Linear Continuous-time System Identification and State Observer Design by Modal Analysis

by

Mohamed Hassan El-Shafey

B.Sc. Ain Shams University, Cairo, Egypt, 1978

M.Sc. Ain Shams University, Cairo, Egypt, 1982

A Thesis Submitted in Partial Fulfilment of

The Requirements for The Degree of

Doctor of Philosophy

in

The Faculty of Graduate Studies

(Department of Electrical Engineering)

We accept this thesis as conforming

to the required standard

The University of British Columbia

November, 1987

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Department of _Electrical Engineering_

The University of British Columbia
1956 Main Mall
Vancouver, Canada
V6T 1Y3

Date _Feb. 5, 1988_

Abstract

A new approach to the identification problem of linear continuous-time time-invariant systems from input-output measurements is presented. Both parametric and nonparametric system models are considered. The new approach is based on the use of continuous-time functions, *the modal functions*, defined in terms of the system output, the output derivatives and the state variables under the assumption that the order n of the observable system is known a priori. The *modal functions* are obtained by linear filtering operations of the system output, the output derivatives and the state variables so that *the modal functions* are independent of the system instantaneous state. In this case, *the modal functions* are linear functions of the input exponential *modes*, and they contain none of the system exponential *modes* unlike the system general response which contains *modes* from both the system and the input. The filters parameters, *the modal parameters* are estimated using linear regression techniques.

The *modal functions* and the *modal parameters* of the output and its derivatives are used to identify parametric input-output and state models of the system. The coefficients of the system characteristic polynomial are obtained by solving n algebraic equations formed from the estimates of the *modal parameters*. Estimates of the parameters associated with the system zeros are obtained by solving another set of linear algebraic equation. The system frequency response and step response are estimated using the *output modal function*. The impulse response is obtained by filtering the estimated step response using *the output first derivative modal parameters*.

A new method is presented to obtain the system poles as the eigenvalues of a data matrix formed from the system free response. The coefficients of the system characteristic polynomial are obtained from the data matrix through a simple recursive equation. This method has some important advantages over the well known Prony's method.

The *state modal functions* are used to obtain a minimum-time observer that gives the continuous-time system state as a direct function of input-output samples in n sampling intervals.

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Aknowledgements

I wish to thank Dr. E. V. Bohn for supervising my research work and for his fruitful suggestions. His financial support through NSERC grant is also appreciated.

Chapter 1

Introduction

1.1 Introduction

The analysis of a physical system starts with the formulation of a mathematical model of that system. Determining the system model may be possible in some cases by detailed analysis using the basic laws of system dynamics. Usually, this is not possible in practice because of the lack of sufficient knowledge of the system and it is therefore necessary to determine the model from observations made on the system. However, the available a priori knowledge of the system is often sufficient to determine the class of models to which a suitable model of the physical system belongs. In other words, to assume that whether a good model of the system is linear or nonlinear, finite dimensional or infinite dimensional, time-invariant or time varying, continuous-time or discrete-time, stochastic or deterministic, is often possible from the available a priori knowledge of the system. A good model should give a close approximation to the physical system according to certain chosen criterian besides being identifiable from the available measurements taken from the system input and output.

The problem of identifying a linear lumped parameter time-invariant system is the simplest and most tractable. This is the main reason why linear time-invariant models have always received considerable interest in the control literature despite the fact that most systems in practical life do not fall in this category. Nonetheless, a linear time-invariant model can be made to serve as a useful approximation to a physically nonlinear system about a nominal operating point. Moreover, the parameters of a time-invariant model can be updated on-line to track possible

variations in the system dynamics under the assumption that these variations occur relatively slowly compared to the parameter updating algorithm.

Since the sixties, with the ready availability of digital computers, most of the research work in system identification was drawn towards discrete-time modeling by difference equations so that the system model matches the serial processing nature of a digital computer. However, most physical systems obey physical laws and have continuous-time outputs so that it is natural to model the system in continuous time by means of differential equations. The identification of a differential equation model of the system is consequently an important problem. This problem is inherently harder than the identification of a difference equation model because of the need to obtain derivatives which are usually unmeasurable.

Modeling a single input-single output linear system by a differential equation of order n can in general take two forms:

1. Input-output model;

$$\mathcal{D}^n y(t) + \sum_{j=0}^{n-1} a_j \mathcal{D}^j y(t) = \sum_{j=0}^{n-1} b_j \mathcal{D}^j u(t) \tag{1.1}$$

where u and y are the input and the output signals respectively and where the operator $\mathcal{D}$ is the time differentiation operator defined by

$$\mathcal{D}^j \equiv \frac{d^j}{dt^j}$$

2. State model;

   (a) State equation;

$$\mathcal{D}\underline{x}(t) = A\underline{x}(t) + \underline{b}u(t) \tag{1.2}$$

   (b) Output equation;

$$y(t) = \underline{c}^T \underline{x}(t) \tag{1.3}$$

where $\underline{x}$ is the state vector, A is an $n \times n$ matrix[1] and $\underline{b}$ and $\underline{c}$ are vectors of dimension n each. The superscript ^{T} means the transpose. Note that the elements of vector $\underline{b}$ in (1.2) are different from $b_o, \ldots, b_{n-1}$ in (1.1). The parametric system identification problem is to obtain estimates of the parameters $a_o, \ldots, a_{n-1}$ and $b_o, \ldots, b_{n-1}$ of the system input-output model (1.1), or estimates of the arrays A, $\underline{b}$ and $\underline{c}$ of the

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[1]Sometimes called the system dynamic matrix

state model (1.2) and (1.3) from the available measurements made on the system. However, it is not possible to obtain all the elements of A, $\underline{b}$ and $\underline{c}$ from the input-output measurements only; some assumptions on the elements of these arrays have to be made which lead to the *canonical forms* which will be discussed in some detail later.

Complementary to the identification problem of a finite dimensional parametric model is the identification of an infinite dimensional nonparametric model, e.g. the frequency response or the impulse response. In this case it is usually required to obtain samples of the frequency response or the impulse response.

The following two sections review the important achievements in parametric and nonparametric system identification respectively. The last section in this introductory chapter gives a overview of the thesis.

1.2   Parametric Identification

A relatively recent survey of the methods of parametric identification of continuous-time systems can be found in [Young 81]. The survey classifies the parameter estimation algorithms to *output error* algorithms and *equation error* algorithms according to the cost function that is minimized. The methods reviewed in the following sections are basically of the *equation error* type which is adopted throughout the thesis. The term continuous systems will be used throughout the thesis to refer to continuous-time systems.

1.2.1   Continuous Models from Discrete Equivalents

In order to make use of the advanced techniques of discrete model identification, the problem of identifying a continuous transfer function model[2].

$$H(s) = \frac{\sum_{j=0}^{n-1} b_j s^j}{s^n + \sum_{j=0}^{n-1} a_j s^j} \tag{1.4}$$

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[2]Equivalent to the input-output model (1.1)

is divided into two parts. The first part consists of identifying a discrete model of the system

$$H_d(z^{-1}) = \frac{\sum_{k=1}^{n} q_k z^{-k}}{1 + \sum_{k=1}^{n} p_k z^{-k}} \tag{1.5}$$

from discrete measurements. The second part is the determination of $H(s)$ from $H_d(z^{-1})$ which is the inverse problem of digital filter design from the equivalent continuous filter [Oppenheim 75, Sec.5.1].

The discrete transfer function $H_d(z^{-1})$ represents the z-transform of $H(s)$ only if the input is a sequence of impulses. Otherwise, and under the assumption that the measurements are discrete, the input has to be approximated between the sampling instants. Different approximations of the input signal, e.g. by steps or ramps, lead to different transformations, step invariant or ramp invariant, from $H(s)$ to $H_d(z^{-1})$ [Sinha,N.K. 72]. One difficulty with this method lies in the need to factorize the denominator polynomial of $H_d(z^{-1})$ in order to obtain the poles of of $H(s)$ through a logarithmic operation

$$s_k = -\frac{1}{T} \ln z_k. \tag{1.6}$$

Other transformations which do not require polynomial factorization or logarithmic operations are given by

$$z^{-1} = 1 - sT \tag{1.7}$$

and

$$z^{-1} = \frac{1 - s(T/2)}{1 + s(T/2)}. \tag{1.8}$$

The latter is known as the bilinear transformation. The above transformations are based on the approximate prediction equations given respectively by

$$r(nT) = r((n-1)T) + \mathcal{D}r((n-1)T)T \tag{1.9}$$

and

$$r(nT) = r((n-1)T) + \frac{\mathcal{D}r((n-1)T) + \mathcal{D}r(nT)}{2}T. \tag{1.10}$$

where r is either the input signal u or the output signal y. As expected, the bilinear transformation usually yields better approximations than the first transformation which is based on approximating the derivative by a finite difference. Therefore, it can be concluded that all the methods of determining a continuous model from the

discrete equivalent model involve approximations either in the input signal or in the transformation from the continuous frequency domain to the discrete frequency domain.

Another method, which assumes a zero input and consequently involves no approximation of the input signal between sampling instants, is known as Prony's method [Marple 87, Ch.11]. Prony's method is a technique for modeling signals as a linear combination of exponentials

$$r(t) = \sum_{k=1}^{n} C_k e^{s_k t} \qquad (1.11)$$

from equally spaced samples. There are three basic steps in Prony's method. Step one determines the linear prediction model that fits the available data, i.e. determines the denominator polynomial of $H_d(z^{-1})$. In step two, the obtained polynomial is factorized to obtain n roots z_k and then obtain s_k as in (1.6). Step three involves the solution of a set of linear equations to yield the estimates of the coefficients $C_1, \ldots, C_n$. The method does not consider estimating the numerator polynomial of $H_d(z-1)$ and therefore it is suitable only for identification of systems with impulsive type inputs [Noharski 85]. Application of Prony's method in identification of electromagnetic systems can be found in [Schaubert 79] and [Poggio 78], and in biological systems can be found in [Crittenden 83].

A state space version of Prony's method is discussed in [Crittenden 83]. The system dynamic matrix A is obtained through a logarithmic operation similar to (1.6) under the assumption that the full state vector is measurable. The method does not give the vector $\underline{b}$. The given reference discusses the choice of the sampling period in conjunction with the finite precision arithmetic, used to represent data samples, and the fastest and the slowest exponentials in the analyzed signal.

All the above discrete models for characterizing continuous systems and signals from discrete measurements are based on the assumption that the sampling time T is fixed, and that the signal is band limited. The choice of T for a wideband signal is critical because a *too small* T could result in numerical instability in the estimation equations of the linear prediction model parameters and in the inverse z-Transform, while a *too large* T gives rise to the problem of aliasing [Åström 83, Sec.2.5].

1.2.2 Modulating Functions

A classical technique, [Shinbort 57], similar to the Fourier transform and the Laplace transform, is used to transform a differential equation to a set of linear algebraic equations in the unknown parameters. Shinbort used the name *method functions* but the name *modulating functions* was used later [Loeb 65]. The differential equation (1.1) is multiplied by a vector $\underline{f}_M(t)$ of modulating functions and integrated over the period of observation $t_o \to t_f$

$$\int_{t_o}^{t_f} \mathcal{D}^n y(t)\underline{f}_M(t)dt + \sum_{j=0}^{n-1} a_j \int_{t_o}^{t_f} \mathcal{D}^j y(t)\underline{f}_M(t)dt = \sum_{j=0}^{n-1} b_j \int_{t_o}^{t_f} \mathcal{D}^j u(t)\underline{f}_M(t)dt. \quad (1.12)$$

Performing the integration by parts and choosing $\underline{f}_M$ such that

$$\mathcal{D}^j \underline{f}_M(t_o) = \mathcal{D}^j \underline{f}_M(t_f) = 0 \quad , j = 0, \ldots, n-1 \quad (1.13)$$

yields an equation in the measured variables only and not in their derivatives

$$(-1)^n \int_{t_o}^{t_f} y(t)\mathcal{D}^n \underline{f}_M(t)dt \;+\; \sum_{j=0}^{n-1}(-1)^j a_j \int_{t_o}^{t_f} y(t)\mathcal{D}^j \underline{f}_M(t)dt$$

$$= \sum_{j=0}^{n-1}(-1)^j b_j \int_{t_o}^{t_f} u(t)\mathcal{D}^j \underline{f}_M(t)dt. \quad (1.14)$$

Unless (1.13) is satisfied, the terminal values of u, y and their derivatives would appear in (1.14).

It is seen from (1.14) that obtaining the estimation equations for the unknown parameters requires only the computation of the scalar products of the recorded signals with the vector of modulating functions and its derivatives. The use of orthogonal functions, e.g. Fourier functions, is preferred to achieve linear independence of the estimation equations, a requirement which may violate the terminal condition (1.13).

The elements of $\underline{f}_M$ can be derived from the impulse response of a set of analog filters such that the output of these filters provide the input data to an estimation algorithm which estimates the unknown parameters from (1.14). Such analog filters are known as *state variable filters* [Young 70].

The Poisson moment functional approach to continuous system identification [Saha 83] is a special case of the state variable method. A Poisson filter chain

is composed of cascaded identical filters with a first order transfer function $\frac{1}{s+\lambda}$, $\lambda > 0$. Beside not being orthogonal, the impulse responses of the cascaded filters obviously do not satisfy the terminal condition (1.14) so that the unknown initial conditions are introduced to the estimation problem. Also, passing the measured signals through many filter stages results in smoothing the signals and consequently reducing the information content of the filtered signals.

Pearson [Pearson 85] used a projection of a vector of Fourier functions in order to satisfy the terminal condition, and applied the method to identification of bilinear systems using sinusoidal probing signals over sequential time intervals.

1.2.3 Multiple Integrations and Orthogonal Functions

In recent years, a new technique has been established for the solution of continuous system identification problem using orthogonal functions. The main feature of this technique is that, like the modulating function technique, it reduces the problem to that of solving a system of linear algebraic equations, thus simplifying the solution. The transformation of the differential equation to an algebraic equation takes place in two steps. First, the differential equation is transformed into an integral equation by means of n integrations[3]. Second, the input-output signals are correlated with a vector $\underline{f}_o(t)$ of orthogonal functions so that multiple integrals of the input-output signals can be obtained in terms of $\underline{f}_o(t)$. The key idea of this technique is based on the following property of a basis vector $\underline{f}_o(t)$ under multiple integration

$$\underbrace{\int_{t_o}^{t} \cdots \int_{t_o}^{t_1} \underline{f}_o(\sigma) d\sigma \cdots dt_{k-1}}_{k} \approx P^k \underline{f}_o(t) \tag{1.15}$$

where P is a constant square matrix, called the operational matrix. The elements of $\underline{f}_o(t)$ are orthogonal on the observation interval $t_o \rightarrow t_f$. Clearly, P depends on the particular choice of the basis vector $\underline{f}_o(t)$.

Since the method was used for optimal control synthesis using Walsh functions [Chen, C. 75], several such basis functions have been in use in solving the continuous system identification problem. Walsh functions [Rao 75] and [Bohn 82], block pulse

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[3]Similar to a Poisson filter chain with $\lambda=0$.

functions [Sannuti 77] and [Chen. W. 87], Laguerre polynomials [Hwang 82], Legen-dre polynomials [Paraskev. 85], generalized orthogonal polynomials [Wang 87a] and Fourier functions [Chung 87] are examples of orthogonal functions used for linear time-invariant system identification. Because of their properties under multiplica-tion, some orthogonal functions have been used in identification of time-varying systems [Tzafestas 77] and [Rangana. 87], and in identification of bilinear systems [Karanam 78] and [Wang 87b]. For example, a set of $N = 2^m$ Walsh functions of sequencies $0, \ldots, N - 1$ form a closed set under multiplication. The product of any two functions in the set results in a third function in the same set. This prop-erty allows simple representation of the products input-output signals and also the products of input-output signals with time functions that correspond to the time varying parameters.

Among these sets of orthogonal functions, Walsh functions have received much of the attention, which is apparent from the relatively large number of publica-tions on the use of Walsh functions [Rao 83]. The reason for this seems to be the simplicity of the Walsh transform and of integration using the Walsh operational matrix. It has been shown, [Bohn 82], that integration using the Walsh operational matrix can be reduced to binary shifts and addition of data samples. However, nu-merical errors accumulate because of multiple integrations while truncating the operational matrix. To reduce these errors, one-shot operational matrices for in-tegration were introduced [Rao 81]. Another drawback of the method is that the unknown initial conditions are introduced into the estimation problem as a result of multiple integrations. These additional unknowns have to be estimated together with the system parameters which may significantly increase the dimension of the estimation problem.

1.3 Nonparametric Identification

Although nonparametric system identification has lost popularity because of the current emphasis on control synthesis and design tools which hinge upon parametric system models, it has gained some ground in recent years because of the growing interest in long-range predictive control, [Richalet 78], [Rouhani 82], [De Keyser 85],

[Bars 85] and [Clarke 87], which mostly utilizes estimates of the system impulse or step response.

1.3.1 Frequency Response

Obtaining the frequency response of the system through sinusoidal input testing is the most robust of all system identification methods. However, the long test time, usually multiples of the response settling time, required to obtain each relevant point on the frequency response curve is the reason why digital spectral analysis using the discrete Fourier transform now replaces the conventional sinusoidal input testing.

To obtain the frequency response of the system, the input signal $u(t)$ and the output signal $y(t)$ are uniformly sampled to obtain $u_s(k)$ and $y_s(k)$. Then, the sampled signals are Fourier transformed to obtain the spectral components $U(k\omega)$ and $Y(k\omega)$ where ω is the fundamental frequency over the data window. Samples of the frequency response are obtained as the ratio of the corresponding input and output spectral components

$$H(k\omega) = \frac{Y(k\omega)}{U(k\omega)} \qquad (1.16)$$

or alternatively as the ratio of the input-output cross-correlation and the input auto-correlation

$$H(k\omega) = \frac{Y(k\omega)U^*(k\omega)}{U(k\omega)U^*(k\omega)} \qquad (1.17)$$

where the astrisk denotes the complex conjugate.

Unless y is the periodic response to u, the frequency response estimates obtained from (1.16) and (1.17) are subject to systematic errors, namely windowing errors and variance errors [Wellstead 81]. One way to reduce these errors is to average the input-output Fourier coefficients over a large number of independent data windows. However, sampling of the input-output signals could give rise to aliasing problems especially if the signals are wideband.

Recently, [Douce 86], a method was suggested to apply a periodic waveform to the system for a time of a few periods and then remove it. It is known that if the system starts from rest, then the transient component of the response due to the input turn-on is equal and opposite to the transient component due to the

input turn-off. The Fourier analysis of the two transient components yields results which sum to zero, and therefore gives correct estimates of the periodic response. However, this method requires response measurements over a relatively long time interval, one equal to the input waveform period plus the response settling time. Advantageous use of this method seems possible only in the case of highly resonant systems where applying a periodic waveform to the system for a long time results in wild system response.

Different methods have been devised since the late fifties [Levy 59] to obtain a parametric model of the system from its frequency response estimates by fitting those estimates to a complex transfer function. A comparison of five related linear least squares methods for transfer function synthesis from frequency response data is given in a recent publication [Whitfield 86].

1.3.2 Impulse Response

The possible damage inflicted by direct use of an impulsive test signal on the system hardware and the presence of output noise have led to the decline in its use. The currently accepted approach is to obtain the impulse response of the system by a process of deconvolution. Deconvolution is the inverse of the convolution of the system output, or the input-output cross-correlation, with the system input, or the input auto-correlation, respectively

$$y(t) = h(t) * u(t) \tag{1.18}$$

$$R_{uy}(\tau) = h(\tau) * R_{uu}(\tau) \tag{1.19}$$

where h is the system impulse response and where R_{uy} and R_{uu} are, respectively, the input-output cross-correlation and the input auto-correlation.

Theoretically, white noise is an optimum test input for estimating the system impulse response because its auto-correlation is an impulse. Thus, R_{uy} is itself an estimate of the impulse response. Unfortunately, white noise is difficult to realize in real life. The simplicity of generating a Pseudo Random Binary Sequence (PRBS), beside having an auto-correlation that approximates an impulse over one period of the sequence, gave a boost to its use in nonparametric system identification.

However, the auto-correlation of a PRBS is periodic, with finite width triangular pulses and nonzero value between pulses which make it *just an approximation* of an impulse. These three differences from the ideal impulse lead to aliasing problems, smearing and bias in the estimated impulse response [Wellstead 81].

1.4 Scope of the Thesis

The thesis considers the problem of identifying a linear continuous-time time-invariant model of a single-input single-output dynamical system from the available measurement taken from the system input and output signals. A deterministic model is used to describe the dynamics of the system. No doubt a stochastic system model that includes a noise model to account for the effect of unknown disturbances on the system could be useful, especially if a priori knowledge of the noise model is available. However, considering a stochastic model makes the identification problem more difficult especially in the continuous-time case. To account for the effect of unknown system disturbances, the method of instrumental variables, which utilizes minimum a priori knowledge of the noise model, is used.

A new approach to the identification problem is considered based on the definition of continuous-time functions, given the name *modal functions*, defined in terms of the system output, the system output derivatives and the system state. The modal functions are defined in chapter 2 and their properties are studied. Associated with the definitions of the modal functions are the modal parameters. Estimating the modal parameters from the available input-output measurements is considered in chapter 3 with numerical examples. The method of instrumental variables is used in simple cases. The relations between the modal functions and the modal parameters with the desired system model parameters are given in chapter 4 in theorem form and then used to identify the model parameters. Use of the modal functions and the modal parameters in nonparametric system identification, namely frequency response, step response and impulse response, is considered in chapter 5. Chapter 6 describes a design procedure to obtain a minimum-time observer for the continuous system state based on the use of the state modal functions and parameters. Finally chapter 7 concludes by summarizing the research results

as well as suggesting some points that are not covered by the thesis and that could be suitable for future research.

The thesis deals mainly with developing and proving theoretical algebraic relations between modal functions, modal parameters, input-output data, and the desired unknowns of the system models. These relations are in the standard linear-in-the-parameters estimation equations form. Consequently, even though some illustrative numerical examples are given, the thesis does not deal extensively with the use of well established numerical techniques for solving linear estimation equations and the known methods for reducing parameter bias due to noise. The thesis places emphasis on showing how the modal functions approach offers new options in parameter estimations and in state observer design.

Chapter 2

Definitions and Properties of Modal Functions

2.1 Introduction

The goal is to identify parametric and nonparametric models of the system; the tool is the modal functions and the modal parameters which will be defined in this chapter. Section 2.2 provides the mathematical background needed in subsequent sections. The modal functions and the modal parameters are defined in section 2.3 and the properties of the modal functions are studied in section 2.4.

2.2 Mathematical Background

It is well known that the general solution of the state equation (1.2) is given by [O'Reilly 83, page 3]

$$\underline{x}(t) = \phi(t)\underline{x}(0) + \int_0^t \phi(\tau)\underline{b}u(t - \tau)d\tau \qquad (2.1)$$

where ϕ is the transition matrix of (1.2) given by

$$\phi(t) = e^{At} \qquad (2.2)$$

The following properties of ϕ will be used in subsequent analysis:

$$
\begin{aligned}
\phi(t_1 + t_2) &= \phi(t_1)\phi(t_2) \\
\phi(-t) &= \phi^{-1}(t) \\
\phi(0) &= I \\
\mathcal{D}\phi(t) &= A\phi(t)
\end{aligned}
\qquad (2.3)
$$

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The vector $\underline{x}(0)$ in (2.1) is the system state vector at time $t = 0$. Similar to the initial conditions $y(0)$, $\mathcal{D}y(0), \ldots, \mathcal{D}^{n-1}y(0)$ required to obtain the solution of the differential equation (1.1), the elements of $\underline{x}(0)$ are necessary to obtain the solution of the equivalent state equation (1.2). The terms *state* and *state variables* are used to refer to $\underline{x}$ and the elements of $\underline{x}$ respectively.

The fact that the input-output relation is invariant under a linear transformation of the state variables given by

$$
\begin{aligned}
\underline{\bar{x}}(t) &= T^{-1}\underline{x}(t) \\
\bar{A} &= T^{-1}AT \\
\underline{\bar{b}} &= T^{-1}\underline{b} \\
\underline{\bar{c}}^T &= \underline{c}^T T
\end{aligned}
\tag{2.4}
$$

implies that the parameters of A, $\underline{b}$ and $\underline{c}^T$ cannot all be determined from the input-output measurements . To be able to obtain unique solutions to state model identification problems, it is necessary to find models which contain the smallest number of parameters, i.e. canonical forms. Two canonical forms that will be used in the thesis are *the Jordan canonical form* and *the observable canonical form.*

(1) *Jordan canonical form:*
The matrix A can be transformed to Jordan canonical form A^* by the transformation

$$
A^* = M^{-1}AM
\tag{2.5}
$$

where M is the *modal matrix* [Sinha,P.K. 84]. The matrix A^* is block diagonal, or *diagonal* for short, with diagonal blocks of the form

$$
A_k = \begin{bmatrix} s_k & 1 & & O \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ O & & & s_k \end{bmatrix} \quad , k = 1, \ldots, n^*
\tag{2.6}
$$

where n^* is the number of distinct eigenvalues of A and the dimension of A_k is equal to the multiplicity of s_k. The vector $\underline{x}^* = M^{-1}\underline{x}$ will be refered to by *the vector of system modes*. Usually, the term *modes* is used to refer to the eigenvectors of a matrix. For the purpose of convenience only, the term *modes* will be used to refer to the state variables of $\underline{x}^*$.

(2) *Observable canonical form:*

If the pair $(A, \underline{c})$ satisfies the observability condition [O'Reilly 83, page 9], a non-singular matrix $\tilde{M}$ can be found that transforms the matrix A into phase variable form[1] given as

$$\tilde{A} = \tilde{M}^{-1}A\tilde{M} = \begin{bmatrix} 0 & 1 & & O \\ 0 & & \ddots & \\ \vdots & & & 1 \\ -a_o & \cdots & \cdots & -a_{n-1} \end{bmatrix} \tag{2.7}$$

such that

$$\tilde{\underline{c}}^T = \underline{c}^T\tilde{M} = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \tag{2.8}$$

The elements of the bottom row of $\tilde{A}$ are the coefficients of the characteristic equation of A given by

$$A^n + \sum_{j=0}^{n-1} a_j A^j = O \tag{2.9}$$

where O is the zero matrix. The corresponding arrays $\tilde{A}$, $\tilde{\underline{x}}$, $\tilde{\underline{b}}$ and $\tilde{\phi}$ are defined by relations similar to (2.4) using the substitution $T = \tilde{M}$.

The state model in the phase variable form has a very close relation to the input-output model (1.1). The elements of the bottom row of A are the parameters $a_o, \ldots, a_{n-1}$ of (1.1). The phase variables, i.e. the elements of $\tilde{\underline{x}}$, are defined in terms of the output y and its derivatives. It can be shown by differentiating (2.1) j times that

$$\mathcal{D}^j y(t) = \underline{c}^T \mathcal{D}^j \underline{x}(t) = \underline{c}^T A^j \underline{x}(t) + \sum_{i=1}^{j} \underline{c}^T A^{i-1}\underline{b}\mathcal{D}^{j-i}u(t) \quad , j > 0. \tag{2.10}$$

Equation (2.10) is invariant under a transformation of the state variables. Then it follows that substituting for A, $\underline{b}$, $\underline{c}$ and $\underline{x}$ by $\tilde{A}$, $\tilde{\underline{b}}$, $\tilde{\underline{c}}$ and $\tilde{\underline{x}}$, respectively, and using (2.7) and (2.8) gives

$$\begin{align} y(t) &= \tilde{x}_1(t) \\ \mathcal{D}^j y(t) &= \tilde{x}_{j+1}(t) + \sum_{i=1}^{j} \tilde{b}_i \mathcal{D}^{j-i}u(t) \quad , 0 < j < n \\ \mathcal{D}^n y(t) &= \underline{a}^T \tilde{\underline{x}}(t) + \sum_{i=1}^{n} \tilde{b}_i \mathcal{D}^{n-i}u(t). \end{align} \tag{2.11}$$

---

[1]Also known as companion form.

Finally, the relation between the elements of vector $\tilde{\underline{b}}$ and $b_o, \ldots, b_{n-1}$ of the input-output model follows by substituting into (1.1) from (2.10) and using (2.9) as

$$b_j = \sum_{k=j+1}^{n} a_k \tilde{b}_{k-j} \quad j = 0, \ldots, n-1, a_n = 1. \tag{2.12}$$

2.3 Definitions of Modal Functions and Modal Parameters

Let f_α be a scalar function defined as a weighted sum of the state $\underline{x}$ at the distinct time instants $t, t - T_1, \ldots, t - T_n$

$$f_\alpha(t) = \underline{\alpha}^T \underline{x}(t) + \sum_{i=1}^{n} p_{\alpha i} \ \underline{c}^T \underline{x}(t - T_i) \tag{2.13}$$

where $\underline{\alpha}$ is a vector of dimension n, and $p_{\alpha 1}, \ldots, p_{\alpha n}$ are constants chosen such that f_α is independent of the instantaneous value of the state $\underline{x}$. From (2.1) it can be shown that

$$\underline{x}(t - T_i) = \phi(-T_i)\underline{x}(t) + \int_{t}^{t-T_i} \phi(t - T_i - \tau)\underline{b}u(\tau)d\tau. \tag{2.14}$$

Substituting for $\underline{x}$ at different time instants from (2.14) into (2.13) gives, after arranging terms

$$f_\alpha(t) = [\underline{\alpha}^T + \sum_{i=1}^{n} p_{\alpha i} \ \underline{c}^T \phi(-T_i)]\underline{x}(t) + \sum_{i=1}^{n} p_{\alpha i} \int_{t}^{t-T_i} \underline{c}^T \phi(t - T_i - \tau)\underline{b}u(\tau)d\tau \tag{2.15}$$

The condition that makes f_α independent of $\underline{x}(t)$ for all time t is

$$\underline{\alpha}^T + \sum_{i=1}^{n} p_{\alpha i} \ \underline{c}^T \phi(-T_i) = \underline{o}^T \tag{2.16}$$

where $\underline{o}^T$ is a vector of zeros. For this condition f_α is given by

$$f_\alpha(t) = \sum_{i=1}^{n} p_{\alpha i} \int_{t}^{t-T_i} \underline{c}^T \phi(t - T_i - \tau)\underline{b}u(\tau)d\tau \tag{2.17}$$

Let

$$\underline{p}_\alpha = \left[\begin{array}{ccc} p_{\alpha 1} & \cdots & p_{\alpha n} \end{array} \right]^T. \tag{2.18}$$

such that (2.16) becomes

$$\underline{\alpha}^T + \underline{p}_\alpha^T \left[\begin{array}{c} \underline{c}^T \phi(-T_1) \\ \vdots \\ \underline{c}^T \phi(-T_n) \end{array} \right] = \underline{o}^T. \tag{2.19}$$

A necessary condition for the existence of a vector $\underline{p}_\alpha$ of dimension n that satisfies (2.19) is given by the following theorem [El-Shafey 87].

Theorem 2.1 :

A vector $\underline{p}_\alpha$, of dimension n, that satisfies (2.19) exists only for observable systems.

Proof:

The existence of $\underline{p}_\alpha$ depends on the invertibility of matrix ψ given by

$$\psi = \begin{bmatrix} \underline{c}^T \phi(-T_1) \\ \vdots \\ \underline{c}^T \phi(-T_n) \end{bmatrix}. \tag{2.20}$$

In order to study the invertability of ψ, the diagonal form of ϕ is used. Substituting for ϕ by $M\phi^* M^{-1}$ and for $\underline{c}^T$ by $\underline{c}^{*T} M^{-1}$ gives

$$\psi = \begin{bmatrix} \underline{c}^{*T} \phi^*(-T_1)M^{-1} \\ \vdots \\ \underline{c}^{*T} \phi^*(-T_n)M^{-1} \end{bmatrix}. \tag{2.21}$$

From matrix multiplication rules, ψ can be put as the product of three matrices

$$\psi = V D(\underline{c}^*)M^{-1} \tag{2.22}$$

where $D(\underline{c}^*)$ is a block diagonal matrix with n^* diagonal blocks; each diagonal block corresponds to a distinct eigenvalue of A and is given by

$$C_k = \begin{bmatrix} c_{k1} & \cdots & c_{kn_k} \\ & \ddots & \vdots \\ O & & c_{k1} \end{bmatrix} \tag{2.23}$$

where n_k is the multiplicity of the corresponding eigenvalue. The matrix V is composed of arrays V_k of dimension $n_k \mathrm{x} n$, $k=1,\ldots,n^*$ given by

$$V_k = \begin{bmatrix} \left(e^{s_k T_1} \ldots e^{s_k T_n} \right) \\ \frac{\partial}{\partial s_k}\left(e^{s_k T_1} \ldots e^{s_k T_n} \right) \\ \vdots \\ \frac{\partial^{n_k-1}}{\partial s_k^{n_k-1}}\left(e^{s_k T_1} \ldots e^{s_k T_n} \right) \end{bmatrix} \tag{2.24}$$

stacked vertically on top of each other.

It is clear from (2.22) that a necessary condition for the invertibility of ψ is that $D(\underline{c}^*)$ is invertible which means that C_k, $k=1,\ldots,n^*$ are invertible. This condition is

exactly the necessary and sufficient condition for the observability of the system in Jordan canonical form [Sinha.P.K. 84, page 173]. The matrix V is of Vandermonde type with known determinant expansion with uniform $T_1, \ldots, T_n$. This completes the proof.

It follows from theorem 2.1 that if the system is observable and ψ is invertible then there exists a vector $\underline{p}_\alpha$ of dimension n given by

$$\underline{p}_\alpha^T = -\underline{\alpha}^T \psi^{-1} = -\underline{\alpha}^T M D^{-1}(\underline{c}^*) V^{-1} \tag{2.25}$$

whose elements satisfy (2.16). Different choices of $\underline{\alpha}$ lead to different f_α and $\underline{p}_\alpha$. Particular choices of $\underline{\alpha}$ are of interest as for example $\underline{\alpha}^T = \underline{c}^T = \underline{c}^{*T} M^{-1}$ which leads to the following definition.

Definition 2.1 :

For an observable system of order n, the output modal function y_o is defined as:

$$y_o(t) = y(t) + \sum_{i=1}^{n} p_{oi} y(t - T_i) \tag{2.26}$$

where $p_{o1}, \ldots, p_{on}$ are the output modal parameters given according to (2.25) by

$$\underline{p}_o^T = -\underline{c}^T \psi^{-1} = -\underline{c}^{*T} D^{-1}(\underline{c}^*) V^{-1}. \tag{2.27}$$

The name *modal time shifts* will be given to $T_1, \ldots, T_n$. In later chapters it will be shown how the output modal function can be used in parametric and nonparametric system identification.

Another choice of $\underline{\alpha}^T$ is $\underline{c}^T A^j$, $0 \le j < n$. It is known that $\underline{c}^T A^j \underline{x}$ is invariant under a linear transformation of the state $\underline{x}$. For example, $\underline{c}^T A^j \underline{x} = \underline{c}^{*T} A^{*j} \underline{x}^* = \tilde{\underline{c}}^T \tilde{A} \tilde{\underline{x}}$. It can easily be verified that

$$\tilde{\underline{c}}^T \tilde{A}^j = \underline{\imath}_{j+1}^T \quad , 0 \le j < n \tag{2.28}$$

where $\underline{\imath}_{j+1}^T$ is the $(j+1)^{th}$ row in the unity matrix. As a consequence of (2.28), $\tilde{\underline{c}}^T \tilde{A}^j \tilde{\underline{x}} = \underline{\imath}_{j+1}^T \tilde{\underline{x}} = \tilde{x}_{j+1}$ the $(j+1)^{th}$ state variable in $\tilde{\underline{x}}$. This leads to the following definition

Definition 2.2 :

For an observable system of order n the state modal function $\tilde{y}_j$, $0 \le j < n$ is

defined as

$$\tilde{y}_j(t) = \tilde{x}_{j+1}(t) + \sum_{i=1}^{n} p_{ji} y(t - T_i) \quad , 0 \leq j < n \tag{2.29}$$

where $p_{j1}, \ldots, p_{jn}$ *are the state modal parameters given according to (2.25) by*

$$\underline{p}_j^T = -\underline{c}^T A^j \psi^{-1} = -\underline{c}^{*T} A^{*j} D^{-1} (\underline{c}^*) V^{-1} \quad , 0 \leq j < n \tag{2.30}$$

It is seen that the state modal function $\tilde{y}_o$ is the same as the output modal function given by definition 2.1. The state modal parameters will play a key role in parametric system identification and in the design of state observers as will be seen in later chapters. Other modal functions of use in system analysis are given in the following definition

Definition 2.3 :

For an observable system of order n, the output derivative modal function is defined as

$$y_j(t) = \mathcal{D}^j y(t) + \sum_{i=1}^{n} p_{ji} y(t - T_i) \quad , 0 < j \tag{2.31}$$

where $p_{j1}, \ldots, p_{jn}$ *are the same state modal parameters given by (2.30).*

The name output derivative modal parameters will be used as well to refer to the state modal parameters $p_{j1}, \ldots, p_{jn}$.

There is a direct relation between a state modal function $\tilde{y}_j$ and the corresponding output derivative modal function y_j for $0 < j < n$. Subtracting (2.29) from (2.31) gives

$$y_j(t) - \tilde{y}_j(t) = \mathcal{D}^j y(t) - \tilde{x}_{j+1}(t) \quad , 0 < j < n. \tag{2.32}$$

From (2.11), relating output derivatives to state variable in $\tilde{\underline{x}}$, and (2.32) it follows that

$$y_j(t) - \tilde{y}_j(t) = \sum_{i=1}^{j} \tilde{b}_i \mathcal{D}^{j-i} u(t) \quad , 0 < j < n. \tag{2.33}$$

Equation (2.33) will prove useful in estimating the vector $\tilde{\underline{b}}$.

2.4 Properties of Modal Functions

After defining different modal functions it is suitable to investigate some of their properties. Equation (2.17) is a general formula for defining different modal func-

tions given by definitions 2.1 or 2.2. The output modal function or the state modal function is obtained by substituting for $p_{\alpha 1}, \ldots, p_{\alpha n}$ by the corresponding modal parameters given by (2.30). Using the substitution of variables $\sigma = t - T_i - \tau$ in the integration in (2.17) and interchanging the limits of integration yields

$$f_\alpha(t) = \sum_{i=1}^{n} p_{\alpha i} \; \underline{c}^T \int_0^{-T_i} \phi(\sigma) \underline{b} u(t - T_i - \sigma) d\sigma. \qquad (2.34)$$

Comparing (2.1) with (2.34), it is seen that the system general response is composed of two parts; one which depends on the system initial state, and the other which is the convolution integral of the input and the system impulse response. Conversely, a modal function is defined to be independent of the state, also it is clear that the integrals in (2.34) are different from the standard convolution integral. These are the reasons that make a modal function possesses properties which are different from the general system response.

Four properties of modal functions in connection with input superposition, scaling, time shifting and time differentiation are given below.

Let the pair (f_α, u) satisfy (2.34). The following properties are obvious from (2.34).

1. *Superposition:* If the pair $(\Delta f_\alpha, \Delta u)$ satisfies (2.34), then $(f_\alpha + \Delta f_\alpha, u + \Delta u)$ also satisfies (2.34).

2. *Scaling:* The pair $(a f_\alpha, a u)$ satisfies (2.34) where a is a scaling factor.

3. *Time shifting:* The pair $(f_\alpha(t + \Delta t), u(t + \Delta t))$ satisfies (2.34) where Δt is a time shift.

4. *Time differentiation:* The pair $(\mathcal{D} f_\alpha, \mathcal{D} u)$ satisfies (2.34).

It is important to note that the state vector, and consequently any linear function of the state variables, e.g. the output, do not possess any of the above properties. The following analysis shows the relation between the modal functions and both the system modes and the input modes.

Let u be the free response of a continuous controller of order n_c given by

$$\mathcal{D} \underline{x}_c(t) = A_c \underline{x}_c(t) \qquad (2.35)$$

$$u_c(t) = \underline{c}_c^T \underline{x}_c(t). \tag{2.36}$$

Substituting for u in (2.34) from (2.36), noticing that $\underline{x}_c(t - T_i) = \phi_c(-T_i)\underline{x}_c(t)$ where ϕ_c is the transition matrix of the controller, yields

$$f_\alpha(t) = \sum_{i=1}^{n} p_{\alpha i} \; \underline{c}^T \int_0^{-T_i} \phi(\sigma)\underline{b} \; \underline{c}_c^T \phi_c(-T_i - \sigma) d\sigma \underline{x}_c(t). \tag{2.37}$$

Let vector $\underline{q}_\alpha$ be

$$\underline{q}_\alpha^T = \sum_{i=1}^{n} p_{\alpha i} \; \underline{c}^T \int_0^{-T_i} \phi(\sigma)\underline{b} \; \underline{c}_c^T \phi_c(-T_i - \sigma) d\sigma \tag{2.38}$$

Note that $\underline{q}_\alpha$ is of dimension n_c as the number of modes in the input. Substituting from (2.38) into (2.37) yields

$$f_\alpha(t) = \underline{q}_\alpha^T \underline{x}_c(t) \tag{2.39}$$

The important conclusion that is drawn from the last equation is that a modal function is just a linear function of the modes of the input, and it contains none of the modes of the system.

Equation (2.39) gives a general modal function for arbitrary $\underline{\alpha}$ in terms of the input modes. It follows from (2.39) and the definitions of the output modal function and the state modal functions (2.29) that

$$\tilde{y}_j(t) \equiv \tilde{x}_{j+1}(t) + \sum_{i=1}^{n} p_{ji} y(t - T_i) = \underline{\tilde{q}}_j \underline{x}_c(t) \qquad , 0 \le j < n \tag{2.40}$$

where $\underline{\tilde{q}}_j$ is given according to (2.38) by

$$\underline{\tilde{q}}_j^T = \sum_{i=1}^{n} p_{ji} \int_0^{-T_i} \underline{c}^T \phi(\sigma)\underline{b} \; \underline{c}_c^T \phi_c(-T_i - \sigma) d\sigma \qquad , 0 \le j < n \tag{2.41}$$

Similarly, an equation like (2.40) can be obtained for the output derivative modal functions. From (2.35) and (2.36), by differentiating j times, it follows that

$$\mathcal{D}^j u(t) = \underline{c}_c^T A_c^j \underline{x}_c(t). \tag{2.42}$$

Substituting from (2.42) into (2.11) yields

$$\mathcal{D}^j y(t) = \tilde{x}_{j+1}(t) + \sum_{i=1}^{j} \tilde{b}_i \underline{c}_c^T A^{j-i} \underline{x}_c(t) \qquad , 0 < j < n. \tag{2.43}$$

Then from (2.40) and (2.43) it follows that

$$y_j(t) \equiv \mathcal{D}^j y(t) + \sum_{i=1}^{n} p_{ji} y(t - T_i) = \underline{q}_j \underline{x}_c(t) \quad ,0 \le j < n \qquad (2.44)$$

where $\underline{q}_j$ is given by

$$\underline{q}_j^T = \left\{ \begin{array}{ll} \underline{\tilde{q}}_j^T + \sum_{i=1}^{j} \tilde{b}_i \underline{c}_c^T A_c^{j-i} & ,0 < j < n \\ \underline{\tilde{q}}_j^T & ,j = 0 \end{array} \right. \qquad (2.45)$$

Equations (2.40) and (2.44) have a continuous time auto-regressive moving average (ARMA) structure. Unlike the well known discrete-time ARMA modal, which describes the system behavior only at the sampling instants, (2.40) and (2.44) are true for continuous time t. Uniform intersample spacing is a characteristic of the discrete-time ARMA model, while as seen from (2.40) and (2.44) $T_1, T_2 - T_1, \ldots, T_n - T_{n-1}$ are not necessarily equal. The intermix between the state variables and the output in (2.40) gives it the form of a state observer which gives the system state as a function of input-output measurements. The following numerical example gives the modal parameter vectors from a known system dynamic matrix.

Example 2.1 :

Consider a third order system for which the dynamic matrix in *diagonal* form and the vector $\underline{x}^{*T}$ are given by

$$A^* = \left[\begin{array}{ccc} -1 + \jmath 5 & 0 & 0 \\ 0 & -1 - \jmath 5 & 0 \\ 0 & 0 & -2 \end{array} \right]$$

and

$$\underline{c}^{*T} = \left[\begin{array}{ccc} 1 & 1 & 1 \end{array} \right]$$

Obviously, the system is observable. The poles of the system are the diagonal elements of A^*. It is desired to obtain the modal parameters of the output and of its first, second and third derivatives for the modal time shifts

$$T_1 = 40T_s$$
$$T_2 = 80T_s$$
$$T_3 = 120T_s$$

where $T_s = \pi/420$. The results of computing $\underline{p}_j$, $j=0,\ldots,3$, according to (2.30) are listed in table 2.1.

j	0	1	2	3
$\underline{p}_j$	$-0.6605E+00$ $+0.6106E+00$ $-0.3022E+00$	$+0.2999E+01$ $-0.2564E+00$ $-0.4330E+00$	$+0.5715E+01$ $-0.1476E+02$ $+0.5721E+01$	$-0.7848E+02$ $+0.3497E+02$ $+0.5819E+01$

Table 2.1: Modal parameter vectors

2.5 Summary

The main results for this chapter are the continuous-time ARMA models (2.40) and (2.44) relating the state and the output derivatives to input-output data. From a practical stand point it is seen that a choice must be made for the system order n and the modal time shifts.

The choice of model order is a problem common to all parametric methods whether discrete or continuous. One approach, commonly used when estimating the model parameters, is to increase the model order until there is no significant decrease in the discrepancy between the real system and the estimated model data. A second approach is to over-parametrize and then eliminate relatively small parameters.

The obtained continuous-time ARMA models are valid for nonuniform time shifts, and this allows greater flexibility in choosing T_i. Such flexibility is not possible in the methods that are based on the standard discrete ARMA models such as Prony's method. The flexibility in choosing the time shifts allows for determining both the fast and the slow modes in the system response.

Chapter 3

Estimation of The Modal Parameters

3.1   Introduction

In chapter 2 the modal functions of the system output, the output derivatives and the state variables were defined and their properties were studied. Associated with each modal function is a modal parameter vector whose dimension is equal to the order of the assumed observable model of the system. So far the modal parameters were obtained through equations (2.27) and (2.30) which assume the knowledge of the system modes. This chapter deals with the problem of estimating the modal parameters assuming a linear time-invariant observable system of order n and the availability of input-output measurements only.

The estimation problem is formulated in section 3.2. Section 3.3 presents with numerical examples the estimation of the output modal parameters. Estimating the output derivative modal parameters introduces the problem of unavailable output derivatives so it is presented separately in section 3.4. The problem of increased dimension of the unknown parameter vector when multi-mode inputs are used is discussed in section 3.5 with a suggestion of a solution using a measurement prefilter to partially annihilate the input modes.

3.2   Problem Formulation

From the definition of modal functions in chapter 2 we arrived at equation (2.44) which has a continuous-time ARMA structure. It is desired to obtain estimates of the parameter vectors $\underline{p}_j$ and $\underline{q}_j$ for $j \geq 0$ from the available input-output

measurements. The approach to the problem is to minimize some scalar function of the equation error which is given according to (2.44) by

$$e_j(t) = \mathcal{D}^j y(t) + \sum_{i=1}^{n} p_{ji} y(t - T_i) - \sum_{i=1}^{n_c} q_{ji} x_{ci}(t). \qquad (3.1)$$

where q_{ji} and x_{ci} are the elements of $\underline{q}_j$ and $\underline{x}_c$ respectively. Let $\underline{\theta}_j$ be the vector of unknown parameters and $\underline{r}(t)$ be the vector of measurements given respectively by

$$\underline{\theta}_j = \begin{bmatrix} \underline{p}_j \\ \underline{q}_j \end{bmatrix} \qquad (3.2)$$

and

$$\underline{r}^T(t) = [y(t - T_1) \cdots y(t - T_n) \quad - x_{c1}(t) \cdots - x_{cn_c}(t)]. \qquad (3.3)$$

Hence, the equation error in (3.1) is given in terms of $\underline{\theta}_j$ and $\underline{r}$ by

$$e_j(t) = \mathcal{D}^j y(t) + \underline{r}^T(t)\, \underline{\theta}_j. \qquad (3.4)$$

An estimate $\hat{\underline{\theta}}_j$ of the unknown vector $\underline{\theta}_j$ is sought such that a scalar loss function J_j defined as some norm of the equation error e_j over the observation interval $t_o \rightarrow t_f$ is minimized. The loss function J_j reflects the discrepancy between the continuous ARMA model (2.44) and the real system so that the estimate $\hat{\underline{\theta}}_j$ that minimizes J_j can be used to obtain estimates of the output derivatives $\mathcal{D}^j y$ as

$$\widehat{\mathcal{D}^j y(t)} = -\underline{r}^T(t)\, \hat{\underline{\theta}}_j. \qquad (3.5)$$

The most common cost function of the continuous-time error e_j is based on the integral of a weighted norm in e_j. However, a cost function that is formulated as the sum of squares of samples of e_j at discrete time instants is possible. In the following two sections estimates of the unknown parameters are obtained by minimizing cost functions which are formulated in terms of discrete-time and continuous-time measurements, respectively.

3.2.1 Estimation from Discrete Measurements

In the case of discrete measurements , let $N \geq n + n_c$ be the number of samples obtained over the observation interval at the discrete time instants $t_k, \; k = 1, \ldots, N$.

The least squares estimates of the unknown parameters are obtained by minimizing the sum of squares of samples of the equation error at t_k, $k = 1, \ldots, N$, that is, it is required to minimize

$$J_{jd} = \sum_{k=1}^{N} e_j^2(t_k) = \underline{e}_{jd}^T \, \underline{e}_{jd}. \tag{3.6}$$

In terms of the measurements , J_{jd} is given by

$$J_{jd} = (\underline{z}_{jd} + R_d \, \underline{\theta}_j)^T (\underline{z}_{jd} + R_d \, \underline{\theta}_j) \tag{3.7}$$

where

$$\underline{z}_{jd} = \left[\begin{array}{ccc} \mathcal{D}^j y(t_1) & \cdots & \mathcal{D}^j y(t_N) \end{array} \right]^T \tag{3.8}$$

and R_d has rows $\underline{i}_k^T R_d$, $k = 1, \ldots, N$, given by

$$\underline{i}_k^T R_d = \underline{r}^T(t_k). \tag{3.9}$$

It is known that the estimate $\hat{\underline{\theta}}_{jd}$ of the unknown parameter vector $\underline{\theta}_j$ that minimizes J_{jd} is given by

$$\hat{\underline{\theta}}_{jd} = -(R_d^T R_d)^{-1} R_d^T \, \underline{z}_{jd}. \tag{3.10}$$

Obtaining estimates of the unknown parameter vector from (3.10) seems possible only for the case of $j = 0$. This is because although R_d is available by direct measurements , samples of the output derivatives in $\underline{z}_{jd}$, $j > 0$, are not available.

3.2.2   Estimation from Continuous Measurements

Because the system is continuous and can be sampled, at least theoretically, at any frequency, a large amount of data could result from the sampling process. Therefore, some form of data compression is needed. Correlating the measured signal with a function f_k over the observation interval results in a single number. Correlating the measured signal with a vector $\underline{f}$ of N functions results in the concentration of the information contained in the collected data into N components, where each component represents a best fit to the measured signal in a least squares sense. Fourier functions and Walsh functions are two suitable candidates for this purpose because each of them forms a set of orthogonal functions.

From the available continuous measurements the loss function is formulated as

$$J_{jf} = \sum_{k=1}^{N} e_{jf_k}^2 = \underline{e}_{jf}^T \, \underline{e}_{jf} \tag{3.11}$$

where $\underline{e}_{jf}$ is the correlation between the equation error e_j and the vector of functions $\underline{f}$ over the observation interval

$$\underline{e}_{jf} = \int_{t_o}^{t_f} \underline{f}(t)e_j(t)dt. \tag{3.12}$$

In terms of the available measurements J_{jf} is given by

$$J_{jf} = (\underline{z}_{jf} + R_f\underline{\theta}_j)^T(\underline{z}_{jf} + R_f\underline{\theta}_j) \tag{3.13}$$

where

$$\underline{z}_{jf} = \int_{t_o}^{t_f} \underline{f}(t)\mathcal{D}^j y(t)dt \tag{3.14}$$

and

$$R_f = \int_{t_o}^{t_f} \underline{f}(t)\underline{r}^T(t)dt. \tag{3.15}$$

The estimate $\hat{\underline{\theta}}_{jf}$ of the unknown parameter vector $\underline{\theta}_j$ that minimizes J_{jf} is given by

$$\hat{\underline{\theta}}_{jf} = -(R_f^T R_f)^{-1} R_f^T \underline{z}_{jf}. \tag{3.16}$$

Equation (3.16) seems suitable for obtaining estimates of the unknown parameter vectors for $j > 0$ as well as for $j = 0$ unlike (3.10) in the discrete case which is suitable only for $j = 0$. This is because of the integration in (3.14) that gives $\underline{z}_{jf}$. Later in this chapter it will be shown that an estimate of $\underline{z}_{jf}$ can be obtained either from an available estimate of $\underline{z}_{(j-1)f}$, or from an estimate of $\mathcal{D}^{j-1}y$ obtained from (3.5) using an available estimate of $\underline{\theta}_{j-1}$.

3.3 Estimating The Output Modal Parameters

The output modal parameter vector $\underline{p}_o$ is estimated as a part of $\underline{\theta}_o$. An estimate of $\underline{\theta}_o$ can be obtained either from discrete data, $\hat{\underline{\theta}}_{od}$ in (3.10), or from continuous data, $\hat{\underline{\theta}}_{of}$ in (3.16). By putting $j = 0$ in (3.8) and (3.14) it becomes clear that $\underline{z}_{od}$ and $\underline{z}_{of}$ required in (3.10) and in (3.16) respectively are obtained from the measured output. The matrices R_d and R_f are obtained from the input-output measurements.

3.3.1 Estimates from Discrete Measurements

This is the simplest case because R_d and $\underline{z}_{od}$ are formed from input-output samples. Input-output samples should be collected at a rate which is at least equal to twice the highest frequency in the output or in the input whichever is greater.

Example 3.1 :

Consider a third order system with a transfer function
$$H(s) = \frac{13s + 52}{s^3 + 4s^2 + 30s + 52}$$
The system has three poles at

$$s_1 = -2$$
$$s_2 = -1 + \jmath5$$
$$s_3 = -1 - \jmath5$$

and a zero at

$$s_4 = -4.$$

The system has the same poles as the system considered in example 2.1. It is desired to obtain the output modal parameters for a third order model of the system using discrete input-output measurements . To start with a simple case let the input u contain a single mode, a constant for example. In this case the continuous ARMA model is given according to (2.44) by

$$y(t) + \sum_{i=1}^{3} p_{oi} \; y(t - T_i) = q_{oo}u_o(t)$$

where u_o is the constant input and where q_{oo} is the single parameter which corresponds to the input single mode and which will be estimated together with the output modal parameters.

Assuming a zero initial state of the system the response of the system to a step input of magnitude $+1$ followed by another step input of magnitude -1 is computed each sampling time $T_s = \pi/420$. The system poles are not far apart so it is reasonable to assume uniform modal time shifts ; $T_1 = T_2 - T_1 = T_3 - T_2 = 40T_s$.

The continuous ARMA model in this example is valid only for data windows w_1 : $T_3 \leq t \leq t_t$, where $t_t = 444T_s$ is the transition instant when the input switches from $+1$ to -1, and w_2 : $t_t + T_3 \leq t \leq t_f$. Therefore, the sampling instants t_k, $k = 1, \ldots, N$ should belong to either w_1 or w_2. Twenty estimation equations are formed

for 20 different t_k; 10 in w_1 and 10 in w_2. In each window the spacing $t_k - t_{k-1}$ is uniform and equals $20T_s$. Estimates of the unknown parameters given by (3.10) are listed in table 3.1. The estimated values of the three output modal parameters can be compared with the true values computed for the same modal time shifts listed in table 2.1.

$\hat{\underline{\theta}}_{od}$	$-0.6605\text{E}+00$
	$+0.6106\text{E}+00$
	$-0.3022\text{E}+00$
	$+0.6480\text{E}+00$

Table 3.1: Estimate of $\underline{\theta}_o$ using discrete data.

3.3.2   Estimates from Continuous Measurements

Estimates of the unknown parameters are obtained from continuous data through (3.16) with $j = 0$. As seen from (3.14) and (3.15), the estimation equations are formed by correlating the available input-output measurements with a vector of functions $\underline{f}$ over the observation interval. As mentioned earlier, Fourier functions and the Walsh functions are suitable for this purpose because each forms a set of orthogonal functions.

Let T_o be the length of the observation interval $t_o \to t_f$ and let $t_o = 0$ such that $t_f = t_f - t_o = T_o$. Fourier functions defined over a time period $0 \to T_o$ occur in pairs, $\sin\omega_k t$ and $\cos\omega_k t$ where $\omega_k = k\omega_o$ and $\omega_o = 2\pi/T$ is the fundamental frequency. A vector of $N = 2m$ Fourier functions defined over the time period $0 \to T_o$ is given by

$$\underline{f}_F(t) = \left[\; \sin\omega_o(t) \quad \cdots \quad \sin m\omega_o t \quad \cos\omega_o(t) \quad \cdots \quad \cos m\omega_o t \;\right]. \tag{3.17}$$

On the other hand, the vector of Walsh functions defined over the time period $0 \to T_o$ is given by

$$\underline{f}_W(t) = \left[\; \text{wal}(0, t/T_o) \quad \cdots \quad \text{wal}(N-1, t/T_o) \;\right] \tag{3.18}$$

where wal(k, σ) is a Walsh function of sequency k defined over the interval $0 \leq \sigma \leq 1$ [Tadokoro 78].

The correlation of a signal with a Walsh function requires no multiplication as opposed to the correlation of a signal with a Fourier function. Only addition and subtraction of samples are needed. A Fast Walsh Transform (FWT) [Shanks 69]reduces the computational burden even further. In this case a number $N = 2^m$ functions, where m is integer, should be used. The time interval of length T_o is divided into N equal subintervals and the samples in each subinterval are averaged to obtain a vector of N elements. This vector of averaged samples is transformed into a vector of N Walsh components using the FWT.

Example 3.2 :

Consider the same system as in example 3.1 with the same input-output measurements. The first step of data compression is to concatenate the data windows w_1 and w_2 where the continuous ARMA model in example 3.1 is valid. The second step is to correlate the concatenated input-output measurements with a vector of N orthogonal functions over the period of orthogonality in order to form matrix R_f and vector $\underline{z}_{of}$. Estimates obtained by employing a vector of six Fourier functions, and estimates obtained by employing a vector of eight Walsh functions are listed in table 3.2.

$\hat{\underline{\theta}}_{oF}$ (using Fourier functions)	$\hat{\underline{\theta}}_{oW}$ (using Walsh functions)
$-0.6605\text{E}+00$	$-0.6605\text{E}+00$
$+0.6106\text{E}+00$	$+0.6106\text{E}+00$
$-0.3022\text{E}+00$	$-0.3022\text{E}+00$
$+0.6480\text{E}+00$	$+0.6480\text{E}+00$

Table 3.2: Estimates of $\underline{\theta}_o$ using continuous data.

Both examples 3.1 and 3.2 use a single input with a single mode in order to keep the dimension of the estimation problem low. A constant input does not provide persistent excitation of the system, a characteristic desired to avoid ill-conditioned parameter estimation equations. To provide persistent excitation of the system,

two different levels of constant input were used in the previous examples. Input transition from one level to the other provides excitation to the system modes such that the response of the system after the transition carries the required information about these modes.

Parameter estimation using discrete or continuous input-output measurements, described in examples 3.1 and 3.2 respectively, can be used with a Pseudo Random Binary Sequence (PRBS); a commonly used test input which provides persistent system excitation, such that the resulting parameter estimation equations are likely to be linearly independent. However, the input does not necessarily have to be of binary type; a piece-wise constant input that remains constant at least for time T_n before a new transition can be used in the same way as in examples 3.1 or 3.2. This restriction on the frequency of the input transitions results from the fact that the continuous ARMA model is valid for $t \geq T_n$ with a constant input. Later in this chapter examples with multi-mode inputs are considered and a solution to the problem of increased number of unknowns is suggested.

Both examples 3.1 and 3.2 assume a deterministic environment where measurements are perfect and there is no unknown disturbances affecting the system. In practice, measurements are often contaminated with noise and the system is subject to uncertainty caused by external unknown disturbances. Therefore it is important to investigate the effect of noise on the estimates of the unknown parameters. It is known that the estimates of the parameters of a regression like (2.44) obtained by least squares using noisy measurements will be biased to a degree dependent on the ratio of the covariance of the noise and the covariance of the noise free signal. The following example shows the problem of biased estimates obtained from discrete and from continuous measurements as described in examples 3.1 and 3.2.

Example 3.3 :

A normally distributed noise signal with zero mean and standard deviation 0.1 is added to the output samples used in both examples 3.1 and 3.2. The noisy samples are used in the estimation equations as has been done in the above examples. The results of least squares estimation using the noisy samples are listed in tables 3.3 and 3.4.

$\hat{\underline{\theta}}_{od}$	−0.5832E+00
	+0.5881E+00
	−0.2825E+00
	+0.7105E+00

Table 3.3: Estimate of $\underline{\theta}_o$ using discrete noisy data.

$\hat{\underline{\theta}}_{oF}$ (using Fourier functions)	$\hat{\underline{\theta}}_{oW}$ (using Walsh functions)
−0.6370E+00	−0.6317E+00
+0.6151E+00	+0.6326E+00
−0.3052E+00	−0.3017E+00
+0.6730E+00	+0.7003E+00

Table 3.4: Estimates of $\underline{\theta}_o$ using continuous noisy data.

By comparing table 3.4 with table 3.2 it can be seen that acceptable estimates resulted when continuous measurements were used. This is because correlating the noisy signal with a Fourier or Walsh function works to average out the noise which has zero mean. Good estimates with low bias are likely to occur if the Fourier or the Walsh functions used have high correlation with the noise free output while they have no correlation with noise.

Conversely, the problem of biased estimate is noticeable in the case of discrete measurements as is clear from comparing table 3.3 with table 3.1. Fortunately, the noise induced bias can be removed by incorporating an instrumental variable which is highly correlated with the true value of the signal, but not with errors in observation caused by various types of additive noise.

One possibility of an instrumental variable is a shifted output [Söderström 81]. The amount of shift is chosen small enough so that the shifted output still has high correlation with the unshifted output, while this amount of shift is large enough so that the noise in the shifted output has no correlation with the noise in the unshifted output. The instrumental variable estimate of the unknown parameter vector is given by

$$\hat{\underline{\theta}}_{ov} = -(V^T R_d)^{-1} V^T \underline{z}_{od} \tag{3.19}$$

where V has rows $\underline{\imath}_k^T V$, $k = 1, \ldots, N$, given by

$$\underline{\imath}_k^T V = [y(t_k - T_1 + \tau) \; \cdots y(t_k - T_n + \tau) \; -x_{c1}(t_k) \; \cdots -x_{cn_c}(t_k)] \qquad (3.20)$$

and τ is the instrumental variable shift. The only difference between $\underline{\imath}_k^T V$ and $\underline{\imath}_k^T R_d$, in (3.9) that gives the least squares estimate, is in the shift τ of the samples of y.

Example 3.4 :

It is desired to obtain an instrumental variable estimate of $\underline{\theta}_o$ using the same noisy input-output data as in example 3.3. Equation (3.19) is used to obtain the desired estimate with the instrumental variable shift $\tau = 25T_s$. Theoretically, any small amount of shift is enough in the case of white noise. The results are listed in table 3.5. It is clear that the instrumental variable method has worked successfully in reducing the estimate bias.

$\hat{\underline{\theta}}_{ov}$	$\begin{array}{c} -0.6650\text{E}+00 \\ +0.6346\text{E}+00 \\ -0.3075\text{E}+00 \\ +0.6537\text{E}+00 \end{array}$

Table 3.5: Estimate of $\underline{\theta}_o$ using an instrumental variable.

3.4   Estimating The Output Derivative Modal Parameters

In this section the problem of estimating the unknown parameter vectors for $j > 0$ is considered. Obtaining least squares estimates of $\underline{\theta}_j$ for $j > 0$ from (3.16) would have been straightforward if $\underline{z}_{jf}$ for $j > 0$ were available by direct measurements . In most practical situations the number of measured variables is less than the number of state variables and therefore it is necessary to have some means of obtaining all state variables from the available measurements . Output derivatives, regarded as state variables, can be obtained by employing so called modulating functions. The use of modulating functions, as will be shown, allows the expressing of high order derivatives of the output in terms of lower order derivatives and ultimately in

terms of the measured output. The following two sections show the use of Fourier functions and Walsh functions as modulating functions.

3.4.1   Estimates Using Fourier Functions

Consider equation (3.14). Let the vector of modulating functions $\underline{f}$ be the vector of $N = 2m$ Fourier functions $\underline{f}_F$ given by (3.17), and let the integration in (3.14) be from 0 to T_o. Performing the integration by parts yields

$$\underline{z}_{jF} = -\int_0^{T_o} \mathcal{D}\underline{f}_F(t)\mathcal{D}^{j-1}y(t)dt + \underline{f}_F(T_o)\mathcal{D}^{j-1}y(T_o) - \underline{f}_F(0)\mathcal{D}^{j-1}y(0). \qquad (3.21)$$

The vector $\underline{f}_F$ satisfies the following relations

$$\underline{f}_F(0) = \underline{f}_F(T_o) = [\overbrace{0\cdots0}^{m} \; \overbrace{1\cdots1}^{m}]^T \qquad (3.22)$$

$$\mathcal{D}\underline{f}_F(t) = \Omega\underline{f}_F(t) \qquad (3.23)$$

where

$$\Omega = \begin{bmatrix} O & \acute{\Omega} \\ -\acute{\Omega} & O \end{bmatrix} \qquad (3.24)$$

and $\acute{\Omega}$ is a diagonal matrix given by

$$\acute{\Omega} = \begin{bmatrix} \omega_o & & O \\ & \ddots & \\ O & & m\omega_o \end{bmatrix}. \qquad (3.25)$$

Substituting from (3.22) and (3.23) into (3.21) gives

$$\underline{z}_{jF} = -\Omega \; \underline{z}_{(j-1)F} + \underline{f}_F(0)c_{j-1} \qquad (3.26)$$

where

$$c_{j-1} = \mathcal{D}^{j-1}y(T_o) - \mathcal{D}^{j-1}y(0). \qquad (3.27)$$

Equation (3.26) is recursive in the index j so that it can be used to obtain the spectral components of the jth derivative of the output in terms of the spectral components of the $(j-1)^{th}$ derivative and c_{j-1}. Except for c_o, c_{j-1} is not available from the measurements . Even in the case of $c_o = y(T_o) - y(0)$ it is not wise to let the result of (3.26) depend heavily on the difference of two measured samples especially in a stochastic environment. However, c_{j-1} can be estimated with other

unknowns in $\underline{\theta}_j$. Hence, only an estimate $\hat{\underline{c}}_{j-1}$ can be used in (3.26) which means that $\underline{z}_{jF}$ obtained this way is just an estimate of the spectral components of the jth derivative of the output and not the true values.

Substituting with the estimate of $\underline{z}_{jF}$ obtained from (3.26) into (3.13) gives

$$\hat{J}_{jF} = (-\Omega \; \hat{\underline{z}}_{(j-1)F} + \bar{R}_F \left[\begin{array}{c} \underline{\theta}_j \\ c_{j-1} \end{array} \right])^T (-\Omega \; \hat{\underline{z}}_{(j-1)F} + \bar{R}_F \left[\begin{array}{c} \underline{\theta}_j \\ c_{j-1} \end{array} \right]) \qquad (3.28)$$

where

$$\bar{R}_F = [R_F \; , \; \underline{f}_F(0)]. \qquad (3.29)$$

Note that a *hat* is put on top of the variable to indicate that it is an estimate of the variable and not the true value. It is obvious that the estimate of $(\underline{\theta}_j^T \; , \; c_{j-1})^T$ that minimizes $\hat{J}_{jF}$ is given by

$$\left[\begin{array}{c} \hat{\underline{\theta}}_{jF} \\ \hat{c}_{j-1} \end{array} \right] = (\bar{R}_F^T \bar{R}_F)^{-1} \bar{R}_F^T \; \Omega \; \hat{\underline{z}}_{(j-1)F}. \qquad (3.30)$$

In order to obtain estimates for $\underline{\theta}_j$ and c_{j-1} for $j > 0$, equations (3.30) and (3.26) are used sequentially. Firstly, $\hat{\underline{\theta}}_1$ and $\hat{c}_o$ are obtained from (3.30) using $\underline{z}_{oF}$ which is available from the measurements, then an estimate of $\underline{z}_{1F}$ is obtained from (3.26) using $\underline{z}_{oF}$ and the estimate of c_o. This estimate of $\underline{z}_{1F}$ is substituted in (3.30) to obtain estimates for $\underline{\theta}_2$ and c_1, and so forth.

Example 3.5 :

Consider the same system as in example 3.1. Let the system this time be excited by a sine wave input $u(t) = \sin 4t$. A sine wave has two imaginary modes thus the dimension of $\underline{q}_j$, $j = 0, \ldots, n$, is two and the continuous ARMA model is given by

$$\mathcal{D}^j y(t) + \sum_{i=1}^{3} p_{ji} y(t - T_i) = \sum_{i=1}^{2} q_{ji} x_{ci}(t) = q_{j1} \sin 4t + q_{j2} \cos 4t.$$

This model is valid for all time $t \geq T_3$ unlike the case of a piece-wise constant input. The vector $\underline{q}_j$ is dependent on the choice of the controller state variables x_{c1} and x_{c2}. A phase variable form is assumed such that

$$\underline{x}_c = \left[\begin{array}{cc} \sin 4t & \cos 4t \end{array} \right]^T$$
$$A_c = \left[\begin{array}{cc} 0 & 4 \\ -4 & 0 \end{array} \right] \quad , c_c^T = \left[\begin{array}{cc} 1 & 0 \end{array} \right].$$

The system output is sampled each $T_s = \pi/420$. It is desired to obtain estimates

j		0	1	2	3
$\underline{\hat{\theta}}_{jF}$	$\underline{\hat{p}}_{jF}$	$-0.6617\text{E}+00$ $+0.6111\text{E}+00$ $-0.3029\text{E}+00$	$+0.2974\text{E}+01$ $-0.2529\text{E}+00$ $-0.4526\text{E}+00$	$+0.5757\text{E}+01$ $-0.1481\text{E}+02$ $+0.5737\text{E}+01$	$-0.7815\text{E}+02$ $+0.3491\text{E}+02$ $+0.6237\text{E}+01$
	$\underline{\hat{q}}_{jF}$	$+0.9608\text{E}+00$ $+0.1448\text{E}+01$	$+0.5126\text{E}+01$ $+0.5592\text{E}+01$	$+0.5580\text{E}+01$ $-0.8856\text{E}+01$	$-0.1107\text{E}+03$ $-0.1557\text{E}+03$

Table 3.6: Estimates of $\underline{\theta}_j$ using Fourier functions.

of $\underline{\theta}_j$ for $j = 0,1,2$ and 3 for the same modal time shifts as in example 3.1 using (3.26) and (3.30) sequentially. A vector of six Fourier functions is used with the fundamental frequency $\omega_o = 2\pi/384T_s$. The results are listed in table 3.6.

3.4.2 Estimation Using Walsh Functions

As mentioned earlier, one advantage of using Walsh functions is the simplicity of computing the Walsh Transform which requires only sample addition or subtraction. A Walsh function is a piece-wise constant function that takes only binary values; $+1$ or -1. A transition from one value to the other may occur only at the distinct time instants $k\Delta$, $k = 1,\ldots,N-1$, where $\Delta = T_o/N$ where N is the dimension of $\underline{f}_W$ and T_o is the time period over which the elements of $\underline{f}_W$ are defined.

Consider equation (3.14). Let the vector of modulating functions be the vector of Walsh functions $\underline{f}_W$ given by (3.18) and let the integration in (3.14) be from 0 to T_o. The integration over the time interval $0 \to T$ can be divided into integrations over subintervals of length Δ such that $\underline{f}_W$ is constant over each of these subintervals

$$\underline{z}_{jW} = \sum_{k=1}^{N} \underline{f}_W(k\Delta^-) \int_{(k-1)\Delta}^{k\Delta} \mathcal{D}^j y(t)dt = \sum_{k=1}^{N} \underline{f}_W(k\Delta^-)[\mathcal{D}^{j-1}y(k\Delta) - \mathcal{D}^{j-1}y((k-1)\Delta)]$$

(3.31)

where $\underline{f}_W(k\Delta^-)$ is the limit of $\underline{f}_W(t)$ as t approaches $k\Delta$ from the left. Define $\underline{z}_{j-1}^+$

and $\underline{z}_{j-1}^-$ as

$$\underline{z}_{j-1}^+ = \begin{bmatrix} \mathcal{D}^{j-1}y(\Delta) & \cdots & \mathcal{D}^{j-1}y(N\Delta) \end{bmatrix}^T$$

$$\underline{z}_{j-1}^- = \begin{bmatrix} \mathcal{D}^{j-1}y(0) & \cdots & \mathcal{D}^{j-1}y((N-1)\Delta) \end{bmatrix}^T. \tag{3.32}$$

Then substituting from (3.32) into (3.31) gives

$$\underline{z}_{jW} = \Psi(\underline{z}_{j-1}^+ - \underline{z}_{j-1}^-) \tag{3.33}$$

where the mth row of Ψ is composed of samples of Walsh function of sequency $(m-1)$ sampled at the discrete time instants $k\Delta^-$, $1 \le k \le N$. The matrix Ψ is the Walsh matrix used to compute the discrete Walsh transform of a sampled signal [Tadokoro 78].

Samples of the output derivative of order $j-1$ required in (3.32) are not available by measurements , so it is required to obtain estimates of these samples by some means. The required samples can be obtained from (3.5) provided that there is an estimate of $\underline{\theta}_{j-1}$ already available

$$\begin{aligned} \hat{\underline{z}}_{j-1}^+ &= -R_d^+ \, \hat{\underline{\theta}}_{j-1} \\ \hat{\underline{z}}_{j-1}^- &= -R_d^- \, \hat{\underline{\theta}}_{j-1} \end{aligned} \tag{3.34}$$

where

$$R_d^+ = \begin{bmatrix} \underline{r}^T(\Delta) \\ \vdots \\ \underline{r}^T(N\Delta) \end{bmatrix}$$

and

$$R_d^- = \begin{bmatrix} \underline{r}^T(0) \\ \vdots \\ \underline{r}^T((N-1)\Delta) \end{bmatrix} \tag{3.35}$$

and $\underline{r}(t)$ is given by (3.3). Substituting for $\underline{z}_{j-1}^+$ and $\underline{z}_{j-1}^-$ in (3.33) by their estimates from (3.34) gives

$$\hat{\underline{z}}_{jW} = \Psi(R_d^- - R_d^+)\hat{\underline{\theta}}_{j-1}. \tag{3.36}$$

Finally, substituting for $\underline{z}_{jF}$ in (3.16) by its estimate $\hat{\underline{z}}_{jW}$ from (3.36) gives

$$\hat{\underline{\theta}}_{jW} = -(R_W^T R_W)^{-1} R_W^T \Psi(R_d^- - R_d^+)\hat{\underline{\theta}}_{(j-1)W}. \tag{3.37}$$

Equation (3.37) shows that it is possible to obtain estimates for θ_j, $j > 0$, sequentially from the available measurements since R_W, R_d^- and R_d^+ are formed from the input-output measurements .

j		0	1	2	3
$\hat{\underline{\theta}}_{jW}$	$\hat{\underline{p}}_{jW}$	$-0.6605E+00$ $+0.6113E+00$ $-0.3022E+00$	$+0.3002E+01$ $-0.2543E+00$ $-0.4325E+00$	$+0.5715E+01$ $-0.1477E+02$ $+0.5718E+01$	$-0.7850E+02$ $+0.3490E+02$ $+0.5870E+01$
	$\hat{\underline{q}}_{jW}$	$+0.8717E+00$ $+0.1370E+01$	$+0.4750E+01$ $+0.5265E+01$	$+0.6102E+01$ $-0.8434E+01$	$-0.1009E+03$ $-0.1470E+03$

Table 3.7: Estimates of $\underline{\theta}_j$ using Walsh functions.

Example 3.6 :

Consider the same system as in example 3.5 with the same input-output measurements . It is desired to obtain estimates of $\underline{\theta}_j$ for $j = 0, 1, 2,$ and 3 for the same modal time shifts considered in the mentioned example using equation (3.37). A vector of eight Walsh functions defined on a period of length $T_o = 384T_s$ is used. The results are listed in table 3.7. In the case of noisy measurements, the instrumental variable method can be used. A continuous-time instrumental variable is needed. A sophisticated method of generating this instrumental variable using the estimated model of the system is given in [Young 70] and [Young 79].

3.5 Estimation Using Multi-mode Inputs

A good test input signal should contain enough modes to excite all the modes of the system in order to obtain well conditioned parameter estimation equations. Such an input is referred to as a persistently exciting input. Examples of persistently exciting inputs are periodic inputs with many frequency components, e.g. square waves and pseudo-random binary sequences. On the other hand, from the computational point of view, it is desirable to keep the number of unknowns to be estimated from a single equation as low as possible. It is seen from (2.44) that there is one unknown to be estimated for each input mode. This could cause a serious problem if the input contains a large number of modes as for example the case of a square wave input which contains an infinite number of modes.

To solve this problem, a prefilter that partially, or totally, annihilates the input modes will be used to filter the input-output measurements before being used in forming the estimation equations [Trivett 81]. The filtered input and output signals have a smaller number of modes than the original signals.

Let u be the response of a continuous controller as given by (2.35) and (2.36). Let the matrix A_c be *diagonal* so that it can be partitioned into square blocks along the main diagonal. Let A_c be partitioned into two diagonal submatrices A_{c1} and A_{c2}, and let $\underline{x}_c$, $\underline{c}_c$ and ϕ_c be partitioned accordingly

$$\mathcal{D} \begin{bmatrix} \underline{x}_{c1} \\ \underline{x}_{c2} \end{bmatrix} = \begin{bmatrix} A_{c1} & 0 \\ 0 & A_{c2} \end{bmatrix} \begin{bmatrix} \underline{x}_{c1} \\ \underline{x}_{c2} \end{bmatrix} \tag{3.38}$$

$$u(t) = \begin{bmatrix} \underline{c}_{c1}^T & \underline{c}_{c2}^T \end{bmatrix} \begin{bmatrix} \underline{x}_{c1}(t) \\ \underline{x}_{c2}(t) \end{bmatrix} \tag{3.39}$$

$$\begin{bmatrix} \underline{x}_{c1}(t+\tau) \\ \underline{x}_{c2}(t+\tau) \end{bmatrix} = \begin{bmatrix} \phi_{c1}(\tau) & 0 \\ 0 & \phi_{c2}(\tau) \end{bmatrix} \begin{bmatrix} \underline{x}_{c1}(t) \\ \underline{x}_{c2}(t) \end{bmatrix}. \tag{3.40}$$

The controller is divided into two subsystems C_1 and C_2 defined by the pairs (A_{c1}, c_{c1}) and (A_{c2}, c_{c2}) respectively such that the controller output is the sum of two components u_1 and u_2 generated respectively by C_1 and C_2;

$$u(t) = u_1(t) + u_2(t) = \underline{c}_{c1}^T \, \underline{x}_{c1}(t) + \underline{c}_{c2}^T \, \underline{x}_{c2}(t). \tag{3.41}$$

The input component u_1 contains modes of C_1 only and the input component u_2 contains modes of C_2 only. Let n_{c1} and n_{c2} be the dimension of C_1 and C_2 respectively such that $n_{c1} + n_{c2} = n_c$. Both pairs $(A_{c1}, \underline{c}_{c1})$ and $(A_{c2}, \underline{c}_{c2})$ are assumed to be observable hence, we can obtain two vectors $\underline{l}_1$ and $\underline{l}_2$ of dimension n_{c1} and n_{c2} respectively whose elements are the output modal parameters of subsystems C_1 and C_2 respectively such that

$$\underline{x}_{c_k}(t) + \sum_{i=1}^{n_{ck}} l_{ki} \, \underline{x}_{ck}(t - T_{ki}) = \underline{o} \qquad , k = 1, 2 \tag{3.42}$$

where $T_{k1}, \ldots, T_{kn_{ck}}$, $k = 1, 2$, are suitable modal time shifts. The vectors of filter parameters l_1 and l_2 are given according to (2.27) by

$$\underline{l}_k = -\underline{c}_{ck} \psi_{ck}^{-1} \qquad , k = 1, 2 \tag{3.43}$$

where ψ_{ck} is given by (2.20) with the substitution $\underline{c}_{ck}$ for $\underline{c}$ and ϕ_{c_k} for ϕ. The right-hand side of (3.43) is *zero* because u_1 and u_2 are assumed to be the free responses

of C_1 and C_2. Equation (3.42) represents a linear filtering operation, hence the linear filters $\mathcal{L}_1$ and $\mathcal{L}_2$ can be defined as as

$$\mathcal{L}_k \underline{x}_{c_k}(t) = \underline{o} \quad , k = 1, 2. \tag{3.44}$$

The filter $\mathcal{L}_1$ annihilates all the modes in $\underline{x}_{c1}$ while $\mathcal{L}_2$ annihilates all modes in $\underline{x}_{c2}$, so if the input u is filtered through $\mathcal{L}_1$, the resulting filtered input will contain none of the modes of $\underline{x}_{c1}$. Conversly, if u is filtered through $\mathcal{L}_2$, the resulting filtered input will contain none of the modes of $\underline{x}_{c2}$. Filtering u through $\mathcal{L}_1$ and $\mathcal{L}_2$ results in annihilating all the modes of the input

$$\mathcal{L}_1 u(t) = \underline{c}_{c1}^T \mathcal{L}_1 \underline{x}_{c1}(t) + \underline{c}_{c2}^T \mathcal{L}_1 \underline{x}_{c2}(t) = \underline{c}_{c2}^T \mathcal{L}_1 \underline{x}_{c2}$$

$$\mathcal{L}_2 u(t) = \underline{c}_{c1}^T \mathcal{L}_2 \underline{x}_{c1}(t) + \underline{c}_{c2}^T \mathcal{L}_2 \underline{x}_{c2}(t) = \underline{c}_{c1}^T \mathcal{L}_2 \underline{x}_{c1}(t)$$

$$\mathcal{L}_2 \mathcal{L}_1 u(t) = \underline{c}_{c1}^T \mathcal{L}_2 \mathcal{L}_1 \underline{x}_{c1}(t) + \underline{c}_{c2}^T \mathcal{L}_2 \mathcal{L}_1 \underline{x}_{c2}(t) = \underline{o}. \tag{3.45}$$

Example 3.7 :

Consider the input

$$u(t) = e^{-2t} + e^{-t} \sin 5t + e^{-t} \cos 5t.$$

This input can be regarded as the free response of a third order continuous controller with one real pole at -2 and two complex conjugate poles at $-1 \pm j5$. The partitioned controller is given by

$$x_{c1}(t) = [e^{-2t}] \qquad \underline{x}_{c2}(t) = [e^{-t} \sin 5t \ \ e^{-t} \cos 5t]^T$$

$$c_{c1} = [1] \qquad \underline{c}_{c2} = [1 \ \ 1]^T$$

$$A_{c1} = [-2] \qquad A_{c2} = [-1 \ \ -1]$$

$$\phi_{c1}(\tau) = [e^{-2\tau}] \qquad \phi_{c2}(\tau) = e^{-\tau} \begin{bmatrix} \cos 5\tau & \sin 5\tau \\ -\sin 5\tau & \cos 5\tau \end{bmatrix}.$$

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two filters which filter out the real mode and the complex modes respectively. Let the modal time shifts for $\mathcal{L}_1$ and for $\mathcal{L}_2$ be given by

$$T_{11} = \pi/10$$
$$T_{21} = \pi/10$$
$$T_{22} = \pi/5$$

For these modal time shifts the corresponding ψ matrices are given according to (2.20) by

$$\psi_{c1} = e^{-\pi/5}$$

$$\psi_{c2} = \begin{bmatrix} e^{-\pi/5} & -e^{-\pi/5} \\ -e^{-2\pi/5} & -e^{-2\pi/5} \end{bmatrix}.$$

Hence the vectors of output modal parameters l_1 and l_2 are given according to (2.27) by

$$l_1 = -e^{-\pi/5}$$

$$l_2 = \begin{bmatrix} 0 & e^{2\pi/5} \end{bmatrix}^T.$$

To filter out the real mode, $\mathcal{L}_1$ is used

$$\mathcal{L}_1 u(t) = u(t) - e^{\pi/5}u(t - \pi/10) = (1 + e^{3\pi/10})e^{-t}\sin 5t + (1 - e^{3\pi/10})e^{-t}\cos 5t.$$

To filter out the complex modes, $\mathcal{L}_2$ is used

$$\mathcal{L}_2 u(t) = u(t) + e^{2\pi/5}u(t + \pi/5) = (1 + e^{-2\pi/5})e^{-2t}.$$

Obviously if $\mathcal{L}_1$ and $\mathcal{L}_2$ are applied the resulting filtered output is *zero*.

Example 3.8 :

Consider a periodic input with period T_o such that $u(t) = u(t - T_o)$. The fundamental frequency is $\omega_o = 2\pi/T_o$. In general a periodic u could have an infinite number of imaginary modes $\pm jk\omega_o$, $k \geq 0$, and consequently the controller that generates u could then have order infinity.

Consider filtering out a single frequency $k\omega_o$. The controller can be partitioned into two parts; one part that generates the frequency $k\omega_o$

$$A_{c1} = \begin{bmatrix} 0 & k\omega_o \\ -k\omega_o & 0 \end{bmatrix}$$

$$c_{c1} = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$$

$$\phi_{c1}(\tau) = \begin{bmatrix} \cos k\omega_o\tau & \sin k\omega_o\tau \\ -\sin k\omega_o\tau & \cos k\omega_o\tau \end{bmatrix}$$

and the other part that generates all other frequencies. To perform the filtering operation, a filter $\mathcal{L}_k$ with the modal time shifts

$$T_1 = T_o/4$$
$$T_2 = T_o/2$$

is used. For these modal time shifts the corresponding ψ_k is given by

$$\psi_k = \left[\begin{array}{cc} \cos k\pi/2 - \sin k\pi/2 & \sin k\pi/2 + \cos k\pi/2 \\ \cos k\pi & \cos k\pi \end{array} \right]$$

For any odd integer k_o, ψ_{k_o} is given by

$$\psi_{k_o} = \left[\begin{array}{cc} -m & m \\ -1 & -1 \end{array} \right]$$

where

$$m = \left\{ \begin{array}{ll} +1 & \text{if } (k_o - 1) \text{ is divisible by } 4 \\ -1 & \text{if } (k_o - 1) \text{ is not divisible by } 4. \end{array} \right.$$

It follows from (2.27) that for odd integers k_o the vector of output modal parameters for filters $\mathcal{L}_{k_o}$ that filters out frequency $k_o \omega_o$ is given by

$$\underline{l}_{k_o} = [0 \ \ 1]^T$$

regardless of the value of k_o. This means that a filter $\mathcal{L}_{k_o}$ with the given modal time shifts and modal parameters, annihilates all the odd frequency components in u. If u_o is a periodic signal with odd symmetry, e.g. a square wave, then $\mathcal{L}_{k_o}$ completely annihilates the modes of u

$$\mathcal{L}_{k_o} u_o(t) = u_o(t) + u_o(t - T/2) = 0.$$

It is clear from the above example that although the input may have an infinite number of modes, it was possible by a simple filtering operation to annihilate all the modes of the input.

Consider the continuous ARMA model (2.44). Let vector $\underline{q}_j$ be partitioned according to the partitioning of $\underline{x}_c$ in (3.38) such that

$$\mathcal{D}^j y(t) + \sum_{i=1}^{n} p_{ji} y(t - T_i) = \underline{q}_j^{1^T} \underline{x}_{c1}(t) + \underline{q}_j^{2^T} \underline{x}_{c2}(t). \tag{3.46}$$

Applying $\mathcal{L}_1$ and $\mathcal{L}_2$, given by (3.44), to (3.46) respectively yields

$$\mathcal{D}^j \mathcal{L}_1 y(t) + \sum_{i=1}^{n} p_{ji} \mathcal{L}_1 y(t - T_i) = \underline{q}_j^{2^T} \mathcal{L}_1 \underline{x}_{c2}(t)$$

$$\mathcal{D}^j \mathcal{L}_2 y(t) + \sum_{i=1}^{n} p_{ji} \mathcal{L}_2 y(t - T_i) = \underline{q}_j^{1^T} \mathcal{L}_2 \underline{x}_{c1}(t). \qquad (3.47)$$

In (3.47) the filtered input-output measurements pairs $(\mathcal{L}_1 y, \mathcal{L}_1 u)$ and $(\mathcal{L}_2 y, \mathcal{L}_2 u)$ are used instead of the direct measurements pair (y, u) such that the number of unknowns in the two cases of filtered measurements is $n + n_{c1}$ and $n + n_{c2}$ respectively, while the number of unknowns if the direct measurements were used is $n + n_{c1} + n_{c2}$.

Example 3.9 :

It is desired to obtain estimates of $\underline{\theta}_j$ for $j = 0, 1, 2$ and 3 of the system considered before in examples 3.5 and 3.6 from the system response to the input

$$u(t) = \cos 6t - \sin 3t.$$

The response of the system to this input starting from zero initial state is sampled each time interval $T_s = T_o/384$ where T_o is the period of the input component of frequency $\omega_3 = 3$. The modal time shifts T_1, T_2 and T_3 are taken such that $T_1 = T_2 - T_1 = T_3 - T_2 = 40T_s$.

The input has four imaginary modes at $\pm j6$ and $\pm j3$ which means that the dimension of each of $\underline{q}_j$, $j \geq 0$, is four. To reduce this number the technique of the measurements prefilter $\mathcal{L}$ is used. Two filters $\mathcal{L}_1$ and $\mathcal{L}_2$ are used to eliminate modes of frequencies 6 and 3, respectively as in (3.47) such that in each case the number of modes in the filtered input is reduced from four to two. The choice of the modal time shifts for each filter completely defines the corresponding filter parameters. The following values for the modal time shifts are assumed;

$T_{11} = T_o/8$

$T_{12} = T_o/4$

$T_{21} = T_o/4$

$T_{22} = T_o/2$

The prefilter parameters which result from the above choice of the modal time shifts are the same for both filters

$$\underline{l}_1 = [0 \ 1]^T$$

$$\underline{l}_2 = [0 \ 1]^T.$$

Hence, the filtered inputs $\mathcal{L}_1 u$ and $\mathcal{L}_2 u$ are given by

$$
\begin{aligned}
\mathcal{L}_1 u(t) &= (\cos 6t - \sin 3t) + (\cos 6(t - T_o/4) - \sin 3(t - T_o/4)) \\
&= (\cos 6t - \sin 3t) + (\cos(6t - \pi) - \sin(3t - \pi/2)) \\
&= -\sin 3t + \cos 3t
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{L}_2 u(t) &= (\cos 6t - \sin 3t) + (\cos 6(t - T_o/2) - \sin 3(t - T_o/2)) \\
&= (\cos 6t - \sin 3t) + (\cos(6t - 2\pi) - \sin(3t - \pi)) \\
&= 2\cos 6t.
\end{aligned}
$$

As expected, $\mathcal{L}_1 u$ has a single frequency $\omega_3 = 3$ and $\mathcal{L}_2 u$ has a single frequency $\omega_6 = 6$. The corresponding filtered outputs are

$$
\mathcal{L}_1 y(t) = y(t) + y(t - T_o/4)
$$

$$
\mathcal{L}_2 y(t) = y(t) + y(t - T_o/2).
$$

Estimates of the unknown parameters are obtained as before by minimizing some function of the equation error defined in this case according to (3.47) by

$$
e_1(t) = \mathcal{D}^j \mathcal{L}_1 y(t) + \sum_{i=1}^{n} p_{ji} \mathcal{L}_1 y(t - T_i) - \underline{q}_j^{2^T} \mathcal{L}_1 \underline{x}_{c2}(t)
$$

$$
e_2(t) = \mathcal{D}^j \mathcal{L}_2 y(t) + \sum_{i=1}^{n} p_{ji} \mathcal{L}_2 y(t - T_i) - \underline{q}_j^{1^T} \mathcal{L}_2 \underline{x}_{c1}(t).
$$

The state vectors $\underline{x}_{c1}$ and $\underline{x}_{c2}$ are chosen such that

$$
\mathcal{L}_1 \underline{x}_{c2}(t) = [\sin t \quad \cos t]^T
$$

and

$$
\mathcal{L}_2 \underline{x}_{c1}(t) = [\sin 4t \quad \cos 4t]^T.
$$

Either Fourier functions or Walsh functions can be used as described in subsections 3.3.1 and 3.3.2 with the exception that the filtered measurements pairs $(\mathcal{L}_1 y, \mathcal{L}_1 u)$ and $(\mathcal{L}_2 y, \mathcal{L}_2 u)$ are used instead of the direct measurements pair (y, u). Two vectors of estimated parameters $\hat{\underline{\theta}}_j^1 = [\hat{\underline{p}}_j^{1^T} \ \hat{\underline{q}}_j^{1^T}]^T$ and $\hat{\underline{\theta}}_j^2 = [\hat{\underline{p}}_j^{2^T} \ \hat{\underline{q}}_j^{2^T}]^T$, for each j, result

	j	0	1	2	3
$\hat{\underline{\theta}}^1_{jF}$	$\hat{\underline{p}}^1_{jF}$	$-0.1390\text{E}+01$ $+0.1188\text{E}+01$ $-0.4190\text{E}+00$	$+0.7137\text{E}+00$ $+0.2994\text{E}+01$ $-0.1701\text{E}+01$	$+0.2271\text{E}+02$ $-0.2640\text{E}+02$ $+0.6527\text{E}+01$	$-0.3997\text{E}+02$ $-0.4291\text{E}+02$ $+0.4671\text{E}+02$
	$\hat{\underline{q}}^1_{jF}$	$-0.3362\text{E}+00$ $+0.4154\text{E}+00$	$-0.1629\text{E}+01$ $+0.2898\text{E}+01$	$+0.4215\text{E}+01$ $+0.6785\text{E}+01$	$+0.7236\text{E}+02$ $-0.4658\text{E}+02$

Table 3.8: Estimates of $\underline{\theta}^1_j$ using Fourier functions.

	j	0	1	2	3
$\hat{\underline{\theta}}^2_{jF}$	$\hat{\underline{p}}^2_{jF}$	$-0.1387\text{E}+01$ $+0.1124\text{E}+01$ $-0.4168\text{E}+00$	$+0.7382\text{E}+00$ $+0.2968\text{E}+01$ $-0.1684\text{E}+01$	$+0.2252\text{E}+02$ $-0.2622\text{E}+02$ $+0.6424\text{E}+01$	$-0.4038\text{E}+02$ $-0.4224\text{E}+02$ $+0.4634\text{E}+02$
	$\hat{\underline{q}}^2_{jF}$	$-0.9154\text{E}+00$ $-0.8869\text{E}-01$	$-0.5276\text{E}+01$ $+0.1354\text{E}+01$	$+0.3682\text{E}+01$ $+0.2380\text{E}+02$	$+0.2279\text{E}+03$ $+0.5318\text{E}+02$

Table 3.9: Estimates of $\underline{\theta}^2_j$ using Fourier functions.

from the filtered measurements pairs $(\mathcal{L}_1 y, \mathcal{L}_1 u)$ and $(\mathcal{L}_2 y, \mathcal{L}_2 u)$ respectively. The results of estimating $\underline{\theta}^1_j$ and $\underline{\theta}^2_j$ using Fourier functions are listed in tables 3.8 and 3.9. Notice that the results obtained in this example for the modal parameters are different from those obtained in the previous examples because the modal time shifts considered in this example are different from those considered in the previous examples. The true values of the modal parameters are listed in table 3.10. Those values are obtained from equation (2.30). The results of estimating the same vectors using Walsh functions are listed in tables 3.11 and 3.12.

Example 3.10 :

Consider a fourth order system with the transfer function

$$H(s) = \frac{400s + 400}{s^4 + 6s^3 + 115.25s^2 + 221s + 338}$$

j	0	1	2	3
$\underline{p}_j$	$-0.1389\text{E}+01$ $+0.1126\text{E}+01$ $-0.4178\text{E}+00$	$+0.7304\text{E}+00$ $+0.2975\text{E}+01$ $-0.1688\text{E}+01$	$+0.2263\text{E}+02$ $-0.2631\text{E}+02$ $+0.6471\text{E}+01$	$-0.4022\text{E}+02$ $-0.4258\text{E}+02$ $+0.4649\text{E}+02$

Table 3.10: True values of $\underline{p}_j$.

$\hat{\underline{\theta}}^1_{jW}$	j	0	1	2	3
	$\hat{\underline{p}}^1_{jW}$	$-0.1388\text{E}+01$ $+0.1126\text{E}+01$ $-0.4179\text{E}+00$	$+0.7306\text{E}+00$ $+0.2975\text{E}+01$ $-0.1688\text{E}+01$	$+0.2263\text{E}+02$ $-0.2632\text{E}+02$ $+0.6474\text{E}+01$	$-0.4024\text{E}+02$ $-0.4256\text{E}+02$ $+0.4649\text{E}+02$
	$\hat{\underline{q}}^1_{jW}$	$-0.3434\text{E}+00$ $+0.4151\text{E}+00$	$-0.1683\text{E}+01$ $+0.2893\text{E}+01$	$+0.4137\text{E}+01$ $+0.6802\text{E}+01$	$+0.7320\text{E}+02$ $-0.4654\text{E}+02$

Table 3.11: Estimates of $\underline{\theta}^1_j$ using Walsh functions.

$\hat{\underline{\theta}}^2_{jW}$	j	0	1	2	3
	$\hat{\underline{p}}^2_{jW}$	$-0.1388\text{E}+01$ $+0.1126\text{E}+01$ $+0.4179\text{E}+01$	$+0.7314\text{E}+00$ $+0.2973\text{E}+01$ $-0.1687\text{E}+01$	$+0.2262\text{E}+02$ $-0.2630\text{E}+02$ $+0.6467\text{E}+01$	$-0.4023\text{E}+02$ $-0.4254\text{E}+02$ $+0.4647\text{E}+02$
	$\hat{\underline{q}}^2_{jW}$	$-0.9128\text{E}+00$ $-0.8730\text{E}-01$	$-0.5325\text{E}+01$ $+0.1364\text{E}+01$	$+0.2981\text{E}+01$ $+0.2371\text{E}+02$	$+0.2261\text{E}+03$ $+0.5283\text{E}+02$

Table 3.12: Estimates of $\underline{\theta}^2_j$ using Walsh functions.

The system has two pairs of complex conjugate poles at $-1 \pm j1.5$ and $-2 \pm j10$, and a zero at -1. A fourth order model is assumed for the system and it is desired to obtain the vectors of modal parameters $\underline{p}_j$ for $j = 0, 1, 2, 3$, and 4 from the input-output measurements for suitable modal time shifts.

The system is excited by a square wave of period $T_o = \pi$ such that the fundamental frequency $\omega_o = 2$. Starting from the following initial state

$$y(0) = 0.2$$
$$\mathcal{D}y(0) = 0.0$$
$$\mathcal{D}^2 y(0) = 0.0$$
$$\mathcal{D}^3 y(0) = 0.0$$

the response of the system to the square wave input is sampled such that 420 samples are taken every time period T_o, so that the sampling time T_s equals $\pi/420$. The values assigned to the modal time shifts are

$$T_1 = 20T_s$$
$$T_2 = 40T_s$$
$$T_3 = 110T_s$$
$$T_4 = 160T_s$$

The differences between the modal time shifts are not uniform in this example.

The square wave input has odd symmetry similar to the case considered in example 3.8. Hence, a filter $\mathcal{L}$ with modal time shifts $T_1 = T_o/4$ and $T_2 = T_o/2$, and modal parameters $l_1 = 0$ and $l_2 = 1$ annihilates all the input modes

$$\mathcal{L}u(t) = u(t) + u(t - T/2) = 0.$$

The filtered output

$$\mathcal{L}y(t) = y(t) + y(t - T/2)$$

represents a free response of the system. In this case the dimension of vector $\underline{q}_j$ in the continuous ARMA model is zero so that for each j it is required to estimate only a vector $\underline{p}_j$ of dimension four.

Similar to the examples considered before estimates of the unknown parameters can be obtained using Fourier functions or Walsh functions. Estimates $\underline{\hat{p}}_{jF}$ obtained using Fourier functions, are listed in table 3.13 and estimates $\underline{\hat{p}}_{jW}$ using Walsh functions are listed in table 3.14.

$\hat{\underline{p}}_{oF}$	$\hat{\underline{p}}_{1F}$	$\hat{\underline{p}}_{2F}$	$\hat{\underline{p}}_{3F}$	$\hat{\underline{p}}_{4F}$
−0.1042E+01	+0.2992E+01	+0.2658E+02	−0.4063E+03	−0.9384E+03
−0.2220E−01	−0.2784E+01	−0.1545E+02	+0.4268E+03	−0.1537E+03
+0.2179E+00	+0.1236E+01	−0.8259E+01	−0.7817E+02	+0.1074E+04
−0.5770E−01	−0.9301E−01	+0.5245E+01	+0.7556E+01	−0.6100E+03

Table 3.13: Estimates of the modal parameters using Fourier functions.

$\hat{\underline{p}}_{oW}$	$\hat{\underline{p}}_{1W}$	$\hat{\underline{p}}_{2W}$	$\hat{\underline{p}}_{3W}$	$\hat{\underline{p}}_{4W}$
−0.1042E+01	+0.2990E+01	+0.2760E+02	−0.4126E+03	−0.1016E+04
−0.2200E−01	−0.2808E+01	−0.1635E+02	+0.4350E+03	−0.9821E+02
+0.2179E+00	+0.1262E+01	−0.8266E+01	−0.8111E+02	+0.1088E+04
−0.5769E−01	−0.1059E+00	+0.5330E+01	+0.8519E+01	−0.6231E+03

Table 3.14: Estimates of the modal parameters using Walsh functions.

$\underline{p}_o$	$\underline{p}_1$	$\underline{p}_2$	$\underline{p}_3$	$\underline{p}_4$
−0.1048E+01	+0.2988E+01	+0.2759E+02	−0.4118E+03	−0.1017E+04
−0.2200E−01	−0.2806E+01	−0.1635E+02	+0.4343E+03	−0.9438E+02
+0.2179E+00	+0.1262E+01	−0.8255E+01	−0.8105E+02	+0.1085E+04
−0.5768E−01	−0.1059E+00	+0.5326E+01	+0.8543E+01	−0.6219E+03

Table 3.15: True values of the modal parameters.

The number of Fourier functions used is eight functions with the fundamental frequency $\omega_o = 2$, and the number of Walsh functions used is eight functions defined over the time period $0 \rightarrow \pi$. The true values of the modal parameters, computed using (2.30), are included in table 3.15 for the purpose of comparison with the estimated values.

3.6 Summary

Chapter 3 is devoted to the problem of modal parameter estimation. Estimation equations, which are linear in the parameters, are formed from discrete data as well as from continuous data. Modulating functions, e.g. Fourier and Walsh, are used in the estimation process. One advantage of estimating the parameters of the continuous-time ARMA model (2.44) over directly attempting to estimate the parameters of the input-output model (1.1) is that only one stage of integration is required. This results in avoiding the introducing of the unknown initial conditions to the estimation problem. Recursive estimation equations, (3.26),(3.30) and (3.37), are used to obtain the vectors of the unknown parameters.

An instrumental variable is used in a simple case to reduce the bias that results from noisy measurements. Signal averaging is another simple way of reducing the effect of noise. In example 3.1, for example, the input could be considered as a segment of a PRBS so that data could be averaged over different segments of the sequence. In general, multi-mode inputs are required to realize persistent excitation and it is shown how modal filters can be used to reduce the order of the parameter estimation equations.

Chapter 4

Parametric Identification

4.1 Introduction

The motive behind the definition of the modal functions and the modal parameters in chapter 3 is to use them in system identification. Obtaining the modal parameters of the output and the output derivatives from input-output measurements was presented in chapter 3 with numerical examples. This chapter will concentrate on obtaining parametric models of the system through the modal functions and the modal parameters. Identification of an input-output model is considered first.

A method to obtain estimates of vectors $\underline{a}$ and $\underline{b}$ of the input-output model through the output modal function is presented in section 4.2. An important relation between the vectors of modal parameters and the model characteristic equation is established in section 4.3 and then used to estimate the coefficients of the characteristic equation. Also in the same section the modal functions of the output and the output derivatives are used to obtain an estimate of vector $\underline{b}$. In section 4.4 a simple method for estimating the system dynamic matrix from the available measurements is described. The coefficients of the system characteristic equation are obtained from the estimated dynamic matrix through a simple recursive relation, and the system poles are obtained as the eigenvalues of that matrix.

4.2 Identification Using the Output Modal Function

The Output modal function y_o is obtained from the system output by a linear filtering operation by adding weighted samples separated by the modal time shifts

$T_1, \ldots, T_n$. Similarly, a filtered input u_o can be defined as

$$u_o(t) = u(t) + \sum_{i=1}^{n} p_{oi} u(t - T_i).$$ (4.1)

It follows from the assumption that the model is linear and time-invariant that the filtered input-output pair (u_o, y_o) satisfies the same differential equation as the original input-output pair (u, y) :

$$\mathcal{D}^n y_o(t) + \sum_{j=0}^{n-1} a_j \mathcal{D}^j y_o(t) = \sum_{j=0}^{n-1} b_j \mathcal{D}^j u_o(t).$$ (4.2)

Equation (4.2) means that the output modal function y_o represents a particular solution to the differential equation model when the input is the filtered input u_o given by (4.1). The particular solution y_o contains modes from the input only and contains none of the system modes while the general solution contains modes from both the system and the input. This distinct characteristic of y_o makes the output modal function very useful for system identification.

Once y_o is obtained from the general system response it becomes straight forward to obtain its derivatives analytically since it is composed of modes of the input which are known. It follows from (2.44) for $j = 0$, and from (2.35) that the jth derivative of y_o is given by

$$\mathcal{D}^j y_o(t) = \underline{q}_o^T A_c^j \underline{x}_c(t).$$ (4.3)

Similarly, u_o and its derivatives can be obtained in terms of the input modes as

$$u_o(t) = \underline{c}_c^T (I + \sum_{i=1}^{n} p_{oi} \phi(-T_i)) \underline{x}_c(t)$$ (4.4)

where ϕ_c is the state transition matrix of the controller. Let

$$\underline{\nu}^T = \underline{c}_c^T (I + \sum_{i=1}^{n} p_{oi} \phi_c(-T_i))$$ (4.5)

so that (4.4) becomes

$$u_o(t) = \underline{\nu}^T \underline{x}_c(t).$$ (4.6)

Then from (2.35) it follows that the jth derivative of u_o is given by

$$\mathcal{D}^j u_o(t) = \underline{\nu}^T A_c^j \underline{x}_c(t).$$ (4.7)

Substituting for y_o, u_o and their derivatives from (4.3) and (4.7) respectively into (4.2) yields

$$\underline{q}_o^T (A_c^n + \sum_{j=0}^{n-1} a_j A_c^j)\underline{x}_c(t) = \underline{\nu}^T \sum_{j=0}^{n-1} b_j A_c^j \underline{x}_c(t). \tag{4.8}$$

Since (4.8) is valid for all t it follows that

$$\underline{q}_o^T A_c^n + \sum_{j=0}^{n-1} a_j \underline{q}_o^T A_c^j = \sum_{j=0}^{n-1} b_j \underline{\nu}^T A_c^j. \tag{4.9}$$

Estimates of the unknown model parameters can be obtained by minimizing the norm of the equation error vector defined as

$$\underline{e}_\theta = A_c^{n^T} \underline{q}_o + R_\theta \underline{\theta} \tag{4.10}$$

where

$$\underline{\theta} = \left[\begin{array}{c} \underline{a} \\ \underline{b} \end{array} \right] \tag{4.11}$$

and R_θ is an $n_c \text{x} 2n$ matrix given by

$$R_\theta = [\underline{q}_o \ A_c^T \underline{q}_o \ldots A_c^{n-1^T} \underline{q}_o \ -\underline{\nu} \ -A_c^T \underline{\nu} \ldots -A_c^{n-1^T} \underline{\nu}]. \tag{4.12}$$

The dimension of θ is in general $2n$, hence in order to be able to obtain an estimate of θ, the number of rows of R_θ should be at least $2n$, i.e. the order of the controller that generates u should be at least $2n$:

$$n_c \geq 2n. \tag{4.13}$$

The estimate of θ that minimizes the norm of $\underline{e}_\theta$ when $n_c > 2n$ is given by

$$\hat{\underline{\theta}} = -(R_\theta^T R_\theta)^{-1} R_\theta^T A_c^{n^T} \underline{q}_o. \tag{4.14}$$

Example 4.1 :

Consider the third order system of example 3.1. It is desired to obtain an estimate of the unknown model parameter vector θ from (4.14). The dimension of θ in this case is six, therefore according to inequality (4.13) the input should contain at least six modes. An input u given by

$$u(t) = \cos 6t - \sin 4t - \sin 2t$$

contains three frequency components and therefore it suits the purpose of estimating θ from (4.14). The output of the system starting from zero initial state is computed every sampling time $T_s = \pi/420$.

The number of the output modal parameters that must be estimated in order to obtain the output modal function from the system output is three which corresponds to the order of the system. The dimension of $\underline{q}_o$ is equal to the number of input modes which is six in this example. The elements of $\underline{q}_o$ are unknown and must be estimated with the three output modal parameters, hence, the total number of unknown parameters is nine. In order to reduce this number of unknowns the measurements prefilter described is section 3.5 will be employed.

Three different prefilters $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$ are used, each one of them annihilates two frequency components. Let $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$ annihilate frequency pairs (6,2), (6,4) and (4,2) respectively, and let the modal time shifts of the three filters be given respectively by

$$105T_s \quad 210T_s$$
$$35T_s \quad 70T_s \quad 120T_s \quad 170T_s$$
$$50T_s \quad 100T_s \quad 150T_s \quad 200T_s.$$

The corresponding filter parameters are given respectively by

$$[1 \quad 0]$$
$$[-0.4146E + 00 \quad 0.1413E + 00 \quad 0.3972E + 00 \quad 0.7438E + 00]$$
$$[-0.1616E + 01 \quad 0.2219E + 01 \quad -0.1656E + 01 \quad 0.1000E + 01]$$

These parameters are computed using (2.27). The reason why $\mathcal{L}_1$ is a second order filter and not a fourth order can be understood from the result of example 3.8. It has been shown in that example that a filter of modal time shifts $(T/4, T/2)$, where T is the period of the fundamental frequency, and modal parameters [0 1], annihilates all the odd multiples of the fundamental frequency. Hence, $\mathcal{L}_1$ annihilates both the fundamental frequency $\omega_2 = 2$ and $\omega_6 = 6 = 3\omega_2$.

Let the filtered input-output pairs (u^1, y^1), (u^2, y^2) and (u^3, y^3) be the result of filtering the input-output measurements pair (u, y) through $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$ respectively. Each filtered input-output pair (u^k, y^k), $k = 1, \ldots, 3$ is used to obtain an estimate of of a vector $\underline{\theta}_o^k$ of dimension five composed of the output modal param-

eter vector $\underline{p}_o$ of dimension three and a vector $\underline{q}_o^k$ of dimension two. Discrete data samples are used to obtain the required estimate of $\underline{\theta}_o^k$ as in example 3.1. The time difference $t_{k+1} - t_k{}^1$ is taken equal to $10T_s$. Fifteen estimation equations are formed from each of the three filtered input-output pairs and least squares estimation is use to obtain an estimate $\hat{\underline{\theta}}_o^k$ in each case. Three sets of results, each corresponds to one filtered input-output pair are listed in table 4.1.

k	1	2	3
$\hat{\underline{p}}_o$	−0.6586E+00 +0.6087E+00 −0.3016E+00	−0.6597E+00 +0.6120E+00 −0.3034E+00	−0.6572E+00 +0.6092E+00 −0.3030E+00
$\hat{\underline{q}}_o^k$	−0.6326E+00 +0.1374E+00	−0.5275E+00 +0.1229E+01	−0.1374E+01 −0.2457E+01

Table 4.1: Estimates of $\underline{p}_o$ and $\underline{q}_o^k$, $k = 1, 2, 3$ from discrete data.

The listed estimates of the output modal parameters are obtained for the following modal time shifts :

$$T_1 = 40T_s$$
$$T_1 = 80T_s$$
$$T_1 = 120T_s$$

Note that the estimates of the output modal parameter vector in each case are the same, which is expected, while the estimates of vectors $\underline{q}_o^k$, k=1,2 and 3, are dependent on k. Each vector $\underline{q}_o^k$, k=1,2 and 3, gives an output modal function y_o^k that corresponds to a filtered input u^k as

$$y_o^k(t) = \underline{q}_o^{k^T} \underline{x}_{ck}(t) \quad , \quad k = 1, 2, 3$$

where $\underline{x}_{ck}$ is the state vector of the subcontroller that generates the frequency component ω_k given by

$$\underline{x}_{ck}(t) = [\sin \omega_k t \quad \cos \omega_k t]^T \quad , \quad k = 1, 2, 3.$$

---

[1]see example 3.1

The filtered input u_o^k obtained as in (4.6) is

$$u_o^k(t) = \underline{\nu}^{k^T} \underline{x}_{ck}(t)$$

where $\underline{\nu}^k$ is given according to (4.5) as

$$\underline{\nu}^k = \underline{c}_{ck}^T (I + \sum_{i=1}^{n} p_{oi} \phi_{ck}(-T_i))$$

where ϕ_{ck} is the state transition matrix of the corresponding subcontroller.

Since the filtered input-output pairs satisfy the system input-output model, it follows from (4.9) that

$$\underline{q}_o^{k^T}(A_{ck}^n + \sum_{j=0}^{n-1} a_j A_{ck}^j) = \sum_{j=0}^{n-1} b_j \underline{\nu}^{k^T} A_{ck}^j \quad , \quad k = 1, 2, 3.$$

For each k, k=1,2 and 3, the last vector equation has dimension two. Therefore, six equations are available in the six unknown model parameters. Solving these equations gives the required estimates of the unknown parameters. The result is listed in table 4.2.

$\hat{\underline{\theta}}$	$\hat{\underline{a}}$	+0.5205E+02 +0.3000E+02 +0.4001E+01
	$\hat{\underline{b}}$	+0.5204E+02 +0.1280E+02 −0.4759E−01

Table 4.2: Estimate of the model parameters.

Example 4.2 :

Consider the same estimation problem as in example 4.1. The conditions of the estimation problem are the same as in example 4.1 with the exception that noisy output samples are used instead of the noise free samples used in that example. The noise added to the output samples is normally distributed with zero mean and has standard deviation 0.01. The bias resulting from using noisy data is evident in the parameter estimates listed in table 4.3.

$\hat{\underline{\theta}}$	$\hat{\underline{a}}$	+0.5671E+02 +0.3945E+02 +0.4170E+01
	$\hat{\underline{b}}$	+0.5596E+02 +0.1266E+02 +0.5583E−02

Table 4.3: Estimates of the model parameters from noisy samples.

The noise added to the output samples affects the estimates of the output modal parameters and the elements of vector $\underline{q}_o$ which are used to obtain the required estimates of the unknown model parameters.

In example 3.4 an instrumental variable was used to reduce the bias in the parameter estimates. A shifted output was used as the instrumental variable. Similarly, the same instrumental variable can be used to obtain unbiased estimates of $\underline{p}_o$ and $\underline{q}_o^k$ which can be used to obtain better estimates of the unknown model parameters. The results listed in table 4.4 shows that the instrumental variable method has worked successfully in reducing the bias in the parameter estimates.

$\hat{\underline{\theta}}$	$\hat{\underline{a}}$	+0.5253E+02 +0.3005E+02 +0.4015E+01
	$\hat{\underline{b}}$	+0.5211E+02 +0.1282E+02 −0.4365E−01

Table 4.4: Estimates of the model parameters using instrumental variables.

4.3 Identification Using Modal Functions and Parameters

In this section the modal functions and the modal parameters of the output and the output derivatives are used to obtain estimates of the parameters of an input-output model of the system, i.e vectors $\underline{a}$ and $\underline{b}$.

4.3.1 Estimation of $\underline{a}$

Theorem 4.1 :

The modal parameter vectors of the output and the output derivatives $\underline{p}_o,\ldots,\underline{p}_n$ satisfy the system characteristic equation :

$$\underline{p}_n^T + \sum_{j=0}^{n-1} a_j \underline{p}_j^T = \underline{o}^T. \tag{4.15}$$

Proof :

It follows from (2.30) that

$$\underline{p}_n^T + \sum_{j=0}^{n-1} a_j \underline{p}_j^T = -\underline{c}^T (A^n + \sum_{j=0}^{n-1} a_j A^j)\psi^{-1}. \tag{4.16}$$

Since the matrix A satisfies its characteristic equation then the sum in brackets on the right-hand side of (4.16) is zero which proves the theorem.

Theorem 4.1 is an important result for parametric system identification. The coefficients of the characteristic equation can be obtained by solving n algebraic equations formed from the modal parameter vectors $\underline{p}_o,\ldots,\underline{p}_n$ as

$$\underline{a} = -P^{T^{-1}} \underline{p}_n \tag{4.17}$$

where

$$P = \begin{bmatrix} \underline{p}_o^T \\ \vdots \\ \underline{p}_{n-1}^T \end{bmatrix} \tag{4.18}$$

Example 4.3 :

Consider the third order system in example 3.1. It is desired to obtain an estimate of $\underline{a}$ using equation (4.17). The modal parameter vectors required in (4.17) has been obtained in examples 3.5 and 3.6 from continuous data using Fourier functions

and Walsh functions respectively. The estimates of the modal parameter vectors resulting from each example are used to obtain an estimate of $\underline{a}$. The results are listed in table 4.5. It is of interest to obtain the poles of the system from the estimate of $\underline{a}$ by finding the roots of the characteristic equation. The roots of the characteristic equation resulting from the estimates of $\underline{a}$ obtained by Fourier functions and by Walsh functions are listed in the same table.

	Using Fourier functions	Using Walsh functions
$\hat{\underline{a}}$	+0.5027E+02 +0.2986E+02 +0.3923E+01	+0.5229E+02 +0.3003E+02 +0.4010E+01
s_1 s_2 s_3	$-0.1932E+01$ $-0.9953E+00 \; +j0.5003E+01$ $-0.9953E+00 \; -j0.5003E+01$	$-0.2011E+01$ $-0.9995E+00 \; +j0.5001E+01$ $-0.9995E+00 \; -j0.5001E+01$

Table 4.5: Estimates of $\underline{a}$ and the system poles using Fourier functions and Walsh functions.

Example 4.4 :

Consider the fourth order system of example 3.10. It is desired to obtain estimates of $\underline{a}$ and the system poles from the estimates of the modal parameter vectors obtained in that example. Two sets of estimates of the modal parameter vectors were obtained using Fourier functions and using Walsh functions. The results of estimating $\underline{a}$ from (4.17) using the results of example 3.10 are listed in table 4.6. The roots of the resulting characteristic equation in both cases are listed in the same table.

4.3.2   Estimation of $\underline{b}$

Theorem 4.2 :

The modal functions of the output and the output derivatives $y_1, \ldots, y_n$ satisfy

$$y_n(t) + \sum_{j=0}^{n-1} a_j y_j(t) = \sum_{j=0}^{n-1} b_j \mathcal{D}^j u(t). \qquad (4.19)$$

	Using Fourier functions	Using Walsh functions
$\underline{\hat{a}}$	+0.3379E+03 +0.2211E+03 +0.1153E+03 +0.5995E+01	+0.3383E+03 +0.2212E+03 +0.1154E+03 +0.6006E+01
s_1	−0.1000E+01 +j0.1499E+01	−0.1000E+01 +j0.1500E+01
s_2	−0.1000E+01 −j0.1499E+01	−0.1000E+01 +j0.1500E+01
s_3	−0.1997E+01 +j0.1000E+02	−0.2003E+01 +j0.1000E+02
s_4	−0.1997E+01 −j0.1000E+02	−0.2003E+01 −j0.1000E+02

Table 4.6: Estimates of $\underline{a}$ and the system poles using Fourier and Walsh functions

Proof:

Equation (4.19) looks quite similar to the input-output model (1.1). Both equations have the same right-hand side. The only difference between both equations is that the output and the output derivatives in (1.1) are replaced by the corresponding modal functions. Let

$$\underline{y}(t) = [y(t - T_1) \ldots y(t - T_n)]^T \tag{4.20}$$

then it follows from the definition of the modal functions of the output and the output derivatives that

$$y_j(t) = \mathcal{D}^j y(t) + \underline{p}_j^T \underline{y}(t) \quad , 0 \leq j \leq n. \tag{4.21}$$

Hence,

$$y_n(t) + \sum_{j=0}^{n-1} a_j y_j(t) = \mathcal{D}^n y(t) + \underline{p}_n^T \underline{y}(t) + \sum_{j=0}^{n-1} a_j (\mathcal{D}^j y(t) + \underline{p}_j^T \underline{y}(t)). \tag{4.22}$$

Rearranging terms yields

$$y_n(t) + \sum_{j=0}^{n-1} a_j y_j(t) = \mathcal{D}^n y(t) + \sum_{j=0}^{n-1} a_j \mathcal{D}^j y(t) + [\underline{p}_n^T + \sum_{j=0}^{n-1} a_j \underline{p}_j^T] \underline{y}(t). \tag{4.23}$$

According to theorem 4.1 the term between rectangular brackets in (4.23) is zero, thus

$$y_n(t) + \sum_{j=0}^{n-1} a_j y_j(t) = \mathcal{D}^n y(t) + + \sum_{j=0}^{n-1} a_j \mathcal{D}^j y(t). \tag{4.24}$$

Equation (4.19) follows directly from (4.24) and the system input-output model (1.1) hence concluding the proof of the theorem.

Equation (4.19) can be used to obtain an estimate of $\underline{b}$ from the modal functions $y_1, \ldots, y_n$ and an estimate of $\underline{a}$. Substituting for $y_1, \ldots, y_n$ and for $u, \ldots, \mathcal{D}^{n-1}u$ in (4.19) by the equivalent in terms of the input modes from (2.44) and (2.42), respectively, yields

$$(\underline{q}_n^T + \sum_{j=0}^{n-1} a_j \underline{q}_j^T)\underline{x}_c(t) = \sum_{j=0}^{n-1} b_j \underline{c}_c^T A_c^j \underline{x}_c(t). \tag{4.25}$$

Equation (4.25) is true for all t, hence it follows that

$$\underline{q}_n^T + \sum_{j=0}^{n-1} a_j \underline{q}_j^T = \sum_{j=0}^{n-1} b_j \underline{c}_c^T A_c^j. \tag{4.26}$$

An estimate of $\underline{b}$ can be obtained by minimizing the norm of the equation error vector which is defined as

$$\underline{e}_b = \underline{q}_t - R_b \underline{b} \tag{4.27}$$

where

$$R_b = [\underline{c}_c \ A_c^T \underline{c}_c \ldots A_c^{(n-1)^T} \underline{c}_c] \tag{4.28}$$

and

$$\underline{q}_t = \underline{q}_n + \sum_{j=0}^{n-1} \underline{q}_j a_j. \tag{4.29}$$

Unless the number of zeros of the system is known a priori, the dimension of $\underline{b}$ is usually assumed to be n. Therefore, the dimenssion of R_b should be at least n by n. The number of rows of R_b is n_c as clear from (2.34) thus the order of the controller generating u should at least be n :

$$n_c \geq n \tag{4.30}$$

The estimate of $\underline{b}$ that minimizes the norm of the equation error vector $\underline{e}_b$ when the strict inequality holds, i.e. $n_c > n$, is given by

$$\hat{\underline{b}} = (R_b^T R_b)^{-1} R_b^T \underline{q}_t . \tag{4.31}$$

Example 4.5 :

Consider the third order system of example 3.1. It is desired to obtain an estimate

of $\underline{b}$ using (4.31). The system has one zero so that there is only two nonzero elements in $\underline{b}$. However, the dimension of $\underline{b}$ will be taken equal to three to illustrate a case where the number of zeros is not known a priori. Estimation of vectors $\underline{q}_j$, $j=0,\ldots,3$, were obtained in example 3.9 using Fourier functions and using Walsh functions. In each case two sets of results were obtained $\hat{\underline{q}}_j^1$ and $\hat{\underline{q}}_j^2$, $j=1,\ldots,3$, each corresponds to a prefiltered input-output measurements . Each of the vectors $\hat{\underline{q}}_j^1$ and $\hat{\underline{q}}_j^2$, $j=0,\ldots,3$, has dimension two which is not adequate to estimate $\underline{b}$ which has dimension three. However, each pair of vectors $\hat{\underline{q}}_j^1$ and $\hat{\underline{q}}_j^2$ can be augmented to obtain a larger vector $\hat{\underline{q}}_j$ of dimension four which is adequate to obtain the required estimate of $\underline{b}$. Estimates of $\underline{q}_j^1$ and $\underline{q}_j^2$, $j=1,\ldots,3$, obtained in example 3.9 and the estimate of $\underline{a}$ obtained in example 4.3 are used to form $\underline{q}_t$ which is needed in (4.31) in order to obtain the required estimate of $\underline{b}$. The results are listed in table 4.7. Two sets of results are listed, each corresponds to using either Fourier functions or Walsh functions in obtaining the estimates of $\underline{p}_j$ and $\underline{q}_j$, $j=0,\ldots,3$.

	Using Fourier functions	Using Walsh functions
$\hat{\underline{b}}$	+0.5191E+02 +0.1291E+02 −0.3879E−01	+0.5181E+02 +0.1266E+02 −0.6689E−01

Table 4.7: Estimates of $\underline{b}$ using Fourier functions and Walsh functions.

4.4 Estimate of $\underline{a}$ from the Free Response

If the input to the system is zero, the vector of unknown parameters $\underline{\theta}_j$ in (3.2) reduces to the modal parameter vector $\underline{p}_j$. In this case, equation (3.37) that gives the estimates of $\underline{\theta}_j$ using Walsh functions reduces to

$$\hat{\underline{p}}_{jW} = W\hat{\underline{p}}_{(j-1)W} \tag{4.32}$$

where

$$W = -(R_W^T R_W)^{-1} R_W^T \Psi (R_d^- - R_d^+). \tag{4.33}$$

From (4.32) it is obvious that a modal parameter vector $\underline{p}_j$ can be given in terms of the output modal parameter vector $\underline{p}_o$ as

$$\hat{\underline{p}}_{jW} = W^j \hat{\underline{p}}_{oW} \tag{4.34}$$

Substituting from (4.34) into (4.15) gives

$$(W^n + \sum_{j=0}^{n-1} a_j W^j) \hat{\underline{p}}_{oW}. \tag{4.35}$$

It can easily be shown by multiplying (4.35) by W^k, $k=0,\ldots, n-1$ and using (4.34) that

$$(W^n + \sum_{j=0}^{n-1} a_j W^j) \hat{P}^T = O \tag{4.36}$$

where $\hat{P}$ is the estimate of P in (4.18). Since P is invertible, it follows from (4.36) that W satisfies the system characteristic equation. This means that W, which is obtained from the system free response, can be used to estimate the system characteristic equation. The following analysis aims at establishing the relation between data matrices, similar to W obtained using Walsh functions, in a more general case.

Let u in (1.2) be zero such that $\underline{x}_f$ and y_f are the free system state and output respectively given as

$$\mathcal{D}\underline{x}_f(t) = A\underline{x}_f(t) \tag{4.37}$$

$$y_f(t) = \underline{c}^T \underline{x}_f(t). \tag{4.38}$$

It follows from (2.1) that for any time shift τ, the shifted output is given by

$$y_f(t + \tau) = \underline{c}^T \phi(\tau)\underline{x}_f. \tag{4.39}$$

Integrating (4.37) from a time instant t_o to another time instant t_f yields

$$\underline{x}_f(t_f) - \underline{x}_f(t_o) = A \int_{t_o}^{t_f} \underline{x}_f(t)dt. \tag{4.40}$$

It is obvious that many data equations (4.40) can be obtained for different pairs of time instants (t_{o1}, t_{f1}), (t_{o2}, t_{f2}), $\ldots$. Writing n data equations (4.40) gives the following

$$[\underline{x}_f(t_{f1}) - \underline{x}_f(t_{o1}) \ldots \underline{x}_f(t_{fn}) - \underline{x}_f(t_{on})] = A[\int_{t_{o1}}^{t_{f1}} \underline{x}_f(t)dt \ldots \int_{t_{on}}^{t_{fn}} \underline{x}_f(t)dt]. \tag{4.41}$$

Let

$$\Xi_n = [\underline{x}_f(t_{f1}) - \underline{x}_f(t_{o1}) \ldots \underline{x}_f(t_{fn}) - \underline{x}_f(t_{on})] \tag{4.42}$$

and

$$\Upsilon_n = [\int_{t_{o1}}^{t_{f1}} \underline{x}_f(t)dt \ldots \int_{t_{on}}^{t_{fn}} \underline{x}_f(t)dt]. \tag{4.43}$$

Then A can be obtained as

$$A = \Xi_n \Upsilon_n^{-1} \tag{4.44}$$

It would be straightforward to obtain A from (4.44) if all the state variables were measurable. The output is seldom the full state so that (4.44) has no practical use. However, the columns of Ξ_n and Υ_n can be transformed from the state space to the output space through an invertible matrix ψ as follows :

$$
\begin{aligned}
Y_n &= \psi\Xi_n = \begin{bmatrix} \underline{c}^T\phi(-T_1) \\ \vdots \\ \underline{c}^T\phi(-T_n) \end{bmatrix} \begin{bmatrix} \underline{x}_f(t_{f1}) - \underline{x}_f(t_{o1}) \ldots \underline{x}_f(t_{fn}) - \underline{x}_f(t_{on}) \end{bmatrix} \\
&= \begin{bmatrix} y_f(t_{f1} - T_1) - y_f(t_{o1} - T_1) & \ldots & y_f(t_{fn} - T_1) - y_f(t_{on} - T_1) \\ \vdots & & \vdots \\ y_f(t_{f1} - T_n) - y_f(t_{o1} - T_n) & \ldots & y_f(t_{fn} - T_n) - y_f(t_{on} - T_n) \end{bmatrix}
\end{aligned} \tag{4.45}
$$

$$
\begin{aligned}
Z_n &= \psi\Upsilon_n = \begin{bmatrix} \underline{c}^T\phi(-T_1) \\ \vdots \\ \underline{c}^T\phi(-T_n) \end{bmatrix} \begin{bmatrix} \int_{t_{o1}}^{t_{f1}} \underline{x}_f(t)dt \ldots \int_{t_{on}}^{t_{fn}} \underline{x}_f(t)dt \end{bmatrix} \\
&= \begin{bmatrix} \int_{t_{o1}}^{t_{f1}} y_f(t - T_1)dt & \ldots & \int_{t_{on}}^{t_{fn}} y_f(t - T_1)dt \\ \vdots & & \vdots \\ \int_{t_{on}}^{t_{fn}} y_f(t - T_n)dt & \ldots & \int_{t_{on}}^{t_{fn}} y_f(t - T_n)dt \end{bmatrix}
\end{aligned} \tag{4.46}
$$

A transformation of A can be obtained from the measured output through (4.45) and (4.46) based on the following theorem:

Theorem 4.3 :

For an observable system, the matrix Λ given by

$$\Lambda = Y_n Z_n^{-1} \tag{4.47}$$

where Y_n and Z_n are given by (4.45) and (4.46) respectively, has the same eigenvalues as A.

Proof :

Substituting for Y_N and Z_n from (4.45) and (4.46) respectively into (4.47) yields

$$\Lambda = \psi \Xi_n \Upsilon^{-1} \psi^{-1}. \tag{4.48}$$

It follows from (4.48) and (4.44) that

$$\Lambda = \psi A \psi^{-1}. \tag{4.49}$$

Equation (4.49) shows that Λ is a transformation of A which maintains the eigenvalues of A unchanged if ψ is invertible which completes the proof.

Although theorem 4.3 does not give the matrix A directly in terms of the available measurements, it is possible to obtain a transformation of A, matrix Λ, from which it is possible to obtain the eigenvalues of A and its characteristic equation. The eigenvalues of A are obtained by solving for the eigenvalues of Λ, and the coefficients of the characteristic equation of A are obtained from the simple recursive relation [Sinha,P.K. 84, page 33]

$$a_{n-k} = \begin{cases} -T_k & , k = 1 \\ -\frac{1}{k}(a_{n-k-1}T_1 + a_{n-k}T_2 + \cdots + a_{n-1}T_{k-1} + T_k) & , k = 2, \dots, n \end{cases} \tag{4.50}$$

where T_k is the trace of Λ^k. As seen from (4.50), only n matrix multiplications are needed to obtain the coefficients of the characteristic equation of the matrices Λ and A which makes it attractive from the computational point of view especially because Λ is obtained from the available measurements through simple averaging and difference operations plus a single matrix inversion of an $n \times n$ matrix. The number of columns in Ξ_n and Υ_n is not restricted to the order of the system n. It is possible to form matrices Ξ_N and Υ_N with a number of columns $N > n$ as follows :

$$\Xi_N = [\Xi_n \; \underline{x}_f(t_{f(n+1)}) - \underline{x}_f(t_{o(n+1)}) \dots \underline{x}_f(t_{fN}) - \underline{x}_f(t_{oN})] \tag{4.51}$$

and

$$\Upsilon_N = [\Upsilon_n \int_{t_{o(n+1)}}^{t_{f(n+1)}} \underline{x}_f(t)dt \dots \int_{t_{oN}}^{t_{fN}} \underline{x}_f(t)dt] \tag{4.52}$$

It follows from (4.41) that a least squares estimate of A obtained from Ξ_N and Υ_N is given by

$$\hat{A} = \Xi_N \Upsilon_N^T (\Upsilon_N \Upsilon_N^T)^{-1}. \tag{4.53}$$

Consequently an estimate of Λ is given by

$$\hat{\Lambda} = \psi \hat{A} \psi^{-1}. \tag{4.54}$$

It can easily be shown that $\hat{\Lambda}$ is given in terms of the available measurements by

$$\hat{\Lambda} = Y_N Z_N^T (Z_N Z_N^T)^{-1} \tag{4.55}$$

where Y_N and Z_N are obtained from equations similar to (4.45) and (4.46) respectively with N replacing n.

Example 4.6 :

Consider the fourth order system in example 3.10. It is required to obtain the coefficients of the characteristic equation of the system and the system poles from the free response. The free response of the system is computed for the following initial state :

$y(0) = 0.2$

$\mathcal{D}y(0) = 10.$

$\mathcal{D}^2 y(0) = 0.$

$\mathcal{D}^3 y(0) = 0.$

Samples of the system response are computed each time interval $T_s = \pi/420$. First, the matrices Y_N and Z_N, $N = 15 > n = 4$, are formed from the free response. The values of the time shifts $T_1, \ldots, T_4$ are taken as follows :

$T_1 = 40T_s$

$T_2 = 80T_s$

$T_3 = 100T_s$

$T_4 = 120T_s$

The time instant pairs (t_{ok}, t_{fk}), $k=1,\ldots,15$, are taken such that $t_{fk} - t_{ok} = 20T_s$ and $t_{o(k+1)} - t_{ok} = 20T_s$ for the specified range of k. An estimate of Λ is obtained as in (4.55). The coefficients of the characteristic equation are obtained recursively from (4.50), and the poles of the system are obtained as the eigenvalues of the estimate of Λ. The results are listed in table 4.8.

Estimation of of the system characteristic equation using theorem 4.3 is based on the assumption that y_f is the unforced response of the system. However, theorem

$\hat{\underline{a}}$	System poles
+0.3388E+03	$-0.2007\text{E}+01\ +j0.1001\text{E}+02$
+0.2216E+03	$-0.2007\text{E}+01\ -j0.1001\text{E}+02$
+0.1156E+03	$-0.1000\text{E}+01\ +j0.1500\text{E}+01$
+0.6014E+01	$-0.1000\text{E}+01\ -j0.1500\text{E}+01$

Table 4.8: Estimates of $\underline{a}$ and the system poles.

4.3 can be extended to include systems driven by the free response of continuous controllers give by (2.35) and (2.36). The system and the controller are augmented to obtain a higher order system given by

$$\mathcal{D}\left[\begin{array}{c} \underline{x}(t) \\ \underline{x}_c(t) \end{array}\right] = \left[\begin{array}{cc} A & \underline{b}\,\underline{c}_c^T \\ O & A_c \end{array}\right]\left[\begin{array}{c} \underline{x}(t) \\ \underline{x}_c(t) \end{array}\right] \tag{4.56}$$

and

$$y(t) = \left[\begin{array}{cc} \underline{c}^T & \underline{o}^T \end{array}\right]\left[\begin{array}{c} \underline{x}(t) \\ \underline{x}_c(t) \end{array}\right]. \tag{4.57}$$

The order of the augmented system is $n + n_c$, and the output y is considered as the free response of the augmented system as clear from (4.56) which does not include a forcing term. The augmented system dynamic matrix is given by

$$A_a = \left[\begin{array}{cc} A & \underline{b}\,\underline{c}_c^T \\ O & A_c \end{array}\right] \tag{4.58}$$

The $n + n_c$ eigen values of A_a are those of A and A_c as can be seen from the *diagonal* form of A and A_c. If matrix M_a is formed from the modal matrices M and M_c of A and A_c respectively as

$$M_a = \left[\begin{array}{cc} M & O \\ O & M_c \end{array}\right]. \tag{4.59}$$

Since M and M_c are invertible it follows that M_a also is invertible and M_a^{-1} is given by

$$M_a^{-1} = \left[\begin{array}{cc} M^{-1} & O \\ O & M_c^{-1} \end{array}\right]. \tag{4.60}$$

A matrix Λ_a given by

$$\Lambda_a = M_a A_a M_a^{-1} \tag{4.61}$$

has the same eigenvalues as A_a. From (4.58) to (4.61) it follows that

$$\Lambda_a = \left[\begin{array}{cc} A^* & M\underline{b}\,\underline{c}_c^T M_c^{-1} \\ O & A_c^* \end{array}\right] \tag{4.62}$$

where A^* and A_c^* are the *diagonal* matrices of A and A_c respectively obtained as $A^* = MAM^{-1}$ and $A_c^* = M_c A_c M_c^{-1}$. It is obvious that Λ_a is upper diagonal, and since the eigenvalues of a triangular matrix are its diagonal elements [Strang 80, page 187], it follows from (4.62) that the eigenvalues of Λ_a are the eigenvalues of A^* and A_c^*.

The characteristic equation of the augmented system and its eigenvalues are obtained as in example 4.6 assuming a system of order $n + n_c$ and a free system response. The eigenvalues of A which correspond to the poles of the actual system can be distinguished from the eigenvalues of A_c which are known. The characteristic polynomial of the augmented system is the product of the characteristic polynomials of the actual system and the characteristic polynomial of the controller. So if $P(s)$, $P_c(s)$ and $P_a(s)$ are the characteristic polynomials of the actual system, the controller and the augmented system respectively then

$$P_a(s) = P(s)P_c(s) \tag{4.63}$$

The characteristic polynomial of the actual system can be obtained from (4.63) either by equating the coefficients of the like powers of s on both sides of the equation or by long division, under the assumption that the characteristic polynomial of the controller $P_c(s)$ is known.

Example 4.7 :

The third order system of example 3.1 is excited by a sinusoidal input of frequency 4

$$u(t) = \sin 4t \ .$$

The order of the controller is $n_c = 2$ and has two imaginary poles at $\pm j4$. The output of the system is computed for zero initial state with the sampling time $T_s = \pi/420$. An augmented system of order $n + n_c = 3 + 2 = 5$ is assumed. The matrices Y_N and Z_N, $N = 20 > 5$, are formed from the system response for the following time shifts :

$T_1 = 40T_s$
$T_2 = 80T_s$
$T_3 = 110T_s$

$$T_4 = 140T_s$$
$$T_5 = 170T_s$$

The time instant pairs (t_{ok}, t_{fk}), $k=1,\ldots,20$, are taken such that $t_{fk} - t_{ok} = 20T_s$ and $t_{o(k+1)} - t_{ok} = 20T_s$ for the specified range of k. An estimate of Λ_a is obtained as in (4.55). The coefficients of the characteristic equation of the augmented system are obtained from (4.50) using the estimate of Λ_a. The estimated characteristic polynomial is

$$P_a(s) = s^5 + 3.992s^4 + 45.93s^3 + 115.5s^2 + 478.4s + 827.1.$$

The characteristic polynomial of the controller is

$$P_c(s) = s^2 + 16 .$$

Substituting in (4.63) and equating the like powers of s on both sides gives the following equations in the unknowns a_o, a_1, and a_2

$$16a_o = 827.1$$
$$16a_1 = 474.4$$
$$16a_2 + a_o = 115.5$$
$$a_2 = 3.992$$

These five equations can in general be solved by least squares to obtain the three unknowns. However a more direct solution can be obtained from the first, the second and the last equations. The results are listed in table 4.9.

$\hat{\underline{a}}$	Eigenvalues of $\hat{\Lambda}_a$
+0.5169E+02 +0.2965E+02 +0.3992E+01	+0.3146E−02 +j0.3995E+01 +0.3146E−02 −j0.3995E+01 −0.1003E+01 +j0.5000E+01 −0.1003E+01 −j0.5000E+01 −0.1993E+01 +j0.0000E+00

Table 4.9: Eigen values of $\hat{\Lambda}_a$.

The poles of the augmented system are obtained by solving for the eigenvalues of the estimate of Λ_a. The results are listed in table 4.9. It is obvious that the first

two eigenvalues correspond to the controller two imaginary poles while the other three correspond to the actual system poles.

The new method for obtaining the poles of the system as the eigenvalues of an output data matrix, as described above, has important advantages over the known methods based on discrete prediction models of the system like Prony's method. The new method uses continuous measurements so that possible aliasing, associated with sampling continuous signals, is not a problem. The flexibility of choosing the modal time shift allows for modeling the signal low frequency components as well as the high frequency components without aliasing. The poles of the continuous system are obtained as the eigenvalues of a data matrix, while if a discrete model is used, the continuous system poles are obtained from the roots of the characteristic equation of the estimated discrete model by a logarithmic operation.

The following example illustrates a case of a wideband signal where Prony's method fails to determine the exponential modes of the signal accurately, while the new method yields accurate results.

Example 4.8 :

Consider a signal

$$y(t) = e^{-t}\sin 3t + \cos 100t.$$

Let ω_l and ω_h be 3 and 100 respectively, so that the period of the low frequency component and the high frequency component are $T_l = 2\pi/3$ and $T_h = 2\pi/100$ respectively. Samples of y are available every sampling interval $T_s = T_l/666 \approx T_h/20$. It is required to determine the complex numbers $s_k, k = 1, \ldots, 4$, so that y can be modeled by

$$y(t) = \sum_{k=1}^{4} C_k e^{s_k t}.$$

It is clear that the complex exponents are

$s_1 = -1 + j3$

$s_2 = -1 - j3$

$s_3 = +j100$

$s_4 = -j100.$

First, estimating the complex exponents is done indirectly through the discrete

prediction model

$$y(mT) + \sum_{k=1}^{4} p_k y((m-k)T) = 0.$$

The parameters of the discrete model are estimated from samples of y using least squares

$$\underline{p} = -(R^T R)^{-1} R^T \underline{y}$$

where R is an $n \times N$ matrix whose m^{th} row is given by

$$\underline{\imath}_m^T R = \left[\begin{array}{ccc} y(m+2) & \ldots & y(m-1) \end{array} \right]$$

and

$$\underline{y} = \left[\begin{array}{ccc} y(4) & \ldots & y(N+3) \end{array} \right]^T.$$

The poles of the discrete model $z_k, k = 1, \ldots, 4$, are obtained as the roots of

$$1 + \sum_{k=1}^{4} p_k z^k = 0.$$

Finally each s_k is obtained from z_k by the logarithmic operation (1.6).

The choice of T in the discrete model is a problem because y has two widely separated frequency components. Choosing T comparable with T_l results in aliasing as seen from the results listed in table 4.10, which are obtained for $T = 33T_s$ and $N = 10$. On the other hand, choosing a small T according to the sampling theory, $T < T_h/2$, results in losing the accuracy of modeling the low frequency component as seen from the results listed in the same table , which are obtained for $T = 4T_s \approx T_h/5$ and the same N. It is seen from the results in the table that the accuracy of the estimate of ω_l is deteriorated by the small T.

The condition of the estimation equations for the discrete model parameters can be measured by testing the condition number, [Strang 80, page 282], of the matrix $(R^T R)$. The higher the condition number is, the more ill-conditioned the matrix is. The condition numbers of the two cases of $T = 33T_s$ and $T = 4T_s$ are 0.6768E+01 and 0.6816E+04, respectively. The condition number in the case of small T is three orders of magnitude larger than the condition number in the case of large T. This can be interpreted as due to the high correlation between the columns of R in the case of small T.

Using $T = 33T_s$	Using $T = 4T_s$
−0.1000E+01 +j0.3000E+01	−0.1187E+01 +j0.2261E+01
−0.1000E+01 −j0.3000E+01	−0.1187E+01 −j0.2261E+01
−0.5744E–06 +j0.2109E+02	−0.5260E–03 +j0.1000E+03
−0.5744E–06 −j0.2109E+02	−0.5260E–03 −j0.1000E+03

Table 4.10: Estimates of s_k from the discrete model.

Applying the new method and using the modal time shifts $T_i = iT$ where $T = 33T_s$, and using time instant pairs (t_{ok}, t_{fk}), $k=1,\ldots,10$, such that $t_{fk} - t_{ok} = t_{o(k+1)} - t_{ok} = 33T_s$, give the results listed in table 4.11. It is obvious that there is no probem of aliasing despite the large modal time shifts.

Using $T = 33T_s$
−0.1000E+01 +j0.3000E+01
−0.1000E+01 −j0.3000E+01
+0.3586E–03 +j0.1008E+03
+0.3586E–03 −j0.1008E+03

Table 4.11: Estimates of s_k using the new method.

4.5   Summary

Chapter 4 deals with the use of modal functions and parameters for determining model parameter estimation equations that do not depend on the system initial conditions or the use of repeated integrations to eliminate derivatives. The use of the output modal function, which is composed of known exponential modes, in place of the general system output gives an algebraic vector equation (4.9) that relates the exponential modes of the output modal function to those of the filtered input through the unknown model parameters and which can be solved for these parameters.

Theorem 4.1 shows that the modal parameter vectors satisfy the system characteristic equation. Using this result yields (4.17) which proves that the a-parameters can be identified independently of the b-parameters. When the a-parameters are

known, theorem 4.2 can be used to identify the *b*-parameters as in (4.26).

Theorem 4.3 proves that the system eigen values can be identified from an output data matrix. By use of an augmented state vector this approach can be extended to the case of nonzero inputs. The output data matrix allows the identification of the coefficients of the system characteristic polynomial recursively from (4.50). It is shown that the new approach yields accurate results with wideband signals whereas Prony's method fails.

Chapter 5

Nonparametric Identification

5.1 Introduction

The identification of nonparametric system models is considered in this chapter. Samples of infinite dimensional models, namely the frequency response, the step response and the impulse response, are obtained.

Obtaining the system frequency response by sinusoidal input testing usually requires long experimental time to let the system response reach the periodic steady state, and therefore it is not suitable for real-time applications. On the other hand, obtaining the frequency response from finite data records has its own drawbacks. Data windowing results in a smearing of the signal spectrum in a manner dependent upon the precise shape of the window, e.g. rectangular, Hanning or Hamming [Marple 87, Sec. 5.3]. It is necessary to average the spectrum estimates over a large number of data windows in order to reduce the variance induced in these estimates by the response transient component. Finite data frames also induce bias in the spectrum estimates which is not decreased by increasing the number of frames. The bias is reduced by increasing the duration of the data frame, i.e. longer test time. Estimating the impulse response by deconvolution using the input auto-correlation and the input-output cross-correlation functions involves windowing, bias and variance errors similar to the case of estimating the frequency response [Wellstead 81]. An advantage of direct nonparametric modeling over parametric modeling of systems is that no a priori knowledge of the system order is required. However, the indirect nonparametric methods described in this chapter do not have this advantage because they are based on the definitions of the modal functions

and the modal parameters which require the knowledge of the system order.

Section 5.2 presents a method of estimating the frequency response of the system from the output modal function for a periodic waveform input. The output modal function is utilized again in section 5.3 to estimate the step response from the system response to a general piecewise constant input. The system impulse response is obtained as the derivative of the estimated step response using the output first derivative modal parameters in section 5.4.

5.2   Estimation of the Frequency Response

The output modal function y_o played a key role in estimating the unknown model parameters in section 4.2. As mentioned earlier the output modal function represents a particular solution to the model differential equation when the input is the filtered input u_o given by (4.1). This particular solution has the property that it contains modes of the input only and the transient component of the system output due to system modes is not present in y_o. Consequently, if the input is periodic, i.e. a summation of complex conjugate imaginary modes, the output modal function is periodic with the same spectral components as the input. The periodic filtered input-output pair (u_o, y_o) can be used to estimate samples of the system frequency response.

Let u be a sinusoidal input with frequency ω_o such that the controller matrix A_c in phase variable form is given by

$$A_c = \begin{bmatrix} 0 & \omega_o \\ -\omega_o & 0 \end{bmatrix} \qquad (5.1)$$

and the controller state vector is given by

$$\underline{x}_c(t) = [\sin \omega_o t \quad \cos \omega_o t]^T \qquad (5.2)$$

The filtered input u_o and the filtered output y_o are given by (4.6) and (2.26), respectively. Choosing the controller state equation in phase variable form makes it possible to obtain vectors $\underline{\nu}$ and $\underline{q}_o$, in (4.6) and (2.44), by correlating u_o and y_o, respectively with the vector $[\sin \omega_o t \quad \cos \omega_o]^T$ over the period of the input.

It follows from equation (4.9), by multiplying both sides by the the inverse of $(A_c^n + \sum_{j=0}^{n-1} a_j A_c^j)$, that

$$\underline{q}_o^T = \sum_{j=0}^{n-1} b_j \underline{\nu}^T A_c^j (A_c^n + \sum_{j=0}^{n-1} a_j A_c^j)^{-1} . \tag{5.3}$$

The diagonal form of A_c in (5.1) is given by

$$A_c^* = \begin{bmatrix} \jmath\omega_o & 0 \\ 0 & -\jmath\omega_o \end{bmatrix} . \tag{5.4}$$

The matrix A_c is related to its diagonal form A_c^* through the modal matrix M as

$$A_c = M A_c^* M^{-1} \tag{5.5}$$

where

$$M = \begin{bmatrix} -\jmath & \jmath \\ 1 & 1 \end{bmatrix} \tag{5.6}$$

and

$$M^{-1} = \frac{1}{2} \begin{bmatrix} \jmath & 1 \\ -\jmath & 1 \end{bmatrix} . \tag{5.7}$$

Substituting from (5.5) into (5.3) gives

$$\underline{q}_o^T = \underline{\nu}^T M \left(\sum_{j=o}^{n-1} b_j A_c^{*j} \right) \left(A_c^{*n} + \sum_{j=0}^{n-1} a_j A_c^{*j} \right)^{-1} M^{-1}. \tag{5.8}$$

The frequency response of a system is given as a complex function of the frequency ω by

$$H(\jmath\omega) = \frac{\sum_{j=0}^{n-1} b_j (\jmath\omega)^j}{(\jmath\omega)^n + \sum_{j=0}^{n-1} a_j (\jmath\omega)^j}. \tag{5.9}$$

Hence, it follows from (5.9) and (5.8) that

$$\underline{q}_o^T = \underline{\nu}^T M \begin{bmatrix} H(\jmath\omega_o) & 0 \\ 0 & H(-\jmath\omega_o) \end{bmatrix} M^{-1}. \tag{5.10}$$

Substituting for M and M^{-1} in (5.10) gives

$$\underline{q}_o^T = \underline{\nu}^T \begin{bmatrix} H(\jmath\omega_o) + H(-\jmath\omega_o) & -\jmath(H(\jmath\omega_o) - H(-\jmath\omega_o)) \\ \jmath(H(\jmath\omega_o) - H(-\jmath\omega_o)) & H(\jmath\omega_o) + H(-\jmath\omega_o) \end{bmatrix} . \tag{5.11}$$

Let $Re(H(\jmath\omega))$ and $Im(H(\jmath\omega))$ be the real part and the imaginary part of $H(\jmath\omega)$, respectively. It follows from (5.11) that

$$\underline{q}_o^T = \underline{\nu}^T \begin{bmatrix} Re(H(\jmath\omega_o)) & Im(H(\jmath\omega_o)) \\ -Im(H(\jmath\omega_o)) & Re(H(\jmath\omega_o)) \end{bmatrix} . \tag{5.12}$$

From matrix and vector multiplication rules, it can easily be verified that (5.12) can be written as

$$\underline{q}_o^T = \left[\begin{array}{cc} Re(H(\jmath\omega_o)) & Im(H(\jmath\omega_o)) \end{array} \right] \left[\begin{array}{cc} \nu_1 & \nu_2 \\ -\nu_2 & \nu_1 \end{array} \right] \qquad (5.13)$$

where ν_1 and ν_2 are the elements of vector $\underline{\nu}$. A sample of the system frequency response evaluated at frequency ω_o is obtained from (5.13) as

$$\left[\begin{array}{cc} Re(H(\jmath\omega_o)) & Im(H(\jmath\omega_o)) \end{array} \right] = \underline{q}_o^T \left[\begin{array}{cc} \nu_1 & \nu_2 \\ -\nu_2 & \nu_1 \end{array} \right]^{-1} . \qquad (5.14)$$

Equation (5.14) gives one sample of the required frequency response. In order to obtain samples at different frequencies, a periodic input with multiple frequencies should be used. For each harmonic frequency $\omega_k = k\omega_o$, two vectors $\underline{\nu}^k$ and $\underline{q}_o^k$, each of dimension two, can be obtained by correlating u_o and y_o, respectively with a vector $[\sin\omega_k t \quad \cos\omega_k t]^T$ over a time period of the fundamental $T_o = 2\pi/\omega_o$. Each resulting vector pair $\underline{\nu}^k$ and $\underline{q}_o^k$ can be used in (5.14) to obtain one sample of the frequency response at frequency ω_k.

Example 5.1 :

From the same input-output measurements of the system considered in example 3.1 it is required to obtain samples of the frequency response of the system. Starting from rest, the system is excited by a square wave with a fundamental frequency $\omega_o=1$. Only odd harmonics of ω_o are present in the spectrum of the input, so it is possible to estimate only the frequency response samples of the system at ω_o and odd multiples of ω_o.

The output modal parameters were obtained in example 3.1. these parameters are used to filter the input-output measurements (u, y) to obtain (u_o, y_o). Pairs of vectors $\underline{\nu}^k$ and $\underline{q}_o^k$, for odd k, are obtained by correlating u_o and y_o, respectively, with a vector $[\sin kt \quad \cos kt]^T$ over the time period of the fundamental which is equal to 2π in this case. Vector pairs obtained for odd k from $k = 1$ to $k = 11$ are listed in table 5.1. These vectors are used in (5.14) to obtain the required estimates of the system frequency response samples at the corresponding frequencies. The results are listed in table 5.2 with the true values of the corresponding samples as computed from (5.9).

k	ν_1^k	ν_2^k	q_{o1}^k	q_{o2}^k
1	+0.1436E+01	+0.2359E+01	+0.1974E+01	+0.1751E+01
3	−0.9008E+00	+0.3559E+00	−0.4778E+00	+0.8423E+00
5	+0.1395E+00	−0.3717E+00	−0.6021E+00	−0.1003E+00
7	+0.4124E+00	+0.6293E+00	−0.1089E+00	−0.3875E+00
9	−0.6618E+00	+0.7798E+00	+0.1795E+00	−0.1675E+00
11	−0.8865E+00	−0.2632E+00	+0.1213E+00	+0.4375E−01

Table 5.1: Spectral components of u_o and y_o.

k	$Re(\hat{H}(\jmath\omega_k))$	$Im(\hat{H}(\jmath\omega_k))$	$Re(H(\jmath\omega_k))$	$Im(H(\jmath\omega_k))$
1	+0.9135E+00	−0.2811E+00	+0.9135E+00	−0.2811E+00
3	+0.7784E+00	−0.6276E+00	+0.7785E+00	−0.6277E+00
5	−0.2964E+00	−0.1509E+01	−0.2974E+00	−0.1509E+01
7	−0.5101E+00	−0.1612E+00	−0.5099E+00	−0.1610E+00
9	−0.2384E+00	−0.2782E−01	−0.2383E+00	−0.2795E−01
11	−0.1392E+00	−0.8015E−02	−0.1393E+00	−0.8181E−02

Table 5.2: Estimated and true values of samples of the frequency response.

One advantage of the method described above to obtain the frequency response of the system is that it uses input-output measurements over a relatively short time interval which makes the method suitable for unstable systems as well as for stable systems. The periodic response of the system is obtained by filtering u and y to get u_o and y_o through the modal filter which annihilates the modes contributed by system which represents the transient response of the system. Practically, to obtain a periodic response the experiment time should be long enough to let the transient component of the response settle down which is not possible for an unstable system. If on the other hand digital spectral estimates of the auto and cross correlation functions of u and y from finite data records are used to obtain the required estimates of the system frequency response, the obtained estimates will suffer from the inherent errors to the digital spectral estimation method, namely windowing errors and aliasing [Wellstead 81].

5.3   Estimation of the Step Response

A simple input that can be applied to a system for the purpose of identification is a step input. If the system is initially at rest when a step input is applied, then the output of the system is the step response. If this is not the case and the system starts from a nonzero state, which is common in practice, the output of the system is the sum of the step response and the response due to the nonzero initial state. A more general input is a piece-wise constant input given by

$$u(t) = \sum_{m=0}^{k} v(m) \Delta(t - mT) \qquad , kT \le t \le (k+1)T \tag{5.15}$$

where

$$v(m) = \left\{ \begin{array}{ll} u(mT) - u((m-1)T) & , m > 0 \\ u(m) & , m = 0 \end{array} \right. \tag{5.16}$$

and $\Delta(t)$ is the unit step function. The impulse response h of the system is given by

$$h(t) = \underline{c}^T \phi(t) \underline{b}. \tag{5.17}$$

Substituting from (5.17) into (2.1) using the piece-wise constant input given by (5.15) yields

$$y(t) = \underline{c}^T \phi(t) \underline{x}(0) + \int_0^t h(t - \tau) \sum_{m=0}^{k} v(m) \Delta(\tau - mT) d\tau. \tag{5.18}$$

The upper limit of the summation in (5.18) is k although τ does not necessarily satisfy $kT \le \tau \le (k+1)T$ as required in (5.15). However, it is possible to use k as the upper limit since the step function $\Delta(t)$ is zero for $t < 0$. Interchanging the order of integration and summation in (5.18) and using the property that $\Delta(t)$ is zero for $t < 0$ yields

$$y(t) = \underline{c}^T \phi(t) \underline{x}(0) + \sum_{m=0}^{k} \int_{mT}^t h(t - \tau) d\tau v(m). \tag{5.19}$$

The step response g is the integration of the impulse response h:

$$g(t) = \int_0^t h(\tau) d\tau \tag{5.20}$$

consequently (5.19) can be written in terms of g as

$$y(t) = \underline{c}^T \phi(t) \underline{x}(0) + \sum_{m=0}^{k} g(t - mT) v(m). \tag{5.21}$$

The fact that $g(0)$ is zero[1] is used to obtain (5.21) from (5.19). It is clear from (5.21) that y is composed of the response due to the nonzero initial state and the convolution summation of the input steps and the system step response.

The determination of the step response from (5.21) by deconvolution is complicated by the existence of the initial state response term. The effect of this term is usually accounted for by considering N terms in the convolution summation with N large enough so that the effect of the nonzero initial state is negligible for $t > NT$. Therefore, for a stable system with an impulse response that *vanishes* for $t > NT$, the output is given approximately by

$$y(t) \approx g(t - (k - N)T)u(k - N) + \sum_{m=k-N+1}^{k} g(t - mT)v(m). \qquad (5.22)$$

An alternative exact approach to the deconvolution problem is considered below using the output modal function.

The output modal function as defined by equation (2.26) is the sum of weighted output samples. Substituting for the output samples in (2.26) from (5.21) and taking the modal time shifts $T_i = iT$, $i=,\ldots,n$, yields

$$y_o(t) = \sum_{m=0}^{k} g(t - mT)v(m) + \sum_{i=1}^{n} p_{oi} \sum_{m=0}^{k-i} g(t - mT - iT)v(m). \qquad (5.23)$$

The response component due to the nonzero initial state is annihilated as a result the output filtering operation, so that it is possible to obtain the step response by deconvolution using y_o as given below.

Each of the summation from $m=0$ to $m=k-i$, $i=0,\ldots,n$, can be divided into two summations; one from $m=0$ to $m=k-n$ and the other from $m=k-n+1$ to $m=k-i$ such that (5.23) becomes, after arranging terms,

$$\begin{aligned} y_o(t) &= \sum_{m=0}^{k-n} [g(t - mT) + \sum_{i=1}^{n} p_{oi} g(t - mT - iT)]v(m) \\ &+ \sum_{m=k-n+1}^{k} g(t - mT)\nu(m) + \sum_{i=1}^{n-1} p_{oi} \sum_{m=k-n+1}^{k-i} g(t - mT - iT)v(m) \end{aligned} \qquad (5.24)$$

The term in rectangular brackets in (5.24) is the output modal function of the system which corresponds to a step input. It has been shown before that the

---

[1]follows from equation (5.20)

modal function which corresponds to a constant input is a constant, hence for $t > nT$ the term in the rectangular brackets in (5.24) is constant and can be taken outside the summation

$$
\begin{aligned}
y_o(t) \quad = \quad & [g(t - (k - n)T) + \sum_{i=1}^{n} p_{oi}g(t - (k - n)T - iT)] \sum_{m=0}^{k-n} v(m) \\
+ \quad & \sum_{m=k-n+1}^{k} g(t - mT)v(m) + \sum_{i=1}^{n-1} p_{oi} \sum_{m=k-n+1}^{k-i} g(t - mT - iT)v(m).
\end{aligned}
$$

Substituting

$$
t = kT + s \qquad , 0 \leq s < T \qquad \qquad (5.25)
$$

and augmenting the last two summations over i and changing the summation argument m to $l=k-m-i$, $i=0,\ldots,n-1$, gives

$$
\begin{aligned}
y_o(kT + s) \quad = \quad & [g(s + nT) + \sum_{i=1}^{n} p_{oi}g(s + (n - i)T)]u(k - n) \\
+ \quad & \sum_{i=0}^{n-1} p_{oi} \sum_{l=0}^{n-i-1} g(s + lT)v(k - i - l). \qquad (5.26)
\end{aligned}
$$

In (5.26) $p_{oo} = 1$. Equation (5.26) can be put in vector form as

$$
\begin{aligned}
y_o(kT + s) \quad = \quad & \begin{bmatrix} v(k) & v(k - 1) & \ldots & v(k - n + 1) & u(k - n) \end{bmatrix} \\
\times \quad & \begin{bmatrix} 1 & & & O \\ p_{o1} & 1 & & \\ \vdots & & \ddots & \\ p_{on} & \ldots & p_{o1} & 1 \end{bmatrix} \begin{bmatrix} g(s) \\ g(s + T) \\ \vdots \\ g(s + nT) \end{bmatrix}. \qquad (5.27)
\end{aligned}
$$

Let the row vector $\underline{u}^T$ be

$$
\underline{u}^T(k) = \begin{bmatrix} u(k) & u(k - 1) & \ldots & u(k - n) \end{bmatrix} \qquad (5.28)
$$

such that

$$
\begin{bmatrix} v(k) & v(k - 1) & \ldots & v(k - n + 1) & u(k - n) \end{bmatrix} = \underline{u}^T(k)\Delta \qquad (5.29)
$$

where

$$
\Delta = \begin{bmatrix} 1 & & & O \\ -1 & 1 & & \\ 0 & -1 & \ddots & \\ & & \ddots & 1 & 0 \\ O & & & -1 & 1 \end{bmatrix}. \qquad (5.30)
$$

Let $\underline{g}(s)$ be a vector of dimension $n+1$ given by

$$\underline{g}(s) = \left[\; g(s) \quad g(s+T) \quad \ldots \quad g(s+nT) \;\right]^T .$$

(5.31)

A least squares estimate of $\underline{g}(s)$ can be obtained from $N > n + 1$ equations like (5.27) for different k as

$$\hat{\underline{g}}(s) = (\Delta L)^{-1}(U^T U)^{-1} U^T \underline{y}_o(s)$$

(5.32)

where

$$\underline{y}_o(s) = \left[\; y_o(s) \quad y_o(s+T) \quad \ldots \quad y_o(s+(N-1)T) \;\right]^T$$

(5.33)

$$L = \begin{bmatrix} 1 & & & O \\ p_{01} & 1 & & \\ \vdots & & \ddots & \vdots \\ p_{on} & \cdots & p_{o1} & 1 \end{bmatrix}$$

(5.34)

and the kth row of U is given by

$$\underline{i}_k^T U = \underline{u}(k) .$$

(5.35)

Equation (5.32) gives estimates of samples of $g(t)$ over the time interval $0 \rightarrow (n + 1)T$ only. Estimates of samples of $g(t)$ beyond this time interval can be obtained by extrapolation as given below.

For $t \geq nT > 0$ the step function is constant and consequently the output modal function of the step response is a constant given by

$$g(t) + \sum_{i=1}^{n} p_{oi} g(t - iT) = q_{oo} \qquad , t \geq nT > 0.$$

(5.36)

The extrapolation relation of $g(t)$ for $t \geq s + (n + k)T$, $k \geq 1$ follows from (5.36) as

$$\begin{aligned} g(s + (n + k)T) &= q_{oo} - \sum_{i=1}^{n} p_{oi} g(s + (n + k - i)T) \\ &= g(s + (n + k - 1)T) + \sum_{i=1}^{n} p_{oi} g(s + (n + k - i - 1)T) \\ &\quad - \sum_{i=1}^{n} p_{oi} g(s + (k + n - i)T) \qquad , k \geq 1. \end{aligned}$$

(5.37)

Example 5.2 :

It is required to obtain estimates of samples of the step response of the third

order system in example 3.1 using (5.32) and (5.37). The system is excited by a piece-wise constant input +1, +2, −1 and −2 such that each constant input level lasts for $80T_s$ where $T_s = \pi/320$. Starting from rest, the output of the system is computed every sampling time T_s. The time interval T is taken equal to $20T_s$. The output modal parameters are obtained as in example 3.1 from discrete data, and (5.32) is used to obtain estimates of $g(t)$ over the time interval $0 \rightarrow 80T_s$. Finally, (5.37) is used to obtain estimates of samples of $g(t)$ for $t > 80T_s$. Estimates of samples of $g(t)$ obtained every $10T_s$ are listed in table 5.3. The true values of the corresponding samples are listed beside the estimated values. From the listed results it is seen that the estimated values are exactly the true values represented by four decimal digits.

5.4 Estimation of the Impulse Response

Theoretically, the impulse response is the system response to a unit impulse input. With some exceptions, such as biological and biomedical experiments where impulsive test signals are often the most practical forms, using an impulsive test signal to estimate the system impulse response is not practical. The impulse response of a system is usually estimated by a process of deconvolution knowing the input auto-correlation and the input-output cross-correlation functions. In this section, a method is considered for obtaining the impulse response as the derivative of the step response using the output derivative modal function.

The system impulse response is related to the step response by

$$h(t) = \mathcal{D}g(t). \tag{5.38}$$

The output derivative modal function that corresponds to a step input is obtained from (2.44) by setting the input single mode to unity as

$$\mathcal{D}g(t) + \sum_{i=1}^{n} p_{1i}g(t - iT) = q_{1o}. \tag{5.39}$$

From (5.38) and (5.39) it follows that

$$h(t) + \sum_{i=1}^{n} p_{1i}g(t - iT) = q_{1o}. \tag{5.40}$$

k	Estimated values	True values
0	+0.6047E−06	+0.8327E−16
1	+0.2291E+00	+0.2291E+00
2	+0.2291E+00	+0.2291E+00
3	+0.1086E+01	+0.1086E+01
4	+0.1154E+01	+0.1154E+01
5	+0.1004E+01	+0.1004E+01
6	+0.8457E+00	+0.8457E+00
7	+0.8207E+00	+0.8207E+00
8	+0.9160E+00	+0.9160E+00
9	+0.1028E+01	+0.1028E+01
10	+0.1071E+01	+0.1071E+01
11	+0.1039E+01	+0.1039E+01
12	+0.9821E+00	+0.9821E+00
13	+0.9542E+00	+0.9542E+00
14	+0.9680E+00	+0.9680E+00
15	+0.1000E+01	+0.1000E+01
16	+0.1021E+01	+0.1021E+01
17	+0.1018E+01	+0.1018E+01
18	+0.1002E+01	+0.1002E+01
19	+0.9804E+00	+0.9894E+00
20	+0.9887E+00	+0.9887E+00
21	+0.9967E+00	+0.9967E+00
22	+0.1005E+01	+0.1005E+01
23	+0.1006E+01	+0.1006E+01
24	+0.1003E+01	+0.1003E+01
25	+0.9981E+00	+0.9981E+00
26	+0.9965E+00	+0.9965E+00
27	+0.9980E+00	+0.9980E+00
28	+0.1001E+01	+0.1001E+01
29	+0.1002E+01	+0.1002E+01
30	+0.1001E+01	+0.1001E+01
31	+0.9999E+00	+0.9999E+00

Table 5.3: Samples of $g(t)$; estimated and true values

Once the samples of g are estimated they can be used in estimating samples of h using (5.40) provided that there are available estimates of the output derivative modal parameter vector $\underline{p}_1$ and the scalar q_{1o}. It is seen that (5.40) can be used for estimating samples of h only where $t > nT$. It is possible, however, to obtain samples of h for $t > 0$ if negative modal time shifts are used. First, the estimates of $\underline{p}_1$ and q_{1o} are obtained for negative modal time shifts and then these estimates are used in (5.40) to obtain $h(t)$ for $t > 0$ in terms of $g(t)$ for $t \geq -T$.

Estimation of the output derivative modal parameters has been considered in chapter 3. The estimation process can be made easier by making use of the step response, already estimated in example 5.2, instead of the general system response. Multiplying (5.40) by a vector of modulating functions $\underline{f}(t)$, e.g. Fourier or Walsh functions, then integrating over the period of orthogonality T_o, making use of the fact that $g(0) = 0$, gives

$$-\int_0^{T_o} g(t)\mathcal{D}\underline{f}(t)dt + g(T_o)\underline{f}(T_o) + \sum_{i=1}^{n} p_{1i}\int_0^{T_o} g(t-iT)\underline{f}(t)dt = q_{1o}\int_0^{T_o} \underline{f}(t)dt. \quad (5.41)$$

Least squares estimates of $\underline{p}_1$, q_{1o} and $g(T_o)$ can be obtained from (5.41) using a vector $\underline{f}$ of dimension $N > n + 2$. It should be noted that employing a vector of N Fourier functions as given by (3.17) is not suitable for estimating the unknown parameter q_{1o} because the integral of $\underline{f}$ over the period of orthogonality is the zero vector. However, it is possible to estimate all other unknowns in (4.44) so that it becomes possible to estimate q_{1o} by correlating (5.39) with a unit function $f_o(t) = 1$ over the time interval $t \rightarrow T_o$ as follows:

$$q_{1o} = \frac{1}{T_o}\left(g(T_o) + \sum_{i=1}^{n} p_{1i}\int_0^{T_o} g(t - iT)dt\right). \quad (5.42)$$

Example 5.3 :

It is required to obtain estimates of the impulse response samples of the system considered in example 5.2 from the estimates of the step response samples obtained in that example. Assuming the following modal time shifts

$T_1 = -20T_s$

$T_2 = -40T_s$

$T_3 = -60T_s$

and using a vector of six Fourier functions with a fundamental time period $T =$

$192T_s$ in (5.41) result in the following estimates of the output derivative modal parameters:

$\hat{p}_{11}=-0.1678E+01$

$\hat{p}_{12}=\ 0.8164E+01$

$\hat{p}_{13}=\ 0.1126E+02$

The estimate of q_{1o}, as obtained from (5.42) is

$\hat{q}_{1o}=\ 0.1774E+02.$

Samples of the impulse response are estimated from (5.40) using these obtained estimates. The results are listed in table 5.4. The true values of the corresponding samples are also listed beside the estimated values. The listed samples are obtained every $10T_s$.

5.5 Summary

Methods of identifying nonparametric models are described in chapter 5. Equation (5.14) gives the system frequency response at frequency ω_o in terms of vectors $\underline{q}_o$ and $\underline{\nu}$ obtained from the input-output measurements. Such a simple direct relationship cannot be obtained when conventional frequency response methods, such as digital spectral analysis with the Discrete Fourier Transform, are used in the general case with output transients due to the system poles. The output transient component is not present when the output modal function is used.

The step response is obtained from the output modal function in the case of a piece-wise constant input through (5.32) and (5.36). The impulse response is obtained by a simple filtering operation of the estimated step response, (5.40).

k	Estimated values	True values
0	$+0.3080E-03$	$-0.9992E-15$
1	$+0.2135E+01$	$+0.2132E+01$
2	$+0.2464E+01$	$+0.2461E+01$
3	$+0.1156E+01$	$+0.1155E+01$
4	$-0.3732E+00$	$-0.3724E+00$
5	$-0.9625E+00$	$-0.9616E+00$
6	$-0.5194E+00$	$-0.5195E+00$
7	$+0.2483E+00$	$+0.2472E+00$
8	$+0.6266E+00$	$+0.6253E+00$
9	$+0.4378E+00$	$+0.4369E+00$
10	$-0.6800E-03$	$-0.1019E-02$
11	$-0.2808E+00$	$-0.2809E+00$
12	$-0.2454E+00$	$-0.2457E+00$
13	$-0.2713E-01$	$-0.2778E-01$
14	$+0.1460E+00$	$+0.1452E+00$
15	$+0.1551E+00$	$+0.1544E+00$
16	$+0.4564E-01$	$+0.4508E-01$
17	$-0.6123E-01$	$-0.6168E-01$
18	$-0.8530E-01$	$-0.8580E-01$
19	$-0.3548E-01$	$-0.3604E-01$
20	$+0.2612E-01$	$+0.2547E-01$
21	$+0.4851E-01$	$+0.4788E-01$
22	$+0.2734E-01$	$+0.2672E-01$
23	$-0.7232E-02$	$-0.7797E-02$
24	$-0.2450E-01$	$-0.2507E-01$
25	$-0.1698E-01$	$-0.1757E-01$

Table 5.4: Samples of $h(t)$; estimated and true values.

Chapter 6

Observer Design

6.1 Introduction

The design of a state observer that gives the internal state of the system in terms of the input-output measurements in case of piece-wise constant inputs is considered in this chapter. It is assumed that there is no available a priori knowledge of the internal structure of the system so that the choice of the state variables is free. The most natural choice of the state variables in this case is the phase variables, which have a close relation with the system output and its derivatives. The state modal functions defined in chapter 2 will form the basis for the design procedure.

In section 6.2, a design procedure for a continuous-time and a discrete-time observers based on the definition of the state modal functions is given. The material given in section 6.3 completes the identification of a state model of the system by introducing a method for estimating the vector $\tilde{\underline{b}}$.

6.2 State Modal Functions as a State Observer

Each of the state modal functions $\tilde{y}_j(t)$, $j = 0, \ldots, n-1$ as given by definition 2.2 is a weighted sum of the state variable $\tilde{x}_{j+1}$ at time t and n earlier samples of the output y. On the other hand, the same state modal function can be derived from equation (2.17), which gives a modal function f_α for arbitrary vector $\underline{\alpha}$, by stting $\underline{\alpha} = \underline{c}^T A^j$ such that $\underline{\alpha}^T \underline{x}(t) = \underline{c}^T A^j \underline{x}(t) = \tilde{\underline{c}}^T \tilde{A}^j \tilde{\underline{x}}(t) = \underline{\imath}_{j+1}^T \tilde{\underline{x}}(t)$ is the state variable $\tilde{x}_{j+1}$. These specific choices of $\underline{\alpha}$ have led to the definition of the state modal parameter vectors $\underline{p}_j$, $j = 1, \ldots, n-1$, given by (2.30). The state modal

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parameter vectors are identical to their respective output and output derivative modal parameter vectors which can be estimated using the methods described in chapter 3. In the following analysis wherever the index j appears, it is assumed that j can take the values from 0 to $n-1$.

Let u be a piece-wise constant input as given by (5.15). Let the modal time shifts be integral multiples of the input sampling interval T such that $T_i = iT$, $i = 1, \ldots, n$. It follows from (2.17) by substituting $p_{\alpha 1}, \ldots, p_{\alpha n}$ for the state modal parameters $p_{j1}, \ldots, p_{jn}$ that the state modal function $\tilde{y}_j$ is given at the time instant $t = kT + s$ by

$$\tilde{y}_j(kT + s) = \sum_{i=1}^{n} p_{ji} \int_{kT+s}^{(k-i)T+s} \underline{c}^T \phi((k-i)T + s - \tau)\underline{b}u(\tau)d\tau. \qquad (6.1)$$

Let

$$z_i(kT + s) = \int_{kT+s}^{(k-i)T+s} \underline{c}^T \phi((k-i)T + s - \tau)\underline{b}u(\tau)d\tau \qquad (6.2)$$

such that

$$\tilde{y}_j(kT + s) = \sum_{i=1}^{n} p_{ji} z_i(kT + s). \qquad (6.3)$$

The analysis given below aims at expressing z_i in terms of the input steps in order to obtain a relation between the state modal function $\tilde{y}_j$ and the piece-wise constant input u.

Substituting for u in (6.2) as a sum of steps yields

$$\begin{aligned} z_i(kT + s) &= \int_{kT+s}^{(k-i)T+s} \underline{c}^T \phi((k-i)T + s - \tau)\underline{b}d\tau u(k-i) \\ &+ \sum_{m=1}^{i} \int_{kT+s}^{(k-i+m)T} \underline{c}^T \phi((k-i)T + s - \tau)\underline{b}d\tau v(k-i+m). \end{aligned} \qquad (6.4)$$

Using the substitution $\sigma = (k-i)T + s - \tau$ in the integrals gives

$$\begin{aligned} z_i(kT + s) &= \int_{-iT}^{0} \underline{c}^T \phi(\sigma)\underline{b}d\sigma u(k-i) \\ &+ \sum_{m=1}^{i} \int_{-iT}^{s-mT} \underline{c}^T \phi(\sigma)\underline{b}d\sigma v(k-i+m). \end{aligned} \qquad (6.5)$$

Each integral over the time intervals $t = -iT \to s - mT$ in (6.5) can be expressed as sum of two integrals over the time intervals $t = -iT \to 0$ and $t = 0 \to s - mT$

such that (6.5) becomes

$$
\begin{aligned}
z_i(kT + s) &= \int_{-iT}^{0} \underline{c}^T \phi(\sigma) \underline{b} d\sigma u(k - i) + \sum_{m=1}^{i} \int_{-iT}^{0} \underline{c}^T \phi(\sigma) \underline{b} d\sigma v(k - i + m) \\
&+ \sum_{m=1}^{i} \int_{0}^{s-mT} \underline{c}^T \phi(\sigma) \underline{b} d\sigma v(k - i + m).
\end{aligned}
\tag{6.6}
$$

Since from the definition of v

$$
\sum_{m=1}^{i} v(k - i + m) = u(k) - u(k - i)
\tag{6.7}
$$

it follows from (6.6) by adding the first two terms on the right-hand side of the equation that

$$
z_i(kT + s) = \int_{-iT}^{0} \underline{c}^T \phi(\sigma) \underline{b} d\sigma u(k) + \sum_{m=1}^{i} \int_{0}^{s-mT} \underline{c}^T \phi(\sigma) \underline{b} d\sigma v(k - i + m).
\tag{6.8}
$$

Substituting back for z_i in (6.3) yields

$$
\tilde{y}_j(kT + s) = \sum_{i=1}^{n} p_{ji} \int_{-iT}^{0} \underline{c}^T \phi(\sigma) \underline{b} d\sigma u(k) + \sum_{i=1}^{n} p_{ji} \sum_{m=1}^{i} \int_{0}^{s-mT} \underline{c}^T \phi(\sigma) \underline{b} d\sigma v(k - i + m).
\tag{6.9}
$$

Introducing

$$
r_m(s) = \int_{s-mT}^{0} \underline{c}^T \phi(\sigma) \underline{b} d\sigma
\tag{6.10}
$$

gives

$$
\tilde{y}_j(kT + s) = \sum_{i=1}^{n} p_{ji} r_i(0) u(k) - \sum_{i=1}^{n} p_{ji} \sum_{m=1}^{i} r_m(s) v(k - i + m).
\tag{6.11}
$$

Let $\tilde{y}_{jo}$ be the state modal function that corresponds to a constant input $u(k)$, then $\tilde{y}_{jo}(kT + s)$, $k \geq n$, can be obtained from (6.11) by setting all the terms $v(k - i + m)$ to zero as follows:

$$
\tilde{y}_j(kT + s) = \sum_{i=1}^{n} p_{ji} r_i(0) u(k).
\tag{6.12}
$$

On the other hand, $\tilde{y}_{jo}$ is given by (2.40) as

$$
\tilde{y}_{jo} = \tilde{q}_{jo} u(k)
\tag{6.13}
$$

where $\tilde{q}_{jo}$ is obtained from (2.41) by setting $\underline{c}_c = \phi_c(t) = 1$ as

$$
\tilde{q}_{jo} = \sum_{i=1}^{n} p_{ji} \int_{-iT}^{0} \underline{c}^T \phi(\sigma) \underline{b} d\sigma.
\tag{6.14}
$$

Then it follows from (6.12) and (6.13) that

$$\tilde{q}_{jo} = \sum_{i=1}^{n} p_{ji} r_i(0).$$ (6.15)

Substituting from (6.15) into (6.11) yields

$$\tilde{y}_j(kT + s) = \tilde{q}_{jo} u(k) - \sum_{i=1}^{n} p_{ji} \sum_{m=1}^{i} r_m(s) v(k - i + m).$$ (6.16)

Equation (6.16) can be written in vector form as

$$\tilde{y}_j(kT + s) = [u(k) \quad -v(k) \quad -v(k-1) \ldots -v(k-n+1)] \Gamma_j \left[\begin{array}{c} \tilde{q}_{jo} \\ \underline{r}(s) \end{array} \right]$$ (6.17)

where

$$\underline{r}(s) = [r_1(s), \ldots, r_n(s)]$$ (6.18)

and

$$\Gamma_j = \left[\begin{array}{cccc} 1 & 0 & \ldots & 0 \\ 0 & p_{j1} & \ldots & p_{jn} \\ \vdots & \vdots & & \\ 0 & p_{jn} & & O \end{array} \right].$$ (6.19)

It can be shown that

$$[u(k) \quad -v(k) \ldots -v(k-n+1)] = \underline{u}^T(k) \Delta^T$$ (6.20)

where $\underline{u}^T(k)$ and Δ are given by (5.28) and (5.30) respectively. Therefore, it follows from (6.17) and (6.20) that

$$\tilde{y}_j(kT + s) = \underline{u}^T(k) \Delta^T \Gamma_j \left[\begin{array}{c} \tilde{q}_{jo} \\ \underline{r}(s) \end{array} \right].$$ (6.21)

Finally, the required relation between the state modal function $\tilde{y}_j$ and the piecewise constant input u follows from (6.21) as

$$\tilde{y}_j(kT + s) = \underline{u}^T(k) \underline{\tilde{q}}_j(s)$$ (6.22)

where

$$\underline{\tilde{q}}_j(s) = \Delta^T \Gamma_j \left[\begin{array}{c} \tilde{q}_{jo} \\ \underline{r}(s) \end{array} \right].$$ (6.23)

Substituting definition 2.2 for $\tilde{y}_j$ in (6.22), and moving all terms with the exception of $\tilde{x}_{j+1}$ to the right-hand side of the equation yields

$$\tilde{x}_{j+1}(kT + s) = -\sum_{i=1}^{n} p_{ji} y((k-i)T + s) + \underline{\tilde{q}}_j^T(s) \underline{u}(k)$$ (6.24)

As seen from equation (6.24) it is necessary to obtain estimates of the vectors $\underline{p}_j$ and $\underline{\tilde{q}}_j$ in order to use the equation as an observer of the state variable $\tilde{x}_{j+1}$. Estimating the modal parameters from the system response to a piecewise-constant input has been considered in chapter 3[1]. It has been shown that a time interval greater than or equal to nT should be allowed between any two successive input transitions. This restriction resulted from the fact that the continuous ARMA model used to estimate the modal parameters is valid only over the time interval of a constant input. An alternative method based on the estimated step response can remove this restriction on the frequency of input transitions. The continuous ARMA model in case of a step input is given by

$$D^j g(t) + \sum_{i=1}^{n} p_{ji} g(t - iT) = q_{jo} \qquad , j \geq 0. \tag{6.25}$$

The methods of obtaining the output derivative modal parameters $p_{j1}, \ldots, p_{jn}$ for $j \geq 0$ using Fourier functions or Walsh functions which are described in section 3.4 are applicable to the case of the ARMA model (6.25). The only difference is that the estimated step response samples will replace the measured system response samples.

The estimates of $\underline{\tilde{q}}_j(s)$ can be obtained from (6.23). It is seen from (6.23) that under the assumption that there is an available Γ_j, i.e. there is an available estimate of $\underline{p}_j$[2], it is possible to obtain $\underline{\tilde{q}}_j(s)$ from the scalar $\tilde{q}_{jo}$ and the vector $\underline{r}(s)$ which is independent of the index j. Actually, it is sufficient to obtain an estimate of $\underline{r}(s)$ in order to obtain $\underline{\tilde{q}}_j(s)$ from (6.23) because the scalar $\tilde{q}_{jo}$ can be obtained from (6.15) as

$$\tilde{q}_{jo} = \underline{p}_j^T \underline{r}(0). \tag{6.26}$$

The required estimate of $\underline{r}(s)$ can be obtained from (6.22) by setting $j{=}0$ as

$$\begin{bmatrix} \tilde{q}_{oo} \\ \underline{r}(s) \end{bmatrix} = (\Delta^T \Gamma_o)^{-1} \underline{\tilde{q}}_o(s) \tag{6.27}$$

A least squares estimate of the vector $\underline{\tilde{q}}_o(s)$ required in (6.27) can be obtained from $N > n + 1$ equations like (6.22) formed from the input-output measurements for different k as

$$\underline{\hat{\tilde{q}}}_o(s) = (U^T U)^{-1} U^T \underline{y}_o(s) \tag{6.28}$$

---

[1]see examples 3.1 and 3.2
[2]see (6.19)

where $\underline{y}_o(s)$ and U are given by (5.33) and (5.35) respectively. Comparing (6.28) with (5.32) gives the relation between the estimate of $\underline{\bar{q}}_{oo}(s)$ and the vector of step response samples $\underline{g}(s)$ as

$$\hat{\underline{\bar{q}}}_o(s) = \Delta L\,\hat{\underline{g}}(s) \tag{6.29}$$

where L is the lower triangular matrix given by (5.34). Equation (6.29) gives the estimate of $\underline{\bar{q}}_o(s)$ as a linear transformation of the estimates of the step response samples. The estimate of $\underline{\bar{q}}_j(s)$ is then obtained from the estimates of $\underline{r}(s)$ and $\tilde{q}_{jo}$ using (6.23) where $\tilde{q}_{jo}$ is obtained from (6.26).

By estimating $\underline{\bar{q}}_j(s)$ equation (6.24) can be used to obtain the state variable $\tilde{x}_{j+1}$ in terms of the available input-output measurements . Equation (6.24) represent a single row in the vector equation

$$\underline{\tilde{x}}(kT + s) = P\underline{y}((k-1)T + s) + \bar{Q}(s)\underline{\bar{u}}(k) \tag{6.30}$$

where $\underline{y}((k-1)T + s)$ is given by (4.20), the matrix P is given by (4.18) and the matrix $\bar{Q}(s)$ is given by

$$\bar{Q}(s) = \begin{bmatrix} \underline{\bar{q}}_o{}^T(s) \\ \vdots \\ \underline{\bar{q}}_{n-1}^T(s) \end{bmatrix}. \tag{6.31}$$

The dimension of P and $\bar{Q}(s)$ are $n \times n$ and $n \times (n+1)$ respectively. The state observer (6.30) gives the system state $\underline{\tilde{x}}$ at time $t = kT + s$ as a direct function of n output samples $y((k-1)T + s), \ldots, y((k-n)T + s)$ and $n+1$ input samples $u(k), \ldots, u(k-n+1)$. To obtain a state observer of the system state at the discrete sampling instants only, the time variable s is set to zero such that (6.30) becomes

$$\underline{\tilde{x}}(kT) = P\underline{y}((k-1)T) + \bar{Q}(0)\underline{\bar{u}}(k). \tag{6.32}$$

The system state at the beginning of the kth sampling interval $\underline{\tilde{x}}(kT)$ is independent of the input $u(k)$ applied to the system during that sampling interval, which means that the first column of $\bar{Q}(0)$ in (6.32) is zero. This can be verified by testing the first element of the rows of $\bar{Q}(0)$. According to (6.31), the $(j+1)^{th}$ row of $\bar{Q}(0)$ is $\underline{\bar{q}}_j(0)$. The first element in $\underline{\bar{q}}_j(0)$ is given by (6.23) as

$$\underline{\imath}_1^T \underline{\bar{q}}_j(0) = \underline{\imath}_1^T \Delta^T \Gamma_j \begin{bmatrix} \tilde{q}_{jo} \\ \underline{r}(0) \end{bmatrix} = \tilde{q}_{jo} - \underline{p}_j^T \underline{r}(0) = 0. \tag{6.33}$$

The last equation follows from (6.26).

Let the n dimensional vector $\underline{u}(k-1)$ be formed from the elements of the $n+1$ dimensional vector $\underline{\bar{u}}(k)$ by deleting the first element, $u(k)$, as follows:

$$\underline{u}(k-1) = [\underline{o}\ I]\underline{\bar{u}}(k). \tag{6.34}$$

Similarly, let the $n \mathrm{x} n$ matrix Q be formed from the elements of the $n \mathrm{x}(n+1)$ matrix $\bar{Q}(0)$ by deleting the first *zero* column such that

$$Q = \bar{Q}(0) \left[\begin{array}{c} \underline{o}^T \\ I \end{array} \right]. \tag{6.35}$$

Thus the discrete-time observer follows from (6.32) as

$$\underline{\tilde{x}}(k) = P\underline{y}(k-1) + Q\underline{u}(k-1). \tag{6.36}$$

The time interval T has been ommited from the discrete-time observer since it is implicitly understood. It is seen from (6.36) that it takes only n sampling intervals to form the vectors $\underline{y}(k-1)$ and $\underline{u}(k-1)$ in order to be able to reconstruct the state vector $\underline{\tilde{x}}(k)$. This kind of observer is called a *dead beat observer* [Åström 83], or a *minimum time observer* [O'Reilly 83]. It should be noted that although the observer (6.36) is discrete, it gives the state $\underline{\tilde{x}}$ of a continuous-time model of the system since $\underline{\tilde{x}}$ is defined in terms of the output and its derivatives which are meaningless in a discrete model case.

To summarize, the design of the state observer in case of a piece-wise constant input takes the following steps:

- Estimation of the output modal parameter vector $\underline{p}_o$.

- Estimation of the vector of step response samples $\underline{g}(s)$.

- Estimation of the state modal parameter vectors $\underline{p}_1$, ...,$\underline{p}_{n-1}$ from the estimated step response.

- Obtaining $\underline{\bar{q}}_o(s),\ldots, \underline{\bar{q}}_{n-1}(s)$ from the estimated step response and the modal parameter vectors using equations (6.29), (6.27), (6.26) and (6.23).

- Forming the matrix P from rows $\underline{p}_0^T,\ldots,\underline{p}_{n-1}^T$.

- If a continuous observer is sought, the matrix $\bar{Q}(s)$ is formed from rows $\underline{\bar{q}}_o(s), \ldots, \underline{\bar{q}}_{n-1}(s)$, while if a discrete observer is sought, the matrix Q is formed as in (6.35).

Example 6.1 :

Consider a third order system with transfer function
$$H(s) = \frac{6.5s + 52}{s^3 + 4s^2 + 30s + 52}$$
The system is identical to the system considered in example 3.1 with the exception that the system in example 3.1 has a zero at $s = -4$ while the system in this example has a zero at $s = -2$. Starting from rest, the system is excited with the piece-wise constant input +1, +2, –1, and –2. Each constant level lasts for a duration $80T_s$ where $T_s = 0.01$. It is desired to obtain a minimum time observer for this system to give the system state at the sampling instants $\underline{\tilde{x}}(kT)$, and midway between successive sampling instants $\underline{\tilde{x}}(kT + T/2)$ where T is the input sampling time taken equal to $20T_s$. Following the design steps listed above results in obtaining three matrices P, $\bar{Q}(0)$ and $\bar{Q}(T/2)$ listed in table 6.1. It is seen from the results that the first column of $\bar{Q}(0)$ is zero as expected.

6.3   Estimation of $\underline{\tilde{b}}$

The identification of the parameters of a system model has been considered in chapter 4. Estimates of the coefficients of the characteristic polynomial have been obtained in sections 4.3 and 4.4. By estimating these coefficients it is possible to obtain an estimate of the system dynamic matrix in phase variable form $\tilde{A}$. Hence, it remains to estimate the vector $\underline{\tilde{b}}$ in order to complete the identification of the state model of the system.

The nonzero elements in $\underline{\tilde{b}}$ result in a difference between the state variables $\tilde{x}_{j+1}$, $j = 1, \ldots, n-1$, and the corresponding output derivative $\mathcal{D}^j y$. The same difference results between the state modal functions and the corresponding output derivative modal functions. The output derivative modal functions and the state modal functions that result from a step input are constants q_{jo} and $\tilde{q}_{jo}$, $j = 1, \ldots, n-1$ respectively. Estimates of these constants are obtained in section 6.2. The relation

Matrix P		
$-0.1555\text{E}+01$	$+0.1264\text{E}+01$	$-0.4494\text{E}+00$
$-0.3142\text{E}+00$	$+0.4343\text{E}+01$	$-0.2163\text{E}+01$
$+0.2621\text{E}+02$	$-0.2842\text{E}+02$	$+0.6032\text{E}+01$

Matrix $\tilde{Q}(0)$			
$+0.4985\text{E}-05$	$+0.1455\text{E}+00$	$+0.1475\text{E}+00$	$-0.3413\text{E}-01$
$-0.9472\text{E}-06$	$+0.1444\text{E}+01$	$+0.5780\text{E}+00$	$-0.1643\text{E}+00$
$+0.2734\text{E}-06$	$-0.2339\text{E}+00$	$-0.2850\text{E}+01$	$+0.4581\text{E}+00$

Matrix $\tilde{Q}(T/2)$			
$+0.3561\text{E}-01$	$+0.2257\text{E}+00$	$+0.9755\text{E}-02$	$-0.1211\text{E}-01$
$+0.7309\text{E}+00$	$+0.1185\text{E}+01$	$+0.1017\text{E}-03$	$-0.5830\text{E}-01$
$+0.1156\text{E}+01$	$-0.3505\text{E}+01$	$-0.4398\text{E}+00$	$+0.1626\text{E}+00$

Table 6.1: Minimum time observer parameters

between q_{jo} and $\tilde{q}_{jo}$ follows from (2.33) by setting the input u to unity and all its derivatives to zero ;

$$q_{jo} - \tilde{q}_{jo} = \tilde{b}_j \quad , j = 1, \ldots, n-1. \tag{6.37}$$

Hence the estimates of $\tilde{b}_j$, $j = 1, \ldots, n-1$, can be obtained from (6.37) as the difference between the corresponding output derivative and state modal functions which are obtained from the estimate of the step response. The nth element in $\underline{\tilde{b}}$ cannot be obtained from (6.37) because, although the output derivative modal functions are defined for $j > 0$, the state modal functions are defined for j up to $n-1$ only. The output derivative modal function of order n is given by definition as

$$q_{no} = \mathcal{D}^n g(t) + \sum_{i=1}^{n} p_{ni}\, g(t - iT). \tag{6.38}$$

The relation between the output derivative of orders $1, \ldots, n$ and the system state variables is given by (2.11). Substituting in (2.11) for y by g, for u by unity and for all the derivatives of u by zero, it follows that the nth derivative of g is

$$\mathcal{D}^n g(t) = \underline{a}^T \underline{x}(t) + \tilde{b}_j. \tag{6.39}$$

From definition 2.2 of the state modal function it followos that the state modal function that results from a step input is given by

$$\tilde{x}_{j+1}(t) + \sum_{i=1}^{n} p_{ji} g(t - iT) = \tilde{q}_{jo} \tag{6.40}$$

Substituting for the state variables in (6.39) from (6.40) it follows that

$$\mathcal{D}^n g(t) = -\sum_{j=0}^{n-1} a_j \left(\tilde{q}_{jo} - \sum_{i=1}^{n} p_{ji} g(t - iT) \right) + \tilde{b}_n. \tag{6.41}$$

Substituting back for $\mathcal{D}^n g$ in (6.38) and rearranging terms yields

$$q_{no} = -\sum_{j=0}^{n-1} a_j \tilde{q}_{jo} + \sum_{i=1}^{n} [p_{ni} + \sum_{j=0}^{n-1} a_j p_{ji}] g(t - iT) + \tilde{b}_n. \tag{6.42}$$

However, from theorem 4.1, it follows that the term in rectangular brackets in (6.42) is zero, and therefore $\tilde{b}_n$ can be obtained as

$$\tilde{b}_n = q_{no} + \sum_{j=0}^{n-1} a_j \tilde{q}_{jo}. \tag{6.43}$$

The constant output derivative modal function q_{no} can be obtained exactly the same way as other output derivatives modal functions q_{jo}, $j < n$. The parameters a_j, $j = 0, \ldots, n-1$ are obtained from the modal parameters as in section 4.3.

Example 6.2 :

It is required to obtain the vector $\underline{\tilde{b}}$ for the system in example 6.1 using (6.37) and (6.43). The estimates of $\underline{p}_j$, q_{jo} and $\tilde{q}_{jo}$, $j = 0, \ldots, n-1$, already obtained in example 6.1 will be used. The estimates of $\underline{p}_n$ and q_{no} are obtained from the estimated step response. Using these estimates in (6.37) and (6.43) gives the estimate of $\underline{\tilde{b}}$ listed in table 6.2. The true values of the elements of $\underline{\tilde{b}}$ are listed with the estimated values for comparison purposes.

6.4   Summary

The main result of chapter 6 is the minimum-time state observer in continuous-time, (6.30), and in discrete-time, (6.36). Although (6.36) is discrete in nature, it gives the continuous-time system state $\underline{\tilde{x}}$ which is defined in terms of the derivatives

$\hat{\tilde{b}}$	$\tilde{b}$
+0.7399E–02	0.0
+0.6453E+01	+0.6500E+01
+0.2624E+02	+0.2600E+02

Table 6.2: Estimated and true values of $\tilde{b}$

of the continuous output. The minimum-time observer gives the system state in terms of the input-output samples in n sampling intervals.

A method is given to determine the vector $\tilde{b}$, of the state model, from the modal functions that correspond to the step response, and the estimates of the a-parameters obtained before in chapter 4.

Chapter 7

Conclusion

7.1 Achievements

A new method for continuous system identification is presented. The new method
uses modal functions which are defined in terms of the system output, the system
output derivatives and the system state. A necessary condition for the existence
of the modal parameters, associated with the modal functions, is established in
theorem 2.1. The properties of the modal functions are studied in section 2.4. It
is shown that these functions are independent of the system poles and the system
instantaneous state and that they are linear functions of the input modes.

Methods of estimating the modal parameters from the available input-output
measurements are presented in chapter 3. Orthogonal functions, Fourier functions
and Walsh functions, are used to estimate modal parameter vectors. It is shown
that only one stage of integration is required to determine one modal parameter
vector at a time resulting in only one unknown added to the estimation problem
in the case of Fourier functions[1], and none in the case of Walsh functions[2]. This
is an advantage over the known methods utilizing modulating functions which
usually result in n additional unknowns [Saha 83], or trade the orthogonality of
the modulating functions for a lower number of unknowns [Pearson 85]. It is also
an advantage over the methods which utilize orthogonal functions and operational
matrices for integration. These methods require the estimation of unknown initial
conditions, and they suffer from accumulated errors due to multiple integrations

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[1]see equation (3.30)
[2]see equation (3.37)

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with truncated operational matrices.

The output modal function is obtained from the system output by a simple filtering operation[3]. After showing that the output modal function is a particular response of the system which is composed of modes from the corresponding filtered input, the filtered input-output pair is used to identify parametric and nonparametric models of the system.

A parametric model is identified in section 4.2 through a method that utilizes the output modal function. It is shown that the unavailable derivatives which used to be a problem in continuous system identification can be obtained from the output modal function analytically[4].

With a periodic waveform input, the output modal function, which in this case is periodic, is used in section 5.2 to obtain samples of the frequency response of the system. The transient component of the system response, which causes the systematic errors in the frequency response estimation methods using digital spectral analysis, [Wellstead 81], is annihilated by the use of the output modal filter.

Estimates of samples of the step response of the system are obtained using the output modal function in section 5.3. Samples of the impulse response of the system are obtained in section 5.4 as the derivatives of the step response by a simple filtering operation that uses the output first derivative modal parameters[5].

Modal functions and modal parameters of the system output and the system output derivatives are used in chapter 4 to identify a parametric input-output model of the system. A nice property of using the modal parameters in parametric system identification is decoupling of the estimation equations with a smaller number of unknowns to be estimated from each equation. By estimating $n+1$ modal parameter vectors, each of dimension n, the vector $\underline{a}$ of the coefficients of the characteristic polynomial of the system is obtained according to theorem 4.1 as the solution of a set of n algebraic equations formed from the estimates of the modal vectors. The vector $\underline{b}$ is obtained according to theorem 4.2 by solving another set of linear algebraic equations formed from the modal functions of the output and

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[3]see Definition 2.1
[4]see equation (4.3)
[5]see equation (5.40)

the output derivatives and the estimate of $\underline{a}$. With the existing methods utilizing modulating functions or multiple integrations, there will be n parameters associated with the system poles, there can be n parameters associated with the system zeros plus n unknown initial conditions all contained in one equation. It can be difficult to realize well-conditioned estimation equations even for low n.

Theorem 4.3 gives a proof that a data matrix formed from the free response of the system possesses the same eigenvalues as the system dynamic matrix A. From this data matrix the coefficients of the system characteristic polynomial are obtained recursively through a simple relation[6] which is well known in matrix algebra but seems not to have been used before in system identification. The system poles are obtained by solving for the eigenvalues of a data matrix. Theorem 4.3 has been extended to include systems driven by continuous controllers.

In section 6.3, a method is presented to estimate the vector $\underline{\tilde{b}}$ thus completing the identification of a continuous state model of the system in phase variable form.

Based on the definition of the state modal functions, a procedure to design a minimum-time observer in case of a piece-wise constant input is given in section 6.2. The observer gives the continuous system state as a direct function of input-output samples in a number of sampling intervals equal to the order of the system.

7.2   Remarks on Future Research

The choice of the modal time shifts affects the linear independence of the columns of ψ. This affects the accuracy of the estimates of the modal parameter vectors and consequently the accuracy of the estimates of the model unknown parameters derived from these vectors. Therefore, by understanding the exact impact of the modal time shifts on the estimation equations, they can be chosen to optimize the conditioning of the estimation problem.

The effect of noise on the performance of the minimum-time state observer when used in a feedback control loop should be investigated. A Kalman filter could be used to obtain *best* estimates of the output samples that are used to obtain the system state.

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[6]see equation (4.50)

Since the modal time shifts T_i, $i=1,\ldots,n$ in the case of a piece-wise constant input is not restricted to $T_i=iT$, it is possible to obtain a parametric discrete model of the system with $T_i=k_iT$, $k_i > i$. Such a model can be useful in designing predictive controllers yielding better responses.

This thesis has dealt with single-input single-output systems. Extensions of the thesis results to multi-input multi-output systems does not seem to pose any theoretical difficulty. Modal functions can be determined for each output, as is the case for single-input single-output systems. The problem appears to be one of ordering output modes and accounting for free subsystem modes and coupling modes between subsystems.

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