SENSITIVITY AND UNCERTAINTY ANALYSIS OF SUBSURFACE DRAINAGE DESIGN

By

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Abstract

Literature on subsurface drainage theories, determination of drainage parameters, and analysis approaches of uncertainty was reviewed. Sensitivity analysis was carried out on drain spacing equations for steady state and nonsteady state, in homogeneous soils and in layered soils. It was found that drain spacing is very sensitive to the hydraulic conductivity, the drainage coefficient, and the design midspan water table height. Spacing is not sensitive to the depth of the impermeable layer and the drain radius. In transient state, spacing is extremely sensitive to the midspan water table heights if the water table fall is relatively small. In that case steady state theory will yield more reliable results and its use is recommended. Drain spacing is usually more sensitive to the hydraulic conductivity of the soil below the drains than to that of the soil above the drains. Therefore, it is desirable to take samples from deeper soil when measuring hydraulic conductivity. A new spacing formula was developed for two–layered soils and a special case of three–layered soils with drains at the interface of the top two layers. This equation was compared with the Kirkham equation. The new formula yields spacings close to the Kirkham equation if the hydraulic conductivity of the soil above the drains is relatively small; otherwise, it tends to give more accurate results. First and second order analysis methods were employed to analyze parameter uncertainty in subsurface drainage design. It was found that conventional design methods based on a deterministic framework may result in inadequate spacing due to the uncertainty involved. Uncertainty may be incorporated into practical design by using the simple equations and graphs presented in this research; the procedure was illustrated through an example. Conclusions were drawn from the present study and recommendations were made for future research.

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Chapter 1

INTRODUCTION

As a key component of the modern agriculture, drainage is playing, and will continue to play, an important role in the food production system. With the rapidly increasing world population, especially in those less developed countries, the strong demand for food and fibre has been putting greater and greater, and at times even explosive, pressures on agriculture. To increase food and fibre production, there are two basic approaches: (i) to expand cultivated area, and (ii) to increase production per unit area by ameliorating the agricultural conditions. Although the scientific and technological advances will contribute greatly to production increase, agricultural drainage will play a very important or even dominant role. By means of drainage, crop yield will be increased by more intensive and efficient use of cultivated land; more and more marshland and other waterlogged lands, estimated at over 3.5 million square kilometers (Bulavko, 1971), could be drained and put into agricultural production to meet the steadily increasing demand for food and fibre.

The primary purpose of agricultural drainage is to remove excess water from the root zone for profitable food production. In humid areas, excess water mainly comes from intensive rainfall and sometimes from surplus irrigation, which may be needed as an anti-frost protection during the frost periods, or in other areas, to supplement and offset the lack of timely or regular rainfall. In irrigated areas, drainage is needed to remove the excess water for salinity control.

The most important design parameters of a subsurface drainage system are drain

depth and drain spacing. Drain depth depends mainly on the soil profile, the installation equipment, and the discharge outlet. Once the drain depth has been determined from actual field conditions, drain spacing is the only design parameter which should be determined from climatic conditions, soil properties and crop requirements. In cases where several drain depths are possible, the combination of drain depth and drain spacing should be such that the total cost of the system (including installation, material and maintenance costs) is minimized.

At present, steady state theory is commonly used in agricultural drainage because of its simplicity and practicability in most situations. In cases where the water table fluctuations must be considered, more complex and relatively more realistic transient theory should be used. In both steady and transient state theories, drainage design is based on a deterministic framework, in which parameters such as soil hydraulic conductivity and drainable porosity are represented by some estimated or equivalent values. However, such soil properties, along with climatic conditions, tend to be both spatially and temporally variable. Their representation by equivalent values is also ambiguous, if not totally misleading. Such equivalent values are selected partly based on the actual nature of the system, but also largely on the experience and professional judgement of the individual designer, which adds to the variability and uncertainty inherent in these properties.

Drain spacing can be calculated from certain drainage models or equations, such as Hooghoudt equation for steady state and Boussinesq equation for transient state, provided adequate information is available for various parameters. Any differences in the estimation of related parameters will eventually change the designed spacing, which in turn affects the economy of the drainage system. In the case of a large well-financed enterprise, the initial installation costs may not be the major concern, but for a farmer who has only limited funds to expend on a drainage facility and whose primary purpose is to minimize the risk of severe crop losses in the event of adverse weather, the initial

cost of a drainage system and the uncertainty about its performance may result in the difference of accepting or declining the drainage design.

The best drainage system design is the one that minimizes cost, while maximizing the monetary return or minimizing the risk of severe crop loss. In practice this usually can not be achieved and a compromise is required. A decrease in cost may result in poorer system performance. A particular design is often dependent on the combination of the amount of money a farmer is willing to expend on drainage system and the risk level of crop failure in the event of adverse weather.

Although many methods are available to determine soil hydraulic properties such as soil hydraulic conductivity and specific yield, these parameters are determined with certain amount of error, due to inherent varibility of the soil and the limitations of accuracy of the methods. For a particular field, the values of the parameters estimated are directly related to the locations and number of measurements performed, which are usually subject to the experience and professional judgement of individual designers. Even after the necessary data have been collected, different methods of interpreting the data may also result in different estimates. While a criterion to decide which estimate is better than others is difficult to set up, it is possible to estimate the effects and uncertainty of each estimate on the drain spacing and performance of a drainage system.

Among the numerous drain spacing formulae (Kirkham, 1966; Mueller, 1967), the Hooghoudt equation is most commonly used in steady state drainage design owing to its simplicity compared with other more complex models. In cases of transient flow, many spacing formulae are obtained by simplifying the governing partial differential equation of groundwater flow, which is obtained by combining Darcy's law with the law of conservation of mass. The Boussinesq equation is developed by applying Dupuit-Forchheimer theory (van Schilfgaarde, 1974). Various linearizations have been applied to solve the Boussinesq equation (Dumm, 1954; Werner, 1957; Krayenhoff van de Leur, 1958;

Maasland, 1959; Polubarino–Kochina, 1962; Dumm, 1964; Terzidis, 1968). Most of the solutions obtained from linearizations, for example, the Glover equation, are restricted to small increment of water table drawdown and time periods. However, van Schilfgaarde (1963,1964) solved the non–linear partial differential equation without linearization, and the van Schilfgaarde solution is not restricted to relatively small increments of time and water table. In addition, the van Schilfgaarde solution is relatively simple and can still provide realistic results. For the above reasons, the van Schilfgaarde equation is used in this study for transient drainage design.

With only minor modification the Hooghoudt equation can also be used in cases of two-layered soils, if the drains are at the interface of the two layers (van Beers, 1976). This is because the Hooghoudt approach primarily distinguishes the flow regions of above and below drains, and only secondarily distinguishes soil layers. However, it is claimed that the Hooghoudt equation can no longer be used in cases where a heavy clay layer of varying thickness overlies a sandy substratum ($K_1 \ll K_2$), and where there are two distinct pervious layers in the soil below drain level (van Beers, 1976). In these cases the use of the Ernst equation is recommended (van Beers, 1976). On the other hand, the Ernst equation is not suitable if a highly pervious layer is above a poorly pervious soil layer ($K_1 \gg K_2$) (van Beers, 1976). A generalized equation applicable to all these cases was obtained by Ernst (1975), who combined the Hooghoudt equation and the Ernst equation for radial flow. But it requires to solve a high order equation, which is sometimes inconvenient in practice. Kirkham (1971a) spacing equations for layered soils neglect the head loss in the region above drains and tend to underestimate the drain spacing if there is a highly pervious soil layer above drain level. A new spacing formula is developed in the study by combining the form of the Hooghoudt equation and the concept of equivalent depth with the radial flow component in the Ernst equation. This new spacing formula can be used in subsurface drainage design of two–layered for any

conditions and a special case of three–layered soils.

A major objective of this study is the sensitivity and uncertainty analysis of subsurface drainage design based on commonly used models and approaches. Sensitivity analysis reveals the relative importance of various parameters in a model to the spacing of a drainage system, which can be used as a guidance in soil surveys and field geohydrologic investigations. Uncertainty analysis can be used to evaluate the performance of a drainage system in a particular field and to adjust drain spacing according to the risk level a farmer is willing to accept. The method employed in the uncertainty analysis is the first and second order analysis of parameter uncertainty. It requires only information about the first two moments (mean and variance) of a random variable, which is much easier to obtain than the complete distributions of the variables (Cornell, 1972). It also requires only basic mathematic and statistical knowledge. These two features provide a strong base for its usefulness in practical drainage design.

The primary aims of this study are to apply simple but useful tools of uncertainty analysis (sensitivity analysis and first and second order analysis) to identify the dominant parameters in subsurface drainage system design, and to provide practicing engineers/designers with ready-to-use formulae and graphs to obtain an overall estimate of the uncertainty involved in the design due to the uncertainty and variability of the parameters.

This thesis is organized in the following way: in Chapter 1 (this chapter), an introduction for this study is provided. In Chapter 2, various approaches of subsurface drainage design are reviewed. Methods currently used in the determination of drainage parameters including the development of drainage criteria and the measurement of hydraulic conductivity are discussed. Techniques of incorporating uncertainty into the study of groundwater flow problems and drainage design are reviewed and discussed. In Chapter 3, the sensitivity analysis of drainage design in homogeneous soils is carried out. The

results from selected steady and transient theories are compared. In Chapter 4, the sensitivity of spacing to various parameters of drainage design in multi-layered soils is analyzed; several models are compared and a new spacing formula is then developed. In Chapter 5, the uncertainty analysis of subsurface drainage design in homogeneous and multi-layered soils is discussed. An example is presented to illustrate how uncertainty analysis can be used in practical design. In Chapter 6, a summary and the conclusions from this study is presented. Recommendations are made for future research in this area. Appendix A includes a discussion on the relationship between equivalent depth and drain spacing. In Appendix B, some methods of measuring and estimating the means and variances of soil properties are reviewed.

Chapter 2

LITERATURE REVIEW

Over the centuries, drainage has evolved from very primitive forms to modern state-of-the-art technology. Extensive literature is available on this and related areas. Current research tends to consider drainage conditions more realistically, such as efforts on applying stochastic approaches to drainage design. While it is extremely difficult, if not impossible, to incorporate natural conditions exactly into design, it is relatively easier to estimate the effects of uncertainty about drainage conditions on drainage system design and its performance under specific situation. This chapter reviews the literature on major subsurface drainage design theories, uncertainty analysis approaches, and drainage parameter measurement and determination methods.

2.1 Subsurface Drainage Design Theory

There are generally two ways to design a drainage system: (a) from empirical data collected through evaluation of existing drainage systems, and (b) from a theoretical analysis of the problem, applying known physical laws and tested theories. Criteria developed by empirical methods are based substantially on experience and assessment of numerous interrelated factors, and special care must be taken in transposing their use from one locality to another.

Theoretical methods are more commonly used in practice: applying proved principles or laws to problems having known limiting conditions. The resulting mathematical

expression explains the observed action of existing drainage systems and permits the rational design of new systems as long as the site conditions are within the limits for which the equation was derived. For example, the Hooghoudt equation (1940) was originally derived for the conditions in the Netherlands, but it is also widely used in Europe and North America (Chieng, 1978).

Drainage design is usually based on one of the numerous spacing formulae with its parameters evaluated from the local conditions. These spacing equations may be derived from two concepts: Darcy's Law, and the equation of continuity. For spacing equations based on the potential theory, these concepts are combined into Laplace's partial differential equation and when this equation is solved for the conditions of the particular problem, the solution yields, simultaneously, spacing equations and detailed information about the flow system. However solutions to Laplace's partial differential equation are usually very difficult to obtain for various initial and boundary conditions. On the other hand, for spacing equations by Dupuit-Forchheimer (D-F) theory (Dupuit, 1863; Forchheimer, 1930), simplifying assumptions can be made along with the use of Darcy's law and the equation of continuity to obtain useful results without solving Laplace's equation. However these results do not give the internal potential of the flow system.

Drainage spacing formulae may be generally categorized as either steady state formulae or non-steady state formulae. In practical drainage design, soils are usually assumed to be homogeneous,(or homogeneous within one layer), isotropic, and only saturated flow is considered. Studies on water flow through heterogeneous soils and in the unsaturated zone are more for research purposes than direct application. In the remainder of this section, the literature on steady and non-steady subsurface drainage design will be reviewed.

2.1.1 Steady State Subsurface Drainage Theory

Steady state spacing formulae are based on a state of equilibrium between recharge and discharge, i.e. discharge equals recharge and the hydraulic head remains constant. In order to apply these formulae, a drainage requirement or design criterion is required, describing the quantity of water to be removed in a given period of time under conditions of a given height of the groundwater table. Steady state conditions are most likely to occur in flat areas of humid regions, where the need for drainage is determined by long duration rainfall with low intensity in winter and spring.

Müller (1967) lists 40 steady state spacing formulae and compares results of calculations with the optimal drain spacings as determined at 35 experimental fields. He concluded that only the formulae of Hooghoudt (1940), Ernst (1956), and Töksoz & Kirkham (1961) are suitable for general practical application.

Hooghoudt (1940) considered only horizontal flow in the region above drains and both horizontal and radial flows in the region below drains. The radial flow caused by the convergence of flow lines near the drains is taken into account by reducing the depth of the flow layer (D) below the drains to the hypothetical depth (d_e) of an "equivalent layer", which is a function of (D), spacing (S), and drain radius (r). The Hooghoudt equation can be written as:

$$S^2 = \frac{8K_2 d_e h}{R} + \frac{8K_1 D_1 h}{R}$$

in which

S = drain spacing (m);

K_1 = hydraulic conductivity of the soil above drains (m/day);

K_2 = hydraulic conductivity of the soil below drains (m/day);

h = water table height above drains midway between two drains (m);

D_1 = average cross-section of flow area above drains (m);

R = drainage coefficient or drainage discharge rate (m/day);

d_e =thickness of the "equivalent layer".

Figure 2.1 is a sketch to show the symbols used in the analyses.

The equivalent depth d_e can be calculated by (Moody, 1966):

For $0 < d/S \leq 0.3$:

$$d_e = \frac{d}{1 + \frac{d}{S}(\frac{8}{\pi}\ln\frac{d}{r} - \alpha)}$$

in which

$$\alpha = 3.55 - 1.6\frac{d}{S} + 2(\frac{d}{S})^2$$

For $d/S > 0.3$:

$$d_e = \frac{S}{\frac{8}{\pi}(\ln\frac{S}{r} - 1.15)}$$

where d is the depth to the impermeable layer, and r is the drain radius.

The Hooghoudt equation can be used for homogeneous soils ($K_2 = K_1$), but also can be used for a drainage situation with two layers of different permeability ($K_2 \neq K_1$), if the drain level is at the interface of these layers. This is because Hooghoudt's approach distinguishes primarily the flow regions of above and below drains, and only secondarily distinguishes the soil layers.

Wesseling (1964) carried out extensive calculations using potential theory and compared the results with the Hooghoudt formula. The comparison shows that results obtained from using Hooghoudt's equation do not vary more than about 5% from those of potential theory for midspan water table heights. Since its appearance, Hooghoudt's equation has been favoured by practicing engineers for design in humid areas because of its accuracy, simplicity of form and ease of use.

However, the Hooghoudt equation is not applicable when there is more than one layer soil below the drains and or the soil above drains has a much lower hydraulic conductivity (van Beers, 1976). In these cases, use of the Ernst equation (1956, 1962) is recommended.

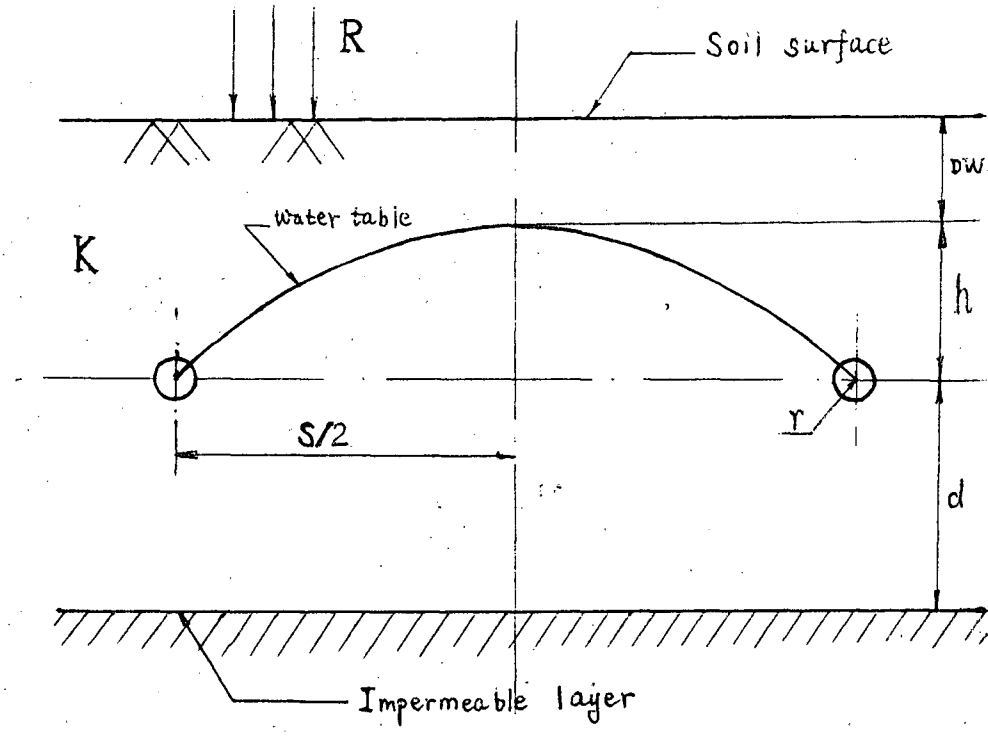

Figure 2.1: Geometry of steady state drainage system.

Ernst (1956, 1962) divided the groundwater flow to parallel drains, and consequently the corresponding available total hydraulic head (h), into three components: a vertical (v), a horizontal (h), and a radial (r) component, or

$$h = h_v + h_h + h_r = qL_v + qSL_h + qSL_r$$

where q is the flow rate and L is the resistance.

Working out the various resistance terms and changing notation of q for drainage rate to R for consistency, we can write the Ernst equation as:

$$h = R\frac{D_v}{K_v} + R\frac{S^2}{8KD} + R\frac{S}{\pi K_2}\ln\frac{aD_2}{u}$$

where

D_v = thickness of the layer over which vertical flow is considered; in most cases this component is small and may be ignored (m);

K_v = vertical hydraulic conductivity (m/day);

KD = the sum of the product of the permeability (K) and thickness (D) of the various layers for the horizontal flow component according to the hydraulic situation:

one pervious layer below drains: $KD = K_1D_1 + K_2D_2$

two pervious layers below drains: $KD = K_1D_1 + K_2D_2 + K_3D_3$

a = geometry factor for radial flow:

$KD = K_1D_1 + K_2D_2$ $a = 1$

$KD = K_1D_1 + K_2D_2 + K_3D_3$ a can be estimated from Graph I

u = wetted section of the drain (m); for pipe drains $u = \pi r$

All other parameters are same defined as in the Hooghoudt equation.

The basic difference between the Hooghoudt and Ernst equations is that Ernst took radial flow into account for the total flow, whereas Hooghoudt considered radial flow only for the flow below drains. For a drainage situation where $K_1 \ll K_2$ (e.g. a clay layer on a sandy substratum), the Ernst equation can provide accurate results. However, for a situation where $K_1 \gg K_2$ (e.g. a sandy layer on a clay layer), the results obtained from the Ernst equation tend to considerably underestimate drain spacing compared with the results obtained from the Hooghoudt equation. According to Ernst (1962) and Van Beers (1965), no acceptable formula has been found for the special case where $K_1 \gg K_2$, and the Hooghoudt equation was suggested to be used for this case.

Efforts have been made to develop an equation which can be used for all K_1/K_2 ratios. Such an equation was obtained by combining the approach of Hooghoudt (horizontal flow only above drains and both horizontal and radial flows below drains) and the expression for the radial flow component in the Ernst equation (Ernst, 1975). Neglecting the vertical resistance, we have:

$$R = \frac{8K_1D_1h}{S^2} + \frac{8K_2D_2h}{S^2 + \frac{8}{\pi}D_2 \ln \frac{aD_2}{u}}$$

By introducing an equivalent drain spacing (S_o), i.e. a drain spacing that would be found if horizontal flow alone is considered:

$$S_o^2 = \frac{8KDh}{R}$$

we can write the above equation as:

$$\frac{8KDh}{S_o^2} = \frac{8K_1D_1h}{S^2} + \frac{8K_2D_2h}{S^2 + \frac{8}{\pi}D_2 \ln \frac{aD_2}{u}}$$

Rearranging the above equation gives the generalized Hooghoudt-Ernst equation:

$$(\frac{S}{S_o})^3 + \frac{8c}{\pi S_o}(\frac{S}{S_o})^2 - \frac{S}{S_o} - B\frac{8c}{\pi S_o} = 0$$

where

S = drain spacing based on both horizontal and radial flow (m);

S_o = drain spacing based on horizontal flow only (m);

c = $D_2 \ln \frac{aD_2}{u}$, a radial resistance factor (m);

B = $\frac{K_1 D_1}{KD}$

Graphs have been prepared to use this generalized equation (van Beers, 1976).

Toksöz & Kirkham (1971a) used potential theory to develop an equation for two-layered soil with drains in the upper layer. The assumption used by Toksöz & Kirkham to simplify the field problem was that the loss in hydraulic head in the region lying below the arch-shaped water table and above the level of the drains is negligible compared with the head loss over the remainder of the flow region. This assumption is acceptable when the horizontal hydraulic conductivity in the soil above the drains is small and when the water table height above the drains is small compared with the depth of the soil profile below the drains. Otherwise the Kirkham equation tends to underestimate the spacing required. The Kirkham equation is also somewhat complicated and its solution requires a large number of nomographs. However, the Kirkham equation provides internal potential and stream functions which might be useful for further reserch.

2.1.2 Non-steady State Subsurface Drainage Theory

Realistically speaking, steady state conditions are seldom, if ever, encountered in the field. Where high intensity rainfall is common and where the field is irrigated, non-steady state groundwater flow theory should be used in the design of subsurface drainage systems.

Non-steady, or time-varying, flow problems are more difficult to solve than time invariant problems. For the viscous flow of fluids through porous media, classical theory

leads to the Navier-Stokes equations (van Schilfgaarde, 1970). However these equations are very difficult to solve in many situations of practical interest and therefore have little direct application. Hence, Darcy's law is applied to replace the momentum balance to obtain more applicable equations.

Darcy's law, originally stated for one-dimensional flow through a sand filter, may be generalized to

$$q = -K\nabla h$$

where q is the rate of flow through the media, h is the hydraulic head, and K is the hydraulic conductivity of the media.

Combined with the equation of continuity for saturated, homogeneous, isothermal and incompressible systems, Darcy's law leads to the Laplace equation:

$$\nabla^2 h = 0$$

Since time does not appears in this equation, to apply it to non-steady state problems the time variable must be introduced by means of the time-variant boundary conditions . However, the determination of such boundary conditions is very difficult, and it is assumed in the derivative of the Laplace equation that the soil is saturated, i.e. no water storage in the flow zone, which is obviously not the case in an actual field.

Another equation that has been widely used for non-steady state groundwater flow problems, is based on Dupuit-Forchheimer theory. The basic D-F assumptions are that in a system of gravity flow towards a shallow sink all streamlines are horizontal and that the velocity along these streamlines is proportional to the slope of the free-water surface but independent of the depth. In addition, it is implied that the free-water surface is a streamline bounding the flow region and, in cases of an open ditch or a dam, that the water table terminates at the water level in the ditch (van Schilfgaarde, 1974).

Based on D-F assumptions, if a constant specific yield or drainable porosity of the soil f is assumed, the partial differential equation governing the groundwater flow towards parallel ditches or drains, which is often referred to as the Boussinesq equation, can be written as

$$K\frac{\partial}{\partial x}(h\frac{\partial h}{\partial x}) + R = f\frac{\partial h}{\partial t}$$

where R is the net recharge rate (or drainage rate). Again, this equation is limited to saturated flow conditions because a constant drainable porosity is assumed. However, drainable porosity is far from constant, depending at least on the proximity of the water table to the soil surface and on the rate and direction of water table movement.

The Boussinesq equation can be used to study the rate of fall of the water table from some initial conditions after the cessation of precipitation or irrigation, or to study fluctuation of the water table as recharge input varies with time.

The Boussinesq equation is non–linear and difficult to solve. Various linearizations have been made to solve this equation (Dumm, 1954; Werner, 1957; Krayenhoff van de Leur, 1958; Maasland, 1959; Tezidis, 1968). Glover, as reported by Dumm (1954), obtained a solution to the Boussinesq equation without source of recharge in a Fourier series:

$$h - d = \frac{4m_0}{\pi}\sum_{n=1,3,5}^{\infty}(\frac{1}{n})\sin(\frac{n\pi x}{S})\exp(-\frac{n^2\pi^2 KDt}{fS^2})$$

starting from an initial flat water table, and the boundary condition:

$h = h_0 = d + m_0$	for $0 < x < S$	at $t = 0$
$h = d$	for $x = 0, S$	at $t \geq 0$
$dh/dx = 0$	for $x = S/2$	$t \geq 0$

If time t is not too small, all but the first term can be neglected, resulting in the

simplified expression of the midspan water table height ($x = S/2$)

$$m_1 = \frac{4m_0}{\pi}\exp(-\frac{\pi^2 KDt}{fS^2})$$

or

$$S^2 = \frac{\pi^2 KDt}{f \ln \frac{4m_0}{\pi m_1}}$$

which is referred to as the Glover equation. Here, $D = d_e + m_0/2$ where d_e is the equivalent depth.

Tapp and Moody (Dumm, 1964) observed that the initial water table shape encountered in the field was more often parabolic than flat. They assumed a fourth degree parabolic initial water table shape of the form:

$$h - d = \frac{8m_0}{S^4}(S^3x - 3S^2x^2 + 4Sx^3 - 2x^4)$$

and $h = 0$ for $x = 0, S$ at $t = 0$.

The solution obtained thereafter is:

$$h - d = \frac{192m_0}{\pi^5}\sum_{n=1,3,5}^{\infty}(\frac{n^2\pi^2 - 8}{n^5})\sin(\frac{n\pi x}{S})\exp(-\frac{n^2\pi^2 KDt}{fS^2})$$

For all but the smallest time periods, all terms but the first are negligible, and the equation becomes:

$$S^2 = \frac{\pi^2 KDt}{f \ln \frac{3.7m_0}{\pi m_1}}$$

which is only a minor modification of the Glover equation.

This modified Glover equation is currently recommended for use by the Bureau of Reclamation. However, it is restricted to small increments of water table recession.

The Boussinesq equation has been solved without linearization for situations where the drains rest on an impermeable layer ($d = 0$) (Boussinesq, 1903; Dumm, 1964), and for situations where the drains do not rest on an impermeable layer (van Shilfgaarde, 1963). The van Schilfgaarde solution is:

$$S^2 = \frac{9Kd_e t}{f \ln \frac{m_0(2d_e+m_1)}{m_1(2d_e+m_0)}}$$

where d in the original equation is substituted with equivalent depth, d_e, to circumvent the D–F restriction that $d \ll S$.

If a water source is presented at the soil surface, an analytical solution becomes very difficult to obtain. In most cases, numerical techniques are employed instead to obtain approximate solutions (Moody, 1966; Skaggs, 1975).

2.2 Determination and Measurement of Drainage Parameters

A successful drainage system design is often more dependent on the investigation and evaluation of local conditions than on groundwater flow theory. This includes the study and analysis of the weather conditions, crop response to water table height and soil properties. Although costs in this stage of design may be very high, it is usually worthwhile for the long term economic returns.

2.2.1 Design Criterion

The design criterion is the difference between the drainage requirement, which is the total desired drainage intensity for a given region or field, and the existing natural drainage intensity of a given field. Development of a drainage design criterion is an indispensable part of the design process. Drainage criteria are developed by local experience or by analysis of soil water balance (Willardson, 1982). As far as the subsurface drainage system costs are concerned, the design drainage criterion should be no larger than necessary to remove excess water adequately and quickly enough to meet crop and trafficability requirements (Chieng, *et al*, 1978). The drainage criterion can be evaluated for a steady

state condition, a non–steady state condition, salinity control, or soil trafficability, depending on the main purpose of the drainage system, which in turn depends on the climatic pattern and crop characteristics.

Steady State Condition

For steady state drainage design, the drainage requirement depends essentially on the rainfall pattern and the effects of water table position on crop yield. However, this effect is difficult to estimate because of the complex nature the crop responses to a number of factors. For areas where there are a considerable number of drainage systems installed, perhaps the most effective way to evaluate drainage criteria would be to observe both farm and experimental systems of different drainage intensities. Chieng *et al* (1978), on the other hand, developed a water balance computer model using a long period of weather record to estimate the minimum drainage rates that would be acceptable. For irrigated areas, the drainage criterion is evaluated from application intensity and uniformity, and evapotranspiration.

Non–steady State Condition

In areas where high intensity rainfall or irrigation is common, the water table may rise very rapidly into the root zone and even to the soil surface. In these cases, drainage criteria are usually expressed as the rate of fall of water table at a certain water table position, or a certain distance of fall from one water table to the next in a given period of time.

The falling water table drainage criterion depends on the tolerance level of crops to temporary high water tables, which in turn depends on crop type, stage of growth, weather conditions, antecedent water table positions, soil fertility level, soil porosity and

other factors (Bouwer, 1974). Observation and experience with local conditions is usually required to estimate the necessary rate of fall of the water table for various crops.

A more realistic view of the situation in the field is a fluctuating water table. If the effect of the water table hydrograph on crop yield is known, and if this hydrograph for a drainage system can be predicted, undesirable patterns of water table fluctuations may be prevented.

Salinity Control

In arid areas, whether or not irrigated, salt may concentrate in the root zone and even at the soil surface because of the high rate of evapotranspiration. Under these conditions, drainage systems must be installed to maintain the salt balance so that the field can be profitable for a long period of time. The drainage system should be able to prevent high water tables in the growing season when leaching of salts is desired, and should keep the water tables sufficiently low between growing seasons to minimize evaporation from the water table and the resulting salt accumulation in the root zone.

Drainage criteria for salinity control are based on the salt balance in the root zone. Excess irrigation water is applied to leach out the salt to maintain a salt balance suitable for crop growth. The amount of leaching water required depends on the salinity level of the irrigation water, salt content in the soil solution, evapotranspiration, irrigation uniformity, deep seepage, and other factors. Successful design of a drainage system for salinity control depends greatly on the accuracy of estimation of these factors.

Trafficability Control

In humid areas, heavy precipitation at inopportune times may seriously interfere with the timeliness of farming operations. Farming operations, such as planting and harvesting, may be impossible due to poor trafficability in the field. The resulting losses may vary

from complete crop failure, if planting is delayed too long, to reduced crop yields if tillage, weed control, or harvesting are not performed on time (Reeve and Fausey, 1974).

The design drainage criteria of drainage systems for the purpose of trafficability control are often based on providing a certain number of consecutive workable days. Skaggs (1978) defines a workable day as a day when the air volume in the soil profile exceeds some limiting value, the rainfall occurring on that day is less than a minimum value, and a minimum number of days have elapsed since that amount of rainfall occurred. The probability of a certain number of "dry" days at a given time can be predicted from the historic weather data of the region, and artificial drainage can be installed to increase the probability of a number of workable days at a given time of year. In some cases, considering the high cost of installing a subsurface drainage system, an alternative to drainage may be to more farm equipment so that planting or harvesting can be completed in shorter period of time,

Soil trafficability depends on the moisture content of the top soil. Oosterbaan and Wind (1979) indicated that most farming operations can be carried out if the tension in the top soil is 100 cm of water or more. However, in practice, agricultural engineers tend to use some simple indicators for workable days, such as depth to the water table, in the design of subsurface drainage for trafficability control.

From the above review of drainage criteria, it is obvious that the steady state drainage criterion is the simplest one to use. Therefore, it may be advantageous and also possible to express all drainage criteria in terms of an equivalent steady state criterion, regardless of whether the primary purpose of the drainage system is to yield a certain fall or fluctuation of the water table, salinity control or an extended workable period (Bouwer, 1974).

2.2.2 Measurement of Soil Parameters

Another factor that is as important as the development of the design drainage criterion, is the determination of soil properties. Physical soil properties which are of significance in subsurface drainage design or in drainage research and investigation are the hydraulic conductivity (saturated or unsaturated), the drainable porosity or specific yield, the soil–water pressure head, the volumetric water content, the water table position, etc. In practical drainage design, only saturated hydraulic conductivity and drainable porosity are of interest; the other information is needed only for further research and detailed investigations.

Measurement of Hydraulic Conductivity

Hydraulic conductivity appears in both steady and non–steady state drain spacing calculations. The performance and costs of the drainage system depend greatly on the value of hydraulic conductivity. The determination of the K value of the soil is thus an important aspect of almost any drainage investigation.

The value of hydraulic conductivity measured from the field or lab should represent as adequately as possible the soil and flow direction in the actual drainage system. Therefore the method, location, and number of measurements should be carefully selected so that adequate data can be collected cost– effectively.

Hydraulic conductivity can be measured in the laboratory on soil samples taken from the field. These samples should be representative of the site under investigation. Due to soil variability, a large number of undisturbed core samples should be taken from representative locations with 2–3 replicates at each location. Laboratory methods are mainly used to determine the K value above the water table, to separate the vertical and horizontal hydraulic conductivities, and to predict how salt affects the hydraulic

conductivity and other physical properties.

Field measurements of K are generally preferred to laboratory measurements. If the water table is sufficiently high, one of the auger–hole techniques can be used. If a water table is not present, or is too low for meaningful K value measurement, the techniques for measuring K in an artificially wetted soil region may be employed. Reliable methods have been developed for measurement below the water table, whereas methods employed above the water table are generally less satisfactory (Smedema & Rycroft, 1983. Land Drainage).

With the auger–hole techniques, a hole, ranging in diameter from a few centimeters to a few decimeters, is dug to some distance below the water table. After the water level in the hole has equilibrated with the water table in the soil, the water level is lowered by quickly removing water from the hole. Then the K value can be measured from the rate of rise of the water level in the hole (van Beers, 1970).

The field methods for measuring hydraulic conductivity in situ above or in the absence of a water table include the shallow well pump–in technique, the cylinder permeameter method, the infiltration gradient technique, the air–entry permeameter, and the double–tube method. For all these techniques, the soil should be as close to saturation as possible and the clogging of pores where water enters the soil should be minimized for relatively reliable measurement of K value.

The selection of measuring method in the field depends on the understanding of the field conditions, including the size of the soil region on which K is to be measured, the flow direction, and the presence of stones in the soil. It also depends on the manpower, equipment, time and funds available. However, even with a reasonable selection of measuring method, a certain degree of inaccuracy is inevitable due to the soil variability, which is a far greater source of inaccuracy in evaluating the K value of a field soil than the errors of inherent to the measuring techniques. It is expected that a more accurate

K value can be obtained with a larger number of measurements. The number of measurements that should be taken to characterize the K of a given field depends on the soil variability, the accuracy desired, and the availability of funds and manpower. Due to other sources of uncertainty, such as entrance resistance of the drains, and water sources and sinks, errors in K of 30% or even more are usually acceptable in drainage design (Bouwer & Jackson, 1974).

The measurement of hydraulic conductivity should be combined with soil surveys, so that the best locations for K measurement can be determined. It may also be possible to establish a correlation between hydraulic conductivity and soil texture and structure. The locations of measurements should be representative of the soil types. It is also desirable that the sampling area be divided into small subareas according to the soil types so that the K value may be better representative of the actual condition in each subarea.

The "true" hydraulic conductivity can be better estimated by the geometric mean than by the harmonic mean and arithmetic mean (Bouwer, 1969). The geometric mean is defined for equally thick layers as:

$$K_g = \sqrt[n]{K_1 K_2 \cdots K_n}$$

where K_1, K_2, ..., K_n are the field data (Bouwer & Jackson, 1974).

Measurement of Drainable Porosity

The drainable porosity, or specific yield, is the volume of water that will be released from a unit volume of soil by lowering the water table. In non–steady state subsurface drainage design, drainable porosity is considered as a constant, while in reality it is far from constant, varying with time and space. It also depends on the rate and direction of fall of the water table.

The technique of estimating drainable porosity from equilibrium water content profiles above two successive water table positions, using the relation between soil water content and negative pressure head, may give a reasonable estimation if the water table falls slowly enough for drainage of the pore space to "keep up" (Talsma & Haskew, 1959; French & O'Callaghan, 1966). A negative pressure head above the water table is often called tension or suction and can be measured with tensiometers. Soil water content can be measured by a gravimetric method, Neutron scattering method, or other methods reviewed by Gardner (1965) and Cope & Trickett (1965).

The drainable porosity determined from the soil water content characteristics becomes less valid when the rate of fall of the water table increases. If the rate of fall of the water table is relatively high, the value of drainable porosity to be used in the non-steady spacing equations is only a fraction of the value determined from the equilibrium water content distributions. Therefore, the value of drainable porosity used in subsurface drainage system design should be chosen cautiously.

2.3 Subsurface Drainage Design Under Uncertain Nature

From the previous section, it can be seen that neither the drainage criterion nor the soil properties are known with complete certainty. It suggests that any drainage system design based on uncertain input information will result in uncertain performance, and sometimes even complete failure. Aware of this, many scientists and researchers have been looking for methods of incorporating uncertainty into the study of groundwater flow and drainage design. The falling water table criterion is a primitive step in this direction. Then a more realistic model of fluctuating water table or intermittent recharge was studied. However, they are still far from the stochastic nature of the groundwater flow in the subsurface drainage problems. Thus, more advanced and complex models and

approaches have been developed. Unfortunately, at present, there are still very few, if any, techniques of satisfactory application in practical designs.

Uncertainties involved in subsurface drainage design can be categorized generally into four groups: (1) forcing function uncertainty, (2) initial condition uncertainty, (3) boundary condition uncertainty, and (4) parameter uncertainty, all of which are usually presented in a specific problem (Sagar, 1978a). Forcing function uncertainty includes the unknown nature and presence of sources and sinks in the groundwater flow region. It is one of the major sources of uncertainty. Initial condition uncertainty can be minimized for the long time period. Unsaturated flow occurring above the water table is a source of boundary condition uncertainty. Parameter uncertainty is caused by the reduction of the real, but unknown, values (which are stochastic in nature), to "equivalent" values, as well as by some other sources of errors such as instrumental, reading and round off errors. The "equivalent" value of a parameter is very often ambiguous because it may not represent the "true" value of the parameter.

In addition to the above uncertainties, there is also a modeling uncertainty. Because of the complexity of the interaction between soil and water, the laws or theories on which drainage and groundwater flow theories are based, are simplified and idealized. Darcy's law, for example, is only an empirical equation, not a theoretical one. Hence the derived equations inevitably involve a certain degree of modeling uncertainty. More uncertainty may enter during the process of derivation by, for example, the Dupuit–Forchheimer assumptions and linearizations.

One obvious step to incorporate uncertainty into the study of groundwater flow is to consider the stochastic precipitation and evapotranspiration process, just as many researchers and scientists have done. In subsurface drainage design, net recharge (recharge minus discharge) is the forcing function and is directly related to the fluctuation of the groundwater table. Both precipitation and evapotranspiration are heavily dependent

on the local meteorological conditions. Hence the uncertainty of the weather condition will result in uncertainty of the groundwater table position and therefore of the performance of a subsurface drainage system. However, analytical solutions to stochastic partial differential equations are very difficult to obtain because these equations are nonlinear. Therefore only comparatively simple problems have been analytically solved with stochastic recharge and discharge processes (Sagar and Kisiel, 1972; Bakr, 1976; Sagar, 1978a).

Intermittent recharge occurs when a drainage system provides relief from precipitation with an irregular pattern and from periodic irrigation. Superposition can be used to study the intermittent recharge problems (Werner, 1957; Maasland, 1959; van Schilfgaarde, 1965a). van Schilfgaarde (1965a) reduced the continuous recharge to a bar histogram with a uniform time increment. Within each time increment the recharge was assumed to be a constant. Choosing a convenient interval, such as a day, climatological data (precipitation, temperature, wind speed, etc.) can be used first to calculate the net recharge and then to calculate the water table positions. In the calculation of net recharge, no surface runoff and deep seepage are taken into account. From the water table positions at the end of each of an arbitrarily long sequence of time increments, which may be as long as a number of years, a frequency distribution of water table heights can be developed, which can be used, in principle, in the selection of a drainage system at an acceptable level of risk. Similar techniques have been used by Chieng (1975) and Skaggs (1978). This kind of method requires a large amount of historical meteorological data and may introduce new uncertainty when historical data is used to predict future precipitation and evapotranspiration.

Sagar and Preller (1980) used a different approach to study the case for drainage under uncertain weather conditions. By linearizing the Boussinesq equation and assuming

different forms of stochastic functions for recharge (precipitation) and discharge (evapotranspiration), they solved the partial differential equation for the first two moments of the hydraulic head (mean and variance). The variance or standard deviation could be used as a indicator or measurement of the uncertainty about the water table position and, therefore, the performance of the drainage system. This approach also requires large amounts of data and its accuracy greatly depends on the selection of the stochastic functional forms of precipitation and evapotranspiration.

By applying the Bayesian statistical analysis technique, Musy and Duckstein (1976) and Fogel *et al* (1979) took into account the crop response to high water table level in designing subsurface drainage systems under uncertainties. Musy and Duckstein's approach requires a probability density function of extreme precipitation and the crop loss model corresponding to the extremes. They limited the problem to situations where only one submergence event can occur in one year and such an event always occurs at the critical growth stage. "Critical" means that the crop is most likely to be damaged by high water table position. Fogel *et al* (1979), on the other hand, solved a similar problem without the above limitations, but still required the probability density function of the extreme precipitation and the crop response model. The disadvantage of the Bayesian technique is due to the difficulty of obtaining the probability density function of the extreme precipitation and the crop response model, which are dependent on the local weather conditions and specific crop type.

The above research only incorporate the uncertainty of climatic conditions. However, other parameters, such as soil properties, are also subject to great variability and uncertainty. Bakr (1979) and Freeze (1975) studied groundwater flow problems under climate and soil parameter uncertainty by employing analytical or numerical methods. Prasher (1982) made an effort to incorporate both climate and soil parameter uncertainty into drainage/subirrigation design.

Methods used in the parameter uncertainty analysis of engineering problems can be categorized into two groups: deterministic methods and probabilistic methods (Uhl & Sullivan, 1982). Deterministic methods assign test values to a variable within some feasible range, but without regard for the likelihood of the respective range. Probabilistic methods incorporate probabilistic information about each of the variables, which yields a probability distribution for the dependent variable. Deterministic analysis is often used to identify dominant variables so that they may be subjected to additional (probabilistic) analysis.

One of the deterministic methods commonly used is sensitivity analysis. Sensitivity analysis involves testing a noncorrelated variable, or a group of correlated variables, at a number of values over a reasonably expected range while fixing other variables at their "best–guess" estimates to observe their effects on the dependent variable. Sensitivity analysis is often undertaken initially to identify dominant variables. These variables can be disaggregated and a sensitivity analysis performed on the effects of the newly introduced variables. As the analysis progresses, this method identifies sensitive variable combinations and gives quantified feedback on assumptions in the appraisal. The primary advantages of sensitivity analysis are its flexibility, versatility and simplicity. It can almost test any number of variables and in almost any desired combination subject to various correlation restrictions. It does not require extensive quantitative background information. The major disadvantages of sensitivity analysis are the problem of correlations and the proliferation of data for all but relatively simple applications. Failure to accommodate correlations can invalidate results. The output of a sensitivity analysis for a large or detailed appraisal can be voluminous. The interpretation of the output can become a time–consuming and costly task. Another disadvantage of sensitivity analysis is that it does not reflect the probability of obtaining any particular value for the dependent variable. In spite of these disadvantages, sensitivity analysis is a powerful tool

to determine the behavior of a function or a model, and to identify the dominant and sensitive variables. It also may serve as a precursor to probabilistic analysis for obtaining likelihood estimates.

The probabilistic methods used in the uncertainty analysis in groundwater flow problems can be generally grouped as those which require full probability distributions of all stochastic inputs and those which require only first and second moments of the stochastic parameters. A comparison and review on these two groups of methods was made by Dettinger and Wilson (1981). A summary is given below.

Full distribution methods are more accurate, provided that the probability distributions of independent variables are correctly established. The derived distribution method and Monte Carlo simulation method are two of the most commonly used full distribution methods. The derived distribution technique analytically derives the probability distribution of the dependent variable which is functionally related to the random independent variables (Benjamin & Cornell, 1970). However, this analysis is very complicated and time consuming for all but simple applications, unless the problem is simplified by assuming that all variables and results are normally distributed. In such case the reliability of these techniques is dependent on the validity of these assumptions. If the function is approximately linear the results obtained from these assumptions are generally acceptable, otherwise another method (e.g. Monte Carlo simulation) should be used (Uhl and Sullivan, 1982). Sagar and Kisiel (1970) applied the full distribution method to examine the parameter uncertainty for aquifer pump tests.

The Monte Carlo method, simulates the groundwater flow system with numerous replications. The simulation involves a sampling technique, usually carried out on a computer, that randomly and iteratively selects sample values for each dominant variable from its respective probability distribution to obtain sample values for the dependent variable. A large number of trials is needed for the resultant frequency distribution for

the dependent variable to converge to the analytical solution. Monte Carlo simulation is a powerful method to analyse uncertainty and can be applied to most uncertainty analysis problems, such as nonlinear and relatively complex functions. However, when many variables are involved and the function is not simple and well-behaved, the Monte Carlo simulation technique can be very complicated and expensive (Uhl and Sullivan, 1982). In cases where the variables are correlated, identifying and accurately representing these interrelationships can be a complex task. Also the results obtained from Monte Carlo method are not readily transferable to new situations. The Monte Carlo method has been employed to study the uncertainty of transport processes through porous media due to the variability of soil properties (Warren and Price 1961; Freeze 1975; Smith and Freeze 1979). Massman (1987) applied the Monte Carlo method to study uncertainty in the travel time of contaminants in the groundwater flow region.

The major disadvantage of full distribution methods is the fact that it is usually very difficult to obtain full probability distributions for all stochastic inputs. In addition, full distribution methods often require advanced mathematic and statistical background and intensive analyses and computations. The results obtained from the full distribution methods are highly dependent on the types of distributions selected to represent each random variable, and if correlationships among variables exist, full distribution methods can be too complicated to be practically applicable.

The first and second moment method uses the mean (first moment) and the variance and/or covariance (second moment) to represent the probability properties of a random variable. The mean is a parameter representing the central tendency of a random variable, and variance is a measure of scatter from the mean. For this representation to be accurate, or in other word, the properties of a random variable can be completely characterized by its mean and variance, the distribution must be normal. In this case it

has zero skewness and other higher moments can be calculated from the variance (Benjamin & Cornell, 1970). If the function is linear or approximately linear, the dependent variable can also be approximately defined by its mean and variance. If the variables are not normally distributed or the function is highly non–linear or the coefficients of variation of random variables are not small enough, the results becomes unreliable. However, these disadvantages can usually be justified by noting that: (1) the data and physical arguments are often insufficient to establish the full probability law of a variable; (2) most engineering analyses include an important component of real, but difficult to measure, professional uncertainty (due, for example, to imperfect physical theories and to engineering approximations); and (3) the final output, namely the decision or design parameter, is often not sensitive to moments higher than the mean and variance (Cornell, 1972). In addition, first and second moment analysis employs only the same, familiar tools and procedures of algebra and calculus that are commonly used in the more deterministic analysis of the same problem. This implies that both students and practicing engineers can learn the technique and apply it more readily. Wider engineering application of stochastic methods by non–specialists will benefit everyone concerned. Also it is far better to approximately model the whole problem than to exactly model only a portion of it (Cornell, 1972).

Cornell (1972) proposed the use of first and second moment analysis based on the Taylor series expansion in the uncertainty analysis of water resource system. He indicated that first and second moment analysis can be used for either unconditional or conditional analyses, and for either parameter or model uncertainty. Sagar (1978b) used this approach to analyze one–dimensional flow in confined aquifers by the finite element method. Prasher *et al* (1982) suggested that first and second moment analysis can be used in uncertainty analysis in the design of a drainage/subirrigation system. The author has employed this approach in the analysis of the uncertainty involved in steady and

transient state subsurface drainage design. Simple, ready–to–use formulae and graphs have been prepared for convenient use of practical application.

Chapter 3

SENSITIVITY ANALYSIS OF DRAINAGE DESIGN IN HOMOGENEOUS SOIL

In agricultural drainage design, input information of system parameters are estimated from historic climatic data, soil geohydrologic properties and crop characteristics. The estimates are more or less subjective and involve a considerable degree of uncertainty due to the complexity of the nature. It is therefore desirable to examine the relative sensitivity of system performance to each parameter, so that certain precautions can be taken in the estimation of parameter values to optimize the design of a drainage system.

This chapter will carry out the sensitivity analysis of drainage design in homogeneous soils. Multi-layered soils will be the subject of the next chapter.

3.1 Steady-state: Hooghoudt's Equation.

For the steady-state condition, Hooghoudt's equation (1940) has been most popularly used for drainage design (Luthin, 1978; Chieng, 1981). Hooghoudt's equation for a homogeneous soil can be described as:

$$S^2 = \frac{4K}{R}(2d_e h + h^2) \tag{3.1}$$

where S is the drain spacing (m);

R is the drainage coefficient (m/day);

K is the saturated hydraulic conductivity (m/day);

h is the water table height midway between the drains;

d_e is the equivalent depth of impermeable layer below the drains.

d_e can be evaluated from tables prepared by Hooghoudt (1940) or graphs (Bouwer & van Schilfgaarde, 1963), or from the equations below:

For $0 < \frac{d}{S} \leq 0.3$

$$d_e = \frac{d}{1 + \frac{d}{S}(\frac{8}{\pi}\ln\frac{d}{r} - \alpha)} \tag{3.2}$$

where $\alpha = 3.55 - 1.6\frac{d}{S} + 2(\frac{d}{S})^2$

and for $\frac{d}{S} > 0.3$

$$d_e = \frac{S}{\frac{8}{\pi}(\ln\frac{S}{r} - 1.15)} \tag{3.3}$$

where r is the radius of the drain pipes, S is the spacing, and d is the actual depth of the impermeable layer below drains.

Although d_e is a function of spacing S , actual depth of impermeable layer d, and drain radius r, it is usually considered as an independent variable when sensitivity analysis is carried out. This can be justified by the fact that results show clearly the sensitivity of drain spacing to other parameters, and that the effect of considering d_e as an independent variable on the sensitivity of spacing is relatively small. For detailed discussion, see appendix A.

3.1.1 Sensitivity of Spacing to Drainage Coefficient R

The drainage coefficient R is the amount of water that the drainage system can remove from the soil per unit time. R is usually estimated from historic rainfall data, evaporation, consumptive water use of crops, surface runoff, deep seepage *etc.* It is obvious that such an estimation can not accurately represent the actual drainage requirement, and different designers make different estimations.

From basic calculus, we know that:

$$\Delta S|_R \approx dS = \frac{\partial S}{\partial R}\Delta R$$

for a small change of R, while all other variables are held constant.

Also

$$\frac{\partial S}{\partial R} = -\frac{\frac{4K}{R^2}(2d_e h + h^2)}{2\sqrt{\frac{4K}{R}(2d_e h + h^2)}}$$

Therefore

$$\Delta S|_R = -\frac{\frac{4K}{R^2}(2d_e h + h^2)}{2\sqrt{\frac{4K}{R}(2d_e h + h^2)}}\Delta R$$

Dividing by S on both sides and rearranging gives:

$$\left.\frac{\Delta S}{S}\right|_R = -\frac{1}{2}\frac{\Delta R}{R} \tag{3.4}$$

where $\frac{\Delta R}{R}$ represents the percentage change of drainage coefficient, and $\left.\frac{\Delta S}{S}\right|_R$ is the corresponding change of the drain spacing. The sensitivity of spacing to the drainage coefficient is approximately a constant.

3.1.2 Sensitivity of Spacing to Hydraulic Conductivity

Hydraulic conductivity K is one of the main soil properties directly related to drainage design. Methods of measuring hydraulic conductivity were reviewed by Bouwer & Jackson (1974). However it is impossible to select a measuring method that can accurately evaluate the K-value because of the soil variability. That means K-values at different locations may not be the same, and even at the same location, the K-value may also change from time to time because of compaction, changes in moisture content and soil structure. Therefore, it is important to know the sensitivity of drain spacing with respect to the $K-$ value, before choosing a certain value to be used in the design.

Again we have:

$$\Delta S|_K \approx dS = \frac{\partial S}{\partial K}\Delta K$$

in which

$$\frac{\partial S}{\partial K} = \frac{\frac{4}{R}(2d_e h + h^2)}{2\sqrt{\frac{4K}{R}(2d_e h + h^2)}}$$

The percentage change in spacing due to a small change in K-value is:

$$\left.\frac{\Delta S}{S}\right|_K = \frac{\frac{4}{R}(2d_e h + h^2)}{2\sqrt{\frac{4K}{R}(2d_e h + h^2)}}\frac{\Delta K}{S}$$

Rearranging the above equation yields:

$$\left.\frac{\Delta S}{S}\right|_K = \frac{1}{2}\frac{\Delta K}{K} \tag{3.5}$$

where $\frac{\Delta K}{K}$ is the percentage change in hydraulic conductivity.

3.1.3 Sensitivity of Spacing to Water Table Height Midway Between Drains

Water table height midway between drains is a very important parameter in agriculture drainage design. Since the depth, D, of drains below the field surface is usually determined by the soil profile, excavating equipment, or water table position, the value of h to be used in Eq. 3.1 is $D - d_{min}$, where d_{min} is the minimum permissible depth of the water table midway between the drains. d_{min} is dependent on the tolerance of crops to waterlogging, and on trafficability for farm machines.

Similar to the previous analysis, the change of spacing due to change in $h-$ value is:

$$\Delta S|_h \approx dS = \frac{\partial S}{\partial h}\Delta h$$

where

$$\frac{\partial S}{\partial h} = \frac{\frac{8K}{R}(d_e + h)}{2\sqrt{\frac{4K}{R}(2d_e h + h^2)}}$$

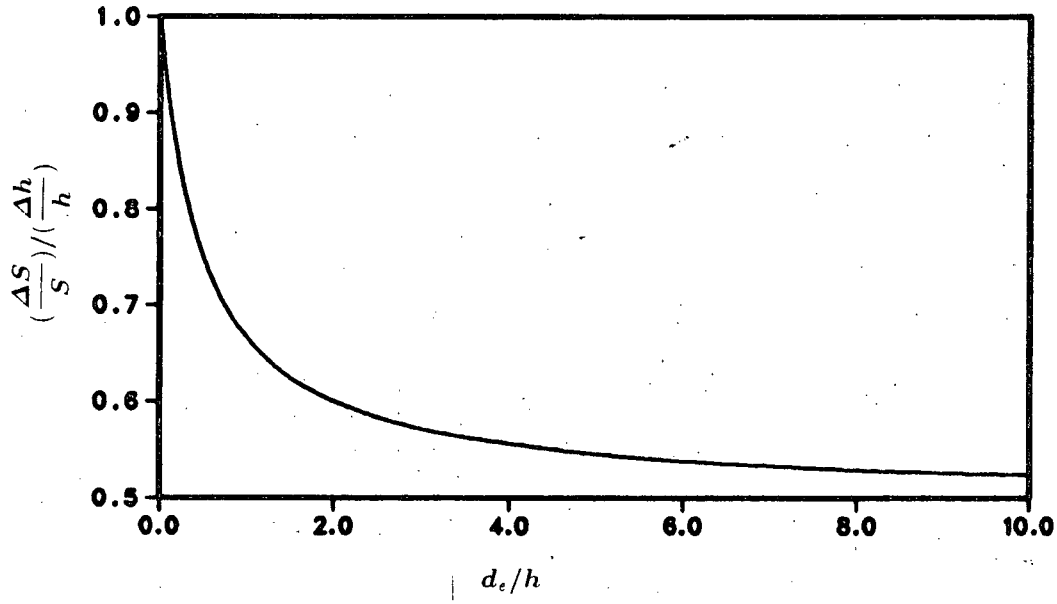

Figure 3.2: Sensitivity of spacing to midway water table height in relation with $\frac{d_e}{h}$

and the percentage change of spacing will be:

$$\left.\frac{\Delta S}{S}\right|_h = \frac{d_e + h}{2d_e + h}\frac{\Delta h}{h}$$

or in the dimensionless form:

$$\left.\frac{\Delta S}{S}\right|_h = \frac{\frac{d_e}{h} + 1}{2\frac{d_e}{h} + 1}\frac{\Delta h}{h} \tag{3.6}$$

where $\frac{\Delta h}{h}$ is the percentage change of h.

Figure 3.2 shows that the sensitivity of spacing to water table height midway between drains decreases as $\frac{d_e}{h}$ increases, the minimum ratio of change of spacing and change of water table height being 0.5. $\frac{d_e}{h}$ represents the relative importance of water flows in the soil region below and above drains. In other words, when the thickness of soil below the drains increases, the water flow in the upper layer becomes less important.

3.1.4 Sensitivity of Spacing to the Equivalent Depth d_e

Equivalent depth is the thickness of an imaginary layer of soil in which the water flow is equal to the water flow in the actual soil profile between drains and the impermeable

layer. Within the equivalent layer, the water flow towards the drains is horizontal, while the actual water flow includes both horizontal and radial components. In a homogeneous soil, d_e is related to the spacing, the depth of the impermeable layer below the drains and the radius of the drain pipe. In this section d_e is treated as a "real" parameter, and in next section its relationship with impermeable layer depth and drain radius will be discussed.

Following the same procedure as in the previous analysis, we have:

$$\Delta S|_{d_e} \approx dS = \frac{\partial S}{\partial d_e}\Delta d_e$$

where

$$\frac{\partial S}{\partial d_e} = \frac{\frac{8Kh}{R}}{2\sqrt{\frac{4K}{R}(2d_e h + h^2)}}$$

The percentage change of spacing due to change in equivalent depth is:

$$\left.\frac{\Delta S}{S}\right|_{d_e} = \frac{d_e}{2d_e + h}\frac{\Delta d_e}{d_e}$$

or in dimensionless form:

$$\left.\frac{\Delta S}{S}\right|_{d_e} = \frac{\frac{d_e}{h}}{2\frac{d_e}{h} + 1}\frac{\Delta d_e}{d_e} \tag{3.7}$$

where $\frac{\Delta d_e}{d_e}$ is the percentage change in d_e.

Figure 3.3 shows that the sensitivity of spacing to equivalent depth increases as $\frac{d_e}{h}$ increases. The ratio of percentage change in spacing to percentage change in d_e, ranges from 0 to 0.5. A larger d_e value implies a greater importance of the equivalent layer, and thus increases the sensitivity of spacing.

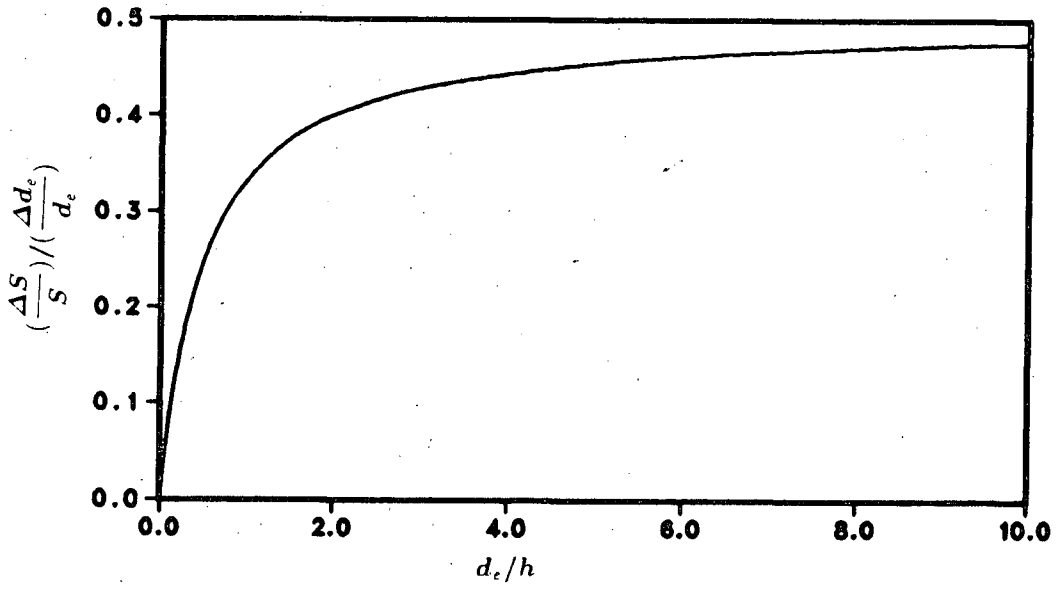

Figure 3.3: Sensitivity of spacing to equivalent depth in relation with $\frac{d_e}{h}$.

3.1.5 Discussion on Relationship between Equivalent Depth d_e and Impermeable Layer depth below drains d and Drains Radius r

Equivalent depth d_e can be calculated from equation 3.2 and 3.3:

For $0 < \frac{d}{S} \leq 0.3$:

$$d_e = \frac{d}{1 + \frac{d}{S}(\frac{8}{\pi}\ln\frac{d}{r} - \alpha)}$$

where

$$\alpha = 3.55 - 1.6\frac{d}{S} + 2(\frac{d}{S})^2$$

and for $\frac{d}{S} > 0.3$:

$$d_e = \frac{S}{\frac{8}{\pi}(\ln\frac{S}{r} - 1.15)}$$

The partial derivatives of d_e with respect to d and r are, respectively:

For $0 < \frac{d}{S} \leq 0.3$:

$$\frac{\partial d_e}{\partial d} = \frac{1 - \frac{8}{\pi}\frac{d}{S} - 1.6(\frac{d}{S})^2 + 4(\frac{d}{S})^3}{[1 + \frac{d}{S}(\frac{8}{\pi}\ln\frac{d}{r} - \alpha)]^2}$$

$$\frac{\partial d_e}{\partial r} = \frac{\frac{8}{\pi}\frac{d^2}{Sr}}{[1 + \frac{d}{S}(\frac{8}{\pi}\ln\frac{d}{r} - \alpha)]^2}$$

and for $\frac{d}{S} > 0.3$:

$$\frac{\partial d_e}{\partial d} = 0$$

$$\frac{\partial d_e}{\partial r} = \frac{\frac{S}{r}}{\frac{8}{\pi}(\ln\frac{S}{r} - 1.15)^2}$$

And for a small change in d and r, the change in d_e will be:

$$\left.\frac{\Delta d_e}{d_e}\right|_d = f_2 \frac{\Delta d}{d} \tag{3.8}$$

$$\left.\frac{\Delta d_e}{d_e}\right|_r = f_3 \frac{\Delta r}{r} \tag{3.9}$$

where

$$f_2 = \begin{cases} \frac{1-\frac{8}{\pi}\frac{d}{S}-1.6(\frac{d}{S})^2+4(\frac{d}{S})^3}{1+\frac{d}{S}(\frac{8}{\pi}\ln\frac{d}{r}-\alpha)} & \text{for } 0 < \frac{d}{S} \leq 0.3 \\ 0 & \text{for } \frac{d}{S} > 0.3 \end{cases} \tag{3.10}$$

$$f_3 = \begin{cases} \frac{\frac{8}{\pi}\frac{d}{S}}{1+\frac{d}{S}(\frac{8}{\pi}ln\frac{d}{r}-\alpha)} & \text{for } 0 < \frac{d}{S} \leq 0.3 \\ \frac{1}{ln\frac{S}{r}-1.15} & \text{for } \frac{d}{S} > 0.3 \end{cases} \tag{3.11}$$

Equivalent depth, d_e, becomes less sensitive to d as $\frac{d}{S}$ or d increases. When $\frac{d}{S} > 0.3$, d_e becomes unaffected by the change of d-value. Equivalent depth becomes more sensitive when $\frac{d}{r}$ decreases or r increases, or when $\frac{d}{S}$ increases. That means that in a deep soil profile, the radius of the drains is a more important factor than in a shallow soil profile.

3.1.6 Sensitivity of Spacing to the Depth of Impermeable Layer below Drains

From the discussions in section 3.1.4 and 3.1.5, we can now carry out a sensitivity analysis of spacing to the depth of the impermeable layer below the drains.

From section 3.1.4, we have:

$$\left.\frac{\Delta S}{S}\right|_{d_e} = f_1 \frac{\Delta d_e}{d_e}$$

where $f_1 = \frac{d_e}{2d_e+h}$.

And from section 3.1.5, we have:

$$\left.\frac{\Delta d_e}{d_e}\right|_d = f_2 \frac{\Delta d}{d}$$

where f_2 can be evaluated from equation 3.10.

Therefore, the percentage change in spacing due to change of depth of the impermeable layer can be approximated as:

$$\left.\frac{\Delta S}{S}\right|_d = f_1 f_2 \frac{\Delta d}{d} \tag{3.12}$$

3.1.7 Sensitivity of Spacing to Radius of Drains

From section 3.1.4, we have:

$$\left.\frac{\Delta S}{S}\right|_{d_e} = f_1 \frac{\Delta d_e}{d_e}$$

where $f_1 = \frac{d_e}{2d_e+h}$.

And from section 3.1.5, we have:

$$\left.\frac{\Delta d_e}{d_e}\right|_r = f_3 \frac{\Delta r}{r}$$

where f_3 can be evaluated from equation 3.11.

Therefore, the percentage change in spacing due to change in radius can be evaluated by:

$$\left.\frac{\Delta S}{S}\right|_r = f_1 f_3 \frac{\Delta r}{r} \tag{3.13}$$

3.1.8 Estimation of Spacing Change due to Changes of All or a Portion of the Parameters

The spacing change, ΔS, is approximately:

$$\Delta S \approx dS = \frac{\partial S}{\partial R}\Delta R + \frac{\partial S}{\partial K}\Delta K + \frac{\partial S}{\partial h}\Delta h + \frac{\partial S}{\partial d_e}\Delta d_e$$

Dividing by S on both sides:

$$\frac{\Delta S}{S} = \frac{\partial S}{\partial R}\frac{\Delta R}{S} + \frac{\partial S}{\partial K}\frac{\Delta K}{S} + \frac{\partial S}{\partial h}\frac{\Delta h}{S} + \frac{\partial S}{\partial d_e}\frac{\Delta d_e}{S}$$

From the analyses in the previous sections, the above equation can be replaced by:

$$\frac{\Delta S}{S} = e_R\frac{\Delta R}{R} + e_K\frac{\Delta K}{K} + e_h\frac{\Delta h}{h} + e_d\frac{\Delta d}{d} + e_r\frac{\Delta r}{r} \tag{3.14}$$

where

$$e_R = -\tfrac{1}{2}$$

$$e_K = \tfrac{1}{2}$$

$$e_h = \tfrac{d_e+h}{2d_e+h}$$

$$e_d = f_1 f_2$$

$$e_r = f_1 f_3$$

$$f_1 = \tfrac{d_e}{2d_e+1}.$$

f_2 and f_3 are evaluated from equation 3.10 and 3.11.

Let $\vec{e}$ be a five dimensional vector and equal to:

$$\vec{e} = (e_r, e_K, e_h, e_d, e_r)$$

and let:

$$\Delta\vec{X} = (\frac{\Delta R}{R}, \frac{\Delta K}{K}, \frac{\Delta h}{H}, \frac{\Delta d}{d}, \frac{\Delta r}{r})$$

The equation 3.14 can be writen as:

$$\frac{\Delta S}{S} = \vec{e} \bullet \Delta\vec{X} \tag{3.15}$$

3.1.9 Conclusions for Sensitivity Analysis of Steady-state Drainage Design in Homogeneous soil

From the analyses in the previous sections, it can be concluded that:

1. Spacing is most sensitive to the water table height midway between drains. Therefore special care should be taken in selecting the design water table height in order that it can meet the crop and leaching requirements, while remaining as high as possible so that the spacing can be greater.

2. Spacing is least sensitive to the depth of the impervious layer below the drains and to the radius of the drains. Thus, it is not necessary to determine the depth to the impermeable layer very accurately, and when choosing drain radius, drain flow capacity should be the major concern (i.e. a drain size should be selected for the necessary flow capacity). A larger diameter for the drains will help little to increase the drain spacing.

3. Spacing is also very sensitive to the drainage coefficient and hydraulic conductivity. It is recommended that more research be done on the determination of drainage coefficients and measurement of hydraulic conductivity so that input information can be more accurate.

3.2 Transient State: van Schilfgaarde Solution of Boussinesq Equation

Based on D-F assumptions, the Boussinesq equation is commonly used in nonsteady subsurface drainage design (Luthin, 1978; Chieng, 1981). It can be expressed as:

$$f\frac{\partial h}{\partial t} = K\frac{\partial}{\partial x}(h\frac{\partial h}{\partial x}) + R \tag{3.16}$$

where h is the height of the water table above the impermeable layer, x is the horizontal distance, t is time, f is drainable porosity, K is saturated hydraulic conductivity, and R is the rate at which water enters or leaves the saturated region (see Figure 3.4.

Eq. 3.16 is nonlinear, and is frequently linearized (Polubarinova- Kochina, 1962) to

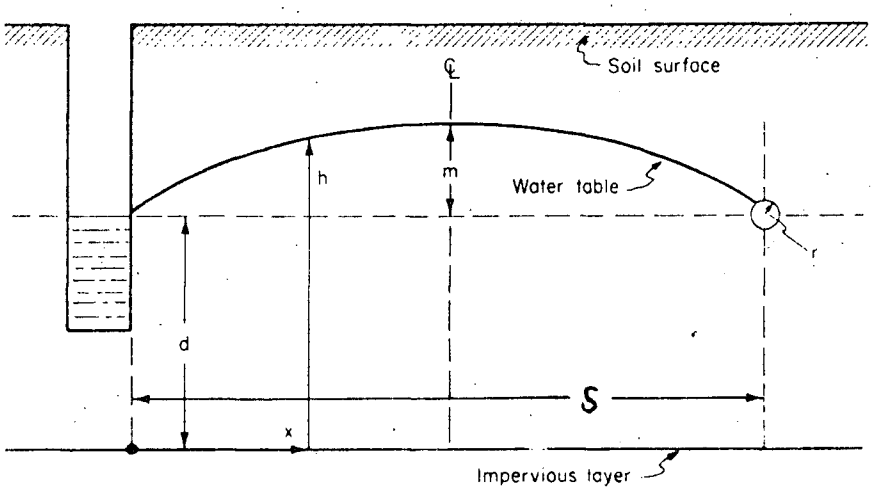

Figure 3.4: Symbols used to describe drainage to a ditch or a tube drain.

obtain its solution. Glover (Dumm, 1954) solved the Boussinesq equation by assuming an initially flat water table:

$$h - d = \frac{4m_0}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n} \sin(\frac{n\pi x}{S}) \exp(\frac{-n^2\pi^2 KDt}{fS^2}) \tag{3.17}$$

For all but the smallest time period, only the first term is considered:

$$S^2 = \frac{\pi^2 KDt}{f \ln \frac{4m_0}{\pi m_1}} \tag{3.18}$$

where D is the average depth of the flow region, $D = d_e + m_0/2$. m_0 is the initial water table height; m_1 is the water table height after fall. All other parameters are as previously defined or in figure 3.4. Equation 3.18 is often referred to as the Glover equation (Luthin, 1978).

Tapp and Moody (Dumm, 1964) observed that the initial water table shape encountered in the field was more often parabolic than flat. They modified the initial conditions used by Glover accordingly, and obtained a solution that had only minor differences with

the Glover equation:

$$S^2 = \frac{\pi^2 K D t}{f \ln \frac{3.7 m_0}{\pi m_1}} \tag{3.19}$$

van Schilfgaarde (1963,1964), on the other hand, solved Eq. 3.16 without linearization:

$$t = \frac{f S^2}{9Kd} \ln \frac{m_0(2d + m_1)}{m_1(2d + m_0)} \tag{3.20}$$

If the equivalent depth concept is used, Eq. 3.20 will provide a practical solution to a complicated problem in simple form.

Eq. 3.20 implies an initial elliptic water table shape. If the observation by Tapp and Moody (Dumm, 1964) that a flattened parabola of the fourth degree is often encountered is correct, the Glover equation would tend to underestimate the required drain spacing, and the van Schilfgaarde equation would tend to overestimate it.

Putting the equation of the initial water table shape aside, Eq. 3.20 can be expected to provide more reliable results than Eq. 3.18 and Eq. 3.19 because it is based on Eq. 3.16 without linearization, and it is not restricted to relatively small increments of time and water table. However, it should be noted that, strictly speaking, Eq. 3.20 does not permit superposition because it is not linear (van Schilfgaarde; 1974).

For the analysis in this section, the van Schilfgaarde equation is used, mainly because it is simple in form and is not restricted to small increments of time and water table. The equivalent depth, d_e, is used to substitute for the impermeable layer depth to provide more realistic results.

Eq. 3.20 can be rewriten as:

$$S^2 = \frac{9 K d_e t}{f \ln \frac{m_0(2d_e + m_1)}{m_1(2d_e + m_0)}} \tag{3.21}$$

with d_e substituting for d in Eq. 3.20. d_e can be evaluated from Eq. 3.2 and 3.3. As in

section 3.1, d_e is considered as functionally related only to the impermeable layer depth and to drain radius, to simplify the analysis and results.

3.2.1 Sensitivity of Spacing to Drainable Porosity

To derive Eq. 3.16, it is assumed that drainable porosity is a constant. Drainable porosity is defined as the volume of water per unit area drained per unit change in water table height. However this quantity is far from constant, depending on the soil uniformity, on the proximity of the water table to the soil surface, and on the rate and direction of water table movement. The derivative of spacing with respect to drainable porosity is:

$$\frac{\partial S}{\partial f} = \sqrt{\frac{9Kd_e t}{\ln \frac{m_0(2d_e+m_1)}{m_1(2d_e+m_0)}}}\left(-\frac{1}{2}\right)\frac{1}{\sqrt{f^3}}$$

This equation can be simplified by substituting:

$$S = \sqrt{\frac{9Kd_e t}{f \ln \frac{m_0(2d_e+m_1)}{m_1(2d_e+m_0)}}}$$

and becomes:

$$\frac{\partial S}{\partial f} = -\frac{1}{2}\frac{S}{f}$$

Therefore, the percentage change in spacing due to a small change in drainable porosity is:

$$\left.\frac{\Delta S}{S}\right|_f = -\frac{1}{2}\frac{\Delta f}{f} \tag{3.22}$$

where $\frac{\Delta f}{f}$ is the percentage change of drainable porosity.

3.2.2 Sensitivity of Spacing to Hydraulic conductivity

The derivative of spacing with respect to hydraulic conductivity is:

$$\frac{\partial S}{\partial K} = \sqrt{\frac{9d_e t}{f \ln[\frac{m_0(2d_e+m_1)}{m_1(2d_e+m_0)}]}} \frac{1}{2} \frac{1}{\sqrt{K}} = \frac{1}{2} \frac{S}{K}$$

Thus the percentage change in spacing is:

$$\left.\frac{\Delta S}{S}\right|_K = \frac{1}{2} \frac{\Delta K}{K} \tag{3.23}$$

This result is the same as in steady state.

3.2.3 Sensitivity of Spacing to Draw-down Time from Water Table m_0 to m_1

If the water table height is m_0 immediately after recharge ends, it will fall to m_1 after a time lapse t. t is the time interval during which the water table falls from m_0 to m_1, and is determined by the tolerance of the crop to waterlogging. A shorter t requires a higher capacity to the drainage system and costs more for the farmer. On the other hand, if t is too long, crop growth may be adversely affected due to excess water in the rootzone. This draw-down time is directly related to the cost–effectiveness of the drainage system.

The partial derivative with respect to draw-down time t is:

$$\frac{\partial S}{\partial t} = \sqrt{\frac{9K d_e}{f \ln[\frac{m_0(2d_e+m_1)}{m_1(2d_e+m_0)}]}} \frac{1}{2} \frac{1}{\sqrt{t}} = \frac{1}{2} \frac{S}{t}$$

And the percentage change in spacing due to change in selected time t is:

$$\left.\frac{\Delta S}{S}\right|_t = \frac{1}{2} \frac{\Delta t}{t} \tag{3.24}$$

Eq. 3.23 and 3.24 are exactly the same in form. This implies that the selection of draw-down time is as important as the determination of hydraulic conductivity to the adequate and economic design of a drainage system.

3.2.4 Sensitivity of Spacing to the Initial Water Table Height after Recharge Ends

If the recharge is natural precipitation, the initial water table height, m_0, is mainly determined by the rainfall intensity and duration with a selected frequency of occurence. In irrigated land, m_0 is determined by the application rate and time of irrigation.

The derivative of spacing with respect to initial water table height, m_0, is:

$$\frac{\partial S}{\partial m_0} = -\frac{S}{2}\frac{2d_e}{m_0(2d_e + m_0)\ln[\frac{m_0(2d_e+m_1)}{m_1(2d_e+m_0)}]}$$

And the percentage change in spacing is:

$$\left.\frac{\Delta S}{S}\right|_{m_0} = -\frac{d_e}{(2d_e + m_0)\ln[\frac{m_0(2d_e+m_1)}{m_1(2d_e+m_0)}]}\frac{\Delta m_0}{m_0}$$

or, in dimensionless form:

$$\left.\frac{\Delta S}{S}\right|_{m_0} = -\frac{\frac{d_e}{m_0}}{(2\frac{d_e}{m_0} + 1)\ln(\frac{2\frac{d_e}{m_1}+1}{2\frac{d_e}{m_0}+1})}\frac{\Delta m_0}{m_0} \tag{3.25}$$

Figure 3.5 shows that spacing becomes more sensitive to the initial water table height as $\frac{m_1}{m_0}$ increases or $m_0 - m_1$ decreases. This indicates that the less the water table drawdown, the more sensitive the spacing to the initial water table. $\frac{d_e}{m_0}$ seems to have very little influence on the sensitivity. When $\frac{m_1}{m_0}$ is greater than 0.5, spacing becomes very sensitive to the selection of initial water table height.

3.2.5 Sensitivity of Spacing to the Water Table Height after Recession

The water table height after recession within a given time period should be such that if the water table is lower than this height, recharge is needed to provide adequate water to meet crop requirement. A higher water table may induce waterlogging and a lower

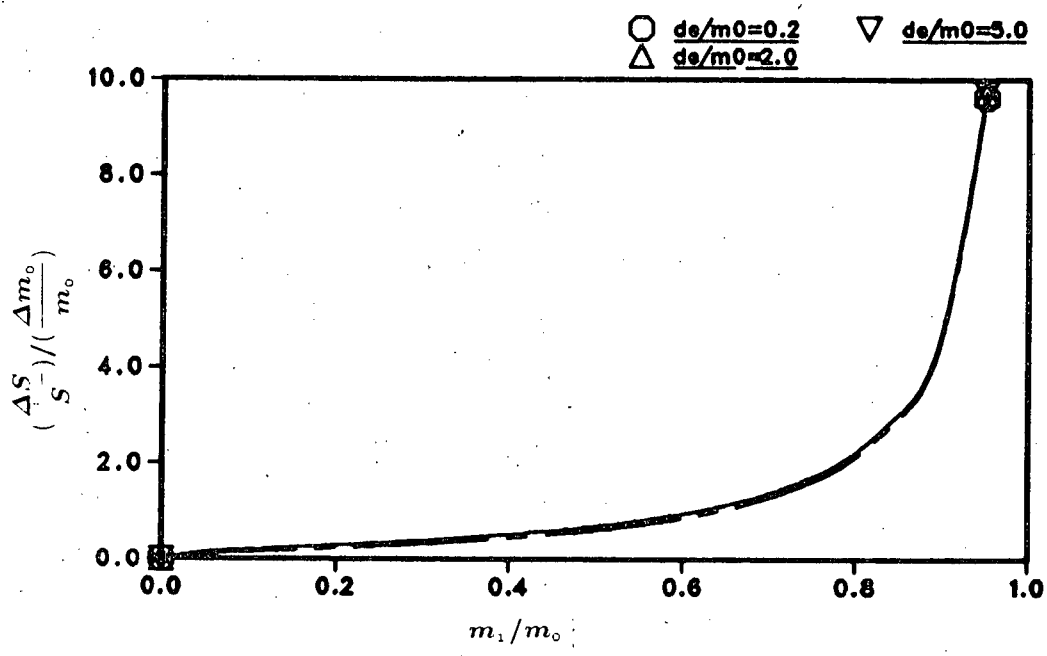

Figure 3.5: Sensitivity of spacing to the initial water table height m_0

water table may impede crop growth. The water table position after recession is directly related to drain spacing.

The derivative of spacing with respect to final water table height, m_1, is:

$$\frac{\partial S}{\partial m_1} = -\frac{S}{2}\frac{2d_e}{m_1(2d_e+m_1)\ln[\frac{m_0(2d_e+m_1)}{m_1(2d_e+m_0)}]}$$

Therefore, the percentage change in spacing is:

$$\left.\frac{\Delta S}{S}\right|_{m_1} = -\frac{d_e}{(2d_e+m_1)\ln[\frac{m_0(2d_e+m_1)}{m_1(2d_e+m_0)}]}\frac{\Delta m_1}{m_1}$$

or in dimensionless form:

$$\left.\frac{\Delta S}{S}\right|_{m_1} = -\frac{\frac{d_e}{m_1}}{(2\frac{d_e}{m_1}+1)\ln(\frac{2\frac{d_e}{m_1}+1}{2\frac{d_e}{m_0}+1})}\frac{\Delta m_1}{m_1} \tag{3.26}$$

Figure 3.6 shows that spacing becomes more sensitive as $\frac{m_1}{m_0}$ increases. Again $\frac{d_e}{m_0}$ has little influence on the sensitivity.

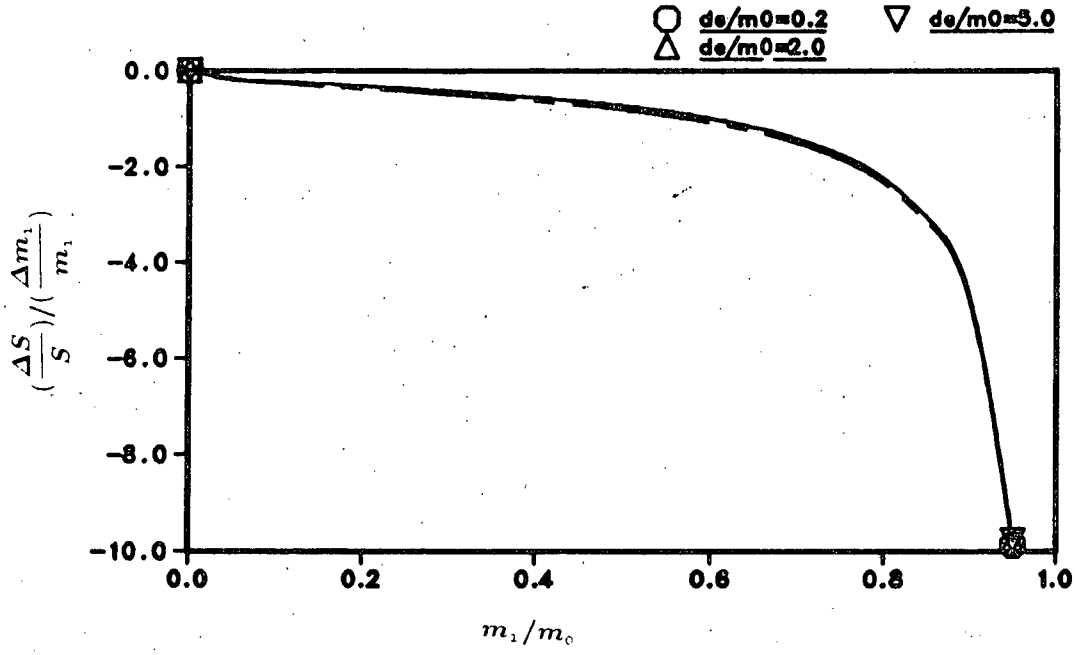

Figure 3.6: Sensitivity of sapcing to water table height after recession.

3.2.6 Sensitivity of Sapcing to the Equivalent Depth

We will follow the same procedure as in steady state analysis. First, the sensitivity of spacing to equivalent depth will be discussed and then the results of section 3.1.5 will be used to establish the relationships between spacing and impermeable layer depth and drain radius.

The derivative of spacing with respect to equivalent depth is:

$$\frac{\partial S}{\partial d_e} = \frac{S}{2}\frac{\ln[\frac{m_0(2d_e+m_1)}{m_1(2d_e+m_0)}] - \frac{2(m_0-m_1)d_e}{(2d_e+m_1)(2d_e+m_0)}}{d_e \ln[\frac{m_0(2d_e+m_1)}{m_1(2d_e+m_0)}]}$$

The percentage change in spacing is:

$$\left.\frac{\Delta S}{S}\right|_{d_e} = \frac{1}{2}\{1 - \frac{\frac{2(m_0-m_1)d_e}{(2d_e+m_0)(2d_e+m_1)}}{\ln[\frac{m_0(2d_e+m_1)}{m_1(2d_e+m_0)}]}\}\frac{\Delta d_e}{d_e}$$

or, in dimensionless form:

$$\left.\frac{\Delta S}{S}\right|_{d_e} = \frac{1}{2}[1 - \frac{2(\frac{d_e}{m_1} - \frac{d_e}{m_0})}{(2\frac{d_e}{m_1}+1)(2\frac{d_e}{m_0}+1)}\frac{1}{\ln(2\frac{d_e}{m_1}+1) - \ln(2\frac{d_e}{m_0}+1)}]\frac{\Delta d_e}{d_e} \tag{3.27}$$

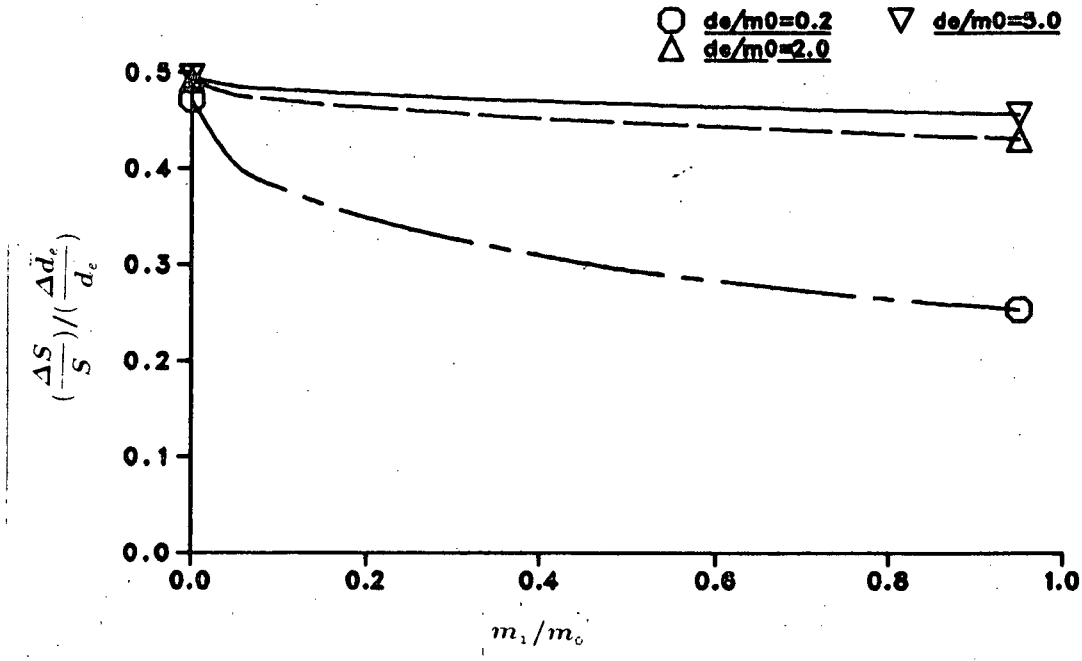

Figure 3.7: Sensitivity of spacing to equivalent depth

Figure 3.7 shows that the sensitivity of spacing to equivalent depth does not vary much, and is less than 0.5. Neither $\frac{d_e}{m_0}$ or $\frac{m_1}{m_0}$ has significant influence on the sensitivity.

3.2.7 Sensitivity of Spacing to Actual Depth of Impermeable Layer below Drains and Drain Radius

From the previous section, we have:

$$\left.\frac{\Delta S}{S}\right|_{d_e} = f_1' \frac{\Delta d_e}{d_e}$$

where

$$f_1' = \frac{1}{2}[1 - \frac{2(\frac{d_e}{m_1} - \frac{d_e}{m_0})}{(2\frac{d_e}{m_1}+1)(2\frac{d_e}{m_0}+1)} \frac{1}{\ln(2\frac{d_e}{m_1}+1) - \ln(2\frac{d_e}{m_0}+1)}]$$

And from section 3.1.5 we have:

$$\left.\frac{\Delta d_e}{d_e}\right|_{d} = f_2 \frac{\Delta d}{d}$$

$$\left.\frac{\Delta d_e}{d_e}\right|_r = f_3 \frac{\Delta r}{r}$$

where f_2 and f_3 are evaluated from Eq. 3.10 and 3.11.

Therefore, the percentage changes in spacing due to the changes of the impermeable layer depth and the drain radius are:

$$\left.\frac{\Delta S}{S}\right|_d = f_1' f_2 \frac{\Delta d}{d} \tag{3.28}$$

and

$$\left.\frac{\Delta S}{S}\right|_r = f_1' f_3 \frac{\Delta r}{r} \tag{3.29}$$

respectively.

The total percentage change in spacing due to changes in all or some of the parameters can be estimated by similar methods to those in section 3.1.8.

3.3 Comparison of Sensitivities between Steady and Transient Drainage Design in Homogeneous Soils

In this part of the chapter, a comparison of the sensitivities of spacing between steady and transient drainage design, based on the analyses of previous sections of this chapter, is made for individual parameters. The parameter values used in the comparison are as follows:

For non-steady state:

Initial water table height at mid–spacing: $m_0 = 0.8$ m;

Water table height at mid–spacing after recession: $m_1 = 0.4$ m;

Time required for water table to recede from m_0 to m_1: $t = 4$ days;

Saturated hydraulic conductivity: $K = 0.5$ m/day;

Drainable porosity: $f = 6\%$;

Depth to impermeable layer below drains: $d = 5.0$ m;

Drain radius: $r = 0.1$ m.

In order to make the comparison, the input values for steady state drainage design should be equivalent to the drainage requirement of a falling water table, which can be approximately achieved by selecting R and h values accordingly. R and h values can be chosen by:

$$R = \frac{(m_0 - m_1)f}{t} = \frac{(0.8 - 0.4) \times 0.06}{4} = 0.006 \text{ m/day}$$

$$h = \frac{m_0 + m_1}{2} = \frac{0.8 + 0.4}{2} = 0.6 \text{ m}$$

K, d and r are the same as for the transient state. The results are shown in table 3.1 (columns 2 and 3). All parameters are assumed to change by 10%. The results shown in column 4 of Table 3.1 were obtined by maintaining the same falling rate of water table while adjusting m_1 from 0.4 m to 0.7 m and required falling time t from 4 days to 1 days.

From the results in Table 3.1, it can be seen that:

1. Transient drainage design (falling water table criterion) can be reduced to the simple steady state drainage design, by appropriately adjusting the values of R and h. Certain combinations of R–h values may result in close or the same spacing as that obtained with the transient state drainage design method.

2. The sensitivities of spacing to hydraulic conductivity, impermeable layer depth, and drain radius in steady drainage design are very close to those in transient drainage design.

Table 3.1: Comparison of sensitivities between steady and transient state drainage design.

Parameter	Steady state	Transient state		
1	2	3		4
d_e (m)	2.44	2.58		2.81
S (m)	33.59	35.24		42.32
R	-5.00			
K	5.00	5.00		5.00
h	5.46	m_0	-6.94	-37.15
		m_1	-7.44	-37.73
d	1.27	1.44		1.70
r	0.80	0.84		0.75
f		-5.00		-5.00
t		5.00		5.00

3. The spacing is least sensitive to impermeable layer depth and drain radius in both steady and non-steady drainage design. Therefore, in both cases it is not necessary to measure the depth to an impermeable layer very accurately, considering the additional cost of more accurate measurement.

4. The spacing is very sensitive to the initial and final water table heights, when they are close to each other. Therefore, it is recommended that for cases where the water table fall or fluctuation is small, steady state drainage theory, such as Hooghoudt's equation, be used instead of transient drainage theory, such as the van Schilfgaarde equation. In these situations, spacing obtained from steady state equations tends to be more stable and therefore reliable than that from transient state spacing equations.

5. If transient drainage theory is used, m_0 and m_1 should be carefully selected, in order to avoid the over or under estimate of spacing for design.

Chapter 4

SENSITIVITY ANALYSIS OF STEADY STATE DRAINAGE DESIGN IN MULTI-LAYERED SOIL

In chapter 3, the sensitivity of drainage design in homogeneous soil is discussed. However, in practice, multi–layered soil is a more common phenomenon. For example, a field under tillage may gradually form a relatively compacted layer below certain depth. Two-layered or three-layered soils are commonly encountered in real fields.

In section 4.1 sensitivity analysis of drainage design in two– layered soil with drains at the interface of these layers will be carried out. In section 4.2, three–layered soil with two layers below the drains, and two–layered soil with drains in the upper layer will be analyzed. Hooghoudt's equation (1940), the generalized Hooghoudt-Ernst equation (van Beers, 1976), and the Kirkham equation (Toköz & Kirkham, 1971a) will be used. A combined equation from the Hooghoudt and Ernst equations for layered soils will be developed. Comparison will be made accordingly.

4.1 Sensitivity Analysis of Steady State Drainage Design in Two-layered Soil

Notations used in this section are: R is the drainage coefficient; K_1 is the hydraulic conductivity of the soil above drains; K_2 is the hydraulic conductivity of the soil below the drains; h is the water table height midway between the drains; S is the drain spacing; $2r$ is the diameter of the drains; and an impermeable layer is at a depth, D_2, below the drains. The concept of equivalent depth will again be used in the Hooghoudt equation and in the combined equation.

4.1.1 Drainage Design Using Hooghoudt's Equation

The Hooghoudt equation for a two-layered soil is (van Beers, 1976):

$$S^2 = \frac{4h}{R}(2K_2 d_e + K_1 h) \tag{4.30}$$

where d_e can be calcualted from Eq. 3.2 and 3.3. d_e is again considered as independent of spacing to simplify the analysis and results (See Appendix A).

Sensitivity of Spacing to Drainage Coefficient

From calculus, we know that

$$\Delta S|_R \approx dS = \frac{\partial S}{\partial R}\Delta R$$

for small ΔR, where

$$\frac{\partial S}{\partial R} = -\frac{1}{2}\frac{\frac{4}{R^2}(2K_2 d_e + K_1 h)}{\sqrt{\frac{4h}{R}(2K_2 d_e + K_1 h)}} = -\frac{1}{2}\frac{S}{R}$$

Therefore

$$\Delta S|_R = -\frac{R}{2}\frac{\Delta R}{R}$$

The percentage change in spacing is:

$$\left.\frac{\Delta S}{S}\right|_R = -\frac{1}{2}\frac{\Delta R}{R} \tag{4.31}$$

where $\frac{\Delta R}{R}$ is the percentage change in the drainage coefficient.

The above result is the same as that in homogeneous soil.

Sensitivity of Spacing to Hydraulic Conductivity of the Upper Soil Layer

Similarly, we have

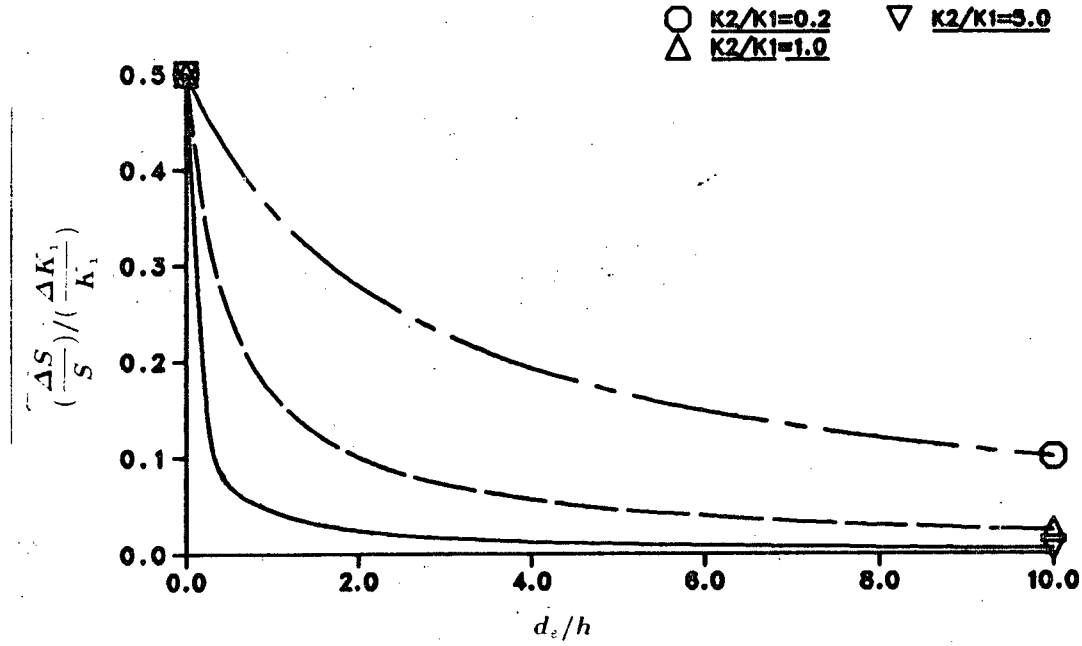

Figure 4.8: Sensitivity of spacing to hydraulic conductivity of upper layer soil.

$$\Delta S|_{K_1} \approx dS = \frac{\partial S}{\partial K_1}\Delta K_1 = \frac{\frac{4}{R}h^2}{2\sqrt{\frac{4h}{R}(2K_2 d_e + K_1 h)}}\Delta K_1$$

Dividing both sides by S gives:

$$\left.\frac{\Delta S}{S}\right|_{K_1} = \frac{1}{2}\frac{K_1 h}{2K_2 d_e + K_1 h}\frac{\Delta K_1}{K_1}$$

or, in dimensionless form:

$$\left.\frac{\Delta S}{S}\right|_{K_1} = \frac{1}{2(1 + 2\frac{K_2}{K_1}\frac{d_e}{h})}\frac{\Delta K_1}{K_1} \tag{4.32}$$

where $\frac{\Delta S}{S}$ and $\frac{dltK_1}{K_1}$ are percentage change in the spacing and in the hydraulic conductivity of the upper layer of soil, respectively.

Fig. 4.8 shows that the sensitivity of spacing to K_1 decreases as $\frac{d_e}{h}$ and $\frac{K_2}{K_1}$ increase.

Sensitivity of Spacing to the Hydraulic Conductivity of the Lower Soil Layer

For the hydraulic conductivity of the lower layer of soil, we have

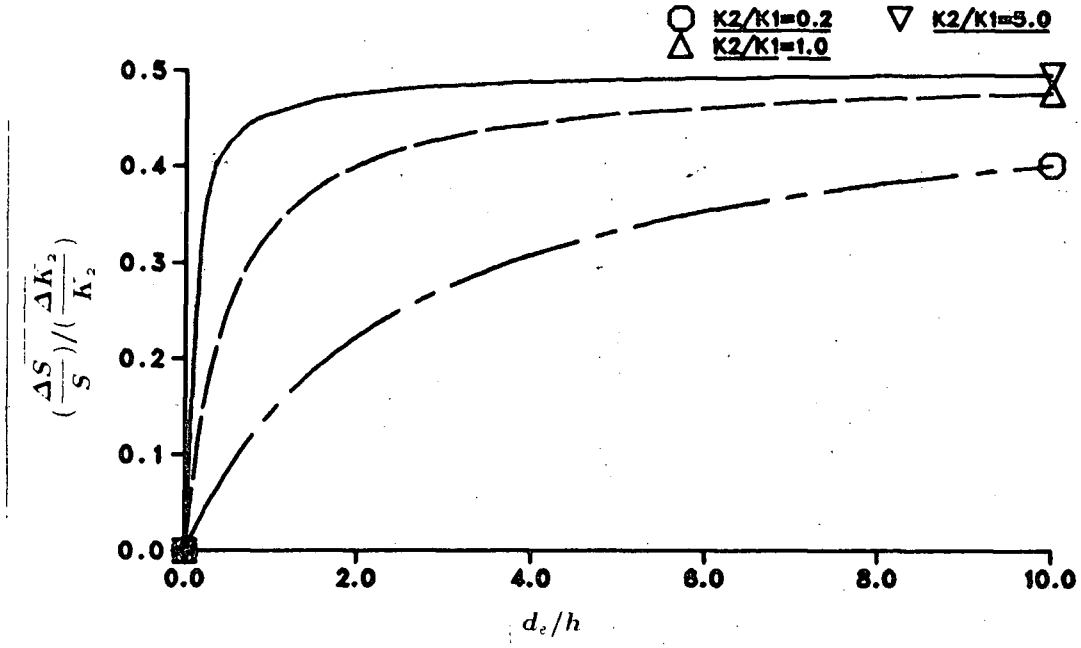

Figure 4.9: Sensitivity of spacing to the hydraulic conductivity of the lower soil layer.

$$\Delta S|_{K_2} \approx dS = \frac{\partial S}{\partial K_2}\Delta K_2 = \frac{\frac{4h}{R}2d_e}{2\sqrt{\frac{4h}{R}(2K_2d_e + K_1h)}}\Delta K_2$$

Dividing both sides by S gives the percentage change in spacing:

$$\left.\frac{\Delta S}{S}\right|_{K_2} = \frac{K_2d_e}{2K_2d_e + K_1h}\frac{\Delta K_2}{K_2}$$

or, in dimensionless form:

$$\left.\frac{\Delta S}{S}\right|_{K_2} = \frac{\frac{K_2}{K_1}\frac{d_e}{h}}{1 + 2\frac{K_2}{K_1}\frac{d_e}{h}}\frac{\Delta K_2}{K_2} \tag{4.33}$$

where $\frac{\Delta K_2}{K_2}$ is the percentage change in the hydraulic conductivity of the lower soil layer.

Fgiure 4.9 shows that the spacing becomes more sensitive to K_2 as $\frac{K_2}{K_1}$ and $\frac{d_e}{h}$ increase.

Comparison of Sensitivity of Spacing to K_1 and K_2

Assuming that $\frac{\Delta K_1}{K_1}$ and $\frac{\Delta K_2}{K_2}$ are equal, then if

$$\frac{1}{2}\frac{K_1h}{2K_2d_e + K_1h} > \frac{K_2d_e}{2K_2d_e + K_1h}$$

spacing is more sensitive to K_1 than to K_2.

The above inequality can be simplified to:

$$\frac{K_2}{K_1}\frac{d_e}{h} < \frac{1}{2}$$

or

$$\frac{K_2}{K_1} < \frac{1}{2}\frac{h}{d_e} \tag{4.34}$$

That is to say, if Equation 4.34 is true, spacing is more sensitive to K_1; otherwise spacing is more sensitive to K_2. In practice, condition 4.34 is not often satisfied there is a highly pervious soil layer above drain level and a very poorly pervious soil layer below drain level ($K_2 \ll K_1$). Thus, the spacing is usually more sensitive to the hydraulic conductivity of the lower layer than to that of the upper layer. In other words, the hydraulic conductivity of the lower soil layer is usually of greater significance to the drainage design; in some extreme cases, the upper layer can be completely neglected without major effects on the spacing, such as in the Kirkham equation (1971a) and the modified Hooghoudt-Ernst equation for layered soils.

Interestingly, the sensitivities of spacing to K_1 and K_2 add to $\frac{1}{2}$, which equals to the sensitivity of spacing to K in homogeneous soils.

Sensitivity of Spacing to Water Table Height h

For the midspan water table height, we have

$$\Delta S|_h \approx dS = \frac{\partial S}{\partial h}\Delta h = \frac{\frac{8}{R}(K_2 d_e + K_1 h)}{2\sqrt{\frac{4h}{R}(2K_2 d_e + K_1 h)}}\Delta h$$

Dividing both sides by S gives:

$$\left.\frac{\Delta S}{S}\right|_h = \frac{K_2 d_e + K_1 h}{2K_2 d_e + K_1 h}\frac{\Delta h}{h}$$

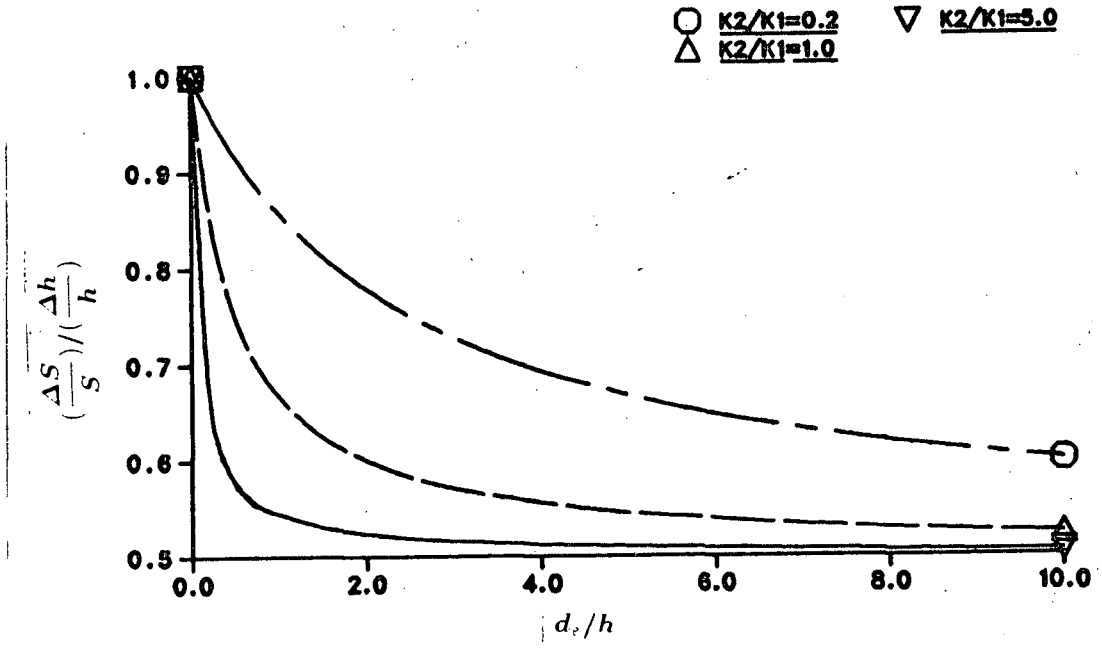

Figure 4.10: Sensitivity of spacing to water table height.

or, in dimensionless form:

$$\left.\frac{\Delta S}{S}\right|_h = \frac{1+\frac{K_2}{K_1}\frac{d_e}{h}}{1+2\frac{K_2}{K_1}\frac{d_e}{h}}\frac{\Delta h}{h} \tag{4.35}$$

Figure 4.10 shows that the sensitivity of spacing to the water table height, h, decreases as $\frac{K_2}{K_1}$ or $\frac{d_e}{h}$ increase.

Sensitivity of Spacing to Equivalent Depth

The derivative of spacing with respect to the equivalent depth is:

$$\frac{\partial S}{\partial d_e} = \frac{\frac{4h}{R}2K_2}{2\sqrt{\frac{4h}{R}(2K_2d_e+K_1h)}}$$

The percentage change in spacing is:

$$\left.\frac{\Delta S}{S}\right|_{d_e} = \frac{\partial S}{\partial d_e}\frac{\Delta d_e}{S} = \frac{\frac{4h}{R}2K_2}{2\sqrt{\frac{4h}{R}(2K_2d_e+K_1h)}}\frac{\Delta d_e}{S}$$

Rearranging the above equation gives:

$$\left.\frac{\Delta S}{S}\right|_{d_e} = \frac{K_2d_e}{2K_2d_e+K_1h}\frac{\Delta d_e}{d_e}$$

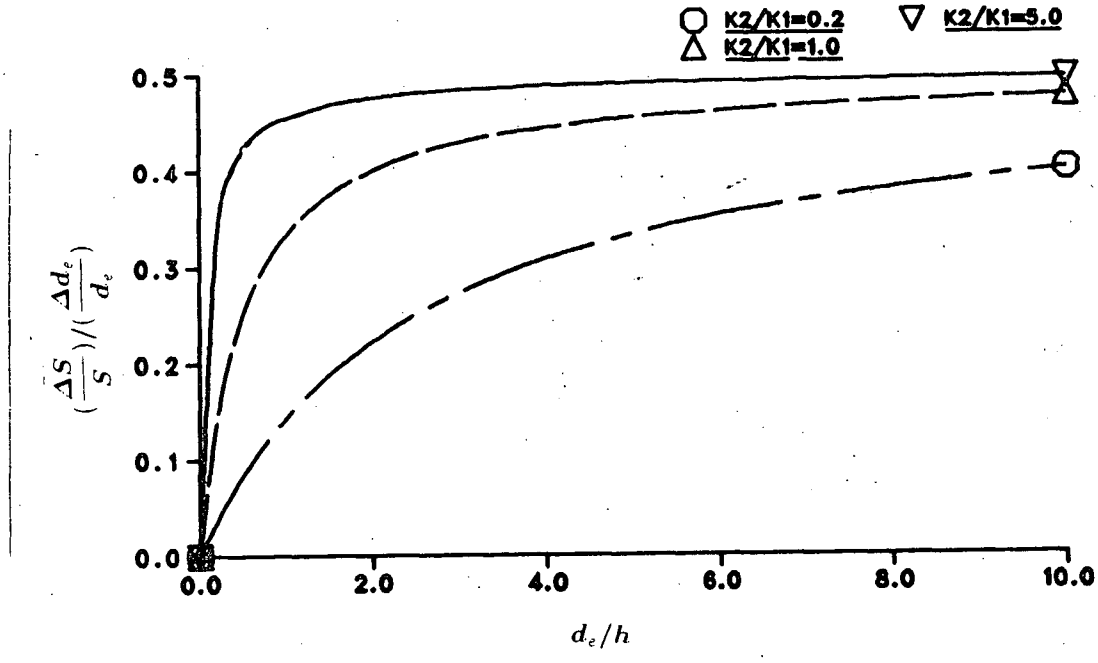

Figure 4.11: Sensitivity of spacing to equivalent depth

or, in dimensionless form:

$$\left.\frac{\Delta S}{S}\right|_{d_e} = \frac{\frac{K_2}{K_1}\frac{d_e}{h}}{1+2\frac{K_2}{K_1}\frac{d_e}{h}}\frac{\Delta d_e}{d_e} \tag{4.36}$$

Eq. 4.36 is exactly the same as Eq. 4.33, which implies that the equivalent depth is as important as the hydraulic conductivity of the lower soil in drainage design.

Figure 4.11 shows that spacing becomes more sensitive to the equivalent depth as $\frac{K_2}{K_1}$ or $\frac{d_e}{h}$ increase.

Sensitivity of Spacing to the Thickness of the Lower Soil Layer and the Drain Radius

From the previous section, we have:

$$\left.\frac{\Delta S}{S}\right|_{d_e} = f_1''\frac{\Delta d_e}{d_e}$$

where

$$f_1'' = \frac{\frac{K_2}{K_1}\frac{d_e}{h}}{1+2\frac{K_2}{K_1}\frac{d_e}{h}}$$

and from section 3.1.5, we have:

$$\left.\frac{\Delta S}{S}\right|_{D_2} = f_2 \frac{\Delta D_2}{D_2}$$

$$\left.\frac{\Delta S}{S}\right|_{r} = f_3 \frac{\Delta r}{r}$$

with f_2 and f_3 evaluated from Eq. 3.10 and 3.11. Here, D_2 is equivalent to d in homogeneous soil.

Therefore, the percentage changes in spacing due to changes of the thickness of the lower soil layer and the drain radius are:

$$\left.\frac{\Delta S}{S}\right|_{D_2} = f_1'' f_2 \frac{\Delta D_2}{D_2} \tag{4.37}$$

$$\left.\frac{\Delta S}{S}\right|_{r} = f_1'' f_3 \frac{\Delta r}{r} \tag{4.38}$$

respectively.

4.1.2 Drainage Design Using the Generalized Hooghoudt–Ernst Equation

A generalized Hooghoudt–Ernst equation was obtained by considering only the horizontal flow above the drains and both horizontal and radial flows below the drains with the radial flow component being from the equation of Ernst (van Beers, 1976).

The generalized Hooghoudt–Ernst equation can be written as:

$$S^3 + \frac{8}{\pi} D_2 S^2 \ln \frac{D_2}{u} - \frac{8(K_1 D_1 + K_2 D_2)h}{R} S = \frac{64 K_1 D_1 D_2 h}{\pi R} \ln \frac{D_2}{u} \tag{4.39}$$

where

S =drain spacing;

K_1 =hydraulic conductivity of the soil above drain level;

K_2 =hydraulic conductivity of the soil below drain level;

h =hydraulic head: the height of the water table above drain level midway between

drains;

D_1 =$h/2$, average depth of flow region above drain level or average thickness of the soil layer through which the flow above the drains takes place;

D_2 =depth to an impervious layer below drain level;

R =drain discharge rate per unit surface area per unit time;

u =wetted section of the drain; for pipe drains: $u = \pi r$

r =drain radius.

The partial derivatives of spacing with respect to other independent variables are as follows:

$$\frac{\partial S}{\partial R} = -\frac{\frac{8(K_1D_1+K_2D_2)hS}{R^2} + \frac{64K_1D_1D_2h}{\pi R^2}}{3S^2 + \frac{16}{\pi}D_2S\ln\frac{D_2}{u} - \frac{8(K_1D_1+K_2D_2)h}{R}}$$

$$\frac{\partial S}{\partial K_1} = \frac{\frac{8D_1hS}{R} + \frac{64D_1D_2h}{\pi R}\ln\frac{D_2}{u}}{3S^2 + \frac{16}{\pi}D_2S\ln\frac{D_2}{u} - \frac{8(K_1D_1+K_2D_2)h}{R}}$$

$$\frac{\partial S}{\partial K_2} = \frac{\frac{8D_2hS}{R}}{3S^2 + \frac{16}{\pi}D_2S\ln\frac{D_2}{u} - \frac{8(K_1D_1+K_2D_2)h}{R}}$$

$$\frac{\partial S}{\partial h} = \frac{\frac{8(K_2D_2+K_1h)S}{R} + \frac{64D_1D_2h}{\pi R}\ln\frac{D_2}{u}}{3S^2 + \frac{16}{\pi}D_2S\ln\frac{D_2}{u} - \frac{8(K_1D_1+K_2D_2)h}{R}}$$

$$\frac{\partial S}{\partial D_2} = \frac{\frac{8K_2hS}{R} + \frac{8}{\pi}(\frac{8K_1D_1h}{R} - S^2)(1+\ln\frac{D_2}{u})}{3S^2 + \frac{16}{\pi}D_2S\ln\frac{D_2}{u} - \frac{8(K_1D_1+K_2D_2)h}{R}}$$

$$\frac{\partial S}{\partial r} = \frac{\frac{8D_2S^2}{\pi r} - \frac{64K_1D_1D_2h}{\pi Rr}}{3S^2 + \frac{16}{\pi}D_2S\ln\frac{D_2}{u} - \frac{8(K_1D_1+K_2D_2)h}{R}}$$

The percentage changes in spacing, due to the individual change in each variable while others are held constant, are:

$$\left.\frac{\Delta S}{S}\right|_R = -\frac{\frac{8(K_1D_1+K_2D_2)h}{R} + \frac{64K_1D_1D_2h}{\pi RS}}{3S^2 + \frac{16}{\pi}D_2S\ln\frac{D_2}{u} - \frac{8(K_1D_1+K_2D_2)h}{R}}\frac{\Delta R}{R} \tag{4.40}$$

$$\left.\frac{\Delta S}{S}\right|_{K_1} = \frac{\frac{8K_1D_1h}{R} + \frac{64K_1D_1D_2h}{\pi RS}\ln\frac{D_2}{u}}{3S^2 + \frac{16}{\pi}D_2S\ln\frac{D_2}{u} - \frac{8(K_1D_1+K_2D_2)h}{R}}\frac{\Delta K_1}{K_1} \tag{4.41}$$

$$\left.\frac{\Delta S}{S}\right|_{K_2} = \frac{\frac{8K_2D_2h}{R}}{3S^2 + \frac{16}{\pi}D_2S\ln\frac{D_2}{u} - \frac{8(K_1D_1+K_2D_2)h}{R}}\frac{\Delta K_2}{K_2} \tag{4.42}$$

$$\left.\frac{\Delta S}{S}\right|_h = \frac{\frac{8(K_2D_2+K_1h)h}{R} + \frac{64D_1D_2h^2}{\pi RS}\ln\frac{D_2}{u}}{3S^2 + \frac{16}{\pi}D_2S\ln\frac{D_2}{u} - \frac{8(K_1D_1+K_2D_2)h}{R}}\frac{\Delta h}{h} \tag{4.43}$$

$$\left.\frac{\Delta S}{S}\right|_{D_2} = \frac{\frac{8K_2D_2h}{R} + \frac{8D_2}{\pi}(\frac{8K_1D_1h}{RS} - S)(1+\ln\frac{D_2}{u})}{3S^2 + \frac{16}{\pi}D_2S\ln\frac{D_2}{u} - \frac{8(K_1D_1+K_2D_2)h}{R}}\frac{\Delta D_2}{D_2} \tag{4.44}$$

$$\left.\frac{\Delta S}{S}\right|_r = \frac{\frac{8D_2S}{\pi} - \frac{64K_1D_1D_2h}{\pi RS}}{3S^2 + \frac{16}{\pi}D_2S\ln\frac{D_2}{u} - \frac{8(K_1D_1+K_2D_2)h}{R}}\frac{\Delta r}{r} \tag{4.45}$$

The total percentage in spacing due to changes of all or some of the variables is:

$$\frac{\Delta S}{S} = E'_R\frac{\Delta R}{R} + E'_{K_1}\frac{\Delta K_1}{K_1} + E'_{K_2}\frac{\Delta K_2}{K_2} + E'_h\frac{\Delta h}{h} + E'_{D_2}\frac{\Delta D_2}{D_2} + E'_r\frac{\Delta r}{r} \tag{4.46}$$

where $E'_R, E'_{K_1}, E'_{K_2}, E'_h, E'_{D_2}$ and E'_r can be evaluated from the coefficients of the right hand side in Eq. 4.40 through 4.45.

Dot product form can be used to write the above equation:

$$\frac{\Delta S}{S} = \vec{E'} \bullet \Delta\vec{X}$$

where

$$\vec{E'} = (E'_R, E'_{K_1}, E'_{K_2}, E'_h, E'_{D_2}, E'_r)$$

$$\Delta\vec{X} = (\frac{\Delta R}{R}, \frac{\Delta K_1}{K_1}, \frac{\Delta K_2}{K_2}, \frac{\Delta h}{h}, \frac{\Delta D_2}{D_2}, \frac{\Delta r}{r})$$

4.1.3 Comparison of Spacings and Sensitivities Obtained from the Hooghoudt and the generalized Hooghoudt-Ernst Equations

The parameter values used in the calculations are:

In case 1: K_1=0.5 m/day; K_2=0.5 m/day; R=0.01 m/day; D_2=5.0 m; h=0.6 m; r=0.1 m.

In case 2: K_1=0.2 m/day; K_2=2.0 m/day; R=0.012 m/day; D_2=5.0 m; h=0.6 m; r=0.1 m.

In case 3: K_1=2.0 m/day; K_2=0.2 m/day; R=0.004 m/day; D_2=3.0 m; h=0.6 m; r=0.1 m.

In case 4: K_1=0.5 m/day; K_2=0.5 m/day; R=0.01 m/day; D_2=6.0 m; h=1.0 m; r=0.1 m.

In Table 4.2, Row (1) represents the spacing obtained from the Hooghoudt equation and the sensitivities obtained from the Hooghoudt equation with simplified d_e–S relationship; Row (2) represents the results obtained from the generalized Hooghoudt-Ernst equation; and Row (3) represents the sensitivities obtained from the hooghoudt equation with unsimplified d_e–S relationship (see Appendix A). All changes in parameters are assumed to be 10% of the original values.

The results calculated (Table 4.2) show that:

1. The Hooghoudt equation gives very close spacings to those obtained from the generalized Hooghoudt–Ernst equation for various conditions ($K_1 = K_2, K_1 \gg K_2, K_1 \ll K_2$). This is in contradiction to the claim that the Hooghoudt equation can not be used in the situation where $K_1 \ll K_2$ (van Beers, 1976).

2. The sensitivity of spacing in the Hooghoudt equation and that in the generalized Hooghoudt-Ernst equation are not greatly different from each other. The source of difference is in part from the assumption used in the analyses that the equivalent

Table 4.2: Comparison of spacing and its sensitivity to individual parameters calculated from the Hooghoudt and the generalized Hooghoudt-Ernst equations.

Parameter		Case 1	Case 2	Case 3	Case 4
Spacing (m)	(1)	23.96	49.25	34.88	36.08
	(2)	24.24	49.48	34.92	36.37
R	(1)	−5.00	−5.00	−5.00	−5.00
	(2)	−6.39	−6.20	−4.85	−6.13
	(3)	−6.76	−6.26	−5.35	−6.53
K_1	(1)	0.63	0.05	2.96	0.77
	(2)	0.82	0.06	3.15	0.98
	(3)	0.85	0.06	3.16	1.00
K_2	(1)	4.37	4.95	2.04	4.23
	(2)	5.85	6.15	2.19	5.48
	(3)	5.91	6.20	2.18	5.52
h	(1)	5.63	5.05	7.96	5.77
	(2)	7.48	6.27	8.48	7.44
	(3)	7.61	6.32	8.51	7.53
D_2	(1)	0.80	2.17	1.09	1.07
	(2)	1.19	2.77	1.19	1.49
	(3)	1.08	2.71	1.16	1.40
r	(1)	0.97	0.77	0.31	0.82
	(2)	1.32	0.96	0.33	1.08
	(3)	1.31	0.96	0.33	1.07

depth is only functionally related to the depth to the impervious layer and the drain radius, and is independent of spacing.

4.1.4 Conclusions from the Sensitivity Analysis of Steady State Drainage Design in Two–Layered Soils

From the analyses in the previous sections, it can be concluded that:

1. Drain pacing is very sensitive to the design water table height midway between the drains, the drainage coefficient, and the hydraulic conductivity of the soil below drain level.

2. Spacing is usually less sensitive to the hydraulic conductivity of the soil above the drain level than that below the drain level. The sensitivities are directly related to the ratios of $\frac{K_2}{K_1}$ and $\frac{d_e}{h}$, both of which represent the relative importance of the upper soil layer and lower soil layer.

3. Spacing is the least sensitive to the depth of the impermeable layer and the drain radius. Therefore, it is not necessary to measure the impermeable layer depth very accurately.

4. The generalized Hooghoudt-Ernst equation and the Hooghoudt equation yield similar spacings and sensitivities. Hooghoudt's equation can be used under most conditions for two–layered soils.

4.2 Sensitivity Analysis of Steady State Drainage Design in Three–layered Soil

Notations used in this section: K_2 is the hydraulic conductivity of the second soil layer with a thickness of D_2; K_3 is the hydraulic conductivity of the third soil layer with a

thickness of D_3. All other parameters are the same as difined in section 4.1. $K_1 = K_2$ represents a special case that there are two layers of soil and the drains is in the upper soil layer.

The original Hooghoudt equation can not be used in this case (van Beers, 1976). Toksöz & Kirkham (1971a,b) developed theories and nomographs for two– layered soils with the drains in the upper layer, and with some modification, for three–layered soils with the drains at the interface of the first two layers (Wesseling, 1964). But the Kirkham equatins are difficult to use and the nomographs can not be accurately used in some situations. The generalized Hooghoudt–Ernst equation for three-layered soils also requires to solve a three order equation, which is sometimes inconvenient in practice. The trail–and–error method used by the Hooghoudt approach is easy to use at any circumstances and is commonly accepted. In this section, a combined equation using the Hooghoudt approach and the concept of equivalent depth with Ernst's component for the radial flow towards drains will be developed. The sensitivity analysis will be carried out based on the newly developed equation. Spacings and sensitivities obtained from the new equation will be compared with those from the Kirkham equation (Toksöz & Kirkham, 1971a).

4.2.1 Development of A Combined Formula for Spacing Calculation from the Hooghoudt and Ernst Equations

The general principle underlying Ernst's basic equation (1962) is that the flow of groundwater towards parallel drains, and consequently the corresponding available total hydraulic head (h), can be divided into three components: a vertical (v), a horizontal (h), and a radial (r) component as:

$$h = h_v + h_h + h_r = qL_v + qSL_h + qSL_r$$

where q is the flow rate, S is the drain spacing, and L is the resistance.

Ernst worked out various resistance terms, and the above equation can be written as:

$$h = q\frac{D_v}{K_v} + q\frac{S^2}{8KD} + q\frac{S}{\pi K_2}\ln\frac{aD_2}{u}$$

where

h, K_2, D_2, S = notation difined in the previous section;

D_v = thickness of the layer over which vertical flow is considered; in most cases this component is small and may be ignored (m);

K_v = hydraulic conductivity for vertical flow (m/day);

KD = the sum of the product of the hydraulic conductivity (K) and thickness (D) of the various layers for the horizontal folw component;

a = geometry factor for radial flow (Appendix C);

u = wetted section of the drain (m); for pipe drains $u = \pi r$.

Now let's consider the flow region below the drains. The total flow towards the drains in this region, from the Ernst equation and neglecting the vertical component, can be written as:

$$q_2 = \frac{8(K_2D_2 + K_3D_3)h}{S^2 + \frac{8(K_2D_2+K_3D_3)}{\pi K_2}S\ln\frac{aD_2}{u}}$$

Borrowing the concept of equivalent depth in the Hooghoudt equation, and assuming an equivalent layer of soil exists such that the horizontal flow within this layer is equal to the total flows of the horizontal and radial components in the whole region below the drain level, with d_e being the thickness of this layer, then the water flow in the equivalent layer is equal to:

$$q_2' = \frac{8\overline{K}d_eh}{S^2}$$

where $\overline{K}$ is the weighted hydraulic conductivity of the equivalent layer,

$$\overline{K} = \frac{K_2D_2 + K_3D_3}{D_2 + D_3}$$

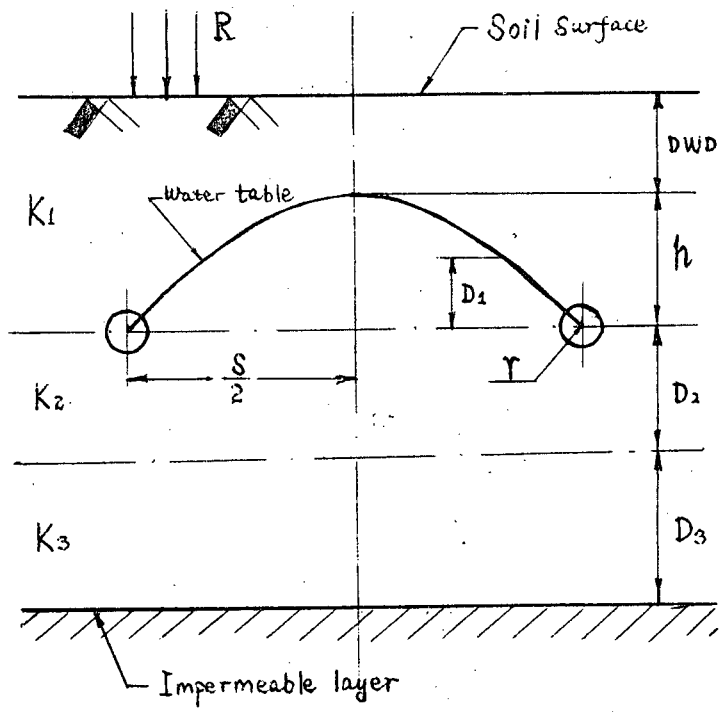

Figure 4.12: Geometry of the subsurface drainage in three–layered soils.

Based on the assumption of an equivalent layer, we have:

$$q_2 = q_2'$$

thus

$$\frac{8(K_2D_2 + K_3D_3)h}{S^2 + \frac{8(K_2D_2+K_3D_3)}{\pi K_2} S \ln \frac{aD_2}{u}} = \frac{8\overline{K} d_e h}{S^2}$$

Rearranging the above equation gives:

$$d_e = \frac{(D_2 + D_3)S}{S + \frac{8}{\pi}(D_2 + \frac{K_3}{K_2}D_3) \ln \frac{aD_2}{u}} \tag{4.47}$$

For the flow region above the drain level, only horizontal flow is considered, an approach used by the Hooghoudt equation (Hooghoudt, 1940). This flow is equal to:

$$q_1 = \frac{8K_1D_1h}{S^2}$$

where

$$D_1 = \frac{h}{2}$$

The total flow towards the drains from both regions should be equal to the drainage coefficient. Therefore, the combined equation becomes:

$$R = \frac{8K_1D_1h}{S^2} + \frac{8\overline{K} d_e h}{S^2}$$

or

$$S^2 = \frac{8h}{R}[K_1D_1 + \frac{(K_2D_2 + K_3D_3)d_e}{D_2 + D_3}] \tag{4.48}$$

with d_e evaluated from Eq. 4.47.

4.2.2 Comparison of Spacing between the New Spacing Equation and the Kirkham Equation

Toksöz and Kirkham (1971a) developed an equation for two-layered soils ($K_1 = K_2$):

$$h(\frac{K_2}{R} - 1) = S\frac{1}{\pi}\{\ln\frac{1}{\sin(\frac{\pi r}{S})} + \sum_{m=1}^{\infty}\frac{1}{m}[-1 + \coth(\frac{2m\pi D_2}{S})][\cos(\frac{2m\pi r}{S}) - \cos(m\pi)]$$
$$[1 - \frac{e^{2m\pi D_2/S}}{\sinh(\frac{2m\pi D_2}{S})}\frac{1}{\frac{K_2}{K_3}\coth(\frac{2m\pi D_3}{S}) + \coth(\frac{2m\pi D_2}{S})}]\} \quad (4.49)$$

Equation 4.49 is based on the assumption that the loss in hydraulic head in the region lying below the arch–shaped water table and above the level of the drains is negligible compared with the head loss over the remainder of the flow region.

In cases where $K_1 \neq K_2$, Wesseling (1964) indicated that Eq. 4.49 still can be used by substituting the factor $(K_2/R - 1)$ in the left side with $(K_2/R - K_2/K_1)$.

The parameter values used in the comparison of spacings are as follows (adopted from van Beers, 1976): h=0.6 m; R=0.006 m/day; $K_1 = K_2$=1.2 m/day; r=0.1 m; D_2=1.6 m. All other parameters are as shown in Table 4.3.

In Table 4.3, (1) is the spacings obtained from the new equation by assuming D_1=0, i.e., the head loss above the drains is neglected, which is the case in the Kirkham equation; (2) is the spacings obtained from the new equation considering the head loss of the region above the drains; (3) is the spacings calculated from the Kirkham equation. The spacings calculated from the Kirkham equation is slightly different from those given by Kirkham (1971a), probably because of the different converging criteria used in the calculations.

The results in Table 4.3 show that the Kirkham equation is likely to underestimate the required spacing. For the new spacing formula, if the head loss above the drains is neglected (i.e. $D_1 = 0$), the spacing obtained from the new spacing formula is very close to that computed by Kirkham's equation. However, if the head loss above the drains is too great to be neglected, the spacing computed from the new equation is always larger

D_3/D_2		0.25			1.5			4.0		
D_3		0.4			2.4			6.4		
K_3/K_2	K_3	(1)	(2)	(3)	(1)	(2)	(3)	(1)	(2)	(3)
0.02	0.024	36.1	40.1	36.0	36.1	40.2	36.3	35.6	39.9	36.7
0.1	0.12	36.4	40.4	36.3	37.9	41.8	37.8	39.8	43.7	39.6
0.2	0.24	36.7	40.7	36.6	39.6	43.5	39.6	43.4	47.1	42.9
0.5	0.60	37.8	41.7	37.7	44.4	47.9	44.1	51.3	54.7	50.9
1.0	1.20	39.4	43.2	39.3	50.4	53.6	50.2	60.7	63.7	60.5
2.0	2.40	42.3	45.9	42.1	59.7	62.7	59.0	73.1	75.8	72.6
5.0	6.00	49.4	52.7	48.9	75.9	78.3	74.4	91.3	93.7	90.2
10.0	12.0	57.7	60.6	57.2	89.5	91.7	87.4	103.3	105.7	102.0
50.0	60.0	84.3	87.6	84.7	113.3	115.9	111.6	119.3	121.6	118.3
∞	∞	124.5	127.3	123.8	124.5	127.3	123.8	124.5	127.3	123.8

Table 4.3: Comparison of drain spacings in two–layered soils obtained from the new spacing formula and the Kirkham equation (1971).

than the result from Kirkham's equation, as the Kirkham equation neglects the head loss above the drains (i.e. all flow toward the drains is assumed to take place in the region below the drains, which is obviously not true). When the hydraulic conductivity of the soil above the drains is relatively small compared with that of the soil below the drains, the results given by the Kirkham equation are acceptable, otherwise it will yield much smaller spacing. The new spacing can be used in three– or two–layered soils with the simple trial–and–error approach. The calculation is much simpler compared with the Kirkham equation and tends to provide more reasonable results.

4.2.3 Sensitivity Analysis of Drainage Design Using the Newly Developed Spacing Formula

From Eq. 4.48, the spacing, S, is a function of R, K_1, h, K_2, D_2, K_3, D_3 and d_e; and from Eq. 4.47, d_e is in turn a function of S, K_2, D_2, K_3, D_3 and r, i.e.

$$S = F(R, K_1, h, K_2, D_2, K_3, D_3, d_e)$$

$$d_e = f(S, K_2, D_2, K_3, D_3, r)$$

The partial derivative of spacing with respect to the drainage coefficient is:

$$\frac{\partial S}{\partial R} = \frac{\partial F}{\partial R} + \frac{\partial F}{\partial d_e}\frac{\partial d_e}{\partial S}\frac{\partial S}{\partial R}$$

Therefore:

$$\frac{\partial S}{\partial R} = \frac{\frac{\partial F}{\partial R}}{1 - \frac{\partial F}{\partial d_e}\frac{\partial d_e}{\partial S}}$$

Similarly, we have:

$$\frac{\partial S}{\partial K_1} = \frac{\frac{\partial F}{\partial K_1}}{1 - \frac{\partial F}{\partial d_e}\frac{\partial d_e}{\partial S}}$$

$$\frac{\partial S}{\partial h} = \frac{\frac{\partial F}{\partial h}}{1 - \frac{\partial F}{\partial d_e}\frac{\partial d_e}{\partial S}}$$

$$\frac{\partial S}{\partial K_2} = \frac{\frac{\partial F}{\partial K_2} + \frac{\partial F}{\partial d_e}\frac{\partial d_e}{\partial K_2}}{1 - \frac{\partial F}{\partial d_e}\frac{\partial d_e}{\partial S}}$$

$$\frac{\partial S}{\partial D_2} = \frac{\frac{\partial F}{\partial D_2} + \frac{\partial F}{\partial d_e}\frac{\partial d_e}{\partial D_2}}{1 - \frac{\partial F}{\partial d_e}\frac{\partial d_e}{\partial S}}$$

$$\frac{\partial S}{\partial K_3} = \frac{\frac{\partial F}{\partial K_3} + \frac{\partial F}{\partial d_e}\frac{\partial d_e}{\partial K_3}}{1 - \frac{\partial F}{\partial d_e}\frac{\partial d_e}{\partial S}}$$

$$\frac{\partial S}{\partial D_3} = \frac{\frac{\partial F}{\partial D_3} + \frac{\partial F}{\partial d_e}\frac{\partial d_e}{\partial D_3}}{1 - \frac{\partial F}{\partial d_e}\frac{\partial d_e}{\partial S}}$$

$$\frac{\partial S}{\partial r} = \frac{\frac{\partial F}{\partial d_e}\frac{\partial d_e}{\partial r}}{1 - \frac{\partial F}{\partial d_e}\frac{\partial d_e}{\partial S}}$$

Various terms in the above equations are:

$$\frac{\partial F}{\partial R} = -\frac{1}{2}\frac{S}{R}$$

$$\frac{\partial F}{\partial K_1} = \frac{2h^2}{RS}$$

$$\frac{\partial F}{\partial h} = \frac{4(K_1 h + \frac{K_2 D_2 + K_3 D_3}{D_2 + D_3} d_e)}{RS}$$

$$\frac{\partial F}{\partial K_2} = \frac{4hD_2 d_e}{RS(D_2 + D_3)}$$

$$\frac{\partial F}{\partial D_2} = \frac{4h(K_2 - K_3)D_3 d_e}{RS(D_2 + D_3)^2}$$

$$\frac{\partial F}{\partial K_3} = \frac{4hD_3 d_e}{RS(D_2 + D_3)}$$

$$\frac{\partial F}{\partial D_3} = \frac{4h(K_3 - K_2)D_2 d_e}{RS(D_2 + D_3)^2}$$

$$\frac{\partial F}{\partial d_e} = \frac{4h(K_2 D_2 + K_3 D_3)}{RS(D_2 + D_3)}$$

and

$$\frac{\partial d_e}{\partial K_2} = \frac{8K_3 D_3 d_e^2 \ln \frac{aD_2}{u}}{\pi K_2^2 (D_2 + D_3)S}$$

$$\frac{\partial d_e}{\partial D_2} = \frac{S + \frac{8}{\pi}(\frac{K_3}{K_2} - 1)D_3 \ln \frac{aD_2}{u}}{(D_2 + D_3)^2 S} d_e^2 - \frac{8d_e^2(K_2 D_2 + K_3 D_3)}{\pi K_2 D_2 (D_2 + D_3)S}$$

$$\frac{\partial d_e}{\partial K_3} = -\frac{8D_3 d_e^2 \ln \frac{aD_2}{u}}{\pi K_2 (D_2 + D_3)S}$$

$$\frac{\partial d_e}{\partial D_3} = \frac{S + \frac{8D_2}{\pi} \ln \frac{aD_2}{u}(1 - \frac{K_3}{K-2})}{D_2 + D_3)^2 S} d_e^2$$

$$\frac{\partial d_e}{\partial r} = \frac{8d_e^2(D_2 + \frac{K_3}{K_2}D_3)}{\pi r(D_2 + D_3)S}$$

$$\frac{\partial d_e}{\partial S} = \frac{d_e}{S}\frac{D_2 + D_3 - d_e}{D_2 + D_3}$$

The percentage change in spacing due to the change in an individual parameter, while other parameters are held constant, can be calculated as:

$$\left.\frac{\Delta S}{S}\right|_R = \frac{\partial S}{\partial R}\frac{R}{S}\frac{\Delta R}{R}$$

$$\left.\frac{\Delta S}{S}\right|_{K_1} = \frac{\partial S}{\partial K_1}\frac{K_1}{S}\frac{\Delta K_1}{K_1}$$

$$\left.\frac{\Delta S}{S}\right|_h = \frac{\partial S}{\partial h}\frac{h}{S}\frac{\Delta h}{h}$$

$$\left.\frac{\Delta S}{S}\right|_{K_2} = \frac{\partial S}{\partial K_2}\frac{K_2}{S}\frac{\Delta K_2}{K_2}$$

$$\left.\frac{\Delta S}{S}\right|_{D_2} = \frac{\partial S}{\partial D_2}\frac{D_2}{S}\frac{\Delta D_2}{D_2}$$

$$\left.\frac{\Delta S}{S}\right|_{K_3} = \frac{\partial S}{\partial K_3}\frac{K_3}{S}\frac{\Delta K_3}{K_3}$$

$$\left.\frac{\Delta S}{S}\right|_{D_3} = \frac{\partial S}{\partial D_3}\frac{D_3}{S}\frac{\Delta D_3}{D_3}$$

$$\left.\frac{\Delta S}{S}\right|_r = \frac{\partial S}{\partial r}\frac{r}{S}\frac{\Delta r}{r}$$

4.2.4 Comparison of Sensitivities of Spacing between the Newly Developed Formula and the Kirkham Equation

Table 4.4 is the summary of results of sensitivities of spacing to the individual parameters based on the new spacing formula, and the Kirkham equation for three–layered soils with the drains at the interface of the top two layers. The parameter values used in the calculations are:

In case 1: R=0.01 m/day; K_1=1.5 m/day; K_2=0.2 m/day; K_3=0.6 m/day; h=0.6 m; D_2=1.0 m; D_3=2.0 m; r=0.1 m.

Parameter	Case 1		Case 2		Case 3	
	(1)	(2)	(1)	(2)	(1)	(2)
Spacing (m)	20.98	12.00	31.93	30.24	27.16	25.08
R	−6.02	−7.60	−6.25	−6.22	−7.56	−7.63
K_1	2.96	0.05	0.18	0.30	0.30	0.37
h	8.98	8.09	6.42	6.22	7.85	7.71
K_2	2.19	6.71	4.53	5.02	5.33	5.60
D_2	−0.72	−1.35	1.05	1.62	−1.80	−1.17
K_3	0.88	1.18	1.53	1.16	1.93	1.87
D_3	0.88	0.56	1.53	0.84	1.93	1.12
r	0.87	2.55	1.00	1.03	2.01	2.12

Table 4.4: Comparison of sensitivities of spacing to individual parameters based on the new spacing formula and the Kirkham equation.

In case 2: R=0.01 m/day; K_1=0.2 m/day; K_2=1.0 m/day; K_3=0.5 m/day; h=0.6 m; D_2=2.0 m; D_3=3.0 m; r=0.1 m.

In case 3: R=0.01 m/day; K_1=0.2 m/day; K_2=0.5 m/day; K_3=1.5 m/day; h=0.6 m; D_2=1.0 m; D_3=3.0 m; r=0.1 m.

All parameters are assumed to change by 10%. In Table 4.4: (1) is the results calculated from the new spacing formula (the head loss above drains is considered) and (2) is the results obtained from the Kirkham equation.

From table 4.4, it can be seen that, if a highly pervious layer is above the drains ($K_1 \gg K_2$ and $K_1 \gg K_3$, Case 1), the Kirkham equation is no longer reliable because the head loss above the drains is much more than negligible. The sensitivities of spacing to individual parameters from the two equations are very close to each other except to K_1 and K_2 when $K_1 \gg K_2$ and $K_1 \gg K_3$. In both equations the spacing is very sensitive to the drainage coefficient and the water table height midway between the drains. In

both equations spacing is not sensitive to the thicknesses of the lower layers, the drain radius, and the hydraulic conductivity of the third layer even when it is highly pervious.

Chapter 5

UNCERTAINTY ANALYSIS OF DRAINAGE DESIGN

In agricultural drainage design, varibility of soil properties and climatic conditions is an inevitable source of uncertainty in the design of a drainage system and in its performance. Such uncertainty incereases the difficulty for a farmer to decide whether to accept or reject the design. Therefore, it is desirable to estiamte the uncertainty involved in a particular design to provide the owner with necessary information, based on which the decision is made.

The amount of water recharge to the soil depends on the balance of precipitation, evapotranspiration, surface runoff and deep seepage, all of which are of stochastic characteristics. Any estimate of the recharge rate is inevitably accompanied by a certain magnitude of variability. Soil properties, such as hydraulic conductivity and porosity, also exhibit a large degree of spatial and perhaps temporal variation (Freeze, 1975; de Vries, 1982). From the sensitivity analyses in previous chapters, it has been found that drain spacing is very sensitive to the soil hydraulic conductivity and the drainage rate. The current practice of drainage design is still mostly based on a deterministic framework by using equivalent climatic and soil parameters. However, the meanings of these equivalent values are not always so clear to agricultural engineers, hydrogeologists, and soil physicists. These equivalent values are chosen partly based on the actual conditions, but also largely on the experience and professional judgement of the individual designer.

Much research has been done to incorporate uncertainty of parameters into stochastic and deterministic analyses (Bakr et al, 1978; Cordova and Bras, 1981; Cornell, 1972;

Dagan, 1979; Delhomme, 1978; Freeze, 1975; Sagar, 1978a and b; Smith and Freeze, 1979). Most of the above studies require either large amount of data, which is seldom, if ever, available in practice, or advanced mathematical and statistical knowledge which is often beyond that of many practicing designers. Some of the methods require excessive computations as prerequisites to solution of the problem. Even so, very few methods can provide an overall analysis of uncertainty.

Cornell (1972) and Benjamin & Cornell (1970) introduced first and second order analysis of model and parameter uncertainty into civil engineering. Prasher (1982) applied first and second order analysis in drainage design. However, he only introduced the idea and very much work is left to be done before this method can be conveniently used by drainage practitioners. In this chapter, first and second order analysis of parameter uncertainty will be briefly illustrated and then applied in the uncertainty analysis of subsurface drainage design in homogeneous and two–layered soils.

5.1 First and Second Order Analysis of Parameter Uncertainty

First and second order analysis only requires the first and second moments of the random variables and the first and second derivatives of functional relationship, which forms its bases of practical usefulness. It is usually very difficult, if not impossible, to obtain full distribution of the random variables as required by more complex and exact models, while it is relatively easier to obtain the first two moments. If the random variables are normally distributed, the first two moments can fully describe the characterizations of the random variables; otherwise this method will only provide approximate results. If coefficient of variation and/or degrees of non-linearity are relatively small, the method will provide quite accurate results (Cornell, 1972).

Consider first the simplest situation. Assume that random variable, y, is functionally

related only to another random variable, x, i.e.

$$y = f(x)$$

in which x has variance σ_x^2 and mean μ_x.

If the function behaves sufficiently well (i.e. it is continuous and differentiable at the mean), it can be expanded by Taylor series about the mean or expected value of x (μ_x):

$$y = f(\mu_x) + f'(\mu_x)(x - \mu_x) + \frac{1}{2}f''(\mu_x)(x - \mu_x)^2 + \cdots$$

where $f'(\mu_x)$ and $f''(\mu_x)$ are the first and second derivatives evaluated at μ_x, respectively.

If the coefficient of variation is relatively small, the above series can be truncated at the third term:

$$y \approx f(\mu_x) + f'(\mu_x)(x - \mu_x) + \frac{1}{2}f''(\mu_x)(x - \mu_x)^2$$

Taking the expectation of both sides gives:

$$\begin{aligned}
\mu_y = E(y) &= E[f(\mu_x) + f'(\mu_x)(x - \mu_x) + \frac{1}{2}f''(\mu_x)(x - \mu_x)^2] \\
&= f(\mu_x) + f'(\mu_x)E[x - \mu_x] + \frac{1}{2}f''(\mu_x)E[x - \mu_x]^2 \\
&= f(\mu_x) + \frac{1}{2}f''(\mu_x)\sigma_x^2
\end{aligned}$$

i.e.

$$\mu_y = f(\mu_x) + \frac{1}{2}f''(\mu_x)\sigma_x^2$$

To obtain the second moment of the random variable y (variance σ_y^2) with the first and second moments of x, the third term in the Taylor expansion is dropped:

$$y \approx f(\mu_x) + f'(\mu_x)(x - \mu_x)$$

Here, $\mu_y = f(\mu_x)$.

Then the variance of y can be estimated as:

$$\begin{aligned}\sigma_y^2 = \mathrm{Var}(y) &= E[y-\mu_y]^2 \\ &= E[f(\mu_x)+f'(\mu_x)(x-\mu_x)-f(\mu_x)]^2 \\ &= E[f'(\mu_x)(x-\mu_x)]^2 \\ &= [f'(\mu_x)]^2\sigma_x^2\end{aligned}$$

So far,the mean, μ_y, and the variance, σ_y^2, of y have been estimated as:

$$\mu_y = f(\mu_x) + \frac{1}{2}f''(\mu_x)\sigma_x^2 \tag{5.50}$$

$$\sigma_y^2 = [f'(\mu_x)]^2\sigma_x^2 \tag{5.51}$$

It should be noted that if the function is non-linear, i.e., $f''(\mu_x)\neq 0$, the mean value of the dependent variable is not equivalent to deterministic solutions, and the vanriance of the dependent variable is only an approximation. The range of error is dependent on the non- linearity and the variances of the independent variables.

More generally, if $\vec{X}$ represents the column vector of independent variables, and y the dependent variable functionally related to $\vec{X}$:

$$y = f(\vec{X})$$

then we have:

$$y \approx f(\vec{\mu}_X) + \vec{B}_1^T\,\Delta\vec{X} + \frac{1}{2}\sum_j(\vec{B}_{2j}^T\,\Delta\vec{X}_j^2) \tag{5.52}$$

where

$\vec{\mu}_X$: column vector of the means of independent variables;

$\Delta\vec{X}$: first order difference column vector of indenpent variables:

$$\Delta X_i = X_i - \mu_{X_i}$$

ΔX^2: second order difference matrix of independent variables:

$$\Delta X_{ij} = \Delta X_i \Delta X_j = (X_i - \mu_{X_i})(X_j - \mu_{X_j})$$

$\Delta \vec{X}_j^2$: jth column of ΔX^2;

$\vec{B}_1^T$: transpose of the column vector of first order derivatives;

B_2^T: transpose of the second order derivative matrix:

$$B_{2ij} = \frac{\partial^2 f}{\partial X_i \partial X_j}$$

$\vec{B}_{2j}^T$: jth row vector of B_2^T.

And first and second moments of y are:

$$\mu_y = f(\vec{\mu}_X) + \frac{1}{2}\sum_j (\vec{B}_{2j}^T \vec{C}_j) \tag{5.53}$$

$$\sigma_y^2 = \vec{B}_1^T \; C \; \vec{B}_1 \tag{5.54}$$

where

C: covariance matrix of the vector of independent variables;

$$C_{ij} = \mathrm{Cov}(X_i, X_j)$$

$$\mathrm{Cov}(X_i, X_j) = \rho(X_i, X_j)\sigma_{X_i}\sigma_{X_j}$$

where $\rho(X_i, X_j)$ is the correlation coefficient of X_i and X_j

$\vec{C}_j$: is the jth column vector of C.

For special cases where $\vec{X}$ are uncorrelated, we have:

$$\mu_y = f(\vec{\mu}_X) + \frac{1}{2}\sum_j (\vec{B}_2'^T \vec{C}') \tag{5.55}$$

$$\sigma_y^2 = \sum_i (\frac{\partial f}{\partial X_i})^2 \sigma_{X_i}^2 \tag{5.56}$$

where

$\vec{B_2'}^T$ is the transpose of column vector of second derivatives:

$$B'_{2i} = \frac{\partial^2 f}{\partial X_i^2}$$

$\vec{C'}$ is the variance column vector of $\vec{X}$:

$$C'_i = \sigma^2_{X_i}$$

As an illustration, let's consider cases where there are two independent variables, U and V, which may or may not be correlated:

$$y = f(U, V)$$

In equation 5.52,

$$\vec{\mu}_X = \begin{pmatrix} \mu_U \\ \mu_V \end{pmatrix}$$

$$\Delta\vec{X} = \begin{pmatrix} U - \mu_U \\ V - \mu_V \end{pmatrix}$$

$$\Delta\vec{X}^2 = \begin{pmatrix} (U-\mu_U)^2 & (U-\mu_U)(V-\mu_V) \\ (V-\mu_V)(U-\mu_U) & (V-\mu_V)^2 \end{pmatrix}$$

$$\vec{B_1}^T = \begin{pmatrix} \frac{\partial y}{\partial U} & \frac{\partial y}{\partial V} \end{pmatrix}$$

$$B_2^T = \begin{pmatrix} \frac{\partial^2 y}{\partial U^2} & \frac{\partial^2 y}{\partial V \partial U} \\ \frac{\partial^2 y}{\partial U \partial V} & \frac{\partial^2 y}{\partial V^2} \end{pmatrix}$$

In equation 5.53 and 5.54,

$$C = \begin{pmatrix} \sigma_U^2 & \mathrm{Cov}(U,V) \\ \mathrm{Cov}(V,U) & \sigma_V^2 \end{pmatrix}$$

Then:

$$\begin{aligned}
\mu_y &= f(U,V) + \frac{1}{2}[\begin{pmatrix} f''_U & f''_{VU} \end{pmatrix} \begin{pmatrix} \sigma_U^2 \\ \mathrm{Cov}(V,U) \end{pmatrix} + \begin{pmatrix} f''_{UV} & f''_V \end{pmatrix} \begin{pmatrix} \mathrm{Cov}(U,V) \\ \sigma_V^2 \end{pmatrix}] \\
&= f(U,V) + \frac{1}{2}[f''_U\sigma_U^2 + f''_{VU}\mathrm{Cov}(V,U) + f''_{UV}\mathrm{Cov}(U,V) + f''_V\sigma_V^2] \\
&= f(U,V) + \frac{1}{2}[f''_U\sigma_U^2 + 2f''_{UV}\mathrm{Cov}(U,V) + f''_V\sigma_V^2]
\end{aligned}$$

and

$$\begin{aligned}
\sigma_y^2 &= \begin{pmatrix} f'_U & f'_V \end{pmatrix} \begin{pmatrix} \sigma_U^2 & \mathrm{Cov}(U,V) \\ \mathrm{Cov}(V,U) & \sigma_V^2 \end{pmatrix} \begin{pmatrix} f'_U \\ f'_V \end{pmatrix} \\
&= \begin{pmatrix} f'_U\sigma_U^2 + f'V\mathrm{Cov}(V,U) & f'_U\mathrm{Cov}(U,V) + f'_V\sigma_V^2 \end{pmatrix} \begin{pmatrix} f'_U \\ f'_V \end{pmatrix} \\
&= (f'_U)^2\sigma_U^2 + 2f'_Uf'_V\mathrm{Cov}(U,V) + (f'_V)^2\sigma_V^2
\end{aligned}$$

5.2 Uncertainty Analysis of Steady Drainage Design in Homogeneous Soils

Among the numerous steady state drainage theories available, the Hooghoudt equation is most commonly used by agricultural engineers (Luthin, 1978; Chieng, 1981). It can be written as:

$$S^2 = \frac{4Kh}{R}(2d_e + h)$$

where d_e is the equivalent depth, evaluated from equation 3.2 or 3.3; K is the saturated hydraulic conductivity; R is the drainage coefficient; h is the permissible midspan water table height above the drain level; and S is the drain spacing.

For the purpose of evaluating the performance of a drainage system, the water table height h is chosen as the dependent variable. The water table height, h, is the most important parameter in drainage design. It is directly related to the soil workability and the moisture content in the root zone. For a given spacing, any uncertainty in evaluating other parameters such as K, R and d_e will result in uncertain performance of the drainage system. The Hooghoudt equation (Eq. 3.1) can be rewritten as:

$$h = \sqrt{\frac{RS^2}{4K} + d_e^2} - d_e \tag{5.57}$$

In the following analyses, only the drainge coefficient, R, and hydraulic conductivity, K, are to be considered as weakly correlated because in most locations where R is high, K is high and vice-versa (Prasher, 1982). All other parameters are to be considered as not correlated.

5.2.1 Drainage Coefficient R as Random Variable

The drainge coefficient, R, is defined as the amount of water removed by a drainage system per unit surface area per unit time. It depends on the climatic conditions, the soil characteristics, the crop type and the trafficability requirement (Chieng, 1978). The proper design drainage coefficient is the one that will remove excess water before crop damage occurs. Drainage coefficients are developed by local experience or by analysis of soil water balance (Willardson, 1982). It is recognised that drainage coefficient determination are uncertain to some extent because of the complexity of nature and limitations of methods.

A practical way of defining the degree of uncertainty in parameters is to estimate their first two moments by whatever means that are available. A brief review of methods to estimate means and variances from limited data can be found in Appendix B. It should

be noted that there is still much work to be done in this aspect. For the analysis in this chapter, the first and second moments of random parameters are assumed to be already known.

The mean and variance of the drainage coefficient are estimated, by some method, as R and $\mathrm{Var}(R)$, respectively. The first and second derivatives of h with respect to R are equal to:

$$\frac{\partial h}{\partial R} = \frac{h^2 + 2d_e h}{2R(h + d_e)}$$

$$\frac{\partial^2 h}{\partial R^2} = -\frac{(h^2 + 2d_e h)^2}{4R^2(h + d_e)^3}$$

Therefore, the mean and the variance of the water table height are:

$$\begin{aligned} E(h)|_R &= h + \frac{1}{2}\frac{\partial^2 h}{\partial R^2}\mathrm{Var}(R) \\ &= h + \frac{1}{2}[-\frac{(h^2 + 2d_e h)^2}{4R^2(h + d_e)^3}]\mathrm{Var}(R) \end{aligned}$$

$$\begin{aligned} \mathrm{Var}(h)|_R &= (\frac{\partial h}{\partial R})^2\mathrm{Var}(R) \\ &= \frac{(h^2 + 2d_e h)^2}{4R^2(h + d_e)^2}\mathrm{Var}(R) \end{aligned}$$

where $|_R$ denoted "due to R", and will be used throughout the thesis.

Rearranging the above equations into dimensionless form yeilds:

$$\frac{E(h)|_R - h}{h} = -\frac{(1 + 2\frac{d_e}{h})^2}{8(1 + \frac{d_e}{h})^3}\frac{\mathrm{Var}(R)}{R^2} \tag{5.58}$$

$$\frac{\mathrm{Var}(h)|_R}{h^2} = \frac{(1 + 2\frac{d_e}{h})^2}{4(1 + \frac{d_e}{h})^2}\frac{\mathrm{Var}(R)}{R^2} \tag{5.59}$$

Figure 5.13 and 5.14 show that the effects of the randomness of the drainage coefficient, R, on the mean water table height decreases, and that on the variance of water table height increases, as $\frac{d_e}{h}$ increases.

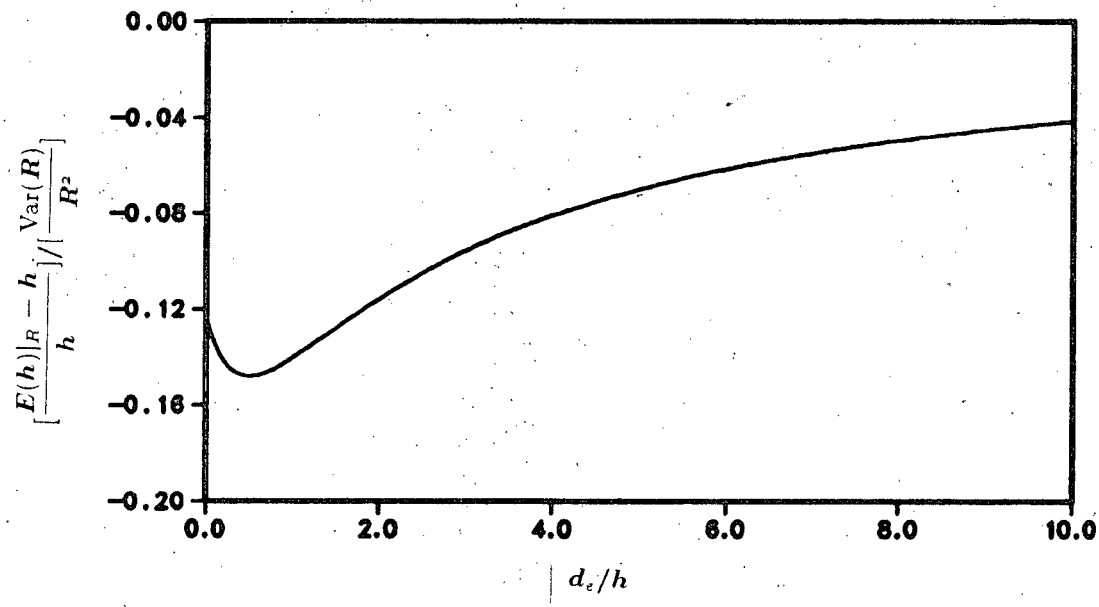

Figure 5.13: Expected water table height due to uncertainty of drainage coefficient.

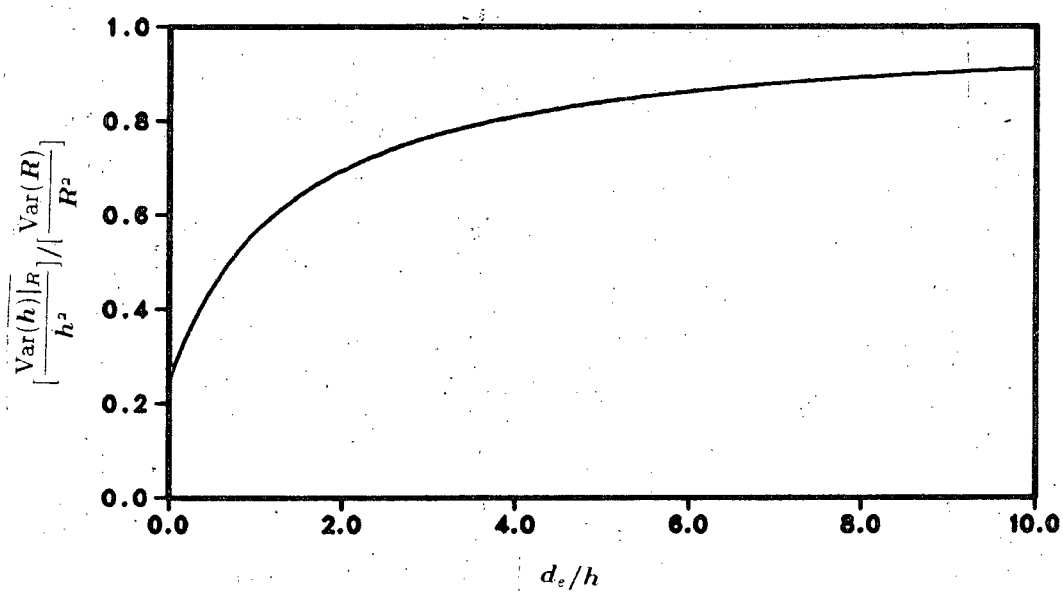

Figure 5.14: Variance of water table height due to uncertainty of drainage coefficient.

5.2.2 Hydraulic Conductivity as Random Variable With Mean of K and Variance of Var(K)

The expected water table height and its variance are:

$$E(h)|_K = h + \frac{1}{2}\frac{\partial^2 h}{\partial K^2}\mathrm{Var}(K)$$

$$\mathrm{Var}(h)|_K = (\frac{\partial h}{\partial K})^2\mathrm{Var}(K)$$

respectively, where

$$\frac{\partial h}{\partial K} = -\frac{h^2 + 2d_e h}{2K(h + d_e)}$$

$$\frac{\partial^2 h}{\partial K^2} = \frac{(h^2 + 2d_e h)(3h^2 + 6hd_e + 4d_e^2)}{4K^2(h + d_e)^3}$$

Therefore, the mean and the variance of water table height in dimensionless form are:

$$\frac{E(h)|_K - h}{h} = \frac{(1 + 2\frac{d_e}{h})[3 + 6\frac{d_e}{h} + 4(\frac{d_e}{h})^2]}{8(1 + \frac{d_e}{h})^3}\frac{\mathrm{Var}(K)}{K^2} \tag{5.60}$$

$$\frac{\mathrm{Var}(h)|_K}{h^2} = \frac{(1 + 2\frac{d_e}{h})^2}{4(1 + \frac{d_e}{h})^2}\frac{\mathrm{Var}(K)}{K^2} \tag{5.61}$$

respectively.

Figure 5.15 and 5.16 show that the effects of uncertainty in the hydraulic conductivity on both the mean and the variance of water table height increase as $\frac{d_e}{h}$ increases.

5.2.3 Correlated Hydraulic Conductivity K and Drainage Coefficient R with a covariance of Cov(R, K)

For the convenience of writing the equations of the mean and the variance when all parameters are considered, only the additional portion of the mean and the variance of the water table height caused by the correlation between K and R will be calculated here. The total effects of K and R will be the sum of the results obtained in the previous two sections and in this section.

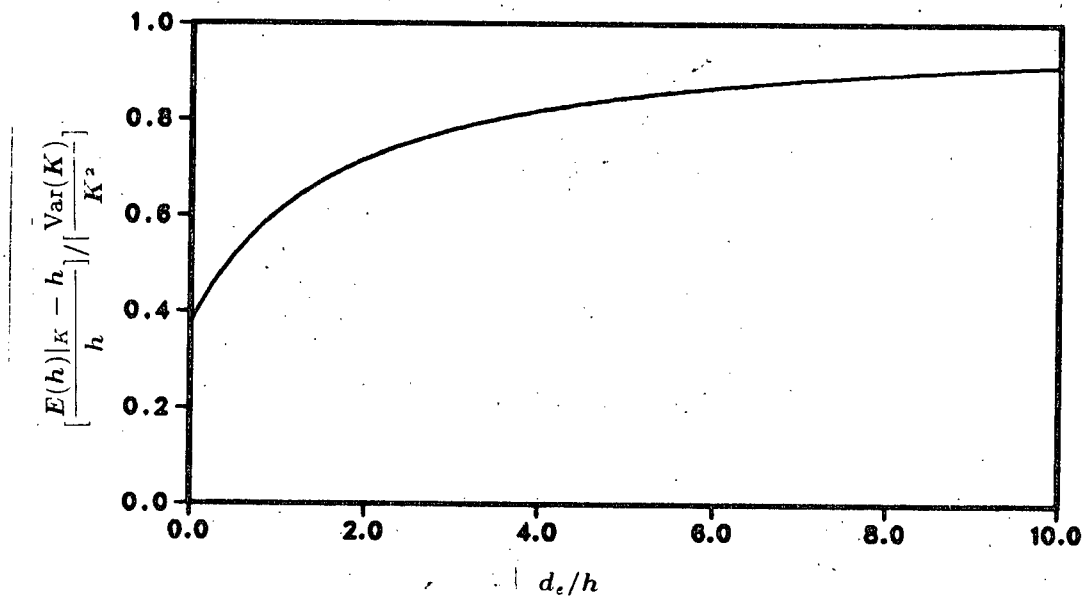

Figure 5.15: Expected water table height due to uncertainty of hydraulic conductivity

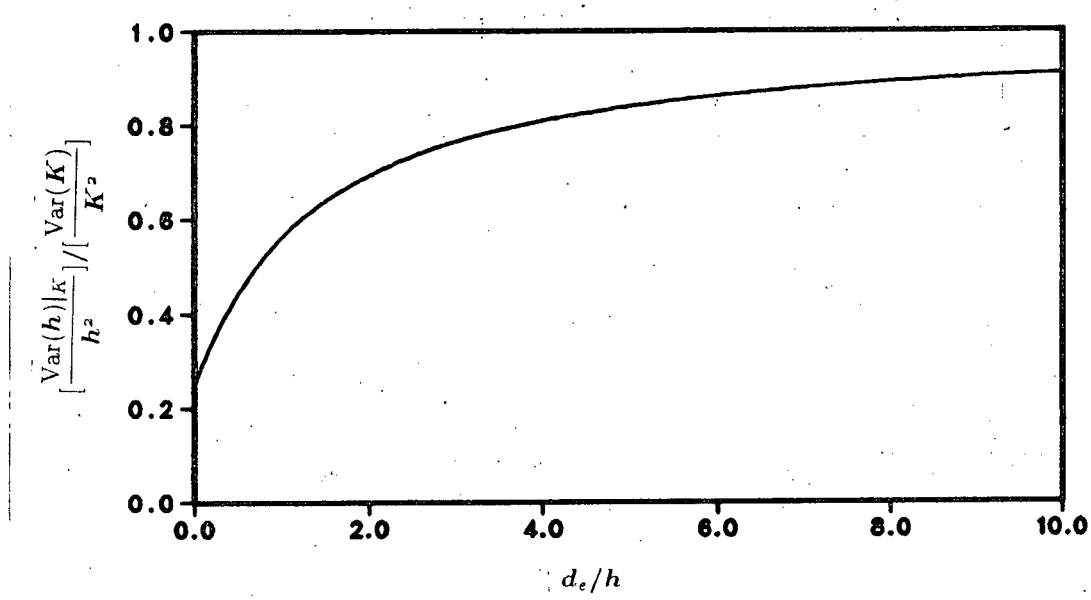

Figure 5.16: Variance of water table height due to uncertainty of hydraulic conductivity

From Eq. 5.57, we have:

$$\frac{\partial^2 h}{\partial R \partial K} = -\frac{h^2 + 2d_e h}{4KR(h + d_e)}$$

$$\frac{\partial h}{\partial R} = \frac{h^2 + 2d_e h}{2R(h + d_e)}$$

$$\frac{\partial h}{\partial K} = -\frac{h^2 + 2d_e h}{2K(h + d_e)}$$

The additional effects on the mean and the variance of the water table height are:

$$\begin{aligned} E(h)|_{(K \times R)} &= h + \frac{\partial^2 h}{\partial R \partial K} \mathrm{Cov}(K, R) \\ &= h - \frac{h^2 + 2d_e h}{4KR(h + d_e)} \mathrm{Cov}(K, R) \end{aligned}$$

$$\begin{aligned} \mathrm{Var}(h)|_{(K \times R)} &= 2(\frac{\partial h}{\partial R})(\frac{\partial h}{\partial K}) \mathrm{Cov}(K, R) \\ &= -\frac{(h^2 + 2d_e h)^2}{2KR(h + d_e)^2} \mathrm{Cov}(K, R) \end{aligned}$$

where $|_{(K \times R)}$ denotes "due to the correlation of K and R".

The relationship between covariance and correlation coeffcient is:

$$\mathrm{Cov}(K, R) = \rho(K, R)\sigma_K \sigma_R$$

in which $\rho(K, R)$ is the correlation coefficient of K and R. If K and R are independent,

$$\rho(K, R) = 0$$

Rearranging the equations of mean and variance gives:

$$\frac{E(h)|_{(K \times R)} - h}{h} = -\frac{1 + 2\frac{d_e}{h}}{4(1 + \frac{d_e}{h})} \frac{\mathrm{Cov}(K, R)}{KR} \tag{5.62}$$

$$\frac{\mathrm{Var}(h)|_{(K \times R)}}{h^2} = -\frac{(1 + 2\frac{d_e}{h})^2}{2(1 + \frac{d_e}{h})^2} \frac{\mathrm{Cov}(K, R)}{KR} \tag{5.63}$$

The negative sign in both equations indicates that R and K have opposite effects on the mean and the variance of the water table height. It can be seen in Figure 5.17 and 5.18 that effects of the correlation of R and K on both the mean and the variance of the water table height increase as $\frac{d_e}{h}$ increases.

5.2.4 Equivalent Depth as a Random Variable with Mean of d_e and Variance of Var(d_e)

The first and second derivatives of h with respect to d_e are:

$$\frac{\partial h}{\partial d_e} = -\frac{h}{h + d_e}$$

$$\frac{\partial^2 h}{\partial d_e^2} = \frac{h^2 + 2d_e h}{(h + d_e)^3}$$

The mean and the variance of the water table height can be estimated as:

$$\begin{aligned} E(h)|_{d_e} &= h + \frac{1}{2}\frac{\partial^2 h}{\partial d_e^2}\mathrm{Var}(d_e) \\ &= h + \frac{h^2 + 2d_e h}{2(h + d_e)^3}\mathrm{Var}(d_e) \end{aligned}$$

$$\begin{aligned} \mathrm{Var}(h)|_{d_e} &= (\frac{\partial h}{\partial d_e})^2 \mathrm{Var}(d_e) \\ &= \frac{h^2}{(h + d_e)^2}\mathrm{Var}(d_e) \end{aligned}$$

or, in dimensionless form:

$$\frac{E(h)|_{d_e} - h}{h} = \frac{(\frac{d_e}{h})^2(1 + 2\frac{d_e}{h})}{2(1 + \frac{d_e}{h})^3}\frac{\mathrm{Var}(d_e)}{d_e^2} \tag{5.64}$$

$$\frac{\mathrm{Var}(h)|_{d_e}}{h^2} = \frac{(\frac{d_e}{h})^2}{(1 + \frac{d_e}{h})^2}\frac{\mathrm{Var}(d_e)}{d_e^2} \tag{5.65}$$

Figure 5.19 and 5.20 show that the uncertainty of equivalent depth becomes more important to the mean and variance of water table height as $\frac{d_e}{h}$ increases.

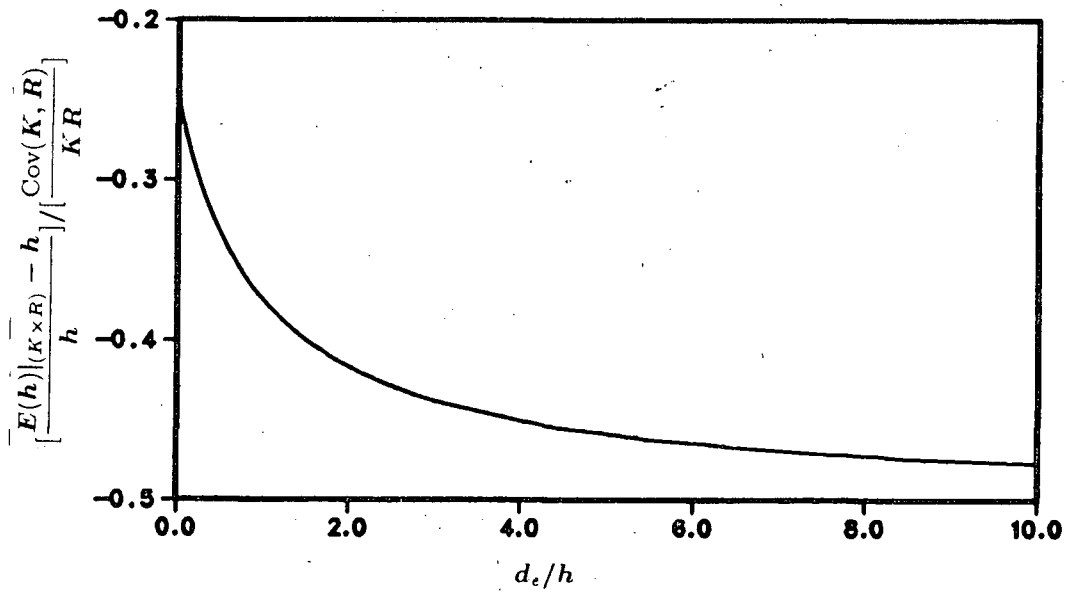

Figure 5.17: Expected water table height due to the correlation of the hydraulic conductivity and the drainage coefficent.

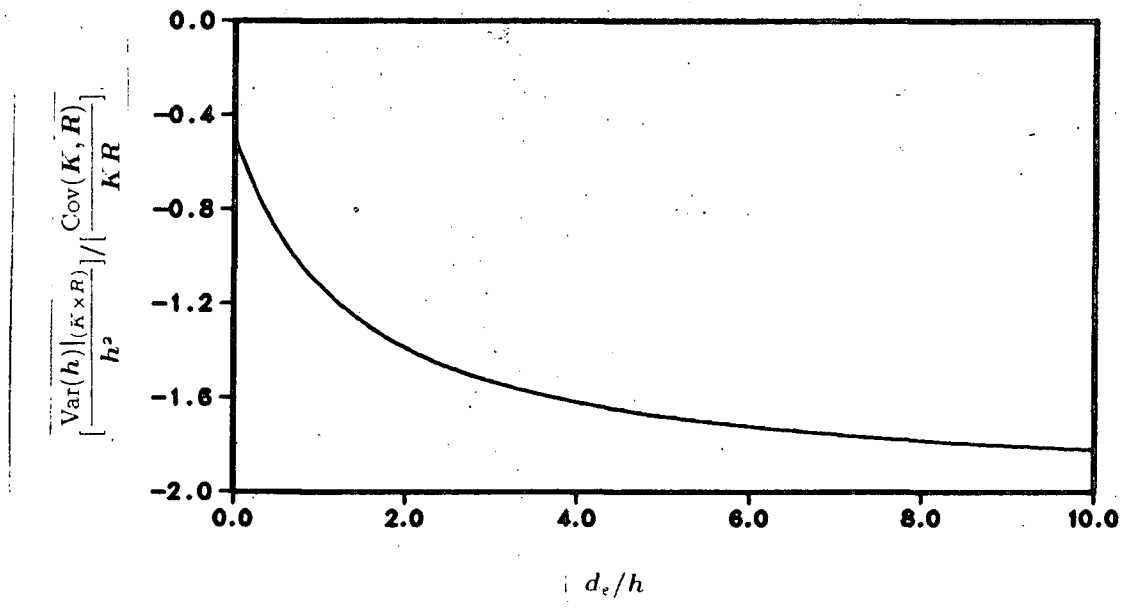

Figure 5.18: Variance of water table hieght due to the correlation of the hydraulic conductivity and the drainage coefficent.

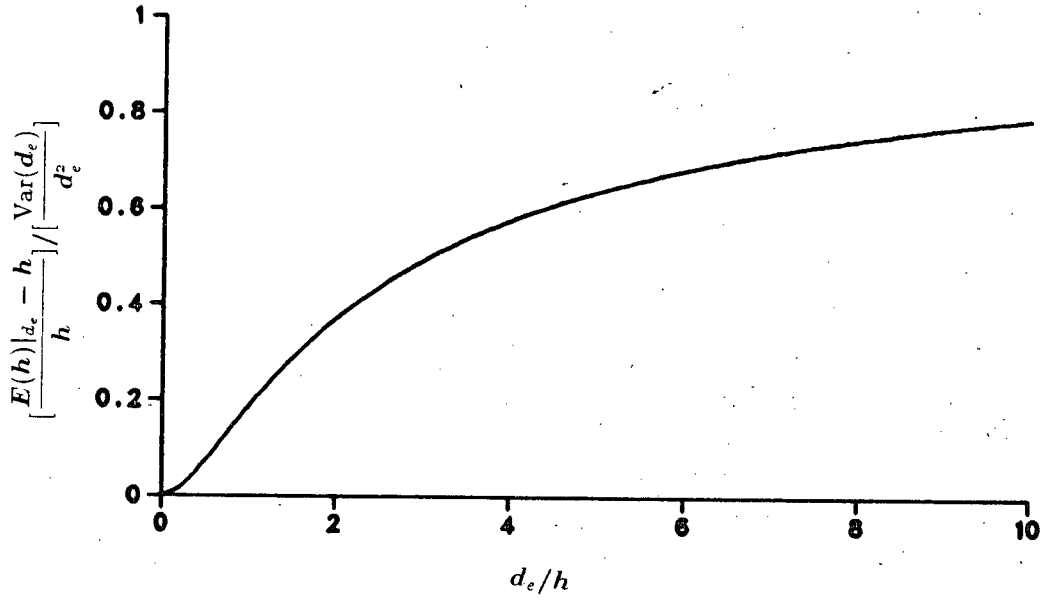

Figure 5.19: Expected water table height due to uncertainty of equivalent depth

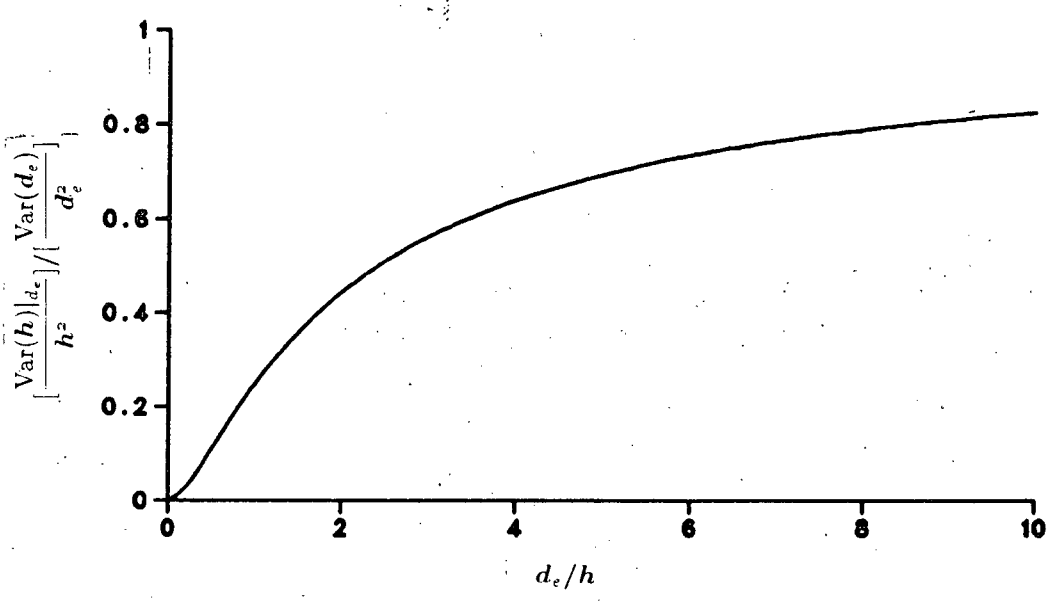

Figure 5.20: Variance of water table height due to uncertainty of equivalent depth

5.2.5 Estimating Variance of Equivalent Depth from the Mean and Variance of the Depth to an Impermeable Layer

Equivalent depth, d_e, is dependent on the spacing, the impermeable layer depth and the drain radius. For a given spacing and a drain radius, the variance of the equivalent depth can be estimated from the mean and the variance of the depth to the impermeable layer. d_e can be calculated by equation 3.2 or 3.3 (Moody, 1966):

For $0 < \frac{d}{S} \leq 0.3$

$$d_e = \frac{d}{1 + \frac{d}{S}(\frac{8}{\pi}\ln\frac{d}{r} - \alpha)}$$

where

$$\alpha = 3.55 - 1.6\frac{d}{S} + 2(\frac{d}{S})^2$$

and for $\frac{d}{S} > 0.3$

$$d_e = \frac{S}{\frac{8}{\pi}(\ln\frac{S}{r} - 1.15)}$$

The first derivative of d_e with respect to d is:

For $0 < \frac{d}{S} \leq 0.3$

$$\frac{\partial d_e}{\partial d} = \frac{1 - \frac{8}{\pi}\frac{d}{S} - 1.6(\frac{d}{S})^2 + 4(\frac{d}{S})^3}{[1 + \frac{d}{S}(\frac{8}{\pi}\ln\frac{d}{S} - \alpha)]^2}$$

and for $\frac{d}{S} > 0.3$

$$\frac{\partial d_e}{\partial d} = 0$$

Therefore, the variance of d_e can be estimated as:

For $\frac{d}{S} > 0.3$

$$\mathrm{Var}(d_e) = 0$$

and for $0 < \frac{d}{S} \leq 0.3$

$$\mathrm{Var}(d_e) = \frac{[1 - \frac{8}{\pi}\frac{d}{S} - 1.6(\frac{d}{S})^2 + 4(\frac{d}{S})^3]^2}{[1 + \frac{d}{S}(\frac{8}{\pi}\ln\frac{d}{S} - \alpha)]^4}\mathrm{Var}(d)$$

Neglecting the square of $\frac{d}{S}$ and higher terms and rearranging the above equation gives:

$$\frac{\mathrm{Var}(d_e)}{d_e^2} = \frac{(1-\frac{8}{\pi}\frac{d}{S})^2}{[1+\frac{d}{S}(\frac{8}{\pi}\ln\frac{d}{S}-3.55)]^2}\frac{\mathrm{Var}(d)}{d^2} \tag{5.66}$$

Let

$$\frac{\mathrm{Var}(d_e)}{d_e^2} = \frac{\mathrm{Var}(d)}{d^2}$$

thus

$$(1-\frac{8}{\pi}\frac{d}{S})^2 = [1+\frac{d}{S}(\frac{8}{\pi}\ln\frac{d}{S}-3.55)]^2$$

Solving the above equation yields:

$$\frac{d}{S} = 0$$

or

$$\frac{d}{r} = 1.483$$

i.e., when $d = 0$ or $d = 1.483r$

$$\frac{\mathrm{Var}(d_e)}{d_e^2} = \frac{\mathrm{Var}(d)}{d^2}$$

and for $0 < d < 1.483r$

$$\frac{\mathrm{Var}(d_e)}{d_e^2} > \frac{\mathrm{Var}(d)}{d^2}$$

which is somewhat unreasonable. This ambiguity can be avoided by assuming that:

$$\frac{\mathrm{Var}(d_e)}{d_e^2} = \frac{\mathrm{Var}(d)}{d^2}$$

for $0 < d < 1.483r$, where r is the radius of the drains.

5.2.6 All or Some of the Parameters as Random Variables with Known Means, Variances and Covariances

From section 5.1, we have

$$E(h) = h + \frac{1}{2}\sum_{j=1}^{3}(\vec{B}_{2j}^T\vec{C}_j)$$

$$\mathrm{Var}(h) = \vec{B}_1^T \; C \; \vec{B}_1$$

where

$$B_2^T = \begin{pmatrix} \frac{\partial^2 h}{\partial R^2} & \frac{\partial^2 h}{\partial K \partial R} & \frac{\partial^2 h}{\partial d_e \partial R} \\ \frac{\partial^2 h}{\partial R \partial K} & \frac{\partial^2 h}{\partial K^2} & \frac{\partial^2 h}{\partial d_e \partial K} \\ \frac{\partial^2 h}{\partial R \partial d_e} & \frac{\partial^2 h}{\partial K \partial d_e} & \frac{\partial^2 h}{\partial d_e^2} \end{pmatrix}$$

$$B_1 = \begin{pmatrix} \frac{\partial h}{\partial R} \\ \frac{\partial h}{\partial K} \\ \frac{\partial h}{\partial d_e} \end{pmatrix}$$

$$C = \begin{pmatrix} \mathrm{Var}(R) & \mathrm{Cov}(R,K) & 0 \\ \mathrm{Cov}(K,R) & \mathrm{Var}(K) & 0 \\ 0 & 0 & \mathrm{Var}(d_e) \end{pmatrix}$$

Rearranging the above equations gives:

$$E(h) = h + \frac{1}{2}[\frac{\partial^2 h}{\partial R^2}\mathrm{Var}(R) + \frac{\partial^2 h}{\partial K^2}\mathrm{Var}(K) + \frac{\partial^2 h}{\partial d_e^2}\mathrm{Var}(d_e) + 2\frac{\partial^2 h}{\partial R \partial K}\mathrm{Cov}(K,R)]$$

$$\mathrm{Var}(h) = (\frac{\partial h}{\partial R})^2\mathrm{Var}(R) + (\frac{\partial h}{\partial K})^2\mathrm{Var}(K) + (\frac{\partial h}{\partial d_e})^2\mathrm{Var}(d_e) + 2(\frac{\partial h}{\partial R})(\frac{\partial h}{\partial K})\mathrm{Cov}(K,R)$$

From the results of analysis in this section, the above equation can be written in the form:

$$\frac{E(h)-h}{h} = \frac{E(h)|_R - h}{h} + \frac{E(h)|_K - h}{h} + \frac{E(h)|_{(K\times R)} - h}{h} + \frac{E(h)|_{d_e} - h}{h} \tag{5.67}$$

$$\frac{\mathrm{Var}(h)}{h^2} = \frac{\mathrm{Var}(h)|_R}{h^2} + \frac{\mathrm{Var}(h)|_K}{h^2} + \frac{\mathrm{Var}(h)|_{(K\times R)}}{h^2} + \frac{\mathrm{Var}(h)|_{d_e}}{h^2} \tag{5.68}$$

in which various terms are correspondingly evaluated from equation 5.58 through 5.66.

5.3 Uncertainty Analysis of Transient Drainage Design in Homogeneous Soils

In the cases where steady state drainage design is inappropriate, transient drainage theory should be used. For saturated flows and for the cases where the D-F assumptions are applicable, the water flow toward drains in the soil can be described by the Boussinesq equation (van Schilfgaarde, 1974):

$$f\frac{\partial h}{\partial t} = K\frac{\partial}{\partial x}(h\frac{\partial h}{\partial x}) + R$$

where f is the drainable porosity, h is the water table height above an impermeable layer, K is the saturated hydraulic conductivity, x is the horizontal distance from the drain center, t is the time and R is the net recharge rate.

The Boussinesq equation is non-linear. Golver (Dumm, 1954) and Tapp & Moody (Dumm, 1964) solved this equation by linearization. van Schilfgaarde (1963; 1964), on the other hand, solved it without linearization and obtained the following equation for the case without recharge:

$$t = \frac{fS^2}{9Kd_e}\ln\frac{m_0(2d_e + m_1)}{m_1(2d_e + m_0)}$$

in which t is the time during which the water table drops from initial height of m_0 to m_1. d in the original van Schilfgaarde solution has been replaced by the equivalent depth d_e to provide more practical results.

For a given spacing, S, the required time, t, for the water table to recede from m_0 to m_1 is determined by the drainable porosity, f, the hydraulic conductivity, K, and the equivalent depth, d_e. However, because of the inevitable uncertainty in the values of these parameters,it can be expected that there will be some uncertainty in the drawdown time, which may result in either a longer period of waterlogging or water deficiency before the

next recharge. It is therefore desirable to estimate the variation of the drawdown time caused by the uncertain input values of parameters. First and second order analysis of parameter uncertainty will be applied to the following cases: (1) the drainable porosity as a random variable; (2) the hydraulic conductivity as a random variable; (3) the correlation of the drainable porosity and the hydraulic conductivity; (4) the equivalent depth as a random variable; (5) all of the above.

5.3.1 Drainable Porosity as Random Variable with Mean of f and Variance of Var(f)

The first and second derivatives of t with respect to f are:

$$\frac{\partial t}{\partial f} = \frac{t}{f}$$

$$\frac{\partial^2 t}{\partial f^2} = 0$$

respectively.

Therefore, the mean and the variance of the required time are:

$$E(t)|_f = t + \frac{1}{2}\frac{\partial^2 t}{\partial f^2}\mathrm{Var}(f) = t$$

$$\mathrm{Var}(t)|_f = (\frac{\partial t}{\partial f})^2\mathrm{Var}(f) = \frac{t^2}{f^2}\mathrm{Var}(f)$$

or, in dimensionless form:

$$\frac{E(t)|_f - t}{t} = 0 \tag{5.69}$$

$$\frac{\mathrm{Var}(t)|_f}{t^2} = \frac{\mathrm{Var}(f)}{f^2} \tag{5.70}$$

5.3.2 Hydraulic Conductivity as a Random Variable with Mean of K and Variance of Var(K)

The first and second derivatives of t with respect to K are:

$$\frac{\partial t}{\partial K} = -\frac{t}{K}$$

$$\frac{\partial^2 t}{\partial K^2} = \frac{2t}{K^2}$$

Therefore, the mean and the variance of required time are:

$$E(t)|_K = t + \frac{1}{2}\frac{\partial^2 t}{\partial K^2}\mathrm{Var}(K) = t + \frac{t}{K^2}\mathrm{Var}(K)$$

$$\mathrm{Var}(t)|_K = (\frac{\partial t}{\partial K})^2\mathrm{Var}(K) = \frac{t^2}{K^2}\mathrm{Var}(K)$$

or, in dimensionless form:

$$\frac{E(t)|_K - t}{t} = \frac{\mathrm{Var}(K)}{K^2} \tag{5.71}$$

$$\frac{\mathrm{Var}(t)|_K}{t^2} = \frac{\mathrm{Var}(K)}{K^2} \tag{5.72}$$

5.3.3 Correlated Hydraulic Conductivity and Drainable Porosity with a Known Covariance of Cov(K,f)

There exists a relationship between the drainable porosity (or specific yield) and the soil hydraulic conductivity (Talsma and Haskew, 1959). Soils with a higher drainable porosity usually have a higher hydraulic conductivity. The correlation coefficient of the drainable porosity and the hydraulic conductivity can be estimated from field data for the region.

The derivatives required for the calculation of the effects of the correlation on the mean and the variance of the drawdown time t are:

$$\frac{\partial^2 t}{\partial f \partial K} = -\frac{t}{Kf}$$

$$\frac{\partial t}{\partial f} = \frac{t}{f}$$
$$\frac{\partial t}{\partial K} = -\frac{t}{K}$$

Therefore, the mean and the variance of the drawdown time are:

$$\begin{aligned} E(t)|_{(K\times f)} &= t + \frac{\partial^2 t}{\partial f \partial K}\mathrm{Cov}(K,f) \\ &= t - \frac{t}{Kf}\mathrm{Cov}(K,f) \end{aligned}$$

$$\begin{aligned} \mathrm{Var}(t)|_{(K\times f)} &= 2(\frac{\partial t}{\partial f})(\frac{\partial t}{\partial K}\mathrm{Cov}(K,f) \\ &= -\frac{2t^2}{Kf}\mathrm{Cov}(K,f) \end{aligned}$$

or, in dimensionless form:

$$\frac{E(t)|_{(K\times f)} - t}{t} = -\frac{\mathrm{Cov}(K,f)}{Kf} \tag{5.73}$$

$$\frac{\mathrm{Var}(t)|_{(K\times f)}}{t^2} = -\frac{2\mathrm{Cov}(K,f)}{Kf} \tag{5.74}$$

where $\mathrm{Cov}(K,f)$ is the covariance of K and f.

If the correlation coefficient is known:

$$\rho(K,f) = \frac{\mathrm{Cov}(K,f)}{\sigma_f \sigma_K}$$

where $\rho(K,f)$ is the correlation of K and f, σ_f is the standard deviation of f, σ_K is the standard deviation of K, and:

$$\sigma_f = \sqrt{\mathrm{Var}(f)}$$
$$\sigma_K = \sqrt{\mathrm{Var}(K)}$$

Then the mean and the variance of the drawdown time can be written as:

$$\frac{E(t)|_{(K\times f)} - t}{t} = -\sqrt{\frac{\mathrm{Var}(K)}{K^2}}\sqrt{\frac{\mathrm{Var}(f)}{f^2}}\rho(K,f) \tag{5.75}$$

$$\frac{\mathrm{Var}(t)|_{(K\times f)}}{t^2} = -2\sqrt{\frac{\mathrm{Var}(K)}{K^2}}\sqrt{\frac{\mathrm{Var}(f)}{f^2}}\rho(K,f) \tag{5.76}$$

5.3.4 Equivalent Depth d_e as a Random Variable with Mean of d_e and Variance of Var(d_e)

d_e is dependent on the spacing, the drain radius and the impermeable layer depth. For a given spacing and a drain radius, the variance of d_e can be calculated from the mean and the variance of the depth of the impermeable layer (See section 5.2.5).

The first and second derivatives of t with respect to d_e are:

$$\frac{\partial t}{\partial d_e} = -\frac{t}{d_e} + \frac{2(m_0 - m_1)t}{(2d_e + m_1)(2d_e + m_0)\ln\left[\frac{m_0(2d_e+m_1)}{m_1(2d_e+m_0)}\right]}$$

$$\frac{\partial^2 t}{\partial d_e^2} = \frac{2t}{d_e^2} - \frac{4t(m_0 - m_1)(m_0 m_1 + m_1 d_e + m_0 d_e)}{d_e(2d_e + m_1)^2(2d_e + m_0)^2 \ln\left[\frac{m_0(2d_e+m_1)}{m_1(2d_e+m_0)}\right]}$$

Therefore:

$$\begin{aligned} E(t)|_{d_e} &= t + \frac{1}{2}\frac{\partial^2 t}{\partial d_e^2}\mathrm{Var}(d_e) \\ &= t + \left\{\frac{2t}{d_e^2} - \frac{4t(m_0 - m_1)(m_0 m_1 + m_1 d_e + m_0 d_e)}{d_e(2d_e + m_1)^2(2d_e + m_0)^2 \ln\left[\frac{m_0(2d_e+m_1)}{m_1(2d_e+m_0)}\right]}\right\}\mathrm{Var}(d_e) \end{aligned}$$

$$\begin{aligned} \mathrm{Var}(t)|_{d_e} &= \left(\frac{\partial t}{\partial d_e}\right)^2 \mathrm{Var}(d_e) \\ &= \left\{-\frac{t}{d_e} + \frac{2(m_0 - m_1)t}{(2d_e + m_1)(2d_e + m_0)\ln\left[\frac{m_0(2d_e+m_1)}{m_1(2d_e+m_0)}\right]}\right\}^2 \mathrm{Var}(d_e) \end{aligned}$$

or, in dimensionless form:

$$\frac{E(t)|_{d_e} - t}{t} = \left[1 - \frac{2(\frac{d_e}{m_1} - \frac{d_e}{m_0})(1 + \frac{d_e}{m_0} + \frac{d_e}{m_1})}{(2\frac{d_e}{m_1} + 1)^2(2\frac{d_e}{m_0} + 1)^2 \ln(\frac{2\frac{d_e}{m_1}+1}{2\frac{d_e}{m_0}+1})}\right]\frac{\mathrm{Var}(d_e)}{d_e^2} \tag{5.77}$$

$$\frac{\mathrm{Var}(t)|_{d_e}}{t^2} = \left[1 - \frac{2(\frac{d_e}{m_1} - \frac{d_e}{m_0})}{(2\frac{d_e}{m_1} + 1)(2\frac{d_e}{m_0} + 1)\ln(\frac{2\frac{d_e}{m_1}+1}{2\frac{d_e}{m_0}+1})}\right]^2\frac{\mathrm{Var}(d_e)}{d_e^2} \tag{5.78}$$

where d_e can be calculated from Equation 3.2 or 3.3 and $\frac{\mathrm{Var}(d_e)}{d_e^2}$ can be estimated from Equation 5.66.

5.3.5 All or Part of the Parameters as Random Variables with Known Means, Variances and Covariances

From section 5.1, we have

$$E(t) = t + \frac{1}{2}\sum_{j=1}^{3}(\vec{B}_{2j}^{T}\vec{C}_j)$$

$$\mathrm{Var}(h) = \vec{B}_1^{T}\ C\ \vec{B}_1$$

where

$$B_2^T = \begin{pmatrix} \frac{\partial^2 t}{\partial f^2} & \frac{\partial^2 t}{\partial K \partial f} & \frac{\partial^2 t}{\partial d_e \partial f} \\ \frac{\partial^2 t}{\partial f \partial K} & \frac{\partial^2 t}{\partial K^2} & \frac{\partial^2 t}{\partial d_e \partial K} \\ \frac{\partial^2 t}{\partial f \partial d_e} & \frac{\partial^2 t}{\partial K \partial d_e} & \frac{\partial^2 t}{\partial d_e^2} \end{pmatrix}$$

$$B_1 = \begin{pmatrix} \frac{\partial t}{\partial f} \\ \frac{\partial t}{\partial K} \\ \frac{\partial t}{\partial d_e} \end{pmatrix}$$

$$C = \begin{pmatrix} \mathrm{Var}(f) & \mathrm{Cov}(f,K) & 0 \\ \mathrm{Cov}(K,f) & \mathrm{Var}(K) & 0 \\ 0 & 0 & \mathrm{Var}(d_e) \end{pmatrix}$$

Rearranging the above equations gives:

$$E(t) = t + \frac{1}{2}[\frac{\partial^2 t}{\partial f^2}\mathrm{Var}(f) + \frac{\partial^2 t}{\partial K^2}\mathrm{Var}(K) + \frac{\partial^2 t}{\partial d_e^2}\mathrm{Var}(d_e) + 2\frac{\partial^2 t}{\partial f \partial K}\mathrm{Cov}(K,f)]$$

$$\mathrm{Var}(t) = (\frac{\partial t}{\partial f})^2\mathrm{Var}(f) + (\frac{\partial t}{\partial K})^2\mathrm{Var}(K) + (\frac{\partial t}{\partial d_e})^2\mathrm{Var}(d_e) + 2(\frac{\partial t}{\partial f})(\frac{\partial t}{\partial K})\mathrm{Cov}(K,f)$$

From the analysis results in this section, we can write the above equations in the forms:

$$\frac{E(t)-t}{t} = \frac{E(t)|_f - t}{t} + \frac{E(t)|_K - t}{t} + \frac{E(t)|_{(K\times f)} - t}{t} + \frac{E(t)|_{d_e} - t}{t} \tag{5.79}$$

$$\frac{\mathrm{Var}(t)}{t^2} = \frac{\mathrm{Var}(t)|_f}{t^2} + \frac{\mathrm{Var}(t)|_K}{t^2} + \frac{\mathrm{Var}(t)|_{(K\times f)}}{t^2} + \frac{\mathrm{Var}(t)|_{d_e}}{t^2} \tag{5.80}$$

in which the various terms are correspondingly evaluated from Eequation 5.69 through 5.78.

5.4 Uncertainty Analysis of Steady Drainage Design in Two-layered Soils

The first and second order analysis of parameter uncertainty can also be applied to drainage design in multi-layered soils. However, more difficulties can be expected in estimating the means and the variances of various parameters and their covariances, if any exist. In this section, only uncertainty in two–layered soils with the drains lying at the interface of the layers will be analyzed, although uncertainty analysis of drainage design using the Kirkham equation, the generalized Hooghoudt-Ernst equation or the new spacing formula developed in Chapter 4 can be carried out by first and second order analysis in a similar way.

The Hooghoudt equation is the most simple and commonly used theory in steady state drainage design in two–layered soils with drains at the interface of the two layers. It mainly distinguishes the flow regions above and below the drain level. It can be expressed as:

$$S^2 = \frac{4h}{R}(2K_2 d_e + K_1 h)$$

where K_1 and K_2 are the saturated hydraulic conductivities of the soils above and below the drains, respectively. All other parameters are as defined in section 5.2.

The midspan water table height, h, is again chosen as the dependent variable for the convenience of evaluating the performance of a drainage system with a selected drain

spacing. The Hooghoudt equation for two layered soils can be rewritten as:

$$h = \sqrt{\frac{RS^2}{4K_1} + (\frac{K_2}{K_1})^2 d_e^2} - (\frac{K_2}{K_1})d_e \tag{5.81}$$

The first and second order analysis will be applied to the following cases: (1) random drainage coefficient R; (2) random hydraulic conductivity of the soil above the drains K_1; (3) random hydraulic conductivity of the soil below the drains K_2; (4) correlation of R and K_1; (5) correlation of R and K_2; (6) correlation of K_1 and K_2; (7) random equivalent depth d_e; (8) all of the above.

5.4.1 Drainage Coefficient as Random Variable with Mean of R and Variance of Var(R)

From equation 5.81, the first and second derivatives with respect to R are:

$$\frac{\partial h}{\partial R} = \frac{K_1 h^2 + 2K_2 d_e h}{2R(K_1 h + K_2 d_e)}$$

$$\frac{\partial^2 h}{\partial R^2} = -\frac{K_1(K_1 h^2 + K_2 d_e h)^2}{4R^2(K_1 h + K_2 d_e)^3}$$

Thus the expected mean of h and its variance are:

$$E(h)|_R = h - \frac{K_1(K_1 h^2 + K_2 d_e h)^2}{8R^2(K_1 h + K_2 d_e)^3}\mathrm{Var}(R)$$

$$\mathrm{Var}(h)|_R = \frac{(K_1 h^2 + 2K_2 d_e h)^2}{4R^2(K_1 h + K_2 d_e)^2}\mathrm{Var}(R)$$

or, in dimensionless form:

$$\frac{E(h)|_R - h}{h} = -\frac{[1 + 2(\frac{K_2}{K_1})(\frac{d_e}{h})]^2}{8[1 + (\frac{K_2}{K_1})(\frac{d_e}{h})]^3}\frac{\mathrm{Var}(R)}{R^2} \tag{5.82}$$

$$\frac{\mathrm{Var}(h)|_R}{h^2} = \frac{[1 + 2(\frac{K_2}{K_1})(\frac{d_e}{h})]^2}{4[1 + (\frac{K_2}{K_1})(\frac{d_e}{h})]^2}\frac{\mathrm{Var}(R)}{R^2} \tag{5.83}$$

Figure 5.21 and 5.22 show that the effect of the randomness of R on the mean of h decreases, and that on the variance of h increases, as $\frac{K_2}{K_1}$ and/or $\frac{d_e}{h}$ increase.

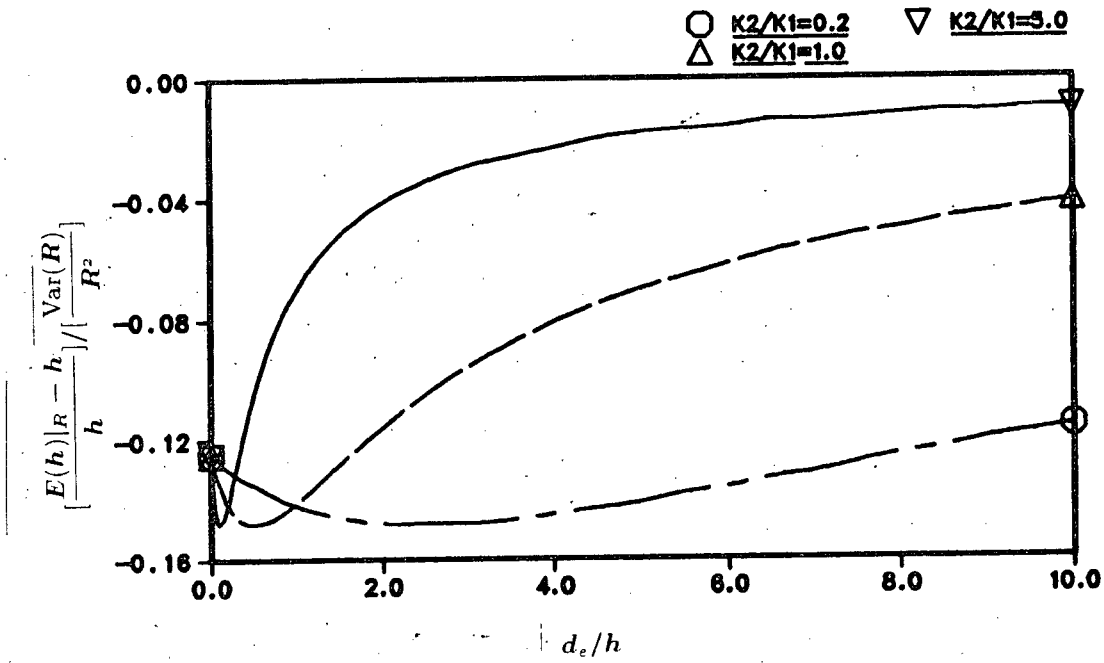

Figure 5.21: Expected water table height due to uncertainty of the drainage coefficient.

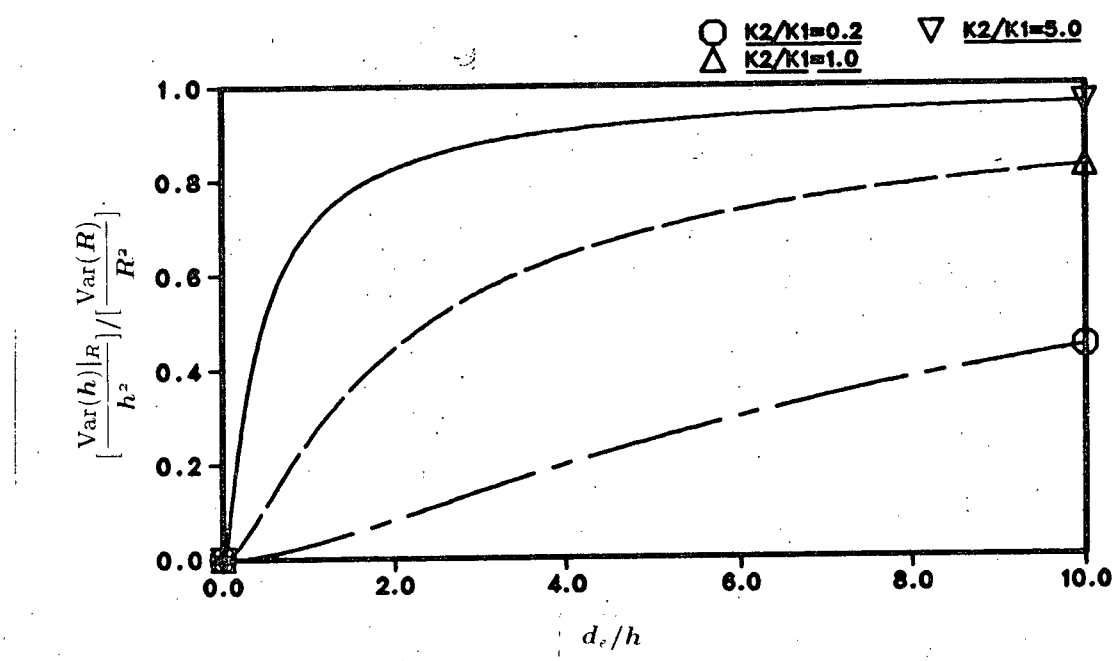

Figure 5.22: Variance of water table height due to uncertainty of the drainage coefficient.

5.4.2 Hydraulic Conductivity of the Soil above Drains as Random Variable with Mean of K_1 and Variance of Var(K_1)

From the Hooghoudt equation, the first and second derivatives with respect to K_1 is:

$$\frac{\partial h}{\partial K_1} = -\frac{h^2}{2(K_1 h + K_2 d_e)}$$

$$\frac{\partial^2 h}{\partial K_1^2} = \frac{3K_1 h^4 + 4K_2 d_e h^3}{4K_1^2(K_1 h + K_2 d_e)^3}$$

Therefore, the mean and the variance of h will be:

$$E(h)|_{K_1} = h + \frac{3K_1 h^4 + 4K_2 d_e h^3}{8K_1^2(K_1 h + K_2 d_e)^3}\mathrm{Var}(K_1)$$

$$\mathrm{Var}(h)|_{K_1} = \frac{h^4}{4(K_1 h + K_2 d_e)^2}\mathrm{Var}(K_1)$$

respectively.

Rearranging the above equations in dimensionless form gives:

$$\frac{E(h)|_{K_1} - h}{h} = \frac{3 + 4(\frac{K_2}{K_1})(\frac{d_e}{h})}{8[1 + (\frac{K_2}{K_1})(\frac{d_e}{h})]^3}\frac{\mathrm{Var}(K_1)}{K_1^2} \tag{5.84}$$

$$\frac{\mathrm{Var}(h)|_{K_1}}{h^2} = \frac{1}{4[1 + (\frac{K_2}{K_1})(\frac{d_e}{h})]^2}\frac{\mathrm{Var}(K_1)}{K_1^2} \tag{5.85}$$

Figure 5.23 and 5.24 show that the effect of the randomness of K_1 on both the mean and the variance of h decreases as $\frac{K_2}{K_1}$ and/or $\frac{d_e}{h}$ increase. This is because that the flow region above the drains becomes less important when $\frac{K_2}{K_1}$ and/or $\frac{d_e}{h}$ become larger.

5.4.3 Hydraulic Conductivity of the Soil below Drains as Random Variable with Mean of K_2 and Variance of Var(K_2)

The first and second derivatives of h with respect to K_2 are:

$$\frac{\partial h}{\partial K_2} = -\frac{d_e h}{K_1 h + K_2 d_e}$$

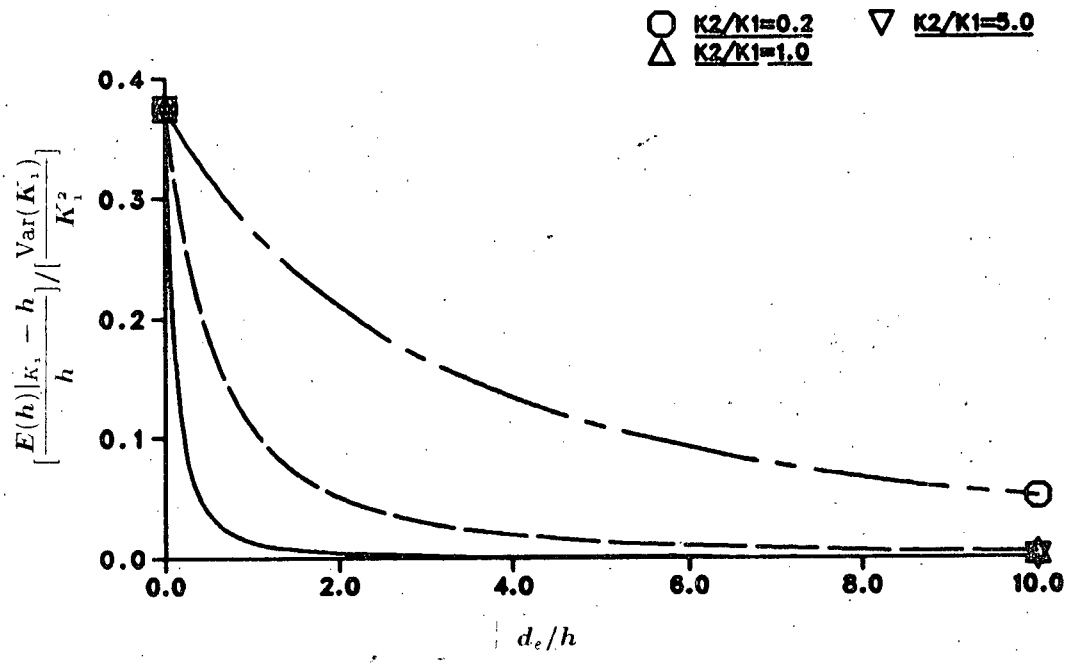

Figure 5.23: Expected water table height due to uncertainty of the hydraulic conductivity of the soil above the drains.

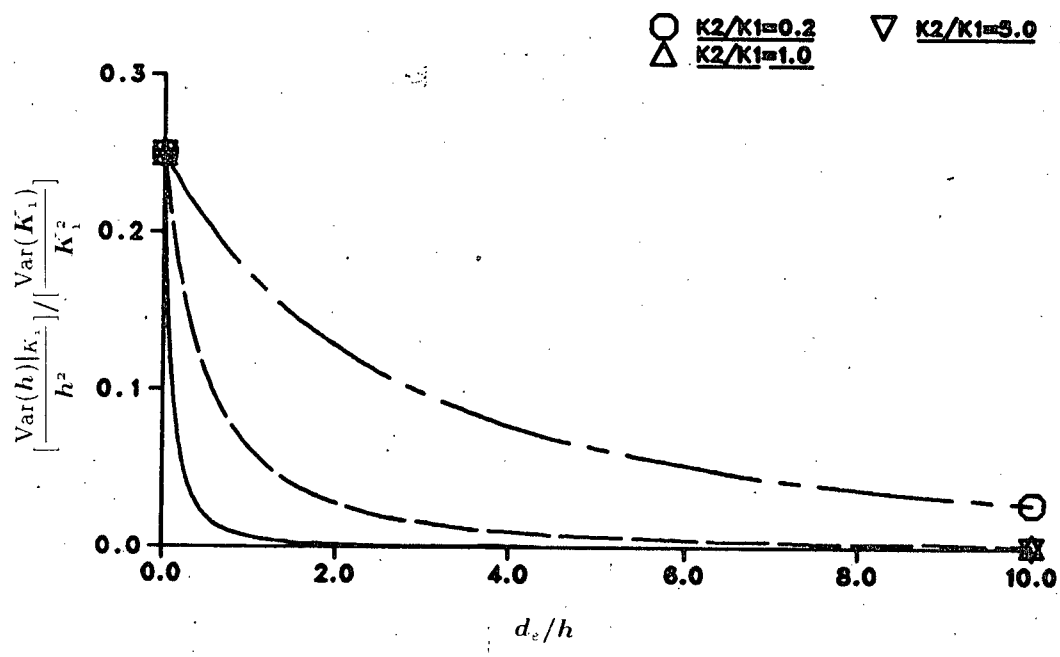

Figure 5.24: Variance of water table height due to uncertainty of the hydraulic conductivity of the soil above the drains.

$$\frac{\partial^2 h}{\partial K_2^2} = \frac{K_1 h^2 d_e^2 + 2K_2 h d_e^3}{(K_1 h + K_2 d_e)^3}$$

Therefore, the mean and the variance of h can be estimated as:

$$E(h)|_{K_2} = h + \frac{K_1 h^2 d_e^2 + 2K_2 h d_e^3}{2(K_1 h + K_2 d_e)^3} \mathrm{Var}(K_2)$$

$$\mathrm{Var}(h)|_{K_2} = \frac{d_e^2 h^2}{(K_1 h + K_2 d_e)^2} \mathrm{Var}(K_2)$$

Rearranging the above equation in dimensionless form:

$$\frac{E(h)|_{K_2} - h}{h} = \frac{[1 + 2(\frac{K_2}{K_1})(\frac{d_e}{h})](\frac{K_2}{K_1})^2(\frac{d_e}{h})^2}{2[1 + (\frac{K_2}{K_1})(\frac{d_e}{h})]^3} \frac{\mathrm{Var}(K_2)}{K_2^2} \tag{5.86}$$

$$\frac{\mathrm{Var}(h)|_{K_2}}{h^2} = \frac{(\frac{d_e}{h})^2(\frac{K_2}{K_1})^2}{[1 + (\frac{K_2}{K_1})(\frac{d_e}{h})]^2} \frac{\mathrm{Var}(K_2)}{K_2^2} \tag{5.87}$$

Figure 5.25 and 5.26 show that the uncertainty of K_2 becomes more significant to both the mean and the variance of h as $\frac{K_2}{K_1}$ and/or $\frac{d_e}{h}$ increase. This is because the flow region below the drains becomes more important when $\frac{K_2}{K_1}$ and/or $\frac{d_e}{h}$ becomes larger.

5.4.4 Correlation of Drainage Coefficient R and Hydraulic Conductivity, K_1, of the Soil above Drains with covariance of Cov(K_1, R)

From the Hooghoudt equation, the necessary derivatives needed for the calcualtion of the mean and the variance of h are:

$$\frac{\partial h}{\partial R} = \frac{K_1 h^2 + 2K_2 d_e h}{2R(K_1 h + K_2 d_e)}$$

$$\frac{\partial h}{\partial K_1} = -\frac{h^2}{2(K_1 h + K_2 d_e)}$$

$$\frac{\partial^2 h}{\partial R \partial K_1} = -\frac{(K_1 h + 2K_2 d_e)^2 h^2}{4R(K_1 h + K_2 d_e)^3}$$

Therefore, the mean and the variance due to the correlation of R and K_1 are:

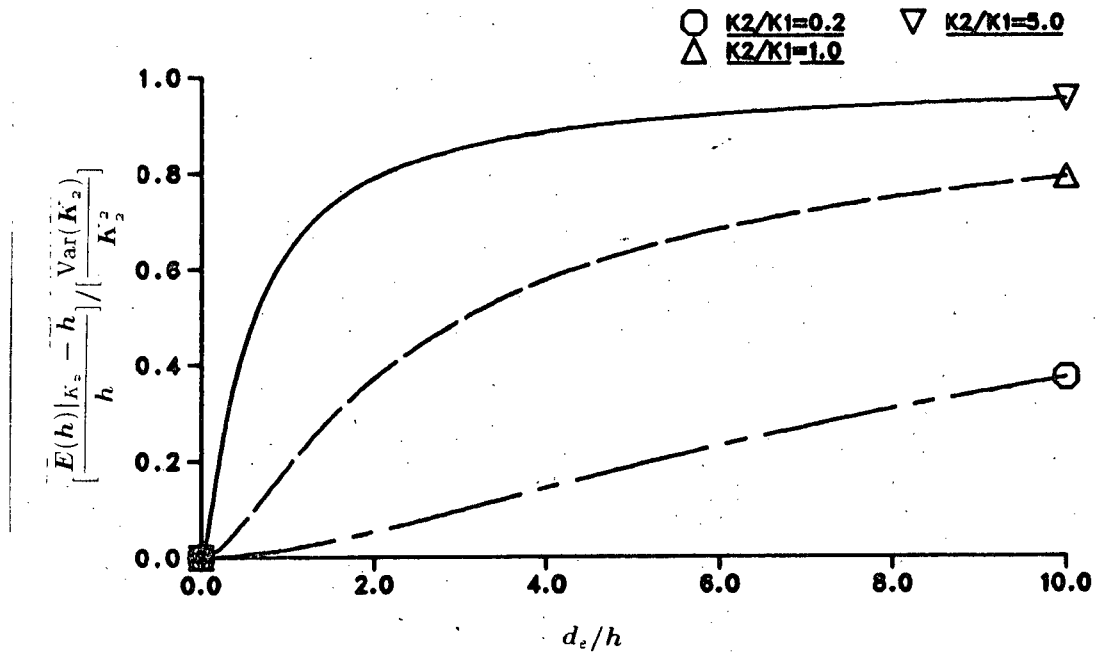

Figure 5.25: Expected water table height due to uncertainty of hydraulic conductivity of the soil below drains.

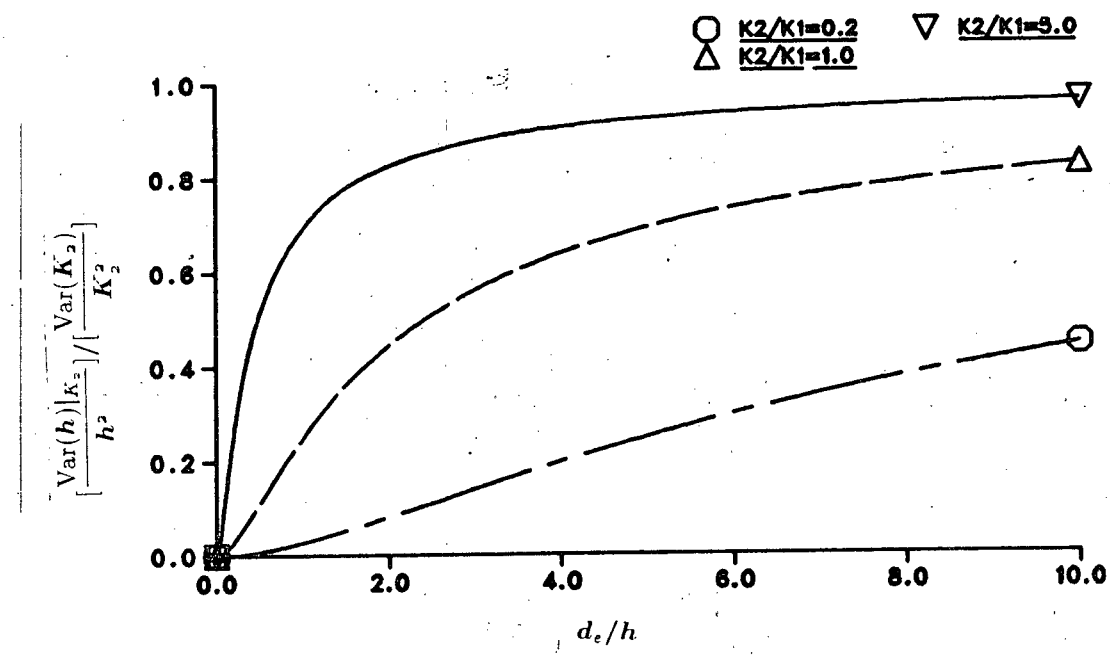

Figure 5.26: Variance of water table height due to uncertainty of hydraulic conductivity of the soil below drains.

$$E(h)|_{(K_1 \times R)} = h - \frac{(K_1 h + 2K_2 d_e)^2 h^2}{4R(K_1 h + K_2 d_e)^3} \mathrm{Cov}(K_1, R)$$

$$\mathrm{Var}(h)|_{(K_1 \times R)} = -\frac{(K_1 h + 2K_2 d_e) h^3}{2R(K_1 h + K_2 d_e)^2} \mathrm{Cov}(K_1, R)$$

or, in dimensionless form:

$$\frac{E(h)|_{(K_1 \times R)} - h}{h} = -\frac{[1 + 2(\frac{K_2}{K_1})(\frac{d_e}{h})]^2}{4[1 + (\frac{K_2}{K_1})(\frac{d_e}{h})]^3} \frac{\mathrm{Cov}(K_1, R)}{K_1 R} \tag{5.88}$$

$$\frac{\mathrm{Var}|_{(K_1 \times R)}}{h^2} = -\frac{1 + 2(\frac{K_2}{K_1})(\frac{d_e}{h})}{2[1 + (\frac{K_2}{K_1})(\frac{d_e}{h})]^2} \frac{\mathrm{Cov}(K_1, R)}{K_1 R} \tag{5.89}$$

Figure 5.27 and 5.28 K_1 and R have opposite effects on both the mean and the variance of h, which generally decrease as $\frac{K_2}{K_1}$ and/or $\frac{d_e}{h}$ increase.

5.4.5 Correlation of Drainage Coefficient R and Hydraulic Conductivity, K_2 of the Soil below Drains with covariance of Cov(K_2, R)

From the Hooghoudt equation, the necessary derivatives needed for the calcualtion of the mean and the variance of h are:

$$\frac{\partial h}{\partial R} = \frac{K_1 h^2 + 2K_2 d_e h}{2R(K_1 h + K_2 d_e)}$$

$$\frac{\partial h}{\partial K_2} = -\frac{d_e h}{K_1 h + K_2 d_e}$$

$$\frac{\partial^2 h}{\partial R \partial K_2} = -\frac{(K_1 h + 2K_2 d_e) K_2 d_e^2 h}{2R(K_1 h + K_2 d_e)^3}$$

Therefore, the mean and the variance due to the correlation of R and K_2 are:

$$E(h)|_{(K_2 \times R)} = h - \frac{(K_1 h + 2K_2 d_e) K_2 d_e^2 h}{2R(K_1 h + K_2 d_e)^3} \mathrm{Cov}(K_2, R)$$

$$\mathrm{Var}(h)|_{(K_2 \times R)} = -\frac{(K_1 h + 2K_2 d_e) d_e h^2}{R(K_1 h + K_2 d_e)^2} \mathrm{Cov}(K_2, R)$$

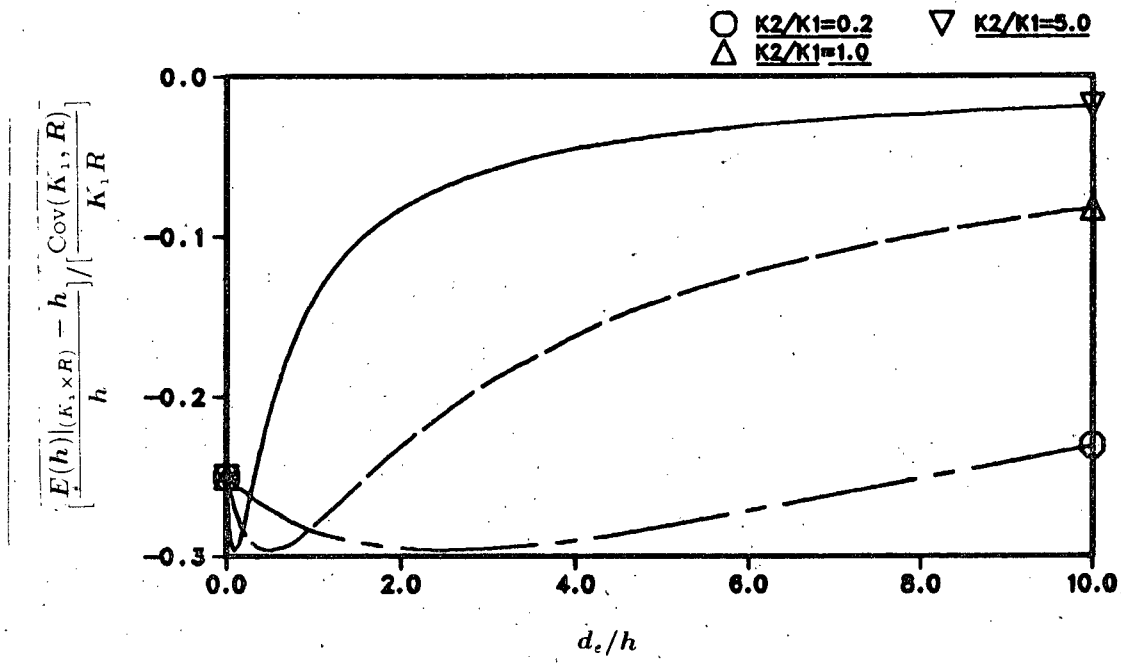

Figure 5.27: Expected water table height due to correlation of drainage coefficient and hydraulic conductivity of the soil above drains.

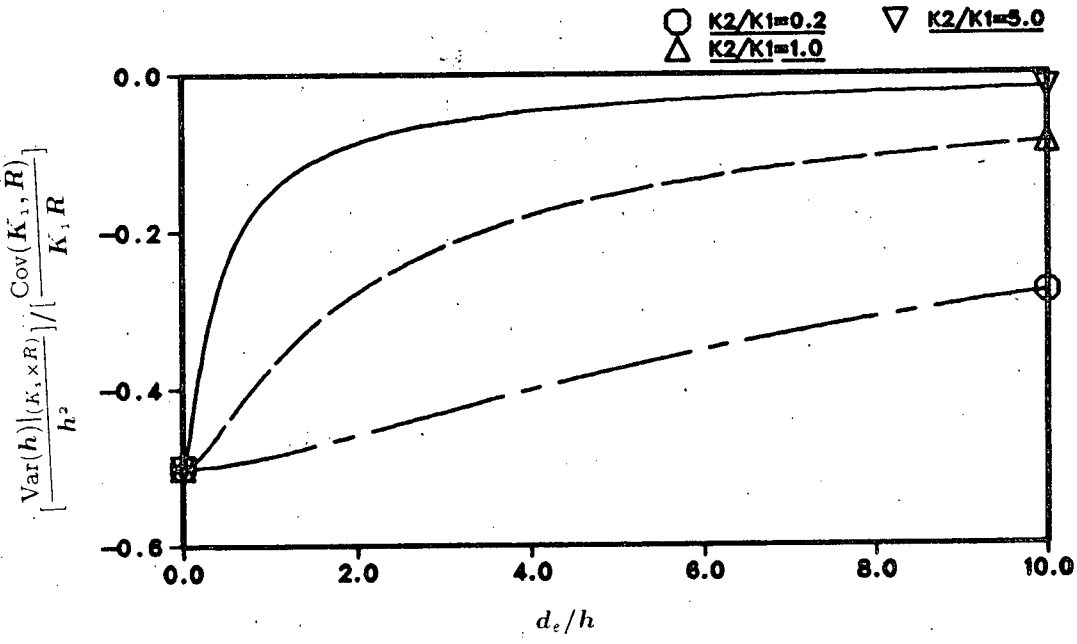

Figure 5.28: Variance of water table height due to correlation of drainage coefficient and hydraulic conductivity of the soil above drains.

or, in dimensionless form:

$$\frac{E(h)|_{(K_2\times R)} - h}{h} = -\frac{(\frac{K_2}{K_1})^2(\frac{d_e}{h})^2[1+2(\frac{K_2}{K_1})(\frac{d_e}{h})]}{2[1+(\frac{K_2}{K_1})(\frac{d_e}{h})]^3}\frac{\mathrm{Cov}(K_2,R)}{K_2R} \tag{5.90}$$

$$\frac{\mathrm{Var}|_{(K_2\times R)}}{h^2} = -\frac{(\frac{K_2}{K_1})(\frac{d_e}{h})[1+2(\frac{K_2}{K_1})(\frac{d_e}{h})]}{[1+(\frac{K_2}{K_1})(\frac{d_e}{h})]^2}\frac{\mathrm{Cov}(K_2,R)}{K_2R} \tag{5.91}$$

Figure 5.29 and 5.30 show that R and K_2 have opposite effects on both the mean and the variance of h, which increase as $\frac{K_2}{K_1}$ and/or $\frac{d_e}{h}$ increase.

5.4.6 Correlation of Hydraulic Conductivities of the Soil above Drains, K_1, and below Drains, K_2, with covariance of Cov(K_1, K_2)

From the Hooghoudt equation, the necessary derivatives needed for the calcualtion of the mean and the variance of h are:

$$\frac{\partial h}{\partial K_1} = -\frac{h^2}{2(K_1h+K_2d_e)}$$

$$\frac{\partial h}{\partial K_2} = -\frac{d_eh}{K_1h+K_2d_e}$$

$$\frac{\partial^2 h}{\partial K_2\partial K_1} = -\frac{d_eh^2(2K_1h+3K_2d_e)}{2(K_1h+K_2d_e)^3}$$

Therefore, the mean and the variance due to the correlation of K_1 and K_2 are:

$$E(h)|_{(K_1\times K_2)} = h + \frac{d_eh^2(2K_1h+3K_2d_e)}{2(K_1h+K_2d_e)^3}\mathrm{Cov}(K_1,K_2)$$

$$\mathrm{Var}(h)|_{(K_1\times K_2)} = -\frac{d_eh^3}{(K_1h+K_2d_e)^2}\mathrm{Cov}(K_1,K_2)$$

or, in dimensionless form:

$$\frac{E(h)|_{(K_1\times K_2)} - h}{h} = -\frac{(\frac{K_2}{K_1})(\frac{d_e}{h})[2+3(\frac{K_2}{K_1})(\frac{d_e}{h})]}{2[1+(\frac{K_2}{K_1})(\frac{d_e}{h})]^3}\frac{\mathrm{Cov}(K_1,K_2)}{K_1K_2} \tag{5.92}$$

$$\frac{\mathrm{Var}|_{(K_1\times K_2)}}{h^2} = -\frac{(\frac{K_2}{K_1})(\frac{d_e}{h})}{[1+(\frac{K_2}{K_1})(\frac{d_e}{h})]^2}\frac{\mathrm{Cov}(K_1,K_2)}{K_1K_2} \tag{5.93}$$

Figure 5.31 and 5.32 show that effects of the correlation on both the mean and the variance of h first increases and then decrease as $\frac{K_2}{K_1}$ and/or $\frac{d_e}{h}$ increase.

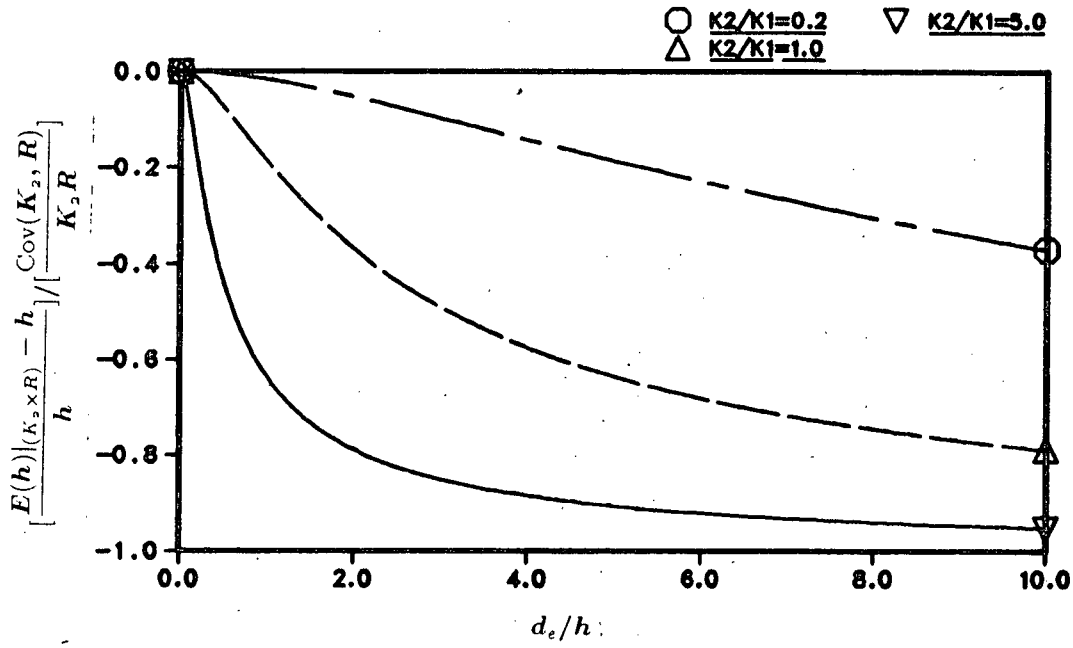

Figure 5.29: Expected water table height due to correlation of drainage coefficient and hydraulic conductivity of the soil below drains.

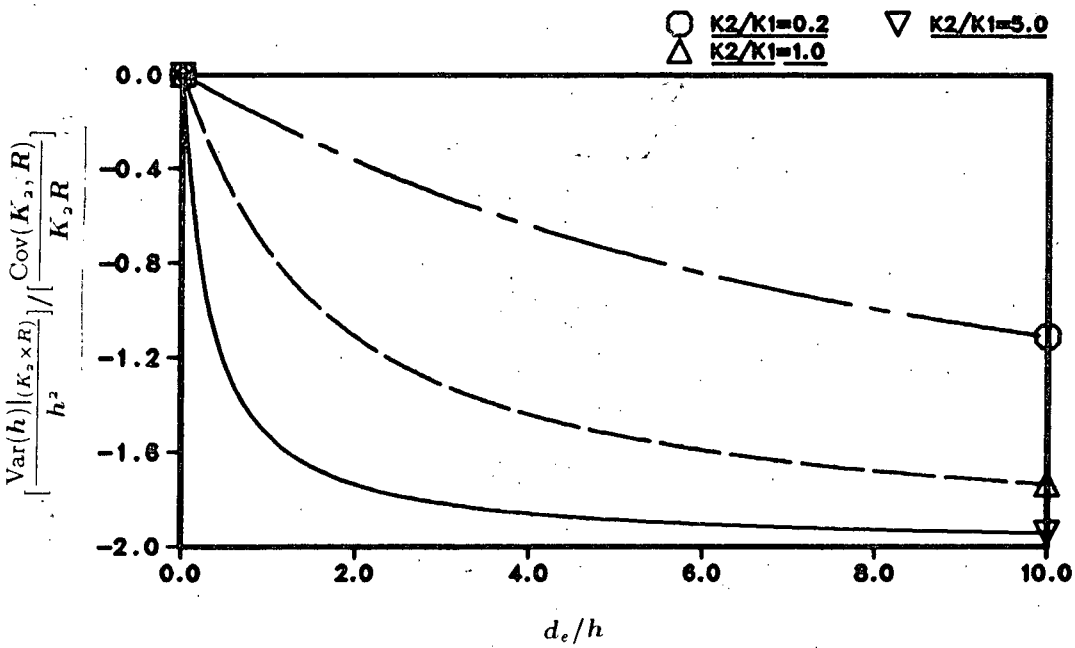

Figure 5.30: Variance of water table height due to correlation of drainage coefficient and hydraulic conductivity of the soil below drains.

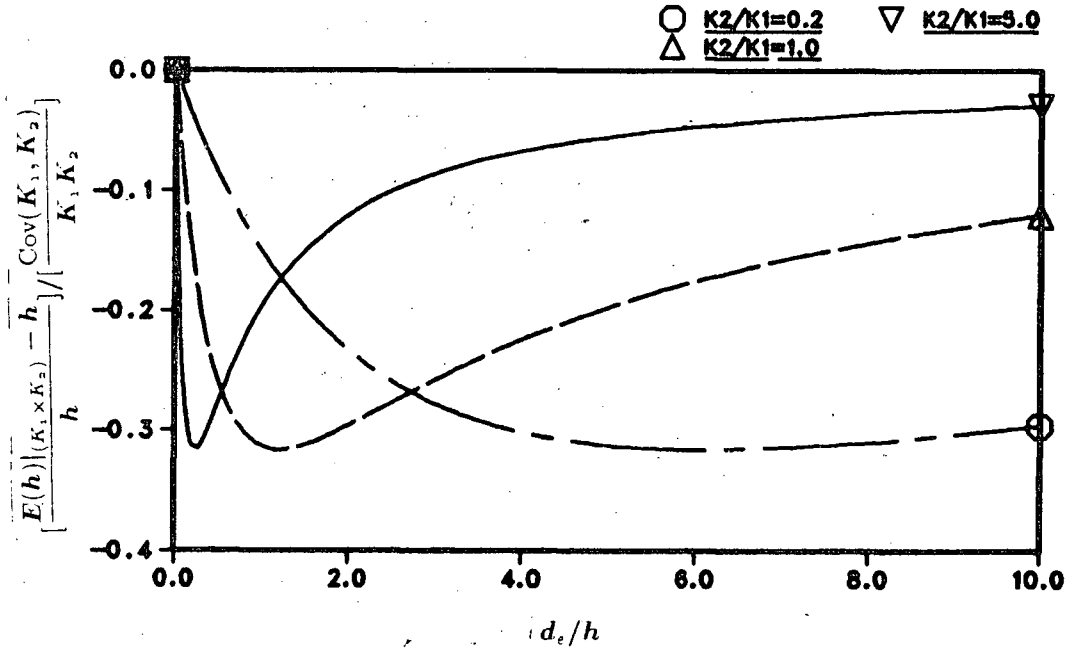

Figure 5.31: Expected water table height due to correlation of hydraulic conductivity of the soil above drains and that below drains.

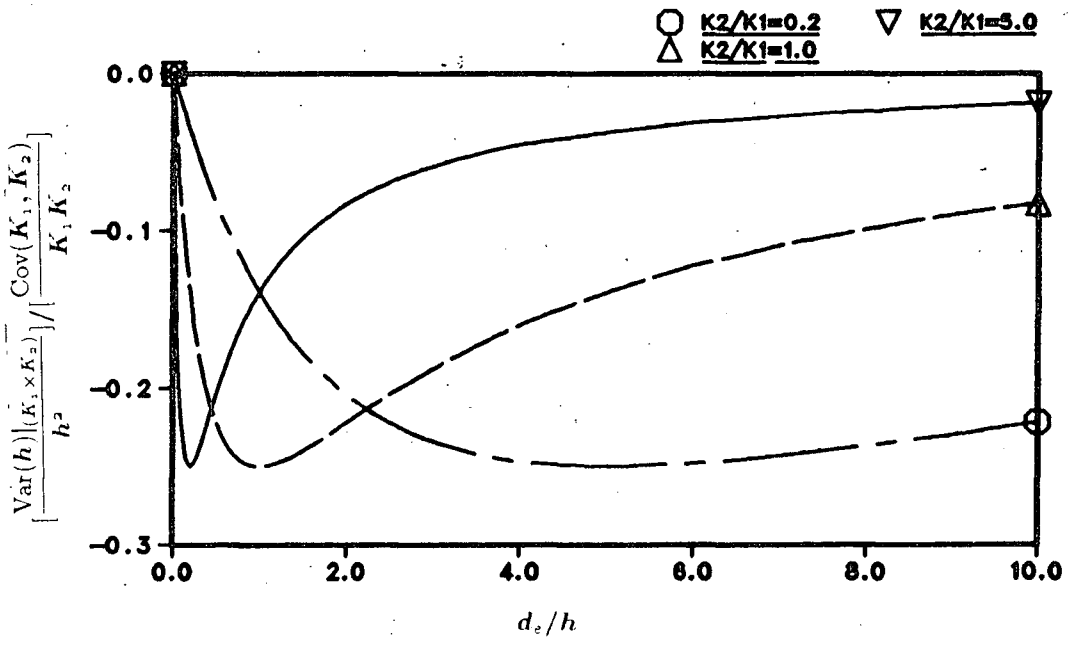

Figure 5.32: Variance of water table height due to correlation of hydraulic conductivity of the soil above drains and that below drains.

5.4.7 Equivalent Depth as Random Variable with Mean of d_e and Variance of Var(d_e)

The first and second derivatives of h with respect to d_e are:

$$\frac{\partial h}{\partial d_e} = -\frac{K_2 h}{K_1 h + K_2 d_e}$$

$$\frac{\partial^2 h}{\partial d_e^2} = \frac{K_2^2(K_1 h + 2K_2 d_e)h}{(K_1 h + K_2 d_e)^3}$$

Therefore, the mean and the variance of h due to the randomness of d_e are:

$$E(h)|_{d_e} = h + \frac{K_2^2(K_1 h + 2K_2 d_e)h}{2(K_1 h + K_2 d_e)^3}\mathrm{Var}(d_e)$$

$$\mathrm{Var}(h)|_{d_e} = \frac{K_2^2 h^2}{(K_1 h + K_2 d_e)^2}\mathrm{Var}(d_e)$$

or, in dimensionless form:

$$\frac{E(h)|_{d_e} - h}{h} = \frac{(\frac{K_2}{K_1})^2(\frac{d_e}{h})^2[1 + 2(\frac{K_2}{K_1})(\frac{d_e}{h})]}{2[1 + (\frac{K_2}{K_1})(\frac{d_e}{h})]^3}\frac{\mathrm{Var}(d_e)}{d_e^2} \tag{5.94}$$

$$\frac{\mathrm{Var}(h)|_{d_e}}{h^2} = \frac{(\frac{K_2}{K_1})^2(\frac{d_e}{h})^2}{[1 + (\frac{K_2}{K_1})(\frac{d_e}{h})]^2}\frac{\mathrm{Var}(d_e)}{d_e^2} \tag{5.95}$$

Figure 5.33 and 5.34 show that the uncertainty of d_e becomes more significant to both the mean and the variance of h as $\frac{K_2}{K_1}$ and/or $\frac{d_e}{h}$ increase. This is because the flow region below the drains becomes more important when $\frac{K_2}{K_1}$ and/or $\frac{d_e}{h}$ becomes larger. Equations 5.94 and 5.95 are exactly in the same form as equations 5.86 and 5.87, respectively. This again shows that the thickness of the equivalent layer is as important as the hydraulic conductivity of this layer to the drainage design.

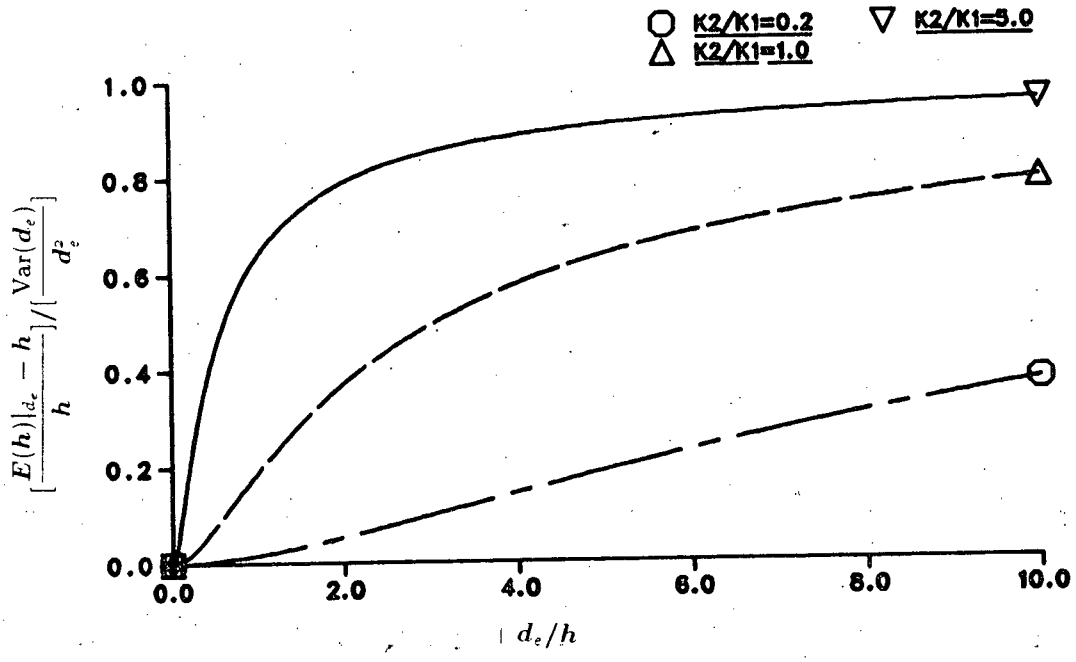

Figure 5.33: Expected water table height due to uncertainty of equivalent depth.

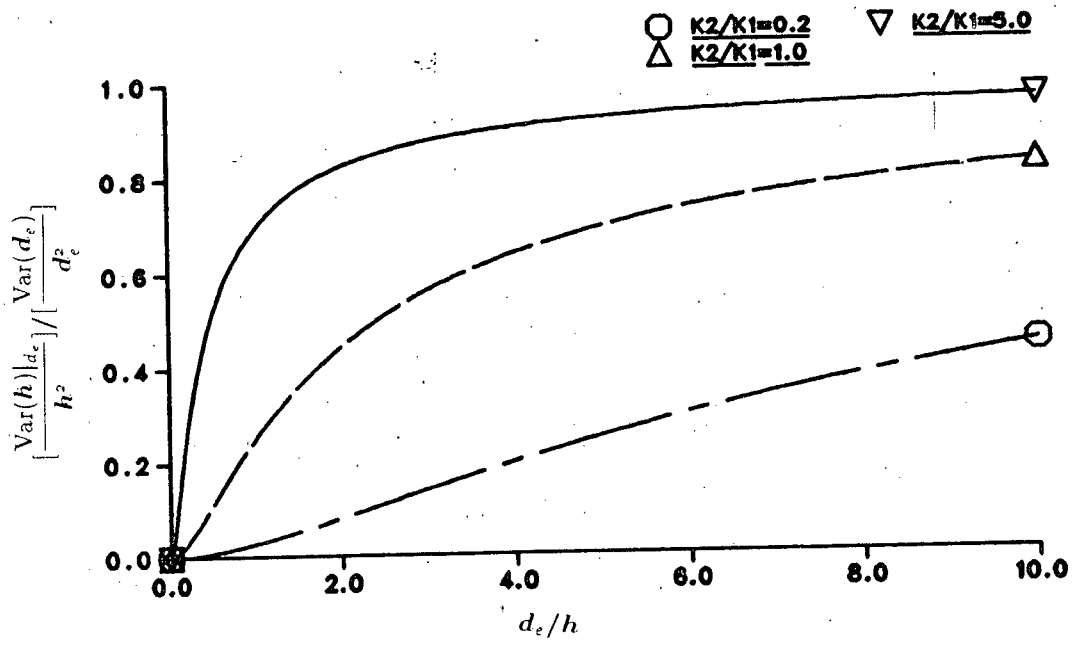

Figure 5.34: Variance of water table height due to uncertainty of equivalent depth.

5.4.8 All or Part of Parameters as Random Variables with Known Means, Variances and Covariances

From section 5.1, we have:

$$E(h) = h + \frac{1}{2}\sum_{j=1}^{4}(\vec{B}_{2j}^{T}\vec{C}_j)$$

$$\mathrm{Var}(h) = \vec{B}_1^T\ C\ \vec{B}_1$$

where

$$\vec{B}_1 = \begin{pmatrix} \frac{\partial h}{\partial R} \\ \frac{\partial h}{\partial K_1} \\ \frac{\partial h}{\partial K_2} \\ \frac{\partial h}{\partial d_e} \end{pmatrix}$$

$$B_2^T = \begin{pmatrix} \frac{\partial^2 h}{\partial R^2} & \frac{\partial^2 h}{\partial K_1 \partial R} & \frac{\partial^2 h}{\partial K_2 \partial R} & \frac{\partial^2 h}{\partial d_e \partial R} \\ \frac{\partial^2 h}{\partial R \partial K_1} & \frac{\partial^2 h}{\partial K_1^2} & \frac{\partial^2 h}{\partial K_2 \partial K_1} & \frac{\partial^2 h}{\partial d_e \partial K_1} \\ \frac{\partial^2 h}{\partial R \partial K_2} & \frac{\partial^2 h}{\partial K_1 \partial K_2} & \frac{\partial^2 h}{\partial K_2^2} & \frac{\partial^2 h}{\partial d_e \partial K_2} \\ \frac{\partial^2 h}{\partial R \partial d_e} & \frac{\partial^2 h}{\partial K_1 \partial d_e} & \frac{\partial^2 h}{\partial K_2 \partial d_e} & \frac{\partial^2 h}{\partial d_e^2} \end{pmatrix}$$

$$C = \begin{pmatrix} \mathrm{Var}(R) & \mathrm{Cov}(K_1, R) & 0 & 0 \\ \mathrm{Cov}(R, K_1) & \mathrm{Var}(K_1) & \mathrm{Cov}(K_2, K_1) & 0 \\ 0 & \mathrm{Cov}(K_1, K_2) & \mathrm{Var}(K_2) & 0 \\ 0 & 0 & 0 & \mathrm{Var}(d_e) \end{pmatrix}$$

Rearranging the above equations gives:

$$\begin{aligned} E(h) \;=\; & h + \frac{1}{2}[\frac{\partial^2 h}{\partial R^2}\mathrm{Var}(R) + \frac{\partial^2 h}{\partial K_1^2}\mathrm{Var}(K_1) + \frac{\partial^2 h}{\partial K_2^2}\mathrm{Var}(K_2) + \\ & \frac{\partial^2 h}{\partial d_e^2}\mathrm{Var}(d_e) + 2\frac{\partial^2 h}{\partial R \partial K_1}\mathrm{Cov}(K_1, R) + \\ & 2\frac{\partial^2 h}{\partial R \partial K_2}\mathrm{Cov}(K_2, R) + 2\frac{\partial^2 h}{\partial K_2 \partial K_1}\mathrm{Cov}(K_1, K_2) \end{aligned}$$

$$\begin{aligned}\mathrm{Var}(h) &= (\frac{\partial h}{\partial R})^2\mathrm{Var}(R) + (\frac{\partial h}{\partial K_1})^2\mathrm{Var}(K_1) + (\frac{\partial h}{\partial K_2})^2\mathrm{Var}(K_2) + \\ &(\frac{\partial h}{\partial d_e})^2\mathrm{Var}(d_e) + 2(\frac{\partial h}{\partial R})(\frac{\partial h}{\partial K_1})\mathrm{Cov}(K_1, R) + \\ &2(\frac{\partial h}{\partial R})(\frac{\partial h}{\partial K_2})\mathrm{Cov}(K_2, R) + 2(\frac{\partial h}{\partial K_1})(\frac{\partial h}{\partial K_2})\mathrm{Cov}(K_1, K_2)\end{aligned}$$

Using the results of the previous analyses in this section, we can write the above equations as:

$$\begin{aligned}\frac{E(h)-h}{h} &= \frac{E(h)|_R - h}{h} + \frac{E(h)|_{K_1} - h}{h} + \frac{E(h)|_{K_2} - h}{h} + \frac{E(h)|_{d_e} - h}{h} + \\ &\frac{E(h)|_{(K_1\times R)} - h}{h} + \frac{E(h)|_{(K_2\times R)} - h}{h} + \frac{E(h)|_{(K_1\times K_2)} - h}{h}\end{aligned} \tag{5.96}$$

$$\begin{aligned}\frac{\mathrm{Var}(h)}{h^2} &= \frac{\mathrm{Var}(h)|_R}{h^2} + \frac{\mathrm{Var}(h)|_{K_1}}{h^2} + \frac{\mathrm{Var}(h)|_{K_2}}{h^2} + \frac{\mathrm{Var}(h)|_{d_e}}{h^2} + \\ &\frac{\mathrm{Var}(h)|_{(K_1\times R)}}{h^2} + \frac{\mathrm{Var}(h)|_{(K_2\times R)}}{h^2} + \frac{\mathrm{Var}(h)|_{(K_1\times K_2)}}{h^2}\end{aligned} \tag{5.97}$$

in which various terms are evaluated from equation 5.82 through 5.95.

5.5 An Example Showing How Uncertainty Analysis Can Be Used in Drainage Design

In this section, an example of uncertainty analysis of drainage design in homogeneous soils by using first and second order analysis of parameter uncertainty is presented. The input information of each parameter is arbitrarily made up but this does not affect the usefulness of the analyses. Appendix B briefly reviews the methods for estimating mean, variance and covariance from limited field data. Then an effort is made to show how the uncertainty analysis can be applied to practical drainage design.

5.5.1 An Example

The values of parameters used as input information are as follows:

Design water table height at middle point between the drains: h=0.6 m;

Hydraulic conductivity: mean K=0.5 m/day, variance Var(K)=0.05 (m/day)^2;

Drainage coefficient: mean R=0.01 m/day, variance Var(R)=1.5 $\times$ 10^{-5} (m/day)^2;

Correlation coefficient of drainage coefficient and hydraulic conductivity: $\rho(K, R)$ = 0.3;

Depth to the impermeable layer: mean d=5 m, variance Var(d)=1.5 m^2;

Drain radius: r=0.1 m.

From the Hooghoudt equation by the deterministic design concept, the drain spacing needed is S=24.0 m, the equivalent depth d_e=2.09 m. Using equations 5.58 through 5.66, we can obtain the following results:

$$\frac{E(h)|_R - h}{h} = -0.013$$

$$\frac{\mathrm{Var}(h)|_R}{h^2} = 0.118$$

$$\frac{E(h)|_K - h}{h} = 0.160$$

$$\frac{\mathrm{Var}(h)|_K}{h^2} = 0.158$$

$$\frac{E(h)|_{K\times R} - h}{h} = -0.023$$

$$\frac{\mathrm{Var}(h)|_{K\times R}}{h^2} = -0.082$$

$$\frac{\mathrm{Var}(d_e)|_d}{d_e^2} = 0.002$$

$$\frac{E(h)|_{d_e} - h}{h} = 0.001$$

$$\frac{\mathrm{Var}(h)|_{d_e}}{h^2} = 0.001$$

Then we can calculate the mean and the variance of h by Equation 5.67 and 5.68 as:

$$\frac{E(h)-h}{h} = -0.013 + 0.160 - 0.023 + 0.001 = 0.125$$

$$\frac{\mathrm{Var}(h)}{h^2} = 0.118 + 0.158 - 0.082 + 0.001 = 0.195$$

i.e.

$$E(h) = 0.675 \text{ m}$$

$$\mathrm{Var}(h) = 0.070 \text{ m}^2$$

The standard deviation of h from the mean is:

$$SD = 0.265 \text{ m}$$

The variance or standard deviation of water table height may be considered as the risk involved in the drainage system design due to uncertainties about the nature of the problem. If we assume a normal distribution of the water table height, it implies that there is a 90% probability that h would not exceed 1.013 m (mean+1.28SD) and a 90% probability that h would exceed 0.337 m (mean−1.28SD), or there is an 80% probability that h is between 0.337 m and 1.013 m if the drainage system is designed based on the deterministic concept.

5.5.2 Application of Uncertainty Analysis in Drainage Design

From the above example, it should be noticed that the mean water table height is actually larger than the design water table height (0.675 m and 0.600 m, respectively), which means the drain spacing based on the deterministic model (the Hooghoudt equation) is inadquate in removing the excess water before damage occurs due to the variability of soil and climatic conditions. The higher water table may result in a longer waterlogging

period and thus reduce the crop yield. Through uncertainty analysis, we can correspondingly adjust the drain spacing so that the drainage system can perform better. On the other hand, even if the mean water table height meet the requirement, there is still a considerable probability that the water table will be higher than required due to the uncertainty of parametes, which may be regarded as a measure of the risk involved in selecting that design.

From the results obtained previously, the increase of h is:

$$\Delta h = E(h) - h = 0.675 - 0.6 = 0.075 \text{ m}$$

In order to lower the mean water table height, h_{new} is used in design ($h_{\text{old}} = h$):

$$h_{\text{new}} = h_{\text{old}} - \Delta h = 0.6 - 0.075 = 0.525 \text{ m}$$

Repeating the procedure in the above example, we have: spacing S=21.64 m, equivalent depth d_e=1.97, and

$$E(h) = 0.592 \text{ m}$$

$$\text{Var}(h) = 0.055 \text{ m}^2$$

Standard deviation SD=0.234 m.

This time the mean water table is close to the required height (0.6 m) (if still not the same procedure may be repeated). If a normal distribution of the water table height is assumed, the probability that the water table would not exceed certain heights are summarized in table 5.5.

Table 5.5 will give the farmer an idea on approximately how much the risk is involved in selecting the design for the particular field. The final spacing can be adjusted to be wider or narrower, depending on the risk level that the owner of the field would like to accept.

Table 5.5: Probability that water table would not exceed certain height.

Mean	0.592 m
Standard Deviation	0.234 m
Height (m)	Probability
0.592	50%
0.710	69%
0.826	84%
0.892	90%
0.943	93%
1.060	98%

Chapter 6

CONCLUSIONS AND RECOMMENDATIONS

This chapter summarizes the study that has been carried out on sensitivity and uncertainty analysis in the design of a subsurface drainage system. Conclusions will be drawn based on the study and recommendations will be made where appropriate for further research work in this or related aspects.

The study started from the simplest subsurface drainage situation with homogeneous soils and steady state criteria. Sensitivity analysis on drain spacing to other design parameters was performed on the most popularly used steady state drainage theory, the Hooghoudt equation, for homogeneous soils to identify the dominant variable or variables among all drainage parameters. Then several transient state spacing formulae were reviewed and compared and the van Schilfgaarde solution of the Boussinesq equation was chosen for the purpose of sensitivity analysis. A comparison was made between sensitivities of spacing in steady state and transient state to corresponding parameters through an example. The Hooghoudt equation and the generalized Hooghoudt–Ernst equation for two– layered soils with the drains at the interface of the layers were analysed and their results compared. A new spacing formula was developed for the situations of two–layered soils with different hydraulic conductivities below the drains. Sensitivity analysis was carried out based on the proposed equation. The spacing and its sensitivities obtained from this new spacing formula were compared with those obtained from the Kirkham equation for multi–layered soils. The first and second moment method was applied to the analysis of parameter uncertainty involved in the subsurface drainage

system design of steady and non–steady state in homogeneous and layered soils. An example was given to illustrate the procedure of the analysis and its application in the practical drainage design. A brief discussion on the methods of estimating the mean and the variance from limited field data of hydraulic conductivity measurements is given in Appendix B.

6.1 Conlusions

1. Sensitivity analysis can be employed to identify the dominant variables in drainage design. The results obtained can be used as guidances in the geohydrological investigations and the determination of parameter inputs in the design of a subsurface drainage system.

2. Drain spacing is very sensitive to the design midspan water table height, hydraulic conductivity, and drainage coefficient. Special care should be taken in the selection or measurement of these parameters for designing a cost– effective drainage system.

3. Drain spacing is not sensitive to the depth of the impermeable layer below drains and drain radius. Therefore high accuracy in the measurement of the depth to the impermeable layer is not desired. This is because most of the flow towards drains will take place in the region near the drains. In the selection of the drain size, the flow capacity should be the major factor to be considered. From the point of view of saving material, the drain size should be as small as possible while meets the requirement of flow capacity.

4. In the case of transient state drainage design, the sensitivities of spacing to the hydraulic conductivity, the depth of the impermeable layer below the drains, and the drain radius are very similar to those of steady state drainage theory.

5. For transient state drainage design, the spacing is very sensitive to the initial water table height and the water table height after falling for a given period of time, particularly when these two positions are very close to each other.

6. Transient drainage design (falling water table criterion) can be reduced to the simpler steady state drainage design by appropriately adjusting the R and h values. Certain combinations of R–h values in steady state design will result in close or the same spacing as obtained from transient state drainage design methods.

7. It is found that that when the water table fluctuation is small, steady state drainage theory , such as the Hooghoudt equation is recommended to use instead of transient drainage theory, such as the van Schilfgaarde equation. In these situations, steady state spacing formulae tend to provide more stable and therefore more reliable results than transient state spacing equations.

8. If transient drainage theory is used, the initial and final water table positions should be carefully selected, otherwise the drain system may be over or under designed.

9. Drain spacing is usually more sensitive to the hydraulic conductivity of the soil below the drains than that above the drains. Therefore, the hydraulic conductivity should be measured from soils deeper than the drain depth.

10. The Hooghoudt equation and the generalized Hooghoudt–Ernst equation give spacings and sensitivities which are very close to each other for different soil layering conditions.

11. A new spacing formula is developed for the situations where there are two soil layers below the drains, by combining the approach and the form of the Hooghoudt equation and the radial flow component of the Ernst equation.

12. This new spacing formula yields spacings close to that obtained from the Kirkham equation of potential theory when the hydraulic conductivity of the soil above the drains is relatively small, and gives more accurate results when there is a highly pervious layer above the drains. In this case, use of the new spacing formula is recommended.

13. Drain spacing is not sensitive to the hydraulic conductivity of the lower layer if there are two soil layers below the drain level. Therefore it is not desirable to stratify the soil below drains into more than two layers.

14. First and second moment analysis can be used to incorporate the parameter uncertainty into the design of a subsurface drainage system. Effects of uncertainty in each parameter on the performance of a drainage system can be estimated from the first and second moment analysis.

15. The actual mean water table height is usually different from the design water table height due to the variability of climate and soil parameters, if the design is based on the conventional deterministic framework.

16. Simple equations and graphs are prepared to encourage the use of uncertainty analysis among the community of practicing agricultural engineers and designers.

17. Drain spacing can be adjusted according to the risk level the farmer is willing to accept and the funds available for the drainage system. The probabilities that the water table would exceed certain heights can be estimated from the uncertainty analysis.

6.2 Recommendations

1. The correlationships between various drainage parameters should be studied to

increase the accuracy of sensitivity and uncertainty analysis.

2. The methods of estimating the means and the variances from limited historic data or field measurements should be studied in more detail. The accuracy of these estimates directly affects the design of a drainage system.

3. Other sources of uncertainty such as the effects of unsaturated flows in the region above water table should be studied and included in the analysis.

4. The correctness and limitation of the first and second moment analysis should be tested by experiments or studies of existing drainage systems. The recharge and soil properties should be better determined and other sources of uncertainty be eliminated or reduced. If the coefficients of variation of parameters are relatively small, the first and second moment analysis should provide very good approximation.

5. The effects of high water table on the crop yield/growth should be studied in more detail to relate the uncertainty in water table position with the uncertainty in the monetary return from agricultural drainage system so that farmers can have more direct criteria to accept or reject a drainage system design.

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Appendix A

DISCUSSION ON THE SIMPLIFICATION OF THE $d_e - S$ RELATIONSHIP

Let's consider the Hooghoudt equation for homogeneous soils. It can be written as

$$S = \sqrt{\frac{4K}{R}(h^2 + 2d_e h)} = F(R, K, h, d_e)$$

where R is the drainage coefficient, K is the hydraulic conductivity, h is the water table height above drains at mid–spacing, and d_e is the equivalent depth.

Equivalent depth, d_e, is a function of the spacing, S, the depth to the impermeable layer, d, and the drain radius, r:

$$d_e = f(S, d, r)$$

The partial derivative of S with respect to R, in a strict mathematic sense, is:

$$\frac{\partial S}{\partial R} = \frac{\partial F}{\partial R} + \frac{\partial F}{\partial d_e}\frac{\partial d_e}{\partial S}\frac{\partial S}{\partial R}$$

i.e.

$$\frac{\partial S}{\partial R} = \frac{\frac{\partial F}{\partial R}}{1 - \frac{\partial F}{\partial d_e}\frac{\partial d_e}{\partial S}} \tag{A.98}$$

Similarly we have

$$\frac{\partial S}{\partial K} = \frac{\frac{\partial F}{\partial K}}{1 - \frac{\partial F}{\partial d_e}\frac{\partial d_e}{\partial S}} \tag{A.99}$$

$$\frac{\partial S}{\partial h} = \frac{\frac{\partial F}{\partial h}}{1 - \frac{\partial F}{\partial d_e}\frac{\partial d_e}{\partial S}} \tag{A.100}$$

$$\frac{\partial S}{\partial d} = \frac{\frac{\partial F}{\partial d_e}\frac{\partial d_e}{\partial d}}{1 - \frac{\partial F}{\partial d_e}\frac{\partial d_e}{\partial S}} \tag{A.101}$$

$$\frac{\partial S}{\partial r} = \frac{\frac{\partial F}{\partial d_e}\frac{\partial d_e}{\partial r}}{1 - \frac{\partial F}{\partial d_e}\frac{\partial d_e}{\partial S}} \tag{A.102}$$

However, in section 3.1, it was assumed that d_e is independent of S to simplify the analysis. Therefore

$$\frac{\partial d_e}{\partial S} = 0$$

and the above equations become

$$\frac{\partial S}{\partial R} = \frac{\partial F}{\partial R} \tag{A.103}$$

$$\frac{\partial S}{\partial K} = \frac{\partial F}{\partial K} \tag{A.104}$$

$$\frac{\partial S}{\partial h} = \frac{\partial F}{\partial h} \tag{A.105}$$

$$\frac{\partial S}{\partial d} = \frac{\partial F}{\partial d_e}\frac{\partial d_e}{\partial d} \tag{A.106}$$

$$\frac{\partial S}{\partial r} = \frac{\partial F}{\partial d_e}\frac{\partial d_e}{\partial r} \tag{A.107}$$

which were used in section 3.1.

It is noted that in Equation A.98 through A.102, the denominators are all the same $(1 - \frac{\partial F}{\partial d_e}\frac{\partial d_e}{\partial S})$, and that Equation A.103 through A.107 are obtained by letting

$$1 - \frac{\partial F}{\partial d_e}\frac{\partial d_e}{\partial S} = 1$$

This simplification will not change their relative order of magnitude, but only their absolute values. Therefore the relative sensitivities of spacing to these parameters will not affected by such simplification. Furthermore conclusions based on the sensitivity analysis will also not affected by such simplification.

Same analysis and conclusion are valid in cases of van Schilfgaarde equation for transient state drainage and Hooghoudt equation for steady state drainage in two–layered soils. For a closer estimation of the spacing change due to changes of other parameters, a coefficient C should be multiplied to the results obtained from the simplified approach. This coefficient equals

$$C = 1/(1 - \frac{\partial F}{\partial d_e}\frac{\partial d_e}{\partial S})$$

Table A.6 is a comparison of results of sensitivities obtained from the simplified approach and the unsimplified one for homogeneous soil. The parameter values used for the computation are:

Case 1: $R = 0.01$ m/day, $K = 0.5$ m/day, $h = 0.6$ m, $d = 5.0$ m, $r = 0.05$ m, and

Case 2: $R = 0.007$ m/day, $K = 1.0$ m/day, $h = 0.6$ m, $d = 3.0$ m, $r = 0.05$ m.

All parameters are assumed to change 5%. Column (1) is the results obtained from the simplified approach; (2) is the results obtained from unsimplified approach; and (3) is the results directly calculated from Hooghoudt equation. It can be seen that the results in column (2) are very close to those in column (3), while there are some differences between results in column (1) and column (3). however, the relative order of sensitivities of spacing is same in all three columns, which justifies the simplification because the primary purpose of sensitivity analysis is to identify the dominant parameters.

	Case 1			Case 2		
Spacing	21.98			39.34		
(m)	(1)	(2)	(3)	(1)	(2)	(3)
R	-2.50	-3.50	-3.36	-2.50	-2.95	-2.84
K	2.50	3.50	3.46	2.50	2.95	2.92
h	2.87	4.02	3.99	2.83	3.34	3.31
d	0.28	0.39	0.36	1.13	1.33	1.30
r	0.42	0.59	0.58	0.27	0.32	0.32

Table A.6: Comparison of sensitivities obtained from different approaches.

Appendix B

ESTIMATION OF MEANS AND VARIANCES OF RANDOM PARAMETERS

In chapter 5, the means and variances of parameters to be used in the uncertainty analysis are assumed already known. However, how to estimate these values from limited data still remains a problem to be solved. While the true mean and variance of a parameter is usually unknown, a reasonable estimation can be obtained from statistical methods with given information about the parameter, although the information is usually limited. In this appendix, some methods to estimate mean and variance of a random parameter will be discussed, and then used in the estimation of the mean and variance of soil hydraulic conductivity.

B.1 Methods to Estimate Mean and Variance of Random Variable

A best estimate of a parameter (mean or variance) should have the properties of unbiasedness, minimum variance, consistency, and sufficiency. The objective in parameter estimation is to determine a statistic

$$\hat{\Theta} = h(X_1, X_2, \ldots, X_n)$$

which gives a good estimate of the parameter θ. A statistic $\hat{\Theta}$ is said to be an unbiased estimator for θ if

$$E[\hat{\Theta}] = \theta$$

for all θ. A statistic is said to be an efficient estimator for θ if the variance of the statistic is minimum among all unbiased estimators. A statistic is said to be a consistent estimator for θ if, as sample size n increases:

$$\lim_{n\to\infty} P[|\hat{\Theta} - \theta| \geq \epsilon] = 0$$

for all $\epsilon > 0$. The definition for sufficiency states that if $\hat{\Theta}$ is a sufficient statistic for θ, all the sample information concerning θ is contained in $\hat{\Theta}$. Exact definitions for these criteria can be found in most statistics books (Miller and Freund, 1977; Larsen and Marx, 1981; Soong, 1981; Ross, 1987).

B.1.1 The Method of Moments

The method of moments is simple in concept. Consider a selected probability density function $f(x;\theta_1,\ldots,\theta_m)$ whose parameters $\theta_j, j = 1,2,\ldots,m$, are to be estimated based on the sample $X_1,\ldots,X_n$ of X. Theoretical or population moments of X are

$$\alpha_i = \int_{-\infty}^{\infty} x^i f(x;\theta_1,\ldots,\theta_m)dx \quad i = 1,2,\ldots \tag{B.108}$$

On the other hand, sample moments of various orders can be found from the sample by:

$$M_i = \frac{1}{n}\sum_{j=1}^{n} X_j^i \quad i = 1,2,\ldots \tag{B.109}$$

The method of moments suggests that, in order to determine the estimators $\hat{\Theta}_1$, $\hat{\Theta}_2$, ..., $\hat{\Theta}_m$ from the sample, we equate a sufficient number of the sample moments to the corresponding population equations (lower–order moment equations are preferred). Estimators can be obtained from solving the resulting moment equations

$$\alpha_i = M_i \quad i = 1,2,\ldots,m$$

The method of moments is straitforward, and the moment equations are seldom difficult to solve. However, this method can not guarantee such desirable properties as unbiasedness and minimum variance (efficiency) for estimators so obtained.

B.1.2 The Method of Maximum Likelihood

Let $f(x;\theta_1,\ldots,\theta_m)$ be the probability density function of population X where $\theta_1,\ldots,\theta_m$ are the parameters to be estimated from a set of sample values $x_1, x_2, \ldots, x_n$. The joint density function of the corresponding sample $X_1, X_2, \ldots, X_n$ has the form

$$f(x_1;\theta_1,\ldots,\theta_m)f(x_2;\theta_1,\ldots,\theta_m)\cdots f(x_n;\theta_1,\ldots,\theta_m)$$

The likelihood function L of a set of n sample values from the population is defined by

$$L(x_1,\ldots,x_n;\theta_1,\ldots,\theta_m) = \prod_{i=1}^{n} f(x_i;\theta_1,\ldots,\theta_m) \tag{B.110}$$

When the sample are given, the likelihood function L becomes a function of variables $\theta_1,\ldots,\theta_m$. The concept of the method of maximum likelihood consists of choosing, as estimates of $\theta_j, j = 1,2,\ldots,m$, the values of $\theta_j, j = 1,2,\ldots,m$, that maximizes the likelihood function L. Hence, the maximum likelihood estimates $\hat{\theta}_j$ of $\theta_j, j = 1,2,\ldots,m$, based on sample values $x_1, x_2, \ldots, x_n$ can be determined from

$$\frac{\partial L}{\partial \hat{\theta}} = 0 \quad j = 1,2,\ldots,m$$

or

$$\frac{\partial \ln L}{\partial \hat{\theta}} = 0 \quad j = 1,2,\ldots,m$$

B.1.3 Bayesian Estimation of Mean

Bayesian methods of estimation have received impetus and much wider applicability in recent years through the concept of personal, or subjective, probability. The idea of Bayesian methods is to regard an unknown parameter θ to be estimated being the value of a random variable from a given probability distribution. This usually arises when, prior to the observance of the outcomes of the data $X_1, X_2, \ldots, X_n$, we have some information about the value of θ and this information is expressible in terms of a probability distribution, which is referred as the prior distribution of θ.

Suppose that based on the knowledge and experiences, we have prior feelings about θ that it can b regarded as being the value of a continuous random variable having probability density function $f(\theta)$; and suppose that we obtained a set of sample values $X_i = x_i, i = 1, 2, \ldots, n$, then the updated, or conditional, probability density function of θ is as follows:

$$f(\theta|x_1, \ldots, x_n) = \frac{f(\theta)f(x_1, \ldots, x_n|\theta)}{\int_{-\infty}^{\infty} f(x_1, \ldots, x_n|\theta)f(\theta)d\theta} \tag{B.111}$$

where the conditional density function $f(\theta|x_1, \ldots, x_n)$ is called the posterior density function; $f(x_1, \ldots, x_n|\theta)$ is the likelihood that data values $X_i = x_i, i = 1, 2, \ldots, n$, when θ is the value of the parameter; and $f(\theta)$ is the prior probability density function.

The best estimate of θ, given the data values $x_1, x_2, \ldots, x_n$, is the mean of the posterior distribution $f(\theta|x_1, \ldots, x_n)$, which is called the Bayes estimator:

$$E[\theta|X_i = x_i, i = 1, \ldots, n] = \int_{-\infty}^{\infty} \theta f(\theta|x_1, \ldots, x_n)d\theta \tag{B.112}$$

As an example, let's consider a sample $X_1, \ldots, X_n$ which are independent normal random variables, each having unknown mean μ and known variance σ^2, and μ itself is a random variable which is normally distributed, with known mean μ_0 and known σ_0^2 (prior

information). In order to determine the Bayes estimator, we need first to determine the conditional density function of μ given the values of $X_i, i = 1, \ldots, n$:

$$f(\mu|x_1, \ldots, x_n) = \frac{f(x_1, \ldots, x_n|\mu)f(\mu)}{f(x_1, \ldots, x_n)}$$

where

$$f(x_1, \ldots, x_n|\mu) = (\frac{1}{\sqrt{2\pi}\sigma})^n e^{-\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma^2}}$$

$$f(\mu) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(\mu-\mu_0)^2}{2\sigma_0^2}}$$

and

$$f(x_1, \ldots, x_n) = \int_{-\infty}^{\infty} f(x_1, \ldots, x_n|\mu)f(\mu)d\mu$$

Working out the above various terms, we obtained the Bayes estimator:

$$\mu_1 = E[\mu|X_1, \ldots, X_n] = \frac{n\sigma_0^2}{n\sigma_0^2 + \sigma^2}\overline{X} + \frac{\sigma^2}{n\sigma_0^2 + \sigma^2}\mu_0 \tag{B.113}$$

which is a weighted average of $\overline{X}$, the sample mean, and μ_0, the a priori mean. The variance of population σ^2 can be estimated from the sample variance. The variance of the population mean can be calculated by

$$\sigma_1 = \sqrt{\frac{\sigma^2\sigma_0^2}{n\sigma_0^2 + \sigma^2}} \tag{B.114}$$

B.1.4 Confidence Interval for Mean and Variance in Normal Distribution

The variance of the population σ^2 is usually not known. However, it can be reasonably estimated by the sample variance S^2. For a sample size n which is large enough to approximate σ with S, the random variable

$$Y = \frac{\overline{X} - \mu}{S/\sqrt{n}}$$

is approximately a unit normal distribution. The interval

$$\overline{X} - Z_{\alpha/2}\frac{S}{\sqrt{n}} < \mu < \overline{X} + Z_{\alpha/2}\frac{S}{\sqrt{n}} \tag{B.115}$$

is an approximate confidence interval for μ having the degree of confidence $(1-\alpha)$. In the above equation $\overline{X}$ is the sample mean, S is the sample standard deviation.

If the sample size n is not large enough, it can be proved that random variable

$$Y = \frac{\overline{X} - \mu}{S/\sqrt{n}}$$

has a t–distribution with $(n-1)$ degrees of freedom. Therefore the interval

$$\overline{X} - t_{n-1,\alpha/2}\frac{S}{\sqrt{n}} < \mu < \overline{X} + t_{n-1,\alpha/2}\frac{S}{\sqrt{n}} \tag{B.116}$$

is a confidence interval for μ having the degree of confidence $(1-\alpha)$.

As mentioned previously, an unbiased point estimator for the population variance σ^2 is the sample variance S^2. For the construction of confidence interval for σ^2, let's introduce the random variable

$$D = \frac{(n-1)S^2}{\sigma^2}$$

It can be proved that D has a χ^2 distribution with $(n-1)$ degrees of freedom. The interval

$$\frac{(n-1)S^2}{\chi^2_{n-1,\alpha/2}} < \sigma^2 < \frac{(n-1)S^2}{\chi^2_{n-1,1-\alpha/2}} \tag{B.117}$$

is a confidence interval for σ^2 having the degree of confidence $(1-\alpha)$.

B.2 Estimation of the Mean and Variance of Hydraulic Conductivity

Hydraulic conductivity is defined by Darcy's law:

$$K = -\frac{Q}{A}\frac{dh}{dx}$$

where

A = area perpendicular to flow direction,

Q = flux through area A,

$\frac{dh}{dx}$ = hydraulic gradient.

From the above definition, hydraulic conductivity is not a point value in a strict sense. It is an average value over a volume of soil ($A \bullet dx$). However, in many agricultural investigations the size of the soil sample used to measure the hydraulic conductivity is so much smaller than the scale of the flow field that the sampled value is essentially a point value, and the hydraulic conductivity can be treated as continuously distributed over the field.

Strictly speaking, hydraulic conductivities do not vary in space in a purely random, unstructured manner. Neighboring values are rather correlated. The scale of correlation can be represented by a parameter called correlation length, which is defined as the distance over which the correlation is positive. However, the distance between two measurements in agricultural investigation is usually too large to consider any meaningful correlation between the measured values of hydraulic conductivity. That is to say that the sample values of hydraulic conductivity can be considered as independent.

There is a fairly extensive literature that suggests that hydraulic conductivities are often lognormally distributed (Massmann, 1987). Suppose we have a sample of size n with n observed values, $K_1, \ldots, K_n$, then $\ln K_1, \ldots, \ln K_n$ are normally distributed. Let

$$x_j = \ln K_j \quad j = 1, \ldots, n$$

and $\overline{X}$ be the sample mean, S the sample standard deviation, $\mu_{\ln K}$ the population mean (to be estimated) and $\sigma^2_{\ln K}$ the population variance.

Now let's estimate the population mean $\mu_{\ln K}$ by the method of moments. $X_j, j = 1, \ldots, n$ are random variables from a normal distribution with mean of $\mu_{\ln K}$ and variance of $\sigma^2_{\ln K}$, which can be estimated from the sample variance S^2. The probability density

function with unknown $\mu_{\ln K}$ is

$$f(x;\mu_{\ln K}) = \frac{1}{\sqrt{2\pi}\sigma_{\ln K}} e^{-\frac{(x-\mu_{\ln K})^2}{2\sigma^2_{\ln K}}}$$

and the first moment of the distribution

$$\alpha_1 = \int_{-\infty}^{\infty} x f(x;\mu_{\ln K}) dx = \mu_{\ln K}$$

On the other hand, the first sample moment is

$$M_1 = \frac{1}{n}\sum_{j=1}^{n} x_j$$

Equating these two moments gives

$$\mu_{\ln K} = \frac{1}{n}\sum_{j=1}^{n} x_j \tag{B.118}$$

The above result can also be obtained from the method of maximum likelihood. The likelihood function L is

$$L(x_1,\ldots,x_n;\mu_{\ln K}) = \prod_{j=1}^{n} f(x_j;\mu_{\ln K}) = (\frac{1}{\sqrt{2\pi}\sigma_{\ln K}})^n e^{-\frac{\sum_{j=1}^{n}(x_j-\mu_{\ln K})^2}{2\sigma^2_{\ln K}}}$$

The first derivative of L to $\mu_{\ln K}$ is

$$\frac{dL}{d\mu_{\ln K}} = (\frac{1}{\sqrt{2\pi}\sigma_{\ln K}})^n \frac{\sum_{j=1}^{n}(x_j-\mu_{\ln K})}{\sigma^2_{\ln K}} e^{-\frac{\sum_{j=1}^{n}(x_j-\mu_{\ln K})^2}{2\sigma^2_{\ln K}}}$$

Let

$$\frac{dL}{d\mu_{\ln K}} = 0$$

then

$$\sum_{j=1}^{n}(x_j - \mu_{\ln K}) = 0$$

or

$$\mu_{\ln K} = \frac{1}{n}\sum_{j=1}^{n} x_j$$

Recall that $x_j = \ln K_j$, $\mu_{\ln K}$ is the mean of the logarithm of hydraulic conductivity. Because of the symmetry of the normal distribution, $\mu_{\ln K}$ is also the median of random variable $X = \ln K$. Let K_m be the median of the random hydraulic conductivity, then

$$\begin{aligned}\mu_{\ln K} = \ln K_m &= \frac{1}{n}\sum_{j=1}^{n} \ln K_j \\ &= \frac{1}{n} \ln K_1 K_2 \cdots K_n \\ &= \ln \sqrt[n]{K_1 K_2 \cdots K_n}\end{aligned}$$

Therefore the median of hydraulic conductivity is

$$K_m = \sqrt[n]{K_1 K_2 \cdots K_n} \tag{B.119}$$

By definition of median, there is a 50% probability that the value of hydraulic conductivity will less or larger than K_m, respectively.

Bouwer (1969) showed that K_m (which is called geometric mean) gives the best estimate of hydraulic mean value of K in the studies with a resistance network analog using various randomized $K-$distributions in a two–dimensional system.

The variance of hydraulic conductivity can be approximately calculated by:

$$\mathrm{Var}(K) = \frac{\sum_{j=1}^{n}(K_j - K_m)^2}{n-1} \tag{B.120}$$

It should be noted that K_m and $\mathrm{Var}(K)$ are not the mean and variance of hydraulic conductivity in a statistical sense, but rather a practical interpretation of some measurement values. From the probability density function of lognormal random variable, the mean and variance of K are

$$\mu_K = K_m e^{\frac{\sigma^2_{\ln K}}{2}} \tag{B.121}$$

$$\sigma^2_K = \mu^2_K (e^{\sigma^2_{\ln K}} - 1) \tag{B.122}$$

where $\sigma^2_{\ln K}$ is the variance of logarithmic hydraulic conductivity, which can be estimated by:

$$\sigma^2_{\ln K} \approx S^2 = \frac{\sum_{j=1}^{n}(\ln K_j - \ln K_m)^2}{n-1}$$

or

$$\sigma^2_{\ln K} = \frac{\sum_{j=1}^{n} \ln^2(\frac{K_j}{K_m})}{n-1} \tag{B.123}$$

As an illustration, let's assume we have 8 sample measurements of hydraulic conductivity over a field under investigation. These sample values are: $K_1 = 0.4, K_2 = 0.6, K_3 = 0.3, K_4 = 0.7, K_5 = 0.4, K_6 = 0.5, K_7 = 0.8, K_8 = 0.6$ (all units are m/day). Suppose these data values are lognormally distributed. The median value of hydraulic conductivity will be $K_m = 0.51$ m/day. From equation B.123, $\sigma^2_{\ln K} = 0.1079$, and from equations B.121 and B.122, we have $\mu_K = 0.54$ m/day, $\sigma^2_K = 0.033$ (m/day)^2. On the other hand, the variance calculated from equation B.120 is Var(K)=0.0293 (m/day)^2. The differences between K_m and μ_K, σ^2_K and Var(K), are fairly small, thus K_m and Var(K) can be reasonably used as the mean and variance of hydraulic conductivity in the uncertainty analysis.

B.3 Application of Bayes Method in the Estimation of the Mean Hydraulic Conductivity

In this section, an example is given to illustrate how the Bayes method of estimation can be used in drainage design. For the purpose of simplicity, we assume that the hydraulic conductivity has a normal distribution. The mean hydraulic conductivity is unknown and can be treated as random variable which is also normally distributed. Based on the knowledge and experience on the field under investigation, the engineer feels that the hydraulic conductivity of this field can best be described by a normal distribution with a mean of μ_0=0.6 m/day and standard deviation of σ_0=0.08 m/day (prior information).

After investigation, a number of, say 8, random sample values have been obtained from measurements. The sample mean $\overline{X}$=0.54 m/day, the sample standard deviation S = 0.17 m/day. The population standard deviation σ can be approximated with S. Then the updated mean of hydraulic conductivity after some measurements is

$$\mu_1 = \frac{n\sigma_0^2}{n\sigma_0^2 + \sigma^2}\overline{X} + \frac{\sigma^2}{n\sigma_0^2 + \sigma^2}\mu_0 = 0.56 \text{ m/day}$$

and the variance of the mean is

$$\sigma_1 = \sqrt{\frac{\sigma^2\sigma_0^2}{n\sigma_0^2 + \sigma^2}} = 0.05 \text{ m/day}$$

The mean of the posterior distribution is based on the direct evidence as well as prior information.

Bayesian method of estimation can be readily used to estimate the means of precipitation and evapotranspiration. The historic record can provide very good information to assume certain kind of distribution, and the recent events can be used to modify the prior distribution to yield more accurate one.

Appendix C

GRAPH OF GEOMETRY FACTOR a FOR RADIAL FLOW

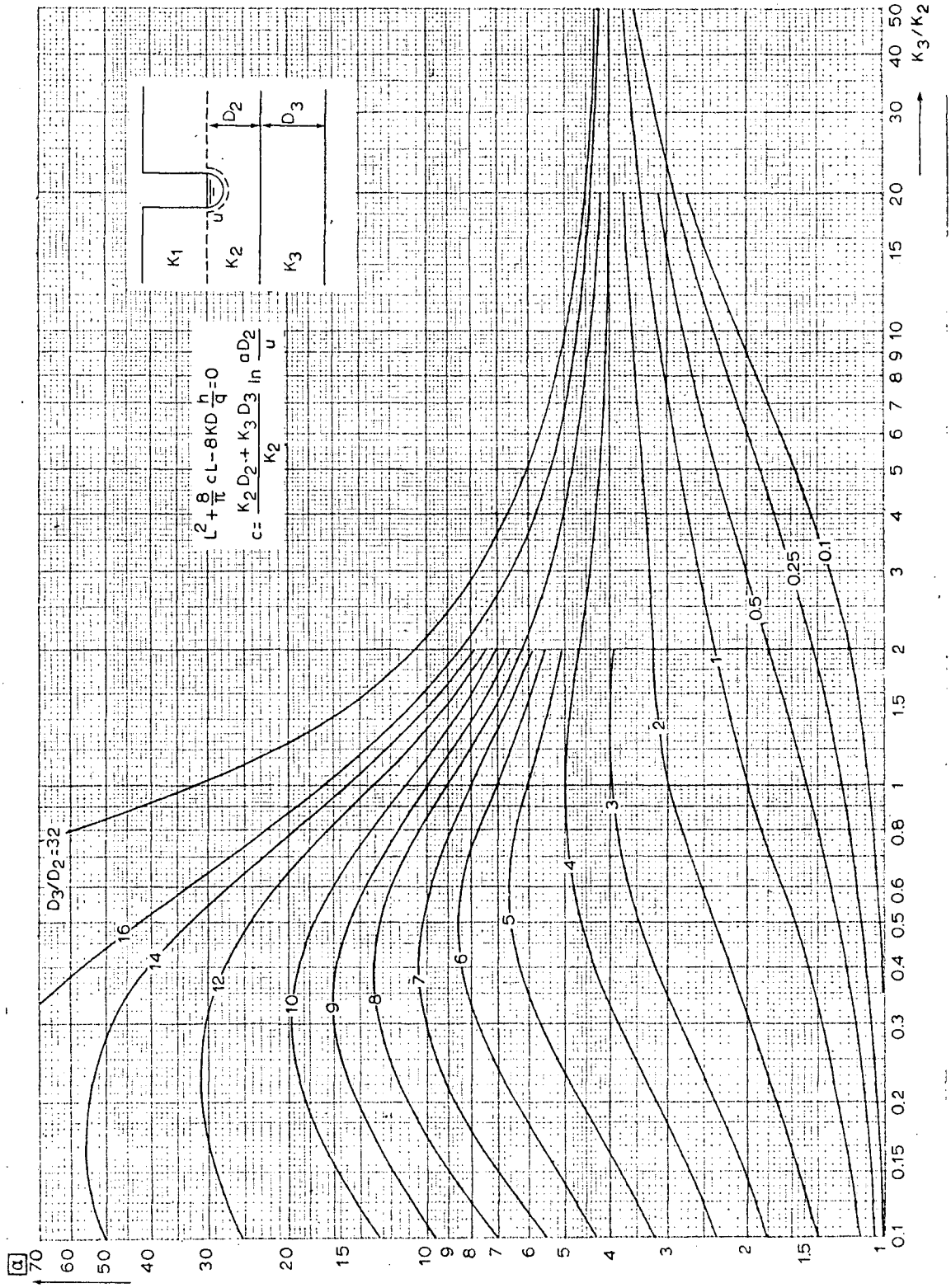

Figure C.35: Geometry factor a for radial flow (taken from van Beers, 1976).