AN EXPERIMENTAL INVESTIGATION OF THE BIFURCATION IN TWISTED SQUARE PLATES

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Abstract

The bifurcation phenomenon occurring in twisted square plates with free edges subject to contrary self-equilibrating corner loading was examined. In order to eliminate lateral deflection of the test plates due to their own weight, a special loading apparatus was constructed which held the plates in a vertical plane. The complete strain field occurring at the plate centre was measured using two strain gauge rosettes mounted on opposing sides of the plate at the centre. Principal curvatures were calculated and related to corner load for several plates with differing edge length/thickness ratios. A Southwell plot was used relating mean curvature to the ratio mean curvature/Gaussian curvature, from which the Gaussian curvature occurring at bifurcation was determined. The critical dimensionless twist $\kappa a$ was then calculated for each plate size. It was found that there is a linear relation between the critical dimensionless twist $\kappa a$ occurring at bifurcation, and the thickness to edge length ratio $h/a$ ratio, specifically: $\kappa a = 10.8h/a$. 
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Chapter 1

Introduction

A state of pure twist occurs when a flat thin square plate with free edges is loaded laterally with equal and opposite corner loads. A linear analysis of this loading suggests that the deformed shape will be of an anticlastic surface (Figure 1.1). (Anticlastic refers to equal and opposite principal curvatures.) With respect to the square plate in question this curvature occurs along the diagonals, creating a saddle-like shape. Since the anticlastic surface is nondevelopable, membrane forces are generated in the mid-surface of the plate. The displacement field provided by linear theory however, is only accurate as long as the mid-surface strains are small compared to strains on the plate surface. More simply, the linear theory is accurate as long as the lateral corner deflections remain small. For larger loads, and increased corner deflection, nonlinear effects, due primarily to the membrane stress, become too great and cause the anticlastic deformed state to become somewhat unstable. At this point the deformed state tends towards one of synclastic shape, with generators parallel to the plate diagonals. This transition from anticlastic to synclastic state is illustrated by a divergence of the absolute principal curvatures with increasing load. Due to the symmetry of the plate and loading configuration, the cylindrical deflection can occur along either plate diagonal, each having equal preference (Figures 1.2 and 1.3). However once the plate has taken one of the two equilibrium states, it will continue to deflect in that state.
Chapter 1. Introduction

Twisted Square Plate

Anticlastic Shape

April 1991

plate as viewed from above

Figure 1.1: Plate showing anticlastic deformed surface

Twisted Square Plate

Synclastic Mode (i)

April 1991

plate as viewed from above

Figure 1.2: Plate deformed to synclastic surface, mode (i)
1.1 Purpose of Research

A plate torsion test, similar to that described, can be used to determine the various constitutive coefficients of thin plate materials. A single square plate torsion test may be used to determine, for example, Poisson’s ratio for an isotropic material. Several variably skewed plates may be used to determine the coefficients of the stiffness matrix, relating bending moments to curvatures, for orthotropic or more generally anisotropic plate materials. With the large number of composite materials currently in use, an accurate assessment of their fitness for use, after some portion of their life is vital. A small sample of a structure could be tested using this method. Such a test would indicate changes in compliance due perhaps to environmental degradation.

The general test, however is limited by:

- the nonlinear effects
- the instability of the deformed state
Chapter 1. Introduction

Nonlinearity can be accommodated. However due to its unpredictability, the instability of the deformed state poses the greatest limitation to the effective use of this test. An accurate prediction of the point of instability would thus provide significantly increased confidence in the test results.

Considerable work has already been performed in trying to accurately predict the point of instability. Much of it, however is of a purely theoretical nature, with little or no experimental validation provided. Lateral plate deflections, critical strains, and critical curvatures have all been used to experimentally predict bifurcation. In the case of critical curvature, problems were encountered with plate deflection due to self-weight [34]. It was felt that an alternative experimental setup, used to determine critical curvature, was worth examining and could provide a unique and valuable contribution to the present pool of experimental data related to this problem.

1.1.1 Scope of Current Thesis

The primary purpose of the current thesis is to examine the instability in the deformed state of the twisted square plate. Since significant numerical and analytical work examining the bifurcation has already been performed, providing conflicting results as to the onset of the instability, it was decided that the most valuable contribution would be through the use of an experimental approach. The production of valid repeatable experimental data would enable the validation of some of the current numerical and analytical models proposed. It would also help satisfy the apparently large contradiction in some of the theoretically predicted points of initiation of bifurcation. This thesis will examine the relation of corner load to curvature for a number of plate sizes, and the critical curvature occurring at bifurcation will be evaluated. Deflections due to plate self weight are eliminated through the use of an alternate experimental apparatus.
1.2 Literature Review

Kelvin & Tait [11] appear to be the first authors to examine the bifurcation phenomenon of the twisted square plate where in 1883 they noted the deviation of the deformed shape from that predicted by linear theory. They did not however elaborate beyond suggesting that the deviations occurred when the corner deflections were on the order of the plate thickness. It is interesting to note their physical interpretation of the collapse of Poisson's three boundary conditions into the two boundary conditions described by Kirchhoff. The collapse of these boundary conditions provide the fundamental description of the loading under consideration in this study.

Fung & Wittrick [5] studied the large deflection of thin plates. Using the concept of a boundary layer to explain the introduction of membrane forces, they examined the bending of rhomboid plates into cylindrical surfaces, of which the square plate is a specific case. Noted in the paper is the bi-stable phenomenon of the deformed square plate, and the similar phenomenon observed in the rhombic plate for sufficiently large curvatures. There is however no discussion of the point of transition between the anticlastic and synclastic modes.

1.2.1 Material Properties

A number of authors have examined the twisting of plates with the motive of determining the various constitutive properties of the plates.

Ramberg & Miller [26] in 1951 evaluated the twisted square plate test as one possible method of determining the shear-stress strain relation in isotropic materials. Considerable effort was expended in choosing an appropriate test specimen size, to minimize the effects not accounted for by linear theory. An a/h ratio of 10 was chosen, which in their opinion provided the best compromise between the nonlinear effects examined.
Based upon conservative estimates by Ramsey [27], the maximum corner loading used, calculated from Figure 6 in [26] was 33% of the bifurcation load. Thus the choice of this small specimen size almost certainly restricted the deflection to the anticlastic mode, and there is no mention of the transformation between modes. Also mentioned by the authors, was the significant difficulty encountered due to the small specimen size.

Miyagawa, Shibuya, & Udea [24, 23, 25] have also done considerable experimental work of a similar nature to that of Ramberg & Miller. The impetus for Miyagawa et. al.'s earlier work appears to be the determination of shear modulus. More recently, they experimented with the stability of square plates. This will be discussed in a subsequent section. Unlike Ramberg & Miller, Miyagawa et. al. experimented with significantly larger a/h ratios, and over a much larger range of ratios. They noted the limiting strains, beyond which the square plate torsion test could no longer be used to obtain valid constitutive data.

The extension of the twisted square plate method for use in the evaluation of elastic constants of anisotropic as well as isotropic plates has been proposed and evaluated by several authors.

Hearmon & Adams [6] in 1951, compared linear theory with results obtained from the testing of plywood plates, which they considered as generally orthotropic. Using 6in square plates with a/h ratios from 19 to 32, cut at various angles with respect to the face grain, 'small' corner loads were applied, and lateral deflections measured. Data observed beyond the linear region was discarded. The authors noted generally close agreement between theoretical and experimental values, although large discrepancies occurred in some instances. Hearmon & Adams pointed out however, that the properties of the wood tend to vary significantly throughout the sheet. This might help explain some of the discrepancies observed.

Tsai [31] also examined the twisted square plate test for use in the determination of
elastic constants. He proposed the twisting of two square plates with fibre orientation at 0° and 45°, and the bending of a beam with 0° fibre orientation. Experimental validation provided favourable comparisons to those predicted by theory. Tsai did not discuss the maximum loading, nor the bifurcation of the square plate.

Whitney [33] in 1967, evaluated the twisted square plate test for use in the evaluation of shear modulus. He demonstrated that a single test is not always sufficient for the determination of shear modulus. In the case where bending and extensional coupling is present, a single twist test will not determine the shear modulus alone, but rather as a combination of elastic constants. Whitney did not comment on the problems encountered with bifurcation.

Chandra [1] in 1976, examined the twisted square plate test for orthotropic materials in the 'large deflection regime', using a numerical approach. He did not examine the bifurcation question, nor did he provide any experimental validation for his results.

Lee & Biblis [13] in 1977 examined the method proposed by Tsai [31]. They used this method to measure the constitutive coefficients of wood plates. They found their experimental results to be 'in excellent agreement' with theory. No comment was made on possible problems that might be encountered for larger deflections.

1.2.2 Plate Stability

A number of authors have studied the question of stability in the twisting of plates into the large deflection regime.

Reissner [28] examined the twisting of rectangular plates using nonlinear Von Karman equations. He demonstrated the existence of some form of instability in the relation between applied moment and resulting curvature.

Foye [4] discussed the limitations of the plate twist test with respect to the post linear behavior of the plate. An analytical approach was used to provide a relation between
corner load and corner deflection, the result of which illustrated a bifurcated path.

More recently, Miyagawa, Shibuya, & Udea [24, 23] studied the bifurcation experimentally over a large range of a/h ratios. They related corner load to principal strains occurring along the plate diagonals. The onset of the instability, which they referred to as critical strain, was illustrated as a function of the a/h ratio.

In later work by the same authors [7, 25] a numerical solution was provided, relating dimensionless corner load to corner displacement. The critical dimensionless corner load producing bifurcation reported was \( Pa^2/Dt = 22.8 \) for \( \nu = 1/3 \). However the authors suggest based upon experimental work also performed, that bifurcation does not occur when a/h ratios are less than 65. This suggestion appears to be in conflict with their earlier work, as illustrated in Figure 5 of [23] where the bifurcation of plates with a/h ratios as low as 35 were reported.

Further evaluated by the authors was the possible occurrence of local yielding. The previous numerical solution was extended to predict the yield load as a function of the a/h ratio. The results suggest yielding will occur prior to bifurcation for a/h ratios less than 55, based upon the dimensionless bifurcation corner load of \( \bar{P} = 22.8 \). The experimental results provided, appear to follow the numerical results in only a general sense.

Lee and Hsu [12] studied the stability of a broader class of similar saddle-like deformed plates using a finite difference scheme based upon the Von Karman nonlinear plate equations. Their study provided displacement fields for plates with different applied twisting moments. In the case of square plates their results suggested that bifurcation occurs when a critical dimensionless moment \( \text{Mc}_{\text{r}}=21 \). Only the case where \( \nu = 0 \) was presented in the paper.

Ramsey [27] also studied the problem and provided an analytical solution using nonlinear shell theory. His results suggest that bifurcation occurs when the dimensionless twist is a linear function of the a/h ratio or specifically, \( \kappa a = 3.29h/a \). He further relates
this result to the critical dimensionless moment of Lee and Hsu [12], where $M_{cr} = 1.42$.

Most recently Williams [34] provided an experimental analysis of the twisted square plate problem, where critical curvature as a function of the $a/h$ ratio was examined. Strain gauge rosettes were used on both sides of the plate, thus providing a full measure of the nonlinear effects. Deflection of the plates due to self weight as a result of their horizontal configuration was cited as a considerable problem.
Chapter 2

Theory

The surface formed by a square elastic plate loaded by equal and opposite twisting moments distributed along opposite edges is anticlastic. Principal curvatures, equal in magnitude, and opposite in sign, occur along the diagonals of the plate (see Figure 2.1). Uniform twisting moments may be replaced by equi-spaced opposing forces $P/2$ distributed along the edges, distance $\delta$ apart, thus forming a series of couples, statically equivalent to the twisting moments (Figure 2.2). The equal distribution of opposing forces $P/2$ produces concentrated corner loads of $2P/2$, or simply $P$. All the other applied forces are canceled by their adjacent pair.

These concentrated loads acting on all four corners of the plate are equivalent to uniform twisting moments $M_{x'y'}$, along the edges. According to St. Venant, loading by a system of concentrated corner forces is equivalent to loading by twisting moments along the edges, at distances of several plate thicknesses away from the edges. This then provides the current system of loading in question (Figure 2.3).

It is convenient to use two sets of rectangular Cartesian coordinates when referring to the plate. As illustrated in Figure 2.1, the $\hat{x}-\hat{y}$ coordinate system has its origin at the plate centre, and axes parallel to the plate edges. The $x-y$ coordinate system, similarly has its origin at the plate centre, but with axes rotated by $45^\circ$ counterclockwise, which are thus coincident with the plate diagonals. In both cases the positive $z$ direction is upward.

Using Mohr's circle, principal bending moments can be shown to occur at $45^\circ$ to the
Chapter 2. Theory

Twisted Square Plate

Figure 2.1: Plate twisted by uniform twisting moments $M$

Twisted Square Plate

Figure 2.2: Plate deformed by uniformly distributed twisting couples $(P/2)\delta$
plate edges, hence along the plate diagonals, the $x$ and $y$ axes. Since these are principal bending moments, and zero twisting moment occurs along these axes, it follows that,

$$M_x = -M_y = M_{\bar{x}\bar{y}} = P/2,$$

$$M_{xy} = M_{\bar{x}} = -M_{\bar{y}} = 0.$$ (2.1)

Using plate theory, it is assumed that,

1. normals to the neutral surface remain normal, and

2. deflections are small.

The moment-curvature relations for an isotropic plate, for a general state of loading, can be written,

$$M_x = D(\kappa_x + \nu \kappa_y),$$

$$M_y = D(\kappa_y + \nu \kappa_x),$$

$$M_{xy} = D(1 - \nu)\kappa_{xy},$$ (2.2)
where κₓ, κᵧ are the curvatures in the x, y directions, κₓᵧ is the plate twist, ν is Poisson's ratio, and D the flexural rigidity is given by,

\[ D = \frac{Eh^3}{12(1 - \nu^2)}. \]  

(2.3)

Here E is Young's modulus, and h is the plate thickness.

Solving equations (2.1), (2.2), and (2.3), it may be shown thus for the twisted square plate,

\[ \kappa_x = -\kappa_y = \frac{6(1 + \nu)P}{Eh^3}, \]

\[ \kappa_{xy} = 0. \]  

(2.4)

Equation (2.4) gives curvatures which are equal in magnitude and opposite in sign. These curvatures are also principal curvatures, since the plate twist is zero.

Again, assuming small deflections, the curvature plate-deflection relations for a general state of loading are,

\[ \kappa_x = \frac{\partial^2 w}{\partial x^2}, \]

\[ \kappa_y = \frac{\partial^2 w}{\partial y^2}, \]  

(2.5)

\[ \kappa_{xy} = \frac{\partial^2 w}{\partial x \partial y}. \]

Positive curvature is concave upward.

The lateral deflection of the plate at any point may be obtained by equating equations (2.3), (2.4) and (2.5), and integrating. If the plane z = 0 is used to define the mid-surface of the plate, then

\[ w = \frac{M_x - \nu M_y}{2D(1 - \nu^2)} x^2 + \frac{M_y - \nu M_x}{2D(1 - \nu^2)} y^2. \]  

(2.6)

For equal and opposite applied moments, in terms of M,

\[ w = \frac{M}{2D(1 - \nu)} (x^2 - y^2). \]  

(2.7)
By equating (2.1), (2.3), (2.4) and (2.7), the deflection of the twisted plate can be expressed as a function of the principal curvature,

$$w = \frac{\kappa(x^2 - y^2)}{2}.$$  \hfill (2.8)

Strains due to bending, according to the Kirchhoff assumption, vary linearly with the distance $z$ from the mid-surface, and for a plate under a general state of loading, are described by,

$$\begin{align*}
\epsilon_x &= -z \frac{\partial^2 w}{\partial z^2} = -z \kappa_x, \\
\epsilon_y &= -z \frac{\partial^2 w}{\partial y^2} = -z \kappa_y, \\
\gamma_{xy} &= -2z \frac{\partial^2 w}{\partial z \partial y} = -2z \kappa_{xy}.
\end{align*}$$  \hfill (2.9)

Surface strains, occurring at $\pm h/2$, are then given by,

$$\begin{align*}
\hat{\epsilon}_x &= -\left(\pm \frac{h}{2}\right) \kappa_x, \\
\hat{\epsilon}_y &= -\left(\pm \frac{h}{2}\right) \kappa_y, \\
\hat{\gamma}_{xy} &= -\left(\pm h\right) \kappa_{xy}.
\end{align*}$$  \hfill (2.10)

The caret $\hat{}$ refers to strains occurring on the surface. For anticlastic bending the curvatures are equal and opposite, and twist is zero. Thus,

$$\begin{align*}
\hat{\epsilon}_x &= -\epsilon_y, \\
\hat{\gamma}_{xy} &= 0.
\end{align*}$$  \hfill (2.11)

Where mid-surface strains are present, the two principal surface strains measured on opposing sides of the plate will contain equal membrane components. Mid-surface strains are signified by the superscript $^o$, and the plate sides are indicated by subscripts $A,B$. Hence,

$$\begin{align*}
\hat{\epsilon}_{xA} &= \epsilon_x^o - \frac{h}{2} \kappa, \\
\hat{\epsilon}_{xB} &= \epsilon_x^o + \frac{h}{2} \kappa.
\end{align*}$$  \hfill (2.12)
Mid-surface strains may be calculated by taking the sum of the surface strains measured on opposing sides of the plate. Then,

\[ \epsilon_{xA} + \epsilon_{xB} = 2\epsilon_z. \] (2.13)

Similar relations hold for \( \gamma_y \) and \( \gamma_{xy} \). Curvatures may be calculated by taking the difference of the surface strains measured on opposing sides of the plate,

\[ \epsilon_{xA} - \epsilon_{xB} = -h\kappa_z. \] (2.14)

Again, similar relations hold for \( \kappa_y \) and \( \kappa_{xy} \).

For isotropic materials the relation between the three elastic constants can be expressed as,

\[ G = \frac{E}{2(1+\nu)}. \] (2.15)

By combining equations (2.4) and (2.15), and solving for the shear modulus \( G \), the shear modulus can be expressed as a function of corner load and resulting principal curvature by,

\[ G = \frac{3P}{\kappa h^2}. \] (2.16)

Alternately Poisson's ratio can be determined as a function of corner load and principal curvature by rearranging equation (2.4),

\[ \nu = 1 - \frac{Eh^2\kappa}{6P}. \] (2.17)

2.1 Limitations of the Linear Theory

The linear theory neglects several significant factors. The most significant factor is due to the exclusion of mid-surface strains. As a plate is twisted by the corner forces and
transverse displacements become of the order of the plate thickness, mid-surface strains become measurable. The relative error in bending strain, as suggested by Ramberg and Miller [26], increases with the square of the surface strain, and by the fourth power of the $a/h$ ratio. Hence for larger $a/h$ ratios, this effect is increasingly important to consider.

Transverse shear effects may cause some error when predicting plate behavior. These effects, however, are generally small for large $a/h$ ratios. Also, since the areas of interest are generally more than one plate thickness away from the edge, transverse shear effects can generally be neglected.

Edge effects due to the assumption that $\tau_{xy}$ is constant along the full length of the plate cross-section will also provide error. The theory assumes that the concentrated corner forces are carried by $\tau_{xy}$ alone. According to St. Venant’s principle, errors resulting from this will be small at distances greater than one plate thickness away from the free edge. Thus for larger $a/h$ ratios, this effect will be negligible.

2.2 Discussion on the Implications of Nonlinearity

Since it is not the purpose of this thesis to provide approximate solutions to the plate twisting problem, and since there is no closed form analytical solution to the nonlinear analysis of the plate twisting problem, nonlinear theory will not be examined beyond a brief discussion of its implications. The classical nonlinear theory of von Karman, Timoshenko [29], provides for the inclusion of mid-surface strains. It is obvious that linear theory cannot adequately describe the real behavior of plates for large deflections. The simplicity of linear theory however, is of considerable value in the initial examination of the behavior of a plate.

In an anticlastic deformed state, in the non-linear range, there is some strain energy stored as a result of plate bending, however there is a large amount of strain energy
stored as a mid-surface strains. In the synclastic deformed state, the majority of the strain energy stored is due to bending strains. At some point, termed the bifurcation point, the amount of strain energy stored due to membrane action starts to exceed that stored in bending, and the deformation mode of the plate changes so as to reduce the amount of strain energy due to mid-surface strain.
Chapter 3

Description of the Experiment

3.1 Apparatus

The loading apparatus consisted of a steel base plate held vertically with one diagonal in the vertical direction. Two horizontal reaction columns were fixed along the vertical diagonal of the base plate and supplied passive compressive loads to opposing corners of the test plate. One passive horizontal tensile cable and one active horizontal tensile cable acted through the horizontal diagonal of the base plate, and supplied tensile loads to the remaining two opposing corners of the test plate. The end of the passive cable was fixed. The active cable was loaded by a series of calibrated weights (see Figure 3.1). This created the test plate self equilibrating loading system with one active tensile load, and three passive loads, two compressive and one tensile.

The \(762\,mm \times 762\,mm \times 6\,mm\) base plate was machined with a series of holes and slots located along both the diagonals. This arrangement allowed the loading cables, columns, pulleys, etc. to be moved to accommodate different sizes of test plates. The test plate was hung from a hanger cable, made from \(1.6\,mm\) multi-strand cable, attached to a \(150\,mm\) cantilever beam that was affixed to the top corner of the base plate. A \(25\,mm\) diameter steel ring was used to attached to the lower end of the hanger cable to the test plate.

A hemispherical seat was machined in the end of the upper horizontal compressive reaction column. A \(6\,mm\) ball bearing was fitted into this seat. When properly aligned,
Figure 3.1: Test plate loading apparatus
Figure 3.2: Upper compressive reaction column for test plate loading apparatus

the ball also fitted into a corresponding seat milled into the test plate (see figure 3.2). This arrangement served to:

1. locate the plate positively with respect to the support;

2. provide a point reaction load;

3. allow for free corner rotation of the test plate.

The end of the lower horizontal compressive reaction column had a 32\text{mm} diameter roller, 10\text{mm} thick, fitted with a small precision ball bearing. The roller was fixed such that the rotation was restricted to the vertical diagonal axis of the base plate. To restrict the loading to a point, a 5\text{mm} radius was turned on the outside of the roller (see Figure 3.3). The roller support:

1. allowed axial movement of the plate due to lateral deflection of the centre portion of the plate;
Lower Compressive Reaction Column
roller point load

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Figure 3.3: Lower compressive reaction columns for test plate loading apparatus

2. provided a horizontal point reaction load;

3. allowed for free corner rotation of the test plate.

The passive horizontal reaction cable passed through a small hole drilled in the test plate corner, and through a 5mm ball bearing, where it was fixed with a crimp in the cable. The ball bearing was matched to a hemispherical seat machined in the test plate corner. The other end of the fixed cable was passed through a hole in the base plate then fixed with a nut and bolt. This arrangement also:

1. allowed some axial movement of the plate due to lateral deflection of the centre portion of the plate;

2. provided a horizontal point reaction load;

3. allowed for free corner rotation of the test plate.
The active loading cable was passed through the plate corner in a manner identical to that of the passive reaction cable. The opposing end was passed through the base plate, over a sheave, such that the end hung vertically. A hanger was hooked through a loop fitted in the end of the active loading cable. Calibrated weights were then placed on the hanger providing the active load. The sheave was attached to the back of the base plate, such that the rotation of the sheave was in a vertical plane.

The apparatus was designed in general to allow the plate corners to rotate freely. A roller fitted at the lower load point allowed free inplane movement of the corners. Precision bearings were used in the sheave and roller to minimize any friction. By mounting the test plate in a vertical plane, rather than a horizontal plane, bending of the test plate due to its weight was eliminated.

3.2 Instrumentation

It was decided, based upon a review of current literature, that the use of strain gauges would be most suitable for the measurement of the data of interest. Several other methods were examined but were judged not to provide the ease, accuracy or consistency of strain gauges. Since the plate deflections were expected to generate significant membrane strains, the simultaneous measurement of strain on both sides of the plate was deemed necessary.

3.2.1 Strain Gauges

The strain gauges selected had the designation, Micro-Measurements CEA-13-125UR-350. The supplier[15] listed this particular gauge as:

“A universal general purpose strain gauge. Constantin grid, completely encapsulated in polyimide, with large rugged copper-coated tabs. Primarily
used for general purpose static, and dynamic stress analysis."
The gauge had the following characteristics:

- Resistance $350\Omega \pm .4\%$
- Gauge length $.125$ in
- Nominal gauge factor 2.0
- Configuration: $45^\circ$ single-plane rosette

The gauge was thermally mated with aluminum for self temperature compensation. A rosette configuration was chosen to allow for accurate measurement of the principal strain directions.

Gauges were mounted according to the manufacture's instructions [20, 22] using the Micro-Measurements group M-bond special grade methyl-2-cyanoacrylate adhesive. The $90^\circ$ arms of the gauge were approximately roughly aligned with the test plate diagonals. Three lead wires per gauge were soldered and strain relief loops were included. Correct gauge installation was checked through the use of a null resistance check. Complete bonding was checked through the use of a pencil eraser tap test. Gauges were then sealed using m-coat air-drying polyurethane coating to prevent gauge degradation due to moisture.

3.2.2 Strain Measurement

Several different strain gauge circuits were used before a half bridge arrangement was settled upon. This bridge wiring arrangement employs one active arm, and one dummy arm, used for temperature compensation. In the final arrangement, another test plate of identical thickness, but different edge length was used to provide the temperature
Chapter 3. Description of the Experiment

compensating dummy gauges. An illustration of the bridge circuit is provided in Figure 3.4.

Two different strain indicators were employed over the course of the experiment to provide strain readings. These were:

- Vishay P-350A Digital Strain Indicator
- Vishay P-3500 Digital Strain Indicator

The first was used in the initial setup of the experiment, and to gather the first sets of data. When it became no longer available, the P-3500 was purchased. The P-3500 provided superior strain resolution, and was used to determine the latter sets of data.

The Vishay SB-1 switch and balance unit was used to switch the indicator between each of the six gauge circuits used on the plate being tested. This arrangement allowed each of the circuits to be independently balanced prior to loading.
Table 3.1: Plate sizes and $a/h$ ratios

<table>
<thead>
<tr>
<th>Plate Number</th>
<th>Edge Length $a$</th>
<th>Plate Thickness $t$</th>
<th>Nominal $a/h$ ratio</th>
<th>Flange Width $e$</th>
<th>Flange Depth $f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>152.4</td>
<td>3.099</td>
<td>48:1</td>
<td>15.24</td>
<td>7.62</td>
</tr>
<tr>
<td>2</td>
<td>203.2</td>
<td>3.216</td>
<td>64:1</td>
<td>20.32</td>
<td>10.16</td>
</tr>
<tr>
<td>3</td>
<td>254.0</td>
<td>3.162</td>
<td>80:1</td>
<td>25.40</td>
<td>12.70</td>
</tr>
<tr>
<td>4</td>
<td>304.8</td>
<td>3.175</td>
<td>96:1</td>
<td>25.40</td>
<td>12.70</td>
</tr>
<tr>
<td>5</td>
<td>609.6</td>
<td>3.099</td>
<td>192:1</td>
<td>25.40</td>
<td>12.70</td>
</tr>
</tbody>
</table>

dimensions in $mm$

3.3 Plate Design

Initial plate design was dictated by previous experiments performed by Williams[34]. Only minor modifications were made to this design. All plates were manufactured from 6061-T6 aluminum sheet, with a nominal thickness of 3.175$mm$. The sizes ranged from edge widths of 635$mm$ to 152.4$mm$. Also see Table 3.1 and 3.2. Dimensions $e$ and $f$ are referred to in Figure 3.5.

Modifications performed to the plates include the machining of hemispherical seats at the plate corners, and the drilling of 1.6$mm$ holes through the seats. Further modifications to the previous design were deemed unnecessary. Figure 3.5 illustrates a typical plate layout.

To enable consistent referencing, each of the plates, and gauges on the plates, were labelled. Plates were labelled 1 through 5, where plate 1 referred to the smallest plate with $a/h$ ratio of 48:1, and plate 5 referred to the largest plate with $a/h$ ratio of 192:1. Each plate had sides labelled A and B, these sides were referenced by some of the plots illustrating the output data. The plates each had six strain gauges in two sets of three.
Table 3.2: Material properties for T6061-T6 Aluminum

<table>
<thead>
<tr>
<th>Material Properties</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ultimate tensile strength</td>
<td>$\sigma_u$</td>
<td>310 MPa</td>
</tr>
<tr>
<td>Ultimate shear strength</td>
<td>$\tau_u$</td>
<td>207 MPa</td>
</tr>
<tr>
<td>Yield tensile strength</td>
<td>$\sigma_y$</td>
<td>276 MPa</td>
</tr>
<tr>
<td>Elastic modulus</td>
<td>$E$</td>
<td>69 GPa</td>
</tr>
<tr>
<td>Shear modulus</td>
<td>$G$</td>
<td>26 GPa</td>
</tr>
<tr>
<td>Poisson ratio</td>
<td>$\nu$</td>
<td>.33</td>
</tr>
<tr>
<td>Density</td>
<td>$\rho$</td>
<td>2.70 Mg/m$^3$</td>
</tr>
</tbody>
</table>

Individual gauges were labelled from one through six, starting on side A. Gauges 1 and 4, 2 and 5, and 3 and 6 were located identically on opposite sides of the plate.

3.4 Experimental Procedure

The following was a typical procedure for an individual test:

1. The gauges were initially balanced to zero by means of the switch and balance;
2. Loads were then incrementally applied by means of a calibrated set of weights;
3. At each step the corner load applied and each of the six indicated strains were recorded;
4. The plates were loaded well beyond the perceived bifurcation point;
5. The load was incrementally reduced, and the strains recorded after each reduction of load, providing an indication of any hysteresis.

This procedure was performed for each plate size, and several trials were run, thus providing good confidence in the recorded data.
Raw strain data from each of the individual gauges was processed by the Fortran routine COR1 (see Appendix C). The program applied several routines to produce error corrected data. The corrections that were applied are as follows:

1. Bridge nonlinearities due to the half bridge configuration were corrected. These errors were minor, since the strain levels encountered were relatively small.

2. Since each individual gauge in a rosette had a different gauge factor, and neither of the strain indicators had individual channel gauge factor adjustments, a common gauge factor was used for each of the gauges. The raw data were then corrected such that the use of the correct gauge factor for each individual gauge was reflected in the result.

3. Transverse sensitivity corrections, based upon the individual gauge’s transverse sensitivity quoted by the manufacturer, were applied to the raw data.
4. The corrected individual strain data was then used to calculate the principal strains, principal curvatures, and principal strain directions.

Data were output to an output file containing corner load, corrected principal strain, corrected principal curvature, and gauge misalignment. These data were used to generate graphical output using the plotting routine Plotdata. Several macro programs were written to run Plotdata in a batch environment. The different macros were used to produce all of the output plots in the thesis.
Chapter 4

Experimental Results

The experimental results gathered throughout the various tests are presented in this section. Results for the individual plates are presented in dimensional form, each on a single plot. Note that all data has been processed by the Fortran routine COR1, described in Chapter 3, and reproduced in Appendix C.

Both the strain and curvature plots refer to strains and curvatures in principal coordinates. In both cases, absolute strains and curvatures are presented. Negative strains are listed as such in the legend included on the plot. Principal strain plots are included in Appendix A, Figures A.1–A.4. In the case of curvature, the choice of coordinate system defines the sign of the curvature. Side A of the plates has been chosen as the positive $z$ direction. Curvatures concave towards side A are thus positive, and curvatures convex towards side A are negative. Curvatures shown on the plots are considered positive unless indicated as negative in the legend.

The #1, and #3 directions described on the plots refer to the $x$–$y$ coordinate system fixed to the test plate. The two directions are thus orthogonal, lie in the plane of the plate, and are aligned with the plate diagonals. Two other plate ‘configurations’ are also referred to on the plots. In the ‘0° configuration’ the #1 direction fixed to the plate and the vertical direction fixed in space are coincident, and the #3 direction fixed to the plate is horizontal. In the ‘90° configuration’ the #3 direction fixed to the plate and the vertical direction are coincident, i.e. the plate has been rotated about the $z$ axis by 90°. The 90° configuration is not presented until later.
Figure 4.1: Plate 1, a/h ratio 48:1, absolute principal curvatures vs corner load
Figure 4.2: Plate 3, a/h ratio 80:1, absolute principal curvatures vs corner load
Figure 4.3: Plate 4, a/h ratio 96:1, absolute principal curvatures vs corner load
Figure 4.4: Plate 5, a/h ratio 192:1, absolute principal curvatures vs corner load
Chapter 4. Experimental Results

The corner load refers to the lateral opposing loads applied to opposite corners of the plate. Plate curvature is defined as the inverse of the radius of curvature of the plate. Absolute principal curvature is displayed so both curvatures may be shown on the same plot.

Figure 4.1, absolute principal curvature vs corner load for plate 1 with a/h ratio of 48:1, has two coincident curves originating at the origin. At small loads the curves approximately follow the line corresponding to linear theory. At larger loads, the slopes increase with increasing corner load, illustrating the presence of nonlinear effects. There is only a slight divergence of the two principal curvatures at the maximum corner load.

Figure 4.4, plate 5 with a/h ratio of 192:1, has a similar pair of coincident curves originating at the origin. At small loads, the curves again approximately follow the line corresponding to linear theory. At larger loads there is a greater increase in slope as compared to linear theory than was apparent with the smaller a/h ratio. This increase also occurs much earlier than that illustrated in Figure 4.1, by the smaller a/h ratio. The divergence of the two principal curvatures for plate 5, is quite dramatic, and begins occurring at only 20% of the maximum load. Figures 4.2, and 4.3, principal curvatures of plates 3 and 4, illustrate intermediate effects between the two extremes.

4.1 Discussion of Errors

During the experiment some initial difficulty was encountered with accuracy requirements. The accuracy of the strain gauges and strain indicator is measured in terms of relative error. In general the gauges used had a relative error of 0.5%, which by itself appears small. However when two large strains of similar magnitude were compared, the error became greater than 0.5%.

One possible solution to this problem was the subtraction of the strains electrically,
using a different bridge circuit. However in this case, only the bending strains would be measured, not the total strain, and thus the mid-surface strains could not be evaluated. Since the measurement of nonlinearity was of considerable importance, the difference was calculated after the strain measurement. The use of the more precise strain indicator, and corrections for individual gauge factors was employed to minimize the error.

Another common problem observed was zero shift and hysteresis in some of the measured results. This can occur for a number of reasons, and is often a property of the material being strained. It is also expected that during the first several strain cycles, the epoxy used to bond the gauges will exhibit some strain hardening. In most cases this effect was very small and caused little problem. Plate 2 however exhibited huge zero shifts especially during the later sets of tests. Initially this problem was attributed to the application of a new set of strain gauges, however continued cycling failed to eliminate the problem. It was eventually noticed that the plate was taking an observable set deformation at the end of each loading cycle.

Prior to testing, the yield loads were calculated based upon von Mises yield criterion. The plates were loaded only to within approximately 70% of this value. Subsequent Rockwell hardness tests on all of the plates revealed a substantially lower hardness in plate 2, as compared to the other plates (see Table 4.1). Under identical measurement conditions a significantly lower hardness value would indicate a significantly lower yield strength. A further review of Williams' thesis revealed that he too had encountered difficulty with this particular plate (see pg. 71 in [34]). It thus appears that plate 2 was manufactured from a material with properties different from the other plates, and had a yield strength much less than the others. Initial testing, as indicated by the marked deformation, presumably caused significant yielding in the plate, and set up a residual strain field in the plate. This caused anomalous results as compared to the other plates. For this reason, data pertaining to plate 2 has been excluded from the body of the report,
Table 4.1: Rockwell hardness test results for the five test plates, () signifies value registered below zero on $R_F$ scale

<table>
<thead>
<tr>
<th>Plate no.</th>
<th>a/h ratio</th>
<th>$R_F$</th>
</tr>
</thead>
<tbody>
<tr>
<td>plate 1</td>
<td>48:1</td>
<td>63</td>
</tr>
<tr>
<td>plate 2</td>
<td>64:1</td>
<td>(10)</td>
</tr>
<tr>
<td>plate 3</td>
<td>80:1</td>
<td>70</td>
</tr>
<tr>
<td>plate 4</td>
<td>96:1</td>
<td>70</td>
</tr>
<tr>
<td>plate 5</td>
<td>192:1</td>
<td>65</td>
</tr>
</tbody>
</table>

and included in Appendix B.

One of the measures used to show confidence in an experimental result is the repeatability of the results. Only plate 2 did not exhibit excellent repeatability, the reasons for which have already been discussed. The maximum deviation of principal curvature data between different trial runs was on the order of 5%. This small deviation provided considerable confidence in the experimental results recorded.
Chapter 5

Discussion of Results

The data in Chapter 4 were presented in a form where the results for an individual plate were shown on an individual figure. To allow a more general interpretation of the results, the graphs presented beyond this point in most cases are presented in dimensionless form. Individual plate results are thus collapsed, such that all results are displayed on a single figure.

5.1 Dimensionless Groups

The dimensionless groups can be determined by evaluating the number of dimensionless quantities, and evaluating various multiples of each quantity until the results collapse. The dimensionless quantities associated with the twisted plate problem include the following,

\[ a/h, \epsilon, \kappa, \frac{P}{Eh^2}. \]  

(5.1)

Since \( \nu \) is identical for all the plates, and is itself dimensionless, it is not included. The groups formed to provide the best collapse of the data are

\[ \frac{Pa^2 \epsilon a^2 \kappa a^2}{Eh^4 h^2 h}. \]  

(5.2)

Figures 5.1 and 5.2 provide the curvature data in dimensionless form. Dimensionless strain data is included in Appendix A, Figures A.5–A.8.

Figures 5.1 and 5.2 illustrate the dimensionless curvature as a function of dimensionless corner load in the #1 and #3 directions respectively. In the 0° configuration, the
Figure 5.1: #1 direction, 0° configuration, dimensionless principal curvature vs dimensionless corner load.
Figure 5.2: #3 direction, 0° configuration, dimensionless principal curvature vs dimensionless corner load
#1 direction corresponds to the vertical direction. The dimensionless groups provide an excellent collapse of the results in the linear region, with a small degree of scatter at the higher corner loads. The points initially follow the linear prediction closely. The deviation from linearity increases with increasing corner load. A comparison of the two directions, #1, and #3 (Figures 5.1 and 5.2), illustrates the general divergence of the two principal curvatures with increasing corner load. This divergence indicates the transition between the anticlastic and synclastic modes that occurs during bifurcation.

One obvious anomaly visible on both Figures 5.1 and 5.2, is the large deviation of the data for plate 5, with a/h ratio 192:1. With closer observation one would notice that the data for plate 5 in Figure 5.1, closely follows the general curve of Figure 5.2 better than 5.1, and vise versa. This observation in itself however is relatively unimportant. Recall that a bifurcating phenomenon is observed where cylindrical deflection can occur along either diagonal, each with equal preference. What is illustrated by plates 1, 3, and 4 with a/h ratios 48, 80, and 96:1, is a bifurcation to the cylindrical mode, where generators are parallel to the #1 diagonal. Thus the curvature in the #1 direction is decreasing and the curvature in the #3 direction is increasing. Plate 5 with a/h ratio of 192:1, however has bifurcated to a cylindrical mode such that generators are parallel to the #3 diagonal. This preferential choice was noted consistently for each of the test trials performed.

To investigate the possible causes of this preferential bifurcation, another series of tests were run, this time with the plates rotated by 90°, to the '90° configuration', causing the #3 direction to be aligned with the vertical, and the #1 direction, the horizontal. This rotation had the effect of reversing the lateral loading directions, with respect to the plate. It was postulated that the cause of this preference was due either to plate initial curvature, plate anisotropy, or bias in the loading apparatus. The reversal of the loading direction was expected to provide insight into the mechanism causing the preference.
Figure 5.3: #1 direction, 90° configuration, dimensionless principal curvature vs dimensionless corner load
Figure 5.4: #3 direction, 90° configuration, dimensionless principal curvature vs dimensionless corner load
Figures 5.3 and 5.4, show the dimensionless curvature results of these tests. Dimensionless strain results are included in Appendix A, Figures A.9–A.12. Plates 1, 3, and 4, bifurcated into the cylindrical mode such that generators were aligned parallel to the #3 direction, plate 5, bifurcated into the cylindrical mode such that generators were aligned parallel to the #1 direction.

Since the generator axis switched in every case, plate anisotropy, was immediately eliminated as a possible cause. A closer comparison of Figures 5.1 and 5.2 with respect to Figures 5.3 and 5.4, reveals a consistent feature. In the 0° configuration (Figures 5.1 and 5.2), plates 1, 3, and 4 all show bifurcation to the cylindrical mode such that the largest curvature was positive or concave to side A of the plate. The largest curvature for plate 5 was negative or convex to side A. In the 90° configuration (Figures 5.3 and 5.4), the largest curvature for plates 1, 3, and 4 was again positive or concave to side A of the plate. The largest curvature for plate 5 was negative or again convex to side A.

Thus for every case, even though the generator axis switched, the plate remained consistently concave towards the same side. This result would be consistent with the assumption that initial curvatures of the plate were influencing the bifurcation path.

When the plates were loaded beyond the perceived bifurcation point, a lateral load was applied to the plate centre, to try to transform the bifurcation from one stable path to the other path. At no time was this successful, and in all cases the plate returned to the initial bifurcated path.

5.2 Mid-Surface Strain

With the large nonlinear response observed, the change in mid-surface strain was of obvious interest. Figure 5.5 is a composite plot showing dimensionless corner load vs a dimensionless form of the mid-surface strain. The data, presented in dimensionless
form, collapsed to provide a single curve with some small amount of scatter. Initial mid-surface strains were small and increased slowly. At a dimensionless corner load of approximately 0.5, the relation became essentially linear, and in an absolute sense, increased with increasing corner load. The difference in the mid-surface strains for the smaller plates, where the curvatures were small, was small. This difference increased with the increased difference in curvature present for the larger plates.

Dimensionless Gaussian curvature and mid-surface strain also formed a relation of considerable interest. Figure 5.6 of dimensionless Gaussian curvature vs mid-surface strain displayed a strongly linear relation with little scatter. A least squares curve fit produced the linear relation,

\[ \epsilon_2^0 = \epsilon_y^0 = \frac{1}{79.1} \kappa_z \kappa_y a^2. \]  

(5.3)

This result compared favourably with results determined by Williams on page 83 in [34].

5.3 Point of Bifurcation

The point at which bifurcation occurs can be experimentally observed by viewing the load vs curvature plots (Figures 4.1-4.4). The bifurcation point is the point at which the curvatures along the two plate diagonals begin to diverge significantly. However if one will recall, in the case of simple column buckling, significant lateral displacement can occur prior to the buckling load. Similar observations have also been made with regard to the buckling of edge-loaded plates [8, 9]. These occurrences are due to the inherent imperfections in the loading and geometry of the structure. Since the presence of imperfections in the test specimens has already been identified in section 5.1, it is certain that deviation from the pre-buckled displacement field occurred prior to the attainment of the buckling load.
Figure 5.5: 0° configuration, dimensionless mid-surface strain vs dimensionless corner load
Figure 5.6: 0° configuration, mid-surface strain vs dimensionless Gaussian curvature
The Southwell plot has been used extensively for both columns and plates as a solution to this problem. A plot of $P/\delta$ vs $\delta$, where $P$ is the load, and $\delta$ is the lateral displacement, collapses the results for a typical end-loaded strut, into a straight line with positive slope. The inverse of the slope is the critical buckling load. Horton et al. [8, 9], have extensively evaluated the use of the Southwell plot for plate, beam, and column structures, and found it to produce excellent results in a wide range of applications, although not explicitly for this particular case.

In the present case, the parameter that best illustrates the deviation from the pre-buckling displacements is the mean curvature. Corner load can be related to curvatures at the corner by equation (2.2). Since mean curvature is measured at a single discreet position, at the centre of the plate, and corner load is related to curvatures at the corner, Gaussian curvature, also measured at a discrete position, at the centre of the plate, was taken as the other parameter. The plot of mean curvature/Gaussian curvature vs mean curvature provides a correspondence to that of the traditional Southwell plot.

Figure A.13, in Appendix A, provides a composite plot of dimensionless mean curvature vs dimensionless Gaussian curvature. The plot illustrates the general characteristic hyperbolic shape associated with the traditional $P$ vs $\delta$ plot for an end-loaded column or edge-loaded plate.

The current version of the Southwell plot is illustrated in Figures 5.7–5.10. The plots all illustrate the characteristic positively sloped straight line of the traditional Southwell plot. The value of the inverse of the slope is listed on the figures, and as previously noted, is a measure of the bifurcation point for the perfect (defect free) structure.

Plate 1, with a/h ratio 48:1, displays the largest degree of scatter, presumably since the mean curvature was very small, even at the largest load. Higher loads, or an induced artificial imperfection might have created a larger initial mean curvature, and pushed the plate closer to the bifurcation point, providing a more distinct result [32]. Plates 3
Chapter 5. Discussion of Results

Figure 5.7: Plate 1, a/h ratio 48:1, Southwell plot

SOUTHWELL PLOT

a/h ratio 48:1
0° Configuration (Test #3)

Bifurcation Gaussian Curvature: 5.68126E-06 mm⁻²

Mean Curvature (mm⁻¹)

Mean Curvature/Absolute Gaussian Curvature (mm)

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Figure 5.8: Plate 3, a/h ratio 80:1, Southwell plot
Chapter 5. Discussion of Results

Figure 5.9: Plate 4, a/h ratio 96:1, Southwell plot
Figure 5.10: Plate 5, a/h ratio 192:1, Southwell plot
Table 5.1: Experimental Bifurcation Points predicted by the Southwell Plots

<table>
<thead>
<tr>
<th>Plate no.</th>
<th>( a/h ) ratio</th>
<th>Critical Twist ( \kappa a_{\text{crit}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>plate 1</td>
<td>48:1</td>
<td>0.3632</td>
</tr>
<tr>
<td>plate 3</td>
<td>80:1</td>
<td>0.1383</td>
</tr>
<tr>
<td>plate 4</td>
<td>96:1</td>
<td>0.1105</td>
</tr>
<tr>
<td>plate 5</td>
<td>192:1</td>
<td>0.0521</td>
</tr>
</tbody>
</table>

through 5 displayed excellent results, providing only a small amount of initial scatter. Experimental results for the four plates are compiled in Table 5.1.

5.4 Comparison of Bifurcation Points

Three authors, as discussed in section 1.2, have provided analytical and/or numerical solutions for the bifurcation point of the twisted square plate. Ramsey [27] found the bifurcation occurred when,

\[
\kappa a = 3.29h/a. \tag{5.4}
\]

Lee & Hsu [12] found bifurcation occurred when a dimensionless twisting moment \( M_{\text{cr}} \) has the value,

\[
M_{\text{cr}} = 21.6. \tag{5.5}
\]

Their result can be placed in the form of Ramsey's by the following manipulation. Rearranging equation (14) of [12], noting the correction by Williams on page 88 of [34], and replacing the edge length of 2a by b the full edge length,

\[
P = \frac{172.4(1 - \nu)Dh}{b^2\sqrt{12(1 - \nu^2)}}. \tag{5.6}
\]
From eq. (2.2),

\[ P = 2D\kappa(1 - \nu). \]  

Equating (5.6) and (5.7) and solving for \( \kappa \),

\[ \kappa = \frac{86.4h}{b^2 \sqrt{12(1 - \nu^2)}}. \]  

Substitution of \( \nu = 0 \), as used by Lee & Hsu, provides the required result of,

\[ \kappa b = 24.94h/b, \]  

where \( b \) is the full edge length. This result is approximately 660% greater than Ramsey's.

Miyagawa et. al. [25] determined the dimensionless corner load to be,

\[ P_{cr} = 22.8. \]  

This may also be placed in the form of Ramsey's solution. Rearranging the equation at the top of page 2266 of [25], replacing the edge length of \( 2a \) by \( b \) the full edge length, and substituting \( F \) for the corner load of \( P/2 \) used in [25],

\[ F = \frac{44.8Dh}{b^2}. \]  

Again using eq. (2.2), this time using \( F \) instead of \( P \),

\[ F = 2D\kappa(1 - \nu). \]  

Equating (5.11) and (5.12) and solving for \( \kappa \),

\[ \kappa = \frac{22.8h}{b^2(1 - \nu)}. \]  

Substitution of \( \nu = 1/3 \) as used by Miyagawa et. al. provides the required result of,

\[ \kappa b = 34.2h/b, \]
where again b is the full edge length. This result is approximately 900% greater than Ramsey and 35% greater than Lee & Hsu.

Equation (2.2) provides the linear relation between corner load and twist. Since the nonlinearities in the plate are strictly geometric, over small distances equation (2.2) provides valid results. The numerical solution by Lee and Hsu provides a dimensionless twisting moment which is related to the applied corner force on page 223 of [12]. In the solution provided by Miyagawa et. al. the dimensionless force $\overline{P}$ is also applied at the corner. Thus the dimensionless twists calculated from these numerical solutions for dimensionless twisting moment and dimensionless corner load are valid near the corners.

This however provides the implication that the curvature across the plate is non-uniform, and is strongly related to the geometric nonlinearity. Comparisons between curvatures or twists at the plate center, as determined experimentally, and of curvatures or twists at the plate corner, as determined numerically will be strongly influenced by the nonlinear effects.

Observations of Figures 4.1-4.4 illustrating corner load vs absolute principal curvature, however provide a method of comparing the curvatures occurring at the plate corner and the plate center. At the bifurcation point predicted by the Southwell plot, the experimental curvature is approximately half the linear prediction. Thus the critical dimensionless twists calculated at the plate corners using linear theory are to a first approximation, double the result occurring at the plate centre.

Further support for this approximation may be found by analyzing the results illustrated in Figure 4 of [23] by Miyagawa et. al. The figure relates load to principal strain for an aluminum plate with edge length, 100mm and thickness, 1.5mm. Since the strain data is not provided for both plate surfaces, the following manipulation must be performed.

Using Mohr's circle, the bending strain can be related to shear strain. Since shear strains vary linearly through the plate thickness, they can be related to plate twist using
Chapter 5. Discussion of Results

equation (2.10), where

$$\kappa = \frac{\gamma_{xy}}{h}. \quad (5.15)$$

This equation can be placed in the familiar dimensionless form by multiplying by the ratio $a^2/h$, providing the result,

$$\frac{\kappa a}{h/a} = \frac{\gamma_{xy}a^2}{h^2}. \quad (5.16)$$

Thus for a critical strain of 455$\mu$e, interpolated from Figure 5 in [23], the dimensionless twist calculated is,

$$\kappa a = 4.04h/a. \quad (5.17)$$

This twist occurs at the plate centre.

The twist at the plate corner may also be evaluated using equation (2.4). Equation (2.4) is made dimensionless by multiplying by the ratio $a^2/h$, thus,

$$\frac{\kappa a}{h/a} = \frac{6(1 - \nu)Pa^2}{Eh^4}. \quad (5.18)$$

Using the plate material properties found in Table 1 of [23], and noting the correction for the exponent on the value for Young's modulus where a 0 is replaced by a 4, a dimensionless twist of,

$$\kappa a = 7.65h/a, \quad (5.19)$$

can be found. This twist occurs at the plate corner. Thus the critical dimensionless twist occurring at the plate centre is approximately 53% of the critical dimensionless twist occurring at the plate corner for the same experimental data.

This indicates that the numerical results of Lee and Hsu, and Miyagawa et. al. may in fact be closer to the experimental results than first indicated.
Chapter 5. Discussion of Results

Ramsey makes the assumption that twist is constant in equation (15) of [27]. The critical dimensionless twist determined through the analytical solution could be interpreted as an average twist over the plate surface. The point of twist measurement is thus immaterial in this situation, and in direct comparisons with this result, the average effect should be noted.

Miyagawa et. al. also provided experimental results for the bifurcation problem [24, 23]. Figure 5 in [23] provides a compilation of these results. In these results however only critical strain occurring at the bifurcation point is recorded. The results may however be related to critical dimensionless twist by equations (5.15) and (5.16). Several points from Figure 5 in [23] were analyzed using equations (5.15) and (5.16). These results indicate curvature occurring at the plate centre, and are illustrated in Figure 5.11.

The results from the three theoretical solutions and bifurcation data, as predicted by the Southwell plot in the current experiment, and data interpolated from experimental work by Miyagawa et. al. are plotted in Figure 5.11. Critical dimensionless twist \( \kappa a \) is plotted as a function of the \( h/a \) ratio. The slope of the resulting straight line is thus the critical dimensionless twist \( \kappa a^2/h \) for any plate size. The current experimental results provide essentially a linear solution and provide an intermediate result as compared to the numerical and the analytical solutions. The large degree of scatter in the Southwell plot results for plate 1 with \( a/h \) ratio of 48:1, caused difficulty in assessing an accurate bifurcation Gaussian curvature. For that reason only the results for plates 3, 4 and 5 were used in the calculation of the relation between \( \kappa a \) and \( h/a \) in the current experimental results. The experimental relation determined was,

\[
\kappa a = 10.8h/a. \tag{5.20}
\]

The current experimental result is approximately 50% greater than the experimental results of Miyagawa et. al. Both of the results indicate curvature at the plate centre.
Figure 5.11: Compilation of theoretical, numerical and experimental results showing dimensionless twist $\kappa a$ vs thickness to length ratio $h/a$
Miyagawa et. al. give no indication of any problems occurring either due to body forces or plate imperfections. The use of the Southwell plot in the interpretation of their data however would have almost certainly increased their predicted bifurcation point, since it is unlikely that their test plates were truly defect-free.

5.5 Determination of Constitutive Coefficients

The onset of bifurcation provides the practical limiting load for the twisted square plate test. Loading into the unstable region will introduce a large degree of error into the resulting data. The approximate corner load causing the onset of instability may be calculated using the experimental results found in Figure 5.11, in combination with equation (2.2). The dimensionless twist occurring at the corner was taken as twice the dimensionless twist occurring at the plate centre, thus providing the solution;

\[ P_{\text{crit}} = \frac{43.2hD(1 - \nu)}{a^2}. \]  

(5.21)

The solution to the experimental result found in Figure 5.11 provides the limiting critical twist at the plate centre, beyond which bifurcation occurs. Thus,

\[ \kappa = \frac{10.80h}{a^2}. \]  

(5.22)

It is important to note that for this case, as in the case of a strut, deviation from the 'perfect' condition in practical cases always occurs prior to reaching the critical load. This implies that bifurcation will occur prior to the theoretical bifurcation point for plates which are other than perfectly flat. Consideration of this must be taken into account when using the twisted plate test, and suitable 'safety factors' must be applied.
Chapter 6

Conclusions

The twisting of a square plate with free edges by self-equilibrating contrary corner loads provides an interesting view of twisting bifurcation. The loaded plate transforms from one with anticlastic shape to one with synclastic shape. Two possible bifurcated paths can occur in the synclastic mode, where generators occur along either of the two plate diagonals. Strain gauges mounted along the plate diagonal were used in the current experiment to record the transition between the deformed states.

Initial experimental results revealed a large nonlinear response in the load curvature relation, with only a small linear region occurring initially for small loads. Larger a/h ratio plates illustrated larger deviations in the magnitude of the principal curvatures at lower loads when compared to plates with smaller a/h ratios. The principal axes were found to be coincident with the plate diagonals.

In the evaluation of the data the use of dimensionless numbers was found to be of considerable value. Dimensionless groups were formed which collapsed the results for the various plate sizes allowing more general conclusions to be drawn about all the plates. Accordingly, the mid-surface strains were found to be compressive, and along the principal axis they were equal. The relation between the mid-surface strains and dimensionless Gaussian curvature was essentially linear, where $e_x^0 = e_y^0 = 0.0126 \kappa_x \kappa_y a^2$.

The use of the vertical plate configuration was found to eliminate any lateral plate deflection due to body forces, and thus eliminated the problems encountered by previous researchers [34]. However initial plate curvatures were found to dominate the choice of
Chapter 6. Conclusions

the bifurcated path. Lateral loads applied to the plate centre at and immediately after bifurcation failed to alter the bifurcation path.

The use of the Southwell plot, relating mean curvature to the ratio mean curvature/Gaussian curvature was found acceptable in evaluating the Gaussian curvature occurring at the theoretical bifurcation point. These values were found to provide an essentially linear relationship between critical dimensionless twist $\kappa a$, and the thickness to length ratio $h/a$, where $\kappa a = 10.8h/a$. This experimental result provided an intermediate result to the numerical results of Miyagawa et. al. [25] and Lee and Hsu [12], and the analytical results of Ramsey [27]. However the position of the curvature measurement was found to be of considerable importance. The critical dimensionless twist measured at the plate corner was found to be approximately twice the critical dimensionless twist measured at the plate centre. This was attributed to the strong geometric nonlinearity illustrated in the corner load/principal curvature relation. Experimental results by Miyagawa et. al. [24, 23] were found to predict a critical dimensionless twist that was 27% of the current experimental result. Both of these experimental results measured curvature at the plate centre.

The theoretical and realistic bifurcation points however may differ greatly and depend largely upon initial imperfections in the plates. Careful thought should be exercised prior to using of the theoretical bifurcation point as the upper limit in a plate torsion test.

Future experimental work should concentrate upon the need for perfectly flat plates, although further evaluation of the Southwell plot could suggest alternate methods of determining the same theoretical bifurcation point. Work should be extended to the examination of anisotropic materials, since this is an area where the plate torsion test could be of considerable value. The testing of variously skewed plate geometries would also be of interest. Efforts should be made to avoid the introduction of yielding since this behavior complicates the results unnecessarily.
Bibliography


Appendix A

Additional Plots

A.1 Principal Strain

The experimental principal strains are presented in Figures A.1 through A.4. The #1, and #3 directions fixed to the plate, were defined previously in Chapter 3, as were the plate sides, A and B. Note that only the $0^\circ$ configuration is presented. The strains are absolute and the sign of the strain is indicated in the plot legend.
Figure A.1: Plate 1, absolute principal strains vs corner load
Figure A.2: Plate 3, absolute principal strains vs corner load
Figure A.3: Plate 4, absolute principal strains vs corner load
Appendix A. Additional Plots

Figure A.4: Plate 5, absolute principal strains vs corner load
Appendix A. Additional Plots

A.2 Dimensionless Strain

Figures A.5–A.12 are plots of dimensionless principal strain vs dimensionless corner load. Figures A.5–A.8 show dimensionless principal strain for the plates in the 0° configuration. A.9–A.12 show dimensionless principal strain for the plates in the 90° configuration. Each plot illustrates the strain on one of the two plate sides, A or B, and along one of the two principal directions on the plate, #1 or #3. Results for all of the plates sizes are collapsed through the use of dimensionless groups, and are thus presented on individual plots.
Figure A.5: \(0^\circ\) configuration, side A, \#1 direction, dimensionless principal strains vs dimensionless corner load
Figure A.6: $0^\circ$ configuration, side A, #3 direction, dimensionless principal strains vs dimensionless corner load
Figure A.7: 0° configuration, side B, #1 direction, dimensionless principal strains vs dimensionless corner load
Appendix A. Additional Plots

DIMENSIONLESS STRAIN

0° Configuration

Side B, #3 Direction (Test #3)

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Figure A.8: 0° configuration, side B, #3 direction, dimensionless principal strains vs dimensionless corner load
Figure A.9: 90° configuration, side A, #1 direction, dimensionless principal strains vs dimensionless corner load
Figure A.10: 90° configuration, side A, #3 direction, dimensionless principal strains vs dimensionless corner load
Figure A.11: 90° configuration, side B, #1 direction, dimensionless principal strains vs dimensionless corner load
Figure A.12: 90° configuration, side B, #3 direction, dimensionless principal strains vs dimensionless corner load
A.3 Other Plots

Figure A.13 of dimensionless Gaussian curvature vs dimensionless mean curvature, is the twisted bifurcating plate counterpart of the P vs δ plot of the end loaded strut. Data is plotted in dimensionless form such that all the results collapse, and may be presented on a single figure. The general hyperbolic nature of the P vs δ plot is in evidence, the only exception being the reversed sign of the data presented here.
Figure A.13: 0° configuration, dimensionless mean curvature vs dimensionless gaussian curvature
Appendix B

Plate 2 Results

Figures B.1 and B.2 are respectively, of the principal plate curvature and principal plate strains for plate 2 with a/h ratio of 64:1. The general results appear similar to the other plates. Figures illustrating mid-surface strain however show significant anomalies, as illustrated in Figure B.3, a composite plot of dimensionless mean curvature vs dimensionless gaussian curvature which includes the plate 2 data.
Figure B.1: Plate 2, absolute principal curvature vs corner load
Figure B.2: Plate 2, absolute principal strain vs corner load
Figure B.3: 0° configuration, dimensionless mean curvature vs dimensionless gaussian curvature, including plate 2
Appendix C

Data Correction Routine

The data correction routine was written to run in a pc environment using the WATFOR77 Fortran compiler. Input and output formats are indicated in the program body. The main program reads the data, calls the various correction subroutines, then writes the data to a file. Input and output file names were entered interactively.
Appendix C. Data Correction Routine

C$NOEX
C
C PROGRAM NAME: COR1.FOR
C AUTHOR: B. HOWELL
C AUGUST 1990
C
C PROGRAM TO CORRECT STRAINS MEASURED FROM TWO RECTANGULAR STRAIN ROSETTES
C
IMPLICIT REAL*8(A-H,O-Z)
C
DIMENSION STR(100,6),RMASS(100),GAUGE(6),TVS(6),SETNG(2),
$PRINC(100,4),PHI(100,2),CURV(100,2)
CHARACTER*20 FIN,FOUT
READ(*,1000) FIN
READ(*,1000) FOUT
1000 FORMAT(A20)
OPEN(UNIT=2,FILE=FIN,STATUS='OLD')
OPEN(UNIT=3,FILE=FOUT)
READ(2,*),N,(GAUGE(I),1-1,6),(SETNG(I),1-1,2),TVS(1),TVS(2),
$ TVS(4),TVS(5),TH
C
READ IN THE RAW STRAIN DATA
C
DO 10 I=1,N
READ(2,*),RMASS(I),(STR(I,J),J=1,6)
10 CONTINUE
C
CALL THE APPROPRIATE SUBROUTINES
C
SUBROUTINE TO CORRECT FOR NONLINEARITIES IN THE BRIDGE
CALL NLINEAR(STR,SETNG,N)
C
SUBROUTINE TO CORRECT FOR INDIVIDUAL GAUGE FACTORS
CALL GFACTOR(STR,GAUGE,SETNG,N)
C
SUBROUTINE TO CORRECT FOR TRANSVERSE SENSIVITY
CALL TRANSVERSE(STR,N,TVS)
C
SUBROUTINE TO CALCULATE PRINCIPAL STRAINS AND GAUGE MISALIGNMENT
CALL PRINCIPAL(STR,N,PHI,PRINC)
C
SUBROUTINE TO CALCULATE PRINCIPAL CURVATURES
CALL CURVATURE(PRINC,N,CURV,TH)
C
GENERATE OUTPUT DATA FILE
C
DO 20 I=1,N
WRITE(3,1010)RMASS(I),(PRINC(I,J),J=1,4),(PHI(I,K),K=1,2),
$ (CURV(I,L),L=1,2)
20 CONTINUE
1010 FORMAT(F6.3,2X,4(F7.2,1X),1X,2(F5.2,1X),1X,2(F10.7,1X))
C
END
C

Figure C.1: Data correction routine, COR1, page 1
SUBROUTINE NLINEAR(STR, SETNG, N)

IMPLICIT REAL*8(A-H,O-Z)
DIMENSION STR(100,6), SETNG(2)
DO 10 I=1,N
   DO 20 K=1,2
      DO 30 J=1,3
         L=J+(K-1)*3
         STR(I,L)=STR(I,L)*(1.0+(SETNG(K)*STR(I,L)/1.D06)/
$            (2.0-SETNG(K)*STR(I,L)/1.D06))
   30 CONTINUE
20 CONTINUE
10 CONTINUE
RETURN
END

SUBROUTINE GFACTOR(STR, GAUGE, SETNG, N)

IMPLICIT REAL*8(A-H,O-Z)
DIMENSION STR(100,6), GAUGE(6), SETNG(2)
DO 10 I=1,N
   DO 20 K=1,2
      DO 30 J=1,3
         L=J+(K-1)*3
         STR(I,L)=STR(I,L)*SETNG(K)/GAUGE(L)
   30 CONTINUE
20 CONTINUE
10 CONTINUE
RETURN
END

SUBROUTINE TRANSVERSE(STR, N, TVS)

IMPLICIT REAL*8(A-H,O-Z)
DIMENSION STR(100,6), TVS(6)
NU=0.285
DO 10 I=1,N
   DO 20 K=1,2
      J=(K-1)*3
      STR(I,J+1)=(1.0-NU/TVS(J+1))/(1.0-TVS(J+1)**2)*
$         (STR(I,J+1)-TVS(J+1)*STR(I,J+3))
$         STR(I,J+2)=((1.0-NU*TVS(J+2))*(1.0+TVS(J+1))*STR(I,J+2)-
$            TVS(J+2)*(1.0-NU*TVS(J+1))*STR(I,J+1)+STR(I,J+3))/
$            (1.0+TVS(J+1))/(1.0-TVS(J+2))
$         STR(I,J+3)=(1.0-NU/TVS(J+1))/(1.0-TVS(J+1)**2)*
$            (STR(I,J+3)-TVS(J+1)*STR(I,J+1))
   20 CONTINUE
10 CONTINUE
RETURN
END

Figure C.2: Data correction routine, COR1, page 2
SUBROUTINE PRINCIPAL(STR,N,PHI,PRINC)

IMPLICIT REAL*8(A-H,O-Z)
DIMENSION STR(100,6),PHI(100,2),PRINC(100,4)

PI=3.1415926
DO 10 I=1,N
   DO 20 K=1,2
      J=(K-1)*3
      L=(K-1)*2
      RAD=SQRT((STR(I,J+1)-STR(I,J+3))**2+(2*STR(I,J+2)-
      STR(I,J+1)-STR(I,J+3))**2)/2
      IF (K .EQ. 1)THEN
         PRINC(I,L+1)=ABS((STR(I,J+1)+STR(I,J+3))/2+RAD)
      ENDIF
      IF (K .EQ. 1)THEN
         PRINC(I,L+2)=ABS((STR(I,J+1)+STR(I,J+3))/2-RAD)
      ENDIF
      IF (TEMP .NE. 0.0) THEN
         PHI(I,K)=(2*STR(I,J+2)-STR(I,J+1)-STR(I,J+3))/
         (STR(I,J+1)-STR(I,J+3))*90.0/PI
      ELSE
         PHI(I,K)=0.0
      ENDIF
   20 CONTINUE
10 CONTINUE
RETURN
END

C
C******************************************************************
C
SUBROUTINE CURVATURE(PRINC,N,CURV,TH)
C
IMPLICIT REAL*8(A-H,O-Z)
DIMENSION PRINC(100,4),CURV(100,2)
DO 10 I=1,N
   CURV(I,1)=(PRINC(I,1)+PRINC(I,3))/TH/1.D06
   CURV(I,2)=(PRINC(I,2)+PRINC(I,4))/TH/1.D06
10 CONTINUE
RETURN
END
C
C*************************************************************************
C
Figure C.3: Data correction routine, COR1, page 3