MOTIVATION IN THE TEACHING OF HIGH SCHOOL MATHEMATICS

by

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Teaching of standard constructions can be motivated by examining numerous methods suggested by the pupils and selecting the best. e.g. drawing perpendicular to a given straight line from a given point outside it. Interest in geometrical language may be created by having pupils try to explain in words what they have done with the instruments - compare constructing geometrical figures to building.

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1. Less formal treatment of theorems - Teach theorem (often by analysis); - build it up together - test following day - develop idea that once key is given to a theorem (or exercise) it is solved once and for all time.
2. Careful treatment of exercises - a carefully selected exercise to be written out and handed in each day - marked - returned - marks recorded - monthly totals read - excellent review daily.
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9. Removal of brackets - explanation of rule (really addition or subtraction) - thrill of getting complicated expressions down to a or o - competition by rows,-

point ahead to equations.
10. Fundamentals in algebra must be thoroughly understood, as all future work in algebra involves the use of these fundamentals - at the beginning of each period a review of the previous lesson is essential - e.g. five short questions on the board, - how many right out of five - pupils stand (relaxation).
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MOTIVATION IN THE TEACHING OF HIGH SCHOOL MATHEMATICS

CHAPTER I.

THE IMPORTANCE OF MOTIVATION IN THE TEACHING OF MATHEMATICS.

What is motivation? In its literal sense it is "the process of inducing movement;" (1) and in its broader sense it is "the process of producing stimuli which initiate, direct and sustain activity" (2). Motivation in connection with teaching might be considered as the introduction of certain factors into teaching, which initiate, direct, and sustain the activity of the individual taught. When considering motivation in relation to the teaching of high school mathematics, we shall consider the numerous ways by which a teacher may induce each pupil in his class to put forth the maximum amount of effort of which he is capable, and to derive the greatest possible benefit from that effort. It is obvious that, for various reasons, a great many pupils do not put forth their best efforts in the study of mathematics, and that they do not derive all the benefits which they might derive from the study of such a wonderful subject.

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(1) F. Goodenough - Developmental Psychology, P. 498.

(2) A. Gates - Psychology for Students of Education, P. 182.

The principal objects of motivation in teaching are two in number:

1. The creation of a maximum amount of interest on the part of the learner; (1) 2. The attainment of maximum effort by each pupil. - In the case of perfect realization of this aim each pupil will have reached an A. Q. of 100. If these two objectives could be reached, the study of mathematics in our high schools would take on a new lease of life, and many of the criticisms levelled at the teaching of mathematics would be made groundless indeed. In the following pages special reference will be made to the forms of motivation suitable for grades IX and XII, with a comparison of the types of motivation used in these grades.

The study of mathematics, when devoid of any motivating forces, becomes a dull and uninteresting task, and yet mathematics is an essential part of a person's education. There is no substitute for it, and no complete education without it. (2) It therefore behoves the teacher of mathematics to bend every effort towards presenting the subject to his pupils in such a way that they will have motivating forces urging them on to their greatest efforts,

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(1) The importance of creating interest is discussed by M.J. Stormzand, and he arrives at the conclusion that "the problem of interest plays such an important part in education because success in all teaching involves the arousing of sufficient interest. <u>Progressive Methods in Teaching</u>, P. 129.

(2) "All scientific education, which does not commence with mathematics, is, of necessity, defective at its foundation." Comte.

and producing most beneficial results. (1) That mathematics is an essential part of a person's education is almost axiomatical. "Our entire present civilization," says Professor Voss, "as far as it depends upon the intellectual penetration and investigation of nature, has its real foundation in the mathematical sciences." (2) The physical laws of the universe are so linked up with mathematics that our innate desire to examine the laws of nature in order to find explanations for all the various natural phenomena, would be doomed to disappointment without a mathematical foundation on which to work. (3) W.A.Millis, when discussing the value of mathematics as a high school subject, makes the statement that "for algebra there is no substitute. The elimination of algebra as a pure science from the curriculum would cut the foundation from under all scientific procedure." (4) If mathematics, therefore, is really an essential part of a person's education, should we not bend every effort to so motivate the teaching of the subject, that each pupil would

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(1) "Euclid has done more to develop the logical faculty of the world than any book ever written. It has been the inspiring influence of scientific thought for ages, and is one of the cornerstones of modern civilization." - Brooks. S.I.Jones - Mathematical Wrinkles, P. 245-6.

(2) A. Schultze - The Teaching of Mathematics in Secondary Schools, P. 17.

(3) "It is when we examine the relation of mathematics to science, both pure and applied, that we see most forcibly its indispensability as a propaedeutic" - Charles De Garmo - The Studies of the Secondary School, P. 65.

(4) W.A.Millis - The Teaching of High School Subjects, P. 240.

derive the greatest possible value from its study, and would apply himself to his work with an eagerness that would bring him to his highest level of efficiency.

Another reason why it is so important that every teacher should make the most of every opportunity to motivate the teaching of mathematics is that in many provinces mathematics is one of the compulsory subjects of the curriculum until the student has completed two years of university work. Whether this regulation is a wise one or not is not for discussion here, but as long as the requirements remain as they are, it is imperative that a student be so instructed in mathematics in high school that he will develop a liking for the subject and not find it a millstone about his neck year after year. Many a student has had the joy taken out of his educational life simply by developing an early hatred for mathematics, and had the subject been properly motivated for those pupils, that abhorrence in many cases may have been entirely eliminated.

Motivation in mathematics is particularly important when dealing with students who are favorably inclined towards scientific study. If a boy or a girl develops a liking for physics, chemistry, geology, astronomy or, in fact, any of the sciences, it is very important that he or she also develop a liking for mathematics. If such a student is given a bad impression of mathematics by lack of motivation in its presentation, then he may abandon the study of a science in which he is really interested simply because of the mathematical calculations involved in it. W.A.Millis in "The Teaching of High School Subjects," points out that mathematics is unpopular with a great many high school students and enumerates reasons for its unpopularity. One of his chief reasons is that the subject is more difficult than most other subjects, and if pupils are given the option of another subject in place of mathematics, many students select another subject instead of mathematics. (1)

Not only do pupils abandon the study of some of the sciences on account of the mathematics involved, but they select subjects in which they have no particular interest, and for which they are not suitably adapted, for the sole purpose of avoiding the only other alternative - mathematics. Undoubtedly a number of these students, at some later date, regret their actions when they suddenly realize to what extent they are handicapped by lacking a fundamental knowledge of mathematics.

E.L.Thorndike made an investigation into the popularity of the various subjects on the high school curriculum. The voting was done by grade XII pupils in the High Schools of New York City. In the final ranking of subjects, algebra ranked 13th. out of 22 with boys and 25th. out of 27 with girls. Geometry ranked 16th. out of 22 with boys, and 26th.

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(1) W.A.Millis, Opcit. Pp. 231-3.

out of 27 with girls. [1] The obvious conclusion to be drawn from the results of these tests conducted by Dr. Thorndike is that in the past the teaching of mathematics has not been carried out in a wholly satisfactory manner. [2] Mathematics is apparently a subject which can become very boresome and distasteful to students if presented in a cold, pedantic manner; [3] but on the other hand, it may become a subject of real and living interest if the teacher uses skillfully motivated methods of presentation.

A further reason for the necessity of motivating the teaching of mathematics in high school is that mathematics is really one of the more difficult subjects of the curriculum. A pupil of inferior intelligence can very often reach the required standard in certain subjects such as French, social studies, grammar, geography and similar subjects by means of frequent repetition and laborious memorization; but such a student has much more difficulty in reaching the required standard in mathematics. The very nature of the subject demands that its teaching be motivated to the fullest possible extent, in order to encourage those who are not especially bright to develop their mathematical skill to the extreme

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(1) E.L.Thorndike, Psychology of Algebra, P. 386.
(2) "One reason for the unsatisfactory status of the subject (mathematics) is poor teaching." - W.A.Millis, Opcit. P. 233.
(3) "The extension of the elective system has revealed that pupils do not like these subjects (Algebra and Geometry)"- Ibid., P. 232.

limit of their ability. Without such motivation mathematics is to these pupils an exceedingly difficult subject which they will drop at the first opportunity. Since the usual regulations for promotion, however, demand that every student, dull or bright, must reach a certain specified standard, it becomes incumbent upon the teacher of mathematics to motivate his teaching in such a way that the duller pupils as well as the brighter ones will develop a sufficient interest in the subject to encourage each one to reach his highest level of efficiency.

Another reason why it is so essential for the teacher of mathematics to motivate his teaching to the fullest extent, is that in this subject a great deal of review work is necessary and unless some form of motivation is adopted, this review work may become very monotonous and uninteresting. Constant practice in the mechanical processes and continual reference to fundamentals are essential, and yet constant repetition in "cut and dried fashion" may take the life out of the subject for pupil and teacher alike. Both in algebra and in geometry, every succeeding topic is built up on the results of what has gone before. There is no possibility of a student making a new start in mathematics beginning at a certain chapter unless he is willing to go back and master all the fundamentals upon which that chapter is based. This state of affairs makes it imperative that even from the very first day when the study of algebra and geometry is begun, the material should be presented in such a way as to create an earnest desire on the part of the pupil to master the subject in every detail.

Every teacher of mathematics finds at some time or other pupils in his class who dislike mathematics because they have a preconceived idea that they are not mathematically inclined. When such a pupil is asked why he dislikes the subject the usual response is such as, "Oh, I can't do mathematics. I never was any good at it." Upon hearing such a statement, one cannot help but wonder what it was that turned that pupil against mathematics; and the logical sequence of thought is to consider whether or not the reason for such dislike lay in the pupil's first introduction to the subject, which was very likely void of any form of motivation whatsoever. Cases such as these are all too numerous, and the utilization of the various forms of motivation to a fuller extent would undoubtedly reduce the number of such cases considerably.

Probably the most important reason why the teaching of mathematics should be motivated to its fullest extent, is that it is the means of opening up a new field of interest to many students. A new and delightful experience is in store for any pupil who develops a liking for mathematics; an experience which may become the chief interest in that student's educational career. The opening to that new and delightful experience is made by the teacher who so motivates his teaching as to create a desire in the student to know more about this wonderful science of mathematics. Even the study of very elementary mathematics, when accompanied by the proper motivating forces, becomes the source of abundant pleasure, a type of pleasure which is different from all other pleasurable experiences. The study of mathematics for its own sake - for the satisfaction and enjoyment derived from its pursuit - is one of the prime reasons for its inclusion in the high school curriculum [(1)], and a student may be led to this new source of pleasure by skillfully motivated forms of presentation on the part of the teacher.

For these several reasons enumerated above it is obviously imperative for the old pedantic methods of instruction to be replaced by highly motivated forms of teaching, if the subject of mathematics is to hold the place it should hold and fulfil the objects it should fulfil in the educational lives of individuals.

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(1) "We study music because it gives us pleasure. ..... So it is with geometry. We study it because we derive pleasure from contact with a great and ancient body of learning that has occupied the attention of master minds during the thousands of years in which it has been perfected and we are uplifted by it." D.E.Smith - The Teaching of Geometry.

CHAPTER II.

TYPES OF MOTIVATION APPLICABLE TO THE TEACHING OF HIGH SCHOOL MATHEMATICS

Having observed the necessity for utilizing motivating forces to their fullest extent in the teaching of high school mathematics, it is now necessary to consider what are the various forms of motivation which are applicable to the teaching of high school mathematics.

It is impossible to compile a list and say, "These are all the forms of motivation which may be used in teaching high school mathematics;" because every pupil is different, and that which is a motivating force to one student may have no effect upon another student whatsoever. Moreover, the motives urging a pupil to do his best are different at the various stages of that individual's school life. A very powerful motivating force to a pupil when in grade IX may have no influence whatsoever upon that same pupil when he reaches grade XII. However, there are certain forms of motivation which may be utilized in the various grades of high school, and the utilization of which may bring new life and interest to the study of mathematics. (1) This present chapter

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(1) H.B.Wilson and G.M.Wilson, The Motivation of School Work, P. 47. In this book there are enumerated eleven different types of motivation. All but two of these are applicable to the teaching of mathematics; - viz. Earning money, and the acquisition of a collection.

contains merely the enumeration of the various types of motivation which may be used in teaching high school mathematics, and the specific methods by which they may be put into effect will be given in succeeding chapters.

1. A natural interest in new experiences. - When a pupil enters grade IX of high school, the novelty of the new subjects which he has not studied before has a great appeal to him. The wise teacher will capitalize on this novelty, and in his teaching of algebra and geometry will try to give the pupil a proper outlook towards these subjects at a time when the pupil is eager to hear about the wonders of these new spheres of knowledge.
2. An interest in the individual topics studies. - After an appropriate introduction to the subject, and a proper attitude towards it has been created in the mind of the pupil, it is essential that the individual topics studied be sufficiently vital to the pupil to maintain the interest which has been aroused. If the material studied is closely associated with the pleasurable aspects of the pupil's experience, then there is a desirable motivating force acting upon the pupil at all times.
3. Desire for Praise. - This is a very powerful motivating force which is particularly strong in grade IX and continues to a large extent throughout a pupil's high school career. Practically every pupil is encouraged to better efforts if he knows that he will receive the praise of his teacher, his

parents or his fellow pupils by making that extra effort. The skilful use of praise by the teacher can be made a very effective motivating force, not only for the brilliant student but also for the dullard who is trying to do his best.

4. <u>Desire to avoid disgrace</u>. - This motivating force is allied very closely to the one immediately preceding, but there are a great many pupils, especially in the group of average intelligence, who are encouraged to better efforts by the fear of being utterly disgraced by failure to reach a certain standard or to gain promotion. It is not so much a desire for praise that urges these pupils on as it is the desire to avoid the shame which would come upon them should they fail to measure up to the standard which they believe they should be able to reach.

5. <u>Desire for good marks</u>. - This is a motivating force which is active throughout the grades of high school, but which can be made especially useful in securing greater effort from a pupil in the earlier grades. Pupils in grades IX and X prize their marks very highly, and a teacher who marks judiciously can make a great deal of the motivating power of marks. Pupils like to receive marks even for the smallest set of questions, and if these marks are recorded and made the basis of careful comparison, the pupils find an added interest even in daily tests.

6. <u>Desire for promotion.</u> - This form of motivation is present

in all the grades of high school, but it is more active toward the end of each school year than at the beginning. At the commencement of a term promotion time seems somewhat distant, and desire for promotion is not a very strong motivating force, but there are ways and means by which the teacher may increase its power even from the beginning of the term.

7. Interest in competitions. - The use of competitions of various kinds is a very useful form of motivation, especially in grades IX and X. The majority of pupils in these grades are very keen on competitions, both against each other and against time. There are numerous ways by which this form of motivation may be used to good advantage.

8. Desire for activity. - One of the dangers in a subject like mathematics is for it to become too inactive. Sitting at one's desk for a considerable time working a long series of questions does not appeal to the ordinary high school student. There is a tendency to boredom which should be overcome by the introduction of more physical activity into the mathematics lesson. The specific methods by which this activity may be introduced will be discussed in a later chapter.

9. Interest in games. - Pupils in all grades of high school derive a great deal of pleasure from games. There are a large number of mathematical recreations which are admirably suited to high school students, and if these are

carefully arranged and properly placed in the mathematics lesson they will have a wonderful motivating effect on the whole subject of mathematics.

10. Interest in humor. - A mathematics teacher, and especially one who has been teaching the subject for a number of years, is very often inclined to become mechanical in his methods of presentation, and overlook some of the possibilities which exist for making a mathematics lesson really enjoyable. Admittedly, the number of opportunities for humorous illustrations and analogies is not as great in a mathematics lesson as in lessons in many of the other subjects. Nevertheless the teacher keenly interested in motivating his teaching to the fullest extent should make the most of every little opportunity that arises for introducing even a slight touch of humor into the mathematics lesson.

11. Desire for change, - variety. - Any experienced mathematics teacher knows that there is a decided tendency on his part to present lessons of a similar nature in almost identical manners. In the teaching of geometry this tendency is particularly noticeable. The treatment of one theorem after another, or one exercise after another in the same manner day after day is certain to become tedious to pupils. Pupils like variety. An essential form of motivation, therefore, especially in the teaching of geometry, is a variety of methods of presentation for lessons of a similar

nature.

12. Interest in constructing. - In grade IX to a great extent, and in the other grades to a smaller extent, pupils find a great deal of pleasure in doing actual constructive work. They enjoy building. There is a very close analogy between constructive geometry and building, and if this fact is fully appreciated by the teacher, a great deal of the work in geometry can be motivated very highly by utilizing the pupil's keen interest in building.

13. The thrill of discovery. - "To discover thrills them," says R.W.Pringle when referring to the nature of adolescents. (1) This trait of the adolescent is active in all grades of high school. When a pupil in the first week of his study of geometry discovers for himself some new (to him) fact about triangles, or when a matriculation student by his own efforts discovers the fundamental nature of an ellipse, the thrill of discovery is sufficient to give that student an ardent desire to delve more deeply into the mysteries of mathematics. The part played by the teacher is to start the pupil on the road to discovery and guide him at difficult crossings.

14. Effect of special privileges. - A pupil in any grade of high school likes to think that he is a specially privileged person. By this it is not meant that he likes

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(1) R.W.Pringle, Methods with Adolescents, P. 126.

to be "teacher's pet," but that as a reward for some special effort on his part he is allowed to emjoy some special privilege not enjoyed by the remainder of the class. There are a great many ways by which good work in mathematics may be rewarded by special privileges, and this type of motivation is very effective with certain types of students.

15. <u>Interest in mathematics for its own sake</u>. - If the teaching of mathematics has been properly notivated by various means during the early part of a pupil's high school career, there should come a time, probably at the end of grade X or the beginning of grade XI, when that pupil discovers that he really enjoys the study of mathematics just for its own sake. He may leave some other things undone, but he will not neglect his mathematics because of the enjoyment he derives simply by its pursuit. When this stage has been reached, and it is reached by a great many high school students, there is added to the motivating forces already at work a new and extremely powerful one; one which may cause the pupil to devote his life to the pursuit of mathematical knowledge.

16. <u>Effect of a challenge</u>. - Most human beings, and especially boys and girls with the red blood of youth in their veins, respond very readily to a challenge and do not rest content until that challenge has been answered. This characteristic of human psychology makes it possible for the teacher

of mathematics to encourage the pupils to put forth their best efforts by making mathematical problems appear as definite challenges to their intelligence and ingenuity. In the advanced grades of high school the motivating force of a challenge issued by a certain algebraical or geometrical problem spurs the students on to efforts far beyond their customary levels.

17. Desire for efficiency in life's work. - Some students in high school are encouraged to better efforts in mathematical study because of their desire to go out into the world better fitted for their life's work by reason of their study of mathematics. If the teacher points out to these students the various possibilities of transfer from their study of mathematics to their intended occupations or professions, then their desire for efficiency may become a real motivating force in the study of mathematics.

18. Satisfaction through mastery. - This form of motivation is one of the most important, if not the most important, of all the forms which can be applied to the teaching of mathematics in high school. If, in the early stages of his study of mathematics, a pupil has the teaching of mathematics motivated for him so as to give him an early liking for the subject, he will in turn put forth his best effort in that subject, and most likely find that he has succeeded in mastering the early part of the work. The satisfaction which that pupil derives from the mastery of

his early work is a compelling influence to further effort. A pupil enjoys a subject which he understands well, and if a teacher can utilize the various minor methods of motivation in order to encourage a pupil to master each section of the work as he goes along, then the very fact of his mastery over a preceding section is sufficient motivation to create in him a desire to proceed to the following section. The motivating force of this satisfaction gained through mastery is indeed a powerful influence in the teaching of high school mathematics.

19. Eagerness for perfection. - The object of a great many forms of motivation in teaching is to encourage the poorer students to put forth greater effort and attain a higher standard of efficiency. While trying to accomplish this aim, the fact must not be overlooked that those pupils who have already attained a very high degree of efficiency might be encouraged to do even better work than they have done. One of the methods of motivation which applies particularly to this top-ranking class of pupils, is the creation of an eagerness for perfection. Mathematics is one subject in which perfection can be reached in a great many cases, and if the teacher can create an eagerness on the part of the pupil to attain absolute perfection in his work, then he will be urging that pupil to extend himself to the limit of his ability and a new interest will be added to that pupil's work in mathematics.

20. <u>Satisfaction through helping</u>. - When a student reaches grade XI or XII he has gained sufficient knowledge of elementary mathematics to enable him to offer some assistance to pupils in grades IX and X. If opportunities are provided by the teacher for the offering of this assistance, then the senior grade pupil will strive to understand his work more thoroughly in order to be able to teach his friends in the lower grades more skilfully. A student gets a great deal of satisfaction out of helping other students, and at the same time he is strengthening his own grasp of the subject by constant review of fundamentals.

21. <u>Desire to be considered mature</u>. - In the upper grades of high school most pupils like to believe that they are getting on into advanced mathematics, and that they will soon be blossoming into mature mathematicians. A certain amount of encouragement to the adoption of this attitude can be given by the teacher to very good effect. If a teacher treats his matriculation students as mature persons of whom much more is expected than is expected of pupils in the lower grades, then those matriculation students will respond and endeavour to show that they are indeed advanced mathematicians capable of concentrated effort.

22. <u>Desire to complete the course selected</u>. - One of the motivating forces urging students in the upper grades of

school to work diligently is the desire on the part of the pupil to complete the course which he has selected. Most pupils in these grades look forward eagerly to the time when they will be graduated from high school, and be able to go out into the world with a successfully completed high school course behind them. The anticipation of this future pleasure has a valuable motivating effect on the student's present efforts.

23. Eagerness to pass matriculation examination. - For pupils who are in the final year of high school and who are taking the matriculation course, there is one very powerful motivating force at work; namely, the desire to pass the matriculation examinations. The motivating force of matriculation examinations is not always the most desirable form of motivation, but, as will be seen in a later chapter, a proper attitude toward matriculation examinations may rid them of most of their undesirable effects, and transform them into admirable motivating forces, not only for pupils in the matriculation grade, but also for pupils working up towards it.

24. Effect of prizes and scholarships. - The motivating power of prizes and scholarships is obviously very limited in its scope. It is only those students who are exceptionally good at their work who have any interest in scholarships, and usually these rewards are offered only to matriculation students. On account of their narrow scope, and also on

account of the fact that the working for the reward may rob a subject of much of its real value, the offering of prizes and scholarships can not be considered one of the valuable forms of motivation for the high school student.

To this list of the various types of motivation applicable to the teaching of high school mathematics, there might be added a great many other forms of motivation which are more limited in their scope or which are applicable only to special types of students. The foregoing list, however, contains most of the more general forms of motivation which are applicable to students attending high school. In the following chapters the specific means by which these types of motivation may be applied to the teaching of algebra and geometry in grades IX and XII will be discussed.

CHAPTER III

MOTIVATION IN THE TEACHING OF GRADE IX GEOMETRY

To the teacher of geometry in grade IX a wonderful opportunity is presented; an opportunity for opening up to his pupils a new field of knowledge which is full of interest and enjoyment. It becomes incumbent upon the teacher to make the most of this opportunity by so motivating his teaching that all the interest and enjoyment latent in the subject of geometry is discovered by the pupils under his care. But what are the various methods of motivating the teaching of grade IX geometry in order to achieve this object?

In the first place, the teacher can capitalize upon the novelty of the subject. People are inherently interested in things new, so while the pupils are in the proper frame of mind the teacher can inform them of some of the wonders of this science of geometry, and give the pupils a proper outlook towards the subject. An introduction might include reference to some of the interesting features about a few of the world's great mathematicians such as Pythagoras, Euclid and Einstein, and also give some indication as to the extent to which modern civilization is built up on a mathematical basis.

After the pupil's interest has been aroused in the subject, it is essential that this interest be maintained by skilfully motivated teaching. There is a danger in grade IX that the novelty of geometry may wear off before the pupil has developed a real interest in the subject for its own sake. One of the means by which this lasting interest may be aroused even at a very early stage is by keeping the first part of the work mostly experimental. A pupil in this grade does not like to sit still in his seat and watch constructions being done on the blackboard by the teacher; but he wants to do the constructive work himself. The blackboard explanations by the teacher should be as short and concise as is conveniently possible, and then the pupil should be allowed to experiment for himself.

Although the early part of the study of geometry should be largely experimental and of a constructive nature, nevertheless it is important that sound geometrical principles be taught in conjunction with these constructive exercises. If this is not done, the pupil will find out in a very short time that his knowledge of geometry has been built up on a rather feeble foundation; and consequently he will lose that early interest which he had in the subject. He will miss the real enjoyment which would have been in store for him had he built his geometrical knowledge on sound mathematical principles. In order to lay a good foundation it is not necessary for the pupil to memorize lists of definitions, axioms and postulates; but rather he should be directed towards gathering accurate information about fundamental geometrical facts. For example, it is essential that a pupil know exactly what is meant by "vertically opposite angles" and what relation they bear to each other; but what profit would there be in demanding that he should be able to recite the definition for vertically opposite angles?

One very important fact to remember in motivating the teaching of grade IX geometry or algebra is the fact that the pupils in this grade require a certain amount of physical activity; - opportunity for actual movement of body and limbs. They are not capable of sustained effort in concentration over a very long period of time, but they require opportunity for periodical physical activity. This need is supplied in part by the introduction of constructive exercises in which the pupils actually do the work. It is satisfied to some extent also by extensive use of the blackboard by the pupils; but there is still another very valuable means of supplying physical relaxation for the pupils, and at the same time satisfying their desire for competitive forms of activity.

A series of short questions (ten for example) is placed on the blackboard and the pupils are instructed to work the questions and on completion to turn their books face down on the desk. The first twenty pupils finished write their names on the blackboard in order as they finish. When all pupils have finished the questions, books are exchanged and marks are assigned to the questions. Extra marks are allotted to students who answer all questions correctly and also finish in time to get their names on the board. These bonus marks are graduated according to the order of the names on the board.

Books are returned and results are compared in the following manner: All the pupils stand, and as the teacher says, "One right"; "two right"; "three right", etc., pupils with the corresponding number of correct solutions take their seats. When "ten right" is about to be reached, only those with perfect solutions are left standing, and these can be ranked in the order in which their names appear on the blackboard.

This at first glance appears to be a very ordinary bit of teaching procedure, but on closer analysis it will be found to contain several valuable features which are very effective forms of motivation.

In the first place, the pupils who succeed in getting most of the questions correct feel a certain sense of mastery over the work that has been covered and they will attack new work with confidence. Also, the very best students will be striving for perfect solutions in order to be able to continue standing until the last. Besides this, if the questions are carefully graded, some of them being comparatively easy, then even the poorest pupils will find that they have developed a certain amount of skill, and on the next occasion they will endeavour to remain standing for a longer time. No pupil wants to be the first to have to take his seat, so even the very poorest pupil in the class has a very strong motive for trying to improve his work.

The very fact that all the pupils in the class have been standing for a few minutes, and that twenty of them have made trips to the blackboard, provides an opportunity for relaxation of muscles which become tired from maintaining a sitting posture. This simple method of procedure, therefore, is extremely useful as a motivating force in the teaching of grade IX geometry. The foregoing methods for motivating grade IX geometry are all of a general nature, but the following are a number of specific methods by which certain topics may be made much more interesting and valuable to the pupil.

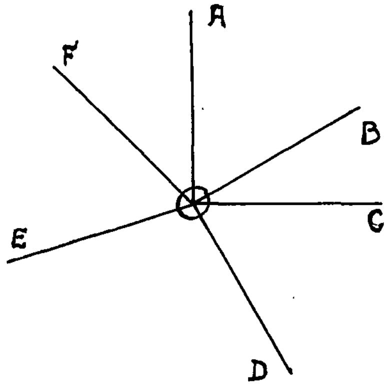
At the very beginning of the course in geometry, when commencing the study of the straight line, practice in drawing and measuring can be made extremely interesting by utilization of the pupil's interest in estimating or guessing. Questions such as the following prove very interesting to grade IX students just commencing the study of geometry:

1. By using the back of your ruler, draw a line which you believe to be 7 in. long. Turn the ruler over and measure. How many are within 1/16 of an inch? how many within 1/8"? within 1/4"? within 1/2"? etc.
2. Using the longest side of your set square, draw a line which you believe to be $4\frac{1}{2}$ in. long. Measure. How many are within 1/16 of an inch? etc.
3. Draw two columns, one for the estimated length and the other for the measured length in each of the following. Use the longest side of your set square for drawing, and your ruler for measuring.
   (a) Draw a line of any length. Estimate its length in inches (in proper column). Measure. Record measurement.
   (b) Draw a line which you believe to be 5 in. long. Measure.
   (c) Draw a line of any length diagonally on the page. Estimate its length in inches and in millimetres. Measure.
   (d) Estimate the length of your geometry book in centimetres and millimetres. Measure.
   (e) Estimate the width of your desk in inches. Measure.

   Mark each answer correct which is within $\frac{1}{4}$ in. or 5 m.m. of the measured length. Series of questions similar to the above may be arranged in competitive form and done from the blackboard.

After the study of the straight line the study of angles is commenced, and there are numerous ways by which the teaching of this topic can be very effectively motivated. During the introduction to the topic, the teacher may ask the pupils to write down the following sentence: "Geometry teaches us to bisect angles," and then see how many have the words "geometry", "bisect" and "angles" spelled correctly. Mention might be made of the little boy who wrote "Geometry teaches us to bisex angels."

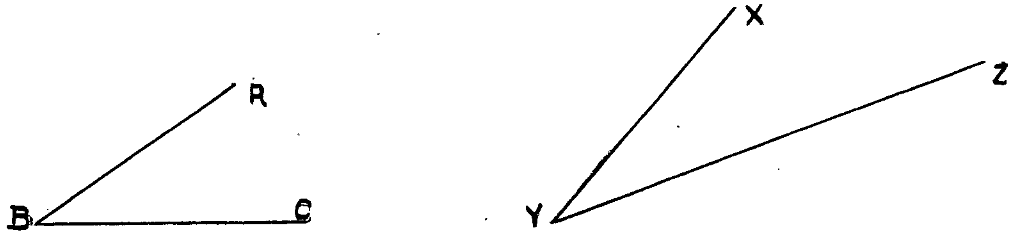
After the method for naming angles has been explained, skill in naming angles correctly can be developed in an interesting manner by seeing who can name all the angles (not reflex) on a figure such as the accompanying one. Each angle must be named only once.



Practice in drawing angles of various sizes, measuring angles already drawn, and developing a good idea of angular sizes can be given in a rather interesting manner by using a method similar to that for giving practice with straight lines. After explaining the use of the protractor, a series of questions such as the following may be given and the pupils instructed to keep a record of the ones which they get correct:

1. Draw any angle ABC. Estimate its size in degrees. Measure.

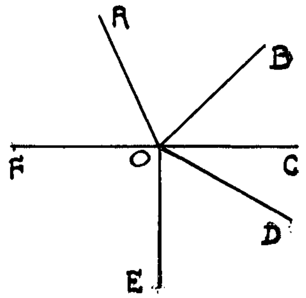
2. Draw an angle which you believe to be 65°. Measure.
3. Of the two angles on the blackboard which is the larger? How many degrees larger? (Angles such as the following may be drawn; - the smaller angle having the longer arms.)



4. Draw any acute angle on your paper. Draw another angle which you believe is exactly twice the first. Measure both angles. Multiply size of first angle by two and compare with the size of the second.
5. Draw a figure on your paper similar to the one on the blackboard. (Accompanying).
   Estimate the size of each of the following angles:

   $\angle$ AOB = $\angle$ BOC =
   $\angle$ BOD = $\angle$ BOE =
   $\angle$ DOE = $\angle$ FOA =
   $\angle$ AOE = $\angle$ AOC =

   Check by measurement.



In marking the above questions, three marks are allowed if the answer is correct to within 1°; two marks if within 2°; and one mark if within 3°. Marks are totalled and compared by the standing method.

In connection with the study of angles, use of the mariner's compass can be made to good advantage. After a short discussion of the mariner's compass, its structure and use, the teacher may enquire if there are any Boy Scouts or Girl Guides in the class. (Members of these organizations are supposed to know the thirty-two points of the compass.) If there are any present they may be allowed to display their knowledge by drawing a diagram of the mariner's compass on the blackboard. From one of these diagrams much practice can be given in angular sizes by asking questions such as the following:

1. A boat is sailing due north, and then changes its course to N N W. Through how many degrees does the keel of the boat turn?
2. Two boats are approaching the same port from different directions. One is coming from a N by N E direction and the other from a N by N W direction. How many angular degrees are there between the lines indicating their routes?

Many other similar questions on angles may be asked using the mariner's compass as a basis, and many interesting facts about navigation may be brought into the discussion of this topic. This all adds interest to a lesson on angles, and it is a typical way of motivating what might otherwise be a rather abstract and uninteresting lesson.

Another practical method by which interest may be aroused in the study of lines and angles is by making use of the pupils' sense of satisfaction at being able to follow a number of rather complicated directions and arrive at the proper destination. Finding the hidden treasure or catching the thief are forms of this type of motivation. A question such as the following is full of interest to a grade IX pupil just beginning the study of geometry:

> A jewel thief stole a diamond ring and was caught by the police after pursuing him over the following course. Can you follow them? How far from the scene of the crime was the thief when caught? (1 mile - 1 in.) From the scene of the crime he travels $1\frac{1}{2}$ miles N.E.; then 1 mile due N.; from there $\frac{3}{4}$ mile N,W.; he then turns and goes $1\frac{1}{4}$ miles S.W.; then 3 miles due E. He turns again and goes 7/8 of a mile S.S.W.; and finally he turns due E. and goes 1 mile before being caught.

Questions such as this can easily be made competitive by seeing who can catch the thief first, or who can find the hidden treasure.

The introduction to the study of parallel lines is very often a rather difficult lesson in which to arouse the interest of the pupils to any great extent. The application of carefully planned methods of motivation, however, may change this situation considerably and convert the lesson into one of exceptional interest to the pupil. After a discussion of the word "parallel" - its spelling, meaning and application to straight lines, common examples of parallel lines may be taken, such as the two edges of the desk; the two edges of the blackboard, or the two edges of a book. Definitions for parallel lines may be suggested by the pupils, and in this connection the teacher may repeat some rather humorous definitions received on examination papers at some time or another. The following are examples of such definitions: "Parallel lines are lines which run along together side by side like the car tracks, sometimes for miles, and never converse." (1) "Parallel lines never meet unless you bend one or both of them." (2) "A parallel line is one that when produced to meet itself does not meet." (3)

When the meaning of "parallel lines" is perfectly understood by the pupils, they may be given questions such as the

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(1) Received on Grade IX examination paper in 1934, Britannia High School.

(2)
(3) Alexander Abingdon, <u>The Omnibus Boners</u>. P. 69.

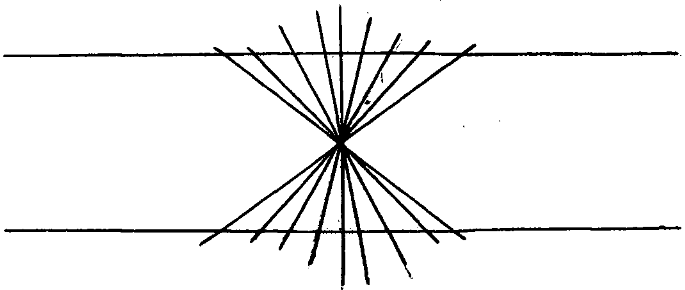
following:

1. On unruled paper draw two straight lines across the page about two inches apart, and which you believe to be parallel. Test by measuring their distance apart at several places.
2. Draw two lines diagonally across the page which you believe to be parallel. Test as before.
3. Draw a line across the page. Draw two other lines which you believe to be parallel to the first one - one about an inch above and the other about an inch below the given line. Test the two new lines to see if they are parallel.

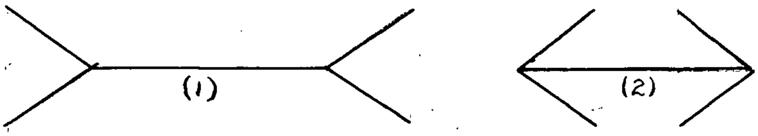
To questions such as these the teacher may add exercises from the blackboard, such as getting the pupils to say whether lines drawn on the blackboard are parallel or not, and if not parallel which way they converge.

Interest may be added to the lesson by the introduction of certain optical illusions, involving parallel lines. The following are some examples of the type of optical illusions which are suitable at this stage:(1)

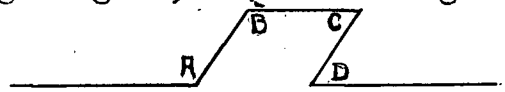
1. Are the two lines in the following diagram parallel?



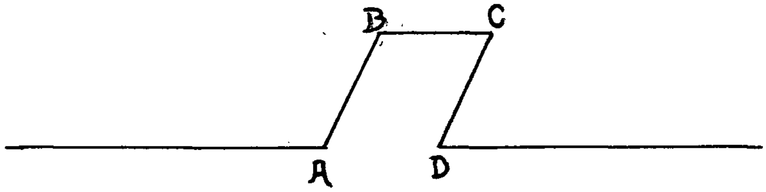
2. Which of two lines below is the longer?



3. In the following diagram, which is longest, AB or BC or CD?



(1) Morgan, Foberg, Breckenridge, Plane Geometry. P. 13.



After the essential facts regarding parallels and transversals have been studied, the inaccuracy of the former method of testing parallels may be pointed out and a more scientific method substituted; namely, the measuring of a pair of alternate or corresponding angles. The old method of measuring the distance between the lines will then be abandoned and the new method adopted.

As a little recreation at the end of a lesson on parallels, the following sentence, suggested by the word "parallel", may be given to the pupils as a spelling test. "In a cemetery an embarrassed cobbler and an harassed peddler were gauging the symmetry of a lady's tomb-stone with unparalleled ecstasy."

When approaching the study of triangles, the teacher is offered a great opportunity for allowing the pupils to experience the "thrill of discovery", and in this way increase their interest and satisfaction in the study of geometry. There are a great many facts about triangles with which the pupil in grade IX is not familiar; and if the material is presented in such a way as to allow the pupil to discover these facts for himself, then he will derive a great deal of satisfaction from so doing. Some of these facts which a pupil may be led to discover for himself in an experimental way are as follows:

1. The sum of the angles of a triangle is equal to 180°.
2. Any two sides of a triangle are together greater than the third side.
3. If the sides of a triangle are produced in order, the sum of the exterior angles so formed is 180°.
4. If one side of a triangle is produced, the exterior angle so formed is equal to the sum of the interior opposite angles.
5. The bisectors of the angles of a triangle meet at a point.
6. The perpendicular bisectors of the sides of a triangle meet at a point.
7. The three medians of a triangle meet at a point. (The pupils may experiment with paper triangles to see if the point at which the medians meet is the centre of gravity.)
8. In a right-angled triangle, the mid-point of the hypotenuse is equidistant from the three vertices.
9. In a 30°, 60°, 90° triangle the longest side is double the shortest.
10. The area of a triangle is less than the area of a square with the same perimeter.

The means of motivating the teaching of the standard constructions in grade IX geometry are many and varied. The main types of motivation which are applicable to this phase of the work are (1) The pupil's interest in building and construction work; (2) The pupil's desire for physical activity; and (3) The pupil's interest in things new. His interest in building can be used to advantage here by attacking each problem experimentally - a certain figure has to be put together, can the pupil find a method for doing it? A straight line has to be erected perpendicular to another straight line from a given point in that line. How can it be done? Can the pupil discover a method? When experimentation is over and results are compared, the comparative values of the various methods adopted by the pupils may be considered, and the best ones studied more carefully.

The physical energy expended by the pupil in working these constructions is a very important part of the grade IX pupil's activity. Without periodic opportunities for physical activity, the pupil is likely to become restless and uneasy. Most youths find it difficult to keep still for any great length of time and opportunities for muscular movement such as the one just mentioned allow the youth to satisfy his desire for physical action.

In the teaching of standard constructions it is well for the teacher to remember that these are new revelations to the grade IX student. The very novelty of such work is an incentive to the pupil to learn. In the manner of expressing in words the various methods of construction, there is a certain amount of novelty also; and interest in this phase of the work may be increased by seeing which pupils can express in the clearest and most concise form the actual work carried out in the process of construction. The pupils may be encouraged in developing an eagerness to be able to express themselves in correct geometrical language. They may be told that a certain professor wired from Vancouver to New York for the publishers to hold up the publication of his new book until a certain change was made. He wished one of the questions which began "P is a point in the line AB" to be changed to "The point P is in the line AB" because it is bad form to begin a line with a symbol. (1) This shows the exactness of

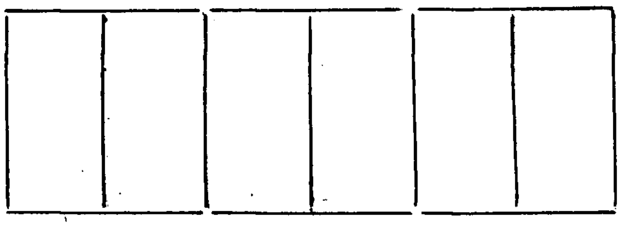
(1) Information obtained from Professor himself.

the science of mathematical expression, and the development of skill in this science sometimes fascinates the young people beginning the study of geometry.

When the essential constructions have been mastered, pupils may be allowed to construct from specifications a few of the more difficult figures involving combinations of the basic constructions. In this part of the work the pupil should be given the impression that he is now tackling an intricate bit of handiwork, and if he is successful he has developed a certain amount of geometrical constructive ability. Problems suitable for this purpose are ones such as drawing the inscribed, circumscribed, and escribed circles to triangles; constructing quadrilaterals necessitating the use of diagonals; and the development of geometrical designs suitable for crests, inlay work, linoleum and tiling patterns.

As mathematical recreations in connection with this phase of the work, the following puzzles are very suitable:

1. A farmer had his prize sheep in six pens, constructed of thirteen sections of fencing, as follows:



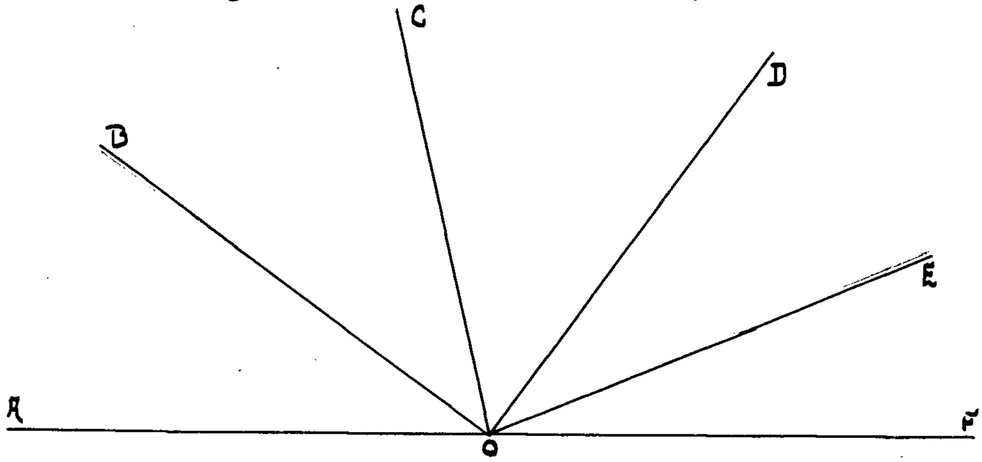
   Somebody stole one section of the fence, so the farmer rearranged the remaining 12 sections so as to still have six pens, all the same size and shape. How did he do it?
2. Given a piece of cardboard 15 inches long and 3 inches wide, how is it possible to cut it so that the pieces when re-arranged shall form a perfect square?

If tests are given at various stages of the work in grade

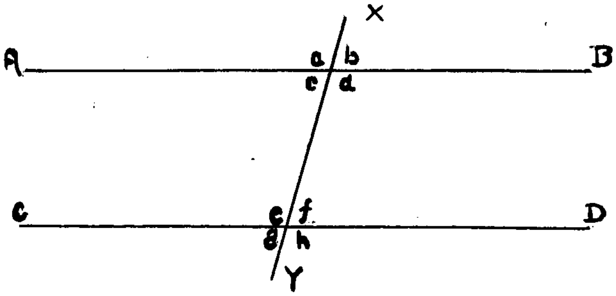
IX geometry, it is essential that the pupils develop a correct attitude toward them. The pupil should regard them as opportunities for showing his constructive and reasoning ability. In each test he has the privilege of performing more new and interesting constructions, as well as the opportunity for reasoning out certain mathematical problems. If the teaching of the work has been motivated so as to arouse the pupil's interest in the subject, and if the tests are skilfully arranged so as to follow up that motivated form of teaching, then the pupil will indeed regard these tests in the proper light and look forward with eagerness toward them.

The following are some examples of tests designed to allow the pupil an opportunity of showing his constructive and reasoning ability, and from the working of which a pupil may derive a great deal of satisfaction and enjoyment.

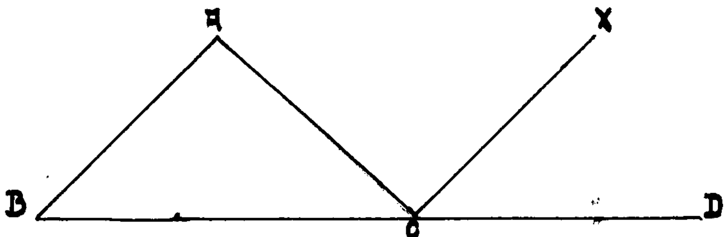
1. Give definitions of the following terms. If you cannot give definitions, explain each term clearly.
   (a) Acute angle (b) Obtuse angle.
   (c) Adjacent angles. (d) Supplementary angles.
   (e) Parallel lines.
2. Name as many angles as you can from the following diagram. Name only angles containing less than 180°. Measure to the nearest degree the size of each angle.



3. (a) Draw $\angle AOB = 42^\circ$; at O make $\angle BOC = 53^\circ$; at O make $\angle COD = 37^\circ$; at O make $\angle DOE = 48^\circ$, making each angle adjacent to the one immediately preceding.
   (b) Without protractor, how could you test quickly to see whether the final result is accurate or not?
4. (a) Draw a straight horizontal line AB 3" in length. Mark X the mid-point of AB. At X, make $\angle BXY = 50^\circ$, drawing this angle on the upper side of AB. Make XY = 1" in length. At Y, make $\angle XYZ$ alternate to $\angle YXB$ and equal to 50°. Make YZ = 2" in length. At point X, make $\angle BXH = 70^\circ$, drawing this angle on the lower side of AB. Make XH = 1" in length. At H, make $\angle XHK$, alternate to $\angle BXH$, and equal to 70°. Make HK = 2" in length.
   (b) What relation exists between the straight lines ZY and AB? Give reason for your answer.
   (c) What relation exists between st. lines AB and KH? Give reason for your answer.
   (d) What relation exists between st. lines ZY and KH? Give reason for your answer.
   (e) By using ruler and protractor only, how could you test this relationship between ZY and KH?
5.



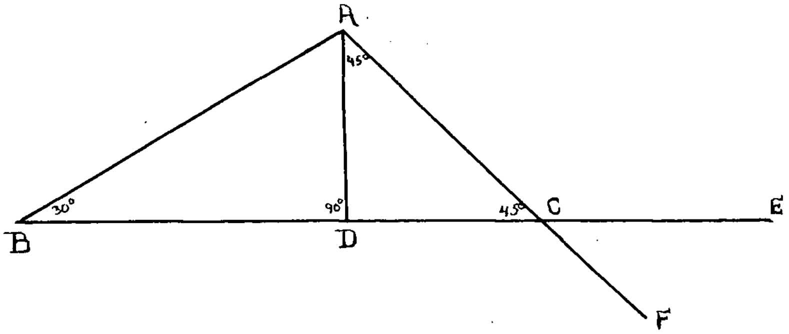
In the diagram above, AB and CD are two parallel lines.
   (a) What name is given to st. line XY?
   (b) Name a pair of alternate angles.
   (c) Name a pair of corresponding angles.
   (d) Name a pair of vertically opposite angles.
   (e) Name a pair of adjacent angles.
   (f) Name a pair of supplementary angles.
   (g) If $\angle a - 112$, what is size of $\angle f$? Do <u>not</u> obtain your answer by measuring.
6.



In the above diagram CX is parallel to BA.
   (a) Name any pairs of angles which you know to be equal, giving the reason in each case.

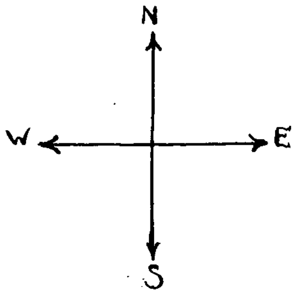
(b) If $\angle A = 85^{\circ}$ and $\angle CD = 48^{\circ}$, without measuring find the size of $\angle ACB$. Show your method of calculation.

7.



In the accompanying diagram, AD is perpendicular to BC. If $\angle DAC - 45^{\circ}$ and $\angle B - 30^{\circ}$, find without measuring the size of each of the angles ACE, BAC, FCE. Show your method of calculation in each case.

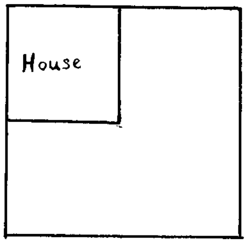
8. (a) Representing 1 mile by half an inch, draw a diagram to illustrate the following journey, using directions as indicated by the accompanying arrows:



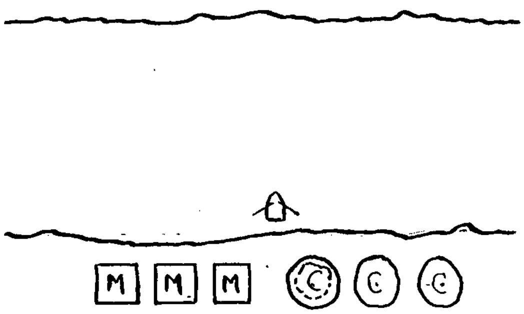
A man starts from a place A and walks to a place B which is 3 miles due east of A. He then walks from B to a place C which is 4 miles north-west of B. From C he goes to D which is 2 miles north-east of C. From D he goes $3\frac{1}{2}$ miles due south to E. From E he walks to place F, which is 5 miles south-west of E. From F he goes $2\frac{1}{2}$ miles due east to a place G.

(b) Measure the direct distance from A to G, and tell how many miles G is distant from A.

An inherent interest in games is one of the motivating forces which may be utilized extensively throughout the entire high school course. The following are a few mathematical recreations connected with the work of grade IX geometry in some way. The judicious use of these recreational problems may add much interest to the study of geometry for many of the pupils.

1. 

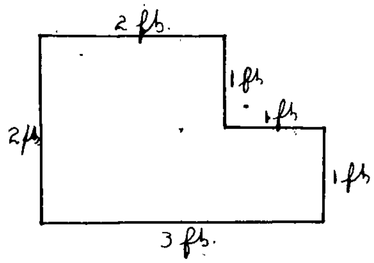
A man who owned a piece of land in the form of a square, died and left the property to his wife and four sons in the following way: The wife was to receive the quarter in the corner of the square where the house stood, and the remaining three-quarters was to be divided evenly among the four sons and all the four parts received by the sons must be the same size and shape. How was the property divided?

2. Why does it take no more pickets to build a fence down a hill and up another than in a straight line from top to top, no matter how deep the gully?
3. Given a plank 12 inches square, required to cover a hole in a floor 9 inches by 16 inches, cutting the plank into only two pieces.
4. A farmer has six pieces of chain, each piece containing five links. If it costs 2¢ for each cut and 3¢ for each weld, what will it cost him to have them made into an endless chain? (i.e. chain in the form of a circle.)
5. 

Three missionaries travelling in cannibal country, came to a river and could not get across. There were three cannibals on the bank of the river and a boat tied at the shore. The missionaries were offered the use of the boat, but the boat could carry no more than two at a time, and it was unsafe to leave more cannibals than missionaries in any one place at any one time, for the cannibals outnumbering the missionaries would devour them. All the missionaries could row, but only one cannibal (Marked Ⓒ) had learned to row. How did all the missionaries and cannibals get across the river, and what is the least number of times the boat need cross the river?

6. A man having a fox, a goose, and a peck of corn is desirous of crossing a river. He can take but one at a time. The fox will kill the goose and the goose will eat the corn if they are left together. How can he get them safely across?
7. A room is 30 ft. long, 12 ft. wide, and 12 ft. high. On the middle line of one of the smaller side walls and one foot from the ceiling is a spider. On the middle line of the opposite wall and eleven feet from the ceiling is a fly. The fly, being paralyzed by fear, remains still until the spider reaches it by crawling the shortest route. How far did the spider crawl?

8.



A girl had a piece of cloth of the shape shown in the diagram. She wished to cut it into three pieces, all of the same size and shape. How could she do it?

9. A man and his wife, each weighing 150 pounds, with two sons each weighing 75 pounds, have to cross a river in a boat which is capable of carrying only 150 pounds' weight. How did they get across?

10. Take two pennies face upward on a table and edges in contact. Suppose one is fixed and the other rolls on it without slipping, making one complete revolution about it and returning to its original position. How many revolutions about its own centre has the moving coin made?

CHAPTER IV

MOTIVATION IN THE TEACHING OF GRADE XII GEOMETRY

Part A. Development of Proper Attitudes Towards Grade XII Geometry

Whether a student in grade XII is influenced by sufficient motivating forces to encourage him to reach his highest level of efficiency in the study of geometry depends very largely upon the attitude which he adopts towards the subject. If a teacher can lead a pupil to develop a proper attitude toward the subject of geometry, then that pupil will have made the first step toward deriving the numerous benefits and pleasures latent in the study of this most fascinating subject. (1)

But what is the proper attitude toward geometry in grade XII? In the first place certain students in their final year of high school may be encouraged to regard themselves as potential mathematicians. If the teacher regards a pupil as an advanced student launching out into the depths of higher mathematics, then that pupil will endeavour to respond to this attitude, and will put forth every effort to show that he is worthy of such recognition. A higher regard for the pupil

(1) "What science can there be more noble, more excellent, more useful to men, more admirably high and demonstrative, than this of mathematics?" - Benjamin Franklin. S.I.Jones, Mathematical Wrinkles, P. 257.

undoubtedly encourages more earnest effort on his part.

Another means of encouraging a proper attitude towards geometry in grade XII, is the development of the forward-looking attitude. The teacher may find many opportunities for making slight reference to some advanced work in mathematics, and thereby show the pupils that they are really laying the foundations for the study of higher mathematics; for example, in the study of graphs the pupils' interest may be aroused to a considerable extent in the study of analytical geometry. Also, in connection with factoring, especially by the grouping method, there is an opportunity for mentioning permutations and combinations. This topic could be mentioned also in connection with evolution and involution. When a pupil is asked to write down the square of an expression such as $3x + 4y + z - 2w$, he has to use the principle of combinations in order to be able to write the terms in the answer involving twice the product of each pair. Longer expressions might be taken, and the number of groups of two calculated without actually pairing them off. By this means a student's interest and curiosity is aroused in the topics which lie just ahead of him, and he is motivated to put forth his best efforts and continue his mathematical studies.

Besides developing the forward-looking attitude in grade XII, it is important that the student become more independent, and less reliant upon the teacher. If the pupil can be led to develop the idea that the teacher is there to guide and not simply to instruct, then that pupil will develop a sense of responsibility and will feel that he himself is the one to do the thinking and the reasoning; and he will find much more satisfaction in doing his work in geometry than if this attitude of independence were not developed.

Probably the most important attitude towards grade XII geometry is the development of the sense of satisfaction through mastery. There is no greater enjoyment in any phase of school work than the thrill derived from obtaining a solution to a difficult geometrical problem which has required a great deal of concentration. If the teacher has a select group of problems at his disposal which are of just the right difficulty for the pupils at each stage of their development, he can use these to wonderful advantage by giving the pupils an opportunity for experiencing that sense of satisfaction through mastery which is such a strong motivating force to further effort. "Success begets success," and every experience of this nature acts as a "stepping-stone to higher things."

In connection with the solution of geometrical exercises, the students in grade XII may be encouraged to develop the power of visualizing geometrical solutions. If a pupil has been working at a problem for some time he may develop a very accurate mental picture of the diagram with which he is working. Even when he ceases to work at the problem he may still visualize the diagram very clearly, and if a student is encouraged to continue working at the problem from his mental picture of the diagram he might discover the solution even while walking along the street, sitting in a street car, or waiting for a friend to keep an appointment. The extreme enjoyment derived from being able to solve exercises in such a manner is a very strong incentive towards further development of that skill.

Still another phase of the development of the proper attitude towards grade XII geometry is the arousal of interest in complicated geometrical diagrams. If the teacher has a selection of exercises which are not very difficult in themselves but which produce a rather complicated looking diagram, the pupils may be led to develop a keen interest in such figures and derive much pleasure both from their construction and from their analysis. There is a great fascination about complicated geometrical diagrams, and when a student finds that he can not only draw the figure from specifications but also analyse it and prove a certain required fact about it, then his faith in his own ability is greatly strengthened, and this in turn is an incentive towards continued activity along these lines.

The following exercises are examples of the foregoing type; they produce rather complicated looking diagrams but their proofs are not particularly difficult:

1. If a triangle is inscribed in a circle and perpendiculars are drawn from any point on the circumference of the circle to the sides of the triangle, the feet of the three perpendiculars are in one straight line. (Simpson's line.)
2. To construct a triangle having a base equal to a given straight line, a vertical angle equal to a given angle, and

an area equal to the area of a given parallelogram.

The following exercises handed in by grade XII pupils indicate that some pupils are indeed interested in drawing complicated looking figures, and that this is one form of motivation in grade XII geometry. (Figures I and II overleaf.)

Another attitude towards grade XII geometry is the development on the part of the pupils of a desire for absolute perfection. There is nothing quite as stimulating to a student's enthusiasm as to be told that his solution of a problem is perfect. In the solution of geometrical exercises it is quite possible for a pupil to reach perfection - perfectly logical reasoning expressed in accurate geometrical terms. When a pupil knows that such an attainment is quite within his reach, he is encouraged to put forth special effort in order to derive the satisfaction of producing a perfect solution.

That students are interested in making their solutions perfect is evident from the accompanying solutions to exercises given to a grade XII class simply as optional home exercises. (Figures III and IV overleaf.)

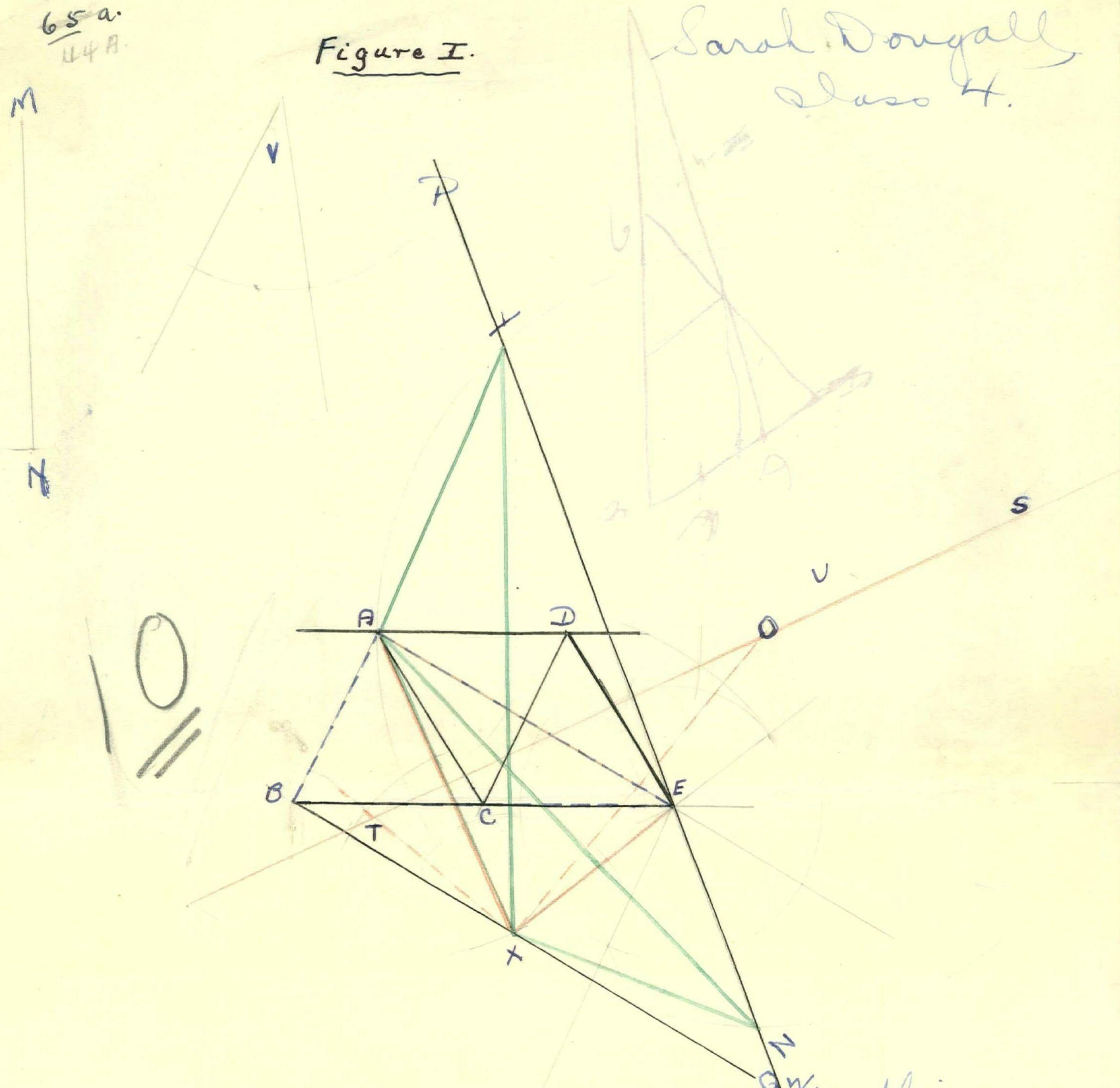
65 a.
44A.

Figure I.

Sarah Dougall
Class 4.

Print & Mount on thesis paper

10



44A

__Data__ ABCD is a parallelogram, MN a given st line and V a given ∠.

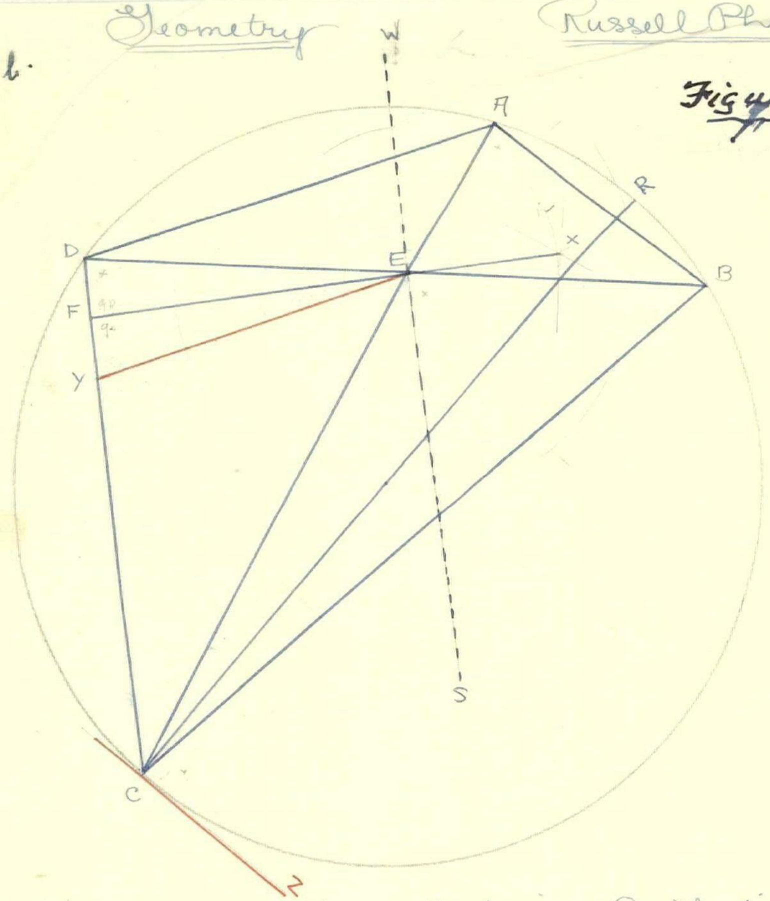
__Required to Construct__ a triangle with a base equal to MN and containing a vertical ∠ equal to ∠V and having an area equal to the ‖~~gram~~ ABCD.

__Construction__ Join AC. Produce BC and draw DE ‖ to AC cutting BC produced at E. Join AE. Draw BQ ‖ to AE and with centre A and radius equal to MN describe an arc to cut BQ at X. Join AX, XE.

Geometry Russell Phillips.

65 b.

Figure II.



10

Data - ABCD is a quad. inscribed in a ⊙. The diagonals AC, BD intersect at E. x is the centre of the circumscribing ⊙ of the △AEB. xE is joined and produced to cut DC at F. CR is a diamter of the ⊙ ABCD.

R.T.P. - ∠ DEF = ∠RCB.

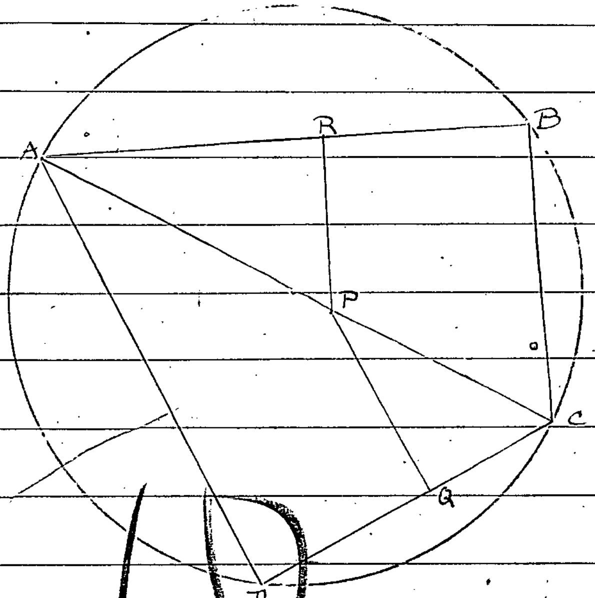
Constr. - At E make an angle FEY = to ∠RCB.
From C draw CZ ⊥ to RC. thus making CZ a tangent.
SW is a tangent of the ⊙ ABE at the point E.

Proof -

∠SEB = ∠EAB — III 14
but ∠CDB = ∠EAB — III 9
∠SEB = ∠CDB.
∴ WS ∥ to DC — I 4
FX cuts ∥'s DC, WS.
∴ ∠YFE = ∠xES.
but xES is a rt ∠.
∴ ∠YFE = 1 rt ∠.
∴ ∠DFE = 1 rt ∠. — I 1.
∠XEB = ∠DEF — I 3

Figure III

Ed Tuley - Cl 4



10

Data: ABCD is a quadrilateral inscribed in a circle of which AC is a diameter. P is any point on AC. PQ and PR are perpendiculars to CD and AB at Q and R respectively.

Req. to Prove: $\frac{DQ}{PR} = \frac{DC}{BC}$

Proof: Since AC is a diameter - - (data)

$\therefore \angle ADC$ and $\angle ABC$ = 1 rt. $\angle$ — III. 10

In the triangle ADC,

$\angle PQC = \angle ADC$ ($\angle PQC$ = 1 rt. $\angle$ - - data) — Axiom

$\therefore$ PQ is // to AD — I. 4 (2)

$\therefore \frac{DQ}{DC} = \frac{AP}{AC}$ — IV. 1 (cor.)

Since $\angle ARP$ = 1 rt $\angle$ - - (data) and $\angle ABC$ = 1 rt. $\angle$ - - Proved

$\therefore$ RP is // to BC — I. 4 (2)

Construction = Draw a straight line CA $4\frac{1}{2}$" in length. From pt. C swing an arc with radius 4" and from A another with radius 3". Call pt. where arcs intersect, B. Join CB, AB. With centres A, B and C respectively, and radii 1" describe three circles. With centre A and radius greater than $\frac{1}{2}$ AC swing an arc, and with centre C and same radius swing another arc. Now draw a straight line to pass thru the two pts. of ~~contact~~ intersection of these arcs. Repeat from pts B and C. From the pt of intersection of these two st. lines to pt A (or B or C) draw a straight line. Call the pt. of intersection of the two st lines O, and the pt. where the line AO cuts the circumference of the circle with centre A, X. Now with centre O and radius OX describe a circle, which is the required circle.

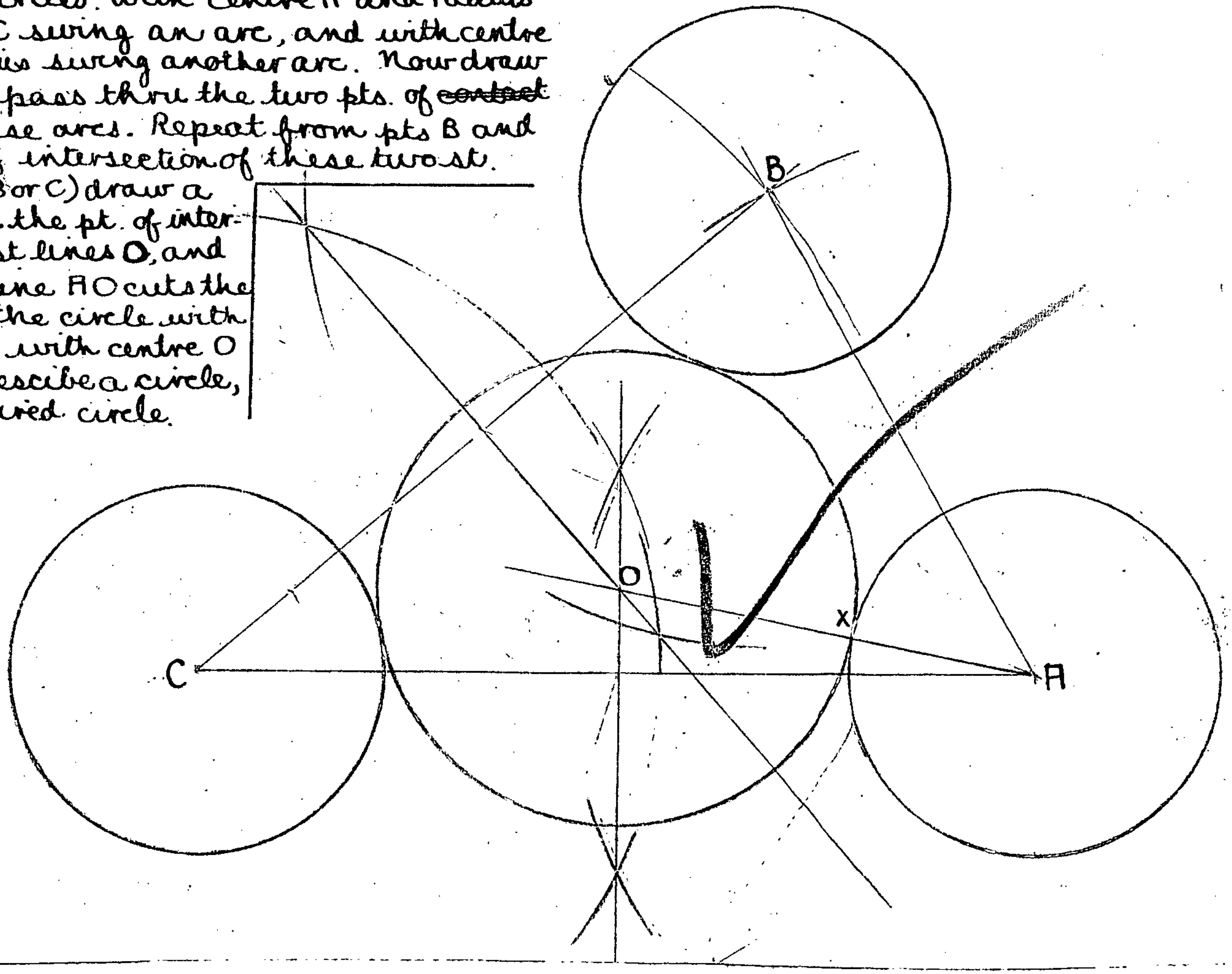


Figure IV

CHAPTER IV

Part B. Motivation in the Methods of Presentation of Grade XII Geometry

In Part A of this chapter it was pointed out that before a grade XII student can derive the maximum amount of satisfaction and enjoyment from the study of geometry he must develop the proper attitude toward the subject, and that this desirable attitude is composed of several different factors. The present section will outline how the material to be studied in grade XII geometry may be presented so as to develop in the pupils a desirable attitude toward the subject, and at the same time enable them to reach a very high level of efficiency.

In grade XII the method of teaching the prescribed theorems can be done in such a way as to motivate this section of the work to a considerable extent. In this grade the theorems may be treated in a much less formal manner than in the previous grades. When a new theorem is about to be commenced, the diagram may be placed on the blackboard, and then thoroughly discussed in order to make sure what facts are given about it and what new truth is to be derived. The pupils then analyse the problem just as in the case of an original exercise, and with a few guiding suggestions from the teacher, if necessary, the desired result is eventually reached. The class as a whole builds up a suitable statement for this new truth, and then it becomes to them a theorem or established fact which may be used in the discovery of other geometrical truths. In the lesson which follows the one in which a theorem is developed, the teacher tests the class on the fact learned in the previous lesson, and the pupils are required to show, first of all, that they understand the logical sequence of reasoning steps necessary to prove the fact; and secondly, that they are able to express this logic in conveniently arranged geometrical terms. If this attitude towards theorems is adopted, it prevents students from regarding them as isolated geometrical facts which have to be remembered, and the proofs for which have to be expressed in stereotyped form. The students should consider each theorem as a newly discovered truth, which, when established, becomes an important foundation stone in the pyramidal structure of geometry.

In grade XII the study of the required theorems is but a small fraction of the work. The larger portion of the time in this grade is spent in the solution of original exercises of various kinds, in order to develop in the pupil a certain skill in geometrical reasoning both of the inductive and deductive type. It is in the treatment of these original exercises that there is the greatest need for utilizing every suitable means of motivation. All interest in this phase of the work may quite easily be killed if the pupils are simply required to attempt the exercises in order as they occur in the text book, and then see these exercises gone over on the blackboard after a certain number of pupils have obtained solutions. This mechanical way of treating exercises robs the subject of geometry of much of the interest of which it is so full, and causes pupils to miss a multitude of pleasurable experiences which might have been theirs had a motivated method of treatment been adopted.

How, then, can the treatment of original exercises in geometry be motivated? In the first place, at the end of almost every theorem there are a number of short and comparatively easy exercises. (1) These can very conveniently be made the subject of rapid solution contests, where each exercise is taken separately and pupils compete to see who can obtain the solution first. When a pupil sees a solution he turns his book face downward on the desk and writes his name on the blackboard. After a large number of names are on the board the winner is asked to explain his solution. If his explanation is not correct, the pupil next in order has his opportunity. Each pupil keeps track of his correct answers, and after several exercises have been given in this manner, results are compared to see which pupils obtained the largest number of correct solutions.

When dealing with exercises of a little greater

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(1) See pages (214, ex. 1119-1126) (236, ex. 1222-1231) (325, ex. 1699-1704) Godfrey and Siddons - <u>Elementary Geometry</u>.

difficulty than those mentioned above [1], a slightly different procedure is desirable. In this case, pupils are allowed a few minutes in which to read the exercise and make a mental note as to what facts are given and what is required to be proved. At the end of the allotted time, all books are turned face down, and one pupil is required to draw a suitable diagram on the blackboard without the aid of the text. If the diagram is not correct, another pupil is selected and so on until a suitable figure is drawn. Pupils then examine the diagram on the board and race for a solution. Upon seeing a solution, a pupil raises his hand. When a number of hands have been raised, the teacher asks certain pupils, who apparently have obtained a solution, to give some hint which will help those not yet successful. In this way the brighter pupils feel that they are helping the slower ones to some extent, and their interest in the exercise is maintained even after a solution has been obtained. The following are some examples of exercises which are suitable for this type of treatment:

1. AD is $\perp$ to the base BC of $\triangle$ABC; AE is a diameter of the circumscribing circle. Prove that $\triangle$'s ABD, AEC are equiangular. (In this case help might be given by a pupil who sees the solution naming the theorems upon which the solution rests.)
2. If two circles touch externally at A, and touched at P, Q by a line PQ, then PQ subtends a right angle at A. (The

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(1) See pages (261, ex. 1351-1353) (277, ex. 1425-1438) (316, ex. 1678-1681) Godfrey and Siddons - Elemtary Geometry.

suggestion in this case might be for the teacher to ask a pupil who sees the solution what is usually the link between two circles which touch externally.)

When dealing with exercises of greater difficulty than either of the types mentioned above, it is still desirable to avoid mechanical treatment of one exercise after another taken consecutively from the text book. The thorough analysis of one good exercise, from the solution of which the brighter pupils can derive a great deal of satisfaction, and from the analysis of which the poorer pupils may gain much assistance in learning how such exercises are attacked, is much better than an incomplete survey of several exercises for none of which a thorough analysis is given. One means of carrying out this plan is for the teacher to assign one suitable exercise each day for home solution. If a solution is obtained, it is written out and handed in at the beginning of the next geometry lesson. The work handed in is examined by the teacher and marked (probably out of 10), and the marks are read out when the papers are returned during the following lesson, at which time a thorough analysis of the problem is carried out. The marked exercises are retained by the pupils, after the marks have been recorded by the teacher. At the end of each month the marks for each pupil are totalled and read to the class.

This method of procedure is full of motivating forces, and it encourages each pupil to do his utmost to develop his skill at solving geometrical problems. The teacher should by no means make the pupil feel that this is compulsory homweork which must be done in order to avoid some penalty, but each exercise should be presented as a new opportunity for the pupil to try his skill at geometrical reasoning.

That pupils will respond favorably to such a method is evidenced by the results of an experiment along these lines conducted by the writer with a grade XII class in the school year 1933-4. The accompanying form indicates that although absolutely no pressure was exerted upon the pupils in order to get them to hand in solutions, the matter being entirely optional, with each one, they were eager to obtain solutions and hand them in to be marked: (Table I.)

Another motivating force can be added to the solution of exercises according to the plan outlined above by keeping before the class models of perfection in reasoning and style. This may be done by picking out examples which have been exceptionally well done and passing them around the class so that each pupil may have an opportunity of examining a model solution. A pupil derives immense satisfaction from having his work passed around the class as a model, and it is a great incentive to him to keep up his good work, and to the others to try to emulate him. That this is actually the case was obvious in the experiment referred to immediately above, and the accompanying solutions are examples of work which undoubtedly is the result of such motivation. Both of the following examples are home exercises handed in by pupils after

Pupil	Daily 50A Mark
A.	5 10 10 10 10 10 10 10 10 10 10 5 10 10 10 7 10 10 10 10 10 10 10 10 10 10 10 10 10 10 10
B.	5 10 10 10 7 10 5 5 5 9
C.	10 10 10 10 10 10 10 10 10 10 5 10 10 10 10 10 5 10 10 10 10 10 10 10 10 10 10 10 10 10 7 10 10 10 10 5 10 10 10
D.	10 10 10 10 10 10 10 10 10 5 10 10 10 10 10 absent 10 8 10 10 8 10
E	10 10 5 10 10 10 10 10 7 10 10 10 10 8
F.	10 10 10 5 10 10 10 10 10 10 10 10 3 10 7 10 10 7 10 10 10 10 10 10 6 7 10 8 6 7 10 9
G.	10 10 10 10 10 5 10 5 10 10 10 10 10 10 5 10 10 10 10 7 10 10 10 7 9 7 7 10 10 6 10 10 10 10 10 10 9 10
H.	10 10 10 10 10 5 10 10 10 10 10 10 10 10 7 10 10 10 7 10 10 10 10 10 10 10 10 10 10 8 10 5 10 10 10
I.	10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 7 10 10 10 10 10 10 7 10 10 10 10 10 10 10 10 10 10 10 10 8 10 5 10 10 10
J.	10 10 10 10 10 10 5 5 10 10 5 10 10 10 10 7 10 5 10 5 10 8 8 10 7
K.	5 10 5 10 10 10 7 10 10 10 10 10 10 10 10 10
L.	10 5 10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 7 10 10 10 10 10 10 7 10 10 10 10 10 10 10 10 7 10 10 absent.
M.	10 10 10 10 10 10 10 10 10 10 10 10 10 10 7 10 10 10 10 10 7 10 10 10 10 10 10 7 10 10 8 10 9 10 10 10 10 10 10 10
N.	10 10 10 10 5 10 7 10 10
O.	5 10 10 5 10 10 10 10 10 10 10 7 10 5 10 absent 7 9 10 8 10 8 5 10 10 10
P.	5 10 10 10 10 10 10 10 10 10 10 10 10 10 10 7 10 10 10 10 7 10 10 10 10 10 10 10 10 10 10 9 10 10 9 8 10 10
Q.	10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 7 10 10 9 10 10
R.	10 10 10 10 10 10 10 10 10 7 10 10 10 10 7 7 10 5
S.	10 10 10 10 5 10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 6 10 10 10 10 10 10
T.	10 10 10 10 10 10 10 10 10 10 10 10 10 5 10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 9 8 10 10 10 9 10 10 10 10 10 10 10 9 10
U.	10 10 10 5 10 10 10 10 5 10 7 10 10 10 10 10 10 10 10 10 10 10 10 10 6 10 8 7
V.	5 5 10 10 10 10 10 10 10 5 5 5 5 10 9 10 10 8 10 10 10 9 10
W.	10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 absent. 10 8 10 10 9 10 10 10 10 10 10 10
X.	10 10 10 10 5 10 7 10 10 10 10 10 10 10 10 10 10 10
Y.	10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 5 7 10 10 10 10 10 10 10 10 10 10 10 10 10 10 9 10 10 10 10 9 10 10 10 10 10 9
Z.	10 10 10 10 10 10 10 7 10 10 10 10 10 10 10 5 10 10 10 7 10 10
a	10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 5 10 7 10 10 10 10 10 10 9 10 7 9 10 10 7 9 10 7 10 9 10 10 10
b	10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 7 10 10 10 10 10 10 10 10 10 10 10 10 10 10 8 9 10 10 10 8 10 6 10 10 10 8 10 10 10 10
c	10 5 10 10 10 5 10 10 10 10 10 10 10 10 10 10 7 10 10 5 5 10 10 10 10 10 10 10 10 10 9 10 10 10 10
d	5 10 10 10 10 10 10 10 10 10 10 10 10 5 5 10 10 10 10 10 10 10 10 10 10 10 10 7 10 5 8 7 9 10 6 10

previous exercises of theirs had been held up to the class as examples of work well done. (Figures V and VI, overleaf)

In the above method of teaching the solution of geometrical exercises, the teacher should encourage each pupil to file his exercises as they are returned to him. At the end of the year, then, the pupil is really amazed to think that he has accomplished such an amount of work. It is a source of great satisfaction to him to think that he was able to solve such a great number of apparently difficult exercises, and it gives him confidence to go forward and accomplish even greater things.

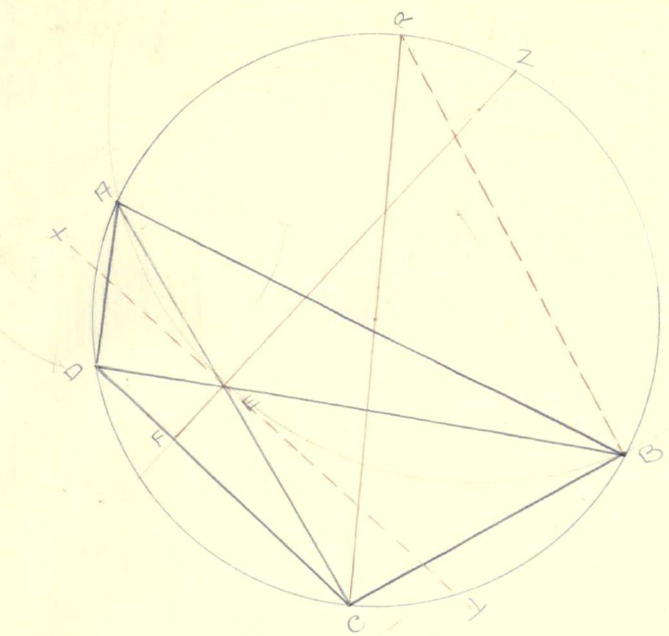
The following is a list of exercises which are suitable for treatment by the method indicated above. The order of the exercises follows the order of the topics studied in grade XII, so that when any particular question is reached sufficient material has been covered to enable the students to obtain a solution:

1. A and B are two points on opposite sides of a st. line CD; show how to find a point P in CD such that $\angle APC = \angle BPC$.
2. A and B are two points on the same side of a st. line CD; show how to find the point P in CD for which $AP + PB$ is least.
3. Show how to draw a straight line equal and parallel to a given st. line and having its ends on two given st. lines.
4. Transform a given triangle into an equivalent isosceles triangle with base equal to a given line.
5. Draw a line through a given point in a side of a triangle to bisectthe triangle.
6. Construct a triangle equal to the sum of two given triangles.
7. Construct a triangle equal to the difference of two given triangles.
8. Construct a square equal to the sum of three given squares.
9. Construct a square equal to the difference of two given squares.

Feb 19/34 ... Geometry Russell Phillips

76a
57 A

Figure V.



Data - ABCD is a cylic quadrilateral,
Its diagonals AC and BD intersect at E.
The line joining E to the circumcentre of $\triangle$AEB
cuts CD at F & ⊙ at Z. CR is the diameter of the ⊙ ABCD.

10

R.T.P. - DE · BC = EF · CR

(that is $\frac{DE}{CR} = \frac{EF}{BC}$)

Constr. - Thru E draw a tangent XEY to
the ⊙ AEB.
Join RB.

Proof - (It must be shown that the
$\triangle$s DEF and BCR are equiangular).

$\angle$CDB = $\angle$CAB = $\angle$CRB. ——— III 9.

Now $\angle$BEY = $\angle$CAB ——— III 14.

∴ $\angle$BEY = $\angle$CDB. ——— axiom

R. Phillips

$\therefore$ xy is $\parallel$ to DC. ——— I 4.

$\therefore \angle ZEY = \angle DFE.$ ——— I 5.

but $\angle ZEY$ is a rt. angle. ——— III 6.

$\therefore \angle DFE$ is a rt. angle

but $\angle RBC$ is a rt angle ——— III 10.

$\therefore \angle DFE = \angle RBC.$

Now in the Δ's DFE, BCR.,

$\angle FDE = \angle CRB$ ——— proved.

$\angle DFE = \angle RBC$ ——— proved.

$\therefore \angle DEF = \angle RCB$ ——— I 8 Cor 5.

$\therefore$ the Δ's DFE and BCR are equiangular.

$\therefore$ their corresponding sides are proportional ——— IV 3.

i.e. $\frac{DE}{CR} = \frac{EF}{BC}$

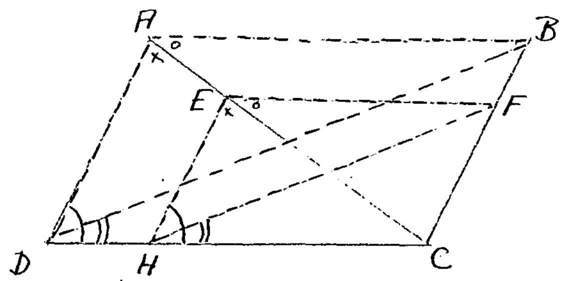
$\therefore DE \cdot BC = EF \cdot CR.$

Q.E.D.

April 2, 1935. Geometry. Figure VI. Beatrice McKenzie.

Page 381

Question 7.



Data:- ABCD is a parallelogram. From any point E in the diagonal AC, EH is drawn parallel to AD to meet DC at H; and EF is drawn parallel to DC to meet BC at F.

Required to Prove:- Δ ABD is similar to Δ EFH

Construction:- Join DB, HF.

Proof:- In the Δ ADC, since EH is $\parallel$ to AD .... data.

$\therefore \frac{DH}{HC} = \frac{AE}{EC}$ .... IV 1.

Since EF is parallel to DC .... data.

and DC is parallel to AB .... I 22

$\therefore$ EF is parallel to AB .... I 6.

In the Δ ABC, since EF is $\parallel$ to AB .... proved.

$\therefore \frac{BF}{FC} = \frac{AE}{EC}$ .... IV 1.

$\therefore \frac{DH}{HC} = \frac{BF}{FC}$

$\therefore$ HF is $\parallel$ to BD .... IV 2.

$\angle ADH = \angle EHC$ .... I 5 (ii)

But $\angle BDH = \angle FHE$ .... I 5 (ii)

$\therefore \angle ADH - \angle BDH = \angle EHC - \angle FHE$ .... axiom.

$\therefore \angle ADB = \angle EHF$

$\angle DAE = \angle HEC$ .... I 5 (ii)

$\angle BAE = \angle FEC$ .... I 5 (ii)

$\angle DAE + \angle BAE = \angle HEC + \angle FEC$ .... axiom.

$\therefore \angle BAD = \angle FEH$

In the Δ's ABD and EHF.

Since $\angle ADB = \angle EHF$ .... proved.

and $\angle BAD = \angle FEH$ .... proved.

$\therefore \angle ABD = \angle EFH$ .... I 8 cor. 5.

$\therefore$ Δ ABD is equiangular to Δ EHF.

$\therefore$ Δ ABD is proportional to Δ EHF .... IV 3

$\therefore$ Δ ABD is similar to Δ EHF.

Q. E. D.

10. Construct a square three times as large as a given square.
11. O is a point inside a rectangle ABCD. Prove that $OA^2 + OC^2 = OB^2 + OD^2$.
12. BE, CF are altitudes of an acute-angled triangle ABC. Prove that $AE.AC = AF.AB$.
13. The sum of the squares on the sides of a parallelogram is equal to the sum of the squares on the diagonals.
14. In a triangle, three times the sum of the squares on the sides is equal to four times the sum of the squares on the medians.
15. The shortest chord that can be drawn through a point inside a circle is that which is perpendicular to the diameter through the point.
16. Show how to draw a chord of a circle, equal to a given chord and parallel to a given straight line.
17. Show how to draw two tangents to a circle making a given angle with each other.
18. The bisectors of the three angles of a triangle meet at a point.
19. Show how to draw three equal circles each touching the other two, and how to circumscribe a fourth circle around the other three.
20. Find the distance between the centres of two circles, their radii being 5 c.m. and 7 c.m. and their common chord 8 c.m. (Two cases.)
21. ABCD is a quadrilateral inscribed in a circle. DA and CB are produced to meet at E; AB, DC to meet at F. Prove that if a circle can be drawn through the points A, E, F, C, then EF is the diameter of this circle; and BD is the diameter of the circle ABCD.
22. The straight line bisecting the angles of any convex quadrilateral form a cyclic quadrilateral.
23. Through a point 2 in. outside a circle of radius 2 in. draw a line to pass 1 in. from the centre of the circle. Calculate the part inside the circle.
24. Circumscribe about a circle of radius 5 c.m. a triangle having its sides parallel to three given lines.
25. Show how to construct a triangle on a given base with a given vertical angle and a given median. When is this impossible?
26. ABC is an equilateral triangle inseeibed in a circle; prove that $PA = PB + PC$.
27. AOB, COD are two chords of a circle, intersecting at right angles. Prove that

$$\text{arc } AC + \text{arc } BD = \text{arc } CD + \text{arc } DA.$$

28. A, B, C are three points on a circle. The bisector of $\angle ABC$ meets the circle again at D. DE is drawn parallel to AB and meets the circle again at E. Prove that $DE = BC$.
29. A, C are two fixed points, one upon each of two circles which intersect at B, D. Through B is drawn a variable chord PBQ, cutting the two circles in P, Q. PA, QC (produced if necessary) meet at R. Prove that the locus

of R is a circle.

30. BE, CF are two altitudes of a triangle ABC, and they intersect at H. BE produced meets the circumcircle at K. Prove that E is the mid-point of HK.

Another means of motivating the teaching of geometry in grade XII is by selecting exercises of special significance. One type of exercise which is of special significance to the pupils is an exercise which has appeared on a matriculation examination paper in a recent year. Matriculation pupils are usually keen to see if they are able to obtain a solution to an exercise which was worthy of a place on a matriculation examination paper. If a teacher compiles a list of such exercises, he can give them to the pupils one at a time during the year as sufficient work is covered to enable the pupils to obtain a solution. If the pupils are informed as to the year in which the question appeared, and the number of marks assigned to it, then a still greater interest is created in the question. The following is a list of questions, compiled from matriculation papers, which would be suitable for use as indicated herein:

1. In the isosceles triangle ABC, having AB = AC, X is any point in AB, and Y is taken in AC so that XY is parallel to BC. Prove: $BY^2 - CY^2$ BC.XY.
2. Two circles touch internally at A. A chord BC of the larger circle touches the smaller circle at D. Prove that AB:AC = BD:DC.
3. The base and vertical angle of a triangle are given. Find the locus of the intersection of lines drawn from the ends of the base perpendicular to the opposite sides of the triangle.
4. With a circle of radius r, draw two circles of radii r_2 and r_3 which touch each other externally and both of which touch the circle of radius r, internally.

5. DEF is a triangle inscribed in a circle with centre O. The diameter perpendicular to EF cuts DE at P and FD produced at Q. Prove that OE is the mean proportional between OP and OQ.
6. AB and CD are two diameters of the circle ADBC at right angles to each other. EF is a chord such that the straight lines CE and CF, when produced, cut AB produced in G and H respectively. Prove that the rectangle contained by CE and GH is equal to the rectangle contained by EF and CH.
7. ABC is an equilateral triangle and D is any point in the base, BC. On the base produced (both ways) points E and F are taken such that the angles EAD and DAF are bisected internally by AB and AC respectively. Show that the triangles ABE and ACF are similar and that $BE.CF = BC^2$.
8. Two circles touch one another externally at A; BA and AC are diameters of the circles. BD is a chord of the first circle which, when produced, touches the second at X, and CE is a chord of the second circle which, when produced, touches the first at Y. Prove that BD.CE = 4DX.EY.
9. If two tangents at the ends of one diagonal of a cyclic quadrilateral intersect on the other diagonal produced, the rectangle contained by one pair of opposite sides of the quadrilateral is equal to that contained by the other four.
10. D, E, F are the mid-points of the sides BC, CA, AB of a triangle ABC. AL is an altitude. Prove D, E, F, L are concyclic.
11. AB is a fixed chord of a circle; CD is a diameter perpendicular to AB. P is a variable point on the circle; AP, BP cut CD (produced if necessary) in X, Y respectively. If O is the centre of the circle, prove OX.OY is constant.
12. ABCD is a quadrilateral inscribed in a circle; its diagonals AC and BD intersect at E. The line joining E to the circumcentre of the triangle AEB cuts CD in F. If CR is a diameter of the circumscribing circle of the quadrilateral ABCD, prove that the rectangle contained by DE and BC is equal to the rectangle contained by EF and CR.
13. A and B are fixed points on a circle, and PQ is any chord of constant length. Find the locus of the point of intersection of AP and BQ. Give a proof.
14. Circles are described on the sides of a right-angled triangle as diameters. Through the right angle at A, between the arms AB and AC, a straight line APQR is drawn cutting the three circles in P, Q, R respectively. Prove that AP is equal to QR.
15. Show how to bisect the triangle whose sides are 2, $2\frac{1}{2}$ and 3 inches long by a straight line parallel to the longest side.
16. The sides BA, CD of a cyclic quadrilateral ABCD are produced and intersect at an angle of 30^o, and the sides DA, CB when produced intersect at an angle of 40^o. Calculate all the angles of the quadrilateral.
17. Two circles intersect at P and Q. Through P a straight

line DPE is drawn terminated by the circumferences at D and E. The bisector of the angle DQE meets DE at F. Prove: (a) The angle DQE is constant. (b) The locus of F is a circle.

18. Let ABC be a triangle. Draw AD, BE perpendicular to BC, CA respectively, meeting at P. Join CP and produce it to cut AB at F. Prove:-
(a) Angle DPC = angle DEC = angle FBD.
(b) CF is perpendicular to AB.

The motivating power of tests is another factor which must be taken into consideration when discussing motivation in the teaching of geometry. If the teaching of theorems and exercises has been motivated sufficiently to bring each pupil up to his maximum level of efficiency, the pupils will regard the working of geometry tests simply as opportunities for showing their skill in geometrical reasoning. However, it is important that the test be such as to encourage the pupils in their good work, and in no way give them a feeling of discouragement. Tests could be set which would be too difficult for even the best in the class, and such tests as these would undermine the confidence of the pupils and have a detrimental effect upon their study of geometry. The ideal test is one which is difficult enough to give the best pupil an opportunity to show his mathematical ability, and at the same time easy enough to allow all pupils in the class to experience a certain amount of satisfaction through mastery. A carefully graded test of a somewhat objective nature appears to satisfy these conditions most suitably. The following are examples of tests which might be given towards the end of the school year, or parts of which might be given at the conclusion of certain sections of the work:

Test I - (May be given at the end of the work in circles.)

Completion Test [1]

1. The greatest chord in a circle is the _______________.
2. The largest central angle has __________ degrees.
3. The line that touches a circle at only one point, however far produced, is a _______________.
4. If AB and CD are two chords of a circle each 10" from the centre, they are _______________.
5. A quadrilateral ABCD is inscribed in a circle. The angle A = 80°, and the angle B = 90°. The angle C = _____ degrees and the angle D = ______ degrees.
6. If A is any point within a circle, the shortest chord through A is _______________ to the diameter through A.
7. AB and CD are two diameters of a circle, then ABCD is a _______________.
8. If the central angle AOB is 60°, and the radius OA is 20", the chord AB is ______ inches.
9. An inscribed angle and a central angle intercept the same arc. The central angle is __________ the inscribed angle.
10. A chord AB meets a tangent AT at an angle of 50°. The angle in the minor segment cut off by AB is _____ degrees.

Test II - (At the end of circles)

True - False Test [2]

1. An angle is inscribed in an arc of 60°. The angle contains 60°. _______
2. If two circles are equal or unequal, angles inscribed in arcs of the same number of degrees are equal. _______
3. An angle inscribed in an arc of 200° is acute. _______
4. An angle inscribed in an arc of 100° is obtuse. _______
5. If one of two arcs intercepted by two parallel lines is 25°, the other is 25°. _______
6. A central angle has the same number of angular degrees as its arc has of arc degrees. _______
7. If two circles touch externally, the line of centres is equal to the sum of the radii. _______
8. If two circles are concentric, all tangents to the smaller circle, cut off by the larger circle are equal chords of

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(1) Morgan, Foberg, Breckenridge, Plane Geometry.
(2) Morgan, Foberg, Breckenridge, Opcit.

the larger circle.

9. If a circle circumscribes a triangle, the triangle is equilateral. ________
10. A diameter which bisects a chord is perpendicular to the chord. ________
11. A quadrilateral inscribed in a circle has its opposite angles supplementary. ________
12. A trapezium inscribed in a circle is isosceles. ________
13. If two chords of a circle bisect each other they are perpendicular. ________
14. Two chords which intersect in a circle are equal. ________
15. An angle inscribed in a semicircle contains 89°60'. ________
16. A parallelogram inscribed in a circle is a rectangle. ________
17. A circle can be circumscribed about a rectangle. ________
18. Two concentric circles have radii 4 in. and 6 in. respectively. A circle of radius 5 in. may be drawn to touch both circles. ________
19. If two circles touch externally, they have but two common tangents. ________
20. If a circle can be inscribed in a quadrilateral, the quadrilateral is a parallelogram. ________
21. Through a point within a circle, it is always possible to draw a chord which is bisected by the point. ________
22. A circle can always be constructed to touch each of three given lines. ________
23. If the angle between two tangents is 60°, the length of the chord joining the points of contact is equal to the length of each tangent. ________
24. If two circles touch a third circle they also touch each other. ________
25. Two circles which touch the same straight line at the same point touch each other. ________
26. A circle of radius 4" has twice the area of a circle with radius 2". ________
27. If two tangents are drawn to a circle from an external point, the angle subtended at the centre by the two points of contact is supplementary to the angle between the tangents. ________
28. The sum of the squares on two tangents drawn from an external point is equal to the square on the line joining the external point to the centre. ________
29. If two circles touch internally they have but one common tangent. ________
30. In a circle of radius 3 in., a chord 2 in. long is twice as far from the centre as a chord 4 in. long. ________

Test III. (At end of circles.)

Multiple-Choice Test. (1)

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(1) Morgan, Foberg, Breckenridge, Opcit.

In each of the following, tell which answer you believe to be correct and why.

1. If an angle is inscribed in an arc of 150°, the angle contains 150°, 75°, 300°.
2. If an inscribed angle intercepts an arc less than a semicircle, the angle is acute, obtuse, right, straight.
3. One angle of a cyclic quadrilateral is 70°. The opposite angle is 102°, 70°, 20°, 110°, cannot tell.
4. Two tangents are drawn to a circle from a point p. They include between their points of contact an arc of 100°. The angle between the tangents is 100°, 200°, 80°, 70°, cannot tell.
5. Two chords are equal if they are parallel, if they are the same distance from the centre, if they bisect each other, cannot tell.
6. Two chords which intersect in a circle are equal, if they make equal angles with the diameter through their point of intersection, if the figure formed by joining their ends is an isosceles trapezium, if they bisect each other.
7. A perpendicular to a diameter at its extremity is a secant, a chord, a sector, a tangent, a segment.
8. Two secants drawn from a point P form an angle of 30°. The larger intercepted arc is 80°. The smaller arc is 40°, 60°, 20°, 30°.
9. An angle formed by a tangent and a chord which passes through the point of contact has as many degrees as the intercepted arc, half the sum of the intercepted arcs, half the intercepted arc, half the difference of the intercepted arcs, cannot tell.
10. A parallelogram inscribed in a circle is a rhombus, a square, a rectangle, a rhomboid.

MIDDLE SCHOOL GEOMETRY (1)

Time $2\frac{1}{2}$ hours

NOTE: Please read carefully the instructions given before attempting the paper.

(1) Do the questions in order.
(2) Do not spend too long at any one question. Pass on to the next question and return to the unsolved questions after completing the paper.
(3) At the close of the examination, hand this paper to the presiding examiner.

A. CONSTRUCTIONS REQUIRED TO SOLVE PROBLEMS:

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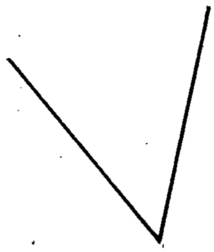
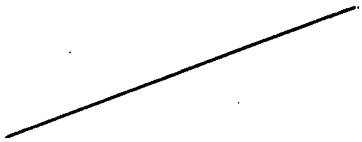
(1) Example of good geometry test as published in findings of a committee appointed to investigate types of examinations suitable for High Schools in Ontario.

NOTE: (1) Make the constructions indicated on the diagrams below each question.
(2) The figures should be neat and approsimately correct; absolute accuracy is not required.
(3) Ruler and compasses are the only instruments to be used.
(4) All construction lines should be clearly shown.
(5) No written statement is necessary in this part of the paper.
(6) In drawing parallel lines use your eye and the ruler; in other constructions show all construction lines.

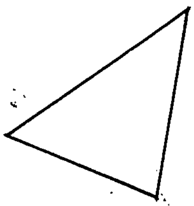
1. Bisect this angle.



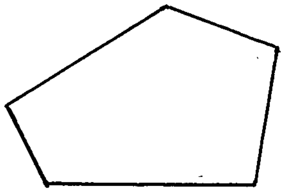
2. Draw the right bisector of this line.



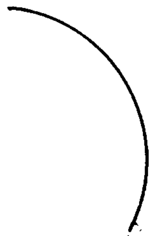
3. Construct a rectangle equal in area to this triangle.



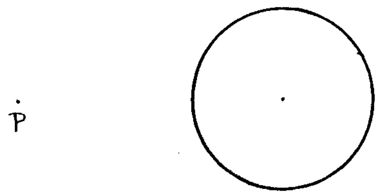
4. Construct a triangle equal in area to this polygon.



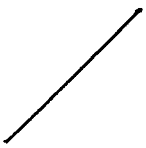
5. Find the centre of the circle of which the arc below is a part.



6. Draw tangents from P to this circle.

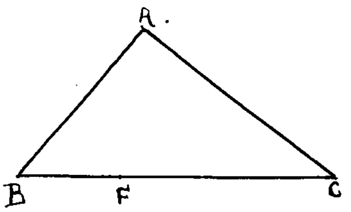


7. Cut off 3/7 of this line.

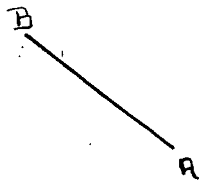


8. Find a line which will be a mean proportional between these two lines.

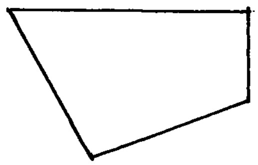
9. Draw a line parallel to BC so that the part between AB and AC may equal BF.



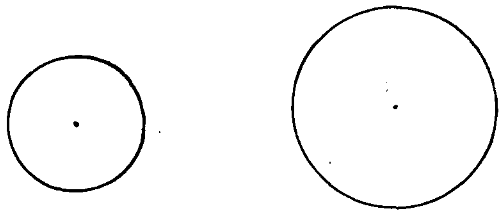
10. Draw a perpendicular to AB at A without producing AB.



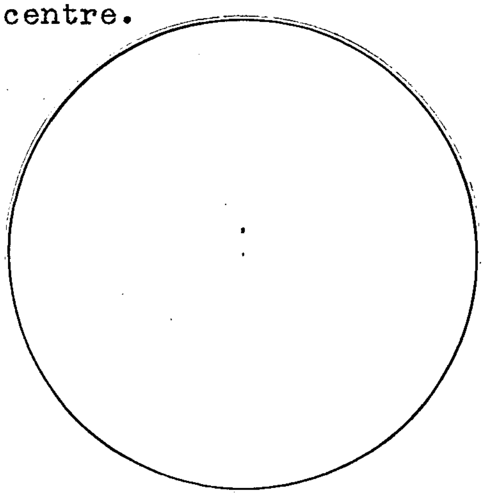
11. From the quadrilateral, cut off a part similar to it and = 4/9 of its area.



12. Draw a transverse common tangent to these circles.



13. The radius of this circle is $1\frac{1}{4}$ inches. Place in it a chord 2 inches long and calculate its distance from the centre.



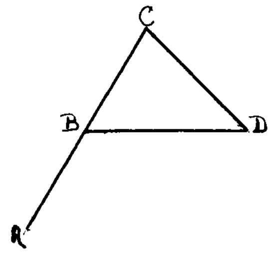
14. Draw a straight line $\sqrt{8}$ inches long.

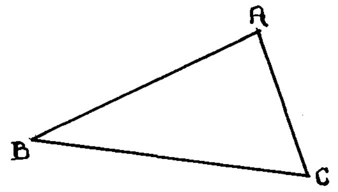
B. CONSTRUCTIONS REQUIRED TO PROVE THEOREMS:

See the instructions under section A. Lines or angles made equal should be so marked on the figure. No proof is required.

1. Make the construction that will prove angle ABD greater than angle BDC.

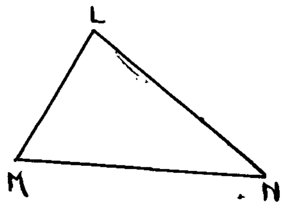
2. If BC is greater than AC then the angle A is greater than the angle B.

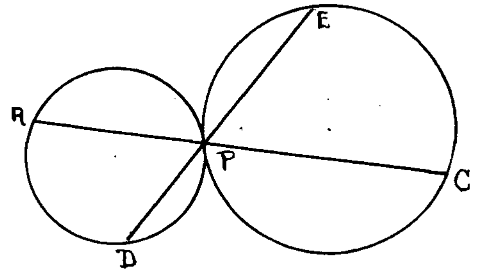




3. ML + MN is greater than LN.

4. The circles touch at P to prove AD parallel to EC.

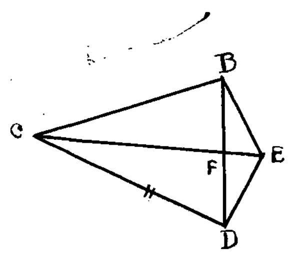




C. PROOFS OF THEOREMS:

NOTE: Pupil will make on the figure any construction necessary and will write the proof only, in as concise a form as possible.

Example: In the figure below BC = DC and BE = DE, prove CE the right bisector of BD.

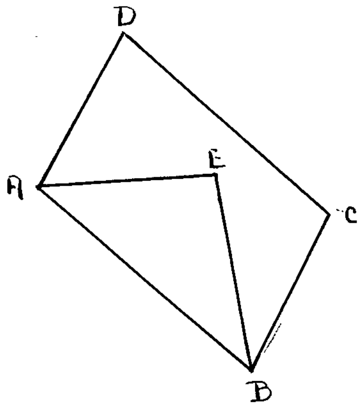


Proof

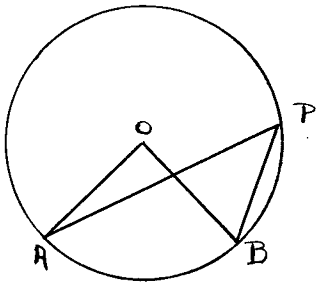
In $\angle^{s}$ BCE and CDE
BC = CD
BE = ED
CE = CE
$\angle$BCE = $\angle$ECD.

In $\angle^{s}$ BCF and DCF
BC = CD
CF = CF
$\angle$BCF = DCF
BF = FD and $\angle$BFC = DFC = 1 Rt. $\angle$.

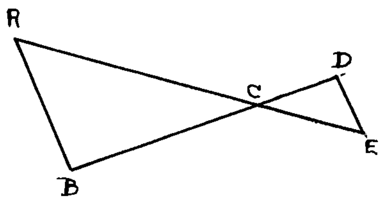
C.1. AE and BE bisect the angles A and B of the parallelogram. Prove angle AEB = right angle.



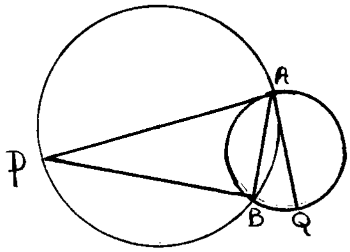
2. Prove that angle AOB is double the angle APB. Given O the centre and P on the circumference.



3. Given DE parallel to AB. Prove $\frac{AE}{CE} = \frac{BD}{CD}$.



4. AP is a tangent to the circle ABQ and AQ is a tangent to the circle ABP. Prove angle ABP = angle ABQ.



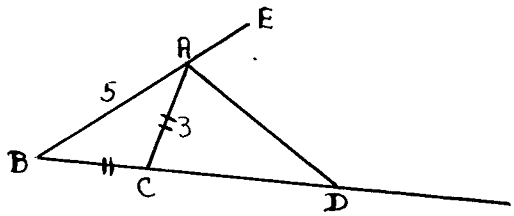
D. NUMERICAL CALCULATIONS:

1. How many degrees are there in one of the interior angles of a regular eight sided figure?
2. The exterior angle made by producing one side of a regular polygon is 20°. How many sides has it?
3. If you were asked to draw triangles (a) 3", 6", 9"
(b) 4", 5", 6"
(c) 5", 6", 8"
(d) 15", 36", 39"

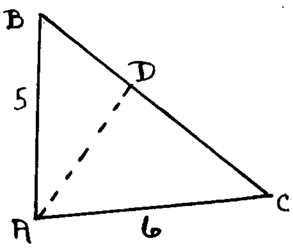
which of these would be ......
(Place a, b, c, or d in brackets)

(1) A right angled triangle ( )
(2) An acute angled triangle ( )
(3) An obtuse angled triangle ( )
(4) Not a triangle ( )

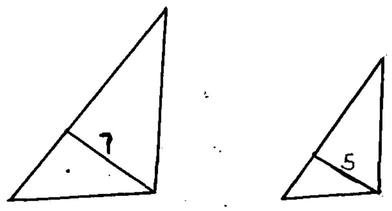
4. In triangle ABC side CB = CA and AD bisects the angle CAE. Also AC = 3 and AB = 5. Find CD.



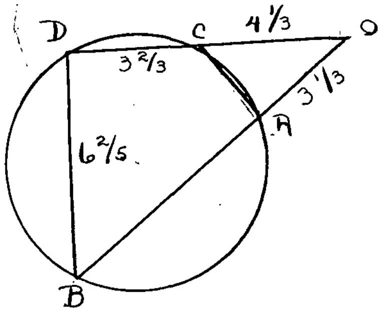
5. In triangle ABC, AB = 5, AC = 6, and BC = 7. AD is at right angles to BC. Find BD and CD.



6. In these similar triangles the perpendiculars are as 7:5. What is the ratio of the areas?



7. Two straight lines CAB and OCD cut a circle. Given that CA = 3 1/3, OC = 4 1/3, CD = 3 2/3 and BD = 6 2/5, find OB and AC.



OB =
AC =

One of the motivating forces mentioned in connection with grade XII work in geometry is the satisfaction derived through helping others. By what means can this form of motivation be brought into operation? The answer is, by instituting a pupil-teacher arrangement between the students in grade XII and those in the lower grades. This arrangement consists of providing an opportunity for pupils in the lower grades to secure help from those in the matriculation year. A teacher who teaches both in grade X and in Grade XII has an excellent opportunity in this regard, because he can refer certain grade X pupils to certain students in grade XII for assistance. If the pupil-teacher idea is encouraged it is found that grade X pupils who have friends in grade XII refer to these friends for assistance.

without actually being instructed to do so by the teacher. This help obtained from pupils in a higher grade is by no means meant to replace the help offered by the teacher but merely to supplement it. In the case where an exercise is assigned for home solution, and a pupil is unable to solve it even after repeated attempts, then instead of simply waiting until the next geometry lesson and seeing the exercise done in class, he can go to his grade XII friend and receive enough assistance to enable him to complete the exercise.

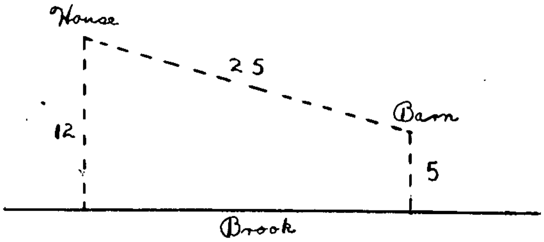
The beneficial effects of the arrangement are shared by both pupils. The senior student finds that he derives a great deal of satisfaction from being able to be of assistance to his less mature friend, and the junior pupil receives extra assistance in his work which he would otherwise not receive, and he is able to come to class prepared to give a solution to the home exercise instead of simply relying on some other member of the class to do so. In the experience of the writer, who has seen such a pupil-teacher plan in operation, it has been distinctly noticeable that pupils who take part in such an arrangement appear to take an added interest in the subject of geometry.

If the teaching of geometry in grade XII is motivated properly, it will be found that the students in this grade take a real interest in discussing geometrical problems. This in itself can by made an additional motivating force by the institution of after school discussion groups. This can be carried out in a very informal way, and the teacher may take a very small part or no part at all in the discussion. The problem for discussion may be one that has been assigned for homework; one which a pupil discovered on an old examination paper; one which was given to a bright pupil as a special privilege; or one from some outside source altogether. The diagram is placed on the board and the students cooperate informally in obtaining a solution. The writer has seen such discussions as these carried on so long after school that the caretaker has had to tell the students either to go home or be locked in the school all night. The number taking part in these discussions may be only a small fraction of the class, but nevertheless the motivating influence upon that few is indeed valuable.

The motivating force of games is one which is extremely powerful in all grades of high school. The types of games may vary considerably in the different grades, but the interest in games is just as much alive in the grade XII student as in the pupil of grade IX. The method of utilizing this motivating force in the teaching of grade XII geometry is chiefly by the judicious use of suitable mathematical recreations. It is, of course, inadvisable to make these recreations a main feature of the course, but they are of immense value in arousing the interest in things of a mathematical nature in those pupils who are less responsive to the other forms of motivation. A good selection of these mathematical recreations in geometry is a valuable asset to the teacher of mathamatics, and if they are presented at suitable times they may produce most desirable effects upon the pupils.

The following are a number of mathematical recreations suitable for use in the teaching of grade XII geometry:

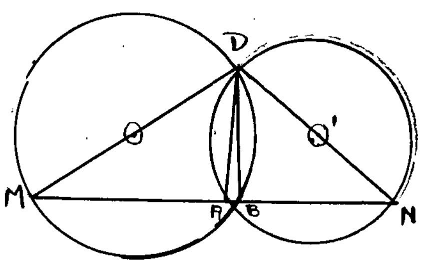
Mathematical Recreations

1. 

A house and barn are 25 rods apart. The house is 12 rods and the barn 5 rods from a brook running in a straight line. What is the shortest distance one must walk from the house to get a pail of water from the brook and take it to the barn?

2. To construct the famous Nine-Point Circle. i.e. If a circle be described about the pedal triangle of any given triangle, it will pass through the middle points of the lines drawn from the orthocentre to the vertices of the triangle, and through the middle points of the sides of the triangle, in all, through nine points.

3. To prove that it is possible to let fall two perpendiculars to a line from an external point.



Take two intersecting circles with centres O, O^1. Let one point of intersection be P, and draw the diameters PM, PN. Draw MN, cutting the circumferences at A, B. Join PA, PB.
Proof: Since $\angle$PBM is inscribed in a semicircle it is a right angle. Also, since $\angle$PAN is inscribed in a semicircle, it is a right angle.
$\therefore$ PA and PB are both $\perp$ to MN.
Where is the fallacy?

4. To prove that part of an angle equals the whole angle.

Take a square ABCD and draw MM'P, the perpendicular bisector of CD. Then MM'P is also the perpendicular bisector of AB. From B draw any line BX equal to AB.
Draw DX, and bisect it by the perpendicular NP. Since DX inter-

sects CD, the perpendiculars cannot be parallel and must meet at P. Draw PA, PD, PC, PX, PB.

Proof: Since MP is the $\perp$ bisector of CD,

$PD = PC$

Similarly $PA = PB$ and $PD = PX$

$\therefore PX = PD = PC$.

But $BX = BC$ by construction

and PB is common to triangles PBX, PBC

$\therefore \triangle PBX \cong \triangle PBC$ (3 sides = 3 sides)

$\therefore \angle XBP = \angle CBP$

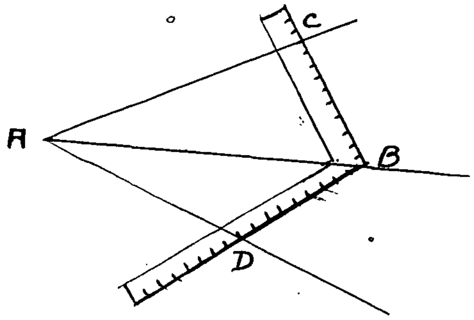
$\therefore$ The whole $\angle XBP$ equals its part the $\angle CBP$.

Find the fallacy.

5. Given a piece of cardboard in the form of an equilateral triangle. Required to cut it into four pieces that may be put together to form a perfect square.
6. Given, five squares of cardboard alike in size. Required, to cut them so that by rearrangement of the pieces you can form one large square.

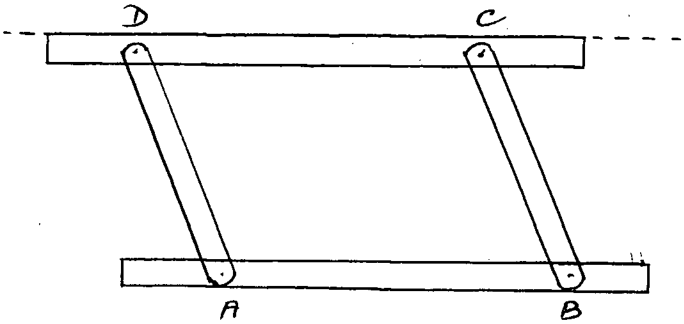
Theory without practice is as faith without works - dead," [1] says James Strachan when writing about the study of geometry in Dr. John Adam's book entitled "The New Teaching", and the statement certainly contains much truth. A boy or girl who never sees any practical application of the facts learned in his theoretical study of geometry will never appreciate the full value of his mathematical education. Moreover, the very fact of being able to apply the theoretical knowledge to real situations adds to the interest of the subject immensely and becomes a genuine motivating force. There are three very appropriate ways by which this motivating force may be applied to the teaching of geometry; namely, (1) by utilizing the geometry of the manual training shops; (2) by utilizing the geometry of outdoor measurements; and (3) by utilizing the geometry of architectural forms. The following list of examples will indicate certain specific methods by which these three forms of motivation might be brought into operation:

1. Bisecting an angle by the carpenter's square. To bisect the angle A, take AC = AD. Place the square so that BC = BD. Prove AB is the bisector of angle A.

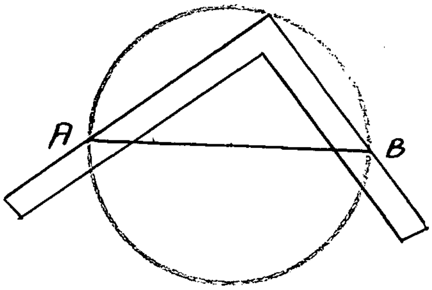


(1) Adams, J., The New Teaching. P. 108.

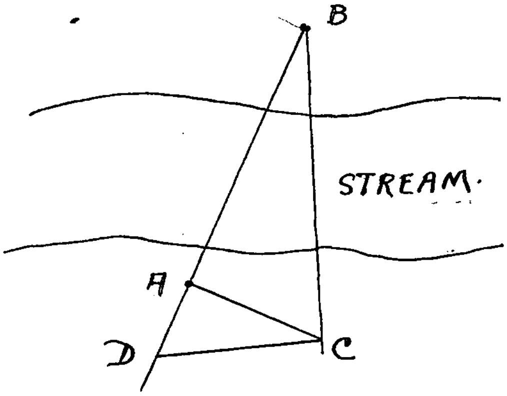
2. Explain how the parallel ruler may be used to draw lines parallel to a given line. The ruler moves freely about the points A, B, C, and D.



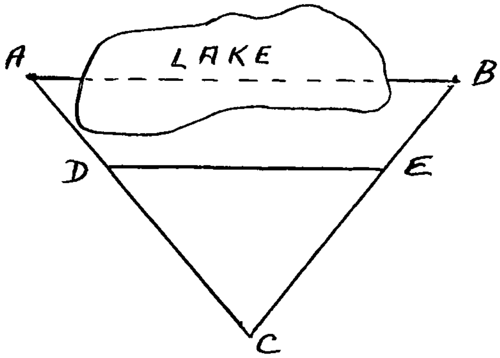
3. To find the centre of a circle the carpenter's square may be used as in the figure and the line AB drawn. Move the square and draw another diameter intersecting AB at the centre.



4. To find the distance between two points on opposite sides of a river. To find AB, lay off any convenient distance AC perpendicular to AB, and CD perpendicular to BC to meet BA produced at D. Prove $AB = AC^2 - AD$.



5. To find the distance between two points when both are accessible. Take $CA = CB$, and $CD = CE$. Prove $AB = \dfrac{DE \times CA}{CD}$



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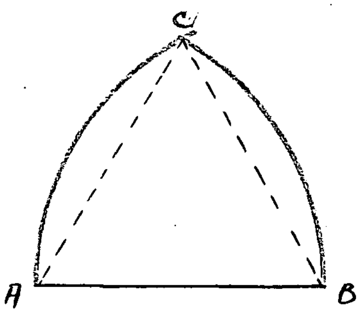
(1) Exercises from Morgan, Foberg, Breckenridge, Opcit. P.413. For more complete list see Morgan, Foberg, Breckenridge, Opcit. Appendix.

6. If A is the centre of the arc BC, B the centre of the arc AC and triangle ABC is equilateral, the figure thus formed by AB, arc AC and arc BC is an equilateral Gothic arch. AB is its span. If a window has the form of an equilateral Gothic arch with a span of 4 ft., find the area of the glass in the window.



CHAPTER V

MOTIVATION IN THE TEACHING OF GRADE IX ALGEBRA

The various forms of motivation suitable for use in the early grades of high school have already been discussed in chapter II. The present chapter will outline specific methods by which the different topics in grade IX algebra may be presented so as to utilize certain motivating forces to their fullest extent. When we know what motivating forces are at our disposal, the question is, "By what means can we bring these motivating forces into full operation?"

When commencing the study of algebra in grade IX, the pupil is naturally interested in it on account of its novelty. It is something entirely new to him to find out that in this peculiar subject he is going to use letters as well as numbers to represent quantities. He is amazed when he is told that in a very short time he will be able to add, subtract, multiply and divide by using letters of the alphabet throughout. The teacher can capitalize upon this novelty of the situation, and arouse an initial interest by allowing the pupils to realize that they are commencing a really strange and interesting study.

In the introduction to the use of symbols, the motivating force of the novelty of the subject may gradually be supplemented by the arousal of the pupils' interest in the actual situations studied. For the illustration of the use of symbols it is necessary to use various types of little problems in which letters are employed instead of numbers. If these problems are made "real" and "living" to the pupil, he will find himself intensely interested in the actual material at hand because of its association with the more pleasurable experiences of his life. Problems involving baseball, swimming, running, skating, building and numerous other pleasurable activities, make an appeal to the pupil which more abstract problems can never make. The following is a comparison between some "living" problems and some of the "lifeless" or abstract types:

1. (a) Divide 84 into three parts so that two of the parts are equal and the other part five times as great as either of the equal parts.(1)
   (b) Three boys went for a drive in an automobile a distance of 84 miles. George and Jack each drove the same distance, but Harold drove five times as far as either of the other two. How many miles did each boy drive?
2. (a) How many square feet are there in a rectangle which has adjacent sides measuring $2p + q$, and $3p-4q$ feet respectively? (2)
   (b) A boy lined out a football field which was $2p + q$ feet long, and $3p-4q$ feet wide. What is the area of the field? How many feet of sawdust did he lay making the outside lines?

It is a great incentive to industry, when a pupil feels that he is mastering something which is really somewhat difficult to master. The early problems in algebra are naturally extremely elementary, but to the beginner some of them appear to be very intricate indeed. If the teacher ~~is~~

(1) H.S.Hall, Elementary Algebra, P. 20.
(2) Ibid, P. 43.

is observant as to which problems seem to the beginner to be of a rather advanced nature, then he can be liberal with his praise when solutions to these problems are reached by the pupil; and the combined effect of the praise of the teacher and the realization that he has really made a definite accomplishment, is an exceedingly strong motivating force inducing further effort on the part of the pupil. The following are examples of problems which might appear to the beginner to be rather complicated, and for the solution of which liberal praise might be given:

1. A man who balances his accounts at the end of every quarter finds that he has three gains followed by a loss. The third gain is 4 times the second, and the second is three times the first. The loss is twice the first gain. If on the whole he gains $1,120, find the amount of the loss.
2. A boy begins to play marbles with x marbles. He wins y more and then loses x. He takes his marbles home and gives his little brother one-third of them. How many marbles has he now?
3. A circular racetrack is m yards around. One boy rides around it on his bicycle n times and another boy rides around it n more times than the first. How many yards did both boys together ride?
4. If Jack can run k miles per hour, and George twice as fast, while Henry can run only half as fast as Jack, how many miles will the three boys together cover in x hours?

When the study of the topic of substitution is reached, there is a grave tendency for the interest of the pupil to wane. The novelty of beginning a new subject has worn off to a large extent, and the topic of substitution does not lend itself very readily to the use of problems which may be made "real" to the pupil. Moreover, there are certain types of substitution questions which may quite easily cause even advanced pupils to become "tangled up".

For these reasons it is advisable to give the beginner the general idea behind the process of substitution, but not cause him to become confused by confronting him with the more technical points involved in substitution. He can proceed to work in the fundamental processes of addition, subtraction, multiplication and division without having wrestled with the technicalities of substitution, and then when these fundamentals have been mastered, he will be able to come back to substitution and understand it much more readily. For example, on page 12 of Hall's School Algebra, there is a whole page of questions on substitution, thirty-three in all, some of which are as follows:

1. If $a = 2$, $b = 1$, $c = 3$, $x = 4$, $y = 6$, $z = 0$, find the value of $\frac{4}{9}y^2 - b^3 - \frac{4}{27}c^2y + \frac{a^4}{x^2}$.
2. With same values as above, find the value of

$$\frac{a^2-b^2}{a^2b^2} - \frac{(a+b+z)^2}{(b+c-z)^2}$$

Questions such as these could very conveniently be omitted until the study of the fundamentals have been completed, and the possibility of a lifeless, uninteresting section of the work killing the enthusiasm of the pupils will be eliminated. Even in the development of the general idea of substitution, specific forms of motivation might be employed in order to maintain the interest of the pupil. Races may be held to find the value of a certain expression, such as:

$$a^2 + 2ab + 4b - 2a + b^2 \quad \text{when } a = 2 \text{ and } b = 3.$$

Also the use of unity and zero may be made rather effective, as a pupil is somewhat amazed to find that some complicated looking expression reduces right down to unity or zero.

In the teaching of addition in algebra, many of the general types of motivation suitable for grade IX may be used, such as interest in competitions, desire for good marks, desire for praise; and these may be supplemented by other methods particularly suited to the teaching of this section of the subject. In the teaching of addition there is an excellent opportunity for the use of problems which are of real interest to the pupil. Here it is possible to introduce questions which are connected with the most interesting phases of the pupil's life. A few examples of questions in addition which are admirably suited to pupils of grade IX are as follows:

1. A man on a motor trip covers x miles the first day, three times as many the second day, y miles the third day and z more miles the fourth day than on the third. Express the number of miles covered in the four days.
2. The hockey team on which Jack plays scored k goals in their first game, two less in their second game, three more in their third game than in their second, and twice as many in their fourth game as in their first. Express the total number of goals scored.
3. A boy, flying his model airplane, found that it flew m feet on its first flight and 57 feet further on its second flight. On its third flight it flew 10 feet less than on its second, and on its fourth flight three times as far as on its first. Express the total distance flown.

The working of the more mechanical types of addition questions can also be motivated somewhat by making use of the fact that a pupil derives great satisfaction from being able to work "big-looking" questions correctly. Some questions in addition occupy a large amount of space, and appear very formidable indeed, but their actual working is comparatively simple. A pupil's confidence in his own ability is greatly strengthened when he discovers that he has obtained the correct answer to such an enormous looking question as the following:

Add together the following expressions:

$$7x^6y^4z-6x^5y^3z^2+5x^4y^4z^3; \quad 5x^6y^4z-8x^4y^4z^3+2x^3y^3z^2;$$
$$-12x^6y^4z+5x^5y^3z^2+3x^4y^4z^3-3x^3y^3z^2.$$

The topic of subtraction is one of the most difficult sections of grade IX algebra to handle satisfactorily. The rule for subtraction is indeed very simple, but the explanation of the rule is a different matter. The method of working subtraction questions can be learned very rapidly by the pupils, and yet it is the source of numerous errors even in advanced classes. For these reasons it is necessary for the teacher to use to their fullest extent any means of motivating this section of the work. The difficulty which pupils experience in understanding the rule for subtraction may be overcome to some extent by the use of fitting illustrations such as the following: (1) Taking away holes in the ground - by adding soil. (2) Taking away debts - by adding money. (3) Taking away the chill from water. (4) Taking away the need for food.

As a means of impressing the rule for subtraction more firmly upon the minds of the pupils, they may be told to go home and tell their parents that they have discovered that there is no such thing as subtraction; and when the parents question the truth of such a statement, it can be explained that it is simply addition with the sign changed. In connection with this the pupils may be asked what must be added to ten to make it nine. After the answer "minus one" has been obtained the teacher may argue that it should be "one", giving his reason thus: Add I to X and you obtain IX.

As a means of overcoming the difficulty presented by the fact that the mechanical method of working subtraction questions seems so very easy to learn and at the same time very likely to cause mistakes, it is necessary to motivate the mechanical process of subtraction so as to induce the pupils to do sufficient practice in it without that practice becoming tiresome. One means of accomplishing this is for the teacher to place a certain number of subtraction questions on the board and have the pupils compete against one another to see who can obtain the largest number of correct answers in a certain specified time. This procedure may be varied somewhat by having the pupils place their names on the blackboard in order as they finish, and then it can be ascertained who was the first one finished with all answers correct. Such methods as these develop the necessary mechanical skill in subtraction and at the same time prevent the topic from becoming dull and lifeless.

In the teaching of multiplication in algebra, the difficulties are very similar to those encountered in the teaching of subtraction, and they can be overcome very largely by the same methods of motivation as were outlined in the preceding section. In multiplication, however, there is more opportunity for developing the pupil's interest in the material at hand, because in this phase of the work there is an excellent opportunity for introducing problems based on interesting life experiences. Problems such as the following give the necessary practice in multiplication and at the same time arouse the interest of the pupil immediately:

1. If an airplane can fly $3x + 2y$ miles per hour, how far will it travel in $3x-4y$ hours?
2. If the distance around a skating rink is $4a-7b$ feet, how far will a boy skate in going around $5a-8b$ times?
3. The circumference of an automobile tire is $7p-8q$ feet. How far will the automobile travel while the wheel is making $6p + 5q$ revolutions?
4. In one baseball there are $2a-3b + 4c$ yards of thread. How many yards of thread would there be in $3a + b-2c$ baseballs?

In the teaching of division the usual forms of motivation can be employed, but here there is an excellent opportunity to add interest to the topic by utilizing the motivating effect of competition. When the principles of long division have been taught, and a certain amount of practice given in its application, the teacher may select a certain fairly long question and require the first person in each row to go to the board, write the question on the board and perform the first set of operations (i.e. writing the first term in the quotient, multiplying and subtracting.) When the first pupil finishes his part, the next pupil in the row goes to the board and performs the second step, providing he thinks the first step is correct. If he has noticed any mistake in the first step, he must correct it before proceeding. This procedure is continued until each of the questions is finished correctly. The row which obtains the correct answer first is declared the winning row, and the others in order as they finish. This form of competition adds very greatly to the interest which pupils take in long division, and it encourages each one to try to develop his skill in this type of question so as to make a good showing when his turn comes to go to the board. The fact that each pupil going to the board has to correct any mistake in the question, keeps pupils following the working very carefully right from the beginning. Since long division involves short division, multiplication and subtraction, this type of contest may be used as a review of these other operations as well as practice in long division. A suitable question for such a competition would be one like any of the following:

1. Divide $6x^5-x^4+4x^3-5x^2-x-15$ by $2x^2-x+3$.
2. Divide $a^{12}+2a^6b^6+b^{12}$ by $a^4+2a^2b^2+b^4$.
3. Divide $x^7-2y^{14}-7x^5y^4-7xy^{12}+14x^3y^8$ by $x-2y^2$.

The motivating effect of special privileges may be utilized to a certain extent in the teaching of long division. The teacher may as a special privilege for work well done, allow certain pupils to try their skill at working a question which will occupy practically a whole page. A question such as, Divide $x^7-2y^{14}-7x^5y^4-7xy^{12}+14x^3y^8$ by $x-2y^2$ might be used for this purpose, and the pupils will derive considerable enjoyment from securing the answer to a question of such length.

When dealing with the removal of brackets, after the process has been explained as really a form of addition and subtraction, the rule for removal of brackets may be impressed upon the minds of the pupils by comparing the removal of brackets to the removal of the wrappings from a Christmas parcel. In the case of brackets, the innermost ones come off first, but in the case of the parcel the outer wrappings are removed before the innermost.

The removal of brackets lends itself very conveniently to the employment of competition by rows such as was carried out in long division. In this case, however, each person going to the board removes one set of brackets, and then he is followed by the next person in the row. This is a very effective method for increasing interest in the longer types of questions which might easily become very mechanical and monotonous. A question such as the following would be very suitable as the subject of a competition by rows:

$$1-a-(1-\overline{a+a^2})-\left\{1-(a-\overline{a^2+a^3})\right\}-\left[1-\left\{a-(a^2-\overline{a^3\ \ a^4})\right\}\right]$$

There is a certain fascination about removing brackets from a long and apparently complicated expression, and finding that is all reduces down to a very simple answer such as a, 1, or zero. If questions such as these are selected by the teacher, the pupils will derive that extra bit of enjoyment which comes from the working of such questions. The following question is an example of the type which might be used for this purpose:

Simplify: $2c^2-d(3c+d)-\left\{c^2-d(4c-d)\right\}+\left\{2d^2-c(c+d)\right\}$
(Answer is 0.)

In the treatment of brackets, the teacher may develop a forward-looking attitude in the pupils by showing them how knowledge of this process is a prerequisite to the study of equations. Pupils may be allowed to glance ahead to grade X work in equations to see how full of brackets some of those equations are. (1)

The various incidental bits of motivation as outlined immediately above all help to stimulate interest in the study of algebra, and encourage the pupil to develop his ability to its highest degree along these lines. Nevertheless, although all these forms of motivation be applied to the teaching of grade IX algebra, and although the pupil be encouraged by these means to develop a keen interest in the subject, this interest will not endure unless it is coupled with the satisfaction that the material covered has been thoroughly mastered. There is no motivating force more conducive to the development of lasting interest in a subject than the sense of satisfaction derived from a thorough knowledge of the subject matter studied. An opportunity for experiencing this sense of satisfaction can by given to the pupils by making use of special review questions at the beginning of each period.

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(1) Cf. Pp. 74, 76, Hall's <u>School Algebra</u>.

A short series of carefully selected questions, bearing on topics previously studied, is placed on the board. The questions should be carefully graded, the first ones being comparatively simple and the following ones of increasing difficulty. The last one or two might be really difficult ones in their class. The following is an example of a set of questions which would be suitable for this purpose after the topics of multiplication and division have been studied:

1. Multiply $3a^3b^2-5a^2b+ab^4$ by $4a^2b^3$.
2. Divide $12x^6y^2-8x^4y^3+20x^3y^8$ by $4x^3y^2$.
3. Multiply $a^2b-3ac^2+7b^2c^3$ by $-3a^3b^4c$.
4. Divide $15x^8y^6-25x^{12}y^9-35x^4y^{12}$ by $-5x^4y^3$.
5. Multiply $3a+4b$ by $5a-2b$.
6. Divide $6x^2-7x-3$ by $2x-3$.
7. Multiply $-ax^2+3axy^2-9ay^4$ by $-ax-3ay^2$.
8. Divide $15+2m^4-31m+9m^2+4m^3+m^5$ by $3-2m-m^2$.

By working a set of questions such as these, each pupil has the opportunity of determining the degree of mastery which he has developed up to this stage. If marks are assigned to these questions, and the marks are compared by the method outlined in chapter III, then the pupils become motivated to greater efforts in their study of algebra at the beginning of each period; and at the same time they are laying for themselves a strong foundation upon which to build their mathematical superstructure.

There are usually a few pupils in each grade IX class who master the work very quickly and are eager to go ahead to new work. Such pupils derive a great deal of enjoyment from knowing that they are considerably ahead of the remainder of the class, and by allowing these pupils to proceed in this manner, the teacher is motivating the work for them to a very considerable degree. In a large class, however, it would obviously be very difficult to put this principle into operation to any great extent, because of the impossibility of one teacher giving individual instruction to a large number of pupils working at different places in the test; nevertheless, the same motivating effect can be obtained if the teacher has at his disposal a large amount of extra material to supplement the work of the text. This material may be in the form of a series of more difficult types of questions on the various topics studied; a number of supplementary texts in which certain questions are marked; mimeographed sheets of review questions; or a supply of old examination papers. When a bright pupil finishes a certain assignment satisfactorily, he may be given the privilege of selecting one of these supplementary groups of questions. The following are a few examples of supplementary questions which are very valuable in bringing into operation the type of motivation referred to here:

1. Collect terms: $5a-6+9-8a+a+8$
2. Collect terms: $2x-5y+3y-3x+x-y$
3. Collect terms: $2a^2b-5ab^2-ab^2+a^2b$
4. Add: $2x+3$; $4-x$; $-3x+1$; $-5+4x$
5. Add: $2c-3d$; $-2d+c$; $5d-2c$
6. Subtract: $3x-2$ from $3-2x$
7. From $-2a+3b$ take $4b-3a$
8. Take $3x^2-5x+1$ from $2x^2+6x-3$

9. Subtract $2a^2b-3ab^2$ from $4ab^2+3a^2b$

10. Find the sum of $3x^2+2xy-y^2$; $-6x^2+3y^2$; $5xy-4y^2$

11. Multiply $(-6x^2y)$ by $(4x)$

12. Multiply $(-4cd^2)$ by $(-3c^2d^3)$

13. Find the product of $-5xy$ and 1

14. Give the square of $-4a^2$

15. Give the cube of $3x$

16. Give the cube of $-2a^4$

17. Simplify $(-5x^3)^2$

18. Simplify $(-4xy^2)^3$

19. Multiply $(-3a+4)$ by $4a$

20. Multiply $(2xy-3)$ by $-4x$

21. Multiply $(3a^2b-2)$ by $(-4ab^2)$

Simplify:

22. $-3a(-4a+6b)$

23. $(6x+3)(6x-3)$

24. $(7x-2y)(5x+y)$

25. $(3a-2b)^2$

26. $(1-3x^3)^2$

27. $(9a^2b^2)-(-9b)$

28. $\dfrac{-5x^2y^3}{-5y^3}$

29. $(-3xy+6y)-(3y)$

30. $\dfrac{x^2-6x+8}{x-4}$

31. $\dfrac{-3a+6a^3}{-3a}$

32. $\dfrac{3a^3-2a^2b}{-a^2}$

33. $\dfrac{30x^2-3x-6}{5x+2}$

34. $\dfrac{8a^3-27}{2a-3}$

35. The number 78 is divided into two parts so that one is five times the other. What is the smaller part?

36. If 90 cents be divided among A, B, and C so that A gets three times as much as C and B twice as much as C, what is A's share?

37. If $x=3$ and $y=4$, find value of $2y-xy^2$

38. If $x=-2$ and $y=1$, find value of $(x-y)^2$

39. If $a=2$, $b=3$, $c=0$, find value of $-3(b^2-ac)$

40. If $x=-1$ and $y=2$, find value of $2x^3-4y^2$

41. Find the sum of $(2x-3)^2$ and $(3x+1)^2$

42. Add $(2a-3b)(a+b)$ to $(2a+b)^2$

43. Take $3x-2y$ from zero

44. Subtract $2a^2-3b^2$ from 1

45. If a boy is 10 years old now, in how many years from now will he be y years old?

46. If a boy will be x years old 4 years hence, how old was he y years ago?

47. How many inches altogether in x yards y feet?

48. At the rate of x miles per hour, how many miles are travelled in b minutes?

49. At the rate of c miles per minute how many minutes are taken to go d miles?

50. What is the total cost in dollars of x books at y cents each. ~~What is my total loss?~~

51. I buy k books at b cents each and sell all of them at y cents each. What is my total loss?

52. Find the sum of three consecutive numbers, the smallest of which is x.

53. Find the sum of two consecutive even numbers if the larger one is y.

54. What must be added to $3x-2y$ to give $2x+5y$?

55. What must be subtracted from $4x-3y$ to leave $5x+2y$?

56. From what must $-3a+b$ be taken to leave $4a+b$?

57. What must be subtracted from zero to leave $3x$?

58. A rectangle is x feet long and y inches wide. Give the number of inches in its perimeter.

59. The perimeter of a square is y inches. Give its area in square inches.

60. Express the area in square feet of a rectangle b feet long and 4 inches wide.

61. In a school with 212 students, $3x+5$ are boys and $4x-3$ are girls. How many girls are there?

Simplify:

62. $(\frac{1}{2}a-3b)+(a-\frac{1}{2}b)$

63. $(x-\frac{1}{4}y)+(\frac{1}{2}x+y)+(x-\frac{1}{2}y)$

64. $(2a^2-\frac{1}{3})-(\frac{1}{3}a^2-1)$

65. $(\frac{x^2}{2}-\frac{3xy}{2}+y^2)(-\frac{1}{2})$

66. $(\frac{3}{2}a-2b)^2$

67. $(\frac{1}{5}x-3y)(\frac{3}{5}x+y)$

68. $(3a^2-2ab)-(\frac{a}{2})$

69. $(2x^2+2x-\frac{3}{2})-(x-\frac{1}{2})$

70. $\frac{1}{3}(5a-6)+\frac{1}{2}(2a-3)$

71. $3x(\frac{1}{2}x-1)-4x(\frac{1}{2}x-\frac{1}{4})$

72. $\dfrac{9(a+b)^2}{3(a+b)}$

73. $2x-\{3-(-x+2)+(3-2x)\}$

74. $2\{5x-2(-x+y)\}$

75. Add $5(a+b)-2(a-b)$ to $4(a+b)+3(a-b)$

76. Subtract $3(x-2)+5(y+1)$ from $2(x-2)-4(y+1)$

77. If $x=\frac{1}{3}$ and $y=-\frac{1}{2}$ find value of $\frac{1}{2}x^2y^3$

78. When $a = \frac{1}{2}$ give value of $1-3a+3a^2-a^3$

79. When $b = -\frac{1}{2}$ give value of b^3-5b^2+1

80. Expand $(x-2)^3$

81. Simplify $\frac{5x-2}{2} - \frac{3-x}{2}$

82. Simplify $\frac{3x-y}{2} + \frac{7x+3y}{4}$

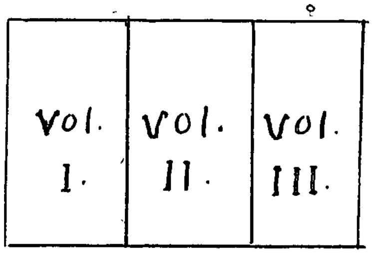
83. Fill in brackets $xa-3ya-a = a(\qquad)$
84. Fill in brackets $ax-2bx-x = -x(\qquad)$
85. If $A = 3x-2$ and $B = 2x+3$ find value of A^2-AB
86. Simplify $(a-2b)^2-3(2a-b)(a+b)$
87. I bought an article for x dollars and sold it at a gain of 5% of cost. Find the selling price.
88. I sold an article for y dollars which cost me x dollars. Find the gain % reckoned on cost.
89. If it takes y seconds to travel c feet, express the rate in yards per minute.
90. If $a = -3$, $b = 1$, find value a^2-5b^2
91. If $x = -3$, $y = 3$, find value of $^3\ (3x^2-2xy+3)$
92. Find value of $(2a+1)^a$ when $a = 2$
93. Find the sum of the numerical coefficients in the expansion of $(2x^2+xy-y^2)^2$
94. A dealer buys x books at c dollars each. He sells all of them at a uniform price, making a total gain of b dollars. For what does he sell each book?

Pupils in grade IX are ever ready to attempt the solution of a puzzle, or to join in some form of mathematical recreation. The use of such material in grade IX, if carefully selected and presented at suitable times, has a very valuable motivating effect upon the pupils. It gives them an opportunity of experiencing that "thrill of discovery" which comes from the solution of mathematical problems, and this leads to the development of that inexplicable feeling of satisfaction which comes from the working of questions of a mathematical nature. The following are some of the simpler forms of mathematical recreations, which are suitable for introduction into grade IX work in algebra:

Mathematical Recreations

1. If you drive from Vancouver to San Francisco at an average speed of 30 miles per hour, and return at an average speed of 20 miles per hour, what is your average speed for the whole journey?
2. Pat and Mike, two painters, were given the contract to paint the lamp-posts on a certain street. Pat was to paint the posts on the North side and Mike the posts on the south side. There were the same number of posts on each side. Pat arrived early and began to paint. When Mike arrived he found that Pat was painting and was just completing the third post on the south side. Mike pointed out the error to Pat and sent him over to his own side where he again commenced to paint at post number one. Mike completed his posts first, so in order to help Pat he went over and painted six posts for Pat. Now, Pat painted three posts for Mike in the morning, and Mike painted six for Pat in the evening; who painted the larger number of posts, and how many more than the other.
3. A farmer died and left his stock to his three sons. The oldest son was to receive one-half of the stock, the second son was to receive one-third of the stock; and the youngest son one-ninth of the stock. When the stock was counted there were seventeen head of cattle. How were they divided, none being killed?
4. Express all the numbers from one to twenty-one, each time using four 4's and arithmetical signs. e.g. $5 = \frac{4 \times 4 + 4}{4}$
5. If a fish weighs 13 lbs. and half its own weight, what is the weight of the fish?
6. Write down any four-digit number. Reverse the order of the digits. Subtract the smaller of these two numbers from the larger. Stroke out one of the digits in the answer. Tell me the remaining digits and I shall tell you which one you struck out.
7. Which would you prefer: a half ton of sovereigns or a ton of half-sovereigns?
8. A man went into a shoe store and bought a pair of shoes. He gave the shoe merchant a ten dollar bill, but the shoes cost only $5.00. The merchant could not change the bill, so he went over to the drug store and changed it. He gave the customer $5.00 change and the shoes. After the customer had gone, the drug clerk came over and said that the $10.00 bill that the shoe merchant had given him was a counterfeit; and it was. The shoe merchant then had to give the drug clerk a good ten dollars for the counterfeit bill. How much money as well as the shoes did the shoe merchant lose?

9. If 6 cats eat 6 rats in 6 minutes, how many cats will it take to eat 100 rats in 100 minutes.
10. The head of a fish is 9 in. long. The tail is as long as the head and one-half of the body, and the body is as long as the head and tail. What is the length of the fish?
11. If a log starts from the source of a river on Friday and floats 80 miles down the stream during the day, but comes back 40 miles during the night with the return tide. On what day of the week will it reach the mouth of the river which is 300 miles long?
12. A king has a horse shod and agrees to pay 1 cent for the first nail, 2 cents for the second, 4 cents for the third, doubling each time. What will the shoeing with 32 nails cost?
13. A hare is 10 rods in front of a hound, and the hound can run 10 rods while the hare runs 1 rod. Prove that the hound will never catch the hare.
14. A and B have an 8-gallon can of milk and wish to divide the milk into two equal parts. The only measures they have are a 5-gallon can and a 3-gallon can. How can they divide the milk?
15. I bought a horse for $90 and sold it for $100, and soon repurchased it for $80. How much did I gain by trading?
16. Three books were placed on a shelf in proper order as shown in the diagram. Each book was three inches thick including the covers, each of which was 1/8 of an inch thick. A bookworm bored a hole from the first page of volume I, straight through to the last page of volume III. How far did he travel?



CHAPTER VI

MOTIVATION IN THE TEACHING OF GRADE XII ALGEBRA

In chapter II, the various forms of motivation used in the higher grades of high school were outlined. In the present chapter those types which are especially applicable to the teaching of grade XII algebra will be discussed more fully, and specific methods will be outlined by which these forms of motivation may be utilized to their fullest extent.

In the first place, we must realize that in this grade there is ever present that great motivating force of matriculation examinations. The very fact that a pupil knows that at the end of the school year he will be required to write departmental examinations is sometimes sufficient to cause him to put forth a maximum amount of effort, and reach his limit of mastery. But the presence of matriculation examinations is by no means an ideal form of motivation. A pupil may work at the highest level of his ability in order to avoid failure, but at the same time he may have very little interest in the work which he is doing. The first step towards motivation in grade XII, therefore, should be the creation of a proper attitude towards matriculation examinations. The teacher should instil in the minds of the pupils the idea that examinations are carried on not for the express purpose of torturing pupils, but rather because they are a desirable means of allowing a pupil the opportunity of satisfying himself that he has reached a high level of mastery in the work which he has been doing. Any pupil who has taken a keen interest in his school work, and has discovered the possibility of securing a great amount of pleasure from attaining mastery over successive units of work, will look on the matriculation examinations as another opportunity for securing that stimulating sensation of satisfaction through mastery.

From this it must not be inferred that the present system of examinations for grade XII pupils is a good one, because it is obvious that matriculation examinations as at present constituted have a great many very undesirable features. Their psychological effect upon high school students is often far from beneficial and sometimes decidedly harmful. However, as long as the present system is in vogue, it is imperative that high school teachers make the best of the situation as it exists and encourage the pupils to regard examinations in such a way that they may be affected by whatever beneficial influences are contained in them.

Having created a proper attitude towards examinations, it remains for the teacher to cause the pupils to develop an interest in their work, simply for the work's sake. This can be done very conveniently in algebra by a special adaptation of the unit system. In grade XII algebra the topics studied are of such a nature that there is very little in common between them. In any one section there is but slight reference to the work of sections immediately preceding; and, consequently, there is practically no opportunity for reviewing a previous unit while studying a new one. Quadratics, indices, surds, ratio and graphs are almost complete units in themselves, and for this reason it is convenient to operate three separate unit systems during the course of the year. During the first part of the year, the various topics are studied from the text book. It would require far too much time, and be too laborious a task for a pupil to work every question in every set, but carefully selected and well-graded examples are chosen to comprise each unit of the text. When the tests at the end of each unit indicate that the majority of the pupils have attained a sufficient degree of mastery in that topic, the following unit is treated in a similar manner. By this means all the units can be covered in approximately two-thirds of the school year.

Since each unit in grade XII algebra is studied as a separate entity, when all have been completed it is necessary to have some form of review and also some means of coordinating the various units. These two objectives - review and coordination - can be reached by making use of two separate sets of unit sheets. The first set consists of a series of carefully graded questions arranged in the same order as they were studied. In this way a pupil can check up on his knowledge of the various units, and thereby ascertain the exact spot where concentrated review is necessary. In the appendix a series of questions is given which were drawn up for this purpose. (1) The pupils are allowed to work the questions as quickly as they wish, with a minimum number to be done each week. If the series is begun twelve weeks before the end of the school year, the teacher may divide the list into about ten equal parts and make one part a minimum for each week. At the end of each week the answers are given to the questions in the section just completed, and special attention is paid to those questions with which the pupils experienced difficulty.

The above-mentioned series of questions, the series for review, is supplemented by a series for coordination. This consists of ten separate sheets designed for the purpose of relating the various topics and providing a general survey of the whole subject. Each sheet contains ten questions dealing with different sections of the work, and the various types are arranged in a different order on each sheet. These questions vary from comparatively easy ones to those of considerable difficulty. A good many of these questions are chosen from previous examination papers, and if the pupils are informed of this fact they show an added interest in the questions on that account. When working on one of these sheets, if a pupil finds that he is unable to master one of the questions, he refers to his review sheets and checks up on his knowledge of

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(1) See p. 113

that particular type of question.

The series for coordination is also given in the appendix, [1] and these questions may be given during the last ten weeks of the school year at the rate of one set per week. The algebra period on one particular day, Friday for example, may be set aside for the discussion of these papers. On the first Friday, sheet number one is distributed. The pupils work these questions during the week, and hand in their solutions on the following Friday. At the end of this period they receive sheet number two. On Friday of the next week the teacher hands back the solutions to sheet number one, marked, collects the solutions to sheet number two, and hands out sheet number three. On each succeeding Friday the same procedure is followed, the teacher collecting one set of answers; handing back another set, marked; and distributing a new set for the following week. Most of the algebra period each Friday is devoted to discussing the papers which have just been returned, and in examining the questions which gave the pupils most trouble.

The adoption of this three-unit method - the unit for learning, the unit for review, and the unit for coordination - has a great motivating effect upon the pupils, and coupled with the development of a proper attitude toward matriculation examinations, it is almost sufficient motivation in itself for

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(1) See p. 124

making the grade XII course in algebra extremely interesting and satisfying. It brings into operation most effectively the motivating forces of satisfaction through mastery, and of the development of an interest in algebraical calculations as a new and delightful experience.

The three-unit system of teaching algebra in grade XII can be made even more successful by overcoming one weakness which is common to most unitary methods of instruction; namely, that when one unit is finished and a new one commenced, the knowledge of the earlier unit gradually fades unless some means is adopted for keeping in constant contact with it. This weakness can be overcome by making use of a set of review questions selected to cover all the salient points in the various units without involving very lengthy calculations. A few of these questions can be assigned for homework each day, handed in by the pupils, marked by the teacher (probably out of ten) and the marks totalled at the end of each month. The first part of each lesson (about ten minutes) may be devoted to going over the questions which the teacher has marked and is returning to the pupils. Such a system keeps a pupil constantly in touch with the material contained in previous units; and when the time comes to commence the hundred review questions, he finds them much more familiar than he would have done had he not had this daily review. The following list of questions will give an idea of the type of question suitable for this purpose:

1. Solve $5x-\left\{\frac{2x-1}{3}+1\right\}=3x+\frac{x+2}{2}+7$

2. A man bought a number of articles for $200. He kept five and sold the remainder for $180, gaining $2 on each article sold. How many did he buy?

3. Solve $5x^2-9x-4=0$.

4. The hypotenuse of a right-angled triangle is 25 in. and the perimeter is 56 in. Find the remaining sides.

5. Solve $x^2-xy=6$

$y^2-3xy=10$

6. Factor (a) $4m^4-21m^2n^2+n^4$.

(b) a^3+b^3+a+b

(c) $12x^4-27a^2x^2$.

(d) $4(2a-3n)^2-(3a-7b)^2$

7. By use of factors find the product of:

$\left[x^2-2(x-1)\right]$ and $\left[x^2+2(x-1)\right]$

8. If $x+y=1$, prove $x^3(y+1)-y^3(x+1)-x+y=0$

9. Solve $x^2-3x=0$.

10. Solve $4-9x=13x^2$.

11. Solve $\frac{6y-4x}{3x-4y}=\frac{5z-x}{2y-3z}=\frac{y-2z}{3y-2z}=1$

12. Solve $4x^2=x-1$.

13. Solve $42x^2-28c^2=25cx$.

14. Solve $x^4+2x=3x^2$.

15. Simplify $\frac{3\times 2^a-4\times 2^{a-2}}{2^a-2^{a-1}}$

16. Simplify $\left(\sqrt[5]{\frac{a^{\frac{1}{2}}x^{-2}}{x^{\frac{1}{2}}a^{-2}}}\times\sqrt[3]{\frac{a\sqrt{x}}{x^{-1}\sqrt{a}}}\right)^{-4}$

17. Solve: $\frac{1}{x^2} + \frac{1}{y^2} = 4\frac{1}{4}$

$$\frac{1}{x} - \frac{1}{y} = 1\frac{1}{2}$$

18. Solve: $7y^2 + 15xy = -68$

$$x^2 + 2xy + 2y^2 = 17$$

19. Express in its simplest form: $\frac{4(\sqrt{3}+1)}{\sqrt{3}-1} - \frac{2+\sqrt{3}}{2-\sqrt{3}}$

20. Rationalize the denominator of:

$$\frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{\sqrt{1+x^2} + \sqrt{1-x^2}}$$

21. Find the square root of: $\sqrt{48} - \sqrt{45}$

22. Solve: $\sqrt{x+a} + \sqrt{x+b} = \frac{a}{\sqrt{x+a}}$

23. The price of photographs is raised \$3 per dozen; and customers consequently receive seven less pictures than before for \$21. What was the original price of the pictures per dozen?

24. Solve: $\frac{x+1}{\sqrt{x}-1} = 3 + \frac{\sqrt{x}+1}{2}$

25. Write down the roots of:

(a) $(x-\overline{a+b})(x-\overline{a-b}) = 0$

(b) $x^2 + 2x = 0$

26. Simplify: $\left\{\frac{32a^{-5}}{b^{-10}}\right\}^{-\frac{1}{5}}$

27. Simplify: $2\sqrt{5} + \frac{10}{\sqrt{5}}$

28. Solve: $25x^2 - y^2 = 84$

$$5x - y = 6$$

29. Show the meaning of a^{-n}

30. Simplify: $\left\{\sqrt[3]{4} \times \frac{1}{\sqrt[6]{8}} \times \sqrt[12]{2^{-1}}\right\}^4$

31. Find the square of: $.2a^0b - .3ab^0$

32. Simplify: $\left\{(x^2)^{-1}\right\}^{-1} \times \left[\left\{(x)^{-1}\right\}^{-1}\right]^{-3}$

33. Rationalize the denominator of $\sqrt{\frac{7}{0.8}}$ and express in simplest form.

34. Find the square root of $17-12\sqrt{2}$

35. Find the square of $e^x + e^{-x}$

36. Simplify: $\frac{a^{\frac{4}{3}} - 2 + a^{-\frac{4}{3}}}{a^{\frac{2}{3}} - a^{-\frac{2}{3}}}$

37. Solve: $x^2 = 14 + xy$
$y^2 = xy - 10$

38. Simplify: $\sqrt{2}(5\sqrt{3} - \sqrt{2}) - \sqrt{3}(2\sqrt{2} - \sqrt{3})$

39. Simplify by removing negative indices: $\frac{a^{-1} + b^{-1}}{a^{-2} + b^{-2}}$

40. Find the value of: $x^3 + x^2 + x + 1$ when $x = \sqrt{3} + 1$

41. Simplify: $\sqrt{1 + \sqrt{1-a^2}} + \sqrt{1 - \sqrt{1-a^2}}$

42. A man has h hours at his disposal. How many miles can he ride out at r miles per hour if he must walk back at w miles per hour?

43. Find the mean proportional between $\sqrt{27} - 3\sqrt{2}$ and $\sqrt{27} + 3\sqrt{2}$

44. If $\frac{a+b+c+d}{a+b-c-d} = \frac{a-b+c-d}{a-b-c+d}$ prove $\frac{a}{b} = \frac{c}{d}$

45. Find the square root of $\frac{9}{4} + \sqrt{5}$

46. Using the scale 1 unit $= \frac{1}{5}$ inch, find the point of intersection of the graphs of $2x-3y = 24$ and $\frac{5x}{3} - \frac{y}{2} = 12$

47. Find the equation for the straight line passing through the points (3,4), (-2,5)

48. Solve for a and for n: $S = \frac{n}{2}(a + l)$

49. Solve for r and for v: $F = \frac{mv^2}{gr}$

50. Solve for x: $x(a-x) = c^2$. Give the numerical value of the roots when $a = 16$ and $c = 6$.

There are several other minor, but nevertheless important, means by which the teaching of grade XII algebra may be motivated to a fuller extent, and one of these is a careful treatment of algebraical problems. A number of students seem to develop somewhat of a dislike for problems, and become possessed with the idea that grade XII problems are tricky bits of mathematical reasoning almost beyond any ordinary student's ability. The development of this erroneous impression can be prevented by prefacing the work on problems with a few carefully selected puzzles which require considerable thought for their solution, but which are by no means out of the range of any average student. If these puzzles are presented so as to give each pupil the idea that they are a challenge to his ingenuity and resourcefulness, then he will respond to the challenge and do his utmost to obtain solutions. The following are examples of rather simple puzzles which might very well be used as a preface to a lesson on problems:

1. A blacksmith had a stone weighing 40 pounds, and a skilled mason broke it into 4 pieces whereby any number of pounds from 1 to 40 could be weighed on scales. Find the weight of each of the four pieces.
2. Two automobiles 20 miles apart are approaching each other, each travelling 10 miles per hour. A bee, which flies at the rate of 15 miles per hour starts at the radiator of one automobile and flies back and forth between their radiators until the automobiles meet. How far does the bee fly?
3. When I was born my sister was one-fourth mother's age, but she is now one-third father's age. I am now one-fourth mother's age, but in four years I shall be one-fourth father's age. How old is each of us?

4. How shall we buy 12 eggs for eighty cents, if hen eggs sell at 5¢ each, duck eggs at 7¢ each, and turkey eggs at 8¢ each, and if we buy some of each?

Puzzles like the ones given above are extremely interesting to the high school student, and he soon begins to realize that real enjoyment can be derived from attempting to solve problems of various kinds. Using such puzzles as these as an introduction, the teacher can lead the class on to the investigation of problems which involve the use of equations in their solution, and once a pupil experiences the "thrill of discovery" derived from the solution of a mathematical problem, he is eager to tackle more problems which will enagle him to experience that thrill more often.

There are many other forms of mathematical recreations, besides the strictly "problem" type, which can be used very conveniently as a means of motivating the work in grade XII algebra. The following are some examples of such:

Mathematical Recreations

1. A ship is twice as old as its engine was when the ship was as old as its engine is now. Their combined ages are 42. How old are they?
2. A farmer bought one hundred head of stock consisting of calves, sheep and lambs. The calves cost $10 each; the sheep $3 each; and the lambs, 50¢ each. Altogether the hundred head cost him one hundred dollars. How many of each did he buy?
3. My father was born on

   June $\left\{\left[\frac{16^{-\frac{1}{4}}}{(\frac{3}{2})^{\frac{1}{3}}}\right]\left[\frac{2^{a(a-1)}}{2^{a+1}}\cdot\frac{4^{a+1}}{2^{a^2-1}}\right]\left[\frac{(3^2.3.2^2)^{\frac{1}{3}}(3)^{\frac{1}{2}}}{(2.3.3^{\frac{1}{2}}.6^2)^{\frac{1}{3}}}\right]\right\}$, 1887

   What was his age on August 10th, 1935?
4. If Dr. Jones loses 3 patients out of 7; Dr. Smith, 4 out of 13; and Dr. Brown, 5 out of 19; what chance has a sick man for his life who is dosed by the three doctors for the same disease?

5. Mr. Dough, a business manager, wished to hire one of three men. In order to decide which was the smartest, he adopted the following system. He called the three men to him and showed them five pieces of paper on his desk. Two of the papers were black and three were white. He then told the men to turn around, and he pinned a paper on each man's back, putting the two remaining papers in his pocket. He told the men to look at the papers on each other's backs, and by so doing decide what color of paper was on their own backs. The first man to decide the color of the paper on his back was to come up to the desk and explain how he made the decision. If his explanation were sound he would get the position. One of the applicants was successful in determining the color of paper on his own back. Which one could it have been, and how did he decide?
6. An eagle and a sparrow are in the air, the eagle 100 ft. above the sparrow. If the sparrow flies straight forward in a horizontal line and the eagle flies twice as fast directly towards the sparrow, how far will each fly before the eagle reaches the sparrow?
7. To prove that $1 = 2$

$$\text{Let } x = 1$$
$$\text{Then } x^2 = x$$
$$\therefore x^2 - 1 = x - 1$$
$$\text{Factoring} \quad (x-1)(x+1) = x-1$$
$$\text{Dividing} \quad x+1 = 1$$
$$\text{but} \quad x = 1$$
$$\therefore \quad 1+1 = 1$$
$$\therefore \quad 2 = 1$$

8. To prove that $-1 = 1$

$$\frac{-a^{\frac{1}{2}}}{a^{\frac{1}{2}}} = \frac{a^{\frac{1}{2}}}{-a^{\frac{1}{2}}}$$
$$\therefore \quad (-a^{\frac{1}{2}})^2 = (a^{\frac{1}{2}})^2$$
$$\therefore \quad -a = a$$
$$\therefore \quad -1 = 1$$

Where is the fallacy?

9. Find the keyword in the following problem in "Letter Division":

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CPN)AOUIERT(PCAAU
    CPN
    PIUI
    PUCN
     RRIE
     RNAN
      REER
      RNAN
       RIRT
       RCUN
        EUT
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A teacher who teaches more than one class of grade XII algebra has at his disposal another means of motivating his teaching to some extent; namely, the stimulation of friendly rivalry between classes. If the same test is given to both classes, and the pupils know beforehand that the results obtained in the two classes are going to be compared, then each pupil will do his utmost to avoid pulling down the average of the class.

The same type of motivation is employed when a teacher has a record of the results obtained by previous matriculation classes, and encourages his present class to strive to surpass even the best of the previous results. If certain former students can be named who obtained 100% in their matriculation examinations, the good students in the class will make up their minds that they, too, will obtain a perfect mark. Care must be taken, of course, to prevent the pupils from developing more interest in examination marks than in the study of the subject itself; nevertheless, the introduction of a slight element of competition is a valuable means of motivating the work in grade XII algebra.

The motivating force of scholarships and prizes is one which is operative to a certain extent in the matriculation grade. This form of motivation affects only a very small percentage of the class, but it is of value in encouraging the few outstanding students to do their very best. These pupils should be informed by the teacher as to what scholarships and prizes are offered to matriculation pupils, and it should be impressed upon them that they are able to obtain one of these awards just as well as any other student in the province. As in the case of rivalry between classes, this form of motivation should not be allowed to become the dominating force behind a student's efforts, but should only supplement the major forms of motivation discussed in the earlier part of this chapter.

CHAPTER VII

SOME EXPERIMENTAL EVIDENCE TO SHOW THE EFFECTS OF MOTIVATION

Every teacher of mathematics has noticed that certain of his lessons have seemed very dull and uninteresting both to himself and to his pupils, while other lessons have been exceedingly enjoyable. When the first type of lesson is completed the teacher experiences a sense of relief, and the pupils lose no time in getting away their mathematics books and preparing for the next lesson; but when the second type of lesson is finished the teacher experiences a sense of great satisfaction, and the pupils continue working even after the bell has rung, trying to complete as many questions as possible before the next lesson begins. The reason for the difference is obvious. The first type of lesson was lacking in motivating value, while the second had some features about it which aroused and held the interest of both pupils and teacher. The beneficial results derived from this second lesson would undoubtedly outnumber those derived from the first; and at the same time, the material studied in the highly motivated lesson would be better understood and more easily remembered than that studied in the dull mechanical lesson.

The writer has recently completed a three-year experiment with methods of motivating the teaching of high school mathematics. Although the results of this experiment cannot be accepted as conclusive because of the smallness of the sampling and because of the fact that a number of variable factors were not controlled, nevertheless they may be considered as evidence to show that efficiency and interest can be developed in the subjects of algebra and geometry to a marked degree by the use of a highly motivated form of teaching.

The writer conducted the experiment between the years 1931 and 1934. He taught the experimental class for three years in succession in grades Ten, Eleven and Twelve. The class was an extremely heterogeneous one, being composed of pupils of many different types and of several different nationalities. Some of the pupils had done their grade IX work in junior high school and others had come directly to the senior high school from grade VIII in the elementary school. There was a wide variation in their academic standing, some advancing to grade X with a very good scholastic record and others being promoted only on trial. The class as a whole was not considered a good class, the principal's opinion being that it was a very weak one.

On being assigned as teacher to the above-mentioned class, the writer attempted to develop the pupils' interest in mathematics and at the same time to bring them to a high level of efficiency. In attempting to do this he adopted most of the methods mentioned in this thesis. The results of the experiment can best be judged from the tables which follow. In the week previous to the matriculation examinations, the pupils of the class were asked to write down the names of the three subjects which they had enjoyed the most during their high school course - these names to be in order of merit. The results of this vote, giving Five for first choice, Three for second choice and One for third, are as follows:

Algebra	54
Chemistry . . .	51
Physics	31
Geometry . . .	24
French	17
Literature . .	19
Social Studies.	16
Composition . .	3
Physiology . .	1
Grammar	0.

The above procedure had its limitations in as much as the writer conducted the voting and the ballots were signed by the pupils. However, the pupils were instructed very emphatically to overlook the personal element entirely and give exact statements of their preferences.

At the same time the pupils were asked to write down the names of the three subjects in which they thought they had obtained the highest degree of efficiency, and in which they were most confident of making good marks in the forthcoming matriculation examinations. The results were as follows:

Algebra	54
Geometry . . .	44
Chemistry . . .	28
Literature . .	22
Grammar	20
Physics	19
Social Studies	13
French	10
Composition . .	4
Physiology . .	1

The marks actually obtained by the pupils of this class in the Junior Matriculation examinations are shown in the table given below.

TABLE II

Class marks - Junior Matriculation examinations

Pupil	Algebra	Geometry
A	92	92
B	75	71
C	19	17
D	74	50
E	95	81
F	83	79
G	84	61
H	83	70
I	85	61
J	44	45
K	80	69
L	84	78
M	87	80
N	63	60
O	83	83
P	94	92
Q	68	67
R	58	74
S	94	83
T	92	72
U	81	69
V	55	57
W	100	82
X	94	89
Y	62	67
Z	56	55
a	82	73

Notes:

1. Pupil C, whose marks brought the class average down considerably, was only a conditioned student, not having passed grade XI; and during the spring term he was injured in a traffic accident which caused him to be absent from school for six weeks.
2. There was one "repeater" in the class.

The following table compares these results with the averages of the other classes in the same school and with the City (Vancouver) and

Provincial (B.C.) averages:

TABLE III

Comparison of Averages in Subjects

	Algebra	Geometry	All Other Subjects
Averages of Experimental Class	76.6	69.5	59.5
Averages of other Classes in Same School	57.2	66.8	60.2
City averages (Vancouver)	63.	65.8	59.8
Provincial Averages (B.C.)	63.5	63.9	60.7

In the above table certain points should be noticed when considering the results of using a specially motivated form of teaching. In the first place, the table indicates very clearly that the class was certainly not a "picked" class. The average of the class in all subjects other than algebra and geometry is 59.5% which is below the average of the three other grade XII classes in the same school; it is below the city average and below the provincial average. This comparison indicates that the class as a whole was rather on the weak side. In contrast with this comparison, the algebra and geometry averages for the experimental class are considerably above the averages for other classes in the same school, ~~as~~ as well as being much higher than both the city and provincial averages for these subjects. It should be noticed still further that the average in algebra for the experimental class is exceptionally high, and when the voting was taken before the matriculation examinations were held, the pupils expressed themselves as feeling better prepared for their algebra examination than for any other. They also indicated by their ballots that they had received more enjoyment from the study of algebra than from the study of any other subject in their high school course.

The evidence seems to indicate, therefore, that the teaching of mathematics can be made much more effective by a wider application of the principles of motivation. It seems safe to infer that, by a careful study of the various specific means of motivating the teaching of mathematics in high school, average and even weak classes may be brought up to a high level of efficiency in both algebra and geometry; and also that the enjoyment of these subjects by the pupils may be increased very greatly.

CHAPTER VIII

GENERAL CONCLUSIONS

In closing this treatise on Motivation in the Teaching of High School Mathematics, it is apparent that the following conclusions might be reached; namely, (1) that mathematics is a very essential part of an individual's education; (2) that the teaching of mathematics in high school is at present not of a sufficiently high standard to enable students to derive the maximum amount of benefit from their mathematical studies; and (3) that the situation can be improved tremendously by a more extensive application of the principles of motivation.

The statement that mathematics is an essential part of an individual's education is corroborated by the statements of numerous authorities on the subject, who commend its study both from a utilitarian and an aesthetic point of view. The assertion of Comte that "All scientific education which does not commence with mathematics is, of necessity, defective at its foundation" [1] is emphatic indeed; while W. A. Millis states that "For algebra there is no substitute. The elimination of algebra as a pure science from the curriculum would cut the foundation from under all scientific procedure." [2] David Eugene Smith speaks of geometry in metaphorical language as follows: "Geometry is a mountain. Vigor is needed for its

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(1) Jones, Mathematical Wrinkles. P.245.

(2) Millis, Opcit. P. 240.

ascent. The views all along the road are magnificent. The effort of climbing is stimulating. A guide who points out the grandeur and the special places of interest commands the admiration of his group of pilgrims." (1) Numerous other quotations might be given emphasizing the same fact - that the study of mathematics is a most essential part of a high school student's education.

In support of the conclusion that the teaching of mathematics is at present not of a sufficiently high standard to enable the students to derive the maximum amount of benefit from their mathematical studies, we have the statement of W. A. Millis that "The reason for the present unsatisfactory status of mathematics is poor teaching." (2) There is also the statistical evidence given by Professor Judd showing that the percentages of failures in mathematics, and withdrawals due to lack of interest in mathematics, is very high.(3) To these might be added statistical evidence given by Millis emphasizing the same point and outlined in his "Teaching of High School Subjects". (4) E. R. Brislock states that "Algebra is a subject difficult to learn and to teach," (5) and Professor Judd writes that "Mathematics must be recognized as one of the most difficult subjects in the high school

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(1) Jones, Ibid. P. 254.
(2) Millis, Opcit. P. 233
(3) Judd, C.H. Psychology of H.S.Subjects. P. 18.
(4) Millis, Opcit. P. 232
(5) Breslich, Problems in Teaching Secondary School Mathematics. P. 107.

course." [1] Both these men follow up their statements by the conclusions that the teaching of mathematics is at present not up to the standard necessary for producing best results in the subject.

The third of our general conclusions, that the teaching of mathematics can be improved tremendously by a more extensive application of the principles of motivation, is supported to a large extent by the evidence presented in preceding chapters of this thesis, but these may be supplemented by the opinions of authorities on these questions. Professor Judd says that "We shall certainly need to inquire, in such negative cases, what is the psychological character of pleasure, and what the possibility of so readjusting the situation as to produce pleasure through the study of geometry." [2] F. W. Westaway also stresses the importance of increasing the interest of students in the subject of mathematics, and he suggests several ways by which this may be brought about. He says that "A young boy's (13-14) natural fondness for puzzles of all kinds may often usefully be employed for furthering his interest in geometry." [3] Westaway also states that "No boy can become a successful mathematician unless he fights hard battles on his own behalf," and that "There is no better means of giving a boy a permanent

(1) Judd. Opcit. P. 17.
(2) Judd. Opcit. P. 89.
(3) Westaway, Craftsmanship in the Teaching of Elementary Mathematics. P. 229

interest in mathematics than to help him to achieve a mastery of the commoner forms of mathematical puzzles and fallacies."(1)

The effects of paying special attention to motivation in t the teaching of high school mathematics may be illustrated graphically as shown on p. 111.

The graphical illustrations given on p. 111 are not based upon the statistical results of experiments, but they give us a vivid comparison between the amounts of interest created. It will be noticed that graph II indicates that even though a teacher of mathematics ignores the possibilities of motivation, some students develop a considerable interest in the subject. These students, however, do not reach the same height of interest which they would reach under more favorable conditions. The majority of students under the conditions of graph II will undoubtedly develop a dislike for mathematics, and in a few cases this dislike may be exceedingly great. In graph III indicating the amount of interest created by a teacher using a highly motivated form of teaching, we notice that the majority of pupils under such conditions develop a real interest in the subject, and some of these reach an exceptionally high level. Only a small percentage of the pupils develop an actual dislike for mathematics. Graph IV shows us the situation which might be reached if an exceptionally well developed system of motivated teaching were employed.

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(1) Westaway, Ibid. P. 11.

In an average class it should be possible to have every member interested to some extent in the subject of mathematics. If the conditions indicated in graph IV were reached, then students would not only gain much positive enjoyment from the study of mathematics, but the level of efficiency would follow to some extent the level of interest, and the whole status of mathematics in our high schools would be tremendously improved.

There is, therefore, an urgent need for a greater attention to motivation in the teaching of high school mathematics, and it has been pointed out in this thesis that there are various specific means by which the principles of motivation may be applied to the teaching of algebra and geometry. By employing certain general methods of motivation throughout all the grades, and by supplementing these by more definite methods especially suitable for certain grades, the subject of mathematics could be made intensely interesting, both to the pupils and to the teacher. [1] If this procedure were followed by more teachers of high school mathematics, then the increase in interest, followed by a natural increase in efficiency, would regain for mathematics some of the prestige it has lost due to the use of mechanical methods of teaching seriously deficient in motivating value.

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(1) The importance of creating interest is discussed by Stormzand, and he arrives at the conclusion that "the problem of interest plays such an important part in education because success in all teaching involves the arousing of sufficient interest." Stormzand, Progressive Methods of Teaching. P. 129.

APPENDIX I

GRADE XII ALGEBRA QUESTIONS

SERIES FOR REVIEW

1. Solve: $\frac{2}{x} - \frac{3}{2y} = \frac{41}{35}$

$$\frac{2\frac{1}{2}}{2x} + \frac{3\frac{1}{2}}{y} = -\frac{73}{70}$$

2. Solve: $2y - x = 4xy$

$$\frac{4}{y} - \frac{3}{x} = 9$$

3. Solve: $\frac{1}{4}x + \frac{1}{6}y + \frac{1}{3}z = 8$

$$\frac{1}{2}x - \frac{1}{9}y + \frac{1}{6}z = 5$$

$$\frac{1}{3}x + \frac{1}{2}y - z = 7$$

4. Solve: $x + z - 1 = \frac{1}{2}(x + 4z - 8) = \frac{1}{3}(x + 9z - 27) = y$

5. Solve: $\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 12$

$$\frac{y}{2} + \frac{z}{3} - \frac{x}{6} = 8$$

$$\frac{z}{3} + \frac{x}{2} = 10$$

6. Solve: $\frac{x-8}{x-10} + \frac{x\ \ 4}{x-6} = \frac{x-5}{x-7} + \frac{x-7}{x-9}$

7. Solve: $\frac{5x-64}{x-13} - \frac{2x-11}{x-6} = \frac{4x-55}{x-14} - \frac{x-6}{x-7}$

8. Solve: $\frac{x}{x-2} + \frac{9-x}{7-x} = \frac{x+1}{x-1} + \frac{8-x}{6-x}$

9. Solve: $\frac{x}{x+b-a} + \frac{b}{x+b-c} = 1$

10. Solve: $\frac{x-bc}{a} + \frac{x-ca}{b} + \frac{x-ab}{c} = 2(a+b+c)$

11. Solve: $\frac{5}{x+2a} + \frac{8}{x-a} = \frac{1}{a}$

12. Solve: $\frac{x}{cd} + \frac{4}{d} = \frac{d}{cx} - \frac{4c}{dx}$

13. Solve: $x^2-6ax+9a^2-b^2 = 0$

14. Solve: $m(x+y)+n(x-y) = 2mn$
$m(x+y)-n(x-y) = mn$

15. Solve: $lx = my$
$\frac{a}{x} - \frac{b}{y} = c$

16. Solve: $y = \frac{x+a}{2} + \frac{b}{3}$
$x = \frac{y+b}{2} + \frac{a}{3}$

17. Solve: $\frac{2x}{x-1} + \frac{3x-1}{x+2} = \frac{5x-11}{x-2}$

18. Solve: (a) $4x^2-10x = -5$

(b) $x^2+6x+3 = c$

19. Write the general equation for quadratics and solve it for x.

20. Solve: (a) $x^2+ax-a^2 = 0$

(b) $x(a-x) = c^2$

21. Find two values for x which will make x(3x-1) equal to 0.362, giving each value to the nearest hundredth.

22. Solve: (a) $x^2+\frac{a^2b^2}{x^2} = a^2+b^2$

(b) $(x^2+2)^2 = 29(x^2+2)-198$

(c) $x(x-2x) = \frac{8a^4}{x^2-2ax} + 7a^2$

23. Solve: (a) $x^3-3x-2 = 0$

(b) $x^4+2x = 3x^2$

(c) $x^3+6a^3 = 7a^2x$

24. Solve: $4x^3-15x^2+1 = 0$ having given one root as $x = -\frac{1}{4}$

25. Solve: $(x-\frac{1}{x})^2 - \frac{77}{12}(x-\frac{1}{x}) + 10 = 0$

26. Solve: (a) $3x + 3y = 4$, $2xy + y^2 = 7y-2$ (b) $xy = 35$, $\frac{1}{x} + \frac{1}{y} = \frac{12}{35}$

27. Solve: (a) $\frac{1}{x^2} + \frac{1}{y^2} = 178$, $39xy = 1$ (b) $\frac{1}{x^2} + \frac{1}{y^2} = 4$, $\frac{1}{x} - \frac{1}{y} = 1\frac{1}{2}$

28. Solve: (a) $x^2-2xy = 24$, $xy-2y^2 = 4$ (b) $4x^2 + xy = 7$, $3xy + y^2 = 18$

29. Solve: (a) $x^2 + 2xy + 10y^2 = 145$, $xy + y^2 = 24$ (b) $2x^2-3xy + 2y^2 = 2\frac{3}{4}$, $x^2-4xy + y^2 + \frac{1}{2} = 0$

30. Solve: (a) $x^4 + x^2y^2 + y^4 = 21$, $x^2 + xy + y^2 = 7$ (b) $4x^2-2xy + y^2 = 31$, $8x^3 + y^3 = 217$

31. Solve: (a) $x^3-y^3 = 243$, $xy(y-x) = 162$ (b) $x + 3xy = 35$, $y + 2xy = 22$

32. Solve: (a) $2\sqrt{x}-1 = \sqrt{4x-11}$ (b) $2\sqrt[3]{5x-35} = 5\sqrt[3]{2x-7}$

33. Solve: (a) $\sqrt{x+a} - \sqrt{x-a} = \sqrt{a}$ (b) $\sqrt{x+7} + \sqrt{x+2} = \sqrt{6x+13}$

34. Solve: (a) $\frac{\sqrt{x}+3}{\sqrt{x}-2} = \frac{3\sqrt{x}-5}{3\sqrt{x}-13}$ (b) $\sqrt{x+a} + \sqrt{x+b} = \frac{a}{\sqrt{x+a}}$

35. Solve: (a) $\frac{6\sqrt{x}-7}{\sqrt{x}-1} - 5 = \frac{7\sqrt{x}-26}{7\sqrt{x}-21}$

(b) $x + \sqrt{a^2+x^2} = \frac{5a}{\sqrt{a^2+x^2}}$

36. Solve: (a) $\sqrt{9+2x} - \sqrt{2x} = \frac{5}{\sqrt{9+2x}}$

(b) $\frac{\sqrt{1+x}+\sqrt{x-7}}{\sqrt{1+x}-\sqrt{x-7}} = 2$

37. (a) Prove that $a^0 = 1$ (b) Prove that $a^{-n} = \frac{1}{a^n}$

38. Multiply: $3m^{1/3} - 3m^{-1/3} + 2m^{-1}$ by $5m^{2/3} + 4$

39. Divide: $x^{3/2} + x^{1/2} y^{2/3} - 2y$ by $x^{1/2} - x^{1/3}$

40. Find the sq. root of: (a) $9x - 12x^{\frac{1}{2}} + 10 - \frac{4}{\sqrt{x}} + \frac{1}{x}$

(b) $12a^{x} + 4 - 6a^{3x} + a^{4x} + 5a^{2x}$

41. Simplify:
(a) $\left\{ \frac{\sqrt[3]{a}}{\sqrt[4]{b^{-1}}} \left(\frac{b^{-1/4}}{a^{1/3}} \right)^{2} \div \frac{a^{-1/3}}{b^{-1/2}} \right\}^{6}$

(b) $\left(a^{-1/2} x^{1/3} \sqrt{a x^{-1/3} \sqrt[4]{x^{4/3}}} \right)^{1/3}$

42. Simplify: $\left(\frac{a^{-3}}{b^{-2/3} c} \right)^{-3/2} \div \left(\frac{\sqrt{a^{-1/2}} \sqrt[6]{b^{3}}}{a^{2} c^{-1}} \right)^{-2}$

43. Simplify:
(a) $(a^{2}-b^{2})^{\frac{1}{2}} \times (a+b)^{-\frac{1}{2}} \times (a-b)^{\frac{3}{2}}$

(b) $\frac{\sqrt{(a^{3}b^{3} + a^{6}}}{\sqrt[3]{(b^{6} - a^{3}b^{3})^{-1}}}$

44. Simplify:

(a) $\frac{3 \cdot 2^{n} - 4 \cdot 2^{n-2}}{2^{n} - 2^{n-1}}$ (b) $\frac{2^{n+3}}{15^{-n-1}} \times \frac{6^{-n+2}}{5^{n+1}}$

45. Simplify: $\frac{3^{n+4} - 6 \cdot 3^{n+1}}{3^{n+2} x + 7}$

46. Factor: (a) $x - 2\sqrt{x} - 15$ (b) $x^{a} - 49$

(c) $3a^{\frac{1}{2}} + 5a^{\frac{1}{4}} + 2$ (d) $x^{3m} + 27$

(e) $x^{-2c} + x^{-c} - 2c$

47. Express in its simplest form free from radical signs:

(a) $\frac{x - 7x^{\frac{1}{2}}}{x - 5\sqrt{x} - 14} - 1 + \frac{2}{\sqrt{x}}^{-1}$ (b) $\frac{a^{\frac{3}{2}} + ab}{ab - b^{3}} - \frac{\sqrt{a}}{\sqrt{a} - b}$

48. Find the value of:
(a) $3\sqrt{147} - 7\sqrt{\frac{1}{27}} - \frac{11}{3}\sqrt{\frac{1}{3}}$ (b) $\sqrt{18p^{3}q^{3}} - p\sqrt{8pq^{3}} - q\sqrt{50p^{3}q}$

49. Find the value of the following to two decimal places:

(a) $\dfrac{5+2\sqrt{3}}{7-4\sqrt{3}}$ (b) $\dfrac{7\sqrt{3}-5\sqrt{2}}{\sqrt{48}+\sqrt{18}}$

50. Find the value of:

$$\frac{1}{3cx}\sqrt{\frac{1}{4cx^4}} \times c^2x^2\sqrt{c^7x^5} \div \frac{x^2}{3c}\sqrt{\frac{x^3}{8c^5}}$$

51. Find the value of:

(a) $\dfrac{\sqrt{2}}{3-\sqrt{2}} - \dfrac{2+x\sqrt{2}}{7}$ (b) $\dfrac{22}{3\sqrt{2}-\sqrt{7}} - (\sqrt{18}+\sqrt{7})$

52. Express with rational denominators:

(a) $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}$ (b) $\dfrac{4}{2+\sqrt{3}+\sqrt{7}}$

(c) $\dfrac{1}{\sqrt{3}+\sqrt{2}-\sqrt{5}}$

53. Express in its simplest form:

(a) $\dfrac{\sqrt{x^2-y^2}+x}{\sqrt{x^2+y^2}+y} - \dfrac{x^2+y^2-y}{x-\sqrt{x^2-y^2}}$ (b) $\dfrac{4(\sqrt{3}+1)}{\sqrt{3}-1} - \dfrac{2+\sqrt{3}}{2-\sqrt{3}}$

54. Find the square root of:

(a) $49-20\sqrt{6}$ (b) $\sqrt{175}-\sqrt{147}$

(c) $3x-1+2\sqrt{2x^2-3x-2}$ (d) $2m+2\sqrt{m^2-9n^2}$

55. Express in its simplest form:

(a) $\sqrt{19+4\sqrt{21}}+\sqrt{7}-\sqrt{12}-\sqrt{29-2\sqrt{28}}$

(b) $\sqrt{27}-\sqrt{8}+\sqrt{17+12\sqrt{2}}-\sqrt{28-6\sqrt{3}}$

56. Show that $\sqrt{a+\sqrt{b}}$ cannot be expressed in the form $\sqrt{x}+\sqrt{y}$ unless a^2-b is a perfect square.

57. (a) Find the ratio compounded of the duplicate ratio 3:7 and the ratio of 35:27.
(b) Find the ratio of $x:y$ from the equation $\dfrac{2ax+by}{2ax-by} = \dfrac{3by}{ax}$

58. (a) If $m:n$ is the duplicate ratio of $m+x:n+x$, prove that $x^2 = mn$.

58. (b) If a and b are unequal, and $ab(c^2+d^2) - b^2c^2+a^2d^2$ shew that the ratio of a to b is the duplicate ratio of c to d.

59. If $\frac{p}{q} = \frac{r}{s} = \frac{t}{u}$ prove that $\frac{r^3-p^2tu}{a^3-q^2u^2} - \frac{prt}{qsu}$

60. If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$ prove that each ratio is equal to

$$\sqrt[5]{\frac{6a^2c^2e-c^4ef+7ac^5}{6b^2d^2f-d^4f^2+7ad^5}}$$

61. If $\frac{p}{bz-cy} = \frac{-q}{cx+az} = \frac{-r}{ay+bx}$, shew that $ap+bq-cr = 0$ and $xp-yq+zr = 0$.

62. If $\frac{2x-3y}{3z+y} = \frac{z-y}{z-x} - \frac{x+3z}{2y-3x}$ prove that each of these ratios is equal to $\frac{x}{y}$; hence show that either $x = y$ or $z = x+y$.

63. (a) If $\frac{a}{b} = \frac{b}{d}$ prove that $\frac{a+b}{b} = \frac{c+d}{d}$

(b) If $\frac{a}{b} = \frac{c}{d}$ prove that $\frac{a}{c} = \frac{b}{d}$

64. (a) Find the fourth proportional to $12x^3$, $9ax^2$, $8a^3x$.

(b) Find the third proportional to $5\sqrt{3}$; $\sqrt{15}$.

(c) Find the mean proportional to $2\sqrt{18}$; $3\sqrt{128}$.

65. If a, b, c are three proportionals, shew that
$(b^2+bc+c^2)(ac-bc+c^2) - b^4+ac^3+c^4$.

66. If $a:b = c:d$, prove that $\frac{ab^3-c^3d}{b^2c^2-c^3d} = \frac{b^3}{ad^2}+1$

67. If a, b, c, d are in continued proportion, prove that
$a:d = a^3+b^3+c^3 : b^3+c^3+d^3$.

68. If $(a+b-3c-3d)(2a-2b-c+d) = (2a+2b-c-d)(a-b-3c+3d)$ prove that a, b, c, d are proportionals.

69. If $b+c$ is the mean proportional between $a+b$ and $c+a$ show that $b+c:c+a = c-a:a-b$.

70. If $\frac{x}{y} = \frac{a}{a+b}$ then $\frac{x^2-xy+y^2}{a^2+ab+b^2} = \frac{x^2}{a^2}$

71. If $12:x = x:y = y:z = z:18$, calculate the value of x to two decimal places, and show that
$x^4+y^4+z^4 - (x^2+y^2+z^2)(x^2-y^2+z^2)$

72. If $a+x:a-x$ is the duplicate ratio of $a+b:a-b$, then $x-b:a-x = b(a+b):a(a-b)$.

73. Resolve into factors:

(a) $c^3d^6e^9-1$ (b) $a^4-3a^3-a^3b+3a^2b$

(c) x^2-y^2+x+y (d) $9(a-2b)^2-(4a-7b)^2$

(e) $2cd-2xy+x^2+y^2-c^2-d^2$ (f) $a^6-6a^3-a^2-2ab-b^2+9$.

(g) $c^2+9(d^2-a^2)+6cd$ (h) $x^4+25y^4-19x^2y^2$

(i) $4m^4-21m^2n^2+n^4$ (j) $30a^2+37ab-84b^2$

(k) x^8-y^8 (l) $3(2b^2-1)-7b$

(m) $(c+d)^3+(c-d)^3$ (n) $x^3-37x-84$

(o) $6a^3+a^2-19a+6$ (p) $a-7\sqrt{a}+12$

(q) x^a-49 (r) $3a^{\frac{1}{2}}+5a^{\frac{1}{4}}+2$

(s) $x^{-2c}+x^{-c}-20$ (t) $x^{3m}+27$

74. Simplify:

(a) $$\frac{1+8x^3}{(2-x)^2} \times \frac{4x-x^3}{1-4x^2} \div \frac{(1-2x)^2+2x}{2-5x+2x^2}$$

(b) $$\frac{x^2(x-4)^2}{(x+4)^2-4x} \div \frac{(x^2-4x)^3}{(x+4)^2} \times \frac{64-x^3}{16-x^2}$$

75. Simplify:

(a) $$\frac{a}{(a-x)^2} + \frac{3a}{x^2+ax-2a^2} + \frac{1}{2a+x}$$

(b) $$\frac{1}{a-x} - \frac{1}{a+x} - \frac{2x}{a^2+x^2} + \frac{4x^3}{(x^2-a^2)^2}$$

76. Find the value of:

(a) $$\frac{-m^2nr}{(m-n)(m-r)} + \frac{n^2rm}{(n-r)(n-m)} + \frac{r^2mn}{(r-m)(r-n)}$$

(b) $$\frac{q+r}{(x-y)(x-z)} + \frac{r+p}{(y-z)(y-x)} + \frac{p+q}{(z-x)(z-y)}$$

77. Find the value of: (a) $$1-\frac{x+y}{x+y-\dfrac{x+\frac{y^2}{x}}{1-\frac{y}{x}}}$$

77\. (b) $2-\cfrac{2}{1-\cfrac{3}{2-\cfrac{3}{1-x}}}$

78\. (a) Divide $x+\frac{16x-27}{x^2-16}$ by $x-1+\frac{13}{x+4}$

(b) Multiply $x+2a-\frac{a^2}{2x+3a}$ by $2x-a-\frac{2a^2}{x+a}$

79\. (a) Simplify: $\frac{3}{8(1-x)}+\frac{1}{8(1+x)}-\frac{1-x}{4(1+x^2)}-\frac{3}{4(x^2-1)}$

(b) Simplify: $\frac{x+2y}{\frac{2}{7}x-y}-\frac{3x^2+63xy+70y^2}{2x^2+3xy-35y^2}$

80\. Find the value of $\frac{x+2a}{2b-x}+\frac{x-2a}{2b+x}+\frac{4ab}{x^2-4b^2}$ when $x=\frac{ab}{a+b}$

81\. Draw quadrilateral ABCD, the coordinates of its angular points being A(12,4); B(-2,8); C(-8,-11); D(9,-9). Calculate its area. If the unit used is 1/10", what is the area of the quadrilateral in square inches?

82\. Draw graphs for the following lines: (a) $x+7=0$ (b) $y-9=0$ (c) $3x=4y$ (d) $2x-3y=6$ (by intercept method) (e) $-3y-4x+5=0$.

83\. Draw a graph of the function $\frac{5x+17}{2}$ and from the graph read the value of the function when $x=5$ and when $x=8$.

84\. Draw a graph to show the variations of the functions $1.2x-3$ and $3.5-3.8x$ between the values 0, 1, 2, 3, 4 of x. Hence find the value of x which satisfies the equation $1.2x-3=3.5-3.8x$.

85\. Solve graphically and prove algebraically:

(a) $\left.\begin{array}{l}2y-5x=20\\4x+3y=7\end{array}\right)$ (b) $\left.\begin{array}{l}2x-5y=16\\4x+y=10\end{array}\right)$

86\. (a) Draw the triangle whose sides are given by the equations $3y-x=9$; $x+7y=11$; $3x+y=13$, and find the coordinates of its vertices.

(b) Find the equation for the line joining the points (4,5), (11,11). Shew that the point (-3,-1) also lies on this line.

(c) Prove that the points (2,4), (-3,8), (12,-4) lie on a straight line which cuts the axis of x at a distance of 7 units from the origin.

87\. (a) Solve graphically: $x^2+y^2=41$, $y=2x-3$

(b) Plot the graph of the function $y=x^2+2x-4$ and give the coordinates of its turning point.

87. (c) Find graphically the roots of the following equation to two decimal places: $4x^2-16x+9 = 0$

88. Draw the graph of:

(a) $y = 2x - \frac{x^2}{4}$ (b) $x^2 - y = 4-2x$

In each case give the maximum or minimum value for x or y.

89. (a) Find graphically the roots of the following equations:

$$x^2-7x+11 = 0$$

(b) What is the minimum value of the expression $x^2-7x+11$?

90. (a) Show graphically that the expression x^2-2x-8 is negative for all values of x between -2 and 4, and positive for all values for x outside these limits.

(b) Solve graphically and test algebraically: $\left.\begin{array}{l} x^2+y^2 = 53 \\ y-x = 5 \end{array}\right)$

91. An income of $160 is derived partly from money invested at $3\frac{1}{2}\%$ and partly from money invested at 3%. If the investments were interchanged the income would be $165. How much is invested at each rate?

92. The profits of a business were $150 in the first year, and half as much in the second year as in the third. In the fourth year they were three times as much as in the first two years together. The total profit in all four years was half as much again as in the first and fourth years together. Find the total profits.

93. A man has a number of coins which he tries to arrange in the form of a solid square. On the first attempt he has 116 over, and when he increases the side of the square by 3 coins he wants 25 to complete the square. How many coins has he?

94. An officer forms his men into a hollow square four deep. If he has 1392 men, find how many there will be in front.

95. Two men, A and B, travel in opposite directions along a road 180 miles long, starting simultaneously from the ends of the road. A travels 6 miles a day more than B, and the number of miles travelled each day by B is equal to double the number of days before they meet. Find the number of miles which each travels in a day.

96. Find two numbers such that their product multiplied by their sum is 330, and their product multiplied by their difference is 30.

97. The interest on a sum of money for 1 year is £31 17s 6d., if the rate of interest were less by $\frac{1}{2}$ per cent it would be necessary to invest £100 more to produce the same amount of interest. Find the sum invested at first.

98. There is a number consisting of two digits such that the difference of the cubes of the digits is 109 times the difference of the digits. Also the number exceeds twice the product of its digits by the digit in the units place. Find the number.

99. (a) A boat goes up-stream 30 miles and then downstream 44 miles in 10 hours, and it also goes upstream 40 miles and downstream 55 miles in 13 hours. Find the rate of the stream and of the boat.

(b) How long will it take each of two pipes to fill a cistern if one of them alone takes 27 minutes longer to fill it than the other and 75 minutes longer than the two together?

100. (a) A man buys 99 oranges at a certain price; they would have cost him 1 shilling less if he had obtained for each shilling spent 4 more oranges than he actually did. What price did he pay?

(b) A rectangular plot of ground is surrounded by a gravel walk 4 ft. wide. The area of the plot is 1200 sq.ft., and the area of the walk is 624 sq. ft. Find the dimensions of the plot.

(c) A man invests some money in 3 per cent stock; if the price were £10 more, he would receive 1 per cent less for his money; at what price did he buy the stock?

SERIES FOR COORDINATION

Grade XII Algebra

I

1. Simplify: $$\frac{\frac{1}{a}+\frac{1}{b+c}}{\frac{1}{a}-\frac{1}{b+c}} \times \left(1+\frac{b^2\quad c^2-a^2}{2bc}\right)$$

2. Solve: $$\frac{x+2}{x-2}-\frac{10-x^2}{4-x^2}-\frac{10}{x^2-4}$$

3. Solve: $$\frac{.32x}{.05}+\frac{.045x}{.125} = 13.52$$

4. Seven years ago a boy was half as old as he will be one year hence. How old is he now?

5. A collection of five-cent pieces and quarters contains 80 coins. Their total value is \$16. How many are there of each?

6. Solve: $$\begin{aligned} 5x+2y-3z &= 160 \\ 3x+9y+8z &= 115 \\ 2x-3y-5z &= 45 \end{aligned}$$

7. Factor: (a) $x^4+y^4-7x^2y^2$.

   (b) $\dfrac{x^3}{512}-\dfrac{64}{x^3}$

   (c) $x^7+x^4-16x^3-16$.

8. (a) Write the general equation for a quadratic.
   (b) Write the solution of your equation.
   (c) By means of this formula solve: $8x^2-4x+3 = 0$

9. A labourer worked a number of days and received for his labour \$36. Had his wages been 20¢ per day more, he would have received the same amount for two days' less labour. What were his daily wages, and how many days did he work?

10. Solve: $$\begin{aligned} x^2+y^2+x+y &= 18 \\ xy &= 6 \end{aligned}$$

II

1. Divide: $16a^{-3}-6a^{-2}+5a^{-1}+6$ by $1+2a^{-1}$

2. Factor: (a) $6x^2+9x-8xy-12y$.

2. (b) $4a^4 + 15a^2b^2 - 4b^4$
   (c) $9r^{2x} - 25x^{6x}$.

3. Simplify: $\left(\frac{2x}{x-2} - \frac{x}{x-1}\right) \div \left(\frac{3x}{x-3} - \frac{2x}{x-2}\right)$ and check your work by letting $x = 4$.

4. Solve:
$$\begin{aligned} 3x-2y-z &= 1 \\ 4x-3y+4z &= -3 \\ 2x+y-5z &= -2 \end{aligned}$$

5. In the formula $T = 2\pi R(R+H)$
   (a) Solve for H in terms of the other letters.
   (b) If $\pi = 3.14$, $R = 11$, $T = 794.42$ find H.

6. A man starts from a certain place and walks at the rate of a miles an hour. b hours later another man starts from the same place and rides in the same direction at the rate of c miles an hour. In how many hours will the second man overtake the first?

7. Solve: $b(a+x)-(a+x)(b-x) = x^2 + \frac{bc^2}{a}$

8. Simplify: $\left(\frac{8x^{-3}}{27x^6}\right)^{-1/3} \left(\frac{9x^2}{4a^{-2}}\right)^{-1/2}$

9. Solve: $2\sqrt{x} - \sqrt{4x-3} - \frac{1}{\sqrt{4x-3}}$

10. Solve: $4x^2 + y^2 = 17$; $2x+y = 5$.

III

1. Divide: $\frac{8}{27}a^5 - \frac{243}{512}ax^4$ by $\frac{2}{3}a - \frac{3}{4}x$

2. A man walks from A to B in h hours. If he had walked "a" miles an hour faster he would have been "b" hours less on the road. Find the distance from A to B and the rate of walking.

3. A grocer spent \$120 in buying tea at 60¢ a pound, and 100 lbs. of coffee. He sold the tea at an advance of 25% on cost and the coffee at an advance of 20% on cost. The total selling price was \$148. Find the number of lbs. of tea purchased, and the cost of the coffee per pound.

4. A man spent \$90 for wood, and finds when the price is increased \$1.50 per load he will get 3 loads less for the same money. What was the price per load?

5. Solve:
$$\begin{aligned} x^4 + x^2y^2 + y^4 &= 21 \\ x^2 + xy + y^2 &= 7 \end{aligned}$$

6. Solve: $3(z-1) = 2(y-1)$
$4(y+x) = 9z-4$
$7(5x-3z) = 2y-9$

7. (a) Find the square root of $83+12\sqrt{35}$.

(b) The area of a rectangle is $16\sqrt{10} - 25$ and the length of one side is $3\sqrt{5} - \sqrt{2}$. Find the other side in its simplest form.

8. Simplify: $$\frac{(x^{\frac{1}{2}}-2x^{\frac{1}{4}}y^{\frac{1}{2}}+y)^2}{(x^{\frac{1}{4}}-y^{\frac{1}{2}})^3}$$

9. If a, b, c are in continued proportion, prove that
$$a^4+a^2c^2+c^4 = (a^2+b^2+c^2)(a^2-b^2+c^2)$$

10. In a certain examination, the number of those who passed was three times the number of those who failed. If there had been 16 fewer candidates and if six more had failed, the total number of candidates would have been to the number who failed as 2 to 1. Find the number of candidates.

IV

1. Simplify: $\frac{3}{8}\left\{\frac{4}{3}(a-b)-8(b-c)\right\} - \left\{\frac{b-c}{2} - \frac{c-a}{3}\right\} - \frac{1}{2}\left\{c-a-\frac{2}{3}(a-b)\right\}$

2. (a) For what value of k does (3,2) lie on the line $4x+ky = 2$?

(b) Solve graphically: $y = \frac{x^2}{4}+x-2$ and $y = 2x-3$.

3. Factor: (a) $x^3-6x^2+12x-8$. (b) $24ax-16a^2-9x^2$
(c) $x^2y+3xy^2-3y^2-y$

4. Supply the missing term so as to make a perfect square:
(a) $9m^2n^2-(\quad\quad)+4$. (b) $25x^2+30x+(\quad\quad)$.

5. Simplify: $$\frac{4a^2-4a+1}{4a^2-1} + \frac{2a^2+a}{8a^3-1} - \frac{a^2-2a}{a^2-4}$$

6. Simplify: $$1-\cfrac{2}{3+\cfrac{2}{5-\cfrac{6}{7}}}$$

7. Solve: $.05x-1.82-.7x = .008x-.504$.

8. The pressure of water on a pipe that will not break the pipe is called the safe working pressure. In cast iron pipes, such as carry water in a city water-system, the safe working pressure is given by the formula
$$P = \frac{7200}{D}\left(T - \frac{1-.01D}{3}\right) - 100$$

in which P = the pressure per square inch,
D = the inside diameter of pipe in inches,
T = thickness of iron shell in inches.
Given D = 3 ft; T = $1\frac{1}{2}$ in., find P.

9. A man receives \$140 per year as interest on \$2,500. \$500 is invested at 5%; part of the remainder at $5\frac{1}{2}$%; and the remainder at 6%. How much has he invested at $5\frac{1}{2}$%?
10. Solve: $$2x^2-xy = 28$$ $$x^2+2y^2 = 18.$$

V

1. Bracket the powers of x so that the signs before all the brackets shall be negative: $$3b^2x^4-bx-ax^4-cx^4-c^2x-7x^4.$$
2. A boy can row 10 miles down stream in 2 hours and return in 3 1/3 hours. Find the rate which he rows in still water, and also the rate of the stream.
3. Find two numbers whose difference is 4 and the sum of whose reciprocals is 3/8.
4. Simplify: $$\frac{4^{n+1}}{2^n(4^{n-1})^n} - \frac{8^{n+1}}{(4^{n+1})^{n-1}} + 3$$
5. If a, b, c, d are in continued proportion, prove that (b+c) is the mean proportional between (a+b) and (c+d).
6. Solve graphically: $$x^2+y^2 = 100$$ $$x-y = 2$$
7. Factor: (a) $12-15a+16x-20ax$ (b) $27y^3-512$
(c) $9x^{5n}-x^{3n}$ (d) x^2-4y^2+9-6x
8. Simplify: $$\frac{2-\frac{2}{8}}{(\frac{1}{2})^2-(\frac{1}{2})^3-12(\frac{1}{2})^4}$$
9. Simplify: $$\frac{2+\frac{a}{b}}{\frac{(a-2b)^2}{4ab}+2}$$
10. Solve: $$\frac{x^2}{2x^3-2} = \frac{2x}{3x^2+3x+3} - \frac{1}{6x+6}$$

VI

1. Solve: $x-(3x-\frac{2x-5}{10}) = \frac{1}{6}(2x-57)-\frac{5}{3}$

2. Solve: $.12(2x+.05) - .15(1.5x-2) = 0.246$.
3. One-sixth of a man's age 8 years ago equals 1/8 of his age 12 years hence. What is his age now?
4. If it costs as much to sod a square piece of ground at 20¢ per sq. yd. as to fence it at 80¢ a yd., find the side of the square.
5. Solve: $\frac{1}{m} + \frac{1}{n} - \frac{1}{p} = 1$
$$\frac{1}{m} + \frac{1}{n} + \frac{1}{p} = \frac{2}{3}$$
$$\frac{1}{m} - \frac{1}{n} + \frac{1}{p} = 0$$
6. Solve: $2x^2 - \frac{11x}{2} - \frac{15}{2} = 0$
7. One side of a right angled triangle is 7 feet shorter than the other, and the area is 30 sq. ft. Find the two sides and the hypotenuse.
8. Simplify: $\sqrt{x^3y^3 - x^2y^2}\sqrt{\frac{1}{xy}} + xy\sqrt{\frac{x^2+y^2}{xy}}$
9. Find a quantity such that when it is added to each of the numbers 11, 17, 19, 23, the results are in proportion.
10. (a) Is the point (3,4) on the line whose equation is $3x-4y = 12$? Give the reason for your answer.
    (b) Solve: $2x + y = 1$
$$y^2 + 4x = 17$$
    Check your result.

VII

1. Solve: $\frac{.25(x-3) + .3(x-4)}{.125} = 5x-19$.
2. Simplify:
$$\frac{xy^{\frac{1}{3}}z}{x^{-2}y^{-1}zy^3} \times \frac{\sqrt{xy}\sqrt[5]{z}}{\sqrt{x^3}\sqrt[5]{x^6}}$$
3. Solve: $(8x)^{\frac{1}{2}} - (8x-15)^{\frac{1}{2}} - \frac{7}{\sqrt{8x-15}}$
4. a. What are the coordinates of the origin?
   b. What is the graph of $x = 2$?
   c. What is the distance of any point $P(x,y)$ from the origin?
   d. Draw the graph of $4y = 8x + x^2$
5. Simplify: $$\frac{6x-11-\frac{7}{x}}{2\quad\frac{11}{x}\quad\frac{5}{x^2}} - \frac{\frac{5-x}{3x}}{x^2-25}$$

6. Solve: $\dfrac{x-2}{.05} - \dfrac{x-4}{.0625} = 56$

7. The denominator of a fraction exceeds the numerator by 4; and if 5 be taken from each, the sum of the reciprocal of the new fraction and 4 times the original fraction is 5. Find the original fraction.

8. Solve: (a) $22x^2 = 2ax + 7a^2$ (b) $5x^2 = 17x - 10$

9. Solve: $7xy - 8x^2 = 10$; $8y^2 - 9xy = 18$

10. Around a rectangular flower bed, which is 3 yds. by 4 yds., there extends a border of turf which is everywhere equal in breadth and whose area is 10 times the area of the bed. How wide is it?

VIII

1. Factor: (a) $2ax^2 - 2bx^2 - 6ax + 6bx - 8a + 8b$.

   (b) $a^2 + b^2 - c^2 - 9 - 2ab + 6c$.

   (c) $ab(x^2 + 1) + x(a^2 + b^2)$

2. Simplify: $\left(x^{\frac{n}{n+1}}\right)^{n^2-1} + \dfrac{\sqrt{x^{2n}}}{x}$

3. Simplify: $\dfrac{\sqrt{3} + \sqrt{2}}{2 - \sqrt{3}} - \dfrac{7 + 4\sqrt{3}}{\sqrt{3} - \sqrt{2}}$

4. Solve: $\dfrac{6\sqrt{x} - 7}{\sqrt{x} - 1} \div 5 = \dfrac{7\sqrt{x} - 26}{7\sqrt{x} - 21}$

5. (a) What do you know about the graph of the following equations: $y = 5x$; $y = 5x - 4$; $y = 5x + 6$.

   (b) Solve graphically: $2x + y = 0$; $y = \frac{4}{3}(x + 5)$

6. Simplify: $\dfrac{1}{x - \dfrac{2}{x + \frac{1}{2}}} + \dfrac{1}{2 + \dfrac{1}{x}} + \dfrac{x}{2x - \dfrac{x+4}{x+1}}$

7. Solve: $.01(2x + .205) - .0125(1.5x - .5) = .01955$

8. Solve: $\left.\begin{aligned} \frac{x}{a} + \frac{x}{b} &= 1 \\ \frac{x}{3a} + \frac{y}{6b} &= \frac{2}{3} \end{aligned}\right\}$

9. In a concert hall, 800 people are seated on benches of equal length. If there were 20 benches fewer, two persons more would have to sit on each bench. Find the number of benches.

10. Solve: $\left.\begin{aligned} \frac{1}{x^2} - \frac{1}{xy} + \frac{1}{y^2} &= 7 \\ \frac{1}{x} - \frac{1}{y} &= 1 \end{aligned}\right\}$

IX

1. The surface of a sphere of radius "r" is given by the formula
$$S = 4\pi r^2$$
Find (a) The surface of a sphere of radius 1.4".
(b) The radius of a sphere whose surface is $38\frac{1}{2}$ sq.ft.
2. Simplify: $\left(a^{1/2} x^{-1/3} \sqrt{a x^{-1/4} \sqrt[3]{x^{5/3}}}\right)^{1/2}$
3. Factor: (a) $\frac{m^3n^3}{729} - 1$ (b) $x^3-8y^3-27z^3-18xyz$
(c) $x^4-15x^2y^2+9y^4$.
4. a. Solve: $22x^2 = 3mn + 7m^2$.
b. The sum of the reciprocals of two consecutive numbers is 5/6. Find the numbers.
5. Find to three places of decimals the value of:
$$\frac{5+\sqrt{10}}{\sqrt[4]{5}-\sqrt{45}-\sqrt{8}+\sqrt{18}}$$
6. Solve graphically: $2x + \frac{x^2}{4} = 0$
7. Factor: (a) $a^4-a^2-9-2a^2b^2+b^4+6a$

(b) $a^3+3a^2b+3ab^2+2b^3$.

(c) $a^4b-a^2b^3-a^3b^2+ab^4$.
8. Find as simply as possible the value of: $\frac{.5433^2 - .4567^2}{.5433 - .4567}$
9. Divide: $\frac{2x}{x+y} + \frac{y}{x-y} - \frac{y^2}{x^2-y^2}$ $\quad \frac{1}{x+y} + \frac{x}{x^2-y^2}$
10. Reduce: $$\frac{x^{2/3} - 4\sqrt[3]{x^{-2}}}{\sqrt[3]{x^2} + 4 + 4x^{-2/3}}$$

X

1. In the formula $F = \frac{mv^2}{gr}$, given m = 12.075, r = 3, g = 32.2, F = 200, find v.
2. Show that $$\frac{1}{\sqrt{16+2\sqrt{63}}} + \frac{1}{\sqrt{16-2\sqrt{63}}} = 3$$
3. (a) Simplify: $\sqrt[3]{21+7\sqrt{2}} \times \sqrt[3]{21-7\sqrt{2}}$

   (b) Solve: $$\frac{6\sqrt{x}-11}{3\sqrt{x}} = \frac{2\sqrt{x}+1}{\sqrt{x}+6}$$
4. Solve: (a) $5x^2-15x+11 = 0$ (b) $x^2-xy = 6$, $y^2+3xy = 10$
5. The length of a field exceeds its breadth by 30 yds. If the field were square, but of the same perimeter, its area would be 1/24 greater. Find the sides.
6. Solve: $$\frac{2x-27}{x-14} + \frac{x-7}{x-8} = \frac{x-12}{x-13} + \frac{2x-17}{x-9}$$
7. A number has three digits, the units being $\frac{1}{2}$ the tens and 1/3 of the hundreds. If 396 be subtracted from the number, the digits are reversed. Find the number.
8. Prove that the points (3,2), (8,8), (-2,-4) lie on a straight line. Find its equation. Prove algebraically and graphically that it cuts the x axis at a distance of 1 1/3 from the origin.
9. Solve: (a) $$\left.\begin{aligned} \frac{9}{x} - \frac{2}{y} &= 4 \\ \frac{10}{z} - \frac{6}{x} &= 8 \\ \frac{21}{y} + \frac{45}{2z} &= 12 \end{aligned}\right\}$$ (b) $$\left.\begin{aligned} x^2-xy+2y^2 &= 4 \\ x^2-3xy &= -2 \end{aligned}\right\}$$
10. Given /5 - 2.23607 find to 4 places of decimals the value of: $$\frac{7\sqrt{5}+15}{\sqrt{5}-1} + \frac{\sqrt{5}-2}{3+\sqrt{5}}$$

APPENDIX II

Answers to Review Questions

1. $x = \frac{7}{2}$, $y = -\frac{5}{2}$

2. $x = \frac{1}{5}$, $y = \frac{1}{6}$

3. $x = 12$, $y = 18$, $z = 6$

4. $x = 6$, $y = 11$, $z = 6$

5. $x = y = z = 12$

6. $x = 8$

7. $x = 10$

8. $x = 4$

9. $x = \frac{c(a-b)}{a}$

10. $x = bc + ca + ab$

11. $x = 13a$ or $-a$

12. $x = -2c-d$ or $-2c+d$

13. $x = 3a+b$ or $3a-b$

14. $x = \frac{3n+m}{4}$, $y = \frac{3n-m}{4}$

15. $x = \frac{ab-bm}{cl}$, $y = \frac{al-bm}{cm}$

16. $x = \frac{7a+8b}{9}$, $y = \frac{8a+7b}{9}$

17. $x = 4$ or $3/2$

18. (a) $x = 1.81$ or $.64$

18. (b) $x = -5.449$ or $-.551$

19. $x = \frac{-b - \sqrt{b^2-4ac}}{2a}$

20. (a) $x = \frac{a}{2}(\sqrt{5}-1)$ or $-\frac{a}{2}(\sqrt{5}+1)$

20. $x = \frac{1}{2}\left(a - \sqrt{a^2-4c}\right)$

21. $x = .55$ or $-.22$

22. (a) $x = -a$ or $-b$

22. (b) $x = \pm 3$ or ± 4

22. (c) $x = 4a$ or $-2a$ or a

23. (a) $x = 2; -1$
(b) $x = 0; 1; -2$
(c) $x = 2a; a; -3a$

24. $x = 3.73; .27$

25. $x = 4; -\frac{1}{4}; 3; -\frac{1}{3}$

26. (a) $x = 5; \frac{5}{4}$, $y = -2; \frac{1}{2}$

26. (b) $x = 5; 7$, $y = 7; 5$

27. (a) $x = \frac{1}{13}; \frac{1}{3}; -\frac{1}{3}; -\frac{1}{13}$, $y = \frac{1}{3}; \frac{1}{13}; -\frac{1}{13}; -\frac{1}{3}$

27. (b) $x = \frac{1}{2}; -2$, $y = 2; -\frac{1}{2}$

28. (a) $x = \pm 6$, $y = \pm 1$

28. (b) $x = \pm 1; \pm\frac{7}{2}$, $y = \pm 3; \pm 12$

29. (a) $x = \pm 5; \pm\frac{19}{3}$, $y = \pm 3; \pm\frac{8}{3}$

29. (b) $x = \pm\frac{3}{2}; \pm\frac{1}{2}$, $y = \pm\frac{1}{2}; \pm\frac{3}{2}$

30. (a) $x = \pm 1; \pm 2$, $y = \pm 2; \pm 1$
(b) $x = 3; \frac{1}{2}$, $y = 1; 6$

31. (a) $x = 6; 3$, $y = -3; -6$

31. (b) $x = 5; -\frac{7}{2}$, $y = 2; -\frac{11}{3}$

32. (a) $x = 9$

32. (b) $x = 2\frac{5}{6}$

33. (a) $x = \frac{5a}{4}$

33. (b) $x = \frac{2}{3}$ ($x = -\frac{5}{3}$ ext.)

34.(a) $x = 49$ 34.(b) $x = -\frac{ab}{a+b}$ 35.(a) $x = 64$

36.(a) $x = 8$ 36.(b) $x = 8$ 35.(b) $x = \frac{4a}{3}$

37.(a)

38. $15m-3m^{1/3}-2m^{-\frac{1}{3}}+8m$ 39. $x+x^{1/2}y^{1/3}+2y^{2/3}$

40.(a) $3x^{\frac{1}{2}}-2+x^{-\frac{1}{2}}$ 40.(b) $a^{2x}-3a^{x}-2$

41.(a) $b^{3/2}$ 41.(b) $x^{1/9}$ 42. $c^{3/2}$ 43.(a) $(a-b)^2$

43.(b) $ab(b^6-a^6)^{1/3}$ 44.(a) 4. 44.(b) 864. 45. 1

46.(a) $(\sqrt{x}+3)(\sqrt{x}-5)$ (d) $(x^m+3)(x^{2m}-3x^m+9)$

(b) $(x^{a/2}+7)(x^{a/2}-7)$ (e) $(x^{-c}+5)(x^{-c}-4)$

(c) $(3a^{\frac{1}{4}}+2)(a^{\frac{1}{4}}+1)$

47.(a) 1. (b) $\frac{a^{\frac{1}{2}}}{b}$ 48.(a) $19\sqrt{3}$ (b) $-4pq\sqrt{2pq}$

49.(a) 117.8897 (b) .4524 50. $\frac{c^7\sqrt{2c}}{x^2}$

51.(a) 1. (b) 2.

52.(a) $\frac{a+\sqrt{a^2-x^2}}{x}$ (b) $\frac{2\sqrt{3}+3-\sqrt{21}}{3}$ (c) $\frac{2\sqrt{3}+3\sqrt{2}+\sqrt{30}}{12}$

53.(a) $\frac{y^2}{x^2}$ (b) 1. 54.(a) $5-2\sqrt{6}$ (b) $\sqrt[4]{7}(\sqrt{\tfrac{7}{2}}-\sqrt{\tfrac{3}{2}})$

54.(c) $\sqrt{2x+1}+\sqrt{x-2}$ 54.(d) $\sqrt{m+3n}+\sqrt{m-3n}$

55.(a) 1. (b) 4. 57.(a) 5:21 (b) $\frac{x}{y} = \frac{b}{a}$ or $\frac{3b}{2a}$

64.(a) $6a^4$. (b) $\sqrt{3}$ (c) $12\sqrt{2}$

73.(a) $(cd^2e^3-1)(c^2d^4e^6+cd^2e^3+1)$
(b) $a^2(a-3)(a-b)$
(c) $(x+y)(x-y+1)$
(d) $(b-a)(7a-13b)$
(e) $(x-y+c-d)(x-y-c+d)$
(f) $(a^3-x-a-b)(a^3-3+a+b)$
(g) $(c+3d-3a)(c+3d+3a)$
(h) $(x^2-5y^2-3xy)(x^2-5y^2+3xy)$
(i) $(2m^2+n^2-5mn)(2m^2+n^2+5mn)$
(j) $(6a-yb)(5a+12b)$
(k) $(x-y)(x+y)(x^2+y^2)(x^4+y^4)$
(l) $3b+1)(2b-3)$
(m) $2c(c^2+3d^2)$
(n) $(x+3)(x-4)(x+7)$

(o) $(a+2)(3a-1)(2a-3)$

(p) $(\sqrt{a}-4)(\sqrt{a}-3)$

(q) $(x^{a/2}-7)(x^{a/2}+7)$

(r) $(3a^{\frac{1}{4}}+2)(a^{\frac{1}{4}}+1)$

(s) $(x^{-c}-4)(x^{-c}+5)$

(t) $(x^{m}+3)(2^{2m}-3x^{m}+9)$

74.(a) $x(2+x)$ (b) $\frac{x+4}{x(x-4)}$ 75.(a) $\frac{x}{(a-x)^2}$

75.(b) $\frac{8a^2x^3}{(a^2+x^2)(a^2-x^2)^2}$ 76.(a) 0.

76.(b) $\frac{-(y-z)+q(z-x)+r(x-y)}{(y-z)(z-x)(x-y)}$ 77.(a) $\frac{x^2+y^2}{2y^2}$ (b) $\frac{6(x-1)}{x-4}$

78.(a) $\frac{x-3}{x-4}$ (b) $(2x+5a)(x-2)$

79.(a) $\frac{2+x+3x^2}{2(1-x^4)}$ (b) $\frac{16x^4}{x^8-256}$ 80. 0.

81. 2.525 sq.in. 83. 21:28.5 84. 1.3.

85.(a) $x = -2$, $y = 5$ (b) $x = 3$, $y = -2$

86.(a) $(-3,2)$; $(4,1)$; $(3,4)$ (b) $7y-6x = 11$

87.(a) $x = 4, -1.6$; $y = 5, -6.2$ (b) $(-1,-5)$ (c) $x = 3.32$ or $.68$

88.(a) max. = 4 (b) min. = -5

89.(a) $x = 2.38$ or 4.62 (b) min. = -1.25

90.(b) $x = 2, -7$; $y = 7, -2$ 91. \$3,000 @ 3% / \$2,000 @ 3½% 92. \$1,800

93. 600 coins. 94. 91 men. 95. A, 18 miles; B, 12 miles

96. 6, 5 97. £750 98. 75

99.(a) 3 m.p.h. and 8 m.p.h. (b) 108 min.; 135 min.

100.(a) 5s. 6d. (b) 30 ft. x 40 ft. (c) £50.

ANSWERS TO ALGEBRA QUESTIONS
FOR COORDINATION

Paper I

1. $8a^{-2}-7a^{-1}+6$
2. (a) $(2x+3)(3x-4y)$
   (b) $(2a-b)(2a+b)(a^2+b^2)$
   (c) $(3r^x-5x^{3x})(3r^x+5x^{3x})$
   (d) $x(1+2ay)(1-2ay+4a^2y^2)$
3. $\frac{x-3}{x-1}$
4. $x = -2;\ y = -3;\ z = -1$
5. (a) $H = \frac{T}{2\pi R} - R$ (b) $H = \frac{1}{2}$
6. $\frac{ab}{c-a}$ hrs.
7. $x = \frac{bc^2}{a^2}$
8. x.
9. $x = 1$
10. $x = \frac{1}{2},\ 2$
    $y = 4,\ 1$

Paper II

1. $\frac{(a+b+c)^2}{2bc}$ 2. $x = -6$ 3. $x = 2$ 4. 15 yrs.

5. 20 @ 5¢
   60 @ 25¢

6. $x = 20$
   $y = 15$
   $z = -10$

7. (a) $(x^2+y^2-3xy)(x^2+y^2+3xy)$
   (b) $(\frac{x}{8} - \frac{4}{x})(\frac{x^2}{64}+\frac{1}{2}+\frac{16}{x^2})$
   (c) $(x-2)(x+2)(x^2+4)(x+1)(x^2-x+1)$
8. (c) $\frac{1\pm\sqrt{-5}}{4}$
9. 20 days @ $1.80.
10. $x = 2;\ 3;\ \sqrt{3}+3;\ \sqrt{3}-3$
    $y - 3;\ 2;\ \sqrt{3}-3;\ -\sqrt{3}-3$

Paper III

1. $\frac{4}{9}a^4 + \frac{1}{2}a^3x + \frac{9}{16}a^2x^2 + \frac{81}{128}ax^3$ 2. $\frac{ah(h-b)}{b}$

3. $133\frac{1}{3}$ lbs. @ 40¢. 4. $6.

5. $x = \pm 2;\ \pm 1$
$y = \pm 1;\ \pm 2$

6. $x = 2;\ y = 4\frac{1}{2};\ z = 3\frac{1}{3}$

7. (a) $3\sqrt{7} + 2\sqrt{5}$ 8. $x^{\frac{1}{4}} - y^{\frac{1}{2}}$

(b) $5\sqrt{2} - \sqrt{5}$ 10. 56.

Paper IV

1. $\frac{3a-13b+10c}{3}$ 2. (a) $k = 5$ (b) $x = 2;\ y = 1$

3. (a) $(x-2)(x-2)(x-2)$

3. (b) $(3x-4a)(4a-3x)$ (c) $(y-3x)(x-y)(x+y)$

4. (a) 12mn. (b) 9. 5. $\frac{10a^3 + a^2 - 2}{(a+2)(2a+1)(4a^2+2a+1)}$

6. $\frac{57}{115}$ 7. $x = -2$

8. $p = 157\frac{1}{3}$ 9. $1,000.

10. $x = \pm 4;\ \pm\frac{7\sqrt{2}}{3}$ (-3.299)
$7 = \pm 1;\ ;\pm\frac{4\sqrt{2}}{3}$ (-1.885)

Paper V

1. $-x^4(a-3b^2+c+7)-x(b+c^2)$ 2. 4 m.p.h. and 1 m.p.h.

3. 8, 4. 4. $3\frac{1}{8}$ 6. $x = 8;\ -6$
$y = 6;\ -8$

7. (a) $(3+4x)(4-5a)$

7. (b) $(3y-8)(9y^2+24y+64)$ (c) $x^{3n}(3x^n+1)(3x^n-1)$

7. (d) $(x-3+2y)(x-3-2y)$ 8. 4. 9. $\frac{4a}{a+2b}$

10. $x - \frac{-2\pm\sqrt{7}}{3}$ $\left(\begin{matrix}.215\\-1.548\end{matrix}\right)$

Paper VI

1. $x = 5$ 2. $x = -4$ 3. 68 yrs. 4. 16 yds.

5. $m = 2$; $n = 3$; $p = -6$. 6. $x = \frac{15}{4}, -1$

7. 12'; 5'; 13'. 8. $(x+y)\sqrt{xy}$ 9. -35.

10. (a) No. (b) $x = 2, -2$
$y = -3, 5$

Paper VII

1. $x = 5\frac{2}{3}$ 2. $\frac{x^{4/5}z^{1/5}}{y^{2/6}}$ 3. $x = 8$ 4. (a) $(0,0)$

4. (c) $\sqrt{x^2+y^2}$ 5. $21x^2-9x^3$ 6. $x = 8$

7. $\frac{8}{12}$ 8. (a) $x = .61a$; $-.52a$ (b) $x = 2.643$; $.7566$.

9. $x = \pm 5$; ± 2
$y = \pm 6$; ± 3 10. 4 yds.

Paper VIII

1. (a) $2(a-b)(x-4)(x+1)$ (b) $(a-b+c-3)(a-b-c+3)$

1. (c) $(a+bx)b+ax)$ 2. $x^{n^2-n}+x^{n-1}$ 3. $2-\sqrt{3}$

4. $x = 64$. 5. (b) $(-2,4)$ 6. $\frac{1}{x+1}$

7. $x = 9$. 8. $x = 3a$; $y = -2b$ 9. 100 benches.

10. $x = \frac{1}{3}$; $-\frac{1}{2}$
$y = \frac{1}{2}$; $-\frac{1}{3}$

Paper IX

1. (a) 24.64 sq.in. (b) $1\frac{3}{4}$ ft. 2. $\frac{a^{\frac{1}{2}}}{x\ \frac{13}{144}}$

3. (a) $(\frac{mn}{9}-1)(\frac{m^2n^2}{81}+\frac{mn}{9}+1)$

(b) $(x-2y-3z)(x^2+4y^2+9z\ +2xy+3xz-6yz)$

(c) $(x^2-3y^2-3xy)(x^2-3y^2+3xy)$

4. (a) $x = \frac{7m}{11}$; $-\frac{m}{2}$ (b) 2;3 5. 2.236.

6. $x = -8$; 0 7. (a) $(a^2-b^2-a+3)(a^2-b^2+a-3)$

7. (b) $(a+2b)(a^2+ab+b^2)$ (c) $ab(a-b)(a-b)(a+b)$

8. 1. 9. x. 10. $\frac{x^{2/3}-2}{x^{1/3}+2}$

Paper X

1. $v = 40$ 3. (a) 7. (b) $x=9$

4. (a) $x = 1.7236;\ 1.2764$

(b) $x = \pm 1;\ \pm 3.$
$y = \pm 5;\ \pm 1.$

5. 60 yd. x 90 yd.

6. $x = 11$

7. 642

8. $6x-5y = 8$

9. (a) $x = 3$
$y = -2$
$z = 1$

(b) $x = \pm\frac{1}{2};\ \pm 2$

$y = \pm\frac{3}{2};\ \pm 1$

10. 1.11803.

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