STUDENTS' CONCEPTUAL UNDERSTANDING OF CALCULUS

By

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The purpose of this study was to identify the nature of students' conceptual understanding of two concepts of calculus namely, derivative and function. As a way of collecting data two methods were employed: (a) modification of Piagetean clinical interview; and, (b) tutorial sessions. Whenever the students seemed to be confused about the issues being discussed, the researcher provided instructions through the tutorial sessions.

The analysis of data was done by developing individual profiles and by response categories. It was found that the interview methodology was effective in revealing some aspects of students' concept images. The students were found to have little meaningful understanding of derivative. A number of students held proper concept images of function which should lead to the development of an appropriate concept definition. It was also evident from the study that students had adequate skill in using algorithm to solve problems.

The results of the study would be useful to the instructors of calculus. It was suggested that introducing a concept by its formal definition would contribute to students' confusions and difficulties. Yet if a concept is presented by means of meaningful examples, students had better opportunity to develop their concept images. Thus leading them to form concept definitions. The researcher strongly recommended that more challenging exercises be posed to the students in problem-solving situations.
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CHAPTER ONE

1.1 Background of the Study

"The teaching of calculus is in a state of disarray and near crisis at most American Colleges and Universities. The principle evidence being a failure rate of about 50%" (Rodi, 1985, p. 1). The calculus course has the greatest drop-out rate at the University of British Columbia (U.B.C.) among first year students and yet is required for the greatest number of students. "At U.B.C., only about 55% of students who enroll in first year mathematics successfully complete both Mathematics 100 and 101" (Walsh, p. 30). There has lately been serious discussions about the problems relating to the teaching of calculus courses to try and determine some of the factors which may have led to this failure rate. Many researchers have tried to identify the possible causes, because as Stein (1985) says: "Before we propose the medicine, we had better agree on diagnosis..." (p. 3). Once these possible causes are known, then suggestions may be made to reduce the failure rate. Among the reasons given are: inexperienced instructors (Maures, 1985; Lax, 1985; Davis, 1985), the compactness of the calculus course (Davis, 1985; Epp, 1985), student's poor high school background (Renz, 1985; Rodi, 1985), the lack of awareness of the students' needs and expectations (Ash, 1985; Epp, 1985), poor textbooks (Stein, 1985; Kenelly, 1985; Rodi, 1985; Stevenson, 1985), and conceptual difficulties in students' understanding (Epp, 1985; Orton, 1980, 1983; Vinner, 1982, 1983, 1986). The literature regarding these possible causes will be reviewed in Chapter 2.

Some suggestions for improvement have also been given by various researchers. Many authors proposed that different calculus courses can be developed to meet the special students' needs (Stein, 1985; Epp, 1985;
Rodi, 1985). Other authors discussed teaching methods where: key concepts are the focus of instruction (Lax, 1985; Davis, 1985), an intuitive approach is used to introduce calculus (Renz, 1985; Ash, 1985), a variety of open-ended problems are presented to students (Stein, 1985), application of mathematical concepts are the focus of instruction (Rodi, 1985; Lax, 1985), and the emphasis is on the development of conceptual not just purely mechanical understanding (Epp, 1985).

Since calculus involves a broad understanding of interrelated concepts, rote memorization which leads to a type of mechanical performance in solving 'type' problems does not lend itself to understanding the nature of the relationships between these concepts. The writer's position is that students experience failure in calculus because they have difficulties in understanding both the concepts and the relationships between the key concepts of the calculus course. It does not mean that other factors are not involved in the students' failure. Perhaps the most important one is the students' understanding of the concepts. Vinner (1981, 1983, 1986) has examined some of these difficulties using a model for cognitive processes that involves the notions of "concept image" and "concept definition". The study will draw upon this model to investigate the nature of students' conceptual difficulties of selected concepts of calculus.

1.2 Definition of Terms

In the following, some of the terms that are being used in this study will be presented very briefly. The mathematical definitions, are based on the Adam's Calculus book which is the text for Math 100 and 101 at U.B.C.
Concept Image

According to Vinner (1983), concept image is a set of properties associated with the concept together with a mental picture of that concept.

Concept Definition

Concept definition, as Vinner (1983) explains, is an accurate verbal definition of a concept in a non-circular way.

Tutorial Session

Tutorial session, as opposed to clinical interview (Piaget, 1929), is a situation in which an interviewer intervenes and acts as a teacher. Cobb and Steffe (1983) indicated that teaching activity helps the researcher to explore the student's construction of mathematical knowledge.

Tangent Line (Adams, 1986, p. 52)

Suppose that the function f is continuous at \( x = x_0 \). If

\[
\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = m
\]

exists, then the straight line having slope m and passing through the point \( P = [x_0, f(x_0)] \), that is, the line with equation \( y = f(x_0) + m(x - x_0) \), is tangent to the graph of \( y = f(x) \) at \( P \).

If \( \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \infty \) (or -\( \infty \)), then the vertical straight line \( x = x_0 \) is tangent to \( y = f(x) \) at \( P \). If \( \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \) fails to exist in in any way other than by being either \( \infty \) or -\( \infty \), then the graph of \( y = f(x) \) has no tangent line at \( P \).
Derivative (Adams, 1986, p. 56)

If \( \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = f'(x) \) exists as a finite real number, we say that the function \( f \) is differentiable at \( x \). In this case the number \( f'(x) \) is called the derivative of \( f \) at \( x \), and the function \( f' \) is called the derivative of \( f \).

Function (Adams, 1986, p. 14)

A function \( f \) is a rule that assigns to each real number \( x \) in some set \( D(f) \) (called the domain of \( f \)) a unique real number \( f(x) \) called the value of \( f \) at \( x \).

1.3 General Statement of the Problem

From a constructivist view, what a student "knows" is dependent in some important ways upon how he has come to know it (Erickson, 1987). It is very important for a teacher to know the student's primitive conceptions, so as to better understand what errors and misunderstandings may follow, how these conceptions may change into wider and more sophisticated ones, through which situations, which explanations, which steps to pursue in the instructional setting (Vergnaud, 1982).

The aim of this study was to reveal some of the difficulties that students were struggling with to understand the concepts of derivative and function. In so doing, this study also hopes to make a contribution to the teaching of the calculus.

1.4 Specific Statements of the Problem

The purpose of this study was to examine some of the conceptual difficulties in calculus that students experience in first year calculus.
courses. Problems related to the concepts of "function" and "derivative" were discussed in interviews with students in order to address the following research questions:

What is the nature of the students' conceptual understanding of the derivative, slope of a tangent line, how derivative and slope are related, differentiability, and function?

Can a category system be developed for these understandings which might provide some insight into the nature of the difficulties experienced by students learning these concepts.

1.5 Overview of Methods of Study

The study was based on individual interviews with volunteer first year students at U.B.C. who were taking Math 100 during the first term. The interviews were tape-recorded. The transcripts of the interviews along with the students' written work were used for the further analyses. The tutorial sessions were used whenever it seemed apparent to the researcher that the students were confused and in need of clarification. The interview was based upon broad questions that will be presented in Chapter 3. The questions were aimed to explore the students' difficulties in understanding the concepts of derivative and function. A strict standardized protocol was not used because the interviewer's first priority was to have the opportunity to study the process of a dynamic passage from one state of knowledge to another. The interviewer did not pre-structure the direction of inquiry. The students had enough time for their responses and to express their ideas toward a question. The students' responses to questions about each one of the concepts were then categorized to facilitate the analyzing procedures. Sixteen Students were interviewed. However, only the last 12 students' interviews' were used in the analysis. As a first step of data reduction, individual
profiles were made for every student and then the profiles were summarized according to defined categories of responses.

The first four students interviewed provided the data for a pilot study. The pilot study gave the investigator the opportunity to revise the questions and to improve her interviewing technique as well.

1.7 Limitations of the Study

The subjects being interviewed were all volunteers. They were solicited by public announcements in calculus classes and tutorial sections. Therefore the sample was not a random sample. All the subjects were first year U.B.C. students. The study was a one shot 60 to 80 minute interview combined with the tutorial sessions. Therefore, there was not enough time for some of the students to reflect on their actions and this might be a potential shortcoming of the interview. The students' weaknesses in algebra, their lack of ability to perform the even simple calculations, etc. were not taken into account in the analysis. This restriction was due to the purpose of the study which focused upon an examination of the students' conceptual difficulties in understanding the concepts of derivative and function.

Also the investigator acknowledges that she is constructing the meaning of the interviewees' verbal and written responses. As Jones (1985) indicates, "different people with different perspectives and different curiosities about the area of investigation will inevitably find different categories with which to structure and make sense of the data" (p. 59). Every investigator could use different methods for analyzing the same data and might arrive at different findings. The
investigator tried to avoid systematic bias although she believes that everyone views the world through one's own perspective.

1.6 Justification of the Study

Two concepts of calculus, namely derivative and function were chosen for this study because not only they are very basic and fundamental in learning calculus and later on in more advanced mathematics, but also these two concepts are quite crucial for almost anybody who needs any level of mathematics in higher education such as engineering, economics, etc.

Most college and university students have conceptual difficulties in understanding the concepts of calculus (Lochhead, 1983; Davis, 1985; Epp, 1985; Zorn, 1985), yet very little research has been done into this important issue.

"In a series of papers (Vinner and Hershkowitz, 1980; Vinner, 1983; Tall and Vinner, 1981; Dreyfus and Vinner, 1982) it was claimed that for some mathematical notions there are conflicts between the concept definition and the concept image. Namely, particular individuals develop concept images which are inconsistent with the mathematical concept definitions. These concept images are quite common and widespread" (Vinner, 1986, p. 2).

He then poses the following question:

"Is this phenomena a result of 'bad pedagogy' or is it inherent to the concept? In other words: Is there a way to teach these concepts, so that such images will not be formed or these images are unavoidable and they will be formed no matter how the concept is taught?" (Vinner, 1986, p. 2).
The present study has addressed this question for the notion of function and derivative in terms of a constructivist view of teacher-researcher, as a model builder.
CHAPTER TWO
REVIEW OF LITERATURE

2.1 Introduction

The four major categories into which the literature in this chapter will be reviewed are as follows:

2.2 The broad problem area of difficulties in teaching-learning calculus
2.3 The more specific problem area of students' difficulties in learning the concepts of calculus.
2.4 Vinner's conceptual model.
2.5 A brief summary on constructivism as a theoretical framework of this study.

2.2 Literature in the broad problem area

This section is divided into two subsections:

2.2.1 Related causes to student failure.
2.2.2 Suggestions to overcome students' failure.

2.2.1 Related causes to students' failure

In December 1985, a conference was held on the teaching of calculus in high school and college. Many eminent instructors of calculus and among them, a few researchers, spoke up about the status of calculus in high school and college. The conference proposal stated that "the teaching of calculus is in a state of disarray and near crisis... [with a] failure rate of nearly half at many colleges and universities" (Stein,
1985). Many researchers criticized the way that calculus was taught in universities and colleges. They suggested several factors which are responsible for the current state of teaching in calculus. These factors are:

- The ability of instructors to teach calculus.
- Rote versus meaningful learning in calculus.
- Students' high school background.
- Instructors' awareness of students' backgrounds.
- Textbooks.
- Students' conceptual ability.

**The ability of instructors to teach calculus**

Maures (1985) complains that calculus is taught by anyone including the least energetic member of the mathematics department or even by a graduate student. Lax (1985) states that calculus is a big enterprise and is taught to a large number of students with diverse needs and backgrounds; and yet it is taught by anyone who is available in a mathematics department. David (1985) has also the same idea about unqualified instructors.

**Rote versus meaningful learning in calculus**

Davis (1985), says that calculus courses have been presented as a series of notions, routine problems, and a few simple applications. This way of presenting calculus does not provide an opportunity for students to understand the concepts but it focuses on learning how to use the techniques and formulas. He continues to say that "...The difficulty comes mainly from the rapid pace of moving through the material, and from
an attempt to cover a large number of details without much focus on main key ideas" (p. 10, emphasis his). Epp (1985) proposes a question relating to the content of calculus and says: "...If it comes to a choice, will we settle for superficial knowledge of a lot or deeper understanding of less? Perhaps less is more" (p. 18).

**Students' high school background**

Some researchers view the students' poor high school background as a potential cause for failure. Renz (1985) says that many students have been taking less mathematics in high school. Rodi (1985) complains about "a generation of certainly unsophisticated, and probably even illiterate, high school graduates" (p. 2). He says that "These students simply are not intellectually ready for it. They do not have the skills in algebra and trigonometry" (p. 4).

**Instructors awareness of students' backgrounds**

Epp (1985) talks about an intellectual gulf between mathematics professors and their students. She says that professors reactions vary from, ignoring this gulf and teaching as though the students were mature enough to understand all the formal proofs, to those whose foci are primarily on skills to enable students to perform certain mechanical computations. Ash (1985) claims that while most of the students in calculus courses are in engineering, the calculus is presented by instructors "...as if the entire audience were planning a career in pure mathematics" (p. 2). She says that because of lack of awareness of students' needs and backgrounds, "...formal mathematical language which
was intended to prevent misunderstanding had precisely the opposite effect" (p. 3).

Textbooks

A great number of researchers talk about poor textbooks which may contribute to students' failure. But it seems that they do not have appropriate suggestions to solve the textbook problem. Steen (1985) asks "why do calculus books weigh so much" (p. 4), and says that the economics of publishing causes encyclopedic textbooks. Stein (1985) proposes that "Authors have less choice, for if they omit someone's favorite topic, their books will not be adopted and soon will be out of print" (p. 3). Kenelly (1985), Rodi (1985), Stevenson (1985), and Masurer (1985) have also presented the same idea about the poor quality of textbooks.

Students' conceptual ability

Hadas Rin (1983) has studied students' difficulties with calculus by examining their spontaneous written questions. She found that students do not know the definitions and they are not able to apply "known" theorems to new situations. Davis (1985) believes that many students do not understand the proofs and do not have a clear conception of the difficulties that must be overcome. Lochhead (1983) wrote that many college students are not able to read or write simple algebraic equations and they also "...seem to lack any well defined notion of variable or of function" (p. 2). Epp (1985) says that the state of most students' conceptual knowledge of mathematics is "abysmal" after they have taken calculus. Zorn (1985) stated that "Instead of learning to
create, verify, and analyze algorithms, calculus students learn mainly how to perform them" (p. 3).

2.2.2 Suggestions to overcome students' failure

Many researchers have tried to not only talk about the causes of students' failure, but also have tried to give productive suggestions in order to deal with students' failure in calculus.

Stein (1985) suggests that if a quarter or semester course of discrete mathematics would be available to first year university students, "Such a course could help develop maturity and thus prepare students for calculus. It could, incidentally, weed out those who are not ready to go on" (p. 4). Epp (1985) is talking about the possibility "...to modify pre-calculus courses to make them include additional work to increase students' logical maturity" (p. 10). Rodi poses the question of: "would it not be more consistent to make normal expectation for entering students in a thorough pre-calculus course before attempting the heady and adult stuff of calculus?" (p. 5, emphasis his).

Davis (1985) suggests that "key ideas are presented very carefully and thoroughly, so that difficulties are clearly perceived and so that students are able to see how these difficulties are met and dealt with" (p. 11). Lax (1985) says that

...for a concept, when presented properly, can be absorbed as a whole, while an algorithm remains a sequence of steps. It is only after a concept has been understood, and made part of one's thinking, that we turn to the intriguing task of devising efficient algorithms (p. 6).

Renz (1985) says that "calculus has been, is, and will continue to be a basic computational and conceptual tool for students studying the
hard sciences and engineering" (p. 6). Then he suggests that more
intuitive foundation and careful imaginative thinking will be needed for
introductory calculus, in order to make it understandable for students.
Ash (1985) calls for a change in mathematicians' style of teaching. She
says:

Mathematics does not come into existence fully developed with
theorems and proofs. It arises from imagination and
intuition aided by physical and geometric reasoning. Students should be taught in a style that reflects the
creation of mathematics and not in style that would satisfy a
professional mathematician tidying things up years after the
fact. It is more important to learn how to formulate and
solve interesting problems than to learn the techniques of
writing formal proofs (p. 4).

Stein (1985) suggests that students should be able to think on
their own and to express their thoughts, and this chance can be given
them by providing open-ended problems with variations and introducing
"...-problem-solving' courses to compensate for the narrowness of our
mission" (p. 10).

Rodi (1985) says that "Applications are a critical part of teaching
and learning calculus precisely because they constitute one of the best
places both to expose and to reinforce intuitive conceptual
understanding" (p. 8). Then he warns that if research mathematicians
teach calculus, they may pay less attention to application and pay more
attention to more abstract generalization.

Lax (1985) seems to have different ideas from Rodi. He says that
the teaching of calculus should be entrusted to those who actively use it
in their own research. Although he sees the same role for the use of
application in calculus when he says: "Teaching of calculus is the
natural vehicle for introducing applications, and that applications give
the proper shape to calculus: They show how, and to what end, calculus
is used" (p. 1). Rodi (1985) believes that applications do not give the
proper shape to calculus if they will not enrich students' intuition and
will not allow them to see each successive stage of generalization. He
says that applications should be carefully selected and expressed by
instructors, since applications "...are as powerful a tool in revealing
it's message as metaphor is to the poet" (p. 10).

Epp states that "...the primary aim of calculus instruction should
be development of conceptual as opposed to purely mechanical
understanding" (p. 9). But she provides some suggestions to help
students and among these suggestions is one that would appear to lead to
mechanical understanding:

Make your students memorize precisely-worded definitions and
perhaps theorem statement also. Memorization is greatly
underrated as a Pedagogical tool. At the least, memorization
of a definition forces students to read it carefully, at
best, it encourages them to understand it (since it is easier
to memorize something intelligible than gibberish) (p. 12).

This writer did not find any study to support this suggestion.

2.3 **The Specific Problem Area**

Calculus is not a course for the elite any more. There is a great
demand for calculus, and yet so little research has been done into the
study of students' difficulties with calculus. All the suggestions
mentioned in the previous section represented the teachers' perspective
but there were no suggestions directly involving students.
Unfortunately, only a few studies have been done in this large area and
many more studies are needed to clarify the nature of students'
difficulties in calculus.
The concepts of derivative and function are two fundamental areas upon which students' understanding of many other concepts of calculus are based. Few researchers have studied the nature of student difficulties in these two particular areas. Among those are Orton (1980,1983), Monk (1987), Vinner (1983) and Markovits et al. (1986).

In 1983, Orton conducted a study to investigate the students' understanding of differentiation. He interviewed 110 students aged 16 to 22 years old. He found out that one of the most difficult questions was concerned with the understanding of differentiation. Students of his study generally found that the application of differentiation was relatively easy. Orton believes that students' errors in dealing with calculus are mostly conceptual.

Markovits et al. (1986) conducted a study with some 400 9th grade students to investigate the students' understanding of some components of the concept of function. The study was designed in a way to give the researchers the opportunity to identify the students' difficulties and the probable causes of these difficulties. The investigators were not, as they said, "interested in the students' overall success but rather in the types of difficulty they encountered" (p. 24). Among their findings were that three types of functions caused difficulty namely, the constant function, a function defined piecewise and a function represented by a discrete set of points. Vinner (1979) also found that students had difficulty with piecewise functions.

Monk (1987) involved 628 first year university students in a study which had as it's objective the investigation of students' understanding of the concept of function. His findings showed that the students were relatively good in answering those questions which did not ask for their
understanding but rather more of their skill, yet they did very poorly in responding to those questions that asked for their understanding of the concept of function.

2.4 Vinner’s Conceptual Model

Vinner has examined students' understanding of "Limit" (1986), "Function" (1983), and "Tangent Line" (1982), by using his model of cognitive processes that involves the notions of concept definition and concept image. Concept definition, as he explains (1981), is an accurate verbal definition of a concept in a non-circular way. According to him, concept image is a set of properties associated with the concept together with a mental picture of that concept. Vinner (1975) used an example to illustrate what he means by a mental picture of a concept. He says that if C is a concept and if P is a person, then "P's mental image of C will be defined as the set of all pictures that have ever been associated with C in P's mind, namely the set of all pictures of objects denoted by C in P's mind" (p. 339). Vinner (1983,1986) claims that the developed concept images by different individuals are inconsistent with the concept definitions that are usually inactive.

For every concept, Vinner (1981) presents a model by assuming the existence of two different cells in the cognitive structure, one cell for the concept definition(s), and the other one for the concept image(s). Depending on how a concept is first introduced to students, different interactions will occur between the two mentioned cells. The interactions vary from the construction of a definition based on one's own experience with the concept which is a description of his concept image to introducing the concept definition prior to the existence of any
concept images. For Vinner (1982, 1983, 1986) the important task is to recognize the students' concept images and try to reveal them whenever it is possible. This revealing will help teachers to not only acquire some understanding of their students' mental activities but also try to find the probable causes for formation of students' wrong concept images.

Fifteen Grade 12 students were involved in a study of "The notion of limit" conducted by Vinner (1986). The concept of Limit was introduced to students in a way that was supposedly to prevent the formation of wrong or inadequate concept images. The students' mathematical ability, as Vinner says, were unquestionable. Yet he arrived at a conclusion that the formation of certain concept images are probably unavoidable. He suggests a reconsideration of the way of introducing new mathematical concepts which is usually presented by the means of concept definition. He also found that the students' concept images play a crucial role in construction of their mathematics yet the textbooks are based on the concept definitions. His finding was supported by his study of function (1980) and study of the tangent line (1982). Vinner then suggested that presenting a concept by it's formal definition may not be the best way of doing that.

2.5 A Brief Summary of Constructivism

This section is divided into three subsection as follows:

2.5.1 Constructivist perspective of knowledge
2.5.2 Constructivism and Mathematics
2.5.3 Teaching and Learning from a Constructive Perspective
2.5.1 Constructivist Perspective of Knowledge

Constructivism as an epistemology concerns the what and how of knowing (VonGlasersfeld, 1984 cited in Erickson, 1987, emphasis is his). Constructivism is "...a general way of interpreting and making sense of a variety of phenomena. It constitutes a framework within which to address situations of complexity, uniqueness, and uncertainty that Schon (1985) calls 'messes' and to transform them into potentially solvable problems. Thus, like any epistemology, constructivism influences both the question posed and the criteria for what counts as an adequate solution" (Cobb; Wood; Yackel, 1988, p. 2).

As stated by Vergnaud (1987), Piaget was the most systematic theorist of constructivism in his time. It seems therefore, appropriate to refer to Piaget. The research of Piaget and his inquiry into cognitive development has been the major influence toward a more constructivist psychology (Magoon, 1977). "According to Piaget, the essential way of knowing the real world is not directly through our senses, but first and foremost through our actions. In this context, action has to be understood in the following way: all behavior by which we bring about a change in the world around us or by which we change our own situation in relation to the world. In other words, it is behavior that changes the knower-known relationship" (Sinclair, 1987, p. 28). The knower should draw upon his existing knowledge and also should be able to extend his knowledge base if he wants to make sense of his experience (Erickson, 1987; p. 22) which is the only reality that he can know (Kilpatrick, 1987). Sinclair (1987) pointed out that in Piaget's view, "new knowledge is constructed from the changes or transformations the subject introduces in the knower-known relationship" (p. 28).
The major assumption of constructivism, as Magoon (1977) states, is that the subjects being studied must be considered as knowing beings and their knowledge has important consequences for how their behavior or actions are interpreted. The constructivist view, as described by Kilpatrick (1987), involves two principles:

1. Knowledge is actively constructed by the cognizing subjects not passively received from the environment.
2. Coming to know is an adaptive process that organizes one’s experiential world; it does not discover an independent, pre-existing world outside the mind of the knower (p. 7).

Kilpatrick argues that the second principle separates trivial constructivism, as VonGlaserfeld calls it, or simple constructivism, as Davis and Mason call it, or empiricist-oriented constructivism, as Cobb calls it, from radical constructivism (1987). He says that radical constructivism rejects metaphysical realism and claims that the search for objective truth should be stopped. Kilpatrick summarizes radical constructivism as:

...an epistemology that makes all knowing active and all knowledge subjective. Following modern physical sciences in its rejection to the possibility of coming to know ultimate reality, it treats the cognizing subject as the organizer of his or her own experience and the constructor of his or her own reality. It views coming to know as a process in which, rather than taking in information, the cognizing subject through trial and error constructs a viable model of the world (p. 10).

The consequences of a radical constructivist position have been identified by VonGlaserfeld (1983, in press, cited in Kilpatrick, 1987) as:
...(a) teaching (using procedures that aim at generating understanding) becomes sharply distinguished from training (using procedures that aim at repetitive behavior); (b) processes inferred as inside the student's head become more interesting than overt behavior; (c) linguistic communication becomes a process for guiding a student's learning, not a process for transforming knowledge; (d) students' deviations from the teacher's expectations become means for understanding their efforts to understand; and (e) teaching interviews become attempts not only to infer cognitive structures but also to modify them.

2.5.2 Constructivism and Mathematics

The fundamental question of whether mathematics is discovered or invented is sometimes viewed as a choice between either the platonist position or the constructivist position. Plato believes that "The concepts of mathematics are independent of experience and have a reality of their own. They are discovered, not invented or fashioned" (Kline, 1972, p. 43). As stated by Thorn, "The mathematical entities exist independently of thought, as Platonic ideas" (1971, p. 696). Hardy (1928, cited in Kline 1985) believes that "...mathematical reality lies outside us, that our function is to discover or observe it, and that the theorems which we prove, and which describe grandiloquently as our 'creations,' are simply our notes of our observations" (p. 205). This means that mathematicians do nothing but to discover the concepts and their properties.

The other view sees mathematics as a product of human thought. "Herman Hankel, Richard Dekekind, and Karl Weierstrass all believed that mathematics is a human creation" (Kline, 1985).

Brouwer's idea gave rise to constructivism as a new school of thought in mathematics. His position as described by Davis and Hersh (1980)" was that the natural numbers are given to us by a fundamental
intuition, which is the starting point for all mathematics. He demanded that all mathematics should be based constructively on the natural numbers" (p. 334). Bishop says that Brouwer and his followers "were much more successful in their criticism of classical mathematics than in their efforts to replace it with something better" (p. ix). Furthermore, Bishop claims that a satisfactory alternative exists although Brouwer did not convince the others that there is an alternative. Bishop, within a constructive framework, develops a large portion of abstract analysis in order to give numerical meaning to classical abstract analysis since he think that "classical mathematics is deficient in numerical meaning" (p. ix).

"The constructivists regard as genuine mathematics only what can be obtained by a finite construction" (Davis & Hersh, 1980, p. 320). Bishop believes that "when a man proves a positive integer to exist, he should show how to find it" (1967, p. 2). As was stated by Goodman (1983), "The emphasis is not on foundational questions, but on the hard work of finding constructive versions and constructive proofs of actual theorem" (p. 61). In constructivist's view, the purpose of proof is to clarify the theorem, "to make the theorem obvious, so that the phenomenon is fully revealed, with nothing hidden" (Goodman, 1983, p. 63).

From a constructivist's view, mathematical objects as Cobb (1987), says are: "phenomenological correlates of systems of conceptual operations rather than elements of a mind-independent mathematical reality. Regardless of the interpretation adopted, it would seem that viable models of learning or problem solving in mathematics must account for the experience of mathematical objects" (pp. 9-10). Kilpatrick is more concerned with the relation between the constructivism and the
practice of mathematics and he consequently expresses his concern for school mathematics and what should be taught.

For Piaget, the evolution of mathematical structure is towards increasing comprehensiveness and rigor since in his view of constructivism, meaning is cumulative. He then sees the process of moving mathematics toward increasing objectivity.

However, other options seem to be available to answer the question of how does mathematics come into being (Wheeler, 1987). Wheeler seems to be in favour of a combination of both as a better answer.

2.5.3 Teaching and Learning From a Constructivist Perspective

The activity of exploring children's construction of mathematical knowledge, as Cobb and Steffe (1983) believe, must involve teaching (p. 33). They make a distinction between the constructivist and nonconstructivist teacher by the emphasis they place on the activity of modelling children's realities. The constructivist teacher aims to see through the overt behavior in contrast to the behaviorist teacher that attempts to see in the overt behavior. By seeing through this behavior, the teacher would help students to reconstruct their mathematical learning contexts.

Teachers should realize that the problem of teaching will not be solved by using mere definitions; "...students' conceptions can change only if they conflict with situations they fail to handle" (Vergnaud, 1982, p. 33). Learning, which is stimulating and relevant to the student is an indispensable part of the constructivist program (Cobb et al. 1988). In the constructivists' view, the core of mathematical learning is the problem solving process (Cobb, 1987, also Cobb, 1986; Confrey,
In this situation, the students try to reach their goals by constructing their solutions to problems which have arisen.

The focus of constructivism in relation to mathematics education is exclusively on the active processes of construction of mathematical realities of individual students (Cobb et al. 1988). The student's process of gaining and constructing knowledge is more important than the structure of the student's knowledge itself (Erickson, 1987).

Pines and West (1986), in referring to Vygotsky (1962), introduces two kinds of knowledge, namely spontaneous knowledge and formal knowledge. The first is the knowledge that children obtain spontaneously from their surroundings, while the second is the knowledge that children gain through formal schooling. "Learning is always an interaction between the learner's current understanding and the new information gleened" (Pine & West, 1986, p. 587).
CHAPTER 3
METHODS

This chapter consists of the following sections:

3.1 The Rationale for Using the Clinical Interview
3.2 The Subjects
3.3 The Interview Procedure
3.4 Method of Analysis
3.5 Specific Interview Questions

3.1 The Rationale for Using the Clinical Interview

Research into mathematical thinking, as Ginsburg (1981) has claimed, has three basic aims which are: the discovery of cognitive processes; the identification of cognitive processes; and the evaluation of competence. Depending on the research purpose, different clinical methods can be used (Ginsburg, 1981). What is important in teaching and learning mathematics is the intellectual process underlying the mathematical knowledge. Since the underlying cognitive processes are numerous and complex, standard tests may be ineffective or at least inefficient (Ginsburg, 1981). This researcher thinks that clinical interviews will help to recognize some of the complexities of the students' cognitive processes. The students being studied are "knowing beings" (Magoon, 1981, p. 652). Clinical interviews will help to investigate some of the processes that these knowing beings use when constructing mathematical concepts. "In order to understand why persons act as they do, we need to understand the meaning and significance they give to their actions. The depth [sic] interview is one way, not the
only way and often used most appropriately in conjunction with other ways-of doing so" (Jones, 1985, p. 46). This researcher believed that she must also act as a teacher in order to explore the nature of students' understanding. Acting as a teacher gave the researcher the opportunity to not only see what the students do in order to answer questions, but also to understand how and why they did it (Cobb and Steffe, 1983).

The interviewees were asked to do a series of questions related to the concepts of "function" and "derivative". They were asked to describe what they were doing while they answered these questions. The investigator's purpose for conducting these interviews was to attempt to find the nature of some of the difficulties that students are struggling with in calculus. Probing questions were used to explore the nature of these difficulties. In the tutorial sessions, the investigator acted as a teacher while using probing questions. As Cobb and Steffe have stated:

The actions of all teachers are guided, at least implicitly, by their understanding of their students' mathematical realities as well as by their own mathematical knowledge. The teacher's mathematical knowledge plays a crucial role in their decisions concerning what knowledge could be constructed by the students in the immediate future. Through reflecting on their interactions with students, they formulate, at least implicitly, models of their students' mathematical knowledge (p. 85).

This investigator had broad questions that the interview was based upon, but the first priority was to have the opportunity to study the process of a dynamic passage from one state of knowledge to another. For this reason, a strict standardized protocol was not used. Jones (1985) suggests that interviewers should not prestructure the direction of inquiry with their own frame of reference in ways that give little time
and space for their respondents to elaborate. In some cases not more than half of the questions were covered, however, the students had opportunities to express their ideas and even to sometimes express their feelings toward a question.

According to Pine et al. (1978), conduction of a pilot study is a necessity prior to every research study that uses clinical interview as its data collection methodology (cited in Aguirre, 1981). Four volunteer students were interviewed for a pilot study. This gave the investigator the opportunity to revise the questions and her interviewing technique as well.

3.2 The Subjects

Sixteen volunteer students participated in the study. They were all first year U.B.C. students and were solicited by public announcement in Math 101 classes and their tutorial sections (see Appendix A). The students were asked to read a consent form prior to interviews (see Appendix B for a copy of the consent form). They also were told that their real names would not be used in the study for the sake of confidentiality.

Only twelve of the students' interviews were used for analysis. First year university students were chosen because they had been introduced to the concepts of function and derivative in the first term of the same year. The sample consisted of four females and eight males. The students' marks for Math 100 varied from 47% to 100%. This gave researcher the opportunity to investigate the conceptual difficulties of students of different mathematical backgrounds. Given the nature of the students being studied along with the small size of the sample, no
attempt was made to generalize the findings of this study. More studies of this nature would be needed to provide supporting evidence for the findings of this study.

3.3 The Interview Procedure

The data for this study were collected during the months of March and April, 1987. The investigator conducted the interviews during a time convenient to the student, in a quiet office located in the Department of Mathematics at U.B.C. Each interview lasted 60 to 80 minutes (which included the tutorial sessions). All the interviews were tape recorded. There were two sets of questions, one on function and the other on derivative. Students were free to start with any of the two sets that they wanted to. They also were asked to explain their work. The first few minutes of each interview was mostly informal conversation until the student seemed to feel comfortable. The students were told that they could terminate the interview whenever they liked, although this did not happen. On the contrary, some of them were eager to continue the discussion. So their wishes were granted, yet they were not recorded since there was not more than one 90-minute cassette tape available for every interview. At last it is worth mentioning that students being interviewed were extremely cooperative.

3.4 Method of Analysis

The investigator was looking for patterns in the data. The data were qualitative and provided by transcripts of the most relevant parts of the interviews and the students' written work. Immediately after each
interview, the interviewer made notes to facilitate further data analysis.

Jones' (1985) advice to listen to the tapes of each interview at least twice was followed. The first time provided a sense of the whole interview while the second allowed for a more detailed analysis of specific questions. Individual profiles were made as a first step data reduction, based on the transcripts of the relevant parts of the interviews, students' written work and the interviewer's notes. Then the students' responses were categorized as the second step of data reduction. Later on the students' responses were "coded" into categories for final analysis. Vinner's model for cognitive processes was used for data analysis.

3.5 Specific Interview Questions

The followings are the specific interview questions. They consist of two sets of questions, the first set is questions related to the concept of derivative and the second set is the questions on function.

1) Derivative

a) Find the derivative of \( f(x) = x^2 + 1 \) at \( x = 1 \).

b) Sketch the graph of \( f(x) = x^2 + 1 \).

2) The diagram shows the graph of the above function and a fixed point P on the curve (Parabola). Lines, PQ are drawn from P to points Q on the Parabola and are extended in both directions. Such lines across a Parabola are called secants, and some examples are shown in diagram.

a) How many different secants could be drawn in addition to the ones already in the diagram?

b) As Q gets closer and closer to P, what happens to the secant?
c) Find the slope of PQ
   Find the slope of PQ
   .
   .
   .

d) Find the slope of L at Point P = (1,2)

Figure 3.1
The Graph of Parabola

3) Give the definition of derivative.

4) What’s the relation between this curve and its derivative?

5) Compute the derivative of above function \( f(x) = x^2 + 1 \) at \( x = 0 \) by using the definition of derivative.

6) (a) compute \( \frac{d}{dx} \left[ 1 + \frac{1}{(7 - 5x)^{1/2}} \right] \)

7) compute the derivative of \( f(x) = \begin{cases} 0 & x \leq -1 \\ x+1 & -1 \leq x \leq 0 \\ -x+1 & 0 \leq x \leq 1 \\ 0 & x \geq 1 \end{cases} \) at:
   a) \( x = -2 \)
   b) \( x = -1 \)
   c) \( x = -1/2 \)
   d) \( x = 1/3 \)
   e) \( x = 1 \)
   f) \( x = 10 \)
Function:

1) Consider the equation \( x^2 + y^2 = 1 \). Sketch the graph of this equation.

2) Determine whether the above equation represents a function \( y = f(x) \) or not.

3) If not, determine a domain and a range such that the above equation is a function.

4) What is the relation between the domain and the range of a function and its inverse? and \( 1/f(x) \).

5) Find the inverse of \( f(x) \) in part (3) if it is invertible. Justify your answer.

6) What is the difference between \( \cos^{-1}(x) \) (\( \cos^{-1}(x) = \arccos(x) \)) and \( \sec(x) \)?

7) If \( f(x) = \sqrt{x} \) and \( g(x) = x + 1 \), compute one of the following.
   
   (a) \( \text{fog}(x) \)
   
   (b) \( \text{gof}(x) \)
   
   (c) \( \text{gog}(x) \)
   
   (d) \( \text{fof}(x) \)
CHAPTER FOUR
RESULTS

4.1 Introduction

The objective of this study was to provide partial answers to the following specific research questions.

1. What is the nature of the students' understandings of derivative, slope of a tangent line, how derivative and slope are related, differentiability and function?

2. To develop a category system for these understandings which might provide some insight into the nature of the difficulties experienced by students learning these concepts.

One method of addressing these questions has been to investigate if there are conflicts between the concept definition and the students' concept images of the concepts of derivative and function. Questions were chosen in such a way so as to give the subjects an opportunity to reveal their concept images and also to help the researcher to see the possible conflicts between students' concept definitions and their concept images.

One way of investigating these issues, in the researcher's view, was to act as a teacher while conducting the interviews. As stated by Cobb and Steffe (1983), "By acting as teachers, and by forming close personal relationships with children, we help them reconstruct the contexts within which they learn mathematics" (P. 85). The "tutorial sessions" are the result of the researcher's teaching action whenever it was appropriate.

Because it is important to see the process of students' thinking and constructing, excerpts of each student's interview were analyzed. This investigator believed that the students' individual profiles will
help the readers in two ways: first to show the nature of the "tutorial sessions" and secondly to help them to capture the essence of the students' understanding.

To facilitate the data analysis, students' responses to the questions on derivative and function have been categorized and excerpts from the students' interviews have been quoted. At the end of this chapter, a summary of results is provided.

4.2 Individual Students' Profiles

This section deals with the Individual Students' Profiles. Only the relevant parts of the students' interviews have been analyzed. An effort has been made to illustrate the process of students' knowledge construction and the role of the tutorial sessions in helping students to do that. Only one example of a tutorial session for a given concept has been discussed. Informal language was used by the researcher to create a more friendly environment for the interview. Some of the profiles are lengthier than the others due to the fact that some of the students were better able to articulate their ideas than others. Profiles are not in any specific order. It should be mentioned that pseudo names have been used for the sake of the students' confidentiality. "I" represents the interviewer in the profiles. Twelve profiles are presented below.

4.2.1 Jenifer's Profile

Jenifer got 77% in math 100. She wanted to major in sociology but she fear of mathematics seemed to affect her decision. She said: "I heard that you have to take math 200, oh, I don't know, I have not really decided yet".
For finding the slope of the secant line, she tried to find the vertical displacement (dy) and horizontal displacement (dx). She had trouble finding the y-component of the point Q. Her attention was drawn to the y-component of a few points with \( x = 1, x = 2, \) etc. I expected her to possibly see a pattern for finding the y values of the point Q. Instead she said:

Jenifer: Ya, but this is not, but this line is not a parabola [she meant the secant PQ]. This \([f(x) = x^2 + 1]\) is the equation of parabola, you can't use this.

She did not realize that the points P and Q are on the parabola. Her main problem was the lack of understanding of the concept of function. Jenifer did not understand that the height (y) of any point (including Q with the length of \( x + h \)) on the parabola was \( y = f(x) = x^2 + 1 \). In a tutorial session, an effort was made to clarify this part. This matter was discussed again during the interview. Later on she was asked:

I: Can you give the definition of derivative?
Jenifer: No, I can't, I can do it. [Emphasis is mine.]

I: o.k., do it please.
Jenifer: \( f(x) = x^2 + 1, f'(x) = 2x \).

I: No, not by using the formula, do you know how you got this formula?

Jenifer: No, I don't, my teacher did the proof, but I didn't understand it. But I can do it. [Emphasis is mine.]

There was a discussion about the slope of a tangent line and the definition of derivative. The question was raised again. This time she
said: "In terms of limit? Oh yes, we did [emphasis is mine] the
definition of derivative, it's from limit..., Oh, I don't know".

Her prior knowledge was inadequate. She mostly used the phrases:
"I remember" or "I don't remember". She was performing (doing)
mathematics, as she said, not understanding it. She was advised to avoid
guessing without thinking. She then was asked to see if there were any
relation between the derivative of a function and the slope of the line
tangent to it.

Jenifer: The derivative is the slope of the tangent line.
I: The derivative of the function at what point?

She was not sure. In a tutorial session, the point P was changed
and different tangent lines were drawn at the different points on the
curve. She then was asked to find the derivative of the function at
those points. She finally defined the derivative correctly and said:

Jenifer: You know, when you do it, you don't think about this. You do
it so mechanical.

On the question of differentiability (#7, b), she skipped the fact
that f(x) = 0 at x = -1 as well. When she was told of her wrong answer,
she said:

Jenifer: Oh ya, you have to do limit or something.
I: What do you mean?
Jenifer: O.K., as x approaches from one side, approaches to 1 and then
x approaches from the other side, approaches to 0. Doesn't
exist...no!
I: Why doesn't it exist? [The derivative.]
Jenifer: Because it approaches to two different numbers.
Tutoring helped her to improve her concept image of derivative and the concept of differentiability. Although she understood the concept of derivative with difficulty, the acquired knowledge helped her to answer the question on differentiability with no difficulty. Not many of the interviewees were comfortable with this question.

On the section on function, she said that the circle \((x^2 + y^2 = 1)\) was a function:

Jenifer: Because it has a certain number of points.
I: Do you think that each set of points represent a function?
Jenifer: Ya.
I: What is the definition of function?
Jenifer: I don’t know.
I: How can we say that this \([x^2 + y^2 =1]\) represents a function? What is function?
Jenifer: What is the function? Oh, I don’t know.

These kinds of answers were widespread among the students. They sometimes did not have any justification for their answers. The tutorial helped her understand the function in a very concrete manner. Then the next question was:

I: Can we restrict the domain and the range of this equation in order to have a function? How can we change this circle to a function?
Jenifer: Make it a line. [Emphasis is mine.]
I: Can we change a circle to a line.
Jenifer: No, we can’t, but that’s the only way.

Her suggestion to change a circle to a line was very interesting. She did not understand the concept of function. She did not believe that
the domain of the equation could be restricted, while the equation itself remained unchanged. She rather aimed to conserve the length of the circumference of the circle. That seemed to be why she suggested changing the circle to a line. After explanation and discussion, she at last said that, "we can restrict it either to the negative y’s or positive y’s". Her last response showed her progress in understanding the concept.

4.2.2 Richard’s Profile

Richard was a commerce major. He took Math 100 and Math 101 as elective courses because he said that he liked mathematics very much*. He said: "I’d like to teach high school mathematics, that’s actually my goal". His reason for majoring in commerce was to have a better opportunity in the future job market. He did very well in Math 100. He got 100%. Richard introduced himself as an eager student who strived so hard to understand the concepts in calculus. He said that he volunteered to participate in the interview to learn something from it.

A minimum amount of time was spent on tutoring in this interview, compared to the amount of time that was spent on tutoring in other interviews. Mostly a hint or suggestion was sufficient to direct his attention (not his memory). For finding the slope of the line PQ, (#2, C) he said:

Richard: If I know the value of the point [he meant Q].
I: You can get it, you have the first coordinate of Q.

*For commerce majors, math 140 and math 141 are requirements which are less theoretical and are mostly based on the applications.
Richard: Oh, I don't know the y value for that, right?...
You mean I can find it by myself and everything?!
[surprisingly.]

I: Yes, first write the x value for Q.

Richard: O.K., Q = (h + 1, (h + 1)^2 + 1).

He then continued and got the slope of the secant line with no
difficulty at all. He then differentiated the f(x) = x^2 + 1 to get the slope
of the tangent line.

Richard: f(x) = y = x^2 + 1
f'(x) = dy/dx = 2x

I: What I'd like you to do is to follow the same procedure that
you did for finding the slopes of the other secants, which
means PQ's, in order to get the slope of the tangent line L.

After a few minutes he said, "I can give you the definition of
derivative". He was advised to forget the derivative for the time being
and to get the slope of the tangent line as if it were any other secant.
He responded as follows:

Richard: The line is tangent to the graph, but there is only one
point, I can't use the same thing here. [Emphasis mine.]

I: What happens to this distance? [Pointing to the distance
between P and Q.]

Richard: h tends to zero, right?

I: Then how can we find the slope?

Richard: Use the same point over again?!

Limit is one of the most problematic concepts for students. They
all experienced some kind of difficulties in understanding this concept.
Although investigating the students' difficulties in understanding the
concept of limit was not the objective of the study, the above
questioning was brought up intentionally. The lack of understanding of the concept of limit was one of the missing chains which prevented students from understanding other concepts such as derivative. Richard saw that the point Q got closer and closer to P (by looking at the graph). He could see that the distance between Q and P (i.e. h) became shorter and shorter. He was then stuck at this point. He did not know what to do and asked "use the same point over again?!" Richard knew that he could not do that, but there was nothing else that he could think of.

He was eager to take a fresh look and try to answer the same question again:

I: What is the slope of this tangent line at this point? [at P(1,2).]

Richard: It's 2 [he used the differentiation formula to get this answer i.e. \( f'(x) = 2x \) and \( f'(1) = 2 \)].

I: What is the slope of the tangent line at any given point [i.e., not this particular case].

Richard: Sorry, Oh no, I don't actually know.

When the question of the slope of the tangent line was raised, his first response was to give the definition of derivative. He did the same thing again. This researcher was wondering if he could see any relation between these two concepts, i.e., the derivative and the slope of the tangent line. In a tutorial session, an effort was made to help him understand what would happen to h as Q got closer and closer to P. This session helped him to get the slope of the tangent line. Then he was asked to compare what he got as a slope of the tangent line with what he gave as the definition of derivative. He gave the definition as:

\[ f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}. \]

His concept image related these two concepts.
together, since he offered the definition of derivative to express the slope of the tangent line. He related these two concepts in a way that he used the definition of derivative as a "method" to find the slope of the tangent line. The tutorial session helped him to understand that derivative was the slope of the tangent line. His response to the next question (#5) was more like the appreciation of the tutorial session, since he knew what he was doing.

I: \( f(x) = x^2 + 1 \), find the \( f'(x) \) at \( x = 0 \) by using the definition of derivative.

Richard: I think I have the answer, because it's gonna be the tangent (he meant \( x = 0 \)), it would be the horizontal line and the slope of that line would be zero. [i.e., \( f'(0) = 0 \).]

His first response to the question on differentiability was not correct. The investigator's intervention helped him to pay more attention to the function and its graph (in #7), and he then said that "there is no...it's singular point. There is no tangent line to it. There is no suitable tangent line to it".

He had an adequate concept image and a correct concept definition for the concept of function. He had difficulties in understanding some of the related concepts to the concept of function. In a tutorial session, the inverse function was discussed. He then restricted the domain and the range of semi-circle such that to make it an invertible function.

At the end of the interview, he liked to express his feelings toward mathematics. He was aware of his difficulties in understanding the concepts of calculus. He said that his 100% mark in Math 100 did not guarantee his understandings of the concepts. He said:
Richard: I went through half of the year and I didn’t know that the dy/dx is actually changing of y over changing of x. So for example, if a question comes up that asks you to visualize the problem, right?! Something that is not straight out of the textbook, it would be difficult if you didn’t know that this is what it meant because you don’t get these difficult questions that asking for really understanding [Emphasis is mine.]

His comment was quite interesting. He said that "you don’t get these difficult questions that ask for real understanding". The fact of the matter was that these questions were selected from the very beginning of the textbook. The first part of his claim was not true that "you don’t get these difficult questions" because these questions were not the difficult ones compared to for example what they got for their homework or their exams. Yet the second part of his claim was true that those questions were not asked for the real understanding since the students could fulfill the course, and the instructor’s expectation as well, by doing them well i.e., getting a good mark!

4.2.3 Brian’s Profile

Brian was a science major. He hoped to get into engineering. He got 70% in Math 100. Brian had a vague and mixed up concept image of derivative. Although, he at last was able to answer the questions, yet he did not have a clear understanding of the discussed concepts. His difficulties were deeply rooted in his understanding of the basic and crucial concepts of calculus, namely the concepts of function and limit. He had an inadequate sense of generalizability. He had to see things to accept their existence. He, like most of the other interviewees, had difficulty to find the y-component of the point. He could not see the
difference between a point and its coordinates. He responded as following in order to find the slope of the secant line:

Brian: So it would be rise over run...Q - 2.
I: Q is a point, right?
Brian: Yes, right, so it would be...but we don't know what the Q is!

The coordinates of a point were discussed in a tutorial session saying that the length of a point was the x-component and the height of a point, i.e., f(x) was the y-component.

I: The height of Q is: f(x₀ + h) which is equal to...
Brian: I can't see that.

The tutorial session went on with more concrete examples. He was told how the value of f(x) was dependent on the value of x, i.e. the changing of f(x) was dependent on the changing of x. He was asked to find the f(x) when x = 1, x = 2, x = 3 and x = x₀ + h. He got f(1), f(2), and f(3). But for f(x₀ + h) he said: "The function...right. You mean...but we are told to use this function [f(x) = x² + 1].

His response revealed an important fact that he did not understand the concept of function. As long as x had numerical value, he was able to get f(x) without any difficulties. Because he learnt the mechanism of doing this sort of problems. For f(x + h), the syntax of the problem was changed. So he was stuck. Brian could not possibly see the x as a variable. For him, it was hard to conceptualize that x could be anything, because x was a variable, and since x was an independent variable, so f(x) would be the subject to change because of x. It was
difficult to see $x$ as just a symbol not a value. After the tutorial he said:

Brian: So the slope would just be, OK, so it would be...
[in the beginning he said that the slope is $\frac{\text{rise}}{\text{run}}$ why he was trying to get the $y$-component of the point].

I: You said the slope is $\frac{\text{rise}}{\text{run}}$

Brian: But...wouldn't it be...the slope would be the derivative of it?!!

His concept image of slope was formed of two discrete elements, namely slope as the derivative and slope as the $\frac{\text{rise}}{\text{run}}$. For him there was no difference between the derivative as slope of the tangent line and as a slope of secant line. In order to find the slope of the tangent line ($#2, d$) he said: "Wouldn't be the derivative of the function".

He was reluctant to proceed with the same procedure that he followed for finding the slope of the secant. Although he mentioned the limit, yet his understanding of the concept was questionable. In a tutorial session, an effort was made to describe to him that how the secant lines became the tangent line. The point of the tangency and the slope of the tangent line were discussed as well. After the tutorial he got the slope of the tangent line at $x = 1$. Yet it took awhile for him to give the definition of derivative. The next question was to find the derivative of $f(x) = x^2 + 1$ at $x = 0$ by using the definition of derivative. He correctly found $f'(x)$ at $x = 0$. This was a good evidence to show that he did not just agree with the interviewer in the tutorial session, but the tutorial helped him to understand the definition of the derivative.
His response to the question of differentiability (#7) was the same as many other interviewees. He first said that \( f'(x) \) at \( x = -1 \) is 1. The interviewer stopped him to draw his attention to the fact that \( f'(x) = 0 \) at \( x = -1 \) as well. He said:

Brian: OK, it would be 0 also.
I: The derivative of function is 1 at \( x = -1 \) and again the \( f'(x) = 0 \) at \( x = -1 \). How can we interpret this?

There was no response to the above question except silence. I asked him if he could recall the definition of derivative. With probing questions and interviewer’s intervening, he understood that there could be infinite tangent lines for \( f(x) \) passing through the point \( x = -1 \).

I: Can we decide which one \([f'(x) = 0 \text{ or } f'(x) = 1]\) is the derivative of \( f(x) \) at \( x = -1 \).
Brian: No, we can’t.
I: Then, what do you suggest? What can you say about it?

There was no answer to this last question. In a tutorial session, he was told about differentiability. Differentiability’s property was only told by the aid of the geometric representation, not in a rigorous language.

He had difficulty understanding the concept of function. Lack of understanding the concept of function, caused him frustration in answering the questions of derivative as well. The section on function started with an intention to see whether or not the circle \( (x^2 + y^2 = 1) \) is a function.

I: Is it [the circle] a function or not?
Brian: Is \( y \) a function of \( x \)? Yes, sure.
I: Why? Why is $x^2 + y^2 = 1$ a function?
Brian: [Silence].
I: What is the definition of a function?
Brian: I don’t know.
I: Why you said that this is a function?
Brian: Because I could draw a circle, that's all.
I: Is circle a function?
Brian: Yes, sure, it has two...I don’t know, of course, you don’t want the definition of the equation.

For him, function was nothing but a few symbols (such as x and y) and few numbers with an equality sign. In a tutorial session, many examples were used to describe the concept of function. The latter question was looking for the possible restriction that could be imposed to the domain and range of the equation $(x^2 + y^2 = 1)$, in order to make it a function.

I: How can we determine the domain and the range of this equation in order to have a function.
Brian: You want to make it into a function?
I: Yes, because we saw that it’s not a function [after the tutorial].
Brian: You could expand it just like that.
I: We are not allowed to do that.

He tried and he could not succeed. The interviewer tried (within the time frame) to help him understand. He unfortunately had no more time to finish the interview. But his last response was very interesting. He said that the circle could be expanded to make it as a function (refer to Jenifer’s profile for the discussion on this matter).
4.2.4 Megan’s Profile

Megan was majoring in biology. She got 50% in Math 100. She chose questions on derivative to start with. When she wanted to get the slope of the secant line, she, like most of the other students, had difficulty to find the \( y \)-component of the point. Probing questions helped her to get the \( y \)-component of \( Q \) and the slope of the secant as well. She did not know how to find the slope of the tangent line. Her knowledge of the matter was not adequate enough to help her to establish the proper concept image for the coordinates of a point. She had trouble understanding the concepts of limit, tangent line and point of tangency. In a tutorial session, the discussions aimed to help her understand that the slope of the tangent line would be the limit of the slopes of the secant lines.

I: We are looking for the slope of the tangent line at \( P [1,2] \).

Megan: So to finding the slope of the tangent line, we get the derivative of the equation.

The interviewer liked to know that why she wanted to get the derivative in order to find the slope of the tangent line? She had no answer. The tutorial session continued for another 20 minutes. Her response after the tutorial was as follows:

Megan: The derivative is the slope of tangent line at point \( P \).

I: Derivative at which \( x \)?

Megan: 1.

Her response was an evidence to show that the tutorial helped her to understand the concept of derivative to some degree.
In answering the question of differentiability (#7), she had the same difficulty that most of the other students had. The responses were almost the same. She did not know the constant function. This was one of the reasons that she was not able to see that whether \( f(x) \) was differentiable at point \( x = -1 \) or not.

I: You said \( f(x) = 0 \), then what is \( f'(x) \)?
Megan: \( f'(x)?!... \) There is no \( x \).

A tutorial session aimed to help her understand the concept of constant function and concept of differentiability. The existence of derivative was discussed by referring to the earlier discussion on the definition of derivative. Although after the tutorial, she thought that she understood the discussed concepts, yet there was not enough evidence to prove that the new knowledge was constructed by her.

The interview continued by her responses to the questions of function. She said that \( x^2 + y^2 = 1 \) was a function for the following reason:

Megan: Because... they are related?!! I don't know [and she did not seem to care].

I: There is a relation between \( x \) and \( y \), so \( x^2 + y^2 = 1 \) is an equation. But why is it a function?
Megan: The equation is a function.

I: Why do we name them two different things if they are the same, why do we say equation and function?

She did not have any answer for it. She had a limited knowledge about function in general. Tutoring helped her to acquire some understanding. Enough to enable her to answer some of the questions. But she had no idea of what the inverse function was.
4.2.5 **Jason's Profile**

Jason was a science major. He aimed to continue his study in dentistry. Jason got 90% in Math 100. He did not at all use such words as **guessing** in the interview. He was quietly thinking unless he certainly had a point to make.

He liked to start with the questions of derivative. He was wondering how he would find the slope of the secant. The interview went on in silence for about 10 minutes. The interviewer necessarily intervened and asked:

I: Do you have any idea that how to find the slope of the line?
Jason: It's rise over run.
I: OK, find rise and run.
Jason: \[ \text{slope} = \frac{Q - 2}{h} \]
I: What do you mean by Q?
Jason: It's distance.
I: Distance from where?
Jason: Distance from p to Q.

He had a misconception about the y-component of a point and the point itself. He could see the distance from the centre to the y-component of P, because it had a numerical value which was 2 (see Figure 3.1). But he was not able to find the distance from 0 to y(Q) since there was no numerical value for it.

I: What is Q?
Jason: Q is a point.
I: What are the coordinates of Q?
Jason: Oh, ya...

I: Can you find the coordinates of Q?

Jason: No, I haven't [have] a height here.

After a bit of thinking he wrote:

Jason: \( Q = (x_0 + h)^2 + 1 \)

I: Why did you write: \( Q = (x_0 + h)^2 + 1 \). Q itself is a point. Every point is indicated by its length and its height. What you got here is the height of Q, it's not Q. Q is a point.

He finally got the slope of the PQ by spending a few minutes on that. Later on he was asked to find the slope of the tangent line at \( p \). He wrote:

Jason: \( f(x) = x^2 + 1, f'(x) = 2x, f'(1) = 2 = \text{slope L [tangent line to the curve at point } p]. \)

I: Why did you take the derivative of \( f(x) \) at \( x = 1 \)?

Jason: Why did I do it? The derivative is slope of line?!!

I: How are they related?

Jason: I'm not sure...

I: Will you find the slope of L in the same way that you got the slope of the other secants.

It took 11 minutes for him to get the answer without any of the interviewer's intervention. He did no mistake in doing that. The questions followed by:

I: What is the derivative? What is the definition of derivative?

Jason: Slope.

I: Slope of what?
Jason: Slope of the line at a point given.
I: Slope of which line?
Jason: $f(x)$.

After a few minutes of thinking, he said:

Jason: Derivative...is the slope of the tangent line to the curve $f(x)$ at a point $x$.
I: Derivative at which point?
Jason: at any point [Emphasis is his].

The following tutorial session, aimed to show that the value of derivative was changed by changing the tangent lines to the curve i.e., by changing the point of tangency. This was done by the means of the geometric representations. After the tutoring, his response was the unique one among all the interviewees.

Jason: The equation of derivative is the same at all point, isn't it?! Putting different numbers for $x$.

This understanding was rare among the students. That the derivative function $f'(x)$ was not changed but the value of derivative was subject to change by changing the points of tangencies.

He got the $f'(x)$ at $x = 0$ by using the definition of the derivative. He also sketched the graph showing the tangent line to the curve at $x = 0$. The correct answer to the above question showed that he seemed to be confident in understanding the concept of derivative. For the section on differentiability, although he had a different approach towards solving the problem (as he said: 'can I do it by inspection'), yet, he was experiencing the same difficulty, i.e., facing two values for
f'(x) at x = -1. With an exclusive discussion and the probing questions, he finally answered the last question of this part:

I: Can a function have two different tangent lines at one point [and still have derivative at that point].

Jason: No, it doesn't have derivative at that point then.

The next set of questions was on function. Jason had an accurate concept image of function. In response to the first question of this set, he said:

Jason: It's \( x^2 + y^2 = 1 \) not a function, because if we draw a perpendicular line [he meant vertical line], there is two values of \( y \) for one \( x \).

For him, \( f^{-1} \) and \( 1/f \) were the same. He said that \( 1/f = f^{-1} \). The tutorial session, in which the researcher aimed to clarify the concepts, of inverse function and reciprocal of a function, was the same in nature with some of the other tutorial sessions with other interviewees that have already been discussed. He then confidently answered all the related questions of the function.

4.2.6 Joe's Profile

Joe was a quiet person. He hardly talked or even wrote during the interview. At least 1/3 of the tape was filled by silent moments. He was mostly thinking. He did not speak unless he thought that he had the answer. He got 90% in Algebra 12. He had no calculus at high school except a very basic introductory at the end of Grade 12. He got 100% in Math 100. Interviewer told him that:

I: It's very good that you got 100%.
Joe: Even though I got 100%, still there are lots of anxiety to understand. When the test came up, I knew how to do everything but when it's going to problems, there are problems that I don't know how to do. [Emphasis is mine.]

The way he expressed himself showed that for him there was a difference between performing calculus and understanding the concepts of calculus. As described by him, he was willing to really understand the concepts but he did not get enough help from the class or the texts.

The interview continued by talking about derivative. His response for finding the slope of the secant was:

Joe: I'm not really sure how to find it.
I: Give it a try, what is the slope?
Joe: I know it's rise over run, but I don't see what the rise is.

He found the rise by himself but he, like many others, had a hard time to substitute the value for $f(x_0 + h)$. His difficulties stemmed from lack of understanding the concept of function. He did not know the behavior of $f(x) = x^2 + 1$ (The behavior of $f(x)$ has been discussed in one of the other profiles). In case of $f(x + h)$, he did many trials and errors till he got the right answer. He did not know how to find the slope of the tangent line (at P). He did not have an adequate understanding of the concept of limit in order to help him to understand the slope of the tangent line. In a tutorial session, it was explained to him that the slope of the tangent line was the limit of the slopes of the secants when $h$ approached to zero. He defined the derivative almost correctly. His concept image of derivative led him easily to the accurate concept definition of derivative.
Joe: The slope of the tangent line at point on the curve is the derivative of function.

I: At where?

Joe: At that point.

In response to the question of differentiability, he thought that \( f(x) \) (#7) was undefined at greater than 2 since \( f(x) = 0 \). The interview with the appropriate interviewer’s intervention went on until he said:

Joe: at \( x = -1 \) is undefined \([f(x) \text{ is undefined}].\)

I: Why is it undefined?

Joe: At that point \( f'(x) \) is between 1 and 0, but undefined at that point because you can’t plug this number into \( f(x) \).

I: What do you mean by: "you can’t plug this number into \( f(x) \)"

Joe: Since your \( f(x) = 1 \), and \( x = -1 \), you can’t put into this formula.

His reason for the derivative of function to be undefined at \( x = -1 \) was interesting. He did not know the concept of constant function which \( f(x) \) is constant for all \( x \) belongs to domain of function. This was partly the reason for his last response as: "you can’t put \([x = -1]\) into this \([f(x) = 1]\)". His main problem was his difficulties in understanding the concept of function.

He gave a list of incorrect conditions for a function to be differentiable. Probing questions were posed to clarify the problem. After the discussion he said: "Oh, this is a smooth curve. I remember this part, I wasn’t quite sure how to do that, I taught myself how to do it and that one worked (Emphasis is mine)".

These honest words must be thrilling for calculus instructors and curriculum makers. Solving and answering problems has become a game.
Some of the students play it smart and some of them do not know the rules of the game.

He had a relatively good understanding of function. No special incident happened while he was answering the questions of this part.

4.2.7 Owen’s Profile

Owen’s major was science. He wanted to continue in engineering but he decided not to "...because Math is one of my difficult subjects, caused me to take it second time through Math 101". He had an introductory to calculus at the end of Grade 12, but as was stated by him, it did not help him to be more prepared for university calculus. "When I came here last year, it was just like...Oh really, what’s going on"? Owen was first in a crowded calculus class and he failed that term. For the second time he changed to a smaller class. As he said: "you can be more relaxed in small classes". He insisted on saying that he did not understand anything and he only repeated his instructor’s words. He could not see himself as being a student who understands the concepts.

For finding the slope of the secant he asked:

Owen: Can I do \( y = m + b \)?
I: How do you find the slope?
Owen: From the tangent, the derivative.
I: What is the relation between the derivative and the tangent line, why do you want to take the derivative?
Owen: The tangent is the derivative at that point.

He had an unclear knowledge of the discussed concepts prior to the interview. He had heard all the notions before, but he did not know how these notions or concepts were related. He could not visualize the
difference between the slope of the secant line, the slope of the tangent line and derivative. He dealt with them as if they all were the same thing. Probing questions were posed to find out more about his primitive knowledge of these concepts. The interview went on until he was again asked to answer the question of the slope of the secant line:

Owen: Oh, Oh, it would be: \( \frac{Q_1 - P}{x_0 + h - x_0} = \frac{Q_1 - P}{h} \)

I: What do you mean by \( Q_1 - P \)?

Owen: \( Q \) is a point on the curve minus \( P \).

I: What is \( x_0 + h - x_0 \)?

Owen: That's the run.

I: OK, what about the rise?

He got "run" because he could see them (see Figure 3.1) but he couldn't see the \( y \)-component of the points (because he had to first find them). In a tutorial session, I talked to him about the coordinates of the point. For \( y \)-component of the point he used the point itself. He did not know clearly how a point was indicated on the plane. It is not reasonable to expect a student to understand derivative while he needs to know many basic concepts prior to that. After the tutorial he said:

Owen: So, \( Q_1 - P \) would be: \( \frac{f(x_0) - f(1)}{h} \), does it make sense to you?

I: I want it to make sense to you, does it?

Owen: Ya, it makes sense, because it's the amount of height.

Interviewer was just about to say something when he said:
Owen: OK, OK, I got it, \( \frac{f(x_0+h)-f(x_0)}{x_0+h-x_0} = \frac{f(x_0+h)-f(x_0)}{h} \)

His last response was more like a jump. I think his response showed the evidence that he had, in his mind, the discrete pieces of the concept either by understanding those pieces or by remembering them.

He was then asked to find the slope of the tangent line at P.

Owen: \( h \) is zero, then \( \frac{x^2+1-x^2+1}{0} \) ...I just plugged in \( h=0 \), because there isn't any distance along x axes, so it would be \( \frac{x^2+1-x^2+1}{0} \), but it's not right.

I: As we move along the curve, what will happen to the secants?

Owen: Secants will become the tangent, it's just the derivative, so that would be 2.

The interviewer asked him if he could consider any given function other than \( f(x) = x^2 + 1 \) and say what will happen if \( Q \) moves toward the fixed point \( P \), he answered:

Owen: The limit...

I: Why \( \infty \)?

Owen: Oh, I just said, the limit \( \frac{f(x_0+h)-f(x_0)}{h-0} \)...

Then he went on and got the slope of the tangent line. But his right solution did not seem to be an evidence of his understanding of the concept. For example, when he was asked that why he took the limit as \( h \) approaches to \( \infty \), he said, "Oh...I just said it". But in investigator's opinion, based on her observations, Owen did not have clear understanding of the concept of limit, approaching to something, etc. He usually had
seen the notion of limit which was accompanied by \( h \) approaching to either \( \infty \) or 0 and he naturally tried them both to see which one worked better. He did not see \( h \) approached to 0 from the graph. After he found the slope of the tangent line, he was asked to give the definition of derivative.

Owen: It's the tangent line at a certain point of that graph, such as the derivative of equation is defined.

I: OK, what is the derivative of function at that point?

Owen: At that point! There...[no answer].

I: What is the slope of the tangent line at this point [at P]?

Owen: 2.

For him, the concepts of secant, tangent line, slope and derivative were all mixed up, although he sometimes used them quite correctly. He knew that these concepts were somehow related, but he wasn't sure how. In another tutorial session, an effort was made; with the aid of graphs; to clarify these concepts as much as it was possible (time constrain). After the tutorial, he defined derivative as: "The derivative is at a certain point defined that it approaches as the same as the limit of the secant line as changing value of \( x \) approaches to zero." The process of tutoring and questioning went on and on a couple of times, until he seemed to understand this specific case of derivative.

His answer to the next question showed his lack of generalizability along with his uncertainty. The question was to compute the derivative of \( f(x) = x^2 + 1 \) at \( x = 0 \) (the \( f'(x) \) at \( x = 1 \) was discussed in detail). He first got it quite right, but he then changed his mind and said: "no, no, it's not right". He tried again and got the same answer (\( f'(x) = 0 \)
and again he repeated that: "it is not right". He tried for the third time and again \( f'(x) = 0 \) was his finding. He said:

Owen: It's wrong.
I: Why is this wrong?
Owen: Because the derivative, the slope should be 2 [he pointed to the derivative at \( x = 1 \)].

What he did was interesting. His difficulties were deeply rooted in his understanding of the concept of derivative. For him, it was hard to believe that the derivative had different numerical values at different points. He did not know the reason that he got two different answers for the \( f'(x) \) was that he was computing the derivative of the function at two different points. He simply thought that there must be something wrong with his computation. He was even asked to "look back" (Polya, 1945) and see if his procedure (and computation as well) was correct. He "looked back" and he did not find any mistakes, yet he was not satisfied with his findings. The interviewer suggested to him that both, he and her could go through the whole procedure again to get the \( f'(x) \) at \( x = 0 \) (by using the definition of derivative). We did it and when we finished, he said:

Owen: It's 0.
I: Then why do you think that it's wrong? You didn't do anything wrong.
Owen: Because for what I've been taught, derivative...Oh, that's right! This is right. Because \( f'(x) = 2x \) and if \( x = 0 \) then \( f'(x) = 0 \). I just wasn't thinking. [Emphasis is mine.]
I: Because at \( x = 1 \), the tangent line is [see Figure 3.1], but at \( x = 0 \), the tangent line is the horizontal line (I sketched the graph).
Owen: Right, right, I just wasn't thinking. [Emphasis is mine.]
Twice he said that he was not thinking but the investigator thinks that his error was rather "structural" (Orton, 1983). He had a vague concept image of the related concepts to the concept of derivative. The interviewer believes that he at last accepted that \( f'(x) = 0 \) at \( x = 0 \) because he plugged the \( x = 0 \) to the formula and he was satisfied with the result. But it did not necessarily prove that he got it right because he understood the concept thoroughly. That was the reason for further tutoring by means of graphs. That helped him to understand the concept better.

For him, the function which was used in the question of differentiability, needed more explanation. His response to question 7 (after the function itself was explained to him) was the common one.

Owen: At \( x = -1 \), \( f(x) = x + 1 \), \( f'(x) = 1 \).

I: What about this: \( f(x) = 0 \) \( x \leq -1 \). \( x = -1 \) is in both intervals \( (x \leq -1 \) and \(-1 \leq x \leq 0)\).

Owen: The \( f(x) \) says that: \( f'(x) = 0 \) at \( x = -1 \) and \( f'(x) = 1 \) at \( x = -1 \), so it's gonna be one of the two.

I: Can you decide which one? Can you say that...[I was interrupted by him].

Owen: You can't say that because it should be both.

I: What?

Owen: It should be both; it can be 0, it can be 1.

The interview went on by asking him whether the function was differentiable at \( x = -1 \) or not.

Owen: Probably, because it's not a smooth curve. Two curves chain together. Is that correct?!!

I: Why because it's not smooth, it's not differentiable?

Owen: Oh, because two different lines cross each other.
His responses showed that he remembered something about differentiability but he was not sure why he was saying those. It is possible that with this much understanding of any concept, a student could get a satisfactory mark for his course, especially those who are experts in getting good marks. But is there any understanding behind the marks is a question that should be answered. The students' responses could be more or less, a part of the answer.

He did not know the concept of differentiability, although he mentioned the "smooth curve". In a short tutorial session, the differentiability was discussed in a very simple way, mainly by the aid of graphs and referring to the discussion at the beginning of the interview (on the definition of derivative).

In order to answer the questions of function, he certainly tried hard to remember all he was told by his instructors. But he did not succeed. His concept image of function was the same as for the relation and equation, although he did not have proper concept image of those two concepts as well. Four times he changed his answers from NO to YES and vice versa. He finally gave up and said: "I don't know how to describe it".

His main idea for f to be a function was: "it defines y" or "it defines points y along the x-axis" and so on. He did not know the concepts of the inverse function, 1/f(x), etc. He was stuck, but he seemed to be anxious to know something about function in general (domain, range, inverse...). That was why he asked me to tutor him. The investigator, within her time frame, tried to use the various examples and sketched different graphs to explain the concept of function to him. Tutoring took about 15 minutes. Unfortunately he had no more time after
the tutoring to see if the tutorial session on function helped him to understand the concept or not.

4.2.8 Kathy’s Profile

Kathy’s major was biology. She was doing well in Math 101, although she had to write the supplementary exam in August since she failed Math 100. She gave following reason for her failure in Math 101:

Kathy: The reason is, because I hate math. I never did my homework, and I usually just cram at last minute and you can’t do that for calculus, calculus is quite different from Algebra.

Kathy had taken Honor Algebra 12 and Geometry. She got ‘A’ in both of the subjects. She decided to start the interview with the concept of function.

I: Determine whether the equation: $x^2 + y^2 = 1$ is a function or not.

Kathy: You mean if I solve it for x? or if I solve it for y? It is a function, you mean if you solve it, what would be the function of x or what could be the function of y,?

And she then continued:

Kathy: Is this a function?? Oh, Oh, I see, isn’t that you draw a horizontal line through it?! That’s not a function, because you can draw a line through it and it intersects both points...or it is a vertical line?!

I: You decide.

Kathy: There is another definition, if you draw a horizontal line through it. There is a word for it, I don’t know, I have forgotten.

Her conception of the function was limited to a line. She only tried to remember that. It seemed to me that she was taught the concept
of the function in such a hurry that no time was spent on the real understanding of the concept. When she was told that the line was vertical, she then said: "Yes, and it intersects the y-axis at two points, so it's not a function".

It is a wishful thinking to expect students to understand the concept by only giving them the definition of the concept (Vinner, 1983) i.e., before constructing any meaning into the concept. Students need many concrete examples in order to build up adequate concept images. If the concept image were established by the means of the familiar examples and geometric representations, they then, would be ready to understand the definition of the concept. Otherwise an abstract definition could be forgotten and all that might be left is a vague memory. Kathy expressed this in her words as:

Kathy: I don't remember doing those in September, you see, I don't have any feeling for math. Everything that I learn it's kind of goes in there and stays there, sometimes, you see, I got confused with what is the function already, like with that definition. I know that it's a business of drawing a line. [Emphasis is mine.]

After the tutoring the researcher asked her that:

I: Don't you think that it would be better to say that if there are two y values for one x, then you couldn't have a function, instead of keeping in your mind that whether you have to draw a vertical line or horizontal line. Say one x can't have two y values.

Kathy: Ya, o.k., that makes sense. I agree with that, because I learned it.

Next, she was asked to respond on the question of inverse function ($f^{-1}$) and reciprocal of function ($1/f(x)$). She certainly had heard about the definition of inverse function, yet she did not understand it. Her
knowledge of the matter was like more of remembering few facts, without even trusting on those memories.

Kathy: Isn’t that a function and it’s inverse are the mirror image of each other?

I: What do you mean by mirror image?

Kathy: That is... that the value of y for one of them is the value of x for the other one and vice versa. And $1/f(x)$ is that you just inverted, is not the inverse. It is the inverse, but not inverse of the function.

She tried to make her point that the $1/f(x)$ and $f^{-1}$ are two different things. But in answering the next question (what is the difference between $\cos^{-1}(x) = \arccos(x)$ and $\sec x$), she could not see them as two different things.

Kathy: Sec is $1/\cos x$, arc $\cos x$?... Could that be also $1/\cos x$?! [With doubt.]

I: You decide.

Kathy: I don’t know, I guess so, I think so, because that would be $\cos^{-1}$ which is $1/\cos$, because if you have anything to negative one [e.g. $(x)^{-1}$], I think it just goes one over that $x$ to the one [she meant: $(x)^{-1} = (1/x)$]. $\cos^{-1}$ would be $1/\cos$ and sec is $1/\cos$, that is the same thing...There must be some difference. Of course. They give you definition like that, and I never ran across it, but I would think, they would be the same thing, although I'm wrong. [Emphasis is mine.]

In a tutorial session, the notions of $1/f$ and $f^{-1}$ were discussed as well as their related concepts. $f^{-1}$ was introduced as the inverse of function in action of the composition of two functions, and $1/f$ as the inverse of function in action of the multiplication of two functions. After the tutorial she said: "Oh, I'm learning something...That would help me in September... I memorized all these things, that's why I'm confused" (Emphasis is mine.).
The questions of derivative started with the sketching graph of $f(x) = x^2 + 1$. Kathy had a hard time to do that. She tried so many points on the axes. She also tried to sketch many different curves without any successions. Kathy had a peculiar way of sketching the curves.

Although sketching the graph of the curve was not in the particular interest of the interview, yet it was revealing in analyzing Kathy's conceptual difficulties in calculus. Kathy was a bright student. It was not fair to put all the blame on her. She got A in both Honor Algebra 12 and Geometry in high school, and she still did not have any reasonable procedure to sketch a simple graph. She did it by trial and error. By locating different points of the graph and join them together to obtain the desired curve. She survived in high school and in university as well. The growing concern is that "How did so many students manage to... successfully escaping their teachers' recognition" (Gorodetsky et al., 1986). She had right to "hate calculus" since she did not understand its basic facts and yet she had to work hard to pass the course. Following passage was the way that Kathy tried to sketch the graph of $y = x^2 + 1$.

Kathy: Would it centre at (1,1) or at (1,0)? [She meant the curve].
I: You can test it.
Kathy: Oh, this one: (0,1), Oh, Gee! I never knew that you can test it.
I: Really? Then how do you sketch the graph? What procedure do you follow?
Kathy: I don't really follow one...I just make sure that I knew it before the test, set of rules! [Emphasis is mine.]
I: If you don't know the procedures, it's hard to keep all the shapes in your mind, this function has this graph, the other function has different one...you know what I mean? But if you try to understand how to do it, then it is much easier.

Kathy: I have a rule, if \( y \) is on this side [i.e. the left side of equality sign, e.g. \( y = x^2 + 1 \)], it will open up, I think that's the way it is, if \( x = y^2 + 1 \) then it opens on the sides, because it looks like an \( x \), or may be it is opposite!

Second question asked for the slope of the secant PQ. She had trouble locating a point on the plane. She did not know that any point was indicated on the plane by its coordinates. She was wondering how to get the y-component of point Q.

Kathy: The Q also lies on the parabola. Can I use the parabola to find the two points x and y of Q.

She immediately added that:

Kathy: I can't find the values. I don't know the relationship between x and y, that comes from the parabola.

In a tutorial session, diagrams were used to show that what would be the y-component of Q. To show that the role of \( f \) was to take every x, square it and add one to it. I also concluded that for every point x, there was one y which was the y-component of that x. She then tried to get the slope of the secant PQ. Nothing significant happened while she was doing that. It took 20 minutes for her to get the slope of the secant. She did it correctly and said: "it's like pulling teeth!"

She moved on to get the slope of the tangent line. Kathy had a vague understanding of the concepts of limit and derivative.

Kathy: I wouldn't know how to find the slope of it, all I know...What I heard is to find the derivative of it.

I: O.K., find the derivative of it.
Kathy: I did already, it is 2.

I: No, I mean by using the definition of derivative, not using a formula. Can you give the definition of derivative?

Kathy: I can give it, but I don't know how to relate it to finding the derivative of it.

I: Can you write it down, then we can try to relate it.

Kathy: No, no, I wouldn't be able to do it.

I: Do you know what is the definition of derivative?

Kathy: Yes, isn't that...like the slope of the line?...but I don't know, I don't know how to find the slope of the line without using the formula.

It was interesting that her concept image of the slope of the secant line, the slope of the tangent line and derivative were tied up together although it seemed that she was not aware of it. She first said that she could find the slope of the tangent line by finding derivative. Later on she said that derivative is the slope of the tangent line. But she said that she couldn't see any relation between the derivative of a function and the slope of the tangent line. Previously, she said that "I don't know how to find the slope of the line without using the formula". She was asked to give that formula.

I: What is the formula?

Kathy: Well, there is some formula, I can't remember it, it had h at the bottom...limit \( \frac{f(h+x_0) - f(x_0)}{h} \).

I: O.K., you already did it.

Kathy: Oh, that's true, it looks familiar... Oh, I remember for exam we had to know how to do that without using chain rule or some rule to finding it.
Her understandings was mostly like disjoint pieces of memories. Sometimes they were easy to recall and sometimes they were not.

I: What happens to h?

Kathy: It gets smaller and smaller, it approaches to infinity?!

I: Infinity is when it's getting bigger and bigger.

Kathy: So minus infinity?! [no trust in her words]. Oh no, it approaches to 0 and that's the formula: \[ \lim_{{h \to 0}} \frac{{(x+h) - f(x)}}{h} \]

Her response to every single question was amazing. She usually responded first and thought later. When she was told that h was not approaching to the infinity, she just changed it to minus infinity. One might think that theoretically, she was right if h was greater than zero \((h>0)\) was not the only condition and h could be less than zero \((h<0)\) as well. But it was not the case. She said it, because the opposite of +\(\infty\) was -\(\infty\). This type of responses was heard all along the interview. It was true that she got 47% in Math 100, but she was smart and had a good memory indeed. One of the reasons for her misunderstanding was, in researcher's view, her naive attitude towards mathematics. She tried to convince her audience that she hated mathematics and she would never be able to understand mathematics. Her negative attitude acted as a barrier which caused her great difficulty in understanding mathematics. Of course many other factors were involved indeed which there was no intention to discuss all of them.
4.2.9 Gary's Profile

Gary was interested majoring into either pharmacy or psychology. He had failed Math 100 twice and he got 86% the third time. Among the reasons that he had for his failure was the size of the classroom. He said: "It was a big class, I didn't show up to the class, now I'm in a class with 20 people. I can stop and interrupt the class...so I just stop the class".

Gary's opinion about the classroom size was expressed by many other students as well. The main objective of this study was to reveal some of the students' conceptual understanding of calculus, yet it was interesting to come across the other understanding barriers that students were dealing with (from their point of view).

Gary liked to start with questions on function. He sketched the graph of $x^2 + y^2 = 1$ and he then said:

Gary: $y=\pm \sqrt{1-x^2}$ is a function.

I: Why is it a function? You solved it for $y$, why is it a function?

Gary: I don't know...[laughter].

I: What do you think a function is?

Gary: Function is just an equation which she would take some number and put it through. It's just like...I don't know, it's like the button of your calculator, you put something in there, you punch that button and use another number which for all...which all numbers put in, give you a related output number.

I: You used a good analogy. You said that as you punch the button of the calculator, you'll get something as an output. When you punch the button, you get one outcome, but look at here. For this one, as you said for example, you punch this one, then you will get two outcomes: $y=+\sqrt{1-x^2}$ and $y=-\sqrt{1-x^2}$. 
Gary: Oh, O.K., I know what you mean. I forgot about that vertical line! \( \sqrt{1-x^2} \) or \( y=-\sqrt{1-x^2} \), two different functions put together.

He used a good example for function. It is not far from the truth to say that Gary was the only one who dared to argue about his views. Also to think of the non-textbook examples to better understanding of the concepts. His concept image of function helped him to be led to a proper concept definition. He evidently did understand the concept of function. His further responses supported this claim. Following is his responses to question #4:

I: What is the difference between the inverse of the function and \( 1/f(x) \)?

Gary: I don’t know.

I: Don’t you want to think about it?

Gary: It’s just whatever \( f(x) \) is, \( 1/f(x) \) is one over that and the inverse is the...I guess what you call opposite of it?! [Emphasis is mine.]

I: What do you mean by that?

Gary: Like..., I cannot be sure about inverse.

I: You could try.

Gary: \( \sqrt{\text{and } ( \text{ })^2}?! \) [square root and the square]

I: What do you mean? Can you be more precise.

Gary: ...[silent]

His immediate responses to many questions were the same in nature. Declaring "I don't know" was the very common and widespread response among students, yet it was irresponsible not a thoughtful one. He needed a push in order to reveal his concept images. Everything seemed blurry to him in the first place. But he came along very nicely. Many students
did not have the habit of thinking about the questions. They usually learned the algorithm of doing the problems, although sometimes there was no understanding behind it. He had a vague but a promising concept image of the inverse of a function. His concept image allowed him to see the function and it's inverse as opposite of each other. His interesting example of √ and (x)² stemmed from his concept images. He had no answer to the question of how one function can be the opposite of the other.

I: How are the domain and the range of the function related to the domain and the range of it's inverse function?

Gary: I don't know, I forgot.

I: Can you think about it?

Gary: I think it was...the domain of...[silent].

I: The domain of what? Go on.

Gary: It's way back, I forgot.

I: You said that the domain of...,you wanted to say something.

Gary: It's supposed to be switched with some...it was equal to the range of something else, but I forget which.

I: We are talking about function and it's inverse and you said that [I was interrupted by him]

Gary: Aha, I remember. It was, the domain of the function is the range of the inverse of the function.

The above questionings could prove a point that students were unwilling to express their thinking because they did not have enough self-confidence and self-respect toward themselves. His answers were used, to show him how the domain and the range were switched. He had a rote learning of the concept. I flipped the page on the space while it was rotated 90°. He saw how the domain of one became the range of the other one (function and it's inverse) since the Y-axis and X-axis were
switched. I described that the function and its inverse were the mirror image of each other with respect to the line Y = X (the text's language). After the tutorial, he answered the rest of the questions correctly. He was surprised to hear that the invertable function was defined by him as: "there is one y for every x and one x for every y".

Gary's answer made it clear that his concept image had necessary components in order to lead him to an accurate concept definition. The barrier was his uncertainty of his understanding. His concept images were disorganized and he was not aware of them.

After finishing the questions on function, he continued with the questions on derivative. It was interesting to see that how many students had difficulty in finding the coordinates of a point. Gary got "(Q-P/h)" for slope of the secant PQ. He was asked why and he then, corrected himself and got it right \(\frac{(x_0 + h)^2 + 1 - (x_0^2 + 1)}{h} = h + 2\).

I: Now, can you find the slope of the tangent line at this point [P(1,2)]?
Gary: \(f(x) = x^2 + 1, f'(x) = 2x\) at \(x = 1\) the derivative is 2.
I: What is the relation between the derivative and this slope? Why did you use the derivative to find the slope of the tangent line?
Gary: The application of derivative is that it defines the slope of the tangent line to curve.

I was anxious to find out how he would define the slope of the tangent line as an application of derivative. Probing questions did not help. He was asked to forget about the derivative for the time being and yet try to find the slope of the tangent line using the same procedure as he did to find the slope of the secant line. The interview was not structured. So mutual discussion was in favor to clarify the matter.
After the tutoring and explaining the slope of the tangent line as the limit of slopes of the secant lines (as Q gets very close to P), he said:

Gary: Ya, but I usually don’t think of that term, I guess you could, but I don’t.

I: How do you think?

Gary: I just think as a road and this has sort of a stationary point and I think of car road that goes, in vision car touches that point \([P]\), and when it gets the point \([\text{pointing } P]\) it stops here and we have a tangent line.

He was the only one who tried to make himself to understand using the concrete examples. The tutoring helped him to state the definition of the derivative as: "The slope of a tangent line at a given point, is the derivative of the function at that point." He then said:

Gary: I was aware of this since I took the course three times, but it’s much easier...if you just use it. It is much easier, much quicker.

Gary had the same difficulty in answering the question of the differentiability as most of the others had. The tutorial helped him to conclude that "Smooth curves are differentiable and there is no derivative at cosps". At the end of the interview, the interviewer said:

I: I am anxious to know if you think that you finally understood the concepts [after taking Math 100 three times] or did you find the way to get a good mark [he got 86% in Math 100 after the third time].

Gary: Ya, more of the latter, it was just to find out what to do in each situation! I think that was most obvious in 'error-estimate tangent approximations', I didn’t learn, I couldn’t understand from the lectures, I saw the examples, I saw what they did and I just did it that way and it worked.

And he then expressed his ideas about the teaching of concept of derivative.
Gary: I think more concrete definition of derivative and to see that the derivative is speed is helping to understand it better. I think we shouldn’t bother this much with the actual derivation or where they come from, and how to get them. More the application. [Emphasis is mine.]

His last comment was important for teachers-researchers to think about it deeply. And may motivate them to do something about it. He said: "In first two terms, I was really frustrated, I didn’t know what the function was, I didn’t know why they are doing those things [Emphasis is mine.]".

Gary’s complaint was expressed in different ways by other students as well. Some of the students were aware of their difficulties and some of them were not. But they all shared the same pain which was their difficulties in understanding the concepts of function and derivative.

4.2.10 Barbara’s Profile

Barbara was a physics major. She got 75% in Math 100. She liked to express herself and she was successful in doing it. Before she started to answer the questions, she changed \( f(x) = x^2 + 1 \) to \( y = x^2 + 1 \) and said: "I call it \( y \). I like \( y \)". In order to get the slope of the secant \( PQ \), she said: "\( y \) of \( Q \) would be equal... Just wild guessing, would be the difference between these two points. many guessings are going on here" (Emphasis is mine.).

She started with the word guessing right from the beginning and this word was used in her interview more than 15 times. She got the slope of \( PQ \) without a major difficulty. She was one of the exceptions that got the slope of the tangent line and the definition of derivative correctly. Yet she still had some difficulties in understanding of the concept. She was asked to investigate the relation between the curve and
its derivative. Her uncertainty could be witnessed by her intention to change her responses from time to time.

Barbara: Well, the derivative will give you so like a tangent and on the curve at a point you pick. So the slope of that point as \( h \) gets very small.

I: What do you mean by the slope of the point?

Barbara: The tangent sort like to the curve.

I: Does a point have a slope?

Barbara: Not really, but the slope of the curve at that point.

I: Does a curve have slope [What she said was right. I just asked to see that if she was trying everything that she knew or if there was any understanding behind it].

Barbara: I guess not.

I: Then the slope of what?

Barbara: Slope of the function, not what the derivative.

Her answers showed that one of her problems was the lack of an accurate (not talking about vigorous) language. In the beginning of the interview she had no trouble getting the slope of the secant and then consequently the slope of the tangent line. But later on, the probing questions revealed that she had trouble understanding the concept of tangency. Same questions about tangency yet, in different contexts were posed on her. The intention was to find out whether she could see the relation between the curve and the line \( L \) tangent to it. She said:

Barbara: How are they [the curve and line \( L \)] related?! It touches it at one point only.

I: What does that mean?

Barbara: I don't know what I mean! [Emphasis is mine.]

I: You said that it touches it at one point, then what do we call this line?
Barbara: What do we call the line? I don't know.

In a tutorial session, the graph was sketched to explain the concept of tangent line and tangency.

I: Let's start from the beginning. As Q gets closer and closer to P, what happens to secant? You said it by yourself [And she was right].

Barbara: The secant is disappeared, it's really disappeared! [emphasis is mine].

I: What do you mean by disappears?

Barbara: Because they are so close at two points. [She did not pay attention to the definition of secant line].

I: We said that these secant lines are extended from both directions, so when P and Q are getting very close, it doesn't mean that the secant PQ disappears, the line is there. The Q is just moved towards P, so it means that it is the same line which has rotated around P. It just hits the curve at one point. What we call this line?

Barbara: I call it the tangent.

The above questions served the writer's purpose, since she believed that the students' prior knowledge were playing significant roles in their further construction of mathematical knowledge. In the beginning, she did not think of the tangent line as a concept, but she remembered how to find its slope by rote. Later on the probing questions put her into a position that she had to think about what she was saying. Then she was frustrated and confused. She did not believe in her words to the degree, that she even said: "I don't know what I mean".

It was hard to understand the concept of derivative without knowing the concepts of tangency, limit, and function. She gave the definition of derivative in response to the "slope of the tangent line", as
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.

I: What is the f'(x)?
Barbara: It is derivative.
I: What is derivative?
Barbara: What is it?... Do you want me to write the definition?
I: No, you gave me the definition as: f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.
What is this f'(x)? You said that derivative is a slope, slope of what?
Barbara: Derivative is the slope of the tangent to a point on the curve.
I: The derivative of function at what point?
Barbara: Any point on the curve, you just pick it and plug it in [Emphasis is mine.]
I: You say any point along the curve. You got the slope of the tangent line at what point? [Emphasis is mine.]
Barbara: At point P there.
I: Which you named it the f'(x) at what point? Is it the f'(x) at any point [Emphasis is mine.]
Barbara: This would be true for any point [Emphasis is mine.]

She did not see the derivative as a dynamic process that will be changed by changing the point on the curve, i.e., changing the lines tangent to the curve. In a tutorial session, I changed the point P(1,2) and drew different tangent lines to the curve. The writer's purpose was to show her that the derivative of function at any point was not the slope of line tangent to curve at a fixed point. Geometric representation helped to establish the fact that derivative has different numerical values at different points.
Barbara: But the tangent changes, going around.

I: Then what happens to the derivative when the tangent changes?

Barbara: Changes with the shape of the curve... ask me the question again.

I: Is it true that the derivative at any point is the slope of the tangent line at a fixed point [emphasis is mine].

Barbara: That is the specific of any which is the general term, that part of any group [emphasis is mine].

I: Means what?

Barbara: The derivative of a function at a point x is the slope of the tangent line at that point (x).

I: This is correct!

It was interesting to see that how the tutorials and probing questions helped her to improve her concept image until the proper (not vigorous) concept definition was reached. In the next questions, she got the derivative of f(x) at x=0 by using the definition of derivative with no trouble.

The question of differentiability caused the same trouble as for most of the other interviewees, the same misunderstandings and the same misconceptions. Her immediate response to the derivative of f(x) at x = -1 (question #7) was: "f'(x) = 1". She was confident in her answer and she wanted to continue. But she was stopped to answer a few questions concerning the f'(x) at x = -1. It was easy to see that f(x) = x + 1 at x = -1 but she did not notice that f(x) = 0 at x = -1 as well. Her attention was drawn to f(x) = 0 at x ≤ -1 but she said:

Barbara: Actually it's nothing to plug in, that's the real thing.

Her answer was quite revealing. She did not know the constant function. She could not imagine that the value of the function was
constant at all points. One of her causing problems was the inadequate concept image of function. It was hard for her to see that there were two different way of writing a function at one point \((x = -1)\).

Barbara: Oh boy, what a question! Something strange coming... Could it be two different answers!?

I: What do you think?

Barbara: I'm not gonna argue with that, it's just there.

I: How do you interpret this?

Barbara: ...[silent].

In a tutorial session, the derivative as a slope of tangent line was explained again. This was done by sketching different graphs. The question was repeated again.

I: What does it mean when the function has two different values for the derivative at one point?

Barbara: Then there is two tangent at that point.

The tutorial went on discussing about the derivative and it's relation with the slope of the tangent line. She was showed that there were so many tangent lines passing through the point \(x = -1\). After all I asked:

I: Is this function differentiable at this point \([x = -1]\)?

Barbara: Yes, something exists, so much exists...so much exists. [Emphasis is hers by changing her tonation.]

Barbara did not know that the existence of the derivative means the existence of limit (even she defined the derivative as the limit of slopes). And the existence of the limit requires that the left limit and right limit should be equal, i.e. there should be only one tangent line
passing through the point in order for function to be differentiable at that point.

I: What is the condition for a function to be differentiable?

Barbara: I guess at point x, there should be only one answer for the derivative [Emphasis is mine.]

I: What is that answer for f'(x) at x = -1?

Barbara: What is that answer?!! We call the slope of the tangent at that point.

I: So you are saying that there must be one tangent line at that point, but there is not one tangent line at that point. There are so many, so is this function differentiable at x=-1 or not.

Barbara: That's weird, doesn't reply that, it should be one, but this guy has more than one.

I: Then what do you conclude?

Barbara: Is not differentiable...[no trust in her words]. It is, here is the answer.

Her doubt was natural. Her eyes agreed that function was not differentiable because she saw that there were so many tangent lines passing through x = -1. But numbers were convincing reason for her to believe that the derivative did exist since she got the numerical value for it.

I: Why is it differentiable and why is it not?

Barbara: I guess it is, because...[silent for few minutes].

Barbara: What does differentiability mean any way?!

I: You just told me!

Barbara: Oh, about the tangent business?! We have so many tangents though, oh, what comprehensive!
She was one of the exceptions that let her concept images to be revealed by themselves.

Barbara: I guess it is, isn't it? Oh, I have to take it overnight and get back to you. I've never seen anything like this, it is good stuff, yes or no?!...no I guess.

I: Why no? If it's no, why not? If it's yes, why yes?

Barbara: I think, if it's yes, because it existed, like you calculate it...

I: We calculated and we came up with two different answers.

Barbara: Oh, my lord..., and it seems to like a range of slopes of that tangent line.

I: What do you mean by the range of slopes?

Barbara: O.K. I say no, there isn't, because there are too many tangent lines.

The existence of derivative was discussed with her in a tutorial session. By showing and proving to her that the right limit and left limit of \( f(x) \) at \( x = -1 \) were not equal (The same discussion in Math 100 text at U.B.C). She saw that the function was differentiable all along the line \( f(x) = 0 \) and all along the line \( f(x) = x + 1 \) as well, but not at point \( x = -1 \) although \( x = -1 \) belonged to both lines. The sharp points and the smoothness of a function were brought up by her. She finally said:

Barbara: Consider at the pick, I guess not [the \( f(x) \) is not differentiable at \( x = -1 \)].

Barbara concluded this section by her following comment:

Barbara: That's neat. I never knew anything like that, nobody told us anything about that. It's pretty neat.**

**This question was very similar to those in math 100 text at U.B.C.
Obviously all the discussed questions were the same (in context) as those that she had in her Math 100 class. But it was interesting to see that how fast students could forget mathematics if there were not the reasonable understanding of concepts. Once she said:

Barbara: Oh, Oh, I remember doing that. That was a long time ago, few months ago, it's ancient history [Emphasis is mine.]

Barbara: We should have done this [interview] at the end of December. There were some kind of memory [interview was conducted at March].

The probing questions and the interviewer's intervention at the appropriate times, allowed her to answer the questions on the sections of function. She said that the circle was not a function. As a reason she drew a vertical line to the circle. The line was hitting the circle at two points. Her logic was quite right. But the point was that she answered the question by remembering the vertical line not by thinking about the function and it's properties. Her concept image of function was accompanied with the notion of "vertical line". Her last comment was interesting. She said: "I got 75% in Math 100, it's good, but that doesn't really reflect like...you know.

It was good to see that she was aware of her weaknesses which is a good start to understand the meanings and concepts.

4.2.11 Nick's Profile

Nick was in first year science. He did well in Math 100 (he got first class). He did not have calculus at high school, yet he was in honor class. He was thinking of majoring in either biochemistry or computer science. He said that he liked mathematics. He was a bit
nervous in the beginning of the interview. We talked about different things till the ice melted between us and he felt comfortable. Then the interview started. The interviewer asked every single interviewee to say the things that they were writing. Some of them did not seem to like this idea and kept quiet. Nick was one of those who mostly wrote rather than talk. At least 1/3 of the tape was filled with the silent moments. He did not like to be interrupted while he was working on problems. His wish was respected unless both Nick and the interviewer felt that the tutoring was necessary.

The interviewees' difficulties were very much alike. Nick like most others had trouble finding the y-component of the point. For finding the slope of the secant line, he was looking for "rise" and "run", but he said: "Here the trouble comes. I don't have the value (she meant y-difference]."

Before he started to find the slope of the secant PQ, he said that he could differentiate the \( f'(x) \) to get the slope of the tangent line. But he had difficulty to understand that the slope of the tangent line at a fixed point \( P \) was the limit of the slopes of the secants \( PQ \) while \( Q \) got closer and closer to \( P \). In a tutorial session, an effort was made to explain that the slope approached to its limit when the distance \( h \) approaches to 0. If he understood the concept of limit, he would have no major difficulty in finding the slope of the tangent line. His answer after the tutorial proved this claim when he found the slope of the tangent line without much of a problem. He only had trouble in substituting for \( f(x+h) \) because he had difficulty in understanding the behavior of function. Then the question of the derivative was raised.

I: What is the relation between this curve and it's derivative?
Nick: The derivative is the slope of the tangent line to a point on the curve, if it's defined in that point.

I: The derivative of function at what point? Can you give the definition of derivative from what you said.

Nick: The derivative of a function at the point tangent to it's secant line is the slope of the secant line.

His vague answer needed to be clarified. Probing questions revealed some aspects of his concept images. His concept images made him to believe that the value of the derivative of a function will not be changed at various points.

Nick: Shouldn't be one derivative for one function! [Emphasis is mine.]

He knew that the derivative was the slope of the tangent line, but he did not pay enough attention to the fact that there were different tangent lines at different points on the curve so there were different values of derivative associated to each of those points. In two tutorial sessions, the geometric representation was used and different tangent lines were drawn to explain that the derivative of function had different values at different points of tangencies. He again said:

Nick: Shouldn't be one derivative for this function?! [Emphasis is mine.]

He did not believe in his words. He had a mixed up concept images of derivative as a function and derivative as a numerical value. After further tutoring and mutual discussion he at last said that: "My definition would be wrong".

His conclusion was a reasonable evidence to prove that the tutoring helped him to develop his concept image toward acquiring of the correct
concept definition. He then believed in (not just accepted) the fact that the derivative of the function had different values at different points. He got the derivative of \( f(x) = x^2 + 1 \) at \( x = 0 \) using the definition of derivative. Because of his lack of understanding of function, he again had a hard time to substitute for \( f(x+h) \).

For the question of differentiability (#7) he was wondering whether to use the definition. He felt confident to use the definition of derivative since he understood it. He then asked if he could answer the question by using the graph.

Nick: Oh, I can do it from the graph.
I: That's why I sketched the graph, you can visualize it.
Nick: At \( x = 0 \) is \( 0[f'(x) = 0] \), at \( x = -1 \) is 0 again \( [f'(x) = 0] \).
I: Look at your function again [see Figure 3.1]. The function is 0 if \( x \leq -1 \) and here [pointing the function and its graph] \( f(x) \) is \( x+1 \) if \( -1 < x < 0 \).
Nick: So anyway, \( f(x) \) is 0, we substitute \( x \) into it.
I: What is the \( f'(x) \) at \( x = -1 \).
Nick: At (a) \( f'(x) = 0 \) (at \( x = 0 \)), and at (b) is 1 (at \( x = -1 \)), [He referred to the parts (a) and (b) of question #7].
I: Then what is the \( f'(x) \) at \( x = -1 \).
Nick: Should we change it a little bit?! [He meant changing the function.]
I: We don't want to change it. We want to keep it this way.
Nick: I don't know.
I: Can you see it from the curve?
Nick: No response.
He was stuck. Silence was his answer to the question. Later on he asked for tutoring. In a tutoring session, I explained to him that the derivative of function at x = -1 had two different values. I related this to the definition of derivative and how the limit \( \frac{f(x+h)-f(x)}{h} \) varied if the approach to the point x = -1 were from left or from right.

I: Then does f(x) have derivative at this point?
Nick: I don't think there is a derivative.
I: Why?
Nick: Adams*** said that it is a sharp point, because there is no tangent line.
I: Actually you can draw so many tangent lines, because by tangent line you mean secant passing through one point on the curve.
Nick: I guess I know it now.
I: What is the condition for a function to have derivative?
Nick: Not be sharp, just one tangent line, Adams said, continuous, smooth interval [Professor Adams is teaching Math 100 and math 101 at U.B.C. His calculus book is the text for Math 100 and 101 at U.B.C.].

He referred to Professor Adams many times. He did not have enough self-confidence. I think that was partly the reason that he referred to Adams as a superior, to put him in-charge of his words. Also to make his audience to believe in his words, because Adams said so! (He referred to him whether he himself was right or wrong).

For the questions of function, he said that: "I forgot what the definition of function is. Is it for only one x? Is it one to one correspondence or that doesn't come to it? I would say it is".

***Professor Adams is teaching calculus at U.B.C.
He tried to relate his concept image to definition of function. He was saying whatever he heard about the function without understanding them. Probing questions did not help and he asked for tutoring. The simple examples were used to explain the concept of the function. Also the textbook definition and geometric representation of function were described. His answer to the next question (inverse of the function) proved that he did not only agree with whatever was told in the tutoring, but he rather understood them. Because he could restrict the domain and the range of the circle in order to have a function. He also said that the semi-circle was not invertible since there would be two y's for one x.

4.2.12  Ted's Profile

Ted was a science major. He got 60% in Math 100 but he took it again. He said that "I really didn't understand it much, so I took it again in summer and I got B, about 70%".

Ted in his words, was an average student at high school (B in Algebra 12). He did not know many of the interview questions, yet he was eager to understand them. He chose to begin with the questions on derivative. He did not get stuck till he was faced with the question of slope of the secant line. He then said:

Ted: The slope is the... I think, the derivative of the function.
I: Without using the derivative.
Ted: O.K., \( \frac{\Delta y}{\Delta x} \)
I: What do you mean by \( \Delta y \) and \( \Delta x \)?
Ted: Changing of rise over the changing of run.
I: What is rise and what is run for PQ_i?
Ted: Rise is Q_i - P, I say it's 2 units.
I: No, Q is an arbitrary point.

What Ted said was rather interesting. He measured the distance from 2 (Y(P): The y-component of point P) to Y(Q) (The y-component of point Q). He did not notice that Q, was an arbitrary point. He assumed each space as one unit. He measured the y-differences according to his diagram which was approximately two centimeter and said that "I say it's 2 units".

The lack of generalizability could be seen in his answer. His response was not the only single incident of this sort. This investigator witnessed many other cases that students' responses were the same with this one in nature. He also had difficulty in understanding the coordinates of point and to indicate the point on the plane. He said that "rise is Q_i - P" which means, he substituted a point itself for y-component of point. In a tutorial session, the coordinates of point and other related matters of this problem were discussed. After the tutorial, he had no difficulty to find the slope of the tangent line as:
\[
\lim_{h \to 0} \frac{(x_0+h)^2+1)-(x_0^2+1)}{h} = \lim_{h \to 0} \frac{2 + h}{h} = 2.
\]
Ted explained this part rather well. He justified whatever he did and it was quite convincing that tutorial helped him to fulfill his understanding gaps. The next question was to define the derivative. He said:

Ted: Slope of tangent line is the derivative of function f(x).
I: O.K. Then can you define the derivative of the function?
Ted: It's the, the derivative is the point...no, the derivative is the...Oh, is the...I guess another definition is the tangent to the slope of the line at point x.

He was a bit confused. He knew that the words derivative, slope and tangent were related. His concept image connected them together. His attention was drawn to what he was saying. Ted noticed his mistake and said: "O.K., you got a function f(x), at a point x, the derivative is going to be the slope of the tangent line that touches the graph of the function f(x) at that point".

His correct answer was surprising. The goal was to lead his concept image towards an adequate concept definition. The tutorial session helped him to reach the goal which was the correct definition of derivative by him.

He had an interesting response to question #7. He said that the f'(x) at x = -1 was 0 and 1. I asked for his reasons. He said that: "Well, I can see it and it is 0 (he pointed the Figure 3.1)".

His answer was interesting. He could look at the graph and see that the derivative of f(x) = 0 was 0 at x = -1 (f'(x) = 0) since f(x) = 0 was a line with a slope of zero. Yet he used formula to get the derivative of f(x) = x + 1 at x = -1, as f'(x) = 1. He got two different values for derivative of function at the same point (x = -1). He did not know what he should do. He believed that the both of his answers were correct. Yet he could not decide which one can be representative of the value of the derivative at x = -1.

I: How can you interpret this? The function has two different derivative at one point, at x = -1 you got f'(x) = 0 and f'(x) = 1.
Ted: It’s discontinuous at this point... Oh no, forgetting what they taught in calculus book, is a corner here [Pointing x = -1 in Figure 3.1].

I asked him the reason that why a function was not differentiable at the corner. His answer was a correct one. That there was no suitable tangent line to the function at x = -1.

Interview continued as we moved to the questions on function. He gave correct answers to the questions on functions. He said that:

\[ x^2 + y^2 = 1 \] is not a function, "because for one value of x is two y values. So it can’t be a function".

He had a good concept image of inverse function and reciprocal of function. He showed, geometrically, the necessary condition for a function to be invertible. He asked for more tutoring about the related concepts to the concept of function, (such as one-to-one function). He was eager to know, even though the time was up and there was no more tape to record on.

4.3 Response Category

This section is divided into two subsections:

4.3.1 Categories of responses to the question of derivative.
4.3.2 Categories of responses to the concept of function.

Each category will be defined and the students’ responses will be quoted whenever it is required. It should be mentioned that probing questions and the interviewer’s intervention helped students to improve their concept images. For example, in response to question #7, almost all the students had difficulties to understand the concept of differentiability yet their later answers were mostly correct.
4.3.1 Categories of responses to the question of derivative

Response categories on concept of derivative and its related concepts are discussed in this section. Following sections define these categories:

Categories of responses to the definition of derivative.

Categories of responses to the concept of slope of the tangent line.

Categories of responses to the question of "How the slope of a tangent line and derivative are related."

Categories of responses to the question of differentiability.

Categories of responses to the question of derivative

Four main categories of students’ responses to the question of derivative have been identified.

Category I Definition of derivative - textbook definition

Two of the students gave the textbook definition of derivative as:

\[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \]

Traditionally, using this definition to get the derivative of a function is one of the questions on the final exam at U.B.C. (The copies of final exams in Math 100 and Math 101 are available from the Department of Mathematics in U.B.C.). Those students who gave this definition, had difficulties understanding the concept of derivative. Their understanding was rote. For them this definition was a tool to express the derivative. This concept definition did not lead them to acquire the proper concept image.
Category II  Derivative as rate of change - velocity

Only one of the student referred to "rate of change" to define derivative. This student had a physical interpretation of derivative. In fact his concept image could lead him to the concept definition if they were developed.

Category III  Derivative as a "slope"

This category consists of those responses in which derivative and "slope" had the same place in the students' minds. One of them defined derivative correctly in terms of slope of the tangent line:

"The slope of a tangent line at a given point, is the derivative of the function at that point".

While some others' responses were not accurate, for example:

"The derivative is the slope of tangent line at point P".

Two of the students simply said that, "derivative is slope of the line at a point given". Seven students' responses have fallen into this category.

Category IV  Derivative as a rule of differentiation

Students know how to use derivative as a tool to do their computation. One of the students gave the formula as definition of derivative. The following quote is self-explanatory:

"No, I don't [defined derivative], I can do it, \( f(x) = x^2 + 1, f'(x) = 2x \). My teacher did the proof, but I didn't understand it, but I can do it" [Emphasis is mine.]
And later on she said:

"in terms of limit?! Oh, Ya, we did the definition of derivative... it's from limit. I don't know... you know, when you do it, you don't think about this, you do it so mechanical [Emphasis is mine.]

Categories of responses to the concept of slope of the tangent line

Two main categories were distinguished based on the students' responses to the questions of slope of the tangent line.

Category I  Slope of the tangent line as derivative of function

Almost half of the students used the formula to get the derivative which they then call that the slope of the tangent line.

"OK, f(x) = y = x² + 1, f'(x) = dy/dx = 2x... There is only one point, I can't use the same thing here, but I can give you the definition of derivative. This is giving you a right answer".

It is interesting to know that the above quote belongs to one of the students that gave the textbook definition of derivative. If he understood that definition, he would not say that "I can't use the same thing here". He defined derivative as: "f'(x) = \( \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} \)" but he was not sure that why h approached to 0, or what was the role of limit in this formula.

Category II  Slope of the tangent line as the limit of slopes of other secants

Many students had difficulty in understanding the concept of tangent line. Only two of them responded correctly. "I guess h approaches to 0, then if we take the limit \( \frac{f(x+h)-f(x)}{h} \) while h→0
we can get the slope of tangent line".

Three others responded correctly after the tutorial session. Some of the students had an idea that the slope of the tangent line should be the limit of the slopes of other secants. Their stumbling block was the lack of understanding the concept of limit and tangency.

Categories of responses to the question of differentiability

The following three major categories present the students’ responses to the question #7 and why \( f(x) \) is not differentiable.

Category I  
**The right limit and left limit are not equal**

Only one of the students’ responses falls into this category.

Oh, Ya, you have to do limit or something... as \( x \) approaches from one side approaches to 1 and then \( x \) approaches from the other side, approaches to 0, [Derivative] doesn't exist, no?! Because approaches to two different numbers.

The above quote shows that he understood the concept of derivative because a function is differentiable if derivative of function exists at that point. By definition of derivative, this existence is equivalent to the existence of limit \( \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} \). He knew that for existence of limit, the right limit and left limit should be equal.

Category II  
**There is more than one tangent line**

"No, there isn't, because there are too many tangent lines".

"No, it doesn't have derivative at that point then?...because it has two different tangent lines at one point".
The responses of this category are not the same as those in category I, since in here they are talking about the existence of tangent line and category I talks about the existence of derivative. Yet the right answer obtains from both categories.

**Category III**  
**Function is not differentiable at sharp points**

"It is a sharp point, because there is no tangent line".

"Probably, because it’s not a smooth curve. Two curves chain together".

"It is a sharp point, because there is no tangent line".

These responses stem from a practical and concrete understanding of the concept of differentiability. It is more convenient for students to look at the graph of function to see that whether it is smooth or it is sharp at some points.

**4.3.2 Categories of responses to the questions of function**

Students’ responses to the question of function is categorized into four major ones.

**Category I**  
**Some elements of the formal definition of a function**

"It’s not a function, because for one value of x is two y values, so it can’t be a function".

"It’s not a function, because for every x there is two y, perpendicular line".

"It’s not a function, because if we draw a perpendicular line, there is two value of y for one x".

"Yes, and it intersects the y axes at two points, so it’s not a function".
"$\sqrt{1-x^2}$ or $\sqrt{1-x^2}$, two different functions put together".

"Something defines the y values, certain values for certain given y's. You are giving value x, and function defines the value of y".

The purpose of the interview has been to investigate the student's understanding of function by means of an example. The responses show that these students have some proper concept images of function which could lead to their developing of an appropriate concept definition; yet the above responses in themselves contain only some of the elements of the concept definition of function.

**Category II Function as a relation between two variables**

This category represents a function as a relation between two variables. In fact function is a relation between x and y such that for each x there is only one y. The following quotes show that the concept images of these students see a function as a relation without having the restriction that for each x there is only one y.

"Yes, sure, because I could draw a circle, that's all".

"Because...they are related?!! I don't know, the equation is a function".

**Category III Function as an algebraic term, an equation**

The concept images of those students whose responses fall into this category, view the function only as an algebraic function. Although they have studied the transcendental functions (i.e., non-algebraic function such as exponential function, trigonometric function and logarithmic function).
"You mean if I solve it for x? or if I solve it for y? It is a function, you mean if you solve it, what would be the function of x or what could be the function of y".

"Function is just an equation".

Category IV \hspace{1cm} Idiosyncratic responses

This category consists of a variety of responses. The following quote: "I know that it's a business of drawing line", shows that her concept image of function was limited to a line without even remembering that whether the line was vertical or horizontal to the graph. Although drawing a vertical line is a good test to check that whether or not a relation is a function (if vertical line hits the graph of relation at two points, it shows that a relation is not a function).

The following is an example of another idiosyncratic responses to the question of function:

Owen: "No, it doesn't, \( y = \sqrt{1-x^2}, y^1 = \frac{1}{2}(1-x^2)^{-\frac{1}{2}} (-2x) = -1(\frac{x}{1-x^2})^\frac{1}{2} \)"

I: Why did you take derivative?

Owen: "I don't know, just instinct".

"Yes, it is a function, because it has a certain number of points?!!"

His response (as he called it instinct) is reminiscent of a quote from Kalmykova who was interviewing a pupil. He said that (the pupil) "When I cannot arrive at the answer to the problem, I begin to add, subtract, multiply, or divide the numbers until I obtain the right answer" (Kalmykova, 1975, p.2).
The question of inverse function and reciprocal of function caused students great deal of difficulties. Five of the interviewees thought that $f^{-1}$ and $1/f$ are the same while four of them said "I don't know". Two of the students did not have extra time to finish this part. Only one of the students responded correctly. Except for two students who did not finish this part, and one student who had trouble with symbols, the rest of the students had no difficulty to answer the questions on composition of function.

4.3.3 Summary of Results

Calculus courses are mostly designed in a way to cover many topics in a limited time. In calculus classes there is not enough time for mutual discussion between the instructor and the students. Many concepts are considered to be known by the students prior to their enrollment in Math 100, such as finding the coordinates of a point.

For example, students were given points $P(1,2)$ and an arbitrary point $Q$ on the parabola and were asked to get the slope of $PQ$. Five out of twelve students whose marks in Math 100 ranged from 60% to 90%, wrote "$Q - 2$" or "$Q - P$" for the difference in $y$ (the rise), while finding the difference in $x$ (the run) caused them no difficulty since it was a matter of a simple subtraction: $(x + h) - x = h$. Probing questions and tutorial sessions helped them to understand how to find the $y$-component of $Q$. Later on all of them got the slope of $PQ$ correctly.

Another difficulty for many students was seeing the tangent line as the limit of other secants and seeing its slope as the limit of the slope of other secants. When $Q$ moved along the graph and got closer and closer to $p$, one of the students said, "The secant has disappeared, it's really
disappeared". In his discussion of students' misconceptions of the tangent line, Orton (1983) also found that the students sometimes viewed the tangent as the disappearance of the secant.

As a first response, the subjects of this study all differentiated \( f(x) \) to get the slope of the tangent line. They were advised to follow the same procedure that they did for getting the slope of the secant lines. Only a few of them, then, got the slope of the tangent line as the limit of other secants.

In order to get the slope of the line tangent to \( f(x) = x^2 + 1 \) at \( P(1,2) \), all the students offered to either give the definition of derivative or differentiate \( f(x) = x^2 + 1 \), yet they mostly did not know the relation between derivative of function and slope of line tangent to it. Many of the students viewed the derivative as a useful tool with many applications. Three out of twelve students did not believe that the derivative of a function at different points has different values. Some of the students were given the graph of \( f(x) = x^2 + 1 \) which had many tangent lines drawn on it. They had trouble finding the derivative of \( f(x) = x^2 + 1 \) at different points of tangency. One of them said: "shouldn't [there] be one derivative for one function"? It was hard for them to conceptualize that the function of derivative remained the same while its numerical values were changed by changing the point of tangency.

Students' difficulties in understanding function, hindered them in their understanding of derivative. They had difficulty finding the y-component of \( Q \) on the parabola \( f(x) = x^2 + 1 \) with arbitrary coordinates, since a majority of them did not know the behavior of this function. In the question on differentiability, (#7), the function
caused them more difficulty since it was not defined under one rule of correspondence (see question #7, Chapter three). Also, it was hard for them to understand the concept of function.

The graph of function (#7) helped some of the students to answer the question correctly. For those subjects who had difficulty in understanding the concepts of tangency and limit, grasping on the concept of differentiability was harder.

None of the students had difficulty in doing straightforward computation of the derivative (#6).

The equation of a circle \((x^2 + y^2 = 1)\) was given and the subjects were asked to determine that whether or not the circle represented a function. The immediate responses of seven students was "yes" to the question. Three of them drew vertical lines and said that circle does not represent a function since the line hits the graph at two points. Only two of them explained that since there are more than one \(y\) for every \(x\), the circle is therefore not a function. After the tutorial sessions, most of the subjects restricted the domain and the range of \(x^2 + y^2 = 1\) (circle with radius 1) in order to have a function. They mostly chose the upper semi-circle as their desired function. Two interesting answers of the same nature were given for this question. One of the students said: "make it [circle] a line, that's the only way" and the other one said: "you could expand it [circle] just like that". They both thought that as long as the length of circumference of circle was conserved, they were allowed to make any changes to the graph. They did not understand the behavior of \(x^2 + y^2 = 1\) which means that this relation between \(x\) and \(y\) should be conserved.
Later research questions drew students' attention to the definition of inverse function. They were asked to check that whether or not the inverse of the semi-circle was a function. A number of subjects said that the function and its inverse are mirror image of each other. Three of them said that the domain and the range of the function must be switched with each other. Those who used the term "mirror images", said that function and its inverse were "mirror image" of each other. Although what they said was true, using this term does not imply that students understood the concept of invertable function. They were asked to show and explain their responses by using the graph of semi-circle. They gave incomplete explanations as to what they meant when they used the term "mirror image". After the tutorial and discussion, many of them restricted the domain and the range of semi-circle and determined the quarter-circle to be a function whose inverse is also a function. After they did it, this writer told them that they had now defined the invertable function as one which has to be a one-to-one function. They were very surprised by this result.

For most of the interviewees, the difference between the inverse of a function \( f^{-1} \) and reciprocal of a function \( 1/f \) was not clear. Most of them could not distinguish between \( 1/f \) and \( f^{-1} \). Their concept images dealt with \( 1/f \) and \( f^{-1} \) as if they were the same. One of the students said that \( f^{-1} \) is like \( x^{-1} \) which can be written as \( 1/x \).
CHAPTER FIVE
CONCLUSIONS, EDUCATIONAL IMPLICATION AND RECOMMENDATIONS
FOR FURTHER RESEARCH

5.1 Summary of the Study

The main objectives of this study have been:

A. To identify the nature of the students' conceptual understanding of the concepts of derivative, function and their other related concepts.

B. To develop a category system for those understandings which might provide some insight into the nature of the difficulties experienced by students learning these concepts.

Twelve first year* university students were interviewed in this study. The collected data were analyzed and the individual profiles were produced for every interviewee. Students' responses to the questions on derivative and function were then categorized to enable the researcher to look at the degree of students' progress in acquiring the proper concept images that may lead them to concept definition.

5.2 Method

The purpose of present study has been to investigate the nature of students' understanding of the concepts of function, derivative, and other related concepts. The researcher's aim has been to choose a method of collecting data that enables her to see the students' processes in constructing their mathematical knowledge. What students did in solving the problems discussed during the interview was not the investigator's only concern, but she also was interested in how they did it and why.

*Some of them were in second year but they were all taking Math 101 at the time that interview was conducted.
Incorporating some instruction along with the conventional aspects of a clinical interview, provided the opportunity to address this concern. The tutorial sessions and probing questions helped students to improve their concept images.

5.3.2 Tutorial Session

This writer believes that teaching has an essential role in students' formation of concept images particularly in an area like calculus. The tutorial sessions, in which the researcher acted as a teacher and provided some instructions for the students, seemed to be effective for revealing the students' concept images of function and derivative. Some of the students preferred to quietly write the answers to the questions, or more often, their immediate responses to specific questions were simply "I don't know". While some aspects of students' concept images were revealed by probing questions, in other instances some students appeared to be quite confused about the issues being discussed. In such a situation, the tutorial sessions were provided to enable the students to better understand the mathematical concepts being discussed. These sessions were judged to be useful if they helped the students to acquire the adequate concept images.

5.3 Conclusions of the Study

A number of tentative conclusions are offered in this section. They are presented in two different sections in the same order of the research questions.
5.3.1 The nature of the students' conceptual understanding

The following conclusions are summary statements obtained from the data analysis presented in chapter 4.

The nature of the students' conceptual understanding of derivative

1) The students had very little meaningful understanding of the concept of derivative as was evident in terms of the types of responses given in the interview setting.

2) The students, except one, did not have a physical interpretation of derivative.

3) For some students, the algorithm of differentiating the function became the definition of derivative.

4) Some of the students did not believe that the derivative of a function at different points has different values.

In general terms, then, their concept images were based upon the notion of derivative as a rule which assigns a number to each function even though derivative is a rule which assigns a new function \( f^{-1} \) to each function \( f \).

The nature of the students' conceptual understanding of the slope of the tangent line

1) Almost half of the students believed that there was a relation between the slope of the tangent line and derivative. Although most did not know that the slope of the tangent line is the derivative of a function at point of tangency.

2) Some of the students viewed the tangent as the disappearance of the secant line.
The nature of the students' conceptual understanding of differentiability

1) The students' difficulties in understanding differentiability was caused by their lack of understanding of the concepts of tangency and limit, also by their lack of knowledge of the properties of a function and, in particular, the constant function.

2) Those who said that a function was not differentiable at sharp points and that the smooth function was differentiable, appreciated the geometric representation since their geometric intuition helped them to do so.

The nature of the students' conceptual understanding of function

1) A number of students held proper concept images of function which should lead to the development of an appropriate concept definition.

2) Few of the students, understood function as a relation between two variables (without having the restriction that for each x there is only one y).

3) For some students, a function was only considered to be an algebraic function.

4) For most students there appeared to be a conflict between the students' concept images and the concept definitions of inverse function \( f^{-1} \) and reciprocal of function \( 1/f \).

5.3.2 A system for categorizing students' understanding

1) The students' understanding of the concepts of derivative and function were varied in nature which enabled the researcher to categorize their responses. The criterion used to categorize the students'
understanding was their degree of closeness of the responses to the concept definition. These categories appear to be useful from an instructional point of view for two reasons:

(a) they provide the researcher with some insight into understanding the conceptual difficulties experienced by the students in the interview setting and hence enhanced the fruitfulness of the tutorial sessions;

(b) they appear to make some intuitive sense from a mathematics point of view since they structure student responses from more primitive conceptions to more sophisticated ones.

5.4 Discussion and Implications - Significant Issues

The concept of limit was quite hard for students to grasp. Furthermore, they need to have an adequate concept image of limit, that is an intuitive understanding, to enable them to understand derivative. This writer believes that limit should not be presented by its concept definition. The abstract definition in the absence of students' proper concept images will not be of great help to students.

Math 100 is concerned primarily with the derivative and its applications. It enables students to do all sorts of differentiation and to mechanically substitute different values of x in the derivative function to get different derivatives. None of the students failed to answer question #6 where they had to use the chain-rule to compute the derivative. The question aimed to check on the students' skill in applying differentiation rules. The subjects seemed to be quite confident working with the derivative. To them, the derivative was a useful tool to do all sorts of things.
In the Math 100 class, the concept of function and limit are usually taught in a short period of time. About two weeks of class is devoted to the teaching of limit, function and all their related concepts (see Appendix C). Function and limit are pillars of calculus. Without them, students will not be able to understand such concepts as derivative. In the interview section on derivative, the students experienced great difficulties because of their lack of understanding of function. One of the educational implications of this study is for instructors to spend more time on the teaching of function and limit in calculus classes. Furthermore, they should use meaningful examples for teaching the concepts. This study shows that students are frequently unable to generalize their understanding since only few routine examples are used by instructors.

Instructors usually only lecture and mark the final exams. Beyond doing the repetitive calculus problems, and getting good marks, students need to be confronted with more challenging and interesting problems. This cannot be done without direct involvement of students and instructors' in the problem-solving process. In this process, the instructors should be focusing on students' difficulties and looking for their causes. Being aware of the nature of students' difficulties, may direct their attention to the possibility of altering their teaching method.

It is important for a teacher to realize the students' prior beliefs and how they might affect their further understanding. Ausubel (1968) says:
If I had to reduce all of educational psychology to just one principle, I would say this: the most important single factor influencing learning is what the learner already knows. Ascertain this and teach him accordingly (p. vi).

The information of students' background and their prior knowledge of concepts should guide an instructor in choosing the way in which to present those concepts to the students. For instance, function is taught in high school in a different way than it is presented in university. It is hard for students to see a connection between the high school definition of function and the one that is presented to them in university. The formal definition of function is not easy for students to understand since for the most part their concept images of function are not adequate.

Concepts are usually presented by their formal definitions. Vinner (1983) said that it is wishful thinking for "the concept image [to be] formed by means of the concept definition and under it's control" (P. 295). This expectation is an idealistic view since many studies show that students usually forget the abstract definitions and, since they do not have a well established concept image, all that remains for them is the technique and some procedural knowledge so that they can do some computational work.

For a majority of students, definitions in calculus are unrelated to their intuition. An example of this is the student who said that, for her, the inverse of a function ($f^{-1}$) and the reciprocal of a function ($1/f$) are the same even though she believed that since there are two different names for them, it means that they are not the same. This contradiction will clearly prevent her from obtaining any real understanding of these concepts.
There are conflicts between students' concept images and concept definitions. An example of this is a student who was asked to inspect that whether or not $x^2 + y^2 = 1$ was a function. She said that the circle $(x^2 + y^2 = 1)$ is a function without giving any reason for it. The interviewer suggested to her to do the "vertical line test" which is used in calculus classes to check whether a graph represents a function. She failed to do the test. The researcher did the test to show her that the circle $(x^2 + y^2 = 1)$ is not a function since the line intersect the circle at two points, which means there are two $y$ values for one $x$. Her response was quite interesting. She drew the vertical line further (with $x > 1$). Her response revealed two aspects of her concept image of function:

1. She ignored the fact that the projection of the vertical line on the $x$-axis should be in the domain of function. In other words, $f(x)$ can be defined for $x$ in the domain of function.

2. She did not know that this "line test" must be true for all lines not some lines. In other words for every $x$ in the domain there should be one $y$ such that $y = f(x)$. It is not enough to show that there are some $x$ and there are some $y$ such that $y = f(x)$. Her concept image of function lead her to view a function as:

   a rule such that for some $x$ in domain (or outside of domain) there is one and only one $y$ in range.

One of the stumbling blocks of students of calculus is their lack of self-confidence in dealing with mathematics. A student computed the derivative of $f(x) = x^2 + 1$ (#5) at $x = 0$ by using the definition of
derivative. He did it correctly but he crossed out his work even though, in question #2, he correctly used the definition of derivative to differentiate \( f(x) = x^2 + 1 \) at \( x = 1 \). He said he was wrong because the answer should be 2. He went through the whole thing again and again (see Appendix D for the copy of his work) without knowing that the value of derivative varies at different points.

Many students complained about the class size. In large classes, there is little mutual discussion between instructor and students. Students passively take their notes and their questions and concerns remain unaddressed. Small classes could provide the opportunity to both students and teachers to have better communications. An alternative model would be a mixture of large classes and tutorial sessions. In this researcher's opinion, the tutorial sessions will be more helpful if they are offered by the same instructor of the large class.

5.5 Recommendations for Further Research

More research should be done to investigate the nature of students' difficulties in calculus. The following are recommendations for further study which aim to identify students' understanding of calculus.

1. None of the students of this subject had taken calculus in high school. The existence of calculus in high school, has been controversial in British Columbia (B.C.) within the last few years. Many studies should be conducted in this area to give some insight into the question as to whether or not calculus should be taught in high school in B.C.

2. The possibility of having a pre-calculus course in high school should be examined. A pre-calculus course could prepare
students for calculus. Since it presents fewer concepts, therefore instructor has more time to elaborate on questions and student has more time to understand them by the means of problem-solving.

3. A majority of students had a weak background in geometry which hindered them in their development of concept images. Research into the nature of students' difficulties in geometry and also the status of geometry in high school is a necessity.

4. Research is needed to investigate the teaching of function in high school.
REFERENCES


APPENDIX B

Oral Consent Form

(To be read or given to students prior to interview)

This project is not connected to your mathematics class in any way. You are not required to respond to all of the questions and if for any reason you feel uncomfortable you can withdraw from the interview at any time without any consequences.
# APPENDIX C

MATH 100 — September - December, 1988

## COURSE OUTLINE

<table>
<thead>
<tr>
<th>Topic</th>
<th>Sections</th>
<th>Hrs</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>I DIFFERENTIATION</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>a functions, domains, ranges, graphs, compositions, odd and even symmetry, inverses</td>
<td>1.2</td>
<td>2</td>
</tr>
<tr>
<td>b limits (intuitive - $\epsilon$-$\delta$ not necessary), properties of limits, one sided limits, infinite limits, limits at infinity</td>
<td>1.3-1.4</td>
<td>2</td>
</tr>
<tr>
<td>c continuity, intermediate value theorem (statement)</td>
<td>1.5</td>
<td>1</td>
</tr>
<tr>
<td>d tangents, normals, derivatives and differentials</td>
<td>2.1-2.2</td>
<td>2</td>
</tr>
<tr>
<td>e differentiation rules</td>
<td>2.3-2.4</td>
<td>3</td>
</tr>
<tr>
<td>f interpretation of the derivative, rate of change, velocity and acceleration, marginals</td>
<td>2.5</td>
<td>1</td>
</tr>
<tr>
<td>g higher order derivatives, differential equations, initial-value problems</td>
<td>2.6</td>
<td>1.5</td>
</tr>
<tr>
<td>h implicit differentiation</td>
<td>2.7</td>
<td>1.5</td>
</tr>
<tr>
<td>i the mean-value theorem (omit proof), increasing and decreasing functions</td>
<td>2.8</td>
<td>2</td>
</tr>
<tr>
<td>j antiderivatives and indefinite integrals</td>
<td>2.9</td>
<td>1</td>
</tr>
</tbody>
</table>

| **II ELEMENTARY FUNCTIONS** | | |
| a trig functions and their derivatives, projectiles, simple harmonic motion | 3.1-3.2 | 4 |
| b inverse trig functions and their derivatives | 3.3 | 2 |
| c natural logarithm and exponential functions | 3.4-3.5 | 3 |
| d general exponentials and logarithms, logarithmic differentiation | 3.6 | 1.5 |
| e exponential growth and decay, logistic growth | 3.7 | 1.5 |

| **III APPLICATIONS OF DIFFERENTIATION** | | |
| a local and absolute extreme values, critical and singular points, first derivative test | 4.1 | 1 |
| b concavity and inflections, second derivative test | 4.2 | 1 |
| c asymptotes, formal curve sketching | 4.3 | 2 |
| d optimization problems | 4.4 | 3 |
| e related rates problems | 4.5 | 2 |
| f tangent line approximation, error estimate, Newton’s method | 4.6 | 3 |
| g indeterminate forms, l'Hôpital’s rules | 4.7 | 2 |

| | Total | 43 |
| | Tests and leeway | 6 |
| | Approximate number of class hours | 49 |

**OPTIONAL TOPICS (if time permits — not examinable)**

| | |
| hyperbolic functions and their inverses | 3.8 |
| parametric curves | 5.1-5.2 |
| vector velocity and acceleration in the plane | 5.3 |

- Calculators, while useful for some parts of the course, will not be required for the final examination.
- Term marks for the course should be based all or mostly on in-class tests, and should be scaled within sections (when the course is finished) so that the final marks for a section have the same median or mean as final exam marks for that section.

ED GRANRER

MATX 1213

Local 3782
APPENDIX D

Written Work From One of the Students' Profiles

\[
\lim_{h \to 0} \frac{\sin(h) - h}{h^3} = \lim_{h \to 0} \frac{\sin(h) - h}{h} \cdot \frac{1}{h^2} = \lim_{h \to 0} \frac{\sin(h) - h}{h} \cdot \lim_{h \to 0} \frac{1}{h^2} = \frac{0}{0} \cdot \frac{1}{0^2} = 0 \cdot \frac{1}{0} = 0
\]
APPENDIX E

Exemplary Transcript From A Student Interview

I: Interviewer   Barbara: Subject

I: Before you start, I like you to read this consent form.

I: How did you do in Math 100?

Barbara: Like marks? You know is good, but...

I: What did you get?

Barbara: I got 75% for the term. But that really doesn't reflect like... you know, so...

I: What about this term?

Barbara: This term? I think [Math] 101 is a bit harder, because you know, in [Math] 100, you just...like chain rule does a chain rule, power rule is power rule, you just gonna do it. That's not so hard, because there is something specific, but in [Math] 101, there are so many methods, you can integrate some functions that they give you, so if your mind having remember what to do or manipulate it, then, oh well, you will make it.

I: What is your major?

Barbara: I don't know. I think I'll gonna go to Physics though.

I: So you will have lots of Math...

Barbara: Ya, I'm taking lots of Math at the same time, because I don't want to go into Biology. I don't like Biology and I don't really like Commerce because I'm not really into the money, so...Math is like...I don't really like it that much, but
it doesn't bother me, but it's more like a tool, you know, when you need it, for...like Physics.

I: You didn’t have calculus in high school, did you?

Barbara: Ya, actually I did about a month of it. My teacher wanted to introduce us to these stuff.

I: Was it in Algebra 12.

Barbara: Ya, just the rich Algebra or something, but he only gave us a month and gave us the test. Because we did differentiating and integrating and it was too much for us, too much. I have forgotten a lot of it.

I: O.K., Now, I like to discuss the concept of function and derivative with you. It's up to you to start with which one. Do you want to start with function or derivative?

Barbara: [Laughing] What is the other choices?

I: Leave it and go [Laughing].

Barbara: Is it [Laughing]?! I was just kidding. You may start with derivative.

Barbara: Do you want me to do it out loud or on paper?

I: It's up to you.

Barbara: I think I need a paper. It's hard to do it mentally.

Question 1: a) Find the derivative of 
   \[ f(x) = x^2 + 1 \] at \( x = 1 \) 
   
   b) Sketch the graph of 
   \[ f(x) = x^2 + 1 \]

Barbara: I love sketching the graphs! I'm good on that.
I: Do you like to go through all of the questions and then come back and discuss each one of them or do you like to finish one question and then move to the other one.

Barbara: When you say discuss it, do you mean my difficulties and...

I: No, I just like to see how you are doing it.

Barbara: To see how I do it!

I: Ya, then if you have any questions, we can discuss it.

Barbara: The questions that I have trouble with?!

I: Ya.

Barbara: O.K., you want me to read this: [question 1], O.K., so...I guess, \( f(x) = x^2 + 1 \), \( f'(x) = 2x \) at \( x = 1 \), \( f'(1) = 2 \). I like that [laughing]. O.K., sketch the graph of it.

I: Yes.

Barbara: O.K., I call it \( y \), because I like \( y \) [She wrote \( y = f(x) = x^2 + 1 \)] and this is a parabola [She sketched the graph correctly].

I: Right on!

She moved to question 2:

2) The diagram shows the graph of the above function and a fixed point P on the curve (Parabola). Lines, PQ are drawn from P to point Q's on the Parabola and are extended in both directions. Such lines across a Parabola are called secants, and some examples are shown in diagram.

a) How many different secants could be drawn in addition to the ones already in the diagram? [See Figure 3.1]

b) As Q gets closer and closer to P, what happens to the secant?
c) Find the slope of PQ.
   Find the slope of PQ₂.
   ...
   ...

d) Find the slope of L at point P = (1,2).

She first read the question.

Barbara: Lines PQ's?

I: Yes, PQ₁, PQ₂, PQ₃...

Barbara: What do you mean [line extended from] both directions?

I: I mean it's not bounded, it's not limited, you know, you can extend as [I was interrupted].

Barbara: Oh, Just a line with arrows.

I: It doesn't have two arrows...

Barbara: Oh, O.K., not a segment, a line.

I: Yes.

Barbara: Such lines across a parabola are called secant [reading a part of the question again], O.K., that's true, ya...How many different secants could be drawn in addition to the ones already in the diagram? Through point P you mean?

I: Yes.

Barbara: How many?! I guess...a lot [Emphasis is hers], infinite numbers I guess, because you can just...you know, angle it or something. Do you want me to write that down?

I: No, it's o.k., because I have it [I am recording it].

Barbara: As Q gets closer and closer to P, what happens to the secant? Oh,
becomes more like a tangent [laughter].
That’s what I think any way.

I: O.K., that’s correct.

Barbara: Find the slope of...here to here?!
[She showed PQ₁ on the graph].

I: Yes.

Barbara: O.K., what is this point
[Pointing P].

I: This is point P = (1,2).

Barbara: That’s (1,2)? O.K., O.K., so this minus this. What is the height of this [Pointing Q₁].

I: The length of this [Q₁] is x₀ + h.

Barbara: This distance? [she showed x₀ + h on the graph, but she was not sure].

I: Ya, the whole thing.

Barbara: O.K., I just redraw it.
And you want the slope of this [PQ₁]?

I: Yes.

Barbara: Oh, it’s that the function? [She meant f(x) = x² + 1

I: Ya, the function is f(x) = x² + 1.
The diagram shows the graph of the above function which is f(x) = x² + 1. [The figure was in front of her].

Barbara: O.K., so I guess y at Q₁ would be equal...Just wild guessing now...I guess y of Q₁. I have no idea...well, I guess it will be difference between these two points I guess. Many guessings are going on here [laughter]. That’s [slope of PQ₁]

\[
\Delta y = Q₁² + 1 - 2
\]

\[
\Delta x = 1 + h - 1
\]

I: What do you mean by Q₁² + 1?
Barbara: What is this \( Q_1^2 + 1 \) for...?!

I: \( Q_1 \) is the point right? Then what do you mean by \( Q_1^2 + 1 \)?

Barbara: Oh, Ya, you're right, you have to put \( x \) in eh...O.K., so that's \( (1 + h)^2 + 1 \), ya, that's more like \( h \) something that we were doing. That's it.

I: This is \( [(1 + h)^2 + 1] \) height of \( Q_1 \).

Barbara: Ya, ya.

I: Then what would be this distance [the amount of rise]. See what did you get for it [she got \( (1 + h)^2 + 1 \)].

Barbara: Oh, I forgot -2, O.K., so:
\[
\Delta y = \frac{(1 + h)^2 + 1 - 2}{h} \\
\Delta x = h
\]
I also make lots of middle errors... [laughter]. Do you want me to finish it?

I: Yes, please.

Barbara: Slope?...Last term memories are coming back. We take the limit I guess.

I: What is this [I showed her what she got as \( \text{rise} = \frac{(1 + h)^2 + 1 - 2}{h} \)]

Barbara: Oh, that's the slope.

I: O.K., then...

Barbara: That's it?!

I: Are you looking for something else.

Barbara: I don't know!
I: Because you told me that this is 
\[ [(1 + h^2 + 1 - 2) \frac{1}{h}] \] slope. You just finish it up.

Barbara: Oh, Oh, I know what are trying me to do, 
\[ l^2 + 2h + h^2 - 1 = 2 + h?! \]

I: O.K., that's fine. What would be the slope of PQ_2.

Barbara: [Laughter] No! No!

I: PQ_2 is another secant.

Barbara: Are they all about the same?!

I: Usually?!

Barbara: What do you mean that they are all the same?

I: What would be the x part of this part?

Barbara: What would be the x part of this part?

I: Just another h. x_0 plus another h. We can name it h_1, h_2, whatever, just another h.

Barbara: Oh, different h?!

I: We named this distance x_0 + h, so when we move down, you know, then this distance will be changed. So this is not the same h, it would be another one.

Barbara: Ya, so I guess slope of line [PQ_2], will be 2 + h_1. I call it h_1.

I: O.K., then what will be the difference between 2 + h and 2 + h_1.

Barbara: The h part.

I: What happened to h part.

Barbara: It gets smaller.

I: O.K. Now find the slope of L at P(1,2).
Barbara: Find the slope of L?! O.K...
\[ f'(x) = 2x \text{ and } f(1) = 2. \]

I: No, you already got that.

Barbara: Oh, Oh, you want me to actually do it!
O.K., I guess \( h \) is approaching zero,
so it would be one of these things again, so it would be \( 2 + h \). We take limit as \( h \) approaching zero, because it's too close like point [Q] coming down down down and then get 2.

[Emphasis is hers.]

I: O.K.

Barbara: So I pointed up there.

I: O.K., Now, can you give the definition of derivative.

Barbara: Oh, O.K., what I do...
\[
* f'(x) = \frac{f(x_0 + h) - f(x)}{(x + h) - h}
\]

I: It's not \( x_0 \), because you said \( f'(x) \).

Barbara: Ya, O.K., O.K., inconsistency!
That was 1 so there is only \( x \), so
\[
f'(x) = \frac{f(x + h) - f(x)}{(x + h) - x} = \frac{f(x + h) - 1}{h}
\]

I: Remember what did you do here!

Barbara: Oh, Ya, where am I?!

I: You just simplify it. It was \((1 + h)^2 + 1 - 2\), you just add them up together and got that.

I: I like you to give me the definition of derivative in general.

Barbara: Oh, in general, it is \( \frac{f(x + h) - f(x)}{(x + h) - x} \)
I guess which is really \( h \) at the bottom.
I: What is the difference between whatever you got here and...[I was interrupted].

Barbara: Oh, \( f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \).

I: O.K. Now can you tell me that what is the relation between this curve and it's derivative?

Barbara: You mean the parabola?!

I: Yes.

Barbara: And it's derivative!...Well, the derivative will give you so like a tangent and on the curve at a point you pick. So the slope of that point as h gets very small.

I: What do you mean by the slope of the point.

Barbara: The tangent sort like to the curve.

I: You mean slope of a point?

Barbara: Yes.

I: Does a point have a slope.

Barbara: Not really, but...the slope of the curve at that point.

I: Does a curve have a slope?

Barbara: I guess not. [Laughter.]

I: Then the slope of what?

Barbara: Slope of the function, not what the derivative.

I: Just look at your graph again.

Barbara: Ya.

I: You have the idea, you just have to put it in precise form. The slope of what? Look, this derivative is equal to the slope of what?
Barbara: Oh, between Q and P?! They are two points on the curve!  

I: It wasn't the derivative. That was just a secant. How did you here [I was interrupted].

Barbara: Just shrunk it!

I: Here, how did you get the slope of that. You said you have to take this limit.

Barbara: Ya, the limit of h approaches to zero.

I: O.K., then what is L, what's the position of this line to the curve.

Barbara: I really don't know [laughter]!

I: You know!

Barbara: I do?! You think I know?!

I: Ya, what's the position of this line to this curve [I showed her the tangent line].

Barbara: I see the line.

I: How this line and this curve are related?

Barbara: How are they related?! It touches it at one point only.

I: O.K., what does that mean?

Barbara: I don't know what I mean [laughter]!

I: You said it touches it at one point, then what we call this line?

Barbara: What we call the line?! Oh...I don't know!

I: Let's go back to this part. You said as Q gets closer and closer to point P, what happens to the secant?

Barbara: What happens to the secant?!
I: You said it by yourself. As Q gets closer and closer to P, then what happens to the secant?

Barbara: The secant is disappeared, does disappear, it really isn't here.

I: What do you mean by disappear?

Barbara: Because they are so close at two points.

I: O.K., we said that these lines are extended from both directions. So it doesn't mean that it disappears. The secant ya, PQ will be disappeared, but the line is there. P is fixed and we are just moving Q towards P, so it means that it is the same line which has rotated around P. It just hits the curve at one point.

Barbara: Ya.

I: Then what we call this line?

Barbara: I call it the tangent, but...

I: O.K., back to here again. You found the slope of the tangent line at this point \([P = (1,2)]\). You got the slope of \([\text{tangent line}]\) by taking the limit of all these slopes. You said that it approaches to its limit. Then you gave the definition of derivative.

Barbara: That! \(f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}\).

I: Ya, how did you get this definition?

Barbara: How did I get the definition?!

I: Yes.

Barbara: I just did this, sort of, only more general term. \(f(x + h) - f(x)\).
I: O.K., then what is the relation between this curve and its derivative? Look at here that what did you get for the slope of the tangent line. Do you see any kind of relation.

Barbara: I should! [Laughter]...

I: How did you come to the idea that this limit \( \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \) is \( f'(x) \).

Barbara: It looks to me... I always call it the slope sort of that line that P and Q like about almost very close.

I: O.K., then call it again...

I: O.K., you are saying that the derivative... you call that the derivative it slope of what?

Barbara: The slope of the secant which I guess is almost like a tangent at a point.

I: O.K., This derivative wasn’t the slope of PQ₁, PQ₂,... you didn’t name the slopes of any of these [secants] as \( f'(x) \). You just got the slope of L [tangent line] as the limit of all those slopes and you called it \( f'(x) \). Now you can see that what the \( f'(x) \) is. Because we just got it, then what is it?

Barbara: Derivative [laughter].

I: What is this derivative?

Barbara: What is it [Emphasis is hers]?! Do you want the definition?!

I: No, you already gave the definition as \( f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \) which is quite right. Then what is this derivative? You said that this derivative is sort of slope. Slope of what?
Barbara: Slope of the tangent?! To a point on the curve!

I: O.K. Yes.

Barbara: [Laughter] O.K., that eased lots of pain.

I: O.K., this is the f'(x) at what point?

Barbara: Any point along the curve. You just pick it and plug it in.

I: You say any point along the curve. But at what point did you get the slope of the tangent line?

Barbara: At point P there.

I: Which you named it f'(x) at what point? Is it the f'(x) at any point?

Barbara: This would be true for any point.

I: You said that f'(x) is the slope of the tangent line at any point. Let's change the point P = (1,2) [Point of tangency] to P₁ = (0,1). Then what would be the derivative of function at this point? [I sketched the graph].

Barbara: Oh!

I: You said that at any point. But at this point [P₁ = (0,1)] you got 0 for derivative and at P = (1,2) you got 2 for derivative.

Barbara: Ya, it's the slope of the tangent at the point. But the tangent changes, going around.

I: Then what happens to derivative when the tangent changes?

Barbara: Changes with the shape of the curve?!...[Laughter]...ask me the question again.
I: O.K., you said that this 
\[ f'(x)=\lim_{h \to 0} \frac{f(x+h)-f(x)}{h} \] 
is the derivative of function at any point. But we said that this derivative at length of any point is equal to the slope of the tangent line at that point.

Barbara: Ya, at that point.

I: O.K., what is that any point and what is this specific point? How can you relate these two points together? At any point and at that point [Emphasis is mine]. You said any point and that point. How are these two related?

Barbara: That is the specific of any which is the general term. That part of that group [Emphasis is hers].

I: Which means?!

Barbara: The derivative of any!

I: Can you restate again? The derivative of a function at...what?

Barbara: Point x.

I: Is equal to the slope of...

Barbara: The tangent at point x.

I: Now, you got it. Before you said that "derivative of function at any point is equal to the slope of the tangent line at that point". I just wanted to be sure that if these two points are the same point or not... which are the same.

Barbara: Ya, I got it.

I: Because for example here [pointing the graph] we got the derivative of function at point \( p = (1,2) \) which is equal to the slope of the tangent line at point \( p = (1,2) \). If we change
the point, we can see that the value of derivative will be changed when the point of tangency is changed.

Barbara: O.K., you wanted a word [laughter].

I: If we don’t put it in precise form, we don’t understand it. Now you saw that by changing the points of tangencies, we got different values for derivative.

Barbara: Oh, what a struggle any way...

I: Do you know that where have all the differentiation formulas come from? Like chain rule etc.

Barbara: Ya, that’s the whole bunch of these stuffs.

I: We derive all these formulas from this definition of derivative

\[ f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \]

I: Now, compute the derivative of above function \( f(x) = x^2 + 1 \) at \( x = 0 \) by using the definition of derivative.

Barbara: \( \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \)

\[ \lim_{h \to 0} \frac{(x + h)^2 + 1 - (x^2 + 1)}{h} \]

That’s really a definition?!

I: Why you doubt it?

Barbara: I don’t know, I have mental elapses right now [laughter].

I: Just look at the curve again and check this definition. See if it makes sense to you.

Barbara: That’s the y part and that’s the h part [pointing the graph]. I mean the x part.

I: O.K. This axis indicates the y values which is \( f(x) \). Then you want to find this \( [f(x + h) - f(x)] \).
Barbara: \[
\lim_{h \to 0} \frac{(x^2 + 2xh + h^2 + 1) - x^2 - 1}{2x + h}
\]

Then \(f'(x) = 2x\) at \(x = 0\) = 0. We should have done this [interview] at the end of December. There was some kind of memory.

I: Right, you check it by the graph too. This derivative is the slope of the tangent line at \(x = 0\) which obviously is...

Barbara: A flat angle.

I: And its slope is 0.

Barbara: Ya.

I: Now we can move to next question.

Please compute \(d/dx (1 + 1/\sqrt{7} - 5x)^5\)

Barbara: Oh, the derivative of that? Not using that way [she meant using the definition of derivative]?!

I: Oh, No, No. Using the formula.

Barbara: \(-50 (1 + 1/\sqrt{7} - 5x)^4 d/dx (1/\sqrt{7} - 5x)\)

I: What is this formula called?

Barbara: The one that I'm using it?

I: Ya.

Barbara: Chain rule, O.K., Woh; what a struggle to remember that stuff.

I: But you are using it again, in Math 101.

Barbara: Ya, Ya, we are in series now, power series, woh! O.K., I have to differentiate this as well, so

\[
d/dx (1 + 1/\sqrt{7} - 5x) = (7 - 5x)^{-1/2 - 2/2}
\]

then \(d/dx (1 + 1/\sqrt{7} - 5x)^5\) =
50 \left(1 + \frac{1}{\sqrt{7 - 5x}}\right)^{49}.
5/2 \left(7 - 5x\right)^{-3/2}

I: Thanks. That's fine. Now compute the derivative of
\[ f(x) = \begin{cases} 
0 & x \leq -1 \\
-x+1 & -1 < x \leq 1 \\
-x+1 & 0 < x \leq 1 \\
0 & x \geq 1 
\end{cases} \]

at: a) x = -2
b) x = -1
c) x = -1/2
d) x = 1/3
e) x = 1
f) x = 10

[See Figure 3.2 for the diagram].

Barbara: a) Oh, O.K., O.K., when x = -2, that's where x \leq -1 so f(x) = 0?!
Therefore the f'(x) = 0. at b) x = -1 still is the same thing, x \leq -1 again and f(x) = 0.  c)

I: [I did not let her to continue] what about here [I showed her that f(x) = x + 1 at x = -1 as well].

Barbara: Oh, Jee, two points! The derivative is f'(x) = 1.

I: O.K., so what is the derivative of function at x = -1?

Barbara: ...[silent]

I: You said 0 and 1 right?

Barbara: Actually there is nothing to plug in, that's the real thing.

I: Why there is nothing to plug in?

Barbara: The derivative of this is 1. Is that for any x?

I: We want to find the f'(x) at x = -1, but we know that at x = -1, f(x) = x + 1 as well and f'(x) = 1. Then how you can interpret this?

Barbara: How we can interpret this?!
I: Yes. Look at the function again. 
\[ f(x) = 0 \text{ at } x = -1, \text{ but } f(x) = x + 1 \text{ at } x = -1 \text{ as well.} \]

Barbara: Oh...what a question! Something strange is coming. Could it be two different answers?!

I: You tell me.

Barbara: Because there is two function for...

I: There is not two functions...

Barbara: One function.

I: This function is defined this way.

Barbara: Ya, it's defined this way. Two definitions of \( x = -1 \) and if you derive it for each of the ways, you define it, then you get two separate answers. I'm not gonna argue with that, it's just there.

I: How do you interpret this?

Barbara: How do I interpret this?!

I: Ya, that the function has two different derivatives at one point. Remember we said that the derivative of a function at a given point is the slope of the tangent line at that point, right?

Barbara: Ya.

I: O.K.

Barbara: Then there are two tangents at that point!

I: Can it be? Can a function have two tangents at one point.

Barbara: In your graph it looks like it sort of [see Figure 3.2].

I: Well, you said there are two tangent lines. I'm arguing that there are
more than two you said two but I like to have this one and this one [I drew many tangent lines to the curve such that they all passed through x = -1].

I: Then what is the derivative of function at that point?

Barbara: I guess from 0 to 1. Such a thing. So I guess derivative is from 0 to 1 like the range goes from 0 to 1.

I: What does that mean. What is the derivative?

Barbara: I don’t know, -1 I guess.

I: No, I mean you said that derivative at a length of a given point, is the derivative of function at that point.

Barbara: Oh, Oh, What a fun!

I: When you say that derivative is from 0 to 1, you mean that the slope of the tangent line is between 0 and 1.

Barbara: Passing through that point.

I: I showed you that we can have many more tangent lines which they pass through that point, they're not only the tangents with the slopes of 0 and 1.

Barbara: I think these may be the highest tangent and lowest tangent, like highest slope and lowest slope.

I: Have you remembered something like this?

Barbara: What? I'm not used to having seen two answers, because we don't do very many of these things...

I: O.K., forget it for now. Is this function differentiable at x = -1.

Barbara: Yes, something exists, so much exists [laughter].
I: What is the condition for a function to be differentiable? In what condition the function is differentiable?

Barbara: Oh... I guess at point $x$, there should be only one answer for the derivative [Emphasis is hers].

I: What is that answer for $f'(x)$ at $x = -1$.

Barbara: What is that answer?! We call the slope of the tangent at that point.

I: So you are saying that there must be one tangent line at that point. But there is not a suitable tangent line at that point. There are so many, so is this function differentiable at $x = -1$ or not.

Barbara: That's weird! I like that. Doesn't reply that, it should be one, but this guy has more than one.

She liked challenging questions.

I: O.K. then, what do you conclude?

Barbara: Is not differentiable [no trust on her words]?!...Yes it is, here is the answer.

I: Why it is [differentiable] and why it is not.

Barbara: Oh, I guess it is, because... What does the differentiability mean any way [laughter]!

I: You just told me [laughter].

Barbara: Oh, about the tangent business?! We have so many tangents though! What a comprehensive!

I: O.K. [laughter], you decide [that whether $f(x)$ is differentiable at $x = -1$ or not].

Barbara: I guess it is, isn't it?! Oh, I have to take it overnight and get back to you. I've never seen anything like this. It is good stuff. Yes or No?! No I guess.
I: Then why no? If it's no, why not.
   If it's yes, why yes.

Barbara: I think, if it's yes, because...it
   existed like you calculate it.

I: O.K., but we calculated and we came up
   with two different answers.

Barbara: Oh my Lord...and...it seems to be
   like a range of slopes of that
tangent line.

I: What do you mean by "the range
   of slopes"?

Barbara: O.K., O.K., I say no, there isn't,
   because there are too many tangent
   lines [laughter].

I: O.K., the f(x) is not differentiable
   at x = -1, because when we move from
   left to right which means from -∞ to
   -1, the slope [of the tangent line]
   would be 0, right?

Barbara: Ya, Ya, right.

I: But when we move from right to left,
   the slope [of the tangent line]
   would be 1, which is the slope of
   this line [pointing on the graph]
   including x = -1. Then this function
   is not differentiable at this point.
   Because the limit from left is not
   equal to the limit from right.

Barbara: Oh, Oh, that...

I: Because the derivative is the limit
   of this slope. When that distance
   [it was showed on the graph]
   approaches to 0. But from here,
   [right limit] we get two
different limits. Which mean
   two different values for limit at
   this point. So the function is
   differentiable all along this line
   [pointing line f(x) = 0] and all
   along that line [pointing line
   f(x) = x + 1], but function is not
differentiable at this point.
Barbara: Ya, because the left and right limit aren't equal.

I: O.K.

Barbara: I remember that, I had so much trouble with that. Because they gave us like...I don't know.

I: Look at the graph again, $x = -1$ belongs to both parts. Can you conclude anything from it that why this function was not differentiable at $x = -1$.

Barbara: Because...didn't you tell me?! The left and right limits are not equal.

I: An in a very simple way, those functions are differentiable that are smooth. Smooth functions are differentiable. Remember the parabola $[f(x) = x^2 + 1]$ we could draw tangent line at any point on the curve. Consider this one. This is differentiable at all the points of the tangencies [I sketched two smooth curves]. But consider this [a graph with a pick] Is this function differentiable at that point [pointing the pick].

Barbara: Oh, at the pick?!

I: Yes.

Barbara: I guess not, because this side is like this and this side is like this. These two are straight lines. [So function is not differentiable at the pick].

I: That's right. When a graph of a function is sharp at some points, so function is not differentiable at picks. And the same thing here again. For example $y = |x|$ is not differentiable at $x = 0$. When the function is smooth, then it is differentiable like lines.

Barbara: This is not continuous.
I: No, it is continuous but it's sharp at that point [pointing the graph], by continuous we mean that when you start from one point, you...[I was interrupted].

Barbara: Oh, Oh, you don't stop. We didn't stop here [at the pick] but...this is a sharp point. To be differentiable, function must be smooth, without a sharpness.

I: And we didn't stop here [at the pick] but...this is a sharp point. To be differentiable, function must be smooth, without a sharpness.

Barbara: O.K.

I: So far so good! Any questions on that.

Barbara: That's neat! I never knew something like that. Nobody told us anything about that. That's pretty neat!

I: We can skip the other parts of the question [Parts c to f]. Because for other parts we have the same reasoning and same logic. Because I'm interested on your responses on function and we don't have more than 10 minutes.

Barbara: Oh, Oh, I have physics [class] right now. But who cares, I'll stay. I won't take more than 5-6 minutes of your time.

Barbara: "Consider the equation \( x^2 + y^2 = 1 \). Sketch the graph of this equation! O.K., this is a circle with radius 1 [She sketched the graph]. Is that it?!"

I: Yes, that's fine.

Barbara: "Determine whether the above equation represents a function \( y = f(x) \) or not". Oh...strange! \( y^2 = 1 - x^2 \) so

\[ y = \pm \sqrt{1 - x^2} \]. Is it \( \pm \)?! I can't remember.
I: Ya.

Barbara: Ya, better be! Is is a function or not...[silent].

I: Can you say it from the graph!

Barbara: No! It's not a function.

I: Why?

Barbara: Because...for certain point of x there are two y's, y value. In function, for a point x, we have to have only one y value.

I: That's right.

Barbara: "Determine..." didn't I answer that [question 2 on functioning].

I: Yes, you did.

Barbara: O.K., "If not, determine a domain and a range such that the above equation is a function". O.K., y...Just the positive values, y ≥ 0, so restrict to only positive y's.

I: Then what is the domain and the range of this function [I meant upper semi-circle].

Barbara: Domain is...-1 to 1 and range is from 0 to 1.

I: Right!

Barbara: O.K., "What is the relation between the domain and the range of a function and its inverse and its reciprocal [1/f(x)]? I guess you can say either you flip the variable around in an equation, for example y = x + 1, it's inverse will be x = y + 1.

I: O.K., now what's the difference between f(x) and 1/f(x)?
The difference between this

\[ y = \sqrt{1 - x^2} \text{ which is upper semi-circle} \] and \( 1/f(x) \)?

Barbara: The inverse is when you switch x's for y's and y's for x's. But \( 1/f(x) \) is the reciprocal, and that's not necessarily equal, so...what am I going to conclude [laughter]?! [I repeated question 4 again].

Barbara: Oh, they are not the same, are they?!

I: Look at the graph. This is the domain and range of your function \( [D = \{-1, 1\} \text{ and } R = \{0,1\}] \), now what would be the domain and the range of its inverse function?

Barbara: Can I redraw it? [She correctly did it.] So was not a function any more if we invert it.

I: Good.

Barbara: O.K., O.K., then no domain! no range!

I: O.K., how we can restrict the domain or the range of the function \[ y = \sqrt{1 - x^2} \] in order to have an invertable function.

Barbara: Say it again.

I: You said that if we flip it over, it is not function any more. Now what can we do to the domain and the range of the semi-circle \[ y = \sqrt{1 - x^2} \] to make it an invertable function.

Barbara: O.K., Let's cut off another quarter off and then you just have a quarter of a circle. Then we flip it over, we get one x, one y. So I restrict the domain to just positive x's or just negative x's.
I: Now you just defined the invertable function. It is a function that for every x [I was interrupted by her].

Barbara: There is only one y and...

I: Remember you restricted again.

Barbara: Then...you mean a function is inverse, better work out both ways?!

I: You said that if we convert the semi-circle, it won't be a function because then for every x we could have two different y's and you restricted to quarter of the circle. Then it means that before you cut off semi-circle you had two x's for one y. But after you cut off one quarter of it, then we have what?

Barbara: One x for one y.

I: O.K., and one y for one x [She got this from the beginning]. We call this [pointing quarter of circle] one-to-one function such that there is one y for every x and there is one x for every y.

I: Now what is the domain and range of inverse function.

Barbara: Oh, [0,1] and [0,1].

I: Let's see what is it in general. You said that x values go for y values and y values go to x values. Then what would be the domain and the range of inverse function with respect to function. If domain of f(x) is D and range of f(x) is R, then what would be the domain and range of inverse function?

Barbara: [laughter]...backwards!

I: What do you mean by backwards?

Barbara: Oh, switch them around.

I: O.K., which means...
Barbara: Ohm... What does that mean?! I'm not very good at words [laughter].

I: You said backwards. What do you mean by backwards?

Barbara: I guess domain and range... D[domain] becomes R[range] and Range becomes Domain.

I: That's fair enough. What about 1/f(x) i.e. the relation between the D and R of f(x) and 1/f(x)?

Barbara: 1/D, 1/R?!...D and R?!

I: Do you have any restriction for it?

Barbara: The bottom [f(x)] better not be 0.

I: Better not be or should not be 0.

Barbara: The f(x) should not be 0.

I: O.K., that's fine. Now compute one of the following please [composing of functions].

Barbara: "If f(x) = \sqrt{x} and g(x) = x + 1, compute one of the following: a) fog(x), b) gof(x), c) gog(x), d) fof(x)". Oh I remember doing that. That was a long time ago, few months or so. I'll try one... woh ... it's ancient history now. O.K.,

a) f(g(x)) = \sqrt{x + 1}.

I: Thank you very much for your time.

Barbara: You're welcome.
APPENDIX F

Written Work from the Exemplary Student Interview

3) $f(x) = x^2 + 1 \\ \text{at } x = 1 \\ f'(x) = f(x + h) - f(x) \\ f'(1) = 2 \\ y = f(x) = x^2 + 1 

2) $y = f(x) = x^2 + 1 \\ f(x) = 2x \\ f'(x) = 2x \\ f'(1) = 2$

$\Delta y = f(1 + h) - f(1) \\ = [(1+h)^2 + 1] - [1^2 + 1] \\ = (1 + 2h + h^2) - 2 \\ = h^2 + 2h - 1$
$y = x^2 + 1$

$\lim_{h \to 0} \frac{(x+h)^2 + 1 - (x^2 + 1)}{h}$

$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + 1 - x^2 - 1}{h}$

$= \lim_{h \to 0} \frac{2xh + h^2}{h}$

$= 2x + h$
7) \[ f'(x) = 0 \]

\[ f'(x) = 1 \]

\[ f'(x) = 1 \]

\[ f'(x) = 0 \]
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