MODELLING THE IMPACT BEHAVIOUR OF
FIBRE REINFORCED COMPOSITE MATERIALS

by

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Abstract

Three analytical models describing the behaviour of composite targets under impact, and suitable for engineering applications are developed herein. Each model assumes that the impacting projectiles are rigid, and that the targets are fibre reinforced laminated plates.

One model is concerned with the non-penetrating impact of hemispherical projectiles. A modal series solution which includes the effects of shear deformation and rotary inertia is developed to describe the target deformations in this model.

Another model predicts the penetration behaviour of blunt projectiles. Results of static penetration tests are used by this model to characterize the damage caused by impacting blunt projectiles.

The third model describes the penetration due to impacting conical shaped projectiles. The progression of damage as it is described by Zhu et al [1992] is used as a basis for the characterization of damage in this model.

Each model is compared with experimental results obtained from low velocity instrumented impact tests, and high velocity ballistic tests.
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CHAPTER ONE

Introduction

Advances in the design and manufacture of fibre reinforced plastics have enabled these materials to be used in place of traditional homogeneous materials. The behaviour of composite materials under extreme load conditions is of great interest as their use in ballistic protection and aerospace applications has increased. Composite materials are also being implemented as critical structural components in which reliability and damage detection is important.

Design with regard to impact performance is obviously necessary for protective systems and aircraft components, however, low velocity impact behaviour is also important for critical components that may experience ‘tool drop’ impacts. An understanding of the damage caused by these incidental impacts, which is often difficult to detect, is required when predicting post impact strength and reliability.

Three types of damage can be expected in fibre reinforced laminated materials: delaminations, matrix cracking, and fibre breakage. Delaminations occur between the layers of the laminate and are the result of shear stresses between these layers. Matrix cracks are often associated with delaminations, as they develop through layers, allowing the delaminations to grow throughout the laminate. Fibre breakage can result from high tensile strains in the plane of the laminate, or high shear loads applied in the transverse directions.

The study of composite materials, including damage states, has garnered much interest recently.

1.1 Previous Work

For information concerning the design and manufacture of composite materials, the reader is directed to any number of introductory texts. Jones [1975] provides insight into the mechanics of fibre reinforced
composites, including micro-mechanical determination of lamina stiffness, and strength theories of lamina.

Selection of fibre and matrix with respect to performance and compatibility is examined by Hull [1981].

**Impact Dynamics**

The mechanics of impacting bodies and isotropic targets was examined early in this century by Timoshenko [1913] and others. The deflection of beams impacted by non-penetrating rigid spheres was analyzed by Timoshenko, using an integral form of Newton’s law of motion and the Hertz contact relation. This simple analysis contrasts with recent work concerning penetrating projectiles with velocities in excess of 1 km/s, and non-homogeneous composite targets. The method used by Timoshenko for the analysis of target response is used extensively in this thesis, for both low and high velocity impact events.

The contact problem has been examined by many authors, most notably Willis [1967], who presented a Fourier transform solution to the indentation problem of a rigid flat punch and an anisotropic half-space. Dahan and Zarka [1977] adapted the Hertzian type indentation relations for an isotropic materials to transversely isotropic media.

Herrman [1955] presented a general solution to the problem of the forced vibration of Timoshenko beams. A series solution and corresponding property of orthogonality is developed, analogous to the shear deformable plate solution presented in Chapter 2.

The application of finite element analysis to impact problems was investigated by Trowbridge [1991]. A non-linear finite element routine is used to model the low velocity impact of a spherical ball on a cylindrical rod. A specialized element is developed which accounts for the non-linear contact relationship and the linear target response. Analysis results are shown to be reasonably close to experimentally measured contact force histories.

den Reijer [15] presented an analytical model in the form of a computer program called ALARM. The impact response of isotropic armour with a ceramic facing is modelled and predictions are compared to experimental results. Mechanisms such as plastic hinging, membrane action, and shear failure of the
backing plate are considered. Model predictions are highly dependent on the behaviour of the ceramic facing and are thus not applicable to uncoated composite targets.

**Impact of Metallic Targets**

Extensive analytical investigations into the behaviour of metals under impact have been published.

Woodward [1978] examined the penetration behaviour of conical shaped projectiles as they impact metallic targets. Ductile hole formation associated with thicker targets is compared to the dishing type failure often seen in thin target plates. The work done by the penetrator in each type of failure is used to predict the impact behaviour of the target.

Woodward and Crouch [1989] developed a simple analytical model for the impact of flat shaped projectiles on layered metallic targets. An energy balance is applied to the projectile and target which includes global deformations of the target, although a contact force history is not generated. The model is presented in the form of a computer program called LAMP.

Awerbuch and Bodner [1974] presented an analytical model for the penetration of metallic targets by blunt projectiles. The process is divided into three distinct phases: initial plug formation, shear growth of the plug, and exit of the plug and projectile. This model is adapted to composite targets in Chapter 3.

A five stage model for the perforation of isotropic metallic targets was developed by Ravid and Bodner [1983] and extended to include the perforation of layered metallic targets by Ravid et al [1987]. Plastic flow of the target material forms a bulge on the impact face of the target in the preliminary stage. The second stage considers the formation of a bulge on the distal face of the target. This distal bulge grows in depth but not in width during the third stage. Plug formation begins in stage four, and stage five considers the exit of plug and projectile. In all stages the projectile is assumed to be rigid, target deflections are ignored, and frictional forces are assumed to be negligible. This five stage model is complex and not suitable for an engineering type of approach. As well the model does not predict a contact force history, making it difficult to compare to experiments.
Impact of Composite Materials

Abrate [1991] presented a survey article of existing experimental studies and analytical models dealing with the impact of composite materials. Articles describing experimental apparatus and procedure for gas gun, drop weight, flyer plate, and other types of testing methods, are listed. Simple spring-mass models, energy balance models, and wave propagation models are described. Studies concerning local indentation laws for hemispherical and flat shaped indenters are reviewed. Cantwell and Morton [1991] presented a similar review.

Sun and Chattopadhyay [1975] presented a complete formulation for the elastic impact of an isotropic sphere and a laminated specially orthotropic plate. The Hertz contact law is used in conjunction with a simplified series type solution which includes shear deformation, but neglects rotary inertia. An incremental approach in time is used to solve the non-linear equations of motion. As well, a simplification is used to perform the time dependent integration, required by the series solution. Results are presented for deflection, bending stress, and shear stress over the duration of impact. These results are used for comparison in Chapter 3.

Zhu et al [1992] developed a penetration model for conical shaped projectiles impacting Kevlar/polyester laminated plates and compared the model predictions to experiment. Experimental projectile velocities were typically about 300 m/s, to a maximum value of 800 m/s. A finite difference analysis is used in the model to predict the global target response. A local penetration model, which includes a prediction of damage, is used. This work is used extensively to develop the penetration model described in Chapter 4.

Langlie and Cheng [1990] developed a comprehensive model for high-velocity penetrations, and incorporated it into a non-linear transient finite element code. A punching shear type of failure is assumed for flat shaped projectiles. Contact force is used to predict the onset of damage, and a shear wave propagation model is used to model the damaged target behaviour.
Experimental Study

Experimental results and observations from tests performed at UBC provide a basis for the impact models developed in this thesis. Three types of tests have been performed: static tests, instrumented low velocity impact tests, and high velocity ballistic tests. A complete and more detailed description of the experimental set-up is contained in Delfosse et al [July 1993].

The instrumented impact tests provide a valuable information about impact events, and are essential for the development and evaluation of impact models. The apparatus consists of a gas gun, and an instrumented projectile which records a contact force history of the impact event. The heavy projectile can be fired at velocities up to 50 m/s, resulting in high impact energies.

A picture and schematic of the instrumented projectile are shown in Figure 1.1 and Figure 1.2. A small, light weight piezoelectric load cell is used to measure loads between an aluminum mass and the projectile shaft. Hardened steel (30Rc) cylindrical impactor heads of diameter 7.6 mm with flat and conical (37° tip angle) nose shapes are threaded onto the tip of the projectile. The total mass of the projectile, depending on the configuration, is either 320 g, or 308 g.

1.2 Present Work

This thesis investigates the behaviour of laminated composite targets as they are impacted by a variety of projectiles. To simplify the modelling of dynamic target behaviour, carbon fibre reinforced epoxy materials, which are rate insensitive, are studied. Both non-penetrating and penetrating events are modelled. Hemispherical shaped projectiles are examined in the context of non-penetrating impact events, while penetrating events are modelled with respect to blunt and conical shaped projectiles. Both ballistic and low velocity 'tool drop' impacts, including the impact of fragments and objects with unconventional shapes, will be applicable to one of the three models presented.

Each model is analytical in nature, and has been implemented into a user friendly computer code, designed to operate in the Windows environment. This type of engineering software will complement other types of
impact studies, including experimental and numerical techniques. A description of the software, *UBC Impact*, is included in Appendix B.

The foundation of each model is the separation of local and global effects. Local effects generally include indentation or penetration, and are unaffected by the target behaviour away from the point of impact. Global effects include the vibrations and elastic deflections of the target plate. Global and local effects are a part of any impact event, however, the local effects tend to dominate at higher impact velocities.

The modelling of global target behaviour is required to predict low energy impact events, and also leads to a better understanding of local effects. The non-penetrating impact model, described in Chapter 2, combines a modal solution for global target deflections, with a well known relation describing the indentation of hemispherical indenters. This model allows validation of the global analysis through comparison with previously published analytical and numerical results, and with experimentally measured results.

The global model developed in Chapter 2 is used in the blunt model presented in Chapter 3, and in the conical model presented in Chapter 4. The blunt and conical models use static penetration behaviour as a basis for dynamic impact predictions. This approach is reasonable due to the strain rate independence of the composite material considered here.

A composite material, for the purposes of this thesis, is defined as a material in which evenly distributed, continuous fibres are aligned within a homogeneous matrix material. Thin slices, or laminae, of such a material are stacked with specific fibre orientations to form a laminate. A detailed examination of the elastic properties and behaviour of such laminates is presented in Appendix A.
1.3 Figures

Figure 1.1: Digital image of instrumented projectile, shown with blunt tip.

Figure 1.2: Schematic drawing of instrumented projectile.
A model describing the non-penetrating impact of rigid, hemispherical projectiles on fibre reinforced laminates has been developed. Analytical techniques are applied to local indentation and global target deflection, which together define the impact problem.

In contrast with existing models, an analytical form of the non-linear Hertzian contact relation is included in the solution. A modal series solution for the target deformation, which includes all first order shear and inertia effects, is developed. The accuracy and convergence of the plate solution and impact model are studied and compared with published results.

The non-penetrating impact model is applicable to low energy impact events, where inelastic effects due to damage are insignificant. This model is important because it serves to validate the modal laminate solution used to calculate target deflections, which can also be used to describe the global behaviour in penetrating models.

2.1 Background

The study of non-penetrating impacts has produced many solutions based on the small increment method first presented by Timoshenko [1913]. This numerical method applies rigid body dynamics to the projectile, and equates projectile displacement to the sum of local deformation and plate bending.

The analysis of hemispherical projectiles allows the well known Hertzian type relation to be used in the solution for local indentation. Sun and Chattopadhyay [1975] applied a recursive relation to deal with the non-linearity introduced by the Hertzian relation. Christoforou and Swanson [1991] used an approximate,
linearized form of the Hertzian relation. Both of these numerical methods simplify the analysis, but introduce an unnecessary error into the solution.

The dynamic response of orthotropic, fibre reinforced targets can be solved with a modal series solution. Dobyns [1980] and others have used the plate equations developed by Whitney and Pagano [1970] to model the target deflection. These equations include first order shear deformation and inertia terms, but the rotary inertia effects are often ignored. It has been shown that shear deformation effects are significant, but rotary inertia terms are not, when considering low velocity impact events. The plate equations apply to rectangular targets with simply supported boundary conditions.

A new model has been developed, which includes a non-linear analytical solution to the Hertzian contact problem. A plate solution which includes shear deformation and rotary inertia effects, has been developed for this model, so that the solution may be applied to high velocity impact events.

2.2 Theory of Non-Penetrating Impact

As in previous approaches, the basis of the new model is the recognition of two distinct responses to impacting projectiles. A hemispherical projectile causes global deformation of the target mid-plane, and local indentation of the target surface. Total projectile displacement is equal to the sum of global deformation and local indentation.

Thus the compatibility condition is:

$$\Delta_p = \alpha + w$$  \hspace{1cm} (2.1)

where $\Delta_p$ is the projectile displacement, $\alpha$ is the indentation, and $w$ is the target deflection (see Figure 2.1).

2.2.1 Indentation

Local indentation of the target is generally assumed to follow a Hertzian type relation. Hertz suggested that contact between a sphere and an elastic medium could be described in terms of the total applied force $F$ and the resulting indentation $\alpha$. 
The Hertzian relation is as follows:

\[ F = k_H \cdot \alpha^{3/2} \]  

(2.2)

The Hertz contact stiffness, \( k_H \), for a transversely isotropic material, is\(^1\):

\[ k_H = \frac{\sqrt{D}}{3 \left( \frac{1-v_p^2}{E_p} + \frac{1-v_{xz}v_{yz}}{E_z} \right)} \]  

(2.3)

where \( E_p \) and \( v_p \) are the modulus and Poisson's ratio of the projectile, \( E_z \) is the transverse modulus of the composite, \( v_{xz} \) and \( v_{yz} \) are the transverse Poisson's ratio of the top layer of the composite, and \( D \) is the tip diameter of the hemispherical projectile.

The force-displacement results for a static indentation test on Material A, are plotted in Figure 2.2. The properties of Material A are listed in Table 2.1. Superimposed on Figure 2.2, is the theoretical indentation curve calculated using Eq. (2.2) and Eq. (2.3). The stiffness calculated according to Eq. (2.3) is 14.1 N/m\(^{2/3}\).

The obvious disagreement shown in Figure 2.2 is not surprising due to the nature of the elastic medium, in this case, a laminated plate. In the static indentation test the plate is placed on a steel backing plate, making it much more stiff than the half space assumed by Hertz. In an impact event, there is a stress free condition on the distal side of the target, making it even less stiff than the Hertz assumption of a half-space. For small amounts of penetration it is assumed that the error due to the stress free condition, is small.

Figure 2.3 shows the same experimental result, with two 'best fit' curves superimposed. Curve 'a)' uses an exponent of 3/2 as in Eq. (2.2), with a stiffness adjusted to match the experiment. Curve 'b)' is fitted with respect to both exponent and stiffness. The best fit parameters are summarized in Table 2.2.

---

\(^1\) See Willis [1967].
The most accurate contact law is curve "b)" in Figure 2.3, which uses an exponent of 1.7, instead of the Hertzian exponent of 1.5. However, the mathematical advantage of using the Hertzian exponent will be apparent when the impact problem is formulated, therefore an exponent of 3/2 is used.

Thus the contact law is:

\[ F = k_e \cdot \alpha^{3/2} \]  

(2.4)

where \(k_e\) is the non-linear contact stiffness obtained as in Figure 2.3.

### 2.2.2 Target Response

The target laminate is modelled as a specially orthotropic, rectangular laminate with simply supported boundary conditions. A dynamic solution for this type of target is available, and is presented in detail in Appendix A.

The laminate equations as developed by Whitney and Pagano, including the effects of transverse shear deformation and rotary inertia are solved.

The assumed displacement field takes the following form:

\[ u = u^0(x, y, t) + z\psi_x(x, y, t) \]  

(2.5a)

\[ v = v^0(x, y, t) + z\psi_y(x, y, t) \]  

(2.5b)

\[ w = w(x, y, t) \]  

(2.5c)

where \(u^0\), \(v^0\) and \(w\) are the laminate displacements in the \(x\), \(y\), and \(z\) directions at the mid-plane, \(\psi_x\) and \(\psi_y\), are the cross-sectional rotations in the \(x\) and \(y\) directions, respectively.
The differential equations of motion are as follows:

\[
\begin{align*}
D_{11} \frac{\partial^2 \psi_x}{\partial x^2} + (D_{12} + D_{66}) \frac{\partial^2 \psi_x}{\partial x \partial y} + D_{66} \frac{\partial^2 \psi_y}{\partial y^2} - kA_{55} \left( \psi_x + \frac{\partial w}{\partial x} \right) &= f\dot{\psi}_x \quad (2.6a) \\
D_{11} \frac{\partial^2 \psi_y}{\partial y^2} + (D_{12} + D_{66}) \frac{\partial^2 \psi_y}{\partial x \partial y} + D_{66} \frac{\partial^2 \psi_x}{\partial x^2} - kA_{44} \left( \psi_y + \frac{\partial w}{\partial y} \right) &= f\dot{\psi}_y \quad (2.6b) \\
kA_{55} \left( \frac{\partial \psi_x}{\partial x} + \frac{\partial w}{\partial x^2} \right) + kA_{44} \left( \frac{\partial \psi_y}{\partial y} + \frac{\partial w}{\partial y^2} \right) + p_z(x, y, t) &= \rho \ddot{w} \quad (2.6c)
\end{align*}
\]

where the overdot indicates differentiation with respect to time.

A solution to the homogeneous form of Eq. (2.6) is obtained by assuming rotations and lateral displacement as follows:

\[
\begin{align*}
\psi_x &= e^{i\omega t} \cdot U_{mn} \cos(m\pi x/a) \sin(n\pi y/b) \quad (2.7a) \\
\psi_y &= e^{i\omega t} \cdot V_{mn} \sin(m\pi x/a) \cos(n\pi y/b) \quad (2.7b) \\
w &= e^{i\omega t} \cdot W_{mn} \sin(m\pi x/a) \sin(n\pi y/b) \quad (2.7c)
\end{align*}
\]

where \(U_{mn}, V_{mn},\) and \(W_{mn}\) are undetermined constant coefficients, and \(a, b\) are the length and width as shown in Figure 2.4.
Substituting Eq. (2.7) into the governing differential equations, results in a set of three linear algebraic equations:

\[
\begin{bmatrix}
L_{11} - I\alpha_{mn}^2 & L_{12} & L_{13} \\
L_{12} & L_{22} - I\alpha_{mn}^2 & L_{23} \\
L_{13} & L_{23} & L_{33} - \rho k\alpha_{mn}^2
\end{bmatrix}
\begin{bmatrix}
U_{mn} \\
V_{mn} \\
W_{mn}
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\quad (2.8)
\]

where

\[
\begin{align*}
L_{11} &= D_{11}(m\pi/a)^2 + D_{66}(n\pi/b)^2 + kA_{55} \\
L_{12} &= (D_{12} + D_{66})(m\pi/a)(n\pi/b) \\
L_{13} &= kA_{55}(m\pi/a) \\
L_{22} &= D_{66}(m\pi/a)^2 + D_{22}(n\pi/b)^2 + kA_{44} \\
L_{23} &= kA_{44}(n\pi/b) \\
L_{33} &= kA_{55}(m\pi/a)^2 + kA_{44}(n\pi/b)^2
\end{align*}
\quad (2.9)
\]

Each solution set m,n, results in three eigenvalues \(\omega_{mnj}\) and their associated eigenvectors \(U_{mnj}, V_{mnj}, W_{mnj}\), where the subscript \(j = 1,2,3\). Only two components of each eigenvector are independent, thus the eigenvectors are normalized with respect to \(W_{mnj}\).

The particular solutions are assumed to be separable into functions of position and time, as follows:

\[
\psi_j = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{j=1}^{3} U_{mnj} \cdot \cos(m\pi x/a)\sin(n\pi y/b) \cdot T_{mnj}(t)
\quad (2.10a)
\]

\[
\psi_j = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{j=1}^{3} V_{mnj} \cdot \sin(m\pi x/a)\cos(n\pi y/b) \cdot T_{mnj}(t)
\quad (2.10b)
\]

\[
w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{j=1}^{3} W_{mnj} \cdot \sin(m\pi x/a)\sin(n\pi y/b) \cdot T_{mnj}(t)
\quad (2.10c)
\]
Substituting Eq. (2.10) into the equations of motion, and applying the homogeneous solution yields:

\[ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{j=1}^{3} U_{mnj} \cos(m\pi x/a) \sin(n\pi y/b) \left[ \omega_{mnj}^2 T_{mnj} + \tilde{T}_{mnj} \right] = 0 \]  
(2.11a)

\[ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{j=1}^{3} V_{mnj} \sin(m\pi x/a) \cos(n\pi y/b) \left[ \omega_{mnj}^2 T_{mnj} + \tilde{T}_{mnj} \right] = 0 \]  
(2.11b)

\[ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{j=1}^{3} W_{mnj} \sin(m\pi x/a) \sin(n\pi y/b) \left[ \omega_{mnj}^2 T_{mnj} + \tilde{T}_{mnj} \right] = \frac{P_z}{\rho h} \]  
(2.11c)

In order to solve for the time dependent variable \( T_{mnj} \), a single equation without summation, i.e. an orthogonal set, is required. The orthogonality condition for the solution sets is obtained by applying Clebsch’s Theorem.

Clebsch’s Theorem, in a general form, is\(^2\):

\[ \int \int \int \rho \left[ u_x u_s + v_y v_z + w_x w_z \right] dx dy dz = 0 \quad r \neq s \]  
(2.12)

where the displacements \( u, v, w \), have solutions of the form:

\[ \{u, v, w\} = \sum \{u_r, v_r, w_r\} \cdot f^n(t) \]  
(2.13)

Using the assumed solutions of Eq. (2.10), which are in the above form, and inserting into Clebsch’s equation, the orthogonality condition becomes:

\[ \int_{0}^{l} \Pi_{efg, mnj} \ dy dx = 0 \quad e, f, g \neq m, n, j \]  
(2.14)

where

\[ \Pi_{efg, mnj} = I U_{efg} \cos(e\pi x/a) \sin(f\pi y/b) \cdot U_{mnj} \cos(m\pi x/a) \sin(n\pi y/b) \]
\[ + I V_{efg} \sin(e\pi x/a) \cos(f\pi y/b) \cdot V_{mnj} \cos(m\pi x/a) \sin(n\pi y/b) \]
\[ + \rho h W_{efg} \sin(e\pi x/a) \sin(f\pi y/b) \cdot W_{mnj} \cos(m\pi x/a) \sin(n\pi y/b) \]  
(2.15)

\(^2\) See Love [1926].
Multiplying Eq. (2.7a) by:

\[ IU_{efg} \cos(\pi x/a) \sin(f\pi y/b); \]  

Eq. (2.7b) by:

\[ IV_{efg} \sin(\pi x/a) \cos(f\pi y/b); \]  

and Eq. (2.7c) by:

\[ \rho h W_{efg} \sin(\pi x/a) \sin(f\pi y/b) \]  

then summing the three results, one equation in terms of \( T_{mnj} \) is obtained:

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{j=1}^{3} \omega_{mnj}^{2} \Pi_{efg, mnj} T_{mnj} + \Pi_{efg, mnj} \bar{T}_{mnj} - \rho h W_{efg} \sin(\pi x/a) \sin(f\pi y/b) \cdot p_{z} = 0 \]  

Integrating over the laminate area meets the orthogonality condition, leaving only one non-zero term when \( e, f, g = m, n, j \) and allowing the summation to be dropped:

\[
\omega_{mnj}^{2} T_{mnj} + \bar{T}_{mnj} = \frac{ab}{4} \int_{0}^{a} \int_{0}^{b} \sin(m\pi x/a) \sin(n\pi y/b) p_{z} \, dy \, dx
\]

where

\[
M_{mnj} = \frac{4}{ab} \int_{0}^{a} \int_{0}^{b} \Pi_{mnj, mnj} \, dy \, dx
\]

\[
= IU_{mnj}^{2} + IV_{mnj}^{2} + \rho h W_{mnj}^{2}
\]

Note that \( W_{mnj} = 1 \) because of the eigenvector normalization, and the integrated term in Eq. (2.18) is simply the Fourier transform of the load, \( p_{z} \).

As well, the load is assumed to be stationary, i.e.:

\[
p_{z} = F(t) \cdot q(x, y)
\]
The Fourier transform of the load is then:

\[ \int_0^a \int_0^b \sin(m \pi x/a) \sin(n \pi y/b) p_z \ dy \ dx = F(t) \cdot q_{mn} \]  

(2.21)

Values of \( q_{mn} \) for different types of loading can be found in Appendix A.

Solving the differential equation of Eq. (2.18), using Eq.'s (2.20) and (2.21), yields:

\[ T_{mnj} = T_{mnj}^0 \cos(\omega_{mnj} t) + \frac{T_{mnj}^a}{\omega_{mnj}} \sin(\omega_{mnj} t) + \frac{ab \cdot q_{mn}}{4 \omega_{mnj} M_{mnj}} \int_0^t F(\tau) \omega_{mnj} \sin(t - \tau) \ d\tau \]  

(2.22)

where the nought superscript indicates a value \( @ t = 0 \).

The complete solution for lateral displacement of the target laminate is then:

\[ w = \sum_{m=1}^\infty \sum_{n=1}^\infty \sin(m \pi x/a) \sin(n \pi y/b) \cdot \sum_{j=1}^3 \left\{ T_{mnj}^0 \cos(\omega_{mnj} t) + \frac{T_{mnj}^a}{\omega_{mnj}} \sin(\omega_{mnj} t) + \frac{ab \cdot q_{mn}}{\omega_{mnj}^2 M_{mnj}} \int_0^t F(\tau) \omega_{mnj} \sin(t - \tau) \ d\tau \right\} \]  

(2.23)

2.2.1 Convergence of Target Response

A closed form convergence criterion for the Fourier solution above is not available. Convergence of the solution is demonstrated numerically for a typical laminate with three loading conditions: point load, load distributed over a small patch, and uniformly distributed load. Properties of the target are shown in Table 2.3.

For each type of loading, responses at the center of the target were calculated using 5, 10, 20, 40, 80, and 160 non-zero modes in both the x and y dimensions. Equation 2.23 was evaluated for a range of \( t \) values, on an Indigo workstation. In each case the load began at \( t = 0 \) and continued to act for the duration. The integration with respect to time in Eq. (2.23) is exact for this type of step loading.
Solution error is calculated using the solution for 160 modes as a reference, i.e.:

\[
\% \varepsilon_\lambda(t) = \left| \frac{w_\lambda(t)}{w_{160}(t)} \right| 100\% \quad \lambda = 5, 10, 20, 40, 80 \tag{2.24}
\]

where

\[
w_\lambda = \sum_{m=1}^{(2\lambda-1)} \sum_{n=1}^{(2\lambda-1)} \sin\left(m\pi/2\right) \sin\left(n\pi/2\right).
\tag{2.25}
\]

Figures 2.5, 2.6, and 2.7 plot this error function vs. the logarithm of a normalized time variable:

\[
t^* = \log\left(\frac{t}{\tau}\right) \tag{2.26}
\]

where

\[
\tau = h \cdot \sqrt{\frac{\rho}{E_3}} \tag{2.27}
\]

The expression of Eq. (2.27) is chosen as a normalizing parameter because it represents the time taken for one transverse shock wave to travel through the thickness of the laminate. Laminate theory assumes that all shock waves travelling through the thickness of the laminate have dissipated and are insignificant. Results calculated at \( t^* \leq 1 \) are likely invalid because of these shock wave effects.

A value of \( t^* = 0 \) indicates \( t = \tau \), which, for the target described by Table 2.3, is 1.2 \( \mu \)s.

For each set in the solution, i.e. each \( m,n \) pair, there is a set of three natural frequencies and three modal amplitudes, as shown by Eq. (2.8). One frequency is associated with the lateral motion of the target, and two frequencies are attributed to the cross sectional rotations. In general the frequency associated with lateral motion is an order of magnitude smaller than the others.
An approximate value of this dominant frequency can be calculated with the following:\(^3\):

\[
\omega_{mn}^2 = \left( Q L_{33} + 2 L_{12} L_{22} L_{43} - 2 L_{43}^2 - L_{11} L_{22}^2 \right) / (\rho h Q) \tag{2.28}
\]

where

\[
Q = L_{11} L_{22} - L_{12}^2 \tag{2.29}
\]

The \((m,n)\)th periods are calculated using the frequencies described by Eq. (2.28), i.e.:

\[
T_{mn} = \frac{2\pi}{\omega_{mn}} \tag{2.30}
\]

Natural frequencies for the material described in Table 2.3 are listed in Table 2.4.

In the target solution, all even numbered modes are zero because of the sinusoidal terms, thus the mode number, \(\chi\), corresponds to the mode indices \(m,n\) as follows:

\[
m, n = 2\chi - 1 \tag{2.31}
\]

The accuracy of a modal solution is often discussed in terms of the last significant natural period of vibration as it compares to the time step. The last significant natural period is shown on Figures 2.5, 2.6, and 2.7 with a vertical line.

**Point Loading**

Figure 2.5 shows the convergence of the laminate solution for a step load applied as a single point force at the centre of the plate. It is apparent that the number of modes required to reach reasonable accuracy, is highly dependent on the time of interest.

Convergence for this type of loading is poor due to the nature of the solution. Early in the event, the point force causes shear and flexural waves to travel outward toward the boundaries. As these waves travel outward, they act over a larger area, and the wave fronts increases in circumference, resulting in reduced

---

\(^3\) From Dobyns [1981].
intensities. However, at a time close to zero, before these waves have time to travel, the point load is supported by an infinitely small region of the target, resulting in infinitely large plate deformations. Thus at times close to zero the solution for target deflection is indeterminate.

Convergence is obtained for normalized times in excess of 3.0, for a reasonably small number of modes. For any given number of modes, the solution converges at a time much greater than the last significant natural period.

This type of loading is not recommended for time steps of the same order of magnitude as the normalizing parameter, $\tau$.

**Patch Loading**

Figure 2.6 shows the convergence of the laminate solution for a patch load applied at the center of the laminate. The patch area is equal to ten percent of the laminate area.

As each curve in Figure 2.6 approaches the ordinate axis, an amount of 'noise' appears. This fluctuation is due to the dynamic nature of the problem. The solutions using a small number of modes are slightly out of phase with the reference solution, causing deceptively high percentage error even after the solution has converged.

Applying the load over a patch eliminates the indeterminate problem seen in the point load solution. As expected the patch solution converges much more rapidly. Each curve shown in Figure 2.6 converges prior to the last significant natural period.

Typically, reasonable accuracy (0.1\%) is obtained at the last significant natural period. The small patch solution is the most appropriate solution for impact events, as reasonable accuracy can be achieved while realistically modelling the force applied to the target.

**Distributed Loading**

For completeness, behaviour of the uniformly distributed solution is presented in Figure 2.7. A unit load has been applied over the entire surface of the laminate.
Convergence of the distributed load is quite similar to the patch load solution, with reasonable accuracy obtained at the last significant natural period. The uniformly distributed load is obviously impractical for modelling impact events, although other dynamic loads such as blasting or fluid pressure may be modelled as such.

2.3 Impact Model

Combining the indentation law and the target deformation solution, a complete impact model can be formulated. The non-linear nature of the indentation law requires a model that is stepwise in the time domain, as in Timoshenko's incremental method.

The impact event is divided into equal segments or time steps, and the contact force is assumed to be constant through each time step. The non-linear equations describing the impactor/target system are solved in terms of the contact force.

The displacement of the rigid impactor, at any given time, in terms of the applied force, is described by:

\[ \Delta_p = V \cdot t - \frac{1}{M} \int_0^t F(\tau)(t-\tau)\,d\tau \]  

(2.32)

where \( V \) is the impactor velocity, and \( M \) is the impactor mass.

During any time increment, in which the contact force is constant, Eq. (2.32) can be rewritten:

\[ \Delta_p(t) = \Delta_p(t-\Delta t) + V(t-\Delta t) \cdot \Delta t - \frac{F(t)\Delta t^2}{2M} \]  

(2.33)

where \( \Delta t \) is the time increment.

In order to solve the displacement constraint equation, Eq. (2.1), using Eq. (2.33), both the target deflection and indentation must be expressed in terms of the contact force.

2.3.1 Stepwise Form of Target Deflection

The modal solution for target deflection requires an integration over the entire history of loading. In order to simplify this integration, the solution is evaluated with a moving time scale.
Evaluating the target deflection is simplified by applying a transformation in the time domain, as follows:

\[ i = \Delta t \]  

(2.34)

The initial conditions then become:

\[ T_{mnj} = T_{mnj}(i = 0) = T_{mnj}(t = t - \Delta t) \]  

(2.35a)

\[ \dot{T}_{mnj} = \dot{T}_{mnj}(i = 0) = \dot{T}_{mnj}(t = t - \Delta t) \]  

(2.35b)

The solution for target deflection is rewritten in a stepwise form by evaluating at \( i = \Delta t \):

\[
w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin(m\pi x/a) \sin(n\pi y/b) \left( \sum_{j=1}^{3} \left[ \frac{T_{mnj}^w}{\omega_{mnj}} \cos(\omega_{mnj} \Delta t) + \frac{\dot{T}_{mnj}^w}{\omega_{mnj}} \sin(\omega_{mnj} \Delta t) + \frac{ab q_{mn}}{\omega_{mnj}^2 M_{mnj}} \int_{0}^{\Delta t} F(\tau) \sin(\omega_{mnj} (\Delta t - \tau)) d\tau \right] \right)
\]  

(2.36)

This stepwise form requires the initial conditions to be used at each interval, and not just at \( t = 0 \). The initial conditions are re-evaluated at the end of each interval. This is equivalent to using initial conditions at \( t = 0 \), and performing the integration over the entire load history.

Equation (2.36) evaluated to reflect the assumption of constant contact force is:

\[
w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin(m\pi x/a) \sin(n\pi y/b) \left( \sum_{j=1}^{3} \left[ \frac{T_{mnj}^w}{\omega_{mnj}} \cos(\omega_{mnj} \Delta t) + \frac{\dot{T}_{mnj}^w}{\omega_{mnj}} \sin(\omega_{mnj} \Delta t) + F(t) \frac{ab q_{mn}}{\omega_{mnj} M_{mnj}} \left[ 1 - \cos(\omega_{mnj} \Delta t) \right] \right] \right)
\]  

(2.37)
Noting that the contact force is independent of the modal summation, a simpler form of Eqn. (2.37) can be written:

\[ w = w^o(t) + C \cdot F(t) \] (2.38)

where

\[ w^o(t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin(m\pi/2) \sin(n\pi/2) \sum_{j=1}^{3} \left[ T_{mnj} \cos(\omega_{mnj} \Delta t) + \frac{q_{mnj}}{\omega_{mnj}} \sin(\omega_{mnj} \Delta t) \right] \] (2.39)

is the displacement due to the loading \( F(t = 0 \rightarrow t - \Delta t) \), and

\[ C = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin(m\pi/2) \sin(n\pi/2) \sum_{j=1}^{3} \frac{abq_{mnj}}{\omega_{mnj} M_{mnj}} \left[ 1 - \cos(\omega_{mnj} \Delta t) \right] \] (2.40)

is the dynamic compliance of the target.

The load is assumed, for optimum convergence, to act over a small patch equal in size to the diameter of the hemispherical projectile. The load term \( q_{mn} \) is evaluated according to Appendix A.

### 2.3.2 Stepwise Form of Indentation

The local indentation is assumed to be independent of the load history, therefore Eq. (2.4) can be used directly. Thus the local response, in terms of the contact force, is:

\[ \alpha(t) = \left[ \frac{F(t)}{k_e} \right]^{2/3} \] (2.41)

### 2.3.3 Impact Model Expressions

The integer fraction exponent in Eq. (2.41), and the linear nature of Eq. (2.37) allow the impact model to be formulated analytically.
Substituting the displacement expressions of Eq.'s (2.33), (2.38), and (2.41) into the constraint condition, Eq. (2.1), one equation in terms of a single unknown variable, $F=F(t)$, is obtained:

$$
\lambda_0 F^3 + \lambda_1 F^2 + \lambda_2 F + \lambda_3 = 0
$$

where

$$
\lambda_0 = \left( C + \frac{\Delta t^2}{2M} \right)^3
$$

$$
\lambda_1 = -\frac{1}{k_H^2} + 3(\Delta_p^* - w^* + V^* \Delta t) \left( C + \frac{\Delta t^2}{2M} \right)^2
$$

$$
\lambda_2 = \sqrt{3}(\Delta_p^* - w^* + V^* \Delta t) \left( C + \frac{\Delta t^2}{2M} \right)^{\frac{3}{2}}
$$

$$
\lambda_3 = \Delta_p^* - w^* + V^* \Delta t
$$

The above cubic equation can be solved as follows:

$$
F = 2\sqrt{-Q \cos(\theta/3)} \frac{\lambda_1}{\lambda_0}
$$

where

$$
Q = \frac{3\lambda_2 - \lambda_1^2}{9\lambda_0}
$$

$$
\cos\theta = \frac{9\lambda_1 \lambda_2 \lambda_0 - 27\lambda_0^2 \lambda_3 - 2\lambda_1^3}{54\lambda_0^3 \sqrt{-Q^3}}
$$

### 2.3.4 Impact Model Convergence

Each of the displacement solutions (indentation and target deflection) are exact for a given contact force. However, because the contact force history is not known a priori, some error is incurred. Convergence of the full impact model is shown in Table 2.6.

---

From Speigel [1968].
Peak force is chosen as the defining characteristic, and the error listed in Table 2.6 is defined as follows:

\[ \% e_{\Delta t}(\tau) = \left( \frac{\bar{F}_{\Delta t}}{F_{0.1\mu s}} \right) \times 100\% \]  

(2.46)

where \( \bar{F}_{\Delta t} \) is the peak force calculated with time step \( \Delta t \), and \( \Delta t = 0.1 \mu s \) is used as the reference solution.

Qian and Swanson [1990] chose to use a 100 mode solution to calculate target deflection. However the present model uses a number of modes appropriate for the time step. The convergence suggested in Section 3.1.1 is applied, thus the last significant natural period is less than the chosen time step. Note that the natural periods listed in Table 2.4 are applicable here.

The different solution method applied to the present model results in different peak contact force values, and an improved convergence rate. Qian and Swanson used a patch of changing size corresponding to the amount of indentation. As well, they used the approximate, non-linear contact law as presented by Christoforou and Swanson [1991].

The present model attains a reasonable level of accuracy (0.1%) with a time step of 1 \( \mu s \), which corresponds to approximately 200 calculations. This is performed in a matter of seconds on a 486-50 MHz personal computer.

2.4 Results of Non-Penetrating Model and Discussion

The impact model is compared to published results and to experimental measurements with good agreement. A comparison with published results is shown in Figures 2.8 and 2.9. Qian and Swanson developed a Rayleigh-Ritz solution in addition to the analytical routine described above, and Sun and Chen [1985] used a finite element code to simulate an impact event. The impact conditions are listed in Table 2.5.

The present model agrees quite well considering the approximations involved. Each of the previous models used a changing patch size when calculating the target deflection; the present model does not. Qian and Swanson used a 100 mode solution, and a time step of 0.1 \( \mu s \). In accordance with the convergence results,
the present model used a 20 mode solution and a time step of 1 µs resulting in slightly more than 200 data points.

Contact force histories collected from instrumented impact tests are compared with model predictions in Figures 2.10 through 2.13. Each impact event involved a 25.4 mm hemispherical shaped projectile, and a 76.2 mm x 127.0 mm target of Material A. The impact tests were carried out with the gas gun and instrumented projectile.

For each simulation the time step was chosen in order to provide sufficient detail in the resulting contact force history. In each case at least two hundred points were generated by the model, with the low mass impact events requiring slightly more points to model the sharp peaks. The number of modes used to calculate target deflection was determined according to Table 2.7, ensuring that the last significant natural period is not larger than the time step. The shock wave parameter for Material A is τ = 2.07 µs, from Eq. (2.27).

Figure 2.10 compares results of a 1.76 m/s, 6.14 kg impact. The simulation used a time step of 20 µs, and a 10 mode solution (T_{m,n} = 11 µs) for target deflection. The analysis agrees closely with the experimental results in most aspects of the contact force history. The rate of loading and unloading as well as the peak force are accurately predicted. Oscillations observed early in the impact event are also indicated, to some extent, in the prediction. Post impact examination of the target indicated that the target incurred no permanent damage.

Figure 2.11 compares results of a 2.68 m/s, 6.14 kg impact. Again, the simulation used a time step of 20 µs, and a 10 mode solution (χ = 10), for target deflection. The analysis agrees closely with the experimental result in the early loading stage of impact. At a contact force of 8 500 N the predicted response is much stiffer than measured by the instrumented projectile. Subsequent examination of the target indicated that some damage, in the form of delaminations had occurred due to the impact. The impact model assumes a perfectly undamaged target, and thus will over-estimate stiffness if softening of the target occurs due to damage.
Figure 2.12 compares results of a 7.70 m/s, 0.314 kg impact. The simulation was performed with a time step of 5 μs, and a 25 mode solution (T_{m,n} = 4.2 μs) for target deflection. This impact event generated a much more articulated contact force history, however the analysis was again quite accurate. Oscillations in the contact force history are the result of higher frequency modes in the target plate being excited by the higher velocity projectile. The target did not incur permanent damage due to the impact.

Figure 2.13 compares results of a 11.85 m/s, 0.314 kg impact. The simulation used a time step of 5 μs, and a 25 mode solution for target deflection. As in Figure 2.11, the analysis agrees closely with the experimental result early in the impact event. At times greater than 0.4 ms, the predicted response does not agree with experiment. This target experienced damage due to the impact event.

The model is not able to predict the dynamic local response of the target in the early stages of the low mass impact events. The measured responses shown in Figures 2.12 and 2.13 show a peak in the contact force at a time less than 0.1 ms, that is not reflected in the model predictions. Early in the impact event, the response is dominated by local indentation, as the target has not had sufficient time to deflect. The amount of indentation increases rapidly in this phase, likely giving rise to large inertial forces. The Hertzian relation used by the model to describe local indentation, does not include any dynamic effects.

Damage in this type of impact may be initiated when the contact force exceeds a critical value. The measured results for impacts where damage has occurred, deviate noticeably from the model predictions after the contact force exceeds 10 000 N. Figures 2.11 and 2.13 provide examples of this. In the impact events not leading to damage, the contact force does not exceed 10 000 N.

The model is able to accurately predict target response due to non-penetrating events, if damage caused by the impact is minimal. Results suggest that a contact force in excess of a critical value will cause damage. The model is able to predict contact forces up until the onset of damage. Agreement with experimental results and with published results validates the solution for target deflection, as well as the non-penetrating impact model. The target solution can now be used to describe the global behaviour in models that simulate penetrating impacts.
2.5 Tables and Figures

Table 2.1: Material used in experiments: Material A.

<table>
<thead>
<tr>
<th>System:</th>
<th>T800H/3900-2 CFRP</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lay-up:</td>
<td>[45/90/-45/0]³s</td>
</tr>
<tr>
<td>$E_{11}$:</td>
<td>129 GPa</td>
</tr>
<tr>
<td>$E_{22}$:</td>
<td>7.5 GPa</td>
</tr>
<tr>
<td>$G_{12}$:</td>
<td>3.5 GPa</td>
</tr>
<tr>
<td>$G_{23}$:</td>
<td>2.6 GPa</td>
</tr>
<tr>
<td>$V_{12}$:</td>
<td>0.33</td>
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<td>$\rho$:</td>
<td>1540 kg/m³</td>
</tr>
<tr>
<td>$h$:</td>
<td>4.65 mm</td>
</tr>
</tbody>
</table>

Table 2.2: Static indentation best fit data, as used in Figure 2.3.

<table>
<thead>
<tr>
<th>Curve</th>
<th>Exponent</th>
<th>Stiffness ($N/m^n$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>1.5</td>
<td>$6.0 \times 10^8$</td>
</tr>
<tr>
<td>b</td>
<td>1.7</td>
<td>$2.5 \times 10^9$</td>
</tr>
</tbody>
</table>

Table 2.3: Data used for target deflection convergence study, from Qian and Swanson [1990].

<table>
<thead>
<tr>
<th>Material:</th>
<th>T300/934 carbon - epoxy</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lay-up:</td>
<td>[0/90/0/90/0]³s</td>
</tr>
<tr>
<td>Size:</td>
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</tr>
<tr>
<td>$E_{11}$:</td>
<td>120 GPa</td>
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<td>7.9 GPa</td>
</tr>
<tr>
<td>$G_{12}$:</td>
<td>5.5 GPa</td>
</tr>
<tr>
<td>$G_{23}$:</td>
<td>5.5 GPa</td>
</tr>
<tr>
<td>$V_{12}$:</td>
<td>0.30</td>
</tr>
<tr>
<td>$\rho$:</td>
<td>1580 kg/m³</td>
</tr>
<tr>
<td>$T$:</td>
<td>1.20 $\mu$s</td>
</tr>
</tbody>
</table>
Table 2.4: Calculated natural frequencies for target described in Table 2.3.

<table>
<thead>
<tr>
<th>Mode # (\chi)</th>
<th>(\omega_{m,n}) (rad/sec)</th>
<th>(T_{m,n}) ((\mu)s)</th>
<th>Normalized (T_{m,n}) (t^*)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>302</td>
<td>3310.77</td>
<td>3.440</td>
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<td>7324</td>
<td>136.54</td>
<td>2.055</td>
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<td>1.490</td>
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<td>11.79</td>
<td>0.991</td>
</tr>
<tr>
<td>40</td>
<td>214679</td>
<td>4.66</td>
<td>0.588</td>
</tr>
<tr>
<td>80</td>
<td>466777</td>
<td>2.14</td>
<td>0.251</td>
</tr>
<tr>
<td>160</td>
<td>955815</td>
<td>1.05</td>
<td>-0.061</td>
</tr>
</tbody>
</table>

Table 2.5: Impact conditions used for convergence study of impact model, from Qian and Swanson [1990].

<table>
<thead>
<tr>
<th>Target</th>
<th>As in Table 2.3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Impactor</td>
<td>Diameter: 12.7 mm (\rho:) 7960 kg/m³</td>
</tr>
<tr>
<td></td>
<td>Mass: 8.537 g (V_0:) 3.0 m/s</td>
</tr>
</tbody>
</table>
Table 2.6: Convergence of peak contact force for Qian and Swanson [1990] and present impact models, as a function of time step (0.1 μs used as reference).

<table>
<thead>
<tr>
<th>Δt (μs)</th>
<th># Modes¹</th>
<th>F_{max} (N)</th>
<th>Error (%)</th>
<th># Modes¹</th>
<th>F_{max} (N)</th>
<th>Error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>100</td>
<td>179.3</td>
<td>37.42</td>
<td>5</td>
<td>302.2</td>
<td>1.10</td>
</tr>
<tr>
<td>5</td>
<td>100</td>
<td>224.8</td>
<td>21.54</td>
<td>10</td>
<td>300.7</td>
<td>0.60</td>
</tr>
<tr>
<td>2</td>
<td>100</td>
<td>255.4</td>
<td>10.86</td>
<td>15</td>
<td>299.5</td>
<td>0.22</td>
</tr>
<tr>
<td>1</td>
<td>100</td>
<td>269.7</td>
<td>5.86</td>
<td>20</td>
<td>299.3</td>
<td>0.13</td>
</tr>
<tr>
<td>0.5</td>
<td>100</td>
<td>279.4</td>
<td>2.48</td>
<td>30</td>
<td>299.1</td>
<td>0.06</td>
</tr>
<tr>
<td>0.2</td>
<td>100</td>
<td>285.3</td>
<td>0.42</td>
<td>50</td>
<td>298.9</td>
<td>0.01</td>
</tr>
<tr>
<td>0.1</td>
<td>100</td>
<td>286.5</td>
<td>0.00</td>
<td>100</td>
<td>298.9</td>
<td>0.00</td>
</tr>
</tbody>
</table>

¹ Indicates number of non-zero modes used in each direction

Table 2.7: Natural frequencies calculated for Material A.

<table>
<thead>
<tr>
<th>Mode #</th>
<th>ω_{m,n} (rad/s)</th>
<th>T_{m,n} (μs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>χ</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>18088</td>
<td>347.36</td>
</tr>
<tr>
<td>5</td>
<td>253048</td>
<td>24.83</td>
</tr>
<tr>
<td>10</td>
<td>572781</td>
<td>10.97</td>
</tr>
<tr>
<td>20</td>
<td>1189696</td>
<td>5.28</td>
</tr>
<tr>
<td>40</td>
<td>4107555</td>
<td>1.53</td>
</tr>
</tbody>
</table>
Figure 2.1: Non-penetrating impact nomenclature.

Figure 2.2: Static indentation experiment compared with Hertzian indentation law (hemispherical indenter, Ø25.4 mm).

Figure 2.2: Static indentation experiment compared with Hertzian indentation law (hemispherical indenter, Ø25.4 mm).
Figure 2.3: Static indentation experiment compared with modified Hertzian indentation law, and with best fit power law (hemispherical indenter, Ø25.4 mm).

Figure 2.4: Target plate nomenclature.
Figure 2.5: Convergence of target deflection, step load applied at center of target, 160 mode solution used as reference.

Figure 2.6: Convergence of target deflection, step load applied over small patch, 160 mode solution used as reference.
Figure 2.7: Convergence of target deflection, step load applied over entire target area, 160 mode solution used as reference.
Figure 2.8: Target deflection history of impact event (see Table 2.5) and resulting free vibration; present model compared with published results.

Figure 2.9: Contact force history of impact event (see Table 2.5); present model compared with Qian and Swanson [1990] results.
Figure 2.10: Contact force history for 6150 g, 1.76 m/s impact event; present model compared with experimental results (no target damage).

Figure 2.11: Contact force history for 6150 g, 2.68 m/s impact event; present model compared with experimental results (target damaged).
Figure 2.12: Contact force history for 314 g, 7.70 m/s impact event; present model compared with experimental results (no target damage).

Figure 2.13: Contact force history for 314 g, 11.85 m/s impact event; present model compared with experimental results (target damaged).
The impact behaviour of blunt projectiles is modelled in this chapter. A blunt projectile is one that penetrates a target by forming a plug of target material, and then ejecting the plug through the distal side of the target. Hemispherical projectiles, as examined in Chapter 2, impacting a target with sufficient energy can behave in a blunt manner, as can projectiles with large angle conical tips. The model presented will deal specifically with cylindrically shaped projectiles.

Awerbuch and Bodner[1974] presented an analytical model describing a plugging type of penetration. This model, intended for metallic targets, is based on failure mechanisms that do not correspond directly with those observed in composite targets. A new model is presented that includes some aspects of the Awerbuch and Bodner model such as inertia force and added mass due to plug formation.

The static penetration behaviour of blunt indenters will be examined and used to model the damage caused by impact events. The target deflection analysis presented in Chapter 2 will be applied to the new blunt model. Predictions made by the new model will be compared to experimental results, with varying degrees of success.

3.1 Background

Blunt projectiles, as they penetrate laminates in a static or impact event, can cause a plugging type of failure. Figure 3.1 provides a good example of such an event, in this case a static penetration test using Material B (see Table 3.1). The plug is visible and still intact at the front of the indenter. The plug length is roughly equal to the laminate thickness, and the diameter of the plug matches that of the indenter. A blunt impact model for this type of material should agree with these observations.
The Awerbuch and Bodner model developed for homogeneous metallic target is simple and appropriate considering Figure 3.1. A three stage penetration process composed of plug initiation, plug formation and plug ejection, is assumed (Figure 3.2). Developed for targets impacted by rigid, high velocity projectiles, the model uses a simple approach to describe the plugging mechanism, and is formulated in terms of the contact force between projectile and target. The Awerbuch and Bodner model does not consider global target deformation.

In stage one (Figure 3.2a), target material in front of the projectile becomes completely sheared away from the surrounding material, and is accelerated to the projectile velocity. Movement of the projectile and plug is resisted with a stress equal to the ultimate compressive stress, $\sigma_u$, of the target material. Through all stages, the diameter of the cylindrical plug is equal to the diameter of the projectile.

Stage two (Figure 3.2b), begins when the sides of the plug are no longer sheared away from the surrounding material. The plug depth, $x$, as stage two begins is:

$$x = h - b$$  \hspace{1cm} (3.1)

where $h$ is the target thickness, and $b$ is defined in Figure 3.2. The value of $b$ must be determined experimentally.

The plug formed in stage two remains continuous with the target, and plastic shear flow at the plug boundary resists plug movement. This shear flow occurs at a stress, $\tau$, equal to the yield stress of the target material in shear. The normal compressive stress continues to act as in stage one.

The plug grows to a maximum size $h$, after which the normal compressive stress no longer resists projectile and plug movement. In this third stage the area over which shear flow acts remains constant, as in Figure 3.2c. When the strain in the material connecting the plug and the target exceeds the ultimate shear strain of the material, the plug separates from the target completely. The penetration process is complete at this point.

Throughout stages one and two, of the Awerbuch and Bodner model, an inertial force is assumed to act over the contact area of the projectile. The inertia arises because as the plug grows, work is done by the
projectile in order to accelerate target material to the velocity of the projectile. This force is a function of the projectile velocity and can be significant at high impact velocities.

A typical force-time history for a high velocity event, as predicted by the Awerbuch and Bodner model is shown in Figure 3.3. The model conditions are for Material B, impacted by a 4.2 g, Ø7.82 mm cylindrical projectile with an initial velocity of 200 m/s. An ultimate stress of 300 MPa, and a shear strength of 200 MPa were used as input to the model (see Table 3.2). Predictions of low velocity events yield results similar to Figure 4.3, and do not correspond to measured results from impact tests. Because predictions of both low and high velocity impact events is desired, a new model has been developed. The concept of a local inertia force, and the increasing projectile mass due to plugging, are included in the new model.

3.2 Theory of Blunt Impact

A force-displacement curve from a static flexure experiment, using Material B and a blunt indenter (Figure 3.4), provides the basis for a penetration model. Similar to Awerbuch and Bodner, the proposed model assumes plug formation, and an inertia force. The new model also includes global target deformations, and an initial elastic loading phase.

As with the non-penetrating model, a displacement constraint defines the impact problem. The constraint equation is:

\[ \Delta_p = \alpha + w \] (3.2)

3.2.1 Target Deflection

Calculation of plate deformation, \( w \), is performed as with the non-penetrating model.

In stepwise form, the solution for target deflection is:

\[ w = w''(t) + C \cdot F(t) \] (3.3)

where \( w'' \), and \( C \) are as defined in Chapter 2.
3.2.2 Penetration Force

In contrast to the non-penetrating model, an analytical relation describing indentation, or in this case penetration, is not available. Instead, force-displacement results from static penetration tests are used to characterize the local behaviour. Although damage will progress at different rates, it is assumed that the damage mechanisms leading to penetration in static events also occur in impact events. It is important to note that the carbon fibre/epoxy system being examined here (Material B), has been shown to be strain rate insensitive.

A typical force-displacement curve, from a static flexure test using a 76.2 mm x 127.0 mm target, is shown in Figure 3.4. The figure shows the experimental results as measured, as well as results adjusted to remove the target deflection component, \( w \).

The deflection measured in the static flexure test is adjusted as follows:

\[
\delta' = \delta - \frac{F}{k_{st}}
\]  

(3.4)

where \( \delta' \) is the adjusted displacement, \( \delta \) is the measured displacement, \( F \) is the contact force, and \( k_{st} \) is the static flexural stiffness of the target.

The static flexural stiffness is calculated according to (see Appendix A):

\[
k_{st} = \frac{1}{\sum m \sum n C_{mn} \sin(m\pi/2)\sin(n\pi/2)}
\]  

(3.5)

By approximating the shape of the static flexure results with a simple relation consisting of linear segments, the characteristic damage can easily be incorporated into the blunt model. Four points, \( A \) through \( D \), define the static indentation force, as shown in Figure 3.4.
Preliminary examination of specimens following static and impact testing resulted in specific assumptions regarding the damage states as they occur in penetration. The initial loading stage, up to point $A$, is considered the elastic loading phase, similar to the indentation found in the non-penetrating model.

Point $A$ is defined by a displacement $\delta_1$, and a stress $\sigma_1$:

$$\sigma_1 = \frac{F_A}{A_f}$$

(3.6)

where $F_A$ is the contact force at point $A$, and $A_f$ is the frontal area of the projectile.

At point $A$ a critical force, $F_A$, is reached, and failure of target material begins. Fibre breakage and matrix cracking cause the material directly under the penetrator to form a plug, and move independently of the surrounding material. The plug formation continues with a driving force $F_A$, from point $A$ to point $B$. The contact force increases in this region because of a shear force acting on the plug surface. This shear force $\tau$, is due to mechanical interlock between the newly formed plug surface and the surrounding target material. As the plug size increases, so does the contact force. Friction between the penetrator and target material also begins to act in this phase of penetration.

Displacement $\delta_2$, and shear stress $\tau$ define point $B$:

$$\tau = \frac{F_B - F_A}{A_c}$$

(3.7)

where $F_B$ is the contact force at point $B$, and $A_c$ is the maximum plug surface area embedded in the target, i.e.:

$$A_c = \pi D \cdot h$$

(3.8)

At point $B$ the damage has progressed to the extent that plug size equals target thickness. The sharp drop in contact force, from point $B$ to point $C$, is due to the removal of $F_A$, which no longer acts once the plug is completely formed. Only the shear stress due to mechanical interlock, and friction resist the penetrator and plug. As the plug is pushed out the distal side of the target, the surface area in contact with surrounding target material is reduced. A corresponding reduction in contact force is seen from point $C$ to point $D$. 

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Point $D$ occurs at a displacement equal to the target thickness, i.e., the plug has been ejected, and at a force equal to the maximum friction load. Friction begins to act at point $A$, and increases to a maximum at point $D$, where the maximum amount of penetrator surface is in contact with target material.

Point $D$ defines the friction stress, $\sigma_f$, as follows:

$$\sigma_f = \frac{F_D}{A_c}$$  \hspace{1cm} (3.9)

An effort has been made in choosing points $A$ through $D$ to ensure that the area under the equivalent characteristic curve, i.e., the energy, is equal to the area under the adjusted static flexure curve. The points chosen for the indentation test shown in Figure 3.4, are listed in Table 3.3. The energy absorbed by the static penetration process, up to point $D$, bending excluded, is 58 J, which corresponds to the area under the assumed model curve. The resulting model parameters are summarized in Table 3.4.

Subsequent examination of specimens has suggested that the damage mechanisms described above may not be completely valid. Delaminations in the target material local to the penetrator may be a significant cause of the softening or change in slope occurring at point $A$. Regardless of what damage is occurring in the target, the indentation force modelled by points $A$ through $D$ is a valid representation.

The parameters listed in Table 3.4 apply only to targets of Material B impacted by 7.84 mm cylindrical projectiles. Changes in lay-up or thickness, or changes in the size of the projectile require new parameters for the model. Using stresses instead of contact force to define the points $A$ through $D$, may reduce the models dependency on geometry, but static testing is required for different impact conditions.

### 3.3 Blunt Impact Model

In the new model, the projectile is treated as a rigid body with a changing mass, similar to the Awerbuch and Bodner model. The plug essentially becomes a part of the projectile, allowing rigid body dynamics to be used to define the motion of the projectile.
The equation of motion of the projectile is:

\[ m\ddot{x} = F \]

(3.10)

where \( m \) is the mass of the combined projectile and plug, and \( F \) is the applied force. This model ignores the acceleration component due to the rate of change of mass.

The total applied force, \( F \), is:

\[ F = F_i(x) + F_i + F_f \]

(3.11)

where \( F_i \) is the force due to indentation, as defined by the static indentation curve, \( F_i \) is the local inertia force, and \( F_f \) is the frictional force.

3.3.1 Inertia Force

The inertia force \( F_i \) is a measure of the force required to accelerate material surrounding the growing plug\(^1\), and is calculated as follows:

\[ F_i = \frac{1}{2} \pi \rho r^2 \dot{x}^2 \]

(3.12)

where \( \rho \) is the density of the target material, \( r \) is the radius of the projectile, and \( x \) is the position of the plug face.

The above expression can be simplified by assuming that \( x \) changes at a rate equal to the projectile velocity, thus the inertia force becomes:

\[ F_i = \frac{1}{2} \pi \rho r^2 \alpha^2 \]

(3.13)

\(^1\) See Awerbuch and Bodner [1974].
3.3.2 Friction Force

Friction acts on the surface of the projectile as it passes through the target, and is calculated as follows:

$$F_f = \sigma_f \cdot A_f$$  \hspace{1cm} (3.14)

where $A_f$ is the surface area of the portion of projectile which is embedded in the target, and $\sigma_f$ is the friction stress.

For each projectile and target, experimental observation provides an estimate of friction stress.

3.4 Results of Blunt Model and Discussion

Low velocity instrumented impact tests were performed using cylindrical $\varnothing 7.82$ mm tips. Two target opening sizes: rectangular 76.2 mm x 127.0 mm, and $\varnothing 25.4$ mm round, with specimens all cut from Material B, were used. Force-deflection results from these tests are shown in Figures 3.5 through 3.8. Ballistic tests using 4.2 g, $\varnothing 7.52$ mm projectiles, and $\varnothing 25.4$ mm round openings (Material B) were also performed. The new impact model was used to predict each of the low velocity and ballistic impact events. In each case the model parameters used, are those listed in Table 3.4. The number of modes and the time step used in the model were chosen according to the criterion described in Chapter 2. The results are summarized in Table 3.5.

Comparison of measured low velocity results with model predictions indicate that the damage mechanisms have been accurately characterized by the model. The contact force histories in Figures 3.5 through 3.8, indicate that the peak values of contact force and the amount of penetration have been predicted by the model. As well, the exit velocities in each low velocity impact were accurately predicted, see Figure 3.10.

The impact events involving the larger 76.2 mm x 127.0 mm targets indicate that target size has an effect on the local behaviour, and/or that delaminations are affecting the global response. Figures 3.5 and 3.6 show an initial peak in contact, at approximately 1.5 mm of projectile displacement, a peak that is not predicted by the model. Results from impact tests involving the smaller, $\varnothing 25.4$ mm targets, show only one peak in the contact force. Impact events using the larger rectangular targets result in an increased amount of target...
deflection, compared to impact events using the \(\phi 25.4\) mm targets. Larger deflections result in larger in-plane strains, which may cause splitting in layers at the distal side of the target. This type of splitting damage will reduce the local stiffness of the target. Delaminations resulting from high shear stresses will reduce the global stiffness of the target. Both sizes of targets experience high shear stresses that can cause delaminations, however the target size can affect how the stiffness changes with such damage.

Each of the impact events resulting in perforation show considerable oscillation in the contact force, at displacements in excess of five millimetres. At this stage in the penetration process it is reasonable to assume that most of the plug has been pushed through the target, and that friction between projectile and target dominates the behaviour. Vibrations in the target are thought to cause both positive and negative friction forces. Projectile velocities remain positive in these events, however the target plate may indeed be moving faster than the projectile. Impulsive types of loading, such as the high contact forces measured early in the impact event, will cause a great deal of vibration in the target. The type of friction assumed by the model does not consider the velocity of the target relative to the projectile, and thus predicts only positive friction forces. The amount of energy absorbed by friction can be significant (note peaks of up to 6000 N), and is thus important for the model.

Although only residual velocities are measured in the ballistic tests, it is apparent that the damage mechanisms are very different compared to the low velocity impact events. The model, which was able to predict the impact response at lower velocities, is obviously misrepresenting the damage at ballistic velocities. Model predictions of absorbed energy are much higher, in some cases 40 J higher, than the absorbed energies measured in the ballistic tests, see Figure 3.10.

Figure 3.11 shows the effects of changing target size on the impact response as predicted by the model. Increasing the impact velocity of the high mass projectiles when a large target is used, leads to an increase in the predicted absorbed energy. In contrast, the predictions involving high mass projectiles and small targets, show almost no change in absorbed energy. This trend can be explained with reference to global target response. The impact events involving large targets absorb more energy at higher velocities because
of the energy stored as elastic strain energy due to bending. The small target absorbs energy mainly via the indentation force and friction force, both of which are not functions of velocity.

The ballistic predictions shown in Figure 3.11 follow a trend similar to the large target, low velocity impact events, while absorbing much less energy. The ballistic predictions are much lower than the low velocity predictions because the ballistic events involve very little target deflection. The ballistic event occurs so quickly that the target does not have time to deflect, thus no energy is stored in bending. The increase in absorbed energy, as ballistic velocity increases, is due to the inertia force described in Eq. (3.13). This force, negligible at lower velocities, becomes significant in the ballistic predictions.

Sections through specimens following a static penetration test and a ballistic impact event are shown in Figures 3.12 and 3.13. Extensive delamination has resulted from the static test indicating that the local damage may indeed be affecting the global behavior. Extensive delamination can also be seen in the ballistic specimen, although the plug formation is much more complete indicating a difference in energy absorption. The ballistic results of this model, although poor, have revealed a lack of understanding concerning damage as it occurs in high velocity events. The low velocity model predictions are reasonably accurate, but have also shown that delaminations and/or splitting in the distal target layers may play a role in the penetration of blunt projectiles.

### 3.5 Tables and Figures

Table 3.1: Material used in experiments: Material B.

<table>
<thead>
<tr>
<th>System:</th>
<th>IM7/8551-7 CFRP</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lay-up:</td>
<td>[-45/90/45/02]4s</td>
</tr>
<tr>
<td>E₁₁:</td>
<td>142 GPa</td>
</tr>
<tr>
<td>G₁₂:</td>
<td>4.1 GPa</td>
</tr>
<tr>
<td>ν₁₂:</td>
<td>0.34</td>
</tr>
<tr>
<td>h:</td>
<td>6.15 mm</td>
</tr>
</tbody>
</table>
Table 3.2: Impact conditions used for Awerbuch and Bodner model simulation.

<table>
<thead>
<tr>
<th>Target</th>
<th>Material B</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\sigma_0$: 300 MPa</td>
</tr>
<tr>
<td><strong>Projectile</strong></td>
<td>Dia.: 7.82 mm</td>
</tr>
</tbody>
</table>

Table 3.3: Characteristic force and displacement data derived from Figure 3.4, and used by model.

<table>
<thead>
<tr>
<th>Point #</th>
<th>Projectile Displacement (mm)</th>
<th>Contact Force (N)</th>
<th>Contact Stress (MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>A</td>
<td>0.64</td>
<td>12820</td>
<td>267</td>
</tr>
<tr>
<td>B</td>
<td>2.23</td>
<td>26110</td>
<td>544</td>
</tr>
<tr>
<td>C</td>
<td>2.23</td>
<td>13290</td>
<td>277</td>
</tr>
<tr>
<td>D</td>
<td>6.15</td>
<td>2370</td>
<td>49</td>
</tr>
</tbody>
</table>

Table 3.4: Model parameters derived from Figure 3.4; used in impact model to define static indentation force.

<table>
<thead>
<tr>
<th>Displacements (mm)</th>
<th>$\delta_1$: 0.64</th>
<th>$\delta_2$: 2.23</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stresses (MPa)</td>
<td>$\sigma_1$: 267</td>
<td>$\tau$: 88</td>
</tr>
</tbody>
</table>
Table 3.5: Experimental and model results for various blunt impact conditions, Material B.

<table>
<thead>
<tr>
<th>Impact Conditions</th>
<th>Analysis</th>
<th>Experiment</th>
</tr>
</thead>
<tbody>
<tr>
<td>M (g)</td>
<td>V₀ (m/s)</td>
<td>E₀ (J)</td>
</tr>
<tr>
<td>1 320</td>
<td>20.5</td>
<td>67</td>
</tr>
<tr>
<td>2 320</td>
<td>20.9</td>
<td>70</td>
</tr>
<tr>
<td>3 320</td>
<td>26.9</td>
<td>116</td>
</tr>
<tr>
<td>4 308</td>
<td>20.1</td>
<td>62</td>
</tr>
<tr>
<td>5 308</td>
<td>28.1</td>
<td>122</td>
</tr>
<tr>
<td>6 308</td>
<td>32.0</td>
<td>158</td>
</tr>
<tr>
<td>7 4.2</td>
<td>148</td>
<td>46</td>
</tr>
<tr>
<td>8 4.2</td>
<td>189</td>
<td>75</td>
</tr>
<tr>
<td>9 4.2</td>
<td>265</td>
<td>147</td>
</tr>
<tr>
<td>10 4.2</td>
<td>299</td>
<td>188</td>
</tr>
</tbody>
</table>

Symbol ‘a’ corresponds to the 76.2 mm x 127.0 mm target, and ‘b’ corresponds to the 25.4 mm target.

² Indicates ballistic limit.
Figure 3.1: Digital image of a Material B specimen following the static penetration test with blunt indenter.

Figure 3.2: Nomenclature for three stage Awerbuch and Bodner model: (a) plug initiation, (b) plug formation, (c) plug ejection.
Figure 3.3: Contact force and velocity histories as predicted by Awerbuch and Bodner model for 400 m/s, 4.2 g impact event.

Figure 3.4: Force-displacement relations for static penetration, including results as measured from static flexure experiment, results adjusted to remove target deflection, and characteristic relation used by model.
Figure 3.5: Force displacement results, 320 g, 20.5 m/s blunt impact: experiment and present model.

Figure 3.6: Force displacement results, 320 g, 26.9 m/s blunt impact: experiment and present model.
Figure 3.7: Force displacement results, 308 g, 28.1 m/s blunt impact: experiment and present model.

Figure 3.8: Force displacement results, 308 g, 32.0 m/s blunt impact: experiment and present model.
Figure 3.9: Contact force history for high velocity blunt impacts, as predicted by impact model.

Figure 3.10: Comparison of predicted vs. measured absorbed energies in various impact events.
Figure 3.11: Energy absorbed versus impact energy for various impact conditions, as predicted by model.
Figure 3.12: Image of section through specimen following a static penetration test; blunt indenter.
Figure 3.13: Image of section through specimen following a ballistic impact event; blunt projectile, $V_0 = 265 \text{ m/s}$.
CHAPTER FOUR

Impact of Conical Projectiles

The impact behaviour of conical tip projectiles is modelled in this chapter. The penetration of sharply pointed projectiles causes a unique type of damage in the target material, characterized by movement of material laterally and along the axis of penetration. Fragments, or other irregularly shaped projectiles can cause similar types of damage, although only regular shaped cones are examined in this thesis.

A model describing the impact of conical projectiles on Kevlar\polyester targets, presented by Zhu et al [1990], is examined. This model assumes plastic type material behaviour with degradation of material strength due to damage. Global target deflections are calculated using a finite difference solution. The Zhu model requires a mainframe computer to perform simulations, thus a new model is developed which runs efficiently on a personal computer.

The new model assumes the same type of plastic material behaviour, and includes the same damage mechanisms as the Zhu model. As well, the bulge mechanism suggested by Zhu will be extended, and damage due to target deflection will be added in the new model. The ‘bluntness’ inherent in all real conical tipped projectiles is also considered. The global model, presented in Chapter 2, and included in the blunt model, will also be used in the new conical model.

Experimental results from low velocity and ballistic impact tests will be compared to predictions made by the new model.
4.1 Background

The Zhu model was developed for unidirectional Kevlar/polyester type laminates with specially orthotropic lay-ups. Three stages of penetration are assumed: initial undamaged penetration, damaged penetration, and perforation. In each stage a perfectly plastic type behaviour is assumed locally in the target material.

Undamaged penetration is governed by a simple force balance. Damaged penetration is initiated by fibre failure at the distal side of the target caused by bulging. Subsequent damage is caused by deformation of the target material surrounding the penetrating projectile. The increasing damage reduces the strength of the target, until only friction resists projectile movement.

Zhu calculated target deflection using a finite difference scheme. This method requires a mainframe computer because of the large amounts of memory needed, and thus cannot be used on a personal computer. Results from experimental studies conducted by Zhu et al were not suitable for comparisons with the Zhu model or the new model presented in this chapter, because of the inherent difficulties in the measurement of high velocity impact events. Projectile displacements were measured by Zhu using an optical technique, and then differentiated twice to produce contact force histories. Sufficient experimental data is available from instrumented impact tests (Delfosse et al) for comparisons with the new model.

4.2 Theory of Conical Impact

The result of a static penetration test involving a conical indenter is shown in Figure 4.1. Material has been pushed away from the indenter, and considerable splitting has occurred at the back face of the specimen. The penetration mechanisms used in the Zhu model have been adapted to the new model because they attempt to describe the type of penetration seen in Figure 4.1.

In the new model damage occurs in the target material throughout the penetration process, instead of only in the latter stages of penetration, as assumed by Zhu. Three types of strain contribute to damage: strain due to target deflection, strain local to the projectile, and strain due to bulging of the target. The analytical solution
for target deflection described in Chapter 2 is used. As well, the model takes into account the “bluntness” of real conical projectiles.

As with the non-penetrating and blunt impact models, the local and global displacements define the impact problem. Therefore, the constraint equation is:

\[ \Delta_p = \alpha + w \]  

(4.1)

where \( \Delta_p \) is the projectile displacement, \( \alpha \) is the penetration, and \( w \) is the central deflection of the target.

### 4.2.1 Target Deflection

Calculation of plate deformation, \( w \), is performed as with the non-penetrating model.

In stepwise form, the solution for target deflection is:

\[ w = w^0(t) + C \cdot F(t) \]  

(4.2)

where \( w^0 \), and \( C \) are as defined in Chapter 2.

### 4.2.2 Penetration Force

The penetrating projectile causes plastic flow in the surrounding target material. The target is assumed to behave in a perfectly plastic manner, resisting penetration with a stress equal to the effective strength of the material.

Resolving the stresses applied to the projectile in the direction of travel (normal to the target surface), the following force balance applies:

\[ F_p = \sigma_e A_p \]  

(4.1)

where \( F_p \) is the force due to penetration, \( \sigma_e \) is the effective strength of the target material, and \( A_p \) is the projected area of the embedded penetrator, as in Figure 4.2.

The projected area determines the contact force to a great extent, and is a function of target and projectile geometries, as well as the amount of penetration.
4.2.3 Effective Strength

The target material resists movement of the projectile with an effective strength, which is a function of the damage in the target. In an undamaged target, the effective strength is equal to the ultimate strength of the material.

The model that assumes damage occurs only in the region directly in front of the projectile, and is the result of tensile strains. The damage mechanisms assumed by the model are much like fibre breakage. However, no attention is paid to the orientation of fibres in the laminate, thus damage is attributed, in a general way, to target material only. Three distinct deformations cause damage in the target: displacement of material directly surrounding the projectile, bulge formation on the distal side of the target, and target deflection. Damage caused by these deformations are considered equal when calculating degradations in the effective strength.

Each of the three strains are calculated at evenly spaced points in the path of the projectile. Figure 4.3 shows how a damage grid, of \( n^2 \) points, is plotted on the target material in the path of the projectile. Each point on the grid represents the surrounding area of material (shaded region of Figure 4.3). The sum of all three strains is compared with an ultimate or breaking strain of the target material. Points experiencing strain in excess of the fibre breaking strain are considered to be damaged, and exhibit no strength. The ability of the target to resist the projectile is due to undamaged material only.

The effective strength of the target material is:

\[
\sigma_e = (1 - d_m) \cdot \sigma_u
\]  

(4.2)

where \( \sigma_u \) is the ultimate strength of the material, and the damage factor \( d_m \) is calculated according to:

\[
d_m = \frac{N_d}{N_f}
\]  

(4.3)

where \( N_d \) is the number of points that are considered damaged, and \( N_f \) is the total number of points.

The ultimate strength of the target material can be estimated from a variety of static tests. Figure 4.4 plots the average stress seen by the indenter in a number of static penetration tests. The early stages of
penetration, i.e., 0-4 mm, are examined so that the maximum, or undamaged, material strength can be estimated. A penetrator tip with one millimetre of bluntness, i.e., one millimetre truncated from the end of the cone, is assumed for each test.

In each case, the measured contact force is used to determine the effective stress as follows:

\[ \sigma_e = \frac{F}{A_p} \]  

The estimated effective strength of Material B is 1350 MPa. This value is highly dependent on the amount of bluntness assumed. Static and dynamic compression tests, have shown the ultimate strength for Material B is approximately 850 MPa. This provides a lower bound for this model because the specimens in the compression tests are small and unconfined, whereas the material resisting conical penetration is confined by rest of the target.

The total strain is calculated as follows:

\[ \varepsilon = \varepsilon_g + \varepsilon_l + \varepsilon_b \]  

where \( \varepsilon_g \) is the strain due to global target deformations, \( \varepsilon_l \) is the local penetrator strain, and \( \varepsilon_b \) is the strain due to bulging.

**Global Strain**

The in-plane strain due to global deformations in the target plate is calculated according to:

\[ \varepsilon_g = \frac{\partial v}{\partial y} \]  

where \( v \) is the in-plane displacement of the target plate in the \( y \) direction.

The in-plane displacement, \( v \), is defined in Chapter 2.

**Local Penetrator Strain**

Strains are induced in the target material directly surrounding the penetrator, as material is displaced laterally and in the direction of penetrator movement.
A line in the target material is examined as it is displaced by the indenter, Figure 4.5. The line is initially straight, passing through points A and B, at a distance \( z \) from the mid-plane of the target, and \( r_o \) from the axis of the penetrator.

The initial length of this line is:

\[
l_o = 2\sqrt{(z_o \tan \beta)^2 - r_o^2}
\]  

(4.7)

where \( z_o \) is the distance of the line from the tip of the conical penetrator, and \( \beta \) is the half-angle of the projectile cone.

The line is assumed to remain fixed at the points A and B, the intermediate length being strained as the penetrator moves into the target.

The length of the line when the penetrator is at a depth \( \alpha \) from the top surface of the target, is:

\[
l = 2\frac{2z_o}{\cos \beta} \sin \left( \sin \beta \cos^{-1}\left( \frac{r_o}{z_o \tan \beta} \right) \right)
\]  

(4.8)

The strain in the target material is then easily found using:

\[
\varepsilon_i = \frac{l - l_o}{l_o}
\]  

(4.9)

**Bulging Strain**

Through the process of indentation, the conical shape displaces target material toward the back face of the target. Experimental observations by Zhu et al and others have shown that a bulge forms on the distal side of the target.

Assuming incompressibility, the volume displaced by the cone, \( V_c \), must be equal to the volume contained in the bulge, \( V_b \). The shape of the bulge is assumed to be spherical with a varying radius.

The volume displaced by the cone, as shown in Figure 4.6, is:

\[
V_c = \frac{\pi}{3} \alpha r_o^2 = \frac{\pi}{3} \alpha^3 \tan^2 \beta
\]  

(4.10)
The volume of the spherical cap is:

\[ V_s = \frac{\pi}{3} z_1^2 (3R_z - z_1) \]  \hspace{1cm} (4.11)

where \( z_1 \) is the height of the bulge, and \( R_z \) is the radius of the sphere, as shown in Figure 4.6.

The target material deforms spherically from the tip of the projectile to the bulge on the distal side of the target. The radius of deformation \( R_g' \) varies linearly from zero at the penetrator tip, to \( R_z \) at the distal side of the target, see Figure 4.7. Thus each fibre lying between the projectile tip and the distal side of the target will experience a unique strain due to bulging, depending on its position.

An initially straight line from points A to B, has a length:

\[ l_0 = 2y_0 = 2\sqrt{r_1^2 - x_0^2} \] \hspace{1cm} (4.12)

where \( r_1 \) is defined in Figure 4.6, and \( y_0, x_0 \) are defined in Figure 4.7.

Once again the end points of line AB remain fixed.

The length of the line as it displaced to the surface of a sphere with radius \( R_g' \) is:

\[ l = 2\gamma \cdot R_g' \] \hspace{1cm} (4.13)

where \( \gamma \) is defined in Figure 4.7.

The strain due to bulging is:

\[ \varepsilon_b = \frac{l-l_0}{l_0} = \frac{R_g' \cdot \gamma}{y_0} - 1 \] \hspace{1cm} (4.14)
4.3 Conical Impact Model

The projectile is assumed to behave as a rigid body, similar to the non-penetrating and blunt models.

The equation of motion of the projectile is:

\[ m \ddot{\Delta}_p = F \]  
(4.15)

The applied force, for this model, is as follows:

\[ F = F_p(\alpha) + F_f \]  
(4.16)

where \( F_p \) is the force due to penetration as described in Section 4.2.2, and \( F_f \) is the friction force.

The concept of an inertial force, as used in the blunt model, can be applied to the conical impactor as well.

Awerbuch and Bodner suggested an inertial force for conical projectiles as follows:

\[ f_i = \phi \frac{1}{2} \rho A_p V^2 \]  
(4.17)

where \( A_p \) is the projected area of the embedded projectile, as in Eq. (4.1), and \( \phi \) is a shape factor defined as:

\[ \phi = \sin^2 \beta \]  
(4.18)

where \( \beta \) is the half-angle of the conical projectile.

The new model is intended for conical projectiles with relatively small half angles, leading to negligible inertia forces. The conical projectiles used for comparisons in this chapter have sharp tips with half-angles of 17.5°, resulting in shape factors of 0.09. The inertia force is ignored in the new model.

4.3.1 Friction Force

Friction acts on the surface of the projectile as it passes through the target. The friction stress is a parameter of the projectile/target system and is assumed to acts only on the shaft of the projectile.
The total friction force is a function of the surface area imbedded in the target:

\[ F_f = \sigma_f \cdot A_f \]  

(4.19)

where \( A_f \) is the surface area of the portion of projectile shaft which is imbedded in the target, and \( \sigma_f \) is the friction stress.

### 4.4 Results of Conical Model and Discussion

Low velocity instrumented impact tests using conical tips with a shaft diameter of 7.62 mm and total cone angle of 35°, were performed. Two target sizes: rectangular 76.2 mm x 127.0 mm specimens, and \( \varnothing25.4 \) mm round specimens, cut from Material B, were used. Ballistic tests using 4.2 g, \( \varnothing7.52 \) mm, 37° projectiles, and \( \varnothing25.4 \) mm round openings (Material B), were also performed. The new conical model is used to predict each of the impact events. In each case the model uses an ultimate strength of 1350 MPa, as obtained from Figure 4.4. All other material parameters can be found in Table 3.1. The number of modes and the time step used in the model were chosen according to the criterion described in Chapter 2.

Experimental results and predictions are summarized in Table 4.1. Low velocity model predictions are compared to the contact force as measured by the instrumented projectile, in Figures 4.8 through 4.11. Contact force histories are not available from the ballistic tests, and are thus only model predictions are shown in Figure 4.12. Figure 4.13 summarizes the model predictions in terms of the energy absorbed. In contrast to the blunt model predictions, the conical model accurately predicts the ballistic events, but not the low velocity impacts.

Examining the force-displacement results of the low velocity impact events, it is apparent that the model does not accurately predict damage. All peak forces are considerably over predicted by the model. Target material effective strength \( \sigma_e \), as predicted by the model, is compared with the average contact stress measured in static experiments (Figure 4.14). Predicted target strengths cannot be compared with results from impact tests because only the projectile displacement \( \Delta_p \), and not the penetration \( \alpha \), is measured by the
instrumented projectile. The predicted target strength, initially agrees with the measured values (about 1350 MPa) because the parameter $\sigma_r$ is obtained from the static flexure test (Figures 4.4 and 4.14).

The disagreement in Figure 4.14, between one and seven millimetres of penetration, indicates that the damage due to bulging as assumed by the model is not valid. A majority of the strength reduction predicted by the model, at these levels of penetration, is due to bulging strain. Experimental observations from static penetration tests have also suggested that bulging on the distal side of the target does not form until three to four millimetres of penetration has occurred. Much of the material displaced by the penetrator forms a bulge on the impact side of the target, and this type of bulge is not considered by the model.

At penetration levels in excess of seven millimetres, the single damage factor used to degrade the target strength, may not be sufficient. According to Figure 4.14, at, the model prediction of effective strength is only slightly higher than the measured value, however examination of the contact force histories reveal how important this error is. The target seems to have a load carrying capacity of approximately 7 000 N to 8 000 N. Although the material in contact with the penetrator likely still has sufficient strength to resist the penetrator, the supporting target structure may be failing due to other mechanisms.

As in the blunt impacts, different damage mechanisms appear to result from ballistic type impacts, compared to low velocity events. This is evident from the increase in measured absorbed energy in the ballistic tests (Table 4.1). The model accurately predicts these higher absorbed energies, however it is difficult to gauge the validity of the model without a method of determining damage in the impacted specimens.

Figure 4.15 shows the effects of changing target size on the impact response as predicted by the model. As with the blunt model, increasing the impact velocity of the high mass projectiles when a large target is used, leads to an increase in the predicted absorbed energy. The predictions involving high mass projectiles and small targets, show almost no change in absorbed energy. As explained in Chapter 3 this trend is due to global target response.
In contrast to the blunt model, the ballistic predictions in Figure 4.15 show a slight decrease in absorbed energy as the impact velocity is increased. Note that the inertia force causing an increasing trend for the blunt model is not used in the conical model. The ballistic predictions are lower than the low velocity predictions because the ballistic events involve very little target deflection.

Examination of specimens following impact events as shown in Figures 4.16 and 4.17, reveal weaknesses in the model assumptions. The damage resulting from the non-perforating impact, Figure 4.16, is very localized and fibre breakage is evident, however the bulge formation assumed by the model is not evident. The specimen shown in Figure 4.17, having undergone a perforating impact, has been damaged in the region beyond the path of the projectile. Assumptions made by the new conical model concerning damage occurring in the path of the projectile, and as a result of bulge formation are not accurate, indicating that the ballistic impact predictions made by the model are not reliable.
4.5 Tables and Figures

Table 4.1: Experimental and model results for various conical impact conditions (Ø25.4 mm opening, Material B).

<table>
<thead>
<tr>
<th>Impact Conditions</th>
<th>Analysis</th>
<th>Experiment</th>
</tr>
</thead>
<tbody>
<tr>
<td>M (g)</td>
<td>$V_0$ (m/s)</td>
<td>$E_0$ (J)</td>
</tr>
<tr>
<td>1</td>
<td>320</td>
<td>20.3</td>
</tr>
<tr>
<td>2</td>
<td>320</td>
<td>26.8</td>
</tr>
<tr>
<td>3</td>
<td>320</td>
<td>30.2</td>
</tr>
<tr>
<td>4</td>
<td>308</td>
<td>26.8</td>
</tr>
<tr>
<td>5</td>
<td>308</td>
<td>29.1</td>
</tr>
<tr>
<td>6</td>
<td>308</td>
<td>31.5</td>
</tr>
<tr>
<td>7</td>
<td>4.2</td>
<td>156</td>
</tr>
<tr>
<td>8</td>
<td>4.2</td>
<td>235</td>
</tr>
<tr>
<td>9</td>
<td>4.2</td>
<td>316</td>
</tr>
</tbody>
</table>

1 Symbol ‘a’ corresponds to the 76.2 mm x 127.0 mm target, and ‘b’ corresponds to the 25.4 mm target.

2 Indicates ballistic limit.
Figure 4.1: Material B specimen following static penetration test with conical indenter.
Figure 4.2: Conical penetration nomenclature and typical contact area projections.
Figure 4.3: Schematic showing damage grid used for conical model.
Figure 4.4: Average contact stress as a function of conical indenter displacement, for various static tests.

Figure 4.5: Nomenclature used for calculation of local strain surrounding conical projectile.
Figure 4.6: Nomenclature used for calculation of bulge radius.
Figure 4.7: Nomenclature used for calculation of bulging strain through the target thickness.
Figure 4.8: Contact force history of 320g, 20.5 m/s, conical impact event; experiment and present model.

Figure 4.9: Contact force history of 320g, 30.2 m/s, conical impact event; experiment and present model.
Figure 4.10: Contact force history of 308g, 29.1 m/s, conical impact event; experiment and present model.

Figure 4.11: Contact force history of 308 g, 31.5 m/s, conical impact event; experiment and present model.
Figure 4.12: Model prediction of contact force history of 316 m/s, 4.2g conical impact event.

Figure 4.13: Comparison of measured and predicted absorbed energies for various conical impacts.
Figure 4.14: Average stress surrounding penetrator, due to conical penetration/impact.

Figure 4.15: Energy absorbed due to impact of conical projectile: experiment and model.
Figure 4.16: Section through specimen following non-perforating impact event; conical projectile, $V_0 = 156 \text{ m/s}$.

Figure 4.17: Section through specimen following ballistic event; conical projectile, $V_0 = 316 \text{ m/s}$, $V_{\text{exit}} = 228 \text{ m/s}$.
Three analytical models describing the impact response of composite targets, have been developed. Each model is simple and efficient, appropriate for use in an engineering environment. To this end all three models have been implemented in a user-friendly computer code.

Each model assumes that the impact event can be divided into two distinct regimes: local behaviour, characterized by indentation or penetration depending on the model, and global target deflection. The local and global solutions are coupled by the models only through the contact force. The target material is also assumed, by the models, to behave in a strain rate independent manner. This assumption allows data gathered from static penetration and low velocity impact tests to be applied directly to high velocity ballistic models. The materials studied in this thesis are carbon fibre epoxy systems which have been shown to be strain rate independent.

One model simulates low energy, non-penetrating impacts, while two other models attempt to predict penetrating impacts. The two penetrating models predict the impact of different shaped projectiles, one dealing with blunt or cylindrical shapes, and the other with sharp conical projectiles. These two shapes are studied in order to provide bounds for the behaviour of projectiles with unknown or irregular shapes.

The first model, which simulates non-penetrating impacts involving hemispherical projectiles, is useful for low energy impacts. This model is more efficient than previous non-penetrating models, and is the first to use a closed form solution that includes the non-linear Hertzian type indentation law. The model is able to accurately predict the force-time histories of impact events up to the onset of substantial damage in the target. The non-penetrating model is also important in the validation of a modal series solution describing
global target behaviour. The global solution developed includes shear deformation and rotary inertia effects, and was found to agree with published data and experimental results.

Convergence of the global solution, and non-penetrating impact model, is much improved in comparison with previously published solutions. Using a patch type of loading, the new model converges if an appropriate number of modes are used in the modal plate solution. A simple criterion, in terms of the natural periods of vibration of the target, is used to ensure convergence.

The validity of the non-penetrating model was found to be limited to impacts where damage in the target is minimal. It was found that impact events where a critical contact force is exceeded resulted in damaged specimens. A validated criteria based on the contact force may enable the model to predict the onset of damage. Prediction of the post-damage behaviour may be possible if a plate solution that is capable of reducing the material properties during a simulation, is developed.

A model describing the impact behaviour of penetrating blunt projectiles has been developed, and compared to experimental results. Aspects of the Awerbuch and Bodner model for metallic targets have been adapted to composite targets, and applied to the new model. A characteristic force-displacement curve, obtained from a static penetration test, is used to describe the local penetration response. The global solution, verified with the non-penetrating model, is used in this model as well.

Model predictions of low velocity impact events agree with experimentally measured results. Absorbed energies, as well as contact force histories are accurately predicted by the new blunt model. The characteristic force-displacement curve, estimated from static tests, is a reasonable representation of damage in low velocity penetration events.

At higher velocities, the blunt model does not predict impact behaviour as it is measured in experiments. The model consistently underpredicts the amount of energy absorbed in high velocity events. It is apparent that different damage mechanisms play a role in high velocity penetration events, making the characteristic force-displacement curve invalid. An examination of targets after they have undergone impact tests, may
provide a method of adapting the force-displacement curve so that it may be used to represent the damage in high velocity impact events.

The effects of friction are also not accurately predicted by the blunt model. An advanced method of analyzing the friction between projectile and target which accounts for the negative friction forces observed in the experiments, is required. These type of friction models have likely been previously investigated and a literature search of this topic may be helpful.

The final model describes the impact behaviour of conical projectiles. A model developed by Zhu et al for the impact of Kevlar/polyester systems, is used as a basis for this new model. A strain based criterion is used to predict the progression of damage as the projectile travels through the target. The initial, undamaged strength of the target is estimated from static penetration tests. The global target solution developed for the non-penetrating model is also applied to the conical model. Predictions made by the blunt model for high velocity impact events are reasonably accurate, compared to experiment.

Low velocity impact predictions made by the model do not agree with experiment, indicating that the damage mechanisms assumed by the model are not valid. In particular, the model predicts a great deal of damage due to bulging in the early stages of penetration. Experimental observations have suggested that the type of bulging assumed by the model does not occur until the later stages of penetration. Accounting for a different type of bulge, one that forms on the impact face of the target instead of on the distal side, may address this problem.

The concept of a degraded effective material strength, as used by the blunt model, may not be sufficient in the later stages of penetration. Evidence suggests that the target structure is capable of sustaining a maximum load, regardless of the residual material strength. The model requires another failure mechanism to predict this type of behaviour.

Further work and improvements to these three models should focus on the understanding of damage. Examination of the damage states in impacted specimens is critical, with the sectioning and visual inspection of specimens being a simple method. As well, the models must be compared to experiments involving a range of target and projectile geometries. This is particularly important for the blunt model,
which at this point requires static testing for each projectile and target combination. The effect of geometrical changes in both the projectile and target should be dealt with by each model.
CHAPTER SIX

References


APPENDIX A

Statics and Dynamics of Laminated Plates

In order to analyze the behaviour of laminated plates we first examine the individual layers, or laminae of the plate.

A.1 Stress-Strain Behaviour of a Lamina

A fibre reinforced lamina is a special case of a generally anisotropic material.

Anisotropic, elastic materials behave according to the following three dimensional stress-strain relationship:

\[
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{23} \\
\sigma_{31} \\
\sigma_{12}
\end{bmatrix} =
\begin{bmatrix}
C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\
C_{22} & C_{23} & C_{24} & C_{25} & C_{26} & C_{27} \\
C_{33} & C_{34} & C_{35} & C_{36} & C_{37} & C_{38} \\
C_{44} & C_{45} & C_{46} & C_{47} & C_{48} & C_{49} \\
C_{55} & C_{56} & C_{57} & C_{58} & C_{59} & C_{60} \\
C_{66} & C_{67} & C_{68} & C_{69} & C_{70} & C_{71}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
\varepsilon_{23} \\
\varepsilon_{31} \\
2\varepsilon_{12}
\end{bmatrix}
\]

(A.1.1)

in which there are 21 independent elastic constants \( C_{ij} \). The subscripts \( i, j \), refer to the principle directions of Figure A.1.
In the case of a unidirectional lamina, i.e. fibres aligned in just one direction, the material has three orthogonal planes of symmetry. The interdependence of normal stresses and shear strains in Eq. (A.1.1) is eliminated due to symmetry, and the number of independent constants is reduced to nine:

\[
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{23} \\
\sigma_{31} \\
\sigma_{12}
\end{bmatrix} =
\begin{bmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\
C_{22} & C_{23} & 0 & 0 & 0 & 0 \\
C_{33} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & C_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{66}
\end{bmatrix}
\begin{bmatrix}
\epsilon_{11} \\
\epsilon_{22} \\
\epsilon_{33} \\
2\epsilon_{23} \\
2\epsilon_{31} \\
2\epsilon_{12}
\end{bmatrix}
\] (A.1.2)

For most applications, a lamina can be considered transversely isotropic. In a transversely isotropic material there exists one plane in which properties are equal in all directions. In a reinforced lamina, the 2-3 plane is isotropic.

The stress strain relation for a transversely isotropic material is as follows:

\[
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{23} \\
\sigma_{31} \\
\sigma_{12}
\end{bmatrix} =
\begin{bmatrix}
C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\
C_{22} & C_{23} & 0 & 0 & 0 & 0 \\
C_{33} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & (C_{22} - C_{23})/2 & 0 & 0 \\
0 & 0 & 0 & 0 & C_{66} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{66}
\end{bmatrix}
\begin{bmatrix}
\epsilon_{11} \\
\epsilon_{22} \\
\epsilon_{33} \\
2\epsilon_{23} \\
2\epsilon_{31} \\
2\epsilon_{12}
\end{bmatrix}
\] (A.1.3)

Note that only five independent constants remain in Eq. (A.1.3).
The stiffness matrix components, $C_{ij}$, can be expressed in terms of standard engineering constants $E_i, v_{ij}, G_{ij}$:

\[
C_{11} = \frac{1-v_{21}^2}{E_2\Delta} \quad (A.1.3a)
\]

\[
C_{12} = \frac{v_{12} \Delta + v_{23}v_{13}}{E_1E_2\Delta} \quad (A.1.3b)
\]

\[
C_{11} = \frac{1-v_{13}v_{31}}{E_1E_3\Delta} \quad (A.1.3c)
\]

\[
C_{23} = \frac{v_{23} + v_{21}v_{13}}{E_1E_2\Delta} \quad (A.1.3d)
\]

\[
C_{66} = G_{12} \quad (A.1.3e)
\]

where

\[
\Delta = \frac{1-v_{12}v_{21}-v_{23}^2-v_{13}v_{31}-2v_{21}v_{32}v_{13}}{E_1E_2^2} \quad (A.1.3f)
\]

In general only in-plane properties of a lamina in plane stress conditions are of interest. This allows us to further simplify the stiffness matrix of Eq. (A.1.3), and define the reduced stiffness, or $[Q]$ matrix:

\[
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{12}
\end{bmatrix} =
\begin{bmatrix}
Q_{11} & Q_{12} & 0 \\
Q_{12} & Q_{22} & 0 \\
0 & 0 & Q_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
2\varepsilon_{12}
\end{bmatrix} \quad (A.1.4)
\]

where the components $Q_{ij}$, in terms of engineering constants, are as follows:

\[
Q_{11} = \frac{E_1}{1-v_{12}v_{21}} \quad (A.1.4a)
\]

\[
Q_{12} = \frac{v_{21}E_1}{1-v_{12}v_{21}} \quad (A.1.4b)
\]

\[
Q_{22} = \frac{E_2}{1-v_{12}v_{21}} \quad (A.1.4c)
\]

\[
Q_{66} = G_{12} \quad (A.1.4d)
\]
Now we consider the behaviour of a lamina in which the direction of reinforcement does not coincide with a natural co-ordinate system \( \{x,y,z\} \), as shown in Figure A.2.

Generating the stress-strain relation involves a co-ordinate transformation, with the result:

\[
\begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{xy}
\end{bmatrix} =
\begin{bmatrix}
\overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\
\overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\
\overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
\varepsilon_{xy}
\end{bmatrix}
\]

(A.1.5)

where the transformed reduced stiffnesses \( \overline{Q}_{ij} \) are as follows:

\[
\overline{Q}_{11} = Q_{11} \cos^4 \theta + 2(Q_{12} + 2Q_{66}) \sin^2 \theta \cos^2 \theta + Q_{22} \sin^4 \theta
\]

(A.1.5a)

\[
\overline{Q}_{12} = (Q_{11} + Q_{22} - 4Q_{66}) \sin^2 \theta \cos^2 \theta + Q_{12} \sin^4 \theta + \cos^4 \theta
\]

(A.1.5b)

\[
\overline{Q}_{22} = Q_{11} \sin^4 \theta + 2(Q_{12} + 2Q_{66}) \sin^2 \theta \cos^2 \theta + Q_{22} \cos^4 \theta
\]

(A.1.5c)

\[
\overline{Q}_{16} = (Q_{11} - Q_{12} - 2Q_{66}) \sin \theta \cos^3 \theta + (Q_{12} - Q_{22} + 2Q_{66}) \sin^3 \theta \cos \theta
\]

(A.1.5d)

\[
\overline{Q}_{26} = (Q_{11} - Q_{12} - 2Q_{66}) \sin^3 \theta \cos \theta + (Q_{12} - Q_{22} + 2Q_{66}) \sin \theta \cos^3 \theta
\]

(A.1.5e)

\[
\overline{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66}) \sin^2 \theta \cos^2 \theta + Q_{66} \sin^4 \theta + \cos^4 \theta
\]

(A.1.5f)

### A.2 Load-Displacement Behaviour of a Laminate

Most applications of fibre reinforced materials require strength and stiffness in more than one direction. Laminates consisting of a series of differently oriented laminae provide a more versatile component. The analysis of these laminates can be performed with varying degrees of complexity, however, a few basic assumptions are required.

First, the thickness of the laminate must be small compared to the in-plane dimensions, and the displacements, in turn, are small compared to the thickness. The laminae are assumed to be perfectly bonded, i.e. the strain field is continuous through layer interfaces. As well, the laminae are considered to be in a plane stress state, and the transverse strain \( \varepsilon_{zz} \) is negligible.
A.2.1 Kirchoff Theory

The study of isotropic plates by Kirchoff led to a simple type of plate analysis, essentially a two-dimensional study of the plate mid-plane. The extension of this theory to laminates is direct.

The in-plane displacement field is of the form:

\[
\begin{align*}
U(X,y,Z,t) &= U^0_Z - z \frac{\partial w}{\partial x} \\
V(X,y,Z,t) &= V^0_Z - z \frac{\partial w}{\partial y} \\
w(x,y,z,t) &= w
\end{align*}
\]  

(A.2.1)

(A.2.2)

(A.2.3)

where \( u \) and \( v \) are the in-plane displacements in the \( x \) and \( y \) directions respectively (the naught superscript indicates a displacement of the mid-plane), and \( w \) is the transverse displacement.

The definition of strain, in terms of the displacement, is as follows:

\[
\varepsilon_{xx} = \frac{1}{2} \left( \frac{\partial u_x}{\partial s} + \frac{\partial u_x}{\partial r} \right)
\]  

(A.2.4)

For Kirchoff plate theory, only the in-plane strains are needed, thus the strain field can be written:

\[
\{\varepsilon_{xx}, \varepsilon_{yy}, 2\varepsilon_{xy}\} = \{\varepsilon^0_{xx}, \varepsilon^0_{yy}, 2\varepsilon^0_{xy}\} + z \cdot \{\kappa_{xx}, \kappa_{yy}, 2\kappa_{xy}\}
\]  

(A.2.5)

where the middle surface curvatures are:

\[
\{\kappa_{xx}, \kappa_{yy}, \kappa_{xy}\} = \left[ \frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial y^2}, \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right]
\]  

(A.2.6)

Given the strains in each lamina we can find the resultant forces and moments in the laminate.
The resultants are calculated in a manner similar to an isotropic analysis, i.e., integrated through the thickness:

\[
\{N_x, N_y, N_{xy}\} = \int_{-h/2}^{h/2} \{\sigma_{xx}, \sigma_{yy}, \sigma_{xy}\} \, dz
\]

\[
{M_x, M_y, M_{xy}} = \int_{-h/2}^{h/2} \{\sigma_{xx}, \sigma_{yy}, \sigma_{xy}\} \cdot z \, dz
\]

The stresses can be expressed in terms of strains using the transformed reduced stiffnesses of Eq. (A.1.5).

For a stacked laminate the stresses are not continuous, thus the integration of Eq.’s (A.2.7) and (A.2.8) must be performed stepwise through each lamina, e.g. for the normal forces:

\[
\{N_x, N_y, N_{xy}\} = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \{\sigma_{xx}, \sigma_{yy}, \sigma_{xy}\} \, dz
\]

where \( n \) is the total number of laminae.

Noting that the mid-plane strains are not functions of \( z \), Eq.’s (A.1.5) and (A.2.5) are substituted into the expression for normal forces, with the result:

\[
\begin{bmatrix}
N_x \\
N_y \\
N_{xy}
\end{bmatrix} = \sum_{k=1}^{n} \begin{bmatrix}
\bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\
\bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\
\bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66}
\end{bmatrix} \begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
2\varepsilon_{xy}
\end{bmatrix} \int_{z_{k-1}}^{z_k} \begin{bmatrix}
\kappa_{xx} \\
\kappa_{yy} \\
2\kappa_{xy}
\end{bmatrix} \, dz
\]

(A.2.10)
Performing the integration and arranging in matrix form, the normal forces and moments are as follows:

\[
\begin{bmatrix}
N_{xx} \\
N_{yy} \\
N_{xy} \\
M_x \\
M_y \\
M_{xy}
\end{bmatrix} =
\begin{bmatrix}
A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\
A_{22} & A_{26} & B_{12} & B_{22} & B_{26} & B_{66} \\
A_{66} & B_{16} & B_{26} & B_{66} & 2\varepsilon_{xy}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
\varepsilon_{xy}
\end{bmatrix}
\]

where

\[
A_i = (Z_k - Z_{k-1}) (A_{ii}) \quad (A.2.12)
\]

\[
B_i = \sum_{k=1}^{n} \overline{Q}_{ij}^k (z_k - z_{k-1}) \quad (A.2.13)
\]

\[
D_i = \sum_{k=1}^{n} \overline{Q}_{ij}^k (z_k^2 - z_{k-1}^2) \quad (A.2.14)
\]

Components of the [A] matrix are referred to as extensional stiffnesses, as they describe the mid-plane extension of a laminate under normal tractions. Bending stiffnesses are contained in the [D] matrix, and the [B] matrix components are referred to as coupling stiffnesses.

For particular lay-ups, in which the lamina properties and orientations are symmetric about the mid-plane, coupling stiffnesses reduce to zero. Even more important, is the case when the coupling stiffnesses and the bend-twist stiffnesses \(D_{16}, D_{26}\), reduce to zero. These laminates are termed specially orthotropic. Normally laminates do not meet the criterion for \(D_{16} = D_{26} = 0\), however, for symmetric lay-ups with \(n > 5\), the bend-twist stiffnesses are often negligible.

It is these specially orthotropic laminates that will be analyzed.

**Equations of Motion**

Governing equations of motion are derived in a manner similar to the isotropic case, by examining a differential element, Figure A.3. For this analysis we will ignore the inertia terms arising from the in-plane
displacements \( u, v \), rotary inertia terms arising from changes in the mid-plane slope \( \frac{\partial w}{\partial x} \), and other higher order terms.

Equilibrium is first satisfied for the forces in the \( z \) direction, with the following result:

\[
\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + p(x, y, t) = \rho \dot{w}
\]  

(A.2.15)

where \( \rho \) is the density of the laminate material, \( h \) is the thickness of the laminate, and the overdot indicates differentiation with respect to time.

Similarly for equilibrium in the \( x \) and \( y \) directions:

\[
\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0
\]  

(A.2.16)

\[
\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = 0
\]  

(A.2.17)

For the purposes of this development, we will assume that no in-plane tractions are applied, thus Eq.'s (A.2.16) and (A.2.17) are identically satisfied by \( N_x = N_y = N_{xy} = 0 \).

Satisfying equilibrium of moments yields:

\[
Q_x = \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y}
\]  

(A.2.18)

\[
Q_y = \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x}
\]  

(A.2.19)

Differentiating Eq.'s (A.2.18) and (A.2.19) and substituting the results into Eq. (A.2.15) yields a single expression for motion of the laminate:

\[
\frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + p(x, y, t) = \rho \ddot{w}
\]  

(A.2.20)
We can now use the load-displacement result of Eq.(A.2.11), along with the definitions of plate curvatures, Eq. (A.2.6), to re-write the equation of motion in terms of the displacement of the middle surface \( w \):

\[
D_{11} \frac{\partial^4 w}{\partial x^4} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w}{\partial y^4} = p(x,y,t) - \rho \ddot{w} \quad (A.2.21)
\]

Compare with the isotropic formulation, in which the isotropic bending stiffness \( D \) replaces the \( D_{ij} \) terms above:

\[
D \left[ \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right] = p(x,y,t) - \rho \ddot{w} \quad (A.2.22)
\]

It now remains to solve the equation of motion for a laminate.

**Static Loading**

In general the loading function \( p(x,y,t) \) is, as indicated, a function of time and space. Static loading is a special case, when the loading function does not vary in time.

A simpler form of Eq. (A.2.21) is then considered:

\[
D_{11} \frac{\partial^4 w}{\partial x^4} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w}{\partial y^4} = p(x,y) \quad (A.2.23)
\]

Boundary conditions are necessary to complete the problem, and identical to the isotropic case, one member of each of the following two pairs must be described along each boundary:

\[
\frac{\partial w}{\partial n}; M_n \quad w; \frac{\partial M_{ns}}{\partial s} + Q_n \quad (A.2.24)
\]

where the subscripts \( n \) and \( s \) indicate normal and tangential to the boundary, respectively.
Common boundary conditions include:

1. Simply Supported

\[ w = M_n = 0 \quad (A.2.25a) \]

2. Clamped

\[ w = \frac{\partial w}{\partial n} = 0 \quad (A.2.25b) \]

3. Free

\[ M_n = \frac{\partial M_n}{\partial s} + Q_n = 0 \quad (A.2.25c) \]

A rectangular plate simply supported on all sides, Figure A.5, will have the following boundary conditions:

at \( x = 0 \) and \( x = a \)

\[ w = M_x = 0 \quad (A.2.26) \]

at \( y = 0 \) and \( y = b \)

\[ w = M_y = 0 \quad (A.2.27) \]

The static solution for a simply supported laminate is a double Fourier series:

\[ w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{p_{mn} \sin(m\pi x/a) \sin(n\pi y/b)}{n^4 \left[ D_{11} \left( \frac{m}{a} \right)^4 + 2(D_{12} + 2D_{66}) \left( \frac{m}{a} \right)^2 \left( \frac{n}{b} \right)^2 + D_{22} \left( \frac{n}{b} \right)^4 \right]} \quad (A.2.28) \]

This solution requires that the load can be expressed in a Fourier series as follows:

\[ p_{mn} = \frac{4}{ab} \int_0^a \int_0^b p(x, y) \sin(m\pi x/a) \sin(n\pi y/b) \, dx \, dy \quad (A.2.29) \]

Loads of common interest include uniform loading, i.e. \( p(x, y) = p_0 \), patch loads, and point loading (see Figure A.6).
The evaluated results of Eq. (A.2.29) for these types of loading follows:

<table>
<thead>
<tr>
<th>Loading Condition</th>
<th>( P_{mn} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>uniform</td>
<td>( \frac{16 p_0}{\pi^2 m n} ; m, n = 1,3,5,... ) (A.2.30a)</td>
</tr>
<tr>
<td>patch</td>
<td>( \frac{16 p_0}{\pi^2 m n} \left[ \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b} + \frac{m \pi x}{a} \sin \frac{n \pi y}{b} \right] ) (A.2.30b)</td>
</tr>
<tr>
<td>point</td>
<td>( \frac{4 P}{a b} \left[ \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b} \right] ) (A.2.30c)</td>
</tr>
</tbody>
</table>

**Dynamic Loading**

If the loading function remains fixed in space, but changes in magnitude, a dynamic analysis must be performed. The loading function is rewritten as the product of a function of space \( p(x,y) \), and a function of time \( F(t) \).

The equation of motion is then:

\[
D_{11} \frac{\partial^4 w}{\partial x^4} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w}{\partial y^4} = p(x,y) \cdot F(t) - p \dot{w} \quad (A.2.31)
\]

Again, the procedure is identical to the isotropic case. The homogeneous form of Eq. (A.2.31) is considered, and a solution is sought that satisfies both the boundary and initial conditions. A rectangular simply supported specially orthotropic laminate will be considered.

A solution of the form:

\[
w = w_0 \cdot e^{j\omega t} \quad (A.2.32)
\]

is assumed, where \( \omega \) is a natural frequency of vibration.

Substitution of the assumed form of solution into Eq. (A.2.31) yields a governing equation independent of time:

\[
D_{11} \frac{\partial^4 w_0}{\partial x^4} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w_0}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w_0}{\partial y^4} - \rho \omega^2 w_0 = 0 \quad (A.2.33)
\]
With regard to the boundary conditions of Eq. (A.2.26) and Eq. (A.2.27), the solution is of the form:

\[ w_o = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \]  

(A.2.34)

which becomes the orthogonality function for the plate.

Substituting this solution into the governing equation of Eq. (A.2.33), an expression for the natural frequency is obtained:

\[ \omega_{mn} = \frac{\pi^2}{\sqrt{\rho h}} \sqrt{D_{11} \left(\frac{m}{a}\right)^4 + 2(D_{12} + 2D_{66}) \left(\frac{m}{a}\right)^2 \left(\frac{n}{b}\right)^2 + D_{22} \left(\frac{n}{b}\right)^4} \]  

(A.2.35)

The homogeneous solution must be complemented by a particular solution involving the loading function \( p(x,y) \cdot F(t) \). The normalizing properties of the orthogonality function are used to find the particular solution.

The particular solution will be of the form:

\[ w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \phi_{mn} T_{mn}(t) \]  

(A.2.36)

where

\[ \phi_{mn} = \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \]  

(A.2.37)

and \( T_{mn}(t) \) is an arbitrary function of time.

Noting that:

\[ \int_0^a \int_0^b \phi_{mn} \phi_{rs} \, dx \, dy = 0; \quad m,n \neq r,s \]  

(A.2.38)

we can normalize the equation of motion with respect to the homogeneous solution.

This requires multiplying each term in Eq. (A.2.33) by the orthogonality function and integrating through the thickness of the plate:

\[ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[ \int_0^a \int_0^b \rho h \omega_{mn}^2 \phi_{mn} \phi_{rs} \, dx \, dy \cdot T_{mn}(t) + \int_0^a \int_0^b \rho h \phi_{mn} \phi_{rs} \, dx \, dy \cdot \tilde{T}_{mn}(t) \right] \]

\[ = \int_0^a \int_0^b \int_0^a \int_0^b \rho(x,y) \phi_{rs} \, dx \, dy \cdot F(t) \]  

(A.2.39)
The orthogonality property allows us to drop the summation in Eq. (A.2.39), and we then only need to solve a series of ordinary differential equations.

The normalized equation is:

\[
\omega_{mn}^2 T_{mn}(t) + \ddot{T}_{mn}(t) = \frac{F(t)}{\rho h} \frac{4}{ab} \int_0^{h a} \int_0^{h b} p(x, y) \phi_{mn} dx dy \tag{A.2.40}
\]

where the homogeneous solution has been applied, and the results of the integration:

\[
\int_0^{h a} \int_0^{h b} \phi_{mn}^2 dx dy = \frac{ab}{4} \tag{A.2.41}
\]

has also been used.

Note that in this case the integral on the right hand side of Eq. (A.2.40) matches the Fourier integral used to evaluate \( p_{mn} \) in the static solution.

The solution for \( T_{mn}(t) \) is the well known form for one degree of freedom dynamic systems:

\[
w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} T_{mn}(0) \cdot \sin \omega_{mn} t + \dot{T}_{mn}(0) \cdot \cos \omega_{mn} t + \frac{p_{mn}}{\rho \omega_{mn}} \int_0^t F(\tau) \sin \omega_{mn} (t - \tau) d\tau \tag{A.2.42}
\]

For the case when the system is initially at rest, and the applied load is constant with respect to time, the following solution applies:

\[
w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{p_{mn}}{\rho \omega_{mn}} (1 - \cos \omega_{mn} t) \tag{A.2.43}
\]

Substituting the expression of Eq. (A.2.35), into the solution above, yields:

\[
w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{p_{mn}(1 - \cos \omega_{mn} t)}{\pi^4 D_{11} \left( \frac{m}{a} \right)^4 + 2(D_{12} + 2D_{56}) \left( \frac{m}{a} \right)^2 \left( \frac{n}{b} \right)^2 + D_{23} \left( \frac{n}{b} \right)^4} \tag{A.2.44}
\]
Comparing this result to the static solution of Eq. (A.2.28), we can write:

$$w = w_{st} \cdot (1 - \cos \omega_{st} t)$$  \hspace{1cm} (A.2.45)

where $w_{st}$ is the equivalent static deflection.

### A.2.2 Whitney and Pagano Theory

The Kirchoff theory reduced the three dimensional elasticity problem of a loaded laminate, to a two dimensional analysis of the laminate mid-plane. Whitney and Pagano [1970] investigated the extension of Mindlin's isotropic plate bending theory to laminated plates.

The assumed strain field takes the form:

\begin{align}
  u(x, y, z, t) &= u^0(x, y, t) + z\psi_x(x, y, t) \tag{A.2.46a} \\
  v(x, y, z, t) &= v^0(x, y, t) + z\psi_y(x, y, t) \tag{A.2.46b} \\
  w &= w(x, y, t) \tag{A.2.46c}
\end{align}

where $\psi_x$ and $\psi_y$ are the cross sectional rotations in the $x$ and $y$ directions respectively (see Figure A.7).

The strains are defined as in the Kirchoff type theory, Eq. (A.2.5), and the transverse shear strains are therefore:

\begin{align}
  2\varepsilon_{xz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \psi_x + \frac{\partial w}{\partial x} \tag{A.2.47} \\
  2\varepsilon_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \psi_y + \frac{\partial w}{\partial y} \tag{A.2.48}
\end{align}
The in-plane forces, and moments are described by the constitutive equation used for the Kirchoff derivation, where the curvatures are redefined as:

\[
\begin{align*}
\kappa_x &= \frac{\partial \psi_x}{\partial x} \\
\kappa_y &= \frac{\partial \psi_y}{\partial y} \\
\kappa_{xy} &= \frac{1}{2} \left( \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right)
\end{align*}
\] 

(A.2.49a) \hspace{1cm} (A.2.49b) \hspace{1cm} (A.2.49c)

In addition the transverse shear force is related to the shear strains by the following:

\[
\begin{pmatrix}
Q_y \\
Q_x
\end{pmatrix} = k
\begin{bmatrix}
A_{44} & A_{45} \\
A_{45} & A_{55}
\end{bmatrix}
\begin{pmatrix}
2\varepsilon_{xx} \\
2\varepsilon_{xy}
\end{pmatrix}
\] 

(A.2.50)

where \( k \) is a parameter first used in the Mindlin isotropic, shear deformation theory.

Satisfying equilibrium in the transverse direction yields the following:

\[
\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + p(x, y, i) = \rho h \ddot{w}
\] 

(A.2.51)

Equilibrium of moments in the \( x \) and \( y \) directions yields:

\[
\begin{align*}
\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x &= I \ddot{\psi}_x \\
\frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_y &= I \ddot{\psi}_y
\end{align*}
\] 

(A.2.52a) \hspace{1cm} (A.2.52b)

where the rotary inertia is defined as \( I = \frac{\rho h^3}{12} \).

Again, we assume that no in-plane tractions are applied thus the in-plane equilibrium is identically satisfied.
Substituting the constitutive relations into Eq.'s (A.2.51) through (A.2.52b) yields three equations of motion:

\[
\begin{align*}
D_{11} \frac{\partial^2 \psi_x}{\partial x^2} + (D_{12} + D_{66}) \frac{\partial^2 \psi_y}{\partial x \partial y} + \frac{1}{D_{66}} \frac{\partial^2 \psi_z}{\partial y^2} - kA_{55} \left( \psi_x + \frac{\partial w}{\partial x} \right) &= f \ddot{y}_x \\
D_{11} \frac{\partial^2 \psi_y}{\partial y^2} + (D_{12} + D_{66}) \frac{\partial^2 \psi_x}{\partial x \partial y} + \frac{1}{D_{66}} \frac{\partial^2 \psi_z}{\partial x^2} - kA_{44} \left( \psi_y + \frac{\partial w}{\partial y} \right) &= f \ddot{y}_y \\
kA_{55} \left( \frac{\partial \psi_x}{\partial x} + \frac{\partial \psi_y}{\partial x} \right) + kA_{44} \left( \frac{\partial \psi_y}{\partial y} + \frac{\partial \psi_z}{\partial y} \right) + p(x, y, t) &= \rho h \dddot{w}
\end{align*}
\] (A.2.53)

**Static Loading**

For a constant applied load, we can ignore the inertial terms of the equations of motion. Solving the equations requires boundary conditions, which we will assume to be hinged on all sides. In this case, the shear rotations normal to the boundary are zero, and the slope is unrestrained.

For the rectangular plate of Figure A.5 the prescribed boundary conditions are:

at \( x = 0 \) and \( x = a \)

\[
w = M_x = \psi_y = 0 \quad (A.2.54)
\]

at \( y = 0 \) and \( y = b \)

\[
w = M_y = \psi_x = 0 \quad (A.2.55)
\]

The following solutions satisfy the simply supported boundary conditions:

\[
\begin{align*}
\psi_x &= \sum_{m} \sum_{n} U_{mn} \cos(m\pi x/a) \sin(n\pi y/b) \\
\psi_y &= \sum_{m} \sum_{n} V_{mn} \sin(m\pi x/a) \cos(n\pi y/b) \\
w &= \sum_{m} \sum_{n} W_{mn} \sin(m\pi x/a) \sin(n\pi y/b)
\end{align*}
\] (A.2.56)
Substituting the above solutions into the equations of motion, Eq. (A.2.53a-c), the following set of linear equations results:

\[
\begin{bmatrix}
L_{11} & L_{12} & L_{13} \\
L_{21} & L_{22} & L_{23} \\
L_{31} & L_{32} & L_{33}
\end{bmatrix}
\begin{bmatrix}
U_{mn} \\
V_{mn} \\
W_{mn}
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]  
(A.2.57)

where the load has again been expressed as a Fourier series.

The components of the \([L]\) matrix are as follows:

\[
L_{11} = D_{11}(m \pi/a)^2 + D_{66}(n \pi/b)^2 + kA_{55} \quad (A.2.57a)
\]

\[
L_{12} = (D_{12} + D_{66})(m \pi/a)(n \pi/b) \quad (A.2.57b)
\]

\[
L_{13} = kA_{55}(m \pi/a) \quad (A.2.57c)
\]

\[
L_{22} = D_{66}(m \pi/a)^2 + D_{22}(n \pi/b)^2 + kA_{44} \quad (A.2.57d)
\]

\[
L_{23} = kA_{44}(n \pi/b) \quad (A.2.57e)
\]

\[
L_{33} = kA_{55}(m \pi/a)^2 + kA_{44}(n \pi/b)^2 \quad (A.2.57f)
\]

Solving for the constants \(U_{mn}, V_{mn}, W_{mn}\), yields the following:

\[
U_{mn} = \frac{(L_{11}L_{23} - L_{22}L_{13})p_{mn}}{\Delta} \quad (A.2.58a)
\]

\[
V_{mn} = \frac{(L_{21}L_{33} - L_{33}L_{23})p_{mn}}{\Delta} \quad (A.2.58b)
\]

\[
W_{mn} = \frac{(L_{11}L_{22} - L_{12}^2)p_{mn}}{\Delta} \quad (A.2.58c)
\]

where

\[
\Delta = L_{11}(L_{22}L_{33} - L_{23}^2) - L_{12}(L_{42}L_{33} - L_{23}L_{43}) + L_{13}(L_{42}L_{33} - L_{22}L_{43}) \quad (A.2.58d)
\]

Recall the solution for lateral deflection:

\[
w = \sum_{m} \sum_{n} W_{mn} \sin(m \pi x/a) \sin(n \pi y/b) \quad (A.2.59)
\]

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**Dynamic Loading**

We begin by solving the homogeneous problem, i.e. \( p(x,y,t) = 0 \).

For the same simply supported boundary conditions we have a solution of the form:

\[
\psi_x = \sum_{m}^{\infty} \sum_{n}^{\infty} e^{i\omega t} \cdot U_{mn} \cos(m\pi x/a)\sin(n\pi y/b) \tag{A.2.60}
\]

\[
\psi_y = \sum_{m}^{\infty} \sum_{n}^{\infty} e^{i\omega t} \cdot V_{mn} \sin(m\pi x/a)\cos(n\pi y/b) \tag{A.2.61}
\]

\[
w = \sum_{m}^{\infty} \sum_{n}^{\infty} e^{i\omega t} \cdot W_{mn} \sin(m\pi x/a)\sin(n\pi y/b) \tag{A.2.62}
\]

When substituted into the equations of motion, the solution yields the following:

\[
\begin{bmatrix}
L_{11} - I_0 \omega_{mn}^2 & L_{12} & L_{13} \\
L_{12} & L_{22} - I_0 \omega_{mn}^2 & L_{23} \\
L_{13} & L_{23} & L_{33} - \rho h \omega_{mn}^2
\end{bmatrix}
\begin{bmatrix}
U_{mn} \\
V_{mn} \\
W_{mn}
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{A.2.63}
\]

where the components of the \( L_{ij} \) are those defined by Eq. (A.2.57a-f).

For each \( m,n \) pair in the series solution there are three eigenvalues, and three eigenvectors:

\[
\omega_{mnj} = (\omega_{mn1}, \omega_{mn2}, \omega_{mn3}) \tag{A.2.64}
\]

\[
\begin{bmatrix}
U_{mn} \\
V_{mn} \\
W_{mn}
\end{bmatrix} = \begin{bmatrix} U_{mn1} & U_{mn2} & U_{mn3} \\ V_{mn1} & V_{mn2} & V_{mn3} \\ W_{mn1} & W_{mn2} & W_{mn3} \end{bmatrix} \tag{A.2.65}
\]

Only two components of each eigenvector are independent, and can be normalized with respect to \( W_{mnj} \).
The particular solution is assumed to be separable into functions of position and time:

\[ \psi_x = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{j=1}^{3} U_{mnj} \cos \left( \frac{m \pi x}{a} \right) \sin \left( \frac{n \pi y}{b} \right) \cdot T_{mnj}(t) \]  

(A.2.66a)

\[ \psi_y = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{j=1}^{3} V_{mnj} \sin \left( \frac{m \pi x}{a} \right) \cos \left( \frac{n \pi y}{b} \right) \cdot T_{mnj}(t) \]  

(A.2.66b)

\[ w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{j=1}^{3} W_{mnj} \sin \left( \frac{m \pi x}{a} \right) \sin \left( \frac{n \pi y}{b} \right) \cdot T_{mnj}(t) \]  

(A.2.66c)

Substituting the above expressions into the equations of motion, and applying the homogeneous solution of Eq. (A.2.63), yields the following:

\[ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{j=1}^{3} U_{mnj} \cos \left( \frac{m \pi x}{a} \right) \sin \left( \frac{n \pi y}{b} \right) \left[ \omega_{mnj}^2 T_{mnj} + \ddot{T}_{mnj} \right] = 0 \]  

(A.2.67a)

\[ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{j=1}^{3} V_{mnj} \sin \left( \frac{m \pi x}{a} \right) \cos \left( \frac{n \pi y}{b} \right) \left[ \omega_{mnj}^2 T_{mnj} + \ddot{T}_{mnj} \right] = 0 \]  

(A.2.67b)

\[ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{j=1}^{3} W_{mnj} \sin \left( \frac{m \pi x}{a} \right) \sin \left( \frac{n \pi y}{b} \right) \left[ \omega_{mnj}^2 T_{mnj} + \ddot{T}_{mnj} \right] = \frac{P_x}{\rho h} \]  

(A.2.67c)

where \( T_{mnj}(t) \) is a time dependent function yet to be determined, and the dot subscript indicates a derivative with respect to time.

In order to solve for the time function \( T_{mnj}(t) \), Eq. (A.2.67a-c) must be orthogonalized. Love [1927] presented a general method of orthogonalizing a three dimensional elasticity problem. Using this method, we apply Clebsch's theorem to obtain the orthogonality condition.
The general conjugate property is:

\[ \int_{-b/2}^{b/2} \int_{-h/2}^{h/2} \rho \left( \overline{u}, \overline{u} + \overline{u}, \overline{u} \right) \, dy \, dx = 0 \quad r \neq s \]  
(A.2.68)

where the displacement fields are in the form:

\[ u = \sum \overline{u}_r(x, y, z) T_r(t) \]  
(A.2.69a)

\[ v = \sum \overline{v}_r(x, y, z) T_r(t) \]  
(A.2.69b)

\[ w = \sum \overline{w}_r(x, y, z) T_r(t) \]  
(A.2.69c)

Examining the displacement fields, Eq. (A.2.46a-c) and the particular solution, Eq. (A.2.66a-c), the displacements, in Love's notation are:

\[ \overline{u}_r = z \cdot U_{mnj} \cos(m\pi x/a) \sin(n\pi y/b) \]  
(A.2.70a)

\[ \overline{v}_r = z \cdot V_{mnj} \sin(m\pi x/a) \cos(n\pi y/b) \]  
(A.2.70b)

\[ \overline{w}_r = W_{mnj} \sin(m\pi x/a) \sin(n\pi y/b) \]  
(A.2.70c)

and \( T_r(t) = T_{mnj}(t) \).

Thus substituting are assumed forms for \( \overline{u}_r, \overline{v}_r, \overline{w}_r \), and integrating through the thickness, the conjugate property becomes:

\[ \int_{-b/2}^{b/2} \int_{-h/2}^{h/2} \Pi_{efg, mnj} \, dy \, dx = 0 \quad e, f, g \neq m, n, j \]  
(A.2.71)

where

\[ \Pi_{efg, mnj} = IU_{efg} \cos(e\pi x/a) \sin(f\pi y/b) \cdot U_{mnj} \cos(m\pi x/a) \sin(n\pi y/b) \]

\[ + IV_{efg} \sin(e\pi x/a) \cos(f\pi y/b) \cdot V_{mnj} \sin(m\pi x/a) \cos(n\pi y/b) \]

\[ + phW_{efg} \sin(e\pi x/a) \sin(f\pi y/b) \cdot W_{mnj} \sin(m\pi x/a) \sin(n\pi y/b) \]  
(A.2.72)

The three equations of motion, Eq. (A.2.67a-c), must now be rearranged to match the above expression.

This is achieved by multiplying Eq. (A.2.67a) by:

\[ IU_{efg} \cos(e\pi x/a) \sin(f\pi y/b) \]  
(A.2.73a)
Eq. (A.2.67b) by:

\[ IV_{eg} \sin(\pi x/a) \cos(\pi y/b) \]  

and Eq. (A.2.67c) by:

\[ \rho W_{eg} \sin(\pi x/a) \sin(\pi y/b) \]  

then summing the three results.

The resulting single equation, in terms of the time dependent variable \( T_{mnj} \), is:

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{j=1}^{3} \left[ \omega_{mnj}^2 \Pi_{efg,nnj} T_{mnj} + \Pi_{efg,nnj} \ddot{T}_{mnj} \right] - \rho W_{eg} \sin(\pi x/a) \sin(\pi y/b) \cdot p_z = 0 \]  

Next, Eq. (A.2.74) is integrated over the area of the plate, with the only non-zero result occurring when \((e,f,g) = (m,n,j)\).

After performing the integration, each set of indices being equal, we can drop the summation, and Eq. (A.2.74) becomes:

\[
\int_{a}^{b} \int_{0}^{\frac{a}{b}} \omega_{mnj} T_{mnj} + \ddot{T}_{mnj} = M_{mnj} \]  

where

\[
M_{mnj} = 4 \int_{0}^{b} \int_{0}^{\frac{a}{b}} \Pi_{mnj, mnj} dy dx \]  

Note that Eq. (A.2.75) is the familiar single degree of freedom equation of motion, and that \( W_{mnj} = 1 \), because of the eigenvector normalization.

As with the Kirchoff type analysis, we consider a dynamic load in the form:

\[ p(x,y,t) = p(x,y) \cdot F(t) \]  

(A.2.77)
The dynamic solution then becomes:

\[
T_{mnj} = T_{mnj}^0 \cos(\omega_{mnj} t) + \frac{\omega_{mnj}^2}{\omega_{mnj}} \sin(\omega_{mnj} t) + \frac{abq_{mn}}{4\omega_{mnj}M_{mnj}} \int_0^t F(\tau)\sin(\omega_{mnj}(t-\tau)) d\tau
\]  

(A.2.78)

where \( p_{mnj} \) is \( p(x,y) \) expressed as a Fourier series.

The solution for lateral deflection of the dynamically loaded laminate is:

\[
\omega = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin(m\pi x/a) \sin(n\pi y/b) \cdot \left\{ \sum_{j=1}^{3} \left[ T_{mnj}^0 \cos(\omega_{mnj} t) + \frac{\omega_{mnj}^2}{\omega_{mnj}} \sin(\omega_{mnj} t) + \frac{abq_{mn}}{\omega_{mnj}^2 M_{mnj}} \int_0^t F(\tau) \sin(\omega_{mnj}(t - \tau)) d\tau \right] \right\}
\]  

(A.2.79)

A.3 Figures

![Figure A.1: Principal material co-ordinate system.](image1)

![Figure A.2: Natural co-ordinate system.](image2)
Figure A.3: Differential plate element - force equilibrium.

Figure A.4: Differential plate element - moment equilibrium.

Figure A.5: Rectangular laminate with general loading.
Figure A.6: Patch loading nomenclature.

Figure A.7: Shear deformation nomenclature.
APPENDIX B

UBC Impact® Software

Each of the impact models developed in this thesis have been implemented in a user friendly computer program called UBC Impact®.

B.1 The Impact Interface

B.1.1 What’s On The Screen

When you start Impact, a new session begins, and the Impact screen is displayed. The following illustration identifies each part of the screen.
The Toolbar

Using the mouse, you can quickly choose commands from the toolbar.

Command buttons may not always be available, depending on the simulation status. For example, while a simulation is running or paused, the New command will be disabled.

The buttons on the toolbar will automatically adjust position when you resize the Impact screen.

The status indicator provides visual information regarding the simulation status. See Simulation Status below.

The Status Bar

The status bar at the bottom of the Impact screen displays information about the current session.
**Input Status**

Information regarding the status of input parameters is shown in the segment on the left side of the status bar.

<table>
<thead>
<tr>
<th>Input Status</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>None</td>
<td>An Impact session has just started, and no parameters have been entered by the user.</td>
</tr>
<tr>
<td>File</td>
<td>An input file has been loaded into the current session, or the existing parameters have been saved to a file.</td>
</tr>
<tr>
<td>Modified</td>
<td>Changes to input parameters have been made since the last open or save operation.</td>
</tr>
</tbody>
</table>

**Laminate Status**

The constitutive properties of the target laminate are only calculated when required, i.e., when you ask to view the laminate properties or when a simulation begins. The status of these calculated laminate properties is shown in the middle segment of the status bar.

<table>
<thead>
<tr>
<th>Laminate Status</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Not calculated</td>
<td>The laminate properties have not been calculated.</td>
</tr>
<tr>
<td>Using current values</td>
<td>The laminate properties have been calculated, and no changes to parameters affecting these properties have been made.</td>
</tr>
<tr>
<td>Using old values</td>
<td>Changes have been made to parameters affecting the laminate properties, but these properties have not been recalculated.</td>
</tr>
</tbody>
</table>

This feature is useful when changing parameters during a simulation.

For example, you may be simulating a non-penetrating impact, and decide that the target plate is too stiff. Pause the simulation, and change the thickness of the target laminate. The status bar will indicate that the old laminate properties are being used.

Restart the simulation and choose to recalculate the laminate properties. The status bar will now show that the current values are being used, i.e., that the properties have been recalculated using the new target thickness,
Simulation Status

The status of the numerical engine used to perform Impact simulations is shown in the right most segment of the status bar, as well as by the status indicator on the toolbar (see above).

<table>
<thead>
<tr>
<th>Simulation Status</th>
<th>Status Indicator</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Idle</td>
<td>Steady grey</td>
<td>No input parameters are available.</td>
</tr>
<tr>
<td>Ready</td>
<td>Steady red</td>
<td>Parameters are available, simulation has not been started.</td>
</tr>
<tr>
<td>Running</td>
<td>Flashing green</td>
<td>Simulation is running</td>
</tr>
<tr>
<td>Paused</td>
<td>Steady yellow</td>
<td>Simulation has been paused</td>
</tr>
<tr>
<td>Stopped</td>
<td>Steady red</td>
<td>Simulation was halted by the user.</td>
</tr>
<tr>
<td>Done</td>
<td>Steady red</td>
<td>Impact event is complete.</td>
</tr>
</tbody>
</table>

B.1.2 Choosing Commands

Commands are grouped in the four main menus of the Impact screen. Some commands carry out an action immediately; others display a dialog box so that you can select options.
Choosing a command by using the mouse  Click the name of a menu on the menu bar, and then click the command name. To close a menu without choosing a command, click outside the menu area.

Choosing a command by using the keyboard  Press ALT or F10 to make the menu bar active, and then press the key corresponding to the underlined letter in the menu name. To choose a command press the key for the underlined letter or number in the command name. To close a menu without choosing a command press ESC.

The arrow keys (ARROW UP, ARROW DOWN, etc.) can also be used to choose menus and commands.

Using the Shortcut Menu

When you point to any of the three output child windows in Impact, a shortcut menu is available, by holding down the right mouse button. This menu is identical to the Edit menu on the menu bar.

Commands in the shortcut menu can be chosen by dragging with the right mouse button, and releasing when the command is selected, or a command can be chosen by clicking the left mouse button over the desired command.

Using Shortcut Keys

You can choose some commands by pressing the shortcut keys listed on the menus to right of the command. For example, to open an input file, press CTRL+O.

B.1.3 Dialog Boxes

The Impact interface can display two types of dialog boxes. Child windows are a type of dialog box that remain on the screen at all times, while other dialog boxes are displayed only when required.
Dialog boxes contain options, commands, and text boxes.

Click a command button to carry out an action or display another dialog box.

Drag the title bar to move the dialog box.

![Dialog Box Example]

Click an arrow to see a list; click or drag to select an option.

Although it is usually easiest to use the mouse while you work in a dialog box, you can also select options or fill in information with the keyboard. To move among the different options, press TAB. To select the current option or command, press enter. To move instantly to a specific option, use the ALT key and the letter underlined in the title of the desired option. For example, to move to the “File Name” text box in the dialog box above, press ALT+N.

**Child Windows**

Child windows are a special type of dialog box because they remain on the screen at all times. Child windows are used both to input parameters, and to display simulation results.
To invoke any changes press the Update command, or to undo any changes press Cancel.

Click here to minimize the dialog box.

Parameters not required by the currently selected model appear in normal type.

Parameters needed by the selected impact model appear in bold type.
**Projectile Window**  The projectile window allows you to edit the dimensions and properties of the projectile.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Units</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Impact Velocity</td>
<td>m/s</td>
<td>Velocity of the projectile at the time of impact.</td>
</tr>
<tr>
<td>Mass</td>
<td>kg</td>
<td>Mass of the projectile. It is assumed not to erode.</td>
</tr>
<tr>
<td>Shape</td>
<td></td>
<td>Shape of the projectile tip. Choose between conical, flat, or hemispherical.</td>
</tr>
<tr>
<td>Friction Stress</td>
<td>MPa</td>
<td>Stress induced by sliding between the surface of the projectile and the target material.</td>
</tr>
<tr>
<td>Diameter</td>
<td>mm</td>
<td>Diameter of the projectile shaft. In the case of a hemispherical shape, the diameter of the tip is assumed to be equal to the shaft diameter.</td>
</tr>
<tr>
<td>Total Length</td>
<td>mm</td>
<td>Length of the projectile from tip to toe.</td>
</tr>
<tr>
<td>Tip Angle</td>
<td>degrees</td>
<td>Half angle of a conical tip.</td>
</tr>
<tr>
<td>Blunt Length</td>
<td>mm</td>
<td>See UBC Conical model.</td>
</tr>
<tr>
<td>Modulus</td>
<td>GPa</td>
<td>Elastic modulus of the projectile material.</td>
</tr>
<tr>
<td>Poisson’s Ratio</td>
<td></td>
<td>Poisson’s ratio of the projectile material.</td>
</tr>
</tbody>
</table>
**Target Window** The target window describes the dimensions, and material properties of the target laminate.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Units</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Target Length</td>
<td>mm</td>
<td>Length of the target laminate in the major or 0° direction.</td>
</tr>
<tr>
<td>Target Width</td>
<td>mm</td>
<td>Length of the laminate in the 90° direction.</td>
</tr>
<tr>
<td>Target Thickness</td>
<td>mm</td>
<td>Total thickness of the laminate.</td>
</tr>
<tr>
<td>Average density</td>
<td>kg/m³</td>
<td>Density of the laminate.</td>
</tr>
<tr>
<td>Bulge Delay</td>
<td>mm</td>
<td>See UBC Conical model.</td>
</tr>
<tr>
<td>Flow Stress</td>
<td>MPa</td>
<td>Yield strength for a perfectly plastic material.</td>
</tr>
<tr>
<td>Shear Strength</td>
<td>MPa</td>
<td>Yield strength, in shear of the target material</td>
</tr>
<tr>
<td>Ultimate Strength</td>
<td>MPa</td>
<td>Ultimate strength of the target material.</td>
</tr>
<tr>
<td>Failure Strains</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Fiber</td>
<td>m/m</td>
<td>Breaking strain of the laminate fibers.</td>
</tr>
<tr>
<td>Matrix</td>
<td>m/m</td>
<td>Ultimate strain of the laminate matrix.</td>
</tr>
<tr>
<td>Lateral Shear</td>
<td>m/m</td>
<td>See Awerbuch and Bodner model.</td>
</tr>
<tr>
<td>Elastic Properties¹</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E_1$</td>
<td>GPa</td>
<td>Modulus of a lamina in the fiber direction.</td>
</tr>
<tr>
<td>$E_2$</td>
<td>GPa</td>
<td>Modulus of a lamina perpendicular to the fiber direction.</td>
</tr>
<tr>
<td>$ν_{12}$</td>
<td></td>
<td>Poisson’s ratio in the plane of a lamina.</td>
</tr>
<tr>
<td>$G_{12}$</td>
<td>GPa</td>
<td>Shear modulus in the plane of a lamina.</td>
</tr>
<tr>
<td>$G_{23}$</td>
<td>GPa</td>
<td>Shear modulus in the 2-3 plane of a lamina.</td>
</tr>
</tbody>
</table>

Other parameters are accessed using the command buttons which display dialog boxes.

¹ The lamina is assumed to be tranversely isotropic, thus $E_2 = E_3$, and $G_{12} = G_{13}$. 
View [A] [B] [D] Dialog Box

Clicking the View [A] [B] [D] command button calculates the laminate properties and displays them in a dialog box.

[A] Matrix In plane extensional stiffnesses $A_{11}$ ... $A_{33}$. The in plane shear stiffness, $A_{44}$ and $A_{55}$, are also shown.

[B] Matrix Coupling stiffnesses $B_{11}$...$B_{33}$. These values are calculated by Impact but not used, as the laminate is assumed to symmetric and orthotropic.

[D] Matrix Bending stiffnesses $D_{11}$...$D_{33}$.

Laminate Dialog Box

The laminate dialog box allows you to describe the lay-up of the target laminate.

Select the number of layers in the laminate.

Type the orientation angle of the selected lamina (in degrees), or click the arrows to adjust by an amount shown in Step Size.

Click here to set the orientation of the currently selected layer to zero.

Use the mouse to select the lamina you want to edit.
Stress Strain Dialog Box

This dialog box allows you to input parameters used by the LAMP model.

Static Indentation Dialog Box

This dialog box allows you to input parameters used by the UBC Flat model.

Picture Window  This window displays a graphic of the impact event, including the damaged target material surrounding the projectile. The picture window operates only when the Zhu or UBC Conical models are selected.

Grid Viewer  Results are shown in a spreadsheet type of display.

Cells within the Grid Viewer can be selected and copied into the clipboard. The entire grid can also be selected, copied to the clipboard, or cleared. When a range of cells is copied to the clipboard, the column labels are automatically included.

Results shown in the Grid Viewer include the following.

<table>
<thead>
<tr>
<th>Result</th>
<th>Units</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>msec</td>
<td>Time elapsed in impact event.</td>
</tr>
<tr>
<td>Force</td>
<td>N</td>
<td>Contact force between projectile and target.</td>
</tr>
<tr>
<td>Velocity</td>
<td>m/s</td>
<td>Velocity of projectile.</td>
</tr>
<tr>
<td>Disp. Projectile</td>
<td>mm</td>
<td>Displacement of projectile. Displacement equals zero at start of the simulation.</td>
</tr>
<tr>
<td>Defl. Target</td>
<td>mm</td>
<td>Deflection of the target mid-plane, at point of impact.</td>
</tr>
<tr>
<td>(1-D)</td>
<td></td>
<td>Damage factor. See Zhu model and UBC Conical model.</td>
</tr>
<tr>
<td>Comments</td>
<td></td>
<td>Information pertaining to the simulation. For example the perforation of the target will be noted in this column.</td>
</tr>
</tbody>
</table>
**Graph Viewer**  Any of the numerical results shown in the Grid Viewer can be plotted in the Graph Viewer.

You can choose the results to be plotted, by double clicking over the Graph Viewer. The Graph Viewer dialog box will be displayed.

![Graph Viewer Diagram]

Choose one parameter for the X-Axis

Choose up to five parameters to plot.

The Graph Viewer requires a large amount of processor time to update. Keep this window minimized while not in use, to speed up the simulation.

**Working With Child Windows**

Initially the five child window icons will be shown at the bottom of the screen, double click these to input data or view output.

**Window Controls**

In the uppermost corner of each window are the window controls. Use the control box in the left corner for menu driven controls, and the control buttons in the right corner for mouse driven controls.
Using the Control Box  Click on the Control Box to access the following menu.

This menu affects the status and appearance of the active window.

<table>
<thead>
<tr>
<th>Command</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Restore</td>
<td>Sizes the window to the default size.</td>
</tr>
<tr>
<td>Move</td>
<td>Allows the user to size the Interface window while displaying the double arrow pointer.</td>
</tr>
<tr>
<td>Size</td>
<td>The active window can be resized by dragging the double arrow mouse pointer on a window border.</td>
</tr>
<tr>
<td>Minimize</td>
<td>The active window is reduced to its icon.</td>
</tr>
<tr>
<td>Maximize</td>
<td>The window is resized to fit the entire screen area. Child windows are sized to fit the Impact Interface window.</td>
</tr>
<tr>
<td>Close</td>
<td>Closing the Impact Interface will have the same effect as using the Exit command. Closing a child window is equivalent to choosing Minimize.</td>
</tr>
<tr>
<td>Switch To</td>
<td>The Switch To Dialog Box will appear allowing the user to transfer control to another application present in the Windows operating environment.</td>
</tr>
</tbody>
</table>
Using the Mouse Controls  You can invoke the Minimize command by clicking on the minimize button on the right side of the title bar. Click the maximize button to invoke the Maximize command. Click the restore button to invoke the Restore command.

B.2 Commands

Commands are used to control the simulation process, open and save files, and to direct output.

B.2.1 Simulation Control

Before starting an Impact simulation, the appropriate parameters should be entered in the Target and Projectile child windows. As well, data controlling the numerical engine is required. Select the Solution and Damage commands from the Options menu to display the appropriate dialog box.

The appearance of warning dialog boxes can be controlled before or during a simulation by choosing the Warnings On command.

Use the command buttons on the toolbar to start stop and pause the simulation.

Solution Dialog Box

This dialog box allows you to adjust the size of the time step, the number of modes, and the type of target loading, all used by the numerical engine.
**Time Step**  The time increment at which results for the impact event are calculated. A large time increment will yield inaccurate results, and a very small time step will cause the simulation to proceed very slowly.

**Number of Modes**  The number of modes, in each direction (x and y), used in the modal solution when calculating the target deflection. Similar to the Time Step, an appropriate number of modes should be chosen to maximize efficiency.

**Loading Type**  The loading pattern used to calculate the target deflection.

This option is only used by the Non-Penetrating model.

Choices include:

<table>
<thead>
<tr>
<th>Loading Type</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Point Load</td>
<td>A one dimensional force applied at the center of the target laminate.</td>
</tr>
<tr>
<td>Patch Load</td>
<td>A patch, equal in length and width to the diameter of the projectile, and centered on the target laminate. The contact force is evenly distributed over this patch.</td>
</tr>
<tr>
<td>Distributed Load</td>
<td>The contact force is evenly distributed over the entire area of the target laminate.</td>
</tr>
</tbody>
</table>
**Damage Dialog Box**

The Zhu model and the UBC Conical model calculate damage in the target laminate according to the parameters entered in this dialog box.

**Picture Window Update**  Determines how often, in relation to the Time Step, the Picture Window is updated. Each time an update is performed a considerable delay to the simulation occurs. Minimizing the Picture Window prevents the window from being updated.

**Size of Damage Grid**  The size of the grid used to calculate the target damage. A grid of size $N$ will require strain calculations to be performed at $N^2$ points.

**Warnings On/Off**

Use this option to control the appearance of warning dialog boxes while a simulation is in progress. These dialog boxes require you to respond before the simulation can continue.

To allow these warning dialog boxes to appear, select the Warnings On option from the Options menu. A check mark to the left of the menu item indicates that the option is on.
**Run**

To begin a simulation, click the Run command button from the toolbar.

First, the Simulation dialog box will be displayed.

![Image of Simulation dialog box]

- Choose this option to save the input file before starting a simulation.
- Choose this option to recalculate the laminate properties using the current target parameters.
- Choose this option to add an informative header to the Grid Viewer.

Simulation dialog options, if grey in colour are not available, or not required. For example if the input file had been saved prior to clicking the Run command, the Save Input option on the Simulation dialog box is not required and will appear grey.

Once a simulation is in progress the Run button will be replaced by the Halt button.

**Halt**

Click this button to completely stop the current simulation. Once halted, a simulation can not be resumed.

After halting the simulation the Halt button will be replaced by the Run button.

**Pause**

Click this button to pause the current simulation. Once paused, the simulation can be resumed by clicking the Resume button which replaces the Pause button.
You can edit target and projectile properties while a simulation is paused.

**Resume**

Click this button to resume a paused simulation. The Pause button will replace the Resume button.

**B.2.2 Edit Commands**

Edit commands are available to manipulate input and output data.

**Copy**

You can copy images of the Graph Viewer or Picture Window to the clipboard. This is useful when preparing summaries of Impact sessions.

Selected cells within the Grid Viewer can be copied to the clipboard in a form that is easily pasted into a Windows based spreadsheet.

**Select All**

Use this command to select the entire contents of the Grid Viewer.

**Clear Window**

You can clear the contents of the active child window with this command.

**Clear Inputs**

Choose this command to clear all the inputs. Data in the Target Window, Projectile Window, and associated dialog boxes will be cleared.

**Clear Outputs**

Use this command to reset the Graph Viewer, and to clear the Grid Viewer and Picture Window.

**Clear All**

Choosing this command is equivalent to choosing Clear Inputs and Clear Outputs.

**B.2.3 File Commands**

Two types of files are used by Impact: input files, and output files.
New

Use this command to start a new Impact session. All existing parameters in the input child windows will be cleared.

Open Input

The data contained in the input child windows can be stored in a file. Use this command to load an existing input file.

The standard Windows dialog box will be displayed.

Save Input

You can save data in the input child window for later use with the Save Input command.

The data is saved as unformatted text. You can edit the file with a standard text editor, such as Notepad, but data is not in an easy to understand format. Editing an input file outside of the Impact interface is not recommended.

If a filename has not been given to the current Impact session, you will be prompted for a filename in which to save the data. Impact input files are normally given the “imi” extension (Impact Input).

Save Input As

This command allows you to save input data to a file with a filename you specify.

Save Output

Results displayed in the Grid Viewer can be saved to a file with the Save Output command.

The data is saved as formatted text, suitable for importing into Excel or other Windows based spreadsheets.

To import into Excel, choose Windows(ANSI), tab delimited format, with no text qualifier.

Save Output As

You can establish a filename for the output data, or change the current filename with this command.
B.2.4 Printing

Images shown in either the Graph Viewer or the Picture Window can be sent to the printer via the Print Manager.

Print Setup

Form the File menu, select Print Setup to change or configure the default printer.

![Print Setup dialog box]

Click here to see more setup options.

For best results choose Landscape

The Impact interface sends data to the printer in a bitmap format. For best results setup your printer to suit image printing. Avoid printing in high resolution Postscript mode.
Print

Choose the Print command from the File menu to display the Print dialog box.

The images will automatically be scaled to fit your printer.

The Damage option will only be available when the currently selected model is either the Zhu model or the UBC Conical model.

B.3 Model Reference

Six models are currently available to simulate impact events in Impact.

Required input parameters, and available simulation results are summarized in this chapter. If any special notes pertaining to these parameters are required, they are listed. For detailed technical information refer to the technical documents included in the Impact documentation package.

The Impact interface can display warnings during a simulation, indicating such things as perforation, excessive damage, and rebound. The types of warnings you can expect are listed for each model.

B.3.1 LAMP Model

This model, developed by Woodward and Crouch, calculates the ballistic limit, of a blunt projectile impacting a layered homogeneous target.
This model differs from others, in that a series of "simulations" are performed by the numerical engine to determine the ballistic limit. An initial velocity is assumed. Then using energy methods calculates the residual velocity of the projectile, after penetration, is calculated. If the residual velocity is less than zero, i.e. the initial velocity was less than the ballistic limit, the initial velocity is incremented and the calculation repeated. The initial velocity used in the first iteration with a residual velocity greater than zero, is chosen as the ballistic limit. Note that there is a non-zero residual velocity at the ballistic limit.

The first iteration uses an initial velocity of 10 m/s, with subsequent iterations using an initial velocity incremented by 5 m/s. In some cases the residual velocity predicted can be substantial as the small incremental velocity steps can result in a change in the penetration mechanics.

The model results are presented in a dialog box, and not in the child windows.

![LAMP Results](image)

Reference

Woodward, R.L. and Crouch, I.G.