THE EXISTENCE OF SINGULAR TERMS AND THEIR EFFECT ON THE VALIDITY OF FERMI LIQUID THEORY IN TWO DIMENSIONS

By

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B. Sc. (Physics) Dalhousie University, 1990.

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

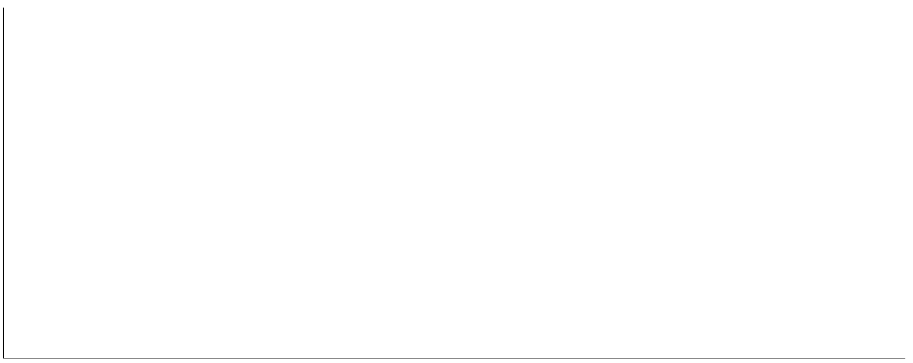
in

THE FACULTY OF GRADUATE STUDIES

PHYSICS

We accept this thesis as conforming

to the required standard



THE UNIVERSITY OF BRITISH COLUMBIA

October 1992

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Date October, 15, 1992

Abstract

The question of the breakdown of Fermi liquid theory in two dimensions is examined in the context of perturbation theory for a dilute interacting Fermi gas. The quasiparticle interaction function, $f^{\sigma\sigma'}_{\mathbf{pp'}}$, is calculated for such a system. The interaction function, calculated to second order in terms of the dimensionless coupling constant, shows various singularities. The most divergent terms appear in the cross channel, but cancel out leaving a much weaker singularity in the limit of two momenta approaching each other ($\theta \rightarrow 0$). As in the case of the three dimensional Fermi gas, the Cooper channel contains a logarithmic singularity in the limit $\theta \rightarrow \pi$. This singularity can be summed and is known to be harmless to the structure of Fermi liquid theory. A different feature in two dimensions is the existence of such a singularity for $\theta \rightarrow 0$. This feature needs further investigation.

Calculations have also been extended to a polarized Fermi gas and the result is equivalent to the unpolarized case and does not show any additional features. In conclusion, the results do not indicate the presence of strong divergences which could cause the breakdown of Fermi liquid theory in two dimensions for a dilute interacting Fermi gas.

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Acknowledgement

First and foremost, I wish to thank my supervisor, Dr. Philip Stamp for suggesting this project and for his continuous guidance and support throughout the course of my research. The many discussions that I had with him were vital in my understanding of the subject and in the completion of this work. I also wish to thank Dr. Ian Affleck for examining this thesis and providing valuable comments.

I also thank the organizing committee of the School of Theoretical Physics at Les Houches and my supervisor for providing me with the opportunity to attend the school in August 1991 and to benefit from the outstanding lectures and the stimulating discussions. I specifically acknowledge useful discussions with Dr. Benoit Douçot.

I appreciate the useful discussions with Dr. Nikolay Prokofiev during his stay at UBC and for his contribution to further investigation of the validity of Fermi liquid theory in two dimensions.[50]

Special thanks should also be given to Richard Szabo for reading the first draft and providing helpful comments. I also appreciate the support of my friends and fellow students while I was writing. Finally, I appreciate the constant love and support that my family has given me.

Chapter 1

Introduction

The motivation for this work partly arises from the recent emphasis on two dimentional physics. Until recently, physics in two dimensions was considered relatively trivial, studied mainly as a matter of curiosity and occasionally to compare and contrast the results to properties in three dimensions. The current interest in two-dimensional physics partly stems from the success in producing thin films: for example, it has become possible[1] to make monolayer films of ^{3}He on superfluid ^{4}He. The need to understand the physics of surfaces, the two-dimensional electron gas and new high-transition-temperature superconductors also has put emphasis on understanding physics in two dimensions.The existence of anyons and their role in the mechanism of quantum Hall effect is just one indication of the subtlety of the phenomena and features in two dimensions.[2]

With the recent discovery of high-T_c copper-oxide superconductors[3] there has been an enormous effort to characterize and understand the physical behavior of these compounds in the normal and superconducting states.[4] The measurements of electrical conductivity and electron tunneling in these materials show their anisotropic behaviour, indicating that the current carriers are confined to the layers of copper-oxide planes[4]. Despite the experimental and theoretical efforts, the nature of the normal state and the mechanism of superconductivity in these materials are still open questions. Almost all candidate theories, such as the Luttinger liquid theory of Anderson[5–7], marginal Fermi liquid theory of Varma, et al.[8] and the anyon mechanism[9] have concentrated on this two dimensional behaviour and almost all propose that electron (electron-pair) tunneling is responsible

for conductivity (superconductivity) in the direction perpendicular to a-b plane (c-axis).

The BCS theory of superconductivity[10] explains the nature and the onset of the superconductivity in normal three dimensional superconductors. The normal state of the electrons in a metal is that of an interacting Fermi system: a Fermi liquid. Such a system is reached by adiabatic continuation from a free Fermi gas and its low-lying energy levels have the same structure as the non-interacting system. There are natural instabilities in a Fermi liquid which lead to the formation of Cooper pairs.[11] The prerequisite for this mechanism is the existence of an effective attractive two-body interaction which in a metal is provided by mediating phonons. Although the phonon-mediated mechanism is not completely ruled out in the cuprate superconductors, the evidence points against it.[12,13]

The normal state properties of the cuprate superconductors are also quite puzzling. These materials show metallic behaviour near their transition temperatures; however, unlike normal metals, they do not obey the predictions of Fermi liquid theory in regard to their decay rate, resistivity and tunneling properties (for a review on this matter refer to ref. [4-7]). Varma, et al.[8,14] have suggested that Fermi liquid theory is obeyed in a "marginal" sense and have proposed a phenomenological postulate from which several features of the normal state could be derived. However, others have claimed that Fermi liquid theory is irrelevant with regard to the ground state of these materials. Specifically Anderson has claimed[15–17] that in two dimensions, Fermi liquid theory breaks down due to the presence of strong singular interaction terms.The question of the relevance of Fermi liquid theory to two dimensional systems is obviously important. Perhaps the crucial test would come from the experiments on the liquid ^{3}He films on the substrate of superfluid ^{4}He. This has been made possible recently[1], and it could be the closest system to a two dimensional Fermi liquid, if there is one.

The fundamental assumptions and the consequences of Fermi liquid theory are explained in the next chapter. We only mention here that a Fermi liquid is the state of an interacting system of fermions reached by adiabatic continuation from a non-interacting Fermi gas. In a Fermi liquid, the role of particles is taken up by quasiparticles which are particlelike excitations of the interacting system. The overlap between the wavefunctions of the bare particle and the quasiparticle is given by the renormalization factor z:

$$z_k = < \psi_k^0 | \psi_k^V > \tag{1.1}$$

Anderson's argument for the breakdown of the Fermi liquid theory begins by suggesting the existence of singular terms of the form

$$f_{pp'} \sim \frac{p' \cdot (p - p')}{|p - p'|^2} \tag{1.2}$$

in the quasiparticle interaction function, and that these interactions produce a finite phase shift for quasiparticles with $p + p' = 2p_F$. (Note that this form is actually asymmetric. One way of justifying this would be to assume that one excitation is above and the other is below the Fermi surface.) He argues that due to these singularities, the renormalization constant z vanishes and it is no longer correct to speak of continuing from the non-interacting system to the interacting system.

Anderson further draws analogy from the problem of a quantum impurity in a static potential that this is also related to the overlap of the initial and final many-body wavefunctions:

$$< 0|V > \simeq \ exp(-(\frac{\delta_0}{\pi})^2 \ln |N|) \tag{1.3}$$

where $|0>$ and $|V>$ are the wavefuctions for the free and interacting systems, respectively, and δ_0 is the forward scattering phase shift. He argues that the scattering of quasiparticles in the vicinity of the Fermi surface results in a finite phase shift which makes this overlap zero and therefore adiabatic continuation is impossible.

Faced with such propositions, there are questions which should be dealt with. The first question is whether there are such singularities and and if so what their origins are. Also one must show that they indeed lead to the breakdown of Fermi liquid theory, since it is well known that many singularities, such as the Cooper pairing, only lead to a modification of Fermi liquid theory.[11] Assuming that these tasks are accomplished, we still have to find an alternative quantum liquid to describe the interacting ground state. This work mainly deals with the first question and examines the validity of Fermi liquid theory in the context of perturbation theory.

Chapter 2

Theoretical background:Fermi Liquid Theory

2.1 The Foundation of Fermi Liquid Theory

Fermi liquid theory is a microscopic theory of systems of interacting fermions at temperatures where quantum effects dominate, i.e. below the degeneracy temperature of the system. Initially proposed in 1957 by Lev Landau[18,19] to be applied to the problem of liquid ^{3}He, Fermi liquid theory has also been successfully applied to the electron liquid in a metal. The theory assumes that the low level excitations of the liquid have fermionic statistics and consequently a necessary but not sufficient condition for Fermi liquid theory to be applicable is that the interacting particles be fermions. Fermi liquid theory has been applied to the dilute solutions of ^{3}He in superfluid ^{4}He[20] and it best succeeds in explaining the quantum behaviour of liquid ^{3}He.[21] Fermi liquid theory predicts low temperature properties of the electron liquid in a metal and provides a conceptual understanding of the success of the free electron approximation.

In constructing his theory, Landau assumed that it is possible to turn on the interaction slowly so that there is an adiabatic continuation from a non-interacting Fermi gas to the interacting Fermi liquid. An adiabatic continuation means that there is an unambiguous and one-to-one correspondence between the low-lying levels of the interacting system and those of a non-interacting Fermi gas. Therefore one is able to label such states of the Fermi fluid by the levels of the initial Fermi gas. The justification for applying this procedure is subtle, and depends on further assumptions appropriate to a

Fermi system.

Even in a strongly interacting Fermi system, the exclusion principle dramatically reduces the phase space available for scattering processes. At $T = 0$, the volume of available phase space for scattering of a particle on the Fermi surface is zero, and therefore its life-time is infinite. A particle with a momentum $k > k_F$, has a volume proportional to $(k-k_F)^2$ available for scattering. At low temperatures, a particle's energy is proportional to the absolute temperature and its decay rate is proportional to the square of the absolute temperature. Therefore, at sufficiently low temperatures it is possible to turn on the interaction in a time that is less than the lifetime of a particle, and one can then speak of one particle states which are approximate eigenstates of the interacting system.

Furthermore it is obvious that there are no single particle stationary states of the interacting system. In the theory of Fermi liquids the role of particles is taken up by quasiparticles which are the particle-like excitations of the liquid and which obey Fermi-Dirac statistics. A quasiparticle can be thought of as a particle in the self-consistent field of all other particles, and it carries the same charge and momentum as the actual particles.[18] With this notion, the problem of interacting particles is replaced by interacting quasiparticles whose number is always equal to the number of actual particles. One should note that the energy of the particle depends on the state of the surrounding particles, and therefore the total energy is no longer the sum of the energies of the individual particles; it is a functional of the distribution function. The energy of a single quasiparticle is defined as the functional derivative of the total energy with respect to the distribution function.

Now assuming that the state of the system remains of the same symmetry, we ask the question: what happens if the distribution fuction,n_k, varies slightly? The energies of the quasiparticles are no longer independent, and by varying n_k, the energy of any *other* quasiparticle changes. The total energy of the system is a functional of the distribution

function and has a perturbation expansion whose first few terms are :

$$E = E_0 + \sum_{\mathbf{k}} \delta n_{\mathbf{k}} \epsilon^0_{\mathbf{k}} + \frac{1}{2} \sum_{\mathbf{k}} \sum_{\mathbf{k}'} f(\mathbf{k}, \mathbf{k}') \delta n_{\mathbf{k}} \delta n_{\mathbf{k}'} \tag{2.4}$$

$\epsilon^0_{\mathbf{k}}$ is the energy of the quasiparticle of wavevector $\mathbf{k}$ and $f(\mathbf{k}, \mathbf{k}')$ is the second functional derivative of the E. The great practical advantage of Landau's theory is that this expansion to second order is sufficient for obtaining the low temperature properties of the system. These properties turn out to depend on some integral of the function f.

There are many reviews of Fermi liquid theory. For further discussions and applications to physical systems see ref.[22-24].

2.2 The Dilute Interacting Fermi Gas

In order to introduce the methods presented in this thesis, we must mention the case of a three dimensional low-density degenerate Fermi gas. This was studied by Lee and Yang[25], Abrikosov and Khalatnikov[26], and others[27].

Lee and Yang considered the cases of Fermi, Bose and Classical Boltzmann gases of hard spheres of diameter a. The method involved finding a two-body pseudopotential to replace the hard sphere potential. An expansion was obtained in terms of the diameter a – which also coincides with the s-wave scattering length – for the ground state of a Fermi gas at a finite density ρ and infinite volume. The energy per particle of such a system was found to be (at $T = 0$):

$$\frac{E}{N} = (\frac{3p_F^2}{5}) + 8\pi a\rho \; J(2J+1)^{-1} \; \left[1 + 6(1 - 2\ln 2)\frac{p_F a}{35\pi} + O(p_F^2 a^2)\right] \tag{2.5}$$

where J is the spin of the particles. To the orders specified, this expansion is exact.

Abrikosov and Khalatnikov approached this problem from a different angle. They also considered a dilute Fermi gas with two-body interactions, and assumed that the interaction range is much smaller than the distance between two particles.This allows

the expansion of the energy in terms of the small parameter $(k_F a)$ where k_F is the Fermi wavevector and a is the s-wave scattering length. Their method was the first to renormalize the potential in terms of a physical quantity such as the s-wave scattering length. As opposed to Lee and Yang, they did not calculate the ground state energy directly, rather they used the quasiparticle approximation and calculated the quasiparticle interaction function for the system. The advantage of their method is that one is able to obtain formulas for thermodynamic and transport properties of the system without further integration, and that the energy can be obtained from the chemical potential. The details of this method are explained in the next two chapters. Here we give a brief outline:

Consider a gas of fermions with two-body interactions as explained. As long as the interaction range and the momentum exchange are small, one can approximate the interaction to be independent of the momenta of the two particles. One can also ignore three-body collisions if one is only interested in the first few terms of the series expansion. This is because such collisions only affect terms of fourth order in $(k_F a)$ and higher. With this method, they obtained the following expression for the quasiparticle interaction function:

$$\begin{aligned} f(\theta) = \frac{2\pi a\hbar^2}{m}[\quad & 1 \quad +2(\frac{3}{\pi})^{\frac{1}{3}} a N^{\frac{1}{3}} (2 + \frac{\cos\theta}{2\sin(\theta/2)} \ln\frac{1+\sin(\theta/2)}{1-\sin(\theta/2)})] \\ & - \quad \frac{8\pi a\hbar^2}{m}(\sigma_1\sigma_2)\left[1 + 2(\frac{3}{\pi})^{\frac{1}{3}} a N^{\frac{1}{3}} (1 - \frac{\sin(\theta/2)}{2} \ln\frac{1+\sin(\theta/2)}{1-\sin(\theta/2)})\right] \end{aligned} \tag{2.6}$$

where θ is the angle between the two momenta. The energy is obtained from the relation:

$$E = \int \mu \, dN \tag{2.7}$$

and coincides with that of Lee and Yang.

From the eq. (2.6), we observe that for angles near π, the function f has a logarithmic singularity and that strictly speaking, the series is no longer meaningful. However, this dilemma is resolved by summing the divergent terms to infinite order. This gives a

non-singular result for $a \geq 0$. However, when $a < 0$, that is in the case of attractive interactions, the scattering amplitude has a pole at a small imaginary value of ϵ where $\epsilon = p^2 + p'^2 - 2p_F$. This pole corresponds to the instability of the Fermi liquid ground state to formation of Cooper pairs and is the cause of (s-wave) superfluidity in a Fermi liquid.

With regard to the recent developements it is worthwhile doing a similar calculation in two dimensions. Not only could this lead us to the formation of bound states as in three dimensions, but it can also reveal divergences which signal the breakdown of validity of the Fermi liquid theory. The latter is actually what happens for a one dimensional gas of fermions and it is well known that the properties of a Fermi gas in one dimension are fundamentally different from that in three dimension.[28] The situation in two dimension is far from clear and we hope to shed some light on it.

Chapter 3

Quantum Mechanical Scattering in Two Dimensions in Free Space

3.1 General Formulations

We are interested in calculating the quasiparticle interaction function and from that the thermodynamical properties of a dilute Fermi gas. The Hamiltonian includes the kinetic energy and a second term for pair interaction of the particles. However, the interaction energy increases at short distances (typical interatomic distances), and perturbation theory is no longer valid. We can overcome this problem by renormalizing the potential in terms of a physical quantity such as the scattering length. That is, we consistently replace the potential with one which has the same scattering amplitude at low energies and is well behaved at short distances. As long as the energies are low, and the calculated quantity includes the interaction only in terms of the scattering amplitude, the result would be the same as the one which uses the actual, non-renormalized potential.

The scattering length is defined as:

$$a = -\lim_{k \to 0} f(\theta) \tag{3.8}$$

where $f(\theta)$ is the *scattering amplitude*. In three dimensions a has the form

$$a = \frac{m u_0}{4\pi\hbar^2} \qquad \text{where} \qquad u_0 = \int d^3r \; V(r) \tag{3.9}$$

and $V(r)$ is the interaction potential. However, as we shall soon see, in two dimensions all scattering amplitudes diverge as $(k)^{-\frac{1}{2}}$ at low energies. The problem is easily fixed by introducing a well behaved dimensionless quantity as is done in the next section.

The following is a calculation of the scattering amplitudes in two dimensions. They are included for completeness. The derivation and notations are mainly followed from Lapidus.[29]. For further discussion see ref. [30-32].

We begin by writing the Schrödinger equation in two dimensions

$$\frac{-\hbar^2}{2m}\nabla^2\psi + V(r,\theta)\psi = E\psi, \tag{3.10}$$

which we can write as

$$\nabla^2 + (k^2 - U(r,\theta))\psi = 0 \tag{3.11}$$

where $k^2 = \frac{2mE}{\hbar^2}$ and $U(r,\theta) = \frac{2m}{\hbar^2}V(r,\theta)$. Furthermore, we assume that the potential is central which means $U(r,\theta) = U(r)$. Now in polar coordinates, the equation has the form

$$(\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2})\psi + (k^2 - U(r))\psi = 0 \tag{3.12}$$

This equation can be separated into radial and angular parts, so we take $\psi(r,\theta) = R(r)T(\theta)$. The two equations are

$$\frac{\partial^2 T}{\partial\theta^2} + m^2 T = 0 \tag{3.13}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial R}{\partial r}) + (k^2 - U(r) - \frac{m^2}{r^2})R = 0 \tag{3.14}$$

If we take the x-axis to be along the direction of the incident beam, it would be an axis of the symmetry of the system. Therefore the probability distribution must be symmetric about this axis. That means $|T(\theta)|^2 = |T(-\theta)|^2$. Then the normalized angular part of the solution would have the form

$$T(\theta) = \sqrt{\pi}\cos m\theta \tag{3.15}$$

m must be an integer so that $T(\theta)$ is periodic in θ(single-valued).

Now, let's look at the asymptotic form of the radial equation. Consider a potential that vanishes at sufficiently large values of r. An example is a potential of the form

$$V(r) = \begin{cases} V(r) & r \leq a \\ 0 & r > a \end{cases} \tag{3.16}$$

The radial equation has the asymptotic form

$$\rho^2 \frac{d^2R}{d\rho^2} + \rho \frac{dR}{d\rho} + (\rho^2 - m^2)R = 0 \tag{3.17}$$

where $\rho = kr$. Eq. (3.17) is in the form of the Bessel differential equation, solutions of which are Bessel and Neumann functions of the first kind. At large values of r their leading asymptotic terms are

$$J_m(kr) \rightarrow (\frac{2}{\pi kr})^{\frac{1}{2}} \cos(kr - (m + \frac{1}{2})\frac{\pi}{2}) \tag{3.18}$$

$$N_m(kr) \rightarrow (\frac{2}{\pi kr})^{\frac{1}{2}} \sin(kr - (m + \frac{1}{2})\frac{\pi}{2}) \tag{3.19}$$

and therefore the radial solution has the asymptotic form

$$R_m(kr) \rightarrow A_m(\frac{2}{\pi kr})^{\frac{1}{2}} \cos(kr - (m + \frac{1}{2})\frac{\pi}{2} + \delta_m) \tag{3.20}$$

The quatity δ_m is called the phase shift of the mth partial wave.

Now consider a free particle of a fixed energy $E = \frac{\hbar^2 k^2}{2m}$. The incoming particle has the wavefunction:

$$\psi_{inc} = e^{ikx} \tag{3.21}$$

For a steady state configuration, conservation of energy requires that the scattered wave have the asymptotic dependence of

$$\psi_{sc}(r, \theta) \sim \frac{e^{ikr}}{\sqrt{r}} f(\theta) \tag{3.22}$$

where $f(\theta)$ contains the angular dependence of the scattered wave and has dimension of $(L)^{\frac{1}{2}}$.

Therefore the asymptotic form of the steady state wavefunction must have the form :

$$\psi(r,\theta) = e^{ikx} + f(\theta)\frac{e^{ikr}}{\sqrt{r}} \tag{3.23}$$

Comparison of the above equation with the previously obtained asymptotic form of the wavefunction (eq. 3.20) gives the scattering amplitude and phase shifts. We proceed by first expanding the incident wavefunction in terms of Bessel functions. The Jacobi-Anger relation (ref. [33], p585) gives

$$\begin{aligned} e^{ikx} = e^{ikr\cos\theta} &= \sum_{m=-\infty}^{\infty} i^m J_m(kr) e^{im\theta} \\ &= J_0(kr) + 2\sum_{m=1}^{\infty} i^m J_m(kr)\cos(m\theta) \end{aligned} \tag{3.24}$$

Then the following equality must be satisfied:

$$J_0(kr)+2\sum_{m=1}^{\infty} i^m J_m(kr)\cos(m\theta)+f(\theta)\frac{e^{ikr}}{\sqrt{r}} = \sum_{m=0}^{\infty} A_m(\frac{2}{\pi kr})^{\frac{1}{2}}\cos(kr-(m+\frac{1}{2})\frac{\pi}{2}+\delta_m)\cos(m\theta) \tag{3.25}$$

Writing this equation in terms of e^{ikr} and e^{-ikr} and putting the coefficient of each to zero gives the following two equations:

$$\frac{\epsilon_m i^m \cos(m\theta)}{\sqrt{2\pi k}} - \frac{A_m}{\sqrt{4k}}e^{i\delta_m} = 0 \tag{3.26}$$

$$f(\theta) + \sum_{m=0}^{\infty}(\frac{\epsilon_m i^m \cos(m\theta)}{\sqrt{2\pi k}} - \frac{A_m}{\sqrt{4k}}e^{i\delta_m})e^{-i\frac{\pi}{2}(m+\frac{1}{2})} = 0 \tag{3.27}$$

where $\epsilon_m = 2, m \neq 0$ and $\epsilon_0 = 1$. These equations give:

$$A_m = 2\epsilon_m i^m (2\pi)^{\frac{1}{2}} e^{i\delta_m} \tag{3.28}$$

$$f(\theta) = \left(\frac{1}{2\pi ik}\right)^{\frac{1}{2}} \sum_{m=0}^{\infty} \epsilon_m \cos(m\theta)(e^{2i\delta_m} - 1) \tag{3.29}$$

For $m \neq 0$, the phase shift vanishes as k^m and therefore ensures that $f(\theta)$ remains regular. However, in the case of $m = 0$, we have (C_1 is a fixed constant):

$$\delta_0 \sim \frac{\pi}{2}(ln\frac{C_1 ka}{2})^{-1} \tag{3.30}$$

which is not enough to make the scattering amplitude finite.[30] The divergence of the scattering amplitude at small energies also appears in the Born approximation.

The analogous quantity to the three dimensional scattering cross section is a length in two dimensions. We define it as

$$\rho = \int_0^{2\pi} |f(\theta)|^2 d\theta \tag{3.31}$$

For a central potential:

$$\rho = \frac{4}{k} \sum_{m=0}^{\infty} \epsilon_m \sin^2 \delta_m \tag{3.32}$$

3.2 Born Approximation

In cases where the scattering potential is weak and the phase shifts are small, one may treat the scattered wave as a perturbation to the incident wavefunction. This is the essence of the Born approximation. (see discussion in ref.[34])

We write:

$$\psi(r) = \psi_0(r) + \phi(r) \tag{3.33}$$

where $\psi_0(r)$, $\phi(r)$ and $\psi(r)$ denote the incident, the scattered and the total wavefunction respectively. They must satisfy :

$$(\nabla^2 + k^2)\psi(r) = U(r)\psi(r) \tag{3.34}$$

or

$$(\nabla^2 + k^2)\phi(r) = U\psi_0(r) \tag{3.35}$$

The Green's function of the Helmholtz equation in two dimensions has the form[33]:

$$G(r, r') = \frac{-i}{4} H_0^{(1)}(k|r - r'|) \tag{3.36}$$

where $H_0^{(1)}$ is the zeroth order Hankel function of the first kind, defined by

$$H_0^{(1)}(k\rho) = J_0(k\rho) + iN_0(k\rho) \tag{3.37}$$

Then the scattered wave must be of the form

$$\begin{aligned}\phi(r) &= -\int G(r,r')U(r')\psi_0(r')dr' \\ &= -\frac{i}{4}\int H_0(k|r-r'|)U(r')\psi_0(r')dr' \end{aligned} \tag{3.38}$$

Using the asymptotic form of the Hankel function[33] :

$$\phi(r) = \int (\frac{1}{8\pi k|r-r'|})^{\frac{1}{2}} e^{i(k|r-r'|+\frac{\pi}{4})} U(r') e^{ikx'} dr' \tag{3.39}$$

where q is the momentum transter, i.e. $q = k' - k$ and k' is the wavevector of modulus k in the direction r'. In the limit of large r and using equation (3.23) for the scattering amplitude, we obtain:

$$f(\theta) = \frac{1}{\sqrt{8\pi k}} \int e^{iq.r'} U(r') d^2r' \tag{3.40}$$

and in the limit $q \rightarrow 0$

$$f(\theta) = \frac{U_0}{\sqrt{8\pi k}} = \frac{2mu_0}{\hbar^2\sqrt{8\pi k}} \qquad \text{where} \qquad u_0 = \int_{-\infty}^{\infty} V(r)\, d^2r \tag{3.41}$$

The analogous quantity to the three dimensional scattering length has the dimension of $(L)^{\frac{1}{2}}$ and is equal to :

$$a = \frac{2mu_0}{\hbar^2\sqrt{8\pi k}} \tag{3.42}$$

This relation can be used for the scattering of two particles by replacing the wavevector with the relative wavevector and the mass by the effective mass.

Chapter 4

Methods of Calculations and Results

4.1 The Quasiparticle Interaction Function for a 2D Dilute Fermi Gas

In the previous chapter, we described two particle scattering in free space. Now, we consider the scattering of quasiparticles in a dilute Fermi gas. As mentioned in section(2.2), this is a generalization of the method of Abrikosov and Khalatnikov to two dimensions. We consider a short-range two body interaction that is independent of particle momenta. The interaction is then renormalized in terms of its low energy scattering amplitude. This procedure removes the difficulty of having to deal with strong interactions at short distances.

The Hamiltonian of a system of particles with pair interaction is :

$$H = \sum_{k,\sigma} \epsilon_{k\sigma} n_{k\sigma} + \sum_{p_1,\alpha_1} \sum_{p_2,\alpha_2} \sum_{p_1',\alpha_1'} \sum_{p_2',\alpha_2'} < p_1'\alpha_1, p_2'\alpha_2|V|p_1\alpha_1, p_2\alpha_2 > a^+_{p_1'\alpha_1} a^+_{p_2'\alpha_2} a_{p_2\alpha_2} a_{p_1\alpha_1} \tag{4.43}$$

where the summation is over all four momenta and four spin indices. Now, one can write:

$$< p_1', p_2'|V|p_1, p_2 >= \frac{1}{\Omega} \int V(r) e^{-iq\cdot r/\hbar} d^2r \tag{4.44}$$

where $q = p_1' - p_1 = -(p_2' - p_2)$, and Ω is the area of the system.Conservation of momentum is implicitly assumed in this expression. Spin indices are suppressed because the interaction is assumed to be independent of spin. Assuming the momentum exchange of the particles to be small, and the interaction to be short ranged one can replace the integral in the above equation with its value at $q = 0$ and the matrix element with $\frac{u_0}{\Omega}$.

We now consider the approximate Hamiltonian

$$H = \sum_{p,\sigma} \epsilon_{p\sigma} n_{p\sigma} + \frac{u_0}{4\Omega} \sum_{p_1,\alpha_1} \sum_{p_2,\alpha_2} \sum_{p_2',\alpha_2'} \sum_{p_1',\alpha_1'} a^+_{p_1'\alpha_1'} a^+_{p_2'\alpha_2'} a_{p_2\alpha_2} a_{p_1\alpha_1} \tag{4.45}$$

We restrict ourselves to s-wave scattering which is dominant in the limit of slow collisions. Then only particles of antiparallel spin can scatter each other, and for particles of spin $\frac{1}{2}$, the Hamiltonian simplifies to :

$$H = \sum_{p,\sigma} \epsilon_{p\sigma} n_{p\sigma} + \frac{u_0}{\Omega} \sum_{1,2} \sum_{1',2'} a^+_{1'+} a^-_{2'+} a_{2-} a_{1+} \tag{4.46}$$

where + and - represent the two possible spin states of the particles.
We consider this Hamiltonian as

$$H = H_0 + \Delta H \tag{4.47}$$

and find its ground state energy to second order of perturbative theory. The first and second order corrections to the energy level E_n are:

$$\Delta^{(1)} E_n = (\Delta H)_{nn} = < \phi^0_n \mid \Delta H \mid \phi^0_n > \tag{4.48}$$

$$\Delta^{(2)} E_n = \sum_{m \neq n} \frac{\mid (\Delta H)_{nm} \mid^2}{E^0_n - E^0_m} = \sum_{m \neq n} \frac{|< \phi^0_n \mid \Delta H \mid \phi^0_m >|^2}{E^0_n - E^0_m} \tag{4.49}$$

where E^0_n and ϕ^0_n represent the eigenvalues and eigenvectors of H_0. The ground state energy of the non-interacting system gives the zeroth order contribution:

$$E^{(0)}_0 = \sum_{p,\sigma} \epsilon_{p\sigma} n_{p\sigma} \tag{4.50}$$

and the first order correction is given by eq. 4.48

$$\begin{aligned} \Delta^{(1)} E_n &= < \phi^0_n | \frac{u_0}{2\Omega} \sum_{\sigma} \sum_{p_1',p_2'} \sum_{p_1,p_2} a^+_{1',\sigma} a^+_{2',-\sigma} a_{2,-\sigma} a_{1,\sigma} | \phi^0_n > \\ &= \frac{u_0}{2\Omega} \sum_{\sigma} \sum_{1,2} \sum_{1',2'} < \phi^0_n \mid (n_{1,\sigma} - 1), (n_{2,-\sigma} - 1), (n_{2',-\sigma} + 1), (n_{1',\sigma} + 1) > \\ &\sqrt{n_{1,\sigma}} \sqrt{n_{2,-\sigma}} \sqrt{1 - n_{2',-\sigma}} \sqrt{1 - n_{1',\sigma}} \end{aligned} \tag{4.51}$$

The matrix elements will not vanish only if

$$n_{1'} = n_1 - 1$$
$$n_{2'} = n_2 - 1 \tag{4.52}$$

so that

$$\Delta^{(1)} E_n = \frac{u_0}{2\Omega} \sum_{\sigma} \sum_{1,2} n_{1,\sigma} n_{2,-\sigma} \tag{4.53}$$

This term represent the shift of the energy levels due to the interaction. Similarly

$$\begin{aligned} < \phi_m^0 \mid \Delta H \mid \phi_n^0 > &= < \phi_m^0 | \frac{u_0}{2\Omega} \sum_{\sigma} \sum_{1,2} \sum_{1',2'} a^+_{1',\sigma} a^+_{2',-\sigma} a_{2,-\sigma} a_{1,\sigma} | \phi_n^0 > \\ &= \frac{u_0}{2\Omega} \sum_{\sigma} \sum_{1,2} \sum_{1',2'} < \phi_m^0 \mid (n_{1'}+1), (n_{2'}+1), (n_2 - 1), (n_1 - 1) > \\ &\quad \sqrt{1-n_{1'}} \sqrt{1-n_{2'}} \sqrt{n_2} \sqrt{n_1} \end{aligned} \tag{4.54}$$

Matrix elements will not vanish only if

$$m_1 = n_1 - 1$$
$$m_2 = n_2 - 1$$
$$m_{1'} = n_{1'} + 1$$
$$m_{2'} = n_{2'} + 1 \tag{4.55}$$

So now we have

$$\Delta^{(2)} E_n = \frac{u_0^2}{2\Omega^2} \sum_{\sigma} \sum_{1,2} \sum_{1',2'} \frac{n_{1,\sigma} n_{2,-\sigma} (1 - n_{1',\sigma})(1 - n_{2',-\sigma})}{E_{1,\sigma} + E_{2,-\sigma} - E_{1',\sigma} - E_{2',-\sigma}} \tag{4.56}$$

with implicit conservation of momentum. This term represents the energy correction due to pair collisions. It is proportional to the occupation number of the initial states and the number of unoccupied final states.

We now recall from the previous section that in two dimensions the "scattering length" diverges as $(k)^{\frac{-1}{2}}$. However, we can renormalize the potential in terms of the dimensionless coupling parameter λ defined by:

$$\lambda = a_0\sqrt{8\pi k} = \frac{mu_0}{\hbar^2} \tag{4.57}$$

For the sake of consistency, we should also include the next term in eq. (3.41) or eq. (4.57) which we have thus far ignored. Fortunately, this is easily done, as the second order Born approximation is obtained by simply changing[35]:

$$\frac{u_0}{2\Omega} \longrightarrow \frac{u_0}{2\Omega} + \frac{u_0^2}{4\Omega^2}\sum_{\sigma}\sum_{1',2'}\frac{1}{E_{1,\sigma}+E_{2,-\sigma}-E_{1',\sigma}-E_{2',-\sigma}} = \frac{\bar{u}_0}{2\Omega} \tag{4.58}$$

$$\begin{aligned} u_0 &= \bar{u}_0 - \frac{\bar{u}_0^2}{2\Omega}\sum_{\sigma}\sum_{1',2'}\frac{1}{E_{1,\sigma}+E_{2,-\sigma}-E_{1',\sigma}-E_{2',-\sigma}} \\ &= \frac{\hbar^2}{m}\lambda - \frac{\hbar^4\lambda^2}{m^2\Omega}\sum_{\sigma}\sum_{1',2'}\frac{1}{E_{1,\sigma}+E_{2,-\sigma}-E_{1',\sigma}-E_{2',-\sigma}} \\ &= \frac{\hbar^2}{m}\lambda(1 - \frac{\hbar^2}{m\Omega}\lambda\sum_{\sigma}\sum_{1',2'}\frac{1}{E_{1,\sigma}+E_{2,-\sigma}-E_{1',\sigma}-E_{2',-\sigma}}) \end{aligned} \tag{4.59}$$

Equations (4.50), (4.53) and (4.56) give the ground state energy to second order in :

$$\begin{aligned} E &= \sum_{p,\sigma}\epsilon_{p\sigma}n_{p\sigma} + \frac{u_0}{2\Omega}\sum_{\sigma}\sum_{1,2}n_{1,\sigma}n_{2,-\sigma} \\ &\quad + \frac{u_0^2}{2\Omega^2}\sum_{\sigma}\sum_{1,2}\sum_{1',2'}\frac{n_{1,\sigma}n_{2,-\sigma}(1-n_{1',\sigma})(1-n_{2',-\sigma})}{E_{1,\sigma}+E_{2,-\sigma}-E_{1',\sigma}-E_{2',-\sigma}} \end{aligned} \tag{4.60}$$

and finally using the renormalized value of the potential from eq.(4.59):

$$\begin{aligned} E &= \sum_{p,\sigma}\epsilon_{p\sigma}n_{p,\sigma} + \frac{\hbar^2}{2m\Omega}\lambda(1 - \frac{\hbar^2}{m\Omega}\lambda\sum_{1',2'}\frac{2m}{p_1^2+p_2^2-p_1'^2-p_2'^2})\sum_{\sigma}\sum_{1,2}n_{1,\sigma}n_{2,-\sigma} \\ &+ \frac{\hbar^4}{2m^2\Omega^2}\lambda^2\sum_{\sigma}\sum_{1,2}\sum_{1',2'}\frac{n_{1,\sigma}n_{2,-\sigma}(1-n_{1',\sigma})(1-n_{2',-\sigma})}{E_{1,\sigma}+E_{2,-\sigma}-E_{1',\sigma}-E_{2',-\sigma}} \\ &= \sum_{p,\sigma}\epsilon_{p,\sigma}n_{p,\sigma} + \frac{\hbar^2}{2m\Omega}\lambda\sum_{\sigma}\sum_{1,2}n_{1,\sigma}n_{2,-\sigma} \end{aligned} \tag{4.61}$$

$$+ \quad \frac{\hbar^4}{\Omega^2}\lambda^2 \sum_{\sigma} \sum_{p_1,p_2} \sum_{p_1',p_2'} \frac{n_{1,\sigma} n_{2,-\sigma}[(1-n_{1',\sigma})(1-n_{2',-\sigma})-1]}{p_1^2+p_2^2-p_1'^2-p_2'^2} \tag{4.62}$$

$$= \sum_{p,\sigma} \epsilon_{p,\sigma} n_{p,\sigma} + \frac{\hbar^2}{2m\Omega}\lambda \sum_{\sigma} \sum_{1,2} n_{1,\sigma} n_{2,-\sigma}$$

$$+ \quad \frac{\hbar^4}{m\Omega^2}\lambda^2 \sum_{\sigma} \sum_{1,2} \sum_{1',2'} \frac{n_{1,\sigma} n_{2,-\sigma}[n_{1',\sigma} n_{2',-\sigma} - (n_{1',\sigma}+n_{2',-\sigma})]}{p_1^2+p_2^2-p_1'^2-p_2'^2} \tag{4.63}$$

The third term is antisymmetric with respect to the transformation $p_1, p_2 \longrightarrow p_1', p_2'$, while the summation is over all four momenta, and therefore the summation of this term vanishes:

$$E = \sum_{p,\sigma} \epsilon_{p,\sigma} n_{p,\sigma} + \frac{\hbar^2}{2m\Omega}\lambda \sum_{\sigma} \sum_{1,2} n_{1,\sigma} n_{2,-\sigma}$$

$$- \quad \frac{\hbar^4}{m\Omega^2}\lambda^2 \sum_{\sigma} \sum_{p_1,p_2} \sum_{p_1',p_2'} \frac{n_{1,\sigma} n_{2,-\sigma}(n_{1',\sigma}+n_{2',-\sigma})}{p_1^2+p_2^2-p_1'^2-p_2'^2} \tag{4.64}$$

The spin dependence of denominator has vanished because in the first approximation

$$E_{\pm\sigma} = E \pm \mu\sigma.B \tag{4.65}$$

Let's rewrite E as :

$$E = \sum_{k,\sigma} \epsilon^0_{k,\sigma} n_{k,\sigma} + \frac{2\hbar^2\lambda}{2m\Omega} \sum_{p_1,\sigma_1} \sum_{p_2,\sigma_2} n_{1,\sigma_1} n_{2,\sigma_2} P_{\sigma_1\sigma_2}$$

$$- \quad \frac{2\hbar^4\lambda^2}{m\Omega^2} \sum_{p_1,\sigma_1} \sum_{p_2,\sigma_2} \sum_{p_3,\sigma_3} \sum_{p_4,\sigma_4} \frac{\delta(p_1+p_2-p_3-p_4)}{p_1^2+p_2^2-p_3^2-p_4^2} n_{1,\sigma_1} n_{2,\sigma_2} (n_{3,\sigma_3}+n_{4,\sigma_4}) P_{\sigma_1\sigma_2} P_{\sigma_3\sigma_4} \tag{4.66}$$

where $P_{\sigma_1\sigma_2}$ is the spin exchange operator.

The energy per quasiparticle $\epsilon_{p,\sigma}$ is the functional derivative of the total energy E:

$$\epsilon_{p,\sigma} = \frac{\delta E}{\delta n_{p,\sigma}} = \epsilon^0_{p,\sigma} + \frac{2\hbar^2\lambda}{m\Omega} \sum_{p_1,\sigma_1} n_{1,\sigma_1} \mathcal{P}_{\sigma\sigma_1}$$

$$- \quad \frac{4\hbar^4\lambda^2}{m\Omega^2} \sum_{p_1,\sigma_1} \sum_{p_2,\sigma_2} \sum_{p_3,\sigma_3} \Big(\frac{\delta(p_1+p_2-p_3-p)}{p_1^2+p_2^2-p_3^2-p^2} n_{1,\sigma_1} n_{2,\sigma_2} P_{\sigma_1\sigma_2} P_{\sigma\sigma_3}$$

$$+ \quad \frac{\delta(p+p_1-p_2-p_3)}{p^2+p_1^2-p_2^2-p_3^2} n_{1,\sigma_1} (n_{2,\sigma_2}+n_{3,\sigma_3}) P_{\sigma_1\sigma} P_{\sigma_2\sigma_3} \Big) \tag{4.67}$$

and the quasiparticle interaction function $f_{pp'}^{\sigma\sigma'}$ is the second functional derivative of the total energy with respect to the quasiparticle distribution function n. So we have

$$\begin{aligned} f_{pp'}^{\sigma\sigma'} &= \frac{\delta\epsilon_{p,\sigma}}{\delta n_{p',\sigma'}} \\ &= \frac{2\hbar^2\lambda}{m\Omega}P_{\sigma\sigma'} - \frac{2\hbar^4\lambda^2}{m\Omega^2}\sum_k \frac{2(n_{k+}+n_{k-})P_{\sigma\sigma'}}{p^2+p'^2-k^2-(p+p'-k)^2} \\ &+ \frac{n_{k,-\sigma}}{p^2-p'^2+k^2-(p-p'+k)^2} + \frac{n_{k,-\sigma'}}{p'^2-p^2+k^2-(p'-p+k)^2}) \end{aligned} \quad (4.68)$$

Note the slight change of notation, namely initial momenta are now denoted by p and p'. For the case of an unpolarized liquid $n_{k,\sigma} = n_{k,-\sigma}$ and the interaction function becomes:

$$\begin{aligned} f_{pp'}^{\sigma\sigma'} &= \frac{2\hbar^2\lambda}{m\Omega}\mathcal{P}_{\sigma\sigma'} - \frac{2\hbar^4\lambda^2}{m\Omega^2}\sum_k (\frac{4\mathcal{P}_{\sigma\sigma'}}{p^2+p'^2-k^2-(p+p'-k)^2} \\ &+ \frac{1}{p^2+k^2-p'^2-(p-p'+k)^2} + \frac{1}{p'^2+k^2-p^2-(p'-p+k)^2})n_k \end{aligned} \quad (4.69)$$

Except for the prefactor, this expression is identical to its form for the $3-d$ case, see ref.[26].

At this point, we could simply assume that $p = p' = p_F$, since important scattering processes are on the Fermi surface. But note that we are also interested in expressions of the form (1.2). Therefore we first evaluate the integrals for general values of p and p', and at the end take the momenta to be on the Fermi surface. The following three terms should be evaluated:

$$I_1 = \sum_k \frac{1}{p^2+p'^2-k^2-(p+p'-k)^2} \quad (4.70)$$

$$I_2 = \sum_k \frac{1}{p^2+k^2-p'^2-(p-p'+k)^2} \quad (4.71)$$

$$I_3 = \sum_k \frac{1}{p'^2+k^2-p^2-(p'-p+k)^2} \quad (4.72)$$

We can use the following geometric identities:

$$(p+p'-k)^2 = p^2+p'^2+2p\cdot p'-2k\cdot(p+p')+k^2 \quad (4.73)$$

$$(p-p'+k)^2 = p^2+p'^2-2p\cdot p'+2k\cdot(p-p')+k^2 \tag{4.74}$$

$$(p'-p+k)^2 = p^2+p'^2-2p\cdot p'-2k\cdot(p-p')+k^2 \tag{4.75}$$

Let's first evaluate the last two integrals:

$$I_2 = \frac{\Omega}{2(2\pi\hbar)^2}\int_0^{p_F}\int_0^{2\pi}\frac{kd\theta dk}{p'\cdot(p-p')-k(p-p')\cos\theta} \tag{4.76}$$

and with a change of variables:

$$I_2 = \frac{\Omega}{2(2\pi\hbar)^2}\frac{p_F}{\mid p-p'\mid}\int_0^1\int_0^{2\pi}\frac{\bar{k}d\theta d\bar{k}}{S-\bar{k}\cos\theta} \tag{4.77}$$

where S is defined by $S=\frac{1}{p_F}\frac{p'\cdot(p-p')}{|p-p'|}$.

The principle value of the integral over the angle can be evaluated by contour integration in the complex plane, and in general:

$$P\int_0^{2\pi}\frac{d\varphi}{a-b\cos\varphi} = \frac{2\pi}{\sqrt{a^2-b^2}}sgn(a)\;\theta(a^2-b^2) \tag{4.78}$$

so :

$$I_2 = \frac{\Omega}{4\pi\hbar^2}\frac{p_F}{|p-p'|}\left(|S|-\theta(|S|-1)\sqrt{S^2-1}\right)\;sgn(S) \tag{4.79}$$

where θ is the step function. Similarly:

$$I_3 = \frac{\Omega}{4\pi\hbar^2}\frac{p_F}{|p-p'|}\left(|S'|-\theta(|S'|-1)\sqrt{S'^2-1}\right)\;sgn(S') \tag{4.80}$$

where S' is defined by $S'=\frac{1}{p_F}\frac{p\cdot(p'-p)}{|p-p'|}$.

It is interesting to notice that I_2 and I_3 both contain terms of the same form as (1.2). But we then see that these two terms actually add up to a constant:

$$\frac{p'\cdot(p-p')}{|p-p'|^2}+\frac{p\cdot(p'-p)}{|p-p'|^2} = -1 \tag{4.81}$$

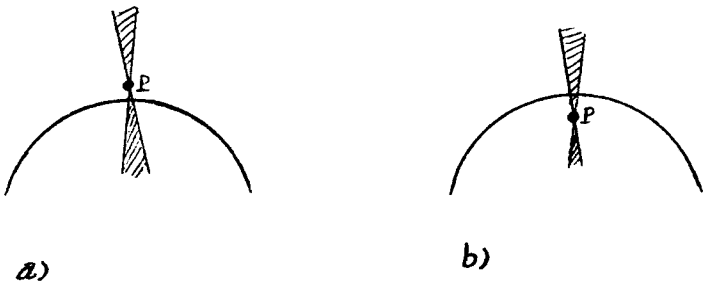

Figure 4.1: The phase space restrictions for the singular terms in $I_2 + I_3$. a) for p outside the Fermi sea, b) for p inside the Fermi sea. The shaded area becomes narrower as $p \to 0$.

Therefore, these terms are harmless! Yet, there are other singularities as well, but the step function puts severe phase space restrictions on them ; see Fig. (4.1).

Finally, the first integral is:

$$I_1 = \frac{V}{(2\pi\hbar)^2}\int_0^{p_F}\int_0^{2\pi}\frac{k dk d\theta}{(k^2 + p\cdot p') - k|p+p'|\cos\theta} \tag{4.82}$$

This integral is equal to:

$$I_1 = \frac{\Omega}{(2\pi\hbar)^2}\int_0^{p_F}\frac{k\,dk}{k|p+p'|}\int_0^{2\pi}\frac{d\theta}{\frac{k^2+p\cdot p'}{k|p+p'|} - \cos\theta} \tag{4.83}$$

$$= 2\pi\frac{\Omega}{(2\pi\hbar)^2}\int_0^{p_F} dk\,\frac{k}{k|p+p'|}\,\frac{sgn(\frac{k^2+p\cdot p'}{k|p+p'|})}{\sqrt{(\frac{k^2+p\cdot p'}{k|p+p'|})^2 - 1}}\;\theta[(\frac{k^2+p\cdot p'}{k|p+p'|})^2 - 1] \tag{4.84}$$

$$= \frac{\Omega}{2\pi\hbar^2}\int_0^{p_F} dk\,\frac{2\pi k\, sgn(k^2+p\cdot p')}{\sqrt{(k^2+p\cdot p')^2 - (k|p+p'|)^2}}\;\theta[(k^2+p\cdot p')^2 - (k|p+p'|)^2] \tag{4.85}$$

Let's define:

$$Q(k) = (k^2 + p\cdot p')^2 - (k|p+p'|)^2 \tag{4.86}$$

The roots of $Q(k) = 0$ are given by :

$$k_1^2 = \frac{1}{4}(|p+p'| + |p-p'|)^2 \tag{4.87}$$

$$k_2^2 = \frac{1}{4}(|p+p'|-|p-p'|)^2 \tag{4.88}$$

and the k values that make $Q(k)$ positive are $0 \leq k^2 \leq k_2^2$ and $k_1^2 \leq k^2 \leq p_F^2$ or $0 \leq k^2 \leq min(k_2^2, p_F^2)$. By a simple argument we can see that $k^2 + p \cdot p'$ is positive when $k_1^2 \leq k^2 \leq p_F$, otherwise $sgn(k^2 + p \cdot p') = sgn(p \cdot p')$.

After taking care of this small detail, we would like to evaluate the integral in eq.(4.84). With a change of variable to $y = k^2$, we have :

$$\int \frac{kdk}{\sqrt{Q(k)}} = \frac{1}{2}\int \frac{dy}{\sqrt{Q(y)}} = \log[2\sqrt{Q(y)} + 2y - (p^2 + p'^2)] \tag{4.89}$$

Now I_1 can be written as:

$$\begin{aligned} I_1 &= \frac{\Omega}{2\pi\hbar^2}\{\log(\frac{p+p'}{p-p'}) \\ &+ (\theta[p_F^2 - k_1^2] + \theta[k_2^2 - p_F^2]sgn(p \cdot p')) \\ &\quad \times[\log \frac{2\sqrt{p_F^4 - (p^2+p'^2)p_F^2 + (p \cdot p')^2} + 2p_F^2 - (p^2+p'^2)}{2|p \cdot p'| - (p^2+p'^2)}]\} \end{aligned} \tag{4.90}$$

Note that the most important contribution in the above expression is the first term. When p and p' are on the Fermi surface, it is this term that describes their interaction.

These equations give the following form for the quasiparticle interaction function in two dimensions:

$$\begin{aligned} f_{pp'}^{\sigma\sigma'} &= 2\lambda(\frac{\hbar^2}{m\Omega})P_{\sigma\sigma'} + \lambda^2(\frac{\hbar^2}{2\pi m\Omega})\{2P_{\sigma\sigma'}[\log(\frac{p+p'}{p-p'}) \\ &+ (\theta[p_F^2 - k_1^2] + \theta[k_2^2 - p_F^2]sgn(p \cdot p')) \\ &\quad \times(\log \frac{2\sqrt{p_F^4 - (p^2+p'^2)p_F^2 + (p \cdot p')^2} + 2p_F^2 - (p^2+p'^2)}{2|p \cdot p'| - (p^2+p'^2)})] \\ &+ [1 + \frac{p_F}{|p-p'|}(\theta(|S|-1)\sqrt{S^2-1} + \theta(|S'|-1)\sqrt{S'^2-1})]\} \end{aligned} \tag{4.91}$$

The cross channel (the last part of the above) has singularities as $p \rightarrow p'$ which are strong, but only appear if $p' > p_F$, while the Cooper channel (i.e. the part proportional

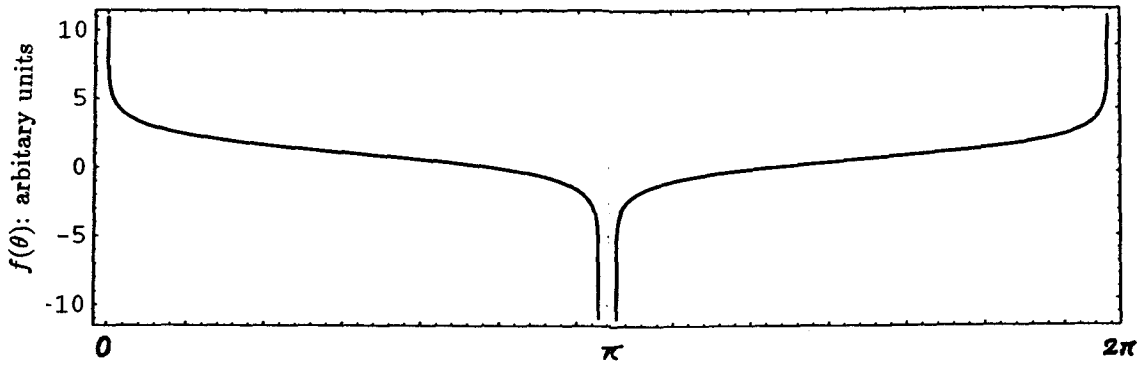

Figure 4.2: The function $f(\theta)$. Note the singular behaviour at $\theta = 0, \pi$. The horizontal scale at π has been slightly magnified for the sake of clarity.

to $P_{\sigma\sigma'}$) has a logarithmic singularity at that limit. In the case when both momenta are on the Fermi surface the latter singularity remains, but the former disappears. In that case the interaction function simplifies to :

$$f_{pp'}^{\sigma\sigma'} = \frac{2\lambda\hbar^2}{m\Omega} P_{\sigma\sigma'} + \frac{\lambda^2\hbar^2}{2\pi m\Omega}(1 + P_{\sigma\sigma'} \log \frac{1+\cos\theta}{1-\cos\theta}) \tag{4.92}$$

where θ is the angle between the two momenta. The above can be separated into the symmetric(parallel spin) and antisymmetric (antiparallel spin) parts:

$$f_{pp'}^{s} = \frac{\lambda^2\hbar^2}{2\pi m\Omega} \tag{4.93}$$

and

$$f_{pp'}^{a} = \frac{\lambda\hbar^2}{m\Omega} + \frac{\lambda^2\hbar^2}{2\pi m\Omega}(1 + \frac{1}{2} \log \frac{1+\cos\theta}{1-\cos\theta}) \tag{4.94}$$

Recall that λ is dimensionless and $\frac{\hbar^2}{m\Omega}$ has the dimension of energy. The behaviour of f as a function of angle is shown in Fig. (4.2).

4.2 The Spin Polarized Case

Comparing equations (4.59) and (4.60), we observe that the form of the interaction function for a spin polarized system is very similar to the unpolarized case. Before proceeding to write the result, we note a few points.

The degree of polarization in the system is given by :

$$\alpha = \frac{N_+ - N_-}{N_+ + N_-} = \frac{p_{F+}^2 - p_{F-}^2}{p_{F+}^2 + p_{F-}^2} \tag{4.95}$$

As the degree of polarization increases in the system, s-wave scatterng becomes less important. This is due to the fact mentioned earlier, that in the limit of s-wave scattering only particles of antiparallel spin can scatter each other. By increasing polarization, the amount of available momentum space for s-wave scattering decreases, and in high magnetic fields, the p-wave scattering dominates. So our result is only valid for weak magnetic fields.

For a polarized Fermi liquid , equation(4.59) gives the result :

$$\begin{aligned} f_{pp'}^{\sigma\sigma'} &= 2\lambda(\frac{\hbar^2}{m\Omega})P_{\sigma\sigma'} + \lambda^2(\frac{\hbar^2}{2\pi m\Omega}\{2P_{\sigma\sigma'}[\log(\frac{p+p'}{p-p'}) \\ &+ [\theta[p_{F+}^2 - k_1^2] + \theta[k_2^2 - p_{F+}^2]sgn(p\cdot p')] \\ &\quad \times(\log\frac{2\sqrt{p_{F+}^4 - (p^2+p'^2)p_{F+}^2 + (p\cdot p')^2} + 2p_{F+}^2 - (p^2+p'^2)}{2|p\cdot p'| - (p^2+p'^2)})] \\ &+ (p_{F+}\rightarrow p_{F-}) + [1 \\ &+ \frac{1}{|p-p'|}(\theta(|S| - (p_{F,-\sigma'}))\sqrt{S^2 - (p_{F,-\sigma'})^2} + \theta(|S'| - (p_{F,-\sigma}))\sqrt{S'^2 - (p_{F,-\sigma})^2}) \end{aligned} \tag{4.96}$$

When the two momenta are on the respective Fermi surfaces, this expression reduces to :

$$f_{pp'}^{\sigma\sigma'} = 2P_{\sigma\sigma'}\left[\lambda(\frac{\hbar^2}{m\Omega}) + \lambda^2(\frac{\hbar^2}{2\pi m\Omega})(\log(\frac{p+p'}{p-p'}) + 1)\right] \tag{4.97}$$

Equation (4.95) does not indicate any pecularities due to polarization. However, the presence of two Fermi surfaces may put restrictions on the scattering events.[36] Therefore, this case needs further investigation.

Chapter 5

Discussion

5.1 Recent Works on the Low Density Fermi Gas in 2D

For a long time, it was implicitly assumed that the structure of Fermi liquid theory was consistent in two dimensions as well as three dimensions. Numerous papers have been devoted to different aspects of the two dimensional Fermi liquids (especially in connection with the experiments on films of ^{3}He), without questioning its validity.[36–40] However, it is well known that interacting Fermions in one dimension do not form a Fermi liquid, but form a state which has been called the Luttinger liquid.[28] It is characterized by separation of charge and spin degrees of freedom and its low-energy excitations have bosonic character.

In connection with new high-T_c superconductors and their anamolous normal state properties,[4] Anderson[5–7,15–17] has claimed that the normal state of these new materials is a new quantum liquid. To support his claim, Anderson asserts that there are two known fundamentally different fixed points (in the context of renormalization group) for systems of interacting Fermions in any dimension and consequently, such systems exhibit Fermi liquid behaviour *or* Luttinger liquid behaviour. Anderson argues that due to singular terms in the quasiparticle interaction function, Fermi liquid theory breaks down in two dimensions and a system of interacting Fermions shows non-Fermi liquid behaviour. The breakdown occurs because as a result of the singular interactions, the quasiparticle wavefunction's renormalization factor vanishes. The last point was explored

in a paper by P. Stamp.[41]

In his paper,[41] Stamp showed that if such singularities as (1.2) are present, they indeed lead to a breakdown of Fermi liquid theory. The proof was based on examining the perturbation series and separating graphs that contained such singularies. The most divergent terms were self-consistently summed, and it was shown that they give a quasiparticle pole vanishing as

$$z(\omega) = (\frac{\omega}{\omega_0})^{\frac{1}{\pi}[\frac{1}{1+\tilde{F}_0}]\frac{\rho\delta_0^2}{\hbar^2\tilde{N}(0)\omega_0}} \tag{5.98}$$

where δ_0 is the phase shift, $\omega = \epsilon - 2\epsilon_F$ and ω_0 is some upper energy cutoff. ρ represents the number density, $\tilde{N}_0$ is the renormalized density of states and $\tilde{F}_0$ is the renormalized zeroth order Fermi liquid parameter: coming from non-singular graphs. This form is valid in the vicinity of Fermi surface ($\omega \to 0$) and signals the breakdown of Fermi liquid theory. The subdominant (less divergent) terms were ignored in this calculation, but later they were summed[42] (by eikonal expansion) and were found to preserve the essential structure of (5.98). It must be emphasized that Stamp's paper did not give a justification for the existence of such singularities, but merely assumed it. Neither did it give any clues to the relevant ground state.

Another work that has examined the validity of Fermi liquid theory in two dimensions is by Fabrizio, et al.[43]. They calculated a nonperturbative, exact solution for the ground state of a finite number of particles in the low density limit. What is of particular interest to us is the Migdal discontinuity which they found to be

$$z(\omega = 0) = 1 - \frac{4\ln 2}{\ln^2(\rho a^2)} \tag{5.99}$$

This indicates that z is well defined and non-zero in the low-density limit. Another work which came to the same conclusion was by Fukuyama et al.[44] This work used t-matrix approximation to study the Hubbard model in the limit of low density and found that

it behaves like a Fermi liquid. However, both these results are shown by Stamp to be consistent with formula (5.98), since z is nonzero in the limit of low density.

Furthermore, one should mention several papers by Engelbrecht and Randeria[45–49] on this subject. Their work[48] on the low density repulsive Fermi gas in 2D is of particular interest to us. Their method was similar to ours in that they extended the method of ref.[26] to 2D. Their dimensionless parameter, although defined differently, is equivalent to ours. They also found the logorithmic singularity that appears in the Cooper channel and signals the superconducting instability for $\theta = \pi$. However, their calculations were done with the momenta on the Fermi surface , and hence is unsuited to detect any singularities of the form (1.2). Note that (1.2) actually is finite ($= \frac{-1}{2}$) for k and k' on the Fermi surface.

5.2 Discussion of Our Results

In this work, we have studied a two dimensional dilute interacting Fermi gas. The quasiparticle interaction function has been calculated to second order in terms of the dimensionless coupling parameter, λ, which characterizes the renormalized potential. The interaction function shows a number of interesting features which are :

I. The appearance of singular terms of the form proposed in equation(1.2). This shows that in essence such terms can exist in the interaction function. However, in our case, the two most divergent terms add up to a harmless constant. What remains is a much weaker singularity which is described below.

II. From eq.(4.78), (4.79) and (4.91), we have other singular terms which are of the form:

$$\Delta \sim \frac{p_F}{|p-p'|}\sqrt{(\frac{p\cdot(p'-p)}{p_F|p-p'|})^2-1}$$

$$= \frac{1}{|p-p'|}\sqrt{p^2\cos^2\gamma - p_F^2}$$
$$\simeq \frac{\sqrt{2}p_F}{|p-p'|}(\frac{|p|-p_F}{p_F})^{\frac{1}{2}}$$

As can be seen from the aforementioned equations, these terms only appear when p or p' or both of them are above the Fermi surface. Even then, they have additional phase space restrictions due to the presence of the step function, (see Fig. 4.1). and they are further weakend by a factor of $(\frac{|p|-p_F}{p_F})^{\frac{1}{2}}$. Therefore, it is unlikely that they could lead to a breakdown of the Fermi liquid ground state. The work by Stamp and Prokofiev[50] shows that this conclusion is indeed correct. We shall look at their work in a bit more detail later.

III.Finally, our result agrees with those of Randeria,et al.[48] that there is a logorithmic singularity in the Cooper channel. This feature is not unique to two dimensional systems and has previously been noted in 3D as well. It can be shown[26] that despite this singularity, the interaction function remains regular for $a \geq 0$.This is done by summing the ladder of such singular terms to infinite order. However, attractive interactions $a < 0$ lead to the appearance of a pole in the scattering amplitude, which is indicative of the instability of the ground state to Cooper pairing. It marks the onset of superfluidity in a Fermi liquid. However, our result also shows the presence of such a singularity for $\theta = 0$. This singularity has the peculiar feature that it is of exactly the same form as $\theta = \pi$ divergence, but with a sign difference. This feature is definitely worthy of further investigation.

5.3 Discussion of the Phase Shift

We have already looked at one or two aspects of Anderson's argument for the breakdown of Fermi liquid theory. So far we have looked at the question of existence of such terms

and their effect on the validity of Fermi liquid theory in two dimensions. Anderson argues in analogy to the problem of a impurity with a static potential. He has argued that in the presence of the Fermi sea, the two particle relative scattering phase shift $\delta(q,\omega)$ (where q is the center of mass momentum and $\omega = \epsilon - 2\epsilon_F$) is finite at $q = 2k_F, \omega = 0$, and that it leads to an orthogonality catastrophe:

$$< 0|V > \sim exp(-(\frac{\delta_0}{\pi})^2 \ln N) \tag{5.100}$$

where $< 0|V >$ represents the overlap of the non-interacting state with the state in the presence of the potential. In the case of a finite phase shift this overlap vanishes. This leads to an orthogonality catastrophe which means we can no longer consider the free Fermi gas as the relavant fixed point of the problem.

Now, one relevant question is whether there is a finite phase shift. Randeria and Engelbrecht have claimed that near the Fermi surface

$$\delta(2k_F, \omega) \simeq \frac{\sqrt{\omega}}{2\ln(k_0 a)} \tag{5.101}$$

which goes to zero as $\omega \to 0$. A work that has studied this problem in detail is that of Stamp and Prokofiev.[50] In their paper, starting from the two-body Schrödinger equation for two particles, they calculated the scattered wave function and the phase shift for such scattering in the presence of a Fermi sea. They found the behavior of $\delta(q,\omega)$ at the limit of $q \to 2k_F$ and $\omega \to 0$ to be peculiar. This was also noted by Fukuyama et al.[44] This dilemna is resolved using the Lippmann-Schwinger equation and applying boundary conditions of a finite box. They noted that a) the answer should be independent of the shape of the box in the thermodynamic limit ($L \to \infty$) and b) that the direction of Q, which is the momentum exchange, should be arbitrary. By angular averaging over all possible directions, one derives an expression for the phase shift that properly and unambiguiously vanishes in the limit of low energy. Therefore they concluded that within

the context of perturbation theory and validity of the Lippmann-Schwinger equation the phase shift remains zero, and Fermi liquid description remains valid.

In conclusion, our calculations show the presence of weak singularities which are limited to a very narrow region of the phase space, and so there is no hint of a breakdown of Fermi liquid theory. This finding is in agreement with other works on the subject, especially that of Stamp and Prokofiev.[50] The conclusion is that perturbation theory is consistent with a two dimensional Fermi liquid and if such strong singularities exist which lead to the breakdown of the adiabatic continuity from a free Fermi gas, they must have a non-perturbative origin.

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