ESSAYS ON FINANCIAL ASSET PRICING

By

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A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

in

THE FACULTY OF GRADUATE STUDIES
COMMERCE AND BUSINESS ADMINISTRATION

We accept this thesis as conforming
to the required standard

THE UNIVERSITY OF BRITISH COLUMBIA

April 1989

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Date **April 13, 1989**
Abstract

The first essay of this thesis is concerned with the pricing of financial assets in positive net supply in a financial market with continuous time trading. A general equilibrium, continuous time model of an economy with production is developed. The model is used to develop a nonlinear partial differential equation whose solution gives the market price of the economy's aggregate long term physical/financial assets in terms of the consumption good. Partial differential equations that are linear are then developed for pricing individual physical/financial asset using the solution for the economy's aggregate of physical/financial assets. A particular solution is presented and applications to the pricing of bonds and the pricing of warrants and stock are made. Unlike previous models, financial assets represent a significant portion of the economy's wealth in this model. The model is generalized to include two kinds of production: risky instantaneous production of the consumption good and riskless production of long term physical/financial assets whose output is risky.

The second essay develops the concept of market risk, which arises from the uncertainty investors have about the economy and the financial markets they trade in. A simple three date model with log and risk neutral investors trading a stock and bond is considered first. Market risk is modeled by introducing economy states which represent the possible economies that traders could be trading in. Two possible equilibria are shown to exist in the model: a separating equilibrium in which market prices reveal the economy state associated with the magnitude of the aggregate endowments of investors and a pooling equilibrium in which first period prices do not reveal the economy state. The concept of market risk, is used to develop a model of market crashes. Market crashes
are modeled as pooling equilibria where the market price initially does not reveal the fundamental value of a financial asset and may not do so until the last trading opportunity takes place. The market price represents an average of possible final market prices. When the market price moves down sharply from a pooling price to a separating price that reveals the economy state the market crash takes place. The market crash occurs when a pooling price that does not reveal the economy state is no longer an equilibrium for the prevailing underlying economy state. Then the price must change and the economy state is revealed. Two methods of triggering the market crash are considered, one based upon information arriving to the market, the other based upon the arrival of new investors.
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Acknowledgement

I would like to thank my advisor, Alan Kraus for all the time, effort and help he has given me. Thanks to the members of my committee David Backus, Jim Brander and Josef Zechn for their time and effort. I would like to thank Ken MacCrimmon who has also been an advisor to me. Without his help I would never have entered the PhD program. Thanks to my parents and Mano for all their encouragement. This work is dedicated to them.
Chapter 1

Overview

The two essays in this thesis are both concerned with the pricing of financial assets. One essay deals with the pricing of financial assets in an economy with continuous time trading when financial assets are a positive fraction of aggregate wealth. The other essay considers the effect on financial asset pricing of market risk, the risk faced by investors due to their imperfect knowledge of the economy/financial markets which determine the prices of assets. An application of the concept of market risk is made in a model of market crashes.

These essays have a number of things in common. All investors in the economies modeled have the same information. Hence, asymmetric information does not play a part in the results obtained. Investors in these models have rational expectations. An essential idea underlying both essays is that investors, in determining their supply or demand for a financial asset and hence determining the current price, make use of their expectations about what the future price of the financial asset is going to be. These expectations are "rational" in the sense that they are consistent with the model i.e. the investor's beliefs about the economy are correct.

In the continuous time financial asset pricing essay investors determine their expectations about future prices by solving a partial differential equation using boundary conditions that depend on the particular asset. The current price is then determined by the condition that demand equals supply for the asset. The demand and supply for the asset depend upon investors' beliefs about future prices. The problem has an inherent
circularity to it because the current price affects wealth and hence consumption which affects future wealth and hence future prices. Thus the current price affects future prices and beliefs about future prices affect the current price. Future prices are determined in the economy in the same way by setting demand equal to supply. Investors have rational expectations because the future prices they solve for using the partial differential equation are identical to the prices that will equate supply and demand at that point. These future prices when determined in the actual economy by equating supply and demand will again be influenced by the effect of investor’s expectations about prices even further in the future. Investors have rational expectations because the model of the economy they solve to determine their beliefs about future prices is a perfect model of the economy they are trading in. The economy they trade in is a model itself which contains investors who solve the same model they are part of. There is an interesting recursive structure to the beliefs of investors and the model itself. It is taken care of by seeking a solution that is a fixed point. The solution to the model, when used to generate the beliefs or expectations of the economy’s participants, leads to the identical solution.

The essay on market risk and market crashes also relies heavily on rational expectations. There are two types of investors having different preferences in these models. The financial assets are traded from one group of investors to the other one at the market clearing price. Again, investors must solve a model of the economy they are trading in to determine their expectations about future prices. These beliefs must only be compatible with the solution to the model of the economy but not identical. Investors do not know the exact economy they are trading in but rather have rational beliefs about the possible economies (called economy states) they might be in. The current market prices (unlike the continuous time models of the other essay) contain information about the possible underlying economies and investors use this information in a rational fashion in determining their beliefs about future prices and hence their supply and demand of the assets.
Since their supply and demand affect the current price there is an unavoidable circularity to the models. The models have multiple equilibria which differ in the amount of information that the market price reveals about the underlying economy. In both essays investors predict future prices using a model which agrees with the way current prices are being determined, i.e., through equating supply and demand for the assets where supply and demand depends upon beliefs about future prices. In both essays these beliefs also depend upon the current price. The circularity implied by the current price affecting beliefs about future prices which in turn affects the current price etc. is resolved by seeking solutions that are fixed points.
A major objective of financial economics is the pricing of risky assets. Partial equilibrium, discrete time asset pricing models include the Sharpe–Lintner–Mossin capital asset pricing model (CAPM) [42] [29] [35] and the Ross [39] arbitrage pricing theory (APT). For pricing one risky asset in terms of another with continuous time trading, there is the Black–Scholes–Merton option pricing model [4], [33], the Intertemporal Capital Asset Pricing Model (ICAPM) of Merton [32] and work by Garman [21]. At the level of general equilibrium, the most successful asset pricing models in terms of application to specific securities are the Arrow–Debreu state preference theory [1] [17] and the continuous time Cox–Ingersoll–Ross (CIR hereafter) model [14]. At the level of pure theory, other models, particularly that of Lucas (1978), [30] and Radner [36] are equally important but have not been applied to the analysis of specific securities as readily.

Continuous time models of asset pricing have become an important part of the theory of finance since the work of Black and Scholes (1973) [4] and Merton (1973) [32], [33]. Their popularity is partly due to the wide variety of problems in finance that can be dealt with in the continuous time framework. The valuation of options is closely associated with continuous time methods. The literature for this area is very large (see [16] for an extensive bibliography). Other topics such as the problem of optimal corporate financial policy [25], [10], optimal regulation [9], bond pricing and the term structure of interest rates [15], [49], [8], [46], [6], international finance [45], natural resource investments [11], and forward and futures markets [37] are just some of the areas where continuous time
models have been used. Another reason for their popularity is the existence of analytic solutions to the valuation equations these models imply, the best known example being that of Black and Scholes. In the far greater number of cases where analytical solutions can not be found, numerical methods can be used to determine a solution. Finally, continuous time models do not suffer from a problem of discrete time models, having to specify an arbitrary time period between trading and decision making.

A number of developments have occurred to enhance the popularity of continuous time models. Breeden in his 1979 paper [5] showed that the risk return relationships derived in the continuous time models of CIR and Merton could be written in the simple form of the CAPM when risk was measured in terms of consumption betas rather than betas based on rate of return on assets. The validity of continuous time models has been rigorously proven using the methods of martingale theory by Duffie and Huang [18][24].

Many continuous time models are not general equilibrium models but rather must specify some price processes exogenously. In these partial equilibrium models the parameters of the exogenously given stochastic process these prices follow are given and arbitrage arguments are used to value other assets or to determine equilibrium relations between the parameters. In the case of the Black-Scholes model the stochastic process the stock price follows is specified, namely geometric Brownian motion. Arguing that in equilibrium with continuous time trading no arbitrage opportunities should exist that allow a risk free profit to be made with no investment, one determines a partial differential equation that the value of a call option written on the stock must satisfy. Given the standard call option boundary conditions the equation is solved by the Black-Scholes equation. In a recent paper Bick [3] shows how the Black-Scholes equation can be derived in a general equilibrium model.

Merton also used a partial equilibrium model rather than a general one in his derivation of the Intertemporal Capital Asset Pricing Model (ICAPM) [33]. In the ICAPM the
stochastic processes that asset prices follow i.e. their rate of return, is specified exoge-
ously but not the prices themselves. Given the stochastic price processes and investor
wealth, the demands for the assets by investors with a given risk averse (i.e. concave)
von-Neumann Morgenstern utility function can be derived. For an investor to hold an
optimal portfolio the expected rate of return of each financial asset minus the riskless
rate of return must equal the covariance of this asset's rate of return with that of the
rate of return on the investor's marginal utility of wealth. The parameters of the exoge-
nously given stochastic price process that a particular financial asset follows are fixed so
that this condition of optimality restricts the covariance. This covariance depends upon
the market values of the various financial assets held. Given the prices of the financial
assets and the wealth of the investor, the number of shares he or she will hold is then
determined so that the value of the covariance is correct.

Making the assumptions necessary for aggregation to hold (either homogeneous in-
vestors or HARA class preferences [40] [7]) the demands of all investors can be added
together. For equilibrium to hold the financial asset prices must be such that the number
of shares in total and the total market value for each asset that investors wish to hold
equals respectively the number outstanding and the market value outstanding. The re-
sulting equations equate the equilibrium expected return on a financial asset minus the
risk free rate to the covariance of the asset's rate of return with the rate of return on the
marginal utility of aggregate wealth. This set of conditions is the ICAPM and gives the
relationship between expected return and risk for the economy. The aggregate amount
invested in each financial asset is also determined.

The ICAPM puts few restrictions on the stochastic price processes and so is not a
general equilibrium specification because the complete details of the stochastic process
can not be derived endogenously from initial assumptions about the economy. Although
the ICAPM gives an equilibrium relation between the expected price changes and the
covariances of the price changes with the change in total marginal utility of wealth, only the absolute level of the prices is determined: the underlying economic process leading to the particular stochastic process is completely unspecified. The model does not specify what economic factors and agents cause demand and supply to balance at a particular price and then to balance at a slightly different price a small time interval later and so forth.

The first general equilibrium continuous time asset pricing model was introduced by Cox, Ingersoll and Ross (1985) (CIR). Instead of specifying exogenous stochastic price processes for financial assets they introduced a production sector for the economy. The production in this economy transforms a single consumption good into a random amount of the consumption good with a stochastic rate of return. As in Merton's ICAPM, the rates of return also depend on the values of certain state variables which follow exogenously given stochastic processes. By considering an individual investor with a given amount of wealth in the consumption good, and specifying the investor's preferences with a risk averse (i.e. concave) utility function, the amount he or she would invest in each production process can be determined. By assuming homogeneous investors it is trivial to aggregate over all investors in the economy to determine equilibrium aggregate investment and a relation between rate of return on a production process and its "risk". "Risk" is measured by the covariance of the return on aggregate production with the change in marginal utility of wealth. In equilibrium investment in a production process occurs until its expected return minus the risk free rate equals the covariance between the production's rate of return and the rate of change in marginal utility. Investment in the production processes must be nonnegative and can be of an unlimited amount. This means that production processes that can not satisfy the risk return relation for the economy will not be used.

Contingent claims, which are financial assets with state variable dependent payoffs,
can also be valued in this economy using the equilibrium risk return relations that hold for the production processes. The expected rate of return of the financial asset minus the risk free rate equals the covariance of its rate of return with the rate of change of per capita marginal utility of wealth. This covariance can be calculated because the stochastic properties of wealth are known after the investments in the production processes that make up wealth are determined. It is assumed that there is zero net supply of all contingent claims so that with homogeneous investors, as is assumed in CIR, the rate of return and risk of these financial assets is such that an individual investor wishes to hold a zero amount although he could be potentially long or short. Based on these assumptions, partial differential equations can be developed whose solution gives the equilibrium price of the contingent claim as a function of time, wealth and the state variables. The solution is determined once sufficient boundary conditions are specified. Since these contingent claims are held in zero net supply, they do not affect the economy wide equilibrium risk return relationship which depends only on aggregate preferences and the parameters of the stochastic production processes.

It is useful to compare CIR with Merton's ICAPM (1973). Merton's ICAPM can also be used to value contingent claims by making use of arbitrage arguments. If the contingent claim is assumed to have a price which is a function of time, wealth and certain state variables which follow known stochastic processes, then Ito's lemma can be used to determine the contingent claim's rate of return. This will itself follow a stochastic process that depends on the form the price function takes. If there are no opportunities for arbitrage profits then the contingent claim's rate of return must satisfy the ICAPM. The ICAPM determines a price of risk based on the aggregate behaviour of investors.

The ICAPM, Ito's lemma and the no arbitrage condition lead to a partial differential equation for the value of the contingent claim which is the same as that of CIR and is referred to by them as the fundamental equation of valuation. This no arbitrage, ICAPM
derivation of the fundamental valuation equation appears in, for example, Brennan and Schwartz (1984) and is used by them in their work on corporate financial structure, the term structure of interest rates, and the valuation of various securities.

When the rate of return is specified as it is for CIR's production processes and the financial assets in Merton's ICAPM then one solves for how much wealth is invested. When either the amount invested or the number of shares held, is specified for a financial asset, e.g. zero in the case of contingent claims in CIR, then the equilibrium rate of return of the asset can be solved for.

In Cox, Ingersoll and Ross, aggregating the preferences of homogeneous investors in an economy with exogenously given production processes determines a price of risk. The exogenously given production processes are in form identical to Merton's exogenously given stochastic asset prices and serve the same function, to determine a price for risk. In the CIR model, the contingent claims are in zero net supply so that because investors are homogeneous none are ever held. This means that they do not affect the equilibrium relation between risk and return. Similarly in the ICAPM-no arbitrage approach to valuation, the presence of contingent claims does not affect the price of risk. In this approach one need not assume that the contingent claims are in zero net supply but only that the presence of the contingent claims does not affect the exogenously given stochastic price processes or the price of risk in the economy. Hence the arbitrage argument assumes that the supply of the exogenously given assets is perfectly elastic in some sense relative to any contingent claim. Then the aggregate market value invested in these assets, their prices and rates of return, and the economy's price of risk are unchanged by the presence of the contingent claims.

Assets in the CIR model are of two varieties: real assets (production possibilities) in perfectly elastic supply with stochastic instantaneous return processes exhibiting stochastic constant returns to scale, and financial assets (contingent claims) written between
investors. One such financial asset is instantaneous riskless borrowing/lending. In aggregate, the net supply of every financial asset is zero. Since individuals are identical in beliefs, endowments and preferences in the CIR model, the expected return on a financial asset in equilibrium is such that desired net demand is zero for each individual; this determines financial asset prices, including the interest rate.

Our purpose in the first essay of this thesis is to investigate the pricing of a type of financial asset that does not exist in the CIR world. We consider the case of a financial asset that represents an ownership claim on some real asset(s) in fixed, nonzero aggregate net supply. For example, the financial asset could be the traded equity of a firm that owns some given set of real assets. A financial claim of this sort differs from either of the types of assets in the CIR model. Unlike their production possibilities with stochastic constant returns to scale, it is not in elastic supply nor does it have an exogenous return process. Unlike their financial claims, its aggregate net supply is positive and represents a claim on a future flow of consumption. As we will show, the positive net supply of our financial asset has a critical effect on the relation determining its equilibrium price.

CIR's model with financial assets in zero net supply is somewhat inconsistent with the main intuition of the CAPM. In the CAPM, financial assets are risky because they are held by investors in their portfolios and form a significant part of wealth. Hence, the rate of return on the portfolio of financial assets investors hold has a significant covariance with the rate of return on wealth and hence with the rate of change in marginal utility of wealth. In CIR the risk of a portfolio of financial assets is due to the covariance of its rate of return with the rate of change in per capita marginal utility of wealth. However, wealth is made up of only the consumption good and not financial assets. The source of this covariance is due to the explicit assumption that both the rate of return on financial assets and the rate of return on the consumption good that makes up wealth depend on the same stochastic processes. The ICAPM retains the CAPM intuition but, unlike CIR,
Chapter 2. Introduction to the Pricing of Financial Assets in Positive Net Supply

it is not a general equilibrium model.

In the model presented in the first essay we develop a generalization of the CIR fundamental valuation equation that is consistent with the CAPM intuition in the way that the lCAPM is; hence it represents a synthesis of the approaches taken by CIR and Merton. Financial assets are priced in a general equilibrium framework. The risk of a financial asset includes the effect of the covariance of the asset's rate of return with the rate of return on wealth, where this covariance is due to financial assets representing a significant portion of wealth.

Chapter 2 deals with a model of an economy with a single state variable representing the uncertainty in the economy, a single consumption good, no production and a single physical/financial asset in nonzero net supply that represents the market portfolio of all physical/financial assets. The amount of the financial asset, its state dependent payoff, and the amount of the consumption good as well as preferences determine the price of risk and produce a downward sloping demand curve for the financial asset. The model implies a nonlinear partial differential equation for the market portfolio price. The stochastic process that the price of the market portfolio of financial assets and its rate of return follow are then determined endogenously from the solution to the equation. A particular example of such a market portfolio of financial assets is considered, and the partial differential equation for its price is solved.

Given the price of the market portfolio, the partial differential equation for the price of an individual financial asset is derived. It is linear and is of the same form as those derived by CIR. A number of important applications are briefly considered. First, the pricing of risk free bonds and the term structure of interest rates when bonds are in nonzero net supply are examined. The pricing of warrants, and other contingent claims is then considered. An appendix generalizes the model to the case of multiple financial assets and multiple state variables.
In Chapter 3 the model is extended to include a production process of the CIR type, i.e., one that takes the consumption good and instantaneously turns it into a risky amount of the consumption good. The partial differential equation for pricing physical/financial assets is again developed. In an appendix the model is generalized to include multiple CIR production processes in perfectly elastic supply, multiple financial assets and multiple state variables. The generalization of CIR’s fundamental valuation equation is determined.

In Chapter 4 the model is extended to include production processes more general than the CIR type that model the “time to build” notion that appears in Kydland and Prescott’s work [28] and Brock’s [12]. This is the idea that when the consumption good is invested it is not returned or available until a finite amount of time has passed. These production processes create the physical/financial assets that are in fixed supply. They convert the consumption good into an asset that returns a risky amount of the consumption good a finite time in the future. We demonstrate that such production processes change the boundary conditions on the partial differential equations for the asset prices but not the equations themselves. The opportunity to create the financial assets always lowers their prices. In the case of long term bonds this implies an increase in long term interest rates when a production process is introduced.
Chapter 3

Pricing A Single Financial Asset

3.1 Assumptions

In this first part of the essay we concentrate on the pricing of a physical/financial asset that is in fixed positive supply. The only source of uncertainty is in the payoff of the physical/financial asset. The only production is instantaneous and riskless. These assumptions allow us to concentrate on the effect of the asset’s characteristics, both the distribution of possible payoffs and the amount outstanding, on its price.

We use a particularly simple general equilibrium setting that includes initial endowments of a consumption good and two assets: the physical/financial asset in which we are particularly interested and the asset created by investing the consumption good in a production process in perfectly elastic supply whose return is instantaneously riskless and thus determines the interest rate.

The form of the financial asset is also very simple: it is an ownership claim to a continuous dividend stream of the consumption good and a single, discrete payment of the consumption good to be received at a specific time in the future; the amounts received are given functions of a single stochastic state variable. In this setting, we derive the partial differential equation that the equilibrium price of the financial asset must satisfy; this equation is seen to differ in an important way from the analogous equation describing the equilibrium price of a zero aggregate net supply contingent claim in the CIR model.

The economy consists of identical individuals, each initially (at \( t = 0 \)) endowed with
Chapter 3. Pricing A Single Financial Asset

$K_m(0)$ units of the single consumption good and fractional share $n_m$ of the financial asset. This asset is an ownership claim on a physical asset that will produce a continuous dividend stream of the consumption good and a single discrete payoff of the consumption good at a known future time ($t = T$) which is the horizon for the individuals. The dividend stream and the amount of the payoff on the financial asset are functions, $D(t, x(t), K_m(t))$ and $f(x(T), K_m(T))$ respectively, of the value of the state variable, $x(t)$, and the per capita holdings of the consumption good $K_m(t)$. The current payoff of the financial asset is always known given the current values of $x(t)$ and $K_m(t)$ which determine the current state of the economy. The infinitesimal dividend $D \cdot \delta t$ received in the next infinitesimal time period $\delta t$ is known given that the current state of the economy is known. The risk in the financial asset comes from not knowing the future values of the dividend stream and final discrete payoff because the future states of the economy are not known.

Between the initial time and the horizon each individual has three possibilities with respect to his or her current holding of the consumption good. The good can be consumed, it can be traded for the financial asset, and it can be invested in a technology, equally available to all individuals, that yields an instantaneous rate of return, $r$. In general, $r$ can be function of $t$ and $x$. These possibilities are not mutually exclusive and can be exercised continuously until the horizon.

The state variable, $x$, evolves according to a stochastic differential equation:

$$dx = \mu_x(t, x)dt + \sigma_x(t, x)dz(t),$$

(3.1)

where $\mu_x(t, x)$ and $\sigma_x(t, x)$ are scalar-valued functions and $z$ is a scalar Wiener process. For convenience, we will frequently suppress the arguments of $x, z, \mu$ and $\sigma$ where no confusion results.
Chapter 3. Pricing A Single Financial Asset

The physical/financial asset, which is continuously available for trading in a frictionless market, has a price per unit share of \( P(t, x(t), K_m(t)) \) units of consumption good at time \( t \). One share is defined to be the amount of the physical/financial asset that pays off the dividend \( D(t, x(t), K_m(t)) \) and \( f(x(T), K_m(T)) \) at time \( T \) and is simply a normalization without any effect on the economy. The amount of this asset that the economy actually has is measured by the per capita holdings of \( n_m \) shares with the per capita payoff of \( n_m f(x(T), K_m(T)) \) and \( n_m D(t, x(t), K_m(t)) \). All individuals act as price takers with respect to \( P \) in their individual optimization decisions.

3.2 An Example of the Model

Consider the following story that represents this model of an economy. Assume a world where the only consumption good is potatoes. Each economic agent/investor has a stock \( K_m \) of potatoes from which he or she eats, i.e. consumes. Potatoes can be consumed, resulting in utility to the investor, or invested in a riskless production process. The riskless production process is the opportunity to plant a kilogram of potatoes in the ground and dig them up a time interval \( \delta t \) later and find that there are \( 1 + r\delta t \) kilograms of potatoes, without any risk. Any investor can do this (land is freely available) so that this riskless, instantaneous production process is in perfectly elastic supply. The real asset represented by one arbitrary unit of potatoes planted in the ground (invested in the riskless production process) always has at that time a price of one unit of potatoes since it always costs this much to create it and the one unit of potatoes can be dug up again (disinvested without loss). Investors then consume at some rate \( C \) from their stock \( K_m \) of potatoes which sit in the ground growing at the rate \( r \). The instantaneous payoff of the riskless production process means that investors have immediate access to their stock of potatoes which includes the increase due to growth at the rate \( r \).
There is also a physical asset in fixed supply in this economy, which is a large field planted with all the potato seeds the economy has. This field of potato seeds is such that a small amount of potatoes representing a dividend stream are produced at some continuous rate that cannot be controlled, but the main harvest from the field occurs at time $T$ and cannot be obtained earlier. This field is assumed fixed in terms of its dividend stream and final payoff when the harvest occurs. The final harvest of the planted potato field depends on the average rainfall that it receives which will be represented by the state variable $x$. Investors know the stochastic process that the average rainfall follows and the effect of the average rainfall on the harvest so that they know the probabilities of various amounts of potatoes being harvested from the field given the current average rainfall. We assume that the harvest from the potato field also depends on the amount of potatoes that will be available to investors just before the harvest occurs i.e. $K_m(T)$. Presumably these potatoes sitting in the ground (perhaps in the field of potato seeds themselves) affect the growing conditions for the potato seeds in the field (this is not meant to be botanically correct) positively or negatively. A very convenient assumption will be that the output of the potato field will be proportional to the amount of potatoes, $K_m(T)$, investors already have at harvest time.

Each investor in this economy holding $K_m(t)$ potatoes also has a claim on the potato field and this claim is represented by $n_m$ shares of a single financial asset, shares in the potato field. (The financial asset could just as easily be a package of debt and equity securities whose payoffs all depend on the amount of potatoes the field produces.) These $n_m$ shares entitle the holder to the dividend stream $n_mD(t, x(t), K_m(t))$ units of potatoes per unit of time and the portion of the harvest $n_tf(x(T), K_m(T))$ at the harvest time $T$. The financial asset, and hence indirectly the underlying physical asset, is traded in a financial market determining an equilibrium price $P(t, x(t), K_m(t); n_m)$ units of potatoes at which investors just want to hold the $n_m$ shares they are endowed with. At a higher
price investors would desire to sell some of their shares for more potatoes creating excess supply. Similarly, investors would try to buy more shares at a lower price and excess demand would result.

Although the financial asset is in positive net supply one can also think of it as being in zero net supply but still being represented by a physical asset. Then when a holder of a physical asset sells the financial asset that represents it he is really selling it short if we assume it does not exist. The short position is then covered by delivery of the physical asset.

Each investor has wealth consisting of $K_m$ potatoes and $n_m P$ in the potato field. The wealth $n_m P$ in the potato field can only be turned into that many potatoes by selling shares in the financial market. Of course, if all investors at once actually tried to do this then the price and wealth held in the shares of the field would be much lower. If there are $N$ investors then the economy has the aggregate amount $NK_m$ potatoes and the size of the potato fields can be measured by the aggregate dividend $Nn_m D$ and final aggregate economy payoff of $Nn_m f(x, K_m)$ potatoes. The fundamental endogeneous quantities of interest in this economy are the consumption rate, $C$, and consumption good holdings, $K_m$, of each individual investor and the price, $P$, of one share in the potato field (the physical/financial asset) measured in units of potatoes. None of these quantities depends on $N$, the number of investors. This number merely scales up the size of the economy. It should be large to make the assumption of price taking investors reasonable.

$n_m f(x(T), K_m(T))$ represents the units of potatoes (consumption good) an investor actually receives due to claims on the potato field (the physical/financial asset) while $f(x, K_m)$ is the payoff of one share in the field. Similarly, $P$ is the price of one share in the potato field that pays off $f(x(T), K_m(T))$ and $n_m P$ is the market value of the shares in the field held by each investor. $n_m D$, $n_m f$ and $K_m$ control the relationship between the proportion of wealth represented by the potato field through the dividend
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\( n_m D \) and final payoff \( n_m f \) and the wealth held in the consumption good \( K_m \) (or the total per capita wealth \( W_m \)). \( n_m \) measures the ratio of the value of an investor's holdings of the physical/financial asset to the value of one share in this asset. In the CIR model this ratio is zero and in our model it is nonzero. The share price \( P(t, x, K_m; n_m) \) can then be parameterized by \( n_m \).

Consider the case of a physical/financial asset that pays no dividend i.e. \( D(t, x, K_m) = 0 \). We would expect that the proportion of potatoes (consumption good) already held, to the expected new potatoes that will be received from the field, at the harvest time \( T \), i.e. the fraction

\[
L = \frac{K_m(t)}{n_mE_t[f(x(T), K_m(T))]},
\]

would be an important determinant of the level of the price of the financial asset. The fraction \( L \) measures liquidity in the economy. The greater \( L \) is, the greater the fraction of wealth represented by the more "liquid" consumption good \( K_m \). The consumption good is more "liquid" than the physical/financial asset, since it can be directly consumed at any time whereas the financial asset can not. We expect that the greater \( L \) is then the lower \( E_t[f(x(T), K_m(T))/P(t, x(t), K_m(t))] \) for a given per capita wealth \( W_m \). The ratio \( E_t[f(x(T), K_m(T))/P(t, x(t), K_m(t))] \) represents the expected value of the payoff of one share in the field relative to the market value of one share in the field now. This ratio is a relative measure of how the financial market currently values a given risky future payoff. An alternative form for this measure is to define a risk adjusted discount rate \( k(t, x(t), K_m(t)) = E_t[f(x(T), K_m(T))/P(t, x(t), K_m(t))] - 1 \). We expect that the greater is \( L \), then the lower is \( k \).

One useful way to think about changing \( L \) is to change \( n_m \) keeping wealth constant. As \( n_m \), the number of shares investors hold, decreases, \( L \) increases. Keeping per capita wealth constant, as we increase the proportion of the economy's wealth coming from the
sure amount of the consumption good already in existence relative to the expected payoff of the risky financial/physical asset, i.e. increase \( L \) by lowering \( n_m \) and raising \( K_m \), the financial asset is "scarcer" in some sense. Being scarcer it should have a higher price. If the "risk" of the financial asset was also held constant then the price would increase as \( n_m \) is lowered so that investors would be just willing to hold the smaller amount of the financial asset.

In fact there is a second effect that also increases the price. As \( L \) or in particular \( n_m \) decreases, the risk of the financial asset also decreases and this also causes the price to be higher. The only source of risk in the economy is due to uncertainty in the final payoff of the financial/physical asset. As will be shown later the magnitude of this risk is measured by the covariance of the financial asset price with marginal utility. For a fixed level of per capita wealth, the lower the fraction of wealth held in the risky financial asset the smaller is the covariance of the financial asset price with marginal utility. This means the financial asset has less risk. Since wealth is held constant the investors' risk tolerance is constant so that the price of risk is constant. This means a lower equilibrium expected return is needed. With \( E_t[f(x, K_m)] \) fixed, this means a lower discount rate \( k \) and a higher current price. Hence the ratio of the financial asset's expected payoff to its current price \( E_t[f(x(T), K_m(T))]/P(t, x(t), K_m(t)) \) and the risk adjusted discount rate \( k(t, x(t), K_m(t)) \) are pushed down by an increase in the liquidity ratio \( L \) for constant wealth both because the supply of the financial asset is lower and because the risk of the financial asset has decreased.

There is a third and more subtle reason why the price might increase with a decrease in \( n_m \) or more generally an increase in \( L \), with wealth constant (this requires a decrease in \( K_m \) to hold wealth constant). The partial derivative \( P_K \) is presumably positive for nonzero \( n_m \). This is because wealth and in general risk tolerance increase with increasing \( K_m \), so that by previous arguments so does the price \( P \). Since \( P \) increases with \( K_m \) we
have the result that $P_K$ is positive. The expected return of the financial asset has a term $P_K(K_m r - C_m)$. Investors are willing to hold the asset when its expected return is high enough. This rate of return can be obtained by the increase of the price $P$ over time with all the economy's other parameters constant, as indicated by the term $P_t$. The rate of return can also be obtained when parameters like $K_m$ or $x$ are expected to increase and $P$ is an increasing function of them. For example the price $P$ gets larger as $K_m$ gets larger through investment returns i.e. $P_K K_m r > 0$. With the final payoff function $f = P(T)$ fixed, the current price $P$ must be larger if $P_t$ must be decreased to give the equilibrium required rate of return on the financial asset. With wealth constant we assume that the consumption rate $C_m$ is constant. Then as the term $P_K(K_m r - C_m)$ increases by an increase in $K_m$ with wealth constant, the current equilibrium price is higher since the instantaneous rate of return obtained when $x$ and $K_m$ are constant given by the term $P_t$ need not be so large.

When $n_m$ is zero then the price $P$ must be high so that investors, although given the opportunity in the financial market, will not wish to buy and hold the physical/financial asset. Furthermore, the financial asset has no risk since investors' wealth is not made up of the financial asset; hence the covariance between the asset price and marginal utility is zero. Finally, for a given level of wealth, when $n_m$ is zero all wealth is represented by $K_m$ and the $P_K$ term is at its largest and hence so is $P$, since $P_t$ is at its smallest.

### 3.3 The Investors' Optimization Problem

Each individual chooses consumption, real investment, and financial asset holding strategies to maximize expected utility until the horizon:

$$E_t \int_t^T U(C(s), s)ds + B(W(T), T),$$

(3.2)
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where \( E_t \) denotes expectation conditional on information at time \( t \), \( U \) is an increasing, strictly concave von Neumann–Morgenstern utility function, \( C(s) \) is the individual’s consumption rate at time \( s \), \( B \) is a bequest function, and \( W(T) \) is the individual’s wealth at the horizon. The individual’s choice is subject to a budget equation, relating consumption, real investment and investment in the financial asset to change in wealth.

To derive the change in wealth resulting from the individual’s actions and market forces, consider a discrete-period version of the problem, where periods are of duration \( h \), and then take limits as \( h \) goes to zero.\(^1\) At the beginning of the period that runs from time \( t \) to time \( t+h \), the individual’s wealth before taking action for the period is

\[
W(t) = K(t) + n(t-h)P(t),
\]

(3.3)

where \( K(t) \) is the amount of consumption good available for use this period and \( n(t-h) \) is the fractional share of the financial asset held during the previous period. The amount of consumption good available at time \( t \) is determined by the amount allocated to riskless real investment, after consumption and financial asset transactions, during the previous period:

\[
K(t) = \{K(t-h) - C(t-h)h - [n(t-h) - n(t-2h)]P(t-h)}(1+rh) + hn(t-h)D(t),
\]

(3.4)

where \( C(t-h)h \) is the consumption by the individual during the previous period, \( hnD \) is the dividend from the financial asset and the final term in the braces is the value of the net purchase of the financial asset at time \( t-h \). Substituting from 3.4 and taking limits as the duration of the period \( h \), goes to zero yields the budget equation for the change in wealth:

\[
dW = [(W - nP)r + nD - C]dt + ndP
\]

(3.5)

The individual chooses \( C \) and \( n \) to maximize expected utility, 3.2, subject to 3.5. At the equilibrium \( K \) must equal \( K_m \) and \( n \) must equal \( n_m \), since individuals are identical, and

this implies that \( C \) must equal per capita consumption, \( C_m \). At the level of individual optimization, however, \( C \) and \( n \) are taken as controllable by the individual.

Applying Ito's Lemma, the change in the price of the financial asset is

\[
\frac{dP}{dt} = P_x dx + P_t dt + PK_m dK_m + \frac{1}{2} P_{xx}(dx)^2.
\]  
(3.6)

The change in the state variable is given by 3.1. The change in the per capita holding of the consumption good is obtained by using 3.4 to express \( K(t) - K(t - h) \), aggregating across individuals, noting that per capita holding of the financial asset is constant over time, and taking the limit as \( h \) goes to zero to give

\[
\frac{dK_m}{dt} = (K_m r + n_m D - C_m) dt.
\]  
(3.7)

To apply stochastic dynamic programming to the optimization problem, define the optimal value (i.e., derived utility of wealth) function:

\[
J(W, x, K_m, t) = \max_{\{C, n\}} E_t \left[ \int_t^T U(C(s), s) ds + B(W(T), T) \right].
\]  
(3.8)

The optimal controls in the problem must satisfy

\[
\max_{\{C, n\}} \left[ U(C(t), t) + L(t) J \right] + J_t = 0,
\]  
(3.9)

where \( L \) is the differential operator:

\[
L(t)J = \left[ J_w E_t (dW) + \frac{1}{2} J_{ww} E_t [(dW)^2] \right] + J_x E_t (dx) + \frac{1}{2} J_{xx} E_t [(dx)^2]
+ J_{wx} E_t (dW dx) + J_{K_m} dK_m) / dt
= J_w \{ Wr - C + n[P_x \mu_x + P_t + D
+ PK_m(K_m r + n_m D - C_m) + \frac{1}{2} P_{xx} \sigma_x^2 - r P)\}
+ \frac{1}{2} J_{ww} n^2 P_x^2 \sigma_x^2 + J_x \mu_x + \frac{1}{2} J_{xx} \sigma_x^2
+ J_{wx} n P_x \sigma_x^2 + J_{K_m}(K_m r + n_m D - C_m) \}.
\]  
(3.10)

\(^2\)See, for example, Cox, Ingersoll, Ross (1985) [14].
Since negative consumption is not allowed but short sales of the financial asset are permitted, necessary and sufficient conditions for an optimal solution to 3.9 are

\[ \begin{align*}
U_C - J_W & \leq 0 \quad (3.11) \\
C(U_C - J_W) & = 0 \quad (3.12) \\
J_W P_x \mu_x + P_t + P_K (K_m r + n_m D - C_m) & + \frac{1}{2} P_{xz} \sigma_z^2 + D - r P \\
+ J_{WW} n P_z^2 & + J_{WX} P_z \sigma_z^2 = 0 \quad (3.13)
\end{align*} \]

The optimal controls must also satisfy a boundary condition at the horizon:

\[ J(W, x, K_m, T) = B(W(T), T) = B(K(T) + n(T)f(x), T), \quad (3.14) \]

since \( f(x) \) is the payoff on the financial asset at the horizon date.

Since individuals are identical, at equilibrium in the economy 3.11-3.13 must hold with \( C = C_m, \ K = K_m \) and \( n = n_m \). The former two quantities are endogenous in the problem, but the per capita fractional holding of the financial asset is an exogenous parameter. If we let \( \alpha_p \) denote the expected rate of return on the financial asset and \( \sigma_{Wx} \) denote the covariance of the change in optimal wealth with the change in the state variable, then equations 3.5 and 3.6 implies that 3.13 can be rewritten as

\[ (\alpha_p - r)P = \left( \frac{-J_{WW}}{J_W} \right) \sigma_{Wx} + \left( \frac{-J_{WX}}{J_W} \right) \sigma_z^2 P_z. \quad (3.15) \]

The above relation is similar to the result of Theorem 2 in CIR [14], the difference being the absence in 3.15 of terms equalling \( \frac{-J_{WW}}{J_W} \sigma_{Wx} P_w \) and \( \frac{-J_{WX}}{J_W} \sigma_{Wx} P_w \), where \( \sigma_w^2 \) denotes the variance of the change in optimal wealth. This difference reflects our simplifying assumption that the return on real investment is riskless. At equilibrium in the CIR model, since their financial assets are in zero aggregate net supply, the stochastic component of the change in optimal wealth is completely due to the stochastic component of the returns on production possibilities (in perfectly elastic supply). If returns on
production possibilities in CIR were nonstochastic (and equal, to avoid arbitrage opportunities), then terms involving, in our notation, \( \sigma_w^2 P_w \) and \( \sigma_w P_w \), would disappear in their Theorem 2; if the aggregate net supply of the financial asset, \( n_m \), were zero in our model, then the term involving \( \sigma_{wx} \) in 3.15 would disappear. Under these conditions, the results of the two models would be identical.

What is more interesting in comparing 3.15 with CIR Theorem 2 is that both equations have a term \( \left( \frac{\sigma_{w,w}}{\beta_w} \right) \sigma_{wx} P_x \) but the basis of this term is not the same in the two models. In both models, this term reflects the premium for bearing risk associated with the degree to which optimal wealth covaries with the state variable. In the CIR model, this covariance arises from the effect of movements in the state variable on the return to real investment that is in perfectly elastic supply. In our model, where real investment is riskless, the covariance arises from the effect of movements in the state variable on the return to holding the financial asset that has fixed supply. Thus, different sources of systematic risk lead to identical equilibrium risk premium terms. However, as is shown below, our model leads to terms in the partial differential equation for the financial asset's price \( P \) that do not appear in the CIR model.

In the case of our illustrative economy based on potatoes the CIR model would have risky instantaneous potato growth, i.e. investors would plant a potato and a short time later dig it up finding it had grown an unpredictable amount. This would be the only source of risk in the economy. Fields planted with potato seeds would not exist but financial assets, in zero net supply, based on such fields would still be priced. The price would depend on the risk of these financial assets and the risk would be due only to how the price covaried with the risky growth of the wealth, \( K_m \), in potatoes planted in the ground. The ultimate source of this covariance would be the dependence on the state variables of both the financial asset's final payoff and the risky growth of the consumption good (net wealth in CIR's model). The variability of the financial asset's price would
not matter since changes in the price would not affect wealth. It is the covariance of the field’s price change with the change in the value of the investor’s net holding of the field that is important and this would be zero since the value actually held would be zero.

On the other hand, in our model the economy has no consumption risk due to the growth of potatoes planted at the riskfree rate (consumption good \(K_m\) invested at \(r\)). The only risk is in the future dividend stream and final harvest of the potato field which depend on the unknown average rainfall. The economy’s risk is measured by the covariance of the field’s price with the change in value of the field and this covariance is greater the greater the value of the field i.e. the more of it that is held.

3.4 Logarithmic Preferences

3.4.1 A Partial Differential Equation for Financial Asset Prices

Next we examine the case of an economy with a single financial asset and consumption good where all investors are identical and have preferences given by the von Neumann–Morgenstern utility function \(U(C,t) = e^{-\rho t} \log(C)\), and the bequest function \(B(W(T),T) = e^{-\rho T} \log W\). Substituting the trial solution

\[
J(W,x,K_m,t) = \frac{A(t)}{\rho} e^{-\rho t} \log(W) + \Psi(x,K_m,t)
\]

into equations 3.11–3.13 results in the following

\[
P(t) = \frac{\rho W}{A(t)}
\]

(3.17)

and

\[
P_t + \mu_x P_x + \frac{1}{2} \sigma_x^2 P_{xx} + P_{K_m}(K_m r + n_m D - \frac{\rho W}{A(t)}) + D - r P
- \frac{P^2 \sigma^2}{W^2} \eta(t,x,W,K_m) = 0
\]

(3.18)
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$C(t)$ is the individual investor’s optimal consumption and $n(t, x, W, K_m)$ is the optimal holding of the financial asset. It is assumed that all investors in the economy have rational expectations and have already solved for $P(s, x, K_m)$ for $s > t$. Then from equation 3.18 it can be seen that $n(t, x, W, K_m)$ is independent of $W$ and depends only on $t, x$ and $K_m$. Substituting the trial solution for $J(W, x, K_m, t)$ equation 3.16 into 3.9 and using the expression for the optimal $C(t)$ 3.17 gives the equation

$$e^{-\rho t}[\log(W) + \log(\rho/A(t))] + [A'(t)\frac{e^{-\rho t}}{\rho} - A(t)e^{-\rho t}]\log(W)$$

$$+ \Psi_t + A(t)\frac{e^{-\rho t}}{W\rho} \{W - \frac{W\rho}{A(t)} + n[P_t + \mu_xP_x + \frac{1}{2}\sigma_x^2P_{xx} + P_{K_m}(K_m r + n_m D - C_m) + D - rP]\}$$

$$+ A(t)\frac{e^{-\rho t}}{2W^2\rho n^2\sigma_x^2} + \Psi_x\mu_x + \frac{1}{2}\Psi_{xx}\sigma_x^2 + \Psi_K(K_m r + n_m D - C_m) = 0$$  \(3.19\)

Using the fact that $\frac{\rho}{W}$ is independent of $W$ allows for separation of the equation into two parts. Each of these two parts must separately equal zero. The terms involving $\log(W)$ must equal zero by themselves and the rest of the equation that is independent of $W$ must also equal zero. This leads to two equations. Setting the $\log(W)$ terms equal to zero results in the equation for $A(t)$:

$$1 - A(t) + \frac{A'(t)}{\rho} = 0$$  \(3.20\)

with the boundary condition $A(T) = \rho$. After substituting for $n/W$ from equation 3.18, setting the other terms equal to zero results in the complicated equation for $\Psi(x, K_m, t)$:

$$e^{-\rho t}\log(\rho/A(t)) + \Psi_t + \Psi_x\mu_x + \frac{1}{2}\Psi_{xx}\sigma_x^2 + \Psi_K(K_m r + n_m D - C_m)$$

$$+ A(t)\frac{e^{-\rho t}}{\rho} \{r - \frac{\rho}{A(t)} + [P_t + \mu_xP_x + \frac{1}{2}\sigma_x^2P_{xx}$$

$$+ P_{K_m}(K_m r + n_m D - C_m) + D - rP]^2/(2P_x^2\sigma_x^2)\} = 0$$  \(3.21\)
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with the boundary condition $\Psi(x, K_m, T) = 0$ and the solution for $A(t)$ is $A(t) = 1 + (\rho - 1)e^{(t-T)}$ so that

$$J(W, x, K_m, t) = \frac{1}{\rho} \left[ 1 + (\rho - 1)e^{(t-T)} \right] e^{-\rho t} \log(W) + \Psi(x, K_m, t) \quad (3.22)$$

and

$$C(t) = \frac{\rho W}{1 + (\rho - 1)e^{(t-T)}}.$$

As has been shown, given the price function for the financial asset $P(t, x, K_m)$ an individual investor’s optimal investment decision can be solved for. Since all investors are identical in equilibrium, the price function $P(t, x, K_m)$ must be such that the optimal holding of the financial asset is $n(t, x, W, K_m) = n_m$ when $W = W_m = n_m P(t, x, K_m) + K_m$, i.e., each investor’s wealth equals the per capita wealth in the economy. Substituting $n = n_m$, and $W = W_m = K_m + n_m P(t, x, K_m)$ into the equation for $P(t, x, K_m)$ gives the following partial differential equation:

$$P_t + \mu x P_x + \frac{1}{2} \sigma^2 x^2 P_{xx} + P K_m \left( K_m r + n_m D - \frac{\rho(n_m P + K_m)}{1 + (\rho - 1)e^{(t-T)}} \right) + D - r P$$

$$= \frac{n_m \sigma^2 P^2}{n_m P + K_m} \quad (3.23)$$

with the boundary condition $P(T, x, K_m) = f(x, K_m)$. The solution to equation 3.23 is the equilibrium price for this economy where all investors have rational expectations and are identical. Investors in the economy at time $t$ solve their optimal investment problem using dynamic programming. They use the fact that all investors are identical and solve the same investment problem as they do and get the same solution. In solving their optimal investment problem, investors also solve for the price function that supports an equilibrium where all investors hold $n_m$ of the financial asset. As investors work backwards from time $T$, they solve for $P$ and $J$ at time $t$ using the solution for time $s > t$ they have obtained so far.
The partial differential equation for \( P(t, x, K_m) \) is nonlinear because of the \( P^2_x \) and \( PP_{K_m} \) terms. With \( n_m = 0 \) the equation becomes linear and corresponds to the fundamental valuation equation of Theorem 3 in CIR when the production technology is riskless, i.e., the return on real investment is the riskless rate \( r \). \( K_m \) and \( P_{K_m} \) in equation 3.23 correspond to \( W \) and \( P_W \) in Theorem 3 of CIR. \( Var(K_m) = 0 \) so that there is no \( P_{K_m,K_m} \) term. The equation 3.23 is of the form \( (\alpha_P - r)P = \frac{n_m \sigma^2 P^2}{n_m P + K_m} \) where \( \alpha_P \) is the expected rate of return on the financial asset. The risk premium which corresponds to \( \frac{-}{\alpha_P} \) of equation 3.15 is \( \frac{n_m \sigma^2 P^2}{n_m P + K_m} \) since \( \sigma_{W_x} = n_m P_x \sigma_x \cdot \sigma_x \) and \( \frac{-}{\alpha_P} = \frac{1}{W_m} = \frac{1}{n_m P + K_m} \). The risk premium is proportional to \( n_m \), the per capita holdings of the financial asset; hence the greater investors' holdings of the risky financial asset, the greater is the risk of the financial asset and thus the larger must be the expected return and hence the lower the price.

### 3.4.2 The Dependence of Asset Prices on the Characteristics of the Economy

Equation 3.23 can also be written

\[
\left[ P_t + \mu_x P_x + \frac{1}{2} \sigma^2_x P_{xx} \right] / P + P_{K_m} \left( K_m r + n_m D - \frac{\sigma^2(n_m P + K_m)}{1 + (\epsilon - 1)X(t - 2)} \right) / P + D / P - r = \frac{n_m P}{K_m} \left( \frac{P}{P} \right)^2 \frac{\sigma^2_x}{(1 + \frac{n_m P}{K_m})} \tag{3.24}
\]

Consider multiplying the physical assets and payoffs of the economy by the factor \( a \) so that \( K_m \) becomes \( K'_m = aK_m \), the dividend stream becomes \( D'(t, x, K'_m) = aD(t, x, K'_m) \) and \( f'(x, K'_m) = af(x, K'_m) \). The financial asset price \( P'(t, x, K'_m) = aP(t, x, K'_m) \) solves 3.23 with the dividend \( D' \) and boundary condition \( f' \) if \( P \) solves 3.23 with the dividend \( D \) and boundary condition \( f \). This can be seen from equation 3.24 by multiplying both sides of the equation by \( a / a \), changing \( aP \) to \( P' \), \( aK_m \) to \( K'_m \), and noticing that \( \partial P'(t, x, K'_m) / \partial K' = \partial aP(t, x, K'_m) / \partial K_m \) solves 3.23. Hence the ratio of the investor's financial asset holdings to his or her consumption good holdings,
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\( n_m P/K_m \), does not depend upon the level of wealth the investor holds but just upon the ratio of the consumption good to the financial asset’s payoffs \( nD \) and \( n_m f \) as measured by such quantities as the liquidity ratio \( L \). Equivalently, this shows that the equation for \( P \) depends only on \( \frac{n_m P}{K_m} \) and not the level of per capita wealth in the economy. This is not a general result and depends on the assumption of log utility. For other preferences \( P \) will also depend on the absolute per capita level of wealth.

Consider the identity

\[
P(t, x, K_m) = f(x, K_m) - \int_t^T P_s ds \]

where \( P_s \) is given by equation 3.23. This identity forms the basis of most methods of numerical analysis for finding solutions to partial differential equations. The general idea is to start from the value of the solution at the boundary condition which is known and in our case given by \( P(T, x, K_m) = f(x, K_m) \). This function of \( x \) and \( K_m \) and its partial derivatives are substituted into equation 3.23 and solved for \( P_t(T, x, K_m) \). An approximate solution for \( t = T - \delta t \) is then obtained using \( P(T - \delta t, x, K_m) = f(x, K_m) - P_t(T, x, K_m) \delta t \). With the new values of \( P \) at \( t = T - \delta t \) the process is repeated to find the value of \( P \) at \( t = T - 2\delta t \). This process is then repeated to find the solution for an interval of time such as \( 0 < t < T \). The numerical solution approaches the true solution as the step size \( \delta t \) approaches zero but the computational cost increases.

We can use these ideas to argue informally what the effect of the parameters of the economy on the financial asset price should be. For example consider again the case of the potato field. If the yield of the harvest is a an upward sloping, concave function \( (f_x > 0, f_{xx} < 0) \) of the average rainfall then a decrease in the average rainfall decreases the yields of the field more than an equal increase in the average rainfall increases them. Then the term \( P_{xx} \) tends to be negative although the \( P^2_x \) term through \( f^2_x \) can eventually offset this. If \( P_{xx} \) is negative because the \( P^2_x \) term is well behaved then the stochastic nature of the average rainfall will tend on average to decrease the price. The more variable the average rainfall as measured by \( \sigma_x \) the lower the price. This is because when \( P \) is concave
in $x$ and $x$ is equally likely to fluctuate an equal amount up as down then on average $P$ will decrease. The greater the magnitude of the fluctuation as measured by $\sigma_x$ the more negative the expected price change given by the term $(1/2)\sigma_x^2 P_{xx}$. This will tend to lower $P$ so that $P_t$ can be large enough as $P$ moves over time towards $f$ to give a risk adjusted expected rate of return of $r$. Similarly a positive drift term to the average rainfall over time ($\mu_x > 0$) and the assumption that a higher average rainfall means higher yields ($f_x > 0$) increases $P$ since $P_t$ need not be so large to give an expected risk adjusted rate of return of $r$.

The risk adjustment term which is the covariance of the financial asset's rate of return with the rate of return on wealth is always positive. This is true as long as the only source of risk is the variance of the price. If there were other sources of risk in the rate of return on wealth, such as risky instantaneous production processes of the CIR type, then they could have a negative correlation with the financial asset price and could lead to a negative risk term.

The greater the uncertainty in average rainfall ($\sigma_x$) and the greater the effect of average rainfall ($f_x$) then, as equation 3.23 shows, the greater the risk adjustment term. The expected rate of return on $P$ must be larger than $r$ by the amount of this risk term. Everything else being equal, this means $P_t$ must be larger and hence current values of $P$ must be lower given that the final value $f$ is fixed.

Similarly, the greater is per capita wealth ($W_m$) the lower are the investor's risk aversion and the risk adjustment term. This tends to increase the price $P$. In general a larger value of $K_m$ means greater wealth $W_m$, and a greater risk tolerance so that the risk adjustment term is smaller and $P$ is larger. This means that $P_K > 0$ since larger values of $K_m$ are associated with higher prices $P$. As $K_m$ changes due to investment at $r$, the dividend stream $D$ and consumption, increases in $K_m$ will result in capital gains in $P$. Thus the nonlinear term $P_K(K_n r + N_mD - C_m)$ is positive when the consumption
good increases over time i.e. $K_m r + N_m D - C_m > 0$. Then $P_i$ need not be so large and hence $P$ will be larger. For logarithmic preferences, consumption is proportional to wealth which depends on $P$. There is an interesting effect from the nonlinear term $-P_K C_m = P_k W_m p/A(t)$. An increase in $P$ due to an increase in $x$ results in greater wealth and consumption which decreases $K_m$ faster and makes the term $-P_K C_m$ more negative. This means $P_i$ must be increased to maintain the required rate of return which lowers $P$. This acts to stabilize the price through negative feedback.

### 3.4.3 CAPM

Equation 3.23 finally can be written in the form $\alpha_P - r = \text{cov}(\tilde{R}_W, \tilde{R}_P)$ where $\tilde{R}_W = \frac{\hat{w}}{W}$ and $\tilde{R}_P = \frac{\hat{P}}{P}$ since $\text{cov}(\tilde{R}_W, \tilde{R}_P) = (\frac{n_m F_m}{W_m})(\frac{\sigma_P}{P})$. Since $\alpha_W - r = \text{cov}(\tilde{R}_W, \tilde{R}_W)$, the equation implies the CAPM equation

$$\alpha_P - r = \frac{\text{cov}(\tilde{R}_W, \tilde{R}_P)}{\text{cov}(\tilde{R}_W, \tilde{R}_W)}(\alpha_W - r).$$  

(3.25)

$\alpha_W - r$ is the price of risk, where $\alpha_W$ is the expected return on wealth on the market and $\alpha_W = \frac{n_m F_m}{W_m} + \frac{r K_m}{W_m}$. Risk is measured by $\frac{\text{cov}(\tilde{R}_W, \tilde{R}_P)}{\text{var}(\tilde{R}_W)}$, which is the "beta" of the financial asset. Equation 3.23 implies 3.25 but 3.25 does not imply 3.23. The CAPM is a more general relationship but does not determine prices. The CAPM only gives a relationship between the rates of return of various assets but not what the prices of these assets should be.

### 3.5 Solving the Equation

Because the equation for $P$ is nonlinear, the well developed theory of linear partial differential equations cannot be used. Nonlinear differential equations do not have unique solutions and in general must be solved by trial and error or numerical methods. The equation for $P$ can be transformed to one that has just one nonlinear term by introducing
$H(t, x, K_m) = \frac{1}{n_m P(t, x, K_m) + K_m}$. $H$ is proportional to $J_w$, the marginal utility. Since

$$\frac{1}{H} = n_m P + K_m, \quad n_m P_x = -\frac{H_x}{H^2}, \quad n_m P_{xx} = -\frac{H_{xx}}{H^3} + \frac{2H_x^2}{H^3},$$

etc. Substituting into the equation for $P$ gives

$$H_t + \mu_x H_x + \frac{1}{2} \sigma^2_x H_{xx} + H_{K_m}(rK_m + n_m D - \rho \frac{1}{H} \frac{1}{A(t)}) - H \cdot \rho \frac{1}{A(t)} + rH = 0 \quad (3.26)$$

where $A(t) = 1 + (\rho - 1)e^{\rho(t-T)}$. Finally substituting

$$G(t, x, K_m) = J_W = H(t, x, K_m)e^{-\rho t}A(t)/\rho$$

gives the equation

$$G_t + \mu_x G_x + \frac{1}{2} \sigma^2_x G_{xx} + G_{K_m}(rK_m + n_m D - e^{-\rho t} \cdot \frac{1}{G}) + rG = 0 \quad (3.27)$$

Relation 3.27 is equivalent to CIR Theorem 1, which states that $r$ equals the negative of the expected rate of change in marginal utility.

Equation 3.27 for $G$ has just one nonlinear term $\frac{G_{K_m}}{G}$. In the special case where consumption only occurs at the boundary condition at time $T$ or when consumption is fixed, the investor's optimal investment problem simplifies. Then the general equilibrium price function is still of the form $P(t, x, K_m)$ although investors have only the control variable $n$ and not $C$ to adjust for an optimum. Equation 3.23 for $P$ is still nonlinear because of the $P^2_x$ term, but with the $P_{K_m} P$ term absent. However, equation 3.27 is linear and hence $P$ can in general be obtained using standard methods for linear partial differential equations ³(See reference [13]). Even with $H_{K_m} \cdot \frac{1}{H}$ absent from the equation, the solution is still dependent on $n_m$.

In addition to the use of numerical methods, other methods for obtaining approximate solutions can be found. One method is to write the solution as a series expansion in a small parameter. If $\frac{n_m P}{K_m} << 1$, then write $P = P_0 + \varepsilon P_1 + \varepsilon^2 P_2 + \cdots$, substitute into the

³For example, the Sturm-Liouville method of eigenfunction expansions can be used for very general boundary conditions.
equation for $P$ and equate to zero separately all terms involving a given power of $\epsilon$ where $\epsilon = 0(\frac{a_0}{K_m})$. The equations for $P_0, P_1$, etc. are linear with the equation for $P_1$ requiring the solution for $P_0$ before $P_1$ can be solved for. Similarly the equation for $P_2$ requires $P_0$ and $P_1$ be known. The perturbation expansion solution gives more insight into general properties of the solution than numerical methods because it is an analytic expression so that the result of varying different parameters is easily found.
3.6 An Example of a Solution

In general equation 3.23 is difficult to solve analytically but analytical solutions do exist for particular assumptions. The assumption that \( f \) and \( D \), the final payoff and dividend from the physical/financial asset, are proportional to \( K_m \) leads to equations that can be solved. For simplicity assume that there is no dividend i.e. \( D = 0 \), and that \( n_m f(x, K_m) = K_m/(\alpha \exp(x) - 1) \) where \( \alpha > 1 \) and \( x > 0 \). A larger \( x \) means a smaller payoff so that we expect that a decrease in \( x \) increases the price of the financial asset.

In the case of the potato field example \( x \) might represent a measure of how dry the field is, e.g. \( x \) is the reciprocal of the average rainfall, or \( x \) could represent how cold the field has been e.g. \( x \) is the reciprocal of the average temperature suitably scaled. We assume that \( x \) follows a stochastic process given by \( \mu_x = \mu \) and \( \sigma_x = \sqrt{x}\sigma \). We have \( x \geq 0 \) for \( \mu > 0 \) since \( \sigma_x = 0 \) for \( x = 0 \) and hence negative \( x \)'s can be ignored. Furthermore, \( x > 0 \) for \( \mu > \frac{1}{2}\sigma^2 \) since Ito's lemma implies that

\[
dln(x) = \frac{\mu}{x} - \frac{1}{2}\sigma^2/x \, dt + \sigma / \sqrt{x} \, dz.
\]

The solution \( n_m P(t, x, K_m) \) can be found most easily by solving equation 3.27 for \( G \) and then using the definitions of \( G \), \( H \) and \( A \) to write

\[
n_m P(t, x, K_m) = -K_m + \frac{e^{-rt} A(t)}{\rho G(t, x, K_m)} = -K_m + \frac{1}{2} \left[ e^{-rt} + (\rho - 1)e^{-\rho T} \right] \frac{G(t, x, K_m)}{G(t, x, K_m)}. \tag{3.28}
\]

\( G(t, x, K_m) \) must satisfy the partial differential equation

\[
G_t + \mu G_x + \frac{1}{2}\sigma^2 x G_{xx} + G_K (r K_m - \frac{e^{-\rho t}}{G}) + r G = 0. \tag{3.29}
\]

A solution of the form

\[
G(t, x, K_m) = \frac{Q(t, x) + \frac{1}{2} \left[ e^{-\rho t} + (\rho - 1)e^{-\rho T} \right]}{K_m} \tag{3.30}
\]

simplifies the equation to

\[
Q_t + \mu Q_x + \frac{1}{2}\sigma^2 x Q_{xx} = 0 \tag{3.31}
\]
and allows us to write

\[ n_{m} P(t, x, K_{m}) = K_{m} \left[ \frac{1}{-1 + \frac{1}{\sigma} e^{-\sigma t} + (\rho - 1)e^{-\rho T}} \right] /Q(t, x) \]  

(3.32)

with \( Q(T, x) = -e^{-x}e^{-\rho T}/a \). The solution

\[ Q(t, x) = -\frac{e^{-\rho T} \exp[-x/(1 + \frac{1}{2}\sigma^2(T - t))]}{a[1 + \frac{1}{2}\sigma^2(T - t)]^{\mu/\sigma^2}} \]  

(3.33)

can be verified by substitution. The resulting solution for \( P \) is then given by

\[ n_{m} P(t, x, K_{m}) = K_{m} \left[ \frac{1}{-1 + \frac{a}{\sigma}(\sigma^2 t - \gamma) + (\rho - 1)[1 + \frac{1}{2}\sigma^2(T - t)]^{\mu/\sigma^2} \exp[x/(1 + \frac{1}{2}\sigma^2(T - t))]} \right]. \]  

(3.34)

It can be seen that the form of \( P \) is intuitively reasonable. A larger \( \mu \) is bad news since the larger is the drift in \( x \) the smaller the payoffs that are likely to be achieved and hence the smaller should be the price \( P \). From equation 3.34 it can be seen that \( P \) decreases as \( \mu \) increases since \( T - t > 0 \).

\( P_t \) can be either positive or negative depending on the values of \( x, \rho \), and \( \sigma^2(T - t) \). If \( x \) is very small then it can be seen from 3.34 that the exponential term is negligible. Then an increase in \( t \) decreases \( T - t \) and hence decreases the two terms multiplying \( a \) in the denominator. Hence for \( x \) small enough but still positive \( P_t > 0 \). For \( x \) very large the exponential term multiplying \( a \) in the denominator dominates the other two terms. This exponential term increases as \( t \) increases which causes the denominator to increase and \( P \) to decrease. Hence for \( x \) large enough we have \( P_t < 0 \).

The market value \( n_{m} P \) increases the larger is the final payoff \( n_{m}f \), i.e. as \( a \) is decreased towards 1. The effect of varying \( n_{m} \) with \( f \) fixed cannot be determined here because the form of \( f \) changes as \( n_{m} \) is varied.

An increase in \( \sigma^2 \) increases \( P \). This can be verified by taking the derivative of \( P \) with respect to \( \sigma^2 \). Equation 3.34 for \( P \) as a function of \( \sigma^2 \) is of the form

\[ P = [-A + B \exp[(2\mu/\sigma^2)\ln(1 + \sigma^2(T - t)/2) + x(1 + \sigma^2(T - t)/2)^{-1}]]^{-1} \]  

(3.35)
with $A$, $B$, $x$ and $T - t$ positive. The derivative of $P$ with respect to $\sigma^2$ is of the form:

$$\frac{dP}{d\sigma^2} = \left[ -\frac{1}{[A + B \exp(\ldots)]^2} B \exp(\ldots) \right]$$

$$\cdot \left[ -\frac{2u}{\sigma^2} \ln(1 + \sigma^2(T - t)/2) + \frac{2u}{\sigma^2} \frac{(T - t)/2}{1 + \sigma^2(T - t)/2} - \frac{u(T - t)/2}{(1 + \sigma^2(T - t)/2)^2} \right]. \tag{3.36}$$

The third term in the final set of brackets is negative. The first term is also negative but the second term is positive. Since $\ln(1 + y) \geq y/(1 + y)$ for $y \geq 0$, and letting $y = \sigma^2(T - t)/2 \geq 0$ it can be seen that the sum of the first and second terms of the last set of brackets is negative. Hence the last set of brackets is negative so that the derivative itself is positive. This is in agreement with our previous nonrigorous argument that since $n_m f(x, K_m) = K_m[-1 + a \exp(x)]^{-1}$ is a convex function of $x$ the price $P$ should increase with an increase in the variance of $x$. A decrease in $x$ increases the financial asset’s payoff more than an equal increase in $x$ decreases it so on average the greater the variability in $x$ the higher becomes the expected payoff and hence the price.

### 3.7 Valuing Financial Assets

The partial differential equation for $P$ is the relationship that determines the total value of all physical/financial assets in the economy in terms of the single consumption good. This corresponds to finding the value in terms of money of the market portfolio that includes all long term investments. In an actual economy $P$ would represent the value of all corporate equity, corporate debt, government debt, real estate not owned by public corporations etc. It would not include short term assets such as bank deposits and T-bills that are close to maturity etc., since these would be included in $K_m$. Commodities such as gold and wheat would also belong in $K_m$ since they are consumption goods. (However, they can not be explicitly handled by the model since it assumes only a single consumption good.)

\[ \ln 1 = 0 \text{ and } d\ln(1 + y)/dy = 1/(1 + y) > 1/(1 + y)^2 = d(y/(1 + y))/dy \text{ for } y > 0 \]
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The model we have developed is most useful in valuing financial assets and their underlying physical assets when they represent a significant proportion of the economy's wealth or equivalently when their dividends and terminating payoffs are a significant fraction of the economy's total holdings of the consumption good. In such circumstances it is not a good approximation to assume that total economy wealth or the market portfolio is made up only of the consumption good invested in instantaneous production processes as CIR do. The CIR approach is inconsistent when it is applied to financial assets that are a significant portion of the economy's wealth. This is the case, for example, with pricing bonds. The solution to the CIR pricing equation for a bond will be a bond price that is quite high since it induces investors not to hold bonds. Hence their model also implies very low interest rates. One cannot then multiply by the number of bonds outstanding in the actual U.S. or Canadian economy and expect to have calculated a reasonable number for the total market value of bonds since the calculation contradicts the original assumption that the market value of bonds was zero. The shape of the term structure may or may not be correct but the level of interest rates will in general be too low. The CIR approach treats financial assets as phantom securities priced so high that in fact they do not exist.

The more general approach to valuing physical/financial assets developed here is not much more difficult to apply than the CIR method. To determine the price of the market portfolio or the total wealth in fixed long term assets a nonlinear partial differential equation must be solved. For the case of an economy with log utility investors it is equation 3.23. Once the solution \( n_m P(t, x, K_m; n_m) \) for the value of all the physical/financial assets in the economy is known it is then substituted into the partial differential equations for the prices of individual financial assets. These equations are always linear as is shown in Appendix 1 where the details of pricing financial assets when there are many state variables and financial assets are worked out. Solving these linear partial differential
equations for the price of financial assets is no more difficult in principle than the zero net supply case of CIR. The valuation equations are linear no matter what percentage of wealth the physical/financial asset represents, once the functional form of the total physical/financial asset value is known. For the case of log utility investors the results of Appendix 1, or simply applying equation 3.25, show that the equation for a financial asset \( F \) where the total value held by investors is \( n_F F \) and the boundary condition \( F(T, x, K_m) \) is given, is

\[
F_t + \mu_x F_x + \frac{1}{2} \sigma_x^2 F_{xx} + F K_m \left( K_m r + n_m D_P - \frac{\rho(n_m P + K_m)}{1 + (\rho - 1)e^{(\mu - \gamma)T}} \right) + D_F - rF = \frac{\sigma_n^2 n_m P F_x}{n_m P + K_m} 
\]

where \( D_F \) is the dividend from the financial asset being priced and \( n_m D_P \) is the total per capita dividend received by each investor. In the next section a number of applications are developed of equation 3.37.

The dependence of \( F \) on \( n_F \) is hidden in the dependence of \( n_m P \) on the dividends to \( F \) and the final payoff \( n_F F(T, x, K_m) \). If this dependence is small relative to \( n_m P \) and the variance term \( \text{cov}(F, n_F F) \) is small relative to the covariance term \( \text{cov}(F, n_m P) \) then the price \( F \) will be nearly independent of small changes in \( n_F \) which also change \( n_m P \). The price of the financial asset is independent of \( n_F \) when \( n_m P \) is unchanged. It only depends on the amount of the physical asset underlying \( P \) that goes to make up \( n_m P \). Hence any change in \( n_F \) that does not affect \( n_m P \) will have no effect on the financial asset price \( F \). In the case of the potato economy developed previously, the physical output of the planted potato field at the harvest time and through dividends can be represented by various packages of financial assets such as debt and equity. Given that the payoff of a “potato field bond” is fixed, the number of potato field bonds issued will not change the price of these bonds but only their total market value, which means that the total value of “potato field equity” will change in the opposite direction. The total value of the
potato field stays the same since its aggregate payoff is unchanged. This is in agreement with the Modigliani-Miller theorem [34].

3.8 A Simple Model of Bond Pricing

In this section a simple bond pricing model is developed for a default free bond in positive net supply as an application of the methods that we have developed. The assumptions are as before, including log utility investors. We now explicitly assume that the riskfree technology is not constant over time but instead is a function of the state variables. For simplicity we will assume $x = r$ so that $\frac{dr}{dt} = \mu_r dt + \sigma_r dz$ where $\mu_r$ and $\sigma_r$ are functions of $r$ and $t$. We assume that $\mu_r$ and $\sigma_r$ are such that $r$ never becomes negative. Using equation 3.37 we can write an equation for the price of a bond as a function of the short term interest rate $r$. Let $n_m P(t, r, K_m ; n_m)$ be the value of all long term fixed assets in the economy where $P$ has already been found by solving a nonlinear partial differential equation of the form 3.23 and total per capita wealth is $W_m = n_m P + K_m$. Let $B(t, r, K_m)$ be the bond’s pricing function and assume the boundary condition $B(T, r, K_m) = 1$. Assume the bond pays nothing before time $T$, i.e. $D_B = 0$. With each investor having the per capita holding $n_B B$ of the bond in equilibrium, equation 3.25 or 3.37 becomes

$$B_t + \mu_r B_r + \frac{1}{2} \sigma_r^2 B_{rr} + B_K (K_m r + n_m D_P - \frac{\rho(n_m P + K_m)}{1 + (\rho - 1)e^{\theta(t-T)}}) - r B = \frac{n_m \sigma_r^2 B_r P_r}{n_m P + K_m} \quad (3.38)$$

This equation is linear and although it will often be difficult to solve analytically, numerical solutions will usually be obtained. Given a solution, a long term interest rate $R(T - t, r, K_m)$ can be calculated using $B(t, r, K_m) = \exp[-R(T - t, r, K_m)(T - t)]$. The long term interest rate $R$ will depend on $n_m$, the per capita holdings of the market portfolio (all long term fixed assets in the economy). With $n_m = 0$ the equation reduces to the form developed in CIR with riskless instantaneous production processes since $B_{K_m}$ terms vanish as well as the $B_r P_r$ term.
3.9 The Pricing of Warrants and Options

In this section we consider the pricing of contingent claims where the price of one financial asset depends upon the price of another. We will consider the case of pricing a warrant written on a stock but the methods generalize to other contingent claims. Again we consider the case of an economy with identical investors having log utility and a single state variable $x$ following the stochastic process in 3.1. The totality of all fixed assets in the economy will be assumed to have a price $P$ with each investor owning an amount $n_m P$ as well as an amount $K_m$ of the consumption good. There is a particular real asset in fixed supply in this economy with a payoff $f(x,K_m)$ at time $T$ but paying no dividend. Consider the case where ownership of this physical asset is represented per capita by $n_S$ common shares and $n_c$ warrants with the price functions $S(t,x,K_m)$ and $C(t,x,K_m)$, respectively. Thus, $n_S S$ represents the total value of common equity per capita and $n_c C$ the total value of warrants per capita. With a change of variables the warrant pricing function can also be written $C(t,S,K_m)$. The warrant pricing function satisfies the boundary condition

$$C(T,x,K_m) = \max[S(T,x,K_m) - S_E, 0]$$

(3.39)

where $S_E$ is the exercise price for a warrant that costs $C$ and when exercised allows one to purchase one share of stock worth $S$. The number of shares and warrants outstanding is somewhat arbitrary in the sense that whether we represent the equity with 100 shares or 10000 does not make an economic difference. Similarly, the number of warrants outstanding and thus the total number of shares they can be converted into is also arbitrary. To represent the exercise price in an economically meaningful way we have used a measure $S_E$ that does not depend on the numbers of shares and warrants outstanding and hence is independent of share splits, for example.

$S_E$ measures the exercise price per share of the warrants in the following way. $n_c$
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represents the number of warrants per capita and with \( n_S \) the per capita holding of the equity, then when all warrants are exercised the total number of shares increases by the fraction \( n_c/n_S \) so that the per capita holding of shares becomes \( n_c + n_S \). The warrants entitle their holders to the fraction \( n_c/(n_c + n_S) \) of the value of the underlying physical asset after exercise. The old shares then become worth the fraction \( n_S/(n_c + n_S) \) of the total value of the equity in this asset. \( S_E \) is the total amount of money per warrant paid by warrant holders and is added to the underlying physical asset, increasing its value by \( S_E \). This amount \( S_E \) fixes the exercise price in a way invariant to the number of shares and warrants outstanding. The actual amount per old share paid to exercise is \( (n_c/n_S)S_E \). Hence the share price after warrants have been exercised must satisfy the boundary condition

\[
f(x, K_m) + \left( \frac{n_c}{n_S} \right) S_E = (1 + \frac{n_c}{n_S}) S
\]

Writing \( n_m P(t, x, K_m) \) without loss of generality as the per capita market value of all the physical/financial assets in the economy we know that \( P(t, x, K_m) \) must satisfy equation 3.23 with \( n_m \) the per capita holdings of \( P \). The boundary condition for \( S(t, x, K_m) \) is

\[
f(x, K_m) = S(T, x, K_m) + \frac{n_c}{n_S} \cdot \max[S(T, x, K_m) - S_E, 0].
\] (3.40)

Alternatively, the boundary condition can be written as

\[
S(T, x, K_m) = \min[(f(x, K_m) + (n_c/n_S)S_E)/(1 + (n_c/n_S)), f(x)].
\] (3.41)

The equations for \( S \) and \( C \) can be derived using equation 3.37 and are:

\[
S_t + \mu_x S_x + \frac{1}{2} \sigma^2_s S_{xx} + S_{K_m} \left( K_m r + n_m D_P - \frac{\rho(n_m P + K_m)}{1 + (\rho - 1)e^{\rho(t-T)}} \right) - r S = \frac{\sigma^2_s n_m P_x S_{xx}}{n_m P + K_m}
\] (3.42)

\[
C_t + \mu_x C_x + \frac{1}{2} \sigma^2_s C_{xx} + C_{K_m} \left( K_m r + n_m D_P - \frac{\rho(n_m P + K_m)}{1 + (\rho - 1)e^{\rho(t-T)}} \right) - r C = \frac{\sigma^2_s n_m P_x C_{xx}}{n_m P + K_m}
\] (3.43)
with the boundary conditions of equation 3.41 and

$$f(x, K_m) = S(T, x, K_m) + \frac{n_c}{n_S} C(T, x, K_m).$$

An alternative and easier way to derive equations 3.42 and 3.43 is to use the equation

$$\alpha_F - r = Cov(\tilde{R}_W, \tilde{R}_S)$$

with $W = n_m P + K_m$ for $F = S$ and $F = C$. Since

$$Cov(\tilde{R}_W, \tilde{R}_S) = \frac{n_m P \sigma_x^2 S_x}{(n_m P + K_m) \cdot S}$$

and

$$Cov(\tilde{R}_W, \tilde{R}_C) = \frac{n_m P \sigma_x^2 C_x}{(n_m P + K_m) \cdot C}$$

we immediately get equations 3.42 and 3.43.

These equations can be simplified if we change variables from $x$ to $S$ in the price function $C(t, x, K_m)$ and use $C(t, S, K_m)$ instead. We are assuming that given $K_m$ and $S$, all compatible $x$'s are consistent with just one $C$. Defining $\sigma_S^2 = Cov(\tilde{R}_S, \tilde{R}_S)$ gives

$$S^2 \sigma_S^2 = S_x^2 \sigma_x^2$$

and permits equations 3.42 and 3.43 to be rewritten as follows

$$(\alpha_S - r)S = \frac{\sigma_x S \sigma_x n_m P_x}{n_m P + K_m}$$

(3.44)

and

$$C_t + S \alpha_S C_S + \frac{1}{2} S^2 \sigma_S^2 C_{SS} + C_{Km}(K_m r + n_m D_P - \frac{\rho(n_m P + K_m)}{1 + (\rho - 1)e^{\rho(t-T)}}) - r C$$

$$= \frac{S \sigma_x \sigma_x n_m P_x}{n_m P + K_m}.$$  

Using equation 3.44 to substitute for $\alpha_S$ in equation 3.45 leads to the more standard form for the equation for $C$.

$$C_t + r S C_S + \frac{1}{2} \sigma_S^2 S^2 C_{SS} + C_{Km}(K_m r + n_m D_P - \frac{\rho(n_m P + K_m)}{1 + (\rho - 1)e^{\rho(t-T)}}) - r C = 0$$  

(3.46)

This equation is similar in structure to the Black-Scholes equation (see [4]) except for the $C_{Km}$ term and the $K_m$ dependence of $\sigma_S^2(t, S, K_m)$. In general, the equation for
$S(t, x, K_m)$ must be solved first so that it can be inverted to find $x(t, S, K_m)$ which allows $\sigma^2_S(t, S, K_m)$ to be found using $S\sigma_S = S_x\sigma_x$. Then using $\sigma_S$ the equation for $C(t, S, K_m)$ is determined and can be solved. This equation is the generalization of the Black–Scholes equation for the case of $\sigma^2_S$ dependent on $t, K_m$ and $S$. The case of $n_c = 0$ when the warrants are in zero net supply, corresponds to the pricing of call options. The difference between the pricing of warrants ($n_c \neq 0$) and call options ($n_c = 0$) is in the effect on the share price $S$. Although both the equation for $S$ and $C$ do not depend directly upon $n_c/n_S$ this ratio affects the boundary condition of the equation for $S$ and hence the solution $S$. This in turn leads to a different $\sigma_S$ which leads to a different equation for $C$ although the boundary condition is the same. Hence the solution $C$ will depend in general upon $n_c/n_S$ albeit indirectly. Since the boundary condition for $C$ does not depend on $K_m$ when $\sigma^2_S$ is independent of $K_m$, the $C_{K_m}$ term vanishes and the equation for $C$ is standard. When $\sigma^2_S$ is constant, the Black–Scholes equation is obtained.
Chapter 4

Asset Pricing with Risky Instantaneous Production

In this chapter we make one addition to the model of the previous chapter. In the previous chapter there were only two investment opportunities, buying the physical/financial asset or investing at the risk free rate. Hence, in the previous chapter all production was riskless. In this chapter risky instantaneous production is introduced of the CIR type. Thus, investors will experience two sources of uncertainty in their expected future consumption, the uncertainty in the payoff of the physical asset and the uncertainty in the instantaneous return from the risky production process. Adding risky instantaneous production to the model allows one to study the effect of production risk on the pricing of financial assets and the effect of financial assets on production levels.

We concentrate on the pricing of a financial asset in a simple general equilibrium setting with instantaneous stochastic production processes in perfectly elastic supply i.e. of the CIR type. We will consider a simple version of the model, again with an initial endowment of a single consumption good but with three assets: the financial asset in which we are interested and two real assets (production processes) in perfectly elastic supply one whose return is instantaneously riskless and thus determines the interest rate, the other whose return is stochastic. We will present the more general version of the model in appendix 2. The physical/financial asset represents the economy's aggregate of all long term assets i.e. that pay off in the consumption good a finite time in the future. The solution to the pricing equations that will be developed in this chapter gives the value of the aggregate of financial assets relative to the consumption good and the value of total
wealth. To price individual financial assets see appendix 2. The method is the same as was used in the previous chapter.

The form of the financial asset is very simple: it is an ownership claim to a continuous dividend stream of the consumption good and a single, discrete payment of the consumption good to be received at a specific time in the future; the amounts received are given functions of a single stochastic state variable. In this setting, we derive the partial differential equation that the equilibrium price of the financial asset must satisfy; this equation is now seen to differ in an important way from the analogous equation describing the equilibrium price of a zero aggregate net supply contingent claim in the CIR model and to include it as a special case. It is also more general than the first model that we derived in Chapter 2 with a single financial asset and state variable.

The economy consists of identical individuals, each initially (at \( t = 0 \)) endowed with \( K_m(0) \) units of the single consumption good and fractional share \( n_m \) of the financial asset. This asset is an ownership claim on a physical asset that will produce a continuous dividend stream of the consumption good and a single discrete payoff of the consumption good at a known future time \( (t = T) \) which is the horizon for the individuals. The dividend stream and the amount of the payoff on the financial asset are functions, \( D(t, x(t), K_m(t)) \) and \( f(x(T), K_m(T)) \) respectively, of the value of the state variable, \( x(t) \), and the per capita holdings of the consumption good, \( K_m(t) \), which determine the current state of the economy. An investor with \( n_m \) shares would receive the dividend stream \( n_m D(t, x(t), K_m(t)) \) of the consumption good and the payoff \( n_m f(x(T), K_m(T)) \) of the consumption good when the physical/financial asset is liquidated.

Between the initial time and the horizon each individual has three possibilities with respect to his or her current holding of the consumption good. The good can be consumed, it can be traded for the financial asset, and it can be invested in one of two technologies, equally available to all individuals. One of them is riskless and yields an instantaneous
rate of return, \( r \). In general, \( r \) can be a function of \( t \) and \( x \). The other instantaneous technology is risky with a rate of return that in general also depends on \( t \) and \( x \). These investment possibilities are not mutually exclusive and can be exercised continuously until the horizon.

The state variable, \( x \), evolves according to a stochastic differential equation:

\[
dx = \mu_x(t, x) dt + \sigma_x(t, x) dz(t),
\]

where \( \mu_x(t, x) \) and \( \sigma_x(t, x) \) are scalar-valued functions and \( z \) is a scalar Wiener process. The rate of return of the risky instantaneous technology is given by the stochastic differential equation:

\[
\frac{dK_1}{K_1} = \alpha(t, x) dt + \gamma(t, x) dz(t)
\]

where \( K_1 \) is the amount of the consumption good invested in this risky technology. For convenience, we will frequently suppress the arguments of \( x, z, \mu \) and \( \sigma \) where no confusion results. The financial asset, which is continuously available for trading in a frictionless market, has a price per unit share of \( P(t, x(t), K_m(t)) \) units of consumption good, where \( K_m \) is the per capita amount of the consumption good in the economy. All individuals act as price takers with respect to \( P \) in their individual optimization decisions.

Each individual chooses consumption, real investment, and financial asset holding strategies to maximize their expected utility until the horizon:

\[
E_t \int_t^T U(C(s), s) ds + B(W(T), T),
\]

where \( E_t \) denotes expectation conditional on information at time \( t \), \( U \) is an increasing, strictly concave von Neumann-Morgenstern utility function, \( C(s) \) is the individual's consumption rate at time \( s \), \( B \) is the bequest function, and \( W(T) \) is the individual's wealth at the horizon. The individual's choice is subject to the budget equation, relating
consumption, the real investments and investment in the financial asset to change in wealth.

The investor's wealth is given by $W(t) = K_0(t) + K_1(t) + n(t)P(t)$, where $K_0$ is invested in the riskless technology and $K_1$ is invested in the risky technology. The change in wealth is given by

$$dW = dK_1 + [(W - nP - K_1)r + nD - C]dt + ndP.$$  \hspace{1cm} (4.50)

Writing $K_1 = bW$, so that $b$ is the control variable, and substituting for $dK_1$ gives

$$dW = [bW(\alpha - r) + (W - nP)r + nD - C]dt + bW^\gamma dz + ndP. \hspace{1cm} (4.51)$$

The individual chooses $C, b$ and $n$ to maximize expected utility, 4.49, subject to 4.51. At the equilibrium $K$ must equal $K_m$ and $n$ must equal $n_m$, since individuals are identical, and this implies that $C$ must equal per capita consumption, $C_m$. The value of $K_1$ chosen will be the same for all investors, because investors are identical, although this is not necessary for equilibrium since the instantaneous technologies are freely available to all investors.

Applying Ito's Lemma, the change in the price of the financial asset is

$$dP = P_x dx + P_t dt + P_{K_m} dK_m + \frac{1}{2} P_{xx} E[(dx)^2]$$
$$+ \frac{1}{2} P_{K_m K_m} E[(dK_m)^2] + P_{xK_m} E[dx dK_m]. \hspace{1cm} (4.52)$$

The change in the state variable is given by 4.47. The change in the per capita holding of the consumption good is given by

$$dK_m = dK_{1m} + (K_{0m}r + n_mD - C_m)dt =$$
$$dK_{1m} + ((K_m - K_{1m})r + n_mD - C_m)dt =$$
$$(K_{1m}(\alpha - r) + K_m r + n_mD - C_m)dt + K_{1m}\gamma dz. \hspace{1cm} (4.53)$$
We can then write
\[
dW = \left[ n(P_t + P_x \mu_x + \frac{1}{2} \sigma_x^2 P_x x + P_K [1 + K_m (\alpha - r)] + K_m r + n_m D - C_m] \\
+ \frac{1}{2} P_{Km} \gamma^2 + P_{Km} \sigma_x \gamma + D - r P \right] + bW(\alpha - r) + Wr - C dt \\
+ \left[ n(P_x \sigma_x + P_{Km} K_1 \gamma) + bW \gamma \right] dz \tag{4.54}
\]

To apply stochastic dynamic programming to the optimization problem, define the optimal value (i.e., derived utility of wealth) function:
\[
J(W, x, K, t) = \max_{\{C, n, b\}} E_t \int_T U(C(s), s) ds + B(W(T), T). \tag{4.55}
\]

The optimal controls in the problem must satisfy
\[
\max_{\{C(t), n, b\}} \left[ U(C(t), t) + L(t)J \right] + J_t = 0, \tag{4.56}
\]

where \( L \) is the differential operator:
\[
L(t)J = \left[ J_{Wt} E_t(dW) + \frac{1}{2} J_{WW} E_t[(dW)^2] + J_{xt} E_t(dx) \right] + \left[ \frac{1}{2} J_{xx} E_t[(dx)^2] + J_{WW} E_t[dx dW] + J_{Km} E_t[dK_m] \right] \\
+ \frac{1}{2} J_{KmKm} E_t[(dK_m)^2] + J_{KmWm} E_t[dW dK_m] + J_{KmKm} E_t[dK_m dx] \right] / dt \\
= J_W \{ Wr + bW(\alpha - r) - C \\
+ n(P_t + P_x \mu_x + P_{Km} (1 + K_m (\alpha - r)) + K_m r + n_m D - C_m) + \\
\frac{1}{2} P_{xx} \sigma_x^2 + \frac{1}{2} P_{KmKm} (1 + K_m (\alpha - r))^2 + P_{Km} K_1 \sigma_x \gamma + D - r P \} \right] \\
+ \frac{1}{2} J_{WW} \{ n^2 P_x^2 \sigma_x^2 + 2n P_x \sigma_x \gamma bW + \gamma^2 b^2 W^2 \\
+ n^2 P_{Km} \gamma^2 K_1 \sigma_x + 2n^2 P_x P_K \sigma_x (\gamma + 2n P_K K_1 \gamma bW \} \\
+ J_{x x} \sigma_x^2 + J_{WW} \{ n P_x \sigma_x^2 + n P_K K_1 \sigma_x \gamma + bW \} \\
+ J_{Km} \{ K_1 (\alpha - r) + K_m r + n_m D - C_m \}
\]

\(^1\)See, for example, Cox, Ingersoll, Ross [1985] [14].
\[ + \frac{1}{2} J_{KmKm} K_{1m}^2 \gamma^2 + J_{xKm} \sigma_x \cdot \gamma K_{1m} \]
\[ + J_{WKm} \{ nP_x \sigma_x \cdot \gamma K_{1m} + nP_{km} \gamma^2 K_{1m}^2 + Wb \gamma^2 K_{1m} \} \]  
(4.57)

Since negative consumption is not allowed but short sales of the financial asset are permitted, necessary and sufficient conditions for an optimal solution to 4.56 are

\[ U_C - J_w \leq 0 \]  
(4.58)

\[ C(U_C - J_w) = 0 \]  
(4.59)

\[ J_W [P_t + P_x \mu_x + P_{Km} (K_{1m}(\alpha - r) + K_m r + n_m D - C_m)] + \]
\[ \frac{1}{2} P_x \sigma_x^2 + \frac{1}{2} P_{km} (K_{1m} \gamma)^2 + J_{Wx} [P_x \sigma_x \cdot \gamma + D - r P] \]
\[ + J_{WW} [nP_x \sigma_x^2 + P_x \sigma_x \cdot \gamma b W + nP_{km} \gamma^2 K_{1m}^2] + 2nP_x P_{Km} K_{1m} \sigma_x \cdot \gamma + P_{km} K_{1m} \gamma^2 b W + J_{Wx} [P_x \sigma_x^2 + P_{km} K_{1m} \sigma_x \cdot \gamma] + J_{WKm} [P_x \sigma_x \cdot \gamma K_{1m} + P_{km} \gamma^2 K_{1m}^2] = 0 \]  
(4.60)

\[ J_W \alpha - J_{WW} [nP_x \sigma_x \cdot \gamma W + \gamma^2 b W^2 + nP_{km} K_{1m} \gamma^2 W] + J_{Wx} W \sigma_x \cdot \gamma + J_{WKm} W \gamma^2 K_{1m} = 0 \]  
(4.61)

The optimal controls must also satisfy the boundary condition at the horizon:

\[ J(W, x, K_m, T) = B(W(T), T) = B(K_m(T) + n(T)f(x(T), K_m(T)), T) \]  
(4.62)

since \( f(x(T), K_m(T)) \) is the payoff on the financial asset at the horizon date and \( K(T) \equiv K_0(T) + K_1(T) \).

Since individuals are identical, at equilibrium in the economy 4.58–4.61 must hold with \( C = C_m, bW = K_{1m}, K_0 + bW = K_m \) and \( n = n_m \). The quantities \( bW, C \) and \( n \) are endogenous in the problem, but the per capita fractional holdings of the financial asset \( (n_m) \) and consumption good \( (K_m) \) are exogenous parameters. Imposing these equilibrium
conditions gives a set of coupled partial differential equations for $K_{1m}, P$ and $J$. These are solvable in theory but difficult in practice. The first order condition for $b$, equation 4.61 with $bW = K_{1m}$ can be used to solve for $K_{1m}$ as:

$$K_{1m} = \frac{J_W W(\alpha - r) + J_W W n_m P_x \sigma_x \cdot \gamma + J_W W \sigma_x \cdot \gamma (\gamma^2)^{-1}}{J_W W (1 + n_m P_{K_m}) + J_W K_m}$$  (4.63)

If we let $\alpha_P$ denote the expected rate of return on the financial asset, $\sigma_{Wx}$ denote the covariance of the change in optimal wealth with the change in the state variable, and $\sigma_{WK_m}$ denote the covariance of the change in optimal wealth with the change in the holding of the consumption good, etc. then 4.51, 4.53 and 4.60 implies that the equilibrium expected return of the financial asset can be written as

$$(\alpha_P - r) P = [(\frac{-J_{WW}}{J_W}) \sigma_{WK_m} + (\frac{-J_{Wx}}{J_W}) \sigma_{xK_m}$$

$$+ (\frac{-J_{WK_m}}{J_W}) \sigma_{K_m}^2] P_{K_m} + [(\frac{-J_{WW}}{J_W}) \sigma_{Wx}$$

$$+ (\frac{-J_{Wx}}{J_W}) \sigma_{x}^2 + (\frac{-J_{WK_m}}{J_W}) \sigma_{xK_m} P_x.$$  (4.64)

The above relation is more general than the result of Theorem 2 in CIR applied to the case of a single state variable, a single risky, instantaneous production process and a single financial asset. An important difference is the necessity to differentiate between $K_m$, the per capita holdings of the consumption good, and $W$, the wealth of the individual investors. In CIR, financial assets are in zero net supply so that the distinction between $W$ and $K_m$ is irrelevant. The $J$ function in their model depends only on $W, x$ and $t$ whereas in our model there is also a $K_m$ dependence. This means that there are terms involving covariances between $x$ and each one of the set $x, W$ and $K_m$ multiplying the term $P_x$ and terms involving covariances between $K_m$ and each one of the set $x, W$ and $K_m$ multiplying the term $P_{K_m}$. If the aggregate net supply of the financial asset, $n_m$, were zero in our model, then the $K_m$ variable in $J$ would be redundant but still correct. Under these conditions, the results of the two models would be identical.
Another important difference between equation 4.64 and CIR Theorem 2 are the sources of some of these terms. For example although both equations have a term \( \frac{-\partial W}{\partial W} \sigma W_x P_x \) the basis of this term is not the same in the two models. In both models, this term reflects the premium for bearing risk associated with the degree to which optimal marginal utility covaries with the price of the financial asset due to changes in the wealth and the financial asset price. In the CIR model, this covariance arises from the fact that both wealth, which is only made up of the real investments of the consumption good (in perfectly elastic supply), and the financial asset price depend on the same state variables. This source of covariance is somewhat indirect and is not the source of covariance considered in the usual presentation of the CAPM. In our model, the covariance arises from both this indirect source found in the CIR model and the direct covariance due to the fact that the financial asset is held in the investor's portfolio and so forms part of wealth. This is the term that corresponds to the CAPM intuition, the covariance between the price of a financial asset and the wealth of the investor held in this financial asset. As is shown below, because of this our model leads to many more terms in the partial differential equation for the financial asset's price \( P \) than appear in the CIR model.

Next, we examine the case of the above economy with a single financial asset and consumption good with the identical investors having preferences given by the von Neumann–Morgenstern utility function \( U(C, t) = e^{-\rho t} \log(C) \), and the bequest function \( B(W(T), T) = e^{-\rho T} \log W \). Substituting the trial solution

\[
J(W, x, K_m, t) = \frac{A(t)}{\rho} e^{-\rho t} \log(W) + \Psi(x, K_m, t)
\]

into equations 4.58–4.61 results in the following

\[
P(t) = \frac{\rho W}{A(t)}
\]
and

\[
[P_t + P_x \mu_t + P_{Km}(K_{1m}(\alpha - r) + K_m r + n_mD - C_m) \\
+ \frac{1}{2} P_{xx} \sigma_x^2 + \frac{1}{2} P_{xKm}(K_{1m})^2 + P_{xKm} K_{1m} \gamma + D - rP] = \\
\frac{1}{W} [nP_x^2 \sigma_x^2 + P_x \sigma_x \gamma b W + nP_{Km}^2 \gamma^2 K_{1m} + \\
2nP_x P_{Km} \sigma_x \gamma K_{1m} + P_{Km} K_{1m} \gamma^2 b W] 
\]

\[ (\alpha - r) - \frac{1}{W} [nP_x \sigma_x \gamma + \gamma^2 b W + nP_{Km} K_{1m} \gamma^2] = 0 \quad (4.68) \]

\( C(t) \) is the individual investor's optimal consumption and \( n(t,x,W,K_m) \) is the optimal holding of the financial asset. It is assumed that all investors in the economy have rational expectations and have already solved for \( P(s,x,K_m) \) for \( s > t \). Then \( n(t,x,W,K_m)/W \) is independent of \( W \) and depends only on \( t, x \) and \( K_m \). Substituting the trial solution for \( J(W,x,K_m,t) \) into 4.56 and using the expression for \( C(t) \) and the fact that \( \Psi \) is independent of \( W \) allows for separation of the equation into one part involving \( W \) and \( t \) but not \( x \) or \( K_m \) and a part that does not involve \( W \) but just \( t, x \), and \( K_m \). Each of these two parts must separately equal zero. As before, the result is a complicated equation for \( \Psi(x,K_m,t) \) with the boundary condition \( \Psi(x,K_m,T) = 0 \) and an equation for \( A(t) \):

\[ 1 - A(t) + \frac{A'(t)}{\rho} = 0 \]

with the boundary condition \( A(T) = \rho \). The solution for \( A(t) \) is

\[ A(t) = 1 + (\rho - 1)e^{\rho(t-T)} \]

so that

\[ J(W,x,K_m,t) = \frac{1}{\rho} [1 + (\rho - 1)e^{\rho(t-T)}]e^{-\rho t}log(W) + \Psi(x,K_m,t) \quad (4.69) \]

and

\[ C(t) = \frac{\rho W}{1 + (\rho - 1)e^{\rho(t-T)}}. \quad (4.70) \]

As has been shown, given the price function for the financial asset \( P(t,x,K_m) \) an individual investor's optimal investment decision can be solved for. Since all investors are identical in equilibrium, the price function \( P(t,x,K_m) \) must be such that the optimal holding
of the financial asset is \( n(t, x, W, K_m) = n_m \) when \( W = W_m = n_m P(t, x, K_m) + K_m \), i.e., each investor’s wealth equals the per capita wealth in the economy. Similarly each investor invests the same amount \( K_{1m} = bW \) of the consumption good into the risky instantaneous production process. Substituting \( n = n_m, bW = K_{1m} \) and \( W = W_m = K_m + n_m P(t, x, K_m) \) into the equations for \( P(t, x, K_m) \) and \( K_{1m} \) and solving for \( K_{1m} \) gives the following partial differential equation:

\[
\begin{align*}
Pt + P_x \mu_x + P_{K_m}(K_{1m}(\alpha - r) + K_m r + n_m D - \frac{\gamma (n_m P + K_m)}{1 + (\alpha - r) \gamma (t - T)}) + \frac{1}{2} P_{xx} \sigma_x^2 + \frac{1}{2} P_{K_m K_m}(K_{1m} \gamma)^2 + P_x K_m K_{1m} \sigma_x \cdot \gamma + D - r P = \\
\frac{1}{n_m P + K_m} \left[ n P^2 x \sigma_x^2 + P_x \sigma_x \cdot \gamma bW + n P_{K_m} \gamma^2 K_{1m}^2 \right] + 2n P_x P_{K_m} \sigma_x \cdot \gamma K_{1m} + P_{K_m} K_{1m} \gamma^2 bW \right] \quad (4.71)
\end{align*}
\]

with

\[
K_{1m} = \frac{((n_m P + K_m)(\alpha - r) - n_m P_x \sigma_x \cdot \gamma) \gamma^{2-1}}{(1 + n_m P_{K_m})} \quad (4.72)
\]

and the boundary condition \( P(T, x, K_m) = f(x, K_m) \). The solution to equation 4.71 and 4.72 gives the equilibrium financial asset price for this economy where all investors have rational expectations and are identical. All investors in the economy at time \( t \) solve their optimal investment problems using dynamic programming. They use the fact that all investors are identical and solve the same investment problem as they do and get the same solution. In solving their optimal investment problem, investors also solve for the price function that supports an equilibrium where all investors hold \( n_m \) of the financial asset. As investors work backwards from time \( T \), they solve for \( P, K_{1m} \) and \( J \) at time \( t \) using the solution for time \( s > t \) they have obtained so far.

The partial differential equation for \( P(t, x, K_m) \) is nonlinear because of the \( P_x^2, P_{K_m}^2, PP_{K_m} \) terms etc. With \( n_m = 0 \) the equation becomes linear and corresponds to the fundamental valuation equation of Theorem 3 in CIR. For example \( K_m \) and \( P_{K_m} \) in
equation 4.71 will then correspond to $W$ and $P_W$ in Theorem 3 of CIR. The equation 4.71 is of the form

$$(\alpha_P - r)P = \frac{P_x P_x (n_m P_x \sigma_x + (n_m P_K + 1) K_{1m} \gamma) + P_{Km} K_{1m} \gamma (n_m P_x \sigma_x + (n_m P_{Km} + 1) K_{1m} \gamma)}{n_m P + K_m}$$

(4.73)

where $\alpha_P$ is the expected rate of return on the financial asset. The risk premium which corresponds to $-\frac{J_{ww}}{J_x} \sigma_{wx} P_x$ of equation 4.64 is

$$\frac{P_x \sigma_x (n_m P_x \sigma_x + (n_m P_{Km} + 1) K_{1m} \gamma)}{n_m P + K_m}$$

since $\sigma_{wx} = \sigma_x (n_m P_x \sigma_x + (n_m P_{Km} + 1) K_{1m} \gamma)$. Similarly the risk premium which corresponds to $-\frac{J_{ww}}{J_w} \sigma_{wKm} P_{Km}$ of equation 4.64 is

$$\frac{P_{Km} K_{1m} \gamma \cdot (n_m P_x \sigma_x + (n_m P_{Km} + 1) K_{1m} \gamma)}{n_m P + K_m}$$

since $\sigma_{wKm} = K_{1m} \gamma \cdot (n_m P_x \sigma_x + (n_m P_{Km} + 1) K_{1m} \gamma)$. For log utility the other terms involving $J_{wx}$ and $J_{wKm}$ are zero. The risk premium includes terms proportional to $n_m$, the per capita holdings of the financial asset; the greater the investor's holding of the risky financial asset, the greater is the portfolio risk of the financial asset proportional to $n_m P_x$ and thus the larger must be the expected return. The riskiness of the investor's overall wealth may decrease, however, if the instantaneous production process has a negative correlation with the financial asset. The financial asset may act as insurance or at least may provide diversification. These effects are reflected in terms such as $P_x K_{1m} \sigma_x \gamma$.

Equation 4.71 can be written in the form $\alpha_P - r = \text{cov}(\tilde{R}_W, \tilde{R}_P)$ where $\tilde{R}_W = \frac{dW}{W}$ and $\tilde{R}_P = \frac{dP}{P}$ since

$$\text{cov}(\tilde{R}_W, \tilde{R}_P) = \frac{(n_m P_x \sigma_x + (n_m P_{Km} + 1) K_{1m} \gamma) (P_x \sigma_x + P_{Km} K_{1m} \gamma)}{W_m}.$$
Chapter 4. Asset Pricing with Risky Instantaneous Production

Since \( \alpha_w = \frac{n_m P_{\sigma x} + \sigma K_m}{W_m} + \frac{r K_m}{W_m} \) these two equations imply \( \alpha_w - r = \text{cov}(\tilde{R}_W, \tilde{R}_w) \), which then implies the CAPM equation

\[
\alpha_P - r = \frac{\text{cov}(\tilde{R}_W, \tilde{R}_F)}{\text{cov}(\tilde{R}_W, \tilde{R}_w)}(\alpha_w - r).
\]

\( \alpha_w - r \) is the price of risk, where \( \alpha_w \) is the expected return on wealth or the market portfolio. Risk is measured by \( \frac{\text{cov}(\tilde{R}_W, \tilde{R}_F)}{\text{var}(\tilde{R}_W)} \), which is the "beta" of the financial asset. Equation 4.71 implies the CAPM but the CAPM does not imply 4.71, i.e., the CAPM is more general but does not determine prices.

By substituting 4.72 into 4.71 we get a single partial differential equation for \( P \):

\[
P_t + P\mu_x +
\]

\[
P_{Km} \left[ (\gamma^2)^{-1} \left( \frac{W_m(\alpha - r) - n_m P_{\sigma x} \gamma}{1 + n_m P_{Km}} \right) \left( \alpha - r \right) + K_m r + n_m D - \frac{\rho W_m}{1 + (\rho - 1)e^{(\rho - 1)}} \right]
\]

\[
+ \frac{1}{2} P_{xx} \sigma_x^2 + \frac{1}{2} P_{Km} \left( \gamma^2 \right)^{-1} \left( \frac{W_m(\alpha - r) - n_m P_{\sigma x} \gamma}{1 + n_m P_{Km}} \right)^2
\]

\[
+ P_{Km} \left( \frac{W_m(\alpha - r) - n_m P_{\sigma x} \gamma}{1 + n_m P_{Km}} \right) \sigma_x \cdot \gamma (\gamma^2)^{-1} + D - r P
\]

\[
= \frac{1}{W_m} \left[ n_m P_{\sigma x}^2 + (P_{\sigma x} \cdot \gamma + 2 n_m P_{\sigma x} P_{Km} \sigma_x \cdot \gamma) \left( \frac{W_m(\alpha - r) - n_m P_{\sigma x} \gamma}{1 + n_m P_{Km}} \right) (\gamma^2)^{-1} \right]
\]

\[
+ (n_m P_{Km} + 1) P_{Km} \gamma^2 \left( \frac{W_m(\alpha - r) - n_m P_{\sigma x} \gamma}{1 + n_m P_{Km}} \right) (\gamma^2)^{-2} \] (4.74)

The left hand side can be simplified to

\[
\frac{1}{W_m} \left[ n_m P_{\sigma x}^2 + P_{\sigma x} \sigma_x \cdot \gamma [W_m(\alpha - r) - n_m P_{\sigma x} \cdot \gamma] (\gamma^2)^{-1} \right]
\]

\[
+ (n_m P_{\sigma x} P_{Km} \sigma_x \cdot \gamma + W_m(\alpha - r) - n_m P_{\sigma x} \cdot \gamma) P_{Km} \cdot \left[ \frac{W_m(\alpha - r) - n_m P_{\sigma x} \gamma}{1 + n_m P_{Km}} \right] (\gamma^2)^{-1}
\]

\[
= \frac{1}{W_m} \left[ n_m P_{\sigma x}^2 + P_{\sigma x} \cdot \gamma [W_m(\alpha - r) - n_m P_{\sigma x} \cdot \gamma] (\gamma^2)^{-1} \right]
\]

\[
+ W_m(\alpha - r) P_{Km} \cdot \left[ \frac{W_m(\alpha - r) - n_m P_{\sigma x} \gamma}{1 + n_m P_{Km}} \right] (\gamma^2)^{-1} \] (4.75)

Further simplification of the left hand side leads to a nonlinear partial differential equation for \( P \):

\[
P_t + P\mu_x +
\]
\begin{align*}
P_{Km} \left( (\gamma^2)^{-1} \left( \frac{(n_m P + K_m)(\alpha - r) - n_m P_x \gamma}{1 + n_m P_{Km}} \right) (\alpha - r) + K_m r + n_m D - \frac{\rho(n_m P + K_m)}{1 + (r - \underline{r})} \right) \\
+ \frac{1}{2} P_{xx} \sigma_x^2 + \frac{1}{2} P_{Km K_m} (\gamma^2)^{-1} \left( \frac{(n_m P + K_m)(\alpha - r) - n_m P_x \gamma}{1 + n_m P_{Km}} \right)^2 \\
+ P_{Km} \left( \frac{(n_m P + K_m)(\alpha - r) - n_m P_x \gamma}{1 + n_m P_{Km}} \right) \sigma_x \cdot \gamma (\gamma^2)^{-1} + D - r P = \\
\frac{1}{(n_m P + K_m)} \left[ n_m P_x \sigma_x^2 - n_m P_x^2 (\sigma_x \cdot \gamma)^2 (\gamma^2)^{-1} \right] \\
+ \frac{[P_{Km}(n_m P + K_m)(\alpha - r)^2 + P_x \gamma (\alpha - r)] (\gamma^2)^{-1}}{1 + n_m P_{Km}} \tag{4.76}
\end{align*}

Equation 4.76 can also be written

\begin{align*}
\alpha P - r = \frac{1}{n_m P} \left( \frac{n_m P}{K_m} \right)^2 \left( \sigma_x^2 + (\sigma_x \cdot \gamma)^2 (\gamma^2)^{-1} \right) \\
+ \frac{[n_m P_{Km}(1 + K_m/n_m P)(\alpha - r)^2 + P_x \gamma (\alpha - r)] (\gamma^2)^{-1}}{1 + n_m P_{Km}} \tag{4.77}
\end{align*}

which shows that the risk premium depends on the ratio of wealth in the financial asset, \(n_m P\), to wealth in the consumption good, \(K_m\). This depends on the fact that terms such as \(n_m P_{Km}\) are invariant to transformations that leave the ratio \(n_m P/K_m\) unchanged for example multiplying both \(n_m P\) and \(K_m\) by the same constant. For more details of this argument see chapter 2. This also shows that the equation for \(P\) depends only on \(\frac{n_m P}{K_m}\) and not the absolute level of wealth in the economy. Hence prices are determined by \(n_m f(x, K_m)/K_m\).

Because the equation for \(P\) is nonlinear, the well developed theory of linear partial differential equations cannot be used. Nonlinear differential equations do not have unique solutions and in general must be solved by trial and error or numerical methods. The equation for \(P\) can be transformed to one that has fewer nonlinear terms by introducing \(H(t, x, K_m) = \frac{1}{n_m P(t,x,K_m)+K_m}\). \(H\) is proportional to \(J_W\), the marginal utility. Since \(\frac{1}{H} = n_m P + K_m\, n_m P_x = -H_x/H^2\, n_m P_{xx} = -2H_{xx}/H^3 + 2H_x^2/H^2\), etc. Substituting into the equation for \(P\) gives
where \( A(t) = 1 + (\rho - 1)e^{\rho(t-T)} \). Relation 4.78 is equivalent to CIR Theorem 1, which states that \( r \) equals the negative of the expected return on marginal utility.

Equation 4.78 for \( H \) still has many nonlinear terms. In the special case where consumption only occurs at the boundary at time \( T \) or when consumption is fixed, the investor’s optimal investment problem simplifies. It can be shown then that the general equilibrium price function is still of the form \( P(t,x,K_m) \) although investors have only the control variables \( n \) and \( b \) and not \( C \) to adjust for an optimum. Equation 4.78 for \( H \) is still nonlinear but with the \( H_{K_m}/H \) term absent. The many other nonlinear terms can only be eliminated by fixing the investment in \( K_m \). If neither \( b \) nor \( C \) are control variables, i.e. \( K_m \) and \( C_m \) are constrained exogeneously, then the equation for \( H \) can be made linear and hence solvable by standard methods. The equation for \( P \) will still be nonlinear, however.
In the previous chapters the physical/financial asset was in fixed supply. In this chapter we add to the model a production process that creates the physical/financial asset. This production process is a long term irreversible investment opportunity. Consumption good invested in it does not pay off for a finite amount of time, unlike the instantaneous production process of the previous chapter. Adding the production of physical/financial assets to the model permits one to more realistically model actual economies. For example the effect of uncertainty on capital formation can be examined, as well as the relationship between capital formation and interest rates. The results presented in this chapter are derived with intuitive arguments and are not rigorous.

We examine the pricing of a financial asset in a simple general equilibrium setting with a general production process in perfectly elastic supply that creates this asset. We will consider a simple version of the model with an initial endowment of a single consumption good and with three assets: the physical/financial asset in which we are interested created by a production process freely available to all investors (in perfectly elastic supply) and two real assets (production processes) in perfectly elastic supply, one whose return is instantaneously riskless and thus determines the interest rate, the other whose return is stochastic and instantaneous.

As before, the economy consists of identical individuals, each initially (at \( t = 0 \)) endowed with \( K_m(0) \) units of the single consumption good and \( n_m(0) \) shares of the financial asset. This asset is an ownership claim on a physical asset that will produce
a single payoff of the consumption good at a known future time \((t = T)\) which is the horizon for the individuals. As before, the amount of the payoff on the financial asset is a function, \(f(x(T), K_m(T))\), of the time \(T\) value of the state variable, \(x(T)\), and the time \(T\) per capita holdings of the consumption good, \(K_m(T)\), which determine the state of the economy. An investor with \(n_m\) shares would receive a payoff of \(n_m f(x, K_m)\) of the consumption good when the physical/financial asset is liquidated. For simplicity, we assume the asset does not pay a dividend.

The production process for the physical/financial asset is instantaneous and riskless in the sense that the consumption good invested immediately becomes a predictable amount of the physical/financial asset. However, the consumption good invested in the physical/financial asset is not returned until a finite time later. We assume that the production process is irreversible so that the physical/financial asset cannot be liquidated into the consumption good before the \(t = T\) horizon when the physical asset actually pays off. The case of reversible investment is not of interest since it is the same as the instantaneous production process. If the physical/financial asset can be immediately liquidated after being created it is a type of instantaneous production process. We further assume that the production process has constant returns to scale so that there is no value in consolidating the production of many investors into a single production process. Investors can always physically increase the amount of the physical/financial asset through investment of the consumption good but they cannot decrease it. Investors are also able to buy and sell the asset in the financial market but this does not change the net amount in the economy.

The production process will represent a ceiling on the price since if the price were to get too high a new supply would immediately be created using the production process and sold. In general, the asset cannot sell at this production cost price for more than an instant since investors will not hold the asset unless it provides the correct rate of return.
If this cost price rises too slowly, no investor will hold the asset and hence none will be created. Therefore, most of the time the price of the physical/financial asset will be less than the cost of creating it. Hence Tobin's q [48] will be less than one in general in this model.

Between the initial time and the horizon each individual has five possibilities with respect to his or her current holding of the consumption good. The good can be consumed, it can be traded for the physical/financial asset, it can be used to create more of the physical/financial asset using the production process, it can be invested in the riskless instantaneous production technology, or invested in the risky instantaneous production technology. The production process for the physical/financial asset and the risky and riskless instantaneous production technologies are equally available to all individuals. As in Chapter three, the riskless instantaneous production technology yields an instantaneous rate of return, \( r \), which is in general a function of \( t \) and \( x \). Similarly, the risky instantaneous technology has a rate of return that in general depends on \( t \) and \( x \). These possibilities are not mutually exclusive and can be exercised continuously until the horizon.

The state variable, \( x \), evolves according to the stochastic differential equation given by 4.47. The consumption good invested in the risky instantaneous technology, \( K_1 \), evolves according to the stochastic differential equation 4.48. The financial asset, which is continuously available for trading in a frictionless market, has a price per unit share of \( P(t, x(t), K_m(t); n_m(t)) \) units of consumption good, where \( K_m \) is the per capita amount of the consumption good in the economy and \( n_m(t) \) is the per capita holding of the physical/financial asset. All individuals act as price takers with respect to \( P \) in their individual optimization decisions.

The production process for the physical/financial asset is of the following form \( \Delta n = I_P / g(T - t) \) for \( I_P > 0 \) where \( I_P \) is the amount of the consumption good invested to
make more of the physical/financial asset. The function \( g(T - t) \) represents the idea that the production process creates different amounts of the physical/financial asset depending upon how much time there is before it pays off. For example a field of wheat or a gold mine takes a period of time to be developed properly. \( g \) represents a long term investment and cannot be prematurely liquidated. We will assume that the longer the time the investment is made before the payoff time the greater the amount of the physical/financial asset that is created. Hence for \( T - t \) large \( g(T - t) \) will be small and for \( T - t \) small \( g(T - t) \) will be large or infinite, i.e. \( g'(\tau) \leq 0 \).

This production process and the absence of arbitrage opportunities ensure that the market price of the financial asset obeys the relation \( P \leq g(T - t) \). If the price \( P \) is ever greater than \( g \) investors will immediately create another share at a cost of \( g \) and sell it for \( P \), making a riskless profit. In general, the financial asset price lies below \( g \) and at these prices no new investment in this asset occurs. We can show that in general production occurs in discrete jumps (delta functions) lasting an infinitesimal amount of time, when for some values of the state variable \( x \) and the per capita consumption good \( K_m \), \( P \) touches the \( g \) boundary at a single point where the slope is \( r \).

If we assume a continuous investment flow of the form \( -dK = dI_p \) this then implies that the price \( P \) stays on the \( g \) boundary for a finite interval of time. In this case the rate of return in general is not consistent with investors being willing to hold the financial asset. Since the price \( P \) is independent of \( x \) and \( K_m \) on this production boundary, the rate of return is riskless. When \( P = g \) the slope of \( P \) as a function of \( t \) must be greater than or equal to \( r \) or the asset will not be held at all. If the slope is greater than \( r \) then investors will invest all of the consumption good \( K_m \) to create more of the financial asset in a discrete jump. This contradicts the assumption of a continuous investment flow. Furthermore the price \( P \) must drop as \( K_m \) decreases (as it is invested) and the consumption good, which is the only source of current consumption, becomes
more valuable. Hence in order for there to be a continuous investment flow with the financial asset price $P$ moving along the production curve $g$ it must be the case that $g$ has a slope of $r$.

$P$ must immediately move off the $g$ boundary when the slope of $g$ becomes greater than $r$ for $g$ differentiable. If it does not we get the following contradiction. Assume that for some $t = s$, $n_m = a$ and $x = b$ we have $P(t = s, x = b, K_m; n_m = a) = g$ and $P$ is tangent to $g$ with a slope greater than $r$. For $n_m < a$ and $x \geq b$ production must occur so that $n_m$ increases and $P \leq g$. Hence for $x \geq b$, $P = g$ and at this point $P$ is not risky. Hence for $t = s - \delta t$ for $x > b$ because there is no risk due to $x$ the slope of $P$ as a function of $t$ must equal $r$. This contradicts our assumption.

If $g$ has a slope greater than $r$ when $P$ touches $g$ at $t = s$ with a slope of $r$, it would mean that $P > g$ for $t = s - \delta t$. This is because $P$ has a slope of $r$ at $t = s$ and the slope $P_t$ is a smooth function of $x$. The only way that $P < g$ for $t = s - \delta t$ is for the slope $P_t$ to jump discontinuously from a value greater than $r$ an infinitesimal time before $t = s$ to the value $P_t = r$ at $t = s$. This is not possible with $P_t$, a smooth function of $x$, which must be assumed for the partial differential equation that $P$ satisfies to be valid. Therefore investment to create the financial asset (when it does not pay a dividend) only occurs when the parameters of the economy are such that $P$ is tangent to $g$ and the slope of $g$ as a function of $t$ equals the riskfree rate $r$.

If the slope of $g$ equals $r$ at a discrete set of points then the investment can only occur in discrete instantaneous jumps at those points. If $g$ is continuous but not differentiable, i.e. it has "sharp points," then financial asset production may occur at these points; $P$ as a function of $t$ is tangent to $g$ there and $\partial P/\partial t = r$ at the point of tangency. For a continuous investment flow, the slope of $g$ must equal $r$ for a finite interval.

Consider the case of $g'(\tau)$ equal to a positive constant, i.e., $g$ is a straight line. The financial asset will not be created when $P < g$. $P > g$ is not an equilibrium since the
financial asset would be created and sold until \( P = g \). When the economy begins at 
\( t = 0 \), if \( x \) and \( K_m \) are such that \( P = g \) it will only be held or created if \( g'(r) > r \) so that \( P \) can have the possibility of a return greater than \( r \) to compensate for its risk. In fact \( P \) can not equal \( g \) unless the slope of \( g \) is greater than \( r \). If \( g'(r) > r \) and \( P = g \) then it will be immediately created once, not be produced again, and from then on \( P < g \). In general for equilibrium to hold it must be created to ensure that \( P \) is not greater than \( g \). If the slope of \( g \) is less than or equal to \( r \) the financial asset will not be created until time \( T \) and its price, \( P \), will be less than \( g \) until time \( T \). This is because investors know that it will never sell above \( g \) given the possibility of creating it at a cost of \( g \). It is only created at time \( T \) if \( x \) and \( K_m \) are such that the payoff \( n f(x, K_m) = (I_P/g)f(x, K_m) \) received is greater than the cost, \( I_P \), of creating it, i.e., \( f(x, K_m) > g(0) \).

As before, each individual chooses consumption, real investment, and financial asset holding strategies to maximize expected utility until the horizon:

\[
E_t \int_t^T U(C(s), s)ds + B(W(T), T).
\] (5.79)

The individual’s choice is subject to the budget equation, relating consumption, the real investments and investment in the financial asset to change in wealth.

The investor’s wealth is given by \( W(t) = K_0(t) + K_1(t) + n(t)P(t, x, K_m, n_m) \), where \( K_0 \) is the amount invested in the riskless technology and \( K_1 \) is invested in the risky technology. The change in wealth is given by

\[
dW = dK_1 + [(W - nP - K_1)r - C]dt + ndP.
\]

Writing \( K_1 = bW \) so that \( b \) is the control variable and substituting for \( dK_1 \) gives

\[
dW = [bW(\alpha - r) + (W - nP)r - C]dt + bW\gamma dz + ndP
\] (5.80)

The individual chooses \( C, b \) and \( n \) to maximize expected utility, 5.79, subject to 5.80. At the equilibrium, \( K \) must equal \( K_m \), and \( n \) must equal \( n_m \), since individuals are identical,
and this implies that $C$ must equal per capita consumption, $C_m$. The optimal amount of the financial/physical asset $n$ to hold is implemented by the investor by determining an amount to buy or sell $\Delta n_{ES}$ at the price $P$ and an amount to produce $\Delta n_C = \frac{1}{P}g \geq 0$ at the effective price of $g$. It is produced with the production process in a discrete amount when for an instant $P = g$. In general $P < g$ and no production occurs.

Applying Ito's Lemma, the change in the price of the financial asset is

$$dP = P_x dx + P_t dt + P_{K_m} dK_m + \frac{1}{2} P_{xx} E[(dx)^2]$$

$$+ \frac{1}{2} P_{K_m K_m} E[(dK_m)^2] + P_{K_m x} E[dK_m dx].$$

(5.81)

$dP$ does not depend on $dn_m$ because the latter changes in discrete jumps which are uncorrelated with $dz, dx$ and $dK_m$.

This means that we can use the results of Chapter 3 and again write the change in the per capita holding of the consumption good as in 4.53 and the change in wealth as in 4.54. The optimal value (i.e., derived utility of wealth) function is:

$$J(W, x, K_m, t) = \max_{\{C, b, n\}} E_t \int^T_t U(C(s), s) ds + B(W(T), T).$$

(5.82)

The optimal controls in the problem must satisfy

$$\max_{\{C, b, n\}} [U(C(t), t) + L(t)J_t] + J_t = 0,$$

(5.83)

where $L$ is the same differential operator as in equation 4.57 The necessary and sufficient conditions for an optimal solution to 5.83 are, as before, 4.58, 4.59, 4.60 and 4.61 with the boundary condition at the horizon:

$$J(W, x, K_m, T) = B(W(T), T) = B(K_m(T) + n(T)f(x, K_m), T)$$

since $f(x, K_m)$ is the payoff on the financial asset at the horizon date.

Since individuals are identical, at equilibrium in the economy 4.58–4.61 must hold with $C = C_m$, $bW = K_{1m}$, $K_0 + bW = K_m$ and $n = n_m$. Furthermore, at the $g$
financial asset production boundary $I_P = I_{P_m}$. The quantities $bW, C, n$ and $I_P$ are endogenous in the problem, but the per capita fractional holdings of the consumption good is an exogenous parameter. Imposing these equilibrium conditions gives a set of coupled partial differential equations for $P$ and $J$. These partial differential equations for $P$ and $J$ are the same as those in Chapter 3. The production process only changes the boundary conditions for these equations. The boundary conditions are given both by the payoff in the consumption good from the physical/financial asset at time $t = T$ and also by the production boundary which determines $n_m$ and $I_P$ as described below. These equations are solvable in theory but difficult in practice. The equilibrium risk return tradeoff is unchanged by the presence of the production process for the financial asset since as far as producing this asset goes it is riskless. Hence relation 4.64 still holds.

The values of $I_P$ and $n_m$ are determined from the physical/financial asset production process given by $g(T - t)$ as follows. Assume that initially $n_m = n_{m0}$ with the price of the financial asset given by $P(t, x, K_m; n_{m0}) < g(T - t)$ so that $n_m = n_{m0}$ for a finite period of time. Investors hold the amount $n_{m0}$ of the financial asset and there is no incentive to create any more of it as long as the slope of $g$ is less than $r$. When the slope first equals $r$ with a positive second derivative, i.e. convex, then at that point discrete investment $I_P$ may occur where $\Delta n_m = I_P/g$ is just such that the price $P(t, x, K_m; n_{m0} + \Delta n_m)$ is an equilibrium where for a finite interval of $s > t$, $P(s, x, K_m; n_{m0} + \Delta n_m)$ is less than $g$, and hence away from the boundary. The $n_m$ dependence of $P$ takes into account the presence of the production boundary given by $g$. The financial asset price $P$ may again touch this boundary, depending upon $x$, so that further investment (and hence an increase in $n_m$) may occur.

In solving the equations for $P$ and $J$ one starts from the boundary condition at time $T$. For times $t$ near $T$ we assume that $g$ is larger than $f(x, K_m)$ for all $x$ and $K_m$, so that $P$ does not touch the $g$ boundary, i.e., $P < g$. In this region near $T$ we solve the
model of Chapter 3 parameterized by $n_m$. For $f(x, K_m)$ bounded and $g$ larger than $f$ the production boundary condition given by $g$ has no influence. Then the solution for $P$ for $t$ near $T$ is unchanged from the $n_m$ constant case. To correctly solve the equations for all $t$ we must take into account the boundary condition that $P \leq g$ and that for $P = g$ we have $P_x = P_{xx} = P_K = P_{KK} = 0$. The latter boundary condition must hold because without the production process we would have $P > g$ for a range of $x$ and $K_m$. With the production process, $P = g$ and production occurs for some range of $x$ and $K_m$ so that the derivatives of $P$ with respect to these variables are zero. Hence for $P$ near the production boundary, $P$ behaves nearly like a bond and must have a rate of return close to $r$ to be held. With the equations for $J$ and $P$ solved for the production boundary condition fixed by $g$, the level of investment in the economy can be found as described below.

At a given time $t$ if the slope of $g$ is less than $r$ no new investment in the financial asset is made and $P < g$. Prices are determined by the partial differential equations with $n_m$ fixed and the $g$ boundary conditions. New investment can only occur at those times $t$ when $P$ can be tangent to $g$ with the slope $\partial P(t, x, K_m; n_m)/\partial t$ equal to $r$. The amount of investment that occurs is given by the solution for $P$ which has $P = g$, i.e. $P$ is continuous at the $g$ boundary. Only one set of values of $n_m$ and $K_m$ is compatible with this tangency condition for a given value of $x$ and $K_m$. The investment $I_P$ is made until $n$ increases and $K$ decreases so that the correct set of values is reached. From then on $P$ is below the $g$ boundary unless there is further investment. If there is, it occurs again by the price approaching the $g$ curve and being tangent to it with a slope of $r$. At the tangency point discrete investment is again made. The prices before this point take into account through the $g$ boundary condition that there will be an increase in $n$ so that a drop in price does not occur as $n$ suddenly increases.

Next we examine the case of the above economy with a single financial asset and consumption good with the identical investors having preferences given by the von
Chapter 5. The Production and Pricing of Financial Assets

Neumann–Morgenstern utility function \( U(C,t) = e^{-\rho t} \log(C) \), and the bequest function \( B(W(T), T) = e^{-\rho T} \log W \). We assume a particularly simple form for the physical/financial asset production function \( g \):

\[
\begin{cases}
\infty, & \tau < T_0 \\
g_0, & \tau \geq T_0
\end{cases}
\]  

(5.84)

Substituting the trial solution

\[
J(W, x, K_m, t) = \frac{A(t)}{\rho} e^{-\rho t} \log(W) + \Psi(x, K_m, t)
\]

into equations 4.58–4.61, imposing the equilibrium conditions and solving as in Chapter 3 results in the solution for \( A(t) \) being \( A(t) = 1 + (\rho - 1)e^{\rho(T_0 - T)} \) so that

\[
J(W, x, K_m, t) = \frac{1}{\rho} [1 + (\rho - 1)e^{\rho(T_0 - T)}] e^{-\rho t} \log(W) + \Psi(x, K_m, t)
\]

and

\[
C(t) = \frac{\rho W}{1 + (\rho - 1)e^{\rho(T_0 - T)}}.
\]

For an equilibrium to hold \( P(t, x, K_m; n_m) \) must satisfy the partial differential equation 4.76 from Chapter 3: with the boundary conditions \( P(T, x, K_m; n_m) = f(x, K_m) \) and \( P \leq g \). Assume that \( n_m \) is the initial holdings of the financial asset, which may change in a discrete jump if production occurs. The solution to this equation gives the equilibrium financial asset price for this economy as a function of \( n_m \) and \( K_m \). Working backwards from time \( T \), one can solve for \( P \) and \( K_{1m} \) at time \( t \) using the solution for time \( s > t \) one has obtained so far. For \( t > T - T_0 \) the solution for \( P \) as a function of \( n_m/K_m \) is the same as it is without production since production of the financial asset does not occur with \( g = \infty > P \). For \( t < T - T_0 \) the solution is changed by the presence of the production boundary condition. Since the slope of \( g \) equals zero, \( P \) cannot be tangent to \( g \) with a slope of \( r \). Hence production cannot occur for \( t < T - T_0 \) which ensures that \( P < g_0 \).
At $t = T - \tau_0$ an interesting singularity occurs. Let $P_0(t, x, K_m; n_m)$ be the solution to the equation with the boundary condition that $P \leq g$ but assuming no production of the financial asset occurs. Assuming for simplicity that $f(x, K_m)$ is increasing in $x$ then so is $P_0$. Let $X_0(n_m/K_m)$ be the smallest $x$ such that $P_0(T - \tau_0, x = X_0, K_m; n_m) = g_0$. Then for all $x > X_0(n_m/K_m)$ the solution $P$ is squeezed together so that $P = g_0$ i.e. $P_x = 0$ and $P_{xx} = 0$ for that value of $n_m/K_m$ at $t = T - \tau_0$. For these values of $x$ and these values only, production of the physical/financial asset occurs at the one time $t = T - \tau_0$. Production occurs for a given $x$ by having $K_m$ decrease by $I_P(x)$ and $n_m$ increase by $g_0I_P(x)$ so that the price $P = g_0$ is an equilibrium. This means that $I_P$ is chosen so that $g_0 = P_0(T - \tau_0, x, K_m - I_P; n_m + g_0I_P)$ and $X_0((n_m + g_0I_P)/(K_m - I_P)) = x$. For $t < T - \tau_0$, the price $P$ is not only below $g_0$ but in fact must be below $g_0 \exp -r(T - \tau_0 - t)$ since $P(T - \tau_0, \ldots) \leq g_0$. Otherwise investors will want to sell the financial asset short. If $f(x, K_m)$ can be made arbitrarily close to 0 for small $x$, there will be values of $x$ small enough that production will not occur. The functional form of $P$ will then have a nontrivial $x$ dependence i.e. $P_x > 0$ and $P(t, x, K_m; n_m) < g_0 \exp -r(T - \tau_0 - t)$. The solution for $P$ is then obtained by solving the standard partial differential equation for $P$ as before with the boundary condition that $P = P_0$ for $x < X_0(n_m/K_m)$ at $t = T - \tau_0$ and for $x > X_0(n_m/K_m)$ investment in more of the financial asset occurs so that $P(t, x, K_m; n_m) = P_0(t, x, K_m - I_P; n_m + I_P/g_0) = g_0$. If one plots $P$ versus $t$ for various values of $x$ one gets a set of flow lines, one line for each value of $x$. Some of the flow lines for higher values of $x$ get squeezed together into a point at $P = g_0$ and $t = T - \tau_0$ where the investment actually occurs. In general the flow lines are squeezed into a point where they are tangent to $g$ with a slope equal to $r$. See figure 5.1.

The above methods are also easily applied to the more general case of a physical/financial asset that can be created with a production process $g(T - t)$ and liquidated back into the consumption good with a liquidation technology $h(T - t)$ where $h < g$. 
The partial differential equations are the same as before but the boundary conditions are modified so that \( h < P < g \). Liquidation occurs at discrete points where \( P \) is tangent to \( h \) with a slope of \( r \).

The case of the creation of physical/financial assets that pay a dividend \( d(t, x, K_m) \) is also of interest. The equations are again the same with the boundary conditions modified. We must have \( P < g \) but the condition for asset creation is that \( P \) is tangent to \( g \) with a slope of \( r - d/g \).

The generalization of the above production process to one that permits physical/financial assets to be created having different payoff times, i.e., boundary condition times \( T \), is not difficult. Assume for simplicity that there are three possible physical/financial assets that can be created with the payoff times \( T_1 < T_2 < T_3 \) using the same production process given by \( g \). As was shown above, a physical/financial asset is created when \( g \) allows \( P \) as a function of \( t \) to be tangent to \( g \) with a slope or \( r \). For \( g \) differentiable, this occurs at those points where the slope of \( g \) equals \( r \). This implies a certain point in time
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$t_i = T_i - \tau_0$ for $i = 1, 2, 3$ when each of the three assets might be created. Asset 1 will be created first, then 2 and finally 3.

Again a physical/financial asset is created at $t_i$ for only some values of the state variable. The equations for the three prices $P_1, P_2$ and $P_3$ are the same as without the production process except that the boundary conditions are modified because of the production process putting a ceiling given by $g$ on each of the prices. In solving the equation for $P_i(t, x, K_m; n_1, n_2, n_3)$ as a function of $n_m$ one needs to know $P_m$, the value of the portfolio of all physical/financial assets in the economy which includes any $P_1$, $P_2$ and $P_3$ that has been created. The partial differential equation for the price of each financial asset is solved in a number of regions where production does not occur so that the per capita asset holdings $n_1$, $n_2$ and $n_3$ are constant. These regions are $t < t_1, t_1 < t < t_2, t_2 < t < t_3, t_3 < t < T$. The solutions give the asset prices parameterized by the asset holdings $n_1, n_2$ and $n_3$. The full solution which gives the price of each physical/financial asset then takes into the account the possibility of the creation of this asset and the other financial assets which will affect its future price. The full solution is obtained by matching the solutions in the four regions so that they are continuous at the boundaries $t_1, t_2,$ and $t_3$ by determining the amount of the three financial assets that should be created as was done in the simpler case of one asset.

This methodology can be extended to the case of a continuum of production possibilities with a continuum of possible physical/financial assets paying off at all possible times. The practical difficulties of solving such a problem are enormous but the theory behind it is the same as in the simpler case. The method still involves solving partial differential equations with modified boundary conditions. Again ranges of the value of the state variable will cause certain physical/financial assets to be produced. As the state variables wander randomly into and out of such production regions particular assets will be created and then cease to be created and remain in constant supply.
Introduction to Market Created Risk

Owners of financial assets are exposed to uncertainty in many ways. Since financial assets are usually valued in terms of the prospective cashflows they offer, the uncertainty of future cash flows and investors' preferences over them must be modelled in order to understand financial markets. Financial theory has been very successful in developing models that explain the relative prices of financial assets as a function of the subjective probability distribution of their cash flows and the preferences and endowments of investors. A widely accepted theoretical framework underlying such models is the Arrow-Debreu state-contingent claims theory. An A-D security pays off, i.e. produces cash flows, in just one state of the world. Financial assets are then modelled as portfolios of A-D securities: they produce various cash flows depending on the state of the world that occurs.

An important feature of the states of the world in the A-D model is that they are independent of the securities price setting process. This means that conditions in the part of the economy that involves investors trading to an equilibrium set of prices are unrelated to the definition of the states of the world. Uncertainty, i.e. low payoffs in certain states and high payoffs in others, is assumed to be exogenous to the financial markets that seek to set prices for insuring against this uncertainty. The specification, probabilities and cash flows of these states are in some sense physically determined and are not the result of strategic behavior by economic actors. It is this strong independence of the state space representation of uncertainty from the decision making of the economic
actors that makes the A-D model especially tractable. Its tractability leads to well defined state prices (and linear pricing operators) which depend on the economy itself through preferences, endowments and beliefs/strategies, as well as the properties of the states. All composite securities can then be priced in terms of the A-D securities' prices.

In this essay of the thesis, we examine some implications of market uncertainty: uncertainty that depends on the economy or financial markets. The financial markets that set a price for insuring against risk are the source of at least part of the risk. This lack of independence between the uncertainty, as represented by the states of the world, and the financial markets that price A-D securities paying off in the different states of the world, is not a feature of most models; Arrow (1964) explicitly ruled it out. Yet to the casual observer of actual financial markets this type of uncertainty is endemic.

Few financial securities are ever liquidated, i.e. paying a large final dividend, and usually only then by being sold to a large buyer such as in a takeover. Most physical assets do not have payoffs that are quickly received, at least before the corresponding financial assets are sold. Hence, most returns on financial assets are based on capital gains where uncertainty arises from not knowing the future market price. Uncertainty about the future price may be due to factors that are unrelated to the market process, such as revisions in beliefs about the probabilities of cash flows based on new, exogenous information. This involves revisions in beliefs about probabilities of states of the world that are defined independently of the market setting the A-D prices.

However, uncertainty about the future price can also arise from the difficulty investors have in predicting the behavior of other participants in the economy. Even when investors know that the probabilities of the final payoffs will not change, there is uncertainty about future prices due to lack of knowledge concerning the true nature of the economy or financial markets. This market uncertainty may be one source of the "excess" volatility in prices that has been discussed in several papers (French and Roll, 1986 [19]; Roll, 1984
[38] and Shiller, 1981 [43]). While the presence of excess volatility in prices is somewhat controversial, one thing is clear: in actual markets, investors have imperfect knowledge of the market’s underlying structure.

A good example of market created risk is the pricing of gold. Abstracting from industrial uses, gold as a financial asset has no liquidating dividends – it only provides a return when it is sold. All uncertainty about its return is due to uncertainty about its future market price. Consider creating A-D securities for states of the world defined by the price of gold one year hence; for example, a price of gold next year between $500 and $501 per ounce could be one such state. Each ounce of gold in existence could be converted to or represented by a combination of A-D securities. One ounce of gold would equal approximately one A-D security paying off $1 if the price of gold is between $.50 and $1.50 an ounce, two A-D securities each paying off $1 if the price of gold is between $1.50 and $2.50,...,500 A-D securities each paying off $1 if the price of gold is between $499.50 and $500.50, etc. The sum of the prices of these A-D securities would equal the current price of gold if the condition of no riskless, costless arbitrage opportunity holds.

The pricing of these A-D securities depends on the future price of gold but, in effect, they are also equivalent in aggregate to the current price of gold. Uncertainty about the future price of gold is due to investors’ uncertainty about future supply and demand for gold, which is uncertainty about the economy. There are no exogenous cashflows independent of the financial markets. As investors set a current price of gold they are implicitly setting prices for a set of A-D securities having payoffs in states defined by outcomes of a similar, future price setting process. The current price of gold provides information concerning the financial markets’ underlying characteristics, which is used to predict the probabilities of the possible states of the world defined by the price of gold next year. Thus, the prices of the A-D securities, which are related to gold’s current price, help determine the probabilities of the various states of the world, based on gold’s future
price, and the prices of claims on these states in turn depend on these same probabilities.

Since prices depend on investors' knowledge about the nature of the economy, new information about the economy can affect prices even when it includes no information about the future cash flows on the underlying assets being priced and all investors are acting rationally. One potential application of this point is in examining the effects of trading, in particular price pressure and information effects in block trading (Kraus and Stoll, 1972 [26]; Scholes, 1972 [41]). For example a large block trade will usually cause a price change which is attributed to "price pressure" if it is temporary. If the price change is permanent, or at least lasts for a long period of time, it is presumed that it is an information effect, i.e. investors have changed their beliefs about future cash flows to the company. A third possibility may exist, however, that a permanent price change does not arise from new external information but is a move to another possible equilibrium. One can think of the different possible equilibrium prices as depending on different sets of price dependent beliefs about the nature of the economy rather than different beliefs about the company's cash flows. A permanent price change unaccompanied by a change in beliefs about future cashflows could then be considered a permanent price pressure effect or a market information effect, where the information is about the market itself. Evidence from the behavior of stocks added to the S&P 500 (Harris and Gurel, 1986 [22]; Shleifer, 1986 [44]; Woolridge and Ghosh, 1986 [50]) also indicates that price pressure is important and may be permanent.

In the next chapter, we present a model of a financial market that contains one form of market uncertainty. In this case, the uncertainty concerns the relative aggregate endowments of other investors. We show that equilibrium in this market may be such that there are price movements even when there is no new information about the underlying cash flows being priced. In order to demonstrate that the intuition that markets themselves are a source of risk is correct, we need only present an example of a reasonable model in
which market uncertainty plays an important role. The model of the next chapter is the example offered.

The model is a two asset economy over three dates. All uncertainty about final payoffs is resolved at the third date and all market uncertainty (if any) is resolved at the second date. The market uncertainty is resolved by prices that reveal the true state of the economy. There are two possible economies with two different second period prices. The market uncertainty can only occur at the first date when investors do not know the future prices. This means that the initial prices are pooling prices; they do not reveal the state of the economy. The pooling equilibrium is not the only equilibrium. In a separating equilibrium, the economy's state uncertainty is revealed as soon as trading occurs. Since the prices at the second date are the same in this separating equilibrium as the initial prices, no trading occurs at the second date.

Relative to the separating equilibrium, the pooling equilibrium appears to exhibit "excess" volatility. In this equilibrium, the pooling prices change to separating prices without any new information about final payoffs. Thus, the fundamentals of the assets, namely the probability and magnitude of final payoffs, are unchanged, but the prices change stochastically.

Endowment uncertainty is important in both equilibria, but it is only in the pooling equilibrium that investors can insure themselves against this risk. Before the initial trading begins, investors do not know the future values of their endowments. Risk averse investors prefer to have the expected value of their endowments for sure rather than face the market created risk. The pooling equilibrium, in which initial trading takes place at pooling prices, allows investors to insure partially against this risk. The pooling equilibrium is not necessarily welfare improving compared to the separating equilibrium. In the separating equilibrium, initial prices reveal the value of the risk averse investors' endowments, giving them no chance to share risk through trading. In the first round
of trading in the pooling equilibrium, fundamental value, i.e., the future trading price, is unknown to investors. Some investors may be better off not knowing it immediately but because expected wealth is not the same in the two equilibria for the two types of investors the equilibria are not Pareto comparable.
Chapter 7

A Three Date Model of Market Created Risk

7.1 Assumptions

Assume that there are two types of investors: risk neutral and risk averse. For simplicity, we assume that the risk averse investors have log utility. There are three time periods, \( t = 1, t = 2 \) and \( t = 3 \), and two possible states of nature affecting the cash flows of productive assets. The actual state of nature is not revealed until \( t = 3 \) which is when financial asset payoffs and consumption occur. There are two financial assets: a bond and a stock. The bond pays off one unit of the consumption good if either state 1 or 2 occurs at \( t = 3 \). The stock pays off nothing if state 1 occurs and one unit of the consumption good if state 2 occurs at \( t = 3 \). The two types of investors in the economy are assumed to be endowed with various amounts of the securities and to trade to an equilibrium at \( t = 1 \). At \( t = 2 \), the market reopens and trading may again occur but a well defined price is established whether or not investors change their portfolios. At \( t = 3 \), the securities pay off and consumption takes place.

Assume that in addition to the two states of nature there are a number of states of the economy (described in more detail below), labeled \( e=A,B,C... \) etc. with probability \( \rho_e \) of occurring and \( \sum_e \rho_e = 1 \). Let \( P \) be the relative price of the stock to the bond. Then at \( t = 2 \) in the state \( A \) economy the price is \( P_A \) and in general in the state \( e \) economy there is a possibly different price \( P_e \). We could consider the states of nature and the states of the economy together as general states of the world but we distinguish them
here to focus on the nature of market risk. Market risk occurs when market participants
do not know some relevant characteristic(s) of the actual economy/financial markets in
which they are trading. Uncertainty about the economy will be shown to have nontrivial
consequences to the nature of equilibria.

The only difference between the economy states is in the aggregate endowments of
the two securities. All investors are assumed to have the same prior probability beliefs
and to learn nothing from observing their own endowments as to what state the economy
is in. There are various ways in which a difference in aggregate endowments that cannot
be distinguished by the individual from observing his or her own endowment could occur.
For simplicity, we assume the difference is due to a difference in the number of investors
having a particular endowment. For each possible endowment there is in each economy
state at least one investor with that endowment. An investor observing his or her own
endowment knows only that there is such an investor and nothing more. The distribution
from which the possible economies are drawn is such that each investor upon observing
his or her own endowments learns the same thing, so that all investors have the same
posterior beliefs.

The philosophical foundations of these assumptions are difficult. Consider a sailor
waking up on a desert island. If he and any other sailors on the island have prior beliefs
that there is a 1/2 chance of there being 10 people on the island and a 1/2 chance of
there being 100 people, should the sailor upon observing himself on the island assume
that since he is there the probability is much higher than 1/2 that there are 100 people?
Not necessarily, since it depends upon the assumed distribution describing how the sailor
was chosen from all possible sailors to occupy the possible 10 or 100 shipwrecked sailor
positions on the island. From the standpoint of our model the important assumption is
that all investors have the same posterior beliefs. This is not essential for the model but
makes it tractable.
The behaviour of investors at $t = 2$ is easy to describe. Because this is the last time at which trading occurs, individual demand and supply decisions are independent of beliefs about the state of the economy because they do not involve forecasts of future prices. Payoffs depend only on the state of nature revealed at $t = 3$. The $t = 1$ decisions are more complicated. One possible equilibrium is of the separating type (see figure 7.2 for an example) where the $t = 1$ price equals the $t = 2$ price ($P_e = P_A, P_B, \ldots$) and the state of the economy is revealed. In this equilibrium, investors trade immediately to their final holdings of the bond and stock. No further trading occurs at $t = 2$.

The other possibility is a pooling equilibrium (see figure 7.3 for an example) with a pooling price $P_p$ at $t = 1$ which does not reveal the economy's state. Given rational expectations, investors predict $P_e = P_A, P_B, \ldots$ based on the alternative $t = 2$ endowments of the A-D securities amongst groups of aggregated investors of different preferences, in our case risk neutral and log utility. For a pooling equilibrium, the price $P_p$ must be an equilibrium $t = 1$ price for all economy states $e = A, B, \ldots$. That is, the different sets of
aggregate endowments corresponding to the different economy states must be consistent with the same $t = 1$ price $P_P$ when investors have just their prior beliefs about which state the economy is in and what the $t = 2$ price $P_A$ is. In general, the $t = 2$ prices $P_e$ in the pooling equilibrium are not equal to the $t = 2$ prices $P_e$ in the separating equilibrium.

One possible pooling equilibrium is to have a pooling price for the $t = 2$ prices as well as the $t = 1$ price. The $t = 2$ prices $P_e$ must be the same for all economy states $e$. The $t = 1$ and $t = 2$ pooling prices lead to an equilibrium in which the economy state is never revealed. We refer to this equilibrium as the permanent pooling equilibrium. Alternatively, the $t = 2$ prices $P_e$ in the pooling equilibria are different in different economy states. We refer to these equilibria as temporary pooling equilibria or more simply as pooling equilibria. Other possible equilibria would include ones in which the $t = 1$ and $t = 2$ prices partially reveal the economy state. These equilibria will not be considered in this chapter.

Assume, for simplicity, that the probability of states 1 and 2 occurring is each $1/2$. 

Figure 7.3: Stock Price Paths in the Pooling Equilibrium
Denote the log utility investors as L types and the risk neutral investors as N types. Let $z_{sk}$ denote the initial holding before trading of bonds and stock ($s = B, S$) held by a type $k$ ($k = L, N$) investor. At $t = 1$ then, type L investors have initial endowments $z_{BL}$ and $z_{SL}$ and type N investors have $z_{BN}$ and $z_{SN}$. In the trading at $t = 1$, a type $k$ investor chooses security holdings $y_{Bk}$ and $y_{Sk}$, which are held until the final round of trading at $t = 2$. At that time, the type $k$ investor chooses final security holdings $x_{Bk}$ and $x_{Sk}$.

We assume that the initial endowment $z_{sk}$ can take on one of two values: $z^a_{sk}$ or $z^b_{sk}$. In economy state A, for example, there are $n_{Ak}^a$ investors of type $k$ who have initial endowments $z_{sk} = z^a_{sk}$ and $n_{Ak}^b$ investors of type $k$ who have initial endowments $z_{sk} = z^b_{sk}$ ($s = B, S; k = L, N$). Similarly, in economy state $e$ there are $n_{ek}^a$ and $n_{ek}^b$ investors of type $k$ who have initial endowments $z^a_{sk}$ and $z^b_{sk}$, respectively.

### 7.2 The Investors’ Optimization Problem

Assume without loss of generality that $P_e < 1/2$ and consider the risk neutral investor’s problem. At $t = 2$, it is

$$
\max_{x_{BN}, x_{SN}} (1/2x_{BN} + 1/2(x_{BN} + x_{SN}))
$$

such that $x_{BN} + P_e x_{SN} = y_{BN} + P_e y_{SN}$ in economy state $e$. Thus, the risk neutral investor’s problem can be rewritten as

$$
\max_{x_{BN}} 1/2x_{BN} + 1/2(x_{BN} + (y_{BN} + P_e y_{SN} - x_{BN})/P_e)
$$

where $e = A, B, \ldots$.

With $P_e < 1/2$ the risk neutral investor puts all wealth into $x_{SN}$, i.e. $x_{BN} = 0$, and $x_{SN} = y_{BN}/P_e + y_{SN}$. This is the same for both the pooling and separating equilibria.

We are assuming that no short sales are allowed. Otherwise, bankruptcy would occur with probability $1/2$, since with short sales of bonds (borrowing) to buy stock the investor
could not deliver in state 1 where the stock pays off zero, the amount contracted for. This could not be an equilibrium, since other traders would not be willing to trade with this investor (assuming that his attempt to sell short were public knowledge).

The $t = 2$ expected utility of the risk neutral investor is

$$\text{max } EU = 1/2(y_{BN}/P_e + y_{SN})$$

(7.86)

where $e = A, B, C \ldots$. The risk neutral investor's $t = 1$ problem in the pooling equilibrium can now be solved. Assuming the $P_e$ are known, the problem is:

$$\text{max } \sum_e \rho_e \left[ \frac{y_{BN}}{P_e} + y_{SN} \right] \cdot 1/2$$

(7.87)

such that $y_{BN} + P_py_{SN} = z_{BN} + Ppz_{SN}$, or

$$\text{max } \sum_e \rho_e \left[ \frac{y_{BN}}{P_e} + [z_{BN} + Ppz_{SN} - y_{BN}] \cdot \frac{1}{P_P} \right] \cdot 1/2$$

(7.88)

If $1/P_P < \sum_e \rho_e/P_e$, then $y_{SN} = 0$, and if $1/P_P > \sum_e \rho_e/P_e$, then $y_{BN} = 0$. At $1/P_P = \sum_e \rho_e/P_e$, $y_{BN}$ can take on any value because the risk neutral investors are indifferent. This means that they will be willing to supply the desired holdings of log investors when they possess enough of the desired security to sell. Because of the simple corner solutions to the risk neutral investor's maximization problem, the model is tractable. A crucial result is that if $1/P_P = \sum_e \rho_e/P_e$ then the risk neutral investors affect the $t = 1$ price $P_P$ in the same way in each economy state $e$. Since they are indifferent no matter what their endowments their actions are to buy and sell the aggregate amount in each state $e$ that meets the log investor's demands. The pooling price $P_P$ will be an equilibrium price in all economy states provided risk neutral investors have aggregate endowments large enough in each economy state to supply the demands of log investors. If so, then investors learn nothing about the economy state from $P_P$ and their beliefs are consistent. The main idea is that prices do not reveal the economy state when different types of
economic agents that make the economy states different (e.g. different endowments), act in the same way.

The volume of trading does not in general reveal the economy's state because trading volume depends on trading between log investors which, given aggregate initial endowments, still depends on the distribution of endowments across the various log investors. For example, if log investors have the same endowments, no trading takes place between them. If half of the log investors have bonds and the other half stock, there will be much more trading. With trading volume not dependent on just aggregate endowments we can have a pooling equilibrium with the log investor's endowments, $z_{BL}$ and $z_{SL}$ not the same for the different economy states $e$. The pooling price $P_P$ is an equilibrium for all economy states if the aggregate net trade between log and risk neutral investors can not be observed. The aggregate net trade between log and risk neutral investors does reveal the aggregate endowments of the log investors which partially reveals the economy state in terms of log investor endowments. If there is significant trading between log investors then trading volume will not reveal log investor endowments. Since this is not crucial, for simplicity we assume a single value for aggregate log investor endowments for all economy states $e$, $z_{BL}$ and $z_{SL}$. We can think of this as assuming that investors know something about the economy state because they know aggregate log investor endowments but not the exact economy state because they do not know risk neutral aggregate endowments.

The $t = 1$ problem in the separating equilibrium is to maximize $t = 2$ wealth and has the same solution as the $t = 2$ problem. Since the price $P_e$ that occurs at $t = 2$ is known at $t = 1$, this must also be the equilibrium price at $t = 1$. Therefore, the risk neutral investors trade at $t = 1$ to their $t = 2$ optimum holdings. At $t = 2$ there will be no trading and the price will be unchanged from $t = 1$. If risk neutral investors did not trade to their $t = 2$ optimum holdings at $t = 1$, the $t = 1$ price would not be the same as the $t = 2$ price and there would be an incentive to keep trading until the prices
were equal. Since the $t = 1$ price equals the $t = 2$ price, any $t = 1$ asset holding is a solution for an individual atomistic investor since $t = 2$ wealth is unchanged but for the aggregate investor only the $t = 2$ optimum holdings lead to the equilibrium, where $t = 1$ asset holdings equal $t = 2$ asset holdings.

Now consider the investment problem of the log utility investors. At $t = 2$ it is

$$
\max_{x_{BL}, x_{SL}} (1/2 \log x_{BL} + 1/2 \log(x_{BL} + x_{SL}))
$$

such that

$$
x_{BL} + P_e x_{SL} = y_{BL} + P_e y_{SL}
$$

(7.89)

$$
= \max_{x_{BL}} 1/2 \log x_{BL} + 1/2 \log \left[ x_{BL} + \frac{y_{BL} - x_{BL}}{P_e} + y_{SL} \right] \quad \text{where} \ e = A, B, C, \ldots
$$

The first order condition for optimality is

$$
\frac{1}{x_{BL}} + \frac{P_e - 1}{P_e [x_{BL} + \frac{y_{BL} - x_{BL}}{P_e} + y_{SL}]} = 0
$$

So

$$
x_{BL} = \frac{y_{BL} + P_e y_{SL}}{2(1 - P_e)} \quad x_{SL} = \frac{(1/2) - P_e}{P_e(1 - P_e)}(y_{BL} + P_e y_{SL})
$$

(7.90)

and expected utility is

$$
\max EU = \log [y_{BL} + P_e y_{SL}] - 1/2 \log(4P_e(1 - P_e))
$$

(7.91)

which can be written in the form that shows the well-known "myopia" characteristic of log utility:

$$
\max EU = \log(W_L) + \text{constant}
$$

The $t = 1$ problem in the pooling equilibrium is then

$$
\max_{y_{BL}, y_{SL}} \sum_{e} \rho_e \log [y_{BL} + P_e y_{SL}] + \text{constant}
$$

such that $y_{BL} + P_P y_{SL} = z_{BL} + z_{SL} P_P$.

Thus, the problem can be rewritten as

$$
\max_{y_{BL}} \sum_{e} \rho_e \log [(P_P - P_e)y_{BL} + P_e(z_{BL} + z_{SL} P_P)] + \text{constant}
$$

(7.92)
The first order condition is

$$0 = \sum_e \frac{\rho_e (P_P - P_e)}{(P_P - P_e) y_{BL} + P_e (z_{BL} + z_{SL} P_P)}.$$  

(7.93)

This implies the demand equation

$$y_{BL} = \alpha (z_{BL} + z_{SL} P_P) = \alpha W_L(t = 1)$$  

(7.94)

and

$$y_{SL} = (1 - \alpha) W_L / P_P,$$

where $\alpha$ is the solution to

$$\sum_e \frac{\rho_e (P_P - P_e)}{(P_P - P_e) \alpha + P_e} = 0$$  

(7.95)

which maximizes the expected log of wealth. Some special cases of importance are when $P_P = \sum_e \rho_e P_e$ which implies $\alpha = 1$, $y_{BL} = W_L(t = 1)$ and $1/P_P = \sum_e \rho_e/P_e$ and $\alpha = y_{BL} = 0$ and $y_{SL} = W_L(t = 1) / P_P$.

In the special case of just two economy states A and B (so that $\rho_A + \rho_B = 1$) the demand equation simplifies to

$$y_{BL} = \left[ \frac{\rho_B P_A}{P_A - P_P} + \frac{\rho_A P_B}{P_B - P_P} \right] (z_{BL} + z_{SL} P_P)$$  

(7.96)

and

$$y_{SL} = \left[ \frac{(z_{BL} + z_{SL} P_P) - y_{BL}}{P_P} \right] = \left[ \frac{-\rho_B}{P_A - P_P} + \frac{-\rho_A}{P_B - P_P} \right] (z_{BL} + z_{SL} P_P).$$

When $P_P = 2 P_A P_B / (P_A + P_B)$ we have $y_{BL} = 0$ and $y_{SL} = W_L / P_P$. When $P_P = (P_A + P_B) / 2$ we have $y_{BL} = W_L$ and $y_{SL} = 0$.

Substituting for $y_{BL}$ into the expression for $EU(t = 1)$ we get

$$J(t = 1) = \max EU(t = 1) = \log(W_L(t = 1)) + \text{constant.}$$

This is an example of the standard result that log investors exhibit "myopia" (see [31]), i.e. they maximize the expectation of the log of next period’s wealth to solve their
intertemporal investment problem. It is easy to show that with $J(t) = \log(W(t)) + constant$ then

$$J(t - 1) = \max E_{t-1}[J(t)] = \log(W(t - 1) + constant.$$  

Since the result holds at the boundary condition where the financial asset pays off, by induction it holds for all previous times.

In the separating equilibrium, the $t = 1$ problem for the risk averse investors is conceptually the same as it was for the risk neutral investors. In equilibrium, the $t = 1$ price equals the known $t = 2$ price because the $t = 1$ price reveals the economy's state. The $t = 1$ problem is to maximize $t = 2$ wealth. Any asset holding at $t = 1$ gives the same wealth at $t = 2$ because of the budget constraint, but only the optimum $t = 2$ holdings are an equilibrium $t = 1$ solution. This leads to $t = 1$ and $t = 2$ prices both equalling $P_e$ depending on the state of the economy $e$.

Using the above results, the pooling and separating equilibria can be calculated. The log investors (type L) can be aggregated together and similarly for the risk neutral investors (type N). The difference between economy states is due to differences in the numbers of investors having particular initial endowments, but all investors have the same priors $\rho_e$ concerning the economy states. For simplicity, assume there is a single log investor (i.e., $n_{eL}^a = 1; n_{eL}^b = 0$) whose $t = 1$ initial endowment of the stock and bond is $z_{sL}^e$ ($s = B, S$) in both states of the economy.

To simplify the notation a little more, let the risk neutral investors' aggregate initial endowments in the economy states be denoted by

$$Z_{sN}^e = n_{eN}^a z_{sN}^a + n_{eN}^b z_{sN}^b(s = B; S; e = A, B, \ldots).$$

Similarly, let the aggregate holdings of risk neutral investors after trading at $t = 1$ and
after trading at \( t = 2 \), respectively, in the two economy states be denoted by

\[
Y_{sN}^e = n_{cN}^a y_{sN}^e + n_{cN}^b y_{sN}^e (s = B, S; e = A, B, \ldots)
\]

\[
X_{sN}^e = n_{cN}^a x_{sN}^e + n_{cN}^b x_{sN}^e (s = B, S; e = A, B, \ldots)
\]

7.3 Separating Equilibria

In a separating equilibrium (see figure 7.2), the risk neutral investors trade at \( t = 1 \) to their \( t = 2 \) optimum holdings and there is no further trade at \( t = 2 \). Assuming \( P_e < 1/2 \) in the separating equilibrium, the risk neutral investors sell at \( t = 1 \) all of their aggregate initial endowment of the bonds. The amount sold is \( Z_{BN}^e \) when the economy is in state \( e \) with price \( P_e \), \( e = A, B, C, \ldots \). To satisfy the log utility investor's \( t = 2 \) first order condition 7.90 with the requirement that they hold all the economy's bonds requires

\[
z_{BL} + Z_{BN}^e = \frac{z_{BL} + P_e z_{SL}}{2(1 - P_e)}
\]

which implies

\[
P_e = \frac{z_{BL} + 2Z_{BN}^e}{z_{SL} + 2z_{BL} + 2Z_{BN}^e}
\]

From this equation it can be seen that the condition that \( P_e < 1/2 \) is equivalent to the condition that \( z_{SL} > 2Z_{BN}^e \).

7.4 Pooling Equilibria

One possible pooling equilibrium of the 3-date model is the permanent pooling one. In the permanent pooling equilibrium \( P_e = P_p = 1/2 \) for all \( e \). Risk neutral investors are always indifferent between stocks and bonds, and prices never reveal anything about the economy. At \( t = 1 \) log investors sell all their stock and hold only bonds. For this to be possible \( z_{SL}/2 < Z_{BN}^e \). We have assumed that \( z_{SL}/2 > Z_{BN}^e \) so that the permanent
pooling equilibrium is not viable. Hence separating prices are required at $t = 2$ and a temporary pooling equilibrium must be considered. In the temporary pooling equilibrium the price starts at $P_P$ at $t = 1$ and then jumps to a new price $P_e$ at $t = 2$ without any new information about final payoffs. One can think of the Walrasian auctioneer or market maker at $t = 2$ trying out the price $1/2$, finding that it is not viable, and being forced to try a separating price.

Assume that there are no $t = 1$ short sale restrictions. Then the only equilibrium price at $t = 1$ is one where risk neutral investors are indifferent between the stock and bond, i.e. $1/P_P = \sum \rho_e / P_e$. (See equation 7.88.) At this price log investors hold exactly zero of the bond and put all of their wealth into the stock. They sell all of their initial endowment of the bond to risk neutral investors and buy more stock. Since the risk neutral investors are indifferent between the bond and stock at $P_P = [\sum \rho_e / P_e]^{-1}$, they are willing to sell all the stock they have for bonds. This equilibrium will exist if $Z^*_{SN}P_P$, the wealth endowment of the risk neutral investors held in stock is equal to or greater than $Z_{BL}$, the wealth endowment held in bonds of the log investors. If this is not true then this pooling equilibrium will in general not be viable. In this case, there is a less general form of the pooling equilibrium which requires a short selling constraint at $t = 1$ and requires that uncertainty about the economy state is due only to uncertainty about the aggregate bond endowment of risk neutral investors since their stock endowment is revealed at $t = 1$. The pooling price $P_P$ is greater than $[\sum \rho_e / P_e]^{-1}$ which makes the log investors willing to hold both bonds and stock but makes the risk neutral investors want to sell an infinite amount of stock and buy an infinite amount of bonds. The no short selling constraint implies that risk neutral investors will hold zero stock so that log investors at $t = 1$ must sell bonds for stock until all the risk neutral investors’ endowment of stock is exhausted. Which of the two possible pooling equilibria holds will depend on whether the risk neutral investor’s wealth endowment in stock, $Z^*_{SN}P_P$, is greater or less
than the log investor’s wealth endowment in bonds, $z_{BL}$, at the price $P_P = [\sum_e \rho_e / P_e]^{-1}$.

We will only be concerned with the pooling equilibrium that has risk neutral investors indifferent at $t = 1$ with the reciprocal of the pooling price $1 / P_P$ equal to the expectation of the reciprocals of the separating prices $\sum_e \rho_e / P_e$. If the initial endowments satisfy the conditions for the general pooling equilibrium, $Z_{SN}^e P_P > z_{BL}$, then $P_e$ and $P_P$ can be calculated as follows. After trading at $t = 1$, investors have their $t = 2$ endowments, which are

$$y_{BL} = 0, \quad y_{SL} = z_{SL} + z_{BL} / P_P,$$

$$Y_{BN}^e = Z_{BN}^e + z_{BL}, \quad Y_{SN}^e = Z_{SN}^e - z_{BL} / P_P > 0 \quad (e = A, B, C \ldots)$$

At $t = 2$, trade again occurs and risk neutral investors end up with $X_{BN} = 0$, so that log investors have $x_{BL} = Z_{BN}^e + z_{BL}$. Trading occurs at the price $P_e = P_A, P_B, \ldots$ depending on the initial endowments for economy state $e = A, B, \ldots$ and so

$$x_{SL} = y_{SL} - (Z_{BN}^e + z_{BL}) / P_e = z_{SL} + z_{BL} / P_P - (Z_{BN}^e + z_{BL}) / P_e$$

and

$$X_{SN}^e = Y_{SN}^e + (Z_{BN}^e + z_{BL}) / P_e = Z_{SN}^e - z_{BL} / P_P + (Z_{BN}^e + z_{BL}) / P_e.$$

For $P_e$ to be an equilibrium price, log investors must be just willing to hold $x_{BL}$ and $x_{SL}$ at $P_e$. From 7.90, this requires

$$Z_{BN}^e + z_{BL} = \frac{W_L(t = 2)}{2(1 - P_e)} = \frac{(z_{SL} + z_{BL} / P_P)P_e}{2(1 - P_e)}$$

which implies

$$P_e = \frac{2(Z_{BN}^e + z_{BL})}{z_{SL} + z_{BL} / P_P + 2(Z_{BN}^e + z_{BL})} \quad (7.99)$$

for $e = A, B, C \ldots$

Substituting $1 / P_P = \sum_e \rho_e / P_e$ and simplifying gives the conditions for equilibrium:

$$\frac{1}{P_e} (2(Z_{BN}^e + z_{BL}) - z_{BL} \rho_e) - z_{BL} \sum_{f \neq e} \frac{\rho_f}{P_f} - 2(Z_{BN}^e + z_{BL}) - z_{SL} = 0 \quad (7.100)$$
plus the condition that $Z_{SN}^e > z_{BL}/P_e$ for all $e$. This is a set of linear equations in the reciprocals of the prices, $1/P_e$. Given the endowments $z_{BL}$, $z_{SL}$ and $Z_{BN}^e$ and the probabilities $\rho_e$, these equations can be solved by standard methods. Again the condition that $z_{SL} > 2Z_{BN}^e$ ensures that $P_e < 1/2$.

The pooling price in this equilibrium has the property that all investors are able to trade to their optimal holdings. With separating prices log investors still can receive their optimal holdings but risk neutral investors can not (without violating short selling constraints). Hence the pooling price should be tried first by a market maker or Walrasian auctioneer. With all investors having rational expectations that this rule will be followed, we have temporary pooling equilibria when permanent pooling equilibria are not viable, that sometimes have sharp price drops not unlike a market crash. The Walrasian auctioneer tries a pooling price at $t = 2$ and finds it is not viable. Separating prices are then tried and in economy states with low separating prices $P_e$, the price drops sharply from the $t = 1$ price $P_P$ without any new information about final payoffs. In the next chapter the model will be generalized to more than 3 dates and the temporary pooling equilibria will be used to model market crashes.

The difference between two $t = 2$ separating prices for two arbitrary economy states $e$ and $f$ in the pooling equilibrium, which we will denote by $|P_e^P - P_f^P|$, is necessarily smaller than the analogous price difference $|P_e^S - P_f^S|$ in the separating equilibrium. This can be seen by substituting from 7.98 and 7.99 to obtain

$$|P_e^P - P_f^P| = \left( z_{SL} + z_{BL}/P_P \right) \cdot \left| \left( z_{SL} + z_{BL}/P_P + 2(Z_{BN}^e + z_{BL}) \right)^{-1} - \left( z_{SL} + z_{BL}/P_P + 2(Z_{BN}^f + z_{BL}) \right)^{-1} \right|$$

(7.101)

and

$$|P_e^S - P_f^S| = \left( z_{SL} + z_{BL} \right) \cdot \left| \left( z_{SL} + 2(Z_{BN}^e + z_{BL}) \right)^{-1} - \left( z_{SL} + 2(Z_{BN}^f + z_{BL}) \right)^{-1} \right|$$

(7.102)
Chapter 7. A Three Date Model of Market Created Risk

Which implies that

\[
\frac{p_e^s - p_j^s}{p_j^s - p_j^p} = \frac{1 + \frac{z_{BL}(1/P_P - 1)}{z_{SL} + z_{BL}}}{\left[1 + \frac{z_{BL}/P_P}{(z_{BL} + z_{BN}) + z_{SL}}\right] \left[1 + \frac{z_{BL}(1/P_P - 1)}{z_{BL} + z_{BN}}\right]^{-1}} > 1
\]

(7.103)

where we have used the fact that \( z_{SL} > 2z_{BN}^e \) for all \( e \) and \( 1/P_P > 2 \). The reason for \( |P_e^p - P_j^p| < |P_e^s - P_j^s| \) is the intermediate “insurance” position of log investors in the pooling equilibrium as they trade in two steps to their final asset holdings.

In both the separating and the pooling equilibrium log investors end up with the payoff of \( z_{BL} + Z_{BN}^e \) in economy state \( e \) when the stock does not pay off i.e. state of nature 1 occurs. In the separating equilibrium, when the stock does pay off (state of nature 2) log investors receive

\[
(z_{BL} + Z_{BN}^e) + (z_{SL} - Z_{BN}^e/P_e^p) = (z_{SL} + z_{BN}/P_e^s)/2
\]

in economy state \( e \), which is different for different economy states. In the pooling equilibrium, when the stock pays off log investors receive

\[
(z_{BL} + Z_{BN}^e) + (z_{SL} + z_{BL}/P_P - Z_{BN}^e/P_e^s - z_{BL}/P_e^p) = (z_{SL} + z_{BN}/P_P)/2
\]

which is the same in all economy states. This illustrates the endowment insurance effect of the pooling equilibrium. The consumption of log investors is less variable in this equilibrium relative to the separating one from the investor’s perspective before trading begins. Unfortunately the log investor is not necessarily better off in the pooling equilibrium as will be seen in a numerical example below. The expected value of the payoff received by log investors can be lower in the pooling equilibrium than the separating equilibrium. Although there is less variability of payoffs in the pooling equilibrium this does not necessarily compensate for the lower expected payoff and log investors may
have a lower expected utility. Risk neutral investors have a higher expected payoff in the pooling equilibrium which means a higher expected utility.

The pooling equilibrium exhibits the property that the price changes from \( t = 1 \) to \( t = 2 \), and trading takes place at \( t = 2 \), although no new information about final payoffs has been received and all investors act rationally at both times.

One interesting feature of the pooling equilibrium is that at time \( t = 1 \) the risk neutral investors always have more of their wealth in the bond than the more risk averse log investors. Intuitively, one would expect the log investors to hold more of the safer investment, which is the bond. At \( t = 1 \) in the pooling equilibrium, however, log investors hold only the stock. This occurs because, while log investors maximize the expected log of wealth, risk neutral investors maximize the expectation of wealth divided by the stock price. Risk neutral investors plan on holding only the stock at \( t = 2 \), so that they are concerned with how much of the stock they can then buy as given by wealth divided by the price. The more risk averse log investors are concerned with wealth as the argument of their utility functions. Thus, the risk neutral investors act as though they were more risk averse because the argument of their utility function is wealth divided by an uncertain price. Log investors receive insurance for their endowment uncertainty by holding more stock. Because the pooling price is so low, stock is preferred over bonds even though it involves greater wealth uncertainty at \( t = 2 \).

7.5 Numerical Example

Consider the following numerical example to illustrate the pooling and separating equilibrium. Assume that there are just two economy states A and B each with a probability of 1/2 occurring.

Let \( z_{BL} = 10, z_{SL} = 70 \), and
\[ z_{BN}^a = 1, n_{AN}^a = 2, n_{BN}^a = 10, \]
\[ z_{BN}^b = n_{AN}^b = n_{BN}^b = 0, \text{ so that} \]
\[ Z_{BN}^A = 2, Z_{BN}^B = 10. \]

\[ Z_{SN}^A = z_{SN}^a n_{AN}^a \text{ and } Z_{SN}^B = z_{SN}^b n_{AN}^b \text{ must be greater than } z_{BL}/P \text{ but are otherwise arbitrary. The equilibrium conditions 7.100 become:} \]
\[
((3/2)z_{BL} + 2Z_{BN}^A)/P_A - z_{SL} - z_{BL}/(2P_B) - 2(z_{BL} + Z_{BN}^A) = 0
\]
\[
((3/2)z_{BL} + 2Z_{BN}^B)/P_B - z_{SL} - z_{BL}/(2P_A) - 2(z_{BL} + Z_{BN}^B) = 0
\]

These conditions are satisfied with \( P_A = 1/6, P_B = 1/4 \) and \( P_P = 1/5 \) since: \(((3/2)10 + 4)6 - 70 - 20 - 2(10 + 2) = 0 \) and \(((3/2)10 + 20)4 - 70 - 30 - 2(10 + 10) = 0 \). With \( Z_{SN}^A > 10/(1/5) = 50 \) and \( Z_{SN}^B > 50 \), but otherwise arbitrary, the equilibrium is completely specified. At \( t = 1 \), log investors sell 10 bonds and buy \( 10/(1/5) = 50 \) shares of the stock. At the end of trading they hold \( 70 + 50 = 120 \) shares of stock and 0 bonds. The risk neutral investors end up with \( 10 + 2 = 12 \) units of bonds in economy state A and \( 10 + 10 = 20 \) units of bonds in economy state B and positive holdings of stock. They sell 50 shares of stock to the log investors.

At \( t = 2 \), trading again occurs and the log investors in economy state A receive and hold 12 bonds, sell \( (12/(1/6)) = 72 \) shares of stock and hold the remaining 48 shares. When the stock pays off in state of nature 2 they receive \( 12 + 48 = 60 \) units of the consumption good. Log investors in economy state B receive and hold 20 bonds, sell \( 20/(1/4) = 80 \) shares of stock, and hold the remaining 40 shares. When the stock pays off they again receive \( 20 + 40 = 60 \) units of the consumption good. Log investors receive 12 in economy state A and 20 in B when only the bond pays off (state of nature 1). Risk neutral investors buy 72 shares of stock, sell 12 bonds and hold 0 bonds in economy state A. They increase their holdings of the stock by \( 72 - 50 = 22 \) from their initial
endowments. Risk neutral investors buy 80 shares of stock, sell 20 bonds and hold 0 bonds in economy state B. They increase their holdings of the stock by \(80 - 50 = 30\) from their initial endowments.

In the separating equilibrium the equilibrium prices given by equation 7.98 are

\[
P_A = \frac{10 + 2 \cdot 2}{70 + (2 \cdot 10) + (2 \cdot 2)} = \frac{14}{94}
\]

and

\[
P_B = \frac{10 + 2 \cdot 10}{70 + (2 \cdot 10) + (2 \cdot 10)} = \frac{3}{11}.
\]

At \(t = 1\) in economy state A log investors buy all of the risk neutral investors bonds which is 2 bonds so that they then hold \(2 + 10 = 12\) bonds. They sell \(2/(14/94) = 13.428\) shares of stock so that they hold \(70 - 13.428 = 56.57\) of the stock. In economy state B they buy 10 bonds so that they then hold \(10 + 10 = 20\) bonds. They sell \(10/(3/11) = 36.667\) shares of stock so that they hold \(70 - 36.667 = 33.333\) of the stock. The risk neutral investors take the opposite side of the trades. In economy state A they sell 2 bonds so that they hold 0 bonds and buy 13.428 shares of stock. In economy state B they sell 10 bonds so that they hold 0 bonds and buy 36.667 shares of stock. At \(t = 2\) the prices are the same and no trading occurs.

The effect on social welfare of moving from a separating to a pooling equilibrium can be determined by calculating expected utilities for this numerical example. In the pooling equilibrium before investors know the economy state or state of nature, log investors have an expected utility of \((\log(12) + \log(12 + 48) + \log(20) + \log(20 + 40)) = 3.417\) and risk neutral investors have an expected utility of \(\frac{22 + 30 + Z^A_{SN} + Z^B_{SN}}{4} = 13 + Z^A_{SN} + Z^B_{SN}/4\).

In the separating equilibrium log investors have an expected utility of \((\log(12) + \log(12 + 56.57) + \log(20) + \log(20 + 33.33))/4 = 3.4212\). and risk neutral investors have an expected utility of \((13.428 + 36.667 + Z^A_{SN} + Z^B_{SN})/4 = 12.5238 + (Z^A_{SN} + Z^B_{SN})/4\). Risk neutral investors prefer the pooling equilibrium while log investors prefer the separating
equilibrium even though their consumption is less variable. Clearly the two types of equilibria are not Pareto comparable.

7.6 Market Completeness with Respect to Economy States

The market is complete in this model with respect to the two states of nature. The market is also complete with respect to the economy states in the pooling equilibrium when there are enough nonredundant securities. In general, if there are \( k \) securities then the market is complete with respect to economy states if there are no more than \( k \) such states. This is implied by the general result that with a maximum number of \( k \) branches leaving any node of the event tree and \( j \) trading periods; then \( k \) nonredundant long-lived securities are enough to span up to \( jk \) Arrow-Debreu states (Kreps, 1982 [27]). Consider the case of just two economy states A and B. By combining the stock and bond, a portfolio can be created that pays off only in economy state A or only in economy state B. The portfolio of \( P_A/(P_B - P_A) \) units short in bonds and \( 1/(P_B - P_A) \) units long in stock pays off 0 in economy state A and 1 in economy state B. Similarly, \( P_B/(P_B - P_A) \) units long in bonds and \( 1/(P_B - P_A) \) units short in the stock pays off 1 in economy state A and 0 in economy state B. With \( P_P = 2P_A P_B/(P_A + P_B) \), since there is no \( t = 1 \) short sales constraint, these portfolios are feasible; hence the market is complete. A lack of completeness with respect to the economy states is thus in general not a factor in the existence of a pooling equilibrium.

7.7 The Role of Aggregation

The role of aggregation turns out to be of major importance in distinguishing between models of financial markets where market uncertainty is an important factor and those
models where it is not. Aggregation is important in standard models of financial markets because when the conditions permitting aggregation hold, demand equations for all investors in the market have the same structure. Asset pricing equations can then be developed of a very tractable form, based on the pricing that would be implied by just a single representative investor in the market.

When the conditions for aggregation hold, however, the importance of market created risk is diminished if not eliminated. The reason is that market uncertainty is concerned with an investor's uncertainty about the true nature of the financial markets that will be setting future prices. When it is common knowledge that aggregation holds, investors have a substantial amount of information about the nature of the financial markets and in models having fully revealing rational expectations equilibria prices can reveal the rest. In the model considered here, imposing the conditions for aggregation has a number of interesting effects. After an initial round of trading to equilibrium, trading volume in the subsequent market opening is zero. The equilibrium prices fully reveal the market uncertainty and equilibrium is unique. Only by breaking the aggregation conditions do trading volume and multiple equilibria, including those of the pooling type, appear.

### 7.8 Generalizations

The model can be extended to more general sets of preferences. Instead of log and risk neutral investors, any two sets of risk averse preferences will work. As before, the pooling price is an equilibrium price for all of the economy states at $t = 1$. At $t = 2$, prices reveal the economy's state, and may or may not require a short selling constraint. Without short selling constraints, however, especially at $t = 1$, the model becomes algebraically tedious. To make the pooling price an equilibrium for different economy states we must require that different groups of investors behave in the same way at the pooling price.
Suppose one economy state has a large number of very risk averse investors with large aggregate holdings of the stock and bond and hence large wealth. They may wish to buy a certain amount of the stock and sell bonds. A less risk averse group of investors with less wealth, if the amount of wealth is chosen properly, may wish to buy exactly the same amount of stock. The two different groups of investors can produce, with some third group of investors that stays the same, two economy states. There can then be a pooling equilibrium. This equilibrium would not be generic since a slight change in endowments or preferences would cause it to fail. If we assume a continuum of economy states produced by a continuous range of possible endowments then for two groups of investors of risk averseness A and B given an endowment x for the A type investors there is always some endowment y for the B type investors from the interval of possible values such that the two groups of investors demand the same amount of the stock and bond. Hence with a continuum of economy states, as occurs in actual financial markets, pooling equilibria are generic.
Chapter 8

A Model of Market Crashes

In the previous chapter we developed a simple three date model that exhibited a number of interesting features. One important feature was the existence of a pooling equilibrium which involved a price that did not reveal at $t = 1$ the true nature of the underlying economy, the economy state. This pooling price equilibrium had the feature that the price changed at $t = 2$ without new information arriving about final payoffs. Instead, the price change itself was the source of information. In this chapter, we want to use the idea behind this three date model and extend it to models involving more time periods and more economy states. The goal of extending the model is to create a theoretical framework for understanding market crashes.

Market crashes have been observed and discussed for hundreds of years. Some of the most famous market crashes were the collapse of the South Sea Bubble, the collapse of the Dutch Tulip Bubble, the great crash of 1929 and more recently the market crash of Oct. 19, 1987. Market crashes have involved many different kinds of markets but they have a number of things in common. First, they involve a market in which a price is set for transactions between buyers and sellers. Second, the price has a steady rise over a period of time. The ongoing price rise attracts speculators whose sole purpose in buying the particular commodity is for its capital appreciation. The price of the commodity rises higher and higher with the amount of trading also increasing until at some point the whole process breaks down. At this point the price starts to fall rapidly with a lot of trading taking place, although if the price drops are too rapid trading can break down.
and cease altogether. If there are a large number of speculators who had borrowed to
buy the commodity as its price rose, they will be forced to sell as the price falls, further
increasing the speed at which prices collapse. The price rise to an unsustainable level
and then subsequent sharp price drop is the market crash.

Although market crashes are not hard to describe and in fact were noted hundreds
of years ago, theoretical models of them have been more difficult to develop. The main
theoretical model of market crashes is based on the idea that they are collapsing specu­
lative bubbles. Speculative bubbles are defined to be asset prices that deviate for a finite
amount of time from the asset’s fundamental value where the deviation is recognized
to exist by market participants. The fundamental value is the correct value that the
asset should have. In the case of a stock it is the sum of all future dividends discounted
at the correct discount rate. The fundamental value can be calculated from knowing
the complete structure of the economy, i.e., having the correct model. The fundamental
value is assumed known by all market participants who are presumed to have a correct
model of the economy, but it is not necessarily the market price. The market price
can exceed the fundamental value and investors can know that this is the case because
they anticipate that future market prices will likely be even higher and the anticipated
capital gains justify the current market price. If there is a final fixed price boundary
condition at some finite future time period then that final price ties down the price an
instant earlier, and by backward induction the whole series of prices from the final one
back to the current one can be deduced if the economy’s structure is known. Hence, in
this theoretical framework, bubbles can exist only with an infinite amount of time or
with irrational investors (see Tirole 1982 [47]). The bubble represents a deviation of the
market price from fundamental value and the market crash is the collapse of the bubble
which returns the price to its fundamental value. There is also the possibility of bubbles
that never collapse in this theory.
In this chapter, we present a different theoretical framework to model market crashes. The basic idea of this approach is found in the previous chapter. We assume that the fundamental value of a financial asset is unknown to investors. By fundamental value we mean the equilibrium market price that would hold if all investors knew the underlying state of the economy. Alternatively, the fundamental value would be the market price if investors were knowingly trading for the last time and would be forced to hold their portfolios until all liquidating dividends were received.

The market price in this theory is an equilibrium price that prevails in an imperfectly known underlying economy (i.e., in an unknown economy state). Hence, it is a pooling price that does not (fully) reveal the true nature of the economy. If the true economy state were known, the correct fundamental value could be calculated and the market price would equal this fundamental value. The market price is an equilibrium price based on the possible future market prices. It is assumed that the market price will become one of the possible fundamental values at some point in the future. If this process occurs quickly and involves a price drop, it will look like a collapsing speculative bubble but will not involve a known deviation from fundamental value being eliminated. Thus, neither irrationality nor an infinite amount of time are required. Hence, we will create a number of models of market crashes by creating multiperiod models with multiple possible economy states with pooling price equilibria. These pooling price equilibria involve the pooling price, as part of the equilibrium, sometimes being forced to move to a price that reveals the state of the economy and fundamental value. When this price move is sharply downwards it corresponds to the market crash.

The event that triggers the market crash in our model can be seemingly insignificant. Certain ranges of desired asset holdings will be compatible with pooling equilibria prices while other ranges will not be. If information arrives to the market or new investors arrive, so that desired asset holdings change from the pooling price compatible range
to the incompatible range the market crash is triggered. Since the change in desired asset holdings can be small and still involve the pooling price becoming nonviable, in our model small events can trigger market crashes. This is in agreement with the intuition most people have about market crashes.

8.1 The General Modeling Methodology

The models we consider are based upon the three date model of the previous chapter. There are two groups of investors, log utility investors with a known aggregate endowment of the stock and bond, and risk neutral investors whose aggregate endowments of bonds and stock are unknown. The unknown aggregate endowments of the risk neutral investors determines the economy state. As before, we assume each risk neutral investor can have one of two possible endowments and the various number of risk neutral investors with one of the two possible endowments determines the aggregate endowment of bonds and hence the economy state. Consumption occurs only in the final period when the true state of nature is revealed and the stock and bond pay off. We allow for the possibility of information about the state of nature arriving in intermediate time periods or for new investors to arrive in the market. The models have more than two trading periods in the financial market but no trading occurs unless something new happens in the economy such as new information or new arrivals.

We also assume that the state of nature and state of the economy are correlated in a particularly simple way. In fact, the distinction between states of nature and states of the economy is somewhat artificial and is made for the sake of understanding the theoretical methodology. "Economy states" should really be thought of as special types of states of the world that affect market participants. "States of nature" by our definition refer to states that determine the payoffs of financial assets and are again special kinds of states
of the world.

The equilibrium we will concentrate on is based on the pooling equilibrium of the three date model. In this equilibrium, the market price is such that risk neutral investors are indifferent as to whether they hold the stock or bond. Log investors have particular optimal holdings of the financial assets. If risk neutral investors have sufficient aggregate endowments of the stock and bond they will be able to supply the log investors with their desired holdings. If risk neutral investors do not have a sufficient aggregate endowment to supply the log investors in a particular economy state then the pooling price is not compatible with this particular economy state. Therefore, the beliefs about the economy state implied by this pooling price must put nonzero probabilities only on economy states that have sufficient risk neutral aggregate endowments to supply the log investors with their desired holdings.

This model represents a market crash as follows. The financial market allows trading at a number of dates. Investors choose their current holdings of the stock and bond based on the current price and their beliefs about the possible future prices of the assets. These beliefs depend upon the current prices. The equilibrium we consider involves pooling prices that switch to separating prices later on. The move from a pooling price to a separating price in the equilibrium represents the market crash. In the initial rounds of trading, the stock price is a pooling price that does not reveal the economy state. It is roughly an average of the possible separating prices in the various economy states. As in the three date model we assume that eventually (before the final trading period) only separating prices that reveal the economy state are viable equilibrium prices.

At some intermediate trading period the actual underlying economy state is not compatible with a pooling price based on current beliefs about the possible underlying economy states. This occurs when supplying the equilibrium holdings of the log investors would require the risk neutral investors in aggregate to sell more of either the bond or
the stock then they possess. The pooling price cannot then be an equilibrium, the stock price must fall to a separating value and the market "crashes". The event that triggers the market crash in our model is the inability of risk neutral investors, at a market price that leaves them indifferent to holding stocks or bonds, to supply the log utility investors with their desired optimal holdings.

We consider two processes by which the market crash may be triggered. In the first process, new log utility investors enter the market in each trading period. These new investors enter with an endowment that is not optimal and attempt to trade to an optimal portfolio that involves holding only stock. Since these new investors are assumed to be endowed with some bonds, the risk neutral investors buy bonds from them and sell them stock until they run out of stock to sell. At that point the market price cannot remain at a value that makes risk neutral investors indifferent to holding stock or bonds. The equilibrium price must then be a separating price whose value is predicted by investors. This separating price reveals the economy state because the aggregate endowments are revealed. The aggregate stock endowment is known because the point in time at which the pooling price is not viable (where only log investors hold stock) is known. The aggregate bond endowment is implied by the separating price. At the separating price risk neutral investors sell all their bonds and buy stock since the separating price of stock is assumed to be lower than the stock’s expected payoff. Log investors sell stock and buy bonds but end up holding both because risk neutral investors are unable to sell them enough bonds. Risk neutral investors are no longer indifferent and because of the no short selling constraint hold 0 bonds (they do not borrow). All subsequent trading until the assets pay off, occurs at separating prices.

In the second process of market crashes, unlike the first scenario, no new investors enter the market, so the make up of the economy’s participants is static. However, information signals about the stock payoff and the economy state arrive in the market
and cause log investors to change their portfolio holdings. Again, the equilibrium prices before the final trading period are such that risk neutral investors are indifferent between bonds and stock. They attempt to satisfy the log investors' desired holdings. As long as the risk neutral endowments are large enough, the pooling price is an equilibrium and the economy state is not revealed. At some point, though, the log investors may demand more bonds than risk neutral investors can supply. This can occur with the arrival of information that changes beliefs by an insignificant amount. If this happens then the current pooling price is no longer an equilibrium. At that point, the price either drops to a fully separating price which represents the market crash if risk neutral investors can not supply the log investors or moves a smaller amount upwards to a new pooling price that involves fewer possible economy states. The two possibilities are anticipated by investors since they have rational expectations. We have the possibility then, of seemingly unimportant information arriving to the market and changing desired asset holdings by just enough to trigger a market crash.

8.2 Market Risk With Many Time Periods

8.2.1 Trigger Method One: New Investors Entering the Market

In this section we examine a model that involves new investors entering the market and show that the equilibrium can act like a market crash. New log investors enter the market each period and buy stock and sell bonds at a pooling price that leaves risk neutral investors indifferent between stock and bonds. The market crash is triggered when risk neutral investors run out of stock to sell.

We consider a model with four dates, two states of nature and three possible economy states (see figure 8.2.1). The probability of the two states of nature, state 1 and state 2, is each 1/2. The economy states are labelled A, B and C. The probability of economy
state A is $1/2$ and the probability of economy states B and C is each $1/4$. As before, there are two financial assets, a stock and a bond. The stock pays off 1 unit of the consumption good only in state of nature 2 whereas the bond pays off 1 unit in both states of nature. The payoffs of the financial assets and consumption occur at the final date $t = 4$. Trading occurs at times $t = 1$, $t = 2$ and $t = 3$; at each trading time an equilibrium price for the stock relative to the bond is determined.

The three economy states differ in the aggregate endowments risk neutral investors have of the stock and bond. The separating prices in the three economy states are determined by the risk neutral investors' aggregate endowment of bonds. The trigger point at which the pooling price is not an equilibrium is determined by the risk neutral
investors’ aggregate endowment of the stock. The model is easily generalized to an \( n \) date model with \( n - 1 \) economy states or to a continuous time version of the model but we will consider only the simple case here.

The aggregate endowments of the risk neutral investors in bonds and stock in economy state A are denoted by \( Z_{BN}^A \) and \( Z_{SN}^A \), respectively, and similarly for economy states B and C. At \( t = 1 \), log investors have aggregate endowments of \( z_{BL} \) and \( z_{SL} \) of the stock and bond, respectively. At \( t = 2 \), new log investors enter the market with additional aggregate endowments of \( z_{BL} \) and \( z_{SL} \) of the stock and bond, respectively, so that the \( t = 2, t = 3, \) and \( t = 4 \) aggregate endowments of the log investors are \( 2z_{BL} \) and \( 2z_{SL} \).

We consider two possible equilibria, the crash equilibrium that can act like a market crash, and the fully pooling equilibrium which given certain initial conditions is not viable. All prices except those that reveal the economy state make the risk neutral investors indifferent between bonds and stock. Consider the fully pooling equilibrium first. At \( t = 1 \) and \( t = 2 \), the equilibrium price is \( P_{ABC} \) which does not reveal the economy state because it is an equilibrium in all three economy states given prior beliefs. At \( t = 3 \), investors trade to their final asset holdings with risk neutral investors holding only stock and log investors holding all the economy’s bonds and the remaining stock. The equilibrium price at \( t = 3 \) reveals the economy state. The \( t = 3 \) price is either \( P_A \) with probability \( 1/2 \) or \( P_B \) or \( P_C \) with probability \( 1/4 \) depending on whether the economy state is A, B or C respectively. We assume that the prices satisfy \( 1/2 > P_C > P_B > P_A \) which means that \( Z_{BN}^C > Z_{BN}^B > Z_{BN}^A \). The \( t = 1 \) equilibrium price \( P_{ABC} \) makes risk neutral investors indifferent between holding stock and bonds based on their expectations that the \( t = 2 \) price will be the same, that the \( t = 3 \) price will be either \( P_A, P_B \) or \( P_C \) and that they will sell all their bonds for stock at \( t = 3 \). It must equal

\[
P_{ABC} = \left[ \frac{1}{2} \left[ \frac{1}{P_A} + \frac{1}{2} \left[ \frac{1}{P_B} + \frac{1}{P_C} \right] \right] \right]^{-1}
\]
as was shown in the previous chapter. At this price, log investors wish to hold all their wealth in the stock. Hence, after trading at \( t = 1 \), log investors hold \( x_{BL} = 0 \) in bonds and \( x_{SL} = z_{SL} + z_{BL}/P_{ABC} \) in stock. If endowments satisfy
\[
z_{BL} < P_{ABC}z_{NS}^A < P_{ABC}z_{SN}^B < P_{ABC}z_{NS}^C
\]
then the indifferent risk neutral investors are able to supply the log investors with their desired holdings and the price \( P_{ABC} \) is an equilibrium at \( t = 1 \). At \( t = 2 \), additional log investors enter the market with the same endowments as the previous log investors (who do not wish to trade again). If
\[
P_{ABC}z_{NS}^A < 2z_{BL} < P_{ABC}z_{NS}^B < P_{ABC}z_{NS}^C
\]
then when the economy is in state A risk neutral investors can not supply log investors with their desired holdings of the stock at \( t = 2 \) and \( P_{ABC} \) is not a \( t = 2 \) equilibrium price. Since it is not an equilibrium in economy state A, it cannot support the beliefs that economy state A is possible. Hence, when condition 8.104 holds, the fully pooling equilibrium is not viable at \( t = 2 \) and investors know that at time \( t = 2 \) the market price will reveal whether or not the economy is in state A.

Consider the crash equilibrium which at \( t = 1 \) has a pooling price \( P_{ABC} \) as above and at \( t = 3 \) has again one of the separating prices \( P_A, P_B \) or \( P_C \). At \( t = 2 \), if the economy is in economy state A then the price is the \( t = 3 \) separating price \( P_A \) and investors trade to their final portfolio holdings. If the economy is in state B or C the equilibrium price is \( P_{BC} \), where
\[
P_{BC} = \frac{2P_B P_C}{P_B + P_C}.
\]
The beliefs necessary to sustain this price are that the \( t = 3 \) price is either \( P_B \) or \( P_C \) with probability 1/2. Log investors hold only stock and risk neutral investors are willing to supply them with it if they have enough. We assume that they can do so in economy
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states B and C but not in economy state A, so that the beliefs implied by the price $P_{BC}$ are consistent. For this to be true we assume that

$$P_{BC}(Z_{NS}^A - z_{BL}/P_{ABC}) < z_{BL} < P_{BC}(Z_{NS}^B - z_{BL}/P_{ABC}) < P_{BC}(Z_{NS}^C - z_{BL}/P_{ABC}).$$

(8.106)

The $t = 1$ price must make risk neutral investors indifferent between bonds and stock so that

$$P_{ABC} = \frac{1}{2} \left[ \frac{1}{P_A} + \frac{1}{P_{BC}} \right]^{-1}.$$

Substituting for $P_{BC}$ from equation 8.105 shows that $P_{ABC}$ in the crash equilibrium has the same form as $P_{ABC}$ in the fully pooling equilibrium, however, they are not equal in general because $P_A$, $P_B$, and $P_C$ differ in the two equilibria.

We assume that the Walrasian auctioneer or market maker who sets prices always starts with a pooling price and so in this case at $t = 2$ would try $P_{BC}$ first. If it is an equilibrium then the new log investors are able to trade so that they hold only stock. By 8.106 this is possible only in economy states B or C and the beliefs are consistent. If the price $P_{BC}$ is not an equilibrium then the price $P_A$ becomes the equilibrium price and investors trade to their final financial asset holdings. The price move from $P_{ABC}$ at $t = 1$ to $P_A$ at $t = 2$, accompanied by the large trading volume of log investors switching from stock to bonds, appears like a market crash since there is no new information about final payoffs to explain the price move.

We can now solve for the prices in the crash equilibrium given the aggregate endowments of all investors. Prices differ from those in the fully pooling equilibrium because the trades of new log investors at $t = 2$ differ. From the previous chapter, we can determine the demand function of log investors at the separating price. At the price $P_e$ where $e = A, B$ or $C$, log investors, knowing that the stock will have payoff 1 with probability
1/2 and 0 otherwise, wish to have
\[
\left( \frac{1}{2} \right) \left( \frac{1}{1 - P_e} \right) W_L
\]
of their wealth in bonds and the amount
\[
\left( \frac{1/2 - P_e}{1 - P_e} \right) W_L
\]
in stock.

With \( P_e = 1/2 \), risk neutral investors are indifferent between stock and bonds and log investors put all their wealth into bonds. If this price were an equilibrium it would again be a pooling price that does not reveal the economy state. We assume that risk neutral investors do not have sufficient bond holdings to supply fully the demands of log investors so as before this equilibrium is not viable. This ensures that a separating price will eventually occur. We must then have \( P_e < 1/2 \) so that risk neutral investors hold only stock and log investors hold all of the economy's bonds and the remaining stock. Since log investors must hold the economy's aggregate endowment of bonds we have the relation
\[
2z_{BL} + z_{BN}^* = \frac{1}{2(1 - P_e)} W_L. \tag{8.107}
\]
Since log investors hold only stock when the price is a pooling price, selling their bond endowments as soon as they can, the value of \( W_L \) can be calculated. In the fully pooling equilibrium, both the log investors at \( t = 1 \) and and those arriving at \( t = 2 \) sell their bond endowment at the price \( P_{ABC} \). In the crash equilibrium, the \( t = 1 \) log investors sell their bond endowment at \( P_{ABC} \). The log investors arriving at \( t = 2 \) in the economy state A keep their bond endowments at the price \( P_A \) whereas in economy states B or C they sell them at the price \( P_{BC} \).

Using the expression 8.107 that equates the log investors' desired holding of bonds to the economy's aggregate bond endowment for the three possible economy states gives
a set of equations that can be solved to determine market prices given endowments. In
the crash equilibrium 8.107 becomes

\[
2 z_{BL} + Z_{BN}^A = \frac{1}{2(1 - P_A)} [P_A(2z_{SL} + \frac{z_{BL}}{P_{ABC}}) + z_{BL}]
\]

\[
2 z_{BL} + Z_{BN}^B = \frac{1}{2(1 - P_B)} [P_B(2z_{SL} + \frac{z_{BL}}{P_{ABC}}) + \frac{z_{BL}}{P_{BC}}]
\]

\[
2 z_{BL} + Z_{BN}^C = \frac{1}{2(1 - P_C)} [P_C(2z_{SL} + \frac{z_{BL}}{P_{ABC}}) + \frac{z_{BL}}{P_{BC}}]
\]

(8.108)

In the fully pooling equilibrium 8.107 becomes

\[
2 z_{BL} + Z_{BN}^A = \frac{1}{2(1 - P_A)} P_A(2z_{SL} + 2 \frac{z_{BL}}{P_{ABC}})
\]

\[
2 z_{BL} + Z_{BN}^B = \frac{1}{2(1 - P_B)} P_B(2z_{SL} + 2 \frac{z_{BL}}{P_{ABC}})
\]

\[
2 z_{BL} + Z_{BN}^C = \frac{1}{2(1 - P_C)} P_C(2z_{SL} + 2 \frac{z_{BL}}{P_{ABC}})
\]

(8.109)

Substituting into these equations $1/P_{BC} = 1/(2P_B) + 1/(2P_C)$ and $1/P_{ABC} = 1/(2P_A) + 1/(2P_{BC})$ and simplifying gives a set of linear equations in the reciprocals of the prices that can be solved given the endowments. For the crash equilibrium they are:

\[
(z_{SL} + 2z_{BL} + Z_{BN}^A) + \frac{1}{P_A}(-\frac{5}{4}z_{BL} - Z_{BN}^A) + \frac{1}{P_B} \frac{z_{BL}}{8} + \frac{1}{P_C} \frac{z_{BL}}{8} = 0
\]

\[
(z_{SL} + 2z_{BL} + Z_{BN}^B) + \frac{1}{P_A} \frac{z_{BL}}{4} + \frac{1}{P_B}(-\frac{13}{8}z_{BL} - Z_{BN}^B) + \frac{1}{P_C} \frac{3z_{BL}}{8} = 0
\]

\[
(z_{SL} + 2z_{BL} + Z_{BN}^C) + \frac{1}{P_A} \frac{z_{BL}}{4} + \frac{1}{P_B} \frac{3z_{BL}}{8} + \frac{1}{P_C}(-\frac{13}{8}z_{BL} - Z_{BN}^C) = 0
\]

(8.110)

For the pooling equilibrium they are:

\[
(z_{SL} + 2z_{BL} + Z_{BN}^A) + \frac{1}{P_A}(-\frac{3}{2}z_{BL} - Z_{BN}^A) + \frac{1}{P_B} \frac{z_{BL}}{4} + \frac{1}{P_C} \frac{z_{BL}}{4} = 0
\]

\[
(z_{SL} + 2z_{BL} + Z_{BN}^B) + \frac{1}{P_A} \frac{z_{BL}}{2} + \frac{1}{P_B}(-\frac{7}{4}z_{BL} - Z_{BN}^B) + \frac{1}{P_C} \frac{3z_{BL}}{4} = 0
\]

\[
(z_{SL} + 2z_{BL} + Z_{BN}^C) + \frac{1}{P_A} \frac{z_{BL}}{2} + \frac{1}{P_B} \frac{z_{BL}}{4} + \frac{1}{P_C}(-\frac{7}{4}z_{BL} - Z_{BN}^C) = 0
\]

(8.111)
In order to ensure that the crash equilibrium is viable but that the pooling equilibrium is not, we require the conditions 8.106 to hold in the crash equilibrium and

\[ P_{ABC}Z_{NS}^A > z_{BL} \]

and

\[ P_{ABC}Z_{NS}^A < 2z_{BL} \]

to hold in the pooling equilibrium.

**Numerical Example**

Consider the following numerical example.

\[ z_{SL} = 45, \ z_{BL} = 8, \]
\[ Z_{BN}^A = 2, \ Z_{BN}^B = 5, \ Z_{BN}^C = 12, \]
\[ Z_{SN}^A = 50, \ Z_{SN}^B = 110, \ Z_{SN}^C = 240. \]

The risk neutral investors' endowments can be created with two endowment patterns a and b:

\[ z_B^a = 1/2, \ z_B^b = 20, \]
\[ z_B^a = 1, \ z_B^b = 10. \]

The number of risk neutral investors with the two endowment patterns is

\[ n_A^a = 2, \ n_A^b = 1; \]
\[ n_B^a = 4, \ n_B^b = 3; \]
\[ n_C^a = 8, \ n_C^b = 8. \]

Consider the crash equilibrium first. Substituting into 8.110 and solving gives \( P_A = 1/6, \ P_B = 1/5, \ P_C = 1/4, \ P_{ABC} = 4/21, \ P_{BC} = 2/9. \) At time \( t = 1 \) the first group of log investors sell their 8 bonds for \( 8/P_{ABC} = 42 \) shares of the stock. Risk neutral investors
buy the 8 bonds and sell 42 shares of stock which they can do in all three economy states since their aggregate endowment of the stock $Z_{SN}$ is greater than 42 for $e = A, B, C$. All investors choose their asset holdings based on their correct beliefs that the $t = 2$ price will be either $P_{BC} = 2/9$ with probability $1/2$ or $P_{A} = 1/6$ with probability $1/2$. At $t = 2$ the Walrasian auctioneer tries the pooling price $P_{BC} = 2/9$ and the new log investors entering the market at $t = 2$ sell their 8 bonds and try to buy $8/P_{BC} = 36$ shares of stock. In economy state A the risk neutral investors would have to have been endowed with $42 + 36 = 78$ shares of stock to satisfy the new log investors’ demands but $Z_{SN}^A = 50$ so $P_{BC}$ (as well as $P_{ABC}$) is not an equilibrium. Instead, the equilibrium price is $P_{A} = 1/6$ and log investors new plus old buy 10 bonds and sell 60 shares to end up holding 18 bonds and 72 shares of stock. The risk neutral investors can only supply the 78 shares in economy states B and C where $Z_{SN}^B$ and $Z_{SN}^C$ are greater than 78. Hence, the beliefs are consistent with the prices. At $t = 3$ in economy state A the price stays the same and no trading occurs. At $t = 3$ in economy states B and C the price is, respectively, $P_{B} = 1/5$ and $P_{C} = 1/4$. In economy state B the log investors buy 21 bonds and sell 105 shares of stock, holding 21 bonds and 63 shares of stock. In economy state C the log investors buy 28 bonds and sell 112 shares of stock, holding 28 bonds and 56 shares of stock. At $t = 4$ the assets pay off and consumption occurs.

The prices in the pooling equilibrium can be obtained from 8.111. The result is $P_{ABC} = 0.203148$, $P_{A} = 0.175820$, $P_{B} = 0.1993$ and $P_{C} = 0.30333$. $P_{ABC}$ is not an equilibrium at $t = 2$ in economy state A since $2z_{BL}/P_{ABC} = 78.76035 < Z_{SN}^A = 50$ however it is an equilibrium at $t = 1$ since $z_{BL}/P_{ABC} = 39.38017 < Z_{SN}^A = 50$. Hence the fully pooling equilibrium is not viable with the endowments given.
8.2.2 Trigger Method Two: Information Arrival

We now consider a general multiperiod model with market risk that has a crash equilibrium in which intermediate prices can change from a pooling price to a much lower separating price with an increase in trading volume. In this model no new investors enter the market. Aggregate endowments, although unknown, remain constant. Trading in this model is generated by the arrival of information concerning a state variable whose value is related to both the payoff of the stock and the aggregate endowments of investors. Information is received by all investors simultaneously at various times. Before a new value of the state variable is received by investors, two rounds of trading occur, setting two prices for the stock. Between any two information events there are two rounds of trading. The first round of trading after the state variable information is received allows prices to adjust to this new information. The second round may be redundant but it is also possible for the price to move from a pooling price to a separating price when a trigger point is reached.

The model is similar to the previous model in other details. There are log investors and risk neutral investors trading in a stock and a bond. The bond always pays off 1 while the stock pays off \( M(s) \) or 0, each with probability 1/2, where \( s \) is the value of the state variable. The log investors are always able to hold their optimal portfolios. Risk neutral investors are always indifferent between holding stock and bonds except at the final trading period where they wish to hold an infinite amount of stock and short the bond (borrow). They are constrained, however, to hold only their wealth in stock (and not borrow).

The final date, \( t = T \), reveals the state of nature. At this point, as before, the financial assets pay off and consumption occurs. At the second last date, \( t = T - 1 \), which is the last trading date, investors trade to their final asset holdings. They do so without the
need to consider future market prices and trading opportunities. All relevant information about the state of the economy is revealed by prices and the value of the state variable if it has not been revealed previously. We assume as before that log investors are unable to hold only bonds at a final price that makes risk neutral investors indifferent between holding bonds and stock. Hence, the stock price, \( P(s) \), is less than \( M(s)/2 \). Risk neutral investors sell all their bonds and hold only stock, while log investors hold both stock and bonds. We assume that the final stock price, \( P(s) \), is such that at all previous dates where trading occurs the risk neutral investors are indifferent between holding stock and bonds and accommodate log investors who have a particular desired allocation of stock and bonds. It is the indifference of risk neutral investors that keeps economy state information, i.e., risk neutral investors’ aggregate endowments, from being revealed. When log investors demand more stock or bonds than risk neutral investors can supply then the risk neutral indifference price is no longer a viable equilibrium price. A trigger point is reached that leads to the revelation of economy state information. This trigger point is the point where the market crashes in our model.

The intertemporal investment problem must be solved for both log and risk neutral investors to solve the model. The results from previous chapters for log investors can be used because log investors always behave myopically, maximizing the expectation of the log of next period’s wealth. The form of the demand function for log investors is independent of future investment opportunities; it depends only on next period’s possible prices and the current price. In general, the problem for risk neutral investors is more complicated, even though they maximize expected wealth rather than a nonlinear function of wealth. By concentrating on equilibrium prices that make the risk neutral investors indifferent between stocks and bonds, however, the problem can be simplified.

With stock paying off either \( M(s) \) or 0, each with probability \( 1/2 \), at time \( T \) and the time \( T - 1 \) price \( P_e(s) \) in economy state \( e \), the expected utility of a risk neutral investor
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with wealth $W$ at the final trading date $t = T - 1$, assuming $P_e(s) < M(s)/2$, is

$$J(T - 1) = EU(T - 1) = \frac{W M(s)}{P_e(s)^2}. \quad (8.112)$$

This is because the wealth $W$ is used to buy $W/P_e(s)$ shares of stock with the expected payoff of $M(s)/2$. The price $P_e(s)$ depends on the economy state $e(s)$ and the final payoff $M(s)$. The economy state $e(s)$ and the stock payoff $M(s)$ both depend upon the final value of the state variable, so that the price also depends upon the state variable. We assume that the particular economy state $e(s)$ can take on different values which depend upon $s$, i.e., $e(s)$ can equal $A(s)$, $B(s)$ or $C(s)$ etc. The economy state is fixed before trading begins and is partially revealed when $s$ becomes known but is completely revealed only when $e$ is revealed by a market price that is a separating price.

In general, the risk neutral indifference price, the price at which risk neutral investors are indifferent between holding stocks and bonds is given by

$$P(t) = \frac{E_t[M(s)/2]}{E_t[M(s)/(2P_e(s))]} = \frac{E_t[P_e(s)M(s)/(2P_e(s))]}{E_t[M(s)/(2P_e(s))]} \quad (8.113)$$

The expected utility of risk neutral investors with wealth $W(t)$ at time $t$ is

$$J(t) = W(t)E_t[M(s)/(2P_e(s))] \quad (8.114)$$

These results are proved by induction starting from the second last date, $t = T - 1$, which is the last trading date, when $J(T - 1) = W(T - 1)M(s)/(2P_e(s))$ and $s$ is known. The only unknown is whether the stock will pay $M(s)$ or 0. The stock price is the separating price $P_e(s)$. At the second last trading date, $t = T - 2$, risk neutral investors solve their maximization problem which is to choose the $t = T - 2$ holding of bonds and stock to maximize the expectation of their $t = T - 1$ expected utility, $E_{T-2}J(T - 1)$. Using equation 8.112 and the $t = T - 2$ budget constraint allows the problem to be written as

$$\max_x E_{T-2}[(x + P_e(s)(W(T - 2) - x)/P_{T-2})M(s)/(2P_e(s))] \quad (8.115)$$
where \( x \) is the holding of bonds. Setting the derivative with respect to \( x \) equal to zero gives the first order condition

\[
E_{T-2}[M(s)/(2P_e(s))] = E_{T-2}[M(s)/2]/P(T-2)
\]

which implies that the price which makes risk neutral investors indifferent between stocks and bonds at \( t = T - 2 \) is:

\[
P(T - 2) = \frac{E_{T-2}[M(s)]}{E_{T-2}[M(s)/P_e(s)]}.
\]  

(8.116)

At a lower price risk neutral investors hold only stock, at a greater price they hold only bonds. Substituting for \( P(T - 2) \) in 8.115 gives an expression for the expected utility at \( t = T - 2 \):

\[
J(T - 2) = W(T - 2)E_{T-2}[M(s)/(2P_e(s))].
\]

Hence the results 8.113 and 8.114 hold at \( t = T - 2 \).

To complete the proof, we assume the results hold at \( t = T - k \) and show that they hold at \( t = T - (k + 1) \). Thus, by induction they are true for all \( t \). With the assumption that the results hold for \( t \geq T - k \), the risk neutral investors' investment problem at \( t = T - (k + 1) \) is

\[
\max_x E_{T-(k+1)}J(T - k) = \max_x E_{T-(k+1)}[(x + P(T - k)(W(T-(k+1))-x))E_{T-k}[M(s)/(2P_e(s))]]
\]  

(8.117)

where \( x \) is the holding of bonds. \( P_{T-k} \) is the stock price at the next trading date, \( t = T - k \). By assumption it makes risk neutral investors indifferent between holding stock and bonds. In general, \( P(T - k) \) will depend upon the current value of the state variable \( s \) because by equation 8.113 it depends upon the expectations of \( M(s) \) and \( M(s)/P_e(s) \) and these depend upon the current value of the state variable.
The first order condition is again obtained by taking the derivative with respect to $x$ and setting it to zero:

$$
E_{T-(k+1)}[M(s)/P_e(s)] = E_{T-(k+1)}[P(T-k)E_{T-k}[M(s)/P_e(s)]]/P(T-(k+1))
$$

$$
= E_{T-(k+1)}[M(s)]/P(T-(k+1))
$$

(8.118)

where we have used the law of iterated expectations and made use of the induction assumption that the result in equation 8.113 holds for $t = T-k$ to substitute for $P(T-k)$.

Hence, the $t = T - (k + 1)$ risk neutral indifference price is

$$
P(T-(k+1)) = \frac{E_{T-(k+1)}[M(s)]}{E_{T-(k+1)}[M(s)/P_e(s)]}
$$

(8.119)

Substituting for $P(T-(k+1))$ into 8.117 gives

$$
J(T-(k+1)) = W(T-(k+1))E_{T-(k+1)}[M(s)/(2P_e(s))]
$$

and this completes the proof.

We have found expressions for prices and the expected utility of risk neutral investors that make them indifferent between bonds and stock, under the assumption that all future prices make them indifferent except at the last trading date, when they hold only stock. These risk neutral indifference prices are equilibrium prices if the endowment of risk neutral investors is sufficient to supply log investors with their optimal holdings. Log investors, because they exhibit myopia, act as if they are solving a two date investment problem to maximize the expectation of the log of next period’s wealth. The previous chapter’s results, equations 7.94 and 7.96, hold, so that given the current price $P(t,s(t))$ and, for example, a probability of 1/2 that the next price will be $P(t+1,s_1(t+1))$ and a probability of 1/2 that it will be $P(t+1,s_2(t+1))$, log investors wish to hold the amount

$$
x_{BL}(t) = \frac{1}{2} \left( \frac{P(t+1,s_1)}{P(t+1,s_1) - P(t,s)} + \frac{P(t+1,s_2)}{P(t+1,s_2) - P(t,s)} \right) W(t)
$$
in bonds and

\[ x_{SL}(t) = \frac{W(t) - x_{BL}(t)}{P(t, s)} \]

in stock, where \( W(t) = x_{BL}(t - 1) + x_{SL}(t - 1)P(t, s) \) and we have made explicit the state variable dependence of the prices. As long as risk neutral investors can supply the necessary stock and bonds so that log investors can attain their optimal holdings then the risk neutral indifference price is an equilibrium. When the risk neutral indifference price is an equilibrium price for a number of economy states then it is a pooling price that does not reveal which of these economy states is the prevailing one. In an economy state where log investors cannot attain their optimal holdings at a pooling price we assume that the equilibrium price is a separating price and reveals the economy state \( e \). This separating price \( P_e(t, s) \) for \( t < T - 1 \) is also a risk neutral indifference price so that it is given by the equation 8.113 for \( P(t) \) and at this price log investors are able to hold their optimal holdings.

The final price at \( T - 1 \) is always a separating price and, unlike the separating prices for \( t < T - 1 \), risk neutral investors are not indifferent, they wish to hold only stock. Log investors hold their optimal holdings at \( t = T - 1 \), as they do throughout, but the final price must be such that they are willing to hold all the economy's bonds. There are two trading dates, at \( t = T - 1 \) and \( t = T - 2 \), after the last arrival of information about the state variable. The equilibrium at these dates is exactly the same as in previous chapters. At \( t = T - 2 \), the stock's potential payoff is known to be \( M(s(T - 2)) \) and if the economy state has not been revealed, a pooling price will be an equilibrium with log investors holding only stock. At \( t = T - 1 \), a separating price reveals the economy state. Since log investors must hold all the bonds in the economy which we denote by \( Z^e_B(s) = z_{BL} + Z^e_{BN} \), we have

\[ Z^e_B(s) = z_{BL} + Z^e_{BN} = x_{BL}(T - 1) = \frac{M(s)}{2(M(s) - P_e(s))} W_L(T - 1) \]  
(8.120)
Since log investors hold only the amount $x_{SL}(T - 2)$ in stock after trading at $T - 2$,
$W_L(T - 1) = P_e(s)x_{SL}(T - 2)$. Substituting into 8.120 and rearranging gives

$$P_e(s) = \frac{M(s)}{1 + \frac{x_{SL}(T-2)M(s)}{2Z_p(s)}} \quad (8.121)$$

In the next section we apply this general modelling approach to a particular case with a small number of time periods and economy states so that the solution is tractable. The method can be applied to models with a large number of time periods but numerical methods must be used to solve the equations. The methods can be extended to continuous time models; however, the continuous time methods of the other essay in this thesis must also be used.

**Special Case of the Model**

Consider an economy with five dates. At the final date, no trading occurs, all financial assets pay off, the values of all state variables are revealed and consumption occurs. At the other four dates, trading can occur in a financial market. The economy has eight possible economy states and two possible states of nature. The economy states are not independent of the states of nature, however. Uncertainty about the states of nature is represented by a state variable $s$ which takes on one of two values, $s_1$ and $s_2$. The economy states that represent the possible types of economy or financial market, have eight possibilities. There are at any given time in the model only four possible types of financial markets but the financial market itself depends on whether the state variable is $s_1$ or $s_2$: there are four possible underlying financial markets present if $s_1$ occurs and a different four if $s_2$ occurs, giving eight in total. (See figure 8.5).

As usual, there are two types of investors, those with log utility and those who are risk neutral. Economy states represent the uncertainty as to the true nature of the underlying financial market, due to an unknown number of risk neutral investors of
Chapter 8. A Model of Market Crashes

Figure 8.5: Stock Price Paths in the Crash Equilibrium
different endowment patterns. This unknown number is assumed to be constant over
time for simplicity. As in the previous chapter, individual investors are initially unable
to determine the economy state and have homogeneous posterior beliefs as to the actual
economy state. The economy state depends upon the numbers of risk neutral investors
having one of two different possible initial endowments of the stock and bond.

Investors trade in two securities: a bond, which always pays off one unit of the
consumption good and has a price of unity, and a stock. The stock has a price of \( P(t,s) \)
and pays off an amount \( M(s) \) or 0, each with probability 1/2, when \( s = s_1 \), and \( M(s_2) \) or
0 when \( s = s_2 \), each with probability 1/2. Consumption and the payoffs of the financial
assets occur at the final date of the model. We denote the price of the stock at the
final trading date (i.e. \( t = 4 \), the second last date) in the 8 possible market states as
\( P_A(s_1), P_B(s_1), P_C(s_1), P_D(s_1) \) and \( P_A(s_2), P_B(s_2), P_C(s_2), P_D(s_2) \). Assume, without
loss of generality, that \( M(s_2) > M(s_1) \) so that \( s_2 \) represents good news and \( s_1 \) bad news.
Assume that the aggregate endowment of bonds of risk neutral investors in any economy
state is greater when \( s = s_2 \) than when \( s = s_1 \). One can think of the good economic
news \( s_2 \) as being associated with a larger economy because of the greater number of
risk neutral investors, as well as a higher payoff from the stock in the final period.

We assume that \( M(s_1)/P_e(s_1) > M(s_2)/P_e(s_2) \), for \( e = A, B, C, D \), although \( M(s_1) < 
M(s_2) \) and \( P_e(s_1) < P_e(s_2) \). Without an increase when \( s = s_2 \) in the aggregate endow-
ment of bonds of the log investors, \( P_e(s_2) \) would increase by a smaller fraction from
\( P_e(s_1) \), than \( M(s_2) \) does from \( M(s_1) \) because log investors are risk averse. Investors are
assumed to have rational expectations and identical beliefs. The probability of \( s_1 \) or \( s_2 
occurring is, for simplicity, 1/2. Similarly, the probability of the stock paying off \( M(s) \) or
0 is also 1/2. The probability that the economy state is of type A, B, C, or D is each
1/4.

As in the previous sections, this model has multiple equilibria, presumably set by a
market maker. The previous chapter's three date model had just two possible equilibria, pooling and separating, with the pooling equilibrium coming in two forms depending upon initial endowments. In this model, there are many more possibilities because of the greater number of economy states and states of nature and because the states of nature are revealed at two different times, with trading occurring before each stage of revelation of the state of nature. The different possible equilibria can be distinguished by the process by which the economy state A, B, C, or D is revealed. When this occurs immediately in the first round of trading we have a separating equilibrium, and if revelation does not occur until the last round of trading we have a fully pooling equilibria. Other possible equilibria involve partial revelation of the economy state. For example, the price at the first round of trading might reveal that the economy state is either A or B.

Depending on initial aggregate endowments, some of the possible revelation patterns may not necessarily be compatible with equilibrium prices. We assume that the market maker who sets the equilibrium price and chooses the equilibrium from the set of possible ones attempts to minimize price changes by choosing the equilibrium with the maximum amount of pooling. Given the aggregate endowments of investors, however, a fully pooling equilibrium may not exist in intermediate rounds of trading and the economy states may have to be partially revealed by the price.

We will concentrate on the fully pooling equilibrium and a crash equilibrium. The crash equilibrium in the first round of trading at $t = 1$ is fully pooling, so the equilibrium price is the same in economy states A, B, C, and D. In the trading at $t = 2$, before the state variable is revealed at $t = 3$ to be either $s_1$ or $s_2$, we assume that aggregate endowments are such that a fully pooling price $P_{ABCD}$ is not an equilibrium for the type A economy, and that only a separating price, $P_A$, that reveals that the economy state is A is an equilibrium. There is still a pooling price, $P_{BCD}$, however, for the B, C and D economy states. At $t = 3$, when the state variable is revealed to be either
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$s_1$ or $s_2$, the price takes on one of two new values. For example, $P_{BCD}$ becomes either $P_{BCD}(s_1)$ or $P_{BCD}(s_2)$, and similarly for $P_A$. In the final trading round at $t = 4$ a pooling price for the B, C, and D economy states is not an equilibrium and the final trading prices must be separating prices $P_B(s_1)$, $P_C(s_1)$, $P_D(s_1)$ or $P_B(s_2)$, $P_C(s_2)$, $P_D(s_2)$ which reveal the true economy state to investors. With $P_A(s_1) < P_B(s_1) < P_C(s_1) < P_D(s_1)$, $P_A(s_2) < P_B(s_2) < P_C(s_2) < P_D(s_2)$, $P_B < P_{BCD} < P_D$, and $P_A < P_{ABCD} < P_D$, the stock price starts off at $P_{ABCD}$ at $t = 1$ and then rises a small amount to $P_{BCD}$ at $t = 2$ if the economy state is B, C or D, or has a large drop to $P_A$ if the economy state is A, before the state variable is revealed as $s_1$ or $s_2$ at $t = 3$. In the final round of trading at $t = 4$, if the price was $P_{BCD}(s)$ at $t = 3$ it changes to $P_B(s)$, $P_C(s)$ or $P_D(s)$. If it was $P_A$ at $t = 2$ it will have moved to $P_A(s)$ at $t = 3$, depending on whether $s = s_1$ or $s_2$, and remains unchanged at $t = 4$. When the initially unknown economy state is A, investors see the price drop from $P_{ABCD}$ to $P_A$ accompanied by a large volume of trading and a change in beliefs about the underlying economy state. Since this seems to match the intuition behind a market crash, we denote this equilibrium the crash equilibrium.

Using the results of the previous sections, we derive a set of linear equations in the reciprocals of the separating prices that can be solved to determine equilibrium prices. Consider first the fully pooling equilibrium. In this equilibrium, the stock price is $P_{ABCD}$ at both $t = 1$ and $t = 2$, risk neutral investors are indifferent between bonds and stock and log investors hold $x_{BL}(t = 1) = x_{BL}(t = 2)$ in the bond and $x_{SL}(t = 1) = x_{SL}(t = 2)$ in the stock. At $t = 3$ and $t = 4$, the model is the same as the simple three date model in the previous chapter since the potential stock payoff $M(s)$ is known. At $t = 3$, the price is $P_{ABCD}(s_1)$ or $P_{ABCD}(s_2)$, each with probability $1/2$. In either case, risk neutral investors are indifferent and log investors hold only stock. Finally, at $t = 4$, if $s = s_1$ the price is $P_A(s_1)$, $P_B(s_1)$, $P_C(s_1)$, or $P_D(s_1)$, each with probability $1/4$ and similarly for $s = s_2$. Risk neutral investors hold only stock and log investors hold all the bonds and
some stock because \( P_e(s) < M(s)/2 \).

From the previous section, we can write the market prices in the fully pooling equilibrium. Using equation 8.113 we have:

\[
\frac{1}{P_{ABCD}} = \frac{M(s_1)/(M(s_1) + M(s_2))}{P_{ABCD}(s_1)} + \frac{M(s_2)/(M(s_1) + M(s_2))}{P_{ABCD}(s_2)}
\]

(8.122)

and

\[
\frac{1}{P_{ABCD}(s)} = \frac{1}{4} \left[ \frac{1}{P_A(s)} + \frac{1}{P_B(s)} + \frac{1}{P_C(s)} + \frac{1}{P_D(s)} \right].
\]

(8.123)

The separating prices \( P_e(s) \) are determined as before by the condition that log investors hold the economy’s aggregate endowment of bonds which implies

\[
Z^*_B(s) = z_{BL} + Z^*_BN(s) = \frac{1}{2} \frac{M(s)}{M(s) - P_e(s)} W_L(t = 4)
\]

Since log investors put all of their \( t = 2 \) holdings into the stock at \( t = 3 \), their \( t = 4 \) wealth is given by

\[
W_L(t = 4) = \left[ x_{BL}(t = 2)/P_{ABCD}(s) + x_{SL}(t = 2) \right] P_e(s)
\]

which implies

\[
Z^*_B(s) = z_{BL} + Z^*_BN(s) = \frac{1}{2} \frac{M(s)}{M(s) - P_e(s)} \left[ x_{BL}(t = 2)/P_{ABCD}(s) + x_{SL}(t = 2) \right] P_e(s)
\]

(8.124)

Given \( z_{BL} \) and \( z_{SL} \), the initial endowments of the log investors, their \( t = 2 \) holdings can be determined. Log investors choose their holdings at the price \( P_{ABCD} \) based on next period’s price at \( t = 3 \) being either \( P_{ABCD}(s_1) \) or \( P_{ABCD}(s_2) \) with a probability of 1/2 each. From the results in the previous chapter concerning the log investors optimal investment decision, equations 7.94 and 7.96, we have

\[
x_{BL}(t = 2) = \frac{1}{2} \left[ \frac{P_{ABCD}(s_1)}{P_{ABCD}(s_1) - P_{ABCD}} + \frac{P_{ABCD}(s_2)}{P_{ABCD}(s_2) - P_{ABCD}} \right] (z_{BL} + P_{ABCD}z_{SL})
\]
and
\[ x_{SL}(t = 2) = \frac{(z_{BL} + P_{ABCD}z_{SL}) - x_{BL}}{P_{ABCD}}. \]

Substituting for \( P_{ABCD} \) in terms of \( P_{ABCD}(s_1) \), \( P_{ABCD}(s_2) \), \( M(s_1) \) and \( M(s_2) \) and simplifying allows us to write:

\[ x_{BL}(t = 2)/P_{ABCD}(s) + x_{SL}(t = 2) = \frac{1}{2P_{ABCD}} \frac{M(s_1) + M(s_2)}{M(s)}[z_{BL} + P_{ABCD}z_{SL}]. \]

Substituting into 8.124, simplifying and substituting for \( P_{ABCD} \) using \( P_e(s) \) for \( e = A, B, C, D \) and \( s = s_1, s_2 \) gives

\[
Z_B^e(s)2(M(s) \frac{P_e(s)}{P_e(s)} - 1) = (1/2)(M(s_1) + M(s_2))[z_{BL}/P_{ABCD} + z_{SL}]
\]

\[ = (1/2)[(M(s_1) + M(s_2))z_{SL}
\]

\[ + z_{BL} \frac{M(s_1)}{4}(1/P_A(s_1) + 1/P_B(s_1) + 1/P_C(s_1) + 1/P_D(s_1))
\]

\[ + \frac{M(s_2)}{4}(1/P_A(s_2) + 1/P_B(s_2) + 1/P_C(s_2) + 1/P_D(s_2))] \tag{8.125}\]

for \( e = A, B, C, D \) and \( s = s_1, s_2 \). This gives a set of linear equations in the reciprocals of the prices, which can be solved to give the prices as a function of the initial aggregate endowments \( Z_B^e(s) = z_{BL} + Z_{BN}, z_{BL} \) and \( z_{SL} \). The fully pooling equilibrium is not viable if \( x_{BL}(t = 2) > Z_B^e(s) \) or \( x_{SL}(t = 2) > Z_{SL}^e(s) \) where \( Z_B^e(s) \) is the economy’s aggregate endowment of the stock. Then \( P_{ABCD} \) is not an equilibrium at \( t = 2 \) in all economy states and, hence, the beliefs that economy states A, B, C, and D are equally likely are inconsistent. We assume that \( x_{BL}(t = 2) > Z_B^A(s) \) for \( s = s_1, s_2 \). Hence, in economy state A, log investors want to hold more bonds than risk neutral investors are able to supply and the fully pooling price is not an equilibrium at \( t = 2 \). Instead, the price is either \( P_A \) in economy state A or \( P_{BCD} \) in economy state B, C, and D and we have the crash equilibrium.

The equilibrium conditions 8.125 in the crash equilibrium turn out to be exactly the
same as in the fully pooling equilibrium. The price $P_A(t = 2)$ is given by

$$\frac{1}{P_A(t = 2)} = \frac{(M(s_1)/(M(s_1) + M(s_2)))}{P_A(t = 3, s_1)} + \frac{(M(s_2)/(M(s_1) + M(s_2)))}{P_A(t = 3, s_2)}.$$ 

Similarly,

$$\frac{1}{P_{BCD}(t = 2)} = \frac{(M(s_1)/(M(s_1) + M(s_2)))}{P_{BCD}(t = 3, s_1)} + \frac{(M(s_2)/(M(s_1) + M(s_2)))}{P_{BCD}(t = 3, s_2)},$$

and

$$\frac{1}{P_{ABCD}} = \frac{1}{4} \left( \frac{1}{P_A} \right) + \frac{3}{4} \left( \frac{1}{P_{BCD}} \right).$$

The equilibrium conditions are derived in exactly the same way as for the pooling equilibrium. As before, log investors hold all the bonds at $t = 4$ so that

$$Z_B^e(s) = z_{BL} + Z_{BN}^e(s) = \frac{1}{2 M(s)} \left[ x_{BL}(t = 2)/P_{BCD}(s) + x_{SL}(t = 2) \right] P_e(s)$$

in economy states B, C and D. In economy state A, we have

$$Z_B^A(s) = z_{BL} + Z_{BN}^A(s) = \frac{1}{2 M(s)} \left[ x_{BL}(t = 2)/P_A(s) + x_{SL}(t = 2) \right] P_A(s)$$

The $t = 3$ price in economy state A is the same as the $t = 4$ price and log investors also hold all the bonds after trading at $t = 3$.

At $t = 2$, the log investor holds an amount in bonds given by

$$x_{BL}(t = 2) = \frac{1}{2} \left( \frac{P_A(t = 3, s_1)}{P_A(t = 3, s_1) - P_A(t = 2)} + \frac{P_A(t = 3, s_2)}{P_A(t = 3, s_2) - P_A(t = 2)} \right) W(t = 2)$$

in economy state A and

$$x_{BL}(t = 2) = \frac{1}{2} \left( \frac{P_{BCD}(t=3,s_1)}{P_{BCD}(t=3,s_1) - P_{BCD}(t=2)} + \frac{P_{BCD}(t=3,s_2)}{P_{BCD}(t=3,s_2) - P_{BCD}(t=2)} \right) W(t = 2)$$
when the economy state is B, C or D. Log investors hold only stock after trading at \( t = 1 \) since \( 1/P_{ABCD} = 1/(4P_A) + 3/(4P_{BCD}) \) and from equation 7.96

\[
x_{BL}(t = 1) = \left( \frac{(1/4)P_{BCD}(t=2)}{P_{BCD}(t=2)-P_{ABCD}(t=1)} + \frac{(3/4)P_A(t=2)}{P_A(t=2)-P_{ABCD}(t=1)} \right) (z_{BL} + z_{SL}P_{ABCD}) = 0 \quad (8.129)
\]

Hence, \( W(t = 2) = |z_{BL}/P_{ABCD} + z_{SL}|P_A \) in economy state A and \( W(t = 2) = |z_{BL}/P_{ABCD} + z_{SL}|P_{BCD} \) when investors know only that the state is B, C or D. Substituting for \( P_{BCD} \) and \( P_A \) and simplifying again allows us to write

\[
x_{BCD}^{BL}(t = 2)/P_{BCD}(s) + x_{SL}(t = 2) = \frac{1}{2} \frac{M(s_1) + M(s_2)}{M(s)} [z_{BL}/P_{ABCD} + z_{SL}]
\]

and

\[
x_{BL}^A(t = 2)/P_A(s) + x_{SL}(t = 2) = \frac{1}{2} \frac{M(s_1) + M(s_2)}{M(s)} [z_{BL}/P_{ABCD} + z_{SL}].
\]

Substituting into 8.126 and 8.127 and simplifying gives the set of linear equations in the reciprocals of the separating prices the same as 8.125. Hence, the separating prices are the same in the crash equilibrium as in the fully pooling one. For the crash equilibrium to be viable it must be the case that risk neutral investors have enough stock to supply log investors in those cases in which the latter hold all their wealth in stock; this is at \( t = 1 \) in all economy states, and at \( t = 3 \) in economy states B, C and D. At \( t = 2 \), the aggregate endowment of bonds in the economy must be small enough in economy state A so that \( x_{BL}^{BCD}(t = 2) \) is too large to be supplied. The aggregate bond endowment in economy states B, C and D, on the other hand, must be large enough to supply log investors' demands \( x_{BL}^{BCD}(t = 2) \). We also assume that the fully pooling equilibrium is not viable which means that aggregate endowments are such that log investors also cannot receive their optimal holdings in economy state A in the fully pooling equilibrium as discussed previously.
Numerical Example

Consider the following numerical example.

\[ M(s_1) = 10, \quad M(s_2) = 12, \]
\[ Z_{BN}(s_1) = 1.2142, \quad Z_{BN}(s_1) = 6.625, \quad Z_{BN}(s_1) = 10.8333, \quad Z_{BN}(s_1) = 14.2, \]
\[ Z_{BN}(s_2) = 3.1818, \quad Z_{BN}(s_2) = 14.2, \quad Z_{BN}(s_2) = 19.25, \quad Z_{BN}(s_2) = 27.6667, \]
\[ z_{BL} = 6.0, \quad z_{SL} = 14.0. \]

We assume that the risk neutral investor's endowment of stock is large enough to supply any amount that log investors might demand. As long as it is large enough, it does not influence prices. The prices that solve equation 8.125 are \( P_A(s_1) = 2/3, \quad P_B(s_1) = 10/9, \quad P_C(s_1) = 10/7, \quad P_D(s_1) = 5/3, \quad P_A(s_2) = 1, \quad P_B(s_2) = 2, \quad P_C(s_2) = 2.4, \quad P_D(s_2) = 3. \)

In the fully pooling equilibrium, we have \( P_{ABCD}(s_1) = 1.0811, \) and \( P_{ABCD}(s_2) = 1.7778 \) at \( t = 3, \) and \( P_{ABCD} = 1.375 \) at \( t = 1 \) and \( t = 2. \) At \( t = 1, \) log investors trade from \( z_{BL} = 6 \) and \( z_{SL} = 14 \) to hold \( x_{BL} = 9.2874 \) and \( x_{SL} = 11.6092. \) Since the aggregate endowment of bonds is \( Z_A(s_1) = 7.2142 \) and \( Z_A(s_2) = 9.1818 \) in economy state \( A, \) log investors can not trade to their desired holdings of the bonds and the fully pooling equilibrium is not viable.

Consider the crash equilibrium with \( P_{ABCD} = 1.375 \) at \( t = 1, \) and \( P_A = 0.8148 \) with a \( 1/4 \) probability and \( P_{BCD} = 1.7838 \) with a \( 3/4 \) probability at \( t = 2. \) This represents the market crash. At \( t = 3 \) in economy state \( A, \) investors trade to their final asset allocation with log investors holding all the bonds and \( P_A(s_1) = 2/3 \) and \( P_A(s_2) = 1. \) At \( t = 3 \) in economy states \( B, C, \) and \( D, \) log investors hold only stock and the stock price is \( P_{BCD}(s_1) = 1.3636 \) and \( P_{BCD}(s_2) = 2.4. \) Finally, at \( t = 4 \) in all economy states, log investors hold all the economy's bonds and some stock, risk neutral investors hold only stock and the price \( P_e(s) \) reveals the economy state.
The crash equilibrium is viable because at $t = 1$ log investors hold only stock which risk neutral investors can supply. At $t = 2$ in economy states B,C,D, with the pooling price $P_{BCD}$ log investors want to hold $x_{BL}(t = 2) = 10.6316$, which is compatible with economy states B, C, and D because the aggregate bond endowment is greater than 12 for either value of the state variable. It is incompatible with economy state A because the aggregate bond endowment is less than 10 for either $s = s_1$ or $s = s_2$ in economy state A. In economy state A at $t = 2$, risk neutral investors are indifferent and log investors want to hold $x_{BL}(t = 2) = 6.7333$ in bonds. This is compatible with economy state A since aggregate bond endowments are greater than 7 for either $s = s_1$ or $s = s_2$. 
Bibliography


Bibliography


Appendix A

More Than One Financial Asset and State Variable

In this appendix we consider an economy with many financial assets and many state variables but with the other assumptions the same as before. Assume that the economy consists of a fixed number of homogeneous investors; hence there is no trading. Assume perfect capital markets without taxes, transaction costs, etc. Uncertainty is represented by an N dimensional vector of state variables $\vec{X}$ which follows a diffusion process given by the following stochastic differential equation

$$d\vec{X}(t) = \mu(\vec{X},t)dt + \Sigma(\vec{X},t)d\vec{z}(t)$$

where $\mu(\vec{X},t) = (\mu_1, \ldots, \mu_N)$ is an N- dimensional vector, $\Sigma$ is an $N \times N$ dimensional matrix with the positive definite matrix $\Sigma'\Sigma$ giving the covariance per unit time of the stochastic process $\vec{X}$. $d\vec{z}(t)$ is an N dimensional Wiener process with

$$E(d\vec{z}(t)'d\vec{z}(t)) = I_{N \times N} dt \quad E(d\vec{z}(t)) = 0$$

There is a single consumption good denoted by K which can be either consumed or invested risklessly at the rate $r$. This consumption good is assumed always to have a price of one. There are a number of financial assets which represent claims on actual physical assets. The physical assets, and hence the corresponding financial assets, are in fixed net supply. At some time $T$ the physical assets are liquidated into the consumption good in an amount depending on the vector of state variables, $\vec{X}(T)$, and the per capita consumption good holdings $K_m(T)$ just before the asset is liquidated. This fixes the prices of the financial assets at time $T$ as functions of $\vec{X}(T)$ and $K_m(T)$. We denote
the liquidating dividend for the \( i \)'th physical asset by \( f_i(\bar{X}(T), K_m(T)) \). The time \( T \) per capita holdings of the consumption good, \( K_m(T) \), does not include the liquidating dividend \( f_i \). The physical assets are also assumed to pay a continuous dividend in the consumption good which depends on time \( t \), the state variables \( \bar{X}(t) \) and the per capita holdings of the consumption good \( K_m(t) \). The dividend from the \( i \)'th physical asset is denoted by \( D^i, \bar{X}(t), K_m(t) \). 

Let the \( i \)th financial asset have a price function \( P_i(t, \bar{X}(t), K_m(t)) \). The price \( P \) is relative to the unit price of the consumption good. Assume that each investor has the endowment \( n_{im} \) of the \( i \)th financial asset. There is no trading because the investors are homogeneous. Investors hold their \( n_{im} \) shares of the \( i \)'th financial asset (assume \( n_{im} \) is a continuous variable) receiving the dividend \( n_{im}D(t, \bar{X}(t), K_m(t)) \) until time \( T \) when they receive the payout \( n_{im}P_i(T, \bar{X}(T), K_m(T)) = n_{im}f_i(\bar{X}(T), K_m(T)) \). Since only the total payout \( nD \) and \( nf \) is of importance, for \( n \neq 0 \) we could set \( n = 1 \) without loss of generality but we omit this so as to make the derivation clearer.

Investors are assumed to have rational (self consistent) expectations and to solve the same model of the economy as we are specifying to determine the price functions \( P_i(t, \bar{X}, K_m) \) which they use to determine expected rates of return and the risk of the financial assets. Based on the expected rates of return and risk, they determine their demands for the assets, which must equal the supply at the equilibrium price \( P_i(t, \bar{X}, K_m) \).

Investors, as in previous chapters, attempt to maximize their expected utility of consumption, which has the functional form

\[
E_t \int_t^T U(C(s), s) ds + B(W(T), T) 
\]

Consider the investor's problem: given wealth \( W \) how best to allocate that wealth into the various financial assets. Assume that the investor puts the fraction \( a_i \) of his or her wealth into financial asset \( i \) with the price \( P_i \). Then \( a_iW = n_{im}P_i(t, \bar{X}, K_m) \). We assume
that there are \( L \) financial assets. The fraction \( 1 - \sum_{i=1}^{L} a_i \) of wealth goes into the riskless investment of the consumption good so that \( W(1 - \sum_{i=1}^{L} a_i) = K \).

\[
W = \sum_{i=1}^{L} n_i P_i(t, \bar{X}, K_m) + K
\]

\[
dW = \left\{ \left( \sum_{i=1}^{L} a_i \frac{(dP_i + D_i dt)}{P_i} \right) + (1 - \sum_{i=1}^{L} a_i) r dt \right\} W - C dt
\]

but

\[
dP_i = \frac{\partial P_i}{\partial t} dt + \sum_{j=1}^{N} \frac{\partial P_i}{\partial X_j} dX_j
\]

\[
+ \sum_{j=1}^{N} \sum_{k=1}^{N} \frac{1}{2} \frac{\partial^2 P_i}{\partial X_j \partial X_k} E(dX_j dX_k) + \frac{\partial P_i}{\partial K} dK_m
\]

\[(A.130)\]

\[
dK_m = (rK_m + D_m - C_m) dt = (r(W - \sum_{i=1}^{L} n_i P_i) + D_m - C_m) dt
\]

\[
dK_m = (rW(1 - \sum_{i=1}^{L} a_i) + D_m - C_m) dt
\]

where \( D_m = \sum_{i=1}^{L} n_i m_i D_i \) is the total per capita dividend from the financial assets. Hence

\[
dW = W \sum_{i=1}^{L} \frac{\partial}{\partial t} \{ P_i dt + D_i dt + \sum_{j=1}^{N} P_{iX_j} dX_j + \sum_{j=1}^{N} \sum_{k=1}^{N} \frac{1}{2} P_{iX_jX_k} E(dX_j dX_k)
\]

\[
+ P_i K (rK_m + D_m - C_m) dt \} + W(1 - \sum_{i=1}^{L} a_i) r dt - C dt
\]

\[(A.131)\]

\[
dW = W \{ \sum_{i=1}^{L} \frac{\partial}{\partial t} [(P_i dt + \sum_{j=1}^{N} P_{iX_j} dX_j + \sum_{j=1}^{N} \sum_{k=1}^{N} \frac{1}{2} P_{iX_jX_k} (\Sigma' \Sigma)_{jk}
\]

\[
+ P_i K (rK_m + D_m - C_m) + D_i - r P_i dt
\]

\[
+ \sum_{j=1}^{N} P_{iX_j} (\Sigma' \tilde{Z})_j (r - C/W) dt \}
\]

\[(A.132)\]

Let

\[
J(W, \bar{X}, K_m, t) = \max_{C, a} \int_t^T U(C(s), s) ds + B(W(T), T)
\]
the indirect utility function, be the solution to the investor's problem of maximizing expected utility. Using the methods of dynamic programming and stochastic control theory we can determine the following partial differential equation that \( J \) must satisfy:

\[
\max_{C,t} L(t)J + U(C(t), t) + J_t = 0
\]

where \( L(t)J \) is the differential generator of \( J \). When written out in full the equation is

\[
0 = \max_{C,a_i} \{U(C, t) + J_W(E[\frac{dW}{dt}]) + \frac{1}{2} J_{WW}(E[\frac{(dW)^2}{dt^2}]) + J_t + \sum_{j=1}^{N} J_{X_j}(E[\frac{dX_j}{dt}]) \\
+ \sum_{j=1}^{N} \sum_{k=1}^{N} \frac{1}{2} J_{X_j X_k}(E[\frac{dX_j dX_k}{dt}]) \\
+ \sum_{j=1}^{N} J_{WX_j}(E[\frac{dW dX_j}{dt}]) + J_{Km}(K_m r + D_m - C_m)\}
\] (A.133)

with the boundary condition \( J(W, \tilde{X}, K_m, T) = B(W(T), T) \). Substituting for \( dW \) and \( d\tilde{X} \) gives

\[
0 = \max_{C,a_i} \{U(C, t) + J_t + \sum_{j=1}^{N} \mu_j J_{X_j} \\
+ \sum_{j=1}^{N} \sum_{k=1}^{N} \frac{1}{2} J_{X_j X_k}[\Sigma' \Sigma]_{jk} + J_{Km}(K_m r + D_m - C_m) \\
+ J_{WW} \{\sum_{h=1}^{L} \frac{a_h}{P_h} [P_{ht} + \sum_{j=1}^{N} P_{hX_j} \mu_j] \\
+ \sum_{j=1}^{N} \sum_{k=1}^{N} \frac{1}{2} P_{hX_j} X_k [\Sigma' \Sigma]_{jk} + P_{hK}(r K_m + D_m - C_m) + D_h - r P_h + r - C/W \}
\] (A.134)

Differentiating with respect to \( C \) and \( a_i \) gives the first order conditions which determine \( C^* \) and the \( a^*_i \), the optimal consumption and investments. These first order conditions are

\[
U_C = J_W
\]

\[
W J_W \frac{1}{P_t} P_{it} + \sum_{j=1}^{N} P_{iX_j} \mu_j + \sum_{j=1}^{N} \sum_{k=1}^{N} \frac{1}{2} P_{iX_j X_k} [\Sigma' \Sigma]_{jk}
\]
Appendix A. More Than One Financial Asset and State Variable

\[ +P_{iK}(rK_m + D_m - C_m) + D_i - rP_i] + W^2 J_{WW} \sum_{h=1}^{L} \sum_{j=1}^{N} \frac{\delta_{kh}}{P_i} \sum_{j=1}^{N} P_{iX_j} P_{jX_k} [\Sigma' \Sigma]_{jk} + W \sum_{j=1}^{N} J_{WX_j} \frac{1}{P_i} \sum_{k=1}^{N} P_{iX_k} [\Sigma' \Sigma]_{jk} = 0 \] (A.135)

The optimization equation for \( J \) and the first order conditions can in principle be solved for \( J \) and \( P_i \) by making use of the equilibrium condition \( W a_i = n_{im} P_i \). The first order conditions A.135 can be rewritten as

\[ J_W [P_{it} + \sum_{j=1}^{N} P_{iX_j} \mu_j + \sum_{j=1}^{N} \sum_{k=1}^{N} \frac{1}{2} P_{iX_jX_k} [\Sigma' \Sigma]_{jk} + P_{iK}(rK_m + D_m - C_m) + D_i - rP_i] + J_{WW} \sum_{h=1}^{L} n_{hm} \sum_{j=1}^{N} \sum_{k=1}^{N} P_{iX_j} P_{jX_k} [\Sigma' \Sigma]_{jk} + \sum_{j=1}^{N} J_{WX_j} \sum_{k=1}^{N} P_{iX_k} [\Sigma' \Sigma]_{jk} = 0 \] (A.136)

These equations can in principle be solved simultaneously to determine the \( P_i \)'s as a function of \( t, X, \) and \( K_m \), and parameterized by \( n_{1m} \ldots n_{Lm} \). Use must be made of the boundary conditions \( P_i(T, X, K_m) = f_i(X, K_m) \) and the homogeneity condition that \( W = W_m = \sum_{i=1}^{L} n_{im} P_i + K_m \). The homogeneity condition requires that the individual investor whose optimization problem is being considered has the same wealth that the other investors in the economy are assumed to have. This is equivalent to the equilibrium condition that demand equal supply. Equation A.136 has the form of equation 18 in CIR,

\[ (\alpha_P - r)P_i = - \frac{J_{WW}}{J_W} \sum_{j=1}^{N} \text{cov}(W_j, X_j) P_{iX_j} + \sum_{j=1}^{N} \sum_{k=1}^{N} (\frac{-J_{WX_j}}{J_W}) \text{cov}(X_j, X_k) P_{iX_k} \]

The major difference between this equation and CIR's equation 18 is again the absence of the \( P_W \) term here. This is because in this model the growth of \( K \) the consumption good, is nonstochastic. \( K \) in this model corresponds to \( W \) in CIR since in their work contingent claims are in zero net supply so that \( W = K \). Similarly \( P_K \) here corresponds to \( P_W \) in CIR. In CIR there are \( N+M \) components of the Wiener process that account for
the uncertainty in the growth of the consumption good as well as determine the equation of motion for \( \tilde{X} \). Here there are just the \( N \) components determining \( \tilde{X} \) so that \( dK \) has no uncertainty associated with it. Thus, all terms involving covariances of \( dK \) with other variables are zero and so there are no \( P_K \) terms present. All uncertainty in wealth derives from the uncertainty in the prices \( P_t \). The stochastic price process is driven by \( \tilde{X} \) and is determined endogenously so that market equilibrium is satisfied. In Appendix B the model is extended to include an exogeneous stochastic set of production processes for the consumption good \( K \) and CIR's equation 18 is derived in a more general setting.

With log utility investors, as in previous chapters we find that \( J_{WX} = 0, -J_W/J_{WW} = W, C_m = \frac{\sigma W_m}{1 + (\rho - 1) \exp(\rho(1 - T))} \) and the partial differential equations for \( P_i(t, \tilde{X}, K_m) \) become:

\[
P_{tt} + \sum_{j=1}^{N} P_{iX_j} \mu_j + \sum_{j=1}^{N} \sum_{k=1}^{N} \frac{1}{2} P_{iX_jX_k} [\Sigma']_{jk} \]

\[
+ P_i K (rK_m + \sum_{h=1}^{L} n_{hm} D_h - \frac{\sigma(K_m + \sum_{k=1}^{L} n_{hm} P_{kh})}{1 + (\rho - 1) \exp(\rho(1 - T))}) + D_i - rP_i =
\]

\[
\frac{\sum_{h=1}^{L} n_{hm}}{K_m + \sum_{h=1}^{L} n_{hm} P_{h}} \sum_{j=1}^{N} \sum_{k=1}^{N} P_{iX_jX_k} [\Sigma']_{jk}
\]

(A.137)

which can be solved independently of \( J \). These equations can be written in the simple form

\[
\alpha_{P_i} - r = \text{cov}(\tilde{R}_W, \tilde{R}_P)
\]

which implies the CAPM.
Appendix B

Instantaneous Production: General Case of Financial Asset Pricing

In this appendix we again consider an economy with many financial assets, many state variables and with the other assumptions the same as before except that there are now many stochastic, instantaneous production processes in perfectly elastic supply that the consumption good can be invested in. Assume as before that the economy consists of a fixed number of homogeneous investors with the same endowments and preferences; hence there is no trading. Assume perfect capital markets without taxes, transaction costs, etc. Uncertainty is represented by an N dimensional vector of state variables $\tilde{X}$ which follows a diffusion process given by the following stochastic differential equation:

$$d\tilde{X}(t) = \mu(\tilde{X}, t)dt + \Sigma(\tilde{X}, t)d\tilde{z}(t)$$

where $\mu(\tilde{X}, t) = (\mu_1, \ldots, \mu_{N+M})$ is an $N+M$- dimensional vector, $\Sigma$ is a $N \times (N + M)$ dimensional matrix with the positive definite $N \times N$ matrix $\Sigma\Sigma'$ giving the covariance matrix per unit time of the stochastic process $\tilde{X}$. $\tilde{z}(t)$ is an $N+M$ dimensional Wiener process with

$$E(d\tilde{z}(t)'d\tilde{z}(t)) = I_{(N+M)\times(N+M)}dt \quad E(d\tilde{z}(t)) = 0$$

There is a single consumption good denoted by $K$ which can be either consumed, invested in a set of $M$ stochastic production processes, or invested risklessly at the rate $r$. The $M$ stochastic production processes as well as the riskless one pay off instantaneously, have constant returns to scale, and are freely available to all investors i.e. in perfectly elastic supply. These stochastic production processes are assumed to have the following rates of
Appendix B. Instantaneous Production: General Case of Financial Asset Pricing

return:

\[
\frac{dK_i}{K_i} = \alpha_i(vX, t)dt + (\Gamma d\tilde{z}(t))_i
\]

where \( K_i \) is the amount of the consumption good invested in the \( i \)'th production process. This can be written in vector notation as:

\[
\left( \frac{dK_1}{K_1}, \ldots, \frac{dK_M}{K_M} \right) = \alpha(\tilde{X}, t)dt + \Gamma \cdot d\tilde{z}(t)
\]

This is the production technology assumed by CIR except that we include riskless production at the rate \( r \). As is done in CIR, if we wish to assume a zero net supply of riskless production a value of \( r \) can be found so that zero is the optimal level of riskless investment and \( r \) is the equilibrium interest rate.

The consumption good is assumed always to have a price of one. There are \( L \) financial assets which represent claims on \( L \) actual physical assets. These physical assets and hence the corresponding financial assets are in fixed net supply. They pay a continuous dividend in the consumption good that depends on time \( t \), the vector of state variables, \( \tilde{X}(t) \), and the per capita holdings of the consumption good, \( K_m(t) \). At the time \( T \) the physical assets are liquidated into the consumption good in an amount that also depends upon the state variables, \( \tilde{X}(T) \), and the per capita holdings of the consumption good, \( K_m(T) \), just before the assets are liquidated. This fixes the assets prices at time \( T \) as a function of \( \tilde{X}(T) \) and \( K_m(T) \). We denote, for the \( i \)'th physical asset, the dividend by \( D_i(t, \tilde{X}(t), K_m(t)) \) and the final liquidating dividend by \( f_i(\tilde{X}(T), K_m(T)) \).

Let the \( i \)'th financial asset have a price function \( P_i(t, \tilde{X}(t), K_m(t)) \). \( K_m(t) \) is the actual consumption good holdings of each investor at time \( t \), a quantity assumed fixed by each investor since it depends on all the other identical investors. The price \( P_i \) is relative to the unit price of the consumption good. Assume that each investor has the endowment \( n_{im} \) of the \( i \)'th financial asset. There is no trading because the investors are homogeneous hence investors hold their \( n_{im} \) shares (assume \( n_{im} \) is a continuous variable)
receiving the dividend \( n_{im} D_i(t, \bar{X}(t), K_m(t)) \) until time \( T \) when they receive the payout \( n_{im} P_i(T, \bar{X}(T), K_m(T)) = n_{im} f_i(\bar{X}(T), K_m(T)) \). Since only the total payout \( nD \) and \( nf \) is of importance, for \( n \neq 0 \) we could set \( n = 1 \) without loss of generality but we omit this so as to make the derivation clearer.

Investors are assumed to have rational (self consistent) expectations and to solve the same model of the economy as we are specifying to determine the price functions \( P_i(t, \bar{X}(t), K_m(t)) \) which they use to determine expected rates of return and the risk of the financial assets. Based on the expected rates of return and risk they determine their demands for the assets which must equal the supply at the equilibrium price \( P_i(t, \bar{X}(t), K_m(t)) \).

Investors as in the previous chapters, attempt to maximize their expected utility of consumption, which has the functional form

\[
E_t \int_t^T U(C(s), s) ds + B(W(T), T)
\]

Consider the investor's problem: given wealth \( W \) how best to allocate that wealth into the various financial assets. Assume that the investor puts the fraction \( a_i \) of his or her wealth into financial asset \( i \) with the price \( P_i \) and the fraction \( b_i \) into risky instantaneous production process \( l \). Then \( a_i W = n_i P_i(t, \bar{X}, K_m) \) and \( b_i W = K_l \) the amount of the consumption good invested in production process \( l \). The fraction \( 1 - \sum_{i=1}^{L} a_i - \sum_{l=1}^{M} b_l \) of wealth goes into the riskless investment of the consumption good so that \( W(1 - \sum_{i=1}^{L} a_i - \sum_{l=1}^{M} b_l) = K_0 \) where \( K_0 \) is the amount of the consumption good invested risklessly at rate \( r \). The investor's wealth is given by:

\[
W = \sum_{i=1}^{L} n_i P_i(t, \bar{X}, K) + K = \sum_{i=1}^{L} n_i P_i(t, \bar{X}, K_m) + K_0 + \sum_{l=1}^{M} K_l
\]

The change in wealth is given by

\[
dW = W\left\{\left(\sum_{i=1}^{L} a_i \frac{(dP_i + D_i dt)}{P_i}\right) + \sum_{l=1}^{M} b_l \frac{dK_l}{K_l} + (1 - \sum_{i=1}^{L} a_i - \sum_{l=1}^{M} b_l) r dt\right\} - C dt.
\]
Applying Ito's lemma, the change in the price of the financial asset can be written

\[ dP_t = \frac{\partial P_t}{\partial t} dt + \sum_{j=1}^{N} \frac{\partial P_t}{\partial X_j} dX_j + \sum_{j=1}^{N} \sum_{k=1}^{N} \frac{1}{2} \frac{\partial^2 P_t}{\partial X_j \partial X_k} E(dX_j dX_k) \]

\[ + \frac{\partial P_t}{\partial K_m} dK_m + \frac{1}{2} \frac{\partial^2 P_t}{\partial K_m \partial K_m} E(dK_m dK_m) + \sum_{j=1}^{N} \frac{\partial^2 P_t}{\partial X_j \partial K_m} E(dX_j dK_m) \]  

(B.138)

The change in the per capita holding of the consumption good is given by

\[ dK_m = \sum_{i=1}^{M} dK_{im} + (K_m - \sum_{i=1}^{M} K_{im}) r dt + D_m dt - C_m dt \]

\[ = (\sum_{i=1}^{M} K_{im}(\alpha_i - r) + K_m r + D_m - C_m) dt + \sum_{i=1}^{M} K_{im}[\Gamma d\bar{z}]_t \]  

(B.139)

where \( D_m = \sum_{i=1}^{L} n_{im} D_i \) is the per capita total dividend received from the financial assets. Hence we have

\[ dP_t = [P_{it} + \sum_{j=1}^{N} P_{iX_j} \mu_j + \sum_{j=1}^{N} \sum_{k=1}^{N} \frac{1}{2} P_{iX_j X_k} (\Sigma')\Sigma]_{jk} \]

\[ + P_{iK_m}(\sum_{i=1}^{M} K_{im}(\alpha_i - r) + K_m r + D_m - C_m) + \frac{1}{2} P_{K_m K_m} \sum_{i=1}^{M} \sum_{n=1}^{M} K_{im} K_{nm}[\Gamma']T]_i \]

\[ + \sum_{j=1}^{N} \sum_{i=1}^{M} P_{iX_j K_m} K_{im}[\Sigma']T]_{ji} + D_i - r P_i] dt \]

\[ + \sum_{i=1}^{N} P_{iX_j}(\Sigma d\bar{z})_j + P_{K_m} \sum_{i=1}^{M} K_{im}(\Gamma d\bar{z})_i \]  

(B.140)

Hence we can write

\[ dW = W \left\{ \sum_{i=1}^{L} \frac{\partial}{\partial P} [P_{it} + \sum_{j=1}^{N} P_{iX_j} \mu_j + \sum_{j=1}^{N} \sum_{k=1}^{N} \frac{1}{2} P_{iX_j X_k} (\Sigma')\Sigma]_{jk} \right\} \]

\[ + P_{iK_m}(\sum_{i=1}^{M} K_{im}(\alpha_i - r) + K_m r + D_m - C_m) + \frac{1}{2} P_{K_m K_m} \sum_{i=1}^{M} \sum_{n=1}^{M} K_{im} K_{nm}[\Gamma']T]_i \]

\[ + \sum_{j=1}^{N} \sum_{i=1}^{M} P_{iX_j K_m} K_{im}[\Sigma']T]_{ji} + D_i - r P_i] dt + \sum_{j=1}^{N} P_{iX_j}(\Sigma d\bar{z})_j \]

\[ + P_{iK_m} \sum_{i=1}^{M} K_{im}(\Gamma d\bar{z})_i + \sum_{i=1}^{N} b_i[(\alpha_i - r) dt + (\Gamma d\bar{z})_i] + (r - C/W) dt \]  

(B.141)

Let

\[ J(W, \bar{X}, K_m, t) = \max_{C_{it}} E_t \int_t^T U(C(s), s) ds + B(W(T), T) \]

the indirect utility function, be the solution to the investor's problem of maximizing expected utility. Using the methods of dynamic programming and stochastic control
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theory we can determine the following partial differential equation that \( J \) must satisfy:

\[
\max_{C(t)} L(t)J + U(C(t),t) + J_t = 0
\]

where \( L(t)J \) is the differential generator of \( J \). When written out in full the equation is

\[
0 = \max_{C(t)} \\{ U(C(t),t) + J_W(E[\frac{dW}{dt}]) + \frac{1}{2} J_{WW}(E[\frac{dW^2}{dt^2}]) \} + J_t
\]

\[
+ \sum_{j=1}^{N} J_{X_j}(E[\frac{dX_j}{dt}]) + \sum_{k=1}^{N} \frac{1}{2} J_{X_k}(E[\frac{dX_k}{dt}])
\]

\[
+ \sum_{j=1}^{N} J_{W Y_j}(E[\frac{dW Y_j}{dt}]) + J_{K m}(E[\frac{dK m}{dt}])
\]

\[
+ \frac{1}{2} J_{K m K m}(E[\frac{dK m^2}{dt^2}]) + J_{W K m}(E[\frac{dW K m}{dt}])
\]

\[
+ \sum_{j=1}^{N} J_{K m Y_j}(E[\frac{dK m Y_j}{dt}])\}
\]

(B.142)

with the boundary condition \( J(W, X, K, m, T) = B(W(T), T) \). Substituting for \( dW, dK_m \) and \( dX \) gives

\[
0 = \max_{C(t)} \\{ U(C(t),t) + J_t + \sum_{j=1}^{N} \mu_j J_{X_j} + \sum_{k=1}^{N} \frac{1}{2} J_{X_k}(\Sigma_\Sigma)_{jk}
\]

\[
+ J_W \{ \sum_{h=1}^{L} \frac{\alpha_{h}}{P_h} [(P_{h t} + \sum_{j=1}^{N} P_{h X_j} \mu_j + \sum_{k=1}^{N} \frac{1}{2} P_{h X_k}(\Sigma_\Sigma)_{jk}
\]

\[
+ P_{h K_m}(\sum_{l=1}^{M} K_{l m}(\alpha_l - r) + K_m r + D_m - C_m)
\]

\[
+ \frac{1}{2} P_{h K_m K_m} \sum_{l=1}^{M} \sum_{n=1}^{M} K_{l m} K_{n m} (\Gamma'\Gamma)_l n
\]

\[
+ \sum_{j=1}^{N} \sum_{l=1}^{M} P_{h X_j K_m} [\Sigma_\Sigma]_{jl} + D_l - r P_h)
\]

\[
+ \sum_{l=1}^{M} b_l (\alpha_l - r) + r - C/W\}
\]

\[
+ \frac{W^2}{2} J_{W W} \{ \sum_{g=1}^{L} \sum_{h=1}^{L} \frac{\alpha_{g h}}{P_g P_h} \sum_{j=1}^{N} \sum_{k=1}^{N} P_{g X_j} P_{h X_k}(\Sigma_\Sigma)_{jk}
\]

\[
+ \sum_{g=1}^{L} \sum_{h=1}^{L} \frac{\alpha_{g h}}{P_g P_h} \sum_{l=1}^{M} \sum_{n=1}^{M} P_{g K_m} P_{h K_m} (\Gamma'\Gamma)_l n
\]

\[
+ 2 \sum_{g=1}^{L} \sum_{h=1}^{L} \frac{\alpha_{g h}}{P_g P_h} \sum_{j=1}^{N} \sum_{l=1}^{M} P_{g X_j} P_{h K_m} (\Sigma_\Sigma)_{jl}
\]

\[
+ \sum_{l=1}^{M} \sum_{n=1}^{M} b_l (\Gamma'\Gamma)_l n + 2 \sum_{h=1}^{L} \frac{\alpha_{h}}{P_h} \sum_{l=1}^{M} b_l [\sum_{j=1}^{N} P_{h X_j}(\Sigma_\Sigma)_{jk}
\]

\[
+ P_{h K_m} \sum_{n=1}^{M} K_{n m} (\Gamma'\Gamma)_{nl} \} + \sum_{j=1}^{N} W J_{W X_j} (\sum_{h=1}^{L} \frac{\alpha_{h}}{P_h} (\sum_{k=1}^{N} P_{h X_k}(\Sigma_\Sigma)_{jk}
\]

\[
+ P_{h K_m} \sum_{l=1}^{M} K_{l m} (\Sigma_\Sigma)_{jl} + \sum_{l=1}^{M} b_l (\Sigma_\Sigma)_{jl} \}
\]

\[
+ \sum_{j=1}^{N} J_{K_m X_j} \sum_{l=1}^{M} K_{l m} (\Sigma_\Sigma)_{jl}.\]
Differentiating with respect to $C$, $a_i$ and $b_i$ gives the first order conditions which determine $C^*$, $a_i^*$ and the $b_i^*$, the optimal consumption and investments. The investor solves his or her own optimal investment problem with assumptions about the behaviour of other investors, namely what their optimal $C_m$, $a_{lm}$ and $b_{lm}$ are (where $a_{lm} W = n_{lm} P_i$ and $b_{lm} W = K_{lm}$). Knowing the amount of the consumption good invested in production process 1, $K_{lm}$, and consumption $C_m$ allows the investor to calculate $E[dP_i]$ and hence $J$. Self consistency then requires that the investor's optimal $a_i$, $b_i$ and $C$ equal the values assumed for the other investors, i.e. $C_m$, $a_{lm}$ and $b_{lm}$ . The first order conditions are

$$U_C = J_W = 0$$

$$0 = J_W W \left[ \frac{1}{P_i} \left( \sum_{l=1}^{N} P_{i,x} \mu_{j} + \sum_{k=1}^{N} \sum_{j=1}^{M} P_{i,x} P_{h,x} (\Sigma' \Sigma)_{jk} \right) + \sum_{l=1}^{N} P_{i,x} P_{h,x} (\Sigma' \Sigma)_{jk} \right]$$

$$+ P_{i,k_m} \left( \sum_{l=1}^{M} K_{lm} (\alpha_l - r) + K_m r + D_m - C_m \right) + \sum_{l=1}^{M} K_{lm} K_{nm} [\Gamma'^T]_{ln}$$

$$+ \sum_{j=1}^{N} K_{lm} [\Sigma'T]_{jl} + D_i - r P_i \right)$$

$$+ W^2 J_W W \left\{ \sum_{h=1}^{L} \frac{a_{hn}}{P_i} \left[ \frac{1}{P_i} \left( \sum_{j=1}^{N} \sum_{k=1}^{N} P_{i,x} P_{h,x} (\Sigma' \Sigma)_{jk} \right) + \sum_{j=1}^{M} P_{i,x} P_{h,x} (\Sigma' \Sigma)_{jk} \right] \right\}$$

$$+ P_{i,k_m} P_{h,x} [\Sigma'T]_{jl} + \frac{1}{P_i} \sum_{l=1}^{M} b_l [\sum_{j=1}^{N} P_{i,x} [\Sigma'T]_{jl}]$$

$$+ P_{i,k_m} K_{nm} [\Gamma'^T]_{nl} \} + \sum_{j=1}^{N} W J_W X_j \frac{1}{P_i} \left\{ \sum_{k=1}^{N} P_{i,x} [\Sigma'T]_{jk} \right\}$$

$$+ P_{i,k_m} \sum_{l=1}^{M} K_{lm} [\Sigma'T]_{jl} + W J_W K_m \left\{ \sum_{l=1}^{M} K_{lm} \left( \frac{1}{P_i} \sum_{j=1}^{N} P_{i,x} [\Sigma'T]_{jl} \right) + P_{i,k_m} \sum_{n=1}^{M} K_{nm} [\Gamma'^T]_{ln} \right\} = 0$$
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\[ J_w W(\alpha_t - r) + W^2 J_{WW} \left\{ \sum_{n=1}^{M} b_n [\Gamma^T]_{in} + \sum_{k=1}^{L} \frac{\partial}{\partial h} \left[ \sum_{j=1}^{N} P_{h} X_j [\Sigma^T]_{ji} \right] \right\} + P_{h K_m} \sum_{n=1}^{M} K_{nm} [\Gamma^T]_{in} \right\} + \sum_{j=1}^{N} W J_w X_j [\Sigma^T]_{ji} \right\} + W J_{WW} K_m \left\{ \sum_{n=1}^{M} K_{nm} [\Gamma^T]_{in} \right\} = 0 \]  

(B.146)

The optimization equation for \( J \) and the first order conditions B.144-B.146 can in principle be solved for \( J, P_l \) and \( K_{lm} \) by making use of the equilibrium conditions \( W a_i = n_{im} P_i \) and \( W b_l = K_{lm} \). The solution determines the \( L \) functions \( P_i \) as functions of \( t, \bar{X}, \) and \( K_m \), and parameterized by \( n_{1m} \ldots n_{lm} \). Use must be made of the boundary conditions \( P_i(T, \bar{X}(T), K_m(T)) = f_i(\bar{X}(T), K_m(T)) \) and the homogeneity condition that \( W = W_m = \sum_{i=1}^{L} n_{im} P_i + K_m \) and \( b_l W = K_{lm} \). The homogeneity condition requires that the individual investor whose optimization problem is being considered has the same wealth that the other investors in the economy are assumed to have. This is equivalent to the equilibrium condition that demand equal supply. With the equilibrium conditions imposed equation B.146 can be solved for \( K_{lm} \) as:

\[
K_{lm} = \left[ -J_w \sum_{n=1}^{M} (\alpha_n - r) [\Gamma^T]_{in}^{-1} - J_{WW} \sum_{n=1}^{M} + \sum_{i=1}^{L} n_i \sum_{j=1}^{N} P_{i} X_j [\Sigma^T]_{jn} \right]
- \sum_{j=1}^{N} J_{WX_j} \sum_{n=1}^{M} [\Sigma^T]_{jn} [\Gamma^T]_{in}^{-1} \left[ J_{WW} (1 + \sum_{i=1}^{L} n_i P_{K_m}) + W J_{WW} k_m \right] \]  

(B.147)

Equation B.145 with the equilibrium conditions imposed can be written in a form similar to equation 18 in CIR,

\[
(\alpha P_i - r) P_i = \\
\sum_{j=1}^{N} \left[ \left( -\frac{J_{WW}}{J_{W}} \right) \text{cov}(W, X_j) + \sum_{k=1}^{N} \left( -\frac{J_{W X_k}}{J_{WW}} \right) \text{cov}(X_k, X_j) \right] \\
+ \left( -\frac{J_{W K_m}}{J_{WW}} \right) \text{cov}(K_m, X_j) P_i X_j + \left[ \left( -\frac{J_{WW}}{J_{W}} \right) \text{cov}(W, K_m) \right] \\
+ \sum_{k=1}^{N} \left( -\frac{J_{W X_k}}{J_{WW}} \right) \text{cov}(X_k, K_m) + \left( -\frac{J_{W K_m}}{J_{WW}} \right) \text{cov}(K_m, K_m) \right] P_i P_{K_m} \]  

(B.148)

since \( \text{cov}(X_k, X_j) = [\Sigma^T \Sigma]_{kj}, \text{cov}(K_m, X_j) = \sum_{i=1}^{M} K_{lm} [\Sigma^T]_{ji} \text{etc.} \) This equation is the full generalization of CIR's equation 18 to the case where financial assets are in nonzero
net supply. Their equation 18 is a special case of B.148. As was discussed in the previous chapters it is necessary to distinguish between $W$ and $K_m$. The CIR derivation is incorrect although it gives the right answer in the special case they consider, financial assets in zero net supply. Their derivation starts off by assuming $n_i \neq 0$ but does not distinguish between $K_m$ and $W$. Later in their derivation they set $n_i$ equal to zero and the correct answer is obtained. However, one cannot simply use their equations and set $n_i$ to a positive value to study the case of financial assets in positive net supply.

With log utility investors, as in the previous chapters we find that $J_{WX} = 0$, 

\[-J_W/J_{WW} = W, \quad C_m = \frac{\rho W_m}{1 + (\rho-1) \exp(\rho(t-T))} \]

and the partial differential equations for $P_i(t, X, K_m)$ become:

\[
[P_{it} + \sum_{j=1}^{N} P_{iX_j} \mu_j + \sum_{j=1}^{N} \sum_{k=1}^{N} \frac{1}{2} P_{iX_jX_k} (\Sigma \Sigma)_{jk} \\
+ P_{iK_m} \left( \sum_{l=1}^{M} K_{lm} (\alpha_l - r) + K_m r + D_m - \frac{\rho W_m}{1 + (\rho-1) \exp(\rho(t-T))} \right) \\
+ \frac{1}{2} P_{iK_m K_m} \sum_{l=1}^{M} \sum_{n=1}^{M} K_{lm} K_{nm} [\Gamma \Gamma]_{ln} \\
+ \sum_{j=1}^{N} \sum_{l=1}^{M} P_{iX_j K_m} K_{lm} [\Sigma \Gamma]_{jl} + D_i - r P_i \\
= \frac{1}{W_m} \left( \sum_{h=1}^{L} \sum_{j=1}^{N} n_h \sum_{k=1}^{N} P_{iX_jX_k} [\Sigma X]_{jk} \\
+ \sum_{h=1}^{L} n_h \sum_{l=1}^{M} P_{iK_m P_h K_m} [\Gamma \Gamma]_{ln} + \sum_{h=1}^{L} n_h \sum_{j=1}^{N} \sum_{l=1}^{M} P_{iX_j P_h K_m} \\
+ P_{iK_m P_h X_j} [\Gamma \Gamma]_{jl} + \sum_{l=1}^{M} K_{lm} [\Sigma X]_{jl} \\
+ P_{iK_m} \sum_{n=1}^{M} K_{nm} [\Gamma \Gamma]_{nl} \right) \\
\text{ with } K_{lm} = \frac{W_m \sum_{n=1}^{M} (\alpha_n - r) [\Gamma \Gamma]_{nl}^{-1} - \sum_{n=1}^{M} + \sum_{h=1}^{L} n_h \sum_{j=1}^{N} P_{iX_j} [\Sigma X]_{jn}}{1 + \sum_{i=1}^{L} n_i P_{iK_m}} \\
\text{ and } W_m = \sum_{i=1}^{L} n_i P_i + K_m \text{ which can be solved independently of } J. \text{ Substituting the equations for } K_{lm} \text{ into the equations for } P_i \text{ gives a set of partial differential equations for } P_i \text{ which in theory can be solved but in practice are extremely difficult. These equations}]
\]
can be written in the simple form

$$\alpha_{R_t} - r = \text{cov}(\tilde{R}_W, \tilde{R}_F)$$

which implies the CAPM.