OPTIMAL TRADING STRATEGIES AND RISK
IN THE GOVERNMENT BOND MARKET

Two Essays in Financial Economics

By

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ABSTRACT

The two main questions arising from the problem of optimal bond portfolio management concern the formulation of an optimal trading rule and the specification of an appropriate dynamic risk measure in which to express portfolio objectives. We study these questions in two related essays: (1) a theoretical study of optimal trading policies in view of, as yet unspecified, portfolio objectives when trading is costly; and (2) an empirical, comparative study of several bond risk measures, proposed in the literature or in use by practitioners, for the government or default-free bond market.

The theoretical study considers a delegated portfolio management setting, in which the manager optimizes a cumulative reward over a finite time period and where the reward rate increases with portfolio value and decreases with deviations from the given risk objectives. Trading is then often not worthwhile, as the possible gains from smaller objective deviations are offset by losses on account of transactions costs. This setting obviates the need for separate ex post performance evaluation.

The trading problem is formulated as one of optimal impulse control in the framework of stochastic dynamic programming; this formulation improves upon prior results in the literature using continuous control theory. A myopic optimal trading rule is characterized, which is also applicable to time-homogeneous problems and more general preferences. An algorithm for its use in applications is derived.

The empirical study applies the usual methods of stock market tests to the returns of constant risk bond portfolios. These portfolios are artificial constructs composed, at varying risk levels, of traded bonds on the basis of six different one or two dimensional risk measures. These risk measures are selected in order to obtain a cross-section of term structure variabilities; they include duration, short interest rate risk, long (13-year) interest rate risk, combined short and consol rate
risks, duration combined with convexity, and average time-to-maturity. The sample period is the 1970s decade, for which parameter estimates for the risk measures—where necessary—are available from source papers. This period is known to be one with wide-ranging term structure movements and is therefore ideally suited for the tests of this paper. Portfolios are formed at two levels of diversification: bullet and ladder selection.

We confirm that all of these risk measures are reasonably effective in capturing relevant bond market risk: the state space of bond returns has in all cases a low dimension (two or three), with only a single factor significantly priced. Best fit is found for portfolios selected by duration, the 13-year spot yield risk, and the two-dimensional short/consol rate risk, all of which consist predominantly of “long” rate risk.

The short rate-based risk measure does not explain portfolio returns as well: it has difficulty discriminating between portfolios with long remaining times-to-maturity. Convexity, furthermore, adds nothing to the explanatory power of duration. Average time-to-maturity compares reasonably well with the above risk measures, provided the portfolios are well-diversified across the maturity spectrum; this lends some support to the use of yield curves.

A strong diversification effect has also been found, to the extent that the returns on ladder portfolios are practically linear combinations of two or three of the portfolios, typically the lowest and highest risk portfolios in the one dimensional risk cases, with an intermediate portfolio added in the two-dimensional cases. Provided that diversified portfolios are used, the comparatively easy to implement duration measure is as good as any of the risk measures tested.
# TABLE OF CONTENTS

Abstract ........................................ ii
List of Tables .................................... vii
List of Figures ................................... viii

INTRODUCTION .................................... 1
1. Portfolio Management Environment ............ 2
2. Research Objectives ............................ 4

AN OPTIMAL PORTFOLIO TRADING RULE ....... 6
1. Introduction .................................... 7
   1.1 Optimal Portfolio Management ............... 7
   1.2 Objective .................................. 10
2. Portfolio Management Setting Defined ....... 11
   2.1 Trading Motives ............................. 11
   2.2 Portfolio Management Setting ............... 12
3. Continuous-Time Framework ..................... 15
   3.1 Portfolio State Variables .................... 15
   3.2 Portfolio Risk Attributes .................... 16
   3.3 Portfolio Risk Target ....................... 18
   3.4 Transactions Costs ........................... 18
4. Manager’s Optimal Control Problem .......... 20
   4.1 Impulse Control Problem ..................... 20
   4.2 Optimal Reward Function ..................... 21
   4.3 A Special Case ................................ 23
5. An Optimal Trade Solution ..................... 26
   5.1 An equivalent ε-perturbed Problem .......... 26
   5.2 A Solution .................................. 27
   5.3 Economic Interpretation of Optimal Control .. 29
6. Trade/No Trade Regions ......................... 30
7. Application of the Optimal Trade Rule in
   Discrete Time .................................. 33
   7.1 Discrete Monitoring .......................... 34
   7.2 Discrete Trading along $u^*(\theta_n)$ .......... 35
8. Summary and Conclusions ....................... 37
References ...................................... 41
Appendices ....................................... 44
   A. Continuous-Time Framework ................. 44
   B. State Variable Dynamics ..................... 47
   C. Derivation of Optimal Control ................ 49
D. Trade/No-Trade Regions ........................................ 55
E. Procedure to Determine Optimal Trade Combination .......... 58
F. Discrete Control of Portfolios following Itô Processes ......... 61
G. Numerical Examples of Applying Impulse Controls ............. 67

RISK AND RETURN IN THE GOVERNMENT BOND MARKET ............. 75
1. Introduction ..................................................... 76
2. Measures of Intertemporal Risk .................................... 82
   1.2 Duration and Single State Variable Models ................. 83
   2.2 Multi-State-Variable Models .................................... 87
   2.3 Risk Measure Comparisons ...................................... 89
   2.4 Constant Risk Measures ........................................ 91
3. Portfolio Returns .................................................. 94
   3.1 Data .......................................................... 94
   3.2 Constant Risk Portfolios ....................................... 96
4. Constant Risk Portfolio Return Distributions ..................... 99
   4.1 Time Series Analysis .......................................... 99
   4.2 Distribution Results ......................................... 100
   4.3 Stationarity Tests ............................................ 102
   4.4 Covariance Matrix and Eigen Value Analysis ............... 103
   4.5 Conclusions ................................................ 105
5. Factor Model ....................................................... 107
   5.1 A.P.T. Factor Model Specification ........................... 107
   5.2 Effect of Portfolio Risk Perturbations ....................... 109
   5.3 Factor Extraction and Rotation ............................... 111
   5.4 A.P.T. Factor Prices .......................................... 112
6. Factor Analysis of Bullet Portfolios ............................. 114
   6.1 Results ..................................................... 114
   6.2 Factor Description ........................................... 116
   6.3 Comparison with Ladder Portfolios ........................... 118
   6.4 Conclusions ................................................ 120
7. A.P.T. Cross-sectional Tests ....................................... 121
   7.1 Tests on Means .............................................. 121
   7.2 Bilinearity Tests ............................................ 123
   7.3 Conclusions ................................................ 125
8. CAPM-style Tests of Asset Pricing ................................. 126
   8.1 Regression Results .......................................... 127
   8.2 Conclusions ................................................ 129
9. Summary and Conclusions ........................................... 130
References ............................................................... 135
Appendices ............................................................... 139
  A. Estimation of Risk Measures ................................. 139
  B. Portfolio Selection Details ................................. 142
  C. Numerical Demonstration of Term Structure Factors ................................. 146
LIST OF TABLES

RISK AND RETURN IN THE GOVERNMENT BOND MARKET

I. Basic Bond Portfolio Composition .................. 149
II. Constant Risk Targets ............................. 150
III. Basic Portfolio Data ................................ 151
IV. Autocorrelations at Lags 1, 2, 6, and 12 Months .................. 154
V. Univariate Statistics of Constant Risk Portfolio Returns ............. 157
VI. Stationarity Tests between Sample Period Halves .................. 163
VII. Correlation Matrices Eigen Value Analysis ....................... 165
VIII. MLFA and PCA Factor Loadings ..................... 167
IX. MLFA Factor Model Goodness-of-Fit ..................... 173
X. Canonical Correlations between Bullet Factors and Ladder Portfolios 174
XI. OLS Regression of Bullet Portfolio Returns on Ladder Portfolios 175
XII. A.P.T. Tests on Mean Returns ......................... 181
XIII. Factor Model Bilinearity Tests ........................ 182
XIV. A.P.T. Bilinearity Tests ............................. 183
XV. Ex-Post Security Market Lines/Planes ....................... 184
LIST OF FIGURES

AN OPTIMAL PORTFOLIO TRADING RULE
1. Trade/No-Trade Regions, $m = 2$ ........................................ 72
2. Trade Regions without Costs, $m = 3$ ................................... 73
3. Trade/No-Trade Regions with Costs, $m = 3$ ......................... 74

RISK AND RETURN IN THE GOVERNMENT BOND MARKET
1. Example of Ex-Post Security Market Line – Bullet
   Portfolios ................................................................. 187
2. Example of Ex-Post Security Market Line – Ladder
   Portfolios ................................................................. 188
INTRODUCTION

to

OPTIMAL TRADING STRATEGIES AND RISK
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The University of British Columbia
1. Portfolio Management Environment

The focus of this dissertation is on the management of portfolios of government bonds in view of risk and return objectives.

Default-free government bonds, with their guaranteed coupon payments and principal repayments, are nevertheless subject to interest rate risk. Promised payments to bondholders are fixed in time and amount, so that changes in the term structure of interest rates then affect the present value of a bond investment and—hence—its market value. Furthermore, as time proceeds, remaining payments come closer in time, some bonds mature eventually, and new bonds with longer remaining times-to-maturity and possibly different coupon rates take their place.

This suggests that the riskiness of a bond portfolio, whichever way defined, and its rate-of-return on investment may fluctuate widely over time, unless an active program of selling and buying is followed by the portfolio manager.

Objectives in terms of risk and return are well-known from portfolio theory and are compatible with the maximization of expected utility of risk averse investors. Such objectives are quite distinct from the, possibly naive, fixed income objectives often found in trust fund management. In the latter, a bond portfolio is managed solely to provide a steady stream of coupon income, while preserving the face value of the principal; such portfolio management is not concerned with current portfolio risk and return. Portfolio objectives studied in this dissertation, however, are limited to those expressed in terms of risk and return.

It is at this stage appropriate for the portfolio owner and/or manager to reflect on a portfolio management objective: what is the preferred risk profile of the portfolio over time and how can it be met while optimizing portfolio value. Such an objective applies equally well to reinvestment of coupon income and to portfolios that are gradually sold off for income maintenance purposes (the latter would be a
preferred alternative to the fixed coupon income stream objective).

This type of objective has been well-known in the management of stock portfolios ever since the advent of the mean/variance Capital Asset Pricing Model (CAPM) in the 1960s. It is more difficult to implement in bond portfolio management, however, because the determinants of bond prices are more complex than the single market index of CAPM. It is only in the past 10 years that consistent theories of term structure dynamics and of bond prices have been advanced based on continuous-time diffusion processes.

Even so, application of such theories to daily portfolio management practice may be difficult. No unanimity exists in bond markets as to an appropriate risk measure for government bonds. Each of the above noted bond pricing theories has its own risk measure, while several others have been developed by practitioners. This presents a problem in the specification of portfolio risk/return objectives.

Because yields on government bonds of all maturities are not perfectly correlated, portfolio management objectives must also take into account benefits from diversification. Diversification objectives must be weighed, however, against the skill of the portfolio manager in taking advantage of market pricing inefficiencies.

A further question in portfolio management practice is the cost of management itself. This includes the transactions costs associated with selling and buying of securities and the agency costs of delegated portfolio management. It is conceivable that a trade-off between costs and benefits will limit the amount of trading worthwhile in pursuit of a stated portfolio risk/return objective. In fact, cost considerations may impinge on the choice of risk parameter itself, both w.r.t. ease of implementation and desired accuracy.

The portfolio management environment described, so far, requires answers to three questions:
(1) what is an appropriate risk measure for bond portfolios?
(2) what is a suitable portfolio management objective?
(3) and, given (1) and (2), what is the optimal trading policy?

The research in this dissertation addresses these questions, as further described in the next subsection.

2. Research Objectives

The above questions are studied in this dissertation in two separate, but related, papers:

(1) a theoretical study of trading rules that optimize a reward function, expressed in terms of portfolio value and risk, in the presence of transactions costs; and
(2) an empirical, comparative study of several bond risk measures proposed in the literature or used by practitioners.

The theoretical study, “An Optimal Trading Rule,” is conducted in a delegated portfolio management setting and also addresses the question of manager’s performance under the stated objectives. The methodology in this study includes the continuous-time framework of stochastic dynamic programming. The results of this study point to necessary elements of the optimal contract between the manager and the portfolio’s owner. The model developed here will tell the manager when to trade and how much.

The empirical “Risk and Return in the Government Bond Market” paper seeks to determine, in a comparative study, which of several known or proposed risk measures for the government bond market are appropriate in our portfolio management setting. It does this by testing the explanatory power of a linear factor model and the significance of the resulting factor prices. The results of this study can help practitioners choose the risk measure in which to specify the portfolio objectives
that were left general in the first paper, but could well include any of the risk measures considered in the second paper.

Risk and return management of bond portfolios can not only be accomplished by trading in the bonds themselves, but also by hedging with bond futures and options on futures. We do not specifically treat such contingent claims in the body of this dissertation. Nevertheless, the optimal trading theory developed here is also applicable to trading in those instruments, even at transactions cost levels different from those for bonds. In any event, much of bond portfolio management practice is enjoined from trading in contingent claims.

The empirical work in this dissertation is based on bonds only for reasons of data availability and because of the complexity of the derivative price relationships of futures and options to those of bonds.
AN OPTIMAL
PORTFOLIO TRADING RULE

An Essay in Financial Economics

By
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1. Introduction

1.1 Optimal Portfolio Management

The problem of optimal portfolio management is well known in financial economics, both at the individual and institutional investor levels.

Merton [27], following Samuelson [32], considers the case of an individual investor who maximizes the utility of lifetime consumption; working in continuous time and with frictionless markets, he shows that the investor faces a stochastic dynamic programming problem whose optimal control requires continuous trading in all assets. This model of investor behaviour also underlies modern theories of equilibrium asset pricing when used in conjunction with no-arbitrage and market clearing conditions, see e.g. Cox, Ingersoll, and Ross [13], Constantinides [7,8], or Breeden [3].

Magill and Constantinides [24] have extended Merton’s work to a market with (proportional) transactions costs, taking price processes as given.¹ Investors will now trade at discrete times: only when the expected utility gain of trading outweighs the cost of trading. Their methodology has been questioned by Duffle [15], however.

The above models focus on maximizing the individual investor’s lifetime utility of consumption, hardly a trivial problem for the investor, whether representative or not. Magill and Constantinides [24] remark (p. 257), that it is “highly unlikely that even the most rational of investors would involve himself with such calculations” as would be required to solve for the trade/no-trade decision at every point in time.

Institutional portfolio management offers, in contrast, a simpler and yet more formal optimization setting, with finite time horizons and exogenous portfolio man-

¹ Garman and Ohlson [19] consider equilibrium asset pricing with transactions costs. This problem is not a simple extension of the individual’s optimal investment policy, as returns differ for investors buying vs. holding an asset. They find that asset prices may vary within a bandwidth from the equilibrium point values of the no-transactions cost case.
agement objectives. It is also an important problem in finance, as most investment and trading activity is done by institutions, e.g. pension funds, mutual funds, and trust funds. Institutional investment differs from the individual problem in that it is carried out by professional portfolio managers on behalf of owners in consideration of a management fee. Not only is such management costly in its own right, it also leads to problems of performance measurement and moral hazard.

Portfolio management performance measures must ideally separate superior skills (e.g. stock picking ability based on fundamental analysis) from lucky market breaks. The Sharpe/Lintner Capital Asset Pricing Model (CAPM) has given rise to a host of performance measures purporting to do this, e.g. the indices by Treynor [34], Sharpe [33], and Jensen [21]; or the work by Fama [17]. Such measures use the ex post security market line as benchmark, reflecting that optimal investment in the CAPM setting must be in the “market portfolio.” Roll [30] has argued that these measures are of no use in practice because the market portfolio cannot be identified, but Mayers and Rice [25] take the satisficing view that superior performance will indeed on average have portfolios plotting above the ex post security market line.

A further problem with the CAPM and its derivative indices is that the model is based on homogeneous beliefs: it is therefore inherently unsuited to judge active performance on the basis of asymmetric information, a point made by Roll [30], Cornell [10], and Dybvig and Ross [16]. A similar argument may be made w.r.t. performance measurement in the framework of the Arbitrage Pricing Theory (APT), see e.g. Connor and Korajczyk [6].

A striking feature of portfolio management using these traditional performance measures is that an explicit portfolio management policy is, in general, not enumerated, although implicitly value maximization is always assumed. An exception is Fama [17]'s measure, which penalizes the portfolio manager for underdiversification.
The moral hazard problem arises because a manager's private utility goals may interfere with those desired by the portfolio's owner. Specifically, the manager may shirk on effort, speculate on lucky breaks, or engage in needless portfolio turn-over if there is little chance of being found out. Problems of this type may be modelled with principal/agent theory, see e.g. Holmstrom [20] and Bhatacharya and Pfleiderer [2]. Here, the portfolio manager's (or agent's) objective function derives from the owner's (or principal's) expected utility maximization problem.

Portfolio management is now optimized from the owner's point-of-view by offering the manager an incentive contract, with penalties for undesirable management behaviour. Undesirable portfolio management may include large deviations from risk or diversification goals, excessive portfolio turn-over leading to high transactions costs, and not spending enough effort on fundamental analysis. Note that these activities are discouraged but not prohibited, allowing the manager a trade-off of reward gains vs. portfolio deviations and value gains. In maximizing the expected incentive reward, he will now simultaneously optimize management performance. It is well known, however, that in any principal/agent problem the principal's utility resulting from the agent's optimized actions lies (weakly) below that of the principal's individual optimal actions (see, e.g. Holmstrom [20]). The actual sharing of economic rents from improved portfolio management will depend on the terms of the incentive contract negotiated between the portfolio owner and manager.

Amershi [1] considers the portfolio management problem in the presence of asymmetric information between manager and owner, i.e. the manager (= fundamental analyst) expends costly effort to attain non-market stock picking information that the owner does not possess. A competitiveness assumption is also made, i.e. actions of the manager do not affect security prices. Amershi's solution of the resulting agency problem shows that an incentive scheme based on target returns, penalties,
and bonuses may be designed to eliminate all moral hazard; but the manager appropriates in equilibrium all economic rent from the use of his expertise. Although the portfolio owner will then on average only experience market returns, Amershi's model provides a rationale for the existence of active portfolio management and fundamental analysis.

1.2 Objective

The objective of this paper is to derive the optimal trading strategy in a generalized institutional portfolio management setting with a finite time horizon. We assume that the principal/agent framework is applicable, with an exogenous incentive scheme resulting from the simultaneous optimization of the portfolio owner's and manager's expected utilities. This framework reduces the portfolio management problem to one of maximization of expected reward: a stochastic dynamic programming problem.

The problem thus formulated is compatible with trading motives originating in differential information (i.e. it encourages the manager to engage in fundamental analysis), and with judicious trading to meet portfolio risk attribute targets in the face of costly transactions. The manager is otherwise free to trade, but is assumed to manage the portfolio so as to maximize expected compensation. This formulation also dispenses with the need for ex post performance measurement, whether for portfolio or manager. In a world with transactions and management costs, optimality will be achieved by an ongoing trade-off of costs and gains in expected management compensation, leading to discrete trading: only when expected gains outweigh the cost of trading.

We state and solve the stochastic dynamic programming problem in the spirit of Magill and Constantinides [24], while addressing Duffie [15]'s critique of their methodology through application of impulse control theory, ref. Menaldi [26].
2. Portfolio Management Setting Defined

2.1 Trading Motives

While the importance of the portfolio management problem is evident from the attention paid to it in research and—not least—in practical portfolio management, it is not immediately clear why this must be so in stock markets with non-zero transactions costs. Investors are expected to hold widely diversified portfolios if markets are efficient (the market portfolio if CAPM holds from period to period). Small incremental changes in asset characteristics or findings from fundamental analysis may then be expected to affect portfolios only marginally, so that any well-diversified portfolio changes its risk/return attributes slowly. This suggests the absence of continuous or frequent trading motives when transactions costs are considered, especially in diversified institutional portfolios.

This is different for portfolios of securities with fixed, finite maturity dates, e.g. those consisting of bonds, options, futures, or leases. Here, the relevant risk attributes\(^2\) of individual securities and diversified portfolios change with time, even if no other changes occur in market fundamentals such as the term structure of interest rates. Here the mere passage of time is sufficient to establish a trading motive in portfolios of fixed maturity securities.\(^3\)

A further trading motive in markets for fixed, finite maturity securities originates in the small number of securities traded, e.g. usually fewer than 100 U.S. government straight coupon bonds are traded at any time. In general, we expect that the amount of trading required to achieve and maintain a portfolio with a

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\(^2\) Following Rothschild and Stiglitz [31], the relevant risk attributes of securities are moments of their return distributions, or measures derived from such moments. An example of the latter in the government bond market is Macaulay [23]'s duration. This matter is further analyzed in the second part of this dissertation.

\(^3\) No trading motive exists if the portfolio risk target profile on \([0, T]\) coincides with one or more traded securities, e.g. a pure discount bond of the exact required maturity in an immunization problem.
specified return distribution varies inversely with the number of traded "atomic" securities.

Finally, certain portfolio objectives, e.g. immunization or insurance, require active trading strategies; this problem is common to life-assurance companies and pension fund administrations that must meet known or anticipated obligations from the proceeds of government bond portfolios. Not having pure discount bonds maturing at exact immunization dates requires periodic trading in coupon bonds with shorter and longer maturities, as these securities change risk attributes at different rates.

Trading motives of this sort derive from well-defined portfolio management objectives. Yet even here a supplementary optimization problem remains, i.e. security selection so as to limit the frequency and necessity of trading in view of transactions and management costs. Active portfolio trading motives thus exist in bond markets even in the absence of differential information, although the latter may also be a motive in bond markets ("portfolio timing").

2.2 Portfolio Management Setting

We focus—for clarity of exposition—on the manager of a portfolio of government (default-free) bonds who has been instructed to manage the portfolio in accordance with a specified policy over some finite time interval \([0, T]\). We assume, without loss of generality, that this policy can be expressed in the form of a "target security", i.e. one with risk/return attributes equal to those desired for the bond portfolio to be managed. Justification for this can be found in the work of Rothschild and Stiglitz [31], who show that investors should be concerned only with return distributions. Possible risk attributes for default-free bonds include Macaulay [23]'s duration, Cox, Ingersoll, and Ross [11]'s stochastic duration, and Brennan and Schwartz [4]'s vector of instantaneous standard deviations.
The portfolio target will often be a hypothetical security or portfolio, such as the pure discount bond maturing at date $T$ in the case of simple immunization (a dynamic target); or a constant risk target for a bond mutual fund.

The portfolio management policy requires that the portfolio be maintained "close to" the target at all times $t \in [0, T]$ in a value maximizing manner. To this end, the manager may trade in the bond market at any time to adjust, or rebalance, the portfolio, thereby incurring transactions costs to be paid *concurrently* from the transactions proceeds. The portfolio is at all times assumed fully invested in bonds and is, therefore, after the initial investment self-financing. We assume that bond holdings are infinitely divisible, with no restrictions on long and short positions, except that portfolio value is assumed to remain non-negative.

The manager's incentive to adhere to the desired policy is a reward function increasing in portfolio value, and decreasing in deviations of the portfolio's risk attributes from those of the policy target. This formulation derives from the principal/agent problem, but is in this paper taken as given.\footnote{There is no apparent restriction on the functional form of the agent's reward, other than that the resulting optimization problem for the portfolio manager be concave in trading activity.} We focus—consequently—on the agent's (manager's) partial optimization problem: optimization of his expected cumulative reward over the period $[0, T]$. We assume implicitly that such behaviour is consistent with the manager's maximization of expected lifetime utility.

The manager's optimization problem is concave, because an increase in trading activity leads to reward gains from lower attribute deviations at the expense of reward losses from increased transactions costs.\footnote{In some cases, the portfolio value maximizing strategy coincides with the one that minimizes attribute deviations, e.g. when a portfolio can be exactly immunized. In such cases, the reward function does not need a penalty component; keeping it, however, does not change the concavity of the resulting optimization problem.}

The manager's optimization problem is specified as a problem in stochastic...
optimal control (dynamic programming) in Section 4. First, however, we formalize market and portfolio dynamics in Section 3 in the framework of continuous time, very much in the tradition of Merton [27], Magill and Constantinides [24], and Cox, Ingersoll, and Ross [13]. We choose the portfolio value vector as the relevant state variable, from which we derive the dynamics of the portfolio value and risk attributes. Although this choice of state variable appears different from that in Cox, Ingersoll, and Ross [12] and Brennan and Schwartz [4,5], we show that they are nevertheless equivalent.
3. Continuous-Time Framework

3.1 Portfolio State Variables

Consider a bond market with \( m(t) \) traded bonds at each time \( t \in [0,T] \). The number of bonds is variable, as new bonds are issued from time-to-time and old ones mature. We shall, for convenience, in the remainder of this paper suppress the dependence of \( m \) on time; it is always understood that all currently traded bonds, pure discount or coupon bearing, are included in \( m \). A portfolio in this market is represented by the \( m \times 1 \) value vector \( s(t) \), i.e. the vector of each bond's dollar contribution to (or position in) the portfolio at time \( t \). Portfolio value is then defined as \( V(t) = s(t)'1 \), and the value fractions vector as \( w(t) = s(t)/V(t) \) (with \( w(t)'1 = 1 \)).

We assume that returns on \( s \) follow a diffusion described by the Itô equation

\[
\frac{ds}{s} = D(s)[\alpha dt + \Delta dz],
\]

where \( D(s) \) is the \( m \times m \) diagonal matrix formed with \( s(t) \),

\( \alpha = \alpha(s,t) \) is an \( m \times 1 \) vector of drift terms,

\( \Delta = \{\delta_{ij}(s,t)\} \) is an \( m \times m \) matrix of diffusion coefficients,

\( dz \) is the increment to an \( m \)-dimensional correlated standard Gauss-Wiener process:

\[
E[dz] = 0, \quad E[dz dz'] = C dt,
\]

\( C \) is the \( m \times m \) instantaneous correlation matrix of \( dz \), and

\( \Delta C \Delta' \) is the \( m \times m \) instantaneous covariance matrix of returns on portfolio positions.

The drift terms \( \alpha \) include adjustments for the continuously earned returns from bond coupons;\(^6\) dependence of portfolio returns on coupon rates and on maturity

\(^6\) This adjustment equals \( x_i c_i/s_i \) for the \( i \)-th bond with coupon rate \( c_i \), when the portfolio contains the quantity \( x_i \) of this bond, valued at \( s_i \).
dates is suppressed in the remainder of this paper.

Diffusion (3.1) will likely be degenerate, i.e. some or even most of the columns of diffusion coefficients matrix $\Delta$ may be zero. This will be the case if the true state-space is of dimension less than $m$. We choose, nevertheless, (3.1) as the state-variable process in this paper, because it is convenient in that portfolio trading affects the portfolio value vector $s(t)$ directly. We show in Appendix A that this choice is not restrictive in any way; in fact, we show that it is equivalent to the more common state variable processes in such bond pricing models as Cox, Ingersoll, and Ross [11,13] and Brennan and Schwartz [4,5], i.e. (3.1) spans the same state space. We take the parameters of (3.1) as given.\(^7,8\)

We also obtain from their definitions the related diffusion processes for $V(t)$ and $w(t)$,

$$dV = s^T \alpha \, dt + s^T \Delta \, dz,$$

and, using Itô's lemma,

$$dw = \alpha_w \, dt + B_w \, dz. \tag{3.3}$$

The coefficients $\alpha_w$ and $B_w$ are derived in Appendix B.1 as known functions of the state variable process parameters in (3.1).

3.2 Portfolio Risk Attributes

The diffusion coefficients matrix $\Delta(s,t)$ is the correct set of risk attributes for the returns on each of the portfolio bond positions,\(^9\) as in the bond pricing models of Cox, Ingersoll, and Ross [11,13] and Brennan and Schwartz [4,5]. Various forms

\(^7\) They may be estimated as in Brennan and Schwartz [5], see also Appendix A.

\(^8\) We also assume a competitiveness condition, where prices are not affected by the trading strategies followed by the individual portfolio manager studied in this paper.

\(^9\) Merton [28] has shown that the diffusion coefficients, or instantaneous return standard deviations, are the relevant risk measures in the Rothschild and Stiglitz [31] sense.
of "duration" risk can be derived from $\Delta$ by means of (1,1) transformations, e.g. the well-known Macaulay [23] duration and the "stochastic" duration by Cox, Ingersoll, and Ross [11]. Formally, let $A(s, t)$ be a mapping of $\Delta$ from $\mathbb{R}^m \times \mathbb{R}^m$ to $\mathbb{R}^m \times \mathbb{R}^l$, where $l < m$. Columns of the $m \times l$ matrix $A$ are then a practitioner's choice of risk attributes relevant to the portfolio management problem in this paper. We require that the mapping preserve the value-weighted addition property of these risk attributes.\(^\text{10}\)

Each $m \times 1$ column of $A$ contains the like risk attributes for all portfolio positions, following the vector-valued diffusion

$$dA_j = B_j \, dt + H_j \, dz,$$

where $A_j$ is the $m \times 1$ vector of $j$-th attributes,

$B_j$ is the corresponding $m \times 1$ vector of drift terms,

and $H_j$ the $m \times m$ matrix of diffusion coefficients for the $j$-th attribute vector.

All coefficients in (3.4) are functions of the parameters of (3.1), with exact specifications depending on the functional form of the mapping $\Delta(s, t) \rightarrow A(s, t)$.

We now define the $l \times 1$ portfolio risk attribute vector $a_p$ as a value-weighted linear compound of the rows of $A$

$$a_p = A' \, w.$$  

(3.5)

Applying Itô's lemma to $a_p$, using also (3.2) and (3.3), leads to

$$da_p = dA' \, w + A' \, dw + dA' \, dw,$$

$$= \beta_p \, dt + H_p \, dz.$$  

(3.6)

Expressions for $\beta_p$ and $H_p$ in terms of the parameters of (3.1) are derived in Appendix B.1.

\(^{10}\) This is the case for each of the risk measures listed above; corresponding diffusion coefficients are additive because their $dz_i$ processes are perfectly correlated.
3.3 Portfolio Risk Target

Portfolio management objectives are specified in terms of an $l \times 1$ target attribute vector $a_T$. We assume that $a_T$ follows a deterministic time-path, or

$$da_T = \beta_T dt,$$

(3.7)

where $\beta_T$ is a known $l \times 1$ vector of drift terms, at most a function of $s(t)$. A popular choice for the process $da_T$ in bond portfolio management derives from the immunization motive, whereby the portfolio is maintained at the risk level of a zero-coupon bond maturing at a given date.$^{11}$

Portfolio risk attribute deviations are next defined as elements of the $l \times 1$ vector $a = (a_p - a_T)$, the difference between the portfolio attribute vector, $a_p$, and the target attribute vector, $a_T$. It follows the diffusion process:

$$da = \beta dt + H dz,$$

(3.8)

where $\beta = (\beta_p - \beta_T)$, $l \times 1$

$\quad H = H_p$, $l \times m$.

3.4 Transactions Costs

Transactions costs applied to portfolio trades result in lower proceeds from sales and higher outlays for purchases. Let $u$ be the discrete change in portfolio value vector $s$ due to a trade; $u_i < 0$ when bond $i$ is sold, $u_i > 0$ when it is purchased, and $u_i = 0$ when it is not traded. Define the functions $\gamma_i(u_i)$ for traded bonds as multipliers, such that

$$u_i \gamma_i(u_i) > u_i.$$  

(3.9)

$^{11}$ The risk profile $a_T$ is then linear and downward sloping, i.e. $\beta_T$ is a negative constant, when the risk is measured in terms of Macaulay [23] duration.

$^{12}$ The manager's reward may be specified as a decreasing function of $a'a$ if signed deviations are not desired.
In words, we require that the proceeds of selling a quantity of bond \( i \) be less than the market value of the quantity sold, while the purchase price of buying a quantity of that bond must be higher. The difference in each case is paid out as transactions costs.

This definition allows transactions costs with both fixed and variable components. Functional forms for the trade multipliers \( \gamma_i \) are assumed given. Definition (3.9) also defines the vector-valued function \( \gamma \) with maximum dimension \( m \), or less when not all bonds are traded. Its elements are, by (3.9), consistent with a particular combination of bonds sold, purchased, or not traded, giving rise in subsequent sections to the terminology of "\( \gamma \)-consistent" trades.

We impose a self-financing constraint on portfolio trades in this paper, i.e. we require that transactions costs be applied concurrently to transactions.\(^{13}\) If we let \( I_b \) and \( I_s \) be the index sets of the bonds bought and sold, respectively, we require that

\[
\sum_{i \in I_s} \gamma_i u_i + \sum_{i \in I_b} \gamma_i u_i = 0. \tag{3.10}
\]

Constraint (3.10) defines a feasible region \( U \subset \mathbb{R}^m \) for any admissible "\( \gamma \)-consistent" trade \( u \). It also implies that portfolio value \( V \) is always fully invested.

Short positions are not prohibited per se, but are constrained by the further requirement that portfolio value \( V(t) \) must remain non-negative on \([0,T]\). This reasonable economic constraint only rules out certain very large offsetting long and short positions that are not of practical interest. Problem parameters can be specified such that this constraint is never binding in practice; this is assumed in the following sections.

\(^{13}\) We assume that prices and risk attributes are not affected by the transactions costs imposed on the model. We thus ignore the pricing "fuzziness" predicted by Garman and Ohlson [19].
4. Manager’s Optimal Control Problem

4.1 Impulse Control Problem

The portfolio is controlled by trading in the \( m \) constituent bonds. We define a trade \( u \) as the \( m \times 1 \) vector of discrete changes in the portfolio value vector \( s \). Trades \( u \) are discrete because the benefits from trading do not always outweigh the associated transactions costs. The optimal trade problem may then be formulated as one in impulse control, i.e. a stochastic dynamic programming problem involving diffusions with jumps, following Menaldi [26].

The portfolio value vector \( s(t) \) follows a continuous path between trading (or stopping) times, but has a discrete jump adjustment \( u \) in the event of a trade. The diffusion process (3.1) for \( s \) is now redefined as one with jumps:

\[
d s(t) = D(s)[\alpha \, dt + \Delta \, dz] + \sum_{n=1}^{N} u_n \delta(t - \theta_n) \, dt,
\]

where \( \alpha \) and \( \Delta \) are known functions of \( s \) and \( t \), \( D(s) \) is a diagonal matrix, and \( \delta(t - \theta_n) \) is the Dirac measure.\(^{14}\)

We define an admissible impulse control \( \nu \) as a sequence of pairs \((\theta_n, u_n)\), where \( \{\theta_n\}_{n=1}^{N} \) is an increasing sequence of stopping times on \([0, T]\), and \( \{u_n\}_{n=1}^{N} \) is a sequence of random discrete (non-zero) portfolio value adjustments taking values in \( U \subset \mathbb{R}^m \). The number of adjustments, or jumps, \( N \) may be infinite. Stopping times \( \theta_n \) are \( \gamma \)-consistent, i.e. the signs of elements \( u_{ni} \) are consistent with the transactions costs implied by the value of the trade operator \( \gamma \) described in Section (3.4). We give in Appendix F a more detailed description of impulse trades, stopping times, and diffusions with jumps, based on Menaldi [26].

\(^{14}\) The Dirac measure is defined such that integration over an interval containing zero yields the Heavyside unit step function, indicating that the value of \( s \) has been changed by \( u \). Intuitively, \( u \, \delta(0) \) may be interpreted as an infinite trade velocity, with zero velocity for any other argument.
We have motivated in Section 2 that the portfolio manager's objective is to maximize a reward function over \([0, T]\). The reward function increases in portfolio value, \(V = s't\), and decreases in portfolio attribute deviations, \(a = \frac{1}{T} A's - a_T\). Each of these is a function of the state variable \(s(t)\) and of the deterministic portfolio target \(a_T(t)\), with dynamics derived in Section 3 and in Appendix B.

We define the manager's reward accrual rate

\[ e^{-\rho t} g(s(t)), \tag{4.2} \]

where the positive constant \(\rho\) is a time patience or discount rate, and where \(g(\cdot)\) is a function to be specified later. The manager's expected cumulative reward at time \(t\), given the current state \(s(t)\) and an admissible control \(\nu(t)\), is then defined as

\[ J_{t,s(t)}(\nu) = E_t \int_t^T e^{-\rho \tau} g(s(\tau)) d\tau. \tag{4.3} \]

Mathematically, the manager's problem is to find the optimal impulse control \(\tilde{\nu}(t)\) that maximizes the expected cumulative reward, or

\[ \tilde{W}(t, s(t)) = \sup \{ J_t(\nu) | \nu \text{ admissible} \}, \tag{4.4} \]

i.e. both the optimal trading times and the optimal trades must be determined.

4.2 Optimal Reward Function

The control problem as defined above is similar to the problem studied by Menaldi \([26]\),\(^15\) who proves existence of the optimal reward function \(\tilde{W}\) and characterizes it.

\(^{15}\) Menaldi studies the optimal control of degenerate diffusions with jumps, i.e. where one or more of the sources of uncertainty may in fact be deterministic. Menaldi's model is specified as a minimum cost problem (requiring sign, min/max, and inf/sup reversals, where appropriate) and is also more general because it associates with each impulse trade a fixed cost element (to the controller) in addition to the continuous cost rate. Finally, he uses an infinite time horizon. None of his results are affected, however, by the simplifications in the portfolio manager's problem presented in this paper.
First, define the Dynkin differential operators
\[ L^0 = \frac{1}{2} \text{trace} \{ D(s) \Delta C \Delta' D(s) \frac{\partial^2}{\partial s^2} \} + D(s) \frac{\partial}{\partial s}, \]  \tag{4.5} 
and \( L = L^0 + \frac{\partial}{\partial t} \). Further define by \( M \) the optimal trade operator
\[ [M\Phi](s) = \sup \{ \Phi(s + u) | u \in U \}, \]  \tag{4.6} 
i.e. the maximum value attainable for some function \( \Phi(s) \) with an admissible jump trade \( u \). The set \( U \) of admissible impulse trades is assumed sufficiently smooth so that \( M\Phi \) is continuous if \( \Phi \) is.\(^\text{16}\)

Then, by Menaldi \([26]\)'s Theorem 0.1, and assuming some mild regularity conditions for the coefficients of (4.1),\(^\text{17}\) the optimal reward function \( \widetilde{W}(t) \) exists and is continuous. It is given by the minimum solution to the following variational inequalities, or V.I.,
\[ W(T, s) = 0, \]  \tag{4.7a} 
\[ W(t, s) \geq MW(t, s), \]  \tag{4.7b} 
\[ -LW(t, s) \geq e^{-\rho t} g(s), \]  \tag{4.7c} 
with
\[ (W - MW)(LW + e^{-\rho t} g) = 0, \]  \tag{4.7d} 
satisfying either (4.7b) or (4.7c) as equality, depending on whether trading or not trading is optimal. Furthermore, an optimal impulse control \( \nu(t) \) exists.

Menaldi \([26]\] also shows that the optimal reward function, \( \widetilde{W} \), is the limit as \( n \to N \) of the minimum solutions \( \widetilde{W}^n \) to a sequence of optimal stopping time problems, each one starting immediately after an impulse trade \( u_{n-1} \). Each \( \widetilde{W}^n \)
\(^\text{16}\) This is satisfied for proportional transactions costs, a case which we shall consider later.\(^\text{17}\) \( D(s)\alpha \) and \( D(s)\Delta \) are Lipschitz continuous, \( g(\cdot) \) is continuous and non-negative.
solves a set of V.I. similar to (4.7), except that the operator $M$ is applied to the previous solution $\overline{W}^{n-1}$ in the sequence:

$$W^n(T, s) = 0, \quad (4.8a)$$

$$W^n(t, s) \geq M\overline{W}^{n-1}(t, s), \quad (4.8b)$$

$$-LW^n(t, s) \geq e^{-\rho t}g(s). \quad (4.8c)$$

Each optimal stopping time is now characterized by condition (4.8b) as an equality, or the marginal gain from trading equals zero.

We do not know how to solve conditions (4.7) in the general case, while the algorithmic implications of (4.8) are beyond the scope of this paper. We are able, however, to characterize the optimal trade control in the special case of myopia.

4.3 A Special Case

The simplifying assumptions of this subsection enable us to derive the optimal impulse control in the special case of myopia, or the portfolio manager will always optimize the expected reward rate $g(s(t))$ over the next instant, rather than its discounted integral. We know from utility theory that this objective is consistent with logarithmic preferences.

(1) Proportional transactions costs are assumed.\(^{18}\)

Self-financing constraint (3.10) on the optimality operator $M$ may now be written as

$$\gamma'u = 0. \quad (4.9)$$

The trade functions $\gamma_i$ are here simplified to positive trade ratios $< 1$ or $> 1$, depending on whether bond $i$ is sold or purchased, respectively. The real vector

\(^{18}\) While a combination of fixed and variable transactions costs occurs more frequently in practice, the case of proportional transactions costs is not as restrictive as appears at first glance. Overhead and management costs will on average be related to the amount of trading or active management of the portfolio. One way of accounting for this is to increase the trade ratio $\gamma$, with the additional “transactions costs” representing a portfolio management fee.
\( \gamma \) is arbitrarily extended to dimension \( m \) for non-traded bonds, for which \( u_i = 0 \). Admissible trade directions are then contained in a manifold of dimension \( m \) or less through the current value of the state variable \( s(t) \in \mathbb{R}^m \). Because infinitesimally small trades are now possible, we have in effect attained a control problem with reflecting barriers, where optimal impulse trades are characterized by their trade direction, or “reflection angle,” at the boundary. This assumption transforms the optimal trade operator \( M \) in (4.6) into a classical maximization problem from which the optimal reflection angle may be determined in Section 5 using the Lagrange multiplier method.

(2) We assume that the optimal value function \( \widehat{W}(s, t) \) has a multiplicatively separable form\(^{19} \)

\[
\widehat{W}(s, t) = e^{-\rho t} f(t) g(s). 
\] (4.10)

This assumption meets the continuity condition of Menaldi’s theorem in (4.7) when trade pulses are infinitesimally small.

(3) The reward rate function \( g(s) \) has a suitable form, such that \( D(t) = L^0 g(s)/g(s) \) is at most a function of time \( t \).\(^{20} \)

Substitution of (4.10) into equality (4.7c) then yields the first-order linear ordinary differential equation (o.d.e.) in \( f(t) \)

\[
\dot{f} + [D(t) - \rho] f + 1 = 0, 
\] (4.11)

with boundary condition \( f(T) = 0 \). The boundary condition ensures that condition

\(^{19} \)Intuitively, the expected future reward is based on the current reward rate (in terms of the current portfolio composition) adjusted for time effects.

\(^{20} \)This assumption is satisfied in the time-homogeneous case by solutions to the second order o.d.e. with non-constant coefficients

\[
L^0 g + kg = 0, 
\]

where \( k \) is a constant. It is well-known that solutions \( g(s) \) can be found in principle as Taylor-series expansions around values of the state variable \( s \).
(4.7a), \( \hat{W}(T) = 0 \), is met. The solution to this o.d.e. is well known, though not trivial.

We need not actually solve for \( f(t) \), however, if we are only interested in the optimal reflection angle, because the term \( e^{-\alpha t} f(t) \) is unaffected by the optimal impulse control: it cancels from the optimization operator \( M \), and also from the optimal control solution found in Section 5, thereby demonstrating the myopia property referred to above.

Solutions of this type have also been studied by Merton [27] and Magill and Constantinides [24] in conjunction with lognormal diffusion processes, i.e. similar to the assumption of constant parameters \( \alpha, \Delta, \) and \( C \) in (3.1) or (4.1). Such assumptions are in general not tenable in bond markets, unless long maturity bonds are considered over a comparatively short time horizon. Our assumption (3) addresses this difference.\(^{21}\)

While a solution in accordance with these assumptions satisfies the optimality conditions (4.7), it may not be a unique solution to the problem. We may rely, however, on a verification theorem in Fleming/Rishel [18] (Theorem 4.1, p. 159), that states (in our terminology): if \( \hat{\nu} \) is an admissible control such that (4.7) hold as equalities, then the reward (4.4) \( \hat{W}(t) = J_t(\hat{\nu}) \) has indeed a maximum and \( \hat{\nu} \) is optimal.

We show in Section 5 how the optimal trade in this special case can be derived.

\(^{21}\) Magill and Constantinides [24] require only assumptions (1) and (2) to derive their equivalent of \( f(t) \). Restatement of their control problem in terms of Menaldi [26]'s work then obviates Duffie [15]'s critique (see further Appendix F).
5. An Optimal Trade Solution

5.1 An equivalent \( \epsilon \)-perturbed Problem

We motivate in Appendix F the similarity at an optimal stopping time \( \theta_n \) between the impulse control problem of Section 4 and a continuous control problem treated by Magill and Constantinides [24], which in our notation transforms process (4.1) into

\[
ds(t) = [D(s)\alpha + v] \, dt + D(s)\Delta \, dz. \tag{5.1}
\]

This process derives from that in (4.1) if we interpret the continuous trading rate \( v(t) \) as \( \sum_n a_n \delta(t - \theta_n) \), see also Appendix B.2. The authors then mimic the on/off nature of the impulse control by introducing an \( \epsilon \)-perturbation in the diffusion term of (5.1)

\[
ds(t) = [D(s)\alpha + v] \, dt + [D(s) + \epsilon D(v)]\Delta \, dz, \tag{5.2}
\]

As \( t = \theta_n \) and \( \epsilon \to 0^+ \), the processes (5.1) and (5.2) coincide, with the optimal trading rate \( v^* \) lying in the same direction as the optimal impulse trade \( u^* \). We do not expect the optimal trading rate \( v^* \) to be finite, since it contains the Dirac function \( \delta(0) \), from (4.1); this does not affect the direction of the trade, however.

By analogy with solutions of continuously controlled diffusions, the optimal trade direction is then determined by the following static maximization problem (the Bellman equation)\(^\footnote{Problem (5.3) is equivalent to the more general V.I. formulation in (4.7), with the maximization taking the place of the sup-operator \( \mathcal{M} \).}

\[
\max_{v} e^{-r \ell} g(s) + L^* W(s, t) + \frac{\partial W}{\partial t} = 0, \tag{5.3}
\]

subject to the transactions cost constraint (4.9) suitably transformed for \( v \)

\[
\gamma' v = 0. \tag{5.4}
\]
The maximization in (5.3) takes place at an optimal stopping time \( \theta_n \), for which the trade ratio \( \gamma \) is not a variable but is uniquely defined by the definition of the optimal stopping time.

The Dynkin operator \( L^v \) is now defined as

\[
L^v = \frac{1}{2} \text{trace} [D(s + \epsilon v) \Delta C \Delta'^t D(s + \epsilon v) \frac{\partial^2}{\partial s^2}] + [D(s) \alpha + v]^t \frac{\partial}{\partial s}. 
\]

(5.5)

5.2 A Solution

Using the special assumptions of subsection (4.3), the static maximization problem in (5.3) can now be restated in terms of the Lagrangean of the myopic optimization problem

\[
\max_{v, \lambda} L^v g(s(t)) - \lambda \gamma^t v,
\]

or

\[
\max_{v, \lambda} \frac{1}{2} \text{trace} [D(s + \epsilon v) \Delta C \Delta'^t D(s + \epsilon v) g_{ss}] + [D(s) \alpha + v]^t g_s - \lambda \gamma^t v.
\]

(5.6)

We show in Appendix C how (5.6) and the subsequent derivation of the optimal control \( v^*(t) \) can be expressed equivalently in terms of the vector \( y(s) = (V, a')' \), where portfolio value \( V = s' \mathbf{1} \) and risk attribute deviation vector \( a = (A's/s') \mathbf{1} - a_T \) are functions of the state variable \( s \) and of the deterministic portfolio target \( a_T \).

The first-order-conditions of (5.6) w.r.t. \( v \) and \( \lambda \) are\(^{23}\)

\[
\frac{1}{2} \nabla_v \text{trace} [D(s + \epsilon v) \Delta C \Delta'^t D(s + \epsilon v) g_{ss}] + g_s - \lambda \gamma = 0,
\]

\[
\gamma^t v = 0.
\]

(5.7)

System (5.7) is solved for the optimal \( v^* \) in Appendix C, where we also give sufficient second-order conditions for a maximum solution.

\(^{23}\) \( \nabla_v \) denotes differentiation w.r.t. the elements of vector \( v \), resulting in a vector or a matrix as the case may be. Derivatives of the reward function \( g(s) \) are indicated by subscripts.
If we write $\Sigma$ for the $m \times m$ matrix $\Delta C \Delta' \mathbb{D} g_{ss}$, and $\xi$ and $\eta$ for the the $m \times 1$ vectors $g_s$ and $\Sigma s$, respectively, then the optimal trading rate solution for the $\epsilon$-perturbed problem is

$$v^*(t) = \frac{1}{\epsilon^2} \left\{ \frac{(\gamma'\Sigma^{-1}\xi) \Sigma^{-1}\gamma - (\gamma'\Sigma^{-1}\gamma) \Sigma^{-1}\xi}{\gamma'\Sigma^{-1}\gamma} \right\} + \frac{1}{\epsilon} \left\{ \frac{(\gamma'\Sigma^{-1}\eta) \Sigma^{-1}\gamma - (\gamma'\Sigma^{-1}\gamma) \Sigma^{-1}\eta}{\gamma'\Sigma^{-1}\gamma} \right\}. \quad (5.8)$$

Appendix C gives an alternative specification of $\Sigma$, $\xi$, and $\eta$ in terms of $V$ and $a$. We give an economic interpretation of the vectors $\xi$ and $\eta$ in the next subsection.

The dimension of (5.7) and (5.8) is adjusted in each trade case for the bonds that are actually transacted, in keeping with the $\gamma$-consistency definition of a stopping time. Non-traded bonds are therefore not considered in the above optimization.\(^{25}\)

The optimal control $u^*(\theta_n)$ for the impulse control problem at stopping time $\theta_n$ equals that of the $\epsilon$-perturbed problem in (5.8) when $\epsilon \to 0^+$. Taking limits as $\epsilon \to 0^+$ shows that the optimal trading rate $v^*(t)$ in (5.8) becomes unbounded for each $\gamma$-consistent non-zero solution. The $\epsilon$-methodology shows thus clearly the impulse quality of the control, i.e. trading at infinite rate.

While expression (5.8) becomes unbounded, we note that its second term becomes insignificant w.r.t. the first term as $\epsilon \to 0^+$, i.e. the first term “blows up” faster. We may therefore in the limit disregard the second term. Because the optimal trade direction is preserved when taking limits, we arrive then at the following expression for the optimal impulse control direction (call it $v$) at optimal stopping

\(^{24}\) $\mathbb{D}$ is the matrix box-product operator for identically dimensioned matrices: elements of the product matrix consist of the products of corresponding elements in the operand matrices.

\(^{25}\) This caveat is necessary to avoid Duffie [15]'s objection that otherwise $v_t = 0$ is not a solution of (5.7) for non-traded bonds $t$. 
time $\theta_n$

$$v^*(\theta_n) = \frac{(\gamma'\Sigma^{-1}\xi) \Sigma^{-1}\gamma - (\gamma'\Sigma^{-1}\gamma) \Sigma^{-1}\xi}{\gamma'\Sigma^{-1}\gamma}.$$  \hfill (5.9)

5.3 Economic Interpretation of Optimal Control

Expressions (5.8) and (5.9) constitute systems of $m$ linear equations in $\xi$ and $\eta$, with a pleasing symmetry in the coefficients, which in turn depend on the $m \times m$ weighting matrix $\Sigma$ and the trade ratio vector $\gamma$.

The $m \times 1$ vector $\xi$ is defined as the first derivative $g_s$ of the reward rate $g(s(t))$. It is therefore, economically speaking, proportional to a vector of elasticities of the reward rate to changes in the portfolio composition $s$. We shall, in the remainder of this paper, refer to $\xi$ as the elasticities vector.

The matrix $\Sigma = \Delta C\Delta' \square g_{ss}$ represents a generalized covariance matrix of the bond positions $s(t)$, weighted by their respective impacts on the reward rate function $g(s(t))$.

The vector $\eta$ is defined as $\Sigma s$, or as $[\Delta C\Delta' \square g_{ss}] s$. It describes an intertemporal or second order effect on the bond positions $s(t)$ that disappears from the optimal control (5.8) as $\epsilon \to 0^+$ and the control is applied instantly.

These interpretations remain valid when $\xi$, $\eta$, and $\Sigma$ are expressed in terms of $V$ and $a$ in Appendix C.
6. Trade/No Trade Regions

We describe in this Section the conditions under which non-zero $\gamma$-consistent admissible trades exist for the myopic problem, i.e. trades $u$ proportional to

$$v^*(\xi; \gamma) = \frac{(\gamma^T \Sigma^{-1} \xi) \Sigma^{-1} \gamma - (\gamma^T \Sigma^{-1} \gamma) \Sigma^{-1} \xi}{(\gamma^T \Sigma^{-1} \gamma)},$$

(6.1a)

while also satisfying

$$\text{sign}(\gamma_i - 1) = \text{sign}(v_i), \quad i = 1, \ldots, m. \quad (6.1b)$$

The trade ratios are $\gamma$-consistent: $0 < \gamma_i < 1$ or $\gamma_i > 1$ when bonds $i$ are sold or purchased, respectively.

Optimal trade directions in accordance with (6.1) define $v^*$ as a function of the elasticities vector $\xi$, written more conveniently as

$$v^*(\xi; \gamma) = \Gamma \xi.$$  

(6.2)

The symmetric $m \times m$ matrix $\Gamma = (\Sigma^{-1} \Sigma^{-1} \gamma \Sigma^{-1} - (\gamma^T \Sigma^{-1} \gamma) \Sigma^{-1})/(\gamma^T \Sigma^{-1} \gamma)$ depends on current market and portfolio characteristics and on the trade ratio vector $\gamma$.

The actual dimension of (6.1) or (6.2) is the number of bonds that participate in the optimal non-zero $\gamma$-consistent trade combination. The $m$ bonds in the market may be traded in $N = \sum_{x=2}^{m} \binom{m}{x} \sum_{y=1}^{x-1} \binom{x}{y}$ possible combinations of bonds sold, purchased, or not transacted; here $x$ is the number of bonds transacted ($2 \leq x \leq m$), of which $y$ bonds are sold.\footnote{The number $N$ grows very large with $m$, i.e. $N \approx 3.5$ billion for $m = 20$.} Each such trade combination involving $x$ bonds has its own version of the matrix $\Gamma$; we write $\Gamma^x(\gamma)$ for this $x \times x$ matrix, which also depends on the inverse of the $x \times x$ weighting matrix $\Sigma^x$ obtained from $\Sigma$ by deleting the rows and columns corresponding to the non-transacted bonds.

Each $\gamma$-consistent trade combination satisfying system (6.1) may now be associated with a set of $x$ strict inequalities $\Gamma^x_i \xi < 0$ or $\Gamma^x_i \xi > 0$, which delineate...
an open trade region $\chi_T^\gamma(\gamma)$ in the space $\chi \subset \mathbb{R}^m$ of elasticities $\xi$, bounded by the hyperplanes $\Gamma^\gamma \xi = 0$. Each inequality, $< \text{or} >$, conforms to the $\gamma$-consistency requirement (6.1b) for traded bonds, as the case may be. The optimal trade $u^*(\theta_n)$, if it exists, satisfies these inequalities by the definition of an optimal stopping time $\theta_n$ (see also Appendix F), i.e. the associated $x \times 1$ vector $\xi(\theta_n)$ of elasticities plots just inside the corresponding trade region.

Conversely, each point $\xi$ in a particular trade region $\chi_T^\gamma(\gamma)$ corresponds to a solution $v^*(\theta_n)$ to (6.1), though not necessarily a unique optimal trade for the impulse control problem of Section 5. This is because the $N$ possible trade regions that may be formed in the above manner can overlap, leaving in question which trade combination is optimal. Overlaps are of two kinds: (i) mutually exclusive (incompatible) and (ii) compatible.

We follow Magill and Constantinides [24] in disallowing all mutually exclusive trade combinations for the economic reason that repeated two-way trades cannot be optimal, because transactions costs are then spent without affecting portfolio attributes.\(^{27}\) In the case of overlaps of trade regions that are not mutually exclusive we favour the one with the greatest number of traded bonds.\(^{28}\)

Determination of a single optimal trade combination, if any, is not a simple task in higher dimensions as the number of possible trade combinations grows very

\(^{27}\) For example: selling bonds 1 and 2, while buying bond 3, or $1 + 2/3$, is then incompatible with the opposite combination $3/1 + 2$. However, $1 + 2/3$ and $1/2 + 3$ are not mutually exclusive, while $1/2$ and $1/3$ are compatible with either choice. Note that combinations $1 + 2/3$ and $1/2$ are not necessarily mutually exclusive.

\(^{28}\) The optimal trade $u^*(\theta_n)$ lies in the hyperplane $\gamma' u = 0$, the self-financing constraint (4.9). It has dimension $x - 1$, where $x$ is the number of traded bonds. As $x$ increases, the optimal trade becomes less restricted by the self-financing constraint, as only the ratio of the totals sold and purchased is fixed, but not the actual amounts sold and purchased within each of these bond groups. The lesser restriction translates into a greater "length", or Euclidean norm for the vector $v^*(\theta_n)$ solved from (6.1). Further to the example of the previous footnote, combination $1 + 2/3$ must then preferred over $1/2$, when both are allowable, because portfolio improvement should then be possible at less cost.
large. We discuss in Appendix E a simple algorithm for obtaining a practical approximation to the optimal trade combination.

The complement to the union of all trade regions \( \chi_T^c(\gamma) \) is the closed region \( \chi_N \subset \mathbb{R}^m \) of no trades. The advantage of trading is outweighed by transactions costs for portfolios whose \( \xi \)-vector plots in this no-trade region. Trading is not optimal for such portfolios.

We show in Appendix D that the no-trade region is a cone, convex with respect to the origin, in the positive and negative orthants of \( \mathbb{R}^m \) arranged with radial symmetry around the diagonal \( \xi_1 = \xi_2 = \ldots = \xi_m \). The cone shrinks to the diagonal itself when there are no transactions costs. In that case, all bond elasticities are all equal, the no-trade region has no interior, and continuous trading will take place in all bonds (without affecting current portfolio value) so as to keep the portfolio exactly on target. With transactions costs, no trading is optimal as long as the elasticities are approximately equal, i.e. when they plot in the no-trade cone around the diagonal.

While from (6.2) the allowable trade combinations depend on the weighting matrix \( \Sigma \), the no-trade cone depends in a lesser way on \( \Sigma \): its convex hull is bounded by all \emph{two-dimensional} hyperplanes \( \Gamma^2 \xi = 0 \) corresponding to trade combinations involving two bonds. Such bonds can only be traded in the proportions \( \gamma_s/\gamma_b \), so that the convex hull of the no-trade cone is independent of market dynamics. Two and three dimensional cases are illustrated in Appendix D and in Figures 1, 2, and 3.
7. Application of the Optimal Trade Rule in Discrete Time

The optimal portfolio impulse control problem consists of two parts: (i) determine the optimal stopping or trading times \( \{\theta_n\} \), and (ii) determine optimal impulse trades. Both can be defined in terms of the trade/no-trade regions described in Section 6.

No trading takes place as long as the portfolio's elasticities vector \( \xi \) plots in the current no-trade region, including its boundary, because the costs of trading outweighs the expected benefits. The very moment the elasticities vector attempts to breach the boundary of the no-trade region, however, impulse trading is optimal and a stopping time \( \theta_n \) is reached; an infinitesimal trade takes then place in the optimal direction \( v^*(\theta_n) \) to keep the elasticities vector on the boundary. The resulting rebalancing scheme ensures that the elasticities vector \( \xi(t) \) will never plot outside the no-trade cone if optimal stopping times are continuously monitored on \([0, T]\).

While the above impulse trades are discrete, an infinite number of infinitesimal "pulses" may nevertheless be required over any finite time interval to keep the \( \xi \) vector on the boundary of the no-trade cone if market forces persist in attempting to move it outside the no-trade cone.\(^{29}\) We conjecture that this may well be a problem in bond markets, whose risk attributes (e.g. duration) tend to decline gradually over time.

Magill and Constantinides [24] have argued, in a general securities market, that the above type of optimal trading leads to the discrete trading behaviour observed in the market place (i.e. a finite number of trades), thereby "obviating a major drawback to Merton's continuous trading". Yet, the gain may only be one of cardinality: an infinite number of discrete trades, as opposed to continuous trades.

\(^{29}\) This can be avoided by introducing a fixed component to the transactions costs of each trade, no matter how small, see e.g. Richard [29].
7.1 Discrete Monitoring

At the very least, the optimal trade rule of this paper requires continuous monitoring of the portfolio and of the market to determine the precise next optimal stopping time at which to trade and to keep the portfolio on the no-trade cone boundary. If this is not believable investor behaviour, then we must consider a truly discrete trading model: optimal stopping times are now taken from a finite set of (possibly stochastic) discrete times $T = \{0 = t_1, t_2, \ldots, t_{n-1}, t_n = T\}$, which are the only occasions on which the portfolio manager will rebalance the portfolio, provided that trading is then optimal. Although such a stopping rule does not appear to invalidate Menaldi [26]'s results quoted in (4.7) and (4.8), the posited solution (4.10) for $W$ and the solution (5.9) for the optimal trade direction do no longer apply in this case, because continuity of the optimal value function is no longer ensured.

Provided that we select discrete time steps in $T$ that are small w.r.t. the portfolio and bond market Brownian motions, however, we may approximate the true optimal impulse control in this case with that of the discretely monitored model.\footnote{Even if we could derive the correct solution to the discretely monitored dynamic programming problem, both speculative and other non-anticipative trades may now exist that outperform it. We know (e.g. from Fleming/Rishel [18]) that this cannot happen when there are no restrictions on application of the control.}

We adopt the following stopping rule in such a discrete market: trading is optimal at the infimum $\theta_n$ of all future times $t \in T$ at which a non-zero, $\gamma$-consistent solution for the optimal trade direction $\nu^*(\theta_n)$ exists using the methodology of Section 6. The difference between this case and that of the continuously monitored portfolio is a loss of performance: in terms of a dynamic programming solution, the optimal value function $W(s,t)$ will have a (weakly) smaller value.\footnote{Denny and Suchanek [14] study properties of approximate trading strategies applicable to the state variable processes in this paper. Their Theorem 2 states that the expected proceeds of an optimal trading strategy are stable in probability}
A consequence of discrete monitoring is that now the portfolio elasticities vector \( \xi(t) \) may be found to plot outside the no-trade cone. This requires finite trades, which would be impossible in the continuously-controlled case. With sufficiently frequent monitoring, however, it will still be close, so that the optimal trade is approximated by a small finite trade along the optimal trade direction \( v^*(\theta_n) \) until the no-trade boundary is reached.\(^{32}\)

7.2. Discrete Trading along \( v^*(\theta_n) \)

Suppose that \( t \) is one of the discrete monitoring times, and that the portfolio elasticities vector \( \xi(t) \) plots outside the no-trade cone. In this case, \( t = \theta_n \), a stopping time, so that we trade in the bond combination indicated by the optimal trade direction \( v^*(\theta_n) \) in the space \( S \) of portfolio values. The initial portfolio value vector \( s \) will at the completion of the trade have moved in the direction \( v^*(\theta_n) \) to a point \( s^* \) such that its accompanying elasticities vector \( \xi^* \) now plots on the no-trade boundary in elasticities space. It is clear from the maximization in (5.6) that this problem is equivalent to maximizing the reward rate \( g(s(t)) \) by a line search along the optimal direction.

Numerical examples in Appendix G use the following simple algorithm to trade back to the no-trade boundary when trading is optimal:

1. Calculate at each discrete monitoring time \( t \) the elasticities vector \( \xi(t) \), and find the optimal trade direction \( v^*(t) \) (see Appendix E) satisfying (6.1), taking against small changes, such as induced by confining stopping times to a set of (possibly random) discrete times. Their Theorem 3 states a similar result for speculative trading strategies vs. non-speculative ones. The implication of their results for the present paper is that using the trade direction \( v^*(\theta_n) \) in the discretely monitored case gives results “close to” those of the original problem, as long as the frequency of monitoring is reasonably high w.r.t. the state-variable Brownian motions.

\(^{32}\) It may now well be optimal to trade to a point slightly inside the no-trade region. With less frequent monitoring, the distance from \( \xi \) to the current no-trade boundary may become so large that the trade direction \( v^*(\theta_n) \) does not intersect the no-trade region in higher dimensional cases. In such cases the continuous-time approximation is not valid, and the frequency of monitoring must be increased.
into account the proper dimensionality of this system. If no trade combination is optimal, \( t \) is not an optimal stopping time and \( \xi(t) \) plots inside the no-trade region. If an optimal trade combination is found, \( \theta_n = t \) is an optimal stopping time, by definition.

(2) If \( t = \theta_n \) is a trading time, perform a line search in the optimal trade direction \( v^*(\theta_n) \). Objective of the line search is to find the scalar \( c \) such that the elasticities vector \( \xi^* \) of the portfolio \( s^*(\theta_n) = s(\theta_n) + cv^*(\theta_n) \) plots on the no-trade boundary, or—alternatively—such that the reward rate \( g(s(t)) \) is maximized.

The line search of step (2) can be any one of standard operations research procedures.

The examples in Appendix G confirm our intuition that trading tends towards equalizing the bond elasticities in the portfolio, and also that trading maximizes the reward rate \( g(s(t)) \) in the best \textit{dynamic} direction \( v^* \), which maximizes the expected change in \( g(s(t)) \) over the next instant, rather than in the best \textit{static} direction.
8. Summary and Conclusions

Optimal institutional portfolio management has been treated in this paper as the residual problem of reward maximization for the portfolio manager. This problem is well-defined and derives from principal/agent interaction, leading to a negotiated incentive reward contract for the manager. The typical form of such a contract rewards value maximizing actions and penalizes deviations of the portfolio's risk attributes from those of a specified policy or "target security." Thus, it allows the manager to act on private information and to deviate from the risk policy if the value gains are deemed worthwhile; it also discourages needless incurring of transactions costs. This specification dispenses with the need to engage in separate ex post performance measurement.

While the model developed in this paper is valid for all financial securities, we apply it specifically to the government (default-free) bond market, as it provides the trading motive often lacking in well-diversified stock portfolios. Portfolio attributes may then, e.g., be expressed as "duration".

We formulate the manager's reward maximization problem as one in stochastic optimal impulse control, using results by Menaldi [26]. This formulation improves on one by Magill and Constantinides [24], who use a continuous control specification and then approach the equivalent of impulse control by means of a limiting argument. They do not, however, define their control problem as a sequence of optimal stopping time problems with infinitesimal portfolio adjustments at the boundary. Duffie [15] has pointed out certain inconsistencies in that approach. Menaldi [26]'s work, however, provides the additional required theory that obviates Duffie [15]'s critique and allows use of Magill and Constantinides [24]'s limiting methodology in the above specified problem.

The optimal trade rule derived in this paper applies to the special case of
proportional transactions costs and myopia, i.e. the manager optimizes the expected change in his reward rate over the next instant. This case also applies more generally, without myopia, to securities whose returns follow time-homogeneous diffusions—as in a stock market. The optimal trade rule is functionally expressed in terms of generalized bond price covariances, transactions costs, and partial elasticities of the reward function w.r.t. the portfolio composition (value fractions). These partial elasticities depend on the current values of the portfolio and its attribute deviations.

Trading is not always optimal: given transactions costs and bond price covariances, we show that open regions exist in the space of elasticities, bounded by hyperplanes, where one or more bond trade combinations improve the manager’s optimal value function. In addition to these trade regions for all possible combinations of traded and untraded bonds, we also have a no-trade region, where trading is not sufficiently valuable to the manager to offset the negative effect of the incurred transactions costs. Predictably, the no-trade region shrinks to the diagonal in elasticities space when trading is costless: now all bond elasticities are the same and continuous trading takes place to keep the portfolio exactly on target: we are back at the Merton [27] case. With transactions costs, however, the no-trade region in elasticities space is a cone arranged around the diagonal; it expands as transactions costs increase.

An optimal impulse trade causes instantaneous rebalancing of the portfolio at the very moment the portfolio elasticities vector breaches the boundary of the no-trade cone. A similar result was found by Magill and Constantinides [24] in the space of portfolio quantity fractions. Because bonds, or other securities with fixed and finite maturities, have—on the whole—gradually declining risk attributes as maturity time draws closer, the portfolio elasticities vector may continuously
attempt to breach the no-trade boundary, leading to a large number of very small
discrete trades over any finite time interval.

In order to mimic more closely actual trading behaviour in the market, we limit
the trading times (or optimal stopping times) to a finite set of discrete times: the
portfolio is now discretely monitored. This type of control will in general be less
efficient as the optimal trading moment is often missed. Application of the optimal
impulse trade at discrete times does now no longer take place exclusively at the
boundary of the no-trade cone, because the portfolio vector of elasticities may have
wandered outside the no-trade region since the previous discrete rebalancing time.
The resulting portfolio management performance will then be an approximation
of that of the continuously monitored portfolio, acceptable if we choose discrete
trading times at intervals that are small compared with the Brownian motion of
bond prices and attributes.

Finding the optimal combination of admissible bond trades (sell/buy/no-trade)
in each case is not easy. We develop in this paper an algorithm for finding the
optimal bond trade combination and trade direction at every point in the space
of reward elasticities; discrete portfolio rebalancings to the no-trade boundary are
then found by means of standard line search methods. Several numerical examples
are given to illustrate application of the algorithm.

In summary, we have shown in this paper that

(1) the institutional portfolio management problem, with its moral hazard po-
tential and non-trivial problem of performance measurement, is reduced to
stochastic dynamic maximization of an incentive reward function for the man-
ager;

(2) optimal trades may then depend on the manager's assessment of current market
dynamics, which may be based on privately held beliefs;
(3) the stochastic dynamic programming problem is formulated as one in impulse control, improving earlier work by Magill and Constantinides [24] and obviating a criticism by Duffie [15]; and

(4) the assumption of myopia allows derivation of an analytical solution to the optimal bond trade problem and implementation of an approximating algorithm.
REFERENCES


APPENDIX A

Continuous-Time Framework

The continuous-time framework usually observed in the bond-pricing literature uses a $k$-dimensional state variable $Y$ describing the dynamics of the term structure of interest rates (or spot-yield curve). The parameter $k$ is chosen sufficiently small (e.g. $k = 2$ in Brennan and Schwartz [4]) so as to make the model tractable in portfolio management practice. The state variable $Y$ is assumed to follow a diffusion process with Itô equation

$$dY = \mu dt + G dz,$$

where $Y = Y(t)$ is a $k \times 1$ vector-valued state variable,

$\mu = \mu(Y, t)$ is a $k \times 1$ vector of drift terms,

$G = G(Y, t)$ is a $k \times k$ matrix of diffusion coefficients,

$dz$ is the increment to a $k$-dimensional correlated standard Gauss-Wiener process:

$$E[dz] = 0, \quad E[dz dz'] = C dt,$$

$C$ is the $k \times k$ instantaneous correlation matrix of $dz$, and

$GCG'$ is the $k \times k$ instantaneous covariance matrix of $dY$.

As Cox, Ingersoll, and Ross [13] and Brennan and Schwartz [5] have also pointed out, the true underlying state variables $Y$ need not be observable or known as long as the relationship between bond returns and the state variables is invertible. In that case, the state space may equally well be spanned by proxy state variables based on bond returns. Cox, Ingersoll, and Ross [11,12,13] choose the instantaneous riskless return on the zero-maturity T-bill as proxy state variable ($r$). Brennan and Schwartz [4,5] add to that the return on the consol bond, i.e. the return on the coupon bond of infinite maturity ($l$). We find it convenient in this paper to use the returns on all portfolio bond positions as proxy state variables; we follow the above
authors in assuming the necessary invertibility. (Alternatively, of course, diffusion processes for bond returns can be posited.)

Let $P = P(Y, t)$ be the $m \times 1$ vector of bond prices at time $t$, expressed as functions of the true state variables. We suppress again the dependence of bond prices on coupon rates and on maturity dates. We assume that bond prices follow continuous sample paths on $[0, T]$, which may be represented by the Itô diffusion processes:

$$dP = D(P) [\alpha dt + \Delta dz],$$

(A.2)

where $D(P)$ is the $m \times m$ diagonal matrix $\{P_i(t)\}$, $i = 1, \ldots, m$,

$\alpha$ is the $m \times 1$ vector of drift terms $\alpha_i(t)$, and

$\Delta$ is the $m \times k$ matrix with rows $\delta_i(t)$ of diffusion coefficients.

The drift terms $\alpha$ include adjustments for the coupon return rates $\{c_i/P_i\}$. Expressions for $\alpha$ and $\Delta$ can be derived from (A.1) using Itô's Lemma.

For a given portfolio composition $x$ (as $m \times 1$ vector of quantities of the traded bonds), the portfolio value vector is defined as $s = D(x) P$, from which we obtain

$$ds = D(x) dP$$

$$= D(x) D(P) [\alpha dt + \Delta dz]$$

$$= D(s) [\alpha dt + \Delta dz].$$

(A.3)

Although (A.3) resembles (3.1) in the text of this paper, $\alpha$ and $\Delta$ are functions of the $k$-dimensional process (A.1), while $\alpha$ and $\Delta$ in (3.1) are functions of $s$ and $t$.

We can now follow Cox, Ingersoll, and Ross [13] or Brennan and Schwartz [5] in substituting bond returns or returns on portfolio positions as proxies for the true state variables $Y$. For example, if the state-space dimension $k = 2$ and if the bonds of zero and infinite maturity are actually traded (call them bonds #1 and #2), then we may substitute their returns as proxies for the state variables $Y$. Process
(A.3) is then identical with that in (3.1) under the assumption of degeneracy for
the standard Gauss-Wiener processes \( dz_i, \quad i = 3, \ldots, m \), or

\[
\frac{ds_i}{s_i} = \alpha_i(s_1, s_2, t) \, dt + \delta_{i1}(s_1, s_2, t) \, dz_1 + \delta_{i2}(s_1, s_2, t) \, dz_2. \tag{A.4}
\]

This process, in turn, is equivalent to that of Brennan and Schwartz [5]: it only
differs by a non-zero scale factor \( x \), which does not affect the spanning of the
state space. State-variable process (A.4), or (3.1), will then for varying portfolio
compositions \( x \) be linear transformations of the Brennan and Schwartz [5] process.

The coefficients of (A.4) may be estimated as in Brennan and Schwartz [5]
by making the substitutions \( P_1 = e^{-rT} \), where we assume \( P_1 \) represents one-month
T-bills, and \( P_2 = c/l \) for the consol bond. The coefficients in (A.4) are then (with
scale vector \( x \) applied),

\[
\begin{align*}
\alpha_1 &= a_1 + b_1 \left( \frac{c}{S_2} + \frac{\log S_1}{T} \right), \\
\alpha_2 &= \frac{c}{S_2} \left( a_2 - \frac{b_2 \log S_1}{T} + c_2 \frac{c}{S_2} \right), \\
\delta_{i1} &= \sigma_1 \frac{-\log S_1}{T}, \\
\delta_{22} &= \sigma_2 \frac{c}{S_2}, \\
\delta_{12} = \delta_{21} &= 0.
\end{align*}
\]

Brennan and Schwartz [5] have estimated parameters \( a_i, b_i, \) and \( c_i, \sigma_i, \) and \( \rho \) for
various time periods between 1954 and 1979. Their method may be extended to
other sets of state variable proxies for \( Y \).
APPENDIX B

State Variable Dynamics

1. Uncontrolled Portfolio

An expression for $dw$ is found by applying Itô's lemma to the definition $w(t) = s(t)/V(t)$:

$$
\begin{align*}
    dw &= \frac{ds}{V} - \frac{s}{V^2} dV + \frac{s}{V^3} dV^2 - \frac{dV}{V^2} \\
    &= \alpha_w dt + B_w dz,
\end{align*}
$$

with

$$
\begin{align*}
    \alpha_w &= \frac{D(s)\alpha}{V} - \frac{s(s'\alpha)}{V^2} + \frac{s(s' \Delta C\Delta' s)}{V^3} \\
    &\quad - \frac{D(s)\Delta C\Delta' s}{V^2}, \\
    B_w &= \frac{D(s)\Delta}{V} - \frac{s s' \Delta}{V^2}. 
\end{align*}
$$

(B.1.1)

(B.1.2)

Here, $\alpha_w$ and $B_w$ are $m \times 1$ and $m \times k$ functions of the state-variable process (3.1), respectively.

Next, using (3.4) and (3.6), each portfolio risk attribute $j$ can be represented by

$$
\begin{align*}
da_{pj} &= \beta_{pj} dt + H_{pj} dz, \quad j = 1, \ldots, l
\end{align*}
$$

where

$$
\begin{align*}
    \beta_{pj} &= B_j w + A_j'^t \alpha_w + \psi_j \quad \text{scalar}, \\
    H_{pj} &= w'H_j + A_j'^t B_w \quad 1 \times k, \quad \text{and}, \\
    \psi_j &= \text{trace} \{H_j'^t B_w C\} \quad j = 1, \ldots, l.
\end{align*}
$$

(B.1.3)

(B.1.4)

All coefficients are functions of the state-variable process (3.1) for $s$. 

2. Portfolio with Jumps

We write for convenience

\[ v(t) = \sum_{n=1}^{N} u_n \delta(t - \theta_n), \]  

and we suppress the \( t \) subscripts in (4.1)

\[ ds = [D(s) \alpha + v] dt + D(s) \Delta dz. \]

Intuitively, \( v \) is akin to a rate of trading, but is defined in terms of the "on-off" Dirac function. Then we find, analogous to subsection B.1,

\[ dV = (s' \alpha + v' 1) dt + s' \Delta dz. \]  

\[ \alpha_w = \frac{D(s) \alpha + v}{V} - \frac{s (s' \alpha + v' 1)}{V^2} + \frac{s (s' \Delta \Delta' s)}{V^3} - \frac{D(s) \Delta \Delta' s}{V^2}, \]  

\[ B_w = \frac{D(s) \Delta}{V} - \frac{s s' \Delta}{V^2}. \]

3. \( \epsilon \)-perturbed Problem

We add a small disturbance to the diffusion term in

\[ ds = [D(s) \alpha + v] dt + [D(s) + \epsilon D(v)] \Delta dz. \]

Then we find, analogous to B.1,

\[ dV = (s' \alpha + v' 1) dt + (s + \epsilon v)' \Delta dz. \]  

\[ \alpha_w = \frac{D(s) \alpha + v + \epsilon D(v)}{V} - \frac{s (s' \alpha + v' 1)}{V^2} + \frac{s (s + \epsilon v)' \Delta \Delta' s}{V^3} \]  

\[ - \frac{[D(s) + \epsilon D(v)] \Delta \Delta'(s + \epsilon v)}{V^2}, \]  

\[ B_w = \frac{[D(s) + \epsilon D(v)] \Delta}{V} - \frac{s (s + \epsilon v)' \Delta}{V^2}. \]
APPENDIX C

Derivation of Optimal Control

We give the following definitions and useful properties of matrix operations for use in the derivations of this Appendix:

a. Define \( \Box \) as the matrix "box-product" operator of two identically dimensioned matrices. The \((i,j)\)-th element of the product matrix is found by multiplying the \((i,j)\)-th elements of the two operand matrices.

b. Define \( D(x) \) as a diagonal matrix with the vector \( x \) as diagonal.

c. The following properties of \( \Box \) and \( D(\cdot) \) are easily verified, with \( A, B, C \) and \( D(x) \) matrices compatible in multiplication:

\[
\begin{align*}
- \quad (A \Box B) + (C \Box B) &= (A + C) \Box B \\
- \quad A \Box B &= B \Box A \\
- \quad AD(x) &= x'A \Box A \\
- \quad D(x)A &= xA' \Box A \\
- \quad D(x) y &= D(y) x
\end{align*}
\]

d. Box-multiplication with the identity matrix, \( (A \Box I) \) retains the diagonal part of a square matrix \( A \). Using this diagonal matrix to pre-multiply a vector of ones, \( (A \Box I) 1 \), turns the diagonal of \( A \) into a column vector.

e. The trace of a scalar equals the scalar, and vice versa.

1. Derivation of Optimal Control \( v^* \)—General Case

Applying the differential and trace operators to the F.O.C. (5.7) leads to

\[
\begin{align*}
\epsilon^2 \Sigma v + \epsilon \Sigma s + g_s - \lambda \gamma &= 0 \\
\gamma' v &= 0,
\end{align*}
\]

where the \( m \times m \) matrix \( \Sigma = [\Delta C\Delta' \Box g_s] \), \( \xi = g_s \), and \( \eta = \Sigma s \). System (C.1) can be solved in a straightforward manner to yield:

\[
\lambda^*(t) = \frac{\gamma' \Sigma^{-1} \xi + \epsilon \gamma' \Sigma^{-1} \eta}{\gamma' \Sigma^{-1} \gamma}, \quad \text{and}
\]
The second-order sufficiency condition for a maximum in the presence of the linear constraint \( \gamma'v = 0 \) reduces to

\[
v' \Sigma v < 0, \quad \forall \ v \in \{ v \mid \gamma'v = 0 \}.
\]  

(C.3)

In practice, we simply check the necessary condition that the objective function must increase on account of \( v^*(t) \), or that \( v^*(t)' \Sigma v^*(t) < 0 \).

2. Alternative Derivation of \( v^* \)

Static maximization problem (5.6) can now be restated in terms of the portfolio value \( V(s) \) and risk attribute deviations \( a(s) \), using the \( \epsilon \)-perturbed process (5.2) and Itô processes for \( V \) and \( a \) as derived in Appendix B.3. The Lagrangean maximization is then

\[
\max_{v, \lambda} L^v g(V(t), a(t)) - \lambda \gamma'v,
\]

or

\[
\max_{v, \lambda} g_V(s'\alpha + v'1) + \frac{1}{2} g_{VV}(s + \epsilon v)' \Delta \Delta'(s + \epsilon v) + g'_a \beta + \frac{1}{2} \text{trace}(g_{aa} HCH') + g'_{av} Hc \Delta'(s + \epsilon v) - \lambda \gamma'v. \quad (C.4)
\]

The F.O.C. with respect to \( v \) are

\[
g_V 1 + \epsilon^2 g_V \Delta C \Delta'v + \epsilon g_{VV} \Delta C \Delta's + \nabla_v \beta' g_a + \frac{1}{2} \nabla_v \text{trace}(g_{aa} HCH') + \nabla_v g'_{av} Hc \Delta'(s + \epsilon v) - \lambda \gamma = 0, \quad (C.5)
\]

and \( \gamma'v = 0 \).

2.1 Determine \( \nabla_v \beta \), an \( l \times m \) matrix of derivatives

Using the derivations in Appendix B.3, we get

\[
\nabla_v \beta = \nabla_v (\beta_p - \beta_T) = \nabla_v \beta_p = \nabla_v (B'w + A' \alpha_w + \psi) = A' \nabla_v \alpha_w + \nabla_v \psi,
\]
where $\nabla_v \alpha_w$ and $\nabla_v \psi$ are $m \times m$ and $l \times m$ matrices, respectively.

For $\nabla_v \alpha_w$ we get

$$\nabla_v \alpha_w = V^{-1} I_m - V^{-2} s' 1' + 2V^{-3} \epsilon^2 s v' \Delta C \Delta' + 2V^{-3} \epsilon s s' \Delta C \Delta'$$

$$- V^{-2} \epsilon [D(s) + \epsilon D(v)] \Delta C \Delta' - V^{-2} \epsilon D[\Delta C \Delta'(s + \epsilon v)].$$

To find $\nabla_v \psi$, we look at $\nabla_v \psi_j$, a $1 \times m$ row vector: $\nabla_v \text{trace}(H_j' B_w C)$, where only $B_w$ is a function of $v$. Substituting for $B_w$ from (B.3.3), and deleting elements not containing $v$, we get

$$\nabla_v \psi_j = \nabla_v \text{trace}(H_j' \left[ \frac{\epsilon}{V} D(v) \Delta \right] C) - \nabla_v \text{trace}(H_j' \left[ \frac{\epsilon}{V^2} s v' \Delta \right] C)$$

$$= \frac{\epsilon}{V} \nabla_v \text{trace}(D(v) \Delta C H_j') - \frac{\epsilon}{V^2} \nabla_v \text{trace}(v' \Delta C H_j' s).$$

Define $\Theta_j = H_j C \Delta'$, $j = 1, \ldots, l$, as the cross-covariance matrix of bond prices and their $j$-th attributes, $m \times m$. Then:

$$\nabla_v \psi_j = \frac{\epsilon}{V} 1'(\Theta_j \otimes I) - \frac{\epsilon}{V^2} s' \Theta_j,$$

which is independent of $v$. We further define the $l \times m$ matrix $\Psi$ by $\epsilon \Psi = \{\nabla_v \psi_j, \ j = 1, \ldots, l\}$.

Then:

$$\nabla_v \beta' g_a = V^{-1} A g_a - V^{-2} 1' s' A g_a + 2V^{-3} \epsilon^2 \Delta C \Delta' s s' A g_a$$

$$+ 2V^{-3} \epsilon \Delta C \Delta' s s' A g_a - V^{-2} \epsilon \Delta C \Delta'[D(s) + \epsilon D(v)] A g_a$$

$$- V^{-2} \epsilon D[\Delta C \Delta'(s + \epsilon v)] A g_a + \epsilon \Psi' g_a.$$
2.2 Determine $\frac{1}{2} \nabla_v \text{tr}(g_{aa} HCH')$

We assume that $a_p - a_T$ has positive elements. With

$$H = (I_l \otimes w')H_v + A'B_w$$
$$= (I_l \otimes w')H_v + V^{-1}A'[D(s) + \epsilon D(v)]\Delta - V^{-2}A's's'\Delta - V^{-2}\epsilon A'sv'\Delta,$$
we let

$$M = (I_l \otimes w')H_v + V^{-1}A'D_2\Delta - V^{-2}A's's'\Delta - HT,$$

where $M$ is an $l \times k$ matrix not depending on the control $v$. We recognize that

$$\lim_{\epsilon \to 0^+} M = H,$$

the $l \times k$ matrix of diffusion coefficients of the process for $a$, the portfolio attribute deviation vector, or

$$H = M - V^{-2}\epsilon A'sv'\Delta + \epsilon V^{-1}A'D(v)\Delta.$$

Next $\nabla_v \text{trace}(g_{aa} HCH')$ becomes

$$\nabla_v \text{tr}\{g_{aa}MC\Delta' + V^{-2}\epsilon^2g_{aa}A'sv'\Delta C\Delta'v's'A$$
$$+ V^{-2}\epsilon^2g_{aa}A'D(v)\Delta C\Delta'D(v)A - 2V^{-2}\epsilon g_{aa}MC\Delta'v's'A$$
$$+ 2V^{-1}\epsilon g_{aa}MC\Delta'D(v)A - 2V^{-2}\epsilon^2g_{aa}A'sv'\Delta C\Delta'D(v)A\}$$
$$= \nabla_v \{V^{-4}\epsilon^2(s'Aga_{aa}A's)v'\Delta C\Delta'v - 2V^{-2}\epsilon s'Aga_{aa}MC\Delta'v$$
$$- 2V^{-3}\epsilon^2v'\Delta C\Delta'D(v)Aga_{aa}A's\}$$
$$+ \nabla_v \text{trace}\{V^{-2}\epsilon^2Aga_{aa}A'D(v)\Delta C\Delta'D(v) + 2V^{-1}\epsilon D(v)Aga_{aa}MC\Delta'\}.$$

Letting $V^{-2}s'Aga_{aa}A's = a'_pg_{aa}a_p =, a$ scalar, and $A's = Va_p$, we get

$$\frac{1}{2} \nabla_v \text{tr}(g_{aa} HCH') = V^{-2}\epsilon^2a'_pg_{aa}a_p[E\square \Delta C\Delta']v - V^{-1}\epsilon \Delta CM'g_{aa}a_p$$
$$- V^{-3}\epsilon^2[(Aga_{aa}A's1' + 1s'Aga_{aa}A')\square \Delta C\Delta']v$$
$$+ V^{-2}\epsilon^2[(Aga_{aa}A')\square \Delta C\Delta']v + V^{-1}\epsilon[(Aga_{aa}MC\Delta')\square I]1.$$
2.5 Determine $\nabla_v g'_{av} H \Delta' (s + \epsilon v)$

Using the definitions of $H$ and $M$ above:
\[
\nabla_v g'_{av} H \Delta'(s + \epsilon v) = m \times 1
\]
\[
= \nabla_v \left\{ g'_{av} M \Delta'(s + \epsilon v) - V^{-2} \epsilon g'_{av} A' s v' \Delta C \Delta'(s + \epsilon v) \right\}
\]
\[
+ V^{-1} \epsilon g'_{av} A' D(v) \Delta C \Delta'(s + \epsilon v) \}
\]
\[
= \nabla_v \left\{ g'_{av} M \Delta'(s - \epsilon v) - V^{-2} \epsilon g'_{av} A' s v' \Delta C \Delta'(s - \epsilon v) \right\}
\]
\[
+ V^{-1} \epsilon g'_{av} A' D(v) \Delta C \Delta'(s - \epsilon v) \}
\]
\[
= \epsilon \Delta C M' g_{av} - V^{-2} \epsilon \Delta C \Delta' s g'_{av} A' s - 2 V^{-2} \epsilon g'_{av} A' s \Delta C \Delta' v
\]
\[
+ V^{-1} \epsilon \gamma [(\Delta C \Delta') \square I] A g_{av} + V^{-1} \epsilon^2 [(1 g'_{av} A' + A g_{av} 1') \square (\Delta C \Delta')] v.
\]

Letting $g'_{av} A' s = V g'_{av} a_p$, a scalar, we get:
\[
\nabla_v g'_{av} H \Delta'(s + \epsilon v)
\]
\[
= \epsilon \Delta C M' g_{av} - V^{-1} \epsilon g'_{av} a_p \Delta C \Delta' s - 2 V^{-1} \epsilon^2 g'_{av} a_p \Delta C \Delta' v
\]
\[
+ V^{-1} \epsilon \gamma D [A g_{av}] \Delta C \Delta' s + V^{-1} \epsilon^2 [(1 g'_{av} A' + A g_{av} 1') \square (\Delta C \Delta')] v.
\]

2.4 Solve F.O.C and Simplify

The equivalent of expressions (C.1) are
\[
g'_{av} 1 + \epsilon^2 g_{vv} \Delta C \Delta' v + \epsilon g_{vv} \Delta C \Delta' s + V^{-1} A g_{a} - V^{-1} a'_{g} a_{1}
\]
\[
+ 2 V^{-2} \epsilon^2 a'_p g_{a} \Delta C \Delta' v + 2 V^{-2} \epsilon a'_p g_{a} \Delta C \Delta' s + \epsilon \Psi' g_{a}
\]
\[
- V^{-2} \epsilon \Delta C \Delta' [D(s) + \epsilon D(v)] A g_{a} - V^{-2} \epsilon D [A g_{a}] \Delta C \Delta'(s + \epsilon v)
\]
\[
+ V^{-2} \epsilon^2 a'_p g_{aa} a_{p} \Delta C \Delta' v - V^{-1} \epsilon \Delta C M' g_{aa} a_{p}
\]
\[
- V^{-3} \epsilon^2 [(A g_{aa} A' s 1' + 1 s' A g_{aa} A') \square \Delta C \Delta'] v
\]
\[
+ V^{-2} \epsilon^2 [A g_{aa} A' \square \Delta C \Delta'] v + V^{-1} \epsilon [A g_{aa} M \Delta \square I] 1
\]
\[
+ \epsilon \Delta C M' g_{av} - V^{-1} \epsilon g_{av} a_{p} \Delta C \Delta' s - 2 V^{-1} \epsilon^2 g_{av} a_{p} \Delta C \Delta' v
\]
\[
+ V^{-1} \epsilon^2 [(1 g'_{av} A' + A g_{av} 1') \square \Delta C \Delta'] v
\]
\[
+ V^{-1} \epsilon D [A g_{av}] \Delta C \Delta' s - \lambda \gamma = 0,
\]
and $\gamma' v = 0$. 

We simplify by defining the negative-valued symmetric weighting matrix:

\[
\Sigma = V^{-2}[(V^2 g_{VV} + 2a_p'g_a + a_p'g_{aa}a_p - 2V g_{aV}a_p) E \\
- (1g_a'A' + Ag_a1') - (V^{-1}Ag_{aa}A's1' + V^{-1}s'Ag_{aa}A') \\
+ Ag_{aa}A' + (V1g_{aV}A' + VA'g_{aV}1')][\Delta C\Delta']
\] (C.7)

We define also the matrix

\[
\Omega = \Delta CM' \quad m \times l,
\]

as the (limit) cross-covariance matrix of portfolio bond prices and attribute deviations. Finally, we define \(m \times 1\) vectors \(\xi\) and \(\eta\) by

\[
\xi = g_V 1 + V^{-1}Ag_a - V^{-1}a_p'g_a 1,
\]

\[
\eta = V^{-2}\{[(V^2 g_{VV} + 2a_p'g_a - V g_{aV}a_p) E \\
- (1g_a'A' + Ag_a1') + V D[Ag_{aV}] \boxdot \Delta C\Delta']\} s
\] (C.8)

\[
= \tilde{\Sigma}s + \Omega'[g_{aV} - V^{-1}g_{aa}a_p] + V^{-1}[Ag_{aa}\Delta CM' \boxdot I] 1 + \Psi'g_a.
\]

The matrix \(\tilde{\Sigma}\) is yet another matrix obtained from the bond price covariance matrix \(\Delta C\Delta'\) by box multiplication with a matrix depending on dynamics and on \(g\). Derivation of the above expressions for \(\Sigma, \xi,\) and \(\eta\) is equivalent to deriving them from those in subsection C.1 using "chain rule" differentiation. See Section 6 for a discussion of the economic interpretation of the vectors \(\xi\) and \(\eta\).

After substitution of the newly defined matrices, vectors, and scalars, system (C.6) has the same form as (C.1) with the same form of solution for \(v^*\), except that \(\Sigma, \xi,\) and \(\eta\) are now specified in terms of \(V\) and \(a\).
APPENDIX D

Trade/No-Trade Regions

1. Trade Combinations—The Matrix \( \Gamma \)

A group of \( x \) bonds may be traded in \( \sum_{y=1}^{x-1} \binom{x}{y} \) combinations of selling \( y \) bonds and buying \( x - y \) bonds \((2 \leq x \leq m)\). Each such trade combination has a unique trade value ratio vector \( \gamma \) and corresponding \( x \times x \) coefficients matrix \( \Gamma^x(\gamma) \).

The rank of each of the matrices \( \Gamma^x \) is \( x - 1 \) because of the self-financing constraint \((4.9)\), \( \gamma'u = 0 \), which makes one of the linear equations in \( u^* = \Gamma^x \xi \) redundant. As a consequence of this rank reduction, the \( x \) hyperplanes \( \Gamma^x \xi = 0 \), which bound each trade region, share a common ray through the origin in the corresponding \( x \)-dimensional subspace \( \chi^x \subseteq \chi \subset \mathbb{R}^m \). These rays may be represented parametrically in the appropriate subspace \( \chi^x \) by

\[
\xi_i = \begin{cases} 
\gamma_s q, & \text{if bond } i \text{ is sold} \\
\gamma_b q, & \text{if bond } i \text{ is purchased.} 
\end{cases} \quad \text{ (D.1)}
\]

Here, \( q \in \mathbb{R} \) is the parameter, while \( \gamma_s < 1 \) and \( \gamma_b > 1 \) are the common trade ratios after transactions costs from selling and buying bonds, respectively.

Expression \((D.1)\) can be proved by considering that the \( ij \)-th element of \( \Gamma^x \) equals

\[
\Gamma^x_{ij} = (\gamma' \Sigma^{-1} \gamma)^{-1} \sum_{k=1}^{x} \sum_{l=1}^{x} (\sigma^{ik} \sigma^{jl} - \sigma^{ij} \sigma^{kl}) \gamma_k \gamma_l,
\]

where \( \sigma^{-1} \) is an element of the inverse of the appropriate weighting matrix \( \Sigma^x \), formed by deleting rows and columns of the non-transacted bonds from \( \Sigma \), and then substituting \((D.1)\) into the \( i \)-th linear compound of \( \Gamma^x \xi \). This yields indeed

\[
\sum_{j=1}^{x} \Gamma^x_{ij} \xi_j = 0.
\]

The tedious details are omitted.
We note immediately that with no transactions costs, when \( \gamma_s = \gamma_b = 1 \), the common rays in (D.1) are the same for each trade combination with the same \( x \) traded bonds, i.e. the diagonal \( \xi_1 = \xi_2 = \ldots = \xi_x \) through the positive and negative orthants of \( \mathbb{R}^x \). With transactions costs, there are \( \sum_{\nu=1}^{x-1} \binom{x}{\nu} \) different rays arranged with radial symmetry around the no-cost diagonal. When \( m = 2 \), there are 2 rays in \( \mathbb{R}^2 \). When \( m = 3 \), we have 6 rays in \( \chi^3 \subseteq \mathbb{R}^3 \), and 2 rays each in 3 subspaces \( \chi^2 \subseteq \mathbb{R}^2 \). The common rays in lesser dimensional subspaces are part of hyperplanes of dimension \( m - x + 1 \) in \( \mathbb{R}^m \), spanned by the common rays and the \( m - x \xi \)-axes of the non-traded bonds.

These common rays or hyperplanes are independent of the appropriate weighting matrix \( \Sigma^x \), but the orientations of the bounding hyperplanes through them depends on \( \Sigma \) as shown in (6.2) (but see the two-dimensional case, next).

2. Two-Dimensional Case \( (m = 2) \)

The simplest case, with \( x = m = 2 \), has only 2 trade combinations: either sell bond 1 for 2 (1/2), or vice versa (2/1). The rays (D.1) and hyperplanes (6.2) coincide, as shown in Fig. 1 where we have assumed for convenience that \( \Sigma \) equals minus the identity matrix. The region to the south-east of \( \xi_1 = \frac{\eta_1}{\eta_2} \xi_2 \) is the 1/2 trade region, while the region to the north-west of \( \xi_1 = \frac{\eta_2}{\eta_1} \xi_2 \) is the 2/1 trade region.

Trading in the overlapping area of the positive orthant is mutually exclusive: it is a no-trade region. The total no-trade region is then a cone bounded by the two hyperplanes. We verify that it is indeed arranged around the no-cost locus, the diagonal \( \xi_1 = \xi_2 \). We note that in this case the no-trade region is independent of the weighting matrix \( \Sigma \). The reason for this is that two bonds can only be traded in proportions that are fixed by the transactions costs: no other degree of freedom exists. This observation has an important implication for the no-trade cone in higher dimensions, as shown in the next subsection.
9. Three-Dimensional Case ($m = 3$)

We have 6 combinations with $x = 3$ traded bonds: $1/2 + 3$, $2/1 + 3$, $3/1 + 2$, $1 + 2/3$, $1 + 3/2$, $2 + 3/1$; and a further 6 combinations with $x = 2$ traded bonds: $1/2$, $2/1$, $1/3$, $3/1$, $2/3$, $3/2$. It is instructive to show the no-cost case first: Fig. 2 shows the trade regions in a cross-section over the diagonal (we have again assumed, for convenience, that $\Sigma = -I$). The trade regions with 3 traded bonds do not overlap or show gaps, and the trade regions with 2 traded bonds are limited to the two-dimensional bounding hyperplanes of the regions with 3 traded bonds.

Fig. 3 shows that with transactions costs a no-trade cone with a complicated cross-section has opened up. Each of the trade regions with 3 traded bonds has rotated and deformed w.r.t. the diagonal, thereby opening up gaps and forming overlaps. The result is that the trade regions where only two bonds are traded are now also three-dimensional.

We also note that the gross outline of the cross-section of the no-trade cone is a hexagon with vertices on the common rays for the $x = 3$ trade cases. The faces of the hexagon are formed by hyperplanes through the common rays from the two-dimensional trade cases, which—as noted there—are independent of the weighting matrix $\Sigma$. We conjecture that in the $m$-dimensional case, the convex hull of the no-trade cone is bounded by the $2 \times \binom{m}{2}$ hyperplanes through the $m$ axes of $\xi$-space and the common rays for each combination of two traded bonds, and that this convex hull is therefore independent of $\Sigma$. The depth of the V-shaped grooves taken out of this convex hull by higher dimensional trade areas are, however, dependent on $\Sigma$. 
APPENDIX E

Procedure to Determine Optimal Trade Combination

Enumeration of all \( N = \sum_{x=2}^{m} \binom{m}{x} \sum_{y=1}^{x-1} \binom{x}{y} \) possible trade combinations is inefficient and wasteful of computer resources, especially when \( m \) is large. The procedure of this Appendix is based on an iterative improvement (or "greedy") method which starts with the best two bonds of which one is sold and the other purchased. Remaining bonds are then added one-by-one, either to be sold or purchased, on the basis of the greatest possible improvement.

We define first what is meant by "best" and "greatest improvement" in this context. These concepts are defined in terms of the length, or Euclidean norm, of the optimal trade direction vector \( \mathbf{v}^* = \mathbf{T} \mathbf{x} \) described in Section 6. The best trade combination is then the one with the highest value of \( |\mathbf{v}^*| \); this choice is motivated by the interpretation of the optimal direction as a rate of trading in Section 6.

Acceptable trade combinations must also be \( \gamma \)-consistent, in the sense that the signs of the elements of the of the resulting \( \mathbf{v}^* \) must match the values of the trade proceed ratios \( \gamma \). In addition, we rule out any trade combination that results in negative portfolio holdings.

Formally, our problem can be defined as

\[
\max_{\gamma} \frac{\Sigma^{-1} \gamma' \Sigma^{-1} - (\gamma' \Sigma^{-1} \gamma) \Sigma^{-1}}{(\gamma' \Sigma^{-1} \gamma)} \xi |, \quad x \times x, \quad x = 2, \ldots, m, \quad (E.1)
\]

subject to the constraints. Elements of the vector \( \gamma \) can take on either of the discrete values \( \gamma_s \) or \( \gamma_b \).

The above problem represents a difficult discrete non-linear programming problem. Even a relaxation to a continuous domain for \( \gamma \) is not helpful, in that the objective function is not necessarily convex. From a practical view point, the large number of matrix inversions (to obtain \( \Sigma^{-1} \) for each trade combination) required
for most types of search algorithms, including complete enumeration, lead us to the use of the simpler, "greedy", algorithm described above.

Although the greedy algorithm converges rapidly to a unique trade combination (if trading is optimal), there is no guarantee that it is indeed the optimal trade combination. There are some indications, however, that the present application may be adequate. First, the starting combination of the two best bonds has been shown in Appendix D to be independent of the program dynamics. Second, our algorithm selects only additional bonds that improve the trade combination, therefore we cannot end up worse than the starting combination. Third, using weighting matrices $\Sigma$ with typical (negative) values, we have been unable to construct counter examples with up to five traded bonds where the algorithm converged on the wrong trade combination (checked with complete enumeration). The procedure described further below may therefore in practice be quite acceptable.

The procedure has the following steps:

(1) Calculate $v^*$ for each of $m(m-1)$ sell/purchase pairs of bonds. Accept only combinations that are $\gamma$-consistent, and that meet the second order conditions for maximum. Of these, choose the pair with highest value of $|v^*|$. If no consistent combinations are found at this stage, we do not trade. (This no-trade decision will be wrong only when the portfolio elasticities vector plots in the small region between the no-trade region and its convex hull described in Appendix D; the error induced by this criterion is, therefore, minimal.)

(2) Augment the system of already decided bonds by one, trying—in turn—each of the undecided bonds. Again, accept only combinations meeting the constraints as in (1), and of the remainder choose the combination with highest norm. If none of the combinations at this step is consistent we stop and the previously determined combination is chosen as optimal.
(3) Repeat step 2 as long as bonds remain undecided.

We note that the above procedure does not require repeated inversions of lead­
ing principal sub-matrices of increasing order of the weighting matrix $\Sigma$: since at
each step only a single bond is added, the inverse of the weighting matrix at that
step may be obtained from that of the previous step by a simple recursive procedure
(see e.g. Johnston [22], p. 93). We do not check the sufficient second order condi­
tions (see Appendix C), as the procedure is quite complicated. Instead, we check
at each stage of the algorithm the simple necessary condition $v^*\Sigma v^* < 0$, i.e. the
objective function must increase following a marginal trade using the combination
considered.
APPENDIX F

Discrete Control of Portfolios following Itô Processes

1. Impulse Controlled Diffusions

Following Menaldi [26], an impulse control is defined as a sequence of pairs $(\theta_n, u_n)$, where $\{\theta_n\}_{n=1}^N$ is an increasing sequence of stopping times on $\mathbb{T}$, and $\{u_n\}_{n=1}^N$ is a sequence of random discrete portfolio adjustments taking values in $\mathbb{R}^m$, adapted w.r.t. the stopping times. The number of jumps $N$ may be infinite.

Although the sequence of stopping times in [26] converges on infinity, with $T = [0, \infty)$, Menaldi’s proofs are not affected by an upper bound on the time domain to $T = [0, T]$. Neither do they appear to depend on a continuous time domain, i.e. we could specify the set of possible stopping times as the discrete set $T = \{0, t_1, \ldots, T\}$. In that case, the maximum number of jumps, $N$, is finite. The number of jumps is not necessarily finite if the stopping times are chosen from a continuous finite time interval, e.g. $[0, T]$. The stopping times $\{\theta_n\}_{n=1}^N$ follow from a stopping rule adapted w.r.t. the current information set. Specifically, in the main text of this paper, the next stopping time is defined as $\theta_{n+1} = \inf \{t \in T\}$ such that $t > \theta_n$ and $u_{n+1}$ is $\gamma$-consistent and non-zero in the sense of Section 4.

Impulse control causes jumps in otherwise smooth diffusion processes. A diffusion with jumps may be thought of as the (serial) sum of a number $(N)$ of smooth diffusions, one for each jump. Again following Menaldi [26], we start with a diffusion for the portfolio value vector $s^0$, where the superscript indicates that the process has no jumps:

$$ds^0(t) = D^0_s[\alpha dt + \Delta dz], \quad t \in [0, T], \quad (F.1)$$

similar to expression (4.1) in the text of the paper, and with the starting value $s^0(0)$ given.

Next, we define the sequence $\{s^n(t)\}_{n=1}^N$ of diffusions with a single jump each
by the Itô equations:
\[ ds^n(t) = D_s^a [\alpha dt + \Delta dz], \quad t \in (\theta_n, T] \]
\[ s^n(t) = s^{n-1}(t) + 1_{t=\theta_n} u_n, \quad t \in [0, \theta_n], \]

where \( 1_{t=\theta_n} \) is the Heavyside unit step function.

It is clear that such diffusions coincide with all other diffusions for which the single jump has not yet taken place, or
\[ s^n(t) = s^i(t), \quad t \in [0, \theta_n), \forall i \geq n. \]  

Next, we focus on the diffusion with jumps
\[ s(t) = \lim_{n \to N} s^n(t), \quad t \in [0, T]. \]

which simplifies to \( s(t) = s^N(t), t \in [0, T] \) when the number of jumps \( N \) is finite. The process \( s(t) \), which is right continuous, satisfies the stochastic equation:
\[ ds(t) = D_s^a [\alpha dt + \Delta dz] + \sum_{n=1}^{N} u_n \delta(t - \theta_n) dt, \quad t \in [0, T], \]
\[ (s(0) = \text{known}), \]

where \( \delta(t - \theta_n) \) is the Dirac measure which—when Lebesgue integrated over any interval containing the origin—yields the unit step function. Heuristically, the step function has infinite velocity at the step, so that the portfolio adjustments \( \{u_n\} \) take place instantly at the corresponding stopping times.

Menaldi’s main results, quoted in the text of this paper, require the further technical assumption that terminal portfolio value composition, \( s(t) \), is bounded below and above by some suitable values \( \underline{s} \) and \( \overline{s} \), respectively.

2. Continuously Controlled Diffusions

The discrete control problem of continuous diffusion processes was treated by Magill and Constantinides [24] by specifying the impulse control as a continuously
applied control rate $v$, thereby preserving continuity with the methodology of Merton [27]. Because they are unable to solve the resulting control problem, for which the control appears in the drift term only, they resort to a related $\epsilon$-perturbed process for the portfolio value vector

$$ds(t) = \left[ D_s \alpha + v \right] dt + \left[ D_s + \epsilon D_v \right] \Delta dz. \quad (F.6)$$

As $\epsilon \to 0^+$, process (F.6) equals expression (F.5) above as long as we interpret the control $v(t)$ as $u_n \delta(t - \theta_n)$, $t \in (\theta_{n-1}, \theta_{n+1})$, i.e. infinitely large at stopping time $\theta_n$ and zero otherwise. Admissible (limit) control rates in this interpretation are then those for which at any time $t \in [0,T]$

$$\sum_{i \in I(t)} \gamma_s v_{in} + \sum_{i \in I(t)} \gamma_b v_{in} = 0, \quad (F.7)$$

defining $V$ similar to constraint (3.10) in the text of the paper.

Magill and Constantinides [24] then proceed to solve their optimal control problem via the classical stochastic dynamic programming formulation, see e.g. Fleming/Rishel [18]. Transposed to the portfolio problem formulation of the present paper, the "Bellman equation" of dynamic programming is derived:

$$0 = \frac{\partial W(t)}{\partial t} + g(t) + \max_{v \in V} L^v W(t), \quad (F.8)$$

where the operator $L^v$ is defined as in (5.4).

The optimal control rate $v^*(t)$ is then found by first carrying out the static maximization in (F.8), subject to the transactions cost constraint. The solution, in terms of the unknown optimal value function $W(s,t)$, is then substituted back into the whole Bellman equation (F.8), giving a partial differential equation (p.d.e.) in the optimal value function $W(s,t)$.

Magill and Constantinides [24] are able to find an analytical solution for the p.d.e. only in the case of proportional transactions costs, equivalent to using the
simplified constraint \( \gamma(t)'v(t) = 0 \), (5.5) in the present paper. They posit a solution for the optimal value function of the type \( W(t) = f(t)g(s(t)) \), i.e. separable in time and in the state variables, and show that in this manner the p.d.e. may be transformed into an ordinary differential equation (o.d.e.) in \( f(t) \) which can be solved.

An alternative method is to solve the initial optimization and the subsequent p.d.e. by numerical methods, which may not be a simple task in the general case with fixed transactions costs and non-constant state variable coefficients.

Substitution of the optimal value function \( W(s, t) \) into the earlier found solution for the optimal control rate \( v^*(t) \) then leads to an expression made up of known parameters and values, multiplied by \( 1/\varepsilon \) or \( 1/\varepsilon^2 \). As \( \varepsilon \to 0^+ \), the optimal trade rate becomes indeed infinite, verifying the instantaneous application of the control. Magill and Constantinides [24] describe then how application of the optimal control rate is governed by the consistency requirement between the trade direction (sell or buy) and the application of transactions costs. A similar consistency requirement in the present paper (\( \gamma\)-consistency) leads to trade/no-trade regions in “elasticities space”, as described in Appendix D. Magill and Constantinides [24] claim that trading takes place instantly from any portfolio composition corresponding to a point in the “trade” region to the boundary of the “no-trade” region, with trades proportional to the optimal trade rate solution of the \( \varepsilon \)-perturbed problem.

3. Control Problem Misspecified

Duffie [15] has criticized the methodology by Magill and Constantinides [24]. His objections are threefold:

(1) the transactions costs (or the trade value ratios \( \gamma \) in the present paper) are assumed to be known constants in the solution of the p.d.e. of dynamic programming, whereas in fact they are not known until after the optimal control
solution has been checked for consistency with transactions costs, and are thus functions of the optimal value function;

(2) a finite discrete trade across the no-trade boundary shows that the optimal value function $W(s,t)$ is not continuous, thereby invalidating it as a solution to the p.d.e. of dynamic programming; and

(3) the optimal trade ratio $(\gamma_i)$ is not zero for portfolio compositions that plot inside the no-trade region.

We address these objections in view of Menaldi [26]'s impulse control theory, and show that they follow mostly from the technical mis-specification of the impulse control problem as a continuous one. Constantinides [9] specifies a discrete trading problem with two assets that meets Duffie [15]'s critique, but without formally relying on stochastic impulse control theory.

The first objection is overcome by the definition of the next stopping time, given in subsection $F.1$ of this Appendix, in which existence of a $\gamma$-consistent solution to the impulse trade is a pre-requisite. At worst, it could mean solving the optimal impulse problem for each of the finite number of trade combinations, in each case holding the trade combination—and therefore the trade value ratios $\gamma$—fixed. A single trade combination, or value of $\gamma$ can always be identified as being optimal, see Appendix E, as long as at least one $\gamma$-consistent trade combination exists.

The second objection is not really applicable, because it is based on finite trades, whereas the trading regime caused by proportional transactions costs leads to infinitesimal trades at the no-trade boundary. With the latter, $W(s,t) = f(t)g(s(t))$ is continuous if $f(t)$ and $g(s(t))$ are, which is assumed.

The third objection can be overcome by considering only those bonds in the problem that are actually traded in the optimal combination found in Appendix D or E: a choice of dimensionality of the local optimal stopping time problem. This is
consistent with the definition of optimal stopping time given in subsection F.1. If the portfolio elasticities vector $\xi$ plots inside the no-trade region, no stopping time can prevail and the portfolio process is not controlled: no optimal impulse trade then exists. Any attempt to solve for an optimal continuous control in such a case is then based on the mis-specification of the problem, rather than on the properly specified impulse control problem.

The above comments apply also to Magill and Constantinides [24].

4. Optimal Impulse Trade Solution

The previous subsection shows that Duffie [15]'s critique can be circumvented with the impulse control problem specification. The similarity between processes (F.5) and (F.6), and between the maximization problems (5.6) and the Bellman equation (F.8), shows that the Magill and Constantinides [24] solution method is applicable to deriving the optimal trade direction of an impulse control at an optimal stopping time $\theta_n$. Intuitively, the two problems coincide at optimal stopping times, so that—with due regard for the impulse control specification—the Magill and Constantinides [24] methodology can be used to obtain the optimal trade proportions for the manager's stochastic control problem.
APPENDIX G

Numerical Examples of Applying Impulse Controls

We assume that the reward rate functions $g(s) = g(V, a)$ used in the following examples can be closely approximated by the Taylor series expansions around values of the portfolio composition $s$ that give rise to the myopic special solution discussed in Section 5. These examples are only intended to show how the optimal control can be implemented.

1. Example 1 ($m = 2, l = 1$)

Our first example is the simplest possible, with only two traded bonds and one relevant risk attribute (e.g. duration). Bonds can only be traded in the fixed proportions $\gamma_b/\gamma_s$ or $\gamma_s/\gamma_b$.

We choose the reward rate $g = .1V - a^2$, the portfolio attribute target $a_T = 3$, the trade value ratios $\gamma_s = .98$ and $\gamma_b = 1.02$, while—for simplicity—the covariance matrix related weight matrix $\Sigma$ equals the negative identity matrix. The starting portfolio composition is

$$s = \begin{pmatrix} 10.5000 \\ 4.5000 \end{pmatrix}, \text{ with } V = 15 \text{ and } w = \begin{pmatrix} .7 \\ .3 \end{pmatrix}.$$  

The bond attributes are

$$A = \begin{pmatrix} 2.0000 \\ 5.0000 \end{pmatrix}.$$  

Then the portfolio attribute vector $a_p = A'w = 2.9000$, giving an attribute deviation $a = a_p - a_T = -.1000$. The initial value of the objective function $g(V, a)$ is then 1.4900. We calculate elasticities from (C.8),

$$\xi_i = V^{-1}\left[V g_v + (A_i - a_p)g_a\right] \quad \forall i = 1, \ldots, m.$$  

This results in

$$\xi = \begin{pmatrix} .0880 \\ .1280 \end{pmatrix}.$$
It is easy to see in this case which of the two trade combinations, if any, is optimal: sell bond 1, buy bond 2, or 1/2, with the optimal trade direction equal to \((-\gamma_0, \gamma_n)\)' or \((-1.02, .98)'\). This may be confirmed with the algorithm described in Appendix E

\[
u^* = \left(\begin{array}{c}
-.01819 \\
.01748
\end{array}\right),
\]

which is the same direction (except for a scale factor).

A line search from the starting portfolio in this direction stops at the no-trade boundary (and maximizes the reward rate \(g(V, a)\)), giving the optimal portfolio composition:

\[
s^* = \left(\begin{array}{c}
10.0382 \\
4.9437
\end{array}\right), \quad \xi^* = \left(\begin{array}{c}
.0987 \\
.1027
\end{array}\right),
\]

\[
ap = 2.9899, \quad a = -.0101, \quad V^* = 14.9819, \quad g = 1.4981.
\]

The impulse trade \(u^*\) is thus the change in \(s\)

\[
u^* = \left(\begin{array}{c}
-.4618 \\
.4437
\end{array}\right),
\]

with transactions costs of .0181.

We note that trading stops short of the no-cost goal of \(a = 0\) and \(\xi^*_1 = \xi^*_2\), which shows indeed the trade-off between portfolio objectives and trading cost. Note also that trading tends to equalize elasticities even with transactions costs.

**2. Example 2 (m = 3, l = 1)**

The starting portfolio in this example is the same as in Example 1, but now a third bond has been added to the traded set:

\[
s_3 = .0000, \quad A_3 = 3.0000,
\]

with \(V = 15.0000, \ a_p = 2.9000, \ a = -.1000, \) and \(g = 1.4900\) remaining the same. Now we get the initial portfolio elasticities

\[
\xi = \left(\begin{array}{c}
.0880 \\
.1280 \\
.1013
\end{array}\right).
\]
The optimal trade direction is found with the algorithm of Appendix E,

\[ v^* = \begin{pmatrix} -.01661 \\ .01912 \\ -.00328 \end{pmatrix}. \]

The trade combination is $1 + 3/2$, so that the additional bond is sold (short).

A line search in this direction results in the optimal portfolio

\[ s^* = \begin{pmatrix} 10.0939 \\ 4.9672 \\ -.0802 \end{pmatrix}, \quad \xi^* = \begin{pmatrix} .0986 \\ .1029 \\ .1000 \end{pmatrix}, \]

\[ a_p = 2.9893, \quad a = -.0107, \quad V^* = 14.9809, \quad g = 1.4980. \]

The impulse trade is then

\[ u^* = \begin{pmatrix} -.4061 \\ .4672 \\ -.0802 \end{pmatrix}. \]

with transactions costs of .0191.

Despite the additional bond available for trading, example 2 sees more trading (higher costs) than example 1, and results in a slightly larger attribute deviation $a$ and lower reward rate $g$. This result appears counter-intuitive, since the additional bond should make it easier, and thus less costly, to achieve portfolio objectives and reward levels.

This example illustrates the errors introduced when the optimal trading rule for infinitesimal impulses in Section 5 is used for large discrete trades, when the portfolio elasticities vector ($\xi$) plots far outside the no-trade boundary. The error appears, because the optimal trade direction does not remain constant along the straight trading line from the original starting point. For example, the point

\[ s^* = \begin{pmatrix} 10.3339 \\ 4.6912 \\ -.0328 \end{pmatrix}, \quad \xi^* = \begin{pmatrix} .0942 \\ .1123 \\ .1003 \end{pmatrix}, \]

has optimal trading direction

\[ v^* = \begin{pmatrix} -.00676 \\ .00720 \\ -.00073 \end{pmatrix}. \]
These trade proportions are slightly different from those above. The resulting error in \( u^* \) is amplified, the further the portfolio is outside the no-trade boundary.

3. Example 3 \((m = 5, l = 2)\)

In this more elaborate example, the reward rate \( g = .1 V - a'a \), the attribute target vector \( a_T = (3, 3)' \), and other parameters are as in the previous examples. Three successive portfolio monitoring dates are considered in this example.

The starting portfolio at the first date is

\[
\begin{pmatrix}
2.0000 \\
6.1000 \\
1.0500 \\
1.9500 \\
4.0000
\end{pmatrix}, \quad \text{and} \quad A = \begin{pmatrix}
2.0200 & 1.0200 \\
4.9800 & 2.0300 \\
3.0000 & 3.0300 \\
2.0300 & 4.0300 \\
1.0200 & 5.0300
\end{pmatrix}.
\]

The value of this portfolio is \( V = 15.1000 \), the reward rate \( g = 1.5092 \), and portfolio attributes and elasticities are

\[
a_p = \begin{pmatrix} 3.0203 \\ 3.0187 \end{pmatrix}, \quad a = \begin{pmatrix} .0203 \\ .0187 \end{pmatrix}, \quad \text{and} \quad \xi = \begin{pmatrix} .1077 \\ .0972 \\ .1000 \\ .1002 \\ .1004 \end{pmatrix}.
\]

The optimal trade direction is found to be

\[
v^* = \begin{pmatrix} .00329 \\ -.00308 \\ -.00024 \\ -.00011 \\ .00000 \end{pmatrix},
\]

and a line search in this direction leads to the optimal portfolio

\[
s^* = \begin{pmatrix} 2.0777 \\ 6.0273 \\ 1.0444 \\ 1.9474 \\ 4.0000 \end{pmatrix}, \quad \xi^* = \begin{pmatrix} .1042 \\ .1002 \\ .1000 \\ .0990 \\ .0981 \end{pmatrix},
\]

\[
a_p = \begin{pmatrix} 3.0059 \\ 3.0130 \end{pmatrix}, \quad a = \begin{pmatrix} .0059 \\ .0130 \end{pmatrix}, \quad V^* = 15.0968, \quad g = 1.5095.
\]

This portfolio is now held without trading till the second date, at which time it and the market have changed to

\[
\begin{pmatrix}
2.1000 \\
6.3000 \\
1.2000 \\
2.1000 \\
4.2080
\end{pmatrix}, \quad \text{and} \quad A = \begin{pmatrix}
2.0000 & 1.0000 \\
5.0000 & 2.0000 \\
3.0000 & 3.0000 \\
2.0000 & 4.0000 \\
1.0000 & 5.0000
\end{pmatrix}.
\]
Portfolio value is now $V = 15.9080$, the reward rate $g = 1.5980$, and portfolio attributes and elasticities are

$$ a_p = \begin{pmatrix} 2.9990 \\ 3.0010 \end{pmatrix}, \quad a = \begin{pmatrix} -.0010 \\ .0010 \end{pmatrix}, \quad \text{and} \quad \xi = \begin{pmatrix} .1001 \\ .1004 \\ .1000 \\ .0997 \\ .0995 \end{pmatrix}. $$

Application of the algorithm shows that no trading is optimal for this portfolio, i.e. the cost of trading outweighs the improvement in the expected reward.

The unchanged portfolio is now held till a third date, at which time it and the market have changed to

$$ s = \begin{pmatrix} 2.2000 \\ 6.5000 \\ 1.3500 \\ 2.2500 \\ 4.4000 \end{pmatrix}, \quad \text{and} \quad A = \begin{pmatrix} 1.9800 & 0.9800 \\ 5.0100 & 1.9800 \\ 3.0000 & 2.9800 \\ 1.9800 & 3.9800 \\ 1.0000 & 4.9800 \end{pmatrix}. $$

The portfolio value is $V = 16.7000$, the reward rate $g = 1.6696$, while attributes and elasticities are

$$ a_p = \begin{pmatrix} 2.9836 \\ 2.9890 \end{pmatrix}, \quad a = \begin{pmatrix} -.0164 \\ -.0110 \end{pmatrix}, \quad \text{and} \quad \xi = \begin{pmatrix} .0954 \\ .1027 \\ .1000 \\ .0993 \\ .0987 \end{pmatrix}. $$

The optimal trade direction is now found to be

$$ v^* = \begin{pmatrix} -.00169 \\ .00162 \\ .00000 \\ .00000 \\ .00000 \end{pmatrix}, $$

i.e. sell bond 1 for bond 2 (1/2), with no trade in bonds 3, 4, and 5 (note the rounding error in the trade direction when $v^*$ is small). A line search in this direction leads to the optimal portfolio

$$ s^* = \begin{pmatrix} 2.1537 \\ 6.5445 \\ 1.3500 \\ 2.2500 \\ 4.4000 \end{pmatrix}, \quad \xi^* = \begin{pmatrix} .0970 \\ .1010 \\ .1000 \\ .1000 \\ .1000 \end{pmatrix}, $$

$$ a_p = \begin{pmatrix} 2.9918 \\ 2.9919 \end{pmatrix}, \quad a = \begin{pmatrix} -.0082 \\ -.0081 \end{pmatrix}, \quad V^* = 16.6982, \quad g = 1.6697. $$

The portfolio adjustments are small close to the no-trade boundary.
Fig. 1 Trade/No-Trade Regions, $m = 2$

Elasticity combinations in shaded area yield optimal trades that are inconsistent with the assumed values of $\gamma$, hence trading is not optimal.
Fig. 2 Trade Regions without Costs, $m = 3$
Cross-section view over Diagonal towards Origin
Fig. 3 Trade/No-Trade Regions with Costs, $m = 3$
Cross-section view over Diagonal towards Origin, Not to Scale
RISK AND RETURN
IN THE
GOVERNMENT BOND MARKET

An Essay in Financial Economics

By
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1. Introduction

The law-of-one-price for risk from the same source is a characteristic property of asset pricing models. It holds both in general equilibrium and in no-arbitrage (partial equilibrium) economies, using either continuous-time or discrete-time models. An asset's expected return at time \( t \), \( E[\tilde{r}_t] \), can then be expressed in terms of the vector \( b_t \) of risk attributes, unique to that asset, and the current vector \( \lambda_t \) of "market prices" of risk,

\[
E[\tilde{r}_t] = \lambda_0 + b'_t \lambda_t,
\]

with appropriate definitions for the holding period in each model. The risk attributes \( b \) have the usual covariance interpretations known from portfolio theory, while \( \lambda_0 \) equals the riskless rate, if one exists.

Empirical tests of the predictive relationship (1.1) have been carried out extensively in the stock markets using the frameworks of the CAPM and the A.P.T., both discrete-time models. As Roll and Ross [50] have argued, relationship (1.1) is void of empirical content without strong assumptions on \( b_t \) and \( \lambda_t \). A common assumption used in all such empirical tests is that of "stationarity" of the return generating process, i.e. the risk attributes \( b \) of the assets are assumed constant over

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1 Starting with Merton [45], continuous-time pricing models have been developed for contingent claims by Black and Scholes [7], Brennan and Schwartz [9], Dietrich-Campbell and Schwartz [24]; and for financial assets by Vasicek [56], Cox, Ingersoll, and Ross [22,23], and Brennan and Schwartz [10,11].

2 Best known are the Capital Asset Pricing Model (CAPM) of Sharpe/Lintner and Black, and the Ross [51] Arbitrage Pricing Theory (A.P.T.).

3 See, e.g., Roll and Ross [50] for a motivation of these risk/price interpretations. Admati and Pfleiderer [1] and Chamberlain [15] show that market prices of risk may be interpreted as excess returns on (portfolios of) securities.

4 Arbitrage pricing models allow relationship (1.1) to be an approximate one as long as the aggregate deviations are bounded in some way as the number of assets grows without bound. See, e.g. Ross [51], Chamberlain and Rothschild [16], and Ingersoll [37].

5 See, e.g., Roll and Ross [50], Chen [17], and Brown and Weinstein [14] for tests of the A.P.T.; Black, Jensen, and Scholes [6], Fama and MacBeth [26], Gibbons [31], Stambaugh [55], and Hess [33] for various tests of the CAPM.
the sample period considered, implying that the assets remain equally risky over time. This assumption enables estimation of the risk attributes $b$ from an ad hoc linear "market" or "factor" model for the return generating process:

$$\tilde{r}_t = E[\tilde{r}_t] + \beta \delta_t + \xi_t,$$

(1.2)

where $\delta_t$ is a vector of known indices, such as the return on the market portfolio in the case of CAPM, or a vector of unknown factor realizations in the case of A.P.T. The assumption of constant risk is the key to estimating relationships (1.1) and (1.2) as bi-linear systems, i.e. $b$ may be estimated given knowledge of $\delta$ or $\lambda$, and vice versa. Hypothesis testing includes tests of significance of risk factor prices and tests of the law of one market price of risk across sub-samples, as shown e.g. in Brown and Weinstein [14] and Hess [33]. The assumption of constancy of the market prices of risk $\lambda$ over all or part of the sample period is usually needed for determinacy of the estimation, to overcome the errors-in-variables problem in the estimation of $b$ from (1.2), or simply in order to gain power in hypothesis testing.

Tests of the bilinear paradigm (1.1) have not been done directly for government bonds, because an immediate difficulty is, of course, violation of the constant $b$ risk assumption for bonds or portfolios of bonds. A considerable body of empirical research has indeed taken relationship (1.1) as given, as in the fitting of continuous-time bond valuation models to observed bond prices, see e.g. Brennan and Schwartz [10,11], Cox, Ingersoll, and Ross [22], Dietrich-Campbell and Schwartz [24], and Jacobs and Jones [39]. The thrust of these models has been towards estimating the dynamics of the term structure of interest rates with non-linear simultaneous

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6. The linearity of the return generating process is then an integral part of any hypothesis test, including that of relationship (1.1).

7. This terminology was introduced into finance by Brown and Weinstein [14].

8. Return distributions collapse as maturity approaches, making bonds and portfolios of bonds on average less and less risky as time goes on.
estimation of the parameters of the continuous-time state variables and market prices of risk $\lambda$, rather than towards tests of the law-of-one-price for risk (1.1).

Other empirical studies of the government bond market have focused on hedging or immunizing strategies in portfolio management, seeking to develop intuitive or ad hoc risk measures, e.g. various forms of “duration”, that describe interest rate or basis risk in the bond market. Bond and portfolio risk is here typically measured as the price volatility w.r.t. certain term structure movements, with duration expressed in relative terms as the remaining time-to-maturity of an equally volatile pure discount bond. Well-known studies in this respect are those of Fisher and Weil [27], Ingersoll, Skelton, and Weil [38], Bierwag, Kaufmann, and Toevs [4], and Fong and Vasicek [28].

Studies combining elements of the term structure models with those of duration strategies include Cox, Ingersoll, and Ross [21], Ingersoll [36], and Brennan and Schwartz [12], who base their work on theoretical risk measures (instantaneous standard deviations or diffusion terms), while Gultekin and Rogalski [32] use ad hoc duration measures in this context.

Empirical studies of government bond markets that come closest to the CAPM and A.P.T. testing literature are those of Oldfield and Rogalski [49], who study portfolios of T-bills, and Gultekin and Rogalski [32], who extend their analysis to bills, notes, and bonds of all maturities. These authors posit linear return generating models, as in (1.2), for portfolios of bonds sorted in each period by a remaining time-to-maturity interval, thereby implicitly choosing maturity as the appropriate risk measure for government bonds. It has also been customary to add diversified bond portfolios to tests of CAPM, as in Stambaugh [55], predicated on an even wider notion of risk for government bonds: all bonds are perfectly correlated w.r.t. term structure changes, which in turn depend only on the market.
Such practices conform well to practitioners' observations, e.g. longer maturity bonds are on the whole more volatile than shorter maturity ones, while diversified bond indexes tend to move with or counter to the market in a largely predictable way. Intertemporal risk, whichever way defined in a proper bond valuation model, may then be expected to stay within a certain bandwidth for diversified bond portfolios that are either periodically rebalanced by remaining time-to-maturity, or for which the traded bond mix does not change too much. This bandwidth, or departure from truly constant risk, may also be affected by the portfolio selection method and holding period, sometimes adding to it and at other times decreasing it. Such portfolios then approximate the constant risk securities for which the usual tests of CAPM and A.P.T. are appropriate. While such methodology appears reasonable, the adequacy of its assumptions has never been demonstrated vis-à-vis some of the available bond pricing models mentioned before.

The objective of this paper is to test government (default-free) bond returns in the spirit of CAPM and A.P.T. using portfolios maintained at constant intertemporal risk, so that the usual empirical assumption on $b$ is satisfied by construction. These portfolios are then artificial securities that lend empirical content to the asset pricing prediction (1.1). This work is an extension of that of Gultekin and Rogalski [32] by controlling more specifically for risk level constancy in the portfolios.\footnote{The constant risk grouping method is also used by Gultekin and Rogalski [32] in a different context. They test the hypothesis of constant variance of time-series returns of bond portfolios with the same constant simple (yield-based) duration, but formed in many different ways from the bonds traded in each period. They interpret rejection of that hypothesis as evidence that duration is not an appropriate risk measure for government bonds. That conclusion is in error, however, since bond returns are not perfectly correlated: depending on portfolio selection method, more or less of portfolio variance is diversified away. This issue is further addressed in Section 4 of this paper.}

Three sets of tests are carried out on these security returns:

1. Distributional tests, both univariate and multivariate, and comparison with
published results on stock and bond returns.

(2) A.P.T.-style tests of (1.2) and (1.1), as guided by the presence of significant (unknown) factors in the return generating model (1.2).

(3) CAPM-style tests of (1.1), taking the risk measures as given.

We base our tests on a set of six risk measures that have been chosen to represent both popular and theoretical bond pricing and immunization models. Eight constant risk portfolios are formed in each case, with portfolios rebalanced monthly so as to keep risk deviations small. Risk measures used and compared in this paper derive from the following models of the term structure of interest rates:

(1) Macaulay [42]’s duration,

(2) the two-dimensional vector of instantaneous standard deviations of the Brennan and Schwartz [11] bond pricing model,

(3) stochastic duration as defined by Cox, Ingersoll, and Ross [21],

(4) the term structure sensitivity to changes in the 13-year spot yield as in Nelson and Schaefer [48],

(5) the two-dimensional extension to Macaulay’s duration by Fong and Vasicek [29], and—by way of contrast—

(6) the traditional grouping of bonds by remaining time-to-maturity.

An interesting question, and an empirical one with possible practical consequences, is the relative “performance” of these risk measures in the bilinear tests of (1.1) and (1.2).\textsuperscript{10} Provided that the linear models (1.1) and (1.2) are well-specified, “goodness-of-fit” results across the various risk measures may yield evidence on their validity.

\textsuperscript{10} Gultekin and Rogalski [32] find little difference between various duration measures and time-to-maturity to describe the risk of portfolios initially sorted by remaining time-to-maturity. Brennan and Schwartz [12] and Nelson and Schaefer [48] find little difference between immunization strategies based on duration and on their own respective risk measures (see Section 2).
An additional question of interest is the effect of portfolio selection methods on the tests. Two extreme portfolio selection methods are colloquially known as "ladder" and "bullet", with greater and lesser degrees of diversification, respectively. Their effect, in conjunction with the choice of rebalancing frequency, will be one of measurement error in the portfolio holding period returns, as the constant risk bond portfolios will be less so at each month's end. This phenomenon is common to studies of securities with finite maturity dates using discrete observations. We expect that the added variance on account of bullet selection and discrete rebalancing is not priced, since it can be diversified away or removed by more frequent trading.

Section 2 contains a detailed discussion of intertemporal risk in bond markets using a continuous-time setting: we motivate our choice of risk measures and show how each relates to underlying state variables governing term structure behaviour. We describe in Section 3 the data sources, term structure estimation, risk measure calculations, constant risk targets, and portfolio selection criteria. Section 4 is dedicated to distributional and time-series analysis of the six sets of constant risk portfolio returns, including covariance stationarity tests; we also analyze differences resulting from portfolio selection methods. We discuss in Section 5 the A.P.T. factor model and methods of factor extraction from the covariance matrices encountered. Results of factor analyses and hypothesis tests are presented in Section 6, together with a comparative study of factors and representative portfolios. Section 7 includes the cross-sectional tests of the A.P.T. law-of-one-price prediction (1.1), while CAPM-style tests are reported in Section 8. Section 9 concludes the paper with a summary of major results.
2. Measures of Intertemporal Risk

We motivate in this section our choice of risk measures to be used in forming constant risk bond portfolios. It can be shown in the continuous-time framework of financial asset pricing\textsuperscript{11,12} that the correct risk measure for a financial asset $P(Y, t)$ is the vector of instantaneous standard deviations of the return process $(dP + c \, dt)/P$,

$$\beta = \frac{G' P_Y}{P},$$  \hspace{1cm} (2.1)

where $P = P(Y, t)$ is the price of the financial asset, with payout rate $c$

$Y = Y(Y, t)$ is the $k \times 1$ vector of state variables,

$G = G(Y, t)$ is the $k \times k$ matrix of diffusion coefficients of the process $dY$,

$P_Y =$ the gradient of $P$ w.r.t. $Y$, a $k \times 1$ vector.

Clearly, any linear transformation of (2.1) is equally usable as a measure of asset risk. E.g., we may use $-G' P_Y/P$ rather than $G' P_Y/P$, as long as this transformation is applied consistently. We shall use it in this paper on occasion without further reference in order to get conveniently positive risk values.\textsuperscript{13} Because the matrix $G$ is shared by all assets subject to the same state variable process, the usual transformation of (2.1), used in portfolio hedging or immunization, omits this

\textsuperscript{11} E.g., Merton [45,46] and Cox, Ingersoll, and Ross [23] in an equilibrium context, or Cox, Ingersoll, and Ross [22] and Brennan and Schwartz [11] using no-arbitrage principles, derive the linear risk/return relationship (1.1), $\alpha - r = \lambda' \beta$, where $\alpha$ is the expected instantaneous asset return, $r$ is the instantaneous riskless return, $\beta$ is the vector of diffusion coefficients (or instantaneous standard deviations) of the asset, and $\lambda$ is the vector of market prices of risk. Clearly, if for two assets the corresponding vectors $\beta$ are identical, the excess returns of the assets must be the same, and hence they must be equivalently priced. See also Merton [46], who shows that the instantaneous standard deviation is the relevant risk measure in the Rothschild and Stiglitz \textsuperscript{52} sense.

\textsuperscript{12} Using Itô’s lemma: $dP = P'_Y \, dY + \frac{1}{2} P_{YY} \, dY \, dY + P_t \, dt = (\cdots) \, dt + P'_Y G \, dz$, after substitution of the dynamics of the state variables $dY = (\cdots) \, dt + G \, dz$.

\textsuperscript{13} This cannot always be achieved, e.g. the short rate risk $P'/P$ in the Brennan and Schwartz [11] model is known to change sign for coupon bonds when their time-to-maturity increases sufficiently.
common matrix\textsuperscript{14} to yield the simplified

$$\beta^* = \frac{P_Y}{P}. \quad (2.2)$$

Expression (2.2) cannot be used in describing intertemporal risk, however, when the elements of $G$ depend on the state variables or on time, as in several of the bond pricing models considered here.

It is instructive to use an exposition by Nelson and Schaefer \cite{48} in expressing (2.2) as

$$\beta^* = \frac{-\sum_{\tau=1}^{T} (\tau - t) c(\tau) B(Y, t, \tau) R_Y(Y, t, \tau)}{\sum_{\tau=1}^{T} c(\tau) B(Y, t, \tau)}, \quad (2.3)$$

where now the asset $P$ has discrete disbursements $c(\tau)$ as in a coupon bond, $B(Y, t, \tau)$ is the price at time $t$ of a unit pure discount bond maturing at time $\tau$, while $R(Y, t, \tau)$ is the corresponding spot yield given by $B(Y, t, \tau) = \exp[-(\tau - t) R(Y, t, \tau)]$.

The $k \times 1$ gradients $R_Y(Y, t, \tau)$ for varying $\tau$ describe the sensitivity of the term structure to changes in the state variables $Y$ at time $t$. The intertemporal equivalent of (2.3) is found by pre-multiplying $R_Y(Y, t, \tau)$ by $G'(Y, t)$.

\textbf{2.1 Duration and Single State Variable Models}

The concept of \textit{duration} of a financial asset has evolved in finance theory\textsuperscript{15} as the time-to-maturity of an equivalent pure discount bond, i.e. a bond with the same “basis” risk as the asset under consideration. Assets with the same basis risk have, by definition, the same return behaviour when changes in the term structure occur. Basis risk is then equivalent to the risk vector $\beta$ in (2.1).

Duration has been used traditionally in bond portfolio management for the purpose of immunization, for which only contemporaneous risk comparisons are

\textsuperscript{14} Portfolio hedging or immunization takes place in comparative terms: the portfolio’s risk measure is matched concurrently with that of a target instrument, both of which are equally affected by the matrix $G$.

\textsuperscript{15} Starting with Macaulay \cite{42}. 

required. Duration $D$ may then be derived from (2.3) by a (1,1) transformation $F$, which—if it exists—gives any pure discount bond a duration equal to its own time-to-maturity:

$$D = F^{-1}(\beta^*),$$

(2.4)

where $F(t) = B_Y/B$, $\forall t \in [t, T]$. Existence of the inverse function $F^{-1}(\cdot)$ depends on the state variable $Y$, which must be (locally) one-dimensional. This necessary requirement is not always sufficient, however, to guarantee a one-to-one mapping between duration and $P_Y/P$.\(^{16}\) The mapping is in practice often one-to-one, so that immunization strategies based on $P_Y/P$ and $D$ must then yield the same results.

Appropriate choices of the single state variable and its stochastic process in (2.4) yield some well-known duration measures.\(^{17}\) For example, a single state variable constraining the term structure to shape-preserving shifts only,\(^{18}\) implying that $R_Y = -1$ for all maturities $r$ in (2.3), makes the function $F^{-1}(\cdot) = F(\cdot)$ the identity transformation. The resulting form of (2.4), using (2.3), is then the well-known Macaulay duration

$$D = \frac{\sum_{\tau=t}^{T} (\tau - t) c(\tau) B(Y, t, \tau)}{\sum_{\tau=t}^{T} c(\tau) B(Y, t, \tau)}.$$  

(2.5)

Intertemporal risk comparisons using Macaulay duration assume implicitly a constant diffusion coefficient $G$ in (2.1).

The "stochastic" duration defined by Cox, Ingersoll, and Ross [21] is based on a mean-reverting stochastic process for the short rate, $r$, as the single state variable

$$dr = \beta (\mu - r) dt + \sigma \sqrt{r} \, dz,$$

(2.6)

\(^{16}\) E.g., it does not exist when the term structure is not quasi-concave or quasi-convex, i.e. has more than one reversal.

\(^{17}\) This discussion follows Cox, Ingersoll, and Ross [21] and Ingersoll, Skelton, and Weil [38]. As these authors have pointed out, the underlying state variable processes are not necessarily realistic, or even consistent with general equilibrium markets.

\(^{18}\) Christensen [18] has shown that no such state variable is consistent with the no-arbitrage requirement of a general equilibrium.
using their variable definitions. This state variable process models the dynamics of near-term spot yields more fully at the expense of long spot yields, which tend to be flat. The corresponding term structure sensitivities $R_Y$ from (2.3) show\(^\text{19}\) a super-linear decay with increasing time-to-maturity, from a high value of 1 down to a typical value of .1 at 15 years. It is furthermore independent of the current value of the state variable $r$.

The transformation (2.4) required to obtain CIR’s stochastic duration is non-linear and (1,1), so that stochastic duration then has the same information content as (2.2). Since the diffusion coefficient $\sigma \sqrt{r}$ in (2.6) is not constant, stochastic duration is not a meaningful concept in intertemporal risk comparisons. We use therefore in this paper the following version of (2.1)

$$\beta = \sigma \sqrt{r} P_r/P.$$  \hspace{1cm} (2.8)

A single state variable model by Nelson and Schaefer \cite{48} is based on the 13-year spot yield (from the term structure), with the simple state variable process

$$dR_{13} = \mu_{13} dt + \sigma_{13} dz.$$  \hspace{1cm} (2.9)

This model has less structure than that of Cox, Ingersoll, and Ross \cite{21}, because it lacks mean reversion and has a homoskedastic stochastic process. Although the 13-year rate has an intuitive advantage over the short rate in the long-term bond market,\(^\text{20}\) it is generally more difficult to estimate. Nelson and Schaefer \cite{48} base their empirical work on estimated discrete values of term structure sensitivity to changes in the 13-year spot yield. Table 4 from Nelson and Schafer \cite{48}, reproduced

\(^{19}\) By inverting CIR’s analytical solution for the price of a pure discount bond maturing at date $\tau$, $B(r,t,\tau) = \exp[-(r - t)R(r,t,\tau)]$, using also parameter values estimated by these authors.

\(^{20}\) This idea has been taken to its logical conclusion by Brennan and Schwartz \cite{11}, who take the consol rate as state variable, together with the short rate.
in Appendix A, shows that these sensitivities are closer to the constant value of 1 implied by Macaulay's duration than to the decaying pattern of the Cox, Ingersoll, and Ross [21] model: values range from a low of .906 (for the six year spot yield) to a high of 1.017 (for the 12 year spot yield). The Nelson and Schaefer model is then basically a refined version of Macaulay's duration.

Because the diffusion coefficient \( \sigma_{13} \) in (2.9) is a constant, intertemporal comparisons of risk in their case may be made on the basis of expression (2.3).

The number of single-state-variable models for the term structure is by no means exhausted by the above described three models. A variant of Macaulay's duration can be constructed by the use of the yield-to-maturity in deriving bond durations. It is well known that this measure is accurate only if the term structure is constrained to be flat at all times, clearly an unrealistic assumption.\(^{21}\) Other single state variable or duration models are due to Cox, Ingersoll, and Ross [23], Vasicek [56], Ingersoll, Skelton, and Weil [38], and—indeed—Merton [45]. A large number of ad hoc duration measures are further reviewed in Ingersoll, Skelton, and Weil [38] and tested in Gultekin and Rogalski [32].

A popular intuitive risk measure in bond trading practice is—of course—the remaining time-to-maturity of a bond, thereby neglecting the effects of varying coupon rates. The pervasiveness of this simple risk measure in bond trading practice is shown by the widespread use of so-called yield curves, i.e. curves drawn through scatter plots of yield-to-maturity against time-to-maturity of all the traded bonds.

The ability of time-to-maturity to describe the variability of bond prices will be no

\(^{21}\) The simple yield-based duration is in practice often a reasonable substitute for Macaulay's duration, especially since the former can be determined without knowledge of the term structure. Furthermore, as Boquist, Racette, and Schlarbaum [8] have shown, the yield-based duration has an interpretation in terms of CAPM \( \beta \), and is therefore certainly a risk measure of interest. Unfortunately, yield-based duration is less suitable in a portfolio setting, since it is not additive in value-weighted proportion. This non-additivity has been ignored by Gultekin and Rogalski [32].
better than that of the yield curve. Equating expression (2.3) to \((T - t)\), we derive the corresponding term structure sensitivity for a bond maturing at date \(T\)

\[
R_T = -\left(\frac{T - t}{T - t}\right),
\]

(2.10)
or a hyperbolic decline down to a value of 1, applicable to the bond’s coupon payments only. No single state variable exists with this term structure sensitivity for all coupon bonds.

### 2.2 Multi-State-Variable Models

Multi-state-variable models of government bond pricing and portfolio immunization have been studied by Brennan and Schwartz [10,11,12], Nelson and Schaefer [48], Dietrich-Campbell and Schwartz [24], Jacobs and Jones [39], and—in a different context—also by Ingersoll [36]. Since the duration concept is meaningless in this context,\(^{22}\) the above authors concentrate on the measure \(\beta^*\) from (2.2). Immunization, or hedging, takes place by matching a portfolio’s \(\beta^*\) vector with a target risk vector, typically that of the hypothetical pure discount bond of the exact required maturity, but more complicated risk targets are also possible.

The theoretical and empirical work done by the above authors is based on two-state-variable models; their methodology, however, can be extended to more than two state variables (but not without greatly increased computational load).\(^{23}\)

The Brennan and Schwartz [11] two-state-variable bond pricing model is based on the short rate, \(r\), and the consol rate, \(l\). The short rate reverts to the consol rate as a moving mean, while the consol rate process depends on its own level and

\(^{22}\) Because the transformation (2.4) cannot be \((1,1)\). Ingersoll [36] calls the elements of his risk measure “durations”, but there is no connection with pure discount bonds of equivalent basis risk.

\(^{23}\) Brennan and Schwartz [11] have reported that U.S. Government bonds are perhaps best priced by a three-state-variable model. Dietrich-Campbell and Schwartz [24] report no obvious arbitrage opportunities or large mispricings when the Brennan and Schwartz model is extended to the bond options market.
on the difference between consol and short rates.\textsuperscript{24} These state variables follow the joint Itô process

\[
d \left( \begin{array}{c}
  r \\
  l
\end{array} \right) = \left( \begin{array}{c}
  a(l - r) \\
  l(l - r + b)
\end{array} \right) dt + \left( \begin{array}{cc}
  r \sigma_1 & 0 \\
  0 & \sigma_2
\end{array} \right) dz, \quad dz
dz' = \left( \begin{array}{c}
  1 \\
  \rho \\
  1
\end{array} \right) dt.
\]

(2.11)

The associated term structure sensitivities, \( R_r \) and \( R_l \), are not constant as they depend on the current values of the state variables. Their values with increasing time-to-maturity follow those of the Cox, Ingersoll, and Ross \textsuperscript{21} and duration models, respectively, i.e. decaying for \( R_r \) and flat for \( R_l \).\textsuperscript{25} As the diffusion coefficients are clearly non-constant, the intertemporally comparable risk vector equivalent to (2.1) is then

\[
\beta = (r \sigma_1 P_r / P, \ l \sigma_2 P_l / P)'.
\]

(2.12)

An alternative two-state-variable model is offered by Fong and Vasicek \textsuperscript{28},\textsuperscript{26} who introduce a measure of dispersion of the portfolio or asset disbursements \( \{c(\tau)\} \) related to the second moment of statistics:

\[
M^{(2)} = \frac{\sum_{\tau=t}^{T} (\tau - t)^2 c(\tau) B(\tau, t)}{\sum_{\tau=t}^{T} c(\tau) B(\tau, t)}.
\]

(2.13)

This measure is clearly an extension of Macaulay’s duration from (2.5). The same authors \textsuperscript{29} and Bierwag, Kaufman, and Toevs \textsuperscript{4} describe \( M^{(2)} \) euphemistically as a measure of “immunization risk” or “stochastic risk” w.r.t. “misspecification of the stochastic process implied by Macaulay’s duration”.

A more recent interpretation of \( M^{(2)} \) in bond portfolio management practice is that of “convexity,” the second partial derivative of price w.r.t. the state variable

\textsuperscript{24} Use of the consol rate allows its market price of risk to be expressed as a function of the remaining parameters. By comparison, Jacobs and Jones \textsuperscript{39} have estimated a two-stage mean-reverting process for the short rate, with the intermediate moving target treated as an unknown second state variable.

\textsuperscript{25} Sensitivities derived on basis of central differences, using discount bond values estimated numerically, see Brennan and Schwartz \textsuperscript{11}.

\textsuperscript{26} These authors do not call it a two-state-variable model.
$Y$ normalized for price, $\frac{1}{P} \partial^2 P / \partial Y^2$. The first two terms of the Taylor expansion of bond or portfolio price $P(Y)$ around some value of the state variable $Y$ can then be written as

$$\frac{\Delta P}{P} = D \Delta Y + \frac{1}{2} M^{(2)} (\Delta Y)^2,$$

giving insight into the role of $D$ and $M^{(2)}$ in proportional price changes.

$M^{(2)}$ is suggested as an additional risk target, in conjunction with Macaulay's duration, in immunization and hedging strategies. Since $M^{(2)}$ is independent from Macaulay's duration, these two risk measures span a two-dimensional state space, if indeed the state space is two-dimensional.\(^{27}\) The implied two-dimensional state variable must impose the following term structure sensitivities\(^{28,29}\)

$$R_Y(Y, t, \tau) = \left( -\frac{1}{(\tau - t)} \right).$$

The first of these term structure sensitivities again relates to shape-preserving changes in the term structure; the second increases linearly with remaining time-to-maturity. As was the case with Macaulay's duration, we also use (2.14) for intertemporal risk comparisons, i.e. we implicitly assume that the diffusion coefficients of the implied unknown state variables are constant.

### 2.3 Risk Measure Comparisons

Although all of the above risk measures have been designed to fit the relatively unsophisticated (in terms of sources of risk) U.S. default-free bond market, considerable differences exist when bonds are ranked with these measures.

Cox, Ingersoll, and Ross [21] and Ingersoll, Skelton, and Weil [38] compare their stochastic durations with Macaulay's duration, finding that the latter may be

\(^{27}\) Frequent portfolio rebalancing can keep changes $\Delta Y$ small in the above Taylor expansion, so that the second term disappears. $M^{(2)}$ risk would then not be priced.

\(^{28}\) But see our earlier comments w.r.t. compatibility with the no-arbitrage property of financial markets.

\(^{29}\) It is easily verified that substitution of (2.14) into (2.3) yields the risk measure $\beta^* = (D, M^{(2)})'$. 
considerably larger, especially for bonds with high remaining time-to-maturity. For example, a typical 10-year 4% coupon bond with a current short rate of 5.623% is found by Cox, Ingersoll, and Ross to have a stochastic duration of 4.29 years, whereas Macaulay’s duration (using the same hypothetical term structure) is 8.2 years. This difference is not surprising in view of the different term structure sensitivities: constant for Macaulay’s duration vs. declining for stochastic duration.

By comparison, Nelson and Schaefer [48]’s single-state-variable model based on the 13-year spot yield, with the 13-year rate at 10%, has an equivalent duration for the same 10-year 4% coupon bond of 8.13 years. Macaulay’s duration (using the same term structure as implied by the Nelson and Schaefer model) is 8.53 years, a difference of only 5%. Again, this is not surprising in view of the similarly shaped term structure sensitivities of these models.

The Brennan and Schwartz model has been tested in the Canadian government bond market [10], in the U.S. government bond market [11,12], and in the U.S. government bond options market (in Dietrich-Campbell and Schwartz [24]). The latter authors, using model parameters estimated for the period 1970–1982, show that predicted prices of bond options in 1982–1984 are sufficiently close to observed prices so as not to admit obvious arbitrage opportunities.

Brennan and Schwartz [12] compare immunization strategies on the basis of Macaulay’s duration with a hedging strategy based on their two-state-variable model, reporting no great differences between the two methodologies. In an interesting aside, they report for their hedge portfolios an almost linear relationship between the two Brennan and Schwartz risk measures and duration.

Nelson and Schaefer [48] also develop a two-state-variable model, working with the one-year and five-year spot yields as state variables, in addition to the 13-year spot yield. Their model is simpler than that of Brennan and Schwartz, however,
as it uses constant diffusion terms in the stochastic processes of the state variables. Empirically, little difference is found by Nelson and Schaefer between immunizing or hedging strategies using Macaulay’s duration and strategies based on the combinations of the (1,5,13), (5,13), and (13) year spot yields as state variables.\textsuperscript{30}

This result parallels that in Brennan and Schwartz [11], which is understandable because a long or consol rate is the dominant state variable in both models, especially for times-to-maturity of more than 2–5 years: Macaulay’s duration is known to perform well for such bonds.

Nelson and Schaefer [48] also test immunization strategies on the basis of a simple remaining time-to-maturity criterion, and report no great difference when compared with strategies based on their own risk measures.

Finally, the results by Ingersoll [36] are not as good, ascribed by the author to the use of a non-standard stochastic process with index data.

Evidently, differences exist between these various risk measures, especially when comparing risk levels across risk measures and in ranking bonds by risk. The operational question of interest to this paper, however, is limited to how well each of these risk measures, when used in conjunction with certain portfolio selection procedures, achieves our constant risk portfolio goal. This, of course, is an empirical question.

\textit{2.4 Constant Risk Measures}

The preceding discussion treats a broad sample of bond market risk measures. Their suitability in this paper for the purpose of constant risk portfolio formation is limited only by the reasonable demand that each adopted measure (with the possible exception of time-to-maturity) be additive in value-weighted proportion.

\textsuperscript{30} They also test various methods of term structure estimation (needed to calculate duration). No great differences are found there either.
This simplifies the portfolio formation and rebalancing process.

This requirement is met by all risk measures discussed\(^{31}\) except the popular yield-based duration measure. This distinction is often ignored by practitioners, who simply assume additivity on the expectation that it makes little difference. We have elected to avoid introducing unquantifiable effects in the constant risk returns, however, using instead durations as determined from term structure estimates.

In addition to Macaulay \([42]\)’s duration, with its sensitivity to parallel movements of the term structure, we choose two additional one-parameter risk measures: the short rate based measure by Cox, Ingersoll, and Ross \([21]\) and the long (13-year) based measure by Nelson and Schaefer \([48]\). Of the two-parameter measures, we choose the Brennan and Schwartz \([11]\) measure (representing a proper pricing model), and the Fong and Vasicek \([29]\) extension of duration (representing the ad hoc and easier-to-use side). To these we add the simplest measure of all: remaining time-to-maturity, arbitrarily treating it as additive in value-weighted proportion.

We summarize our choice of (intertemporal) risk measures for the formation of constant risk bond portfolios, and show abbreviations to be used in the remainder of this paper:

1. Macaulay \([42]\)’s duration \(D\), or DUR.
2. Brennan and Schwartz \([12]\) \((rP_r/P, LP_l/P)'\), or B/S = (BSR, BSL)'.
3. Nelson and Schaefer \([48]\) \(P_{13}/P\), or N/S.
4. Cox, Ingersoll, and Ross \([21]\) \(\sqrt{r} P_r/P\), or CIR.
5. Fong and Vasicek \([29]\) \((D, M^{(2)})'\), or F/V = (DUR, DURVAR)'.
6. Remaining time-to-maturity \((T - t)\), or MAT.

We describe in Section 3 the data sources, the term structure estimation pro-

\(^{31}\) All risk measures have been cast in terms of diffusion terms or instantaneous standard deviations. Since corresponding Gauss-Wiener terms \(dz\) are perfectly correlated, the standard deviations are additive.
cedure, and the portfolio selection methods used in calculating the risk measures for the constant risk bond portfolios. Appendix A contains details of analytical or numerical procedures used in calculating these risk measures.
3. Portfolio Returns

3.1 Data

Empirical work uses the 1984 CRSP tape of monthly U.S. government bond prices. The sample period is 1 December 1969 to 1 December 1979, yielding 121 observations of monthly bond returns. This period has been chosen because it overlaps, or coincides with, the sample periods in Brennan and Schwartz [11,12], Cox, Ingersoll and Ross [21], and Nelson and Schaefer [48], so that state variable parameter estimates from these papers may be used in calculating the corresponding risk measure values. The instruments selected from the tape are bonds and notes, fully taxable, with semi-annual coupons. Furthermore, bond prices must be listed at least for two successive months in the sample period prior to maturity. Averages of bid/ask prices are used when no quoted trades are recorded.

The discount function is used in each period to derive the risk measures listed in Section 2. We define the discount function as the present value function of a unit discount bond with variable time-to-maturity. It is estimated from the above coupon bond selection, using the tax-adjusted method of McCulloch [44]. For this purpose, an average income tax rate of 26% and an average capital gains tax rate of 13% have been used, as suggested by the work of Babbel [3]. The estimation method is based on a least squares curve fitting technique with cubic spline functions.  

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32 We have not extended the sample period beyond 1979, in view of a change in tax withholding on coupon interest in the U.S. effective in late 1979. The effect has been a great increase in term structure volatility, with a presumed discrete jump in market prices of risk factors, see e.g. Constantinides and Ingersoll [20]. An opposing view may be obtained from Dietrich-Campbell and Schwartz [24], who estimate parameters of the Brennan and Schwartz [11] bond pricing model for the period 1970–1982, and who then apply the model to find only insignificant mispricings in the bond options market for the next two years.

33 Alternative methods exist of estimating the discount function. McCulloch [43] uses quadratic spline functions and ignores taxes. Houget [34] fits exponential spline functions, but his results are not greatly different from those of McCulloch, and hence seem not worth the extra computational effort. Constantinides and Ingersoll [20] find significantly different estimates when taking into account the timing option.
extrapolation beyond the highest available bond maturity, where necessary, assumes a flat continuation of the yield curve. A maximum of 10 cubic spline functions has been used in each estimation.

Short rates are estimated more accurately as the (annualized) yields for those T-bills on the CRSP tape that have on the respective first days of each month as close as possible to 30 days left till maturity. The consol rate is derived from each discount function as the yield on a 20-year semi-annual annuity.\(^{34}\)

Constant risk portfolios have been selected in a two-stage procedure: the traded bonds in each period are first combined into a number of “basic” portfolios, which are then in turn used as the atomic securities for the constant risk portfolios. This procedure has been adopted for two reasons: (1) to manage the wide fluctuations in the number of traded bonds over the sample period, and (2) to compensate for the effects of pricing errors, non-synchronicity, and thin trading, on the individual bond prices listed on the CRSP tape.\(^{35}\)

Each basic portfolio is—in each period—equally weighted in the bonds that mature in the same non-overlapping six-month interval, with bonds of more than 10 years to maturity together comprising the last of the 20 basic portfolios. Table I

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\(^{34}\) Brennan and Schwartz [12] estimate the consol rate as the yield of the highest yielding bond with time-to-maturity of at least 20 years remaining, or 15 years if no 20+ year bond is available. Inspection of the bonds listed on the CRSP tape for the sample period shows that even this criterion cannot always be used because of a dearth of long-maturity bonds in the 1970s. The average yield implied by the discount function, while easier to obtain once the discount function is available, will in general not equal the Brennan and Schwartz [11] estimate. The difference is not believed to be important, because we use the consol rate only to form constant risk portfolios, not to predict returns.

\(^{35}\) The existence of these pricing inaccuracies is well-known to bond market researchers (ref. J. Bierwag in private conversation, 1985).
shows the minimum, maximum, and average numbers of bonds in each of the basic portfolios, with on average less than 7 (range from 0 to 13). Portfolio returns and risk measures are value-weighted averages of those of the constituent bonds. The value-weighted addition property for the risk measures (except time-to-maturity) was shown in Section 2. The same property for bond returns is predicated on calculating bond returns simply as \((P_t + C/12)/P_{t-1}\), where \(P_{t-1}\) and \(P_t\) are the bond’s (without interest) prices at beginning and end of month \(t\), and \(C\) is the bond’s coupon rate. This formula is not quite correct, as it does not allow for the semi-annual compounding of coupons, but the deviations are not large and tend to offset each other in the resulting basic portfolio returns.

The naive time-to-maturity risk measure for each of the basic portfolios has been taken as the upper limit of the maturity range by which it was formed.

### 3.2 Constant Risk Portfolios

Constant risk portfolios are formed each month from the current non-empty basic portfolios, using the risk targets and portfolio selection methods described below.

The risk levels in this paper have been chosen with the objective of spanning the range of risk parameter values observed in the basic portfolios. Inspection reveals that their risk measure values rise only occasionally beyond those of a nine-year pure discount bond,\(^{36}\) and at times the maximum observed risk is only that of an equivalent six-year pure discount bond. We have not attempted to stay at all times within this minimum range of risk measures, thereby admitting short positions in some of the constant risk portfolios.

Short positions are also often difficult to avoid with the two-dimensional Bren-
nan and Schwartz [11] and Fong and Vasicek [29] risk measures, even when stay­ing within the above minimum risk measure range. Chen [17] draws attention to the danger of forming non-representative portfolios, e.g. with large long and short positions: such portfolios may affect the observable presence of latent factors—a phenomenon also documented by Shanken [53]. The presence of large offsetting short and long positions may also be expected to amplify measurement errors in observed returns. For these reasons, we have attempted to minimize the absolute values of portfolio fractions, as further described in Appendix B.

Portfolio selection from the basic bond portfolios can be done in many ways, witness such descriptive terms from immunization practice as “bullet”, “barbell”, and “ladder” selection methods. Diversified portfolio selections, as in the ladder method, may be expected on the one hand to reduce measurement error in observed portfolio returns, and on the other hand to mask the presence of non-linear dependencies in the return data.\(^{37}\)

In contrast, the bullet method selects from the basic portfolios whose risk parameter values are as close as possible on either side of the risk target values: the resulting portfolios come closest to preserving the variation in the basic portfolio returns. Even bullet portfolios are somewhat diversified, however, because the basic portfolios are. The barbell method selects at any time from the basic portfolios with the highest and the lowest risk levels. It is not suited for the present analysis, as it would result in covariance matrix singularity and would not not use all observations.

We have chosen to use both the ladder and the bullet method, in order to study the effect of diversification on the dependency structure of constant risk portfolio returns. The selection methods have been designed so as to achieve stabilized

\(^{37}\) Portfolio selection should have no effect on the observability of linear dependencies, thereby creating the potential for Type II error in the tests of this paper: acceptance of the linear price/risk relationship (1.1), where it should be rejected.
variances from one period to the next, so that constant risk portfolio returns then resemble those of stocks. This accords with the purpose of this paper, which is after all to apply traditional asset pricing tests to the bond market via the use of artificial securities with stationary return distributions. The selection algorithms are further discussed in Appendix B.

The final question centres on the number of constant risk portfolios to be formed. This is governed by the expected dimension of the bond market state space and by the informational content of the basic portfolios. Preliminary work, not further reported, with constant duration portfolios with up to 20 different risk levels, showed (near)-singularity of the covariance matrix of returns even for undiversified bullet portfolios. The empirical work by Brennan and Schwartz [10,11], Nelson and Schaefer [48], and Dietrich-Campbell and Schwartz [24] suggests furthermore a high degree of explanatory power for only a two-dimensional bond pricing model, indicating that perhaps as few as three or four constant risk portfolios suffice to capture all information in the government default-free bond market.

In view of these considerations, we have chosen eight different constant risk levels in each risk case to reflect the variation in the government bond market. This paper’s choice of eight constant risk levels for each of the six risk measures is presented in Table II. The risk levels across these measures are not exactly comparable, but they each span the range of values for each risk measure, arranged in order of increasing riskiness in the time-to-maturity sense. For comparison, we give in Table III typical return and risk values for basic bonds at selected dates from the sample period.

38 Expected returns are not affected by selection variation if the underlying risk measure is true. This cannot be guaranteed, however, if the risk measure is only approximate.
4. Constant Risk Portfolio Return Distributions

Testing of the bilinear risk/return paradigm (1.1) and (1.2) relies on stationary multivariate normal distributions of asset returns. The data sets consist of constant risk portfolio returns for the six different risk measures, each represented by bullet and ladder portfolios, for a grand total of 12 data sets. Each data set consists of 120 monthly return observations on the constant risk portfolios selected in accordance with the eight chosen risk levels.

4.1 Time Series Analysis

Visual inspection of the data shows few outliers: unrealistic portfolio returns occur only in two periods of the B/S bullet portfolios, and in one period of the F/V bullet portfolios. We show in Appendix B that such outliers result from the error amplification associated with large offsetting long and short positions in these portfolios. The portfolio selection algorithm produces occasional outliers of this type, depending on the constant risk levels chosen (as in Section 3), and usually not in the same time periods. The above cases have simply been omitted from all further tests, as it proved difficult to avoid such occasional outliers even with variations in the selection method. The number of monthly return observations is, therefore, reduced to 118 and 119 for B/S and F/V bullet portfolios, respectively.

Box-Jenkins time-series analysis of all return sets shows (in Table IV) insignificant (at the 5% level) autocorrelation at all lags, although a hint of negative autocorrelation at a lag of six months, and of positive autocorrelation at a lag of 12 months is present in all panels of Table IV. We also find some positive autocorrelation, at lags of one and two months, in the returns of the lowest risk (#1) bullet portfolios for all risk measures, while this is not observed for the corresponding ladder portfolios. This finding parallels the positive serial correlation that is known to exist in returns on T-bills.
These autocorrelations—if at all present—are not very significant, because au-
tocorrelations from the differenced series (of order 1), not further reported, have
values of approximately -.5 at lag 1 (ranging from -.44 to -.55, with standard error
approximately .10), while cutting off thereafter. This result is typical of "white
noise with constant trend," i.e. the return series are consistent with random draw-
ings from stationary probability distributions. This finding supports the use of
deviations from means as "innovations," or as unexpected returns, of constant risk
portfolios.

The surprising result is that there is no appreciable difference in these interpre-
tations across the various risk measures and portfolio selection methods: the result
holds equally well for MAT bullet portfolios as for B/S ladder portfolios, or for any
of the other risk/selection portfolio combinations.

4.2 Distribution Results

Univariate statistics for all return series are reported in Table V, including
mean, standard deviation, skewness, kurtosis, and studentized range. We recall
that the eight constant risk portfolios in each risk case are ordered by increasing
risk. One of the objectives of this subsection is to scan the results for patterns that
conform to this risk ordering.

We note that mean ladder portfolio returns show a uniform monotonicity (a
declining one in the time period considered), whereas this is not present in the
bullet portfolio means. Since the risk levels in bullet and ladder portfolios are the
same for each risk measure, the result illustrates dramatically the effect of portfolio
selection, i.e. diversification, on mean returns. The strength of this effect is evident
when considering that even bullet portfolios consist of several traded bonds (two
basic portfolios, each with on average about seven constituent bonds, but ranging
from one to 13), which tends to reduce the effects of pricing errors. The effect may
indicate non-linear pricing relationships for bonds, such as mis-specification of the risk measures used, which tend to disappear when diversifying across the whole spectrum of traded bonds.

Standard deviation is not an appropriate risk measure for individual assets, or even small portfolios, when assets are not perfectly correlated (see Table VI-3 and 4 for typical correlations). Nevertheless, the high positive correlation between all default-free bonds should lead to considerable parallels between standard deviation and the true risk measure. Such parallelism is indeed observed for all risk measures, and for both bullet and ladder portfolios. In fact, the differences between corresponding bullet and ladder portfolio standard deviations are remarkably small (except for B/S portfolios). This similarity supports the presence of the non-linear pricing effects already alluded to, since otherwise diversification would have reduced standard deviations more in the ladder portfolios. The large reduction in standard deviations of B/S ladder portfolios, as compared with bullet portfolios, may indicate the absence of non-linear risk mis-specification effects, which then allows the regular diversification effect to be manifested.

Skewness statistics reveal that none of the return distributions show severe asymmetry. Most of the skewness statistics stay within the 5% (-.43,.43) confidence intervals from D'Agostino and Pearson [2], indicating no severe asymmetry of the return distributions. It appears, though, that most of the skewness statistics are negative (i.e. more left-tailed), more so for the bullet portfolios than for the ladders. Skewness of ladder portfolios appears more uniform across risk levels, with in general the skewness of the lowest risk (#1) portfolios being reversed (right-tailed) from that of the remaining portfolios. Again, the B/S portfolios buck this trend, with all of the ladder portfolios showing negative skewness.

Kurtosis statistics are of the order of magnitude of 6, where normal distribu-
tions have a theoretical value of 3. The constant risk portfolio return distributions are therefore a little more peaked and fat-tailed than the normal distribution. There is no clear pattern in kurtosis values across risk and diversification levels, although higher values occur often for higher risk levels.

Finally, we report in Table V studentized ranges of the order of magnitude of 6 or 7 for all data sets. Such values are marginally consistent with the normal distribution assumption: 99% confidence interval is (-6.5, 6.5), see e.g. Fama [25]. As with kurtosis, no clear pattern of studentized range values exist across all risk measures, although some (e.g. DUR) show an increasing pattern.

The above distributional results are not unlike those encountered for stock returns, see e.g. Fama [25]. By comparison, Gultekin and Rogalski [32] and Bildersee [5], using maturity-sorted portfolios of Treasury securities over a sample period (1947–1976) almost three times as long as the one used in the present paper, report mostly positive skewness (right-tailed distributions), while kurtosis statistics for maturities greater than one year are quite similar, as are studentized ranges. We conclude that our constant risk portfolio returns are sufficiently similar to other financial securities so as to allow meaningful interpretation of the asset pricing model tests in Sections 6, 7, and 8.

4.3 Stationarity Tests

Stationarity of the return distributions may be tested with the multivariate test of equality of means between two sample period halves. Splitting the sample period into two half periods of five years each, we report in Tables VI-1 and 2 the results of Hotelling $T^2$ tests on bullet and ladder portfolios, respectively. The hypothesis of equal means in the two half periods is accepted for all portfolios at the 1% level. The results for bullet and ladder portfolios are not directly comparable, however, because only two or three independent portfolios were used in the tests.
of ladder portfolios due to singularity of the covariance matrices involved (see also next subsection 4.4).

Inspection of the covariance and correlation matrices between the half periods shows that variances remain stationary, but that off-diagonal correlations vary quite widely. Tables VI–3 and 4 show a typical set of first and second period correlation matrices illustrating this point. No apparent pattern is discernible between risk measures: sometimes correlations of like portfolios increase, sometimes they decrease. A possible explanation for this phenomenon may be found in the term structure slope reversals that were prevalent in the second half of the 1970s sample period.

Consequently, we reach a weak conclusion as to the multivariate distributional stationarity of all return series: the means and variances are very similar between the two sample period halves, while the correlations between portfolios of different constant risk levels appear to have changed during the period.

4.4 Covariance Matrix and Eigen Value Analysis

Analysis of the covariance/correlation matrices for the bullet and ladder portfolios reveals that the state space for ladder portfolio returns is of low dimension: covariance matrix rank = 2 for the one-dimensional risk measures DUR, CIR, N/S, and MAT; and rank = 3 for the two-dimensional ones, B/S and F/V. Covariance matrices for the bullet portfolios, however, are of full rank = 8, although condition numbers are, on the whole, rather high. We report in Table VII rank, condition number, and eigen values for all correlation matrices. The number of large eigen

39 Morrison’s [47, p. 252] test of equality of half-period covariance matrices (not further reported) is soundly rejected; this test has low power, though, and is highly susceptible to departures from multivariate normality.

40 Observations from term structure estimates as described in Section 3. Downward sloping term structures were, e.g., observed in the period from mid-1978 to end-1979.

41 Similar statistics, after scaling, were found for the covariance matrices. Matrix
values for bullet portfolio correlation matrices matches in general the rank of the corresponding ladder portfolio correlation matrices.

Likely explanations for the singularity of the ladder covariance matrices, vis-à-vis the non-singularity for bullet portfolios, lie in diversification: ladder portfolios are less subject to the effects of bad data, asynchronicity, and thin trading; and diversified ladder portfolios deviate more predictably at the end of each month’s holding period from the assumed constant risk level. In other words, bullet portfolio returns may contain more noise, as also suggested by the uniformly small values of the remaining eigen values of bullet correlation matrices. In addition, as we will also argue below for the two-dimensional risk measures, possible non-linearities in the return generating process may be preserved in bullet portfolios, while being diversified away in ladder portfolios.

The trend towards a more pronounced dependency structure, when going from bullet to ladder portfolio selection, is not observed in the CIR portfolios: the first eigen value of the ladder return correlation matrix is less than the corresponding eigen value for bullet returns, while the ordering is just the opposite for the second eigen values. This reflects the lack of distinction between the CIR risk measure values for basic portfolios (and bonds) of higher maturities, see Table III for examples. The ladder portfolio selection algorithm in this case arrives at more arbitrary portfolio compositions, that mask the presence of common factors.

Although the two-dimensional risk measures B/S and F/V lead to an additional non-zero eigen value also in the diversified ladder portfolio returns, the effect is not overwhelming. If we recall that the eigen value decomposition is based on “percent variance explained”, the additional eigen value explains variance of the order of rank decisions are based on computational accuracy criteria within the BMDP statistical software used for these analyses. The eigen values and condition numbers reported indicate more fully the nature of the near-singularity.
magnitude 8–10%. This could be caused by non-linearities, but could also reflect the lingering noise from the larger short/long positions associated with portfolio selection on the basis of two risk measures. For all practical purposes, one would not be far wrong to interpret the results of Table VII-3 as showing that the state space for these risk measures is mostly one-dimensional.

4.5 Conclusions

Application of the A.P.T. and CAPM empirical methodology to the artificial constant risk security returns appears justified in view of the results of this Section. We have shown that the constant risk bond portfolio returns behave much like the stock returns they were intended to resemble. The difference between bullet and ladder portfolios appears to be mainly one of added noise in the bullet portfolio returns, although the univariate B/S portfolio statistics suggest that a non-linear effect may also be present. We are unable to distinguish between these effects with tests based on the general linear statistical model, however.

Except for the B/S portfolio standard deviations, no significant differences exist between distributional results across the different risk measures, at least not ones that can be argued to lead back to the risk measures per se. The comparative results for the simple MAT risk measure are interesting, especially in view of the fact that the parameters for the B/S, CIR, and N/S risk measures were fitted to the bond observations of the sample period.

Another interesting result is the low effective rank of the covariance matrices: one plus the dimension of the corresponding risk measures. Bullet portfolios have full rank, but the additional eigen values are quite small and may signify mainly noise. Again, the result is the same across all risk measures, except that CIR ladder portfolios show some effect from lack of distinctive risk measure values for bonds and basic portfolios of higher remaining time-to-maturity.
A cautionary note in interpreting the results of this Section across risk measures must be that they are not independent, but based on return series derived from the same traded CRSP listed bonds. It is, therefore, not possible to draw strong conclusions from the similar distributional and test results across these various return series.
5. Factor Model

Our first test of the linear pricing relationship (1.1)

\[ E[\tilde{r}] = \lambda_0 + \lambda'b \]

is based on the Arbitrage Pricing Theory (A.P.T.). The (unknown) factor structure of the bullet portfolio returns is analyzed by means of factor analysis and principal component analysis of the return covariance matrix.\(^{42}\)

5.1 A.P.T. Factor Model Specification

Following Ross [51] or Huberman [35], we posit a linear generating model for the returns of the \(i\)-th constant risk bond portfolio

\[ \tilde{r}_{it} = \mu_{it} + \delta_t\lambda_i + \tilde{\varepsilon}_{it}, \quad (i = 1, \ldots, N) \quad (t = 1, \ldots, T) \] (5.1)

where \(\mu_{it}\) is the expected return over the next holding period \(t\),

\(\tilde{\delta}_t\) is a \(k \times 1\) vector of unobserved factor realizations in period \(t\),

\(\lambda_i\) is a \(k \times 1\) vector of factor loadings for the \(i\)-th constant risk portfolio, and

\(\tilde{\varepsilon}_{it}\) is a residual not explained by the factor model.

Without loss of generality, the factors \(\tilde{\delta}_{tj}, j = 1, \ldots, k\), are assumed orthogonal, with mean zero and unit variance; they are uncorrelated with the residuals \(\tilde{\varepsilon}_{it}\). The factor loadings \(\{\lambda_i\}\) have then the natural interpretations as covariances between returns and factors familiar from e.g. the CAPM. We assume further that the above generating process is stationary over the sample period with return covariance matrix \(\Sigma\) and residual covariance matrix \(\Omega\). The \(N \times N\) covariance matrix \(\Sigma\) then has the following well-known factoring property

\[ \Sigma = B_k'\Lambda_k + \Omega, \] (5.2)

\(^{42}\) This is not possible for ladder portfolio covariance matrices, as these are singular.
where \( B_k \) is the \( k \times N \) matrix of factor loadings \( \{b_i\} \). This factoring is not unique: orthogonal rotations of \( B_k \) satisfy (5.2) equally well.

The principal assertion of A.P.T. for the sample period is

\[
\mu_{ti} \approx \lambda_{t0} + \lambda'_t b_i, \quad \left( i = 1, \ldots, N \right), \quad \left( t = 1, \ldots, T \right),
\]
equating the expected excess return\(^{43}\) on the \( i \)-th asset in each period with the inner product of its loading vector \( b_i \) and the "market price" of risk vector \( \lambda_t \).

This relationship is approximate in the sense that in an arbitrage-free economy the deviations in the cross-section may not lead to unbounded riskless returns; A.P.T. bounds the sum of the squared deviations in the cross-section as \( N \to \infty \). Tests of condition (5.3) gain power when the riskless return and the vector of market prices of risk are assumed constant over the sample period.\(^{44}\)

The strict factor model described by Ross [51], Huberman [35], and also Connor [19], is based on a diagonal residual covariance matrix \( \Omega \). If furthermore asset returns are assumed distributed as multivariate normal, then the above factor model is specified in accordance with the classic factor model, for which the factoring (5.2) may be carried out by means of Maximum Likelihood Factor Analysis (MLFA). The Jöreskog [40] MLFA procedure has been used by Roll and Ross [50], Chen [17], and Brown and Weinstein [14].

Chamberlain and Rothschild [16] show that the A.P.T. prediction (5.3) also holds for an "approximate" factor structure, requiring only that the \( N \times N \) residual covariance matrix \( \Omega \) be restricted to positive semi-definite with bounded eigenvalues, as the number of securities \( N \to \infty \). They show then, furthermore, the asymptotic equivalence of all factoring methods of the asset return covariance ma-

\(^{43}\) We assume that \( \lambda_0 \) equals the riskless return.

\(^{44}\) This assumption has been standard in the finance empirical literature, e.g. Gibbons [31].
trix $\Sigma$, as in (5.2), that are based on solving a generalized eigen value problem.\(^{45}\) This includes the classic MLFA method, and the spectral decomposition based on Principal Components Analysis (PCA). The authors recommend that PCA be used to extract factors, as it does not require the multivariate normality assumptions of MLFA, and is easier to calculate.\(^{46,47}\)

5.2 Effect of Portfolio Risk Perturbations

Because the risk measures studied in this paper differ in important aspects, as shown in Section 2, most or all of the constant risk portfolios analyzed in this paper must deviate from being (artificial) securities in the sense of the A.P.T. model. To this must be added the effect of the one-month holding periods, during which portfolio risk levels may change (or decay) at different rates. The resulting returns and covariances will then be perturbed. It is not possible to describe the nature of these perturbations without assuming a lot more structure in these risk deviations. A possible way of highlighting this problem is by re-writing the return generating

\[^{45}\text{They formulate the following eigen value problem}
\]

$$\Sigma J = \Omega J \Phi,$$

where $J = \Omega^{-1}B_k$ is the $N \times N$ matrix with the eigen vectors as columns, and $\Phi = I + B_k^T\Omega^{-1}B_k$ is an $N \times N$ diagonal matrix with eigen values on the diagonal. The diagonal restriction of $\Phi$ allows a unique solution to the eigen value problem. The $k \times N$ factor loadings matrix $B_k$ has rows $\{\Phi^{1/2}\Omega J_j\}$, $j = 1, \ldots, k$. With $\Omega = I$, we have the usual eigen value or principal component analysis (PCA), and with $\Omega = D$ the classic MLFA problem. Chamberlain and Rothschild show that $B_k^T B_k$ converges on $\Sigma$ as $N \to \infty$, no matter what $\Omega$ specification.

\[^{46}\text{Chamberlain and Rothschild [16] also show that the first } k \text{ eigen values of } \Sigma \text{ must be unbounded, and the remaining eigen values bounded, for a } k\text{-factor model to hold. Linn and Chang [41] use this prediction to test the possible dimensionality of a factor model for common stock returns, using increasing sets of securities. This type of test is not well possible in the present paper, as the number of independent securities in the government bond market at any time is small (see Table I).}
\]

\[^{47}\text{Franke [30] claims that the MLFA and PCA factor extraction methods imply restrictions on the residual covariance matrix } \Omega \text{ that are at variance with the classic A.P.T. model specification. However, Franke’s result is derived from the A.P.T. prediction (5.3) holding exactly, and does not recognize that only the deviations from it must be bounded.} \]
process (5.1)

\[ \tilde{r}_{ti} = \mu_i + \bar{u}_{ti} + (b_i + \tilde{b}_i)'\tilde{\delta}_t + \tilde{\epsilon}_{ti}, \]

(5.4)

where both the mean and the factor loadings are perturbed by random amounts. We may write this in turn as

\[ \tilde{r}_{ti} = \mu_i + b_i'\tilde{\delta}_t + \tilde{v}_{ti}, \]

(5.5)

where now \( \tilde{v}_{ti} = \tilde{\epsilon}_{ti} + \bar{u}_{ti} + b_i'\tilde{\delta}_t. \)

If we assume that \( \tilde{b}_i \) and \( \tilde{\delta}_t \) are uncorrelated, then \( E[\tilde{v}_{ti}] = 0. \) But these disturbances \( \tilde{v}_{ti} \) are now no longer uncorrelated with each other and with the factors \( \tilde{\delta}_t, \) so that the standard assumptions of the factor model are no longer satisfied. It is clear that, in general, the disturbances \( \tilde{v}_{ti} \) will have a full covariance matrix, even when the original disturbances \( \epsilon \) have only a diagonal covariance matrix.

The perturbed case is similar to the generalized factor model described by Ingersoll [37], and also resembles the approximate factor structure described by Chamberlain and Rothschild [16]. Ingersoll [37] proves (as Theorem 6) that the A.P.T. prediction still holds under these conditions, as long as a distance function of the data is bounded as the number of securities grows without bound. Unfortunately, the limited number of independent securities in the government bond market precludes a meaningful test of this condition.

We assume in the remainder of this paper that the A.P.T. prediction (5.3) also holds in cases perturbed by inaccurate risk measures and discrete holding periods. The assumption of diagonal covariance matrices is not likely to hold in this case, however, so that PCA may be the more appropriate factor extraction method here. Furthermore, misspecification of the linear generating model may bias the factor extraction, such as by introducing additional factors.
5.3 Factor Extraction and Rotation

Due to the singularity and low rank of ladder covariance matrices, factor extraction using MLFA and PCA is limited to the full rank bullet covariance matrices. Of course, if the ladder portfolios are merely bullet portfolios without extraneous noise, the two or three independent portfolios in each case may themselves be interpreted as factors, though not orthogonal ones as in MLFA and PCA.\footnote{48 It is appropriate to associate factors with returns on portfolios, see Admati and Pfleiderer [1].}

After the factor loading matrix has been found, either using MLFA or PCA, we follow prior practice in empirical finance by estimating the factor scores from cross-sectional GLS regressions in each time period. We write (5.1) as

$$\hat{\gamma}_t - \mu = \hat{\gamma}_t^* = \delta'_t B_k + \varepsilon_t, \quad t = 1, \ldots, T,$$

where now the $1 \times N$ mean vector $\mu$ has been assumed constant, and $r_t^*$ is the $1 \times N$ vector of return deviations from the mean. The $N \times N$ covariance matrix $E[\varepsilon_t \varepsilon_t']$ is also assumed to be constant and equal to the diagonal part $D$ of the residual covariance matrix $\Omega$ from the factorization of $\Sigma$.\footnote{49 The diagonality assumption fits the MLFA model specification, but not the PCA model. In the latter, the residual covariance matrix derives from the $N - k$ unused eigen vectors $P_{N-k}$ and eigen values $\Lambda_{N-k}$ as $\Omega = P_{N-k} \Lambda_{N-k} P_{N-k}'$, which is not a diagonal matrix. Our procedure is asymptotically correct, however, by Chamberlain and Rothschild [16].} The resulting regression procedure is also known as weighted least squares (WLS). The (transposed) WLS estimator

$$\hat{\delta}_t' = r_t^* D^{-1} B_k' (B_k D^{-1} B_k')^{-1},$$

is maximum likelihood, given the factorization (5.2). The joint estimation of factor loadings, residual covariance matrix, and factor scores on the basis of MLFA factorization is then also maximum likelihood, which bypasses the errors-in-variables
problem, as Gibbons [31] has also argued in the context of a test of CAPM. This is the case only asymptotically for the PCA factorization.

A further issue concerns rotation of the factor structures found. It is well known that the factor model is indeterminate when the factors are not known. Any orthogonal rotation of the factor scores and loadings is then equally capable of satisfying the matrix factorization (5.2). Factor analysts have long used rotation as a means of obtaining factors that have a greater intuitive explanatory power.

While orthogonal rotation is itself an arbitrary procedure, it should be realized that not rotating may lead to equally arbitrary conclusions. It is well known, for example, that MLFA and Principal Component algorithms, when performed on the type of highly correlated securities common in the government bond market, lead to a first factor or principal component representing an average of the variables, and explaining a dominant percentage of the covariance matrix. It would be erroneous to conclude that then "obviously" an all-pervasive market factor was present.\(^{50}\)

We perform in the present paper varimax rotation on all MLFA and PCA\(^{51}\) factors, for the purpose of enabling better comparison of factors derived from these two extraction methods. Varimax rotation is known to maximize the differences between factor loadings, which may aid in interpreting the otherwise unknown factors. While the matter of rotation will remain arbitrary, we shall let the results and their interpretations in Section 6 speak for themselves.

5.4 A.P.T. Factor Prices

The estimation of factor prices \(\lambda\) is also affected by the rotational indeterminacy of factor analysis. This is evident from the substitution of the A.P.T. prediction

\(^{50}\) This point has also been made by Brown [13] in a test of eigen value boundedness, ref. Chamberlain and Rothschild [16].

\(^{51}\) Rotated principal components then do no longer possess the property of orthogonal variance partition.
(5.3) as an equality into the linear return generating model (5.1)

$$\tilde{r}_t = \lambda_0 t + (\lambda_t + \delta_t)'B_k + \tilde{\epsilon}_t,$$

$$= a_t' B + \tilde{\epsilon}_t, \quad t = 1, \ldots, T,$$

(5.8)

where $a_t' = (\lambda_0 t, (\lambda_t + \delta_t)')$ is a $(k + 1)$ row vector and $B$ is a $(k + 1) \times N$ matrix derived from $B_k$ by adding a row of ones as the first row. Using the same procedure as in the previous sub-section, estimates of $a_t$ may be obtained in each period by cross-sectional WLS regression:

$$\hat{a}_t' = r_t D^{-1} B'(BD^{-1}B')^{-1}.$$  \hspace{1cm} (5.9)

This estimator is maximum likelihood, given $B$ and $D$. The values of the last $k$ elements of $\hat{a}_t$ depend on the particular rotation used in deriving $B_k$ from MLFA or PCA, while the value of the first element equals the riskless rate and is always determinate.\textsuperscript{52} We deal in Section 6 with the unknown factor scores $\delta_t$.

\textsuperscript{52} The E.I.V. problem prevents us from deriving estimates of the factor prices by subtracting $\hat{\delta}_t$ in (5.7) from $\hat{a}_t$ in (5.9).
6. Factor Analysis of Bullet Portfolios

6.1 Results

We report in Table VIII the results of MLFA and PCA on the return covariance matrices for the six risk measures. The factor loadings are shown in scaled form, divided by the respective return standard deviations. BMDP-4M software has been used, which in the case of MLFA is based on Jöreskog [40]'s algorithm. The factor loadings have been rotated with the normalized varimax method. Factors have been ordered by similarity of loadings, showing also the partition of the total (scaled) variance associated with each MLFA and PCA factor.

The MLFA factor loadings in Table VIII are based on two or three pre-specified factors, i.e. one more than the dimension in each case of the risk measure by which the constant risk portfolios have been formed. This choice has been motivated by a desire to test whether the bullet portfolio selection method adds only noise to the returns when compared with ladder portfolios, or whether an additional factor (non-linearity, e.g.) is masked by the diversified ladder selection method.

The goodness-of-fit of the MLFA factor models is already evident from the high variance explained. The reported cases in Table VIII explain total variance ranging from 85% (for the B/S portfolios) to 94.5% (for the CIR portfolios), with the lowest values for the two-dimensional B/S and F/V portfolios indicating the effects of the bullet selection algorithm (see also Appendix B).

We report in Table IX-1 the results of classic L.R. \( \chi^2 \)-tests of the factor model with one to four factors in each risk case. None of these cases is acceptable even at the 5% level, but it is known that this test is not powerful in the presence of

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53 This is equivalent to factoring of the correlation matrix in the case of MLFA, but not (necessarily) for PCA. The scaled factor loadings, now with interpretations as correlations between portfolios and factors, show better the relative importance of each factor in explaining the variation in a portfolio's return.
non-normality or non-linearity, ref. Joreskog [40]. These results are much worse than those reported by Brown and Weinstein [14] and Chen [17] for MLFA on groups of stocks, notwithstanding the high degree of variance explained. These bad \( \chi^2 \) statistics are, ironically, caused by the high degree of fit of the model itself, resulting in the high condition numbers of the covariance matrices already reported in Table VII. This is bound to lead to numerical accuracy problems in the statistic

\[
\chi^2 = \text{constant} \times \log \frac{|\hat{D} + \hat{B}_k^t \hat{B}_k|}{|\Sigma|},
\]

especially in the calculation of the determinants. Consequently, the “variance explained” measure is more useful in comparing the respective factor models.\(^{54}\)

We report in Table IX-2 the marginal improvement in total variance explained by each subsequent factor in MFLA. It is apparent that a first factor explains the lion’s share of the variance in all but the B/S risk cases. Only the B/S returns show clear signs of being two-dimensional, with addition of the second factor improving total variance explained by as much as 26%, as compared with only about 7% for the F/V portfolios. F/V portfolios differ from DUR portfolios by having the added DURVAR risk measure: it does not appear to add much explanatory power and may be worse on account of the added difficulty of forming constant risk bullet portfolios with two risk targets, see Appendix B.

An additional factor beyond each risk measure’s dimension explains a maximum of about 9% (for CIR portfolios); while this may be due to omitted non-linear economic factors, the evidence is not overwhelming. We return to this matter in the next sub-section.

\(^{54}\) This can nevertheless only be an approximate measure of goodness-of-fit, as the MLFA algorithm seeks mainly to explain off-diagonal elements of the covariance matrix, rather than the diagonal as in PCA. Comparisons between MLFA and PCA in Table VIII, however, show that MLFA performs about as well as PCA in this respect.
The alternative factor loadings derived from PCA for the six risk measures are based on the same number of factors as in the MLFA, and have been similarly rotated. The marginal improvement in total variance explained by each subsequent PCA factor is evident from the eigen value analysis in Table VII-2, showing that eigen values are declining in value rather rapidly after the first two or three.

The similarity between the MLFA and PCA factor loadings is obvious, which is interesting in view of Chamberlain and Rothschild [16]'s result of the asymptotic equivalence of the two methods. PCA factor loadings are of the same order of magnitude and preserve the ranking of the MLFA factor loadings in Table VIII. This similarity has been obtained after varimax rotation, as the unrotated factor loadings appeared very different. Although we do not report residual covariance matrices for the MLFA and PCA cases, these differed for the off-diagonal elements more than for the variances, but this is as expected on the basis of the MLFA and PCA algorithms.

6.2 Factor Description

Varimax rotation attempts to maximize the differences between factor loadings: the observed patterns in the rotated factor loadings, with emphasis on different groups of portfolios, is then not unexpected. It is interesting, though, that these patterns follow monotonically the original ordering of the portfolios by risk/maturity level in all cases. The first factor has loadings with an increasing pattern, the second factor a decreasing one, and—for the two-dimensional risk cases—the third factor a pattern with emphasis on middle risk portfolios. Such patterns suggest interpretations of the factors as bond portfolios. The patterns are, moreover, not compatible merely with noise in bond prices (such as from thin or asynchronous trading, or from bid/ask price differentials): that would require bias depending on the risk level of the bonds.
Whatever its identification, factor #1 loads in all cases more heavily on higher risk portfolios (even relatively speaking, because of the scaling involved). It may be related to a long-maturity bond or portfolio. Likewise, factor #2 loads more heavily on low risk portfolios, and may be related to a short-maturity bond or portfolio. The third factor in B/S and F/V portfolios, with its highest loadings for middle risk levels, is then—perhaps—related to a middle maturity bond or portfolio.

These interpretations of the factors as long, middle, and short maturity bonds or portfolios are speculative, of course. It is also possible to relate these factors to changes in the shape of the term structure, following the analysis in Section 2. In this interpretation, the “long” factor is associated with shape-preserving changes in the level of the term structure; the “short” factor then resembles changes in the slope of the term structure; while the “middle” factor has the traits of change in the curvature of the term structure. We demonstrate in Appendix C by numerical example how level, slope, and curvature changes of the term structure have exactly the above described effect on bond returns.

Risk changes on account of level perturbations in the term structure are, by Section 2, exactly described by the DUR risk measure. The N/S risk measure is similarly designed to capture the “long” or “level” risk, as is—in a naive sense—the MAT risk measure. It comes as no surprise then that all of these risk measures show similarities in explanatory power for the first or “long” factor. Note that the DURVAR component of the F/V risk measure, while not a restatement of DUR, nevertheless covers the same type of term structure perturbation as DUR. We are, therefore, not surprised to see the variance explained by additional factors beyond the first one (in Table IX-2) rather small. These results confirm that “long” risk is dominant in default-free bond markets.

This conclusion is also supported by the results for the remaining risk measures.
The dominant BSL (consol rate) component of the B/S risk measure is related to changes in average spot yields, or to the level of the term structure. The CIR risk measure, with its emphasis on modelling term structure dynamics in the near term, shows the highest variance explained by an additional (second) factor: it does apparently not discriminate sufficiently between bonds of higher maturity, resulting in increased noise in the bullet portfolios selected with this risk measure.

6.3 Comparison with Ladder Portfolios

We may derive further insight about the derived factors from a comparison with the corresponding independent (two or three) ladder portfolios. These independent portfolios are those ladder portfolios that are the most different from the remaining ones on the basis of a multiple correlation criterion. We are not surprised that such portfolios consist typically of ladder portfolios #1 (short), #8 (long), and—in the case of the three dimensional risk measures—#3 or 4 (middle).

We carry out the following tests:

(a) canonical correlations analysis (CCA) between the independent ladder portfolio returns and the MLFA factor scores of the bullet portfolios; and

(b) regressing the bullet portfolio returns on the independent ladder portfolios.

Canonical correlations analysis is the multivariate equivalent of regressing each independent ladder portfolio on the bullet factors and vice versa. The methodology of CCA is to form orthogonal canonical variates in each group that partition the correlation between the groups in a manner similar to the variance partition in PCA. The method is especially useful to test for correlations where the two sets of variates are not both orthogonal (the independent ladder portfolios are not). If the independent ladder portfolios and the bullet portfolio factors largely describe the same factor space, we would expect large correlations between the corresponding canonical variates.
The CCA results are reported in Table X, using the factor scores obtained from (5.7) for the two and three factor models of Table VIII. As expected, the canonical correlations are very high indeed, although no conclusion may be drawn about the explanatory power of each of the canonical variates. This result confirms the interpretation of the factor scores as returns on certain portfolios, an interpretation familiar from CAPM and also treated in Roll and Ross [50] and Admati and Pfleiderer [1]. It is interesting that the third canonical variate in the F/V case appears to be just noise.

The factor interpretation of sub-section 6.2 in terms of short, long, and perhaps middle portfolios may also be tested by regressing bullet portfolio returns on ladder portfolio returns. Table XI shows for each risk measure the results of the OLS regressions of \( N = 8 \) bullet portfolio returns \( \tilde{r}_{bi} \), \( T \times 1 \) vectors, on the associated independent ladder portfolio returns \( \tilde{R}_t \), a \( T \times 2 \) or 3 matrix:

\[
\tilde{r}_{bi} = \alpha_i + \tilde{R}_t \beta_i + \epsilon_i, \quad i = 1, \ldots, N.
\] (6.1)

The explanatory power of these regressions is again high, judging from \( R^2 \)-values. We do not report the intercepts, as these have no ready economic interpretation as a riskless return. While the independent ladder portfolios are not orthogonal, the regression coefficients show similar patterns as those for the rotated factor loadings in Table VIII and the interpretation discussion in Appendix C: tell-tale increasing and decreasing patterns when viewed against the constant risk portfolio risk levels. The values of \( t \)-statistics generally decrease for the coefficients of the low-risk ladder portfolios, increase for those of the high-risk ladder portfolios, and have highest values for middle risk levels with middle risk ladder portfolios. This finding also supports the interpretation, in subsection 6.2, of the factors in terms of bond portfolios.
6.4 Conclusions

The results of MLFA and PCA in this section confirm the low dimension of the state space in the default-free bond market. Return data sets for all six risk measures show a high degree of fit with only two or three specified factors. The dominant factor, after varimax rotation, is in all cases a "long" factor, which may be associated with changes in the general level of the term structure. Only for the two-dimensional B/S risk measure does a second ("short") factor show high explanatory power. We have also shown, that the factors have high canonical correlations with long, short, and middle ladder portfolios. The interpretation of bond market factors in terms of portfolios is reinforced by directly regressing bullet portfolio returns on independent ladder portfolio returns: goodness-of-fit appears as high in these cases as in the MLFA, PCA, and CCA analyses.

While up to three factors have been identified in bullet portfolio returns, it remains to be seen whether they are all priced. As it is possible to diversify by means of ladder portfolios, we would expect that these additional factors are not significantly priced. We return to this matter in Section 7.
7. A.P.T. Cross-sectional Tests

Cross-sectional tests of A.P.T. concern the law-of-one-price for each common source of risk, or—in the context of the A.P.T. prediction (5.3)—tests of the significance and uniqueness of \( \lambda_0t \) and \( \lambda_t \) in each cross-sectional \( 1 \times N \) regression model

\[
\mu_t = \lambda_0t + \lambda_t'B_k + \eta_t, \quad t = 1, \ldots, T. \tag{7.1}
\]

This regression cannot be performed directly, because the expected return vectors \( \mu_t \) are unknown.

Two types of tests have been used in the empirical A.P.T. literature\(^{55} \) based on the cross-sectional regression model (5.8)

\[
\tilde{r}_t = \lambda_0t + (\lambda_t + \tilde{d}_t)'B_k + \tilde{\eta}_t, \quad t = 1, \ldots, T,
\]

and its accompanying WLS estimator (5.9). The problem here is the presence of the unknown factor scores \( \tilde{d}_t \). Roll and Ross [50] and Brown and Weinstein [14] deal with this problem in different ways.

7.1 Tests on Means

Roll and Ross [50] test factor price significance by collapsing the time-series into a single \( 1 \times N \) cross-sectional regression, similar to Black, Jensen, and Scholes [6] in tests of CAPM, which in this case gets rid of the factor scores because they are mean-zero by construction, or

\[
\mu = \lambda_0 + \lambda'B_k + \epsilon, \tag{7.2}
\]

with estimator

\[
(\hat{\lambda}_0, \hat{\lambda})' = \mu D^{-1}B_k' (B_kD^{-1}B_k')^{-1}. \tag{7.3}
\]

\(^{55} \) A third test was developed by Shanken [53] and Shanken and Weinstein [54] for identified factor models. They test for equality of mean risk prices in different cross-sectional sub-samples by means of Hotelling's \( T^2 \)-test.
Values of $B_k$ and $D$ are as derived from MLFA, and $\epsilon$ is the vector of mean residuals not explained by the factor model. Significance of risk prices is then tested with the usual $F$-tests (or $t$-tests) of restrictions on the factor prices and riskless return.

A draw-back with this method is that only one of the risk prices $\lambda$ needs to be non-zero, as shown e.g. by Chen [17]. Non-significance of some of the $\lambda$ may thus be a result from the specific rotation involved. We conjecture, though, that the varimax rotation used in the MLFA and PCA dispels this danger, especially in view of the factor interpretations in Section 6. Rotation does not affect the estimate of the riskless return $\lambda_0$, which is always determinate.

Results of the tests are given in Table XII. The intercept riskless return is in all cases significantly different from zero, as expected. All values are reasonably alike and there is no obviously wrong value amongst them. Evidence on the remaining risk prices is less clear. The price $\lambda_1$ of risk for the "long maturity" risk factor (#1 in the ordering of the factors in Table VIII) has the highest $F$-statistic in all risk cases, but is significantly different from zero at the 5% level only for the one-dimensional DUR, CIR, and N/S risk measures, with highest $F$-statistics for DUR and N/S and lower for CIR.

The two-dimensional risk measures B/S and F/V, as well as the MAT risk measure have no risk prices significantly different from zero at the 5% level. This result is probably due to a lack of power of the test, based as it is on only eight variates in the cross-section. The difference in significance for $\lambda_1$ in the DUR and F/V cases is puzzling, as the latter includes the DUR risk measure as well. A likely explanation is again the increased noise in bullet returns resulting from difficulties in portfolio selection with two or more risk targets. The low $F$-statistics for the B/S case are then likewise related to the increased noise in the corresponding bullet portfolio returns.
The low $F$-statistics for the risk prices in the case of MAT, the simple time-to-maturity risk measure, is the first obvious difference we find in behaviour of these constant risk portfolios when compared with those associated with the other one-dimensional risk measures DUR, CIR, and N/S.

7.2 Bilinearity Tests

Brown and Weinstein do not collapse the time-series, but test instead the law-of-one-price directly in each cross-section by splitting the sample in two parts. The presence of the factors is taken care of by realizing that their scores $\tilde{\delta}_t$ must also be the same in each cross-sectional sub-sample. Their methodology then is a two-stage one: (i) first test the equality of the scores $\tilde{\delta}_t$ in the factor model, then—if confirmed—follow it by (ii) tests of the equality and significance of the combined factor scores and prices.

The first stage uses the cross-sectional regression model (5.6)

$$\bar{\tilde{r}}_t^* = \delta'_i B_k + \bar{\epsilon}_t, \quad t = 1, \ldots, T,$$

and its WLS estimator (5.7)

$$\hat{\delta}_t = \bar{\tilde{r}}_t' D^{-1} B_k' (B_k D^{-1} B_k')^{-1}.$$

If the cross-section is considered as two sub-samples, the above represents a restricted regression, i.e. the factor score estimates are restricted to be the same in the entire cross-section. The corresponding unrestricted model consists of performing the above regression on each of the two sample halves, represented in the present paper by the even and odd numbered portfolios. The appropriate test of equality is then a multivariate Chow $F$-test, for which the $F$ statistic can be stated in terms of (weighted) sums of squares

$$\hat{F} = \frac{SSR - SSU}{SSU} \frac{\text{d.f.}}{q}. \quad (7.4)$$
SSR and SSU are the error sums of squares, weighted by the covariance matrix $D$, of the restricted and unrestricted models, respectively; and d.f. and q the degrees of freedom and the number of restricted parameters between the models, respectively.

The second stage uses a similar procedure on regression model (5.8)

$$\tilde{r}_t = a_t' B + \tilde{e}, \quad t = 1, \ldots, T,$$

and its WLS estimator (5.9)

$$\hat{a}_t' = \tilde{r}_t D^{-1} B' (B D^{-1} B')^{-1}.$$

This model is always considered the restricted model, whereas unrestricted models are represented by allowing one or more of the risk prices to differ in the two sample halves. Again the appropriate test is the Chow $F$-test.

Our Chow $F$-tests differ from those of Brown and Weinstein [14], in that the restrictions are true assumed: $B_k$ and $D$ are then the estimates derived from the MLFA on the whole sample, as in Table VIII.\footnote{Brown and Weinstein [14] assume—implicitly—that the restrictions are not true, because they use separate estimates of the partitions of $B_k$ and $D$ derived from MLFA on each sub-sample. The resulting tests of cross-sectional restrictions then may suffer from rotational indeterminacy, in addition to not being tests solely of the cross-section.}

Table XIII shows our results for the first stage, the bilinearity test of the factor scores. The tests have been done both for two and three factors in all cases. All $F$-statistics are less than 1.00, so that the hypothesis of one cross-section of factor scores is accepted, thereby providing the basis for second-stage testing of the equality of the combined factor scores and prices in the cross-section.

The second-stage test has been done only for the two-factor case, in order to avoid a degree of freedom problem when estimating a half cross-section reduced to three portfolios after throwing out a Heywood case in the original MLFA of Table
VIII. Table XIV shows the results of these A.P.T. tests. Again, most $F$-statistics are sufficiently small so as to accept the hypothesis of equal risk prices in the cross-sections of all six risk measures. The DUR risk measure stands out in that it appears to go against the bilinearity hypothesis; the $F$-statistics are not large, however, and the evidence is not strong considering the low power of these tests.

7.9 Conclusions

Our concern in this section was with the pricing of explanatory factors resulting from MLFA after varimax rotation. We have not separately considered the PCA-derived factors, because these appeared so similar to the MLFA ones.

If we accept the factor interpretations of Section 6, our results for the Roll/Ross type tests on means show that only one risk factor, #1 or "long", is priced, and then only in the DUR, CIR, and N/S cases. The difference between the DUR and DURVAR cases suggest, however, that this test is not well specified considering the small number of constant risk portfolios. A further difficulty may have been the high degree of fit of the factor models themselves: the resulting small residuals problem makes inversion of the covariance matrix $D$ a source of error in the estimators and the significance tests. The near alike estimates for the riskless rate in all cases confirms that all six risk measures are at least unbiased in their description of risk in the default-free bond market.

The Brown/Weinstein type bilinearity tests show that both factor scores and prices are the same for cross-sections consisting of the odd and even numbered portfolios. Only the DUR results are somewhat different, but not strongly so since the above comment about small residuals also applies to this test.
8. CAPM-style Tests of Asset Pricing

CAPM-style estimation of the asset pricing relationship (1.1) for our constant risk portfolios proceeds from knowledge of the appropriate risk measure, i.e. an identified model, as opposed to the A.P.T. where the factors remain unknown.

We wish to estimate and compare the asset pricing relationship (1.1), with the usual stationarity assumptions,

$$\text{E}[\tilde{r}] = \lambda_0 + b'\lambda,$$

(8.1)

for each of the six risk measures used in forming the constant risk portfolios. As before, $b$ represents either the scalar risk measures DUR, CIR, N/S, and MAT; or the $2 \times 1$ risk measures B/S and F/V. The parameters $\lambda_0$ and $\lambda$ represent the riskless rate (if it exists) and the market price(s) of risk for $b$. Equation (8.1), fitted in each risk case to the return data set, is an ex-post security market hyperplane.

We expect relationship (8.1) to hold if $b$ is the correct risk measure, by the theory of Section 2. But because we do not know how well each of these risk measures approaches the "true" risk measure, we can only resort to an empirical comparison of the model fit in each case.

An important aspect of the CAPM testing literature is the handling of the errors-in-variables (EIV) problem in the "known" values of the risk measure $b$. This problem is also relevant in the present paper, despite the fact that portfolio risk measures are constant by construction. Two sources of errors may be identified in our constant risk portfolio return series:

(1) errors in the risk measure calculations, whether based on the estimated term structures of Section 3, or originating in published parameters from source papers;

(2) pricing errors originating in thin trading and asynchronous observation times
and dates; and

(3) errors from using discrete monthly holding periods, during which the portfolios will show decaying risk profiles.

Maximum likelihood estimation of the risk measures, risk prices, and covariances has become the preferred method of dealing with the EIV problem. Gibbons [31], Stambaugh [55], and Hess [33] have shown how this procedure may be applied to tests of the CAPM restrictions of a market model regression on market portfolio returns. Because we cannot utilize a similar market model for our constant risk bond portfolios, however, we rely instead on the earlier method by Black, Jensen, and Scholes [6], who use grouping and cross-sectional OLS estimation on return time series means.\[^{57}\]

The grouping procedure—while not as formal as in Black, Jensen, and Scholes [6] or in Fama and MacBeth [26]—is the one described earlier in Section 3: “basic” portfolios sorted by maturity interval from which the constant risk portfolios of this paper are formed using the selection algorithms of Appendix B. Ladder portfolios are more diversified than bullet portfolios, and thus less prone to the EIV problem.  

8.1 Regression Results

Table XV-1 shows the results of ordinary least squares regression of ladder mean returns on their respective constant risk target values from Table II: near perfect perfect fit for all risk measures. Highest $R^2 = 1.0000$ occurs for the two-

\[^{57}\] An alternative method is that by Fama and MacBeth [26], who perform OLS estimation of $\lambda_0$ and $\lambda_t$ in each cross-section, and then use means of the resulting time series to obtain estimates of $\lambda_0$ and $\lambda$. An extension is the use of the iterated estimated general least squares (EGLS) method: an improved estimate of the residuals covariance matrix is obtained from the residuals of (8.1) using the latest EGLS estimate of $\lambda_0$ and $\lambda$, which then in turn leads to an improved EGLS estimate. The method is started with an identity covariance matrix, say, and stopped upon satisfactory convergence. This method has not been used in this paper, however, because of the problem of small residuals, which makes inversion of the successive covariance matrices a source of numerical inaccuracy. Besides, the OLS estimator is unbiased.
dimensional risk cases B/S and F/V, while the lowest $R^2 = .9885$ occurs for the CIR portfolios. The $t$-statistics for all coefficient estimates show them to be highly significantly different from zero, with highest values for the B/S portfolios, and lowest for the CIR and F/V portfolios. All estimates of the intercepts (= average riskless return, possibly) are close, ranging from a low .566% for B/S to a high .581% for CIR per month.

We show in Fig. 1 a typical plot of these results for the DUR risk measure. The ex-post security market lines are clearly downward sloping when plotted against the one-dimensional risk measures DUR, CIR, N/S, and MAT. This is also the case for the two-dimensional risk measures, when considering the dominance of the "long" or second risk measures DURVAR and BSL (see Table II) over the "short" or first measures DUR and BSR, respectively. The downward slope confirms the experience of low yields in the face of unexpected inflation and increasing interest rate uncertainty during the 1970s sample period.

Table XV-2 shows the results for similar regressions involving bullet portfolios. The model fit is evidently much worse: the highest $R^2$ value equals .68 for the DUR portfolios, while the lowest value is an outlier of .16 for the F/V portfolios. It is interesting to see that the coefficient estimates (market prices of risk) in the two-dimensional B/S and F/V cases are insignificantly different from zero at the 5% level, especially so for F/V portfolios. This shows again the error amplification effect of portfolio formation with two simultaneous risk targets. Also in evidence is the misspecification of the CIR risk measure: its coefficient $t$ and $R^2$ values are lower, reflecting the lack of discriminatory power of this risk measure for the higher maturity range. The intercepts are of the same order of magnitude as with the ladder portfolios, although also here with greatly reduced $t$-values. Fig. 2 shows some of these results for the DUR risk measure.
We also report in Table XV-3 the result of $F$-tests of restricting intercept and coefficient values of bullet mean return regressions to those of their ladder counterparts. Most of these $F$-statistics are small, consistent with bullet mean returns being noisy versions of ladder returns. Exceptions are the F/V and CIR measures, which display signs of misspecification. The good performance of the B/S risk measure here shows that it is a consistent risk measure across different levels of diversification, and therefore well-specified as a bond risk measure.

8.2 Conclusions

Perhaps not surprisingly, the relationship between the mean returns of highly diversified ladder portfolios and their constant risk measures is strongly linear. One might suspect a circular argument here, since the risk measures, or their parameters, were themselves estimated from the same CRSP tape bonds that appear in the portfolios. Arguing against this interpretation, though, is less impressive results for the less diversified bullet portfolios. The high degree of fit in ladder portfolios is then a result of diversification.

Diversification covers both the effects of risk measure misspecification and portfolio selection algorithm. Especially, the CIR and F/V risk measures seem misspecified by the results of Table XV-2, while the low significance of the coefficients for the long risk measure components in the B/S and F/V cases show the difficulty of forming constant risk portfolios with two simultaneous targets. The highest internal consistency between bullet and ladder portfolios is then achieved by the B/S and N/S risk measures, with B/S providing the better fit.
9. Summary and Conclusions

We apply in this paper the empirical methodology of A.P.T. and CAPM to government bond returns: we test for the law-of-one-price for risk and for the linearity of the risk/return relationship. Tests use the returns of up to eight constant risk portfolios, i.e. artificial securities that are constructed over time from traded bonds to resemble the characteristics of stocks, with risk levels spanning the observed risk range in the bond market. We use a broad representation of risk measures in the portfolio formation, each representing a well-known academic bond pricing model: Macaulay [42] (DUR), Brennan and Schwartz [11] (B/S), Cox, Ingersoll, and Ross [21] (CIR), Nelson and Schaefer [48] (N/S), and Fong and Vasicek [29] (F/V); we use, in addition, a simple time-to-maturity criterion (MAT). We also study the effects of diversification through the use of "bullet" and "ladder" portfolio selection methods.

Our first concern, in Section 4, is the distribution of the constant risk portfolio returns. We show that, for all cases of risk measure and diversification level, the artificial security objective is well met. Return time-series follow random walks, have stabilized variances, and show return distributions that resemble those for stocks, i.e. reasonably symmetric and only slightly more peaked and fat-tailed than the normal distribution. Return series for the two-dimensional risk measures B/S and F/V exhibit higher variances than their one-dimensional counter-parts, especially for the less diversified bullet portfolios. This is caused by the difficulty in the portfolio selection algorithm of matching two risk targets simultaneously: large offsetting long and short positions then amplify the variation in the basic portfolio returns. This affects all subsequent test results for these risk measures.

Eigen value analysis of the covariance matrices shows that the dependency structure of the constant risk portfolio returns is of low dimension, with the number
of large eigen values matching the dimension of the risk measure used. Ladder return covariance matrices are singular (with rank two or three), while bullet return covariance matrices are not, but have high condition numbers.

The eigen value analysis also shows that the second eigen value in the two-dimensional F/V case is rather small: the addition of the "duration-squared" (DURVAR) risk measure to Macaulay’s duration (DUR) does not add much explanatory power, as both cover the volatility of the longer maturity end of the term structure. DURVAR also describes portfolio "convexity," for which our findings indicate that it is not a factor in bond portfolio returns.

Results for the one-dimensional CIR measure show just the opposite: here the second eigen value is surprisingly large, indicating that this short-rate based measure does not discriminate sufficiently between the higher maturity bonds.

Both factor analysis (MLFA) and principal component analysis (PCA) are used to extract factors from the full rank bullet return covariance matrices. These methods result—after varimax rotation—in very similar factor loadings, which is in part due to the presence of small residuals. The goodness-of-fit of the factor models is evident from the high percent variance explained; the classic L.R. goodness-of-fit test is not well-specified, however, due to the small residuals effect mentioned.

An analysis of the marginal variance explained by additional factors shows that the DUR, N/S, and MAT bullet portfolios are probably best explained by a one-factor model; this matches the dimension of these risk measures. The B/S bullet portfolios are likewise best explained by a two-factor model, again verifying the dimension of this risk measure. A second factor for F/V returns has little explanatory power, echoing a similar conclusion drawn from eigen value analysis and with a similar conclusion as to the explanatory power of the "convexity" property. The CIR case requires a second significant factor, i.e. one beyond the dimension of
this risk measure, again indicating that this measure is not well-specified.

The rotated factor loadings possess in all cases patterns that are suggestive of factor interpretations as long, short, and middle maturity bond portfolios, i.e. securities that show greatest sensitivity to changes in the level, slope, and curvature of the term structure, respectively. This view meshes with the finding that the diversified ladder portfolios have only two or three independent constant risk portfolios, typically the ones with highest, lowest, and middle risk levels. High canonical correlations between rotated bullet factor scores and independent ladder portfolio returns confirm this interpretation, as do the high $R^2$ coefficients of regressing bullet portfolio returns on those of the corresponding independent ladder portfolios.

The significance of mean factor prices in the A.P.T. framework has been tested with a method also used by Roll and Ross [50]. The time series are collapsed into a single cross-sectional regression, and $F$-tests of significance of the mean factor price estimates are then performed. The one-dimensional risk cases DUR, CIR, and N/S have one risk factor priced significantly different from zero: the “long” factor associated with level changes in the term structure. The remaining one-dimensional risk measure, MAT, has no risk factors priced significantly different from zero. This is also the case for the two-dimensional risk measures B/S and F/V, although the $F$-values suggest that this may just be a matter of lack of power in these tests with only eight variates in the mean cross-section.

We have also carried out Chow tests on the factors for two halves of the cross-section of eight variates, i.e. bilinearity tests following Brown and Weinstein [14]. We find that the same factors exist in the two cross-sections consisting of the odd and even numbered constant risk portfolios. A second (nested) Chow test is then performed on the combined factor scores and prices; we find, again, that the bilinearity constraint is acceptable at the 5% level.
None of these tests possesses great power, however, and the $F$-statistics in all cases suffer from numerical inaccuracy on account of the small residuals problem referred to above. At best, these tests do not disprove the low dimensionality conclusion based on eigen value analysis alone.

Finally, we perform CAPM-style tests by regressing (OLS) returns of the constant risk portfolios on the known risk measure values by which the portfolios were formed. The linear return/risk model (1.1) is properly specified if the corresponding risk measure is. We find almost perfect fit for all diversified ladder portfolios. Similar tests for bullet portfolios reveal again the effect of diversification (much lower $R^2$ and $t$ values for coefficient estimates), which confirm the above conclusions as to the misspecification of the F/V and CIR risk measures. Especially, the B/S and N/S risk measures show a high degree of consistency in explanatory power across levels of diversification.

The main conclusions of this paper are:

1. the risk/return relationship (1.1) for default-free bonds is of low dimension, with in most cases only a single "long" risk measure significantly priced;
2. a strong diversification effect exists in the risk/return relationship w.r.t. diversification across the maturity spectrum—bond portfolio managers should be encouraged to diversify accordingly—;
3. surprisingly little difference exists between the results for the Macaulay duration (DUR) risk measure, which may be determined from current bond price quotations, and those for the fitted B/S and N/S models, confirming that bond portfolio managers can safely use Macaulay duration (although no conclusion is drawn about use of the yield-based duration measure);
4. even the naive MAT risk measure seems to perform well in conjunction with basic portfolios, lending some support to the use of yield curves in this case;
(5) there is no advantage in adding the DURVAR risk measure to DUR, as in the F/V risk measure: no additional explanatory power is achieved, while portfolio variance actually increases on account of the more difficult portfolio selection process;

(6) as a consequence of (5), "convexity" is not a (priced) factor in bond portfolio returns;

(7) the fitted CIR risk measure, based on the short rate, appears misspecified and performs less well than the DUR, N/S, and B/S risk measures;

(8) the B/S risk measure is capable of extracting portfolios with the greatest (two-dimensional) variability from the basic portfolios and it exhibits the greatest degree of consistency in explanatory power across maturity diversification: this does not appear to result in superior hedging or immunization performance, however, on account of the error amplification associated with the portfolio selection process.
REFERENCES


APPENDIX A

Estimation of Risk Measures

The six risk measures are estimated in each monthly period for the currently traded bonds. These values are then used in value-weighted proportion to obtain risk measures for the basic portfolios, and from them—in turn—the constant risk portfolios are obtained. We showed in Section 2 that value-weighted addition is appropriate for the risk measures DUR, B/S, CIR, N/S, and F/V, because they are each derived from the instantaneous standard deviations of perfectly correlated stochastic Gauss-Wiener variables. The MAT measure does not fit this definition, but we assume that this naive risk measure also follows this rule.

Risk measure calculations proceed in several stages, whereby each eligible bond is at the first stage regarded as a small portfolio of pure unit discount bonds, whose risk measures are then each derived in accordance with the underlying model, as detailed further below. These unit discount bonds are next aggregated to the respective coupon bonds using the value-weighted addition property; the value weights (or present values of the bond disbursements) at this stage are in all risk cases derived from the estimated discount function for the period. Next, the coupon bonds are aggregated to basic portfolios, each of which contains equal quantities of those bonds maturing in a six-month interval; value-weighted average risk measures now use the observed bond values (from the CRSP tape) as weights, because these values are more accurate than those estimated from the discount function. The final aggregation takes place when forming the constant risk portfolios from the basic portfolios: the portfolio selection algorithms of Appendix B are again based on value-weighted averaging, where now the weights are the portfolio value fractions.

Duration DUR of a unit discount bond is simply its time-to-maturity. Likewise, duration variance (DURVAR) for a unit discount bond is then defined as the square
of time-to-maturity, or duration-squared. This simple relationship does not hold for coupon bonds or portfolios.

The B/S risk measures BSR = rP_r/P and BSL = lP_l/P for unit discount bonds are derived from a table of discount values generated from a numerical solution of the Brennan and Schwartz [11] partial differential equation

\[ P_{rr} r^2 \sigma_1^2 / 2 + P_{rl} r \rho l \sigma_1 \sigma_2 + P_{ll} l^2 \sigma_2^2 / 2 + P_r [a(l - r) - \lambda_1 r \sigma_1] + P_l [l \sigma_2^2 + l^2 - rl] - P_r - rP = 0. \]

The boundary condition for this p.d.e. is \( P(T) = 1 \), so that the resulting table is one of discount values with parameters \( r, l, \) and time-to-maturity.\(^{58}\) Parameter estimates (see also Section 2.2) used in the solution were \( a = .560, b = 0, \sigma_1 = .447, \sigma_2 = .142, \rho = .516, \) and the market price of short rate risk \( \lambda_1 = .260, \) all in annualized terms. The table contains discount factors for a grid of short and consol rates with 1% increments; it is entered with the current values of \( r \) and \( l, \) derived from T-bills and from the current term structure, respectively, as described in Section 3. Linear interpolation is used for \( r \) and \( l \) values between the grid values. The partial derivatives \( P_r \) and \( P_l \) are determined numerically with central differences, perturbing \( r \) and \( l \) by 1% both up and down.

The CIR risk measure \( \sqrt{r} P_r/P \) for a unit discount bond is derived from the analytic function given in the source paper [21], using the parameter estimates \( \beta = .692, \sigma^2 = .00608, \) and \( \pi = 0: \)

\[ \text{CIR} = \frac{2\sqrt{r}}{.692 + .7007 \coth^{-1}(0.0292m)}, \]

in annualized terms, where \( m \) is time-to-maturity in months. Multiplication by \( \sqrt{r} \) reflects the intertemporal form of this risk measure, with \( r \) derived from T-bills as described in Section 3.

\(^{58}\) I thank Bruce Dietrich-Campbell for making this table available to me.
The N/S measure is based on the term structure sensitivities w.r.t. the 13-year spot yield, as given in the following table reproduced from Table 4 of the source paper [48]:

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>$R_{13}(\tau)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.947</td>
</tr>
<tr>
<td>2</td>
<td>0.976</td>
</tr>
<tr>
<td>3</td>
<td>0.967</td>
</tr>
<tr>
<td>4</td>
<td>0.940</td>
</tr>
<tr>
<td>5</td>
<td>0.917</td>
</tr>
<tr>
<td>6</td>
<td>0.906</td>
</tr>
<tr>
<td>7</td>
<td>0.909</td>
</tr>
<tr>
<td>8</td>
<td>0.927</td>
</tr>
<tr>
<td>9</td>
<td>0.955</td>
</tr>
<tr>
<td>10</td>
<td>0.985</td>
</tr>
<tr>
<td>11</td>
<td>1.009</td>
</tr>
<tr>
<td>12</td>
<td>1.017</td>
</tr>
<tr>
<td>13</td>
<td>1.000</td>
</tr>
</tbody>
</table>

Linear interpolation is used for values within the range of the table (13 years), and higher maturities have been treated as 13 years. Unit discount bond risk measures are then

$$N/S = (\tau - t)R_{13}(\tau),$$

from (2.3) in the text of this paper, while omitting the $- -$ sign.

The final risk measure is the naive time-to-maturity risk measure MAT. Basic portfolios, sorted by six month maturity intervals, are assumed to be of constant risk here, with MAT risk equal to the upper bounds of their respective maturity ranges, expressed in years. The precision of this risk measure depends on the diversification within each basic portfolio, see also Table I for details on the number of bonds in each basic portfolio.
APPENDIX B

Portfolio Selection Details

Our objective is to form constant risk portfolios from the bonds listed on the CRSP tape, using monthly rebalancing or updating as required by new and maturing issues, or merely by the passage of time. Atomic securities to be used in the algorithms of this Appendix are the basic portfolios described in Section 3. The number of non-empty basic portfolios in each period varies considerably, from a low of 13 (occurring in 1970-1971) to a maximum of 20, see also Tables I, II, and III.

Constant risk portfolios have target values for the one or two elements of the relevant risk measure. Because the risk measures are additive in value-weighted proportion (see also Section 2), the portfolio value fractions stand in linear relation to the basic portfolio risk values and the target values. The selection problem is usually not determinate, as more basic portfolios are available than needed. The minimum number of basic portfolios needed to achieve a target risk value is the dimension of the risk measure plus one, i.e. two or three for the risk measures in this paper. The required algorithms must therefore also deal with a degrees-of-freedom problem.

It is not difficult to see that different selection algorithms may lead to larger or smaller variances for constant risk portfolios of the same risk\(^59\). In addition, the portfolio selection method may affect the deviations from constant risk at the end of an interval.\(^59\) This is well-known from portfolio theory. A simple example can be constructed as follows: three securities have durations of 1, 2, and 3; and variances of .25, .36, and .49, respectively. The securities are imperfectly correlated: \(\rho_{12} = .9, \rho_{13} = .6, \) and \(\rho_{23} = .8\). Portfolio risk target is a duration of 1.5.

Using only securities 1 and 2, portfolio variance is .288; with only securities 1 and 3, portfolio variance is .250; and use of only securities 2 and 3 leads to portfolio variance .429. The diversified ladder portfolio choice, to be explained later in this Appendix, yields portfolio variance of .273.

This is also a counter example to Gultekin and Rogalski [32]'s claim of constant variance in view of varying selection methods. The portfolio selection method does not affect expected return, as long as the risk measure used is the correct one.
of each monthly holding period, as different bonds will show risk decay at different rates.

It is not immediately clear, how this affects the time-series behaviour of constant risk returns: a bias in the absolute values of returns will not materially affect the tests in this paper, while a possible effect on the variance will make a difference in the power of the bilinearity tests and the factor model tests of goodness-of-fit. Compounding the issue is the well-known lack of accuracy of data on the government bond CRSP tape, with its mix of asynchronicity and thin-trading problems. As we are forced to use discrete monthly data, this issue is an empirical one.

Two popular methods in bond portfolio management, representing extreme cases of diversification and the lack of it, are the so-called "ladder" and "bullet" methods.

1. Ladder Portfolio Choice

This selection method uses all available bonds through the basic portfolios. We follow prior practice, e.g. Brennan and Schwartz [12], in determining portfolio fractions \( w \) as the solution to a quadratic programming problem

\[
\min \ w'w, \quad \text{s.t.} \quad A'w = a_T, \quad w'1 = 1.
\]

The vector \( w \) has dimension \( m \times 1 \), where \( m \) is the current number of non-empty basic portfolios. The \( k \times m \) matrix \( A \) contains the risk measures of the basic portfolios, and the \( k \times 1 \) vector \( a_T \) is the constant risk target. The determinate solution to this programming problem tends toward equal weights (short or long), and does not prohibit short positions. Addition of a non-negativity constraint to the above problem would make it much more cumbersome to solve, while not improving the empirical work of this paper: although short positions in government bond portfolio management may be rare, allowing them will result in smaller portfolio
variances without affecting the linear risk and return relationships to be tested.

2. Bullet Portfolio Choice

Here we attempt to form constant risk portfolios using the pair of basic portfolios whose risk measures straddle the target risk values in the closest possible manner. In the case of the two-dimensional risk measures B/S and F/V, the straddling basic portfolios must do so for both risk measure elements. Such straddling basic portfolios may not always exist, in which case two other basic portfolios are chosen as further described below.

Two basic portfolios are sufficient to form a target constant risk portfolio only if the risk measure used is one-dimensional. Positive portfolio fractions result only from straddling basic portfolios. In the two-dimensional risk case, we must add a third basic portfolio to choose from in order to make the linear portfolio fractions problem determinate. It is now no longer possible to avoid short positions, however; nor is it possible to avoid short positions at those times, where no straddling basic portfolios exist, regardless of risk measure dimension.

We minimize the possibility of very large short or long positions by employing in those cases a modified "barbell" strategy:

(1) in the case of a two-dimensional risk measure, we reserve the first basic portfolio (less than six months remaining time-to-maturity) as the third atomic security in addition to the two straddling bonds;

(2) if no straddling basic portfolios exist in the one-dimensional case, we choose the two basic portfolios with the highest and lowest risk values, consistent with trying to keep absolute values of portfolio fractions small;

(3) if no straddling basic portfolios exist in the two-dimensional case (not counting the first basic portfolio), we choose the basic portfolios with the lowest and highest remaining time-to-maturity, again trying to avoid unrealistically large
short and long positions.

This modified barbell method leads of necessity to less structure in the constant risk bullet portfolio returns, because portfolio returns are now not always drawn from the same probability distribution, but rather include returns drawn from ones with an added mean-preserving spread in the Rothschild and Stiglitz [52] sense. This method has been chosen, because the alternative of very large long and short positions (of order of magnitude of 500%) has been judged more unpalatable.

3. Results

Portfolio risk target levels in Table II were chosen so as to perform well with the above two methods. Ladder portfolio selection presented no problems, as was to be expected from the nature of the quadratic programme. Bullet portfolio selection, on the other hand, proved somewhat troublesome for the two-dimensional cases B/S and F/V. Some of the resulting portfolios had a few very large short and long positions, the result of near-dependence in the three linear equations to be solved. As they only occurred in two periods for B/S portfolios, and in one (different) period for F/V portfolios, these observations were thrown out before performing the main tests in this paper.
APPENDIX C

Numerical Demonstration of Term Structure Factors

We wish to demonstrate in this Appendix that certain term structure perturbations can generate the factor loading patterns observed in Section 6, i.e. increasing or decreasing with risk. We consider five bonds $P_n$, maturing in periods $n = 1, \ldots, 5$, respectively. The bonds have promised payments of one dollar per period until maturity. The risk ordering of these bonds then parallels their time-to-maturity, $n$. We posit a simplified term structure, $i_n$, that is initially flat.

We mimic the factor loadings with price sensitivities, i.e. the $\Delta P_n/P_n$ resulting from a discrete perturbation $\Delta i_n$ of the term structure.

Our first example demonstrates the increasing factor loadings pattern. It is associated with a parallel movement of the term structure, i.e. the type of perturbation for which the risk measure duration is exactly specified. We show in the following table the sensitivities of the bond prices to a parallel shift of the flat term structure from 9% to 10% (per period).

<table>
<thead>
<tr>
<th>Period $n =$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta i_n$%</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$-(\Delta P_n/P_n)100% =$</td>
<td>.9047</td>
<td>1.3416</td>
<td>1.7580</td>
<td>2.1576</td>
<td>2.5427</td>
</tr>
</tbody>
</table>

We see clearly that the sensitivities increase monotonically with bond risk. We could call this factor a “level” factor or a “long” factor. It is the type of term structure factor that dominates term structure perturbations, judging by the good performance of duration-based hedging strategies in bond portfolio management. A high risk/maturity bond or portfolio would mostly be subject to this risk.

The second example demonstrates a decreasing factor loading pattern. It can be generated by perturbing the slope of the term structure. We must, however, extract the “slope” effect net of the “level” effect which is also present. We do
this by a form of differencing. We start with two flat term structures, one at 10% and the other at 11%. Both are given the same slope perturbation, subtracting 3% in period 1, adding 1% in period 5, and linear in between. We now calculate the price sensitivities for both cases, and then their differences which are reported in the following table.

<table>
<thead>
<tr>
<th>Period n =</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta i_n %$</td>
<td>-3</td>
<td>-2</td>
<td>-1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$-(\Delta P_{1n}/P_{1n})100%$</td>
<td>2.7750</td>
<td>3.2175</td>
<td>3.0773</td>
<td>2.4239</td>
<td>1.3285</td>
</tr>
<tr>
<td>$-(\Delta P_{2n}/P_{2n})100%$</td>
<td>2.8050</td>
<td>3.2498</td>
<td>3.1084</td>
<td>2.4386</td>
<td>1.3164</td>
</tr>
<tr>
<td>$\Delta i_{12}100%$</td>
<td>.0300</td>
<td>.0323</td>
<td>.0311</td>
<td>.0147</td>
<td>-.0121</td>
</tr>
</tbody>
</table>

Subscript 1 perturbed from 11%, subscript 2 from 10%.

We see indeed a generally decreasing pattern for the corresponding factor loadings. A low risk/maturity bond or portfolio would best capture the slope variability of the term structure.

The last pattern, with highest loadings on intermediate portfolios, can be recreated by a change in term structure curvature. Again, we must correct for the change in term structure level by taking differences. We start with two flat term structure levels at 8% and 9%, respectively. We perturb both by increasing the value at the end of period 3 by 2%, leaving values at either end unchanged, and having a piecewise linear form in between.

<table>
<thead>
<tr>
<th>Period n =</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta i_n %$</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$-(\Delta P_{1n}/P_{1n})100%$</td>
<td>.0000</td>
<td>.8748</td>
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<td>$-(\Delta P_{2n}/P_{2n})100%$</td>
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<td>.8698</td>
<td>2.2242</td>
<td>2.5218</td>
<td>2.1005</td>
</tr>
<tr>
<td>$\Delta i_{12}100%$</td>
<td>.0000</td>
<td>.0050</td>
<td>.0304</td>
<td>.0356</td>
<td>.0209</td>
</tr>
</tbody>
</table>

Subscript 1 perturbed from 8%, subscript 2 from 9%.

The table shows differences of the return sensitivities of the five bonds, with a pattern of high intermediate loadings clearly in evidence. The peak sensitivity
would shift forward in time when using coupon bonds. The corresponding factor could well be called a "curvature" factor, and its variability would then be captured by an intermediate risk/maturity bond or portfolio.

Even in the simple numerical examples presented here, with term structure perturbations of similar order of magnitude, the values of the price sensitivities $\Delta P/P$ are clearly largest for the first example, demonstrating the level or duration effect. Successively smaller sensitivities apply to the slope and curvature effects. Although not shown, the same effects can be demonstrated with more complicated coupon bonds and portfolios. These effects mimic those found in the empirical results of this paper.
<table>
<thead>
<tr>
<th>Portfolio No.</th>
<th>Average No. of Bonds</th>
<th>Minimum No. of Bonds</th>
<th>Maximum No. of Bonds</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6.0</td>
<td>1</td>
<td>13</td>
</tr>
<tr>
<td>2</td>
<td>6.7</td>
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<td>4.1</td>
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<td>4</td>
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<td>2</td>
</tr>
<tr>
<td>17</td>
<td>.4</td>
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<td>0</td>
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<tr>
<td>19</td>
<td>.5</td>
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<td>2</td>
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<tr>
<td>20</td>
<td>4.0</td>
<td>0</td>
<td>13</td>
</tr>
</tbody>
</table>

**TABLE I**

Basic bond portfolios are the atomic securities from which the constant risk portfolios in this paper are formed. Basic portfolios are constituted every month in the sample period 1 Dec/1969–1 Dec/1979 from the currently traded bonds on the CRSP tape, see Section 3 for description of eligible bonds. Each basic portfolio is equally weighted in bonds with remaining time-to-maturity falling in non-overlapping six-month intervals, i.e. 0–6, 6–12, 12–18,.....,114+ months maturity remaining, numbered in that order. Coupon rates are simple averages, while risk measures and returns are value-weighted averages of those of the individual bonds. Basic portfolios are sometimes empty; the minimum number of non-empty basic portfolios is 13 occurring in 1970 and 1971 when few high maturity bonds were outstanding.
### TABLE II

Risk measure target values are expressed in annualized terms; intertemporally constant terms have been omitted. The target values in each row refer approximately to portfolios with the same risk level, see e.g. Table III for observed values in basic portfolios. The acronymic column headings in this and other tables have the following meaning:

- **DUR**: Macaulay [42] duration, in years
- **DURVAR**: Fong and Vasicek [29]'s duration variance, in years-squared
- **BSR**: Brennan and Schwartz [11] short rate risk measure $rP_t/P$
- **BSL**: —do—consol rate risk measure $LP_t/P$
- **CIR**: Cox, Ingersoll, and Ross [21] short rate risk measure $\sqrt{rP_t}/P$
- **N/S**: Nelson and Schaefer [48] 13-year rate risk measure $P_{13}/P$
- **MAT**: Time-to-maturity of basic portfolios, in years.

<table>
<thead>
<tr>
<th>Level</th>
<th>DUR</th>
<th>DURVAR</th>
<th>BSR</th>
<th>BSL</th>
<th>CIR</th>
<th>N/S</th>
<th>MAT</th>
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</thead>
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<tr>
<td>1</td>
<td>.75</td>
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<td>.030</td>
<td>.01</td>
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</tr>
<tr>
<td>2</td>
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<td>2.00</td>
<td>.050</td>
<td>.04</td>
<td>.15</td>
<td>1.25</td>
<td>2.00</td>
</tr>
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<td>6.00</td>
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<td>.07</td>
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<td>4.25</td>
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<td>8.00</td>
<td>70.00</td>
<td>.015</td>
<td>.30</td>
<td>.32</td>
<td>6.00</td>
<td>8.00</td>
</tr>
</tbody>
</table>
### Table III-A

Example of basic portfolio data. Numbers in the first column are those of the basic portfolios described in Table I, with in column 2 the number of constituent bonds (may be empty). Column 3 contains the one-month holding period returns on the basic portfolios, expressed as a net percentage change. The risk measures in the remaining columns are those described in Table II.

(Table III continues)
### BASIC PORTFOLIO DATA – APRIL 1975

<table>
<thead>
<tr>
<th>No.</th>
<th>Bonds</th>
<th>Return</th>
<th>DUR</th>
<th>DURVAR</th>
<th>BSR</th>
<th>BSL</th>
<th>CIR</th>
<th>N/S</th>
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<td>1</td>
<td>4</td>
<td>.625</td>
<td>.40</td>
<td>.16</td>
<td>.017</td>
<td>.002</td>
<td>.077</td>
<td>.37</td>
</tr>
<tr>
<td>2</td>
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<td>.73</td>
<td>.032</td>
<td>.010</td>
<td>.142</td>
<td>.80</td>
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<td>3</td>
<td>8</td>
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<td>1.26</td>
<td>1.63</td>
<td>.041</td>
<td>.021</td>
<td>.185</td>
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<td>4</td>
<td>9</td>
<td>1.891</td>
<td>1.76</td>
<td>3.21</td>
<td>.049</td>
<td>.037</td>
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<td>5</td>
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<td>.057</td>
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**TABLE III-B**

Example of basic portfolio data. Numbers in the first column are those of the basic portfolios described in Table I, with in column 2 the number of constituent bonds (may be empty). Column 3 contains the one-month holding period returns on the basic portfolios, expressed as a net percentage change. The risk measures in the remaining columns are those described in Table II.

*(Table III continues)*
Example of basic portfolio data. Numbers in the first column are those of the basic portfolios described in Table I, with in column 2 the number of constituent bonds (may be empty). Column 3 contains the one-month holding period returns on the basic portfolios, expressed as a net percentage change. The risk measures in the remaining columns are those described in Table II.
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**TABLE IV-2**

Selected values of autocorrelation functions estimated with Box-Jenkins timeseries analysis of eight constant risk portfolio returns, based on bullet and ladder portfolio selection, respectively. The standard errors are approximately .10 in all cases.

*(Table IV continues)*
Selected values of autocorrelation functions estimated with Box-Jenkins timeseries analysis of eight constant risk portfolio returns, based on bullet and ladder portfolio selection, respectively. The standard errors are approximately .10 in all cases.

(Table IV continues)
### AUTOCORRELATIONS at Lags 1, 2, 6, and 12 Months

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**TABLE IV-6**

Selected values of autocorrelation functions estimated with Box-Jenkins timeseries analysis of eight constant risk portfolio returns, based on bullet and ladder portfolio selection, respectively. The standard errors are approximately .10 in all cases.

*(Table IV ends)*
BULLET PORTFOLIO RETURNS – DURATION

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TABLE V – 11

LADDER PORTFOLIO RETURNS – DURATION

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TABLE V – 21

Univariate statistics of constant risk portfolio returns. Portfolio numbers refer to risk levels of Table II.

(Table V continues)
### BULLET PORTF. RETURNS – BRENNAN/SCHWARTZ

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**TABLE V – 12**

### LADDER PORTF. RETURNS – BRENNAN/SCHWARTZ

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**TABLE V – 22**

Univariate statistics of constant risk portfolio returns. Portfolio numbers refer to risk levels of Table II.

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**TABLE V - 23**

Univariate statistics of constant risk portfolio returns. Portfolio numbers refer to risk levels of Table II.

*(Table V continues)*
### TABLE V - 14

BULLET PORTF. RETURNS - NELSON/SCHAEFER

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### TABLE V - 24

LADDER PORTF. RETURNS - NELSON/SCHAEFER

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Univariate statistics of constant risk portfolio returns. Portfolio numbers refer to risk levels of Table II.

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**TABLE V – 15**

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**TABLE V – 25**

Univariate statistics of constant risk portfolio returns. Portfolio numbers refer to risk levels of Table II.

*(Table V continues)*
### BULLET PORTFOLIO RETURNS – MATURITY

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### LADDER PORTF. RETURNS – MATURITY

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**TABLE V – 16**

**TABLE V – 26**

Univariate statistics of constant risk portfolio returns. Portfolio numbers refer to risk levels of Table II.

*(Table V ends)*
Multivariate stationarity tests of constant risk portfolio returns between sample period halves. Each test uses returns on $m \leq 8$ independent portfolios, formed with the six risk measures and two portfolio selection methods analyzed in this paper. The statistic

$$\frac{n_1 + n_2 - m - 1}{(n_1 + n_2 - 2)m} T^2$$

is distributed as the central $F$-distribution with degrees of freedom $m$ and $n_1 + n_2 - m - 1$, where $m =$ the number of independent portfolios in each set; $n_1$ and $n_2$ are the number of observations in each sample sub-period. The critical $T^2$-value at the 1% level is approximately 20 for the above problem.

(Table VI continues)
**CORRELATION MATRIX – 1st Half Period of DUR Bullet Portfolio Returns**

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**TABLE VI-3**

**CORRELATION MATRIX – 2nd Half Period of DUR Bullet Portfolio Returns**

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**TABLE VI-4**

Typical example of correlation matrix dynamics between two sample period halves, December 1969–December 1974 and December 1974–December 1979. Results shown are for the returns on eight constant risk portfolios selected by Macaulay’s duration and bullet portfolio choice. Similar results were obtained for the other five risk measures.

(Table VI ends)
### Analysis of Correlation Matrices of Constant Risk Portfolio Returns

Columns 1 and 2 describe the risk measures used in forming eight constant risk portfolios in each case, see Table II for risk levels. Results are shown for two diversification levels, bullet and ladder.

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**Table VII-1**

*(Table VII continues)*
Eigen value analysis of correlation matrices of constant risk portfolio returns. Each data set consists of eight variates; the eight eigen values are listed in decreasing order for each of the risk measure/diversification combinations.

(Table VII ends)
Factor loadings from maximum likelihood factor analysis (MLFA) and principal components analysis (PCA) of the returns covariance matrix of eight constant risk portfolios selected with the bullet method and with the risk measure shown in each Table. Factor loadings are scaled by the respective standard deviations of each portfolio variate, and may now be interpreted as correlations between portfolios and factors. Factors have been rotated with the normalized varimax method. VP is the part of total variance (scaled to equal 8) explained by each factor.

(Table VIII continues)
Factor loadings from maximum likelihood factor analysis (MLFA) and principal components analysis (PCA) of the returns covariance matrix of eight constant risk portfolios selected with the bullet method and with the risk measure shown in each Table. Factor loadings are scaled by the respective standard deviations of each portfolio variate, and may now be interpreted as correlations between portfolios and factors. Factors have been rotated with the normalized varimax method. VP is the part of total variance (scaled to equal 8) explained by each factor.

(Table VIII continues)
### MLFA Factor Loadings – CIR

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<td>4</td>
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*Total variance explained by 2 factors is 94.5%.

#### TABLE VIII–13

### PCA Factor Loadings – CIR

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*Total variance explained by 2 factors is 94.0%.

#### TABLE VIII–23

Factor loadings from maximum likelihood factor analysis (MLFA) and principal components analysis (PCA) of the returns covariance matrix of eight constant risk portfolios selected with the bullet method and with the risk measure shown in each Table. Factor loadings are scaled by the respective standard deviations of each portfolio variate, and may now be interpreted as correlations between portfolios and factors. Factors have been rotated with the normalized varimax method. VP is the part of total variance (scaled to equal 8) explained by each factor.

(Table VIII continues)
### TABLE VIII–14

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VP: 3.513 3.788

*Total variance explained by 2 factors is 91.3%.

### TABLE VIII–24

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</table>

VP: 3.327 3.942

*Total variance explained by 2 factors is 90.9%.

Factor loadings from maximum likelihood factor analysis (MLFA) and principal components analysis (PCA) of the returns covariance matrix of eight constant risk portfolios selected with the bullet method and with the risk measure shown in each Table. Factor loadings are scaled by the respective standard deviations of each portfolio variate, and may now be interpreted as correlations between portfolios and factors. Factors have been rotated with the normalized varimax method. VP is the part of total variance (scaled to equal 8) explained by each factor.

(Table VIII continues)
Factor loadings from maximum likelihood factor analysis (MLFA) and principal components analysis (PCA) of the returns covariance matrix of eight constant risk portfolios selected with the bullet method and with the risk measure shown in each Table. Factor loadings are scaled by the respective standard deviations of each portfolio variate, and may now be interpreted as correlations between portfolios and factors. Factors have been rotated with the normalized varimax method. VP is the part of total variance (scaled to equal 8) explained by each factor.

*(Table VIII continues)*
Factor loadings from maximum likelihood factor analysis (MLFA) and principal components analysis (PCA) of the returns covariance matrix of eight constant risk portfolios selected with the bullet method and with the risk measure shown in each Table. Factor loadings are scaled by the respective standard deviations of each portfolio variate, and may now be interpreted as correlations between portfolios and factors. Factors have been rotated with the normalized varimax method. VP is the part of total variance (scaled to equal 8) explained by each factor.

(Table VIII ends)
MLFA FACTOR MODEL GOODNESS-OF-FIT

\[ \chi^2 = M \log \frac{|D + \hat{B}_k|}{|E|} \]

<table>
<thead>
<tr>
<th>Risk</th>
<th>1 Factor</th>
<th>2 Factors</th>
<th>3 Factors</th>
<th>4 Factors</th>
</tr>
</thead>
<tbody>
<tr>
<td>DUR</td>
<td>444.4</td>
<td>145.6</td>
<td>37.9</td>
<td>8.4</td>
</tr>
<tr>
<td>B/S</td>
<td>352.3</td>
<td>106.9</td>
<td>40.8</td>
<td>11.0</td>
</tr>
<tr>
<td>CIR</td>
<td>674.5</td>
<td>188.2</td>
<td>46.1</td>
<td>3.8</td>
</tr>
<tr>
<td>N/S</td>
<td>481.4</td>
<td>159.5</td>
<td>19.2</td>
<td>4.3</td>
</tr>
<tr>
<td>F/V</td>
<td>434.6</td>
<td>126.5</td>
<td>48.3</td>
<td>2.6</td>
</tr>
<tr>
<td>MAT</td>
<td>254.4</td>
<td>80.7</td>
<td>24.6</td>
<td>1.8</td>
</tr>
<tr>
<td>d.f.</td>
<td>20</td>
<td>13</td>
<td>7</td>
<td>2</td>
</tr>
<tr>
<td>( \chi^2 )</td>
<td>31.4</td>
<td>22.4</td>
<td>14.1</td>
<td>6.0</td>
</tr>
</tbody>
</table>

TABLE IX-1

Goodness-of-fit tests for MLFA factor models described in Table VIII. The multiplier \( M \) in the Likelihood Ratio \( \chi^2 \) statistic equals \( N - 1 - (2m + 5)/6 - 2k/3 \); the degrees of freedom are \( (m - k)^2 - m - k)/2 \), see Morrison [47], p. 314. \( N \) is the number of observations, \( m \) the number of constant risk portfolio variates, and \( k \) the number of factors. Bold face entries correspond to factor models of like dimension as the risk measure used in the portfolios. The factor loadings of Table VIII are based on one additional factor in each case.

VARIANCE EXPLAINED BY MARGINAL FACTOR

(out of scaled maximum = 8)

<table>
<thead>
<tr>
<th>Risk</th>
<th>1 Factor</th>
<th>2 Factors</th>
<th>3 Factors</th>
<th>4 Factors</th>
</tr>
</thead>
<tbody>
<tr>
<td>DUR</td>
<td>6.984</td>
<td>.501</td>
<td>.175</td>
<td>.073</td>
</tr>
<tr>
<td>B/S</td>
<td>4.224</td>
<td>2.125</td>
<td>.453</td>
<td>.356</td>
</tr>
<tr>
<td>CIR</td>
<td>6.834</td>
<td>.725</td>
<td>.173</td>
<td>.083</td>
</tr>
<tr>
<td>N/S</td>
<td>6.843</td>
<td>.458</td>
<td>.320</td>
<td>.042</td>
</tr>
<tr>
<td>F/V</td>
<td>6.569</td>
<td>.567</td>
<td>.050</td>
<td>.350</td>
</tr>
<tr>
<td>MAT</td>
<td>6.863</td>
<td>.363</td>
<td>.190</td>
<td>.093</td>
</tr>
</tbody>
</table>

TABLE IX-2

Alternative goodness-of-fit test based on variance explained by the model. Bold face entries are again for factor models corresponding to the dimension of the corresponding risk measure. The results for the PCA factor models were virtually identical, see also the results in Table VIII.
CANONICAL CORRELATIONS between
BULLET FACTORS and LADDER PORTFOLIOS

<table>
<thead>
<tr>
<th>Risk</th>
<th>Number of Factors</th>
<th>Ladder Portfolios</th>
<th>C.V.#1</th>
<th>C.V.#2</th>
<th>C.V.#3</th>
</tr>
</thead>
<tbody>
<tr>
<td>DUR</td>
<td>2</td>
<td>1, 8</td>
<td>.99</td>
<td>.90</td>
<td></td>
</tr>
<tr>
<td>B/S</td>
<td>3</td>
<td>1, 3, 8</td>
<td>.99</td>
<td>.80</td>
<td>.54</td>
</tr>
<tr>
<td>CIR</td>
<td>2</td>
<td>1, 8</td>
<td>.99</td>
<td>.62</td>
<td></td>
</tr>
<tr>
<td>N/S</td>
<td>2</td>
<td>1, 8</td>
<td>.99</td>
<td>.80</td>
<td></td>
</tr>
<tr>
<td>F/V</td>
<td>3</td>
<td>1, 4, 8</td>
<td>.99</td>
<td>.81</td>
<td>.08</td>
</tr>
<tr>
<td>MAT</td>
<td>2</td>
<td>1, 8</td>
<td>.99</td>
<td>.78</td>
<td></td>
</tr>
</tbody>
</table>

TABLE X

Canonical correlations analysis in each risk case between the factor scores of the MLFA model of Table VIII and the returns on the independent constant risk portfolios selected with the diversified ladder method. Bullet factor scores are derived with WLS estimator (5.7), using the rotated MLFA factors as in Table VIII. Canonical correlations are those between corresponding orthogonal canonical variates (C.V.) in each risk case.
OLS REGRESSION OF BULLET PORTFOLIO RETURNS ON LADDER PORTFOLIOS
Risk=DUR — Ladder Portfolios=1, 8

\[
\tilde{r}_i = \alpha_i + \beta_{i1}\tilde{r}_{l1} + \beta_{i2}\tilde{r}_{l8} + \tilde{\eta}_i
\]

<table>
<thead>
<tr>
<th>Portfolio (i)</th>
<th>(\beta_{i1})</th>
<th>(\beta_{i2})</th>
<th>(R^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.57</td>
<td>.02</td>
<td>.84</td>
</tr>
<tr>
<td></td>
<td>(15.6)</td>
<td>(2.1)</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>.83</td>
<td>.12</td>
<td>.95</td>
</tr>
<tr>
<td></td>
<td>(21.7)</td>
<td>(12.1)</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1.07</td>
<td>.19</td>
<td>.96</td>
</tr>
<tr>
<td></td>
<td>(21.6)</td>
<td>(15.3)</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>.96</td>
<td>.34</td>
<td>.94</td>
</tr>
<tr>
<td></td>
<td>(12.8)</td>
<td>(18.0)</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>.84</td>
<td>.50</td>
<td>.95</td>
</tr>
<tr>
<td></td>
<td>(11.1)</td>
<td>(26.1)</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>.37</td>
<td>.68</td>
<td>.94</td>
</tr>
<tr>
<td></td>
<td>(3.8)</td>
<td>(27.9)</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>-.12</td>
<td>.85</td>
<td>.96</td>
</tr>
<tr>
<td></td>
<td>(-1.3)</td>
<td>(37.7)</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>-.49</td>
<td>1.02</td>
<td>.97</td>
</tr>
<tr>
<td></td>
<td>(-5.5)</td>
<td>(45.2)</td>
<td></td>
</tr>
</tbody>
</table>

(Intercepts not shown; \(t\)-values in parentheses.)

TABLE XI–1

Test of factor identification as diversified (ladder) low and high constant risk portfolios. Returns on each of eight constant risk bullet portfolios are regressed on the same ladder portfolios, for the risk measure shown.

(Table XI continues)
OLS REGRESSION OF BULLET PORTFOLIO RETURNS ON LADDER PORTFOLIOS

Risk = B/S — Ladder Portfolios = 1, 3, 8

\[ \tilde{r}_{bi} = \alpha_i + \beta_{i1} \tilde{r}_{i1} + \beta_{i2} \tilde{r}_{i3} + \beta_{i3} \tilde{r}_{i8} + \tilde{\eta}_i \]

<table>
<thead>
<tr>
<th>Portfolio i</th>
<th>( \hat{\beta}_{i1} )</th>
<th>( \hat{\beta}_{i2} )</th>
<th>( \hat{\beta}_{i3} )</th>
<th>( R^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.66</td>
<td>.07</td>
<td>-.35</td>
<td>.77</td>
</tr>
<tr>
<td>2</td>
<td>(8.8)</td>
<td>(2.8)</td>
<td>(-2.55)</td>
<td>.92</td>
</tr>
<tr>
<td>3</td>
<td>.63</td>
<td>.36</td>
<td>.05</td>
<td>.92</td>
</tr>
<tr>
<td>4</td>
<td>(7.0)</td>
<td>(11.6)</td>
<td>(2.8)</td>
<td>.86</td>
</tr>
<tr>
<td>5</td>
<td>.26</td>
<td>.83</td>
<td>.06</td>
<td>.92</td>
</tr>
<tr>
<td>6</td>
<td>(1.7)</td>
<td>(15.8)</td>
<td>(2.2)</td>
<td>.63</td>
</tr>
<tr>
<td>7</td>
<td>(-35)</td>
<td>.97</td>
<td>.16</td>
<td>.86</td>
</tr>
<tr>
<td>8</td>
<td>(-1.6)</td>
<td>(13.2)</td>
<td>(4.0)</td>
<td>.92</td>
</tr>
<tr>
<td>9</td>
<td>-.71</td>
<td>.97</td>
<td>.28</td>
<td>.63</td>
</tr>
<tr>
<td>10</td>
<td>(1.7)</td>
<td>(6.8)</td>
<td>(3.8)</td>
<td>.69</td>
</tr>
<tr>
<td>11</td>
<td>.13</td>
<td>.73</td>
<td>.49</td>
<td>.72</td>
</tr>
<tr>
<td>12</td>
<td>(.3)</td>
<td>(4.8)</td>
<td>(6.0)</td>
<td>.73</td>
</tr>
<tr>
<td>13</td>
<td>-.24</td>
<td>.31</td>
<td>1.02</td>
<td>.79</td>
</tr>
<tr>
<td>14</td>
<td>(-.6)</td>
<td>(2.3)</td>
<td>(14.4)</td>
<td>.72</td>
</tr>
<tr>
<td>15</td>
<td>1.13</td>
<td>-.51</td>
<td>1.42</td>
<td>.72</td>
</tr>
<tr>
<td>16</td>
<td>(2.0)</td>
<td>(-2.7)</td>
<td>(13.9)</td>
<td>.72</td>
</tr>
</tbody>
</table>

(Intercepts not shown; t-values in parentheses.)

**TABLE XI-2**

Test of factor identification as diversified (ladder) low and high constant risk portfolios. Returns on each of eight constant risk bullet portfolios are regressed on the same ladder portfolios, for the risk measure shown.

(Table XI continues)
### TABLE XI-3

Test of factor identification as diversified (ladder) low and high constant risk portfolios. Returns on each of eight constant risk bullet portfolios are regressed on the same ladder portfolios, for the risk measure shown.

*Table XI continues*
### TABLE XI-4

Test of factor identification as diversified (ladder) low and high constant risk portfolios. Returns on each of eight constant risk bullet portfolios are regressed on the same ladder portfolios, for the risk measure shown.

\( \tilde{r}_{hi} = \alpha_i + \beta_{i1}\tilde{r}_{1i} + \beta_{i2}\tilde{r}_{8i} + \tilde{\eta}_i \)

<table>
<thead>
<tr>
<th>Portfolio i</th>
<th>( \hat{\beta}_{i1} )</th>
<th>( \hat{\beta}_{i2} )</th>
<th>( R^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.46</td>
<td>.02</td>
<td>.78</td>
</tr>
<tr>
<td></td>
<td>(13.6)</td>
<td>(2.9)</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>.80</td>
<td>.14</td>
<td>.94</td>
</tr>
<tr>
<td></td>
<td>(20.1)</td>
<td>(14.4)</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>.95</td>
<td>.26</td>
<td>.97</td>
</tr>
<tr>
<td></td>
<td>(25.1)</td>
<td>(27.1)</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>.97</td>
<td>.36</td>
<td>.95</td>
</tr>
<tr>
<td></td>
<td>(15.9)</td>
<td>(23.6)</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>.82</td>
<td>.49</td>
<td>.91</td>
</tr>
<tr>
<td></td>
<td>(8.5)</td>
<td>(20.1)</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>.25</td>
<td>.74</td>
<td>.94</td>
</tr>
<tr>
<td></td>
<td>(2.5)</td>
<td>(29.7)</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>-.09</td>
<td>.85</td>
<td>.94</td>
</tr>
<tr>
<td></td>
<td>(-.88)</td>
<td>(32.0)</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>-.31</td>
<td>1.01</td>
<td>.95</td>
</tr>
<tr>
<td></td>
<td>(-3.0)</td>
<td>(39.1)</td>
<td></td>
</tr>
</tbody>
</table>

(Intercepts not shown; \( t \)-values in parentheses.)

(Table XI continues)
OLS REGRESSION OF BULLET PORTFOLIO RETURNS ON LADDER PORTFOLIOS

\[
\tilde{r}_{bi} = \alpha_i + \beta_{i1}\tilde{r}_{i1} + \beta_{i2}\tilde{r}_{i4} + \beta_{i3}\tilde{r}_{i8} + \tilde{\eta}_i
\]

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>(\hat{\beta}_{i1})</th>
<th>(\hat{\beta}_{i2})</th>
<th>(\hat{\beta}_{i3})</th>
<th>(R^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.50</td>
<td>.01</td>
<td>.04</td>
<td>.69</td>
</tr>
<tr>
<td>2</td>
<td>1.11</td>
<td>.20</td>
<td>.01</td>
<td>.95</td>
</tr>
<tr>
<td>3</td>
<td>(1.24)</td>
<td>(7.1)</td>
<td>(.8)</td>
<td>.97</td>
</tr>
<tr>
<td>4</td>
<td>(3.1)</td>
<td>(20.8)</td>
<td>(1.3)</td>
<td>.96</td>
</tr>
<tr>
<td>5</td>
<td>1.00</td>
<td></td>
<td>.17</td>
<td>.95</td>
</tr>
<tr>
<td>6</td>
<td>(.30)</td>
<td>(17.4)</td>
<td>(7.3)</td>
<td>.92</td>
</tr>
<tr>
<td>7</td>
<td>(.46)</td>
<td>1.12</td>
<td>.302</td>
<td>.83</td>
</tr>
<tr>
<td>8</td>
<td>(.72)</td>
<td>1.46</td>
<td>.30</td>
<td>.65</td>
</tr>
</tbody>
</table>

(Intercepts not shown; \(t\)-values in parentheses.)

TABLE XI–5

Test of factor identification as diversified (ladder) low and high constant risk portfolios. Returns on each of eight constant risk bullet portfolios are regressed on the same ladder portfolios, for the risk measure shown.

(Table XI continues)
OLS REGRESSION OF BULLET PORTFOLIO
RETURNS ON LADDER PORTFOLIOS
Risk=MAT — Ladder Portfolios=1, 8
\[ \tilde{r}_i = \alpha_i + \beta_{i1}\tilde{r}_{1i} + \beta_{i2}\tilde{r}_{8i} + \tilde{\eta}_i \]

<table>
<thead>
<tr>
<th>Portfolio i</th>
<th>( \hat{\beta}_{i1} )</th>
<th>( \hat{\beta}_{i2} )</th>
<th>( R^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.57</td>
<td>.03</td>
<td>.84</td>
</tr>
<tr>
<td></td>
<td>(14.2)</td>
<td>(2.0)</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>.84</td>
<td>.18</td>
<td>.95</td>
</tr>
<tr>
<td></td>
<td>(19.5)</td>
<td>(13.2)</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1.10</td>
<td>.21</td>
<td>.93</td>
</tr>
<tr>
<td></td>
<td>(16.9)</td>
<td>(10.1)</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1.07</td>
<td>.35</td>
<td>.91</td>
</tr>
<tr>
<td></td>
<td>(11.1)</td>
<td>(11.6)</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>.74</td>
<td>.51</td>
<td>.85</td>
</tr>
<tr>
<td></td>
<td>(5.3)</td>
<td>(11.8)</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>.33</td>
<td>.75</td>
<td>.94</td>
</tr>
<tr>
<td></td>
<td>(3.5)</td>
<td>(24.8)</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>.20</td>
<td>.87</td>
<td>.93</td>
</tr>
<tr>
<td></td>
<td>(1.7)</td>
<td>(23.7)</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>-.33</td>
<td>1.01</td>
<td>.95</td>
</tr>
<tr>
<td></td>
<td>(-3.4)</td>
<td>(32.5)</td>
<td></td>
</tr>
</tbody>
</table>

(Intercepts not shown; t-values in parentheses.)

TABLE XI-6

Test of factor identification as diversified (ladder) low and high constant risk portfolios. Returns on each of eight constant risk bullet portfolios are regressed on the same ladder portfolios, for the risk measure shown.

(Table XI ends)
### TABLE XII

Tests are similar to those in Roll and Ross [50], using generalized least squares (GLS) estimation. Tests are carried out for the factor models in Table VIII for each of the risk measures shown in column 1. Rotated factor loadings $B_k$ and residual covariance matrices $D$ from MLFA are used, with factors ordered in each risk case as per Table VIII. Number of factors $k$ is two for one-dimensional risk measures and three for two-dimensional ones. Null hypotheses are listed in column headings.
### FACTOR MODEL BILINEARITY TESTS

F-tests of factor score equality in odd/even cross-section partitions

\[ \tilde{r}_t^* = \delta_t B_k + \tilde{\epsilon}_t, \quad t = 1, \ldots, T \]

<table>
<thead>
<tr>
<th>Risk</th>
<th>( k = 2 )</th>
<th>( k = 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>DUR</td>
<td>.78</td>
<td>.97</td>
</tr>
<tr>
<td>B/S</td>
<td>.75</td>
<td>.85</td>
</tr>
<tr>
<td>CIR</td>
<td>.90</td>
<td>.72</td>
</tr>
<tr>
<td>N/S</td>
<td>.76</td>
<td>.92</td>
</tr>
<tr>
<td>F/V</td>
<td>.76</td>
<td>.73</td>
</tr>
<tr>
<td>MAT</td>
<td>.91</td>
<td>.89</td>
</tr>
</tbody>
</table>

**TABLE XIII**

Tests as described in Brown and Weinstein [14]. Partitioned rotated factor loadings \( B_k \) and residual covariance matrices \( D \) from MLFA are used, similar to those in Table VIII. Factor scores are forced equal in the two sub-samples as the restricted model. Boldface results refer to cases reported in Table VIII. Though not reported, similar tests using PCA factors and residuals showed comparable results. Critical F-values at the 5% level are all above 1.00.
A.P.T. BILINEARITY TESTS

F-tests of equality of prices+scores
in cross-section sub-samples of
\[ \tilde{r}_t = (\lambda_0, (\lambda_t + \delta_t)')B + e_t \]

<table>
<thead>
<tr>
<th>Risk</th>
<th>( \lambda_0t )</th>
<th>( (\lambda_t + \delta_t) )</th>
<th>( (\lambda_0, (\lambda_t + \delta_t))' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>DUR</td>
<td>1.71</td>
<td>1.45</td>
<td>1.53</td>
</tr>
<tr>
<td>B/S</td>
<td>.76</td>
<td>.92</td>
<td>.78</td>
</tr>
<tr>
<td>CIR</td>
<td>.21</td>
<td>.51</td>
<td>.53</td>
</tr>
<tr>
<td>N/S</td>
<td>.81</td>
<td>.85</td>
<td>.77</td>
</tr>
<tr>
<td>F/V</td>
<td>.94</td>
<td>1.06</td>
<td>.85</td>
</tr>
<tr>
<td>MAT*</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

TABLE XIV

Bilinearity tests as described in Brown and Weinstein [14]. Rotated factor loadings \( B \) and residual covariance matrices \( D \) are derived from MLFA, with two factors only. Matrices are partitioned appropriately for each cross-section sub-sample. Prices and scores are forced equal as the restricted model. The critical F-values at the 5% level are approximately equal to 1.00. Though not reported, the results for PCA factors were similar.

* Results for the MAT risk case are not available due to near-singularity of the covariance matrix of the estimate \( (BD^{-1}B') \).
EX POST SECURITY MARKET LINE (PLANE)  
FOR LADDER PORTFOLIOS  
OLS Regression of Mean Returns on Risk Measure Values  
\[ \bar{R} = \lambda_0 + \lambda_1 b_1 (+\lambda_2 b_2) + \eta \]

<table>
<thead>
<tr>
<th>Risk</th>
<th>(\hat{\lambda}_0)</th>
<th>(\hat{\lambda}_1)</th>
<th>(\hat{\lambda}_2)</th>
<th>(R^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>DUR</td>
<td>.569E-2</td>
<td>-.378E-4</td>
<td></td>
<td>.999</td>
</tr>
<tr>
<td></td>
<td>(2389)</td>
<td>(-74.8)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>B/S</td>
<td>.566E-2</td>
<td>.360E-2</td>
<td>-.171E-2</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td>(4100)</td>
<td>(174.2)</td>
<td>(-423.8)</td>
<td></td>
</tr>
<tr>
<td>CIR</td>
<td>.581E-2</td>
<td>-.344E-3</td>
<td></td>
<td>.989</td>
</tr>
<tr>
<td></td>
<td>(1613)</td>
<td>(-22.7)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>N/S</td>
<td>.571E-2</td>
<td>-.494E-4</td>
<td></td>
<td>.999</td>
</tr>
<tr>
<td></td>
<td>(2492)</td>
<td>(-78.1)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>F/V</td>
<td>.564E-2</td>
<td>.593E-4</td>
<td>-.193E-4</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td>(2037)</td>
<td>(39.8)</td>
<td>(-130.4)</td>
<td></td>
</tr>
<tr>
<td>MAT</td>
<td>.574E-2</td>
<td>-.332E-4</td>
<td></td>
<td>.999</td>
</tr>
<tr>
<td></td>
<td>(2463)</td>
<td>(-72.0)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(t-values in parentheses)

TABLE XV-1

Mean returns for eight constant risk portfolios in each risk case are regressed on the values of their constant risk levels from Table II. Risk measures \(b_1/b_2\) are ordered as BSR/BSL and DUR/DURVAR in the B/S and F/V risk cases, respectively. The estimation method deals with the errors-in-variables problem in a manner similar to that in Black, Jensen, and Scholes [6].

(Table XV continues)
EX POST SECURITY MARKET LINE (PLANE)
FOR BULLET PORTFOLIOS

OLS Regression of Mean Returns on Risk Measure Values

\[ \hat{R} = \lambda_0 + \lambda_1 b_1 (+\lambda_2 b_2) + \eta, \]

<table>
<thead>
<tr>
<th>Risk</th>
<th>( \lambda_0 )</th>
<th>( \lambda_1 )</th>
<th>( \lambda_2 )</th>
<th>( R^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>DUR</td>
<td>.579E-2</td>
<td>-.740E-4</td>
<td></td>
<td>.685</td>
</tr>
<tr>
<td></td>
<td>(59.9)</td>
<td>(-3.61)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>B/S</td>
<td>.472E-2</td>
<td>.244E-1</td>
<td>-.155E-2</td>
<td>.496</td>
</tr>
<tr>
<td></td>
<td>(4.56)</td>
<td>(1.57)</td>
<td>(-.51)</td>
<td></td>
</tr>
<tr>
<td>CIR</td>
<td>.544E-2</td>
<td>.184E-2</td>
<td></td>
<td>.389</td>
</tr>
<tr>
<td></td>
<td>(24.4)</td>
<td>(1.95)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>N/S</td>
<td>.584E-2</td>
<td>-.100E-3</td>
<td></td>
<td>.453</td>
</tr>
<tr>
<td></td>
<td>(35.9)</td>
<td>(-2.23)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>F/V</td>
<td>.558E-2</td>
<td>-.274E-4</td>
<td>.921E-6</td>
<td>.152</td>
</tr>
<tr>
<td></td>
<td>(35.3)</td>
<td>(-.32)</td>
<td>(.11)</td>
<td></td>
</tr>
<tr>
<td>MAT</td>
<td>.579E-2</td>
<td>-.584E-4</td>
<td></td>
<td>.426</td>
</tr>
<tr>
<td></td>
<td>(41.4)</td>
<td>(-2.11)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(t-values in parentheses)

TABLE XV-2

Mean returns for eight constant risk portfolios in each risk case are regressed on the values of their constant risk levels from Table II. Risk measures \( b_1/b_2 \) are ordered as BSR/BSL and DUR/DURVAR in the B/S and F/V risk cases, respectively. The estimation method deals with the errors-in-variables problem in a manner similar to that in Black, Jensen, and Scholes [6].

(Table XV continues)
### F-Test of Forcing Bullet Mean Returns Onto Ladder Ex Post SML

<table>
<thead>
<tr>
<th>Risk</th>
<th>d.f.</th>
<th>F</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>DUR</td>
<td>2, 6</td>
<td>1.95</td>
<td>.22</td>
</tr>
<tr>
<td>B/S</td>
<td>3, 5</td>
<td>.80</td>
<td>.54</td>
</tr>
<tr>
<td>CIR</td>
<td>2, 6</td>
<td>4.39</td>
<td>.07</td>
</tr>
<tr>
<td>N/S</td>
<td>2, 6</td>
<td>.69</td>
<td>.54</td>
</tr>
<tr>
<td>F/V</td>
<td>3, 5</td>
<td>13.77</td>
<td>.01</td>
</tr>
<tr>
<td>MAT</td>
<td>2, 6</td>
<td>.92</td>
<td>.45</td>
</tr>
</tbody>
</table>

**TABLE XV–3**

Values of intercepts and coefficients of the bullet portfolio returns are in all risk cases restricted to those of the diversified ladder portfolio returns. An F-test of these joint restrictions indicates whether bullet portfolio returns are just noisy versions of ladder returns, which null hypothesis is supported by large p-values.

*(Table XV ends)*
BULLET PORTFOLIOS

Fig. 1
Ex-Post Security Market Line – DUR

OLS regression of bullet portfolio mean returns (from Table V-11) on constant duration levels (from Table II). Returns are simple annualized percentages: values in Table V have been multiplied by 1200. Graph is typical for that of the one-dimensional risk measures studied.
**Fig. 2**  
Example of Ex-Post Security Market Line – DUR  
OLS regression of ladder portfolio mean returns (from Table V–21) on constant duration levels (from Table II). Returns are simple annualized percentages: values in Table V have been multiplied by 1200. Graph is typical for that of the one-dimensional risk measures studied.