THE GRAMMAR OF SYMBOLIC ELEMENTARY ALGEBRA

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ABSTRACT

In the practise of algebra education it is generally assumed that the rules for manipulating symbols are explicitly presented by text and teacher, and that acquisition of skill depends upon successful apprehension and application of these rules. Research into the psychology of algebra generally starts from the assumption that algebraic rules are consciously accessible and rationally employed. The view adopted here is that algebraic symbol skill is based upon procedural rules which are acquired informally (and often unconsciously) through interaction with algebraic symbols and which may be only peripherally related to the rules presented in instruction.

The principle purpose of the present research is to describe the procedural knowledge which underlies algebraic symbol skill. Knowledge of algebra is viewed as of-a-kind with the highly structured yet unconscious systems of rules which underlie natural-language competence. Formal methods of generative transformational linguistics are adapted for analysis of algebra. The model of algebraic symbol skill is captured in a 'grammar' which details the various components of skill.

In several instances alternative formulations are offered which equally well fulfill the formal requirements of the grammar. For some of these, techniques of psycholinguistics are used to guide selection on the basis of psychological considerations. Many questions remain in need of further elaboration and resolution.

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DISSERTATION FORMAT

This document is written as a series of Papers rather than as a series of Chapters which is more typical of doctoral dissertations. A unitary Table of Contents as well as a List of Figures and a List of Tables precedes the main body; however, Bibliographies and Appendices are presented within the individual Papers. Except for the final one, each Paper is intended to be complete and coherent independently of the others (though Paper #2 is less readable outside the context of the whole). As a result of this format, an occasional quotation or table is repeated in different Papers.

The principal reason for this departure from convention is that the dissertation does not present just a single, focussed study, but rather develops a new orientation for the psychology of algebra. Indeed, the scope of the dissertation is rather broad, including a rationale for a new approach (Paper #1); development of new methods and a comparison with current methods (Paper #2); application of the new methods and generation of specific research hypotheses (Paper #3); experimental and empirical evaluation of hypotheses as well as consideration of practical consequences for educational application (Papers #4 and #5); a brief overview of basic issues, and a discussion of further directions (Paper #6).

The scope of the project makes it of potential interest to several different audiences including methodologists, psychologists of algebra, and classroom teachers. The individual Papers are targeted for distinct audiences: Paper #1 for a general audience of mathematics educators; Paper #2 for methodologists; Paper #3 and #4 for psychologists of algebra, and Paper #5 for classroom teachers. (Of course there is no attempt here to dissuade readers from crossing these subdisciplinary boundaries.)

The first five Papers are independent and mutually referential. The reader wishing to become familiar with all of the Papers can procede in the logical order of presentation. Alternatively, a reverse order of reading will have the advantage of familiarizing the reader with implications and consequences before tackling the more difficult early Papers. In either case, the sixth Paper should be left until the end.

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I. THE AUTONOMY OF ALGEBRAIC SYMBOL SKILL

Paper #1

> Is there a semantic component in the language of mathematics, or is it a symbolic system that has only syntax?
>
> (Nesher 1981, p. 28)

Mathematics is a unique element in the cultural endowment of humankind. It is at once a private legacy of deftly woven argument and image, and a public vehicle for precise description. In the former capacity it is an enduring edifice of disciplined conjecture, in the latter, an engine of technology, enabling control of the material environment.

As a result of its utility, the private aspect of mathematical thought is drawn into the domain of public concerns. The presence or absence of mathematical knowledge and skill is a matter of consequence in the life choices available to an individual. When such skill and knowledge is unequally distributed among races, social classes or genders, intergroup dynamics are affected. Understanding the nature of mathematical thought and directing its equitable dissemination are important social undertakings.

Mathematics derives its utility in application to science. Science is the investigation of relationships which occur in nature. The precision and clarity embodied in mathematical language are hallmarks of scientific investigation. Even more importantly, mathematics, as a formal system, allows derivation of new statements from old ones according to formal procedures. That such derivation is truth preserving means that new truths in substantive domains can be achieved through formal means in the symbolic domain. It is this feature of derivation which makes mathematics so powerful an instrument of science.

The process of applying mathematics in science helps set the agenda for school mathematics. The mathematics curriculum is substantially concerned with (a) the identification of critical elements in problem situations and analysis of their interrelations; (b) the coding of these elements and their relationships in mathematical notations; (c) the manipulation of mathematical expressions and equations according to formal procedures; and (d) the appropriate translation of new mathematical statements back into the context of origin.

The first of these concerns has wide applicability beyond the mathematics curriculum. The structural analysis and comprehension of situations is a key element in any form of rational argument. Rational analysis is an important component in many subject-matter domains. In mathematics education, the general importance of such problem analysis skills is reflected in substantial research into the psychology of problem representation and in the call of educational leaders in the 1980's to make problem solving — beyond purely manipulative skills — the principal focus of mathematics instruction (NCTM, 1980; Position statement of NCSM, 1978).

Symbol translation and especially symbol manipulation skills, however, have traditionally been the mainstay of the mathematics curriculum. Goldin (1983) suggests that "notations available for mathematical problem solving are highly structured formal systems. Learning their use probably comprises more than 95% of present-day school mathematics" (p. 117).

Despite the undeniable importance of general problem solving, the school emphasis on symbol skills is understandable. Their specialized nature insures that if symbol skills are not acquired in the mathematics classroom they will, likely, never be attained. Furthermore, ability to manipulate symbols is necessary for the completion of most school mathematics

problem-solving tasks. Thus the crucial internal and external incentives which derive from task completion are not available to the student who is unable to successfully manipulate mathematical symbols. A lack of competence and confidence in manipulating symbols must act as a tremendous disincentive to serious engagement in the more generally applicable skills of problem representation and analysis. Symbol skills are justly prominent in school mathematics.

Given the major curricular emphasis, high levels of mastery of symbol manipulation might be expected. At the arithmetic level, most students are eventually able to acquire whole number skills (NAEP, 1983). Algebraic symbol skills, however, remain beyond the grasp of most students. Researchers observed from the Second National Assessment of Educational Progress

> that between one-third and two-thirds of the 17-year-olds could simplify algebraic expressions that are not particularly complex. On ... exercises ... requiring more complex algebraic manipulations, less than one-fourth of the 17-year-olds were successful. (NAEP, 1979, p. 32)

[Multiplying two binomials of the form $ax + b$ is an example of a "more complex" task.] In British Columbia, the 1985 mathematics assessment (Robitaille & O'Shea, 1985) revealed that only 25% of grade 10 students could successfully simplify the expression $(r + s) - (r - s)$. The design of more successful approaches to algebraic symbol skills must be considered a priority in mathematics education.

1. Psychological Representation

How can a curriculum be structured to better promote the acquisition of algebraic symbol skill? Clearly, the answer to such a question is dependent upon the character of this skill as a psychological entity. If algebraic symbol skill can be analysed as an autonomous, procedural subsystem consisting of straightforward application of rules then it should be possible to implement the skill through a linear, one-step-at-a-time, curriculum. Alternatively, if algebraic

symbol skill involves the interaction of complex conceptual domains, then a more flexible curriculum may be needed to allow for individual differences in the acquisition and coordination of these domains.

Curriculum theories point in both directions. The "mastery learning" model in which the curriculum is rationally organized into accessible units based on detailed task analyses seems to reflect the former psychological perspective as do, perhaps, behavioural objectives and programmed instruction materials. On the other hand, the "spiral curriculum" in which topics are repeated through successive years of instruction, each time at a slightly more sophisticated level, seems to reflect the latter psychological perspective (as does the "holistic" model advocated by Seymour, 1984).

Unfortunately, the split in curricular theory mirrors an equally profound division among psychological theorists. VanLehn (1982) reporting on substantial research into the nature of whole number subtraction errors reflects:

> The procedural skill chosen for investigation was ordinary multi-digit subtraction. Its main advantage, from a psychological point of view, is that [it] is a virtually meaningless procedure. Most elementary school students have only a dim conception of the underlying semantics of subtraction, which are rooted in the base ten representation of numbers. When compared to the procedures they use to operate vending machines or play games, subtraction is as dry, formal and disconnected from everyday interests as the nonsense syllables used in early psychological investigations were different from real words. This isolation is the bane of teachers but a boon to the psychologist. It allows one to study a skill formally without bringing in a world's worth of associations. (p. 15)

However, the position that symbol skills are based upon rote procedures is often contested. For example, Lewis (1980) states:

> We are accustomed, I think, to deplore "blind symbol manipulation" as an unproductive approach to mathematics for students. We may perhaps also note (as I will below) that viewing mathematical operations as just operations on symbols throws away redundancy in the system that can be used to detect errors. But I want to suggest that seeing algebra as symbol manipulation is even more pernicious, in that it actually promotes wrong ideas.
>
> The key notions in symbol manipulation are deletion, rearrangement, replacement: those are the things you do with symbols.... (p. 5, emphasis added)

As another example, Steffe and Blake (1983) respond to Gagné's (1983) procedural position:

> We can find no justification for the claim that for children the "operations of computation are entirely concrete. The objects with which they deal are numerals and operations signs printed or written on a page" (Gagné, 1983, p13). The only possible philosophical justification for this position comes from formalism. Hilbert himself does not, however, go as far as Gagné. Hilbert (1964, p. 143) takes the symbol 3 as an abbreviation of |||. In other words, it has meaning. (p. 212)

Thus the autonomy of symbol skills is a matter of vigorous controversy within the mathematics education community.

The importance of this controversy should not be underestimated. If mathematical symbol skills are founded upon broad conceptual reasoning skills then, ultimately, only broad, relational methods of instruction should succeed in promoting these skills in a reliable fashion. In this case, the complexities underlying conceptual development need to be addressed before low achievement in algebraic symbol skills can be effectively remedied. Alternatively, if algebraic symbol skill is an autonomous, procedural cognitive function, then it may be possible to

implement that skill through instrumental methods. Thus rationales for the form of substantial portions of the mathematics curriculum rest on this issue.

A. RESEARCH

How might research shed light on this difficult area? Since there are no means available for direct inspection of mental processes, it is necessary to construct a theoretical description or model of the skill. The usual scientific method is to propose a model of the skill; consider various empirical consequences of the model; and devise means to test the consequences. Since two competing conceptions of algebraic symbol skill exist in mathematics education, attempts to piece together distinct theoretical models might be expected. Indeed, this would seem to be a necessary first step to resolving a theoretical impasse.

An examination of the research record shows that a dualistic program has not been undertaken. Most studies of algebra skill are intensive in that they look at some aspect of symbolism and attempt a thorough analysis of the psychological representations for that domain. As examples, Herscovics and Kieren (1980) have considered the meaning and use of the equation symbol, Kuchemann (1978) has examined understandings of literal symbols, and Wagner (1981) has studied the conservation of equations. (See Wagner, Rachlin & Jensen, 1984 for a more complete list.) Typically these studies are not directed specifically to manipulative aspects of symbolism, but may involve perspectives from symbol translation or problem representation as well. In any event, there is no attempt to generalize beyond the domain of intensive inquiry. The bigger picture of algebraic symbol skill as an integrated system is not addressed.

A number of extensive studies have been explicitly concerned with providing a uniform

theoretical perspective for a broader scope of mathematical behaviours (e.g. Davis Jockusch & McKnight, 1978; Carry, Lewis & Bernard, 1980; Matz, 1980; Wagner, Rachlin & Jensen, 1984). All of these studies were expressly directed towards explaining algebraic symbol skills, however, many of them had far more ambitious objectives.

Davis, Jockusch and McKnight (1978) set out to provide a partial answer to the question: "What thought processes are involved when students begin the study of algebra" (p. 10)? They catalogue a long list of key phenomena and discuss observations and applications of these with examples from arithmetic, algebra, geometry, matrices, graphing, trigonometry, vectors and calculus. Their actual intention is to found a general theory of mathematical problem solving.

Wagner, Rachlin and Jensen (1984) examined ability to manipulate algebraic symbols in tandem with the comprehension of verbal instructions. Their theoretical foci were "generalization" and "reversibility", "two well-established problem-solving processes" (p. 7). They observe:

> A basic premise of this study was that the learning of algebra, beyond the level of rote memorization of formulas and algorithms, can be regarded as a kind of problem-solving process. That, even the application of formulas to "routine" textbook exercises involves some degree of problem-solving activity on the part of most students, at least initially. (p. 7)

For studies in which algebraic symbol skill is treated in the context of broader aspects of problem solving, it is natural that symbol skills are viewed as integrated with and of-a-kind with conceptual domains of cognition.

The studies by Carry, Lewis and Bernard (1980) and Matz (1980) stand out for the specification of algebraic symbol skills as the domain of investigation. For Matz (1980), however, "a theory of mathematical competence would explain how new rules are constructed from familiar knowledge, or more particularly, how existing rules are extended to handle a wider range of inputs" (p. 94). Matz (1980), mainly focussed on modelling *acquisition of skill*, does not provide a detailed characterization of skillful performance or even suggest how such a model might be structured.

The study of Carry, Lewis, and Bernard (1980) is alone in postulating an explicit, independent model of skillful behaviour in its discussion of algebraic symbol skill. This model, proposed by Bundy (1976) in a computer implementation of algebraic manipulation, provides a solid framework for analysis of algebraic competence. In it is postulated a pool of "operators" which transform one algebraic structure into another by means of a "pattern matcher."

The Bundy model, however, is incomplete in that it applies to "tree representations" of equations rather than to strings of symbols. Though acknowledging the necessity, Carry, *et al* do not specify the details of implementation of the model on strings. Furthermore, the pool of operators representing the knowledge of mathematical rules available to the algebraist is not presented. Rather than elaborating the model of competence, Carry, *et al* were primarily interested in (a) evaluating the psychological validity of the (incompletely specified) Bundy model, (b) extending the model to deal with common errors, and (c) exploring the strategies or heuristics employed by students in selecting an operator or sequence of operators from the pool of possible operators. Apparently Bundy too shared this latter concern: "The objective of Bundy's program is the solution of the search problem presented by the large number of operators applicable to a given problem" (Carry, Lewis, & Bernard, 1980, p. 25).

1. Cognitive Science

A feature of many current studies of algebraic symbol skills (Brown *et al*, 1975; Davis, Jockusch and McKnight, 1978; Davis and McKnight, 1979; Carry, Lewis & Bernard, 1980; and Matz, 1980) is that they adopt a cognitive science or information processing (IP) framework. In IP studies, intellectual skills are modelled using terms, concepts and metaphors derived from computer science.

The overwhelming tendency in IP models has been to describe the global picture of cognition. The terminology employed by the cognitive scientists is revealing. In contrast to mathematics education definitions, "problem solving" refers to a whole gamut of intellectual skills from conceptual to rote. As Kilpatrick (1986) observes, there is no traditional distinction between cognitive domains in cognitive science:

> Cognitive science has devoted considerable attention over the past several decades to attempts to develop general models of intelligent performance--in particular, models of human problem solving (Newell & Simon, 1972). In such models, the mind is seen as essentially unitary, and mental structures tend to be viewed as primarily "horizontal" in nature, cutting across a variety of contents of thought. (pp. 16-17)

Even though he goes on to observe "signs that a counter movement is developing" (p. 17), cognitive scientists have not yet identified algebraic symbol skill as a domain for *independent* analysis. As a result cognitive science has contributed principally conceptually based models of algebraic skill. Indeed, with the exception of the incompletely specified Bundy model (Carry, Lewis & Bernard, 1980) procedural explanations of skillful algebraic symbol behaviour have been almost entirely ignored.

B. PERSPECTIVES

The point of this discussion is not that researchers regard algebraic symbol skills as an entirely conceptual activity. Most researchers recognize a multiplicity of types of understanding involved in mathematics (see, for example, Davis, 1982). The point is simply that few authors have considered the possibility that algebraic symbol manipulation may be essentially procedural in nature and gone on to examine the skill from that perspective.

Undoubtedly, frustration with the poor results of instructional methods perceived to be largely procedural in nature is part of the reason for abandoning procedural explanations of skillful behaviour and focussing on conceptual or mixed models. But this abandoning of procedural explanations ignores precisely the possibility that procedural knowledge underlying skillful behaviour may require deep and intensive study to uncover, and that present instruction fails, not because it is procedural, but because the procedures involved have not been sufficiently explicated.

The present paper is introductory to a set of six papers in which a procedural account of algebraic skill is elaborated. Before proceeding onto these, however, it is useful to consider some of the prevailing influences on mathematics education thought which may have contributed to previous neglect of procedural explanations.

1. Introspective Limitations

The scope and complexity of non-conscious, procedural elements of thought are not accessible to introspection. This is probably a fundamental and necessary limitation of human consciousness even as the auditory faculty is just sufficiently dull that one cannot hear the pounding of blood through the capillaries. Jaynes/Kilpatrick (1986) elegantly express the

difficulty in apprehending this limitation on consciousness.

> Consciousness is a much smaller part of our mental life than we are conscious of, because we cannot be conscious of what we are not conscious of. How simple that is to say; how difficult to appreciate! It is like asking a flashlight in a dark room to search around for something that does not have any light shining upon it. The flashlight, since there is light in whatever direction it turns, would have to conclude that there is light everywhere. And so consciousness can seem to pervade all mentality when actually it does not. (p. 12)

The psychology of mathematics is not the first domain to be hampered by the illusion of introspective access. Chomsky (1968) reports a general tendency to gloss over complex, non-conscious processes:

> One difficulty in the psychological sciences lies in the familiarity of the phenomena with which they deal. A certain intellectual effort is required to see how such phenomena can pose serious problems or call for intricate explanatory theories. One is inclined to take them for granted as necessary or somehow "natural." (p. 12)

Mathematics education research joins distinguished company in its neglect of rigour in detailing familiar skill.

2. Mathematical Theory

Apart from the general tendency to misapprehend the limits of introspection, mathematics, in particular, is usually regarded as the epitome of conscious, rational intellectual activity. This view is undoubtedly sound for the creating and comprehending of mathematical theory. The further assumption (generally held within the mathematics education community) that symbol manipulation is rigorously accounted for by mathematical theory leads unerringly to the conclusion that underpinnings of skillful behaviour are to be found in conscious and rational

cognitions.

The generally held assumption that symbol manipulation is rigorously accounted for by mathematical theory, it should be understood, is not usually dealt with as an empirical claim demanding of careful exposition and reasoned support. Indeed, the statement is rarely made in an explicit form at all, but remains an article of faith underlying much thinking about mathematical thought. As an example, this view can be discerned in the distinction made by Brown *et al* (1975) between the "abstract logical structure of the [algebraic] knowledge" and the "reorganized, learner oriented structuring of how he is to use the knowledge for solving algebra problems" (p. 84). Apparently they see some definite body of theory which underlies the knowledge actually used to manipulate symbols. It is only the occasional lone voice which gives challenge to the prevailing wisdom:

> The apparent view of Steffe and Blake that understanding involves some aspects of the "structure of mathematics" is what I would be inclined to question. I realize that this is an extremely common view among mathematics educators, and it is exactly this view I should most strongly like to urge them to examine critically. Is this view actually based on solid evidence, or does it merely reflect a traditional statement by those who like to reassure themselves that they are on the side of the angels? (Gagné, 1983b, p. 215)

Despite its wide acceptance and intuitive appeal, the view that mathematical theory underlies algebraic symbol manipulation does not stand up well under scrutiny. At least three distinct aspects of knowledge must be accounted for in a theory of algebraic symbol manipulation: Knowledge of notational conventions; knowledge of rules which are applied to transform equations or expressions; and the pragmatics of rule selection.

To call the first of these "mathematical theory" is misdirected. The notational forms and conventions currently employed in mathematical expression have not been the product of systematic design, but of gradual evolution over a millenium in response to changing needs and circumstances (Cajori, 1928).

The pragmatics of rule selection also is outside the realm of mathematical theory. The fact that it is convenient to use certain selections of transformations to accomplish specific goals (e.g. reducing fractional expressions, rationalizing denominators, etc.) is not a function of principled mathematical theory but of contingent human requirements.

It is transformational rules themselves which are the main contenders in the search for a link between mathematical theory and algebraic skill. The abstract algebraic theory of integers or rational numbers does provide some possible rules (e.g. the "Distributive Law", the "Associative Law", etc.) for inclusion within a psychological account of algebraic symbol manipulation. But that algebraic accounts of integers or rational numbers provides an "abstract logical structure" for algebraic knowledge is untenable as these theories cannot account for properties of exponents and radicals —operations which lie outside the systems of integers or rationals. These operations are studied as part of the systems of real numbers or of algebraic numbers. There is no evidence (and more importantly there has been no argument) that any particular mathematical theory is the "correct" one.

Presumably, there is some set of 'real number properties' which the successful manipulator of algebraic symbols has learned and which underlies his or her fluency.† It is startling that

†Single quotations have been used in 'real number properties' because there is no evidence that the interpretation *as number* plays any role in the psychological representation of these rules, or that they are connected in any way to established mathematical theory.

there has been no previous attempt in mathematics education to actually catalogue and examine these rules as a set. In one study, Matz (1980) refers to "base rules" which

> usually takes the form of a rule a student has extracted from a prototype or gotten directly from a textbook. For the most part these are basic rules (such as the distributive law, the cancellation rule, the procedure for solving factorable polynomials using the zero product principle) that form the core of the conventional textbook content of algebra. (p. 95)

Mathematics educators are thus in the untenable position of inferring the psychology of algebra skill from the contents of textbooks rather than directing the contents of textbooks on the basis of objective inquiry into algebraic skill.

3. Arithmetic

A final possible factor to have mitigated against more serious investigation of procedural aspects of algebraic symbol skill is comparison with arithmetic skills. Formal models for rules of arithmetic manipulation are available from abstract algebraic theory (since arithmetic uses only the four operation, +, −, x, ÷, of the rational number system). Even if this formal treatment does not correspond to psychological representations, it at least delimits the scope of explanation required of a psychological model. In algebra there is no comparable delimitation for the problem of transformational rules.

More importantly, procedural explanations of arithmetic skill can be presented as relatively straightforward manipulations of physical tokens in fixed spatial arrangements on a page. For example, in a highly regarded study Brown and VanLehn (1980) are able to provide an adequate formal account of multidigit subtraction in essentially a single page (page #8) – an account which matches "a standard version of subtraction taught in the United States" (p. 9).

That arithmetic procedures are straightforward and accessible to introspection, however, should not be taken to mean that algebra skill can be so easily accounted for. Petitto (1979) suggests that far greater complexity may be required for explanations of algebraic skill:

> ... ninth-grade elementary algebra puts all of the basic arithmetic which the students have previously learned into a formal system. The learning taking place is not a matter of learning new manipulations of numbers, as is arithmetic, but rather of learning to formulate and manipulate formal statements involving arithmetic operations whose numerical content is relatively incidental. Further, and perhaps as a result of this new formality, many previously good arithmetic students start to have trouble with mathematics at just this stage. (p. 70)

Difficulties in acquisition of symbol skill in algebra may stem from the incomplete procedural analysis provided at the secondary level and the correspondingly greater cognitive load imposed on all students.

C. LINGUISTIC METHODS

Whether because of misdrawn analogies to arithmetic skill, assumptions of the adequacy of "mathematical theory" explanations, or just a natural tendency to oversimplify procedural, non-conscious elements of thought, a need for complex, abstract explanations of algebraic symbol skills has not previously been recognized. In many respects, the present state of theoretical analysis of algebra can be likened to the state of natural language theory before the introduction of formal methods of inquiry in that field. Natural language too was regarded as obvious, simple and unneeding of detailed or deep explanation. Chomsky (1968) comments on the psychology of language prior to his own contributions:

> It seems to me that the essential weakness in the structuralist and behaviorist approaches to these topics [language and mind] is the faith in the shallowness of explanations, the belief that the mind must be simpler in its structure than any known physical organ and that the most primitive of assumptions must be adequate to explain whatever phenomena can be observed. Thus, it is taken for

> granted without argument or evidence (or is presented as true by definition) that a language is a "habit structure" or a network of associative connections, or that knowledge of language is merely a matter of "knowing how," a skill expressible as a system of dispositions to respond. Accordingly, knowledge of language must develop slowly through repetition and training, its apparent complexity resulting from the proliferation of very simple elements rather than from deeper principles of mental organization that may be as inaccessible to introspection as the mechanisms of digestion or coordinated movement. (Chomsky 1968, pp. 24–26)

A pressing analogy between pre–Chomskian linguistics and current mathematics education research lies in the focus on obtrusive features of performance instead of on the more pervasive regularities. Chomsky (1965) discusses the nature of linguistic theory prior to the introduction of formal and rigorous methods of analysis:

> The limitations of traditional and structuralist grammars should be clearly appreciated. Although such grammars may contain full and explicit lists of exceptions and irregularities, they provide only examples and hints concerning the regular and productive syntactic processes. (p. 5)

In mathematics education research there has also been a concern with obtrusive aspects of performance. It can be observed that virtually every study of algebraic skill (including all of the above mentioned and many others: Budden, 1972; Davis & Cooney, 1977; Hammer, 1978; Laursen, 1978; Meyerson & McGinty, 1978; Davis, 1979; Davis & McKnight, 1979; Davis & McKnight, 1979a; Sacher, Baker & Miller, 1979; Lewis, 1980; Clement, 1982; Davis, and Young, & McLoughlin, 1982, Sleeman, 1986, to cite just a few) has focussed substantially on the analysis of errors. As in linguistics, the regularities underlying successful performance are rarely central to investigation. Perhaps in mathematics education research it is now time to follow linguistic research into the study of the "regular and productive" elements of performance.

The present paper is introductory to a set of six papers in which this task is undertaken. It is pure serendipity that in a limited technical sense, as required for formal linguistic analysis, "the set of 'sentences' of some formalized system of mathematics can be considered a language" (Chomsky, 1957, p. 2). Starting from the assumption that the syntax of algebra is unknown and inaccesible to introspection, formal methods of linguistics and psycholinguistics are adapted for the analysis of the *language of algebra.*

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II. LINGUISTIC ANALYSIS OF ALGEBRAIC SYMBOL SKILL

Paper #2

The Study of Algebra may be pursued in three very different schools, the Practical, the Philological, or the Theoretical, according as Algebra itself is accounted an Instrument, or a Language, or a Contemplation; according as ease of operation, or symmetry of expression, or clearness of thought, (the agere, *the* fari, *or the* sapere,*) is eminently prized and sought for. (Hamilton 1837, p. 293)*

A. MATHEMATICS AS LANGUAGE

Many authors have drawn parallels between mathematical language and natural language. Most often what is attempted is to relate mathematical structure to established linguistic categories. For example, Hogben (1959) pronounces:

> The nouns of mathematical grammar are called numbers. Just as we can recognize different kinds of nouns which we call proper nouns, common nouns, abstract nouns, collective nouns, and pronouns, we can recognize corresponding ways in which numbers can be classified. (p. 74)
>
> There are various ways of classifying mathematical verbs. Two of them are worth recognizing early. ... The first way of classifying mathematical verbs depends on their relation to the rest of the sentence. One class of operators might be loosely compared with reflexive verbs like "I *wash myself* when I get up." The phrases "3 + 4" or "3 X 4" mean just the same thing as "4 + 3" or "4 X 3" respectively. (p. 95)
>
> The other important distinction in classifying operators depends on their relation to one another. ... In one way it is rather like our distinction between the active verb and the passive verb. In another way it is rather like the distinciton between a verb like "to assert" and a verb like "to deny." We say that the operator "÷" is the *inverse* operator to "X", and the operator "−" is the inverse operator to "+". (p. 96)

Aiken (1972, pp. 25–28) reviews a selection of similar attempts.

More recent reviews have expressed little optimism about the value of such approaches for

mathematics education:

> It has frequently been pointed out that mathematics itself is a formalized language and it has been suggested that it should be taught as such ...
> Such statements possess a degree of validity, but would appear to be somewhat dangerous and potentially confusing. Mathematics is not a language – a means of communication – but an activity and a treasure house of knowledge acquired over many centuries. (Austin and Howson 1979, p. 176)

Wheeler (1983) is more succinct: "I shall keep well away from the region signposted Mathematics is a Language. I believe it to be uninhabited" (p. 86).

Rather than impose linguistic categories (the results of linguistic analysis) on mathematics it is possible to apply the methods of linguistic analysis to mathematical systems. At least two recent studies have made reference to such a project. Matz (1980) introduces a theory of algebraic skill:

> This article employs two classic methods for probing the content and mechanisms underlying a competence: analyzing the errors people make in its use and studying the acquisition of that competence. With evidence from these two sources we can begin to piece together a theory of algebraic competence in much the same manner as linguistic researchers who have formulated a theory of linguistic competence using grammaticality judgments and considerations of children's acquisition of language. (pp. 94–95)

Similarly, Greeno, Riley and Gelman (1984) relate their analytic methods for the study of children's counting competence to linguistic methods.

The trend to thinking about linguistic methods in psychological analysis is at least partly attributable to the prevalence of information processing theories in mathematics education. As discussed below, linguistics and information processing theories share a number of important methodological features. The study by Matz (1980), for example, is preliminary to a

computational theory of algebraic skill.

Previous references to linguistic methods in mathematics education, however, have been intended more to draw analogies to linguistics, than to actually apply a linguistics-based methodology to mathematics. This paper presents an adaptation of linguistic methods for analysis of the "language" of symbolic elementary algebra.

The paper has two major sections. The first part describes the formal, linguistics-based methodology developed as a vehicle for algebraic skills investigation. The second part offers a comparison of linguistic methods and information processing modelling more commonly used in psychological analysis of algebraic skill.

1. Adapting Linguistic Methods

The methods developed herein are derived from the generative transformational approaches which dominated linguistics since their inception in the late 1950's until the present decade. That mathematics can be considered a language in a limited technical sense as prescribed for these formal methods is explicitly acknowledged. Chomsky (1957) begins his seminal work:

> From now on I will consider a *language* to be a set (finite or infinite) of sentences, each finite in length and constructed out of a finite set of elements. All natural languages in their spoken or written form are languages in this sense, since each natural language has a finite number of phonemes (or letters in its alphabet) and each sentence is representable as a finite sequence of these phonemes (or letters), though there are infinitely many sentences. Similarly, the set of 'sentences' of some formalized system of mathematics can be considered a language. (Chomsky, 1957, p. 2)

Even when language is reduced simply to strings of elements, however, fundamental differences between natural language and mathematical language remain. For example natural languages possess the characteristic of (sentence level) homonymity (eg. Flying planes can be dangerous) whereby a single surface string may represent more than one sentence. Mathematical languages generally avoid such ambiguity. Chomsky (1957, p. 86) establishes homonymity as a central feature of language to be accounted for in a linguistic theory. Thus linguistic methods have evolved, partly, to accomodate features of natural language which are not characteristic of mathematical language.

A second consideration in applying linguistic methods to mathematical language is the goals of the analysis. Chomsky (1965) sets as a fundamental objective of linguistic inquiry the discovery of the basic inherited core of language competence (universal grammar) which enables a child to deduce the particular grammar of his/her native language on the basis of the fragmented and degenerate data provided by early language experience. The explanatory goals of the present algebraic theory are more modest: To discover the set of rules to which the successful manipulator of algebraic symbols has access. Thus generative transformational linguistic methods have been shaped by goals which differ from those of the present analysis.

Finally generative transformational linguistics does not refer to a unitary procedure or set of procedures. There have been numerous versions, some evolving to replace previous versions, others proceeding in parallel. (See Newmeyer (1980) for a historical development, and Sells (1986) for a discussion of several models). Disputes, controversies and differences in terminology are ongoing. Fodor, Bever and Garrett (1974) observe that "Contemporary linguistics is an unfinished science. Its methodology is, in many respects, more interesting than its firm results" (p. 112). There is no unitary set of guidelines to apply when adapting

generative transformational linguistics to mathematical language.

For all these reasons, generative transformational linguistics cannot provide a ready recipe for application to mathematical language. What it does provide is the framework for an approach to mathematics. Beyond this the methodological details must be improvised as seems suitable for an inquiry into algebraic symbol skills. In the next section, the skeleton of the linguistic framework is presented. The details of the Adapted Linguistic Paradigm (ALP) are subsequently presented. Occasional footnotes point out only some of the deviations from standard linguistic paradigms.

As a final introductory note, it should be stressed that the connection between the present theory and linguistics is methodological. There is no reason to anticipate or plan for any connection between a model of algebraic language and a model of natural language. The advice of Jackendoff & Lerdahl (1983) in developing a generative linguistic model of music is equally appropriate for mathematics:

> Many previous applications of linguistic methodology to music have foundered because they attempt a literal translation of some aspect of linguistic theory into musical terms—for instance, by looking for musical "parts of speech," deep structures, transformations, or semantics. But pointing out superficial analogies between music and language, with or without the help of generative grammar, is an old and largely futile game. One should not approach music with any preconceptions that the substance of music theory will look at all like linguistic theory (pp. 5-6).

B. GENERATIVE TRANSFORMATIONAL LINGUISTICS

As noted above, a language is idealized as "a set (finite or infinite) of sentences, each finite in length and constructed out of a finite set of elements" (Chomsky, 1957, p. 2). Clearly, not all combinations of elements are sentences of a language (i.e. grammatical). The sentences are only a subset of the possible combinations. In its broad outlines, the formal objective of linguistic analysis is to devise a set of rules, called a grammar, that "generate[s] all of the grammatical sentences of a language and no ungrammatical sentences" (Baker, 1978, p. 8).

At first glance such a problem may seem insignificant or uninteresting. If, for example, a language were to consist of only a finite number of sentences, the grammar could simply rewrite this finite list, thereby generating all and only sentences. A requirement that the grammar itself be finite insures, however, that such trivial solutions fail for more interesting languages such as natural languages and algebraic language where the number of sentences is unbounded.

In practise, natural language linguistics has involved analysis at a variety of levels. In Chomsky's (1965) "Standard Theory" model, a formal system of phrase structure rules generates structural descriptions for deep structures of sentences, where deep structures are related to sentence meanings. Transformations carry deep structures into surface structures which specify the spoken (or written) organization of sentence constituents. Finally a phonological component produces sentences as strings of phonetic elements. Thus a linguistic analysis may detail many functions which would seem to be relevant to psychological analysis of language skill. It is the possible connection between grammars and psychological functions which has motivated psychologists' interest in linguistic models and which motivates the present investigation of algebraic language.

C. THE ADAPTED LINGUISTIC PARADIGM

In the following sections, after defining algebraic language, the evaluation of grammars both as formal instruments and as psychological theories is discussed. The details of these sections have been developed in consideration of the nature of algebraic language and of the goals of the inquiry. These details, therefore, differ from standard natural language treatments.

1. Algebraic Language

For the purposes of the present analysis, the "elements" of algebra are taken to be the usual symbols "x"'s, "y"'s, "+"'s, "−"'s, ")"'s, etc., and numerals of standard notation, as well as positional features such as horizontal juxtaposition, diagonal juxtaposition, etc.

The inclusion of positional features should not be regarded as a frivolous or unnecessary concession to graphological convention. Certainly, 2^x is different from 2x, and in a written language such as symbolic algebra, accounting for such differences is fundamental to the linguistic inquiry.

The "sentences" of algebra are taken to be the standard simplifications of expressions, and solutions of equations (or systems of equations) such as are normally produced by individuals engaged in the manipulation of algebraic symbols. Sample sentences are displayed in Table 1.

Other characterizations of "sentence" in the algebraic grammar are possible. More elementary units could be selected as sentences. For example, $3x + 6$, x, $\frac{(x - 2)^3}{5y} - 2$ or $f = ma$, $3x + 2y = 15$, $x + 3 = 2$, etc., might be classed as sentences for this analysis; however, since the objective of study is the skill of manipulating symbols, it makes sense to incorporate a manipulative act within the unit of investigation. Indeed, individual expressions or

Table 1
Sample Sentences

a. $25 - 16x^2 + 32xy - 16y^2 = 25 - 16(x^2 - 2xy + y^2) =$

$= 25 - 16(x - y)^2 = [5 - 4(x - y)][5 + 4(x - y)] =$

$= (5 - 4x + 4y)(5 + 4x - 4y)$

b. $\dfrac{28 - 7x^2}{6 - 5x + x^2} = \dfrac{7(4 - x^2)}{6 - 5x + x^2} = \dfrac{7(2 - x)(2 + x)}{(3 - x)(2 - x)} = \dfrac{7(2 + x)}{3 - x}$

c. $3x - 5 = -7x$

$10x = 5$

$x = \dfrac{1}{2}$

d. $\dfrac{\sqrt{x + 1} - 4}{\sqrt{x + 1} + 3} = \dfrac{\sqrt{x + 1} - 4}{\sqrt{x + 1} + 3} \times \dfrac{\sqrt{x + 1} - 3}{\sqrt{x + 1} - 3} = \dfrac{x + 13 - 7\sqrt{x + 1}}{x - 8}$

e. $3x + 2y = 8$

$2y = -3x + 8$

$y = -\dfrac{3}{2}x + 4$

f. $\dfrac{16 - 4x}{8y} = \dfrac{4(4 - x)}{8y} = \dfrac{4 - x}{2y}$

equations may be accounted for in a grammar of more broadly defined "sentences," even as phonemes, words and phrases are accounted for in natural language grammars.

Algebraic sentences, as defined above, can be divided into two categories corresponding to equation (or system of equation) solving sentences (c and e in Table 1), and expression simplifying sentences (a, b, d and f in Table 1). Such a division is standard in the literature (Wagner, Rachlin & Jensen, 1984). It turns out that expression simplification sentences are more elementary than equation solving sentences in that solving equations may involve, as a

subprocedure, simplifying expressions, while the converse is false. In the research conducted to date only expression simplification sentences are considered. An extension to equation solving sentences is intended.

D. DESCRIPTIVE ADEQUACY

The foundation of ALP is the segregation of the formal properties of grammar from possible psychological interpretations. In this way, the fit between the grammar and the language (*descriptive adequacy* of the grammar) can be considered apart from the fit of the grammar to the language speaker (*explanatory adequacy* of the grammar.) Each of these aspects of evaluation is an important and complex undertaking. Also, they would seem, in principle, to require quite different analytic tools to accomplish.

A grammar fulfills the formal requirements of ALP if the set of sentences generated by the grammar corresponds to the set of sentences of the language. In ALP, the grammatical sentences are defined simply to be such sentences as would normally be produced by persons who have fluent mastery of the language. (The term "algebraist" will refer, throughout, to one who is fluent in the correct manipulation of algebraic symbols. Students, teachers, scientists, mathematicians or just members of the general public may be algebraists in this sense.)

Assuming, momentarily, consensus among investigators as to the contents of a language, evaluation of descriptive adequacy is relatively straightforward. The set of sentences, G, generated by a rigorous formal grammar is completely determined. The set of sentences, L, of the language is also completely determined (consensus). To establish the inclusion of G in L, sentences produced by the grammar are evaluated as to their grammaticality. For the inclusion

of L in G, the grammar is analysed to determine if it can generate sentences which are known to be part of L. If there is deficiency in either direction the grammar may be adjusted so as to bring G into greater accord with L.

In practise, however, there is little reason to expect complete consensus among investigators as to the contents of a language. The number of grammatical sentences is unbounded whereas in the entire history of a language only a finite number of sentences have been produced. Thus direct verification as to the contents of a language is precluded.

As Chomsky (1957) notes, perfect accord is not a necessary precondition for linguistic analysis:

> Notice that in order to set the aims of grammar significantly it is sufficient to assume a partial knowledge of sentences and non-sentences. That is, we may assume for this discussion that certain sequences of phonemes are definitely sentences, and that certain other sequences are definetely non-sentences. In many intermediate cases we shall be prepared to let the grammar itself decide, when the grammar is set up in the simplest way so that it includes the clear sentences and excludes the clear non-sentences. This is a familiar feature of explication. (pp. 2-3)

Nevertheless, a high degree of consensus is required to prevent substantive discussions about the adequacy of a grammar degenerating into a debate about the contents of language.

There are grounds to expect a reasonable degree of consensus for languages such as English or algebra. Knowing English or algebra includes the possibility of producing and interpreting unlimited numbers of acceptable sentences. This aspect of language as a shared artifact within a community is an important ingredient for productive linguistic analysis.

E. ISSUES OF GRAMMATICALITY

When disagreements about grammaticality do arise, the undertaking in ALP is to resolution on the basis of sentences which have actually been produced by fluent users of the language.† Several issues which may form a basis for disagreement in algebraic language are discussed below and tentative positions are staked out in advance.

1. Prescriptive and Descriptive Grammars

In most language communities, a distinction can be made between the way in which a language is actually used, and the way in which authorities decree the language *should* be used. For example in English a decision is required as to whether or not "*Who* did you meet" or "I *will* go" should be considered grammatical sentences. Natural-language linguists in general have handled such cases by attempting to describe the language as it is actually written or spoken, rather than as the prescriptive authorities say that it should be used.

The issue of prescription versus description is important for an academic, school based skill such as elementary algebra symbol manipulation. In the authoritarian environment of the school, the admonitions regarding the form of algebraic expression are bound to be prominent and effective. Thus, for example, students may be instructed to never "cancel" from numerator and denominator of a fractional expression, $\frac{a\cancel{x}}{b\cancel{x}} = \frac{a}{b}$, but rather to proceed according to *sound mathematical theory*: $\frac{ax}{bx} = \frac{a}{b} \cdot \frac{x}{x} = \frac{a}{b} \cdot 1 = \frac{a}{b}$. Similarly, "crossmultiplying" or "transposing" may be taboo in equation solving.

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†Performance of language users is not the criterion to which natural-language linguists subscribe as a basis for the determination of grammaticality. Instead, linguists generally accept judgements of native speakers regarding the acceptability of proposed sentences. This is one aspect of a competence/performance distinction which has provided a major source of controversy in linguistics, psycholinguistics and psychology (Greene, 1972).

The violation of these regulations, however, is endemic to the normal practise of algebraic symbol manipulation. What is more, it is the fluent algebraist who is most likely to violate these normative standards. Thus the same teacher who meticulously takes three steps to derive $\frac{a}{b}$ from $\frac{ax}{bx}$ in the classroom, will unhesitatingly cancel the "x"'s in the privacy of his or her study. This is an indication that such normative strictures are often conceived for pedagogic purposes and not intended for general consumption. Since the criterion for grammaticality in ALP is the performance of competent algebraists, pedagogical prescription will be ignored in evaluation of grammaticality, whenever it conflicts with actual adult performance.

2. Competence/Performance

The general concern under the heading of competence/performance is whether the sentences actually produced by users of a language (performance) should form the basis for a linguistic theory. Chomsky (1968) has noted that as well as a speaker's knowledge of language (competence), actual performance reflects extralinguist factors including "memory limitations, distractions, shifts of attention and interest, and errors (random or characteristic)" (p. 4).

Two types of instances emerge in the competence/performance consideration: (A) There may be sentences which do not occur in performance, but which should be allowed in a competence model; and (B) There may be sentences which do occur in performance but which should be excluded from a competence model.

An example of Type (A) would be sentences comprised of vast numbers of elements. Imposing a cut-off on the length of algebraic expression would be an arbitrary and unrevealing limitation on the grammar —though an unrestricted grammar will generate sentences which no algebraist could produce or decode in a single lifetime.

The resolution of this dilemma requires some latitude in the interpretation of the criteria for grammaticality. The statement that grammatical sentences are simply "such sentences as would normally be produced by persons who have fluent mastery of the language" (page 31) is of necessity vague. The need for a basic consensus as to the contents of a language is real. In comparing the finite sample of available sentences of some corpus to the unlimited sentences of a language, there will always be a need for judgement in the assessment of which sentences are grammatical. The problem of arbitrary length would seem to be an aspect of sentences about which existing corpuses will provide little direct guidance.

Case (B) above concerns the presence of errors in performance. Of course, any substantial corpus of actual algebraic manipulations may contain errors. But a linguistic theory of algebra is primarily a theory of the knowledge of the *fluent* manipulator of algebraic symbols. Thus intra-individual data which display systematic errors can be ignored in the consideration of grammaticality.

The problem of errors arises in earnest if the occasional errors produced by competent algebraists turn out to have inter-individual consistencies. A study by Lewis (1980) suggests that this may indeed be the case, but little empirical investigation of fluent algebraists has actually been undertaken to date. It is too early to predict the extent to which consistent error patterns will pose difficulties for linguistic analyses of algebra.

A thorny problem which can arise in relation to inter-individual errors is the identifying of competent algebraists. In the natural language paradigm, all normal adults within a language community are taken to be fluent speakers. This is clearly not the case for algebraic language

nor is it likely that any particular credential will provide useful necessity and sufficiency criteria (though some credentials will provide sufficiency criteria). This can lead to disputes about the acceptability of some observed data.

3. Pragmatics/Syntax

A third issue in the identification of the sentences of a language concerns syntactic versus pragmatic grammaticality. Consider the sentences

$3x^2 - 27x + 42 = 3(x^2 - 9x + 14) = 3(x - 2)(x - 7)$ and

$3x^2 - 27x + 42 = 3x^{1+1} - 27x + 27 + 15 = 3x^{1+1} - 27(x - 1) + 5{\cdot}3.$

Each is a "correct" (syntactically grammatical) mathematical statement, however, the second lacks some quality of utility or meaningfulness which the first possesses. Clearly a decision to constrain the grammar to "meaningful" sentences has tremendous implications for the character of a linguistic theory and for the level of explanation which it will provide.

It is useful for algebra (as for natural-language linguistics) to distinguish pragmatics from semantics within the broad area of linguistic meaning. Demarcation of a clear boundary between the two domains is far from complete (Leech, 1983; Levinson, 1983); nevertheless, there is general agreement that "meaning in pragmatics is defined relative to a speaker or user of the language, whereas meaning in semantics is defined purely as a property of expressions in a given language" (Leech, 1983, p. 6). As the issue illustrated by the above examples concerns the purposes to which an algebraist might put manipulations in algebra, the use of the term "pragmatics" seems indicated. The pragmatic analysis of algebra is, thus, an exploration of terms such as "simplify," "reduce," "rationalize," "factor," etc., which characterise the goals of expression simplification.

What is proposed for algebraic language is the analysis of sentence types into canonical forms, each described in terms either of syntactic elements, or of more elementary canonical forms. These canonical forms will operationalize notions of "simplify," "reduce," "factor," etc. Sentences, then, will be considered pragmatically grammatical precisely if they conform to any one of a number of predefined "profiles" or forms. In this way, the class of syntactically grammatical sentences may be appropriately constrained. In the present linguistic research, elaboration of the pragmatic component has not yet been undertaken with the result that the whole range of syntactically grammatical sentences are generated by the grammar.

Pragmatics have been studied in algebra before (for example by Carry, Lewis & Bernard, 1980, under the title "Strategic Choices"). Mostly, however, semantic notions of meaning have dominated in mathematics education research. Generally, what is meant by "meaningful" is that the symbols employed have reference beyond the symbol scheme. Thus "x" may represent, say, "Sally's age," or perhaps just "a quantity"; "+" an action of combining, etc. It is by no means clear that such referential aspects of meaning (which would seem to be fundamental to the skills of translating between real world contexts and algebraic forms) need be invoked for explanations of manipulative skill. The syntactic and pragmatic components of the grammar taken together, if successful in meeting the formal requirements of the grammar, will constitute a detailed hypothesis that algebraic manipulation skill does not depend upon semantic, referential knowledge.

F. EXPLANATORY ADEQUACY

Once its descriptive adequacy is established, the validity of a grammar as a psychological theory can be considered. There is no guarantee, however, that any psychological claims can even be advanced for a grammar. A linguist may discover some means of generating sentences

in a way which seems to defy possible connection to processes underlying language skill. On the other hand, as noted for the Standard Theory (page 28), a grammar may provide a description of many functions which would seem to be relevant to psychological functions.

In so far as a grammar can be associated (implicitly or explicitly) to a psychological model of language processing, it may be postulated that the rules employed in the grammar actually reflect speakers' knowledge of the language. (In the present linguistic theory of algebra an explicit process model is outlined and related to the structure of the grammar.) To the degree that the rules of the grammar do reflect the knowledge of the language, the grammar will be said to have *explanatory adequacy*.†

Two distinct empirical approaches to evaluation of explanatory adequacy are possible. The first approach concerns evaluation of the processing model which underlies the psychological claims for the grammar. A well known example from the natural language literature is the study by Miller and McKean (1964). It was postulated that sentences which received more complex analysis in the linguistic theory should require more resources, including time, for processing. Response latencies were measured for subjects assigned various language tasks. Initially results seemed to support the psychological reality of the linguistic complexity measures; however, subsequent analysis yielded disconfirmatory results (see Fodor, Bever & Garrett, 1974, pp. 226–241).

Such analyses can provide useful support for a linguistic theory, however, problems arise when disconfirmatory results are used to falsify a linguistic theory. Disconfirmatory evidence indicates

†The explanatory objectives of the ALP are far more modest than those of Chomsky's (1965) theory.

either that the grammar is inadequate as a reflection of the language users' knowledge, *or* that the processing model upon which the experimental design hinges is misconceived in some small or large way. In general, there is no commitment within a linguistic theory to the details of any particular processing model. Thus such approaches fail to provide conclusive evidence of the deficiency of the grammar.†

A second category of psycholinguistic research has more direct application to refinement and development of linguistic models. Often, it is possible that more than a single grammar can be designed which fulfills the formal linguistic requirements. These versions may reflect importantly different views of the psychological basis for language use. In such cases, psychological evidence may be marshalled in support of one of the competing alternative grammars.

In cognitive science, the crucial importance of "competitive argumentation" —a means for selecting from among plausible alternative theories— is recognized by VanLehn, Brown and Greeno (1984) as "the only realistic alternative to [problematic] necessity arguments" (p. 240). In linguistics, psycholinguistic techniques are already available for selecting from among competing descriptively adequate grammars, the one with greatest explanatory adequacy. Two

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†The objection might be raised that a linguistic model lacks force as a scientific theory because it cannot be refuted. As Kuhn (1970) suggests, however, refutation of scientific theory is not nearly so straightforward as contradiction-implies-refutation. Rather, refutation requires a number of concommitant factors, of which disconfirmatory evidence is only one. (Others include the presence of some viable alternative theory.)
Empirical research is most vital in "normal science" as a vehicle for the refinement and elaboration of theoretical models. The objective of the present theory is to produce grammars which are useful to applications in educational psychology and especially curriculum theory. Failure to produce results within these domains will undoubtedly lead to abandonment of these theories, and it is in these domains also that success should be measured. There is no attempt in the present theory to anticipate or meet external criteria for "psychological reality."

such techniques have been coopted for *psychoalgebraic*† research (Papers #4 and #5). The discussion of linguistic methods is concluded with a brief description of these psycholinguistic paradigms.

1. The Nonce Forms Paradigm

Languages often incorporate redundant features. In such cases, it may be possible to devise alternative observationally adequate grammars which differ as to the feature used. It may, however, be the case that one feature predominates as an active cognitive agent. In such a case, a nonce form experiment may be used to aid in selection.

A nonce form is a contrived or artificial language form. Nonce forms may be devised which incorporate only one of the normally co-occuring features. In a nonce forms experiment, competent speakers are directed to operate upon such a specially designed nonce form as if it were a normal part of the language. The ability of the subjects to operate successfully supports the position that the excluded feature is not essential to cognitive processing. Conversely, inability to function successfully illustrates cognitive dependence upon the excluded feature. In either case, the results may aid in the selection of one of the competing grammars.

The Berko-Gleason (1958) study is a prominent example of the nonce forms paradigm in natural language. In that study, Berko-Gleason was interested in finding out whether children acquire morphological rules or whether their competence in, for example, supplying correct plural endings is simply a result of rote memorization. The ability of children to function

†The term "psychoalgebra" is used to refer to the evaluation of the psychological claims made for an algebraic grammar.

appropriately using contrived nonce words allowed her to conclude that morphological rules are indeed acquired by children.

2. The Acquisition Grammar (Error Analysis) Paradigm

When alternative observationally adequate grammars are postulated for a language it may be useful to examine acquisition data to aid in selection. Acquisition data result from the use of the language by novices who have not yet achieved full mastery. In mathematics as in natural language it has been noted that learners' errors tend to be systematic rather than random deviations from competent use. Thus it may be possible to formulate an "acquisition grammar" which generates such sentences as are observed in the learner's language use. If a strong developmental argument can be advanced linking the acquisition grammar with one of the competing 'adult' grammars, then this supports the descriptive adequacy of both the acquisition grammar and that version of the 'adult' grammar. In this way, acquisition data can aid in the selection from competing observationally adequate grammars.

It is important to distinguish the use of acquisition grammars in local arguments as described above from their use in a general theory of skill acquisition. A generative linguistics approach to skill acquisition through the analysis of a "language of acquisition" (presumably incorporating patterns of common errors as well as "correct" algebraic sentences) would be problematic. The principal problem would be compatibility of different errors within such a language. The usual assumption that any sentence of a language can be spoken by any speaker of the language probably does not hold for a language of errors. The presence of some errors for an individual may preclude the presence of other errors. Thus it is unlikely that a grammar which generates some general class of errors would be of use as a model of a "speaker's" knowledge.

PART II

G. LINGUISTIC AND INFORMATION PROCESSING PARADIGMS

Many recent studies of the psychology of algebra have adopted cognitive science or information processing (IP) orientations. Why then develop new linguistics-based methods when established paradigms are already in place? This section examines some similarities and differences between linguistic and IP methods and considers advantages and disadvantages of each.

Information processing psychology is the attempt to understand human information processing through analogy to computer information processing. Generally, an IP theory for a particular skill begins by positing certain *higher level* constructs (e.g. "procedure", "sub-procedure", "VMS-sequence", "frame", "assimilation paradigm", "meta-language", etc., [Davis & McKnight, 1979a]) which can be implemented as computational procedures (i.e. in elementary computer science terms). These constructs form the basis for a model of human performance in some domain. When the model is specified in sufficient detail, it can be implemented as a computer simulation of performance thus providing a test of the internal consistency of the theory.

Newell and Simon (1972) point out that linguistic grammar bears comparison to such a simulation paradigm:

> As a matter of historical fact, the basic concepts that have entered into our description of an IPS [information processing system] have almost the same origins as the concepts that underlie the formalized transformational grammars that linguists have developed over the past fifteen years (Chomsky, 1957). What is the relation, then, of language and the concepts of modern linguistics to our concept of an

information processing system?

> In modern linguistics, a language is a formal system containing a set of elements (cf. our "symbols") that can be arranged in linear strings (cf. our "lists"). A set of rules, or grammar, determines which strings, from among all possible concatenations of elements, are to be regarded as admissible in the language. The admissible strings are called "well-formed" or "grammatical...". (p. 38)

Despite the methodological connection, however, three fundamental differences between IP and linguistics bear upon the study of algebra for the purposes of educational application: The nature of the models as formal objects; the scope and generalizability of theories; and the status of the models as psychological theories. In the remaining sections of the paper these are considered in turn.

1. The Nature of the Formalism

On a microanalytic level, an IP theory which is realized as a computer program can be considered a formal model. The elements of computer science are rigorously defined terms and procedures. To the extent that an IP theory is defined directly in terms of basic computer science constructs the theory is a formal theory. Part of the tremendous optimism which has greeted IP psychology is the only rarely challenged conviction (e.g. Putnam, 1980) that ultimately human and computer processing can be described within the same formal framework:

> The basic idea of cognitive science is that *intelligent beings are semantic engines*—in other words, automatic formal systems with interpretations under which they consistently make sense. We can now see why this includes psychology and artifical intelligence on a more or less equal footing: people and intelligent

> computers (if and when there are any) turn out to be merely different manifestations of the same underlying phenomenon. (Haugland, 1985, p. 31)

While a computer simulation, in principle, may be considered a psychological theory, that is not usually a productive level for theory *formulation.* At that level, there is a "general tendency of simulation-writing ... to be pushed toward dealing with minute details - and frequently details that are more characteristic of computers than they are of human thought" (Davis & McKnight, 1979, p. 31). Rather it is the higher level constructs which provide the substantive basis for psychology. Davis & McKnight (1979a) expresss it thus:

> We ... focus ... on another key aspect of recent studies: the interpretation of observed behavior in terms of metaphoric models of human information processing - the analogue of the physics/chemistry metaphoric models of atoms, molecules, and other particles or aggregates of particles. (p. 95)

At the higher level of these constructs, rigorous formality deserts IP psychology. Formal interrelationships can not simply be defined into existence for these constructs. The metaphoric connection implies that any statement of formal relationships is perforce a substantive knowledge claim. Psychological theory cannot yet support systematic treatment of these higher level information processes and, indeed, the set of constructs in IP is itself subject to evolution as progress in modeling and simulation advances theory. For this reason the formal status of the higher level constructs is never clear *vis a vis* logical consistency, logical independence, etc.; so, while incorporating some aspects of formalism in modeling and testing, cognitive science lacks the underpinnings of a truly formal theory.

The status of linguistics as a formal study is more secure. Lacking the direct connection to mind, a grammar employs formal means in an unconstrained way to achieve its generative goals. As will be seen in Paper #3, the grammar of algebra is manipulated as a formal object to yield new models which form the basis for further psychological theory. This aspect of formal manipulation is hardly viable in IP modelling and represents a significant advantage of ALP.

2. Scope and Generalizability of Theories

Part of the enormous appeal of IP psychology is the breadth of phenomena which can be modelled. Just within mathematics there have been studies of arithmetic, algebra, word problems, experts and novices, meaningful and rote processes, etc. (See Paper #1 for a more detailed discussion of some studies.) Just within a single study, Davis, Jockusch and McKnight (1978) have dealt with examples from arithmetic, algebra, geometry, matrices, graphing, trigonometry vectors and calculus. IP psychology, by providing a broad theoretical framework offers the possibility of relating diverse skills in deep ways.

Computational theories can cut such a wide swath because they are primarily theories of individuals' performances, grounded in specific protocols. As Matz (1980) righteously declares in a footnoted quotation of Sussman's statement: "Statistics is the science of populations, artificial intelligence is the science of the individual" (p. 97). But there are problems of generalizability which cannot be ignored:

> In particular, when such detailed analyses [IP theories] aspire to be empirical theories, they face difficulties in achieving an adequate treatment of individiual differences. In most analyses, there has been considerable obscurity in the boundary between what is meant to be true of all subjects, and what is meant to be true of a particular subject. (VanLehn, Brown & Greeno, 1984, pp. 235–236)

Linguistics shares with IP psychology neither these benefits of scope nor hazards of generalizability. As described in the first part of this paper, the formal nature of linguistics imposes severe constraints as to what can be considered a "language" for the purposes of inquiry. Besides the requirement that the performance investigated be interpretable as consisting of "sentences" composed of finitely many identifiable "elements," there must be some possible basis for consensus among investigators as to which combinations of elements are actually sentences (page 31). Thus a formal linguistic approach can only apply to the investigation of a very restricted class of phenomena.

While the scope of a linguistic theory is of necessity limited, there are some grounds to expect generalizability to a large population:

> Every human being who speaks a language knows the grammar. When linguists wish to describe a language they attempt to describe the grammar of the language that exists in the minds of its speakers. There may of course be some differences between the knowledge that one speaker has and that of another. But there must be shared knowledge because it is this grammar that makes it possible for speakers to talk to and understand one another. (Fromkin & Rodman, 1974, p. 12)

This observation holds equally for algebraic language which is not generally used to communicate information, but which, presumably, could be.

Of course, individual differences must be addressed within either a linguistic or IP framework. In linguistics the extension is from a general theory to variations for individuals. IP theory has the less enviable task of formulating generalizations from individual, and possibly idiosyncratic, observations.

3. Theories of Mind

As noted above, an IP model is a metaphoric model of mind. The psychological connection for linguistics is much less direct. A grammar is a formal mechanism for generating sentences of a language. It is not a model of a human 'generator of sentences' (speaker). Chomsky (1965) explains:

> ... a generative grammar is not a model for a speaker or a hearer. It attempts to characterize in the most neutral possible terms the knowledge of the language that provides the basis for actual use of language by a speaker-hearer. (p. 9)

More specifically, a grammar may characterize elements of knowledge of language. But a grammar is not organized as a process model.

Linguistic theories have been the subject of deep interest and occasional severe criticism from psychologists (Green, 1972) on account of the tenuous connection to psychology. The charge has been made that although linguists insist on making psychological claims for grammar, they often refuse to accept the evidence of psychological experiment as intrinsically relevant to their theories. The specific details of these charges are complex and only partially relevant to ALP, however, in general terms, the criticism stems from the indirectness of the connection between grammar and mind. Process models which may form the basis for psychological interpretation of a grammar are rarely explicated in a linguistic theory. Problems of associating the grammar to a process model are even less frequently addressed by linguists. In the grammar of algebra (Paper #3) an informal process model related to the grammar is made explicit and problems in associating the grammar to the process model are discussed and dealt with through ammendment of the grammar.

If the absence of a direct connection between model and mind is a problem in linguistics,

the presence of such a connection in IP theory is not an unmixed blessing. The necessity of implementing higher level skills in terms of atomic constructs leads to overly complex theories:

> The capability of analyzing the details of processing specific information is clearly an advance. For example, it enables psychological analyses of human behaviors that one would label "understanding" that are much more detailed than those provided previously. However, there is a well known danger to such an approach. Analyses can become mired in their increased precision and detail, with the result that it is extremely difficult to separate fundamental principles from their supporting detail. (VanLehn, Brown, & Greeno, 1984, p. 235)

Marr (1981) distinguishes between the detailed model as a whole, and the "computational theory" which is the principle (if any) being implemented in the model. He notes that linguistic models may represent the guts of the psychological theory embodied in a computer simulation:

> Chomsky's (1965) notion of a competence theory for English syntax is precisely what I mean by a computational theory for that problem. Both have the quality of being little concerned with the gory details of algorithms that must be run to express the competence (i.e., to implement the computation). That is not to say that devising suitable algorithms will be easy, but it is to say that before one can devise them, one has to know what exactly they are supposed to be doing, and this information is captured by the computational theory. (p. 131-132)

Winograd (1983) expresses the questions motivating an IP theory (of language) as follows:

> *What knowledge must a person have to speak and understand language?*
> *How is the mind organized to make use of this knowledge in communicating?*
> (p. 1)

For psychology which has a direct investment in questions of specific skills as well as general properties of mind, IP methods have been a development of immense value. For educational applications, however, it is necessary to consider whether answers to the second question, in

themselves, will contribute manipulable variables for curriculum or instruction. In the limited domain of algebraic symbol manipulation skills, the adapted linguistic paradigm provides a means to address the first question without the necessity of entering the extremely difficult territory demarcated by the second.

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III. A GRAMMAR OF ALGEBRAIC SYMBOL SKILL

Paper #3

This three part paper presents a model of symbolic elementary algebra using a methodology adapted from generative transformational linguistics. The first part gives a brief rationale for the study, provides an introduction to the linguistics-based methods; and gives an overview of the algebraic grammar. The grammar itself is presented in the second part. In the third part a variety of alternatives for selected components of the grammar are proffered and psychological bases for selection are considered.

Part I summarizes previous papers. A more complete rationale can be found in Paper #1. Paper #2 provides an introduction to the adapted linguistic paradigm and describes its relationship to cognitive science methods more commonly used in the psychological analysis of mathematical skill. More detailed discussion of many points raised here is available from these sources.

Part I

A. RATIONALE

A great deal of research has been conducted into the processes whereby algebraic knowledge is acquired and into the deviations from competent performance which characterise acquisition (Davis & McKnight, 1979; Carry, Lewis & Bernard, 1980; Matz, 1980; Clement, 1982; Wagner, Rachlin & Jensen, 1984; and Sleeman, 1986; for example). In contrast, very little research has attempted to specify a detailed account of the rules which are being acquired and from which deviations occur.

At least part of the inattention to competent performance can be attributed to an implicit belief that mathematical theory (in some sense) underlies or generates or explains knowledge

used to manipulate algebraic expressions. This belief is evidenced, for example, in the distinction which Brown, Burton, Miller *et al* (1975) make between the "abstract logical structure of the [algebraic] knowledge" and the "reorganized, learner oriented structuring of how he is to use the knowledge for solving algebra problems" p. 84. Apparently they believe that some abstract structure (presumably a formal mathematical model) underlies the psychological representations, however, they do not provide a detailed account of the presumed connection.

The prevailing notion that mathematical theory underlies mathematical competence is only rarely challenged. In one such instance Gagné (1983) responds to Steffe and Blake (1983):

> The apparent view of Steffe and Blake that understanding involves some aspects of the "structure of mathematics" is what I would be inclined to question. I realize that this is an extremely common view among mathematics educators, and it is exactly this view I should most strongly like to urge them to examine critically. Is this view actually based on solid evidence, or does it merely reflect a traditional statement by those who like to reassure themselves that they are on the side of the angels? (Gagné, 1983b, p. 215)

In general, however, this implicit belief has received little explicit attention.

Does the view that mathematical theory underlies algebraic symbol manipulation rules stand up under scrutiny? At least three distinct aspects of knowledge must be accounted for in a theory of algebraic symbol manipulation: Knowledge of notational conventions; knowledge of rules which are applied to transform equations or expressions; and the pragmatics of rule selection. To call the first of these "mathematical theory" is misdirected. The notational forms and conventions currently employed in mathematical expression have not been the product of systematic design, but of gradual evolution over a millenium in response to changing needs and circumstances (Cajori, 1928). Mathematical theory makes use of notational convention but

cannot be said to account for it in any systematic way.

The pragmatics of rule selection also is outside the realm of mathematical theory. The fact that it is convenient to use certain selections of transformations to accomplish specific goals (e.g. reducing fractional expressions, rationalizing denominators, etc.) is a function of mathematical application, not of mathematical theory.

It is transformational rules themselves which are the main contenders in the search for a link between mathematical theory and algebraic skill. The abstract algebraic theory of integers or rational numbers does provide some possible rules (e.g. the "Distributive Law", the "Associative Law", etc.) for inclusion within a psychological account of algebraic symbol manipulation. But that algebraic accounts of integers or rational numbers provides an "abstract logical structure" for algebraic knowledge is untenable as these theories cannot account for properties of exponents and radicals —operations which lie outside the systems of integers or rationals.

The real numbers (which do admit the operations of exponentiation and radical) are normally studied within analysis rather than algebra. There are no proposals known to this researcher that analytic concepts such as least upper bound, Dedekind cut, or Cauchy sequence underlie knowledge of the rules used in manipulating algebraic equations and expressions. Perhaps some *algebraic* treatment of real numbers (e.g. Macintyre, 1979/1980) can provide a solution.† Such

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†Macintyre himself is not overly optimistic. He concludes:

> The most interesting problem provoked by the above is that of showing that there are no "exotic" laws, i.e. that every law is a consequence of the laws of $+, \cdot, -, ^{-1}, 0, 1$ together with $x^1 = 1$ $\quad x^{y+z} = x^y \cdot x^z$ $\quad x^{yz} = (x^y)^z$ $(xy)^z = x^z \cdot y^z$.

prospects should be critically examined.

For the present purposes, however, the important observation is that, despite a widespread belief, two of three components of algebraic knowledge are clearly not derived from mathematical theory, and the prospects for the third component are dubious. This observation leads inexorably to the need for a rigorous accounting of algebraic rules; a need to which the present paper attempts to respond.

B. LINGUISTIC METHODS

As a result of its vast familiarity to its investigators, natural language too has provided the illusion that detailed and rigorous accounts of regularities are unnecessary. Even as in the study of the psychology of algebra today, the focus in linguistics prior to the modern period was on obtrusive, irregular elements of language use:

> The limitations of traditional and structuralist grammars should be clearly appreciated. Although such grammars may contain full and explicit lists of exceptions and irregularities, they provide only examples and hints concerning the regular and productive syntactic processes. (Chomsky, 1968, p. 5)

It was only with Chomsky's (1957) introduction of generative transformational linguistic methods that the full complexity of grammar could be appreciated. It is serendipitous that as an artifact of the characterization of language in that paradigm that "the set of 'sentences' of some formalized system of mathematics can be considered a language" (Chomsky, 1957, p. 2). Following is a brief outline of linguistic methods as adapted for the "language of algebra."

Generative transformational linguistics is a formal enterprise. A *language* is idealized as "a set

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†(cont'd) It seems difficult to prove such a theorem by the methods of real algebra used above. (p. 97)

(finite or infinite) of sentences, each finite in length and constructed out of a finite set of elements" (Chomsky, 1957, p. 2). For natural languages, the sentences are taken to be the statements, exclamations and questions of ordinary speech. The *elements* are the phonemes, or sound units, of which speech is composed.

Clearly, not all sequences of elements are well formed (grammatical) sentences. Most sequences are ungrammatical. The formal objective of a linguistic analysis is the specification of a set of rules, called a *grammar*, which generates all of the sentences of the language but none of the ungrammatical sequences of elements.

An adequate grammar must be sketched out in sufficient detail and clarity so that its scope (the set of sentences which it generates) can be clearly established through logical analysis. The sentences generated by the grammar can then be compared to the sentences of the language, and in cases of deficiency, adjustments made.

A grammar, however, is not a model of a speaker or a listener. Psychological relevance is obtained through possible connection between the components of the grammar and the components of some process model to which the grammar —either explicitly or implicitly— is linked.† In the present case, a process model related to the algebraic grammar is made explicit.

†As Green (1972) points out, linguists have usually been content to let the process model remain implicit, resulting in considerable confusion about the status of grammar as a psychological theory.

1. Algebraic 'Language'

In the present theory, the "sentences" of symbolic elementary algebra are taken to be the simplifications of expressions, solutions of equations, etc., as are ordinarily produced by competent manipulators of algebraic symbols. The elements of which these sentences are composed are the explicit symbols =, -, +, $\sqrt{}$, —, ., 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, a, b, c, ... x, y, z, (,), [,], {, }, as well as the positional designators of horizontal juxtaposition, vertical juxtaposition, and diagonal juxtaposition.† Sample sentences appear in Table 2.

It is standard to distinguish expression simplification sentences from equation solution sentences (Wagner, Rachlin, & Jensen, 1983). In the present theory, only expression simplification sentences (such as a, b, d and f in Table 2) are considered (although the extension to equation solving sentences should require no basic methodological innovation). The algebraic grammar, then, must provide a formal means to generate all possible expression simplification sentences while avoiding the myriad possible nonsentence combinations of symbols.

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†A couple of technical points are related to this list. The "language of algebra" as comprised of these elements is a slight idealization of algebra as it is actually written. First of all, capital letters which are often used as variables in algebra serve another function in the grammar and so do not appear in the list. Secondly, there seems to be some instances where an explicit symbol for multiplication (either "X" or "·") is required (see sentence d in the accompanying table). The necessary contextual analysis for these symbols is avoided by their ommission.

A second point arises with respect to the vincula used as grouping indicators in division and radical operations. There are, technically, an unlimited number of vincula of different sizes possible —a circumstance which violates the finiteness condition on number of elements. A solution to this problem could be achieved by allowing only a minimal unit length vinculum and regarding vincula in use as finite concatenations of copies of this unit This solution is not adopted in the present grammar. Vincula of arbitratry length are simply used as required.

In this instance, as in others, the formality of the grammatical treatment is sacrificed in favour of its simplicity and conciseness. Whenever an issue of psychological interest hinges on a technical point, care is taken to proceed with due rigor. In other instances where there is no particular psychological concern, the presentation is only sufficiently formal to insure clear and unambiguous interpretation.

Table 2
Sample Sentences

a. $25 - 16x^2 + 32xy - 16y^2 = 25 - 16(x^2 - 2xy + y^2) =$

$= 25 - 16(x - y)^2 = [5 - 4(x - y)][5 + 4(x - y)] =$

$= (5 - 4x + 4y)(5 + 4x - 4y)$

b. $\dfrac{28 - 7x^2}{6 - 5x + x^2} = \dfrac{7(4 - x^2)}{6 - 5x + x^2} = \dfrac{7(2 - x)(2 + x)}{(3 - x)(2 - x)} = \dfrac{7(2 + x)}{3 - x}$

c.
$$3x - 5 = -7x$$
$$10x = 5$$
$$x = \frac{1}{2}$$

d. $\dfrac{\sqrt{x + 1} - 4}{\sqrt{x + 1} + 3} = \dfrac{\sqrt{x + 1} - 4}{\sqrt{x + 1} + 3} \times \dfrac{\sqrt{x + 1} - 3}{\sqrt{x + 1} - 3} = \dfrac{x + 13 - 7\sqrt{x + 1}}{x - 8}$

e.
$$3x + 2y = 8$$
$$2y = -3x + 8$$
$$y = -\frac{3}{2}x + 4$$

f. $\dfrac{16 - 4x}{8y} = \dfrac{4(4 - x)}{8y} = \dfrac{4 - x}{2y}$

C. THE LINGUISTIC MODEL

A fundamental distinction is made in the grammar between the abstract underlying structure of a sentence (the deep form), and the conventional notational form in which algebraic language currently happens to be expressed (the surface form). For example, the surface form, "$3x^2$", implicitly conveys operations of multiplication and exponentiation, as well as a parse for the expression. The deep form, 3M[xE2], displays these operations and parse explicitly, where "M" and "E" represent multiplication and exponentiation respectively and parentheses indicate grouping in the usual way.

The task of generating the sentences of algebra is divided into two parts. Sentences are generated in the neutral, deep form, notation. Then a series of *translation rules* are used to attain the standard surface form of ordinary notation.

A second major distinction in the grammar is between the full sentences of the language, which involve sequences of algebraic expressions, and the individual expressions themselves. For example, $\frac{16 - 4x}{8y} = \frac{4(4 - x)}{8y} = \frac{4 - x}{2y}$ would be classified as a sentence of symbolic elementary algebra, but $\frac{16 - 4x}{8y}$, by itself, is an expression, not a sentence. In this grammar, the generation of individual expressions is dealt with first, and then *transformational rules* are incorporated to generate compatible deep forms for sequencing.

The organization of the grammar is as follows: A phrase structure component generates the deep form for only and all expressions. A series of translation rules carry deep forms (DF) into surface forms (SF). Finally, transformations are postulated which carry the deep form of one expression into the deep form of another.

A sentence such as $\frac{(2x)^2}{x} = \frac{4x^2}{x} = 4x$ is generated by the grammar as follows: The phrase structure grammar generates the DF, [[2Mx]E2]Dx. Translations carry this DF to the SF, $\frac{(2x)^2}{x}$. Successive transformations are applied to deep forms carrying [[2Mx]E2]Dx ——> [[2E2]M[xE2]]Dx ——> [4M[xE2]]Dx ——> [4M[xMx]]Dx ——> [[4Mx]Mx]Dx ——> [4Mx]M[xDx] —> [4Mx]M1 —> 4Mx. Translations of *some* of the intermediate DF's into SF completes the process.

Figure 1
The Linguistic Model

$$\begin{array}{ccccccccccc} & SF_1 & = & & SF_2 & = & SF_3 & = & \cdots & = & SF_n \\ & \Updownarrow & & & \Updownarrow & & \Updownarrow & & & & \Updownarrow \\ PSG+(OT) \longrightarrow & DF_{1,1} & \longrightarrow & DF_{1,2} & \longrightarrow \cdots \longrightarrow & DF_{2,1} & \longrightarrow \cdots \longrightarrow & DF_{3,1} & \longrightarrow \cdots \longrightarrow & & DF_{n,1} \end{array}$$

"SF" represents surface form.
"DF" represents deep form.
"PSG" stands for phrase structure grammar.
"OT" stands for obligatory transformation.

a. Omitted Translations

Each of the deep forms derived through transformation can be associated through translation to a surface form. In the above example, the translation of each DF expression would yield the SF sentence $\frac{(2x)^2}{x} = \frac{2^2x^2}{x} = \frac{4x^2}{x} = \frac{4(x \cdot x)}{x} = \frac{(4x)x}{x} = 4x \cdot \frac{x}{x} = 4x \cdot 1 = 4x$. In practise, however, algebraists† rarely express each DF in a written SF. Rather, only selected intermediate steps are chosen for transcription.

To some degree, this selection is idiosyncratic, reflecting an individual's memory capacity, experience, or even the caution being exercised in a particular instance. Consequently, a variety of related sentences may share the same basic DF derivation. This consideration poses little problem, since the grammar is able to produce any desired combination of intermediate steps through translation.

Beyond such idiosyncratic variation, however, it can be observed that some of the intermediate

†The term "algebraist" refers here and throughout to one who is fluent in the correct manipulation of algebraic symbols. Students, teachers, scientists, mathematicians or just members of the general public may be algebraists in this sense. This usage differs from the standard.

steps which might be hypothesized to underlie a manipulation, *never* surface in actual sentences. For example, no algebraist will write $4x + 4 = 4x + 4 \cdot 1 = 4(x + 1)$ (other than for pedagogical purposes) even though this may reflect the underlying transformations. In the present theory, this has been accounted for by marking certain transformations (eg. $a \longrightarrow a \cdot 1$) with the "&" symbol as a prohibition against translation of any intermediate DF's which they may produce, into surface form.

Figure 1 displays the model of the algebraic grammar. An obligatory transformation is required for stylistic adjustments to expressions (eg. $x3 \longrightarrow 3x$). The dotted line-breaks indicate that transformations may occur in DF which are not represented in SF. The double arrows between DF and SF indicate bidirectional translations. Bidirectionality serves no formal function within the grammar, however, it is a key notion in the psychological interpretability of the model. (See below.)

b. Pragmatic Component

The above model, if successful, will generate the sentences of algebra, such as would be produced by an algebraist, but it will also produce sequences of expressions which, while "correct" mathematical derivations, lack some quality of directedness or coherence. Consider the sentences $3x^2 - 27x + 42 = 3(x^2 - 9x + 14) = 3(x - 2)(x - 7)$ and $3x^2 - 27x + 42 = 3x^{1+1} - 27x + 27 + 15 = 3x^{1+1} - 27(x - 1) + 5 \cdot 3$.

Each is a "correct" (syntactically grammatical) mathematical statement, however, the second lacks some quality of directedness or "meaningfulness" which the first possesses. A pragmatic analysis of algebra is an exploration of terms such as "simplify," "reduce," "rationalize," "factor," etc., which inform "meaningfulness" in this sense. Elaboration of the pragmatic component of the grammar has not yet been undertaken (though this is intended) with the

result that the whole range of syntactically grammatical sentences are generated by the grammar.

1. Psychological Interpretation

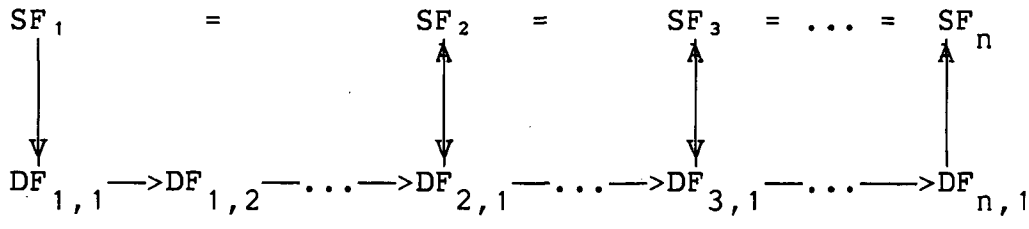
A grammar is not a process model. An actual instance of algebraic simplification would presumably start off with an expression already given.† It would not be generated through the internal processes of the algebraist, by a phrase structure grammar or any other device. In its overall shape, however, the grammar does characterize certain structures (deep and surface forms) which might be hypothesized to underlie mental representations. As well, it details translations and transformations to mediate between these structures. Thus, while not itself a psychological theory, a grammar provides for many of the operations which could be included in a process model.

A process model which is closely related to the linguistic model might appear as in Figure 2. Presumably an expression given in surface form (SF_1) would be translated into deep form and subsequently transformed into a series of further deep forms with occasional translation for transcriptive purposes into surface form as an aid to short term memory. The double arrows between DF and SF are required in the process model in order to account for the decoding as well as the encoding of SF's. For the most part, the rules for translation from DF ——> SF can be reversed both in order and direction to translate SF ——> DF. For one rule, however, a new formulation is required for the reversal.

In evaluating a grammar, it is necessary to distinguish between the formal objectives of the

†This might be derived from translation of a "real world" situation; represent an application of a scientific formula; have arisen in a calculus computation; or simply been presented in an algebra text.

Figure 2
Informal Process Model



"SF" represents surface form.
"DF" represents deep form.

linguistic enterprise and the psychological validity of the constructs. To the extent that a grammar meets its formal objectives it is said to be *descriptively adequate.* To the extent that the rules established in the grammar correspond to the rules which can be attributed to fluent algebraists, the grammar will be said to have *explanatory adequacy.*†

2. Arithmetic Anomaly

At some point in the discussion of algebra, it is necessary to account for the knowledge of the competent algebraist which allows avoidance of expressions such as $\frac{x}{0}$, $\sqrt{-1}$,‡ etc. In the present model, this knowledge is classed as arithmetic knowledge. Arithmetic knowledge involving numerical combination is presumed to be accounted for by a grammar of arithmetic which is not the concern of the present research.

There is some justification for this classification aside from the convenience to the investigator. The object of the inquiry is the skill which underlies the competent manipulation of algebraic symbols. Certainly there are other related skills which are routinely required in any sensible

†Descriptive and explanatory goals for the present theory differ from usual natural-language goals. See Paper #2.
‡At least in the algebra of real numbers.

application of algebra. These might include the ability to interpret 'real world' situations in algebraic notation; the ability to evaluate the results of a manipulative procedure in terms of external goals and criteria, etc. A critical assumption in generative linguistic analysis, however, is that it is sensible to study the syntax of language as an independent object in its own right. In algebra, for instance, the manipulative rule that $\sqrt{ab} = \sqrt{a}\sqrt{b}$ is taught, whereas in arithmetic one teaches (or at least should teach) that this is not so.† The present study, therefore, is an investigation of algebraic manipulative skill, undertaken with the full knowledge that more than the skill of manipulating symbols underlies useful algebraic application.

3. Linguistic Terminology

Many of the terms employed in this algebraic grammar have been borrowed from natural language linguistics. This should not be construed as an indication that the present grammar is structured similarly to standard natural language grammars. On the contrary, the organization of the algebraic grammar is quite different from standard natural language model. For example, transformations perform an entirely different function in the present theory than they do in natural-language theories.

†E.g. $\sqrt{-1 \cdot -1} = \sqrt{1} = 1$, but $\sqrt{-1} \cdot \sqrt{-1} = (\sqrt{-1})^2 = -1 \ (\neq 1)$.

Part II: THE GRAMMAR OF ALGEBRA

D. PHRASE STRUCTURE GRAMMAR

The first task in the algebraic grammar is the generation of the deep form for every possible algebraic expression. The deep form of an expression explicitly displays the operations and parse which may be only tacit in surface form. A system of capital letters and parentheses is used to accomplish this. A, S, M, D, E, and R abbreviate the six binary operations. "N" represents the unary operation, negation. The parse of an expression is indicated by the use of parentheses. Thus, for example, the expression $\frac{\sqrt{-x^2 + 3} - 5}{4y}$ is represented in deep form as [[[2R[[N[xE2]]A3]]S5]D[4My]].†

A phrase structure grammar is employed to generate all such possible deep forms. A phrase structure grammar is a series of rewrite rules which allows an initial symbol to be replaced by other symbols, and those to be replaced by still others etc., until only members of a special set of terminal symbols remain. In linguistics, the traditional initial symbol is "S", however a "Z" is used here since the "S" is reserved as a terminal symbol. A production a —> b indicates that the symbol string represented by "a" may be replaced by that represented by "b".

†The use of "[" and "]" as symbols in both deep and surface form introduces a measure of ambiguity into the notation. The terms *deep level parentheses* and *surface level parentheses* will be used in circumstances where misinterpretation seems possible.

The productions of the grammar are

Z —> [ZOZ]

Z —> [NZ]

Z —> L

Z —> Q

O —> A, S, M, D, E, R

L —> a, b, c, ... x, y, z

Q —> ., 0, 1, 2, 3, 4, 5, 6, 7, 8, 9†

The symbol "O" may be interpreted as (binary) "operation." "N" can be interpreted as the unary operation "negation." "L" and "Q" may be interpreted as "literal" and "quantity" respectively. "Q" can be replaced by any rational number constructed from the symbols ., 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. A deep form derivation is completed when all occurences of Z, O, L, and Q have been replaced by terminal symbols. The terminal symbols are N, A, S, M, D, E, R, ., 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, a, b, c, ... x, y, z, [,].

1. Example of Phrase Structure Derivation

The expression $\frac{\sqrt{-x^2 + 3} - 5}{4y}$ is a well formed algebraic expression. It's deep form, [[[2R[[N[xE2]]A3]]S5]D[4My]], is derived from the phrase structure grammar as follows:

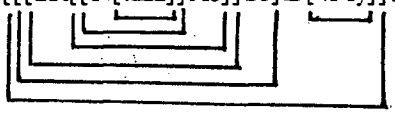
Z —> [ZOZ] —> [ZDZ] —> [ZD[ZOZ]] —> [ZD[ZMZ]] —> [ZD[QMZ]] —> [ZD[4MZ]] —> [ZD[4ML]] —> [ZD[4My]] —> [[ZOZ]D[4My]] —> [[ZSZ]D[4My]] —> [[ZSQ]D[4My]] —> [[ZS5]D[4My]] —> [[[ZOZ]S5]D[4My]] —> [[[ZRZ]S5]D[4My]] —> [[[QRZ]S5]D[4My]] —> [[[2RZ]S5]D[4My]] —>

†Here again the formality of the grammar is compromised. The intended replacements for "Q" are decimal numbers constructed from the given symbols according to a presumed "grammar of arithmetic." That grammar is not supplied here.

[[[2R[ZOZ]]S5]D[4My]] —> [[[2R[ZAZ]]S5]D[4My]] —> [[[2R[ZAQ]]S5]D[4My]] —> [[[2R[ZA3]]S5]D[4My]] —> [[[2R[[NZ]A3]]S5]D[4My]] —> [[[2R[[N[ZOZ]]A3]]S5]D[4My]] —> [[[2R[[N[ZEZ]]A3]]S5]D[4My]] —> [[[2R[[N[LEZ]]A3]]S5]D[4My]] —> [[[2R[[N[xEZ]]A3]]S5]D[4My]] —> [[[2R[[N[xEQ]]A3]]S5]D[4My]] —> [[[2R[[N[xE2]]A3]]S5]D[4My]]

The "[" and "]" symbols occur in matched pairs as indicated in

[[[2R[[N[xE2]]A3]]S5]D[4My]].



2. Syntactic Structure

As can be seen from the phrase structure rules, parentheses are generated exclusively from the initial symbol, "Z", and appear at the boundaries of the derived symbol string. It follows, then, that each parenthesis pair, together with its contents is a well formed expression in its own right (since it has replaced the initial symbol). This expression is called a *subexpression* of the original.

Each parenthesis pair is introduced into DF in company with a single operation, unary or binary. This operation can be identified as the only operation of the subexpression which is not further embedded within parentheses. It is called the *dominant* operation and is said to *dominate* any other operations of the subexpression. Alternatively, the embedded operation is said to be *precedent* to the dominating one.

As an example, consider the subexpression [xE2] of the DF [[[2R[[N[xE2]]A3]]S5]D[4My]]. It contains just a single operation, E, which is therefore its dominant operation. The subexpression [N[xE2]] has two operations, N and E. N is contained in fewer parentheses and

therefore dominates E. The expression [[N[xE2]]A3] contains three operations, N, E and A. In this case, A is the dominant operation since it is contained within just a single parenthesis pair.

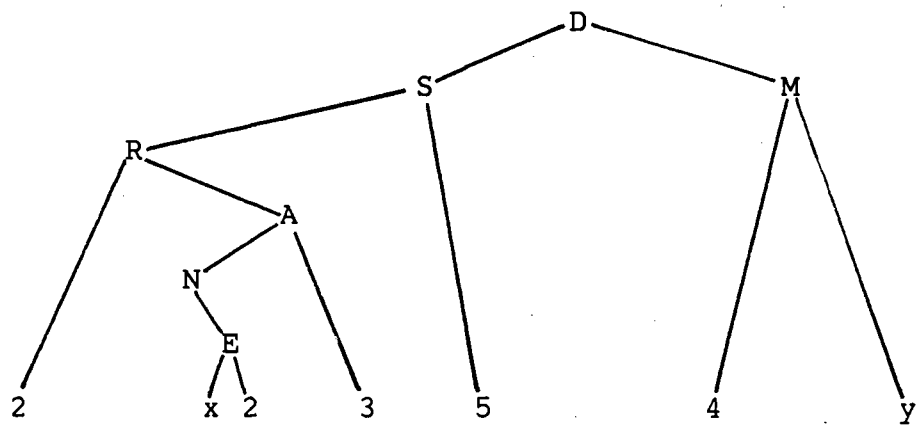
The pattern of nesting of subexpressions determines a hierarchy of dominance. The dominant operation of an expression is said to *directly dominate* the operation(s), if any, which are contained within just a single additional set of parentheses. Since each subexpression is itself an expression, a fixed hierarchy of direct domination is established recursively. In the above example, A directly dominates N which in turn directly dominates E. The dominance hierarchy is called the *syntactic structure* of an expression.

Syntactic structure can be visualized as a tree diagram (Carry, Lewis & Bernard, 1980; Matz, 1980) by placing the dominant operation at the top "node" of the tree and linking it by "branches" to its directly dominated operation(s) —continuing downwards from there in similar fashion. Figure 3 displays the tree diagram for the DF [[[2R[[N[xE2]]A3]]S5]D[4My]]. Tree diagrams and the linear parenthesized notations will be used interchangeably. Lower case Greek letters near the beginning of the alphabet, (e.g. α, β, γ, δ, ϵ, ζ, η, θ) will be used to refer to expressions, or to subexpressions within the dominance hierarchy. Numbers will be represented as lower case Greek letters near the end of the alphabet (e.g. ρ, σ, τ, υ, ϕ, χ, ψ, ω).

3. Obligatory Transformation

The claim for the proposed phrase structure component is that any derivation employing the productions and ending with only terminal symbols results in a valid deep form for an algebraic expression. Furthermore, the deep form for every possible algebraic expression is so

Figure 3
Tree Diagram



derivable. Since expressions such as $x^2 3$ are not produced by algebraists, it is necessary to constrain the grammar accordingly. An obligatory transformation is applied to the output of the phrase structure grammar to commute subexpressions for stylistic purposes:

$$\alpha\rho \longrightarrow \rho\alpha$$

E. TRANSLATIONS OF DEEP FORM TO SURFACE FORM

The next component of the grammar translates deep forms into standard surface form notation. As noted above, bidirectionality of the translation component is vital to the psychological interpretability of the grammar. Consequently the present translation component has been designed for the translation of SF ——> DF as well as DF ——> SF.

The system of translations is divided into four stages, each of which must be completed before the next is begun. The first stage involves the deletion of parentheses made redundant by a conventional hierarchy of operations. For example, in Stage 1, the deep form, [[[2R[[N[xE2]]A3]]S5]D[4My]] of $\frac{\sqrt{-x^2 + 3} - 5}{4y}$ is reduced to [[2R[NxE2A3]S5]D[4MY]].

The next stage involves the translation of operations to surface form. For example, in this Stage [[2R[NxE2A3]S5]D[4MY]] becomes $[\frac{[\sqrt[2]{[-x^2 + 3]} - 5]}{[4y]}]$. Stage 3 effects the removal of parentheses made redundant by physical artifacts of the representation of operations: $[\frac{[\sqrt[2]{[-x^2 + 3]} - 5]}{[4y]}]$ becomes $[\frac{\sqrt[2]{-x^2 + 3} - 5}{4y}]$. Finally, Stage 4 performs adjustments to SF such as the deletion of "2" in square root signs.

The Phrase Structure Component of the grammar is presented as a standard context free phrase structure grammar (see, for example, Gross & Lentin, 1970). Stages 2, 3, and 4 of the Translation Component also consist of standard rewrite rules, however, Stage 1 is more conveniently presented in natural language. This may influence the difficulty of machine implementation of the grammar, however, the present purposes require only sufficient clarity and precision to determine, unambiguously, the class of sentences producible through the grammar. This condition does not preclude natural-language formulations.

1. Stage 1 Translation

The first stage effects the removal of parentheses which occur in deep form but are rendered redundant by grouping conventions. Redundancy of parentheses is based on a hierarchy of operations. It is useful to use a classification of operations introduced by Schwartzman (1977).

Operation Hierarchy

Level 1 operations are "A" (addition) and "S" (subtraction)
Level 2 operations are "M" (multiplication) and "D" (division)
Level 3 operations are "E" (exponentiation) and "R" (radical)

In this classification, Level 3 operations are said to be *higher* than Level 2 operations which in turn are *higher* than Level 1 operations. The two operations contained within each level

are said to be *inverse* to each other. Notationally, if O is an operation, then O' represents its inverse, and $|O|$, its level.

The process of parenthesis deletion entails repeated passes over the expression starting at the most dominant operation and proceding down the dominance chain. At each step, tests are required to determine whether the parentheses at that level should be deleted or retained. These tests are provided for in the following rule with the inclusion of negation (N) as a Level 2 operation.†

1. Syntactic Rule

Parentheses are deleted from around a subexpression if

(a) the dominant operation of the subexpression is directly dominated by a lower level operation, or

(b) the dominant operation of the subexpression is directly dominated by an operation of equal level, provided that the latter is to the right of the former.

The dominant operation of an expression effectively partitions the expression into two subexpressions both of which it dominates. One of these needs to be chosen for immediate

†Negation functions as a Level 2 operations in regards to conventions of grouping. It has precedence over Level 1 operations [$-x + 2$ is interpreted as $(-x) + 2$], but Level 3 operations have precedence over it [$-x^2$ is interpreted as $-(x^2)$]. The ambiguity in relation to Level 2 operations [$(-x)y$ is equivalent to $-(xy)$] is similar to associativity of multiplication.

A consequence of defining negation as a Level 2 operation for the purposes of the Syntactic Rule is that $-(xy)$ must be transformed to $(-x)y$ before parentheses deletion. This is an assertion for which there is no independent corroboration.

Table 3
Stage 1 - Parenthesis Deletion

Expression	Dominant Operation	Dominating Operation	Action
[[N[xE[yD2]]]S[xA[[3Mx]My]]]	S	none	
[{N[xE[yD2]]}S[xA[[3Mx]My]]]	N	S	delete
[N[xE[yD2]] S{xA[[3Mx]My]}]	A	S	retain
[N{xE[yD2]} S[xA[[3Mx]My]]]	E	N	delete
[N xE[yD2] S[xA{[3Mx]My}]]	M	A	delete
[N xE{yD2} S[xA[[3Mx]My]]]	D	E	retain
[N xE[yD2] S[xA {3Mx}My]]	M	M	delete
[N xE[yD2] S[xA 3Mx My]]			

Result: [NxE[yD2]S[xA3MxMy]]

Curly braces are under consideration.

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analysis, and the other stored for later analysis. A more detailed implementation would have to specify selection and recall rules.

As an example of this stage, consider the deep form [[N[xE[yD2]]]S[xA[[3Mx]My]]] of the expression $-x^{y/2} - (x + 3xy)$. It is necessary to reduce the deep form to [NxE[yD2]S[xA3MxMy]]. (The parentheses around the whole expression and around yD2 will be deleted in subsequent stages of translation.) Parenthesis deletion progresses as in Table 3. The left hand column, "Expression", shows the progressive change to DF as parentheses are deleted or retained at each step. The next column names the dominant operation of the subexpression under consideration. The operation which directly dominates the subexpression under consideration is named in the next column. Comparison of the dominant operation and dominating operation according to the Syntactic Rule results in the decision to retain or delete the parentheses surrounding the subexpression. For instance, line 3 shows that the subexpression contained in curly braces has addition as the dominant operation, and it is

directly dominated by subtraction. Since addition and subtraction are of equal level, and the dominant operation is to the left of its directly dominating operation, the Syntactic Rule mandates the retention of the parentheses.

2. Stage 2 Translations

The second stage involves the translations of operator symbols (unary and binary) into their surface form counterparts. These are either explicit symbols such as " + " for A, or positional features such as concatenation for multiplication:

Unary and Binary Operations

2. $[\mathrm{N}\alpha \longrightarrow [-\alpha$

3. $\alpha\mathrm{A}\beta \longrightarrow \alpha + \beta$

4. $\alpha\mathrm{S}\beta \longrightarrow \alpha - \beta$

5. $\alpha\mathrm{M}\beta \longrightarrow \alpha\beta$

6. $\alpha\mathrm{D}\beta \longrightarrow \frac{\alpha}{\beta}$

7. $\alpha\mathrm{E}\beta \longrightarrow \alpha^{\beta}$

8. $\alpha\mathrm{R}\beta \longrightarrow \sqrt[\alpha]{\beta}$

Note that the parentheses in the production [Na —> [-a are not strictly necessary for translation to SF from DF. They are required to ensure bidirectionality of translation. The ambiguity of the "-" symbol (subtraction and negation) necessitates the specifying of a syntactic context for translation of surface forms.

3. Stage 3 Translation

The third stage of translation involves the further deletion of parentheses made redundant by spatial markers in surface notation.

9. $\frac{[a]}{\beta}$ —> $\frac{a}{\beta}$

10. $\frac{a}{[\beta]}$ —> $\frac{a}{\beta}$

11. $a^{[\beta]}$ —> a^{β}

12. ${}^{a}\sqrt{[\beta]}$ —> ${}^{a}\sqrt{\beta}$

13. ${}^{[a]}\sqrt{\beta}$ —> ${}^{a}\sqrt{\beta}$

14. [ɨ] —> ɨ †

For example, these rules would be used for parenthesis deletion in "$x^{[-2]}$".

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† "ɨ" is used here to represents either a variable or a quantity symbol (decimal number). "[ɨ]" cannot be produced directly from the phrase structure grammar, but it may occur as a result of transformation of one deep form into another —for example when [3A2]Mx is transformed into [5]Mx.

4. Stage 4 Translation

The fourth stage concludes some minor adjustments to surface form such as the deletion of "^{2}" in "$\sqrt[2]{}$" and the variation of parentheses types (eg. "(" and "{" as well as "[").

15. $\#[a]\# \longrightarrow a$ †

16. $\sqrt[2]{} \longrightarrow \sqrt{}$

17. $[\frac{a}{\beta}] \longrightarrow \left(\frac{a}{\beta}\right)$

18. $[a] \longrightarrow (a)$

19. $[a] \longrightarrow \{a\}$

There must be rules which govern the actual selection of parentheses type; for example * $(x + [y - z]^3)^2$ would more normally be written $[x + (y - z)^3]^2$.‡ These rules have not been detailed in the grammar.

5. Reverse Translation

The process of translation from DF ——> SF can be reversed to SF ——> DF by the simple expedient of reversing the order in which the stages are traversed and by reversing the direction of productions within each stage. For example Rule #6,

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†The octothorpe is used here, as in linguistics, to mark the boundary of an expression. The phrase structure grammar produces each DF containing an operation with surrounding parentheses. This rule is needed for the deletion of parentheses which surround an entire expression.

‡As in linguistics, the "*" is used to mark an ungrammatical sentence.

$\alpha D\beta \longrightarrow \frac{\alpha}{\beta}$, becomes $\frac{\alpha}{\beta} \longrightarrow \alpha D\beta$. For Stage 1 translation, however, the Syntactic Rule must be rephrased by substituting "inserted" for "deleted from."

Reversed Syntactic Rule

Parentheses are *inserted* around a subexpression if

(a) the dominant operation of the subexpression is directly dominated by a lower level operation, or

(b) the dominant operation of the subexpression is directly dominated by an operation of equal level, provided that the latter is to the right of the former.

In the case of DF ——> SF translation, the Syntactic Rule is applied to an expression whose syntactic structure is entirely determined by parentheses. In this case, the dominant operation is simply the one contained within just a single set of parentheses. In the reversal, however, a new rule must be formulated to find the dominant operation when parentheses are not fully descriptive of syntactic structure:

Dominant Operation Rule

The dominant operation is:

the operation embedded within the fewest number of parenthesis pairs, or in case of a tie:

the lowest level of the competing operations, or in case of a tie:

the right-hand most operation.

F. TRANSFORMATIONS

In the first section, rules which generate the deep forms of algebraic expressions were presented. The second section showed a method for translating deep forms to surface forms. This final section of the grammar focuses on transforming the deep form of expressions into deep forms of any possible compatible expressions. The rules of this section generally correspond to properties of real numbers.

Previous treatments of algebra (e.g. Carry, Lewis & Bernard, 1980; Matz, 1980) have alluded to a set of operators or transformations, and made reference, by example, to one or two specific cases. There has been no previous attempts in mathematics education research to compile the comprehensive list of transformational rules to which fluent algebraists have access. (Various lists are incorporated within computer algebra programs; however, these have generally not been proposed as models of algebraists' knowledge.) As a result, the following list represents only a tentative first step in what will be a long and difficult analysis. Some of the problems and considerations in compiling and classifying members of such a list are presented prior to the list itself.

As discussed in the introduction, there is no convenient minimal set of independent rules from which the larger operational set can be generated by combination. Thus an immediate problem in the designation of transformational rules is the wide redundancy among potential candidates. For example, $(a^{\beta}\gamma^{\delta}\ldots\epsilon^{\zeta})^{\eta} = a^{\beta\eta}\gamma^{\delta\eta}\ldots\epsilon^{\zeta\eta}$ could be classed as a transformation in its own right, or it could be interpreted as a result of application of transformations $(a\beta\ldots\gamma)^{\delta} = a^{\delta}\beta^{\delta}\ldots\gamma^{\delta}$ and $(a^{\beta})^{\gamma} = a^{\beta\gamma}$ viz $(a^{\beta}\gamma^{\delta}\ldots\epsilon^{\zeta})^{\eta} \longrightarrow (a^{\beta})^{\eta}(\gamma^{\delta})^{\eta}\ldots(\epsilon^{\zeta})^{\eta} \longrightarrow a^{\beta\eta}\gamma^{\delta\eta}\ldots\epsilon^{\zeta\eta}$. The result is that there are many possible versions of the transformational component with equal descriptive adequacy. The problem, then,

is to select the set of transformations which most closely matches the properties of real numbers to which the competent algebraist has access in manipulating algebraic symbols.

As an added complication, there is no guarantee that this set is the same for all algebraists. It is possible that dependent, say, on educational experiences, one algebraist may derive $x^2 + 2xy + y^2$ directly from $(x + y)^2$ whereas another may transform $(x + y)^2 \longrightarrow (x + y)(x + y) \longrightarrow x^2 + 2xy + y^2$. The present list reflects the author's bias as to which transformations may be elementary for a signficant number of algebraists. Reasoned argument for alternative selections is anticipated as an important component of psychoalgebraic† research.

A variety of factors condition the utility of potential transformational rules. An obvious prerequisite is that the rule must state a correct relationship among real numbers (or some subset thereof). This, however, is clearly not a sufficient condition, since there are an infinite number of such statements possible. The rule must also fill some need which originates in the tasks for which transformations are used —i.e. in the pragmatic component of the grammar. For example, the rule $(1/a)^{1/\beta} = a^{-\beta^{-1}}$ might be of great utility if only there were more demand for transformations involving reciprocal bases to reciprocal powers. For this reason, it can be anticipated that the transformational component will undergo considerable ammendment during the process of elaboration of the pragmatic component.

A second factor conditioning utility is the "notation boundedness" of real number properties. Utility is not just a function of the underlying DF, but also of surface characteristics of

†The term "psychoalgebra" is used to refer to the evaluation of the psychological claims of an algebraic grammar.

notational representations. For example, there would be no reason to suppose a transformational rule $(\alpha - \beta) + (\gamma - \delta) \longrightarrow (\alpha + \gamma) - (\beta + \delta)$, but the rule $\frac{\alpha}{\beta} \cdot \frac{\gamma}{\delta} = \frac{\alpha\gamma}{\beta\delta}$ is a standard feature of instruction in algebra. Note that the underlying structure of these two rules is identical except for the level of the operations. The greater utility of the latter rule appears to reside in surface differences between Level 1 and Level 2 representation. Kirshner (1985) suggests that natural language grammar may provide a key to understanding the salience of notational features. Further investigation of this notion of notational utility needs to be undertaken.

a. Algebra and Arithmetic

Transformations can be thought of within two distinct categories: arithmetic transformations (*computations*) and algebraic transformations (*transformations proper*). Their demarcation in this grammar is not standard. Usually, the algebra designation centres around the use of variables whereas the arithmetic designation is for numbers. For example, Davis (1985) reports on the group discussion "Algebraic Thinking in the Early Grades" at ICME-5 (The Fifth International Congress on Mathematics Education):

> Early in the discussions the working definition was this: It's *algebra* if it includes: work with *variables* ..., *notation for variables*, work with *functions*, some *metacognitive modeling* of functions, and *negative numbers*. (p. 198)

In the present study the criterion used is whether the syntactic structure of the expression is altered (Transformations Proper) or merely reduced (Computations). For example the sentence $\sqrt[3]{2} + \sqrt[3]{16} = \sqrt[3]{2} + 2 \cdot \sqrt[3]{2} = 3 \cdot \sqrt[3]{2}$ is classified as an algebraic sentence, whereas the sentences $3x^2 + 1 = 3 \cdot 4 + 1 = 12 + 1 = 13$ for $x=2$, and $(x + y^2)^0 + \frac{x^{2-1}}{x} = 1 + 1 = 2$ are arithmetic.

The computational (or arithmetic) part of the transformational component can itself be divided into three sections. The first section concerns constraints on combinations: $\frac{5}{0}$, $\sqrt{-5}$, etc. The second section involves numerical combinations $2 + 3 = 5$, etc. The numerical component cannot consist of just a list of substitutions since such a list would have to be infinite in length. A grammar of arithmetic, not delineated in the present theory, is presumed to account for numerical substitutions. The third computational component considers generalizations of numerical computations: $x + 0 = x$, etc. These rules are arithmetic in the sense that they reduce the syntactic tree rather than alter it, structurally.

b. Notation

Recall that subexpressions are indicated by lower case Greek letters α to θ, that quantity symbols are indicated by lower case Greek letters ρ to ω, and that the level of the operation O is denoted $|O|$, and its inverse, O'. The symbol "<——>" is used for transformations to indicate that the expressions which flank it may be substituted one for the other. Thus "<——>" specifies two rules (e.g. $\alpha - \beta$ <——> $\alpha + (-\beta)$ refers to $\alpha - \beta$ ——> $\alpha + (-\beta)$ and $\alpha + (-\beta)$ ——> $\alpha - \beta$). These will be refered to as "Transformation a" and "Transformation b", respectively. Occasionally, the symbol "——>" will be used for transformations that are unidirectional. Several transformations are marked with the "&" symbol. As discussed on page 63, the "b" form of these properties cannot directly precede translation from deep to surface form. When "&&" is used, then both the "a" and "b" form of the rule are so marked.

Transformations mediate between deep forms of expressions. In this section, however, transformations are presented in a mixed deep and surface notation. This has been done in order to ameliorate readability. This advantage is partially offset by the danger of missing

relationships in deep form between apparently unrelated transformations. An attempt has been made to group structurally similar transformations under informal headings. A catalogue of transformations in DF is supplied in Appendix A. Of course, the correct organization of these properties is ultimately of concern in the explanatory adequacy of the grammar.

1. Arithmetic Component

a. Arithmetic Constraints

1. $\frac{a}{\beta}$ is disallowed whenever $\beta = 0$

2. $\sqrt[a]{\beta}$ is disallowed whenever a is not a natural number

3. $\sqrt[a]{\beta}$ is disallowed whenever $\beta < 0$, and a is even †

b. Arithmetic Computations

4.& $\rho O \sigma \longleftrightarrow \tau$ ‡

c. Computational Generalizations

5.& $a + 0 \longleftrightarrow a$

† *Arithmetic* terms like "natural number", "<" and "even" are not defined in the grammar.
‡Recall that these lower case letters near the end of the Greek alphabet represent quantity symbols (numbers).

6.& $a - 0 \longleftrightarrow a$

7.& $1a \longleftrightarrow a$

8.& $-1a \longleftrightarrow -a$

9.& $\frac{a}{1} \longleftrightarrow a$

10.& $a^1 \longleftrightarrow a$

11.& $a^0 \longleftrightarrow 1$

12.& $a - a \longleftrightarrow 0$

13.& $\frac{a}{a} \longleftrightarrow 1$

2. Transformations Proper

Rerepresentation Rules

14. $a - \beta \longleftrightarrow a + (-\beta)$

15. $a^{\frac{\beta}{\gamma}} \longleftrightarrow \sqrt[\gamma]{a^{\beta}}$

16. $a^{-\delta} \longleftrightarrow 1/a^{\delta}$

17.& $\frac{a}{\beta} \longleftrightarrow a \cdot \frac{1}{\beta}$

18. $\frac{a}{\beta} \cdot \frac{\gamma}{\delta} \longleftrightarrow \frac{a\gamma}{\beta\delta}$

19. $\frac{a/\beta}{\gamma/\delta} \longleftrightarrow \frac{a}{\beta} \cdot \frac{\delta}{\gamma}$

Commutativity Rules

20.& $a + \beta \longleftrightarrow \beta + a$

21.& $a - \beta \longleftrightarrow -(\beta - a)$

22.& $a\beta \longleftrightarrow \beta a$

Associativity Rules

23.&& $(a + \beta)\ \mathrm{O}\ \gamma \longleftrightarrow a + (\beta\ \mathrm{O}\ \gamma)$ where $|\mathrm{O}| = 1$ †

24.&& $(a\beta)\gamma \longleftrightarrow a(\beta\gamma)$

25.& $\frac{a\beta}{\gamma} \longleftrightarrow a \cdot \frac{\beta}{\gamma}$

26. $\sqrt[a]{\beta^{\gamma}} \longleftrightarrow (\sqrt[a]{\beta})^{\gamma}$

27.&& $(-a)\beta \longleftrightarrow -(a\beta)$

28.& $\frac{-a}{\beta} \longleftrightarrow -\frac{a}{\beta}$

29.& $\frac{a}{-\beta} \longleftrightarrow -\frac{a}{\beta}$

†Recall that "|O|" refers to the level of operation "O".

30. $(a^{\beta})^{\gamma} \longleftrightarrow a^{\beta\gamma}$ †

Distributivity Rules

31. $a(\beta\ O_1\ \gamma\ O_2 \ldots O_n\ \delta) \longleftrightarrow a\beta\ O_1\ a\gamma\ O_2 \ldots O_n\ a\delta$
where $|O_i| = 1$ ‡

32. $\chi a\ O_1\ \psi a\ O_2 \ldots O_n\ \omega a \longleftrightarrow (\chi\ O_1\ \psi\ O_2 \ldots O_n\ \omega)a$
where $|O_i| = 1$

33. $-(a\ O_1\ \beta\ O_2 \ldots O_n\ \delta) \longleftrightarrow -a\ O_1'\ \beta\ O_2' \ldots O_n'\ \delta$
where $|O_i| = 1$ *

34. $a - (\beta\ O_1\ \gamma\ O_2 \ldots O_n\ \epsilon) \longleftrightarrow a - \beta\ O_1'\ \gamma\ O_2' \ldots O_n'\ \epsilon$
where $|O_i| = 1$

35. $(a\beta\ldots\epsilon)^{\delta} \longleftrightarrow a^{\delta}\beta^{\delta}\ldots\epsilon^{\delta}$

36. $\sqrt[\delta]{a\beta\ldots\epsilon} \longleftrightarrow \sqrt[\delta]{a}\sqrt[\delta]{\beta} \ldots \sqrt[\delta]{\epsilon}$

37. $\sqrt[\delta]{a/\beta} \longleftrightarrow \sqrt[\delta]{a}/\sqrt[\delta]{\beta}$

38. $(a/\beta)^{\delta} \longleftrightarrow a^{\delta}/\beta^{\delta}$

---

†Thanks to Professor Hoechsmann at the University of British Columbia for pointing out the associative character of this transformation.

‡The three dots within this transformation (and others) are an informal indicator of distribution over an indeterminate number of operations. More formally correct alternatives would be to define the transformation as $a(\beta\ O\ \gamma) \longleftrightarrow a\beta\ O\ a\gamma$ and to extend to multiple operations by repeated application: $a(\beta\ O_1\ \gamma\ O_2\ \delta) \longrightarrow a[(\beta\ O_1\ \gamma)\ O_2\ \delta] \longrightarrow a(\beta\ O_1\ \gamma)\ O_2\ a\delta \longrightarrow a\beta\ O_1\ a\gamma\ O_2\ a\delta$ (where $|O_i| = 1$), or else to initiate some sort of formal recursive device to achieve the required generality. The discussion of *non-binary operations* (the final section of this paper) expresses skepticism about the explanatory adequacy of the first alternative. If the second has any explanatory validity, it still remains to determine the nature of the correct formal device (as a psychological theory).

*The symbol "O'" refers to the inverse operation of "O"

39. $\dfrac{\alpha\ O_1\ \beta\ O_2\ \ldots\ O_n\ \gamma}{\text{where } |O|^{\delta} = 1} \longleftrightarrow \dfrac{\alpha}{\delta}\ O_1\ \dfrac{\beta}{\delta}\ O_2\ \ldots\ O_n\ \dfrac{\gamma}{\delta}$

40. $\alpha^{\delta}\alpha^{\epsilon}\ldots\alpha^{\gamma} \longleftrightarrow \alpha^{\delta+\epsilon+\ldots+\gamma}$

41. $\alpha^{\delta}/\alpha^{\epsilon} \longleftrightarrow \alpha^{\delta-\epsilon}$

Cancellation Rules

42. $\dfrac{\alpha\beta}{\alpha\gamma} \longleftrightarrow \dfrac{\beta}{\gamma}$

43. $\dfrac{\alpha/\beta}{\gamma/\beta} \longleftrightarrow \dfrac{\alpha}{\gamma}$

44.& $(-\alpha)(-\beta) \longleftrightarrow \alpha\beta$

45.& $\dfrac{-\alpha}{-\beta} \longleftrightarrow \dfrac{\alpha}{\beta}$

46.& $-(-\alpha) \longleftrightarrow \alpha$

47.& $(\sqrt[\alpha]{\beta})^{\alpha} \longleftrightarrow \beta$

Polynomial Rules

48. $(\alpha + \beta + \ldots + \delta)(\epsilon + \zeta + \ldots + \theta) \longrightarrow$
$\alpha\epsilon + \alpha\zeta + \ldots + \alpha\theta + \beta\epsilon + \beta\zeta + \ldots + \beta\theta + \ldots.$
$+ \delta\epsilon + \delta\zeta + \ldots + \delta\theta$ †

†Note that this transformation is stated in terms of addition. In order to apply it to polynomials consisting of sums and differences of terms, there seems to be no way to avoid defining subtraction as the sum of a negation ($\alpha - \beta = \alpha + (-\beta)$) and then applying a series of transformations to return to subtractions in the derived expression.

49. $\chi a^2 \ \mathrm{O} \ \psi a\beta + \omega\beta^2 \longrightarrow (\rho a \ \mathrm{O} \ \sigma\beta)(\tau a \ \mathrm{O} \ \upsilon\beta)$
where $\rho\tau = \chi$, $\sigma\upsilon = \omega$, $\rho\upsilon + \sigma\tau = \psi$, and $|\mathrm{O}| = 1$

50. $\chi a^2 \ \mathrm{O} \ \psi a\beta - \omega\beta^2 \longrightarrow (\rho a \ \mathrm{O}_1 \ \sigma\beta)(\tau a \ \mathrm{O}_1' \ \upsilon\beta)$
where $\rho\tau = \chi$, $\sigma\upsilon = \omega$, $|\rho\upsilon - \sigma\tau| = \psi$, $|\mathrm{O}| = |\mathrm{O}_1| = 1$, and
O_1 is addition if $\rho\upsilon > \sigma\tau$; subtraction if $\rho\upsilon < \sigma\tau$ †

51. $(a \ \mathrm{O} \ \beta)^2 \longleftrightarrow a^2 \ \mathrm{O} \ 2a\beta + \beta^2$ where $|\mathrm{O}| = 1$

52. $a^2 - \beta^2 \longleftrightarrow (a + \beta)(a - \beta)$

53. $a^3 \ \mathrm{O} \ \beta^3 \longleftrightarrow (a \ \mathrm{O} \ \beta)(a^2 \ \mathrm{O}' \ a\beta + \beta^2)$ where $|\mathrm{O}| = 1$

†Trinomial factoring seems naturally to break down into two cases.

Part III

G. ALTERNATIVE COMPONENTS

An obvious advantage of a rigorous and formal model is that it requires specification of all of its components in exacting detail. A more subtle advantage is that, if tightly woven, the seemingly autonomous components of the model can be seen to be mutually interdepedent.

The grammar, as presented above, is in many respects tentative and immature. It is, however, tightly enough drawn so that a suggested alteration in one component can be traced through to its implications in other components. In this way a more adequate grammar can be developed as an integrated and powerful explanatory theory.

The following section sketches out various proposed alternatives for the grammar. The descriptive adequacy is first established and then psychological issues are taken up. In some cases, psycholinguistic methods have been used to evaluate the psychological validity of various proposals. Two such studies are reported in the subsequent papers comprising this series. Many of the questions reported here, as well as many other questions, remain in need of further explication and further research.

H. PRECEDENCE VS DOMINANCE

Recall that the first stage of translation from DF to SF involves making repeated passes over the expression until parentheses (made redundant by the hierarchy of operations) are deleted. As shown in Table 3 (page 73) operations are tested starting from the most dominant to least dominant. (The dominant operation partitions the expression into two subexpressions, and, recursively, each of these becomes "the expression" for subsequent application of the Syntactic

Rule.)

Alternative to a dominance-to-precedence analysis, it is possible to ascend the syntactic tree from most precedent to most dominant operations. The most precedent operations† are simply those contained in the greatest number of parenthesis pairs. Each of these operations is directly dominated by the next most precedent operation, identifiable because it is adjacent to the subexpression containing the original operation and contained in just one fewer set of parenthesis pairs. The Syntactic Rule (page 73) can then be applied to decide the deletion or retention of parentheses for that subexpression. A recursive procedure is established by the continued identification of the next most precedent operation, and subsequent reapplication of the Syntactic Rule. (A more complete implementation of the grammar would have to specify rules for selecting a most precedent operation for immediate analysis and storing other most precedent operations for later recall.)

As with dominance-to-precedence, the situation is slightly complicated in SF ——> DF translation by the fact that syntactic structure is not fully determined by parentheses. In this circumstance an additional rule is needed to determine the most precedent operations when parentheses are absent:

†Unlike "most dominant," "most precedent" is not a unique designation (e.g. [[xE2]S[3My]]).

Operator Precedence Rule

The most precedent operation in an expression is:

the operation embedded within the greatest number of parenthesis pairs, or in case of a tie:

the highest level of the competing operations, or in case of a tie:

the left-hand most operation.

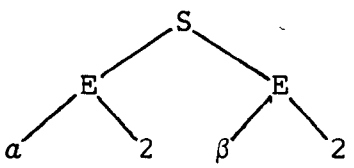
Having described two observationally adequate alternatives for expression parsing, it is sensible to ask whether the syntactic decisions of algebraists are based upon operator precedence or operator dominance. Operator precedence seems to be more closely allied to standard curricular strategies of syntactic instruction than is operator dominance. Typically, students are taught that multiplication is "done before" (i.e. precedent to) addition, etc. Common pedagogical devices such as "BOMDAS", an acrynom for Brackets, Of, Multiply, Divide, Add, Subtract illustrate this. The study of algebraic syntax is generally restricted to evaluating expressions (eg. evaluate $3x^2 - 4x - 1$ when $x = 2$) where, indeed, precedence of operations does determine the order of activity.

Aside from expression evaluation tasks (which, technically speaking, are classified as arithmetic in the present theory rather than algebraic [page 81]), syntactic skill is also required for application of transformations proper. In order to apply a transformation to an expression it is necessary to compare the syntactic structure of the expression to the structure of the anticedent expression of the transformational rule. For example, the transformation

Figure 4
Transformational Application

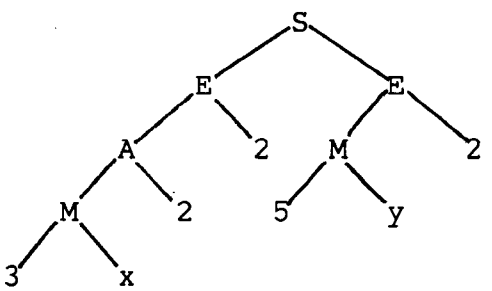
Anticedent Expression for Transformation

$\alpha^2 - \beta^2$



Expression

$(3x + 2)^2 - (5y)^2$



$\alpha^2 - \beta^2 \longrightarrow (\alpha - \beta)(\alpha + \beta)$ can be applied to the expression $(3x + 2)^2 - (5y)^2$ because the dominant operation matches the dominant operation of the anticedent expression of the transformation, the second most dominant operations match, etc. (see Figure 4).

These considerations circumstantially favour the explanatory adequacy of a dominance procedure. A dominance analysis provides the information relevent to the application of transformations in the order in which it is to be used. In fact for complex expressions (e.g. $(3x^2 - \sqrt{4 + x^4})^2$) it is not necessary to actually complete a dominance based syntactic analysis to determine that the "difference of squares" transformation does not apply, and indeed to determine exactly which transformations can be applied. A precedence analysis would require completion of the parse. Thus the dominance analysis of syntax would seem to reduce significantly the cognitive load on the algebraist. For this reason, the precedence alternative is tentatively rejected. An independent test of this question is nevertheless desirable.

I. RADICAL REVERSAL

In translating operations from DF to SF, the order of the subexpressions is preserved. For example, $\alpha A\beta \longrightarrow \alpha + \beta$, $\alpha S\beta \longrightarrow \alpha - \beta$, $\alpha M\beta \longrightarrow \alpha\beta$, etc. The proposed alteration to the grammar is that for the radical operation, there be a commutation between DF and SF. That is, Translation Rule #8 (page 75),
$\alpha R\beta \longrightarrow \sqrt[\alpha]{\beta}$, becomes $\alpha R\beta \longrightarrow \sqrt[\beta]{\alpha}$.

There are several kinds of criteria upon which to base adoption of this proposal, however, not all of them are acceptable within the present methodological framework. The opportunity is taken here to briefly consider a range of possible criteria as a vehicle to exploring some delicate methodological issues.

1. Semantic Representation

For all of the noncommutative binary operations, subtraction, division, exponentiation and radical, there is a strong semantic sense of an active and passive constituent. For example, the sense for a - b, $\frac{a}{b}$, and a^b is that the second constituent, "b", is *acting upon* the first constituent, "a". The radical operation is an anomoly in that $\sqrt[a]{b}$ has "a" acting upon "b".

Acceptance of the radical reversal rule would explain the apparent anomoly for radical as a feature of SF only. In DF, the expression $\sqrt[b]{a}$ would be represented as aRb. Thus the semantic principle that in aOb "b" *acts upon* "a" for every binary operation "O" would hold in DF.

This argument touches upon deep referential aspects of operation "meaning." Even if the above analysis is correct, however, it is not a suitable criterion for adoption of the radical

reversal rule. Within the present methodological framework, an attempt is being made to distinguish between semantic reference (so important for translating between 'real world' and symbolic representations) and the mental structures which underlie manipulation of symbolic representations. In particular, there is no *a priori* reason to accept as linked, referential structures and any components of the present grammar. Independent grounds for postulating some connection need to be found.†

2. The Generalized Inverse Operation Relation

The relationship between inverse operations can be described by the following rules:

$$(a + b) - b = a \qquad (a - b) + b = a$$

$$\frac{ab}{b} = a \qquad \frac{a}{b} \cdot b = a$$

$$\sqrt[b]{a^b} = a \qquad (\sqrt[b]{a})^b = a$$

In DF these properties appear as

$$(aAb)Sb = a \qquad (aSb)Ab = a$$

$$(aMb)Db = a \qquad (aDb)Mb = a$$

$$bR(aEb) = a \qquad (bRa)Eb = a$$

If the radical operation is reversed from DF into SF, then the DF for these properties becomes:

†At the same time, if independent grounds are established for accepting radical reversal, this could be cited as evidence in support of a connection between semantic reference and manipulative skill.

(aAb)Sb = a	(aSb)Ab = a
(aMb)Db = a	(aDb)Mb = a
(aEb)Rb = a	(aRb)Eb = a

These six properties, in DF can then be expressed as a single property, The Generalized Inverse Operation Relation (GIOR):

(aOb)O'b = a for any binary operation O.

A problem with this proposal is that the particular rules which are summarized by GIOR do not appear as transformations within the grammar. It is, therefore, not the case that the grammar is simplified through this device.

There are certainly rules, closer to transformational rules, which depend upon properties of inverse operations. For example, $a - \beta \longleftrightarrow a + (-\beta)$, $\frac{a}{\beta} \longleftrightarrow a \cdot \frac{1}{\beta}$ and $\sqrt[\beta]{a} \longleftrightarrow a^{\frac{1}{\beta}}$ are close to *Rerepresentation Rules* (page 84). If radical reversal is adopted, then the DF's of these rules appear to have some commonality:

$$a\mathrm{S}\beta \longrightarrow a\mathrm{A}(\mathrm{N}\beta),\ a\mathrm{D}\beta \longrightarrow a\mathrm{M}(\mathrm{I}\beta),\ \text{and}\ a\mathrm{R}\beta \longrightarrow a\mathrm{E}(\mathrm{I}\beta)$$

where Nb is the additive inverse of b and Ib is the multiplicative inverse. While combining these three transformations would certainly simplify the grammar, despite an apparent relationship, there is no obvious rule underlying the three.

The rules of algebra to which the fluent manipulator of symbols has access may or may not include some abstract or generalized inverse relationship. This is an important question for further theoretical and empirical investigation. It seems likely that any generalization which

might eventually emerge will require the reversal of radical representation between DF and SF.

3. The Generalized Distributive Law

Consider the following distributive rules:

Level 2 over Level 1	Level 3 over Level 2
$(a + b)c = ac + bc$	$(ab)^c = a^c b^c$
$(a - b)c = ac - bc$	$\left(\frac{a}{b}\right)^c = \frac{a^c}{b^c}$
$\frac{a + b}{c} = \frac{a}{c} + \frac{b}{c}$	$\sqrt[c]{ab} = \sqrt[c]{a}\,\sqrt[c]{b}$
$\frac{a - b}{c} = \frac{a}{c} - \frac{b}{c}$	$\sqrt[c]{\frac{a}{b}} = \frac{\sqrt[c]{a}}{\sqrt[c]{b}}$

Schwartzman (1977) has noted that in each case, the operation distributed over is exactly one level lower than the distributed operation.

In DF these properties are written as

(aAb)Mc = (aMc)A(bMc)	(aMb)Ec = (aEc)M(bEc)
(aSb)Mc = (aMc)S(bMc)	(aDb)Ec = (aEc)D(bEc)
(aAb)Dc = (aDc)A(bDc)	cR(aMb) = (cRa)M(cRb)
(aSb)Dc = (aDc)S(bDc)	cR(aDb) = (cRa)D(cRb)

If the proposal for radical reversal is adopted, then the two radical rules appear as follows: (aMb)Rc = (aRc)M(bRc) and (aDb)Rc = (aRc)D(bRc). This enables consolidation of these eight properties in a single Generalized Distributive Law (GDL):

$$(aOb)O_1c = (aO_1c)O(bO_1c) \text{ where } |O_1| = |O| + 1$$

These rules are closely related to rules #31, #35, #36, #37, #38, #39 of the Transformational Component (page 86). The GDL can be rewritten to summarize these transformations as follows:

$(\alpha O_1 \beta O_2 \ldots O_n \gamma) O_{n+1} \delta \longleftrightarrow (\alpha O_{n+1} \delta) O_1 (\beta O_{n+1} \delta) O_2 \ldots O_n (\gamma O_{n+1} \delta)$ where $|O_1| = |O_2| = \ldots = |O_n|$ and $|O_1| + 1 = |O_{n+1}|$.

This version of the Generalized Distributive Law (GDL) is somewhat broader in scope than the previous version. For example this version permits the transformation of $\left(\frac{x}{y/z}\right)^2$ to $\frac{x^2}{y^2/z^2}$ directly in one application, whereas it would require two applications of the earlier version. Beyond number-of-application differences, however, the GDL permits the same range of transformation as the original productions. Therefore, this reformulation of the GDL Transformation does not diminish the descriptive adequacy of the proposal. Independent evidence in support of the GDL proposal is contained in Paper #5, Part II.

4. Generalized Associativity

Consider the following two transformational rules:

$(\alpha^{\beta})^{\gamma} \longleftrightarrow \alpha^{\beta\gamma}$ and $\sqrt[\gamma]{\alpha^{\beta}} \longleftrightarrow \alpha^{\frac{\beta}{\gamma}}$. In DF these transformations are $(\alpha E \beta) E \gamma \longleftrightarrow \alpha E(\beta M \gamma)$ and $\gamma R(\alpha E \beta) \longleftrightarrow \alpha E(\beta D \gamma)$ respectively. If radical reversal is adopted, then the underlying relationship between the two transformations is visible: $(\alpha E \beta) E \gamma \longleftrightarrow \alpha E(\beta M \gamma)$ and $(\alpha E \beta) R \gamma \longleftrightarrow \alpha E(\beta D \gamma)$. Thus some abstraction of associativity is also enabled by adoption of the radical reversal proposal.

5. Simplicity Criteria

The prospect of a substantially simplified Transformational Component makes the radical reversal proposal appealing. In natural-language linguistics, where explanatory adequacy is defined to be the extent to which the grammar of a particular language reflects general characteristics of all languages, there is a premium on the simplicity and abstractness of proposals. Indeed several attempts have been made to operationalize a "simplicity measure" so that selection of the "simplest" alternative can proceed without argument.

In the present adapted linguistic paradigm, however, the explanatory goals are different: to discover the rules underlying fluency in algebraic skill. With this definition, simplicity and abstraction do not have *a fortiori* desirability as attributes of grammar. Knowledge structures may turn out to be simple and abstract or complex and superficial. Some independent evidence should be sought in each case. Pending such evidence, however, and in accord with accepted criterion for the evaluation of any scientific theory, the simpler formulation of the grammar (incorporating radical reversal) is accepted.

J. SPATIAL CUES IN SYNTACTIC ANALYSIS

In the translation from SF ——> DF there are two stages which are directly concerned with assignment of syntactic structure. Stage 3 assignment is made on the basis of surface features of operations. Stage 1 assignment is based on abstract categories of operations. For example in translating y^{-5x^3} to its deep form, [yE[N[5M[xE3]]]], Stage 3 carries [y^{-5x^3}] to [$y^{[-5x^3]}$] because of the surface characteristics of exponentiation, Stage 2 carries [$y^{[-5x^3]}$] to [yE[N5MxE3]], and and Stage 1 carries [yE[N5MxE3]] to [yE[N[5M[xE3]]]] based on abstract hierarchies of operations.

In this section, an alternative model is suggested. Instead of syntactic analysis in separated stages on the basis of differing representations, it is proposed that syntactic analysis is accomplished in adjacent stages and soley on the basis of concrete spatial features. More specifically, it is proposed that Stage 1 and Stage 2 translation are reversed and that Stage 1 (now the second stage) functions on the basis of spatial cues rather than abstract categories. Thus y^{-5x^3} is translated to [yE[N[5M[xE3]]]] via

$$[y^{-5x^3}] \xrightarrow{3} [y^{[-5x^3]}] \xrightarrow{1} [yE^{[-[5[x^3]]]}] \xrightarrow{2} [yE[N[5M[xE3]]]]$$

instead of via

$$[y^{-5x^3}] \xrightarrow{3} [y^{[-5x^3]}] \xrightarrow{2} [yE[N5MxE3]] \xrightarrow{1} [yE[N[5M[xE3]]]].$$

The implementation of this proposal requires that the Syntactic Rule (page 73) be reformulated in terms of concrete representation of operations. The only reference to operations in the Syntactic Rule, however, is to "operation level." The regularity of spacing features within levels allows the following alternative formulation of the operation hierarchy definition (page 72):

Operation Hierarchy: Alternative

Level 1: wide spacing; a b (a + b and a − b)

Level 2: horizontal or vertical juxtaposition; ab, $\frac{a}{b}$ (ab, $\frac{a}{b}$)

Level 3: diagonal juxtaposition; a^b , $^a b$ (a^b, $\sqrt[a]{b}$)

With this concrete specification of operation level, the Syntactic Rule can be used as before to generate correct syntactic decisions.

In the original formulation of the grammar, an expression such as x^{2y} is parsed as $x^{[2y]}$ on the basis of spatial charactistics, whereas $2x^y$ is parsed as $2[x^y]$ based on propositional

information about operation hierarchies. This seems an unlikely cognitive accomodation. Also, the inadequacy of standard pedagogical accounts of propositional hierarchies (see Paper #4) weighs in favour of a spatially based system of parsing. Nevertheless, an objective assessment is required.

One possible test is to see if algebraists (who are competent with surface form) can correctly parse expressions presented in a deep form notation. Success would support the existence of propositional knowledge structures concerning operation hierarchies. Failure to transfer syntactic skill to a DF environment would suggest a dependence upon surface cues in syntactic decision, thereby supporting the present alternative. Such a test was conducted by the author in a study reported in Paper #4. Some subjects were unsuccessful in parsing algebraic expressions without the benefit of surface cues, but others were able to successfully transfer their syntactic skill to the deep form notation. This raises the possibility of individual differences in the psychological instantiation of deep forms.

K. SPATIALLY CUED TRANSFORMATIONS

The observation that algebraic expressions may be correctly parsed on the basis of the spatial relations among symbols, without reference to operations, raises the question as to whether operations, as abstract representations, can be dispensed with entirely from the grammar. Formally this hypothesis is a variation on that expressed in the last section. Beyond just reversing Stage 1 and Stage 2 translations, it is proposed that Stage 2 is removed from the Translation Component entirely, and appended to the Phrase Structure Grammar. Deep forms (if that term is still appropriate) are then generated directly with surface attributes of operations.

This move is a radical revision of the grammar. The transformational component which mediates between DF's would have to be entirely reformulated in terms of spatial features. A consequence of this reformulation is that level-dependent transformations such as the Generalized Distributive Law (page 96) can use the spatially defined hierarchy introduced above. This consequence is (presumably) empirically verifiable although no work has been done. A second consequence of the proposal is that the pragmatic component of the grammar (see page 63) governing the selection of sequences of transformations would have to be formulated without explicit reference to operations. The difficulties which this entails have not yet been assessed.

L. NON-BINARY REPRESENTATION

A fundamental precept of elementary algebra is that the basic operations of addition, subtraction, multiplication, etc. (excluding negation) are binary operations: That is they map a pair of numbers into a single number. As a consequence, every expression involving these operations has a binary parse. Thus, for example, the expression $x + y^2 + z$ is parsed as $(x + y^2) + z$ according to the Syntactic Rule (see Figure 5).

The Associative Law for Addition (Transformational Rule #23) however, determines that $(x + y^2) + z$ can be transformed to $x + (y^2 + z)$. Thus there would be no serious consequence if one were to parse $x + y^2 + z$ as $x + (y^2 + z)$. This further raises the possibility that $x + y^2 + z$ is not, psychologically, represented in binary form at all, but rather in non-binary form with the two addition operations being of equal dominance (Figure 6). This would permit the parsing of $x + y^2 + z$ either as $(x + y^2) + z$ or $x + (y^2 + z)$ as context requires without the necessity of an associative transformation. More generally it is proposed that all operations at any given operation level are equally dominant.

Figure 5

Syntactic Tree (Binary Representation)

for $x + y^2 + z$

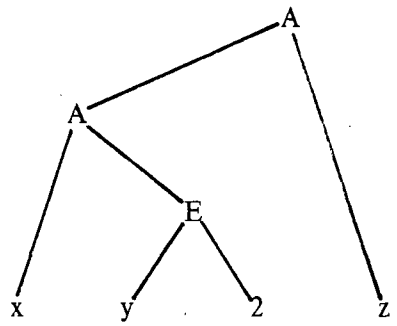
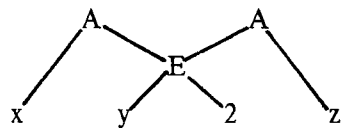


Figure 6

Syntactic Tree (Non-binary Representation)

for $x + y^2 + z$



A formal statement of this hypothesis is obtained through an adjustment to the Reversed Syntactic Rule (page 78):

Reversed Syntactic Rule

Parentheses are *inserted* around a subexpression if

(a) the dominant operation of the subexpression is directly dominated by a lower level operation, or

(b) the dominant operation of the subexpression is directly dominated by an operation of equal level, provided that the latter is to the right of the former.

The required adjustment is the simple deletion of (b).

There are obvious problems with the descriptive adequacy of this proposal. In particular, not all operations are associative. Therefore, the proposed rule allows multiple dominance in cases where the alternative binary representations are not equivalent.

Of the six binary operations, only two are associative: addition and multiplication. Of the remaining binary operations, division, exponentiation and radical all have implicit parentheses associated with their surface forms. Thus the translation of SF ——> DF never progresses beyond Stage 3. For example, the ambiguous DF xDyMz cannot arise because each of the related SF's, $\frac{x}{yz}$ and $\frac{x}{y}z$., would have been assigned a parse (either $\frac{x}{[yz]}$ or $[\frac{x}{y}]z$), in Stage 3 Translation. The unary operation, negation, does associate with the other Level 2 operations, so it is unproblematic. This leaves only subtraction which can be inadequately parsed by the non-binary assignment of dominance [$(x - y) - z \neq x - (y - z)$].

Despite the apparent problem with descriptive adequacy, this hypothesis warrants serious consideration. Under the binary interpretation a polynomial such as $5x^4 + 3x^3 + 4x^2 + 2x + 1$ is parsed as $\{(\{[5(x^4)] + [3(x^3)]\} + [4(x^2)]) + [2x]\} + 1$ —a very complex construction in comparison to the non-binary option $[5(x^4)] + [3(x^3)] + [4(x^2)] + [2x] + 1$. According to a binary representation hypothesis, the position of a term in a polynomial determines the depth of syntactic nesting. This implies, for example, that transforming $5x^4 + 3x^3 + 4x^2 + 2x + 1$ to $5x^4 + 3x^3 + 4x^2 + 1 + 2x$ requires more processing (associativity, commutativity, associativity) than transforming it to $3x^3 + 5x^4 + 4x^2 + 2x + 1$ (just commutativity). This may be true, but it certainly has no *a priori* plausibility. Presumably further research can address this question.

A second reason to hesitate in rejection of the non-binary operation theory is the inconsistent treatment of subtraction in the grammar. In some transformations it was necessary to omit subtraction entirely thereby implying that the representation of subtraction is addition of a negative ($x + y = x + (-y)$) (see the footnote for Transformation 48). It is unlikely, however, that subtraction is always so represented. For example it would be difficult to ascribe explanatory adequacy to the reinterpretation of, say, the "Difference of Squares" rule ($\alpha^2 - \beta^2 \longleftrightarrow (\alpha - \beta)(\alpha + \beta)$), as the "Sum of a Square plus the Negation of a Square" rule. Thus the status of subtraction is, to some extent, unresolved.

The crux of the problem is that the binary representation theory imposes a cumbersome syntactic structure on polynomials. Non-binary representation avoids the incumberance, but leads either to incorrect sentences (eg. * $x - y + z = x - z + y$ —commutativity of addition), or else entails the unpalatable conclusion that subtraction is not an operation at all, the psychological representation being addition of a negative.

The interpretation that the author prefers at the moment is a compromise. It is proposed that expressions involving subtraction receive a non-binary parse (so that $x - y + z$ has equally dominant operations) but that certain *ad hoc* constraints on the *application* of transformations apply to polynomials. (For example, a constraint would require that $x - y + z$ be transformed to $x + (-y) + z$ prior to rearrangement of terms.)†

In tentative support of this hypothesis are some data which were collected by the author in investigating a quite different question, but which inadvertantly bear on the present concern. A sample of 137 fourth year engineering students were asked to evaluate each of the following

†These constraints need to be formulated in detail.

expressions for $x=2$:

1) $5x + 7 =$	2) $5x^2 =$	3) $4(6 + x) =$	4) $3 + 4x =$
5) $x^3 - 2 =$	6) $2^4 - x + 1 =$	7) $3 + 2x^2 =$	
8) $19 - 4x + 2 =$	9) $3 + (7x - 2) =$	10) $5 - x^2 + 1 =$	

Such problems are very simple for students at this educational level. Indeed only 14 students in the sample *did not* score perfectly (a total of 14 incorrect responses and one ommitted response). Clearly these errors are a marginal phenomena, however, they are not random. Twelve of the 15 errors (including the missing response) occured with the trinomial expressions, #6, #8 and #10 (the lion's share going to #8). In each of these cases the response given (if any) was compatible with the incorrect assignment of dominance in the expression; e.g. $19 - 4x + 2 = 19 - (4x + 2)$.

There are many explanations possible for these errors. It could be that unlike their peers whose syntactic representations are binary these students construct an non-binary representation. (Of course it still remains to explain that thirteen of these fourteen students got two of three similar questions correct.) Alternatively, it could be that subtraction, which for their peers is represented as addition of a negative, for these students is just subtraction. This, however, would force one to predict that these fourth year engineering students would be unable to correctly rearrange terms in simple polynomials.

A third possibility does not require hypothesizing such major differences in the cognitive organization of the students. Questions #6, #8, and #10 are nonstandard problems in that

usually there is (at most) only a single constant term in a polynomial. It could be that the need for initial focussing on the middle term of the polynomial (for substitution purposes) embedded between constant terms overrode some procedural constraint on the left-to-right evaluation of polynomial expressions. This explanation entails only a slight adjustment to some set of *ad hoc* rules governing *transformation* of polynomials to explain the students' errors, but leaves the syntactic structure of expressions and the representation of subtraction homogeneous for the entire sample.

Clearly these are difficult, subtle questions requiring substantial additional research. Equally clearly, their resolution is of direct relevance to the ways in which learning is structured for new initiates to the language of symbolic algebra.

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N. APPENDIX A

TRANSFORMATIONAL RULES IN DEEP FORM NOTATION

1. Arithmetic Component

a. Arithmetic Constraints

1. $\alpha D\beta$ is disallowed whenever $\beta = 0$

2. $\beta R\alpha$ is disallowed whenever α is not a natural number

3. $\beta R\alpha$ is disallowed whenever $\beta < 0$, and α is even †

b. Arithmetic Computations

4.& $\rho O\sigma \longleftrightarrow \tau$ ‡

c. Computational Generalizations

5.& $\alpha A0 \longleftrightarrow \alpha$

6.& $\alpha S0 \longleftrightarrow \alpha$

7.& $1M\alpha \longleftrightarrow \alpha$

8.& $[N1]\alpha \longleftrightarrow N\alpha$

9.& $\alpha D1 \longleftrightarrow \alpha$

10.& $\alpha E1 \longleftrightarrow \alpha$

---

† *Arithmetic* terms like "natural number", "<" and "even" are not defined in the grammar.
‡Recall that these lower case letters near the end of the Greek alphabet represent quantity symbols (numbers).

11.& $\alpha E0 \longleftrightarrow 1$

12.& $\alpha S\alpha \longleftrightarrow 0$

13.& $\alpha D\alpha \longleftrightarrow 1$

2. Transformations Proper

Rerepresentation Rules

14. $\alpha S\beta \longleftrightarrow \alpha A[N\beta]$

15. $\alpha E[\beta D\gamma] \longleftrightarrow [\alpha E\beta]R\gamma$

16. $\alpha E[N\beta] \longleftrightarrow 1D[\alpha E\beta]$

17.& $\alpha D\beta \longleftrightarrow \alpha M[1D\beta]$

18. $[\alpha D\beta]M[\gamma D\delta] \longleftrightarrow [\alpha M\gamma]D[\beta M\delta]$

19. $[\alpha D\beta]D[\gamma D\delta] \longleftrightarrow [\alpha D\beta]M[\delta D\gamma]$

Commutativity Rules

20.& $\alpha A\beta \longleftrightarrow \beta A\alpha$

21.& $\alpha S\beta \longleftrightarrow N[\beta S\alpha]$

22.& $\alpha M\beta \longleftrightarrow \beta M\alpha$

Associativity Rules

23.&& $[\alpha A\beta]O\gamma \longleftrightarrow \alpha A[\beta O\gamma]$ where $|O| = 1$ †

24.&& $[\alpha M\beta]O\gamma \longleftrightarrow \alpha M[\beta O\gamma]$ where $|O| = 2$

25. $[\beta E\gamma]R\alpha \longleftrightarrow [\beta R\alpha]E\gamma$

26.&& $[N\alpha]O\beta \longleftrightarrow N[\alpha O\beta]$ where $|O| = 2$

27.& $\alpha O[N\beta] \longleftrightarrow N[\alpha O\beta]$ where $|O| = 2$

28. $[\alpha E\beta]E\gamma \longleftrightarrow \alpha E[\beta M\gamma]$

Distributivity Rules

29. GDL (See page 96)
$[\alpha O_1\beta O_2 \ldots O_n\gamma]O_{n+1}\delta \longleftrightarrow [\alpha O_{n+1}\delta]O_1[\beta O_{n+1}\delta]O_2 \ldots O_n[\gamma O_{n+1}\delta]$

where $|O_1| = |O_2| = \ldots = |O_n|$ and $|O_1| + 1 = |O_{n+1}|$. ‡

30. $[\chi O_1\psi O_2 \ldots O_n\omega]M\alpha \longleftrightarrow [\chi M\alpha]O_1[\psi M\alpha]O_2 \ldots O_n[\omega M\alpha]$
where $|O_i| = 1$

31. $N[\alpha O_1\beta O_2 \ldots O_n\delta] \longleftrightarrow [N\alpha]O_1'\beta O_2' \ldots O_n'\delta$
where $|O_i| = 1$ *

32. $\alpha S[\beta O_1\gamma O_2 \ldots O_n\epsilon] \longleftrightarrow [\alpha S\beta]O_1'\gamma O_2' \ldots O_n'\epsilon$
where $|O_i| = 1$

33. $\alpha E[\beta A\gamma A \ldots A\delta] \longleftrightarrow [\alpha E\beta]M[\alpha E\gamma]M \ldots M[\alpha E\delta]$

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†Recall that "|O|" refers to the level of operation "O".
‡The three dots within this transformation (and others) indicate distribution over an indeterminate number of operations.
*The symbol "O'" refers to the inverse operation of "O"

34. $\alpha E[\beta S\gamma] \longleftrightarrow [\alpha E\beta]D[\alpha E\gamma]$

Cancellation Rules

35. $[\alpha M\beta]D[\alpha M\gamma] \longleftrightarrow \beta D\gamma$

36. $[\alpha D\beta]D[\gamma D\beta] \longleftrightarrow \alpha D\gamma$

37.& $[N\alpha]O[N\beta] \longleftrightarrow \alpha O\beta$ Where $|O| = 2$

38.& $N[N\alpha] \longleftrightarrow \alpha$

39.& $[\alpha R\beta]E\beta \longleftrightarrow \alpha$

Polynomial Rules

40. $[\alpha A\beta A...A\delta]M[\epsilon A\zeta A...A\theta] \longrightarrow$
$[\alpha M\epsilon]A[\alpha M\zeta]A...A[\alpha M\theta]A[\beta M\epsilon]A[\beta M\zeta]A...A[\beta M\theta]A....$
$A[\delta M\epsilon]A[\delta M\zeta]A...A[\delta M\theta]$ †

41. $[\chi M[\alpha E2]]O[\psi M\alpha M\beta]A[\omega M[\beta E2]] \longrightarrow [[\rho M\alpha]O[\sigma M\beta]]M[[\tau M\alpha]O[\upsilon M\beta]]$
where $\rho M\tau = \chi$, $\sigma M\upsilon = \omega$, $[\rho M\upsilon]A[\sigma M\tau] = \psi$, and $|O| = 1$

42. $[\chi M[\alpha E2]]O[\psi M\alpha M\beta]S[\omega M[\beta E2]] \longrightarrow [[\rho M\alpha]O_1[\sigma M\beta]]M[[\tau M\alpha]O_1[\upsilon M\beta]]$
where $\rho M\tau = \chi$, $\sigma M\upsilon = \omega$, $|[\rho M\upsilon]S[\sigma M\tau]| = \psi$, $|O| = |O_1| = 1$, and
O_1 is addition if $\rho M\upsilon > \sigma M\tau$; subtraction if $\rho M\upsilon < \sigma M\tau$ ‡

43. $[\alpha O\beta]E2 \longleftrightarrow [\alpha E2]O[2M\alpha M\beta]A[\beta E2]$ where $|O| = 1$

44. $[\alpha E2]S[\beta E2] \longleftrightarrow [\alpha A\beta]M[\alpha S\beta]$

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†Note that this transformation is stated in terms of addition. In order to apply it to polynomials consisting of sums and differences of terms, there seems to be no way to avoid defining subtraction as the sum of a negation ($\alpha S\beta = \alpha A[N\beta]$) and then applying a series of transformations to return to subtractions in the derived expression.
‡Trinomial factoring seems naturally to break down into two cases.

45. $[\alpha E3]O[\beta E3] \longleftrightarrow [\alpha O\beta]M[[\alpha E2]O'[\alpha M\beta]A[\beta E2]]$
where $|O| = 1$

IV. SPATIAL CUES IN ALGEBRAIC SYNTAX

Paper #4

Spatial Cues in Algebraic Syntax

Every language, be it natural or artificial, employs a system of rules, called a syntax, to govern the arrangement and interrelation of its elements as they occur in "sentences." It is the syntax of English, for example, that determines for the sentence "The man Bill grabbed fell" that it is the man that fell, Bill who did the grabbing, the man who was grabbed, etc. Similarly, it is the syntax of the computer language LOGO that determines that in the command "FORWARD :CHARGE", :CHARGE will be evaluated as a number of units for the turtle to progress. As well, it is the syntax of algebra that determines that in "$3x^2$" it is the x which is squared and the result tripled, rather than the x tripled and the result squared. In general, syntax provides a definite internal structure for the elements of the "sentences" of a language.

Clearly, syntax must be mastered before any sensible use can be made of a language. In the case of natural languages, syntax is learned unconsciously through informal exposure to a language community. A speaker need not be consciously aware of the "rules" which comprise syntax. For example, the fluent speaker of English, unschooled in linguistic theory, would be hard pressed to identify a rule in which the stress applied to a pronoun determines its reference. Yet every native speaker, will unhesitatingly recognize that pronominal reference in the sentence "Bill hit John and then Frank hit him" is determined by the presence or absence of stress on "him." Indeed, a major project of linguistic research is to discover exactly what the rules of syntax are for natural languages.

For artificial languages, such as computer languages, the situation is reversed. Rules of syntax

are deliberately laid out in the original formulation of the language. They remain fixed (or subject to controlled development) throughout the life of the language. For the novice, syntactic rules are learned explicitly from manuals or through structured teaching. The computer insists on explicit formulations for commands, and those who design computer languages and those who "speak" to computers through these languages are well apprised of the fact.

Mathematical languages are usually considered to reside within the artificial camp rather than the natural language camp. Indeed, the very essence of mathematics may be seen as the presentation of rigorously derived rules through a rational and determinate notation. Like computer languages, mathematical languages have been designed for specific technical or scientific purposes. Their rules are completely circumscribed by conscious and rational consideration at the time of their inception. They are communicated to novice users from textbooks or through structured pedagogy.

As compelling as this view of mathematical language may seem, it is assailable on several grounds. Unlike computer languages, mathematical languages such as the system of algebraic notation are not the creation of a single integrated effort. Algebraic results originally were coded in careful natural language, rather like legal language is today. The following translation from the great Mohammed ibn-Musa al-Khowarizmi in his ninth century book Al-jabr wa'l muqabalah (from which "algebra" got its name) illustrates this point:

> You ought to understand also that when you take the half of the roots in this form of equation [quadratic] and then multiply the half by itself; if that which proceeds or results from the multiplication is less than the units above-mentioned as accompanying the square, you have an equation. (Boyer, 1968, p. 253).

(This statement expresses the fact that in a quadratic equation the discriminant must be

positive in order for there to exist a real solution). Specialized mathematical symbols were only gradually introduced, and as Cajori (1928) observed, the systematization of notations was halting and evolutionary rather than decisive and final. Thus the assumption that mathematical languages are like computer languages in having a rationally accessible syntax should not be accepted unquestioningly.

Careful examination of pedagogical methods and materials for algebraic syntax proves particularly damning to this thesis. A survey of several textbooks (Brown, Snader & Simon, 1970; Vannatta, Goodwin & Crosswhite, 1970; Johnson, Lendsey & Slesnick, 1971; Sobel & Maletsky, 1974; Dolciani & Wooton, 1975; Johnson & Johnson, 1975; Travers, Dalton, Brunner, & Taylor, 1976) found only 1 to 6 pages devoted to instruction in algebraic syntax. More importantly, the rules presented were, in most cases, inadequate to the actual requirements of syntactic skill. In many instances exponentiation and radical were entirely omitted from the discussion (which typically is near the beginning of the text) since these operations had not yet been introduced. Occasionally the left-to-right precedence of related operations was unstated [$5 - 3 + 1 = (5 - 3) + 1$, not $5 - (3 + 1)$].

Similar deficiencies can be discerned in classroom-based syntactic strategies. In the present study many grade 9 and grade 11 students tested reported the use of the acronym "BOMDAS" which stands for Brackets, Of, Multiply, Divide, Add, Subtract. This indicates that operations within parentheses are precedent to those outside of parentheses and that multiplication and division have precedence over addition and subtraction. "Of" is a relic of older notations as illustrated in "3/8 of 24." A similar technique uses the mnemonic "My Dear Aunt Sally" for which the initials correspond to the MDAS of BOMDAS. Such methods are inadequate even to the parsing of such simple examples as $3x^2$, and $x - y + 1$.

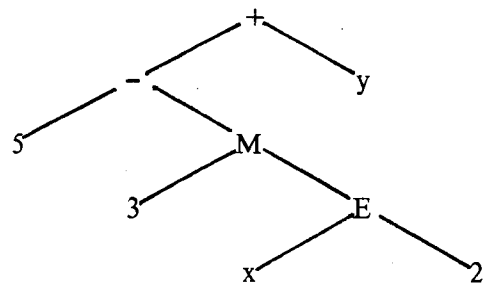
It is apparent from these observations that the assumption that syntactic knowledge is transmitted to the neophyte in an explicit form through the propositions espoused by text and teacher is highly suspect. Thus one may ask what is the nature of syntactic skill, if it is not based on textbook rules? How are these skills actually acquired by students if pedagogical methods are incomplete? Are all students equally successful in acquiring this syntactic knowledge which is fundamental to any sensible use of algebraic symbolism?

A. ALGEBRAIC SYNTAX

In order to begin to address these questions, it is necessary to provide a comprehensive account of algebraic syntax. The syntax of algebra assigns a definite internal structure to the elements of each well-formed algebraic expression. A tree notation adapted from linguistic theory (Carry, Lewis & Bernard, 1980; Matz, 1980) is used to display this structure. This notation is illustrated in Figure 7 for the expression $5 - 3x^2 + y$. The operations closer to the bottom of the tree are said to be "precedent" to those above, while the higher ones are "dominant" to those below. In keeping with standard practise, syntactic conventions will be formulated as precedence rules, although in Paper #3, the author has argued that cognitively, such rules are represented in terms of dominance. That discussion, however, is orthogonal to the present concerns.

There are two kinds of syntactic conventions in elementary algebra. The first concerns the physical presence of syntactic markers. Besides the usual parentheses, there are two other types of syntactic markers. The vinculum as it occurs in both fractions and radicals is an indicator of grouping. (Compare, for example, $\sqrt{x + y}$ and $\sqrt{x} + y$). Similarly, the raising of symbols in exponents carries syntactic import (compare x^{5y} and x^5y). Inclusion of an operation within parentheses, over or under a vinculum, or within an exponent gives it precedence which it

Figure 7
Syntactic Tree Diagram for $5 - 3x^2 + y$



"M" represents multiplication
"E" represents exponentiation

otherwise would not have.

The second kind of syntactic convention operates in the absence of these physical markers. A comprehensive treatment of this syntactic component utilizes a hierarchy of operation levels introduced by Schwartzman (1977). The six operations are grouped into three levels of two inverse operations each. The operations at each successive level are repeated applications of those at the previous level:

Operation Hierarchy

Level 1:	addition	subtraction
Level 2:	multiplication	division
Level 3:	exponentiation	radical

(In this hierarchy, Level 3 is said to be "higher" than Level 2 which is "higher" than Level 1).

Using this hierarchy, a syntactic convention can be stated for instances where the above mentioned physical markers are absent.

Syntactic Convention

1. Higher level operations have precedence over lower level operations, and
2. In case of an equality of levels, the left-most operation has precedence.

As an example, an expression such as $5 - 3x^2 + y$ is analysed as in Figure 7 with exponentiation (Level 3) having first precedence and multiplication (Level 2) having next precedence. Addition and subtraction are of the same level (Level 1) therefore subtraction precedes addition because of its leftwards position.

B. HYPOTHESIS

If there is a need to explain the incompleteness of typical instuctional strategies relating to operation hierarchies, then the total absence of explicit instruction regarding the spatial characteristics of the vinculum and exponent certainly deserves some comment. Apparently the syntactic import of these spatial cues is sufficiently salient so that not a word need be uttered to initiate the neophyte. What is more, it seems that conscious awareness is essential neither for the successful acquisition of this spatial knowledge, nor for its successful application.

Is it possible that some unconscious system of spatial cues also underlies the syntactic processes related to operation hierarchies? Careful examination of standard algebraic notation reveals definite regularities associated with operation levels. Indeed, operation levels can be reformulated without reference to the operations within each level, but purely on the basis of

spatial regularities:

Alternative Definition of Operation Level

Level 1: wide spacing; a b ("a + b" and "a - b")

Level 2: horizontal or vertical juxtaposition; ab, $\begin{matrix} a \\ b \end{matrix}$ ("ab" and "$\frac{a}{b}$")

Level 3: diagonal juxtaposition; a^b , ${}^a b$ ("a^b" and "$\sqrt[a]{b}$")

The syntactic convention (page 121) assumes a quite different character under these new definitions. For example, the first rule "Higher level operations have precedence over lower level operations" becomes "diagonally juxtaposed symbols have higher precedence than horizontally or vertically juxtaposed symbols, which have higher precedence than spaced symbols," instead of "exponents and radicals precede multiplication and division which precede addition and subtraction." It thus appears that there are at least two separate bases upon which operation hierarchies can be determined: as a propositional system of knowledge about operation levels; or as an unconscious system of spatial relations.

The delineation of these two characterizations of "operation level" raises the question as to which version actually underlies the syntactic knowledge of the fluent algebraist. Is syntactic knowledge coded as information about "operations," or is it coded in terms of the spacing and positioning of symbols? Are the degenerate rules of syntax which are presented in classrooms and in textbooks somehow the vehicle for a propositional syntactic knowledge, or does the student respond, untutored, to the positional cues in the notation?

C. METHOD

The task of evaluating algebraic expressions was chosen as the means to investigate the psychological basis of syntactic skill. A typical example of such a task is to evaluate $3x^2$ for $x=2$. A result of 12 is taken as an indication that the expression has been analysed appropriately as $3(x^2)$. A result of 36 is taken as an indication that the expression has been analysed inappropriately as $(3x)^2$.

An overview of the basic strategy of investigation in this study is as follows: Firstly, it is necessary to verify that the subjects can perform evaluative tasks appropriately using standard algebraic notation. Then similar tasks are presented using a nonce, or artificial, notation. This notation is specially devised to display the propositional character of algebraic expressions while distorting the surface cues of ordinary notation. The ability of subjects to perform appropriate syntactic analyses in the nonce notation is taken as an indication that syntactic knowledge is coded as propositional information about operations. Inability to transfer competent behaviours to the nonce setting indicates a dependence upon the surface cues found in ordinary notation.† (Several possibly confounding factors to this simplified inferential scheme are considered throughout this report.)

1. The Nonce Forms Study

A nonce notation was devised which displays the propositional information about algebraic

†The use of nonce forms is a standard paradigm in psycholinguistics. See, for example, Braine, (1971); Marslen-Wilson & Welsh, (1978).

expressions, but distorts the usual spatial arrangement of the symbols:

aAb	=	$a + b$	aSb	=	$a - b$
aMb	=	ab	aDb	=	$\frac{a}{b}$
aEb	=	a^b	aRb	=	$\sqrt[a]{b}$

In each case a capital letter abbreviation is used to identify the operation. Thus the notation may communicate, in propositional form, the operations within an algebraic expression. The nonce form, however, spaces all of the symbols equally in a horizontal array. This distorts the surface, spatial cues available in standard notation for the determination of operation levels.

Three kinds of tasks were devised using the nonce forms notation: arithmetic; simple algebraic; and complex algebraic. The arithmetic tasks were simple binary combinations such as 5E2 (5^2). These items were included to insure that subjects could indeed retrieve the appropriate propositional information from the nonce notation.

The algebraic items, simple and complex, consisted of algebraic expressions expressed in nonce form. Each expression contained a single occurrence of the variable "x." The task was to evaluate the expression for $x=2$. Simple algebraic items were of the form 5MxS2A1 (i.e. $5x - 2 + 1$), in which the correct precedence of the operations is from left to right. Complex algebraic items were of the form 1A3MxE2 ($1 + 3x^2$) in which the correct order is other than from left to right.

This distinction was made because of the nature of the spatial cueing hypothesis. If subjects do in fact rely upon such cues, then they might be expected to transfer their response

patterns to the nonce notation. But the crucial indicator according to the hypothesis is the relative spacing of the symbols. Since the nonce notation spaces all symbols equally, this would be construed as indicating that the levels of all of the operations are equal. In this event, Syntactic Convention 2 (page 121) would dictate a left-to-right assignment of operation precedence. Thus for the items identified as "simple algebraic," subjects using spatial cues, as well as subjects using propositional information, would be expected to perform correctly. For the complex algebraic items however subjects using spatial cues would tend to continue with the left-to-right assignment of precedence, while those employing propositional strategies would adjust their precedence procedures appropriately.

The instrument contained three ten-item subtests. The first subtest consisted of nonce arithmetic items. The second subtest was an arbitrary arrangement of five nonce algebra simple and five nonce algebra complex items. The third subtest featured regular notation algebra items which correspond in syntactic structure to the nonce algebra items. Thus item #7 in subtest 2 was 1A3MxE2 and item #7 in subtest 3 was $3 + 2x^2$. These sections of the instrument are included as Appendix A.

a. Multiple Choice Distractors

Five multiple choice options were presented with each item, but a blank space was also provided for subjects' answers not corresponding to the given choices. The five options for the algebra items included all of the possible parsings of the algebraic expression. For example, the nonce item 1A3MxE2 (x=2) had 13, 16, 37, 49, and 64 provided as possible answers. These answers correspond to the parsings 1A[3M(xE2)], (1A3)M(xE2), 1A[(3Mx)E2], [1A(3Mx)]E2, and [(1A3)Mx]E2 respectively.

2. Alternate Nonce Notation

It was anticipated that performance using a new and unfamiliar notation might decline somewhat relative to performance using the usual notation just because of the unfamiliarity, and quite apart from any particular hypothesized causes. To guard against the possibility that poor performance on the nonce items might be attributed to the general unfamiliarity of the notation, rather than specifically to the distortion of spatial cues, an alternate form of the nonce notation was devised. This notation differs from the first nonce notation only in that the spacing of the symbols more closely resembles spacing in the regular notation.

$$a\ A\ b = a + b \qquad a\ S\ b = a - b$$

$$a\ M\ b = ab \qquad a\ D\ b = \frac{a}{b}$$

$$aEb = a^{b} \qquad aRb = \sqrt[a]{b}$$

(The Level 1 operations are more widely spaced than the Level 2 or Level 3 operations). It was reasoned that inferior performance on the first nonce forms relative to performance on the spaced form could not be explained in terms of the general unfamiliarity of the notation, but only in terms of the differential spacing characteristics of the notations.

Two versions of the instrument were devised differing only in the type of nonce notation used. (See Appendix B for the spaced form version of the instrument). The two versions were randomly distributed within each of the groups to which tests were administered.

D. SUBJECTS

The instruments were administered to 562 subjects.† Twenty-two of these subjects failed to provide information on at least one independent or covariate variable (see "Covariates") and

†Additionally, 68 subjects in grade 9 remedial mathematics classes were tested. These subjects, in the main, lacked even rudimentary exposure to elementary algebra and their scores were not included in the analysis.

their scores could not be included in the analyses. Of the remaining 540 subjects, 23 (4.4%) were eliminated because of failure to obtain at least seven correct answers out of nine nonce arithmetic items.† It was reasoned that subjects either lacking basic arithmetic skills, or unable to grasp the representational system of the nonce abbreviations would not be fairly tested by the instrument.

Of the 517 subjects (352 male, 165 female) whose responses were analysed, 133 were students in grade 9 (75 male, 58 female), 159 were students in grade 11 (93 male, 66 female), 81 were students in first year calculus classes (49 male, 32 female), 124 were students in fourth year engineering (115 male, 9 female), and twenty were professional engineers (all male). The grade 9 and 11 students were drawn from two predominantly middle and lower middle class secondary schools of the public school system in Vancouver, British Columbia. The calculus and engineering students were enrolled at the University of British Columbia, and the professional engineers were in attendance at a meeting of the Professional Engineers Association of British Columbia. Due to the small number of professional engineers participating, their scores were grouped together with those of the graduating engineering students.

The broad range of "grade" levels was included to determine whether syntactic reasoning skill develops with increased mathematical experience and maturity, as well as to provide a reference group of unquestionably competent algebraists with which to anchor the study. All subjects, however, were expected to be reasonably proficient in the syntactic analysis of the fairly simple, standard notation algebraic expressions presented.

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†A tenth item dealing with the radical operation was dropped because this operation does not appear in any of the subsequent algebra items, and because many of the grade 9 students had not yet been introduced to computation with radicals.

E. TEST ADMINISTRATION

Equal numbers of the two forms of the instrument were randomly distributed within each gender grouping in each of 16 participating "classes." Participants were instructed to circle or write down only final answers for the items, since a transcription of the nonce items into regular notation would undermine the intended effect of the notation. Compliance was enforced by the use of pens rather than pencils to prevent erasure, and by prohibition on the use of scrap paper. (The possibility that subjects would make a mental transcription in regular notation is considered below.)

Subjects were given adequate time to complete, check and correct the current subtest before proceeding to the next. Correcting of answers in the previous subtest, however, was not permitted. This was monitored by requiring the initial of the test administrator next to each authorized (within subtest) correction.

F. COVARIATES

Two covariates were used in the analysis: Computing Experience and Past Algebraic Achievement. In most instances the data for these covariates were obtained on the title page of the instruments. A different title page was used for the Professional Engineers than for all other subjects. The two versions of the title page are given in Appendix E.

1. Computing Experience

Several calculator and computer languages employ syntactic rules for arithmetic evaluation which differ from the standard of algebraic notation. For example, in the programming language APL, the expression 3*2+5 is evaluated as 3*(2+5), 21, rather than as (3*2)+5, 11. (The "*" represents multiplication). Also, virtually all computer languages employ explicit

Table 4
Computing Experience

	GRADE 9	GRADE 11	CALCULUS	ENGINEERS	TOTAL
MALE	1.49*	1.71	1.63	2.29	1.88
	75**	93	49	135	352
FEMALE	1.48	1.50	1.34	2.22	1.50
	58	66	32	9	165
TOTAL	1.49	1.62	1.52	2.28	1.76
	133	159	81	144	517

*Mean computing experience based on 3 point scale having 1 as "none", 2 as "some", and 3 as "quite a bit".

**Number of subjects within cell.

symbols for multiplication and exponentiation which, in standard algebraic notation, are only positionally marked. Thus subjects with experience in computer languages have already learned to function with more than one system of surface representations and might be expected to have developed a more flexible syntactic rule. Accordingly, each subject was asked to indicate whether he or she had "none," "some," or "quite a bit" of experience in programming a computer.

This measure provides only a crude indicator of programming experience, since "quite a bit" to a grade 9 student undoubtedly means something different to a professional engineer. Nevertheless, the vastly greater computing experience of the fourth year engineering students is reflected in the responses to this question. (See Table 4).

Because of the likelihood that the measure used tends to inflate the relatively lesser computer exposure of the younger subjects, developmental trends would be difficult to evaluate. A covariate adjustment will only partly compensate for the greater computer experience of the

engineering students. The computing experience scores were entered as a covariate in the analyses of nonce algebra scores in order to obtain this partial control.

2. Past Algebraic Achievement

The tendency for men to outperform women in higher level mathematical tasks such as algebra is well documented (e.g. NAEP, 1981). In a pilot study, the gender of the respondent had appeared to be related to nonce notation performance. Data were collected on the past algebraic achievement of the subjects in order to insure that apparent gender differences were not merely an artifact of general algebraic aptitude. Post secondary school subjects were asked to record the final mark in their final high school algebra course. The most recent report card marks in algebra were obtained directly from the schools for the secondary student participants. The data were entered as a three point covariate measure against nonce algebra scores.

G. RESULTS

Univariate analyses of covariance were performed on each of the dependent variables; regular notation algebra score, simple nonce notation algebra score, and complex nonce notation algebra score. The analyses were performed hierarchically on the ordered independent variables; grade, gender, and test form. The heirarchical analysis adjusted gender effects for grade effects, and test form effects for both grade and gender effects.

There were several reasons for choosing this ordering. First of all, significant grade effects could be predicted on the basis of the greater academic stature of the older subjects. If nothing else, better work habits and greater arithmetic experience would be expected to improve performance for the advanced students. The disproportionate distribution of gender

through grade (the vast majority of the engineering students were male) demanded that grade effects be controlled for in the gender analysis in order that the latter be unambiguously interpretable. Finally, since test form was the critical variable in the study, the hierarchical ordering of grade, gender, test form provided a more conservative test of significance for this variable.

1. Regular Algebra Subtest

As anticipated, most of the subjects demonstrated competence in the regular notation algebra tasks. The mean score on this subtest was 93.6% across all grades. There was a significant grade effect ($p < 0.001$). Cell means ranged from 86.4% for the grade 9 subjects to 96.8% for the engineers and graduating engineers. Men and women performed about equally well on the regular notation items (93.8% and 93.3% respectively). The interaction of sex with grade was not statistically significant.

2. Nonce Algebra-Simple Subtest

The expectation was that both subjects using propositional referents and those using surface cues in syntactic decision making would be generally successful on the five simple nonce algebra items. This expectation was borne out by the analysis, with the overall mean score being 91.0%. Furthermore there were no significant deviations in this performance between men and women or between those who received the spaced and the unspaced form of the instrument. There was, however, a statistically significant grade effect. None of the interaction effects were significant (see Table 5).

Table 5
Dependent variable: NONCE ALGEBRA-SIMPLE PERCENTAGE
Independent variables: TEST FORM, SEX and GRADE
Covariates: ALGEBRA ACHIEVEMENT and COMPUTER EXPERIENCE

TOTAL POPULATION				
91.03%				
(517)				
TEST FORM				
closed	spaced			
91.39[1]	90.66			
92.00[2]	90.04			
(260)[3]	(257)			
SEX				
male	female			
90.91	91.27			
90.05	93.11			
(352)	(165)			
GRADE				
9	11	1'st yr	4'th yr+	
86.17	92.20	88.64	95.55	
82.55	95.25	84.93	97.61	***
(133)	(159)	(81)	(144)	

[1]mean
[2]adjusted mean
[3]cell size
*** significant at 0.001 level

3. Nonce Algebra-Complex Subtest

The complex nonce algebra items proved more diffcult for most subjects than the other items. The mean percentage correct for all subjects was 66.3%. The analysis of covariance also indicated that these items were significantly more difficult when presented with the unspaced form of the notation than when presented with the spaced form ($p < 0.001$). Also, men scored significantly better than women on this subtest ($p < 0.05$), and there was a significant interaction between gender and form ($p < 0.05$). Women using the unspaced form of the instrument experienced greater difficulty than other subjects (Table 6). There were no

Table 6
Dependent variable: NONCE ALGEBRA-COMPLEX PERCENTAGE
Independent variables: TEST FORM, SEX and GRADE
with Covariates: ALGEBRA ACHIEVEMENT and COMPUTER EXPERIENCE

TOTAL POPULATION
66.27%
(517)

TEST FORM

closed	spaced	
59.54[1]	73.07	
53.41[2]	79.27	***
(260)[3]	(257)	

SEX

male	female	
70.06	58.18	
72.31	53.39	*
(352)	(165)	

GRADE

9	11	1'st yr	4'th yr+	
55.79	62.77	65.93	80.00	
50.23	66.05	62.49	83.45	*
(133)	(159)	(81)	(144)	

FORM	SEX male	SEX female	
closed	65.78	45.50	
	(180)	(80)	
			*
spaced	74.53	70.12	
	(172)	(85)	

[1]mean
[2]adjusted mean
[3]cell size
* significant at 0.05 level
** significant at 0.01 level
*** significant at 0.001 level

significant interactions other than gender and form.

A significant grade effect was noted for this subtest as for the previous ones. In order to obtain a clearer picture of this source of variance the data were reanalysed using the regular notation algebra scores as an additional covariate. In this way it could be determined whether the nonce notation introduced a significant new source of variance beyond that which was present in the regular notation scores. The grade effect was not significant in the reanalysis. Table 7 presents the two ANOVA tables for comparison.

a. Distractor Selection

There had been an expectation that subjects unable to apply propositional rules of syntax to the nonce notation would process the symbols from left to right (page 124). In each case, the left-to-right solution was the most frequently chosen distractor. For the item 1A3MxE2 ($1 + 3x^2$), for example, 140 subjects (53.8%) chose the solution corresponding to the correct parse, 1A[3M(xE2)], and two subjects offered no response. Of the 118 subjects who chose an incorrect response, 51, (43.3%) selected the response corresponding to the left-to-right parse, [(1A3)Mx]E2. The pattern of distractor selection for this item is shown in Table 8.

This propensity towards left-to-right processing, however, is difficult to interpret. It could result from the assumption (page 124) that the uniform spacing provided by the nonce notation was construed as indicating a single level for all operations. Alternatively, subjects may have "read" the symbols from left to right as English text. But more than one half of the respondents selecting an incorrect response rejected the left-to-right option indicating that they actively grappled with the syntactic question though without adequate tools.

Table 7 - Two ANOVA Tables

ANOVA TABLE for NONCE ALGEBRA-COMPLEX SCORES

SOURCE OF VARIATION	SUM OF SQUARES	DF	MEAN SQUARE	F	P VALUE
COVARIATES	133123	2	66562	67.4	.000
ACHIEVEMENT	104500	1	104500	105.9	.000
COMPUTER EXPER	28623	1	28623	29.0	.000
MAIN EFFECTS	31027	5	6205	6.3	.000
GRADE	7857	3	2619	2.7	.048
SEX	4035	1	4035	4.1	.044
FORM	19135	1	19135	19.4	.000
2-WAY INTERACTIONS	9334	7	1333	1.4	.224
GRADE SEX	995	3	332	.3	.799
GRADE FORM	2440	3	813	.8	.481
SEX FORM	5895	1	5895	6.0	.015
3-WAY INTERACTIONS	4880	3	1627	1.6	.177
GRADE SEX FORM	4880	3	1627	1.6	.177
EXPLAINED	178364	17	10492	10.6	.000
RESIDUAL	492517	499	987		
TOTAL	670881	516	1300		

ANOVA TABLE for NONCE ALGEBRA-COMPLEX SCORES
with REGULAR ALGEBRA SCORE AS ADDED COVARIATE

SOURCE OF VARIATION	SUM OF SQUARES	DF	MEAN SQUARE	F	P VALUE
COVARIATES	152841	3	50947	53.3	.000
REGULAR ALGEBRA	71349	1	71349	74.7	.000
ACHIEVEMENT	55587	1	55587	58.2	.000
COMPUTER EXPER	25905	1	25905	27.1	.000
MAIN EFFECTS	31255	5	6251	6.5	.000
GRADE	3624	3	1208	1.3	.286
SEX	4494	1	4494	4.7	.031
FORM	23137	1	23137	24.2	.000
2-WAY INTERACTIONS	6732	7	962	1.0	.425
GRADE SEX	479	3	160	.2	.918
GRADE FORM	2106	3	702	.7	.531
SEX FORM	4428	1	4428	4.6	.032
3-WAY INTERACTIONS	4440	3	1480	1.6	.201
GRADE SEX FORM	4440	3	1480	1.6	.201
EXPLAINED	195268	18	10848	11.4	.000
RESIDUAL	475613	498	955		
TOTAL	670881	516	1300		

Table 8
DISTRACTOR SELECTION for
1A3MxE2 $(1 + 3x^2)$ with $x=2$

RESPONSE	PARSE	N	%
16	(1A3)M(xE2)	19	16.1
37	1A[(3Mx)E2]	29	24.6
49	[1A(3Mx)]E2	16	13.6
64(expected error)	[(1A3)Mx]E2	51	43.3
other		3	2.5
		118	

H. CONCLUSIONS

Almost all of the subjects participating in the study were able to evaluate expressions such as $1 + 3x^2$, for $x=2$, when presented in standard notation. It proved, however, to be significantly more difficult to transfer this ability to the closed nonce notation, 1A3MxE2, than to the spaced nonce notation, 1 A 3 M xE2. These two notations differ only in the spacing of the symbols. The latter notation was devised specifically to mimic spacing features of ordinary notation. Thus it seems necessary to conclude that for some students, at least, surface features of ordinary notation provide a necessary cue to successful syntactic decisions.

1. Individual Differences in Nonce Notation Processes

Although a general tendency for subjects to score less well on the complex nonce items than on similar standard notation items was observed, variation among respondents was substantial. Many subjects were able consistently to render correct syntactic decisions using the nonce notation, while others were consistently unable to do so.

What do these results indicate about differences in the cognitive strategies employed by the successful and unsuccessful participants? Can the conclusion be drawn that the more successful

subjects had access to correct propositional rules of syntax which were unavailable to their cohorts?

An alternative explanation of these differences is that the successful subjects created a mental picture, in standard notation, of the algebraic expression presented in nonce notation. They could then "read" the correct syntax from the spatial cues in their mental image of the items. The prohibition against physically transcribing the nonce items into regular notation (page 128) could not be extended to the imagination of the subjects. Subjects employing such a strategy might achieve success because of a superior ability to visualize and manipulate mental images rather than through access to correct propositional rules of syntax.

a. Post Hoc Analysis

In an attempt to discover whether success was due to such mental gymnastics, or if successful subjects were truly using explicit rules to guide their responses, each subject was asked to provide an introspective account of his or her mental processes for one of the complex nonce items with which he or she had been successful (if any).

Subjects were asked to record "yes," "no," or "not sure" to a question such as "For the problem 1A3MxE2, did you imagine or visualize or picture in your mind $1 + 3x^2$?" Similarly, subjects were asked if they used rules of order of operations in solving the problem and if so, which rules. Space was provided for students to elaborate on their response. The portions of the instruments dealing with this part of the study are included as Appendices C & D.

Ideally, this kind of datum should be acquired through careful interview to assure that

Table 9
SELF REPORT of
VISUALIZATION STRATEGY versus RULE-BASED STRATEGY

		Rule used?		
		YES	NO	TOTAL
Visualization used?	YES	120 (36.0%)	15 (4.5%)	135 (40.5%)
	NO	184 (55.3%)	14 (4.2%)	198 (59.5%)
	TOTAL	304 (91.3%)	29 (8.7%)	333 (100%)

subjects fully understand the intent of the questions. Of the 449 subjects who had correctly solved a complex nonce algebra item without guessing or using a flawed procedure, 116 were unable to provide a definitive "yes" or "no" response for each of these questions. The majority (55.3%) of the remaining 333 subjects claimed to have used precedence rules, and not to have used the visualization technique (Table 9). Only 15 subjects (4.5%) claimed to have visualized the problem in ordinary notation and *not* applied rules of operation precedence. Thirty-six per cent of the subjects claimed to have used both strategies.

Verbal corroboration for subjects in this last category was skewed. None of these subjects provided a verbal account of the visualization strategy, however, many did report the use of propositional rules. For example, one such grade 9 student wrote "exponentiation (a^b) is first in order of operations, then multiplication and addition last," and said nothing about forming a mental image in standard notation.

In fact, only one or two subjects in the study made a clear statement of using the

visualization technique. One of these, a grade 9 student, wrote "I tried to visualize the capital letters as being math symbols and worked on the exponent part of the question which was the hardest part." It does not appear from these self reports, however, that success on the nonce item tasks was often accomplished through the visualization of nonce expressions in standard notation. This seems to confirm that access to adequate propositional information about operation precedence is truly variable from one individual to the next.

2. Explaining Individual and Group Differences

To some degree, differences in nonce item performance can be accounted for by differential instruction. Students in some grade 11 classes had been taught the BOMDAS rule (page 118) in the more complete form of EBOMDAS, where "E" stands for exponentiation. These students tended to apply their rule of operation precedence successfully to a greater range of nonce tasks than did other subjects. Additionally, the propensity for better nonce task performance at higher grade levels might be attributable to a cumulative exposure to more adequate instructional fragments over time.

Such explanations, however, do not easily account for the fact that within each class, the range of performance was great. In grade nine classes, for example, the likelihood is that the students have experienced the same instructional situation for their entire algebra careers. Thus differential instruction does not likely account for the entirety of individual differences. Another possible explanation concerns cognitive style differences of the subjects. See "Implications For Research."

3. Individual Differences in Regular Notation Processes

The study has pointed to differences in the rule structures which subjects have available to them. Some subjects have adequate propositional rules available and others do not. Those who do not must be relying upon the spatial cues in order to perform correct syntactic analyses with ordinary notation. But what of the subjects who do have adequate propositional rules available to them? Do they make use of their propositional knowledge, or, like their cohorts, do they exploit surface cues in ordinary notation syntactic decision making? This study has provided no direct evidence about these students' regular notation processes.

It is this researcher's opinion that all (or nearly all) people proficient in the manipulation of algebraic symbols do normally make use of the spatial cues in ordinary notation to assist in syntactic decisions. There are two main reasons for this view. Firstly, the presence of surface rules for which there are no usual propositional counterparts is suggestive. For example, x^{5y} is interpreted, syntactically, as $x^{(5y)}$ despite the fact that normally exponentiation has precedence over multiplicaton. A propositional formulation of such a rule might be "operations within an exponent have precedence over the associated exponentiation." The presence of such a formulation, even if only in limited usage, would suggest that some portion of the population requires propositional constructs in order to function syntactically. Its absence (at least in the limited experience of this author) opens the possibility that some aspects of surface processing of syntactic cues may be universal.

The second reason is that spatial cues in standard algebraic notation tend to mimic syntactic cues in natural language. Consider, for example, the interpretation of the morphemic string *light house keeping*. Its ambiguity ({light + house keeping} versus {light house + keeping}) is resolved on the basis of several factors including temporal spacing of its lexical units. (In

written form, of course, the parse is indicated directly by a physical gap.) Thus the untutored predisposition to interpret physical spaces in algebraic notation as syntactic markers may result from an adaptation of a learned linguistic response.†

This final observation provides an answer for one question which has hitherto remained unasked in this report. Why, if students do not develop the fragmentary rules provided by instruction into a viable propositional system, would they be predisposed to discover a complex surface form syntax completely unaided by pedagogic assistance? It may be the case that virtually any surface pattern is more easily apprehended than an incompletely specified propositional system. This explanation should be considered, but it seems to this author that a truly arbitrary surface system would provide little prospect for spontaneous discovery. It seems more likely that students are predisposed to "discover" the system of surface notation cues because of patterns of syntactic analysis which have already been established in natural language. The hypothesis that natural language competencies underlie algebraic syntax skills calls for additional theoretical and empirical investigation.

In its long and gradual evolution the system of algebraic notation has acquired surface features which permit automatic and unconscious parsing of algebraic expressions. There is no evidence in the historical record that such features were deliberately incorporated to facilitate syntactic analysis. Rather, the syntax of algebra, as of natural language, has derived from a spontaneous relationship between the speaker and the language even as it is spontaneously mastered by each new generation of speakers.

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†In fact the temporal spacing between words in oral language is not achieved by an actual break in the flow of speech, but rather by a slight lengthening of word initial and/or word final speech sounds in accordance with intricate but unconscious rules of speech. Additionally, phonetic features such as word initial aspiration for voiceless stops, voicing for glides, and syllabic stress contribute to the identification of internal juncture (Stageberg, 1966, pp. 69–71).

I. IMPLICATIONS FOR RESEARCH

1. Constructivist Perspectives

The constructivist epistemology in mathematics education, as advocated in recent works (eg. Houlihan & Ginsburg, 1981; Cobb & Steffe, 1983; and von Glasersfeld, 1983), holds that the student is to be seen as an active agent in his/her own learning rather than as a passive recipient of predigested knowledge. The present study falls within the purview of constructivist research in mathematics education, since the student has been shown to take an independent (if unconscious) initiative in the development of syntactic skill.

Support for a constructivist epistemology has been provided in such focal mathematics education studies as Erlwanger (1974) and Brown and VanLehn (1980). In such studies, constructive involvement is evidenced in the persistent misconceptions or errors of the neophyte.

Constructivist epistemology, however, applies beyond just the modelling of learning processes. As Cobb and Steffe (1983) observe,

> ... the adult cannot cause the child to have experience qua experience. Further, as the construction of knowledge is based on experience, the adult cannot cause the child to construct knowledge. In a very real sense, children determine not only how but also what mathematics they construct. (p. 88)

In the present study, students were successful in their standard notation syntactic decisions. The differences in nonce performance reflected differences in the nature of the mathematical competence which was acquired, rather than just in the way that it was acquired. Thus, in

the limited sphere of evaluating algebraic expressions, a measure of support has been found for a more radical constructivism.

2. Computer Algebra

Computer manipulation of algebraic formulae has been available on main frame computers for some time. Advances in hardware and software, however, are rapidly bringing advanced technologies to the microcomputer level (e.g. MU MATH, MAXIMA). Some universities (e.g. University of Waterloo) already provide computer manipulation facilities to their students. Anticipating the further availability of symbol manipulators, perhaps at the secondary school level, participants at ICMI V have asked "What effect will the automation of symbolic as well as numeric calculation have on mathematical thought processes" (Proceedings of ICME V, 1986, p. 164)?

Although the technology to manipulate formulae is developing rapidly, graphic representation of symbols is making a less rapid advance. Present programs use such artificial notations as "*" for multiplication "↑" for exponentiation and "↓" for radical. Undoubtedly, further advances in hardware will make accessible to microcomputers the vast stores of memory needed to obtain more natural graphic representation of algebraic expressions. In the meantime, there is the possibility that notational variations will interfere with syntactic skills. For example, Sleeman (1986) reporting on a study of computer algebra notes that:

> Many of the pupils were using consistent mal-rules. Just over half of the 24 pupils we saw mishandled precedence in equations of the form:
>
> $$2+3*X = 9$$
>
> ...(Indeed, more recently we have discovered that 90% of a sample of 13-year-olds had precedence difficulties with *arithmetic* expressions involving the "+" and "*" operators.) (p. 42)

The results of the present study suggest that the distortion of the spatial cues of ordinary notation in the computer notation may account for the observed student difficulties.

Some proposals regarding computer notations have been particularly unmindful of spatial regularities within algebraic notation. For example, Iverson (1980), developer of the computer language APL and implementor of APL algebra utilities, has recommended adoption of an APL-like notation for paper and pencil computations based on perceived inconsistencies:

> Mathematical notation provides perhaps the best-known and best-developed example of language used consciously as a tool of thought. ... Nevertheless, mathematical notation has serious deficiencies. In particular, it lacks universality, and must be interpreted differently according to the topic, according to the author, and even according to the immediate context. (Iverson, 1980, pp. 444-445)

This study suggests the need to also consider the powerful spatial consistencies within standard notation prior to implementing computer algebra in the secondary schools.

3. Field Dependence/Independence

The study revealed individual differences in the bases for syntactic decisions among subjects. Why should some students respond to their instructional situation by constructing a propositional knowledge of syntactic rules which allows transfer of syntactic skill to the nonce notations, while their cohorts are reliant upon surface forms? Individual differences in the ability to translate skills from a familiar context to a new one have been studied by psychologists involved in cognitive style research. Witkin and Goodenough (1981), pioneers in such research, note:

> Subjects identified as field dependent in perception of the upright were found to have greater difficulty in solving that particular class of problems in which the solution depends on taking an element critical for solution out of the context in which it is presented and restructuring the problem material so that the element

is now used in a different context. (p. 17)

The task of transferring syntactic skill to the new nonce environment appears to fall within the "class of problems" which Witkin and Goodenough describe.

The terms "field dependence/independence" originally derived from variations in judgements of vertical self-orientation on the basis of the visual field (field dependence), or alternatively, on the basis of the vestibular stimulation caused by the action of the gravitational force on the human body (field independence). In ordinary circumstances, external and internal stimuli are mutually reinforcing since objects in the visual field are usually aligned to gravity. In many ingenious experimental settings, however, the visual and vestibular cues were displaced from each other and subjects were measured as to which stimulus predominates in decisions of vertical orientation. Eventually, other cognitive indices such as spatial/visualization ability and the ability to find hidden figures embedded within a compelling visual format were correlationally associated with field independence (Maccoby & Jacklin, 1974).

The nonce forms experiment has some apparent similarities to field dependence studies. In ordinary syntactic processing, spatial cues and propositional frameworks are mutually reinforcing and usually lead to adequate decisions. The nonce notation, however, has displaced these cues from one another and the effect upon subjects decisions has been observed. The embedded figures correlations and similar studies have already shown that intellectual processes can mediate in the process of field independent self-orientation. Therefore, the propensity to structure notational stimuli so as to make them transportable to the new context of the nonce notation seems, almost by definition, to be a measure of field independence.

The sex differences found in nonce notation performance support this hypothesis, since

cognitive style and gender are well-documented correlates (Fennema, 1975; Witkin & Goodenough, 1981). The hypothesis that syntactic representation style is a correlate of cognitive style could be empirically tested by appending a test such as the Embedded Figures Test in a replication of the nonce forms study.

4. Sex Differences in Mathematical Achievement

Sex differences in mathematical achievement (especially for more abstract content domains) have consistently been observed in mathematics education studies (eg. NAEP, 1975; NAEP, 1981). Observed correlations between cognitive style and mathematical inclination (Witkin, Moore, Oltman, *et al*, 1977), as well as an apparent relationship between mathematical processes and cognitive style correlates, have provided a source for intriguing conjecture about the nature of these gender differences. However, verifiable explanations have not been forthcoming. For example, Fenemma (1975) speaking with regard to spatial visualization skill notes:

> It appears reasonable, therefore, to hypothesize that since there is a concurrent developmental trend and since tests of spatial visualization ability contain many of the same elements contained in mathematics, the two might be related. Perhaps less adequate spatial visualization ability may partially explain girls inferior performance in mathematics. However, there are no data available which enables (sic) one to accept or to reject this hypothesis. (p. 37)

What is lacking is a firm demonstration that specific mathematical processes actually depend upon the generally operative cognitive structures associated with field independence.

If, as hypothesized above, the propensity to formulate propositional syntactic rules from surface notational cues is a function of cognitive style, then this establishes a link between cognitive style and the specific mathematical task of syntactic analysis. Of course, such a link provides

an explanation of sex differences in mathematical achievement only to the extent that the presence or absence of such propositional rules can be shown to be a determinant of algebraic competence. To some degree, this result is contraindicated by the present study since even subjects dependent upon surface cues were shown to be reasonably proficient at the standard notation expression evaluation tasks. In the concluding section, however, this question is considered in more depth.

5. Spatial Cues in Transformational Rules

The present study has examined syntactic representation in the context of evaluating particular expressions. It is through such evaluation tasks that syntax is normally taught in the schools (when it is taught at all). Syntactic skill, however, is required not only for expression evaluation tasks, but also for application of real number properties in deriving one expression or equation from another.

Consider, for example, the derivation, $(3x)^2 - y^2 = (3x - y)(3x + y)$. In order to produce such a sentence of algebra a number of components of knowledge are necessary. First, the initial expression $(3x)^2 - y^2$, must be correctly parsed, $[(3x)^2] - [y^2]$. Next, the "difference of squares" rule, $a^2 - b^2 = (a - b)(a + b)$, must be known. Finally, strategic knowledge is required to govern the selection and sequencing of transformational rules in order to accomplish specific tasks (for example, if more than one transformational rule can be applied to a given expression, it is necessary to select from among the possibilities).

"Knowing" the difference of squares rule, however, is an ambiguous designation. One may know "$a^2 - b^2 = (a - b)(a + b)$" as a sequence of symbols assigned a positive truth-value in the algebra game (the symbols have no external reference). Alternatively, one

may know that the sentence expresses a true relationship among real numbers ("a" and "b" represent real numbers). Use of the difference of squares rule for transforming expressions, however, requires the ability to match up the particular expression to be transformed with an abstractly represented *syntactic context of application*: $(\text{expression1})^2 - (\text{expression2})^2$ ("a" and "b" are place holders for well-formed expressions within a particular syntactic configuration). It is only if this more abstract representation of syntactic contexts is available that transformational rules can be appropriately applied.

There is abundant evidence that this more abstract knowledge of syntactic contexts is problematic for many students. Students who have learned to simplify fractional expressions (e.g. $\frac{a\cancel{x}}{b\cancel{x}} = \frac{a}{b}$) will often tend to 'cancel' in the wrong syntactic context, $\frac{a + \cancel{x}}{b + \cancel{x}} = \frac{a}{b}$ (Matz, 1980). Similarly, the common occurence of "linearity errors" (Matz, 1980) (e.g. $(x + y)^2 = x^2 + y^2$ and $\sqrt{x + y} = \sqrt{x} + \sqrt{y}$) can be understood as the application of correct distributivity rules in the wrong syntactic context.

These considerations raise the powerful hypothesis that inability to correctly apply transformational rules is the result of a dependence upon spatial cues in syntactic analysis and a corresponding absence of propositional knowledge to support the abstract representation of syntactic context. If this analysis is correct, there is a clear implication for practise: The propositional content of syntactic instruction must be dramatically increased.

As appealing as this hypothesis may be, however, it is not inconceivable that syntactic contexts are represented (at least for some competent algebraists) as generalized spatial configurations, and that application of transformational rules is, essentially, a "shape manipulation" activity.

Consider, for example, the Generalized Distributive Law (GDL) which states: "Any operation right distributes over any one lesser level operation."† In Paper #5 it is argued that the GDL is a transformational rule used in algebraic manipulation. Since this rule employs the notion of "operation level" it is legitimate to ask, as in the present study, whether propositional or spatial versions of the construct underlie psychological representations for fluent algebraists. Presumably, equally precise hypotheses can be formulated for other transformational rules.

One way to obtain correlational evidence on the relationship between algebraic competence and mode of syntactic knowldege would be to replicate the present study taking care to assess each subject's competence on a range of algebraic transformation tasks.‡ The discovery of subjects with a clear dependence upon spatial cues (and a clear absence of sound propositional knowledge of syntax) but with relative competence in application of transformational rules would provide strong support for the adequacy of spatial cues in successful algebraic performance.

We may well find, however, that dependence upon spatial cues is the hallmark of the failing

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†The GDL incorporates the following previously independent real number properties under the rubric of a single law.

Level 2 over Level 1		Level 3 over Level 2
$(a + b)c = ac + bc$		$(ab)^c = a^c b^c$
$(a - b)c = ac - bc$	○	$\left(\frac{a}{b}\right)^c = \frac{a^c}{b^c}$
$\frac{a + b}{c} = \frac{a}{c} + \frac{b}{c}$		$\sqrt[c]{ab} = \sqrt[c]{a}\,\sqrt[c]{b}$
$\frac{a - b}{c} = \frac{a}{c} - \frac{b}{c}$		$\sqrt[c]{\frac{a}{b}} = \frac{\sqrt[c]{a}}{\sqrt[c]{b}}$

‡The letter grade assessment of the Past Algebraic Achievement covariate used in the present study is too crude a measure for this purpose.

student. That the successful algebraist is the one who has internalized sound propositional rules despite the inadequacies of standard instructional practise. That a curriculum which relies on the powerful spatial regularities of standard notation as a vehicle for syntactic learning is a recipe for the continued domination of mathematics by that portion of the population of a more field independent cognitive style.

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K. APPENDIX A

CLOSED NONCE FORMS TEST

Section 1 CAPITAL LETTER Arithmetic

In each of the following ten examples an arithmetic problem has been translated using the following CAPITAL LETTER notation:

aAb = a + b		aSb = a - b	
aMb = ab		aDb = $\frac{a}{b}$	
aEb = a^b		aRb = $\sqrt[a]{b}$	

eg. 3M5 means 3 is multiplied by 5

Calculate the answer to each of these problems in your head. If your answer is one of those listed after the problem then circle that answer in the test booklet. If your answer is not in the list, then write it in the space following.

1) 3E4 = 7, 12, 64, 81, __

2) 5A16 = -11, 11, 21, 80, __

3) 3R8 = 2, 3, 5, 11, __

4) 7M12 = -5, 5, 19, 84, __

5) 22S17 = 5, 13, 14, 39, __

6) 18D6 = -12, 3, 12, 24, __

7) 4E3 = 7, 12, 64, 81, __

8) 26A14 = -40, -12, 12, 40,__

9) 12M4 = 3, 8, 48, 60, __

10) 7E1 = 1, 6, 7, 8, __

Please do not look ahead when you have finished.

Wait for instructions from the tester.

Section 2 CAPITAL LETTER Algebra

In each of the following ten examples an algebraic expression has been translated using the CAPITAL LETTER notation:

aAb = a + b		aSb = a - b	
aMb = ab		aDb = $\frac{a}{b}$	
aEb = a^b		aRb = $\sqrt[a]{b}$	

Evaluate each of the algebraic expressions in your head. If your answer is one of those listed after the problem then circle that answer in the test booklet. If your answer is not in the list, then write it in the space following.

In all of these algebraic expressions x = 2.

1) 3MxA4 = 10, 13, 18, 48, __

2) 2MxE3 = 10, 16, 27, 64, __

3) 5M(2Ax) = 12, 20, 100, 625, __

4) 5A3Mx = 4, 11, 14, 16, __

5) xE4S2 = 3, 4, 6, 14, __

6) 3E4SxA1 = 3,10,27,78,80, __

7) 1A3MxE2 = 13,16,37,49,64, __

8) 10S3MxA1 = 1,3,5,15,21, __

9) 6A(3MxS2) = 0,6, 9, 10, 16,__

10) 6SxE2A1 = -2,1,3,17,64,__

Please do not look ahead when you have finished.

Wait for instructions from the tester.

Section 3 Algebra(usual notation)

Evaluate each of the following algebraic expressions in your head. If your answer is one of those listed after the problem then circle that answer in the test booklet. If your answer is not in the list, then write it in the space following.

In all of these algebraic expressions $x = 2$.

1) $5x + 7 =$ 3, 17, 32, 45, 70, __

2) $5x^2 =$ 5, 10, 20, 49, 100, __

3) $4(6 + x) =$ 14, 24, 26, 28, 32, __

4) $3 + 4x =$ 9, 11, 13, 14, 24, __

5) $x^3 - 2 =$ 1, 2, 6, 7, 16, __

6) $2^4 - x + 1 =$ 2, 5, 8, 13, 15, __

7) $3 + 2x^2 =$ 11, 19, 20, 49, 100, __

8) $19 - 4x + 2 =$ 3, 9, 13, 32, 60, __

9) $3 + (7x - 2) =$ 0, 3, 15, 18, 21, __

10) $5 - x^2 + 1 =$ -3, 0, 2, 10, 27, __

Please do not look ahead when you have finished.

Wait for instructions from the tester.

L. APPENDIX B

SPACED NONCE FORMS TEST

Section 1 CAPITAL LETTER Arithmetic

In each of the following ten examples an arithmetic problem has been translated using the CAPITAL LETTER notation:

a A b = a + b

a S b = a − b

a M b = ab

a D b = $\frac{a}{b}$

aEb = a^b

aRb = $\sqrt[a]{b}$

eg. 3 M 5 means 3 is multiplied by 5

Calculate the answer to each of these problems in your head. If your answer is one of those listed after the problem then circle that answer in the test booklet. If your answer is not in the list, then write it in the space following.

1) 3E4 = 7, 12, 64, 81, __

2) 5 A 16 = -11,11,21,80,__

3) 3R8 = 2, 3, 5, 11, __

4) 7 M 12 = -5, 5, 19, 84, __

5) 22 S 17 = 5, 13, 14, 39, __

6) 18 D 6 = -12, 3, 12, 24, __

7) 4E3 = 7, 12, 64, 81, __

8) 26 A 14 = -40,-12,12,40,__

9) 12 M 4 = 3, 8, 48, 60, __

10) 7E1 = 1, 6, 7, 8, __

Please do not look ahead when you have finished.

Wait for instructions from the tester.

Section 2 CAPITAL LETTER Algebra

In each of the following ten examples an algebraic expression has been translated using the following CAPITAL LETTER notation:

a A b = a + b	a S b = a − b
a M b = ab	a D b = $\frac{a}{b}$
aEb = a^b	aRb = $\sqrt[a]{b}$

Evaluate each of the algebraic expressions in your head. If your answer is one of those listed after the problem then circle that answer in the test booklet. If your answer is not in the list, then write it in the space following.

In all of these algebraic expressions x = 2.

1) 3 M x A 4 = 10,13,18,48,__

2) 2 M xE3 = 10,16,27,64,__

3) 5 M (2 A x) = 12,20,100,625,__

4) 5 A 3 M x = 4,11,14,16,__

5) xE4 S 2 = 3,4,6,14,__

6) 3E4 S x A 1 = 3,10,27,78,80,__

7) 1 A 3 M xE2 = 13,16,37,49,64,__

8) 10 S 3 M x A 1 = 1,3,5,15,21,__

9) 6 A (3 M x S 2) = 0,6,9,10,16,__

10) 6 S xE2 A 1 = -2,1,3,

Please do not look ahead when you have finished.

Wait for instructions from the tester.

Section 3 Algebra(usual notation)

Evaluate each of the following algebraic expressions in your head. If your answer is one of those listed after the problem then circle that answer in the test booklet. If your answer is not in the list, then write it in the space following.

In all of these algebraic expressions $x = 2$.

1) $5x + 7 =$ 3, 17, 32, 45, 70, __

2) $5x^2 =$ 5, 10, 20, 49, 100, __

3) $4(6 + x) =$ 14, 24, 26, 28, 32, __

4) $3 + 4x =$ 9, 11, 13, 14, 24, __

5) $x^3 - 2 =$ 1, 2, 6, 7, 16, __

6) $2^4 - x + 1 =$ 2, 5, 8, 13, 15, __

7) $3 + 2x^2 =$ 11, 19, 20, 49, 100, __

8) $19 - 4x + 2 =$ 3, 9, 13, 32, 60, __

9) $3 + (7x - 2) =$ 0, 3, 15, 18, 21, __

10) $5 - x^2 + 1 =$ -3, 0, 2, 10, 27, __

Please do not look ahead when you have finished.

Wait for instructions from the tester.

M. APPENDIX C

POST HOC QUESTIONNAIRE: CLOSED NONCE FORM

Section 4 Answer Check PAGE 1

Recall that question 7 in Section 2 was 1A3MxE2 (x = 2).
Without changing it, please go back and check your answer.
1) Was your answer 13? Yes __ or No __. [Check yes or no].
If your answer was 13, then please proceed to Page 2 now.

If your answer was not 13, then please go on to the next question.

Recall that question 8 in Section 2 was 10S3MxA1 (x = 2).
Without changing it, please go back and check your answer.
1) Was your answer 5? Yes __ or No __. [Check yes or no].
If your answer was 5, then please proceed to Page 4 now.

If your answer was not 5, then please go on to the next question.

Recall that question 10 in Section 2 was 6SxE2A1 (x = 2).
Without changing it, please go back and check your answer.
1) Was your answer 3? Yes __ or No __. [Check yes or no].
If your answer was 3, then please proceed to Page 6 now.

If your answer was not 3, then please go on to the next question.

Recall that question 2 in Section 2 was 2MxE3 (x = 2).
Without changing it, please go back and check your answer.
1) Was your answer 16? Yes __ or No __. [Check yes or no].
If your answer was 16, then please proceed to Page 8 now.

If your answer was not 16, then please go on to the next question.

Recall that question 4 in Section 2 was 5A3Mx (x = 2).
Without changing it, please go back and check your answer.
1) Was your answer 11? Yes __ or No __. [Check yes or no].
If your answer was 11, then please proceed to Page 10 now.

If your answer was not 11, then please put your pen down now and close your test booklet.

Than you for having participated in the study.

Section 4 PAGE 2

If you don't understand the questions on this page then please ask for assistance from the tester.

I would like to find out how you got 13 as your answer for

1A3MxE2.

Did you guess? yes __ or no __. [Check yes or no]

If you did guess, then please close your test booklet now.

Thank you for your cooperation in the study.

If you didn't guess the answer, then did you use this order of operations?

xE2 = 4, 3M4 = 12, 1A12 = 13.

yes __ or no __ [check yes or no]

If you did use this order of operations, then please turn to the next page now.

If you didn't use this order of operations then how did you figure out the answer? (explain)

_____

_____

_____

Please put your pen down now and close your test booklet.

Than you for having participated in the study.

Section 4 PAGE 3

I would like to know what went through your mind as you figured out the order of operations xE2 = 4, 3M4 = 12, 1A12 = 13.

Think Back To When You Were Doing The Problem A Few Minutes Ago.

1) For the problem 1A3MxE2, did you imagine or visualize or picture in your mind $1 + 3x^2$?

yes __ or no __ or not sure __ [check one]

2) Did you consciously remember rules that tell you what the order of operations is supposed to be?

yes __ or no __ or not sure __ [check one]

If yes then what rules did you use? _____

_____

_____

3) Please try to explain in your own words what went through your mind as you figured out the order of operations.

_____

_____

_____

_____

Please put your pen down now and close your test booklet.

Thank you for having participated in the study.

Section 4 PAGE 4

If you don't understand the questions on this page then please ask for assistance from the tester.

I would like to find out how you got 5 as your answer for

10S3MxA1

Did you guess? yes __ or no __. [Check yes or no]

If you did guess, then please close your test booklet now.

Thank you for your cooperation in the study.

If you didn't guess the answer, then did you use this order of operations?

3Mx = 6, 10S6 = 4, 4A1 = 5.

yes __ or no __ [check yes or no]

If you did use this order of operations, then please turn to the next page now.

If you didn't use this order of operations then how did you figure out the answer? (explain)

_____

_____

_____

Please put your pen down now and close your test booklet.

Than you for having participated in the study.

Section 4 PAGE 5

I would like to know what went through your mind as you figured out the order of operations 3Mx = 6, 10S6 = 4, 4A1 = 5.

Think Back To When You Were Doing The Problem A Few Minutes Ago.

1) For the problem 10S3MxA1, did you imagine or visualize or picture in your mind 10 - 3x + 1 ?

yes __ or no __ or not sure __ [check one]

2) Did you consciously remember rules that tell you what the order of operations is supposed to be?

yes __ or no __ or not sure __ [check one]

If yes then what rules did you use? _____

_____

_____

3) Please try to explain in your own words what went through your mind as you figured out the order of operations.

_____

_____

_____

_____

Please put your pen down now and close your test booklet.

Thank you for having participated in the study.

Section 4 PAGE 6

If you don't understand the questions on this page then please ask for assistance from the tester.

I would like to find out how you got 3 as your answer for

6SxE2A1.

Did you guess? yes __ or no __. [Check yes or no]

If you did guess, then please close your test booklet now.

Thank you for your cooperation in the study.

If you didn't guess the answer, then did you use this order of operations?

xE2 = 4, 6S4 = 2, 2A1 = 3.

yes __ or no __ [check yes or no]

If you did use this order of operations, then please turn to the next page now.

If you didn't use this order of operations then how did you figure out the answer? (explain)

_____

_____

_____

Please put your pen down now and close your test booklet.

Than you for having participated in the study.

Section 4 PAGE 7

I would like to know what went through your mind as you figured out the order of operations xE2 = 4, 6S4 = 2, 2A1 = 3.

Think Back To When You Were Doing The Problem A Few Minutes Ago.

1) For the problem 6SxE2A1, did you imagine or visualize or picture in your mind $6 - x^2 + 1$?

yes __ or no __ or not sure __ [check one]

2) Did you consciously remember rules that tell you what the order of operations is supposed to be?

yes __ or no __ or not sure __ [check one]

If yes then what rules did you use? _____

_____

_____

3) Please try to explain in your own words what went through your mind as you figured out the order of operations.

_____

_____

_____

_____

Please put your pen down now and close your test booklet.

Thank you for having participated in the study.

Section 4 PAGE 8

If you don't understand the questions on this page then please ask for assistance from the tester.

I would like to find out how you got 16 as your answer for

2MxE3.

Did you guess? yes __ or no __. [Check yes or no]

If you did guess, then please close your test booklet now.

Thank you for your cooperation in the study.

If you didn't guess the answer, then did you use this order of operations?

xE3 = 8, 2M8 = 16.

yes __ or no __ [check yes or no]

If you did use this order of operations, then please turn to the next page now.

If you didn't use this order of operations then how did you figure out the answer? (explain)

_____

_____

_____

Please put your pen down now and close your test booklet.

Than you for having participated in the study.

Section 4 PAGE 9

I would like to know what went through your mind as you figured out the order of operations xE3 = 8, 2M8 = 16.

Think Back To When You Were Doing The Problem A Few Minutes Ago.

1) For the problem 2MxE3, did you imagine or visualize or picture in your mind $2x^3$?

yes __ or no __ or not sure __ [check one]

2) Did you consciously remember rules that tell you what the order of operations is supposed to be?

yes __ or no __ or not sure __ [check one]

If yes then what rules did you use? _____

_____

_____

3) Please try to explain in your own words what went through your mind as you figured out the order of operations.

_____

_____

_____

_____

Please put your pen down now and close your test booklet.

Thank you for having participated in the study.

Section 4 PAGE 10

If you don't understand the questions on this page then please ask for assistance from the tester.

I would like to find out how you got 11 as your answer for

5A3Mx.

Did you guess? yes __ or no __. [Check yes or no]

If you did guess, then please close your test booklet now.

Thank you for your cooperation in the study.

If you didn't guess the answer, then did you use this order of operations?

3Mx = 6, 5A6 = 11.

yes __ or no __ [check yes or no]

If you did use this order of operations, then please turn to the next page now.

If you didn't use this order of operations then how did you figure out the answer? (explain)

_____

_____

_____

Please put your pen down now and close your test booklet.

Than you for having participated in the study.

Section 4 PAGE 11

I would like to know what went through your mind as you figured out the order of operations 3Mx = 6, 5A6 = 11.

Think Back To When You Were Doing The Problem A Few Minutes Ago.

1) For the problem 5A3Mx, did you imagine or visualize or picture in your mind 5 + 3x ?

yes __ or no __ or not sure __ [check one]

2) Did you consciously remember rules that tell you what the order of operations is supposed to be?

yes __ or no __ or not sure __ [check one]

If yes then what rules did you use? _____

_____

_____

3) Please try to explain in your own words what went through your mind as you figured out the order of operations.

_____

_____

_____

_____

Please put your pen down now and close your test booklet.

Thank you for having participated in the study.

N. APPENDIX D

POST HOC QUESTIONNAIRE: SPACED NONCE FORM

Section 4 Answer Check PAGE 1

Recall that question 7 in Section 2 was 1 A 3 M xE2 (x = 2).
Without changing it, please go back and check your answer.
1) Was your answer 13? Yes __ or No __. [Check yes or no].
If your answer was 13, then please proceed to Page 2 now.

If your answer was not 13, then please go on to the next question.

Recall that question 8 in Section 2 was 10 S 3 M x A 1 (x = 2).
Without changing it, please go back and check your answer.
1) Was your answer 5? Yes __ or No __. [Check yes or no].
If your answer was 5, then please proceed to Page 4 now.

If your answer was not 5, then please go on to the next question.

Recall that question 10 in Section 2 was 6 S xE2 A 1 (x = 2).
Without changing it, please go back and check your answer.
1) Was your answer 3? Yes __ or No __. [Check yes or no].
If your answer was 3, then please proceed to Page 6 now.

If your answer was not 3, then please go on to the next question.

Recall that question 2 in Section 2 was 2 M xE3 (x = 2).
Without changing it, please go back and check your answer.
1) Was your answer 16? Yes __ or No __. [Check yes or no].
If your answer was 16, then please proceed to Page 8 now.

If your answer was not 16, then please go on to the next question.

Recall that question 4 in Section 2 was 5 A 3 M x (x = 2).
Without changing it, please go back and check your answer.
1) Was your answer 11? Yes __ or No __. [Check yes or no].
If your answer was 11, then please proceed to Page 10 now.

If your answer was not 11, then please put your pen down now and close your test booklet.

Than you for having participated in the study.

Section 4 PAGE 2

If you don't understand the questions on this page then please ask for assistance from the tester.

I would like to find out how you got 13 as your answer for

1 A 3 M xE2.

Did you guess? yes __ or no __. [Check yes or no]

If you did guess, then please close your test booklet now.

Thank you for your cooperation in the study.

If you didn't guess the answer, then did you use this order of operations?

xE2 = 4, 3 M 4 = 12, 1 A 12 = 13.

yes __ or no __ [check yes or no]

If you did use this order of operations, then please turn to the next page now.

If you didn't use this order of operations then how did you figure out the answer? (explain)

_____

_____

_____

Please put your pen down now and close your test booklet.

Than you for having participated in the study.

Section 4 PAGE 3

I would like to know what went through your mind as you figured out the order of operations xE2 = 4, 3 M 4 = 12, 1 A 12 = 13.

Think Back To When You Were Doing The Problem A Few Minutes Ago.

1) For the problem 1 A 3 M xE2, did you imagine or visualize or picture in your mind $1 + 3x^2$?

yes __ or no __ or not sure __ [check one]

2) Did you consciously remember rules that tell you what the order of operations is supposed to be?

yes __ or no __ or not sure __ [check one]

If yes then what rules did you use? _____

_____

_____

3) Please try to explain in your own words what went through your mind as you figured out the order of operations.

_____

_____

_____

_____

Please put your pen down now and close your test booklet.

Thank you for having participated in the study.

Section 4 PAGE 4

If you don't understand the questions on this page then please ask for assistance from the tester.

I would like to find out how you got 5 as your answer for

10 S 3 M x A 1

Did you guess? yes __ or no __. [Check yes or no]

If you did guess, then please close your test booklet now.

Thank you for your cooperation in the study.

If you didn't guess the answer, then did you use this order of operations?

3 M x = 6, 10 S 6 = 4, 4 A 1 = 5.

yes __ or no __ [check yes or no]

If you did use this order of operations, then please turn to the next page now.

If you didn't use this order of operations then how did you figure out the answer? (explain)

_____

_____

_____

Please put your pen down now and close your test booklet.

Than you for having participated in the study.

Section 4 PAGE 5

I would like to know what was in your mind as you figured out the order of operations 3 M x = 6, 10 S 6 = 4, 4 A 1 = 5.

Think Back To When You Were Doing The Problem A Few Minutes Ago.

1) For the problem 10 S 3 M x A 1, did you imagine or visualize or picture in your mind 10 - 3x + 1 ?

yes __ or no __ or not sure __ [check one]

2) Did you consciously remember rules that tell you what the order of operations is supposed to be?

yes __ or no __ or not sure __ [check one]

If yes then what rules did you use? _____

_____

_____

3) Please try to explain in your own words what went through your mind as you figured out the order of operations.

_____

_____

_____

_____

Please put your pen down now and close your test booklet.

Thank you for having participated in the study.

Section 4 PAGE 6

If you don't understand the questions on this page then please ask for assistance from the tester.

I would like to find out how you got 3 as your answer for

6 S xE2 A 1.

Did you guess? yes __ or no __. [Check yes or no]

If you did guess, then please close your test booklet now.

Thank you for your cooperation in the study.

If you didn't guess the answer, then did you use this order of operations?

xE2 = 4, 6 S 4 = 2, 2 A 1 = 3.

yes __ or no __ [check yes or no]

If you did use this order of operations, then please turn to the next page now.

If you didn't use this order of operations then how did you figure out the answer? (explain)

_____

_____

_____

Please put your pen down now and close your test booklet.

Than you for having participated in the study.

Section 4 PAGE 7

I would like to know what went through your mind as you figured out the order of operations xE2 = 4, 6 S 4 = 2, 2 A 1 = 3.

Think Back To When You Were Doing The Problem A Few Minutes Ago.

1) For the problem 6 S xE2 A 1, did you imagine or visualize or picture in your mind $6 - x^2 + 1$?

yes __ or no __ or not sure __ [check one]

2) Did you consciously remember rules that tell you what the order of operations is supposed to be?

yes __ or no __ or not sure __ [check one]

If yes then what rules did you use? _____

_____

_____

3) Please try to explain in your own words what went through your mind as you figured out the order of operations.

_____

_____

_____

_____

Please put your pen down now and close your test booklet.

Thank you for having participated in the study.

Section 4 PAGE 8

If you don't understand the questions on this page then please ask for assistance from the tester.

I would like to find out how you got 16 as your answer for

2 M xE3.

Did you guess? yes __ or no __. [Check yes or no]

If you did guess, then please close your test booklet now.

Thank you for your cooperation in the study.

If you didn't guess the answer, then did you use this order of operations?

xE3 = 8, 2 M 8 = 16.

yes __ or no __ [check yes or no]

If you did use this order of operations, then please turn to the next page now.

If you didn't use this order of operations then how did you figure out the answer? (explain)

_____

_____

_____

Please put your pen down now and close your test booklet.

Than you for having participated in the study.

Section 4 PAGE 9

I would like to know what went through your mind as you figured out the order of operations xE3 = 8, 2 M 8 = 16.

Think Back To When You Were Doing The Problem A Few Minutes Ago.

1) For the problem 2 M xE3, did you imagine or visualize or picture in your mind $2x^3$?

yes __ or no __ or not sure __ [check one]

2) Did you consciously remember rules that tell you what the order of operations is supposed to be?

yes __ or no __ or not sure __ [check one]

If yes then what rules did you use? _____

_____

_____

3) Please try to explain in your own words what went through your mind as you figured out the order of operations.

_____

_____

_____

_____

Please put your pen down now and close your test booklet.

Thank you for having participated in the study.

Section 4 PAGE 10

If you don't understand the questions on this page then please ask for assistance from the tester.

I would like to find out how you got 11 as your answer for

5 A 3 M x.

Did you guess? yes __ or no __. [Check yes or no]

If you did guess, then please close your test booklet now.

Thank you for your cooperation in the study.

If you didn't guess the answer, then did you use this order of operations?

3 M x = 6, 5 A 6 = 11.

yes __ or no __ [check yes or no]

If you did use this order of operations, then please turn to the next page now.

If you didn't use this order of operations then how did you figure out the answer? (explain)

_____

_____

_____

Please put your pen down now and close your test booklet.

Than you for having participated in the study.

Section 4 PAGE 11

I would like to know what went through your mind as you figured out the order of operations 3 M x = 6, 5 A 6 = 11.

Think Back To When You Were Doing The Problem A Few Minutes Ago.

1) For the problem 5 A 3 M x, did you imagine or visualize or picture in your mind 5 + 3x ?

yes __ or no __ or not sure __ [check one]

2) Did you consciously remember rules that tell you what the order of operations is supposed to be?

yes __ or no __ or not sure __ [check one]

If yes then what rules did you use? _____

_____

_____

3) Please try to explain in your own words what went through your mind as you figured out the order of operations.

_____

_____

_____

_____

Please put your pen down now and close your test booklet.

Thank you for having participated in the study.

O. APPENDIX E

NONCE TEST TITLE PAGES

MENTAL ARITHMETIC AND ALGEBRA:

A Special Notation Questionnaire

Location: (school or organization) _____

Grade: (if applicable) _____

Section: (if applicable) ____

Sex: male__ or female__ [check one]

Computer Experience: Do you have experience in programming a computer? none__, some __, quite a bit __. [check one]

Algebra Experience (high school graduates only):

In your last high school algebra course, did you do above average ("A" or "B")__, average ("C+" or "C")__, or below average ("C-", "P" or "F")__? [check one]

Please Do Not Look Ahead Until Instructed To.

MENTAL ARITHMETIC AND ALGEBRA:

A Special Notation Questionnaire

Organization: Are you a member of the A.P.E.B.C.?
Yes__ or No__ [check one]

In what year did you qualify as an engineer? _____

Sex: male__ or female__ [check one]

Computer Experience: Do you have experience in programming a computer? none__, some __, quite a bit __. [check one]

Algebra Experience:
In your last high school algebra course, did you do above average ("A" or "B")__, average ("C+" or "C")__, or below average ("C-", "P" or "F")__? [check one]

Algebra Use:
Are you called upon to utilize your elementary algebra skills

at the work place? Never__, Less than 4 or 5 times a year__, Once or twice a month__, Almost every week or more__. [check one]

outside of work? Never__, Less than 4 or 5 times a year__, Once or twice a month__, Almost every week or more__. [check one]

Please Do Not Look Ahead Until Instructed To Do So.

V. THE CULTURE OF THE CURRICULUM

Paper #5

Note

As outlined in the introductory pages of the Dissertation, the research perspective adopted in the dissertation differs from those which are standardly held in mathematics education research, and in mathematics education more generally. The previous papers have explored this perspective within a research context. The present paper explicates the linguistic perspective of algebra within a curricular context.

In keeping with the dissertation format of quasi-independent "chapters," the discussion in this paper is centred around a single question taken from the grammar (Paper #3, page 96). The perspective elucidated, however, is more generally representative of the algebraic grammar.

The broad goals and objectives of mathematics education are determined by political, social, and ecomomic forces in society. The implementation of these objectives, however, is the domain of the community of mathematics educators including teachers, administrators, teacher trainers, researchers, textbook authors and such mathematicians as have entered the discussion. As in any community, the mathematics education community has its own traditions, belief systems, and social organization through which it expresses itself and accomplishes its objectives. The structure of a community, however, need not be obvious even to its own members. Indeed, acculturation into a community is largely accomplished through unconscious adaptations.

Seymour Sarason (1971) introduced the phrase, "The Culture of the School" in his examination of the school community and its apparent resistance to innovations initiated from without. The present paper examines some of the dominant belief systems within the mathematics education community which help shape the elementary algebra curriculum. As with Sarason, the main attempt here is to explore some of the implicit assumptions in the mathematics education community so that change may be facilitated.

The paper is divided into three parts, each of which addresses a major theme. The theme of Part I (Instruction) concerns the structuring of educational goals within the mathematics curriculum. To what degree is a specific instructional intervention directed towards a specific goal? To what degree is the curriculum "holistic," addressing a broad range of goals simultaneously?

The second Part (Psychology) concerns the acquisition of skill and knowledge. The term "constructivism," more and more in vogue at research and professional meetings, expresses the

notion that the student is the author of his or her own knowledge. What, then, should be the teacher's role in structuring learning for students? Can we direct, or even uncover, the knowledge which the student must construct for himself or herself?

The third Part (Mathematical Theory) concerns the role of mathematical theory in the mathematics curriculum. Does the study of theory enhance or retard the acquisition of skill? The 'new math' era of the 1960's and 1970's saw the elevation of abstract representations of number systems to the fore in the mathematics curriculum. Did this movement reflect implicit commitments regarding a connection between theory and skill? Has that belief system been analysed and evaluated?

Despite the generality of these questions, the discussion to follow is rooted in specific instances and problems. Indeed, the $(x + y)^2 = x^2 + y^2$ problem so characteristic of beginning algebra wends its way throughout the whole paper. Part I begins with a brief immersion into the culture of classroom remediation.

Part I: INSTRUCTION

Francis has just made that perennial algebraic gaffe, $(x + y)^2 = x^2 + y^2$. Three hypothetical teachers bend over to offer assistance.

1. Teacher A

> "No, $(x + y)^2$ is not equal to $x^2 + y^2$. Consider for example when x has the value 1 and y has the value 2. Then the left hand side of the equation is 9,

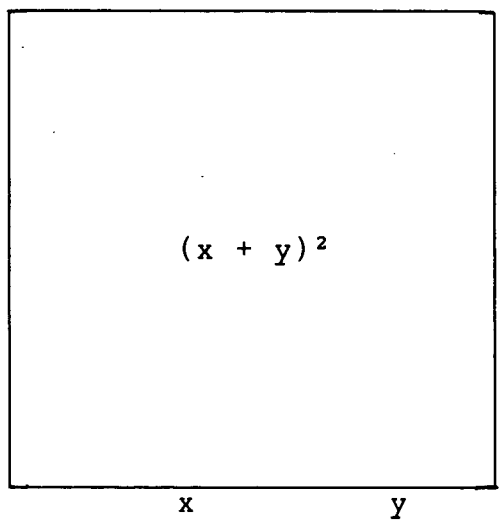
and the right hand side is 5. They are not the same. Try it with x and y values of your own choosing. ... See they are not the same."

2. Teacher B

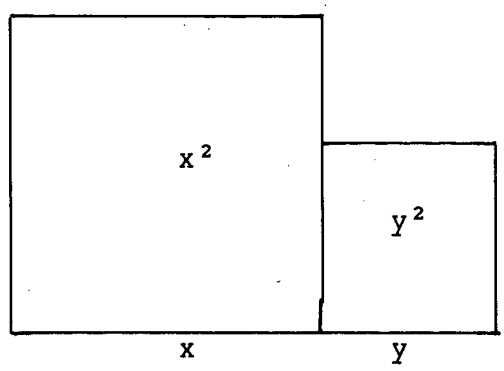
"Imagine that you had a length

x = _____ and a length

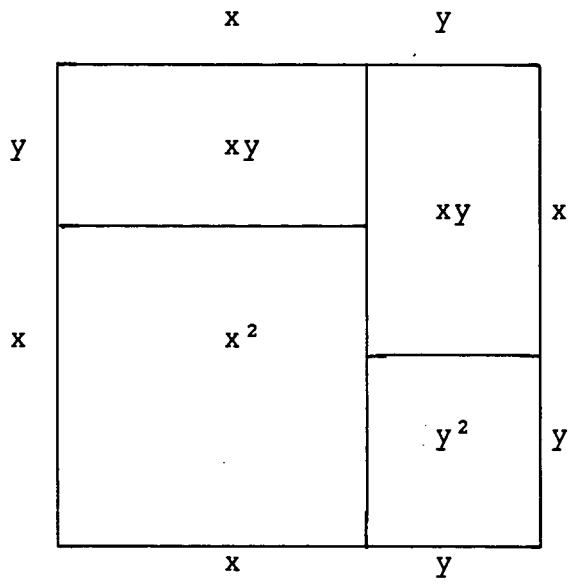
y = _____ . Can you show me a picture of $(x + y)^2$?



Good. And $x^2 + y^2$?



Good. You can see from the diagram that $(x + y)^2$ is greater than $x^2 + y^2$. In fact $(x + y)^2$ is two rectangles bigger than $x^2 + y^2$, each one of area xy.



So you see that $(x + y)^2$ is equal to $x^2 + 2xy + y^2$, not to $x^2 + y^2$."

3. Teacher C

"Consider $(x + y)^2$. By the definition of squaring this is equal to $(x + y)(x + y)$. Recall that the distributive property allows the replacement of an expression "a(b + c)" by the expression "ab + ac". Considering the first $(x + y)$ as "a", and the second $(x + y)$ as "(b + c)", we get $(x + y)(x + y) = (x + y)x + (x + y)y$. By right distributivity, this is equal to $x^2 + yx + xy + y^2$. But this is just another way to write $x^2 + 2xy + y^2$."

This selection of remedial strategies outlines some of the methods used for the $(x + y)^2 = x^2 + y^2$ problem. It is not meant to provide an exhaustive list. It is merely an illustration of the diversity of educational techniques employed.

A. THE DIVERSITY OF EDUCATIONAL PRESCRIPTION

Imagine if instead of the above scenario, Francis had a medical problem and Teachers A, B, and C were actually doctors. Would we not be a bit concerned about the wide variation in prescription? We might attribute it to differential diagnoses —the doctors have conflicting interpretations regarding the state of the patient. Alternatively, there may be a single diagnosis, but physiological or biochemical theory may point to several distinct methods for treating the condition. Finally, in charity, we might hope that the practitioners have comprehensive malpractice insurance coverage.

In the case of mathematics education, what explains the profusion of remedial and instructional methods for dealing with a single topic or problem? It would be comforting to think that the different methods of the mathematics curriculum address different conceptions of the student's underlying problems. But perhaps it is unrealistic to imagine that with no formal diagnostic procedures teachers base their minute-to-minute classroom interventions on the character of each individual's cognitive situation.

Could alternative analyses of the task, rather than of the student, underlie our diversity of method. But then what are these analyses? In the above example, for instance, where is the detailed analysis of algebraic symbol skills that assigns a particular role to geometric conceptions, or to axiomatic procedures? Why do such models not figure more prominently in mathematics education journals and conferences where techniques are discussed or exchanged?

Educational practice tends to lack the rigorous theoretical infrastructure which would make this explanation more plausible.

Incompetence is not the explanation either. For every Teacher A, B, or C, there are Teachers X, Y, and Z who will only say: "Francis, $(x + y)^2$ is not equal to $x^2 + y^2$. It is equal to $x^2 + 2xy + y^2$. Try to remember that in the future." No. It is uninspired teaching which might have a remarkable uniformity.

If alternative diagnoses, alternative task analyses and incompetence are not adequate explanations, then how *can* one explain wide variations in educational prescription? Let's examine the three remedial samples above for clues.

B. ANALYSIS OF THE THREE METHODS

Teacher A focuses the student on the referential character of algebraic variables. "x" and "y" represent numbers, not words or just meaningless marks on a page.

In translating a real life problem, or a textbook word problem into algebraic equations, it is important that the referential nature of variables be well understood. Furthermore, as research on the now legendary "students and professors" problem has shown (Clement, Lochhead & Monk, 1981; Rosnick, 1981; Clement, 1982), this understanding is absent for a large proportion of high school and university mathematics students.

Teacher A also provides practise in the evaluation of algebraic expressions. This too is an important task with which many students experience difficulty.

Teacher B also addresses referential aspects of algebraic symbols. In this case, the variable symbol is tied to a physical referent —length. That abstract ideas should be tied to concrete representations is perhaps the most widely advocated precept of mathematics education. (See for example, Brownell, 1928; Bruner, 1960; and Suydam & Dessart, 1976, p. 6)

Teacher B also presents the student in a most elegant manner with evidence of the interconnectedness of algebra and geometry. The call for a unified approach to the mathematical sciences goes back at least as far as the "new math" era (College Entrance Examination Board, 1959).

Teacher C illustrates axiomatic procedures and deductive reasoning, and provides an introduction to the formal theory of rational numbers. Each of these are well established goals of mathematics education. The development of logical reasoning skills scores high on inventories of teachers' instructional objectives. Furthermore, historical records show that when Euclidean Geometry was dropped as a compulsory course by many North American jurisdictions, abstract algebra was intended to fill the need for rigorous axiomatic treatment.

> One way to foster an emphasis upon understanding and meaning in the teaching of algebra is through the introduction of instruction in deductive reasoning. The Commission [on Mathematics] is firmly of the opinion that deductive reasoning should be taught in all courses in school mathematics and not in geometry alone. (College Entrance Examination Board, 1959, p23)

These analyses demonstrate that the techniques introduced for remediation of the

$(x + y)^2 = x^2 + y^2$ problem actually address a whole range of vital objectives in the mathematics curriculum. But does some deficit regarding these particular objectives actually lie at the root of students' difficulty? Suspend your disbelief momentarily as the novel answer "No!" is considered.

C. THE "HOLISTIC" CURRICULUM

Educators are accustomed to thinking of mathematics education as a rational endeavour. The usual notion of a linear curriculum or even a spiral curriculum implies an orderly progression through the content. The assumption is that students are grappling with a rational subject matter. Teachers possess a rational understanding of that subject matter. The attempt is to assist students to create a similar rational structure of knowledge within their own cognition. Teaching, therefore, is an organized and directed activity. Students' questions have answers. Teachers' actions have conscious intentions.

Consider an alternative viewpoint. Algebraic symbol manipulation is accomplished by means of covert knowledge. Skillful symbol manipulators have highly structured knowledge; however, introspective access to this knowledge is extremely limited. One knows how to do algebra, but not necessarily how algebra is done.

From this viewpoint teaching and learning are not rational exchanges. Each individual grapples with algebraic language on his or her own. Some students eventually master algebra, others do not. A supportive learning environment provides plenty of opportunity to grapple with algebraic language. It does not provide an expository account of algebraic knowledge, because that account is largely unavailable.

Publisher Dale Seymour (1984) in a Soundoff Column for the Mathematics Teacher has advocated a more diffuse, "holistic approach... [which] requires teachers continually to tie topics together" (p. 498). It is noteworthy that, as reflected in his title, We Can Dream, Can't We?, Seymour believes that this desirable state of affairs is remote from present educational practise.

The present viewpoint holds that the holistic curriculum is largely in place. Good teachers such as Teachers A, B & C above have an intuitive grasp of key themes and objectives of the algebra curriculum. Their teaching is enriched by the continuous interweaving and interrelating of these diverse themes and topics. The implication of this viewpoint, however, is that mathematics instruction is less of a rational, directed activity than may previously have been assumed.

To support this viewpoint, a fourth remedial strategy for the $(x + y)^2 = x^2 + y^2$ problem will be presented in Part II. In this case, however, it will be argued that the method really does address the underlying root cause of students' difficulty. The psychological analysis of the problem points to the covert nature of mathematical rules and mathematical learning. In Part III this leads to reassessment of the connection between mathematical theory and mathematical skill.

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The Culture of the Curriculum

Part II: PSYCHOLOGY

In Part I of this article, three remedial techniques for dealing with the $(x + y)^2 = x^2 + y^2$ problem were presented. An analysis of these methods revealed that they actually address important yet diverse concerns and objectives in the mathematics curriculum. But what of the actual causes of the $(x + y)^2 = x^2 + y^2$ problem? Is it possible to precisely understand and correct the underlying confusion of which this error is an indication?

In this Part, a fourth remedial technique is shown. This method has appeared in a previous issue of the Mathematics Teacher, except that whereas Schwartzman (1977) apologized for the "wordiness" of his technique and suggested that others might be equally valid, it will be argued here that this method really does address itself to the source of students' difficulty. His method raises basic questions about the nature of mathematical thought and mathematical learning which are discussed below, and about the influence of mathematical theory on curriculum which is pursued in Part III.

E. THE GENERALIZED DISTRIBUTIVE LAW

Schwartzman's method for dealing with the $(x + y)^2 = x^2 + y^2$ error, is to teach what amounts to a Generalized Distributive Law. This law is based on his simple hierarchy of operations:

Level 1 operations are addition and subtraction
Level 2 operations are multiplication and division
Level 3 operations are exponentiation and radical

(In this hierarchy, Level 3 is said to be higher than level 2 which is higher than level 1.)

The Generalized Distributive Law (GDL) states, simply:

"Any operation right distributes over any one lesser level operation."

Specifically, exponentiation and radical (Level 3) right distribute over multiplication and division (level 2), which in turn, right distribute over addition and subtraction (Level 1).

The GDL incorporates the following previously independent real number properties under the rubric of a single law.

Level 2 over Level 1	Level 3 over Level 2
$(a + b)c = ac + bc$	$(ab)^c = a^c b^c$
$(a - b)c = ac - bc$	$\left(\frac{a}{b}\right)^c = \frac{a^c}{b^c}$
$\frac{a + b}{c} = \frac{a}{c} + \frac{b}{c}$	$\sqrt[c]{ab} = \sqrt[c]{a}\,\sqrt[c]{b}$
$\frac{a - b}{c} = \frac{a}{c} - \frac{b}{c}$	$\sqrt[c]{\frac{a}{b}} = \frac{\sqrt[c]{a}}{\sqrt[c]{b}}$ †

Schwartzman's remedial strategy is to teach the GDL, and then to describe the error $(x + y)^2 = x^2 + y^2$ as a case of misapplied distributivity. The level 3 operation, exponentiation, has been distributed over the level 1 operation, addition.

† The properties involving radicals appear to be left distributive laws rather than right distributive laws. The resolution of this anomoly involves a commutation of the radical operation, $\sqrt[a]{b}$, in its psychological representation. For a discussion of this point see the author's doctoral dissertation (unpublished, page 93).

F. PSYCHOLOGICAL CLAIM

While the technique is surely elegant, what is involved in the claim that it really addresses the source of students' underlying confusion? The claim is true if

1. The GDL is actually part of the knowledge which underlies successful manipulation of algebraic symbols, and

2. The $(x + y)^2 = x^2 + y^2$ problem is the result of a deficient mental representation of the GDL on the part of novices.

Such claims are difficult to establish. There is no direct way to observe the inner machinations of the mathematical mind. It is necessary to rely on such indirect evidence as may be externally manifest in performance and to formulate arguments based on such observations.

The form of argument employed in the present case is commonly used in psycholinguistics —the study of the underlying basis for natural-language performance. To support the existence of a structure which is proposed to be part of the knowledge underlying language use, empirical evidence is marshalled for a structure which, developmentally, may be a predecessor or precursor to the one in question. Then a potential line of cognitive development is sketched linking the precursor structure and the proposed structure. To the extent that this developmental argument is compelling, the plausibility of the two structures is mutually enhanced. In this way, it is possible to use evidence from acquisition of skill to support a theory of attained skill.

a. Predecessor Structure

Consider the following "overgeneralized distributive property" which states that: "Any operation right distributes over any lesser level operation."

Overgeneralized distributivity incorporates the eight real number properties of the GDL however it also incorporates four additional non-properties in which level 3 is distributed over level 1:

$$(x + y)^n = x^n + y^n \qquad \sqrt[n]{x + y} = \sqrt[n]{x} + \sqrt[n]{y}$$

$$(x - y)^n = x^n - y^n \qquad \sqrt[n]{x - y} = \sqrt[n]{x} - \sqrt[n]{y}$$

It is proposed that overgeneralized distributivity is a developmental precursor to generalized distributivity.

b. Empirical Support

There has been little systematic collection of algebra students' errors. There are, however, a substantial number of anecdotal and clinical reports of the prevalence of all four level 3 over level 1 distributivity errors (e.g. Budden, 1972, p. 8; Schwartzman, 1977, p. 595; Laursen, 1978, p. 194; Davis & McKnight, 1979, p. 37 and p. 98; Matz, 1980, p. 98-99; and Smith, 1981, p. 310). In fact some of these educators have also reported errors such as $a(bc) = ab \cdot bc$, $a^{mn} = a^m a^n$, $a^{m+n} = a^m + a^n$, and $\frac{a}{b + c} = \frac{a}{b} + \frac{a}{c}$. This suggests that overgeneralized distributivity may itself descend from a more generalized rule, or that it may require a more generalized formulation. (Suitable candidates might be "Any operation right distributes over any *lesser or equal* level operation" or "Any operation *left and right* distributes over any lesser level operation.")

c. Developmental Argument

The developmental sequence which is hypothesized to link the GDL and the overgeneralized distributive law is common in psychological theories. It is proposed that exposure to correct instances of distributivity leads to overgeneralization of the rule, which is gradually honed down to its maximally appropriate domain. Psycholinguists explain children's use of words like "goed" as the past tense of "go" by just such an argument (see Crystal, 1976). In fact Matz (1980, p. 115) independently arrived at the same analysis of these algebra errors as has been presented here.

One might even venture that the general inclination to generalize a rule beyond its appropriate context is reinforced and focussed by a tendency to impute transitive relations: Level 3 operations distribute over level 2 operations, and level 2 operations distribute over level 1 operations, therefore level 3 operations distribute over level 1 operations — hence the apparently greater frequency and tenacity of Level 3 over Level 1 distributivity errors than the other variants.

d. Summary

The above argument lends support to the contention that the GDL is part of the knowledge underlying the fluent manipulation of algebraic symbols. Common errors of the sort

$$(x + y)^n = x^n + y^n \qquad \sqrt[n]{x + y} = \sqrt[n]{x} + \sqrt[n]{y}$$

$$(x - y)^n = x^n - y^n \qquad \sqrt[n]{x - y} = \sqrt[n]{x} - \sqrt[n]{y}$$

appear to be the result of overgeneralization of some distributivity principle. If this is the case, and the path to fluency in algebraic manipulation passes through a phase of overgeneralizaton, then it seems only sensible that for the successful student, distributivity has

been honed down to its *maximal appropriate domain*; the Generalized Distributive Law.

G. IMPLICATIONS

a. Introspective Power

If the analysis is correct and the GDL is actually a part of the knowledge which enables successful symbol manipulation, then there are important implications for the way in which one views oneself as a mathematician and as mathematics educator. First of all, it must be stressed that the GDL is not proposed merely as a description of *students'* knowledge or learning. The above argument is intended to support the proposal that fluent manipulators of algebraic symbols have acquired GDL as part of *their* knowledge of algebra.

The immediate consequence of this is that neither as educators of mathematics nor as fluent manipulators of algebraic symbols does one have introspective access to the rules which govern one's own successful performance. One operates upon algebraic symbols without conscious knowledge of rules one uses. This result conflicts with many of the usual assumptions about mathematics as a rational, deductive system in which solutions are reached on the basis of explicit reasoning from known premises. Rather, mental processes in algebra employ covert knowledge of rules.

b. Learning

The second implication of the GDL concerns student learning. If the GDL does underlie successful manipulation of algebraic symbols, then the relationship between instruction and learning must be reassessed. A student who has not been *taught* the GDL must have discovered, or invented, it on his or her own.

This constructivist view of mathematics learning complements the notion of "holistic" instruction introduced in Part I. The curriculum supplies the materials which will be required for algebraic mastery, —not in a directed, predigested sequence, but as a diffuse, intuitive gestalt. The student receives the curriculum, —not as a ready made prescription for action, but as raw material from which to fashion the tools of algebraic skill.

c. Mathematical Theory

The third implication of the GDL is somewhat more speculative. The GDL does not appear to be related to any mathematical treatment of rational or real numbers. What, then, is the basis of algebraic skill? Is the manipulation of algebraic symbols not to be viewed as an application of mathematical theory? These questions which touch deeply upon basic assumptions within the mathematics education community are pursued in Part III.

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The Culture of the Curriculum

Part III: MATHEMATICS

In Part II of this paper, the Generalized Distributive Law (GDL) was presented. The rule states that:

"Any operation right distributes over any one lesser level operation," where addition and subtraction are "level 1" operations (the "lowest" level), multiplication and division are "level 2", and exponentiation and radical are the "level 3" operations. This rule embodies eight previously independent real number properties:

Level 2 over Level 1	Level 3 over Level 2
$(a + b)c = ac + bc$	$(ab)^c = a^c b^c$
$(a - b)c = ac - bc$	$\left(\frac{a}{b}\right)^c = \frac{a^c}{b^c}$
$\frac{a + b}{c} = \frac{a}{c} + \frac{b}{c}$	$\sqrt[c]{ab} = \sqrt[c]{a}\,\sqrt[c]{b}$
$\frac{a - b}{c} = \frac{a}{c} - \frac{b}{c}$	$\sqrt[c]{\frac{a}{b}} = \frac{\sqrt[c]{a}}{\sqrt[c]{b}}$ †

It was noted in Part II that

$$(x + y)^n = x^n + y^n \qquad \sqrt[n]{x + y} = \sqrt[n]{x} + \sqrt[n]{y}$$

$$(x - y)^n = x^n - y^n \qquad \sqrt[n]{x - y} = \sqrt[n]{x} - \sqrt[n]{y}$$

have been cited as common student errors by several investigators. These errors seem to have the form of an overgeneralization of distributivity.

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† The properties involving radicals appear to be left distributive laws rather than right distributive laws. The resolution of this anomoly involves a commutation of the radical operation, $\sqrt[a]{b}$, in its psychological representation. For a further discussion of this point see the author's doctoral dissertation (unpublished), pages 93

If this is the case and students do tend to overgeneralize distributivity, then it seems only sensible to conclude that the successful student, in overcoming this tendency, will have honed down distributivity to its *maximally permissible domain*: The Generalized Distributive Law. Thus it was argued that the GDL is actually present as a factor in the *psychological* organization of real number properties.

The GDL is unlike other rules which are presumed to underlie mathematical skill in that it is derived from psychological rather than mathematical theory. What is the appropriate role for a psychologically-based rule, such as GDL, in the curriculum? Should the GDL be standardly presented to students within textbooks *as mathematics*? These are difficult questions. In the past it may have been possible to imagine that mathematical theory and mathematical skill possess some deep intrinsic relationship. But can such an assumption be supported? The discussion of these questions begins with a detailed analysis of the mathematical credentials of the GDL.

I. THE MATHEMATICAL STATUS OF THE GDL

The GDL is a true statement about real numbers. Unlike its prototype, the (regular) distributive law, however, the GDL is not an axiom of field theory. More critically, it is not a theorem of field theory.

The reason for this is straightforward. A field is a mathematical system defined on two operations, addition and multiplication. The inverse operations, subtraction and division, are easily constructed in terms of identities and inverse elements. The "level 3" operations of exponent and radical, however, cannot be defined in terms of the existing operations. (In particular, the definition of a^b as a·a·a....a ("b"-times) is a non-algebraic technique, since it

attributes to "b" a quality of numerousness which conflicts with the formal, attribute-free character of abstract algebraic variables.†) The mathematical treatment of the *real number system* (which does include exponentiation and radical) is usually accomplished through analytic rather than algebraic means using such devices as suprema, Cauchy-Sequences or Dedekind cuts. The GDL is surely not a construct of analytic theory.

More than just the disavowal of a basis in mathematical theory for the GDL, this analysis suggests that no algebraic rules involving exponents or radicals are systematically related to established mathematical theory. As properties of real numbers, the rules underlying algebraic manipulation are true mathematical statements. That, however, is the extent of the connection to mathematical theory. The rules of algebra remain an *ad hoc* collection rather than a systematically linked and theoretically derived collection.

J. THE "NEW MATH"

Despite the validity of these observations, the notion that the rules used in symbolic manipulation are 'explained by' or 'derived from' mathematical theory has been a force of tremendous influence in mathematics education. During the 'new math' era of the 1960's and 1970's abstract algebraic theory was introduced into the secondary school curriculum. The "field axioms" including the commutative, associative and distributive laws were routinely presented in textbooks and classrooms. It was anticipated that the abstract algebraic approach might provide a unifying, logical rationale for the myriad algebra rules which previously had been learned by rote.

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†Indeed, if exponentiation could be formally introduced in abstract algebra as repeated multiplication, then wouldn't multiplication have been defined as repeated addition?

Authors for the major funded curriculum projects of the day were careful to avoid obviously false claims for the new math, preferring to support their expectations with generalities. Less exposed publishers were not quite as careful. For example the following passage appears in the introduction of a prominent "new math" textbook:

> ... a prior logical basis has been developed for all mechanical processes. Deductive proof is emphasized throughout the course. Attention to techniques of proof is emphasized early in the book. In the early chapters all deductive arguments are given in formal two-column proof style. Later, the arguments are given in a less formal style, but deductive reasoning is employed throughout, both in the development of theory and in applications. (Banks, Sobel & Walsh, 1965, p. vi)

The statement here is not completely explicit but the implication is that the two-column proofs (which turn out to be standard abstract algebraic proofs) are possible for all the algorithms introduced in the text. Of course the "less formal style" is introduced precisely at the point that properties of exponents are discussed.

This position is more succinctly stated in the introduction to a textbook by Johnson, Lendsey and Slesnick (1971, p. T2).

> Because it is important for students of elementary algebra to realize that general properties of a number system can be deduced from the basic assumptions, students are given ample opportunity to deduce the properties of the real number system from the basic assumptions. In this way, deductive reasoning is introduced.

Although misstatements about mathematical theory are unfortunate, they must be understood within the larger context of expectations and values within the mathematics education community. Mathematical thinking is generally perceived to be a conscious and rational activity

involving powers of explicit reasoning. If algebraic symbol manipulation *feels* to be an automatic and unconscious process, then this can only be because its logical, systematic essence has been obscured by long familiarity and truncation or telescoping of reasoning patterns. Its essence, however, must reside within some explicit body of theory, and mathematical theory is the only likely candidate.

From this traditional viewpoint, it is clear that acceptance of the *ad hoc*, atheoretical status of algebraic rules demands renunciation of basic values long entrenched within the community of mathematics educators. It is only an occasional lone voice which has called for reassessment. For example, Robert Gagné (1983), in response to critics makes this appeal:

> The apparent view ... that understanding involves some aspects of the "structure of mathematics" is what I would be inclined to question. I realize that this is an extremely common view among mathematics educators, and it is exactly this view I should most strongly like to urge them to examine critically. Is this view actually based on solid evidence, or does it merely reflect a traditional statement by those who like to reassure themselves that they are on the side of the angels? (p. 215)

The reassessment of assumptions regarding algebraic skill is long overdue.

K. THE BASIS OF ALGEBRAIC SKILL

The title of this section promises more than it can deliver. Attempts to assess the underlying structure of algebraic knowledge are still tentative and preliminary. Perhaps some existing algebraic treatment of real numbers in which there are three basic operations, +, X, and exponentiation (e.g. Macintyre, 1979/1980) can provide insight into the psychological structure of real number properties used in elementary algebra?† Perhaps a new algebraic theory of

†Macintyre, himself, is not too optimistic about this prospect. He concludes:

> The most interesting problem provoked by the above is that of showing that

real numbers could be discovered in which GDL appears as an axiom. These conjectures, however, are entirely speculative at the present time. Algebraic rules underlying manipulative skill may remain inscrutable to formal mathematical theory.

If mathematical theory turns out not to be a good guide to the rules underlying skilled performance it may be necessary to look beyond the purely mathematical experience and knowledge of students. In the case of the GDL, for example, distributivity is a feature of natural language with which speakers have vast experience.

A simple example illustrates the point. Consider the ambiguity of "I like cake and ice cream" viz "I like cake and I like ice cream and, in fact, I like all sweets" or "I like cake and ice cream, but I can't stand ice cream by itself." Capps (1970) provides a more complex example:

> In a sentence such as "He saw a black dog and wagon" the application [of the distributive property] is more subtle. Black modifies dog and wagon, thus indicating a black wagon. If the wagon were a color other than black, we would insert "a" before black, having "He saw a black dog and a wagon." (P. 330)

These examples evidence the fact that students in secondary mathematics are already expert in analysing contexts in which distributivity applies. What is more, this expertise has come about through informal experience within a language community rather than through explicit instruction. Thus the predisposition of students to spontaneously seek out contexts for

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†(cont'd) there are no "exotic" laws, i.e. that every law is a consequence of the laws of

$+, \cdot, -, {}^{-1}, 0, 1$ together with $x^1 = 1$ $\quad x^{y+z} = x^y \cdot x^z$ $\quad x^{yz} = (x^y)^z$ $\quad (xy)^z = x^z \cdot y^z$.

It seems difficult to prove such a theorem by the methods of real algebra used above. (p. 97)

mathematical distributivity may relate to deeply entrenched processes of natural-language acquisition.

L. CONCLUSIONS

As mentioned above, rules like GDL pose difficult choices regarding the mathematics curriculum. Should *psychologically* based rules be included within *mathematics* texts? Should abstract algebra rules be taught in the secondary schools even if it can be demonstrated that such rules are almost entirely irrelevant to the rules which underlie algebraic performance? How should the mathematics education community express its commitment to imparting something of mathematical theory and what should be the balance between theory and skill in the curriculum?

These are questions which will require wide debate and critical reassessment within the mathematics education community. Regardless of the outcome, however, it will be necessary to pursue the quest for the psychological basis underlying algebraic skill. As long as developing skill in algebra remains an objective of the curriculum, it will be desirable to understand the nature of the rules which students must assimilate.

The attempt in this three part article has been to raise these issues of curriculum and to suggest some alternative perspectives to those which have been so influential upon the mathematics curriculum in the past. Rather than a rational endeavour towards specified goals, remediation and instruction in algebra can be viewed as a diffuse, "holistic," and intuitive approach towards a complex matrix of objectives. Rather than a passive receiver of predigested knowledge, the successful learner can be seen as an active synthesisor and constructor of the rules required to manipulate formulae. Rather than application of some presumed (but

unspecified) "mathematical theory," the manipulation of formulae can be seen as based upon a hidden, and still largely unexplored, complex of rules.

The discovery of the generalised distribution technique by Schwartrzman (1977) is all the more remarkable because he didn't know that he had found a psychologically secure foundation upon which to base instruction. This is evident from his apology for the "wordiness" of his method and his suggestion that other methods could also be used.

As a practitioner aware of the limitations of introspective access to algebraic symbol knowledge; as a practitioner who cares deeply about what goes on in students' minds —and especially the hurdles which remain for them after instruction has been received— one can begin to chart the remaining unexplored regions of algebraic knowledge, and in so doing reshape the school algebra experience for students.

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VI. ISSUES IN THE LINGUISTIC ANALYSIS OF ALGEBRA

Paper #6

In the previous five papers a linguistic approach to the study of algebraic symbol skills has been motivated, developed, partly implemented and partly tested. This final paper briefly overviews the major issues involved in adopting a linguistic approach and then goes on to consider some new issues which have arisen from the implementation.

A. SEMANTICS VERSUS SYNTAX

A basis of the linguistic approach was the attempt to answer Nesher's question asked at the very start of the first paper:

> Is there a semantic component in the language of mathematics, or is it a symbolic system that has only syntax?
>
> (Nesher 1981, p. 28)

The method has been to demonstrate (by example) the viability of a model of algebraic symbol skills based only upon syntactic and pragmatic knowledge. The pragmatic component of the grammar has not yet been elaborated (although the character of that component is briefly outlined on page 63). The syntactic component for expression manipulation sentences has been elaborated in detail.

The ultimate success of this attempt will depend upon the degree to which its arguments and models are found to be logically coherent and upon the degree to which the linguistic approach provides access to psychologically and educationally relevant variables. In the meantime, the linguistic approach has provided a challenge to implicit but extremely pervasive assumptions in mathematics education regarding the connection of mathematical theory to the psychological underpinnings of skillful performance (Paper #5). More generally, the complex, detailed syntactic model provides a specific hypothesis regarding a knowledge structure which is

complex, abstract, and essentially inaccessible to introspective investigation. It was argued in Paper #1 that the assumption of introspective access as well as the assumption of a link between mathematical theory and mathematical skill have been important influences in the previous neglect of syntactic explanations of algebraic symbol skills and in the corresponding emphasis on semantic models. Thus, if successful, the adoption of a procedural (and specifically a linguistic) perspective will constitute a major paradigm shift in mathematics education research.†

That a syntactic/pragmatic approach to the psychology of algebraic symbol skills has been advocated here, however, does not mean that syntactic and semantic representations necessarily turn out to be disjoint. For example, the hypothesis that the radical operation is reversed between DF and SF is supported by both syntactic and semantic evidence (page 93). The extent and character of such connections needs to be carefully considered. Perhaps some syntactic characteristics will turn out to have evolved from semantic predecessors (either within individuals or within the language as a whole). Until more solid grounds for connection are established, however, it is methodologically prudent to proceed on the assumption of unconnected syntax and semantics.

B. NATURAL LANGUAGE AND ALGEBRAIC LANGUAGE

Following the advice of Jackendoff and Lerdahl (1983) —formulators of a linguistic theory of musical "language"— (see Paper #2, page 27) a relationship between natural and algebraic grammar has not been assumed in the present research. Linguistics has supplied a

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†Again, the reminder that the definition of algebraic symbol skills adopted here meticulously excludes the skills of translating between natural language and mathematical language, or of representing "real world" problem situations which mathematics may help solve. These latter skills undeniably depend upon semantic or referential knowledge in algebra.

methodological framework to approach the covert matrix of mathematical knowledge underlying algebraic skill as a "grammar." No substantive connection between natural language grammar and mathematical grammar has been anticipated or sought.†

Despite the initial intention, possible substantive connections have emerged for both of the questions which have been intensively analysed (see Paper #4, page 140 and Paper #5, page 213). In each case natural language grammar provided a possible source for algebraic rules which are acquired in interaction with algebraic language, but without benefit of instructional direction. More generally, the devaluation of theoretical mathematics as a basis for algebraic rules suggests that more organic sources may need to be found to explain the ontology of algebraic rules. Natural language as a prototypical (and possibly innate—Chomsky, 1965) symbol skill is an obvious source to consider. In order to pursue the connections already advanced, and to facilitate further investigation, it is useful to overview Chomsky's Standard Theory and look at possible reformulations of the algebraic grammar to be compatible with it.

1. The Standard Theory

A language, for the purposes of linguistic analysis, is a collection of sentences each of which is composed of finitely many phonetic elements. It is an artifact of the complexity of natural language that there is no simple way to distinguish the sequences of phones which are sentences from the myriad non-sentence sequences imaginable. Linguistic analysis must proceed at a variety of levels: Phonetics is the study of basic vocal units (phones) which the human

†Chomsky's (1965) "Standard Theory" has contributed technical devices (e.g. phrase structure grammar) and some terminology (e.g. transformations, deep structure, etc.), however, these terms and devices have performed different functions within the grammar of algebra than within natural language models.

animal is capable of uttering, and of the distribution of phones into phonemes, the distinctive sound units within a given language. Morphology is the study of the minimal meaning bearing sequences of phones within a language. Syntax is the study of the organization of morphs into phrases or sentences. These elements are coordinated within a grammar which generates all and only sentences.

In the Standard Theory, a basic distinction is made between the deep structure and surface structure of a sentence. The deep structure of a sentence represents an abstract organization of sentence constituents from which a logical structure can be computed. The surface structure represents the organization of the units one would actually say or write.

A well-known example illustrates the distinction. The sentences *John is eager to please* and *John is easy to please* have nearly identical surface structures: John, the subject, is followed by a verb phrase and then by an infinitive phrase. The underlying logical relations, however, are quite different for these two sentences. The first sentence receives an analysis along the lines of *John is eager* [*that John pleases someone*]. The analysis for the second sentence is more like *It* [*for someone to please John*] *is easy.* Thus "John," which plays the role of subject for both surface structure representations, is analysed as the subject both of the sentence and of an embedded clause for the first sentence, but as the direct object of an embedded clause for the second sentence.

In the Standard Theory a phrase structure grammar is used to generate structural descriptions for the deep structures of sentences. A fragment of a phrase structure grammar and a resulting tree diagram for the deep structure of the sentence *A child eats the banana* is shown in Figures 8 & 9. The insertion of lexical items into the tree diagram completes the

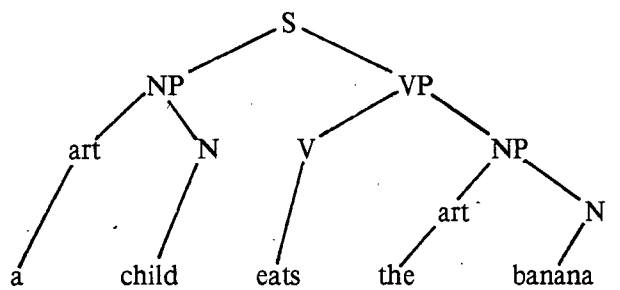
Figure 8
Fragment of a Phrase Structure Grammar

S ——> NP VP;
NP ——> Art N;
VP ——> V NP;
Art ——> the;
Art ——> a;
N ——> child;
N ——> banana;
V ——> picks;
V ——> eats.

S stands for sentence, NP stands for noun phrase,
VP stands for verb phrase, Art stands for article,
N stands for noun, V stands for verb.

From Gross and Lentin (1970, p. 34)

Figure 9
Syntactic Tree Diagram for
a child eats the banana



deep structure for the sentence.

The next phase in generating sentences is a series of transformational rules which reorganize, add to, and delete from, tree diagrams. These transformations carry the deep structure through intermediate structures to the surface structure. (In the case of "A child eats the banana" the transformational component has relatively little work to do. "John is easy to please," however,

requires substantial transformation between deep structure and surface structure.) Finally the phonological component expresses the surface structure as a string of phonetic elements.

2. Relating the Algebraic and "Standard Theory" Models

Sentences in natural-language linguistics are like (and may often coincide with) the utterances produced by fluent speakers of a language in the normal course of events. In the Standard Theory sentences of natural language are (in part) the result of transformational processes.

In parallel with natural-language linguistics, the sentences of algebra were defined to be sequences of symbols such as those produced by fluent algebraists in the normal course of events (page 29). In order to obtain better accord between natural language grammar and the algebraic grammar, however, it is necessary to interpret algebraic sentence not only as the *product* of transformations (as is the case for natural language), but also, in part, artifacts of the process of transformation itself.

This is most easily seen by means of an example. Consider the sentence $\frac{(2x)^2}{x} = \frac{4x^2}{x} = 4x$. The previous analysis of this sentence is illustrated on page 61. In the reinterpretation, [[2Mx]E2]Dx (the deep form of $\frac{(2x)^2}{x}$) is taken to be the *deep structure*, and 4Mx, the *surface structure* of the sentence. In this case, the transformational component uses intermediate structures to mediate between deep and surface structure just as in the Standard Theory; [[2Mx]E2]Dx ——> [[2E2]M[xE2]]Dx ——> [4M[xE2]]Dx ——> [4M[xMx]]Dx ——> [[4Mx]Mx]Dx ——> [4Mx]M[xDx] —> [4Mx]M1 —> 4Mx. In this interpretation, the translation component (e.g. [[2Mx]E2]Dx into $\frac{(2x)^2}{x}$) is considered analogous to the work of the phonological component.

When the grammar is structured in this way, an algebraic sentence is a graphic trace of the transformational journey from deep structure to surface structure. The psychological implication is that mathematical language is an externalized version of processes which are completely internalized in natural language processing. This might be a productive line to take in comparative psychological analysis of natural language and algebraic language.

C. PEDAGOGICAL IMPLICATIONS

The grammar of algebra provides a theory of the psychology of fluent algebraic skill; it presents the rules which guide successful algebraic performance. In this final section the curricular and educational value of uncovering these rules is considered.

First of all, it should be reiterated that a grammar is not a theory of the acquisition of skill. As discussed in Paper #2 (page 41), acquisition of skill is not amenable to the same kind of formal analysis as is the acquired skill. At the same time, elements of a theory of acquisition can be instrumental in supporting formal hypotheses within the linguistic framework. Thus the adapted linguistic paradigm is of direct relevance to issues of acquisition even if a comprehensive theory of acquisition is beyond its purview.

As desirable as a comprehensive theory of acquisition might be, it must be noted that it is neither necessary nor sufficient for application of grammar to curriculum and instruction. A theory of acquisition would, presumably, still require a theory of how to *support* acquisition in order to be of direct utility. Conversely, knowledge of the grammar of algebra might prove a valuable asset to teachers engaged in developing algebraic fluency in students. Skemp (1976) points out that many students seem to hunger for the rules of algebra and that many teachers are inclined to give them those rules with a minimum of "theory" or explanation.

As the grammar of algebra *is* the rules which the novice manipulator of symbols must master, such students and teachers might be well served by an explicit account.

On a somewhat more structured level, the grammar of algebra may suggest some logical task analysis. A task analysis differs from a theory of acquisition in that it is subject-centered rather than learner-centered. In the absence of immediate prospects for a rigorously structured theory of acquisition, a formal analysis of the task might serve well as a model for curricular organization. It is this project which has motivated the theoretical research presented in these Papers, and which the author is still intent upon pursuing.

At the same time, linguistics may provide other avenues of approach to the problems of educational application. There is a vast literature on second-language instruction. This literature has yet to be culled from the point of view of algebraic applications. What seems clear on the basis of the research presented so far is that successful students acquire (at least portions of) the grammar of algebra through immersion in the algebraic community of the classroom — much as children acquire their first language. The educational goals of the linguistic approach should be to take advantage of explicit knowledge of grammar and make the learning of algebra more like acquisition of a second language.

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