of the structure of the problem and the action of division as sharing allowed for a great deal of observable data in terms of drawing as problem-solving. Children had to share out the cookies, and in so doing had to mirror a physical process of distribution through drawing. Since 6 of the 18 cookies had to be split in half, the action of cutting was replicated in the children's processing of the problem through drawing. Overall, the actions observed in the students' performance included: counting and partitioning (12 cookies and 6 cookies), splitting remaining cookies and distributing whole and partial cookies to children. These actions, specific to division or sharing problems, provided rich observable data from a research perspective. As well, the complexity of the problem (18+12) was such that children could not easily solve the problem without some sort of scaffold in the form of virtual manipulative or recorded system. Drawing provided children with this support for thinking, allowing them to keep track of the elements of the problem by operating on smaller sets of objects or by using a system like the process of elimination.

In contrast, the Wheels Problem did not provide as rich an array of drawing as problem-solving strategies to observe. The nature of the wheels problem in terms of conceptual understanding and complexity (finding multiple addends to make 18) lead these children to use drawing as a memory support and a way to record their solution, as opposed to using drawing as a way to think about the problem. In the Wheels Problem, students were more often drawing their solutions rather than employing drawing to process information as they had in the Cookie Problem. Consequently, I did not observe as much of the children's thinking – or as divergent a range of thinking – as I had done earlier in the study.
Although the problem was open-ended and featured many possible answers, the number of drawing as problem-solving processes for finding an answer were limited. Many children used pictorial representations – detailed, realistic pictures for their solution – that in some cases distracted children from the mathematical thinking. Trent and others (Orson, Lorne) were concerned about the quality of their pictures, commenting, "I don't know how to draw a __". These comments suggest that physically drawing the vehicle was creating a more difficult problem for these students, compromising their thinking and getting in the way of the math. Although no child was able to solve this problem without drawing I would suggest this was not because of the complexity of the mathematical concept involved, but rather the size of the number (18) being broken up into pieces and reassembled. What distinguished the Wheels Problem from the Cookie Problem in terms of drawing as problem-solving was the lack of action required to solve it. Children did not need to partition, split or distribute wheels as they had with the cookies. Instead, they had to select and assign wheels to vehicles in groups, effectively counting up to 18 in chunks and recording their solution as a cumulative total.

This would imply that certain types of problems are more suited to drawing as problem-solving, providing for more observational data with which to assess thinking. Problems that feature action, are complex in terms of number of steps, difficulty of the mathematical concept and/or the quantity being manipulated lend themselves more readily to drawing as problem-solving. The mathematical reasoning involved in this kind of problem requires a tool for thinking through it, and the combination of virtual manipulative use, imagery and systems is well suited to solution finding. In designing opportunities for children to demonstrate and
experience this type of mathematical process, the data suggest one would need to ensure that the problems:

a) include an action in the conceptual structure (joining, separating, sharing, distributing, comparing),

b) are sufficiently complex (multi-step, large numbers, multiple addends or factors) to require the application of a system.

A task that is too simple for the solver can be solved by similar mental processes (a more developed evolution of the drawing as problem-solving schema) and does not require the student to manipulate the elements of the problem through representation. While the internalizing of these drawing as problem-solving processes is not an undesired outcome, in order to provide opportunities for children to practice, talk about and reflect on how drawing can support their mathematical thinking, care must be taken with the design of the problems posed to include action and complexity. Action in the problem structure requires action in the drawing; complexity in the problem requires a system for keeping track of the elements of the problem during problem-solving. In this way, a student’s drawing as problem-solving has a purpose – the elaboration and determination of a solution.
What is drawing as problem-solving?

Creating a definition
In developing my research question and pursuing clarification through research, I had to create some language to describe what it was I hoped to observe. With a focus on how students solve mathematical problems through the act of drawing, I needed to distinguish between during problem-solving and after problem-solving data; focusing on the act of drawing, not the drawing itself as a measure of mathematical reasoning.

I did not want to assess student drawings after they had solved a problem and make assumptions about their thinking during the task. This type of assessment would not provide me with observational or performance data, only artifacts. Instead, I wanted to be able to observe and describe student thinking while it happened, in particular the act of drawing, and connect that observable action to a mathematical thinking process or line of reasoning. As suggested by Tall (1994) I wanted to observe children doing and thinking about the math – I wanted to observe students in the act of converting their visual images to representations and then reasoning through them.

The term I chose to use – drawing as problem-solving – grew from my research study and careful processing of the study’s data. The definition itself evolved over time, and proved helpful in refocusing my attention on process as opposed to product in terms of student thinking. Assessing drawing as problem-solving requires video footage or first hand observation; assessing drawing of a solution requires only student artifacts. The term drawing as problem-solving served as a reminder that it was the process-based observational data that I sought to explore.
The operational definition that informed and shaped my observations, then, is as follows.

Drawing as problem-solving describes the process of thinking about a problem, its context and elements and processing that information through the act of drawing. Drawing as problem-solving can result in a complete solution, with all parts of the problem described and elaborated—a finished product that accounts for both the process and product of the thinking. Alternately, the drawing itself can be only partially fleshed out; in this situation reasoning is supported through the act of putting pencil to paper and a response determined without a completely drawn solution. As a process, however, the reasoning through pictures is complete, even if the solution itself is not fully drawn.

For example, each of the following students used drawing as a means to solve the Cookie Problem, but the degree to which the students manipulated the image on the page varied. In drawing to solve the problem, children created a partial or complete picture as product. In Anthony’s work (Figure 32—Anthony’s drawing), the process of solving the problem was evident in the lines on the page. He used drawing as a means of connecting cookies to children, and his work illustrates a complete solution—both in process and product. Anthony’s use of drawing as problem-solving continued until he had distributed all the cookies and partial cookies to the 12 children. Anthony solved the problem by drawing.

Shauna, however, solved the problem while drawing (Figure 33—Shauna’s drawing). Within the process of drawing sets of cookies and children and
considering the elements of the cookie problem, Shauna solved the problem. The drawing was a conduit for her thinking, a means to an end; she processed the information in the problem and recorded her answer in numbers and words.

While Anthony's thinking is explicit in his drawing (connect cookies to children with a line, distribute all the whole cookies then cut up all the leftovers and distribute them), Shauna's thinking is implicit in terms of what she recorded on the page (Figure 33). Shauna drew, numbered and then compared sets, found 6 cookies remaining and split those 6 cookies in half. She mentally calculated one and a half cookies per child, taking into account the whole and partial cookies. She explained: "I tried one cookie each but there was 6 left, and then I tried 2, so I just tried one and a half and then it worked." Shauna is less reliant on drawing as problem-solving than Anthony; she was able to use the act of drawing to scaffold her thinking and mentally distribute the cookies, while Anthony worked through all of the elements of the Cookie Problem by physically moving them.
Drawing as problem-solving: What it is, what it isn't

Drawing as problem-solving is not intended to focus on the analysis of a child's representation of the answer to a problem after a child has completed the thinking process. It is not intended to describe the examination of the details included in the drawn solution or the accuracy of that solution. Drawing as problem solving is the process of thinking through representation that leads to a solution – a process which might only be partially recorded on paper. Drawing as problem-solving strategies also include and are supported by visualization and mental image-making; for some young children in this study these latter strategies were the only ones used in the solution of the problems presented.

New understandings given video observations:
From "drawing the solution" to "drawing thinking"

Before beginning this study, I had examined student work and interpreted this product for a range of tasks – whether that finished product was in the form of a drawing, writing, numbers or a combination of all three. Like Diezmann (2000), Smith (2003) and Woleck (2001), I understood that drawing supported children in describing their process for solving a problem, but imagined that a student's drawing was a way to represent that thinking after the fact. I discovered, through the analysis of the video data and through interviews with the students, that in fact a great deal of processing of information happens while children put pencil to paper – so much so that the complexity of the process itself cannot truly be seen in the student's marks on the page. Like the notion of a procept coined by Tall (1994), drawing as problem-solving provides children with both a way to do the math and a way to think about the math.
As described in the drawing as problem-solving framework, student thinking and processing of the problem – acting on it – can be observed through a variety of behaviours (creating sets, check and recheck, distribution, visualization) while children are actively engaged in thinking through the problem. Some of these behaviours lead to pictorial drawing (elaborate and detailed pictures); some lend themselves to iconic or abstract representations (tallies, lines, dots); and still other behaviours do not produce any physical representations at all, but rather mental images. Regardless of the product (pictorial, iconic or mental images), drawing as problem-solving, (or reasoning through representation), is a continuous process as a child works through a mathematical task. At its most efficient, this is a flexible process involving many overlapping processes or strategies. And, at its most evolved level, these processes take place mentally.

After problem-solving to during problem-solving

Drawing as problem-solving happens while a child is engaged in the act of reasoning through the problem – working with the elements of the problem to make sense of it. It is a “during problem-solving” strategy, and must be observed as a process or performance. The framework is intended to provide a series of observable behaviours to support teachers and researchers in seeing these strategies as they happen, to assess student thinking during problem-solving rather than assessing student solutions after problem-solving. Assessing during problem-solving gives us formative performance data based on student understanding and strategic thinking; it is dynamic and active by definition. After problem-solving assessment provides a static representation of the solution; it allows for evaluative data gathering afterwards.
The Drawing as Problem-solving Framework

Some questions and reflections
The drawing as problem-solving framework as outlined in Chapter 4 highlights the strategies students used and characteristics of their drawings as children worked through both the division as sharing problem (the Cookie Problem) and the Wheels Problem. The intention of the framework is to identify and describe a general list of drawing as problem-solving behaviours and characteristics. In developing the framework, several questions were raised for me, especially around the broader applicability of the descriptors beyond these two problems and the range of students’ developmental approaches to drawing as problem-solving. These questions, regarding the use of virtual manipulatives, the notion of a developmental continuum to describe drawing as problem-solving, and the relative sophistication of students’ representations will be raised in each subsequent section.

Virtual Manipulatives
Through observation during problem solving I established that children were using their drawings like physical objects. The notion of action was key in the description of virtual manipulatives as a set of behaviours in the framework; as documented in the work of Carpenter and Fennema’s Cognitively Guided Instruction (Carpenter, Fennema & Franke, 1996), children’s solution finding was supported by modeling the action in the problem. Like manipulatives, drawings of the elements of the problem served a purpose in scaffolding mathematical thinking. Children were able to operate on their drawings in much the same way they would with a block or counter, moving, counting and distributing them. This is consistent with the work of Woleck (2001) and Smith (2003) who described
their students' use of pictures as manipulatives. With the support of their pictures and a strategy for organizing their drawn counters into sets, students were able to mirror the action in the problem to solve it. Even the most sophisticated thinkers (those who accessed mental imagery or visualization) used virtual manipulatives to support their solution finding – only the action performed on the manipulatives was mental rather than physical. (Iris, Shauna)

From physical models to mental images

I believe that this active use of drawings as a virtual manipulative may be a precursor to dynamic imagery. Dynamic imagery (Gray, Pitta, & Tall, 2000; Owens & Clement, 1998; Owens, Mitchelmore, Outhred, & Pegg, 1996; Wheatley & Cobb, 1990) is a complex thinking process that involves the creation of a mental image and the manipulation of that mental image. In mathematical research, the capacity to create and operate on a mental image is touted as a powerful mathematical problem-solving tool (Thomas, Mulligan, and Goldin, 2002). Developing dynamic imagery requires that the user has had meaningful experiences with a variety of physical models, and that these models become internalized as visual models or images.

In the progression from concrete to abstract – from physical manipulatives to mental images for them – virtual manipulatives may provide an intermediary step towards the construction of a mental representation of a concept or procedure. Children’s drawings of cookies or wheels and the manipulation of those drawings through connecting lines, slice marks, circles and boxes seem to support children in transitioning from physical models to a more abstract form of imagery-based thinking. Talking about these images, their movement on the page and how they match and
connect to the images children form in their heads can provide an important next step instructionally, since children’s language for the images they create falls short of describing the visualization that takes place. Interview results support this; although children acknowledged that drawing helped them to solve these problems, few were able to articulate how, and many lacked the language necessary to describe the process at all. Anthony was an exception; he said that drawing was “like acting what’s happening.”

More recognition is being paid to the power of imagery in mathematical problem-solving. By supporting children in connecting metacognitively to their mental images, they will be better able to understand both the structures and processes necessary for more abstract – and therefore more generalizable – mathematical thought.

**Sophistication of the drawing**

The following is a comparison of the relative sophistication of the representations made by the students as they solved the Cookie and Wheels problems. A rationale for supporting children in developing structures and systems for their images follows.

Drawing in math class poses issues for classroom teachers, who are concerned that too much student emphasis on highly detailed drawings and artistic representations will detract from the mathematics. Conversely, a child’s iconic representations are considered more indicative of higher-level thinking; since the level of abstraction is greater, iconic representations indicate more developed mathematical notions. How then to support children in making the most of drawing as problem-
solving? How can student representations support reasoning most effectively?

In this study, children's drawings demonstrated a range of sophistication, from highly detailed pictorial representations to abstract iconic sketches. Children who drew complex and detailed drawings – not surprisingly – spent more time on the pictures than those who made tallies or circles. As a result, children who laboured over their drawings had less time to manipulate those drawings as virtual counters, and consequently spent less time engaged in the math of the problem.

If a mathematical concept is explored and the problem resolved through the manipulation of the images on the page, then it follows that children's time on task should be spent in the manipulation of their images rather than on the construction of them. Children who concentrated on the drawing of their images (Livena, etc) did not have the same degree of success as those children who were more abstract in their representation. My question is whether children resort to pictorial drawings when they are unclear on the mathematics needed (Avril's solution) or whether children get lost in the act of drawing and are distracted from the mathematics. Which of these (or both) is responsible for the gap in achievement of those children who drew pictorially? Diezmann (2000) suggests that students who experience difficulty in representing their thinking diagrammatically may also struggle with spatial and/or number sense. In terms of this study and drawing as problem-solving, did either of these have a role to play? Is the drawing of elaborate pictures a response to a problem that is simply too hard, or a means of doodling towards an answer?
I am curious about the valid reasoning and sense-making that occurs while children draw - even pictorially. Woleck (2003) and Smith (2001), in their classroom-based research, speak to the developmental and individualized use of representation in their studies of students' pictures as solutions, stating that the image on the page was a perfect match for the child in terms of meaningful representation. They drew what made sense for them, at a level of complexity and sophistication that matched (Woleck, 2001, Smith, 2003). In this study, students made sense through their drawing of pictures: Avril's storytelling with the images on the page, Jason's stroke marks, Anthony's stickmen all made sense for that individual child - although not all of these lead to a response of 1 and a half cookies.

Children in this study recognized that it was not necessary to draw detailed pictures in order to be understood or to solve the problem. Although I gave no direction around the drawing, children had a sense that there were short cuts that could be taken in the representation. Some had been taught strategies in their classrooms to support them (see Figure 35 - Pricilla's lines) and substituted this strategy. Other children supported one another to simplify the task: Lorne suggested to Cathy that she could "just draw faces" and Norman offered his advice to a peer: "Try to do it with sticks". Jason monitored this process for himself. He addressed the cookie problem by writing his ideas about the distribution of the first whole cookie. Then he began to draw 12 children - sketching their faces only - to operate on them. After 5 faces, he groaned, crossed them out and switched to stroke marks to represent children. (Figure 34 - Jason switches) Jason's groan indicated his frustration with drawing all the elements, and the switch to iconic representations midway through his solution-finding increased his efficiency in solving the problem.
Two things are of note here. First, although children knew they did not have to draw detailed pictures to represent their thinking and gave others advice to this effect, they drew nonetheless (eg. Lorne, Norman). Second, although some children had been taught alternate strategies for representing mathematical thinking, they chose not to apply that strategy, or, in the case of Pricilla, the application of the strategy did not help. Diezmann and English (2001) and Pantziara et al (2004) recommend that students be explicitly taught and develop an iconic system of representation over time and with practice; that in so doing children will come to a more sophisticated sense of understanding – both of the problem itself and the mathematical concepts behind it.

I believe that children may get distracted from the mathematics by over-attention to detail in their representations, but I would stress that children's development be carefully considered with regards to the introduction of more iconic images. Like the introduction of an algorithm, the premature teaching of iconic images for mathematical situations can lead to misconceptions on the part of students – and false assumptions of understanding on the part of teachers observing them. Pricilla provides an interesting example (see Figure 35 – Pricilla’s lines). While solving both the Cookie Problem and the Wheels Problem, she attempted to use the strategy of “make tallies and cross them off” as she had been taught by her classroom teacher. Pricilla’s response was to over-generalize a
strategy better applied to a situation involving subtraction. After several attempts, she abandoned the strategy and returned to writing as her preferred mode of information processing. When I asked Pricilla if drawing had helped her, she responded in a way that confirmed her reliance on this particular strategy and her understanding of its use:

It helped instead of in your head and get mixed up.
What did the drawing do to make it easier?
By looking at it and thinking how much I can cross off.

![Figure 35 - Pricilla's lines](image)

Cookies and Kids

Perhaps rather than valuing certain types of drawings over others or introducing children to iconic representations too early, (as suggested by Diezmann and English, 2000; Novik, Hurley and Francis, 1999) it would be worthwhile to support children in pairing up their drawings with a system for operating on them. Lome suggests this very idea when he talked about how drawing helped him:

What's easier? Pictures or no pictures?

They’re both the same. I thought not drawing was like 100 - like 98% hard, but actually it's fairly easy. Like I can’t tell - maybe this is slightly harder, that is slightly harder. It sure is shorter doing this (indicates number sentence). But I need pictures to guide me. So it’s best if I team up. (indicates pictures and number sentence).

In the solution of the problem, Lome speaks to his use of multiple strategies to support his solution-finding. A meta-cognitive and verbal child, Lorne is
aware of and can articulate his thinking about pictures, systems, numbers and the interactivity between them. Asking children to stop and talk about what they're doing, why they are doing it and how that action is helping them are powerful prompts for teachers to use during the act of problem-solving – particularly if we want to encourage connection-making between the picture on the page and the dynamic image in the brain for use in mathematical thinking. Gifford (2005) suggests that rich networks of visual images, and a multi-modal approach support children in developing mental structures for problems-solving. “Teaming up” (to quote Lorne) is indeed a powerful strategy.
What makes for successful drawing as problem-solving?

In addressing the research question namely, "How does drawing as problem-solving support students in making sense of and reasoning through a mathematical problem?" it was important to identify the behaviours displayed by children while they reasoned through the problems. The drawing as problem-solving framework organizes and provides examples for these strategies. Determining the characteristics of effective use of drawing as problem-solving helps to describe how children use these strategies to advantage. I used the framework and specific students as exemplars to describe success in drawing as problem-solving. Knowing what it looks like when children are experiencing success in this area may support teachers in assessment.

The active use of virtual manipulatives
Students who used their pictures as virtual manipulatives were able to solve the problems, provided there was action applied to those manipulatives. The act of moving the images or operating on these pictures in some way was critical to success with the problems posed. While the drawings could be either iconic or pictorial, the important contributor to success with the problems was the act of movement of the images created. Actions like distributing, using the pictures as counters, and creating sets are examples. Although students who drew more iconic, less detailed representations were more efficient with their solutions, those who drew pictorially still experienced success with the problems, given time.
John **distributes** half cookies to each person (represented by a stick). He sees there are 36 halves to hand out.

Jason **distributes** wheels to vehicles using an elimination system, crossing off wheels used.

Joanna **connects** people to cookies using lines to match child to whole and part cookies.

Joanna uses her eraser to point and **count** cookies to ensure they have been matched.

Martin **circles to create sets** of wheels into virtual vehicles, then draws his ideas.

Ruby numbers cookies to 12 then **isolates a set of 6** remaining cookies with her fingers.

In much the same way as a manipulative benefits children in working through a problem, virtual manipulatives support children in emulating the structures of a problem (creating sets, joining, separating) and in processing its elements (splitting, distributing, counting, estimating). Like a manipulative, though, these images must be used with meaning attached; that is, they must be connected developmentally to the students’ own mental image of the problem. This includes the child’s construction of the problem’s elements, shown in the iconic or pictorial form in which the child elects to draw them. The representation of the elements must match the child’s level of abstract thinking to make the use...
of virtual manipulatives a meaningful and useful strategy. Unlike Pricilla's attempt to echo her teacher's more abstract way of representing cookies and wheels with tallies, which ultimately confused her, the symbolic representation with which students do the math and through which they think about the math (Tall, 1994) must then be cognitively and developmentally aligned to the student's understanding.

So too must the children manipulate their pictures to make the effort of drawing them worthwhile and meaningful. It is critical here that the manipulation of the image be connected to the child's experience. In this study, 3 children drew the cookies and the children, but had not made sense of the problem as sharing equal parts. As such, they did not manipulate the images (split and then distribute them) and therefore did not solve the problem. Virtual manipulatives, then, like other manipulative representations require that a child has experiences modeling problem situations, and that these situations are connected to both a mental image of the objects (ranging from detailed, real objects to abstract representations) and an action to be performed on them. Meaning-making is key. Smith's research recommendations (2003, p.273) are consistent with this. He indicates that in order to bridge from idiosyncratic (pictorial or case-specific representations) to mathematical (more generalizable) representations in a developmentally appropriate way, students should access language, and have the opportunity to communicate their mathematical ideas by sharing, discussing and analyzing peers' solutions.
The flexible use of strategies

Having a set of counters drawn on the page afforded students the same scaffold as a physical manipulative would; that is, students were able to count, check and recheck their ideas, try out solutions and then self correct errors if they occurred. It also allowed students the opportunity to move beyond the virtual manipulative and use a visualization strategy. Some, like Iris and David, drew only one element of the problem (only the cookies) and visualized the children who would receive them. This scaffold of having one element of the problem represented and visible supported children in solution-finding. While creating virtual manipulatives to solve the problem was helpful, being able to use these manipulatives in flexible ways was more so. Sara’s work shows this flexibility clearly. (Figure 37 – Sara’s flexibility)

Sara began by creating a set of children (drawn pictorially in Figure 37a) on the left hand side of her page, then drew a line to divide it from her 19 cookies. She re-drew a cookie beside 6 of the children and then lost track of how many she had distributed, so she began to “X” out cookies, eliminating the ones she had given away so far (Figure 37b). Next, she used her virtual manipulatives as counters to check, establishing that she had given out six already (Figure 37c). When each child had a cookie next to them, (Figure 37d) she counted again to be sure her system had accounted for them all; at this point she discovered an error and erased one extra cookie (Figure 37e and 36f).
Figure 37 – Sara’s flexibility

<table>
<thead>
<tr>
<th>Figure 37a</th>
<th>Figure 37b</th>
<th>Figure 37c</th>
</tr>
</thead>
<tbody>
<tr>
<td>Making a set. Note the line drawn in the middle of the page as a separator.</td>
<td>Sara has begun to assign cookies to children. The first 6 kids have a circle drawn next to them. Now Sara adopts a system for keeping track, crossing off 1 cookie...</td>
<td>...and then matching it to a child. This is one phase of check and re-check – note that the circle representing the cookie is already present.</td>
</tr>
<tr>
<td>Figure 37d</td>
<td>Figure 37e</td>
<td>Figure 37f</td>
</tr>
<tr>
<td>Crossing off the next cookie in the set to eliminate it from the set.</td>
<td>Once done, Sara re-counts her set of cookies to ensure she has eliminated 12...</td>
<td>...and finds an error, which she erases.</td>
</tr>
</tbody>
</table>

When she saw all 12 had been distributed, she counted leftover cookies and said, “Now I can split some.... That’s going in a half cookie.” She split up the remaining 6 cookies by drawing a line through each one, (Figure 37g) then re-drew a half cookie beside each child, showing clearly that each child would get one whole cookie and a half cookie (Figure 37i). While she did not systematically eliminate these half cookies, she did distribute them in pieces to each child. (Figure 37h)
<table>
<thead>
<tr>
<th>Figure 37g</th>
<th>Figure 37h</th>
<th>Figure 37i</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dividing the 6 remaining cookies in half with a line</td>
<td>Sara’s cookies – all used up and accounted for; 12 given away, 6 cut in half.</td>
<td>Showing pictorially the amount each child will get. (One and a half cookie.)</td>
</tr>
</tbody>
</table>

In Sara’s approach we see several things: first, the flexible use of multiple drawing as problem-solving strategies, and the repetition of these strategies over the course of resolving the problem; Sara’s inclination to check and re-check her work for accuracy, which allowed her to catch an error in her drawing; and the system she adopted in order to keep track. As Sara applied these strategies and moved flexibly between them, she both drew to solve the problem and drew her solution concurrently – showing each child’s portion of the shared cookies. To view Sara’s flexible drawing as problem-solving work, please see Video Clip 5, “Sara’s Flexibility”.

Strategic competence and flexibility in mathematical problem-solving are powerful thinking tools. The same is true in terms of drawing as problem-solving strategies. In this study, fluency was demonstrated in knowing when to apply a particular strategy (distribution, creating sets), using self-monitoring scripts (check and re-check, keeping track) and recognizing the need to shift to a new strategy (elimination, iconic representation). Children in this study who employed multiple drawing as problem-solving
strategies and who were flexible in their application were able to solve the problem more efficiently. The inclination to move between strategies and to shift from one to another prevented children from getting stuck, over-emphasizing or over-relying on a particular method.
The power of a system

These problems, despite the degree of student success, were not easy to solve. Children struggled with the complexity and the large numbers in both of the problems. What was observed over time was that students who used a system to support their drawing as problem solving were more likely to solve the problem.

Norman’s work is a good example of this (Figure 38 – Norman’s system). Norman persisted with the cookie problem for over 11 minutes. He began by constructing sets of virtual manipulatives on his page – a set of cookies and a set of children (Figure 38a, 38b). Then he counted and attempted to distribute the pictures of cookies to the children, touching a single cookie and then 2 children (Figure 38c). He seemed to be dividing the cookies up into 2 parts in order to share them out, but was doing so without physically splitting them – rather he seemed to split them mentally, using dynamic imagery.

Figure 38 – Norman’s system

<table>
<thead>
<tr>
<th>Figure 38a</th>
<th>Figure 38b</th>
<th>Figure 38c</th>
</tr>
</thead>
<tbody>
<tr>
<td>Drawing kids</td>
<td>Making faces</td>
<td>Set of cookies added. Using pencil to match 1 cookie to 2 faces.</td>
</tr>
</tbody>
</table>

Next, he began to touch, and then draw faint lines to attach whole cookies to children (Figure 38d, 38e, 38f), then paused and asked for help. He told me what he was trying to do: “I was trying to do a half and then
trying to match which pieces go to which", so I reiterated the steps he had followed to that point, and he asked for some more time to think: "Come back for 5 minutes..." (Figure 38g)

<table>
<thead>
<tr>
<th>Figure 38d</th>
<th>Figure 38e</th>
<th>Figure 38f</th>
</tr>
</thead>
<tbody>
<tr>
<td>Touching one cookie...</td>
<td>...to one face.</td>
<td>Drawing faint lines to connect a single cookie to a single child.</td>
</tr>
<tr>
<td>Figure 38g</td>
<td>Figure 38h</td>
<td>Figure 38i</td>
</tr>
<tr>
<td>Lines drawn for 12 cookies. &quot;I'm stuck. I need some help.&quot; I gave him 3 minutes of think time...</td>
<td>...then a new method - numbering was applied. Numbers 1-18 in the cookies and numbers 1-12 over the kids.</td>
<td>Note that Child 1 and Child 2 have 1/2 written over them.</td>
</tr>
</tbody>
</table>

After approximately 2 minutes, Norman developed a more deliberate method - numbering his cookies from 1-18 and the cookies from 1-12. (Figure 38h) He recognized that there were going to be some cookies left over. In his head, he visualized cookie 13 broken into two pieces and wrote 1/2 over Child 1 and Child 2 (Figure 38l) - but after trying to do the same with cookie 14 (Figure 38j, 38k) and 15, Norman became confused, not recalling which child had been allotted a half cookie. (Figure 38m,
38n) Halfway through his mental distribution, he exclaimed, “I lost count!” Norman’s flexible use of strategies – and his positive sense-making disposition - resurfaced here as I walked away.

<table>
<thead>
<tr>
<th>Figure 38j</th>
<th>Figure 38k</th>
<th>Figure 38m</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image_url" alt="Image of child pointing to a cookie" /></td>
<td><img src="image_url" alt="Image of child pointing to a cookie" /></td>
<td><img src="image_url" alt="Image of child distributing cookies" /></td>
</tr>
<tr>
<td>He explains why the 1/2 over Child 1 and Child 2. “This one, this one, those 2 cracked in half.” (pointing to #13)</td>
<td>When asked for clarification, Norman says, “It means this one and this one (indicating only #14) goes...”</td>
<td>“...here.” Norman splits the cookies and distributes them mentally.</td>
</tr>
</tbody>
</table>

I returned several minutes later when Norman cried out “I did it! I solved the problem!” He had finally applied a system of elimination to the task by colouring, and succeeded in finding an answer. Norman had coloured over cookies 13-18 as he distributed 1/2 of each cookie to the next 2 children in his numbered set. He explained his idea: “These are just colouring ‘cuz if these are all white I keep losing tracks and so I make a good decision to colour it so that’s why I don’t get lost track.” (Figure 38q)

<table>
<thead>
<tr>
<th>Figure 38n</th>
<th>Figure 38p</th>
<th>Figure 38q</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image_url" alt="Image of child pointing to a cookie" /></td>
<td><img src="image_url" alt="Image of child pointing to a cookie" /></td>
<td><img src="image_url" alt="Image of child distributing cookies" /></td>
</tr>
<tr>
<td>“Then this one (#15) goes...”</td>
<td>Hesitates... “Oh! I lost count!”</td>
<td>Norman develops a system of elimination, colouring in the “used” cookies.</td>
</tr>
</tbody>
</table>
Norman demonstrated at least 8 different drawing as problem-solving strategies in the solution of this problem. Although he was flexible in applying different strategies when he encountered a roadblock (and also asking for think time to be able to do that important meta-cognitive work), Norman’s drawings did not support him in finding a solution until he developed a system – namely elimination – for negotiating of the last elements of the problem. To view Norman’s drawing as problem-solving work involving a system, please see Video Clip 6, “Norman’s system”.

<table>
<thead>
<tr>
<th>Figure 39 – Norman’s final work</th>
<th>Norman explains:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cookies and Kids</td>
<td>“I draw 18 cookies and I accidentally draw 22 so I crossed it out, and I draw 12 kids and I put every number in the cookies and then I put the same amount of cookies and when I got to 12, ... and then I crack two and I got one half... and I shade it in like this.”</td>
</tr>
<tr>
<td></td>
<td>How did the shading in help you?</td>
</tr>
<tr>
<td></td>
<td>“It doesn’t make me lost count...”</td>
</tr>
</tbody>
</table>

Systems such as numbering, keeping track, and elimination were very supportive of children in successfully solving the problems. Like Norman, several other children accessed systems (Cathy, Sara, Karl) and were able to arrive at an answer because of the structure these strategies allowed them. Processing the elements of the problem in a systematic way (eliminating used wheels, crossing off whole cookies that had been
distributed, colouring in cookies broken up and handed out) forced children to attend to each small piece of the problem and record their actions. Although many had a system in mind, those who drew, made marks or physical representations on the page were more successful in completing the problem. Their printed representations allowed them to reduce the load on their working memory and provided them with something to act upon in terms of a manipulative.
What about children who don’t draw?
Successful problem-solving did not necessarily require that students draw. Within this study, there were several children who did not draw at all to solve the problems, but managed to solve them successfully. These children, whom I call “visualizers”, are spatial thinkers, who seem to be able to handle a large amount of information and manipulate that information mentally. Interestingly, it was noted that some visualizers felt the need to draw something after the fact; that is, although they solved the problem mentally they felt compelled to draw a picture to show their solution. This would be classified as drawing of problem-solving, rather than drawing as problem solving, since the actual process took place mentally, rather than on the page. This group of students in particular helped me to address my second research question, to understand how young children access visualization and imagery in their resolution of mathematical tasks.

The only way to access this kind of mental image making was to observe carefully and to ask questions of the children. For those able to explain what they were seeing in their images, it was evident that multiple processes were happening simultaneously. The complexity of the thinking required not only visual spatial capacity but also strong working memory. Sets were created, cookies and children counted and compared, cookies broken up and distributed while students kept both the number in each set, then number distributed and the half pieces allotted to children separate in their heads. Clearly these children had a plan, a process and a system for solving the problem - but because nothing was written on the paper, these processes were invisible. It is easy to misinterpret this lack of print on the page. While filming the very first group of children I made this error myself, with Mark.
Mark and the others at his table were presented with the Cookie Problem. While all the others began to draw and talk about their ideas, Mark did not. He picked up his pencil several times but seemed, at first glance, to be confused about where to start (Figure 40a). I filmed him for a while, watching him making false starts, but then moved on to other children. About 3 minutes into the problem I asked if he understood what he was supposed to do, which he assured me he did – even telling me he was "...trying to count by like stuff – 3’s and 2’s and ones" so I left him again.

When another 5 minutes had passed and Mark still had nothing on his page, I offered him a simpler problem to solve, thinking he was not able to process the question. Fortunately he ignored me and persisted with the original task. I was startled, at the end of 12 minutes of work time, when Mark orally presented an elaborate, clear and correct solution for the problem. He explains:

If you cut six cookies in half everyone gets half a cookie, but if you cut six cookies three times (cuz 6 plus 6 plus 6 is 18), so you cut six cookies in half and you cut 6 cookies in half and you cut 6 cookies in half so everybody gets 3 halves of a cookie.

Where did you get that idea from?

It just came in my head.

Intrigued, I watched the video again and observed the following postures and behaviours while Mark reasoned through the problem. These images indicated to me that Mark was not only thinking, but thinking deeply about the problem – his persistence and multiple attempts to process the information are clear in the movement of his eyes, fingers and mouth.
Figure 40 – Mark uses imagery

<table>
<thead>
<tr>
<th>Figure 40a</th>
<th>Figure 40b</th>
<th>Figure 40c</th>
</tr>
</thead>
<tbody>
<tr>
<td>Everyone writing, Mark without his pencil in hand.</td>
<td>Looking up, finger counting, watching cookies and children.</td>
<td>Leaning back from the table, grouping of 2’s and 3’s with his fingers (note they are held together in sets).</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Figure 40d</th>
<th>Figure 40e</th>
<th>Figure 40f</th>
</tr>
</thead>
<tbody>
<tr>
<td>Making a set of three...</td>
<td>Then dismissing that idea. Still, persisting to solve the problem.</td>
<td>Counting one group of 6 cookies, then counting to 12, then 18 in sets of 6's.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Figure 40g</th>
<th>Figure 40h</th>
<th>Figure 40i</th>
</tr>
</thead>
<tbody>
<tr>
<td>Looking up and finger counting</td>
<td>Scanning his mental image while finger counting, head bobbing</td>
<td>Explaining his solution aloud, by looking up to describe the 3 sets of 6 cookies cut in half.</td>
</tr>
</tbody>
</table>

To view Mark’s visual working through this problem, please see Video Clip 7, “Mark’s use of imagery”.

137
There were many behaviours here that I had not attended to during the process itself. Mark was using his visual image of 18 and 12, and trying to determine how many pieces he needed to break each cookie into in order to give everyone an equal share. Unlike the other children in this study (with the exception of John), Mark split all of the cookies in half and distributed those halves fairly. Imagine the complexity of managing 36 half cookies mentally! Mark's finger counting provided him with an important scaffold; but surprisingly, he did not always look at his fingers while touching them (Figures 40b, 40c, 40g). Instead, he looked upwards, toward his mental picture and touched his fingers simultaneously (Figure 40h). The visual image was the key piece here, and his fingers operated like Sara's X's – providing him with a system for keeping track of the cookies he had split. This is an important behaviour to note. In order to visualize effectively, students must also employ a system. In this study, however, that visual system was not acknowledged as it was not "drawing" per se. There was no written tally or elimination of items on the page that gave a physical indicator of his processing.

Mark's performance and solution gave me a great deal to wonder about, and to question about my own practice. What supports a child in creating visual images strong enough to solve a complex problem without paper and pencil? What systems are the most effective for students who are visualizers? And what types of problems are too difficult to be understood and processed in this way? Most importantly, this piece of video and my own reaction to it raised the question of classroom practice. If I was intently focused on the children and did not notice Mark's thinking, how is a classroom teacher to become aware of this kind of mental activity?
What about children for whom drawing is not helpful?

Not all children were successful in solving the problems. Below are reflections after observing video footage of a group of 3 children who were not able to solve the Cookie Problem. My concern with this group of students was that perhaps drawing was too abstract for them, that the lack of a mental image for the process of dividing and sharing made drawing inaccessible – or unhelpful – to them as a problem-solving strategy.

Mona, for instance, drew two sets of pictures on her sheet of paper – 18 cookies and 12 children – and then added the two sets together to arrive at 30 as a response. When asked, “What does 30 mean?” Mona responded, “It’s what the answer is for the whole thing.” Avril drew two tables – one with cookies on it and one with glasses of milk, divided 9 of the 18 cookies in half and distributed only those halves to the children. “Everyone has half of a cookie”, she wrote on her page – although she created 18 halves and there were 12 children. Rebecca struggled with multiple attempts, erasing her pictures and starting over each time. She began with: “I’m giving everyone 18 cookies.” And in the end, she drew 18 cookies, cut each one in half and wrote: “They are each going to get a half a cookie”. When asked if drawing helped her, Rebecca responded:

...not really, because when I tried to draw it really didn’t make sense, but when I wrote it, it really made sense. I only wrote pictures, not all those children...

Although each of these children drew while processing the problem, their drawings did not help them. This raised questions for me regarding both their background knowledge and the appropriateness of drawing as a
problem-solving strategy for these students. Was it just too early to introduce them to a problem like this without manipulatives? And would they have been able to solve it even with physical models? Students' mathematical understanding is supported by the opportunity to model problems and talk about their reasoning. I suspect that a lack of connected mathematical experience might have impacted both their ability to reason through this problem and their capacity to use drawing as a problem-solving strategy to advantage.

Diezmann (2000) states that the inability to visualize a mathematical situation affects a student's ability to represent that situation in a diagram. In addressing this area of need, Gifford (2005) recommends instructional support, particularly for students with dyscalculia, to create networks of images and vivid associations to support memory and meaning-making. While I am not suggesting these three students suffer from dyscalculia, I would echo that for all students, language, visualizing and the development of number sense are inter-relatedness processes. Drawing as problem-solving (e.g. using virtual manipulatives) can only support mathematical problem solving when connected to meaningful prior experiences, a mental image for the concept involved and an action to be performed on the representational object. Gifford suggests providing visual reasoning support to students who struggle to call up a mental image of the process; allowing them to do the math and think about the math through their representations.
Dispositions and mathematical sense making:
How they influenced drawing as problem-solving behaviours

Asking permission to draw
During the study, each of the problems was posed of the children orally, and a blank paper with only the problem written on it was presented to the students. There were no manipulatives on the table, and no directives given by me for solving the problem. What was surprising was that, within a very few seconds of being presented the problem sheet, children asked permission to use pictures as a problem-solving strategy. "Can we draw?", "Is it ok if we use pictures?" or other similar phrases were asked in 6 of the 9 problem-solving groups. It was interesting to consider the children's queries about drawing as an acceptable form of problem-solving. I had thought that a blank page (as opposed to one with lines on it) would suggest freedom to draw or sketch a response to the problem, but children's prior experience or some notion of the inappropriateness of pictures in math prevented them from employing the strategy without approval. Rather than being leading, my response was either, "Would drawing help you?" or "Do what would help you." Once drawing was deemed acceptable, almost all of the children chose drawing as a strategy for solving the problems.

Those children who did not draw to solve the problem (the visualizers) found themselves in an awkward position. Since they did not draw to process the information but used mental imagery instead, they drew afterwards, recording their response in numbers, pictures or words. Mark and Trent wrote out their ideas in long form, describing their product (3 halves each) and their process respectively (I counted by ones to 12...). Charlie and Stephanie created pictures to match the solution they had
generated in their heads – drawing their solution rather than drawing to solve.

For the first group of children, there was a sense that drawing might not be an appropriate means of solving the problem. For the visualizers, drawing was not needed, but in watching the other children it was established as the correct or desired way to record a response. Classroom expectation and norms – even a child’s construction of “what is mathematically appropriate” – clearly played a part in students’ response to the problem.

“But what’s division answer going to be?”

Classroom culture and student expectations

Lome and Norman had a fascinating discussion about “what is mathematically appropriate” after solving the Cookie Problem. Both boys worked hard to solve the problem. Lome applied a range of strategies (set creation, counting, distribution, circling, drawing lines) and Norman persisted through several attempts, finally succeeding upon application of a system (“I got it I got it! I solved the problem!”). Both boys found 1 and a half cookies as a response, and compared their ideas afterwards. Lome saw that they had the same answer, yet asked Norman, “But what’s the division answer going to be?” He pointed to his page, where he had earlier recorded $18 \div 12$. Norman looked over at Lome’s work and replied “12 of 18. So hard. Try to do it with sticks.” Although the boys had successfully – painstakingly – solved this problem, they did not see the connection between the solution they generated (1 and a half cookies each) and the algorithm they were so keen to write and respond to. In fact, when I asked Lome to tell me about his equation on the page, he said, “That’s just for extra work. Like if I figure that out first (pointing to his drawing of 18 cookies and kids) I can just write the answer.”
Even though he generated the equation to solve for “extra work” and saw that by drawing to solve the problem he could “just write the answer”, Lorne could not connect back to that equation and saw it as a new problem to solve rather than a representation of the same problem. Norman’s response to his cognitive dissonance was to request a bathroom break and to ask the custodian for help...! “What’s $18 + 12$?” Norman asked. The custodian scratched his head. “$18 + 12$? What grade are you in?” Lome and Norman could work through the problem with drawings, but still sought a “real math” way to find and illustrate the solution. Even in the second grade, norms for what makes “real math” have been constructed and understood by students in our classrooms. How children interpret and apply this understanding can be examined in part in the analysis of the interview data that follows.
Understanding how drawing helped:
Insights from the interview results

In order to address the research questions: “How do young children access visualization and imagery in their resolution of mathematical tasks?”, and “How does drawing as problem-solving support students in making sense of and reasoning through a mathematical problem?”, I had to actually ask the children what they thought. Interview data proved extremely illuminating and supported insights gained from examination of the framework.

Children described a variety of ways in which drawing had helped them to solve the problem. Several of their reasons were well thought out and well articulated. Themes in their responses highlighted the notion of the drawing as a virtual manipulative to be operated on (“It helps because I can try different things”; “I saw the 6 leftover and I knew that I had to cut them into pieces”; “You can draw faces and draw cookies to the face”), or as a memory aid (“It doesn’t make me lost count”; “…instead of doing it in your brain and making it explode kinda.”; “Doing it in our heads is harder”). These children have made the connection between their drawings and the mathematical purpose for them. Like a physical manipulative or other tool, these students understand that their drawings assist them with problem-solving.

John spoke about the parts of his drawing and explained their purpose. “The lines...tell you how many you need and the lines that connect tell you which cookies you get.” His response to the problem included cutting all the cookies in half and distributing halves to each child: “[Y]ou break all the cookies in half in the middle and it gives you 36 halves. Then you give the 36 halves to the children until there’s no more.” After drawing lines
to connect each half cookie to a line representing a child. (Figure 41 – John distributes) John concluded that each person would get “one whole and one half” cookie. (circling with his pencil tip, in Figure 42 – John explains) For John, the act of drawing allowed him to connect his idea of breaking up the cookies to a final solution. He describes not only his drawings (the cookies cut in half) but also how the act of drawing supported him in finding an answer.

Another set of children spoke explicitly about the act of drawing, and explained how the act itself assisted them. Anthony said that drawing was “like acting what’s happening.” And Orson agreed: “It’s like acting because it’s like what people do.” In their reasoning through the problem, these boys saw that the act of drawing was mirroring the act of dividing and giving out cookies – sharing them equally among children. They were able to connect a mental action to their representational action, reasoning through their drawings in a dynamic way. They referred to action in telling about drawing to solve problems (what’s happening and what people do) and their solutions themselves complemented this thinking. Anthony’s solution to the cookie problem included lines to distribute the cookies (acting what’s happening); Orson’s solution included groupings of cookies, circled together with the children who
would eat them (what people do). In their responses, these boys echoed Tall's notion of a procept (1994) – for them, their drawings were a way of doing and thinking about the math.

<table>
<thead>
<tr>
<th>Figure 43 – Jessie works it out</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Cookies and Kids</strong></td>
</tr>
<tr>
<td>There are 16 cookies on a plate and 12 children who want to share them. How can the children share the cookies? How much will everyone get?</td>
</tr>
<tr>
<td><img src="image" alt="Image of Jessie's drawing" /></td>
</tr>
</tbody>
</table>

Students spoke about math moving and having action as expressed through their drawings. Actions like “taking away”, “giving to people”, “crossing off” and “drawing cookies to the faces” were all interpreted by the children as mathematical events. Jessie said that “drawing is sort of like doing the math... like drawing the math” and cited grouping and take-aways as examples of the math he was drawing. In solving the problem (see Figure 42 – Jessie works it out), you can see partially erased attempts at the problem before Jessie settled on half cookies for today and whole cookies for tomorrow. In working through the Cookie Problem, Jessie counted, created sets, divided, and checked and rechecked his thinking. The act of drawing for him was synonymous with solving the problem – he was drawing the math. This notion is reminiscent of Tall's procept, in which students' symbolic representation encapsulates both
the concept and the process – allowing the interpreter of that symbol to do the math and think about it concurrently. For Jessie, doing the math and thinking about it happened through the act of drawing. Not all children indicated that drawing had helped them to solve the problems, but rather there were some children who responded that drawing had not helped them. Interview statements for this group of children revealed either confusion around the problem or the purpose of the pictures as a problem-solving tool. Accessing the potential for reasoning through the drawings seemed difficult. Mona commented, “I knew it didn’t work and I started starting to the pictures again and again”; Sammy said, “Whenever I tried to make 18 it just went to 32.” Unlike the other children who explained that drawing had helped, these children were not able to act on their representations in a way that allowed them to reason through the pictures they created.

What the visualizers reported

There were 4 children (Trent, Mark, Charlie and Stephanie) who did not draw to solve the problem, but rather drew their solutions or wrote about their thinking. As such, Trent and Mark did not answer the interview question at all. They acknowledged that in fact they had not drawn to solve the problem and so could not say that drawing had helped.

Charlie tried though, and gave an interesting response. He said that drawing helped “...so I could remember the answer.” For Charlie, solving the problem was a mental process involving imagery. In fact, he recorded the answer “1 and a half” before ever drawing anything. Drawing the cookies wasn’t helpful in solving the problem – only in representing the solution he had arrived at in his head. This was the point at which I began to distinguish between drawing to solve and drawing a
solution, recognizing that visualizers do not record their images in order to operate on them. Instead, visualizers operate on the images in their heads and then translate that mental action to pictures or words after the fact.

I watched Stephanie doing this very thing, so I asked her “Do you ever see things in your head?” Stephanie nodded. “What do you see in your head when you do this problem?” I asked. “I just see like cookies and kids... and see like how can you share them in your head.” Stephanie and I had a separate conversation about problems in math and her representations of them. She admitted that she knew the answers long before she recorded anything on the page, and that in fact she only recorded to satisfy the teacher.

The responses from the visualizers in the group distressed me. A group of capable mathematicians and flexible thinkers, these children seemed reluctant to share their knowledge and thinking – even purposefully hiding their processes. When Trent pulled me away from the camera to ask if his idea of counting was ok I was amused at first – then was concerned to think that he should question his developed and sensible method. I understood better why children might disguise their thinking when I misinterpreted Mark’s attempts at problem solving and offered him a simpler problem. My own assumptions about the way to solve the problem through pictures lead me to believe that Mark was struggling when in fact he was actively problem-solving. Stephanie and Charlie’s inclination to draw the solution rather than to draw to solve speaks to an implicit product-based expectation as well. And yet if I couldn’t see them thinking, how could I recognize their cognitive work?
I return then to the question of how young children access visualization and imagery in their resolution of mathematical tasks. Data from this study suggests that students who access imagery in their reasoning through the problems (and then drew or did not draw their solution) did so in a visually active way, moving images around in their heads, manipulating those images in dynamic ways and showing outward signs of this manipulation through gestures and upraised eyes. Students who were able to access language for these processes described likewise them in active terms.

Visualization might be considered an invisible form of drawing as problem-solving, the culmination of experiences and meaning-making in mathematical terms. As a valid and complex strategy, it needs recognition on the part of teachers and students in order to be used confidently and efficiently by students. Research highlighting the efficient use of imagery among high achieving students – imagery that is dynamic, flexible and fluid in its application – celebrates this kind of mental activity. (Gray, Pitta & Tall, 2000; Reynolds & Wheatley, 1997; Thomas, Mulligan & Goldin, 2002). Classroom teachers would benefit from support to do the same.

In observing this group of children, I learned to recognize – and acknowledge aloud – the outward signs of visualization, and make mention now of these behaviours when I speak to children. ("I see you’re looking up. You are thinking about the parts of the problem. What do you see in your head? Is anything moving? How?") I learned that spatial sense and its role in mathematical problem-solving should be recognized, named and honoured as a valid and powerful strategy. Further suggestions for classroom practice follow.
Implications for practice

Educational researchers have a responsibility to both explore questions regarding the enhancement of learning and to translate research findings into accessible instructional guidelines and tasks for the classroom teacher. To this end, the following section begins to address notions of implementation by considering four general areas: types of problems, types of learners, metacognitive and instructional tasks.

Types of problems
In order to support drawing as problem-solving or reasoning through the act of drawing, the tasks presented to students should meet certain criteria around complexity and movement. That is, the problems posed of the children must be complex enough (e.g. multi-step, non-routine, problematic) to ensure that reasoning towards an answer requires cognitive effort on the part of students. In this study, one student (Stephanie) reported that often the questions she was asked to solve were too simple, and that she had arrived at the answer long before she picked up a pencil to represent the solution. Open-ended tasks requiring creativity and generative thinking are examples of non-routine problems; these may lend themselves to drawing as problem-solving and visualizing behaviours in students.

Problems should also feature some movement or distribution of quantity in their structure to encourage drawing as problems-solving or visualizing activity. Mental math tasks involving mathematical operations in which chunks of number are moved, balanced or set aside and replaced are good examples.
Types of learners
Within this study there were three general groupings or types of learners, described by the ways in which they interacted with the problems and the manner in which they reasoned through their drawings.

Interview data and observations referenced against the framework indicated there were students who were not yet ready to access drawing as a problem-solving strategy. I am unclear as to whether this was owing to confusion around the problem context or their difficulty is making their drawings “work” for them. Nonetheless, there are students for whom drawing as problem-solving is not helpful. Their particular needs with respect to drawing as problem-solving and the understanding of the mathematical structures within the problems should be recognized and, as Gifford (2005) suggests, a multi-modal approach used to build spatial thinking.

Like the majority of the children involved in this study, I would suggest that children in the regular classroom will also respond to problems (like those described in the previous section) by drawing and/or visualizing. In order to develop and hone this capacity, support children who reason through drawing to reflect on the act of drawing and to ask themselves – how does it help me? And if drawing is not helping, then students should be encouraged to switch gears, to refine or alter their drawing as problem-solving strategy, or apply a system to the process of reasoning.

The third group of children who identified themselves through this study included those children who did not need to draw to solve the problem; children who reasoned only visually, and elected if they drew at all to record a solution to the problem. Students like these in the regular
classroom, despite their seeming facility with problem-solving may still benefit from metacognitive explorations and the development and explicit naming of systems to support mental or image-based problem-solving (e.g., finger counting, stroke marks, scaffolding tools like a numberless number line).

**Addressing and supporting metacognition**

**Questions and prompts for classroom teachers**

Supporting students to reason mathematically is encouraged by the development of metacognition, the ability to think about thinking and reflect on the mental processes being used. These prompts are especially effective when used during the act of thinking itself. The following list provides some suggested questions classroom teachers might consider asking to both assess the mental process taking place and to make the act of reasoning through drawing explicit for the students who are using it.

Suggested prompts and a rationale for their use include:

- What are you doing? Why are you doing it? How does it help you? (addresses metacognition)
- Tell me about your picture/mental image. What are the parts for? (suggests the need for structure in mental imagery)
- Close your eyes. What do you see? What’s happening? How does this help you solve the problem? (focuses attention on the mental image; highlights the notion of action)
- Stop and share with a partner. How are your drawings/images alike? How are they different? (encourages language and expands thinking)
- What does your partner’s picture tell you? What can you see happening? (promotes reading of pictures, representational action)
- I see you looking up. What do you see there? How are you keeping track? How are all the pieces organized? (recognizes outward signs of visualizing, encourages application a system to visual images)
**Classroom strategies:**

Classroom-based tasks that encourage visual reasoning and capitalize on students' spatial strengths would be of benefit to students in their application of drawing as problem-solving strategies. Supporting students to access multi-layered systems (pictures, mental models, manipulatives, numbers, words, language) for representing and thinking though problems and encouraging them to "team up" will help to strengthen networks of visual thinking and allow for flexibility between them.

Teacher modeling of metacognitive talk related to drawing as problem-solving practices through think-alouds ("I am picturing a tree with steps on the side, and a squirrel climbing up and down the stairs. I'll need to use my fingers to keep track of the action here...") is a powerful instructional technique that supports the development of language to describe this form of mathematical reasoning as well as raising the awareness of its existence. After all, teachers don't "just know" the answers. Caution should be exercised so that students do not simply echo the teachers' strategy; providing opportunities for children to "think aloud" validates developmentally appropriate ways of reasoning through drawing. Using dry-erase or chalkboards during lessons and partnering students over them to encourage sharing and talk about their reasoning is one suggested strategy.

The effect of classroom culture and expectation cannot be underestimated. It is important to create a classroom culture that allows for multiple representations (including iconic, pictorial and mental images), flexibility of strategies (manipulating and acting upon drawings or mental images) and the application of systems (keeping track, elimination, check and double check) where helpful.
Summary and further research directions

In addressing the research question, "How does drawing as problem-solving support students in making sense of and reasoning through a mathematical problem?", student’s interview data proved extremely helpful. Two purposes for drawing emerged in the students’ response to the question, “How did drawing help?”. According to the children, drawing helped in one of two ways – by providing them with something to see and operate on, and in allowing them to process their thinking through movement. For those who did not draw, I would suggest that visual images provided them with exactly the same scaffold – something to see and a way to move the elements of the problem – product and process both, although invisible to the observer.

Observational data gathered and organized in the drawing as problem-solving framework supported student statements during the interviews; that is, the clusters of behaviours observed while the students were in the act of problem-solving were consistent with what the students themselves reported. In considering how drawing as problem-solving supports students in reasoning through a problem, the video data collated in the framework suggests that several behaviours emerge across problems, regardless of problem structure. The use of virtual manipulative behaviours, including the creation of sets, the use of pictures as counters and the act of distribution were ways in which students in this study applied an action to their pictures. Through virtual manipulative use, students were able to reason through the mathematical problem by acting on their pictures – whether those pictures were iconic or pictorial in nature. System use emerged in student interview data as well as video data observed during student problem-solving. Systems might be defined
as self-defined structures through which students refined and clarified their thinking about the problem. In the case of drawing as problem-solving, this involved applying a system or structure to the drawings – using drawings as the object of a system and “operating on them” in a systematic way. Elimination, check and recheck and keeping track behaviours were observed in student work and were mentioned by students during the interview as a way in which drawing (or the drawings) had helped them reason through the task.

The development of capacity in drawing as problem-solving is an issue of interest. Students’ representational objects can take one of three forms – pictorial, iconic or mental – or even a combination of the three. Supporting students in accessing and applying a developmentally appropriate representation that provides a fit cognitively while allowing the student to approach drawing as problem-solving in an efficient way is worth pursuing. Although not linear or hierarchical in its design and application, a continuum of development of mental imagery is suggested here – that the meaningful use of physical manipulatives may precede the development of pictorial representations, that pictorial, problem-specific representations may pave the way for more iconic or generalizable ones, and that the use of iconic representations and an exploration of the relationships between the elements of the problem over time may help to create mental images. Data in this study suggest that these categories exist; how they develop and are subsequently implemented by students is an area for further research.

As observed during this study, these interrelated representational tools can be used in problem-solving with success; they can likewise be used in combination to reason through problems. Lorne shared his thinking about
the use of drawings and other symbolic representations to solve problems: "I need pictures to guide me. So it’s best if I team up." Lorne’s comment goes a long way to answering one of the key research questions: “How do young children access visualization and imagery in their resolution of mathematical tasks?” These visual activities, evidence of which was noted during the video analysis, supported children in reasoning through the problems; allowing them to “team up” their drawn representations with visual images.

Further research is warranted to fully explore this notion of interrelated and interactive strategies for drawing as problem-solving. I propose a more in-depth study of individual children working through complex, multi-step problems to both validate the drawing as problem-solving framework as a more broadly applicable tool and to determine the extent to which drawing as problem-solving strategies apply to different problem types and structures beyond the 2 problems used here. I suggest that students who are visualizers be identified and targeted for inclusion in a similar study to address questions related to drawing as problem-solving – for these students who tend not to draw at all. To what degree are their mental strategies sufficient in solving complex problems, and at what point are supplemental strategies necessary?

This drawing as problem-solving investigation provided me as teacher-researcher with a way to observe and describe visual-spatial strengths as they relate to mathematical problem-solving. A rewarding exploration, this study has allowed me to pose and consider several new research directions. I welcome the opportunity to collaborate with other researchers in further exploring these questions.
References


Appendix A - Interview data

Organized by degree of help provided by drawing
Colour coded by type of help provided by drawing

Drawing didn't help

TAPE 3B

Rebecca – "...not really, because when I tried to draw it really didn't make sense, but when I wrote it, it really made sense. I only wrote pictures, not all those children. I had to draw all of those children so I thought writing is better than drawing."

** NOT SUCCESSFUL IN SOLVING THE PROBLEM **

Mona: "It didn't help for me for all the pictures because I counted it all together there were 30 and then I figured and I knew it didn't work and I started starting to the pictures again and again."

** NOT SUCCESSFUL IN SOLVING THE PROBLEM **

TAPE 4B

Susie: Did the picture help you? “Not really. Cuz whenever I tried to make 18 it just went to 32.” What would have helped you? If you could have had anything in the room, what would have helped? "Um... ummm..." Cookies? “Maybe.”

SUMMARY

<table>
<thead>
<tr>
<th>TYPE OF SUPPORT (HOW DRAWING HELPED)</th>
<th>NUMBER OF REFERENCES TO EACH TYPE OF SUPPORT</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subject of an action</td>
<td>0</td>
</tr>
<tr>
<td>Distribution</td>
<td>0</td>
</tr>
<tr>
<td>Memory aid</td>
<td>0</td>
</tr>
<tr>
<td>Elimination system</td>
<td>0</td>
</tr>
<tr>
<td>Pictures as iconic representations</td>
<td>0</td>
</tr>
</tbody>
</table>
Not clear on how drawing had helped – or could not explain clearly how

**TAPE 1B**

Susanne: “You can count like the ones that are first, then the ones that are left we just put it down there. It just helps.”

**TAPE 2A**

Larry: “Helping me um make lines and ... and the kids were happy and they had to get crumbs and chocolate and 2 of them shared it.”

**TAPE 2B**

Avril: I counted how many cookies they should get. Drawing gives you an idea.... Acting was breaking.”

**NOT SUCCESSFUL IN SOLVING THE PROBLEM**

**TAPE 3A**

Lorne – “it showed me like half and half or whole and stuff. And I could like draw math and stuff, like circles, except... they were cookies.” Tell me more about draw the math. “Like if there were 15 houses and 5 probably, I could probably do that, only like I wouldn’t draw full houses with so much detail, I’d use up all my time just using one house... I actually just draw circles for like everything. Sometimes squares, but rarely ever.”

**SUMMARY**

<table>
<thead>
<tr>
<th>TYPE OF SUPPORT (HOW DRAWING HELPED)</th>
<th>NUMBER OF REFERENCES TO EACH TYPE OF SUPPORT</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subject of an action</td>
<td>2</td>
</tr>
<tr>
<td>Distribution</td>
<td>1</td>
</tr>
<tr>
<td>Memory aid</td>
<td>0</td>
</tr>
<tr>
<td>Elimination system</td>
<td>1</td>
</tr>
<tr>
<td>Pictures as representations</td>
<td>1</td>
</tr>
</tbody>
</table>
### TAPE 1B

**Lisa:** "I drew faces... and cookies. I don't know why. I just wanted to. So when we don't have to do it in our heads and doing it in our heads is harder."

**Jason:** "I just make every children have one cookie and then there were some cookies left and so I divided them in 2's so everyone would have one and a half."
  "...to remind you."
  "It's easier to write it down."

Do you sometimes make a picture in your head? "Yeah."

**Grant:** "Cuz you can draw faces and draw cookies to the faces and you can... instead of just keeping it in your head."

### TAPE 2A

**John:** "...it helped us notice um how many... kids get the cookies and how they can share them all." How did it help you? "The lines... tell you how many you need and the lines that connect tell you which cookies you get."

**Martin:** "...to help me count and stuff. Because like I pretend these are like cookies, and then I cross off some cookies because the kids eat them. Then there were 6 more left, then I crossed them out because 6 more children ate them..."

**Pricilla:** "It helped instead of in your head and get mixed up."

What did the drawing do to make it easier?

"By looking at it and thinking how much I can cross off."

### TAPE 3A

**Cathy:** "It helps me draw so like, uh, in case I do like four in half and it doesn't work then like sometimes it can help me, so I try to cut it in half or just leave it..." Ok. Does it help in any other way? "It helps me solve the problem..."

**Norman:** "I don't really know. I draw 18 cookies and I accidentally draw 22 so I crossed it out, and I draw a 12 kids and I put every number in the cookies and then I put the same amount of cookies and when I got to 12... and then I crack two and I got one half... and I shade it in like this..." How did the shading in help you? "It doesn't make me lost count..."
Ruby: “I drawed one cookie for each children then I saw the 6 leftover and I knew that I had to cut them into pieces and I cut them into 2 pieces. Because then it will all give one and a half.”

David: “So we don’t get.” When did it help most? “When I put the line.”
Charlie: “So I could remember the answer.”

Hannah: “It just helped me count the cookies.”

Shauna: What strategy did you use? “I, um drawing.” How did it help you? “Good, cuz I couldn’t do it in my head. Cuz there was 18 cookies.” “It helps because I can try different things, like 3’s or ones if that works.”
Iris: “It helped me how to get the problem solved in an easier way.” What makes it easier? “Like I use the picture to help me with the problem. Like to see if I have to make some of them a half.”
Stephanie: Do you ever see these things in your head? “Yeah.” What do you see in your head when you do this problem? “Just see like cookies and kids and like and see like, how can you share them in your head.”

**SUMMARY**

<table>
<thead>
<tr>
<th>TYPE OF SUPPORT (HOW DRAWING HELPED)</th>
<th>NUMBER OF REFERENCES TO EACH TYPE OF SUPPORT</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subject of an action</td>
<td>6</td>
</tr>
<tr>
<td>Distribution</td>
<td>3</td>
</tr>
<tr>
<td>Memory aid</td>
<td>6</td>
</tr>
<tr>
<td>Elimination system</td>
<td>3</td>
</tr>
<tr>
<td>Pictures as iconic representations</td>
<td>0</td>
</tr>
</tbody>
</table>
Drawing Helped – Clear response

TAPE 2B

Anthony: What made you want to draw? “Because drawing helps.” “It’s like acting what’s happening. Drawing’s like acting because it… [interrupted by Orson]” How is it like acting? When you were drawing, what part were you acting? “The kids that take cookies.” So it’s the taking. That’s what we were acting out?

Orson: “Drawing gives you more ideas to write about or think about. …like how much you give other people. It’s like acting because it’s like what people do.” Did you have any acting in yours, Orson? “Yes, like taking away, like giving people.”

TAPE 4B

Zarah: “It helped me so when I look at it, it would be a little bit easier. … it’s like you could look at the picture and do it with the picture instead of doing it in your brain and making it explode kinda.” (smiles) So how does a picture help? “It helps you put your ideas down on the paper so that your brain won’t feel like it’s going to explode.”

Jessie: “It helped me um get smarter. By getting harder questions.” So can you work on harder questions if you can draw it? Is that what you mean? “Yeah, by like um, the drawing is sort of like doing the math. Like drawing the math.” What part of the math were you drawing, Jessie? “It’s like we’re doing take-aways. Like I did one, there’s 1, 2, 3 and 1, 2, 3, and it makes six. And it makes 12.”

TAPE 3A

Cathy: What’s easier? Pictures or no pictures? “Pictures are easier – cuz then you just draw and then you know how much stuff… how many wheels are on a toy car or instead of just writing writing writing and you take up the whole space.”

Lorne: What’s easier? Pictures or no pictures? “They’re both the same. I thought um not drawing was like 100 - like 98% hard, but actually it’s fairly easy. Like I can’t tell – maybe this is slightly harder, that is slightly harder. If sure is shorter doing this (indicates number sentence). But I need pictures to guide me. So it’s best if I team up. (indicates pictures and number sentence)”
**SUMMARY**

<table>
<thead>
<tr>
<th>Type of Support (How Drawing Helped)</th>
<th>Number of References to Each Type of Support</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subject of an action</td>
<td>3</td>
</tr>
<tr>
<td>Distribution</td>
<td>1</td>
</tr>
<tr>
<td>Memory aid</td>
<td>1</td>
</tr>
<tr>
<td>Elimination system</td>
<td>0</td>
</tr>
<tr>
<td>Pictures as iconic representations</td>
<td>0</td>
</tr>
</tbody>
</table>

**SUMMARY OF ALL RESULTS:**

<table>
<thead>
<tr>
<th>Type of Support (How Drawing Helped)</th>
<th>Number of References to Each Type of Support</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subject of an action</td>
<td>6</td>
</tr>
<tr>
<td>Distribution</td>
<td>5</td>
</tr>
<tr>
<td>Memory aid</td>
<td>7</td>
</tr>
<tr>
<td>Elimination system</td>
<td>3</td>
</tr>
<tr>
<td>Pictures as iconic representations</td>
<td>1</td>
</tr>
</tbody>
</table>
Dear Ms. XXXXX:

This letter is to seek permission to conduct educational research at your school, XXX Elementary. As the co-investigator in a study entitled “Drawing as problem-solving: Young children’s mathematical reasoning through pictures”, I would like to present to you the purpose and overview of procedures being proposed.

I am currently pursuing a Masters of Arts in Mathematics Education; the study described below will form the basis for my graduate research. Along with my faculty advisor and graduate research supervisor Dr. Cynthia Nicol, I hope to investigate how young children think through mathematical problem-solving. We are interested in finding out how children respond when presented with a mathematical problem to solve, what kinds of pictures they draw spontaneously, the things they are thinking while they draw, and how to support children in developing these skills in order to become better mathematical problem-solvers.

In terms of procedures, I would like to work with approximately twenty grade 2 students from XXX, drawn from the different grade 1/2 and 2/3 classes. I will work with small groups of 4-5 children at a time, preferably in a small resource room or other quiet space that will allow for videotaping with quality audio. Children will be presented with a mathematical problem to solve. I will videotape the small group while they solve the problem and ask questions about students’ thinking while working through the problem. Afterwards, the children’s work will be collected. This small group problem-solving session will last approximately 45 minutes. The videotapes and student work will be examined afterwards, looking for illustrative pieces that will help to show the nature of drawing as a problem-solving tool in mathematics.
additional mathematical problems. This individual problem-solving interview will last about 45 minutes. It will also be videotaped, and will follow the same format as outlined above – children will be presented with a problem, then while solving it, be asked to talk about their thinking and drawing. If your child is selected and you and your child are willing for your child to participate in this individual problem-solving interview, notification and further information will be given.

Throughout this project we will be very happy to show participants the videos of themselves. The only people who will see the videos apart from them will be your child’s classroom teacher and the investigators. No one else will see the tapes. Your child will of course be free to opt out of the project at any time and such withdrawal will not affect his/her grades or relationship with the school in any way. Students who do not choose to be part of the project will simply not be videotaped.

**Confidentiality:** Any information resulting from this project will be kept strictly confidential. All documents and videotapes will be kept in a locked cabinet. Any digital video will be kept in secure electronic files. Any report that is written concerning this project will, by giving false names, preserve the complete anonymity of all participants and their school. However, you and your child are welcome to view at any time, upon request, any video clips that include your child. Once the co-investigator, Carole Saundry, selects pieces of video to create a digital illustration of “drawing as problem solving”, we will ask you and your child if you are willing for these video clips to be used in a more public forum to help other teachers or researchers. But you and your child do not need to make that decision until after you have seen the video clips.

**Contact:** If you have any questions or desire further information about this study you many contact Dr. Cynthia Nicol at 822-5246. If you have any concerns about your child’s rights as a research participant you may contact the Director of Research Services at the University of British Columbia, at 822-8598.

**Consent:** Students’ participation in this study is entirely voluntary and they may refuse to participate or withdraw from the study at any time without jeopardy to their class standing, grades, or relationship with the school.

Please indicate your child’s participation or non-participation by completing and detaching the consent slip below. Please keep this description of the study and detach the consent slip below. Thank you.
Please check the box indicating your decision:

☐ I CONSENT to my child participating in the videotaping of small-group problem-solving sessions, and, if chosen, I CONSENT to my child participating in the individual problem-solving interview. I CONSENT to the copying of materials produced during the lessons as described in the above form.

☐ I DO NOT CONSENT to my child participating in the study as described in the form.

☐ I acknowledge that I have received a copy of this consent form.

Name of student (please print) ___________________________ Date: ___________________________

Signature of parent or guardian ____________________________________________
Please check the box indicating your decision:

☐ I CONSENT to my child participating in the videotaping of small-group problem-solving sessions, and, if chosen, I CONSENT to my child participating in the individual problem-solving interview. I CONSENT to the copying of materials produced during the lessons as described in the above form.

☐ I DO NOT CONSENT to my child participating in the study as described in the form.

☐ I acknowledge that I have received a copy of this consent form.

Name of student (please print) ___________________________ Date: _________________________
Signature of parent or guardian __________________________
Please check the box indicating your decision:

☐ I CONSENT to the sharing of digital video clips that include my child in a more public forum – in teacher education coursework, graduate student coursework, and at educational and research oriented conferences. I understand that my child may be identified in the video clips, however the school and classroom will not be named.

☐ I DO NOT CONSENT to the sharing of digital video clips that include my child in a more public forum.

☐ I acknowledge that I have received a copy of this consent form for my own records.

Name of student (please print) ___________________________ Date: ______

School:__________________________________________________________

Teacher:________________________________ Div: __________

Signature of parent or guardian________________________________________
Script for Introducing Study to Grade 2 Students

Today I will be asking you to take a letter home to your parents. This letter tells about a study that a teacher is doing. She is curious about how grade 2 children think when they solve math problems. She would like to work with small groups of children in grade 2, give them some math problems to solve and ask them questions about their strategies. This teacher will videotape children while they work so that she can keep a record of their thinking. The children will work in another small room in our school where they can have some space to spread out and then talk about their ideas and strategies. She will read you the problems and give you paper to record your thinking.

This letter talks about the study, and asks your parent if it is ok for you to participate. It's all right if you don't want to. This is your choice. If you don't want to work on the problems with the teacher, no one will be upset. You can stay in the classroom with the rest of the class. If you want to participate, your parent should sign the form and send it back to the school. After that, the teacher will begin to work with small groups to solve math problems.

Thank you.