THE IMPACT OF PERSONAL TAXES ON TWO AREAS IN THE THEORY OF FINANCIAL MARKETS

by

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A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

in

THE FACULTY OF GRADUATE STUDIES
(Commerce and Business Administration)

We accept this thesis as conforming
to the required standard

The University of British Columbia
August, 1987

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To my parents,

A. Lakshminarayanan and Visalakshi Ammal.

Acknowledgement

I thank Robert Heinkel, Espen Eckbo, Josef Zechner, Gordon Sick, and the participants of the finance workshop at the University of British Columbia for their assistance, guidance and suggestions without which this thesis may not have come to be. I also wish to thank Michael Brennan for his invaluable suggestions when I was working on the first of the following essays. I am indebted to Usha Mittoo, Venkatraman Sadanand and C. E. Sudhakar for their invaluable assistance in collating, binding and distributing copies of the thesis. Last but by no means the least, I must thank my wife Mandira for helpful discussions which always gave me a fresh perspective on the problem at hand. In addition, I am indebted to her for all the inputs, too numerous to enumerate, that carried me through the Ph.D experience.

ABSTRACT

This thesis considers the impact of taxation on two problems in the theory of financial markets. The first paper deals with the optimal choice of debt made by value-maximising firms. We consider a one-period world with personal and corporate taxation and distinguish between the repayment of principal and the payment of interest on corporate debt. It is shown that at optimum, a value-maximising firm may choose to issue multiple debt contracts with differing seniorities. In addition, the impact of a change in the tax rates (corporate or personal) on the optimum level of debt is seen to be ambiguous. Unambiguous statements can, however, be made about the impact of a change in the corporate tax rate on firm value, the value of the equity and on the required rate of return on risky corporate debt.

The analysis borrows heavily on a framework that we develop early in the paper which permits us to visualise the value-maximising firm's choice of an optimal capital structure, graphically.

The second essay examines the impact that taxes have on the pricing of call options on corporate stock. It is demonstrated that the process of replication can be influenced by the basis of the stocks used for the replication process as a result of the capital gains taxes involved. Consequently, the equilibrium price for an option is some average of the various costs of replication that different investors face.

We find that the equilibrium price for the option can be influenced by investor preferences and by the history of the stock price. The empirical findings of an apparently unpredictable strike-price bias that have been observed in the past literature is examined and duplicated numerically. In addition, one explanation is given for the rationale behind covered option positions that consist of an option position and the corresponding hedge.

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Introduction

During the early years of the theory of financial markets, Modigliani & Miller(1958) suggested that in perfect capital markets value-maximising firms should be indifferent amongst the choice of different debt-equity ratios. Soon after, (1963), they introduced corporate taxation into their model, and argued that debt provided tax savings at the corporate level. This implied that firms should all be completely debt financed – a result that was not an acceptable representation of reality to any observer who believed that managements act so as to maximise the wealth of the owners of firms. In order to explain this inconsistency, subsequent writers introduced "bankruptcy costs". Robichek & Myers(1965) and Kraus & Litzenberger(1973) are two examples. The empirical study of Warner(1975) with the railroad industry, however, indicated that these costs were rather small in comparison with the other assets of the firm – a finding that raised concern about the importance of such costs in the choice of an optimal debt-equity mix. On a parallel road, Donaldson(1963) and Jensen & Meckling(1975) introduced other agency costs and hinted at the possibility that management may in reality be more concerned with their own future than in shareholders wealth. They were hence reluctant to raise the probability of bankruptcy (and the resulting unemployment) that results from high debt levels. Though plausible, this theory was not completely satisfactory and so the controversy persisted. Some viewed internal capital structures as arising from a signalling model of firm value. For example, Leland & Pyle(1977) show that an enterpreneur who wishes to raise capital (Debt) for his project, signals the worthiness of the project through his willingness to invest in the project himself (Equity). Ross(1977) uses debt as a signal by managers of the future cash flows of the firm. Heinkel(1982) develops a non-dissipative model of debt signalling that captures the essence of the Ross and Leland & Pyle models. On a different track, Brennan & Schwartz(1978) recognised that one of the consequences of bankruptcy was the termination of

the tax savings that debt generated. Hence, they reasoned that the issue of debt would no doubt increase the tax savings, but it would also increase the probability of termination of such tax savings within any finite period of time. By this time, however, reasearchers began to look more closely at the tax code in order to explain the debt puzzle.

Financial economists have, for some time now, been aware of the tradeoffs between the issuance of debt and equity resulting from the differential tax treatment of these instruments at the corporate and personal levels. In his seminal paper, Miller(1977) demonstrated, however, that in a world of certainty, the value of any individual firm was independent of the level of financial leverage employed. To obtain this result, Miller assumed capital markets to be perfect with the sole exception of corporate and personal taxation. He argued that in equilibrium, the prices of debt and equity income realised in the market would be such that the advantage to debt resulting from tax savings generated at the corporate level would be completely offset by the disadvantage of debt resulting from increased taxes paid on personal account. The existence of personal tax rates on either side of the corporate tax rate, however, resulted in an optimum quantity of debt for the entire economy, in his model. The implication of his model, amongst other things, was the existence of a *"clientele effect"*, whereby, investors with low tax rates (below the corporate tax rate) held only debt instruments whereas investors in high tax rates (above the corporate tax rate) held tax exempt securities such as equity.

DeAngelo & Masulis(1980) were amongst the first to introduce uncertainty into Miller's framework. In keeping with previous work, however, they were more concerned with the implication of bankruptcy and agency costs coupled with more realistic assumptions of the tax code, on the debt irrelevance question. In particular, they showed that the existence of non-debt corporate tax shields was sufficient to overturn the leverage irrelevancy theorem. However, they assumed personal tax rates to be determined exogeneously, independent of the state of nature revealed at the resolution of uncertainty, and consequently, the clientele effect for debt that was present in Miller's model, persisted.[1] Their major contribution was in demonstrating that interior optimum capital structures could be realised at the level of an individual firm, solely as a consequence of the interaction of the different tax rates involved. The clientele effect was, nevertheless, inconsistent with casual observation of reality.

Dammon(1984) eliminated this clientele effect by endogenising personal tax rates. He linked the tax rate to the investor's finally realised taxable income which in turn was dependent on the final (state-contingent) payoffs of the investor's portfolio. As a result, it was now possible for the investor to have a tax rate *below* the corporate tax rate (assumed invariant) in some states and *above* the corporate tax rates in other states. This implied that in some states the investor would

[1] They did, however, identify the "marginal investor" who was indifferent between debt and equity.

prefer debt income, whereas in others he would prefer equity. He may consequently hold both debt *and* equity in his optimal portfolio. Dammon assumed that markets were not complete in the sense that it was not possible to obtain state-contingent claims on debt and equity income. If such claims were to exist, however, then, in equilibrium, **all** investors would end up with identical state-contingent personal tax rates[2]. However, since firms are not able to issue such state-contingent claims (being restricted to conventional debt & equity instruments which are essentially portfolios of the state-contingent claims), they might still obtain interior optimum leverage ratios. This is due to the fact that an individual firm could find its profile of state-contingent cash flows to be such that, by issuing finite quantities of debt and equity, it ends up paying out debt income in those states where debt income is more valuable than equity income as well as paying out equity income in states where equity income has a greater value. In essence, the value-maximising firm obtained an interior optimum leverage ratio as a result of its trying to "tailor" (as best as possible) its state-contingent payout vector of debt and equity income to existing state-contingent price vectors of debt and equity income, so as to maximise the dot product of the payout and price vectors, subject to the constraint that total payouts in each state be equal to the after-tax cash flow of the firm in that state. Clearly, if the firm were allowed to issue *state-contingent claims* on debt and equity, it would choose to payout all its cash flow as debt in states where debt was more valuable and as equity in those remaining states where equity was more valuable.[3]

In the studies mentioned above, a complete specification of the states of nature had been assumed. In other words, states were defined by a vector of cash flows realised by each and every firm in the economy. Ross(1985) realised that such a detailed specification was not necessary, and that it is sufficient for us to distinguish states by aggregate consumption (or some other aggregate price standard). In other words, he considered only the states that were *relevant* in the pricing of state-contingent claims on income. In his model, it does not matter whether markets are complete or not (for state-contingent claims on debt and equity income), since in any case, the firm is not allowed[4] to issue claims whose payoffs are contingent on its cash flow being at some prespecified level. By imposing some restrictions *(Monotone Likelihood Ratio Property)* on the joint distribution of the firm's cash flow and the underlying state of nature, he obtains functional relationships between firm value and the level of financial leverage, and illustrates one case where an interior optimum capital structure is realised.

All the studies mentioned above, with the exception of Park & Williams(1985), treat debt as being completely exempt from corporate taxation. This allowed the studies to focus on the tax savings generated by debt. In reality, however, debt has principal and interest components which

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[2] For, if not, tax arbitrage would be possible.

[3] See Park & Williams (1985).

[4] Implicit prices are however, assumed to exist for such claims.

are treated differently for tax purposes[5]. In our study, we incorporate this into our model since it has much to offer. By incorporating this single feature, we learn that

1) Value maximisation may call for the issue of multiple debt contracts with differing seniorities.

2) Optimal debt levels and corporate tax rates are not necessarily monotonically related in the sense that a rise in the corporate tax rate does not necessarily lead to higher debt levels at the optimum.

At times, however, the assumption that the debt payout is completely tax exempt at the corporate level, lends intuitive appeal to our results and allows us to isolate the effects that arise from the principal component of debt. Consequently, we shall refer to this simplification frequently. In order to obtain these results, we develop a graphical representation that allows us to easily visualise the firm's choice of an optimal capital structure.

We shall assume that markets are sufficiently complete to span the space of payoffs generated by the vector of aggregate consumption *and* the output of every firm in the economy[6], that investor beliefs are homogeneous and that implicit state prices for debt and equity income exist. In addition, we ignore the existence of non-debt related tax shields. This is to ensure that our focus is concentrated on the more interesting question of *when* firms will issue debt. The introduction of non-debt-related tax shields, would alter the algebra and the form of the results, but the essence of the analysis would remain. We address the impact of this modification, on our results, in the conclusion. We also assume that the corporate tax rate is exogeneously determined at an invariant rate to retain mathematical tractability. It does not appear that the essence of our arguments would be substantially altered by allowing the corporate tax rate to be an increasing function of the firm's cash flow, but this is still an open question. Finally, we assume that equity income is completely tax exempt at the personal level. It is a simple matter to allow for partial deductibility, but the gains in terms of understanding are not expected to be substantial.

In Section 2, we develop the framework which we shall use for our analysis. By appropriately transforming the market valuation equation for securities, we obtain a graphical interpretation of the value-maximising firm's choice of an optimal capital structure. Conditions that are sufficient to induce firms to issue debt are then derived. It is noted that even though a "sufficiently high" corporate tax rate is *sufficient* to induce firms to issue debt, it is not *necessary*. This is due to the fact that a firm's optimal capital structure is intimately linked to the joint distribution of its cash flows with aggregate consumption. In fact, it turns out that interior optimal capital structures

[5] Refer to Talmor, Haugen & Barnea(1985) for a review of how this distinction has been treated in the past literature.

[6] This implies that state-contingent personal tax rates will be identical for all investors.

could be the result of particular kind of cash flow distribution, rather than any specification of macroeconomic parameters.

In Section 3, we integrate some of the results obtained by Ross (1985) into our framework. The graphical framework developed here makes his results considerably easier to visualise. Section 4 demonstrates that value-maximisation could require the use of multiple debt instruments with various levels of seniority. Once again, the need for more than one type of debt instrument is dictated largely by the joint distribution of cash flows and aggregate consumption.

Section 5 examines the effect of a change in marginal tax rates on the optimal debt choice. It is observed that an increase in the corporate tax rate increases the risk of the corporate bond whilst increasing the tax-savings generated by it. Consequently, no unambiguous statements can be made of the direction in which optimal debt levels will move in response to a change in the corporate tax rate. Once again, the joint distribution of the firm's cash flows and aggregate consumption turns out to be an important factor. Section 6 concludes the paper and suggests some directions for further research.

Every section has been provided with a summary at the end, which highlights the main features contained in that section. The proofs of the major results have been delegated to the appendix. In addition, an additional appendix contains all the notation used in the study.

Chapter 1

The firm's debt choice

1.1 A framework to analyse the firm's debt choice

In this section, we shall develop the graphical framework which is at the heart of our entire analysis. In addition we shall obtain conditions under which individual value-maximising firms will issue debt. A summary is provided at the end of the section. The model we shall develop is set in a one period framework. To start off, let

$x =$ *Pre-tax* cash flow of a firm.

$c =$ Aggregate end-of-period consumption in the economy.

$\Psi(c, x) =$ Joint Density Function of aggregate consumption and firm's cash flow

$g(x|c) =$ Conditional probability density of the firm's cash flow (x) when aggregate consumption $= c$

$f(c|x) =$ Conditional probability density of aggregate consumption (c) when firm's cash flow $= x$

$p_1(c) =$ Equilibrium value of a claim that pays \$1 of taxable income when aggregate consumption $= c$

$p_0(c) =$ Equilibrium value of a claim that pays \$1 of non-taxable income when aggregate consumption $= c$

$\tau =$ Corporate tax rate $(0 < \tau < 1)$

$\mathcal{C} =$ Set of possible aggregate consumption levels attained at the end of the period. i.e.$c \in \mathcal{C}$

Note : *We have defined $p_1(c)$ and $p_0(c)$ rather loosely. They are actually price densities and technically, $p_1(c)dc$ is the value of a claim that pays \$1 of income that is taxable at the personal level when aggregate consumption lies in the infinitesimal interval $[c, c+dc]$. $p_0(c)$ is similarly defined. Notice, however, that the earlier interpretation is valid in a discrete framework.*

Individuals are assumed to be taxed at a rate that depends on their taxable income. In particular, when a person's taxable income is π, the total taxes paid are assumed to be

$$T(\pi) = \int_0^{\pi} t(\omega)d\omega \tag{1}$$

where $t(\pi)$ is the marginal tax rate of the individual. We assume that $0 < t(\pi) < 1 \; \forall \pi$.
By specifying the tax function as above, we have ensured that $T(\pi)$ is a continuous function of π. This excludes situations where $T(\pi)$ is a step function. However, by appropriately specifying $t(\pi)$, we can approximate this case. Further, we assume that $T(\pi)$ is a convex function[7]. We also assume that individuals are expected-utility maximisers with state-independent and time-additive utility functions of consumptions. In addition, they all are in agreement in respect to their assessments of $\Psi(c,x), g(x|c)$ and $f(c|x)$. Under these assumptions, Breeden & Litzenberger(1978) have shown that the value, today, of a security that pays $h_T(x)$ dollars of income that is taxable at the personal level and $h_{NT}(x)$ dollars of income that is not taxable at the personal level, when the firm's cash flow is x, is given by

$$V(h_T, h_{NT}) = \int_C \left(\int_0^{\infty} h_T(x)g(x|c)p_1(c)dx + \int_0^{\infty} h_{NT}(x)g(x|c)p_0(c)dx \right) dc \quad ; \tag{2a}$$

$$= \int_0^{\infty} h_T(x) \left(\int_C p_1(c)f(c|x)J(x,c)\, dc \right) dx$$

$$+ \int_0^{\infty} h_{NT}(x) \left(\int_C p_0(c)f(c|x)J(x,c)\, dc \right) dx. \tag{2b}$$

In the above equation, $J(x,c)$ is the appropriate Jacobian, which in this case reduces to the ratio of the marginal densities of x and c. Though it is possible to work with the above valuation, a simple transformation of the elementary pricing functions makes it considerably simpler to view the value-maximising firm's choice of an optimal capital structure. It turns out that we are made better off by viewing the market as being comprised of elementary securities whose cash flows are contingent on the firm's cash flow, rather than viewing it as being comprised of securities whose cash flows are contingent on aggregate consumption. Such elementary securities may not in fact

[7] Refer to Dammon and Green (1987). Footnote 7 contains a discussion of this.

exist, but they are implicitly priced, nevertheless, in our market. In particular, let,

$$\phi_1(x) = \int_C p_1(c) f(c|x) J(x, c)\, dc \quad ; \tag{3a}$$

$$\phi_0(x) = \int_C p_0(c) f(c|x) J(x, c)\, dc \quad ; \tag{3b}$$

$\phi_1(x)dx$ is the value of a claim that pays \$1 of fully taxable income when the firm's cash flow lies in $[x, x+ dx]$ and $\phi_0(x)dx$ is the value of a claim that pays \$1 of non-taxable income in this state. In a discrete world, $\phi_1(x)$ and $\phi_0(x)$ can be interpreted as the values of claims that pay \$1 of taxable and non-taxable income when the firm's cash flow equals x. Since $p_0(c)$ and $p_1(c)$ are non-negative[8], $\phi_0(x)$ and $\phi_1(x)$ must also be non-negative.
From (2) and (3)

$$V(h_T, h_{NT}) = \int_0^\infty h_T(x)\phi_1(x)dx \; + \int_0^\infty h_{NT}(x)\phi_0(x)dx \tag{4}$$

The problem can be simplified further when $h_T(x)$ and $h_{NT}(x)$ are piecewise-linear in x as are payouts made by the firm to debtholders and equityholders. In order to do this, let $P_1(x)$ and $P_0(x)$ be the values of claims that pay \$1 of taxable and non-taxable income respectively, in the states when the firm's cash flow **exceeds** x. Hence,

$$P_1(x) = \int_x^\infty \phi_1(s)ds \tag{5a}$$

$$P_0(x) = \int_x^\infty \phi_0(s)ds \tag{5b}$$

$$P_1(\infty) = P_0(\infty) = 0 \tag{5c}$$

$$P_0(0) = (1 + R_0)^{-1} \tag{5d}$$

$$P_1(0) = (1 + R_1)^{-1} \tag{5e}$$

R_0 and R_1 are the riskless rates of return on non-taxable and taxable income respectively. In equilibrium, $R_1 \geq R_0 \geq 0$. Observe that the functions $P_1(x)$ and $P_0(x)$ have non-positive slopes and that they meet <u>at least once</u> (at infinity). Using (5) we obtain

$$V(h_T, h_{NT}) = -\int_0^\infty h_T(x)P_1'(x)dx \; - \int_0^\infty h_{NT}(x)P_0'(x)dx$$

Integrate the right-hand-side by parts to obtain

[8] Negative state prices provide arbitrage opportunities.

$$= - h_T(x)P_1(x)\Big|_0^\infty + \int_0^\infty h'_T(x)P_1(x)\,dx - h_{NT}(x)P_0(x)\Big|_0^\infty + \int_0^\infty h'_{NT}(x)P_0(x)\,dx$$

$$= \frac{h_T(0)}{(1+R_1)} + \frac{h_{NT}(0)}{(1+R_0)} + \int_0^\infty h'_T(x)P_1(x)\,dx + \int_0^\infty h'_{NT}(x)P_0(x)\,dx\,; \qquad (6)$$

We have assumed that $P_1(x)$ and $P_0(x)$ approach zero faster than $h_T(x)$ and $h_{NT}(x)$ approach infinity[9]. The above valuation equation is but a modification of (2) and is hence not new in itself. Though it can be used to value any security, its benefits show up when $h_T(x)$ and $h_{NT}(x)$ are piecewise-linear functions of x, with $h_T(0) = h_{NT}(0) = 0$. Since debt and equity are options on x, their payoffs satisfy these conditions, and consequently, we shall work with (6) for the rest of our analysis.

To use (6) in the context of a firm, we need to identify $h_T(x)$ and $h_{NT}(x)$. As indicated in the introduction, we shall ignore the existence of non-debt related tax shields and assume that firms issue debt contracts with a "principal" and an "interest" component. In addition, we shall assume the following:-

a) Principal repayment takes precedence over interest payment.[10].

b) Principal repayment is *not* tax-deductible at the corporate level, but neither is it subject to personal taxation.

c) The interest component of debt is tax-deductible at the corporate level, but is subject to personal tax.

d) Equity payments, like Principal repayments, are *not* tax-deductible at the corporate level, but are not subject to personal taxes.[11]

e) In the event that the firm issues many classes of debt with seniority & juniority provisions, **all** payments on senior debt must be made **before** the junior debtholders get anything[12]. This is

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[9] We could assume that the distribution of x, has a finite upper support point also, to justify equation (6).

[10] This is relevant to our analysis only in bankrupt states because now the firm will pay taxes, even though it does not pay interest. Refer Zechner and Swoboda (1986) for a discussion of principal-first versus interest-first doctrines in the context of single and multi-period models. They demonstrate that in a single-period framework, the principal-first doctrine is the appropriate assumption to make.

[11] Allowing for partial deductibility on personal account is rather straightforward, but the gains are not substantial to our analysis.

[12] Senior bond-holders will ensure this by writing it up as a covenant at the time that senior debt is issued.

to ensure that having two bonds is in fact different from having one big bond which is just the sum of the two.

In this scenario, if a firm has a single debt instrument with principal component D and interest component I, then the payouts to the bondholders are as follows when the firm's pre-tax cash flow equals x :

Cash Flow x	*Principal Repayment* $h_{NT}^{Debt}(x)$	*Interest Payment* $h_{T}^{Debt}(x)$	*Equity Payment* $h_{NT}^{Equity}(x)$
$0 \leq x \leq \frac{D}{1-\tau}$	$(1-\tau)x$	0	0
$\frac{D}{1-\tau} \leq x \leq \left(I+\frac{D}{1-\tau}\right)$	D	$x-\frac{D}{1-\tau}$	0
$\left(I+\frac{D}{1-\tau}\right) \leq x$	D	I	$(1-\tau)\left(x-I-\frac{D}{1-\tau}\right)$

Payouts to Debt and Equity holders

(Single Debt Instrument outstanding)

Table 1

It is easy to verify from equation (6) that

$$\text{Value of debt } = \int_0^{\frac{D}{1-\tau}} (1-\tau)P_0(x)\,dx + \int_{\frac{D}{1-\tau}}^{I+\frac{D}{1-\tau}} P_1(x)\,dx\,; \tag{7}$$

$$\text{Value of equity } = \int_{I+\frac{D}{1-\tau}}^{\infty} (1-\tau)P_0(x)\,dx\,; \tag{8}$$

If we assume that the corporate bond is issued at par[13], then D solves

$$D = \int_0^{\frac{D}{1-\tau}} (1-\tau)P_0(x)\,dx + \int_{\frac{D}{1-\tau}}^{I+\frac{D}{1-\tau}} P_1(x)\,dx\,. \tag{9}$$

[13] This is not a severe assumption in a one-period world. The assumption also precludes the firm from issuing bonds that pay out principal or interest exclusively. Refer to Green and Talmor (1985).

Now, let E be the value of the equity of the firm. Hence,

$$E = \int_{I+\frac{D}{1-\tau}}^{\infty} (1-\tau)P_0(x)dx \,. \tag{10}$$

The value of the firm, $V(D)$ is given by

$$\begin{aligned} V(D) &= D + E \quad ; && (11) \\ &= \int_0^{\frac{D}{1-\tau}} (1-\tau)P_0(x)dx + \int_{\frac{D}{1-\tau}}^{I+\frac{D}{1-\tau}} P_1(x)dx + \int_{I+\frac{D}{1-\tau}}^{\infty} (1-\tau)P_0(x)dx \,. && (12) \end{aligned}$$

Looking at figure 1, we note that the value of the firm is the sum of three areas under the two curves $(1-\tau)P_0(\bullet)$ and $P_1(\bullet)$. D equals the sum of areas $\mathcal{A}$ and $\mathcal{B}$, and this relation uniquely determines I. A value maximising firm will choose D so as to maximise the sum of the three shaded areas. In order to obtain an intuitive handle on the value-maximising firm's choice of debt, consider the following special example:
With reference to figure 1, let a_1 and a_2 be the values of x at which the two curves $P_1(x)$ and $(1-\tau)P_0(x)$ intersect. Assume that

$$\left\{ \begin{array}{l} (1-\tau)P_0(x) \geq P_1(x) \qquad \forall x \geq a_2 \\ \quad a_1(1-\tau) = \int_0^{a_1} (1-\tau)P_0(x)dx + \int_{a_1}^{a_2} P_1(x)dx \end{array} \right\} \tag{13}$$

Obviously, the firm would benefit by paying out its pre-tax cash flow as interest, whenever $P_1(x) \geq (1-\tau)P_0(x)$. In this particular example, the value of the firm is maximised by issuing debt with face value $a_1(1-\tau)$, because now the value of the firm is the area under $Max\{P_1(\bullet), (1-\tau)P_0(\bullet)\}$ between 0 and ∞.

If, as is most likely, (13) does not hold, then the solution to the firm's maximisation problem becomes considerably more complex. The firm maximises (12) by choosing D, subject to (9). In order to analyse this problem in more detail, it is useful to rewrite (12) in a couple of different forms with the help of (9). In particular, we shall identify the gains from leverage as a function of the level of debt issued. First of all, define,

$$\nabla(x) = P_1(x) - (1-\tau)P_0(x) \,; \tag{14}$$

and

$$\theta(x) = \int_0^x \nabla(s)ds \,. \tag{15}$$

$\nabla(x)$ represents the benefits of paying out the $(x+1)^{\text{th}}$ dollar of the firm's cash flow as interest over paying it out as equity. $\theta(x)$, on the other hand, represents the benefits of paying out the first 'x' dollars of the firm's cash flow as interest over paying it out as equity.

Using (9), (12), (14) and (15) we obtain that

$$V(D) = \left[\theta(I + \tfrac{D}{1-\tau}) - \theta(\tfrac{D}{1-\tau})\right] + \int_0^\infty (1-\tau)P_0(x)\,dx\,; \tag{16}$$

$$= D - \int_0^{I+\frac{D}{1-\tau}} (1-\tau)P_0(x)\,dx + \int_0^\infty (1-\tau)P_0(x)\,dx\,; \tag{17}$$

$$= \text{Gains from leverage} + \text{Value of Unlevered firm}\,. \tag{18}$$

The gains from leverage is seen to be the benefits of paying out every dollar of cash flow in excess of $\frac{D}{1-\tau}$, as interest rather than as equity income, subject to a maximum of I dollars being so paid. Since we treat principal repayment in a fashion identical to equity, this then is the only difference between a levered and an unlevered position. In Figure 1, the gain from leverage of (18) is the difference in the areas $(\mathcal{G} - \mathcal{F})$. From (16) we can easily notice that the value-maximising firm will issue debt if and only if there exists a debt level D and corresponding level of promised interest payment I, such that $\theta(I + \frac{D}{1-\tau}) > \theta(\frac{D}{1-\tau})$.[14] This is essentially the idea that is presented in the following theorem.

Theorem 1

(a) *A sufficient condition for* **all** *value-maximising firms to issue debt is*

$$\tau > \frac{R_1 - R_0}{1 + R_1}\,.$$

(b) *In the event that the above condition did not hold, a sufficient condition for any individual value-maximising firm to issue debt is*

$$\exists\, x^* > 0 \qquad \textit{such that} \qquad \theta(x) < 0 \;\; \forall\, x < x^* \quad \textbf{AND} \quad \theta(x^*) = 0\,.$$

(c) *A necessary condition for* **any** *value-maximising firm to issue debt, on the other hand is*

$$\exists\, x \qquad \textit{such that} \qquad \nabla(x) > 0\,.$$

[14] The first order conditions for the firm are easily derived and have been delegated to Appendix 1.

The proof of this theorem uses the following lemmas.

Lemma 1 *All debt requires interest.*

$$D > 0 \implies I > 0$$

Lemma 2 *Any increase in debt calls for increased interest payments.*

$$\frac{dI}{dD} > 0 \qquad \forall \quad D > 0$$

Proof Refer to Appendix 2.

The above lemmas are but straightforward statements of common intuition that all debt requires interest and that higher debt levels call for higher interest charges. A diagrammatic representation of theorem 1(a) is given in figure 2. Observe that area $\mathcal{G}$ represents the increase in firm value due to debt D. The positive increase was realised since $P_1(x) > (1-\tau)P_0(x)$ in some interval $(0, a_1)$.

Part (b) is represented diagrammatically in figure 3. In there,

$$area(ABG) = area(GKJ) \cdots \text{ by definition of } x^*$$

and

$$V(D) - V(0) = area(GJK) - area(CGF) \ .$$

Since $D > 0$, $area(CGF) < area(ABG)$, and hence, $V(D) > V(0)$.
Another way to appreciate this condition is to look at $\theta(\bullet)$. This representation is given in figure 4. It is clear there that

$$\theta(I + \tfrac{D}{1-\tau}) > \theta(\tfrac{D}{1-\tau})$$

and consequently, firm value is increased with debt. The intuition behind this result is as follows. Since $\theta(x^*) = 0$, there is no advantage in terms of firm value, to paying out the first x^* dollars as interest over paying it out as equity. In addition, since $\theta(x) < 0$ for all $x < x^*$, equity is preferred

to interest at low levels of cash flow. However, this means that paying out the first $(x^* - x)$ dollars of cash flow **in excess of** x, as interest, is preferred to paying it out as equity. Hence, if the firm chooses a debt level D, with corresponding interest I, such that $I + \frac{D}{1-\tau} = x^*$, the gains from leverage resulting from the benefits of paying interest (when cash flow exceeds $\frac{D}{1-\tau}$) are positive. This will induce a value-maximising firm to issue debt.

The obvious question at this juncture, is what types of firms will have such a $\theta(\bullet)$ function. To answer this, consider a firm whose cash flows are negatively correlated with aggregate consumption. When the firm has very small cash flows, aggregate consumption is likely to be large and if personal tax rates are progressive, it would be reasonable to expect high personal tax rates. With high personal tax rates, it might be preferable for the firm to have its cash flows taxed at the corporate level rather than have them taxed at the personal level – in other words, equity payouts would be preferable. This is a scenario where $\theta(x)$ is negative. On the other hand, when the firm has large cash flows, aggregate consumption would be "low" and so would personal tax rates. Consequently, interest payments may be preferred in this region and $\theta(x)$ will be positively sloped. Whether or not it will rise sufficiently to reach zero, will depend upon the structure of the personal tax code as well as the exact distribution of x and c. In any case, a firm whose cash flows are negatively correlated with aggregate consumption may have a $\theta(\bullet)$ function that satisfies condition (b) of the theorem.

The intuition behind (c) of the theorem is simple. All that we have said there is that if there is no advantage to paying the firm's cash flow as interest, then the firm will not issue debt. In our framework, this translates into $P_1(\bullet) > (1-\tau)P_0(\bullet)$.

Coming back to part (a) of the theorem, we observe that

$$\tau > \frac{R_1 - R_0}{1 + R_1}$$

is a sufficient condition for firms to issue debt, but is in general not necessary for the issue of debt. When the condition holds, however, we would expect **all** firms to issue debt. The values of $\tau, R_1, R_0, p_0(c), p_1(c)$ can be, however, reasonably assumed to be determined by macroeconomic factors, and consequently,

$$\tau - \frac{R_1 - R_0}{1 + R_1}$$

could be positive or negative[15]. In order to obtain an intuitive handle on why firms would issue

[15] Casual observation of reality indicates that this term is likely to be positive. τ, the corporate tax rate is around 0.5, whereas riskless interest rates are observed to rarely exceed 20% or so.

debt when this term is positive, we need to rewrite it in a slightly different form. Recall that

$$\begin{aligned} P_0(0) &= \int_0^\infty \int_C p_0(c) f(c|x)\, dc\, dx \\ &= \int_C p_0(c) \left(\int_0^\infty g(x|c)\, dx \right) dc \\ &= \int_C p_0(c)\, dc \\ &= \frac{1}{1+R_0}. \end{aligned}$$

and similarly

$$\begin{aligned} P_1(0) &= \int_C p_1(c)\, dc \\ &= \int_C ((1-t(c))\, p_0(c)\, dc \\ &= \frac{1}{1+R_1}. \end{aligned}$$

where $t(c)$ is the tax rate of the investor who is indifferent between debt and equity income when aggregate consumption equals c.

$$p_1(c) = (1-t(c))p_0(c) \quad \forall\, c \in C$$

We can interpret this investor as the *"indifferent (marginal) investor"* (Refer to DeAngelo & Masulis[1980]). He is indifferent between receiving taxable and non-taxable income when aggregate consumption is c, at the existing prices. Now define t^* as

$$(1-t^*)P_0(0) = P_1(0).$$

The investor who has a *constant* tax rate of t^*, will be indifferent between the purchase of riskless bonds that pay \$1 of taxable and non-taxable income at the end of the period. It is easy to verify from the above that

$$t^* = \int_C t(c) q(c)\, dc$$

Where

$$q(c) = \frac{p_0(c)}{\int_C p_0(w)\, dw}$$

If we interpret $q(c)$ as a probability measure, then t^* is in some sense, the *"expected tax rate"* of

the "indifferent" investor. Now from the proof of the theorem, we note that

$$\tau > \frac{R_1 - R_0}{1 + R_1} \iff P_1(0) > (1 - \tau)P_0(0)$$

Using our definition of t^*, this yields

$$\tau > \frac{R_1 - R_0}{1 + R_1} \iff t^* < \tau$$

This is essentially Miller's result. As long as the tax rate of the marginal investor lies below the corporate tax rate, there is an advantage in the issue of debt. In a world of certainty, firms will issue debt until the tax rate of the marginal investor equals the corporate tax rate. With uncertainty, however, equilibrium does not require that t^* be equal to τ. If it turns out that t^* is lower than τ, then we expect all value-maximising firms to issue debt. In part b) of the theorem, however, t^* is greater than τ (since $\nabla(0) < 0$) and in a world of certainty, we would expect to see no debt issued. With uncertainty, the same is not necessarily true. The form of $\theta(x)$, like $P_1(x)$ and $P_0(x)$ depends upon the joint distribution of aggregate consumption and firm's cash flow, $\Psi(c, x)$. Consequently, it maybe possible to have a firm that chooses to issue debt, no matter what macroeconomic variables we allow for.

In reality we observe firms with interior capital structures. The next theorem gives sufficient conditions for this to be true.

Theorem 2 *A sufficient condition for a value maximising firm to have an interior optimal capital structure of stock and bonds is*

$$\exists\, x^* \qquad \textit{such that} \qquad \forall\, x \gtrless x^*, \quad \nabla(x) \lessgtr 0$$

Proof Refer to Appendix 2.

Figures 5 and 6 provide a graphical interpretation of Theorem 2. In figure 5, if the debt level is increased beyond D^+, then area G will decrease. In addition, to make things worse, we will be taking the area under the lower curve $P_1(x)$ above x^*. Hence, value declines. In this case, since $\nabla(\bullet) < 0$ beyond x^*, equity is preferred to interest whenever cash flows are sufficiently high. In

addition, at low cash flow levels, interest payments are more valuable. Consequently, the firm would elect to make equity payments when its cash flows are large, and yet when its cash flows are small, it would prefer to make interest payments. This means that the firm would have *both* debt *and* equity in its capital structure.

In figure 6, $G(D^+)$ is the gain from leverage. Clearly, if debt level was increased from D^+, $G(D^+)$ will decline. We can also note that the condition of Theorem 2 can be relaxed. A less stringent condition that would work just as well is that $\theta(x)$ be increasing in $\left[\frac{D^+}{1-\tau}, x^*\right]$ and be decreasing in $[x^*, \infty]$. Yet another relaxation is that $\theta(x)$ be decreasing in $[x^*, \infty]$ where

$$x^* = \text{Arg Max}_{x} \left[\theta(x)\right] .$$

However, the condition was introduced corresponding to Figure 6 since this is precisely the characteristics of the $P(\bullet)$ functions of a firm that satisfies the *Increasing Monotone Likelihood Ratio Property*.

Summary

In this section we started by defining our notation. In equation (1) we specified how personal taxes are determined and then went on to equation (2) which specifies the value of any portfolio. This valuation equation uses price vectors that are defined on aggregate consumption. Since we are interested in the firm's cash flow, we transformed the valuation equation to value claims that are contingent upon the cash flows of the firm. This required that we obtain the values of some elementary claims whose payoffs were contingent on the cash flow of the firm. In equations (3) and (5), we defined such price vectors. $\phi_1(x)$ was defined as the price today, of a claim to a dollar of taxable income if the firm's cash flow is x. $\phi_0(x)$ was the price of a corresponding claim that paid a dollar of non-taxable income. $P_1(x)$ was then defined as the value of a claim that paid a dollar of taxable income when the cash flow of the firm exceeds x, and $P_0(x)$ was correspondingly defined for non-taxable income. Equation (6) is the modified valuation equation. We then obtain a table that shows the payoffs to bond-holders and stock-holders. From this table and equation (6), we learn that the value of the firm can be represented as the sum of three areas under the two curves $P_1(x)$ and $(1-\tau)P_0(x)$, where τ is the corporate tax rate. To make the problem facing the value-maximising firm more tractable, we further define $\nabla(x)$ and $\theta(x)$ in equations (14) and (15), as the benefits of paying interest. Equation (16) then gives us the gains to firm value as a function of the quantity of debt issued.

Theorem 1, which follows, specifies two mutually exclusive conditions under which the gain will be positive. The last part of the theorem states that $\theta(x)$ must slope upwards *somewhere* for there to be any potential gains from leverage. We then examine the first part of the theorem in greater detail and note that it is very similar to Miller's result in that as long as the corporate tax rate exceeds the tax rate of the marginal investor, firms will issue debt.

The section ends with Theorem 2. Here we obtained a condition that was sufficient to ensure an interior optimum capital structure of stocks and bonds. It turns out that this condition specifies a positive beta firm.

In the next section, we shall graphically depict the results obtained by Ross(1985). After that, we shall examine conditions under which firms may elect to issue multiple debt contracts and finally observe the effect of a change in tax rates on optimal debt levels.

1.2 The Monotone Likelihood Ratio Property

The form of the value function could be studied in more detail by assuming specific forms of the joint distribution of aggregate consumption and the firm's cash flow – $\Psi(c, x)$. Ross(1985) assumes that $\Psi(c, x)$ satisfies the *Monotone Likelihood Ratio Property* (MLRP). By doing so, he narrows down the value function to a few possible shapes. The framework that we developed in the earlier section makes his results easier to visualise and that is what we shall do here in this section. The main objective of this section is to demonstrate the ease with which an otherwise complex problem can be comprehended when we express it in terms of the framework we have developed. Consequently, all of this section, with the exception of equations (24) and (25) can be skipped without any loss of continuity. Equations (24) and (25) yield the value of the firm when we do not distinguish between the principal and interest components of debt. A summary is provided at the end of the section which contains the essential points that we make here.

In this section we make the same assumptions as Ross(1985), and integrate his results with our $P(\bullet)$ functions. In keeping with Ross, we shall also assume that all debt is tax-exempt at the corporate level and taxable at the personal level. In other words, we shall not distinguish between principal and interest components of debt. The validity of Theorems 1 and 2 under these new assumptions is also demonstrated.

The MLRP requires that the conditional distributions $f(c|x)$ **and** $g(x|c)$ be monotone functions of the conditioning variable. If the monotonicity is positive, in the sense that higher levels of c are likely to result in higher levels of x, then the firm is said to be a MLRP positive beta firm. Similarly, if the monotonicity is negative, the firm is said to be a MLRP negative beta firm. In particular, for the MLRP positive (negative) beta firm, if $x' > x''$, then

$$\frac{g(x'|c)}{g(x''|c)}$$

is monotone increasing (decreasing) in c. In addition, for the positive (negative) beta firm, if $c' > c''$, then

$$\frac{f(c'|x)}{f(c''|x)}$$

is monotone increasing (decreasing) in x. The MLRP is sufficient to identify the sign of a firm's beta, but is in general not a necessary condition for the sign of a firm's beta. This is to say that all firms that satisfy the "increasing" ("decreasing") MLRP condition are positive (negative) beta firms, **but** all positive (negative) beta firms *do not* satisfy the "increasing" ("decreasing") MLRP condition.

Recall from the earlier section that $p_1(c)$ and $p_0(c)$ are the values of claims that pay \$1 of taxable and non-taxable income respectively, when aggregate consumption equals c and that $t(c)$ is the tax-rate of the *indifferent investor*.

Assumption *(Progressivity)*

$$\frac{\mathbf{d}}{\mathbf{d}c}\big(t(c)\big) > 0$$

Hence, $\exists\ c^*$ such that

$$t(c) \gtrless \tau \qquad \Longleftrightarrow \qquad c \gtrless c^*$$

or equivalently,[16]

$$\big(t(c) - \tau\big)p_0(c) \gtrless 0 \qquad \Longleftrightarrow \qquad c \gtrless c^*$$

Lemma 3 *For the MLRP positive beta firm, there exists a cash flow level x^+ such that*

$$P_1'(x) - (1-\tau)P_0'(x) = \int_0^\infty \big(t(c) - \tau\big)p_0(c)f(c|x)J(x,c)\ dc \lessgtr 0 \quad \Longleftrightarrow \quad x \lessgtr x^+$$

Similarly, for the MLRP negative beta firm, there exists a cash flow level x^+ such that

$$P_1'(x) - (1-\tau)P_0'(x) = \int_0^\infty \big(t(c) - \tau\big)p_0(c)f(c|x)J(x,c)\ dc \gtrless 0 \quad \Longleftrightarrow \quad x \lessgtr x^+$$

This lemma and its proof is identical to Lemmas 1 and 2 in Ross(1985) and so we shall not reproduce the proof here. From the lemma we note that for the MLRP positive beta firm, there exists a cash flow level x^+ such that[17]

$$Slope\,(P_1(x)) \lessgtr Slope\big((1-\tau)P_0(x)\big) \quad \forall\ x \lessgtr x^+$$

[16] If c^* does not exist in the interior of C, then only *one* of the inequalities will be valid. To ensure that c^* lies in the interior of C we need that $t_{min} < \tau < t_{max}$

[17] A similar condition can be obtained for the MLRP negative beta firm by simply interchanging the inequalities.

The intuition behind this is simple. Recall that $P'(\bullet) = -\phi(\bullet)$. Hence the above result is equivalent to

$$\phi_1(x) \gtrless (1-\tau)\phi_0(x) \quad \forall\, x \lessgtr x^+$$

Now, since lower cash flows for the MLRP positive beta firm go hand in hand with lower expected aggregate consumption and consequently, by the assumption of progressivity, lower personal tax rates, it is preferable to pay out the firm's cash flow as debt rather than as equity, when the firm's cash flow is "low". Similarly, "high" cash flow states dictate that it is better to pay out the cash flow as equity.

From (5) we know that the curves $P_1(x)$ and $(1-\tau)P_0(x)$ meet at $x = \infty$. In addition, $P_1(\bullet)$ has a greater slope than $(1-\tau)P_0(\bullet)$ for all $x > x^+$. Hence, if $x^+ < \infty$, then the two curves will not meet at any finite point *above* x^+. Again, since $P_1(\bullet)$ has a lesser slope than $(1-\tau)P_0(\bullet)$ for all $x < x^+$, they can meet at most once in $(0, x^+)$. A similar argument can be used for MLRP negative beta firms. This implies that *for all* firms that satisfy the MLRP, the curves $P_1(x)$ and $(1-\tau)P_0(x)$ meet *at most* twice; one of the meetings being at the upper support point of the distribution of cash flows. If we assume that for the MLRP positive beta firm, the $P(\bullet)$ curves cross at x^* $(< x^+)$, then,

$$\nabla(x) \gtrless 0 \qquad \forall\, x \lessgtr x^*$$

and we obtain figures 5 and 6.

The assumption that **all** payments made to debtholders are tax deductible at the corporate level and taxable at the personal level, modifies the after-tax payouts received by debt-holders and equity holders as follows :

$$\begin{aligned} h_T^{Debt}(x) &= Min\{x, F\} \\ h_T^{Equity}(x) = h_{NT}^{Debt}(x) &= 0 \\ h_{NT}^{Equity}(x) &= (1-\tau)Max\{x - F, 0\} \end{aligned}$$

where F is the face value of debt issued by the firm. Therefore from (6), we obtain that

$$\begin{aligned} V(F) &= \int_0^F P_1(x)\,dx + \int_F^\infty (1-\tau)P_0(x)\,dx \qquad (24) \\ &= \theta(F) + \int_0^\infty (1-\tau)P_0(x)\,dx \\ &= \theta(F) + V(0)\,. \qquad (25) \end{aligned}$$

Maximising value now involves maximising $\theta(F)$. Taking the derivatives of $\theta(\bullet)$, we note that at

optimum $P_1(F) = (1-\tau)P_0(F)$. For the MLRP positive beta firm, from figures 5 and 6, the optimum is clearly at x^*.

The results of theorem 2 are valid in this framework [MLRP firms and valuation as per (25)] as well, since all that we have stated there is that if the first and second order conditions needed for an optimum be satisfied at a finite debt level, then the firm will obtain an interior optimal capital structure of stocks and bonds.

The results of theorem 1 are valid as well, but this is not so obvious. Firstly, observe that if condition (a) of the theorem held, then there is a region $(0, x^*)$ where $P_1(x) > (1-\tau)P_0(x)$. By issuing debt with face value $F \in (0, x^*)$, the firm's value increases. Hence, it would issue some debt. To see part (b), note that since for the MLRP firm, $P_1(\bullet)$ and $(1-\tau)P_0(\bullet)$ cross at most once in the interior of $(0, \infty)$, $\nabla(\bullet)$ changes sign at most once. This implies that $\theta(\bullet)$ changes slope at most once. Figures 7, 8, 9, 10, 11 and 12 show the various forms that $\theta(x)$ can have.
With figures 7 and 8, there is an internal optimum[18]. With figures 9 and 11,the firm keeps on issuing debt, whereas, with figures 10 and 12, no debt is issued.
The condition suggested in theorem 1(b), restricts us to figure 11. This is a MLRP negative beta firm which issues debt with infinite face value[19].

[18] These are MLRP positive beta firms with a finite x^*.

[19] Debt value may be bounded since $\int_0^\infty P_1(x)dx$ may be finite.

Summary

In this section, we consider firms that satisfy the *Monotone Likelihood Ratio Property* (MLRP). With the additional assumption of progressive personal tax rates, we obtain that for such firms, $\theta(x)$ is restricted to a few possible forms. These may be seen in figures 7 through 12. All the cases where $\theta(x)$ is negatively sloped at "high" cash flow levels refer to positive beta firms whereas the others refer to negative beta firms. In keeping with Ross(1985), we relax the distinction between the principal and interest components of debt contracts and assume that *all* payments made to bond-holders are tax-deductible at the corporate level and taxable at the personal level. Under this simplification, the value of the firm is given by equation (25). Since value-maximising firms are now maximising $\theta(\bullet)$, we can easily see which firms will issue debt and which ones will not by simply viewing figures 7 through 12.

In the section that follows, we shall examine cases where firms will issue multiple debt contracts.

Chapter 2

2.1 The case for multiple debt issue

The case for multiple debt issue with various levels of seniority has no relevance when we fail to differentiate between principal and interest components since debt is anyway homogeneous. The distinction between firm value with a single debt issue and multiple debt issues takes on relevance when we differentiate between principal and interest components, however. We shall assume that senior bondholders protect their interests by ensuring that any junior issue, however large, will not affect their profile of state contingent payoffs. This requires that they be paid in full *before* the junior bondholders get anything.

Denote by D^s, I^s, D^j, I^j the principal and interest components of senior and junior debt respectively. The payments received by the various concerned parties can easily be verified to be as follows:-

	$0 \le x \le x_1$	$x_1 \le x \le x_2$	$x_2 \le x \le x_3$	$x_3 \le x \le x_4$	$x \ge x_4$
Senior Debt Principal	$(1-\tau)x$	D^s	D^s	D^s	D^s
Senior Debt Interest	0	$x - x_1$	I^s	I^s	I^s
Junior Debt Principal	0	0	$(1-\tau)(x - x_2)$	D^j	D^j
Junior Debt Interest	0	0	0	$x - x_3$	I^j
Equity	0	0	0	0	$(1-\tau)(x - x_4)$

where

$$x_1 = \frac{D^s}{1-\tau} \qquad x_2 = I^s + \frac{D^s}{1-\tau} \qquad x_3 = I^s + \frac{D^s + D^j}{1-\tau} \qquad x_4 = I^s + I^j + \frac{D^s + D^j}{1-\tau}$$

Payouts to Debt and Equity Holders

(Multiple Debt issues outstanding)

Table 2

If the two bonds are issued at par, then from equation (6) it follows that interest payments I^s and I^j are determined as follows:-

$$D^s = \int_0^{\frac{D^s}{1-\tau}} (1-\tau)P_0(x)\,dx + \int_{\frac{D^s}{1-\tau}}^{I^s + \frac{D^s}{1-\tau}} P_1(x)\,dx \tag{26}$$

$$D^j = \int_{I^s + \frac{D^s}{1-\tau}}^{I^s + \frac{D^s + D^j}{1-\tau}} (1-\tau)P_0(x)\,dx + \int_{I^s + \frac{D^s + D^j}{1-\tau}}^{I^j + I^s + \frac{D^s + D^j}{1-\tau}} P_1(x)\,dx \tag{27}$$

The firm's value is given by

$$V(D^s, D^j) = D^s + D^j + E \tag{28}$$

where

$$E = \text{Value of Equity} = \int_{I^j + I^s + \frac{D^s + D^j}{1-\tau}}^{\infty} (1-\tau)P_0(x)\,dx \tag{29}$$

As before, we can simplify this to obtain

$$V(D^s, D^j) = \left\{\theta(I^j + I^s + \tfrac{D^s+D^j}{1-\tau}) - \theta(I^s + \tfrac{D^s+D^j}{1-\tau})\right\} + \left\{\theta(I^s + \tfrac{D^s}{1-\tau}) - \theta(\tfrac{D^s}{1-\tau})\right\} + V(0,0) \tag{30}$$

The value maximizing firm will maximize (30) subject to (26) and (27). It turns out that the issue of two (or more) debts *may* be preferred to the issue of single debt by a value maximizing firm. To see this refer to figure 13. In this figure, we have drawn a particular $\theta(\bullet)$. Were the firm to issue two debts, the gain from leverage is $G1 + G2$. An upper bound on the gain from leverage when the firm issues only one debt instrument is easily seen to be[20] $\left\{ \underset{x}{\text{Max}} \left(\theta(x)\right) - \underset{x}{\text{Min}} \left(\theta(x)\right) \right\}$. If $G1 + G2$ is greater than this bound, it is clear that the value-maximizing firm would prefer to issue two bonds. However, much weaker conditions are sufficient to ensure that the firm would prefer to issue two bonds as we see below[21].

Theorem 3 *If D^* and I^* maximize (16) subject to (9) and (10), then either one of the following two conditions is sufficient to induce a value-maximising firm to issue two bonds at optimum*

a) $\nabla(I^* + \frac{D^*}{1-\tau}) > 0$
b) $\exists x^* > \left(I^* + \frac{D^*}{1-\tau}\right)$ *such that* $\theta(x) < 0 \ \forall x \in \left[\left(I^* + \frac{D^*}{1-\tau}\right), x^*\right]$, *and* $\theta(x^*) = 0$

Proof Refer to Appendix 3.

Theorem 2 of section 2, which outlines a sufficient condition a firm to have an optimal interior capital structure, is easily observed to be valid as well when we allow multiple issue. Essentially, the value-maximising firm does not wish the bankruptcy determining state $\left(I^j + I^s + \frac{D^s + D^j}{1-\tau}\right)$ to be in the region where $\theta(x)$ is downward sloping, and hence ends up issuing some equity.

The issue of interest, next, is as to whether it is necessary for the $P(\bullet)$ functions to cross more than once for us to obtain multiple debt issue. The answer to this can be obtained if we look at the first order condition at optimum when only one debt is issued.

$$\begin{aligned} \frac{dV}{dD} &= 0 \\ &\Updownarrow \\ \nabla(I + \tfrac{D}{1-\tau}) &= \nabla(\tfrac{D}{1-\tau}).P_0(I + \tfrac{D}{1-\tau}) \end{aligned} \tag{31}$$

[20] This bound may of course never be realised.

[21] The intuition behind this theorem is identical to the intuition behind Theorem 1 and is not explicitly outlined here. A brief version of the intuition is presented in the summary that follows this section.

Clearly, at optimum $\nabla(I + \frac{D}{1-\tau})$ need not equal zero[22]. If it turns out to be positive, then, from Theorem 3, the firm would issue junior debt. Observe that the $\theta(\bullet)$ function is uni-modal when the $P(\bullet)$ functions cross each other just once. Figures 14 and 15 show two cases where the $\theta(\bullet)$ function has but a single mode. In both cases, the issue of a junior debt instrument raises the gain from leverage by $G2(> 0)$.

[22] In other words, the $P(\bullet)$ functions need not intersect. What is needed, however, is that $\nabla(I + \frac{D}{1-\tau})$ have the same sign as $\nabla(\frac{D}{1-\tau})$.

Summary

In this section, we obtain conditions that will induce a value-maximising firm to issue debt contracts to two classes of bond-holders - the senior and the junior. Equation (30) gives us the value of a firm that has issued two such debt contracts, where the superscripts 's' and 'j' refer to the senior and junior debt respectively. The last term in that equation is the value of an all-equity firm and the first and second terms are the gains from leverage from issuing the junior and senior debt respectively. The intuition behind why the firm may issue junior debt is can be best appreciated if we start off by assuming that the firm is only issuing senior debt. Let us assume that the second term in (30) is maximised at D^*, I^*. Now if $\theta(x)$ is positively sloped at $I^* + \frac{D^*}{1-\tau}$, then by issuing a very small amount of junior debt, the firm makes the first term in (30) positive. This will increase its value beyond the maximum value it could have realised when there was only one group of bond-holders, and consequently, any value-maximising firm would choose to issue junior debt.

The next question that we analyse in this section is whether we need a firm whose $\theta(\bullet)$ function is multi-modal for the existence of junior debt. It turns out that this is not so. In fact, even a firm that satisfies the MLRP may choose to issue junior debt contracts.

In essence, multiple seniority levels of debt allow the firm to make interest payments (instead of principal payments) in the low payoff (and low consumption) states of the world where personal taxes (on interest) are lower, but the corporate interest tax shield is still high (corporate tax rate is flat).

It is important to note that though our analysis here considered only two debt contracts, the analysis is rather similar when we consider more than two types of debt contracts that are differentiated by seniority. This opens up the problem of an optimal ***number*** of debt contracts. To see this, let V_N be the value of a firm that has ***optimally*** issued N different debt contracts. It is rather easy to verify that V_N is non-decreasing in N, (and bounded as N increases, since V_∞ is at best the area under Max $[P_1(x)\,,\,(1-\tau)P_0(x)]$) but the curvature properties of V_N are an unexplored area. If V_N turns out to be a concave function of N, then the imposition of an issue cost to debt would yield an optimum number of debt contracts as well as an optimum level of debt.

Chapter 3

The effect of the tax rates

3.1 The tradeoff between tax-savings and riskiness

Conventional wisdom tells us that if the corporate tax rate were to rise, then the present value of tax shields generated by debt would also rise and consequently debt becomes more attractive to value-maximising firms. This would result in increased optimum debt levels at the level of individual firms as well as at the aggregate level of the economy. This intuition is valid when we fail to distinguish between the principal and interest components of debt, but it turns out that the effect of a rise in the corporate tax rate on the firm's optimal level of debt, is not unambiguous when we distinguish between principal and interest components of debt. The ambiguity stems from the interaction of two effects –

1) The increased value of tax-savings generated.

2) The resulting increase in the riskiness of debt calling for greater interest payments.

The second component comes up solely as a consequence of the fact that debt principal is taxable at the corporate level. This component is consequently not present in a framework where debt is homogeneously treated for tax purposes. To better understand the first component, we shall begin our analysis of the effect of a change in the corporate tax rate on the firm's optimal debt choice by assuming away the principal-interest distinction for now.

When we do not distinguish between principal and interest, the effect of changing the corporate tax rate, τ, on the firm's optimal choice of debt is rather clear to visualise. Intuitively, if the corporate tax rate were to rise, the benefit realised from every dollar of debt in the form of tax savings generated goes up and we would consequently expect the firm to issue larger quantities of debt at optimum. This intuition is presented in Figure 16. Originally with the tax rate at τ_1, the firm issues debt with face value F^* since its value would be maximised at this level. If the tax rate rose to τ_2 ($\tau_2 > \tau_1$), the firm would issue debt with face value F^{**} ($F^{**} > F^*$) at optimum.

Even when we do not have such simple curves, (which intersect only once) the result follows through. However, we must be careful since the value of the firm may obtain local optima at a multitude of interior debt levels, and we are interested in the *global* optimum. The intuition is clearer if we look at $\theta(\bullet)$. Recall that value maximising was equivalent to [Equation (25)]

$$\underset{F}{\text{Max}}\, \theta(F) \;=\; \int_0^F \left[P_1(x) - (1-\tau)P_0(x)\right] dx\,.$$

We also know that

$$\frac{d}{d\tau}\left(\theta'(F)\right) \;=\; P_0(F) \;>\; 0\,.$$

This suggests that the marginal benefit of issuing debt rises as the corporate tax rate rises. The question, however, is why can't we have a global optimum at any point *below* F^*, the original global optimum ? The intuition behind the answer to this lies in the fact that as τ rises, $(1-\tau)P_0(x)$ falls, *but,* the magnitude of its decrease is larger at lower levels of x. In order to show this technically, let F^* be the optimum when the tax rate was τ_1. Therefore,

$$\theta(F^*) \;\geq\; \theta(F) \qquad \forall\, F. \tag{32}$$

Now, when τ_1 goes to τ_2 ($> \tau_1$), let $\theta(F)$ go to $\theta_2(F)$.

Lemma 4 *The increase in $\theta(x)$, resulting from an increase in τ, is larger at higher cash flow levels.*

$$\left\{\theta_2(x_2) - \theta(x_2)\right\} \;>\; \left\{\theta_2(x_1) - \theta(x_1)\right\} \qquad \forall\, x_2 > x_1$$

Proof Refer to Appendix 4.

From the above Lemma,

$$\theta_2(F^*) - \theta(F^*) \ > \theta_2(F) - \theta(F) \qquad \forall\, F \leq F^* \tag{33}$$

From (32) and (33)

$$\theta_2(F^*) \ \geq \theta_2(F) \qquad \forall\, F \leq F^*$$

Hence the new optimum, F^{**} must satisfy $F^{**} \geq F^*$. A similar argument can be made for the case when personal tax rates rise at all levels of consumption. In this case, the curve $P_1(x)$ shifts down and the consequence is to have lower optimal debt levels.

When we distinguish between principal and interest, we introduce a new dimension to the analysis however. Consider the effect of a rise in the corporate tax rate τ. The benefits of debt rise since the present value of tax shields generated go up. On the other hand, debt becomes more risky since the repayment of debt principal (which is <u>not</u> tax deductible) calls for higher pre-tax cash flows. As a consequence, debtholders will demand more interest. This further raises the pre-tax cash flows required to prevent bankruptcy. Equity value is reduced due to both the lowering of $(1-\tau)P_0(\bullet)$ and the raising of the cash flow level required to start payments to stockholders. Consequently, at any fixed debt level, D, the value of the firm declines with τ. Since a rise in the corporate tax rate increases the benefits of debt resulting from increased tax shields per dollar of interest payments while increasing the costs of debt owing to increased interest payments, it is not clear as to whether or not more debt will be issued. The intuition that increasing the corporate tax rate makes debt riskier requiring greater promised interest payments is confirmed below.

Theorem 4 *The promised interest payments required to support any level of debt, rises as the corporate tax rate rises.*

Proof Refer to Appendix 4.

Corollary *A firm that maintains its debt level when the corporate tax rate rises will suffer a reduction in value.*

Reason The value of the equity in the firm will decline since greater cash flows are now needed to

begin payments to equity holders. Analytically, using (10):

$$V = D + E\,.$$

Hence

$$\begin{aligned}\left.\frac{dV}{d\tau}\right|_{D\ fixed} &= \left.\frac{dE}{d\tau}\right|_{D\ fixed} \\ &= -(1-\tau)P_0(I+\tfrac{D}{1-\tau})\left[\frac{D}{(1-\tau)^2}+\frac{dI}{d\tau}\right] - \int_{I+\frac{D}{1-\tau}}^{\infty} P_0(x)\,dx \\ &< 0\end{aligned}$$

The results of theorem 4 and the resulting corollary are clarified in figure 17. At $\tau = \tau_1$ and debt level D, let interest payable be I_1. From (9) we have

$$Area[ACGO] + Area[FJLG] = D$$

Now, if we increase τ to τ_2, let $(1-\tau)P_0(\bullet)$ shift down to the dotted version shown. Since $P_0(\bullet) < 1$ and $P_0'(\bullet) \leq 0$, the area under $(1-\tau_2)P_0(x)$ to the left of $\frac{D}{1-\tau_2}$ will be less[23] than the area under $(1-\tau_1)P_0(x)$ below $\frac{D}{1-\tau_1}$. Hence, for (9) to hold, we must increase the interest component (because we need greater area from under $P_1(\bullet)$). Also, since both $\frac{D}{1-\tau}$ and I have risen as a consequence of increased taxes, the area under $(1-\tau)P_0(\bullet)$ beyond $\left(I+\frac{D}{1-\tau}\right)$ (viz. the value of Equity), drops[24]. Even though the higher tax rate provides for increased tax shields, they are insufficient to cover the reduction in value resulting from increased interest payments.

To get a handle on the effect of an increase in τ on the optimal level of debt, let the optimum be at D^*, *prior* to the increase in the corporate tax rate. Now from the corollary, we know that the value of the firm will decrease *after* the increase in τ. If the *magnitude* of decrease were larger at all levels of debt, $D < D^*$, then it must be that the new optimum level of debt cannot be lower than D^*. Analytically stated, a sufficient condition for optimal debt levels to rise with an increase in the corporate tax rate is that

$$\frac{d^2 V}{dD\,d\tau} > 0 \qquad \forall\ D,\tau$$

If we compute the required second derivative, and impose the above condition, and simplify, this yields the following condition as being sufficient to ensure that optimal debt levels will rise with

[23] This can easily be verified by integrating the relevant terms

[24] To make things worse, the curve also has shifted down!

the corporate tax rate:

$$\mathcal{T}.\mathcal{U}.\mathcal{V} > \mathcal{W} \qquad \forall\, D,\tau$$

where

$$\begin{aligned}
\mathcal{T} &= \left[\frac{D}{(1-\tau)^2} + \frac{dI}{d\tau}\right] \\
\mathcal{U} &= \left[1 - \tau + \nabla(\tfrac{D}{1-\tau})\right] \\
\mathcal{V} &= \frac{P_1'(I + \frac{D}{1-\tau})}{P_1(I + \frac{D}{1-\tau})} - \frac{P_0'(I + \frac{D}{1-\tau})}{P_0(I + \frac{D}{1-\tau})} \\
\mathcal{W} &= \left[\nabla'(\tfrac{D}{1-\tau})\left(\frac{D}{(1-\tau)^2}\right) - 1\right].
\end{aligned}$$

$\mathcal{T}$ and $\mathcal{U}$ can easily be verified to be unambiguously non-negative, but in general nothing can be said about $\mathcal{V}$ and $\mathcal{W}$. They could well take on values that will violate the condition cited. Hence, the direction in which the optimum will move, is not clear.

The effect of an increase in personal tax rates can be analysed in a way similar to the above. If personal tax rates were to rise at <u>all</u> levels of taxable income, ceteris paribus, the present value of a dollar of taxable income would decrease. In our framework, an increase such as this would imply an increase in $t(c)$, the marginal tax rate of the "indifferent investor". This in turn implies a decrease in $p_1(c)$ and consequently in $\phi_1(x)$. Equation (5a) tells us that the increase in personal tax rates would lower $P_1(\bullet)$. Let the resulting decrease in $P_1(x)$ be denoted by $\delta(x)$. Thus

$$P_1^{new}(x) = P_1^{old}(x) - \delta(x) \tag{34}$$

It is easily verified that

$$\delta(x) > 0\,;\ \delta'(x) < 0\ \forall\, x \tag{35}$$

Again, let $\nabla(x)$ go to $\nabla_2(x)$ and $\theta(x)$ go to $\theta_2(x)$. Hence

$$\begin{aligned}
\nabla_2(x) &= \nabla(x) - \delta(x) \\
\theta_2(x) &= \theta(x) - \int_0^x \delta(s)\,ds
\end{aligned}$$

When we do not distinguish between principal and interest components of debt, it is rather straightforward to observe that this results in lowered optimal debt levels. To get a feel for this, refer to figure 16. If $P_1(x)$ were to move down, it would intersect $(1-\tau)P_0(x)$ earlier, resulting in

a lower optimum F^*.
Analytically, this can be shown as follows ·····

Let F^* be the old optimum <u>before</u> the increase in tax rates. Hence,

$$\theta(F^*) \geq \theta(F) \quad \forall F \tag{36}$$

Now,

$$\theta_2(x) - \theta(x) = -\int_0^x \delta(s)ds \tag{37}$$

Also since $\delta(s) \geq 0 \, \forall \, s$, this implies that

$$\theta_2(x_2) - \theta(x_2) \leq \theta_2(x_1) - \theta(x_1) \quad \forall \, x_2 > x_1 \tag{38}$$

hence

$$\theta_2(F) - \theta(F) \leq \theta_2(F^*) - \theta(F^*) \quad \forall \, F > F^* \tag{39}$$

From (36) and (39) we obtain

$$\theta_2(F) \leq \theta_2(F^*) \quad \forall \, F > F^* \tag{40}$$

Hence the new optimum $F^{**} \leq F^*$.

As with the case with τ, when we differentiate between principal and interest, it is no longer necessary that increasing personal taxes will induce lower optimal debt levels. It is true, however, that since the value of every dollar of tax shields generated is lowered (due to the fact that these payouts will be taxed at higher levels - evident by the fact that $P_1(\bullet)$ falls), firms will have to promise to pay increased interest charges in order to maintain the same debt level. Though at first sight this appears to be sufficient cause for lowered optimal debt levels, closer inspection indicates otherwise. To get an intuitive feel for the tradeoffs present here, refer to figure 18 and consider an increase in $t(c) \, \forall \, c \in \mathcal{C}$. From (34) and (35), we notice that the decline in $P_1(\bullet)$ is greater at lower cash flow levels than at higher cash flows. Let (D^+, I^+) be the optimum prior to the increase in personal tax rates. If after the change in tax rates, the firm were to maintain debt at D^+, let the new increased interest charges be I_+^{++}. On the other hand, if the firm were to reduce its debt level to D^{++}, let the interest be $I_{++}^{++} (< I_+^{++})$. From the figure

$$V(D^+) = V(D^{++}) + \alpha - \beta + \text{Area } \mathcal{B}$$

Clearly, the firm would be better off by leaving the debt level at its old level. It is also evident that some firms may in fact raise their debt levels[25].

[25] For example, if Area $\mathcal{B}$ were negative

The tradeoff arises from the following :-

a) The lowering of the attractiveness of tax shields in the vicinity of $\frac{D}{1-\tau}$ and $I + \frac{D}{1-\tau}$.

b) By lowering D, however, what we gain near $\frac{D}{1-\tau}$ may be more than offset by what we lose at $I + \frac{D}{1-\tau}$ since the <u>*change*</u> in the relative values of taxable and non-taxable income has been less at $I + \frac{D}{1-\tau}$ than at $\frac{D}{1-\tau}$ (and also since $\frac{dI}{dD}$ has changed).

An analytical evaluation of this scenario is made easier when we impose a specific form to the function $\delta(\bullet)$. In particular, assume that $\delta(x) = \alpha P_1(x)$. This implies that

$$P_1^{new}(x) \;=\; (1-\alpha)P_1^{old}$$

By differentiating the appropriate equation and simplifying the terms we obtain

$$\left.\frac{dI}{d\alpha}\right|_{D\ fixed} = \frac{\int_{\frac{D}{1-\tau}}^{I+\frac{D}{1-\tau}} P_1(x)\,dx}{(1-\alpha)P_1(I+\frac{D}{1-\tau})} > \frac{I}{1-\alpha} > 0 \tag{41}$$

Hence, as in the analysis with τ, we can look at $\frac{d^2 V}{d\alpha\, dD}$ to obtain a sufficient condition for reduced debt levels

This yields

$$\frac{d^2 V}{d\alpha\, dD} \;=\; -(1-\tau)\{\mathcal{U} + \mathcal{V}\mathcal{W}\} \tag{42}$$

where

$$\mathcal{U} \;=\; P_0'(I+\tfrac{D}{1-\tau})\left.\frac{dI}{d\alpha}\right|_{D\ fixed} \cdot \frac{1-\tau+\nabla(\frac{D}{1-\tau})}{(1-\tau)(1-\alpha)P_1(I+\frac{D}{1-\tau})}$$

$$\mathcal{V} \;=\; \left\{\frac{P_0(I+\frac{D}{1-\tau})}{(1-\tau)(1-\alpha)^2\left(P_1(I+\frac{D}{1-\tau})\right)^2}\right\}$$

$$\mathcal{W} \;=\; \left\{\left[\alpha P_1(\tfrac{D}{1-\tau})P_1(I+\tfrac{D}{1-\tau})\right] - \left[(1-\tau).\mathcal{Z}.\left(1-P_0(\tfrac{D}{1-\tau})\right)\right]\right\}$$

$$Z = (1-\alpha)P_1'(I+\tfrac{D}{1-\tau}) \left.\frac{dI}{d\alpha}\right|_{D\ fixed} - P_1(I+\tfrac{D}{1-\tau})$$

$$\nabla(\bullet) = (1-\alpha)P_1(\bullet) - (1-\tau)P_0(\bullet)$$

It is easy to verify that

$$\mathcal{U} < 0 \ ; \ \mathcal{V} > 0 \ ; \ \mathcal{W} > 0$$

This suggests that the second derivative in (42) could be of any sign. Hence, no definite conclusions can be drawn of the direction that optimal debt will move in, in the absence of any further assumptions.

Summary

In this section, we have analysed the impact that a change in tax rates (corporate and personal) will have upon the optimal level of debt in a firm's capital structure. It is observed that there are two forces that come into play that determine the attractiveness of debt when tax rates change. It turns out that these forces are in opposing directions and consequently the resultant could be in any direction. For example, consider an increase in the corporate tax rate. This will make the tax shields generated by debt more valuable and consequently the attractiveness of debt will rise. However, the increased corporate tax rate also makes debt riskier since higher pre-tax cash flows are needed to fully repay debt principal. Bond holders will therefore demand more interest and this will reduce equity value and thus be a decrease in the attractiveness of debt. It turns out that no single force dominates the other and therefore in general we cannot unambiguously predict the direction in which optimal debt levels will move.

If we do not distinguish between the principal and interest components, however, it is true that an increase in the corporate tax rate will result in increased debt levels at optimum.

Conclusion and future research potential

What we have obtained here is a purely tax motivated rationale for value-maximising firms to issue multiple debt instruments along with equity. In addition, we observed that debt issue is not the prerogative of firms with any specific characteristics, nor is it of economies with specific parameters. We were unable, however, to establish a monotonic relationship between tax rates and optimal debt levels at the level of an individual firm.[26]

One obvious direction for future research is to examine the firm's optimal capital structure choice in a multi-period framework. In a multi-period world, the firm could be assumed to face the problem of issuing debt which pays interest every period, as well as issuing equity that pays dividends each period. This choice will be made subject to the prevailing $P(\bullet)$ functions in the market. At the end of any future period, it is very likely that the $P(\bullet)$ functions will have changed, in which case the firm reoptimises. This process of reoptimisation *may* call for the issue of new debt and in case some earlier debt contract has matured, the firm may choose to rollover its debt. Even dividend payout structures may have to be changed. Though this extension appears to be very rewarding, there are a few major hurdles that one must cross.

a) Very severe assumptions may have to be made on the way the $P(\bullet)$ functions change.

b) Determination of bankruptcy

[26] If the relationship is *not* monotonic, then it should be possible to simulate scenarios where the lack of monotonicity is evident.

The latter problem,[27] does require some very careful modelling. In a one-period world, it is a simple matter to determine bankruptcy. Whenever the cash flow falls short of meeting debt repayment, the firm is bankrupt. In a multi-period framework, on the other hand, the firm may be allowed to defer payment to a future date, depending on the markets' perception of future cash flows. All that may happen is that the firm incurs a penalty for delaying payment. Considering the number of firms around us that incur losses without going bankrupt, this is probably something that *must* be incorporated into any model. The markets' estimates of future cash flows will be, however, incorporated into the $P(\bullet)$ functions and it may thus be possible for the firm to issue fresh debt and use the proceeds to pay off existing liabilities. In any case, once a working definition of bankruptcy is modelled, it may be possible to simulate examples of firms that change debt/equity ratios frequently, pay dividends, etc., due to purely tax motivated reasons. Such a result would be ample reward to the researcher who chooses to look in this direction.

Another modification that warrants further study is the introduction of non-debt-related tax shields. This will alter the form of our results somewhat. However, the framework that has been developed allows us to extend the current analysis to this modification in a quite straightforward manner. Since tax-shields of the firm allow it to pay out its initial cash flows as income that is not taxable on any account (corporate or personal), whereas debt-interest allows the firm to pay out its cash flows as income that is taxable only on personal account, the benefits of paying interest (over equity) would have to extend to substantially higher cash flows in the presence of non-debt-related tax shields. Consider for example, a firm whose cash flows are negatively correlated with aggregate consumption. When the firm's cash flow is "low", aggregate consumption is likely to be "high", and if personal tax rates are progressive, it is likely that tax shields provided by interest are not valuable. However, at high cash flows and low levels of aggregate consumption, it may be preferable to have taxes paid on personal account rather than on the corporate account. Hence, for this firm, interest tax-shields are attractive, and we would expect to see debt issued. Notice, also that all that we need is an appropriate specification on the joint distribution of x and c at "high" cash flow levels. This translates into a specification on the slope and level of the *tail* of the $\theta(\bullet)$ function.

Technically, let δ be the level of non-debt-related tax shields. There are essentially three different levels of debt that we must be considered for any analysis.

(a) $I + D < \delta$

(b) $D \leq \delta \leq I + D$

(c) $\delta < D$.

[27] The former is self-evident, and needs no explanation.

In (a) above, the value of the firm will be less than the corresponding all-equity firm, since by using debt at this level, there are taxes being paid by individuals on interest income, whereas in the absence of debt, no taxes would have been paid when the pre-tax cash flow of the firm fell short of δ. Hence, we should *not* observe firms that issue such low levels of debt.

Under (b) and (c), the relative magnitudes of $P_1(\bullet)$ and $(1-\tau)P_0(\bullet)$ will once again determine the optimum level of debt. In (b), what is lost in terms of taxes paid when x is between D and δ, must be compensated for by what is gained (in terms of having taxes paid at the personal rather than at the corporate level) between δ and $I+D$. Theorem 1 and 2 will hence have to be accordingly modified. In (c), the full benefits of interest can potentially be realised. However, the conditions for the issue of debt will become weaker as with (b) above.

Theorems 3 and 4 go through essentially unaltered by the inclusion of non-debt-related tax shields. Recall that in theorem 3, D^* was the optimum level of debt, were only one debt contract to be issued. From the discussion above, we know that the pre-tax cash flow corresponding to the solvent states, must exceed δ. If at this level, $\theta(\bullet)$ were positively sloped (or satisfied the second condition of Theorem 3), the firm benefits from the issue of some junior debt. However, the bankruptcy-dividing state may no longer be the one where the firm's pre-tax cash flow equals $I+\frac{D}{1-\tau}$. If we change all occurences of $I+\frac{D}{1-\tau}$ to read as "bankruptcy dividing pre-tax cash flow", then theorem 3 is still valid. Finally, observe that the result regarding the ambiguity regarding the direction in which optimum debt levels will move in response to a change in the tax rates also carries through since that argument is not dependent upon the absence of non-debt-related tax shields.

Appendix 1

Proofs

From (16) we obtain the first-order-conditions for the firm's problem as

$$V'(D) = \nabla(b)\left(\frac{dI}{dD} + \frac{1}{1-\tau}\right) - \nabla(a)\left(\frac{1}{1-\tau}\right) \tag{16a}$$

where $a = \frac{D}{1-\tau}$ and $b = I + \frac{D}{1-\tau}$. Differentiate (9) and rearrange to obtain

$$\left(\frac{dI}{dD} + \frac{1}{1-\tau}\right) = \frac{1-\tau+\nabla(a)}{(1-\tau)P_1(b)}$$

Use this to get

$$\begin{aligned}
Sign\left(V'(D)\right) &= Sign\left\{\frac{\nabla(b)[1-\tau+\nabla(a)]}{(1-\tau)P_1(b)} - \frac{\nabla(a)}{1-\tau}\right\} \\
&= Sign\left\{\nabla(b)[1-\tau+\nabla(a)] - \nabla(a)P_1(b)\right\} \\
&= Sign\left((1-\tau)\nabla(b) - \nabla(a)[P_1(b) - \nabla(b)]\right) \\
&= Sign\left\{(1-\tau)\nabla(b) - (1-\tau)P_0(b)\nabla(a)\right\} \\
&= Sign\left(\nabla(b) - \nabla(a)P_0(b)\right) \qquad (16b)
\end{aligned}$$

Lemma 1 *All debt requires interest.*

$$D > 0 \implies I > 0$$

Proof Since $P_0'(x)$ and $P_1'(x)$ are non-positive,

$$P_0(x) \leq P_0(0) \qquad \forall\, x \quad ; \tag{19}$$

$$P_1(x) \leq P_1(0) \qquad \forall\, x \quad . \tag{20}$$

From equations (5d), (5e) and the assumption that $R_0, R_1 > 0$, we obtain that

$$P_0(0), P_1(0) < 1 \tag{21}$$

Hence,

$$P_0(x), P_1(x) < 1 \qquad \forall\, x \tag{22}$$

and

$$(1-\tau)\int_0^{\frac{D}{1-\tau}} P_0(x)\,dx < (1-\tau)\int_0^{\frac{D}{1-\tau}} dx = D \tag{22a}$$

From (22a) and (9) we obtain

$$\int_{\frac{D}{1-\tau}}^{I+\frac{D}{1-\tau}} P_1(x)\,dx = D - \int_0^{\frac{D}{1-\tau}} (1-\tau)P_0(x)\,dx$$
$$> 0$$

Now since by our earlier assumption of non-negative state prices, $\phi_1(x) \geq 0\ \forall\, x$ we also have that $P_1(x) \geq 0\ \forall\, x$. Therefore,

$$\int_{\frac{D}{1-\tau}}^{I+\frac{D}{1-\tau}} P_1(x)\,dx > 0 \implies I > 0$$

■

Lemma 2 *Any increase in debt levels calls for increased interest payments.*

$$\frac{dI}{dD} > 0 \qquad \forall \quad D > 0$$

Proof Differentiate (9) with respect to D to obtain

$$1 = P_0(\tfrac{D}{1-\tau}) + P_1(I + \tfrac{D}{1-\tau})\left[\frac{dI}{dD} + \frac{1}{1-\tau}\right] - P_1(\tfrac{D}{1-\tau})\left[\frac{1}{1-\tau}\right]$$

$$\frac{dI}{dD} = \frac{\left[1 - P_0(\frac{D}{1-\tau})\right] + \left(\frac{1}{1-\tau}\right)\left[P_1(\frac{D}{1-\tau}) - P_1(I + \frac{D}{1-\tau})\right]}{P_1(I + \frac{D}{1-\tau})} \quad . \tag{23}$$

Now, $P_0(\frac{D}{1-\tau}) < 1$ from (22) and

$$P_1(\tfrac{D}{1-\tau}) \geq P_1(I + \tfrac{D}{1-\tau}),$$

since $P_1'(x) < 0 \; \forall \, x$ <u>and</u> from Lemma 1, $I > 0$. In addition,

$$P_1(I + \tfrac{D}{1-\tau}) \geq P_1(\infty) = 0.$$

Hence the right hand side of (23) is positive. ■

Theorem 1

(a) *A sufficient condition for* **all** *value-maximising firms to issue debt is*

$$\tau > \frac{R_1 - R_0}{1 + R_1} \, .$$

(b) *In the event that the above condition did not hold, a sufficient condition for any individual value-maximising firm to issue debt is*

$$\exists \, x^* > 0 \qquad \textit{such that} \qquad \theta(x) < 0 \;\; \forall \, x < x^* \quad \textbf{AND} \quad \theta(x^*) = 0 \, .$$

(c) *A necessary condition for* **any** *value-maximising firm to issue debt, on the other hand is*

$$\exists \, x \qquad \textit{such that} \qquad \nabla(x) > 0 \, .$$

Proof

Part (a)

Observe from (5), (14) and (15) that $\theta(\bullet)$ is a differentiable function. With (16) this implies that $V(D)$ is also a continuous and differentiable function. From (16) we also obtain

$$V'(D) \;=\; \nabla(I+\tfrac{D}{1-\tau})\left[\frac{dI}{dD}+\frac{1}{1-\tau}\right] \;-\; \nabla(\tfrac{D}{1-\tau})\left[\frac{1}{1-\tau}\right]$$

Set $D=0$ in (9) to obtain $I=0$. Therefore,

$$V'(0) \;=\; \nabla(0).\left[\frac{dI}{dD}\right]_{D=0}$$

From (23)

$$\begin{aligned}\left[\frac{dI}{dD}\right]_{D=0} &= \frac{1-P_0(0)}{P_1(0)}\\ &> \;\; 0 \quad \text{since } R_0>0\end{aligned}$$

Hence, $sign\{V'(0)\} = sign\{\nabla(0)\}$. Now, if $V'(0)>0$, then $\exists D>0$ such that $V(D)>V(0)$ and the value maximising firm will issue debt.

$$\begin{aligned}V'(0)>0 &\Longleftrightarrow \nabla(0)>0\\ &\Longleftrightarrow P_1(0)>(1-\tau)P_0(0)\\ &\Longleftrightarrow \frac{1}{1+R_1}>\frac{1-\tau}{1+R_0}\\ &\Longleftrightarrow \tau>\frac{R_1-R_0}{1+R_1}>0\end{aligned}$$

Part (b)

Consider a value of debt D such that $(I+\frac{D}{1-\tau})=x^*$. Now,

$$V(D) \;=\; \theta(x^*)-\theta(\tfrac{D}{1-\tau})+V(0)$$

Thus, $V(D)-V(0) \;=\; -\theta(\frac{D}{1-\tau})$
From Lemma 1, $\frac{D}{1-\tau}<x^*$. Hence,

$$\theta(\tfrac{D}{1-\tau}) \;<\; 0$$

and

$$V(D) \;>\; V(0)$$

Part (c)

The firm will issue debt only if it benefits by doing so. Hence we need that

$$V(D) \;>\; V(0)$$

for some D. This requires that there exist a level of debt,D (> 0), and corresponding interest I (> 0), such that

$$\theta(I + \tfrac{D}{1-\tau}) \;>\; \theta(\tfrac{D}{1-\tau})\,.$$

For this to hold, at the very least we need that $\theta(\bullet)$ slope upward *somewhere* between $\frac{D}{1-\tau}$ and $I + \frac{D}{1-\tau}$. Since $\nabla(\bullet)$ is the slope of $\theta(\bullet)$, this completes the proof. ■

Theorem 2 *A sufficient condition for a value maximising firm to have an interior optimal capital structure of stock and bonds is*

$$\exists\, x^* \qquad \textit{such that} \qquad \forall\, x \gtrless x^*, \quad \nabla(x) \lessgtr 0$$

Proof Let the maximum amount of debt that a firm can issue be D^*. Hence D^* solves

$$D^* = \int_0^{\frac{D^*}{1-\tau}} (1-\tau)P_0(x)\,dx + \int_{\frac{D^*}{1-\tau}}^{\infty} P_1(x)\,dx$$

Now let D^+ solve

$$D^+ = \int_0^{\frac{D^+}{1-\tau}} (1-\tau)P_0(x)\,dx + \int_{\frac{D^+}{1-\tau}}^{x^*} P_1(x)\,dx$$

From Lemma 2, $x^* < \infty \Rightarrow D^+ < D^*$. From Lemma 1, and the assumption that $x^* > 0$, we learn that

$$\frac{D^+}{1-\tau} < x^*$$

Now since $\nabla(x) > 0 \quad \forall\, x < x^*$,

$$\begin{aligned} V(D^+) &= \theta(x^*) - \theta(\tfrac{D^+}{1-\tau}) + V(0) \\ &> V(0) \end{aligned}$$

Hence, the firm will issue some debt. To show that the firm will not issue more than D^+, consider a debt level $D(> D^+)$ with an interest payment of I.

CASE 1

$$\boxed{\frac{D^+}{1-\tau} < \frac{D}{1-\tau} < x^*}$$

From Lemma 2, $I + \frac{D}{1-\tau} > x^*$. Hence,

$$\theta(x^*) > \theta(I + \tfrac{D}{1-\tau}) \qquad \text{Since } \nabla(x) < 0 \quad \forall\, x > x^*$$

Also

$$\theta(\tfrac{D^+}{1-\tau}) < \theta(\tfrac{D}{1-\tau}) \qquad \text{Since } \nabla(x) > 0 \quad \forall\, x < x^*.$$

Therefore

$$\theta(x^*) - \theta(\tfrac{D^+}{1-\tau}) > \theta(I + \tfrac{D}{1-\tau}) - \theta(\tfrac{D}{1-\tau})$$

In other words

$$V(D^+) > V(D)$$

CASE 2

$$\boxed{\frac{D}{1-\tau} > x^*}$$

Now,

$$\theta(I + \tfrac{D}{1-\tau}) < \theta(\tfrac{D}{1-\tau}) \qquad \text{Since } \nabla(x) < 0 \quad \forall\, x > x^*.$$

In other words

$$V(D) < V(0)$$

Hence, $\forall D > D^+$, $V(D) < V(D^+)$. This concludes the proof. ■

Theorem 3 *If D^* and I^* maximize (16) subject to (9) and (10), then either one of the following two conditions is sufficient to induce a value-maximising firm to issue two bonds at optimum*

a) $\nabla(I^* + \frac{D^*}{1-\tau}) > 0$
b) $\exists x^* > \left(I^* + \frac{D^*}{1-\tau}\right)$ *such that* $\theta(x) < 0 \ \forall \left(I^* + \frac{D^*}{1-\tau}\right) \leq x \leq x^*$, *and* $\theta(x^*) = 0$

Proof

Part (a)

Differentiate (28) with respect to D^j holding D^s constant. We obtain the derivatives of I^j with respect to D^j from (27). Simplify to obtain

$$Sign\left(\frac{dV}{dD^j}\right) = Sign\Big(\nabla(I^j + I^s + \tfrac{D^s+D^j}{1-\tau}) - \nabla(I^s + \tfrac{D^s+D^j}{1-\tau})P_0(I^j + I^s + \tfrac{D^s+D^j}{1-\tau})\Big)$$

Hence at $D^j = I^j = 0$, issuing junior debt will increase firm value if $\nabla(I^s + \frac{D^s}{1-\tau}) > 0$. Since in the theorem, senior debt is already at maximum, and issue of junior debt increases value further, the firm will wish to issue junior debt.

Part (b)

Refer to the proof of Theorem 1(b). The proof here is very similar to the one presented there. In Theorem 1, we showed that there exists a positive level of debt that is preferred to zero debt.

Following exactly the same logic, we can show that there exists a positive level of junior debt that is preferred to no junior debt. ■

Lemma 4 *The increase in $\theta(x)$, resulting from an increase in τ, is largest at higher cash flow levels.*

$$\big(\theta_2(x_2) - \theta(x_2)\big) \;>\; \big(\theta_2(x_1) - \theta(x_1)\big) \qquad \forall\, x_2 > x_1$$

Proof

$$\begin{aligned}\theta_2(x_2) - \theta(x_2) &= \int_0^{x_2} \big[P_1(y) - (1-\tau_2)P_0(y) - P_1(y) + (1-\tau_1)P_0(y)\big]\,dy \\ &= [\tau_2 - \tau_1]\int_0^{x_2} P_0(y)\,dy\end{aligned}$$

and similarly

$$\theta_2(x_1) - \theta(x_1) \;=\; [\tau_2 - \tau_1]\int_0^{x_1} P_0(y)\,dy$$

Now since $P_0(y) \geq 0\,\forall\, y$, we obtain

$$\theta_2(x_2) - \theta(x_2) \;>\; \theta_2(x_1) - \theta(x_1) \quad \forall\, x_2 > x_1, \tau_2 > \tau_1$$

■

Theorem 4 *The promised interest payments required to support any level of debt, rises as the corporate tax rate rises.*

Proof Differentiate both sides of (9) with respect to τ, while holding D fixed. This yields

$$\begin{aligned}0 \;=\; &(1-\tau)P_0(\tfrac{D}{1-\tau})\frac{D}{(1-\tau)^2} - \int_0^{\frac{D}{1-\tau}} P_0(x)\,dx \\ &+ P_1(I + \tfrac{D}{1-\tau})\left[\frac{D}{(1-\tau)^2} + \frac{dI}{d\tau}\right] \\ &- P_1(\tfrac{D}{1-\tau})\frac{D}{(1-\tau)^2}\end{aligned}$$

We solve this to obtain

$$P_1(I+\tfrac{D}{1-\tau})\frac{dI}{d\tau} = \frac{D}{(1-\tau)^2}\left[P_1(\tfrac{D}{1-\tau}) - P_1(I+\tfrac{D}{1-\tau})\right]$$
$$+\int_0^{\frac{D}{1-\tau}} P_0(x)dx - \left[\frac{D}{1-\tau}\right] P_0(\tfrac{D}{1-\tau})$$

Now since $I \geq 0$ and $P_1'(\bullet) \leq 0$,

$$P_1(\tfrac{D}{1-\tau}) \geq P_1(I+\tfrac{D}{1-\tau})$$

hence

$$P_1(I+\tfrac{D}{1-\tau})\frac{dI}{d\tau} \geq \int_0^{\frac{D}{1-\tau}} P_0(x)dx - \left[\frac{D}{1-\tau}\right] P_0(\tfrac{D}{1-\tau})$$
$$\geq \int_0^{\frac{D}{1-\tau}} \left(P_0(x) - P_0(\tfrac{D}{1-\tau})\right) dx$$

But we know that

$$P_0(x) \geq P_0(\tfrac{D}{1-\tau}) \quad \forall x \leq \frac{D}{1-\tau}$$

and

$$P_1(I+\tfrac{D}{1-\tau}) \geq 0$$

Hence

$$\left.\frac{dI}{d\tau}\right|_{D\ fixed} \geq 0$$

■

Appendix 2

Figures

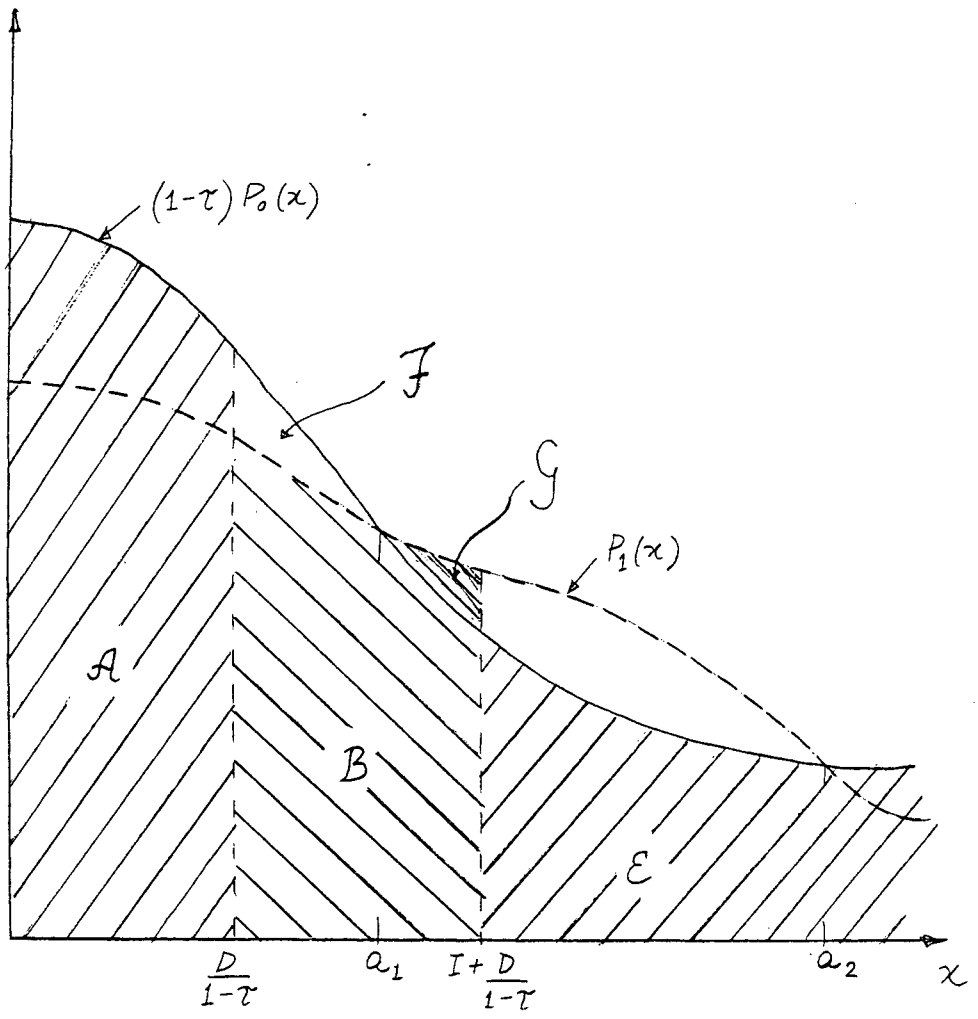

$(1-\tau)P_0(x)$
$\mathcal{F}$
$\mathcal{G}$
$P_1(x)$
$\mathcal{A}$
$\mathcal{B}$
$\mathcal{E}$
$\frac{D}{1-\tau}$
a_1
$I+\frac{D}{1-\tau}$
a_2
x

FIGURE 1

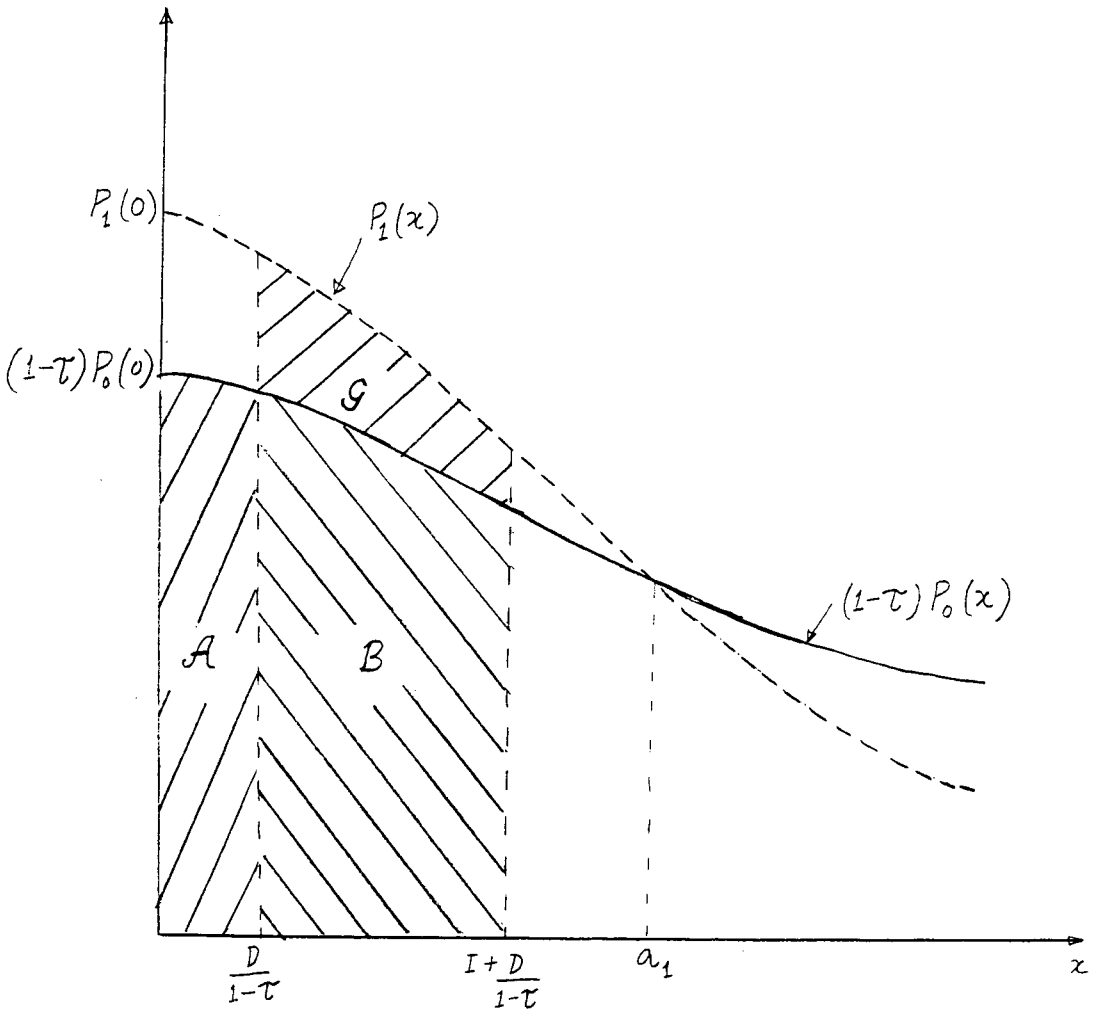

$P_1(0)$
$P_1(x)$
$(1-\tau)P_0(0)$
$\mathcal{G}$
$(1-\tau)P_0(x)$
$\mathcal{A}$
$\mathcal{B}$
$\frac{D}{1-\tau}$
$I+\frac{D}{1-\tau}$
a_1
x

FIGURE 2

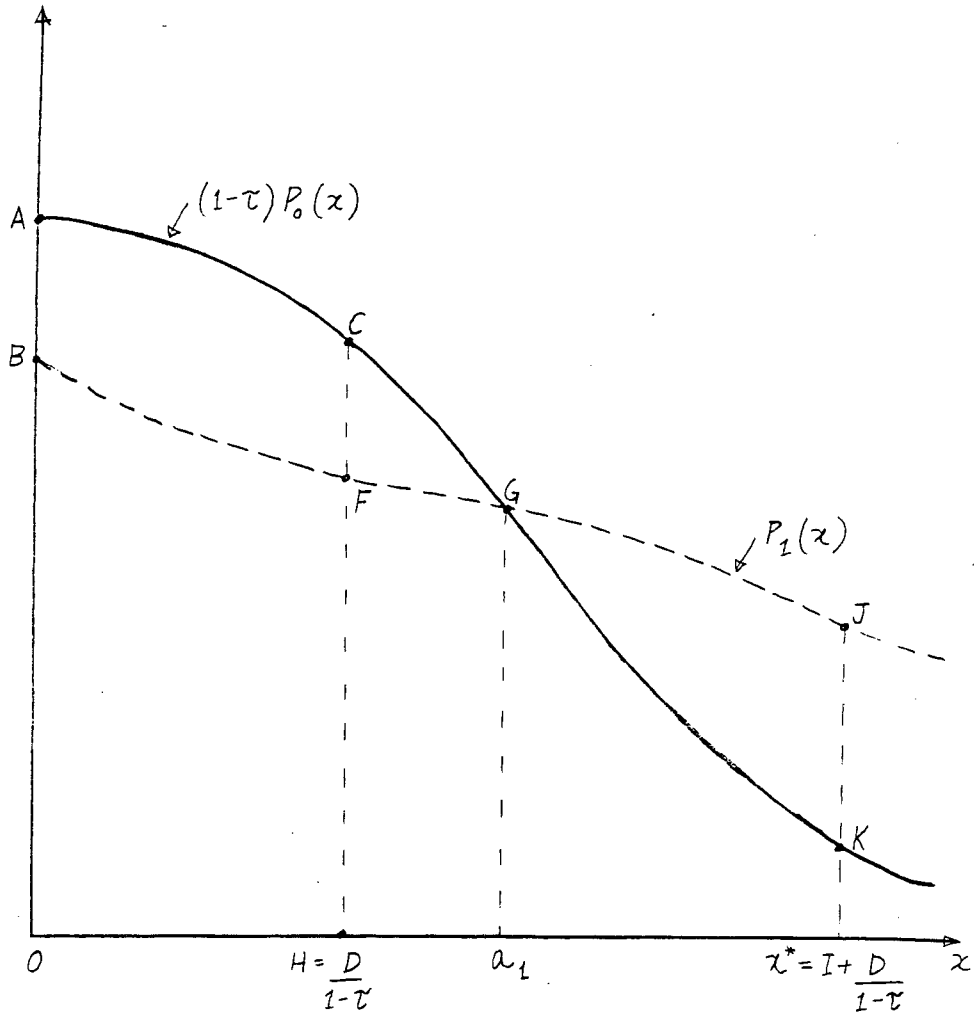


FIGURE 3

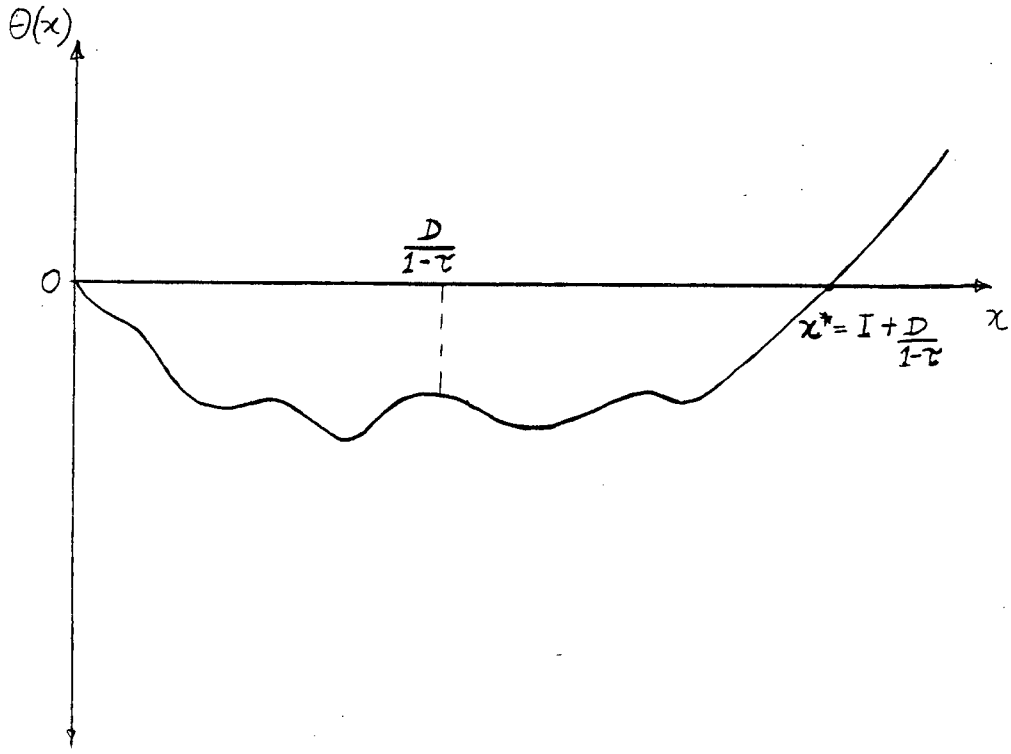
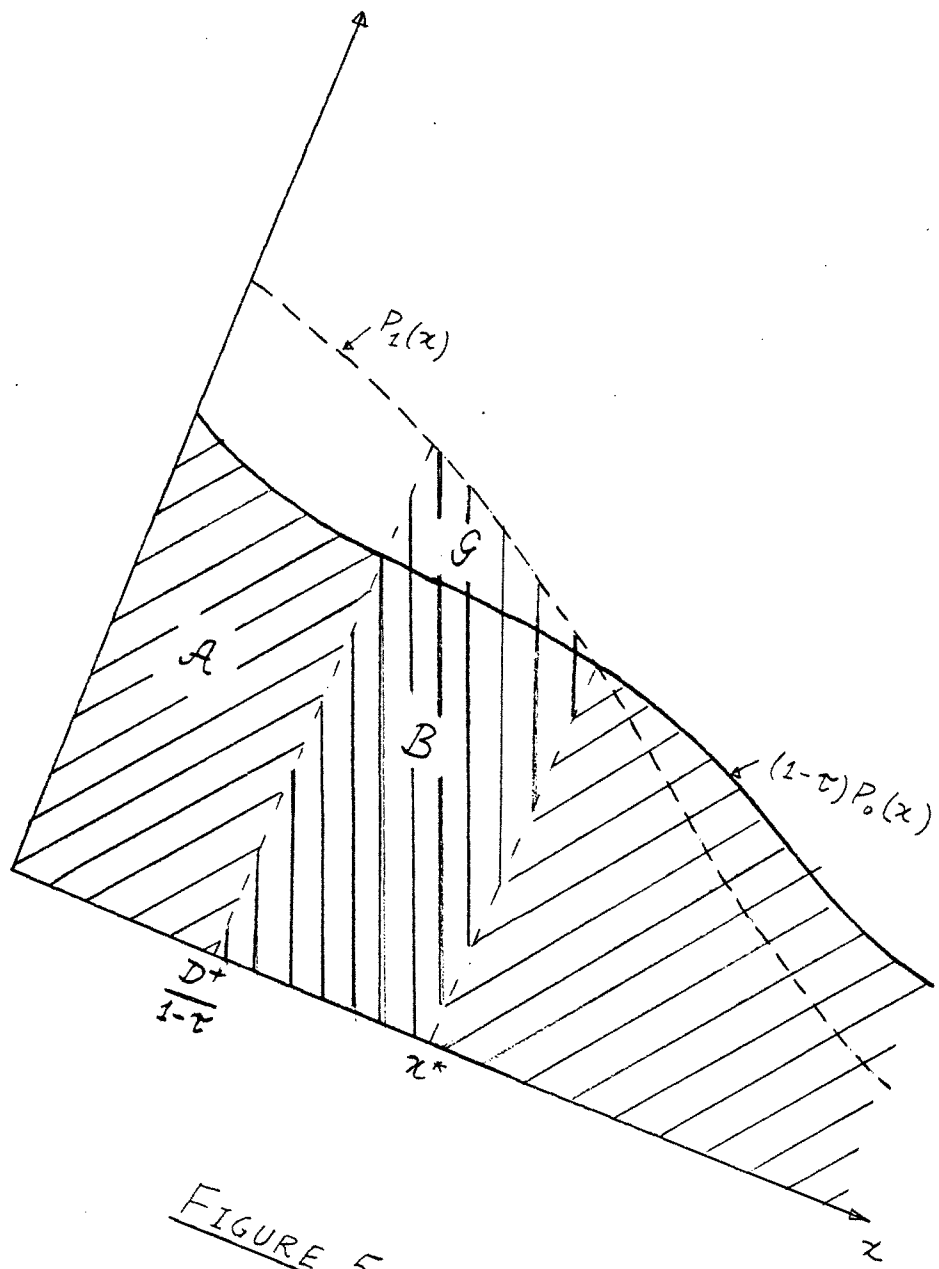


FIGURE 4



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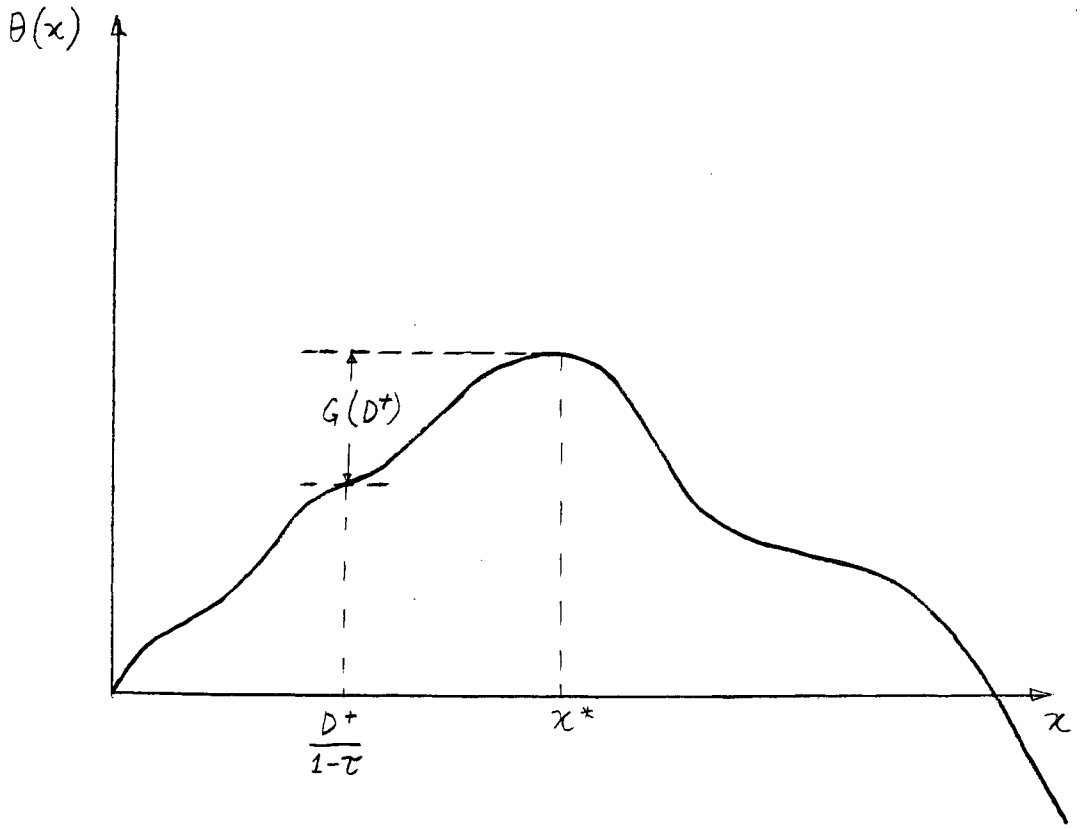

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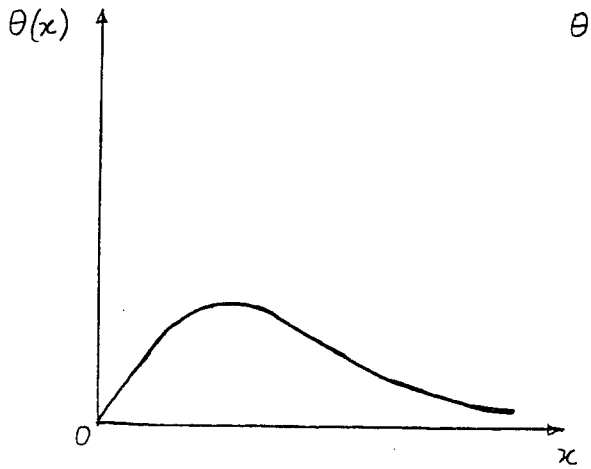


FIGURE 7

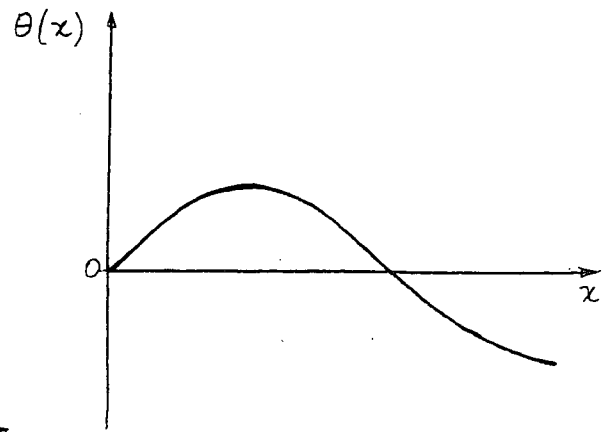
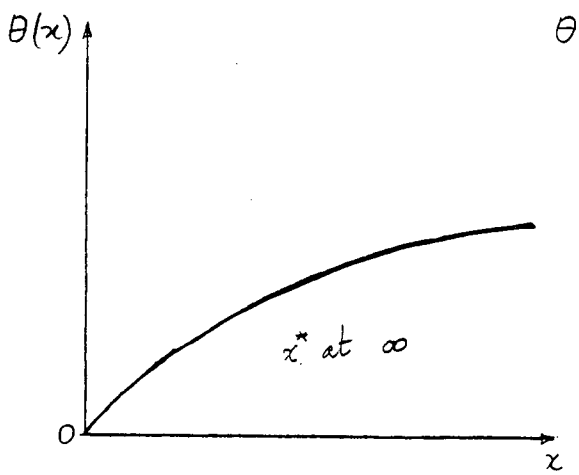


FIGURE 8



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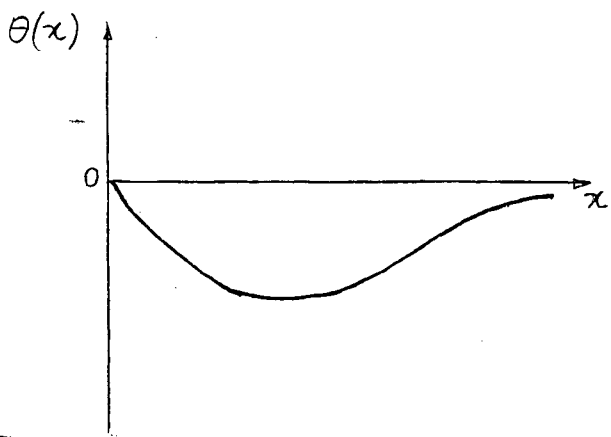

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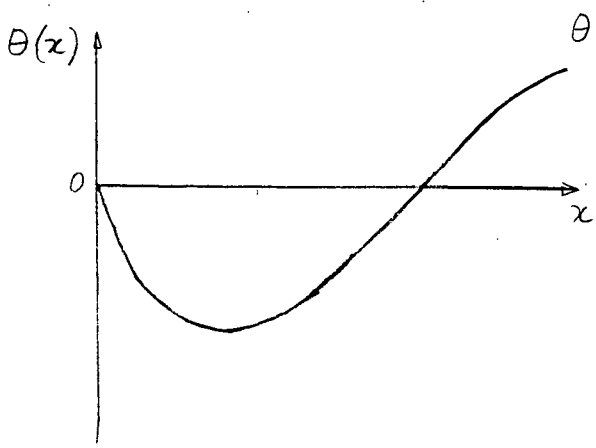
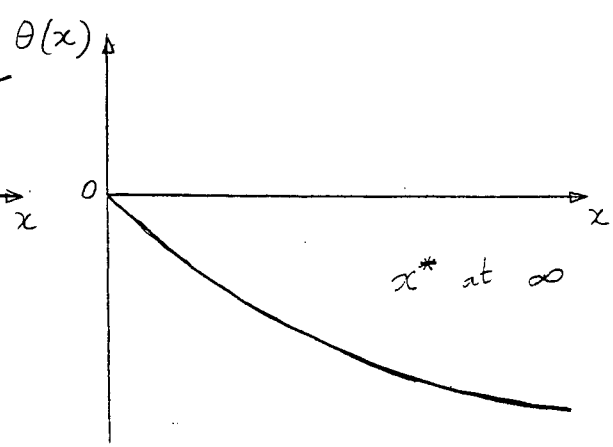


FIGURE 11



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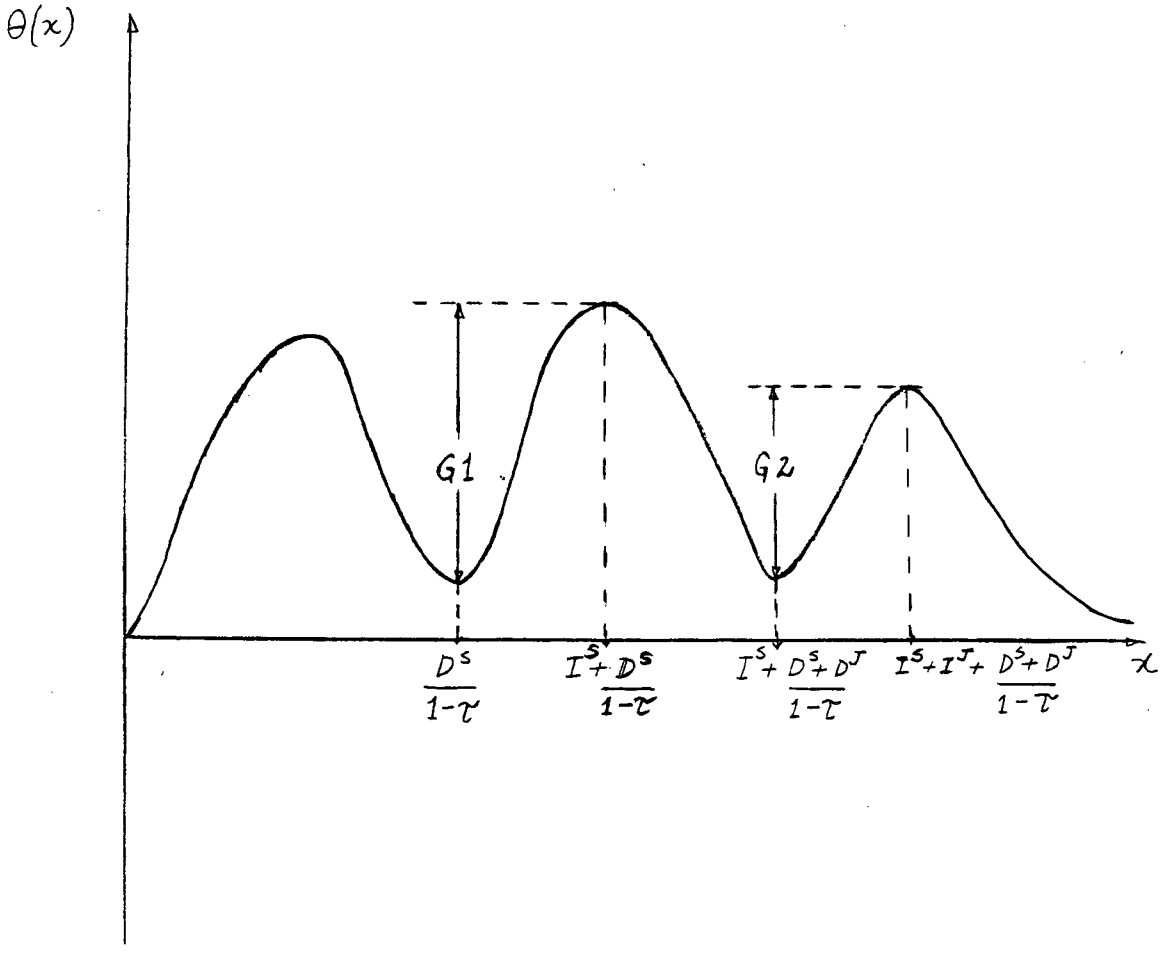
θ(x)
G1
G2
$\frac{D^s}{1-\tau}$
$\frac{I^s+D^s}{1-\tau}$
$\frac{I^s+D^s+D^J}{1-\tau}$
$\frac{I^s+I^J+D^s+D^J}{1-\tau}$
x

FIGURE 13

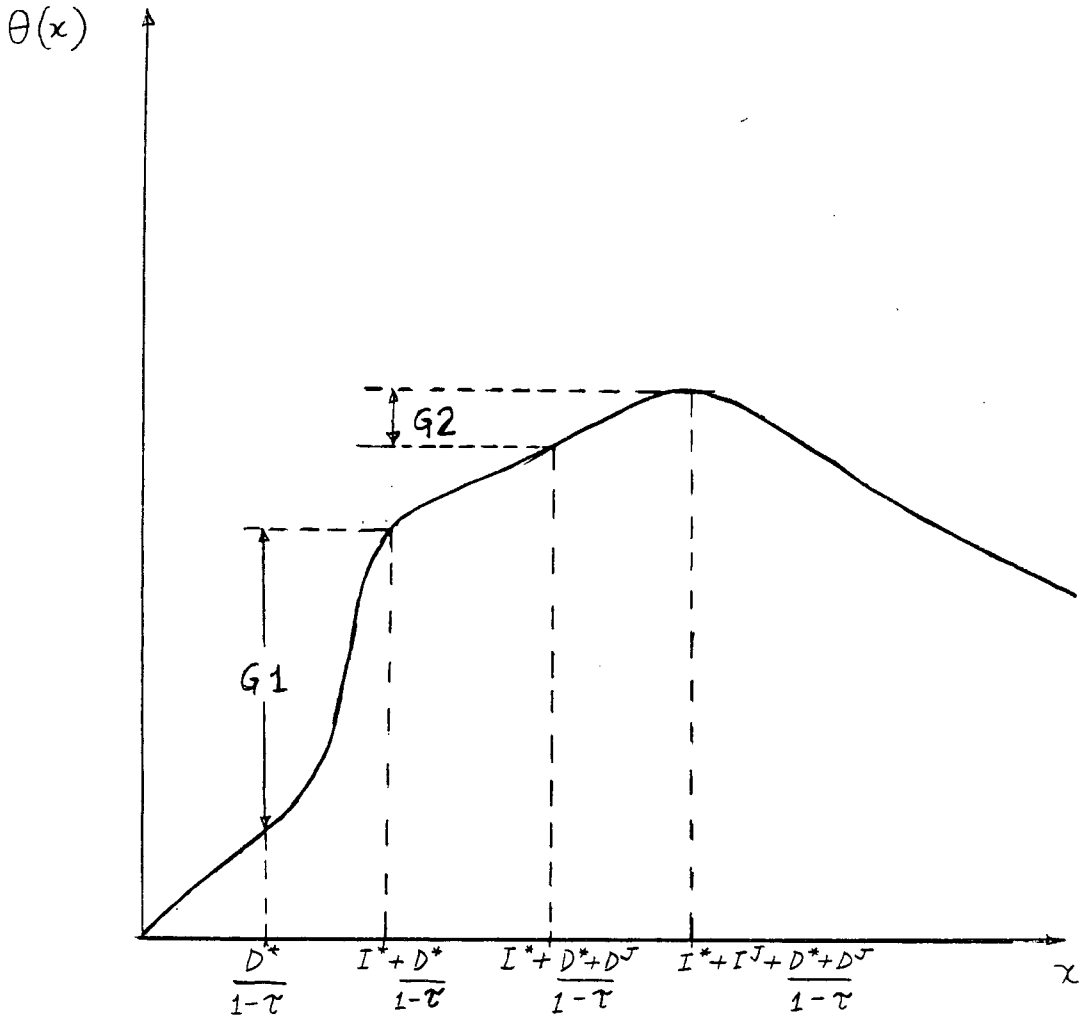

FIGURE 14

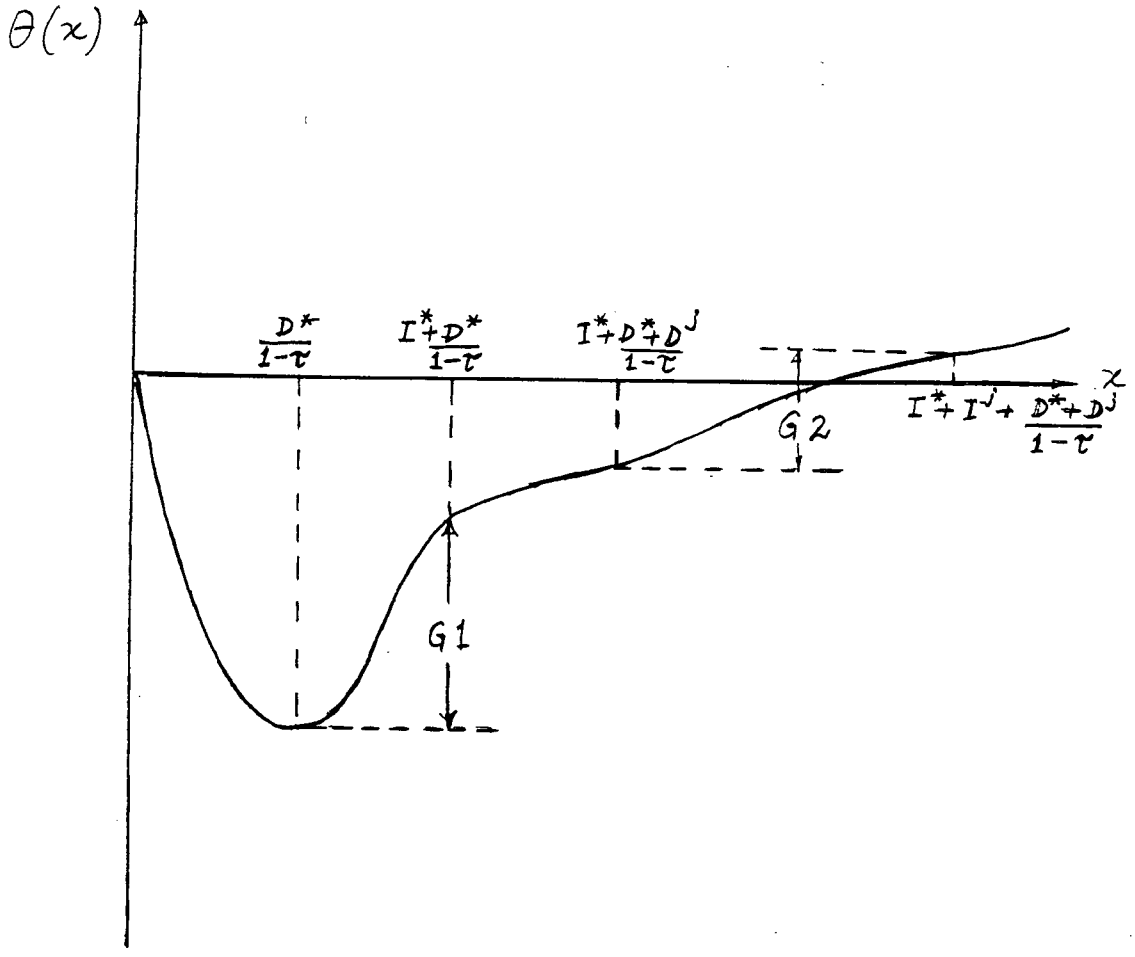
$\theta(x)$
$\frac{D^*}{1-\tau}$
$\frac{I^*+D^*}{1-\tau}$
$\frac{I^*+D^*+D^J}{1-\tau}$
x
$\frac{I^*+I^J+D^*+D^J}{1-\tau}$
G2
G1

FIGURE 15

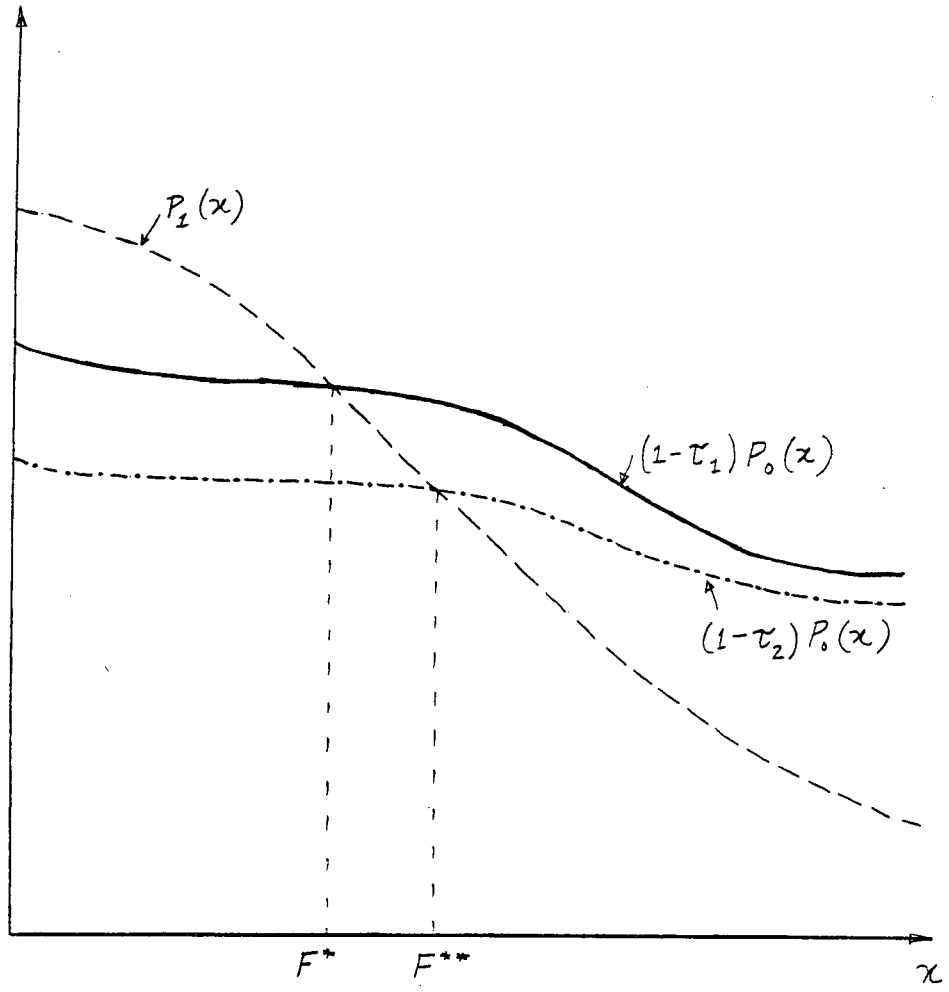

FIGURE 16

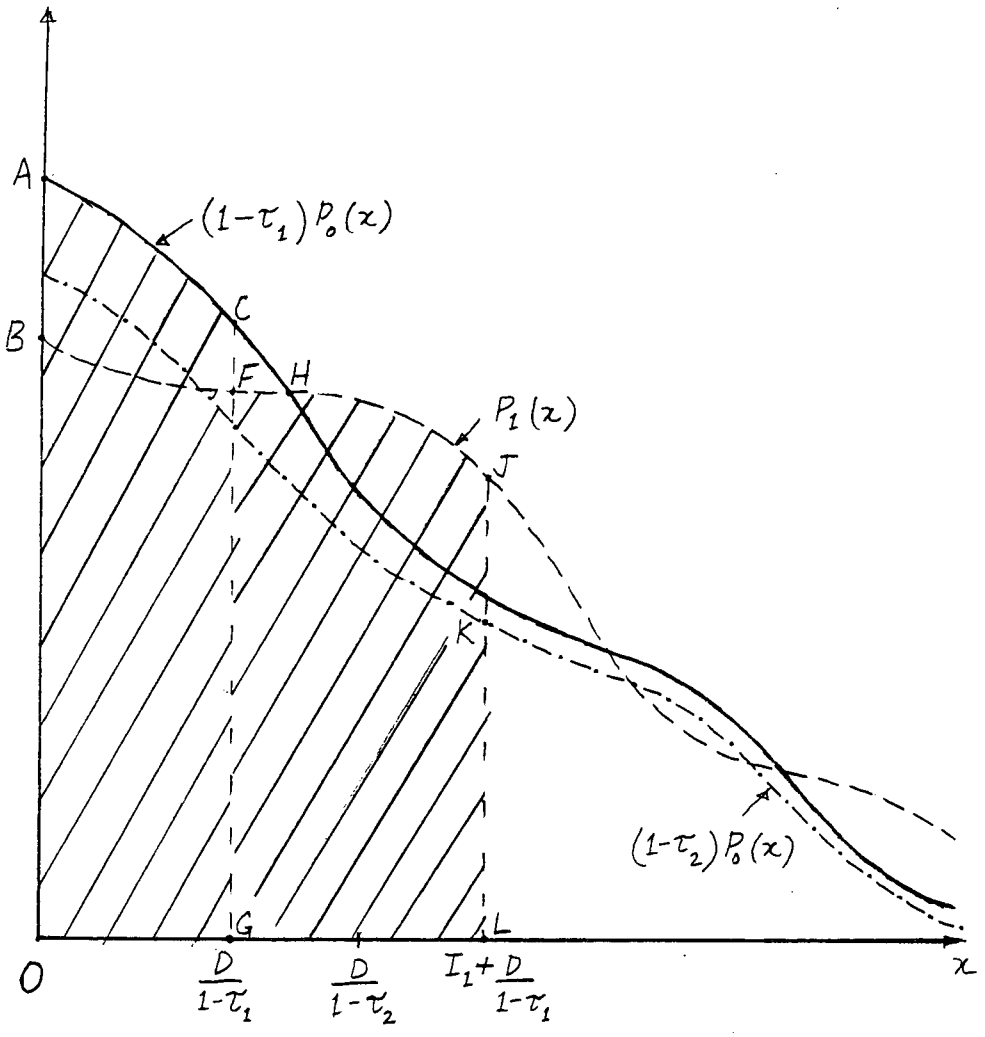
A
B
C
F
H
J
K
G
L
O
x
$(1-\tau_1)P_0(x)$
$P_1(x)$
$(1-\tau_2)P_0(x)$
$\frac{D}{1-\tau_1}$
$\frac{D}{1-\tau_2}$
$I_1+\frac{D}{1-\tau_1}$

FIGURE 17

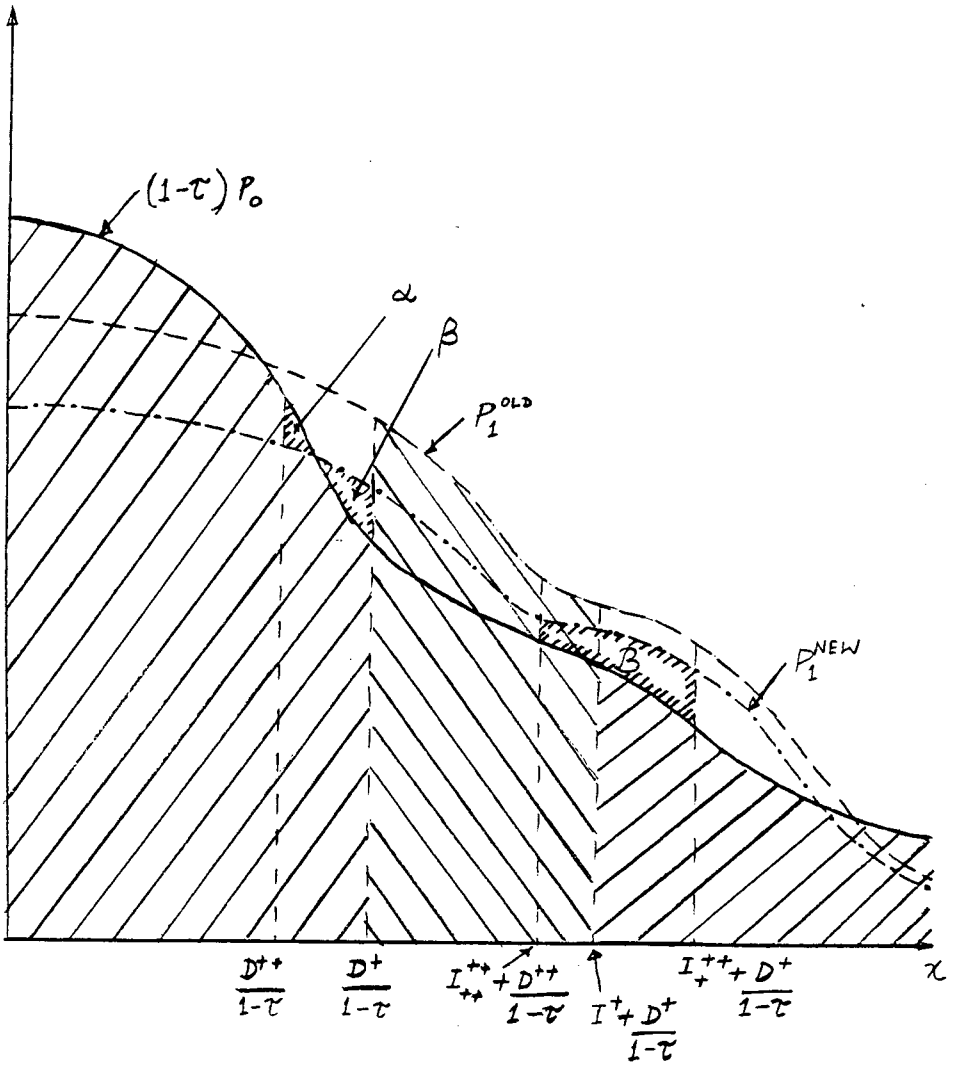
$(1-\tau)P_0$
α
β
P_1^{OLD}
P_1^{NEW}
$\frac{D^{++}}{1-\tau}$
$\frac{D^{+}}{1-\tau}$
$I_{++}^{++}+\frac{D^{++}}{1-\tau}$
$I^{+}+\frac{D^{+}}{1-\tau}$
$I_{+}^{++}+\frac{D^{+}}{1-\tau}$
x

FIGURE 18

Appendix 3

Notation used in **"Debt, Taxes and Uncertainty"**

$\Psi(c, x)$ = Joint Density Function of aggregate consumption and firm's cash flow

$g(x|c)$ = Conditional probability density of the firm's cash flow (x) when aggregate consumption $= c$

$f(c|x)$ = Conditional probability density of aggregate consumption (c) when firm's cash flow $= x$

$p_1(c)$ = Equilibrium value of a claim that pays \$1 of taxable income when aggregate consumption $= c$

$p_0(c)$ = Equilibrium value of a claim that pays \$1 of non-taxable income when aggregate consumption $= c$

τ = Corporate tax rate $(0 < \tau < 1)$

$\mathcal{C}$ = Set of possible aggregate consumption levels attained at the end of the period.

i.e.$c \in \mathcal{C}$

π = An individual's taxable income.

$t(\pi)$ = The marginal tax rate of an individual whose taxable income is π.

x = The end of period cash flow, before taxes, of a firm.

$V(h_T, h_{NT})$ = The value today of a claim that pays $h_T(x)$ dollars of income that is taxable at the personal level and $h_{NT}(x)$ dollars of income that is exempt from personal taxes, when the firm's cash flow is x.

$\phi_1(x)$ = The present value of a claim that pays a dollar of income that is taxable at the personal level when the firm's cash flow is x.

$\phi_0(x)$ = The present value of a claim that pays a dollar of income that is *not* taxable at the personal level when the firm's cash flow is x.

$P_1(x)$ = The present value of a claim that pays a dollar of income that is taxable at the personal level when the firm's cash flow **exceeds** x.

$P_0(x)$ = The present value of a claim that pays a dollar of income that is *not* taxable at the personal level when the firm's cash flow **exceeds** x.

R_1 = The riskless rate of return on income that is taxable at the personal level.

R_0 = The riskless rate of return on income that is *not* taxable at the personal level.

D = The face value of the debt issued by the firm. By assumption, this is also the market value of the debt.

I = The interest promised by the firm to its debt-holders.

E = The market value of the equity in the firm.

$\nabla(x)$ = The benefits, in terms of firm value, of paying the $(x+1)^{\text{th}}$ dollar of the firm's cash flow as interest rather than as equity.

$\theta(x)$ = The benefits, in terms of firm value, of paying the *first* x dollars of the firm's cash flow as interest rather than as equity.

$V(D)$ = The market value of a firm that has a debt level of D.

D^s = The face value of the senior debt of a firm.

D^j = The face value of the junior debt of a firm.

I^s = The interest promised by the firm to its senior bond-holders.

I^j = The interest promised by the firm to its junior bond-holders.

F = The face value of debt issued by a firm, when we do not distinguish between principal and interest components of debt.

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OPTION MARKET EQUILIBRIUM WITH TAXES

Introduction

An *American option* is a security that offers its owner the opportunity to trade in a fixed number of shares of a pre-specified asset at a pre-determined price (the *strike* or *exercise price*) on or before a pre-specified date in the future (*expiration date*). The owner may choose to *exercise* the option at any time upto and including the expiration date (*the maturity period*), if he or she so desires. A *European* option differs from its American counterpart in that with a European option, the owner is permitted to exercise the option on the expiration date and no sooner. Options come in basically two forms – the *call* and the *put*. A call option gives its owner the right to *buy* (to "call" into his possession) whereas, the put option gives its owner the right to *sell* ("put" out his possession). An option is said to be *in-the-money, at-the-money,* or *out-of-the-money*, depending on whether the profits from a hypothetical immediate exercise are positive, zero or negative respectively.

Options have been around for a long time, but prior to the formation of organised exchanges for their trade, markets were thin and owners ran the risk of the *writer* of the option reneging from his obligation. With the advent of the Chicago Board of Option Exchange (CBOE) in 1973, however, all this changed. Since then, the growth of options markets has been unprecedented in U.S. securities markets. Understandably, this growth has been accompanied by intense activity on the part of financial economists to try and understand the "hows" and "whys" of this growing giant.

Option-pricing theory can be traced back as far as 1900 to Louis Bachelier. Half a century later, in 1956, Kruizenga made another attempt while doing his doctoral dissertation at the Massachusetts Institute of Technology. The world, however, had to wait till 1973 when Fisher Black

and Myron Scholes revolutionized the theory of option pricing by presenting the first satisfactory equilibrium model. In the derivation, Black and Scholes, with incredible insight, paved the way for future researchers who wished to price *any* contingent claim. They noted that in perfect capital markets, any secondary security could be replicated perfectly by an appropriate combination of the underlying asset and a riskless security. This was the key that was needed to use arbitrage arguments to price an option in terms of the prices of the underlying stock and riskless bonds.

The very same year, Merton(1973) enriched the theory with some general arbitrage relationships that today, every serious student of financial theory is intimately familiar with. Significant advances were made soon after, most notably by Sharpe(1978), Cox, Ross & Rubinstein(1978) and Rendlemann & Bartter(1979) who independently simplified the Black-Scholes valuation formula by a process of discretization. Like its analogue of digitalization in the natural sciences, this crucial step of viewing the world, as being comprised of an infinity of discrete steps where only two outcomes were possible, permitted us to glimpse and comprehend fully the intuition of a riskless hedge and appreciate the concept of preference-free valuation. This method of valuing is often referred to as the *Binomial Option Pricing Model* (BOPM). The Black-Scholes valuation, it turns out, is the limiting case of the BOPM under appropriate assumptions.

Paralleling the development of the BOPM, researchers began to relax some of the stringent assumptions that Black and Scholes had to make in order to obtain their result. Cox(1975), for example, noted that the variance of the stock price is observed[1] to be inversely related to its price. He incorporated this and obtained what has come to be known as the *Constant-Elasticity-of-Variance* formula. Cox and Ross(JFE 1976) and Merton(JFE 1976), on the other hand, modified the distributional assumptions made by Black and Scholes on the price of the underlying stock. In particular, they relaxed the assumption that stock prices follow continuous paths by allowing finite jumps. The resulting formula, though considerably more complex, could be, once again, reduced to the Black-Scholes formula. Geske(1979) noted that the limited liability of stock-holders of a firm would make their payoff distribution dependent on the degree of leverage employed, provided the returns of the assets of the firm were unchanged. Consequently, the variance of the stock price would also be directly affected by any change in the debt-equity ratio of the firm's capital structure. In addition, since the stock is itself like a call option on the assets of the firm with a strike price equal to the face value of the bonds issued by the firm, a call option on the stock is essentially a call option *on a call option*. In other words, the call option is essentially a *compound option*. As expected, the valuation formula becomes more complex than its Black-Scholes counterpart.

As in any human endeavour, not only was the question "how ?" tackled, but so was the more philosophical question of "why ?". In particular, why options ? Ross(1976) demonstrated

[1] Refer to Black(1976), Becker(1980) and Christie(1982).

that options would assist in completing capital markets. He showed that even though there may be scenarios wherein simple options on stocks may not suffice for this task, the use of complex options will do so. In addition, he proved that simple options on an appropriately constructed portfolio would assist in completing markets. Given that complete markets provide for efficient wealth transfers amongst investors, it hence follows that options assist in attaining a Pareto efficient wealth allocation. Hakansson(1976) noted that although stocks and bonds identified with particular corporations may provide the most efficient way to raise capital, it does not follow that most investors are directly interested in these basic securities. He introduced the notion of a "superfund" that issues "supershares". Such a superfund would essentially be an appropriately constructed portfolio of the existing assets. In addition, a supershare is basically a state-contingent claim on the superfund that has a payoff that is linear in the value of the superfund, *provided*, this value lies in a pre-specified interval. Otherwise, the supershare is worthless. He shows that the use of such instruments provides a very powerful mechanism to achieve Pareto Efficiency. Since options can be viewed [2] as a type of supershare, it follows that they serve a beneficial function. A very simple example of the benefits provided by options comes about when we consider an investor who wishes to borrow or lend money. Now, not all investors can obtain the best lending/borrowing rates. Some of us just face larger spreads. The use of options and stocks[3], via the Put-Call parity relationship[4], however, allows a larger group of investors to transact at close to the *true* riskless rate. There are several other plausible benefits stemming from the existence of options. The interested reader is directed[5] to the book *"Options Markets"* by Cox and Rubinstein (pages 44 through 59) for an integrated treatment.

In financial theory, all analytical characterisations of reality are subjected to rigorous empirical testing. As with all research, this process, too, starts with a substantial degree of naivety and rapidly progresses to heights of sophistication using all the modern tools at our disposal. The analytical models of option-pricing are no exception. In the words of Black(1975),

> *"The actual prices on listed options tend to differ in certain systematic ways from the values given by the* (Black-Scholes) *formula. Options that are way out of the money tend to be overpriced* (by the formula), *and options that are way into the money tend to be underpriced. Options with less than three months to maturity tend to be overpriced."*

Simpler tests conducted by Trippi(1977) which involved the purchase of an option when its price fell 15% below the Black-Scholes price and selling it when the price rose 15% above the

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[2] Especially those on the market portfolio.

[3] Or the use of options and futures contracts.

[4] Refer to Merton(1973) for this relationship.

[5] See also Banz and Miller(1978), Breeden and Litzenberger(1978), Arditti and John(1982).

Black-Scholes price, led him to conclude that the options market was inefficient. His conclusion was reached by viewing the return from such a strategy. However, he failed to adjust for the risk of the constructed portfolio. Nevertheless, it was a start. Researchers were looking for opportunities to test this new model and the opportunities were not restricted to the United States. To give an example, as far away as Australia, Brown(1977) tested the validity of the model for the newly formed options market there. He realised that one of the problems in using the Black-Scholes formula was the need to estimate the variance of the rate of return on the stock. So he shifted into reverse gear and used the formula to infer the markets' estimate of the variance, *given* that the model is correct. Using observed option prices he obtained the *Implied Standard Deviation* (ISD), which he then compared to other estimates of the standard deviation. In conclusion, he says,

> *"It is shown that the model produces biased estimates and that part of this bias might be explained by incorrect pricing possibly due to a lack of experience with exchange traded options in Australia. It is also shown that the bias has decreased substantially, indicating that a learning process may have occured in the market."*

The use of ISD's had been explored in the US already, the previous year, by Latané and Rendleman(1976). Using data for the period October 1973 to June 1974, they reached the same conclusion as Black(1975). However, given that the market had had sufficient time to get used to this new instrument, they were forced to conclude that while the model could be used effectively to determine whether individual options are properly priced, *"... the model may not fully capture the process determining option prices in the actual market."*(page 375). Macbeth and Merville(1979) repeated the study using 1976 data and found that the actual prices of options that are in the money (out of the money) are generally greater (less) than the model price. This was in direct contrast to the findings of Black(1975) and of Merton(1976). The very next year, Macbeth and Merville(1980), using data from 1977, obtained results that were consistent with Black(1975). In addition, they reported that by using the Cox(1975) model, they obtained prices that were closer to reality. Thorp and Gelbaum(1980) entered the scene and using 1979 data, found that for this period, the deviations between the model price and the listed price for options were not systematically related to the strike price. Meanwhile, Macbeth(1981) repeated his earlier study (with Merville), but this time around, he used data from 1978. He found that for part of the year, the model correctly priced options, but for the remaining part, it behaved as noted by Black(1975). In addition, he concluded that the model parameters of both the Black-Scholes *and* Cox models were non-stationary. Using data for the period July 1977 through June 1978, Blomeyer and Klemkosky(1982) tested the Black-Scholes models against the Roll(JFE 1977) model for dividend unprotected American call options. As expected, they found their results with the Black-Scholes model to be consistent with the observations of Black(1975). Surprisingly, though, the Roll model yielded similar results. In addition, the Roll model did not outperform the Black-Scholes model in

any significant way. They thus concluded that deviations from the Black-Scholes model were *not* likely to be due to early exercise opportunities of unprotected American call options. Blomeyer repeated the study with Resnick using the same data to compare the Geske compound option model to the Black-Scholes model. They [Blomeyer and Resnick(1982)] found that the results using the Geske model were in contrast to the results using the Black-Scholes model. In short, their findings using the Geske model were similar to the results obtained by the Macbeth and Merville(1979) study using the Black-Scholes model. In addition, the magnitude of the bias using the Geske model was greater then the corresponding bias obtained using the Black-Scholes model.

On a slightly different track, Schmalensee and Trippi(1978), Beckers(1980), and, Brenner & Galai(1981) focused on the nature of the ISD. They found that the ISD was extremely volatile and that there were significant deviations between ISD's computed using prices of the *last* transaction of the day and ISD's computed using the daily average price. To cut a long story short, their findings led to the rejection of the joint hypothesis that the Black-Scholes model is valid — that the stocks and options markets are efficient and *synchronous*, and that the estimation procedures are correct.

Rubinstein(1981) conducted a thorough and painstaking study comparing the models of Cox-Ross (pure jump), Merton (Diffusion/Jump), Geske, and Cox against the Black-Scholes, to find out which was appropriate. He finds that the strike price biases had different signs in different periods and that the bias appeared to be similar for all stocks in any given period. He concluded that *none* of the existing models captures all the effects, and that a composite model is needed through which one can relate the biases to macro-economic parameters.

We conclude this brief report on the empirical track record of the Black-Scholes model with the following observations:

1) The Black-Scholes model performs relatively well, especially for at-the-money options. Deviations from the model are consistently observed for deep-in and deep-out-of-the-money options. These deviations are not systematic and change over time. However, in any given period, the deviations appear to be similar for options on all stocks.

2) No alternative model consistently offers better predictions of market prices than the Black-Scholes model. There is some evidence to prefer the constant elasticity of variance model, but it is not conclusive[6].

3) No one model accounts for transaction costs and taxes, which may affect the prices of traded

[6] Refer to Merville and Macbeth(1980) and the discussion of their paper by Manaster(1980).

options.

As far back as 1975, Black(1975) stated, (page 64)

> *"One possible explanation for this pattern is that we have left something out of the formula. Perhaps if we assumed a more complicated pattern for changes in a stock's volatility, or for changes in interest rates, we would be able to explain the overpricing of out-of-the-money options. But this seems unlikely to give a complete explanation. The underpricing of way-in-the-money options is so extreme that they often sell at "parity", where the option price is approximately equal to the stock price minus the exercise price. This means that the market is not giving the remaining time to maturity any value at all. Only tax factors, as dicussed above, seem to have any chance of explaining this."*

Scholes(1976) made the first attempt to incorporate taxes into the Black-Scholes formula. He assumed that all investors were in identical tax brackets, τ, and that capital gains taxes were incurred continuously on the stock. The basis for computation of capital gains taxes at the end of any interval $(t,\ t+\ dt)$ was assumed to be the stock price at the start of the interval, S_t. The return from the stock position of a portfolio replicating the option was hence, $(1-\tau)$ times the return on the stock, *had we not paid taxes.* This yielded, as expected, the Black-Scholes formula modified to allow for continuous dividends that are proportional to the stock price. However, instead of the dividend rate, we now had the tax rate, τ. Clearly, this formula will not explain the observed deviations from the Black-Scholes formula[7]. To better understand what is needed to explain these observed deviations, consider the following abstract provided by Dybvig and Ross(1986)

> *"Taxation of assets can create various clientele effects. If every agent is marginal on all assets, no clientele effects arise. If some (but not every) agent is marginal on all assets, there arises a clientele effect in quantities, but none in prices. If no agent is marginal on all assets, there arise clientele effects in both quantities and prices. In the first two cases, standard asset pricing and martingale results extend to analogous after-tax results. In the third case, linear asset pricing works only on subsets of assets, and the standard martingale results become after-tax supermartingale results."*

Further, on page 752, they state that

> *"In the marginalist and marginal investor cases, there exist* risk-neutral probabilities *and implicit tax rates under which all assets have the same after-tax extected return. This extends the usual no-arbitrage result to a world with taxes. In the general case, no single set*

[7] This was not Scholes' intention, anyway.

of risk-neutral probabilities and associated implicit tax rates can price all assets. Nonetheless, such pricing rules can govern subsets, and the additional assets appear inferior. In technical terms, this gives us a super-martingale *result that under each of these pricing rules all assets have expected return less than or equal to the shared constant."*

In all the models that have been developed so far, including the Scholes(1976) model, there exists at least one marginal agent. Even if we relax the assumption that all investors face identical tax rates in the Scholes model, we will not eliminate this marginal agent, for the payoffs from the stock for any investor will be a constant multiple of the payoffs to any other investor. Since this multiple is not affected by either the level of the stock price *or* by the strike price of the option that investors are replicating, we will end up with the result that the investor with the lowest cost of replication will be marginal and will determine the equilibrium price. No other investor will consider replicating the option, for it is cheaper to buy it. Since short sales are possible with options, all investors whose cost of replication are lower than the equilibrium price will choose to write the option and hedge it at a lower cost. These arbitrage opportunities will persist unless *all* investors end up in identical tax brackets. This will lead us to the original Scholes model as being the one that is consistent with equilibrium.

The observed strike-price bias from existing models that obtain a "formula" for option prices, could be explained in one of two ways

a) There exists a hypothetical agent who is marginal on all options on any particular[8] stock, and we just have not isolated him.

b) There does not exist any agent who is marginal on all options, even if they are on the same stock. Consequently, the pricing operator that prices options of identical maturity on the same stock, recognises differences between options *other* than the strike price. Any "formula" that fails to identify and incorporate such differences will suffer from a strike-price bias.

In this study, we focus on the second possibility. We hence need a structure that will allow us to conclude that no single investor is marginal on *all* options. Heterogeneity amongst investors is introduced by allowing the *taxable income* of two investors with identical wealths to be different. This difference can come about if the investors have traded in stocks at different points in time, and the basis of their stock holdings for the computation of capital gains taxes, is different. It turns out that under this assumption, we obtain a clientele in prices and in quantities. This is true, even if all investors were taxed at a fixed rate, τ. Since investors *do* purchase/sell stocks at different

[8] Options on different stocks may have different marginal investors.

points in time at different prices, this is a perfectly reasonable assumption. In fact, it is the crucial assumption in this study.

The above assumption was arrived at independently while working with the problem at hand, but it is interesting to note that Dybvig and Ross(1986) were familiar with the assumption. On page 761 they state

> *"Much work remains. In particular, little of practical use is known about the general case (discussed at the end of Section II) in which each agent may have a different basis for assets already owned. This case has some interesting features and may very well be useful for explaining tax-related empirical anomalies, provided we can develop a tractable and reasonable model."*

In equilibrium, the market price of the option is the one that equates supplies and demands for options that stemmed, in part, from investors who were originally in a position where they could create portfolios containing options whose returns dominate the riskless bond. We demonstrate that in equilibrium, there could be investors who take up naked positions in options as well as investors who take up covered positions. The existence of a tax-free investor who can indulge in infinite trading yields, as expected, a clientele in quantities, but not in prices. In addition, the presence of such an investor will ensure that the Black-Scholes price *is* the equilibrium price. The absence of such an investor, however, yields a clientele in prices.

The term "covered option position" in general refers to any portfolio that consists of an option and some quantity of stock, as long as the variance of the return from the resulting portfolio is not greater than the variance of the return of at least one of the components. In other words, the two positions must be negatively correlated. Examples of a covered option position are long call/short stock, long put/long stock, etc. If the variance of the covered position is zero, then the term "neutral covered position" is used. In our study, we refer only to neutral covered positions. Consequently, we refer to them simply as *covered positions*. The rationale for such positions has not been reasonably explained in the past. We suggest one explanation for their existence.

The beauty of the Black-Scholes model stems from its sheer simplicity. The existence of a hypothetical marginal investor who by arbitrage determines the equilibrium price for all options allows us to make direct inferences about this investor's estimates of unobservable variables in the pricing process. Having estimated this variable (the volatility of the stock price), we can easily test the validity of the formula. In addition, even though deviations from the model are observed, the model in itself gives us reasonable approximations to reality. To borrow an analogue from the physical sciences, it is like the laws of Newtonian mechanics. Though incomplete, it serves well

for everyday life, and is simple to use. The introduction of imperfections into the model, while having considerable intuitive appeal, also destroys its simplicity. This is a cost of a sort that all such modifications must absorb. For example, in the model developed here, since we do not have a hypothetical agent who is marginal on all options, we cannot use one subset of options to compute the value of another subset. We find that the value of any option in the market is unique, and that the price is determined in equilibrium by the net supplies and demands for the option. These supplies and demands which are in turn determined by the composition of investors in terms of preferences and endowments, are themselves unique. Consequently, in order to determine the price, we need to obtain a complete specification of the economy in terms of these parameters. While, at first sight, it appears that we cannot obtain any testable implications, this is not so. We outline one testable hypothesis which, though is not of the absolute level of option prices, is on the direction of the observed bias of option prices from their Black-Scholes counterparts.

Our study borrows heavily from the work of Cox, Ross and Rubinstein(1978) and the model is set in a discrete framework[9].

We provide a numerical example to illustrate the main results from the model, at the end. In order to obtain the equilibrium price, a hypothetical economy is set up and we describe the make-up of the individuals in the market. The solution for the equilibrium price is dependent on this specification and the solution is iterative. It is unlikely that any simple modifications of this model will ever yield closed-form solutions for the equilibrium price, since that price will be the one that equates supplies and demands, which in turn are dependent on the composition of the economy in terms of size, preferences and endowments.

Notwithstanding the limitations of this study, it represents a first step towards a more integrated model of option and stock prices[10]. Given that the size of the options market is as massive, if not more so, than the size of the market for primary stocks, it is believed that determination of option prices is bound to have an impact on the stock prices. Though this is not obvious from our analysis, we can get a feel for this by noting that the investors who were endowed with stocks at certain bases, prior to the determination of the option price, end up making excess profits while moving towards an equilibrium. If the knowledge, that the possession of stocks at "low" or "high" bases yields excess profits via option markets, is publicly available, then this may, in turn, affect the demands and supplies of traded stocks. This will lead to a stock price different from the one that would have prevailed in the absence of options. Consequently, to fully understand the mechanics of this market, we should strive towards more integrated models of pricing.

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[9] It is hoped that an extension to continuous time will simplify the analysis considerably, but at this point in time, this is merely a conjecture.

[10] See also Detemple and Selden(1986).

Chapter 1 starts by presenting the intuition of this study, and then outlines the assumptions of the model that we develop. The supply and demand for naked option positions is then determined and a preliminary equilibrium is outlined. Chapter 2 introduces trading in covered option positions and the resulting equilibrium. We then discuss the strike-price bias and obtain a prediction on the directions of this bias resulting from different price histories of the stock price. Chapter 3 contains a few numerical examples to illustrate the strike-price bias and concludes the study. An appendix is provided at the end that outlines the major notation used in this paper, for easy reference.

Chapter 1

The Model

1.1 To replicate or not

To appreciate the processes at work, we shall confine our analysis to call options on corporate stock. Extensions of the analysis to other types of options are relatively straightforward [11]. For mathematical simplicity we shall assume that there are no transaction costs. Further we partition the set of investors as follows :

$\mathcal{A}$ = The set of investors who desire the payoffs generated by holding a particular call option.
$\mathcal{B}$ = The set of investors who desire the payoffs generated by writing a particular call option.
$\mathcal{C}$ = Those who do not belong to $\mathcal{A}$ or $\mathcal{B}$.

Another partition of the set of investors that we shall consider is :

$\mathcal{D}$ = The set of investors who are holding onto stocks that they bought sometime ago at prices below the current stock price. They desist from selling them for fear of realising capital gains taxes at the capital gains tax rate which is assumed to be less than the ordinary income tax rate. [12]

$\mathcal{E}$ = The set of investors who have a short position on stocks that they sold at prices in excess

[11] The results with other options may of course be neither trivial nor identical.

[12] Refer to Constantinides - ECONOMETRICA(1983)

of the current market price. They desist from closing out their position for fear of realising capital gains taxes.

$\mathcal{F}$ = The set of investors who do not belong to $\mathcal{D}$ or $\mathcal{E}$.

The actual composition of the above sets is assumed exogeneous to our analysis.

Now, an investor in $\mathcal{A}$ can satisfy the demand for the payoffs resulting from holding a call option in two ways.

(i) Buy the option.

(ii) Create one *synthetically*, using stocks and bonds.

The cost of creating such a synthetic option, will depend on the usual parameters[13] as well as, on the investor's tax rate and the basis of the stocks used for the process of replication. If an "inventory" of long stocks are available, then these can be used for the process of replication. If, in addition, these stocks qualify for the computation of capital gains taxes at the long-term capital gains tax rate, then any differences between the long term and short term tax rate on capital gains and losses will also affect the cost of replication. We assume that if an inventory of long stocks is available, the basis of the stocks in this inventory will be less than the current market price for the stock[14]. If in the process of replication, the stock needs to be sold, and if as a result taxes are paid, then such taxes will affect the cost of replicating the long option position. Also notice that the lower the basis of the stocks in the replicating portfolio, the greater the magnitude of the capital gains taxes paid. Consequently, we would expect that the lower the basis of the stocks in the inventory, the higher the cost of replicating a long option. In figure 1, we have drawn a hypothetical schedule that describes the costs of replicating a long option, as a function of the basis of the stocks used in the inventory. Consider an investor with an inventory of stocks that have a basis in excess of the current stock price. Such an inventory would consist of shorted stocks which, if liquidated, would imply capital gains taxes. To replicate a long option in the presence of this inventory, the investor would have to first liquidate the inventory, pay taxes, and then buy new stocks to use for the replication process. However, now capital gains and losses will be taxed at the higher short-term rate. From the above, we note that the costs of replicating a long option for the investor with an inventory of short stocks, will not be less than the costs of replication for an investor in $\mathcal{F}$. In figure 1, we have drawn these costs to be increasing in the basis, when the basis exceeds the current stock price, S_0. If an investor with an inventory of short stocks can purchase fresh stocks *without* having to pay any capital gains taxes, then the schedule describing the costs

[13] Current stock price, excercise price, interest rates, etc.

[14] If the basis is in excess of the current stock price, then a capital loss tax shelter can be immediately realised.

of replication as a function of the basis of the inventory, would be horizontal to the right of S_0 in figure 1. The investor who desires the payoffs from an option, will choose to buy the instrument provided his cost of replication exceeds the price of the option.

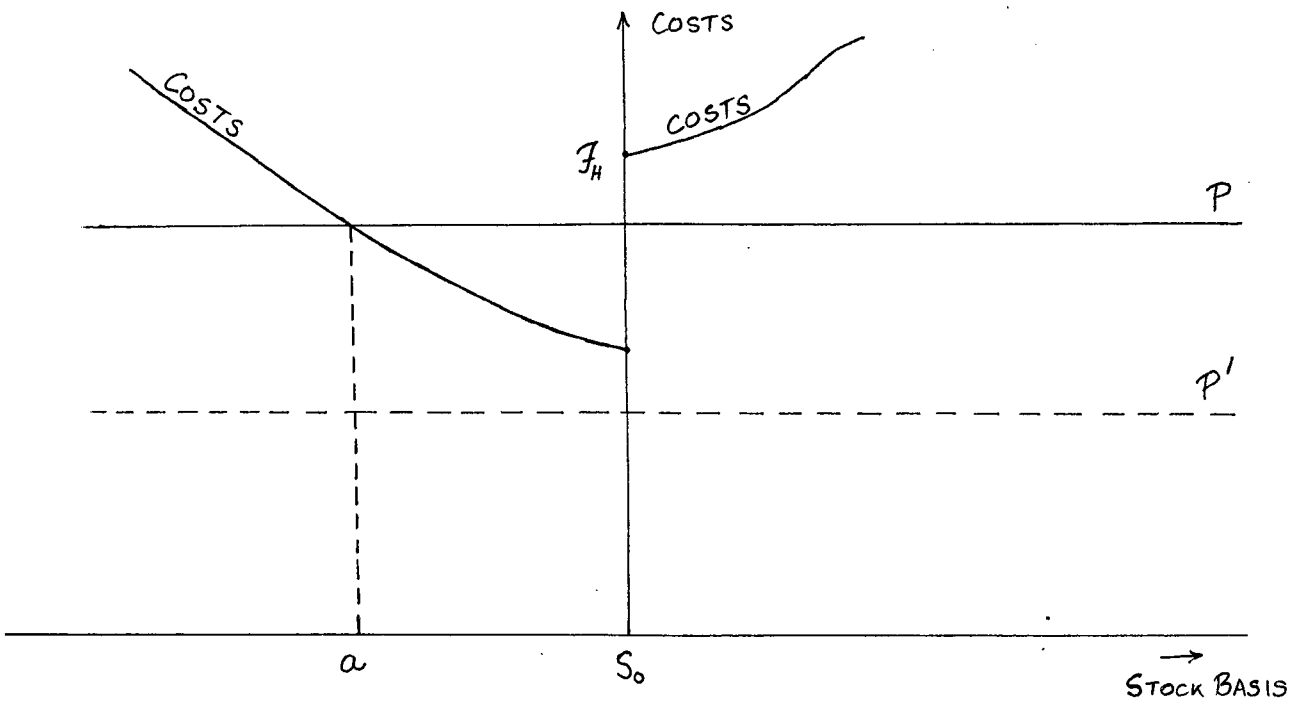

Figure 1

Following an identical thread of reasoning, we obtain that investors in $\mathcal{B}$ who desire the payoffs generated by writing an option, could either write the option, or replicate the payoffs using stocks and bonds. To replicate the payoffs to a short option, the replicating portfolio must be *short* in stocks. Investors who have inventories of short stocks (basis in excess of the current stock price), can use the stocks in their inventories to replicate these payoffs. The higher the basis of the short inventory, the lower will be the revenue from replication, since a higher basis implies greater taxes for short stocks. In addition, investors with inventories of long stocks with bases lower than the current stock price, will obtain even lower revenues since they may have to liquidate their long inventories, pay taxes, and then replicate to receive a revenue similar to those received by investors in $\mathcal{F}$. Figure 2 shows a hypothetical schedule that describes the revenues from replicating a short option position, as a function of the basis of the stocks in the inventory.

Consider figure 1 once again, and assume that the market price for the option is currently at P. $\mathcal{F}_H$ is the cost of replicating a long option payoff for an investor in $\mathcal{F}$. Observe that for

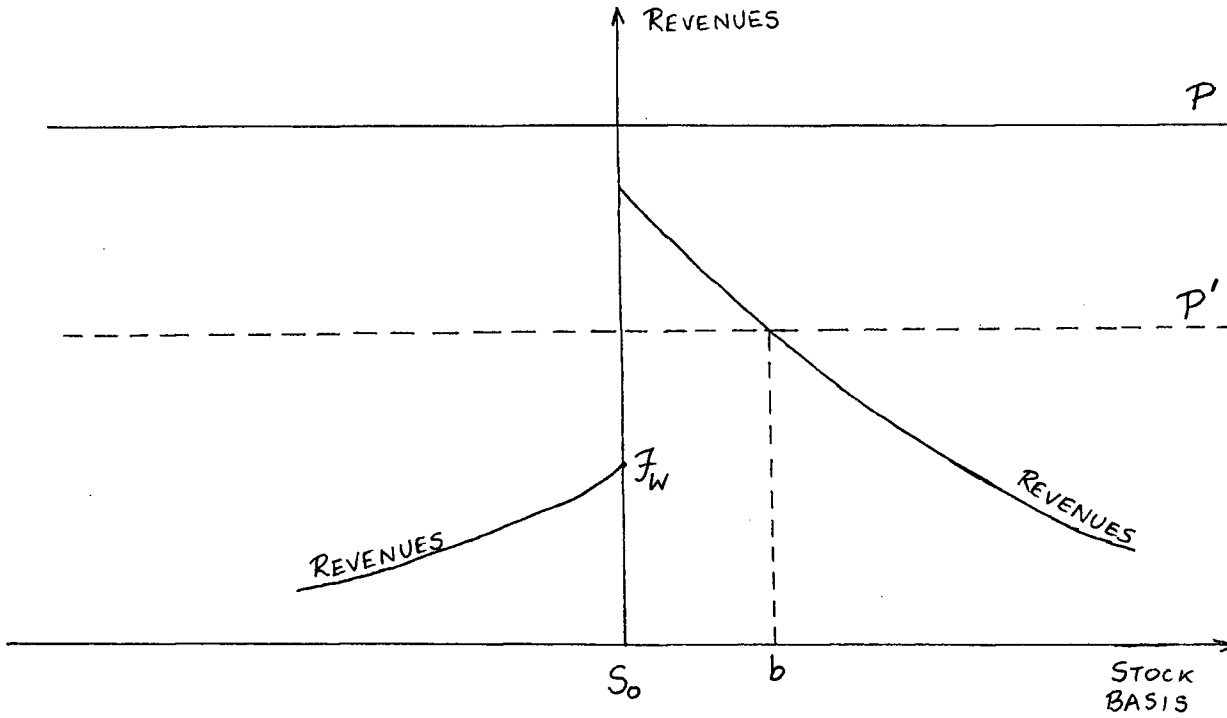

Figure 2

such investors, the cost of replication exceeds[15] the price of the option. Consequently, any investor in $\mathcal{F}$ and in $\mathcal{E}$, who desires the payoffs from a long option will prefer to actually buy the option rather than replicate it using stocks and bonds. Similarly, investors with inventories of long stocks with a basis lower than a, will prefer to purchase the option, if they desire such payoffs. On the other hand, investors who have long inventories with a basis in the interval (a, S_0), would prefer to replicate the payoffs rather than actually buy the option. In the event that the option price were at P', all investors who desire the payoffs from a long option, would actually buy the option. In other words, the *demand for naked options* would be higher at P' than at P.

Consider figure 2 now. $\mathcal{F}_W$ represents the revenue from creating a short synthetic option for the investor in $\mathcal{F}$. Investors who have long inventories with a basis below the current stock price will, as argued earlier, have revenues that are not in excess of $\mathcal{F}_W$. If the market price for the option were at P, every investor who desires the payoffs resulting from a short option position will find that his revenues from replicating the payoffs using stocks and bonds will be less than the revenues from actually writing the option[16]. Consequently, all investors who desire short option payoffs, will actually write options. On the other hand, if the current option price were at P', then the investors who have inventories of short stocks with a basis in the interval (S_0, b), will find that the

[15] This need not be the case, however.

[16] In which case, he would receive P dollars.

revenues from replication will be in excess of P'. Consequently, they will prefer to replicate a short option, rather than actually write options. In other words, the *supply of naked options* is lower at P' than at P. Finally consider investors who do not want the payoffs generated by options, but who possess inventories of stocks. Refer to figure 1 and assume that the inventory of some such investor consists of long stocks with a basis in the interval (a, S_0). Also assume that the current option price is at P. This investor can now *write* an option for P, and hedge its payoffs by creating a long synthetic position at a cost *below* P. Consequently, the investor can make some arbitrage profits. He cannot do so indefinitely, for after a while his inventory will be exhausted. Nevertheless, the existence of such investors creates a *supply* of covered option positions. Observe that had the option price been at P', there would not have been any profitable opportunities for covered option positions. In other words, the supply of covered option positions increases with the option price. The *demand* for covered option positions becomes clearer when we look at figure 2. If the current option price were at P', and if there is an investor who has an inventory of short stocks with a basis in the interval (S_0, b) (and if he does not desire option payoffs), then this investor can buy an option for P' dollars and hedge the payoffs by creating a short synthetic position that yields a revenue in excess of P'. Also observe that if the current option price were at P, there are no opportunities for profitably covering the payoffs to a long option.

The existence of profitable covered positions for an investor with no pre-existing "inventory" is not consistent with equilibrium, for if such opportunities were to exist, infinite arbitrage is possible. Recall that $\mathcal{F}$ is the set of investors with no pre-existing inventories of stocks. Denote by $\mathcal{F}_{CSO}$, the cost of covering a short option for such investors. $\mathcal{F}_{CSO}$ represents the costs of creating a long option position using stocks and bonds that exactly nullifies the future after-tax payoffs to the writer of a call option. Similarly, denote by $\mathcal{F}_{CLO}$, the revenue from covering a long option. It is the revenue from shorting stocks and bonds in such a way so as to nullify the future after-tax payoffs to the buyer of a call option. An investor who takes up such covered positions is subject to *wash-sale rules*, whereby capital gains are taxed, but capital losses do not qualify for tax shelters. Consequently, $\mathcal{F}_{CSO}$ will exceed the "no-tax" cost of covering a short option and $\mathcal{F}_{CLO}$ will be less than the "no-tax" revenue from covering a long option. There will be no arbitrage opportunities as long as the equilibrium price for the option, P, satisfies

$$\mathcal{F}_{CSO} \geq P \geq \mathcal{F}_{CLO} \quad .$$

We shall not explicitly compute $\mathcal{F}_{CSO}$ and $\mathcal{F}_{CLO}$, but shall restrict our analysis to the location of the option price within these no-arbitrage bounds. It should be noted, however, that the supply of options becomes perfectly elastic at $\mathcal{F}_{CSO}$ while the demand becomes perfectly elastic at $\mathcal{F}_{CLO}$.

The option price, in equilibrium, must be the one that equates all supplies (naked and covered)

to demands (naked and covered). Since these demands and supplies are influenced by investor preferences, endowments and beliefs in the economy, so will option prices. In particular, observe that the endowment of inventories depends to a large degree on the past trading behaviour of the market as well as, on the history of the stock price. Hence, these factors, some of which can be observed, will also have a direct impact on the option price and on the observed trading behaviour in options markets.

Even in the simple analysis above, it is clear that $\mathcal{F}_H$ need not equal $\mathcal{F}_{CSO}$. Observe that the process of replicating a long option for investors in $\mathcal{F}$ is not subject to *wash-sale rules*, whereas if the investor were to take up a covered position, he would be subject to these rules. Consequently, $\mathcal{F}_{CSO}$ will not be less than $\mathcal{F}_H$. Similarly, $\mathcal{F}_{CLO}$ will not exceed $\mathcal{F}_W$. In the example above, we assumed that any individual will face identical costs of covering a short option and creating a long option provided that he is not in $\mathcal{F}$. However, given the possible inter-relationship between the option and the stock in the covered position, it turns out that the optimal decision rules in a covered position could be somewhat different from those in a naked position. Consequently, we should actually be looking at the following four schedules:

1) Cost of creating a synthetic long option position.

2) Revenue from creating a synthetic short option position.

3) Cost of covering a short option.

4) Revenue from covering long option.

One final point that must be highlighted is that the location of the schedules will also be a function of the option price. The final payoffs from the option depend on the initial premium paid (or received) since the computation of capital gains and losses on the option will involve this premium. Consequently, the cost of (or the revenue from) replicating the payoffs to options, will depend on the market price for the option. The higher the premium, the higher the capital gains taxes paid by writers of options, and the larger the capital loss shelters realised by holders of options. This will raise the cost of replicating a long position, and at the same time it will raise the revenues from replicating a short position. However, changing the option premium by one dollar, only changes the final payoffs by the investor's short-term capital gains tax rate. Since the schedules will change by the present value of these tax induced changes to the final payoffs, computed under the appropriate martingale measure[17], the schedules themselves will shift by *less* than the change in the option premium. Nevertheless, they do change with the option price. Further, the determination of the

[17] The one that makes the expected after-tax return on the stock equal to the after-tax return on the riskless bond.

option price is itself dependent upon the location of these schedules, and on the distribution of investors along various bases. Hence, the determination of the price and of the location of the schedules must be done jointly. In our study, we iteratively[18] solve for the equilibrium price. To do this, we first start with some price[19] and compute the above schedules and the resulting net demand and supplies after assuming a distribution of bases in the economy. If we find an excess demand (or supply) at this price, we revise the price estimate in the appropriate direction, and start all over again. It turns out that the difference between the equilibrium price of our model and the Black-Scholes price is not linked in any monotonic manner to the strike price of the option. The remainder of this chapter is devoted to deriving the above schedules.

1.2 Assumptions and Notation

A1 *Stock price follows a Binomial Multiplicative Jump Process.*

A2 *There exists a risk-free asset that costs* \$1 *today and which yields a payoff of* $\$R$ $(R > 1)$ *in one period. We shall call this asset the BOND.*

A3 *No dividends are paid on the stock.*

A4 *Markets are efficient and frictionless* i.e. *there are no transaction costs or short sale restrictions.*

A5 *Long-term capital gains/losses are taxed at* τ_g. *Ordinary income and short-term capital gains/losses are taxed at* τ_o $(\geq \tau_g)$. *In addition, the trader can offset all tax losses against other tax liabilities.*

Given the above assumptions, there are essentially two ways by which we can compute the costs of (or revenues from) replicating an option position. One way, for example, would be to assume that the trader purchases (or sells short) new stocks in the market for replication, and that the basis of his pre-existing "inventory" of stocks will affect the taxes paid when he has to rebalance the composition of the replicating portfolio. However, this will have an effect on the basis of his pre-existing stocks, and the change in the basis of these pre-existing stocks must be appropriately modelled by describing the investor's portfolio choice problem. By assuming that these changes in

[18] It is not clear as to whether a closed-form solution exists for the equilibrium price.

[19] The Black-Scholes price appeared to be a good place to start.

the basis of the investor's pre-existing stock are of little or of no concern to him, however, we can solve the problem of computing these costs of replication. In the section that follows, we present an attempt of this sort. We also provide a few numerical examples in a two-period setting to illustrate the intuition presented in the previous section.

1.3 One approach to computing replication costs and revenues

In this section, we shall explore the derivation of the cost of and revenues from replicating long and short option positions, for an investor who wishes to *add* an option to his portfolio, *without liquidating the underlying stock*. We shall restrict the analysis to a 2-period model, with the stock price following a binomial jump process. In addition to assumptions 1 through 5, we shall make a few assumptions here, but these will be local to this section and will not necessarily spill over to the model developed in the following sections. To start off, let

S_N = The stock price at $t = N$.
S_0 = The stock price today.
u = The magnitude of an "up" move.
d = The magnitude of a "down" move.
E = The exercise price of the call option.
P = The market price at $t = 0$ for the option.
α_0 = The market value of the stock in the replicating portfolio at $t = 0$.
y_0 = The number of dollars invested in the riskless bond in the replicating portfolio at $t = 0$.
α_1^u = The market value of the stock in the replicating portfolio at $t = 1$ if the stock price moves to uS.
y_0^u = The number of dollars invested in the riskless bond in the replicating portfolio at $t = 1$ if the stock price moves to uS.
α_1^d = The market value of the stock in the replicating portfolio at $t = 1$ if the stock price moves to dS.
y_0^d = The number of dollars invested in the riskless bond in the replicating portfolio at $t = 1$ if the stock price moves to dS.
C^{uu} = The payoff to the call option, if the final stock price is u^2S.
C^{ud} = The payoff to the call option, if the final stock price is udS.
C^{dd} = The payoff to the call option, if the final stock price is d^2S.

1.3.1 Replicating a long option

We assume that the investor has one stock[20] held long in his existing portfolio with a basis of S_B, which qualifies for long term capital gains taxes, if liquidated. In addition, we assume that $S_B \leq S_0$, for otherwise, it may be optimal to liquidate the stock immediately, and realise the capital loss. The investor who wants a long option, could either buy it at the current market price of P, or he could replicate it using stocks and bonds. In order to replicate the option, the investor must buy fresh stock in the market at the current price. Now, once this is done, he could replace the stock in his pre-existing portfolio with the newly purchased stock and use the stock with a basis of S_B for replicating the option. The advantage (or disadvantage) of doing this stems from the observation that the fresh stock will be subject to short term capital gains/losses whereas, the older stock would qualify to be taxed at the lower long-term rate. Clearly, this action of switching stocks will affect the basis of the stocks in the pre-existing portfolio[21], and will in turn affect the after-tax state contingent payoffs from that portfolio. In fact, *if* the process tends to lower the cost of replicating an option, *then* it is quite likely to lower the utility stemming from the older portfolio. If these two effects[22] exactly cancel each other out, then there will be no benefits from this action. However, it is possible to visualise situations where these two effects do not cancel each other completely[23]. To completely analyse the problem in its entirety, we shall have to examine in substantial detail situations where this exchange of stocks between the replicating portfolio and the pre-existing portfolio increases the utility of the investor. However, to keep the analysis simple, we shall assume that changes in the basis of the stocks in the pre-existing portfolio have little or no impact on the utility of that portfolio to the investor.

Before we actually write the equations that the replicating portfolio must satisfy, it is worthwhile to describe the replication process completely, in words. At time, $t = 0$, the portfolio replicating a long option will contain a certain number of stocks held long. If the stock price rises to uS_0, then the number of stocks would have to be raised, since the call option is positively correlated to the stock. This will involve purchasing some more stocks, but no taxes are involved. We assume that the investor can replace the new additions with old stocks from his pre-existing portfolio. If the stock price were to drop to dS_0, on the other hand, the investor will have to *reduce* the number

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[20] The existence of just one stock is for convenience alone and is made with no loss of generality. We do need that the number of stocks be finite, however.

[21] It will now be higher than S_B.

[22] *Advantages* resulting from a lowered replication cost, and *disadvantages* resulting from a reduction in utility as a result of a change in the basis of the pre-existing portfolio.

[23] For example, the liquidation of the stock in the pre-existing portfolio may be called for only when the investor's wealth level is "high", whereas the portfolio replicating the option may require liquidation of the stock at "low" wealth levels as well. In this case, the transfer of a dollar of tax-liability from the replicating portfolio to the pre-existing portfolio will be preferred by any risk-averse investor.

of stocks in the replicating portfolio. This will require that he sell some of them, and realise capital gains/losses. If he sells the stocks that he bought fresh, then he will realise a short-term capital loss, but if he sells some of the old stock that he had, he may realise long-term capital gains[24] or losses. To keep our algebra simple, we shall assume that *all sales of stocks from the replicating portfolio will involve the old stocks with a basis of* S_B[25].

Finally, we assume that at maturity of the option, the option (if held) would be sold and the replicating portfolio (if formed) would be liquidated[26]. Under these assumptions, we can outline the equations that the replicating portfolio must satisfy. Let

$$R^* = R - (R-1)\tau_o \quad .$$

Recall that R is the pre-tax return on the riskless bond and that τ_o is the investor's tax rate for ordinary income. R^* is hence, the investor's after-tax return from a dollar invested in the riskless bond. The replicating portfolio must satisfy

$$C^{uu} = \alpha_1^u u + y_1^u R^* - \tau_g \frac{\alpha_1^u}{uS_0}(u^2 S_0 - S_B) \tag{i}$$

$$C^{ud} = \alpha_1^u d + y_1^u R^* - \tau_g \frac{\alpha_1^u}{uS_0}(udS_0 - S_B) \tag{ii}$$

$$C^{ud} = \alpha_1^d u + y_1^d R^* - \tau_g \frac{\alpha_1^d}{dS_0}(udS_0 - S_B) \tag{iii}$$

$$C^{dd} = \alpha_1^d d + y_1^d R^* - \tau_g \frac{\alpha_1^d}{dS_0}(d^2 S_0 - S_B) \tag{iv}$$

$$\alpha_0 u + y_0 R^* = \alpha_1^u + y_1^u \tag{v}$$

$$\alpha_0 d + y_0 R^* = \alpha_1^d + y_1^d + \tau_g \left[\frac{\alpha_0}{S_0} - \frac{\alpha_1^d}{dS_0}\right](dS_0 - S_B) \tag{vi}$$

Equations (i) through (iv) are the terminal conditions that the replicating portfolio must satisfy. The left sides of the equations are the after-tax payoffs from the option. On the right sides, we have the payoff from the replicating portfolio after taxes. Taxes are assumed to be paid at the long term rate of τ_g and observe that we have sold the stock with a basis of S_B always. The terms

$$\frac{\alpha_1^d}{dS_0} \quad \text{and} \quad \frac{\alpha_1^u}{uS_0}$$

[24] It will depend on whether S_B is larger or smaller than dS_0.

[25] This assumption simplifies the algebra considerably, and the objective of this example is not to obtain a complete model, but rather to illustrate some of the intuition behind the determination of the equilibrium option price.

[26] This may, once again, not be the optimal action always.

represent the number of stocks in the hedge portfolio if the stock price at $t = 1$ was dS_0 and uS_0 respectively. Equation (v) and (vi) are self-financing conditions. Equation (v) requires that the return from the replicating portfolio when the stock price moves from S_0 to uS_0 is the same as the investment required at $t = 1$ when the stock price is uS_0. Equation (vi) is similar, but refers to the case when the stock price moves from S_0 to dS_0. In this case, there are some taxes to be paid since the number of stocks in the hedge portfolio is reduced.

The cost of creating a long synthetic option for the investor is given by the value of the replicating portfolio at $t = 0$. Hence,

$$\alpha_0 + y_0 = \text{Cost of creating a long synthetic option}$$

We can solve these equations in a straightforward manner to yield

$$\alpha_0 + y_0 = \frac{1}{R^*}\left[p\phi^u + (1-p)\phi^d\right]$$

where

$$\begin{aligned}
p &= \frac{R^* - \Gamma}{u - \Gamma} \\
\Gamma &= d(1-\tau_g) + \tau_g \frac{S_B}{S_0} \\
\phi^u &= \alpha_1^u + y_1^u \\
\phi^d &= \frac{\alpha_1^d}{d}\Gamma + y_1^d \\
\alpha_1^u &= \frac{C^{uu} - C^{ud}}{(u-d)(1-\tau_g)} \\
y_1^u &= \frac{\theta^u C^{ud} - \theta^d C^{uu}}{(u-d)(1-\tau_g)R^*} \\
\theta^u &= u(1-\tau_g) + \tau_g \frac{S_B}{uS_0} \\
\theta^d &= d(1-\tau_g) + \tau_g \frac{S_B}{uS_0} \\
\alpha_1^d &= \frac{C^{ud} - C^{dd}}{(u-d)(1-\tau_g)} \\
y_1^d &= \frac{\Lambda^u C^{dd} - \Lambda^d C^{ud}}{(u-d)(1-\tau_g)R^*} \\
\Lambda^u &= u(1-\tau_g) + \tau_g \frac{S_B}{dS_0}
\end{aligned}$$

$$\Lambda^d = d(1-\tau_g) + \tau_g \frac{S_B}{dS_0}$$

C^{uu}, C^{ud} and C^{dd} are the payoffs from the option. We assume that at maturity $(t = 2)$, the option is sold in the market. The price of the option at maturity must be

$$\text{Max}(S_N - E, 0) \quad ,$$

where S_N is the price of the stock in the market. The option was purchased for P at $t = 0$, and, if $S_N \leq E$, then it is considered to have been sold for zero dollars. In this event, the investor obtains a capital loss tax shelter. However, since most options have maturities of less than a year, this tax shelter is computed at the investor's tax rate for short term capital gains, τ_o. Similarly, if $S_N > E$, then the option is treated as a asset that was purchased for P dollars and sold for $(S_N - E)$ dollars. This may lead to a capital gains or a loss. Once again the relevant tax rate is τ_o. Hence, the payoffs from the option can be outlined as follows:

$$C^{uu} = \begin{cases} P\tau_o & \text{if } u^2 S_0 \leq E \ ; \\ (u^2 S_0 - E) - \tau_o(u^2 S_0 - E - P) & \text{otherwise.} \end{cases}$$

$$C^{ud} = \begin{cases} P\tau_o & \text{if } udS_0 \leq E \ ; \\ (udS_0 - E) - \tau_o(udS_0 - E - P) & \text{otherwise.} \end{cases}$$

$$C^{dd} = \begin{cases} P\tau_o & \text{if } d^2 S_0 \leq E \ ; \\ (d^2 S_0 - E) - \tau_o(d^2 S_0 - E - P) & \text{otherwise.} \end{cases}$$

1.3.2 Replicating a short option

The stock position of a portfolio that replicates the payoffs to the writer of an option must be short in stocks. As with the case of the portfolio that replicates a long option, assume that all trades involving the stock in the replicating portfolio use some pre-existing stock position with a basis of S_B. In this case, the pre-existing position is assumed to consist of some short stocks which qualify for long term capital gains and losses. In addition, let (α_0, y_0), (α_1^u, y_1^u) and (α_1^d, y_1^d) be the stock and bond investment in a portfolio replicating the payoffs to a short option when the stock price is S_0, uS_0 and dS_0 respectively. Observe that α_0, α_1^u and α_1^d are negative, since the replicating portfolio in this case will be short in stocks. In addition, since a short call-option is negatively correlated with the price of the underlying asset, we have that

$$\frac{|\alpha_1^d|}{dS_0} \leq \frac{|\alpha_0|}{S_0} \leq \frac{|\alpha_1^u|}{uS_0} \quad .$$

Hence, if the stock price were to rise from S_0 to uS_0, the investor would have to increase the number of short stocks in the replicating portfolio. This can be achieved by selling some additional stocks short with no immediate tax consequences. However, if the stock price were to move to dS_0, some of the short stocks in the replicating portfolio must be liquidated. This will involve a capital gain or loss, and the resulting tax. For simplicity, we shall assume that the investor liquidates the short stocks in his pre-existing portfolio with a basis of S_B, and realises long-term capital gains or losses. Hence the self-financing conditions that the replicating portfolio must satisfy at the end of the first period are given by

$$\alpha_0 u + y_0 R^* = \alpha_1^u + y_1^u \tag{vii}$$

$$\begin{aligned}\alpha_0 d + y_0 R^* &= \alpha_1^d + y_1^d + \tau_g \left[\frac{|\alpha_0|}{S_0} - \frac{|\alpha_1^d|}{dS_0}\right](S_B - dS_0) \\ &= \alpha_1^d + y_1^d - \tau_g \left[\frac{\alpha_0}{S_0} - \frac{\alpha_1^d}{dS_0}\right](S_B - dS_0) \\ &= \alpha_1^d + y_1^d + \tau_g \left[\frac{\alpha_0}{S_0} - \frac{\alpha_1^d}{dS_0}\right](dS_0 - S_B)\end{aligned} \tag{viii}$$

At termination, the replicating portfolio is assumed to be liquidated. We assume that the short stock in the investor's pre-existing portfolio is used for this purpose, so as to avail of the long-term capital gains tax rate. Since the final payoffs from the replicating portfolio must be identical to the payoffs from the option, we note that

$$\begin{aligned}C^{uu} &= \alpha_1^u u + y_1^u R^* - \tau_g \frac{|\alpha_1^u|}{uS_0}(S_B - u^2 S_0) \\ &= \alpha_1^u u + y_1^u R^* - \tau_g \frac{\alpha_1^u}{uS_0}(u^2 S_0 - S_B)\end{aligned} \tag{ix}$$

$$\begin{aligned}C^{ud} &= \alpha_1^u d + y_1^u R^* - \tau_g \frac{|\alpha_1^u|}{uS_0}(S_B - udS_0) \\ &= \alpha_1^u d + y_1^u R^* - \tau_g \frac{\alpha_1^u}{uS_0}(udS_0 - S_B)\end{aligned} \tag{x}$$

$$\begin{aligned}C^{ud} &= \alpha_1^d u + y_1^d R^* - \tau_g \frac{|\alpha_1^d|}{dS_0}(S_B - udS_0) \\ &= \alpha_1^d u + y_1^d R^* - \tau_g \frac{\alpha_1^d}{dS_0}(udS_0 - S_B)\end{aligned} \tag{xi}$$

$$C^{dd} = \alpha_1^d d + y_1^d R^* - \tau_g \frac{|\alpha_1^d|}{dS_0}(S_B - d^2 S_0)$$

$$= \alpha_1^d d + y_1^d R^* - \tau_g \frac{\alpha_1^d}{dS_0}(d^2 S_0 - S_B) \qquad \text{(xii)}$$

As before, C^{uu}, C^{ud}, and C^{dd} are the payoffs to the option. The revenue from replication at $t = 0$, is given by

$$-(\alpha_0 + y_0) \quad .$$

We assume further that the writer of the option repurchases the option just prior to maturity. Since he wrote the option for P dollars at $t = 0$, capital gains or losses are computed on the difference between the repurchase price and the original premium received, P. The tax rate used in computing taxes is the short-term capital gains rate, τ_o. This gives us the payoffs to the writer of the option at maturity as

$$C^{uu} = \begin{cases} -P\tau_o & \text{if } u^2 S_0 \leq E \ ; \\ (E - u^2 S_0) - \tau_o(P - u^2 S_0 + E) & \text{otherwise.} \end{cases}$$

$$C^{ud} = \begin{cases} -P\tau_o & \text{if } udS_0 \leq E \ ; \\ (E - udS_0) - \tau_o(P - udS_0 + E) & \text{otherwise.} \end{cases}$$

$$C^{dd} = \begin{cases} -P\tau_o & \text{if } d^2 S_0 \leq E \ ; \\ (E - d^2 S_0) - \tau_o(P - d^2 S_0 + E) & \text{otherwise.} \end{cases}$$

Now observe that equations (vii) through (xii) are identical to equations (i) through (vi). Hence the solutions will be similar. Consequently, the cost of replicating a long option for the investor who has an inventory of long stocks with a basis of S_0 will be the same as the revenue from replicating a short option for the investor who has an inventory of short stocks with a basis of S_0.

In this example, we needed to solve a total of six equations to obtain the cost of replication. In general, with a N-period model, we will need to solve a total of

$$2^N + 2^{N-1} + \cdots + 2^1 \ = \ 2^{N+1} - 2$$

equations. The reason for this becomes apparent when we consider the fact that there are 2^M different paths that lead to any node that could be realised after the stock price has jumped M times, and that since the hedge ratio is dependent upon the price path, it is unique to each node. The number of equations that must hence be solved is equal to the number of potentially different nodes. We disregard the *first* node[27], since we are interested in obtaining the sum $\alpha_0 + y_0$, and not the value of each one of these terms. The problem was simplified considerably, by assuming that all trades in the interim period took place using stocks with a basis of S_B. If we relax this assumption, then the problem becomes considerably more complex, since the decision of which stock position

[27] The starting point.

to liquidate will depend upon the basis of the newly purchased stocks, the basis of the pre-existing stocks, the relative magnitudes of the short-term and long-term capital gains tax rates, and on the level of the stock price at the relevant node. It does not appear that the number of equations that must be solved in this general case will increase, but the complexity of the equations that must be solved will increase substantially, since we will now have to choose between different trading strategies *at each node*.

We end this section by presenting a few numerical examples for the model that we developed here, to demonstrate that the costs and revenues from replication are different for different inventory bases. In the numericals, the costs and revenues are decreasing in the basis of the inventory, but we do not have a formal proof for this observation.

Example 1

Consider an option with the following parameters :

$$\begin{array}{lll} u = 1.5 & S_0 = 10 & \tau_o = 0.4 \\ d = 0.5 & E = 10 & \tau_g = 0.25 \end{array} \qquad R = 1.10$$

The Black-Scholes price for this option is \$ 3.7190. In figure 13 of Appendix 3, we have plotted the costs and revenues from replicating this option for investors who have inventories of stocks with different bases. The schedules have been obtained by assuming that the current market price for the option is the Black-Scholes price. Observe that investors who have inventories of long stocks with bases in the range [9.27,10] will prefer to replicate a long option, rather than purchase the option. The very same investors also stand to gain by *writing* an option for \$ 3.7190 and hedging its payoffs at a lower cost, thereby pocketing the difference. All investors who have inventories of short stocks, on the other hand are better off by trading in naked options, if they desire option payoffs. The net demand or supply at any price depends partly on the distribution of investors with different inventories, and it is not necessary that demands and supplies exactly cancel each other at the Black-Scholes price. Consequently, the actual price could be higher or lower.

Example 2

Consider the impact of changing τ_o and τ_g, on the schedules. Let

$$\tau_o = \tau_g = 0.3 \quad ,$$

and let all other parameters remain the same. The Black-Scholes price for this option is still \$ 3.7190, however, the costs and revenues of replication change. In figure 14 of Appendix 3, we have plotted the costs and revenues from replicating this option for different bases. Once again,

the schedules have been obtained by assuming that the current market price for the option is the Black-Scholes price. Observe that investors who have inventories of short stocks with bases in the range [10,10.3] will prefer to replicate a short option, rather than write the option. The very same investors also stand to gain by *buying* an option for \$ 3.7190 and hedging its payoffs for a higher revenue, thereby pocketing the difference. All investors who have inventories of long stocks, on the other hand are better off by trading in naked options, if they desire option payoffs. Once again, it is not necessary that demands and supplies exactly cancel each other at the Black-Scholes price. Consequently, the actual price could be higher or lower.

1.4 An alternative approach to computing replication costs and revenues

In deriving the costs/revenues in the model of the previous section we assumed that the investor had in his possession, a portfolio comprised of stocks (of a particular firm) that qualified for long-term capital gains/losses. Let us denote this as portfolio **X**. In order to attain his desired (optimal) state-contingent wealth profile, the investor had to *add* the payoffs generated by an option to portfolio **X**. He did this by either trading in the option, *or* by trading in an appropriately constructed portfolio of stocks and bonds. The costs/revenues involved with this portfolio of stocks and bonds was influenced by the basis of the stocks in portfolio **X**. Consequently, different investors faced different costs/revenues from replicating the same option.

Alternatively we could consider an investor who has in his possession, some portfolio (let us call it portfolio **Y**) *and* a stock that qualifies for long-term capital gains/losses. To attain his desired optimal state-contingent wealth profile, this investor may need to liquidate the stock *and* obtain a call option on the stock[28]. Denote by U_1^*, the investor's expected utility from this activity, *if the process of liquidating the stock did not involve any taxes.* However, the process of liquidating the stock may well be costly since capital gains taxes may have to be paid. In the light of this cost, a second alternative for the investor would be to hold portfolio **Y**, obtain an option, *and* hold on to the stock position as well. Denote his expected utility from this portfolio as U_2^*. Since this portfolio is sub-optimal by assumption[29], U_2^* will not exceed U_1^*. Finally, let

$$\nabla = U_1^* - U_2^* \quad .$$

∇ represents the maximum benefits to the investor from liquidating the stock position. If the costs

[28] Any excess funds could be invested further, or consumed.

[29] He is holding on to the stock.

of liquidating the stock position (in terms of taxes paid) exceed ∇, the investor would not liquidate the stock position. We shall focus on such investors who have in their possession, stock positions that they have not liquidated since the benefits from liquidation fall short of the capital gains taxes that need to be paid. Now, instead of purchasing an option, such an investor could replicate the option payoffs by using a fraction of the stock and by investing an appropriate amount in a riskless bond[30]. As a consequence, the investor obtains the desired option payoffs and in addition reduces the quantity of undesirable stocks in his portfolio. We label this process as the creation of a *long synthetic option*.

The "cost" of creating such a long synthetic option will be the value of the replicating portfolio *less* the benefit that stems from having reduced the quantity of undesirable stocks. The composition of the replicating portfolio (and consequently its value) will depend on, amongst other factors, the basis of the stocks used since the stocks may have to be liquidated (and taxes paid) at termination of the option. The benefit that stems from reducing the quantity of undesirable stocks will depend on the basis of the stocks, the quantity of stocks used for replication, the investor's long-term capital gains tax rate as well as on the shape of his utility function. If we let ρ denote the magnitude of this benefit, then ρ is bounded from below by zero[31], and from above by the magnitude of the long-term capital gains taxes that will be paid were the trader to liquidate the stocks[32]. The cost of creating a long synthetic option is hence *not greater than* the value of the replicating portfolio. We shall assume that $\rho = 0$ for now, and consider the implications of relaxing this assumption later in the paper.

Under the assumption that $\rho = 0$, the cost of creating a long synthetic option is the current value of the replicating portfolio. We could interpret this cost as the investor's reservation price for the option. If the market price for the option exceeds this reservation price, the investor would prefer to replicate the option payoffs rather than purchase the option itself. On the other hand, if the market price for the option were below the investor's reservation price, he would prefer to purchase the option.

If the investor did not desire the option payoffs, he could still write the option and hedge its payoffs using the stock and bonds *provided* this is a profitable venture. In essence, he can cover the long stocks with short options. We view such investors as taking up covered option positions. Observe that such investors can profitably trade in covered options as long as they do not exhaust their "inventory" of stock. Once this inventory is entirely locked up in covered positions, any

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30 We assume that a riskless bond exists.

31 Since these stocks are undesirable.

32 If the benefits from liquidation exceed the taxes that must be paid, then the trader will liquidate them and, by assumption, he is not doing this. This upper bound is in turn proportional to the difference between S_0, the current stock price, and S_B, the basis of the stocks.

further covered positions must involve the purchase or sale of fresh stock at current market prices. In equilibrium, this cannot be a profitable venture any more.

Similarly, investors who have short stock positions that they are holding on to, can create *short synthetic options* that replicate the payoffs from a short option position. Under the assumption that $\rho = 0$, we will be underestimating the revenues from such synthetic positions.

To summarise, we have briefly discussed two different ways of computing the costs/revenues from replicating option positions. The first method involved the purchase of fresh stock in the market for the replication process, whereas the second method used an inventory of pre-existing stock that the investor wishes to liquidate, for the replication process. Which of these two methods is appropriate depends to some extent on the composition of investors at any point in time, in the market. The first method would be appropriate if we believe that the market for options is dominated by investors who wish to *add* options to their existing portfolios, whereas the second method would be more appropriate if we believe the options market to be dominated by individuals who wish to *switch* from stocks to options. In other words, these investors wish to trade the payoffs that the stock in their portfolio offers, for the payoffs resulting from an option. In this study, we focus on the second approach, primarily for mathematical tractability[33]. However, in the process of choosing a tractable model, we make a sacrifice. To identify this limitation, consider the costs and revenues from replication for investors who do not have a pre-existing inventory of stocks that qualify for long-term capital gains and losses. These investors will have to purchase stocks at the current market price and in addition, the process of rebalancing the composition of the replicating portfolio may involve capital gains taxes or capital loss shelters. Consequently, in order to compute $\mathcal{F}_H$, $\mathcal{F}_W$, $\mathcal{F}_{CSO}$ and $\mathcal{F}_{CLO}$ properly, we need a model of the first sort, which computes the costs of and revenues from *adding* option payoffs to the investor's portfolio, rather than a model which computes the costs/revenues of replacing stock payoffs with option payoffs. In this paper, the best we can do is to acknowledge the existence of these costs and revenues. In particular, we assume that they form bounds for the option price.

It is believed that the two approaches will have similar implications on trading behaviour, though the equilibrium prices could differ. Since investors of both categories (those who wish to "add" options and those who wish to "switch" to options) will most likely exist in any market, to obtain a complete picture we must include them in a common framework. We shall briefly talk about the implications of including the first approach into our framework, in the event that a reasonable and tractable model can be developed, towards the end of this paper.

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[33] It turns out that this specification makes the problem mathematically simpler. The number of simultaneous equations that need to be solved reduces to $(2N + 1)$ in a N-period model.

A6 *A trader may have an inventory of stocks held long*[34]*, or sold short*[35]*, which is sufficiently large to enable him to enter into any position required during the process of replication of an option without having to buy/sell any stocks from/in the open market. This inventory is homogeneous, infinitely divisible and has a basis price S_B for computation of capital gains tax. In addition, this inventory consists of stock positions that have not been closed out for fear of incurring capital gains taxes.*

The assumption that the investor has a sufficiently large inventory is not a restrictive as it first appears. Even if the investor has only a fraction of stock left, he can still replicate an option. It just means that he can now only replicate the payoffs to a fraction of an option.

The assumption that the inventory is homogeneous is once again not very restrictive. If the investor has two stocks in his inventory with different bases, then we can still view him as two different investors, each with one stock at different bases.

A7 *The trader can buy or sell stocks from his inventory and thus adjust the proportion of stocks in the replicating portfolio without incurring any capital gains tax. At maturity of the option, he takes an optimal decision and may consequently be affected by capital gains taxes.*

This assumption is critical to the analysis, since it actually allows the trader to transfer stocks to and from his replicating portfolio, without having to trade in the market and thereby modify the basis of the stocks in the replicating portfolio. Such modifications will make the hedge ratio a function of the stock price path[36]. It should be noted, however, that this assumption is appropriate *only* when we are computing the costs/revenues for an investor who is contemplating on *switching from stocks to options*. Any investor who is contemplating the addition of an option to his existing portfolio, will end up having to pay taxes in the interim period. The income from the bond is assumed to be taxed at the end of every period.

A8 *The final payoffs from the option are $f(S_N, S_B, R, \tau_g, \tau_o, E, \pi^*, P)$, where*

S_N = Final stock price

[34] To create a synthetic option

[35] To write a synthetic option

[36] Refer to Boyle and Emanuel(1980), Goldman, Sosin and Gatto(1979) for path-dependent option pricing. In section 1.3 we outlined some of the complications with this approach.

P = Original premium paid/received for the option. This price is set in the market at time, $t = 0$.

E = Set of non-stochastic option parameters. (E.g. Exercise price, time to maturity, etc)

Π = Set of feasible policies that the trader has available to him. (E.g. Exercise, Don't Exercise, etc.)

π^* = $\arg \underset{\pi \in \Pi}{\text{Max}} \{ f(S_N, S_B, R, \tau_g, \tau_o, E, \pi, P) \}$

S_B = Basis of investor's stock holding.

R = Riskless rate of return ($R > 1$).

τ_o = Investor's income tax rate for ordinary income

τ_g = Investor's tax rate for capital gains.

Under the above assumptions, the model to compute costs and revenues from replication can be split into two sub-models which together yield the reservation prices for each trader. The first sub-model which we shall refer to as Model $\mathcal{X}$ operates from period 1 to $(N-1)$ where N is the "maturity" period. During its regime, no capital gains taxes are paid. The bond income, however, is subject to taxation. The second sub-model which we shall refer to as Model $\mathcal{Y}$ operates during the maturity period at the end of which an optimal decision is taken and taxes are paid.

Model $\mathcal{Y}$ will yield the value of the replicating portfolio at time $t = (N-1)$ as a function of S_{N-1}, R, S_B, etc. Let us assume that these values are

$$\Phi\{S_{N-1}, S_B, R, \tau_g, \tau_o, E, \Pi_N^*, P\}$$

where

$$\Pi_N^* = \text{the set of optimal policies available at maturity.}$$

Model $\mathcal{X}$ takes the $\Phi\{\bullet\}$'s as the final payoffs and obtains the value of the replicating portfolio today.

1.5 Model $\mathcal{X}$

In general, during any single period when Model $\mathcal{X}$ operates, let

S = Stock price at start of relevant period.

u = Magnitude of an *up* move. i.e. Stock price could move to uS.
d = Magnitude of a *down* move. i.e. Stock price could move to dS.
q = Probability of an *up* move.
R = The riskless rate of return before taxes. ($R \geq 1$.)
C = Implicit value of call option to trader. This need not be the market price for the option, but rather represents the price that makes the trader indifferent between the option and the replicating portfolio. It is the cost of replication.
C^u = Implicit value of call option when S moves to uS.
C^d = Implicit value of call option when S moves to dS.

The assumption that the proceeds from bonds are taxed every period yields the net payoff from \$1 invested in bonds as

$$R - \tau_o(R-1) \;=\; \tau_o + R(1-\tau_o) \;=\; R^* \text{ (say)}$$

In addition we need that $d < R^* < u$, in order to avoid the bond dominating (or being dominated by) the stock.

Under these conditions, it is easy to observe that Model $\mathcal{X}$ is identical to the Binomial Option Pricing Model of Cox, Ross and Rubinstein. This implies that

$$C = \frac{pC^u + (1-p)C^d}{R^*}$$

where

$$p = \frac{R^* - d}{u - d}$$

Proceeding further, we have that the value of the option at time $t = 0$ in terms of the values at time $t = (N-1)$ can be written as

$$C = (R^*)^{(1-N)} \sum_{i=0}^{N-1} \Psi(i)$$

where $\cdots (1)$

$$\Psi(i) = \binom{N-1}{i} p^i (1-p)^{(N-1-i)} \, \Phi\left\{ u^i d^{(N-1-i)} S, S_B, R, \tau_g, \tau_o, E, \Pi^*_{N,i}, P \right\}$$

Note *The assumption that the trader can offset any interest paid out to a negative bond holding against his other incomes so as to utilise fully the tax shield generated, will ensure that (1) holds for both writers and buyers of the option when $R^* \neq R$.*

1.6 Model $\mathcal{Y}$

Let the stock price at time $t = (N-1)$ be S_{N-1}, and let the value of the replicating portfolio be Φ_{N-1} (for notational simplicity) and its payoffs be C^u and C^d when the stock price moves to uS_{N-1} and dS_{N-1} respectively. Also let,

$$\alpha = \text{Dollar value of stock in replicating portfolio}$$
$$y = \text{Dollar value of Bonds in replicating portfolio}$$

Hence,

$$\Phi_{N-1} = \alpha + y = \text{Value of claim}$$

The notation that we shall use is as follows :

S_0 = The current (at $t = 0$) market price of the stock.
S_N = The stock price at maturity of the option.
E = The exercise price (also known as the *strike price*) of the option.
N = The life of the option. In other words, the option expires in N periods.
$\mathcal{F}_H$ = Initial cost of creating a long synthetic option for someone in $\mathcal{F}$.
$\mathcal{F}_W$ = Initial revenue from creating a short synthetic option for someone in $\mathcal{F}$.
P = Market price of a call option with exercise price E and maturity of N periods.
S_B = The basis of the investor's stock holding.
η = The cost of creating a long synthetic option for someone who has an inventory of long stocks with a basis of S_0.
μ = The revenues from creating a short synthetic option for the person who has shorted stocks with a basis of S_0.

η is different from $\mathcal{F}_H$, since the person in $\mathcal{F}$ who buys fresh stock will pay capital gains taxes at the ordinary income tax rate whereas, the person with a cost of η pays such taxes at the capital gains tax rate which is assumed to be lower. In addition, since the person in $\mathcal{F}$ does not have an inventory of stocks, he must purchase stocks from the market, as and when he needs it. This will make his basis, and consequently the taxes paid by him, dependent on the price path of the stock. Similarly, μ is different from $\mathcal{F}_W$.

1.6.1 The Buyer

At maturity, the holder of a call option has available to him the following feasible policies[37]

(i) Let the option expire unexercised.

(ii) Exercise the option and sell the stock immediately.

(iii) Exercise the option and hold on to the stock.

The payoffs under these three policies are :-

(i) $P\tau_o$. This is the tax shelter due to the capital loss incurred. The option was purchased for $\$P$ and "sold" for zero dollars. However, since almost all options have maturities of 9 months or less, the eventual tax paid is at the ordinary income tax rate, τ_o.

(ii) $\{ (S_N - E) - (S_N - E - P)\tau_o \}$. The first term represents the benefit to the trader. He pays $\$E$ to acquire the stock, then sells it for $\$S_N$. As per tax laws, the stock is assumed to have a basis equal to the original premium paid on the call *plus* the exercise price i.e. $P + E$. Hence, the second term is the capital gains tax paid (or tax shelter).

(iii) $(S_N - E)$. This represents the value of the stock less the price paid to acquire it, **provided** capital gains tax can be deferred indefinitely.

The trader chooses the policy with the best payoff.

A9 *There is no "forced liquidation" and the trader can defer capital gains tax indefinitely. In addition, τ_d, the tax rate on accumulated capital gains after death is zero.*

The assumption of no forced liquidation (Constantinides [1983]) ensures that there is no need for insurance on probable capital gains tax. $\tau_d = 0$ is equivalent to saying that the present value of capital gains taxes paid after death is zero.

This assumption justifies the payoff under (iii) above. Constantinides [1983] shows that under such assumptions the optimal policy would be to realise all capital losses[38] immediately, and to

[37] In addition to these three courses of action that we have outlined, the holder has also the option of selling the instrument in the open market at the prevailing price which must be Max[0, $(S_N - E)$]. However, this will not add to the possible payoffs that we have listed here.

[38] We assume that they can be realised

defer capital gains. (He also dicusses ways in which this may be done.)
Capital loss requires that

$$(S_N - E - P)\tau_g < 0$$

which in turn implies that $(S_N - E) < C$. This gives us the optimal policy as[39]

State of the World	Optimal policy
$(S_N - E) \leq 0$	Do not exercise the option
$0 < (S_N - E) < P$	Exercise the option and sell the stock immediately
$(S_N - E) \geq P$	Exercise the option and hold on to the stock

This policy is also consistent with the no-tax case where we exercise if $(S_N - E) > 0$.

1.6.2 The Writer

At maturity, the writer of a call option is subject to the following :

(i) The call expires unexercised. As seen in the buyer's case, this comes about whenever $S_N < E$. The net payoff to the seller is a *tax liability* in the amount of $P\tau_o$. This is the tax payable on the original premium which is treated as ordinary income.

(ii) The option is exercised against him. The writer is now subject to capital gains tax. Since he will have to deliver one unit of stock to the option holder, it is treated as the sale of one unit of stock at $(P + E)$. The value of the stock delivered is S_N and so the net payoff is

$$E - S_N - \tau_o(E + P - S_N) \quad .$$

Notice that in this formulation, the writer does not really have any decision to make. If repurchase was considered, he could repurchase the option if he so desires, whenever the option is about to be exercised. He may wish to do so since in this case, he is subject to capital gains tax in the amount of $\tau_o(P - RP)$ where RP is the repurchase price. Whenever this tax amount is less than the capital gains tax realised, were the option to be exercised, he will choose to repurchase.

The payoffs to the buyer and writer are presented in Table 1 in appendix 2.

[39] We can get the the optimal policy independently by considering the maximum of the three payoffs above

1.6.3 Cost of holding a synthetic option

Let HSO be the cost of creating a long synthetic option. It is a function of S_B, S_0, u, d, N, τ_o, τ_g, R, E, and P. It can be interpreted as the cost to the investor of the portfolio that replicates the payoffs resulting from holding an option. At $t = (N-1)$, the trader will have a certain quantity of stock and a certain quantity of bonds in the replicating portfolio. Let

$$\begin{aligned} \alpha &= \text{Dollar value of stocks in the replicating portfolio} \\ y &= \text{Dollar value of bonds in the replicating portfolio} \end{aligned}$$

The value of this portfolio at $t = N$ will be either

$$\alpha u + yR^* \quad \text{or} \quad \alpha d + yR^* \ ,$$

depending on how the stock price moved in that last period. However, depending upon whether the option was exercised or not (had he held the option instead of the replicating portfolio), the trader may have to further adjust his portfolio to ensure that it is identical to the position that he would have been in *after* expiration of the option, had he bought an option. For example, were the option to expire unexercised, he would have to liquidate the replicating portfolio. This requires that he sell the stock position. As a consequence, he will be affected by capital gains taxes. If it is a tax loss, we cannot attribute it to his replication cost, since had he *not* been replicating, he would anyways have optimally realised this loss. Any taxes paid on the other hand represent a deviation from an otherwise optimal strategy, and must be attributed to the option. Hence, whenever $S_N < E$, the payoff from the required portfolio is

$$\alpha\left(\frac{S_N}{S_{N-1}}\right) + yR^* - \tau_g\left(\frac{\alpha}{S_{N-1}}\right)[S_N - S_B]\mathcal{I}(S_N > S_B) \ .$$

where

$$\mathcal{I}(S_N > S_B) = \begin{cases} 1 & \text{if } S_N > S_B; \\ 0 & \text{otherwise.} \end{cases}$$

In general, we shall follow the convention that $\mathcal{I}$(condition) is an indicator function that takes on a value of unity whenever the condition is true and a value of zero otherwise.

An identical situation arises in the case where the option would have been exercised and the resulting stock sold. i.e. when $E < S_N < P + E$. The payoff from the replicating portfolio is as above even in these states.

Finally, had the option been exercised and the resulting stock held on to, the trader would not

have to sell any stock in the market[40]. Putting all this into one function, we obtain the payoff from the replicating portfolio as

$$\alpha\left(\frac{S_N}{S_{N-1}}\right) + yR^* - \tau_g\left(\frac{\alpha}{S_{N-1}}\right)[S_N - S_B]I(S_B < S_N < P + E) \quad .$$

If the payoffs from the option are C^u and C^d resulting from an up or down move respectively, then the replicating portfolio must also satisfy :

$$\left\{\begin{array}{l} C^u = \alpha u + yR^* - \tau_g\alpha\left[u - \dfrac{S_B}{S_{N-1}}\right] I(S_B < uS_{N-1} < P + E) \\ C^d = \alpha d + yR^* - \tau_g\alpha\left[d - \dfrac{S_B}{S_{N-1}}\right] I(S_B < dS_{N-1} < P + E) \end{array}\right\}$$

where we have used the fact that either

$$S_N = uS_{N-1} \quad \textbf{or} \quad S_N = dS_{N-1} \ .$$

In addition,

$$\alpha + y = \Phi_{N-1} \quad .$$

The solution to this system of equations is given by

$$\left\{\begin{array}{rl} \Phi_{N-1} &= \dfrac{1}{R^*}\{\varrho C^u + (1-\varrho)C^d\} \\ \varrho &= \dfrac{R^* - \theta(d)}{\theta(u) - \theta(d)} \\ \theta(u) &= u - \tau_g\left[u - \dfrac{S_B}{S_{N-1}}\right] I(S_B < uS_{N-1} < P + E) \\ \theta(d) &= d - \tau_g\left[d - \dfrac{S_B}{S_{N-1}}\right] I(S_B < dS_{N-1} < P + E) \end{array}\right\} \tag{2}$$

To obtain HSO, we now employ Model $\mathcal{X}$ and compute the present value of the vector Φ_{N-1} at $t = 0$ using equation (1).

Recall that η is the cost of creating a long synthetic option for someone who has an inventory of long stocks with a basis of S_0, the current market price of the stock (at time, $t = 0$). To obtain η, set $S_B = S_0$ in (2).

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[40] He may have to buy some, but no capital gains taxes are involved.

1.6.4 Revenue from writing a synthetic option

Let WSO be the revenue from creating a short synthetic option. It is a function of S_B, S_0, u, d, N, τ_o, τ_g, R, E, and P. Let

$$\alpha = \text{Dollar value of stocks in the replicating portfolio}$$
$$y = \text{Dollar value of bonds in the replicating portfolio}$$

When the option expires unexercised, the writer of an option has a net stock position of zero. To replicate this, the replicator (who has a short position in stocks) will have to buy the stocks back. If he realises a capital loss tax shelter in the process, this is something that he would have anyways have realised had he not been replicating, and thus we cannot attribute this benefit to the revenues from the synthetic option. On the other hand, all capital gains taxes paid will serve to reduce the revenues from the replicating portfolio. The payoff from the replicating portfolio is hence,

$$\alpha\left(\frac{S_N}{S_{N-1}}\right) + yR^* - \tau_g\left|\frac{\alpha}{S_{N-1}}\right|[S_B - S_N]I\,(S_N < S_B \text{ and } S_N < E)$$

The absolute value of α is used since α will be negative here[41], and we wish to compute the *number of units* of stock that are bought. In other words, the payoff is

$$\alpha\left(\frac{S_N}{S_{N-1}}\right) + yR^* + \tau_g\frac{\alpha}{S_{N-1}}[S_B - S_N]I\,(S_N < S_B \text{ and } S_N < E)$$

When the option is exercised, the writer will have to deliver one unit of the stock. To replicate this, the trader will have to increase his short position to a single share. No taxes are involved. The payoff in this case is

$$\alpha\left(\frac{S_N}{S_{N-1}}\right) + yR^*$$

As before, let C^u and C^d denote the final payoffs to the writer of the option at $t = N$. Hence,

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[41] This is proven later in a more general framework

$$C^u = \alpha u + yR^* + \tau_g \alpha \left| \left(\frac{S_B}{S_{N-1}} - u \right) \right. I\left(uS_{N-1} < S_B \text{ and } uS_{N-1} < E\right)$$

$$C^d = \alpha d + yR^* + \tau_g \alpha \left| \left(\frac{S_B}{S_{N-1}} - d \right) \right. I\left(dS_{N-1} < S_B \text{ and } dS_{N-1} < E\right)$$

and

$$\Phi_{N-1} = \alpha + y$$

The solution to this system is given by

$$\left\{ \begin{array}{rcl} \Phi_{N-1} & = & \frac{1}{R^*}\left(\omega C^u + (1-\omega)C^d\right) \\ \omega & = & \frac{R^* - \lambda(d)}{\lambda(u) - \lambda(d)} \\ \lambda(u) & = & u + \tau_g \left[\frac{S_B}{S_{N-1}} - u\right] I\left(uS_{N-1} < S_B \text{ and } uS_{N-1} < E\right) \\ \lambda(d) & = & d + \tau_g \left[\frac{S_B}{S_{N-1}} - d\right] I\left(dS_{N-1} < S_B \text{ and } dS_{N-1} < E\right) \end{array} \right\} \tag{3}$$

The positive sign in $\lambda(u)$ and $\lambda(d)$ arises due to the fact that $\alpha < 0$.

To obtain WSO, we use Model $\mathcal{X}$ as with HSO. However, this time, the "cost" will be negative. To get WSO, use $-\Phi_{N-1}$ in Model $\mathcal{X}$.

Recall that μ is the revenue from writing a synthetic option for someone who has an inventory of short stocks with a basis of S_0 the current market price of the stock (at time, $t = 0$). To obtain μ, set $S_B = S_0$ in (4).

1.7 The Equilibrium Price - a simple case

Consider first an economy where **all** investors belong to $\mathcal{A}$ or $\mathcal{B}$. In other words, everyone in this economy desires either the payoffs realised by holding a call option, or the payoffs realised by writing one. Also, let P be the equilibrium price of a call option (with a strike price of E) on some particular stock. Define $\mathcal{F}_W$ to be the revenue from creating a short synthetic option for someone in $\mathcal{F}$, and $\mathcal{F}_H$ to be the cost of creating a long synthetic option for the same person. Hence $\mathcal{F}_H$ is

your cost of creating a option with stocks and bonds if you are not fortunate to be endowed with an inventory of stocks and $\mathcal{F}_W$ is your revenue from replicating the payoffs generated from writing an option. Though at first sight it appears that $\mathcal{F}_H$ is equal to $HSO(S_B, \tau_o, \tau_o)$, it turns out that this is not so. In the computation of HSO, we assumed that the investor is endowed with an inventory of stocks that is "sufficiently large" to permit replication. The investor in $\mathcal{F}$, however, does not have such an inventory. Consequently, he must purchase stocks from the market as and when they are needed. Such transactions in the market will alter the basis of his stock holdings and consequently, his tax liabilities. Thus his cost of creating a long synthetic option will have to be computed using a model akin to the one we developed in section 1.3. Similarly, $\mathcal{F}_W$ cannot be obtained from WSO directly. We shall not obtain the form that $\mathcal{F}_H$ and $\mathcal{F}_W$ have, in this study, but shall assume that they exist[42] In addition, we assume that $\mathcal{F}_H \geq P \geq \mathcal{F}_W$.

Let us focus on the problem faced by an investor in $\mathcal{A}$. His cost of creating a long synthetic call is $HSO(S_B, \tau_g, \tau_o)$, provided that he has an inventory of long stocks with a basis of S_B. If he has an inventory of short stocks, then the cost will exceed $\mathcal{F}_H$, since he must first clear this short position, pay capital gains taxes and then replicate at a cost of $\mathcal{F}_H$. Since P is less than $\mathcal{F}_H$, he would prefer to buy the option. In any case, if the cost of creating a long synthetic option, HSO, exceeds P, the investor will be better off purchasing the option at the market price P. If, on the other hand, HSO is less than P, the investor is made better off by creating a long synthetic option.

Similarly, an investor in $\mathcal{B}$ will compare his $WSO(S_B, \tau_o, \tau_g)$ with P. If $WSO(S_B, \tau_o, \tau_g) < P$ for this investor, his revenues from writing a call option exceed the revenues from creating a short synthetic option. Hence, he would prefer to write at P. Alternatively, if $WSO(S_B, \tau_o, \tau_g) > P$, he would prefer to replicate the payoffs of a short call option. Hence, associated with any market price P, there is a demand and a supply of options. The equilibrium price must be the one that clears supplies and demands.

The presence of a tax-free investor who can arbitrage infinitely leads us to a special case. Consider HSO and WSO for such an investor. Since the investor pays no taxes, the basis of his stock holding is irrelevant. In addition, it is not necessary for us to assume that he has an inventory of stocks, since the purchase of new stocks at the market price only affects the basis, which is irrelevant for this person. To obtain HSO and WSO for this investor, set $\tau_g = \tau_o = 0$ in

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[42] If P lies outside $(\mathcal{F}_W, \mathcal{F}_H)$., this does not imply arbitrage opportunities since the portfolio that replicates a short option position does not hedge a long option position. This can be verified in Table 1. The no-arbitrage condition that is relevant in our study is outlined in the next chapter

(2) and (3). This yields

$$HSO = WSO = \frac{1}{R^N} \sum_{i=0}^{i=N} \binom{N}{i} p^i (1-p)^{(N-i)} \text{Max} \left\{0, \left(u^i d^{(N-i)} S - E\right)\right\}$$

$$p = \frac{R-d}{u-d}$$

This is the option price suggested by Cox, Ross and Rubinstein, which in the limit is the Black-Scholes price. Let us call it the **BS** price. Whenever P deviates from this, the tax-free investor is presented with an arbitrage opportunity. If $P >$ **BS**, he can write at P and hedge the resulting payoffs at **BS**. Alternatively, if $P <$ **BS**, he can buy the option for P, hedge it and realise a revenue of **BS** dollars. Hence, in equilibrium, $P =$ **BS**. Any excess supply or demand at this price, from the investors with a non-zero tax rate, will be absorbed by the tax-free investor. A finite supply and demand for options from investors in $\mathcal{A}$ and $\mathcal{B}$ will, however, still exist.

In the absence of such a tax-free investor who can arbitrage infinitely, however, the equilibrium price P could deviate from **BS**. It will be higher or lower than **BS** depending upon the distribution of investors in $\mathcal{A}$ and $\mathcal{B}$ and on their costs and revenues resulting from long and short synthetic options. However, we could impose certain bounds on P. One such bound is that

$$P \geq \text{Max}\left(0, (S_N - E)\right)$$

In the absence of taxes, this bound is well-known and can be obtained in any textbook in the theory of modern finance that has a section on option pricing. When we introduce taxes, however, this bound is not so obvious and needs to be proven. The proof can be viewed in Appendix 1. Another bound on P comes from the realisation that it should not be possible for an individual to profit by the following exercise :

a) Buy (or write) an option.

b) Cover the option payoffs by creating a hedge with stocks and bonds, bought/sold at the current market price.

Clearly, if the above strategy was profitable, every individual would jump in at the opportunity. Their action, will ensure that the price P adjusts to eliminate such opportunities. The next chapter deals with the costs and revenues from hedging option payoffs

Chapter 2

Determining the Equilibrium Price

2.1 Covered option positions

In the example of the previous chapter, we assumed that all investors belonged to either $\mathcal{A}$ or $\mathcal{B}$. In other words, we assumed that the set $\mathcal{C}$ was empty. If this were not the case, and if there were some investors who do not particularly desire the option payoffs, then it is possible that one such investor may find himself in a position whereby he can write an option at P and hedge its payoffs at a cost below P owing to the characteristics of his inventory of stocks. Alternatively, he might find it profitable to buy the option at P and hedge it. Such investors will take up covered positions in options.

For a tax-free investor, the cost of covering a short option is $WSO(S_B, 0, 0)$ and the revenue from covering a long option is $HSO(S_B, 0, 0)$[43]. In general if $\tau_o \neq 0$, the cost of covering a short option position is not $HSO(S_B, \tau_o, \tau_g)$ and the revenue from covering a long option is not $WSO(S_B, \tau_o, \tau_g)$. To see this, look at Table 1. Observe that if $\tau_o \neq 0$, the payoffs from a long option position and short option position do not add up to zero whenever S_N, the final stock price, exceeds $(P + E)$. Hence, the investor who buys an option and creates a short synthetic option is not completely

[43] Observe from equations (2) and (3) that

$$WSO(S_B, 0, 0) = HSO(S_B, 0, 0) = \text{BS}$$

and that they are independent of S_B.

hedged. Similarly, the investor who writes an option and creates a short synthetic option will not be completely hedged. Consequently, the first step in the process of introducing investors in $\mathcal{C}$, into the equilibrium that we developed in the earlier section, involves computing their costs and revenues related to hedging short and long options.

2.1.1 Cost of hedging a short option

Let CSO be the cost of hedging a short option using stocks and bonds. In particular, it is the cost that will be incurred by the trader who writes an option and decides to cover it with a hedge portfolio of stocks and bonds. The hedge portfolio will be long in stocks[44]. Let

$$\begin{aligned} \alpha &= \text{Dollar value of stocks in the hedge portfolio} \\ y &= \text{Dollar value of bonds in the hedge portfolio} \end{aligned}$$

Now if the option expires unexercised, the hedge must also be liquidated. This may entail capital gains taxes. The payoff from the hedge portfolio in these states is hence,

$$\alpha\left(\frac{S_N}{S_{N-1}}\right) + yR^* - \tau_g\left(\frac{\alpha}{S_{N-1}}\right)[S_N - S_B]I\,(S_B < S_N < E)$$

and the payoff from the option is $-P\tau_o$. Once again, we have only attributed the capital gains taxes paid to the cost of hedging, and not the capital gains tax shelters since these would have been optimally realised in the absence of the hedge.

If the option is exercised, the writer has to deliver one unit of the stock. He can use the long stock in the hedge to do this. The payoff from the hedge in this case is hence,

$$\alpha\left(\frac{S_N}{S_{N-1}}\right) + yR^* - \tau_g[P + E - S_B]I\,(S_N > E)$$

However, if $S_N < S_B$, the hedger could have realised a capital loss tax shelter in the amount of $\tau_g[S_B - S_N]$, by just selling the stock in the hedge portfolio. Since this is not being realised, we must subtract it from the payoffs under the appropriate states. The net payoff with exercise from

[44] Proven later in this section.

the hedge is hence[45]

$$\alpha\left(\frac{S_N}{S_{N-1}}\right) + yR^* - \tau_g[P+E-S_B]I\,(S_N > E) - \tau_g[S_B - S_N]I\,(E < S_N < S_B)$$

The payoff from the option is $-(S_N - E)$.

Putting all this into one function, we obtain that if the portfolio is to hedge the payoffs,

$$\begin{aligned}
C^u &= \alpha u + yR^* - \alpha\tau_g\left(u - \frac{S_B}{S_{N-1}}\right) I\,(S_B < uS_{N-1} < E) \\
&\quad - \tau_g[P+E-S_B]I\,(uS_{N-1} > E) - \tau_g[S_B - uS_{N-1}]I\,(E < uS_{N-1} < S_B) \\
C^d &= \alpha d + yR^* - \alpha\tau_g\left(d - \frac{S_B}{S_{N-1}}\right) I\,(S_B < dS_{N-1} < E) \\
&\quad - \tau_g[P+E-S_B]I\,(dS_{N-1} > E) - \tau_g[S_B - dS_{N-1}]I\,(E < dS_{N-1} < S_B)
\end{aligned}$$

In addition, we need that

$$\Phi_{N-1} = \alpha + y$$

In this case, since the hedge portfolio is hedging the payoffs from the written option, C^u and C^d, the final payoffs are given by

$$\text{Payoff from hedge portfolio} = \begin{cases} P\tau_o & \text{if } S_N < E \\ (S_N - E) & \text{if } S_N > E \end{cases}$$

The solution to this system is

$$\left\{\begin{aligned}
\Phi_{N-1} &= \frac{1}{R^*}\{\varsigma[C^u + F^u] + (1-\varsigma)[C^d + F^d]\} \\
\varsigma &= \frac{R^* - \Gamma(d)}{\Gamma(u) - \Gamma(d)} \\
\Gamma(u) &= u - \tau_g\left(u - \frac{S_B}{S_{N-1}}\right) I\,(S_B < uS_{N-1} < E) \\
\Gamma(d) &= d - \tau_g\left(d - \frac{S_B}{S_{N-1}}\right) I\,(S_B < dS_{N-1} < E) \\
F^u &= \tau_g\Big([P+E-S_B]I\,(uS_{N-1} > E) + [S_B - uS_{N-1}]I\,(E < uS_{N-1} < S_B)\Big) \\
F^d &= \tau_g\Big([P+E-S_B]I\,(dS_{N-1} > E) + [S_B - dS_{N-1}]I\,(E < dS_{N-1} < S_B)\Big)
\end{aligned}\right\} \tag{4}$$

[45] The trader could have sold the stock for a loss and delivered cash on the option. This case will add an extra term. To keep the analysis relatively simple, we have dropped this term. Its inclusion, will lower the costs of hedging, since in some states, this may be an prefered action, implying lower costs.

To obtain CSO, we now employ Model $\mathcal{X}$ and compute the present value of the vector Φ_{N-1} to $t = 0$ using equation (1).

2.1.2 Revenues from hedging a long option

Define CLO as the revenue from the hedge portfolio that hedges the payoffs generated by holding an option. The hedge portfolio will have a short position in stocks[46]. Let

$$\begin{aligned} \alpha &= \text{Dollar value of stocks in the hedge portfolio} \\ y &= \text{Dollar value of bonds in the hedge portfolio} \end{aligned}$$

If the option expires unexercised, then the payoff from the option is $P\tau_o$. The hedge must be liquidated and capital gains taxes may have to be paid. The payoff from the hedge is hence,

$$\alpha\left(\frac{S_N}{S_{N-1}}\right) + yR^* - \left|\frac{\alpha}{S_{N-1}}\right| \tau_g[S_B - S_N] I\,(S_N < E;\ \text{and};\ S_B > S_N) \quad .$$

Once again, we have only considered capital gains taxes that are paid and not attributed capital loss tax shelters to the hedge. In addition, α is negative, so the above expression reduces to

$$\alpha\left(\frac{S_N}{S_{N-1}}\right) + yR^* + \frac{\alpha}{S_{N-1}}\tau_g[S_B - S_N] I\,(S_N < E;\ \text{and};\ S_B > S_N) \quad .$$

If the option is exercised and the stock that is received is sold, then the hedge portfolio must also be liquidated. The payoff from the option is $\{(S_N - E) + \tau_o(P + E - S_N)\}$ and the payoff from the hedge portfolio is

$$\alpha\left(\frac{S_N}{S_{N-1}}\right) + yR^* - \left|\frac{\alpha}{S_{N-1}}\right| \tau_g[S_B - S_N] I\,(E < S_N < P + E;\ \text{and};\ S_B > S_N)$$

which reduces to the following since α is negative

$$\alpha\left(\frac{S_N}{S_{N-1}}\right) + yR^* + \frac{\alpha}{S_{N-1}}\tau_g[S_B - S_N] I\,(E < S_N < P + E;\ \text{and};\ S_B > S_N)$$

Finally, were the option to be exercised, and the resulting stock held, the hedge portfolio could be modified to have one share sold short and no taxes would be paid. The payoff from the option

[46] Proven later in this section.

would be $(S_N - E)$ and from the hedge portfolio

$$\alpha\left(\frac{S_N}{S_{N-1}}\right) + yR^*$$

Putting all this together, we see that the hedge portfolio must solve

$$\begin{aligned} C^u &= \alpha u + yR^* + \alpha\tau_g\left(\frac{S_B}{S_{N-1}} - u\right) I\left(uS_{N-1} < P + E, \text{ and } S_B > uS_{N-1}\right) \\ C^d &= \alpha d + yR^* + \alpha\tau_g\left(\frac{S_B}{S_{N-1}} - d\right) I\left(dS_{N-1} < P + E, \text{ and } S_B > dS_{N-1}\right) \end{aligned}$$

In addition,

$$\Phi_{N-1} = \alpha + y$$

The final payoffs, C^u and C^d are the negative of the payoffs from the option, and they are as follows:

$$\text{Final Payoffs to the hedge portfolio} = \begin{cases} -P\tau_o\ ; & \text{if } S_N < E, \\ (E - S_N) - \tau_o(P + E - S_N)\ ; & \text{if } E < S_N < P + E, \\ (E - S_N)\ ; & \text{if } S_N > P + E. \end{cases}$$

The solution to this system is given by

$$\left\{\begin{aligned} \Phi_{N-1} &= \frac{1}{R^*}\left(\psi C^u + (1-\psi)C^d\right) \\ \psi &= \frac{R^* - \Lambda(d)}{\Lambda(u) - \Lambda(d)} \\ \Lambda(u) &= u + \tau_g\left(\frac{S_B}{S_{N-1}} - u\right) I\left(uS_{N-1} < P + E, \text{ and } S_B > uS_{N-1}\right) \\ \Lambda(d) &= d + \tau_g\left(\frac{S_B}{S_{N-1}} - d\right) I\left(dS_{N-1} < P + E, \text{ and } S_B > dS_{N-1}\right) \end{aligned}\right\} \tag{6}$$

Lemma 1 *The hedge portfolio that replicates a long option* **or** *covers a short option position will be long in stocks.*

Proof Refer to Appendix 1.

Lemma 2 *The hedge portfolio that replicates a short option* **or** *covers a long option position will be short in stocks.*

Proof Refer to Appendix 1.

2.2 The equilibrium with covered option positions

Let $\mathcal{F}_{CSO}$ be the cost of covering a short option for the investor who is in $\mathcal{F}$. This investor must purchase fresh stock in the market and also pay taxes at τ_o. Now an investor who has an inventory of short stocks (with $S_B \geq S_0$) will incur a cost higher than $\mathcal{F}_{CSO}$ since he must first liquidate his short position, pay capital gains taxes and *then* cover at a cost $\mathcal{F}_{CSO}$. Similarly the revenues from covering a long option for the investor with an inventory of long stocks (with $S_B \leq S_0$) will be less than $\mathcal{F}_{CLO}$, where $\mathcal{F}_{CLO}$ is the revenue from covering a long option for an investor in $\mathcal{F}$. To preclude arbitrage opportunities, the equilibrium price P must satisfy

$$\mathcal{F}_{CLO} \leq P \leq \mathcal{F}_{CSO} \quad .$$

If P exceeded $\mathcal{F}_{CSO}$, one could write an option for P dollars and hedge the payoffs for $\mathcal{F}_{CSO}$ dollars. Similarly, if $\mathcal{F}_{CLO}$ exceeded P, one could purchase an option for P dollars and hedge the payoffs for a revenue of $\mathcal{F}_{CLO}$ dollars. Since all investors can "enter" $\mathcal{F}$ by liquidating their existing inventories, the existence of such arbitrage opportunities is not consistent with equilibrium.

Any investor who has $CSO(S_B, \tau_o, \tau_g)$ less than P, will profit by writing an option at P dollars and hedging it for $CSO(S_B, \tau_o, \tau_g)$ dollars. Similarly, the investor with $CLO(S_B, \tau_o, \tau_g) > P$ will buy an option at P and cover it for a revenue of $CLO(S_B, \tau_o, \tau_g)$. Table 2 in appendix 2 describes the trading behaviour of the various participants in our market.

To visualise these choices graphically, let us remove one dimension of heterogeneity amongst investors. In particular, assume that all investors have identical tax rates of τ_o and τ_g and let the only heterogeneity be along S_B. This may come about in an economy where the inventories

of different individuals were constructed at different stock prices. In addition, we shall make an assumption that will render the functions HSO, WSO, CSO, and CLO monotonic in S_B.

Look at equations (2), (3), (4), and (5). The terms ϱ, ω, ς, and ψ are like the preference free probability measure of Cox, Ross and Rubinstein. It is a simple exercise to verify that the after-tax expected return on the stock position in the hedge portfolio is R^* under the assumption that these terms represent the probability of an *up* move[47]. It is not necessary that ϱ, ω, ς, and ψ lie in the interval $(0,1)$. We shall discuss this further, but first notice that if they do lie in $(0,1)$ and thus be interpretable as probabilities, then

Lemma 3 *If $\varrho \in (0,1)$, then HSO is a non-increasing function of S_B.*

Lemma 4 *If $\omega \in (0,1)$, then WSO is a non-increasing function of S_B.*

Lemma 5 *If $\varsigma \in (0,1)$, then CSO is a non-increasing function of S_B.*

Lemma 6 *If $\psi \in (0,1)$, then CLO is a non-increasing function of S_B.*

Proof Refer to Appendix 1.

In order to appreciate the implication of the assumption that ϱ, ω, ς, and ψ lie in the interval $(0,1)$, we need to look at the derivation of the schedules HSO, WSO, CSO, and CLO. Observe that they all have a common structure, for valuation during the maturity period, that can be characterised as follows:

1) There is a stock portfolio whose value is H dollars today.

2) If the stock price moves *up* by a factor u, the value of the portfolio goes to H^u. If the stock price were to move *down* by a factor d, the value of the portfolio goes to H^d.

3) There exists a measure 'p' (corresponding to ϱ, ω, ς, and ψ) such that

$$pH^u + (1-p)H^d = R^* H \quad .$$

[47] Substitute the after-tax return on the stock into C^u and C^d. Φ_{N-1} will then equal unity.

Viewed another way, H is the current value of the portfolio that yields H^u or H^d after-tax dollars tomorrow. Alternatively, H is the "present value" of this payoff vector $[H^u, H^d]^T$. Now consider another portfolio that has H dollars invested in the bond. The payoff from this portfolio, after taxes, will be HR^* tomorrow. If $HR^* > \text{Max}[H^u, H^d]$, or if $HR^* < \text{Min}[H^u, H^d]$, then p will not lie in the interval $(0, 1)$. In other words, if the bond dominates the stock portfolio, or is dominated by it, p will lie outside of $(0, 1)$. To avoid such dominance, we need that

$$\frac{H^d}{H} \leq R^* \leq \frac{H^u}{H}$$

A condition that will ensure this is

$$\frac{dS_{N-1} + \tau_g(S_B - dS_{N-1})I(S_B > dS_{N-1})}{S_{N-1}} \leq R^* \leq \frac{uS_{N-1} + \tau_g(S_B - uS_{N-1})I(S_B > uS_{N-1})}{S_{N-1}}$$

where S_{N-1} is the stock price at the start of the period. This can be rewritten as

$$d + \tau_g\left(\frac{S_B}{S_{N-1}} - d\right) I\left(\frac{S_B}{S_{N-1}} > d\right) \leq R^* \leq u + \tau_g\left(\frac{S_B}{S_{N-1}} - u\right) I\left(\frac{S_B}{S_{N-1}} > u\right)$$

Now, since $u > R^*$, the inequality on the right side will always be true. Hence, were the probability measure p to lie outside $(0, 1)$, it must be because of a violation of the inequality on the left. Observe also that a violation will cause p to become negative, as long as H^u exceeds H^d when H is positive and H^d exceeds H^u whenever H is negative. In other words, as long as the payoff resulting from a stock portfolio, that is long (short) in stocks, is higher when the stock price moves up (down) than when it moves down (up), p will not exceed unity. We can further analyse the left inequality by noting that whenever the indicator function is zero, the inequality reduces to $d \leq R^*$, which is true always by assumption. Hence, we need only examine the cases when the indicator function is not zero, i.e. when $S_B > dS_{N-1}$. In these cases, the inequality can be reduced to

$$\frac{S_B}{S_{N-1}} \leq d + \frac{R^* - d}{\tau_g}$$

But we know that S_{N-1} cannot be less than $d^{N-1}S_0$. Hence, a sufficient (but not necessary) condition to ensure that p (and consequently ϱ, ω, ς, and ψ) will lie in $(0, 1)$, is given by

$$\frac{S_B}{S_0} \leq d^N + d^{N-1}\left(\frac{R^* - d}{\tau_g}\right) \tag{7}$$

Now, if there were some investors whose bases were high enough to violate the above condition, given the parameters of the distribution of the stock, then it is conceivable that the schedules may not be negatively sloped.

We shall, for now, assume that the schedules are negatively sloped. This assumption is not essential to our results, but is being made so that we may be able to *view* the problem better. We shall comment on the lack of monotonicity at the end of the current section. Now let,

$$\begin{aligned} HSO(S_0, \tau_o, \tau_g) &= \eta \\ WSO(S_0, \tau_o, \tau_g) &= \mu \\ CSO(S_0, \tau_o, \tau_g) &= \nu \\ CLO(S_0, \tau_o, \tau_g) &= \xi \end{aligned}$$

Here η is the cost of creating a long synthetic option for the person who has an inventory of long stocks with basis S_0 and ν is his cost of covering a short option. Similarly, μ is the revenue from creating a short synthetic option for the person who has an inventory of short stocks with basis S_0 and ξ is his revenue from covering a long option. Given the forms of equations (2), (3), (4) and (5), it turns out that we cannot order the points η, μ, ν and ξ. As τ_o changes, the direction in which the schedules move will depend largely on the distributional characteristics of the stock, the parameters of the option, and on the interest rate. Hence the **BS** price, obtained by setting τ_o and τ_g to zero, could lie anywhere relative to η, μ, ν and ξ. However, as τ_g changes, the movement of the schedules can be predicted. But first, let us put the equilibrium in a graphical framework.

In Figure 3, we have plotted the schedules HSO, WSO, CSO, and CLO against S_B. Individuals who have inventories of long stocks will have a basis below S_0, the current market price of the stock, and individuals with short stocks to spare will have a basis above S_0. In figure 3, we have drawn the schedules so that

$$\mu > \xi > P > \nu > \eta$$

where P is the equilibrium price of the option. The Black-Scholes price for this option (**BS**) could be anywhere relative to P. With reference to figure 3, let

$$\begin{aligned} HSO(S_B, \tau_o, \tau_g) &= P \quad \text{at} \quad S_B = a \\ CSO(S_B, \tau_o, \tau_g) &= P \quad \text{at} \quad S_B = b \\ CLO(S_B, \tau_o, \tau_g) &= P \quad \text{at} \quad S_B = c \\ WSO(S_B, \tau_o, \tau_g) &= P \quad \text{at} \quad S_B = d \end{aligned}$$

Recall that in addition, to preclude arbitrage, we need that

$$\mathcal{F}_{CSO} \geq P \geq \mathcal{F}_{CLO} \quad .$$

We continue to assume that $\mathcal{F}_H \geq P \geq \mathcal{F}_W$.

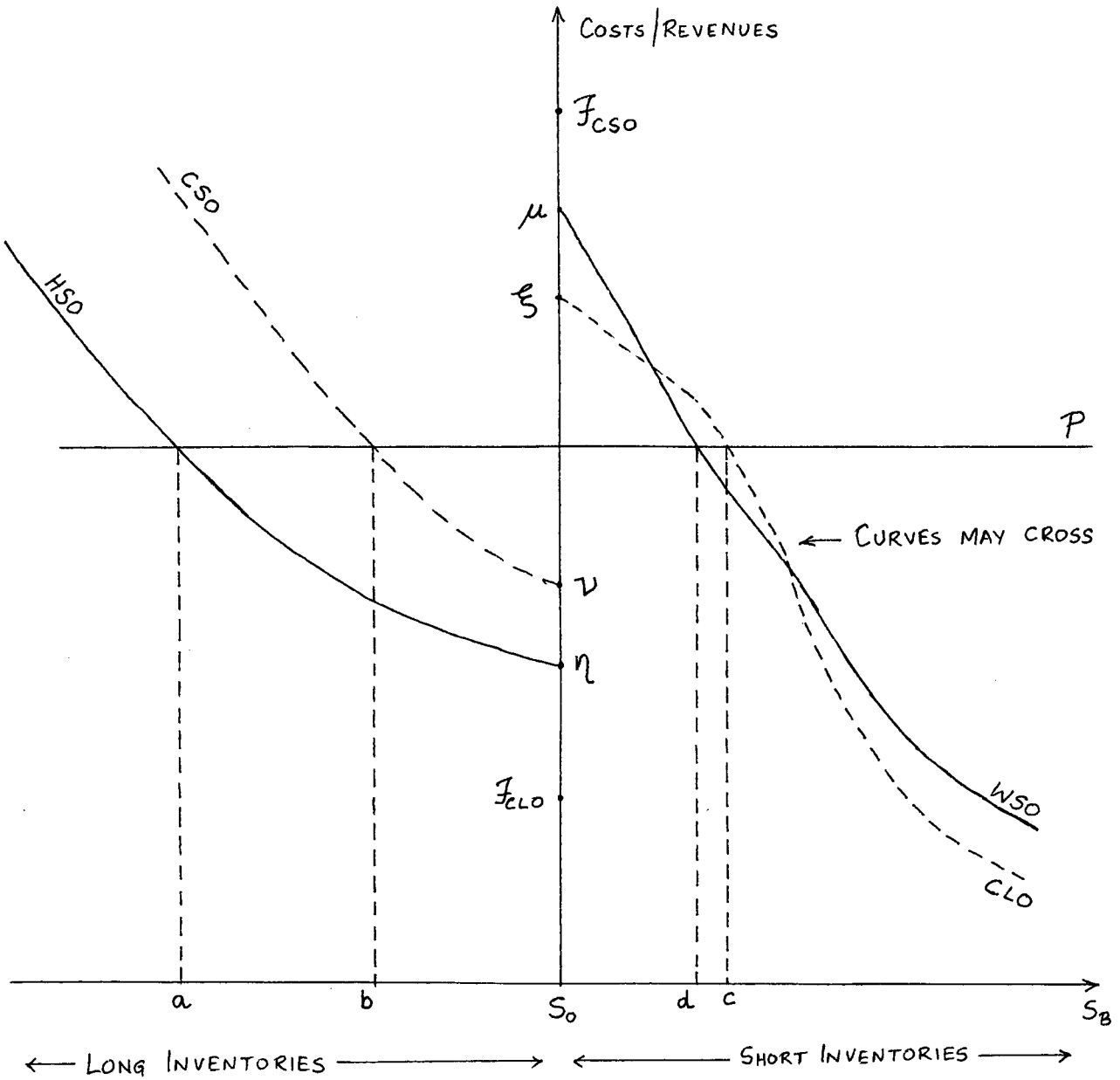

Figure 3

Now individuals in $\mathcal{A} \cap (\mathcal{D} \cup \mathcal{E})$ with $S_B \notin (a, S_0)$ will buy options, whilst those with $S_B \in (a, S_0)$ will create long synthetic options. Similarly, people in $\mathcal{B} \cap (\mathcal{D} \cup \mathcal{E})$ with $S_B \in (S_0, d)$ will create short synthetic options whereas those with $S_B \notin (S_0, d)$ will write options. Investors in $\mathcal{C}$ with $S_B \in (b, S_0)$ will write covered options whereas those with $S_B \in (S_0, c)$ will buy covered options. Finally, investors in $\mathcal{A} \cap \mathcal{F}$ will buy options, whereas those in $\mathcal{B} \cap \mathcal{F}$ will write them. Investors in $\mathcal{C} \cap \mathcal{F}$ will not enter the market for options.

Define $\mathcal{A}(x)$ as a population density function so that $\mathcal{A}(x)dx$ represents the *number* of investors in $\mathcal{A}$ who have inventories of stocks with bases in the interval $(x, x+dx)$. Consequently,

$$\int_{x_1}^{x_2} \mathcal{A}(x)\, dx$$

represents the *number* of investors in $\mathcal{A}$ for whom $x_1 \leq S_B \leq x_2$. Similarly, define $\mathcal{B}(x)$ and $\mathcal{C}(x)$ to be the corresponding population density functions describing investors in $\mathcal{B}$ and $\mathcal{C}$. Also let $\mathcal{A}_{\mathcal{F}}$ and $\mathcal{B}_{\mathcal{F}}$ be the cardinality of the sets $\mathcal{A} \cap \mathcal{F}$ and $\mathcal{B} \cap \mathcal{F}$, respectively. Hence $\mathcal{A}_{\mathcal{F}}$ and $\mathcal{B}_{\mathcal{F}}$ represent the *number* of investors in $\mathcal{A} \cap \mathcal{F}$ and in $\mathcal{B} \cap \mathcal{F}$ respectively. The net demand and supply of options can then be characterised as follows

$$\text{Demand for options} = \int_0^a \mathcal{A}(x)\, dx + \int_{S_0}^{\infty} \mathcal{A}(x)\, dx + \int_{S_0}^{c} \mathcal{C}(x)\, dx + \mathcal{A}_{\mathcal{F}}$$

$$\text{Supply of options} = \int_d^{\infty} \mathcal{B}(x)\, dx + \int_0^{S_0} \mathcal{B}(x)\, dx + \int_b^{S_0} \mathcal{C}(x)\, dx + \mathcal{B}_{\mathcal{F}}$$

The points a, b, c, and d are in themselves functions of P, as also HSO, WSO, CSO, and CLO. The equilibrium price P is the one that equates demands and supplies. Clearly, the actual location of this price will depend upon a variety of factors such as the distribution of individuals with various stock bases, the demand for naked option positions, the distribution of tax rates in the economy[47], and so on. These factors will, in turn, be affected by other factors such as the price history of the stock. In any case, the existence of covered option positions is consistent with rational investor behaviour and the lack of arbitrage, as also is any deviation of the equilibrium price from the Black-Scholes price, given **any** distributional assumptions on the stock. The introduction of transaction costs will serve to reduce the volume of trade in covered options and increase the volume of trade in naked options at any given price, but does not necessarily eliminate covered option trading.

Consider now the equilibrium that will result in the event that the schedules did not slope downwards. In figure 3 and in the analysis above, we assumed that all investors in $\mathcal{C}$ with bases in excess of c would not enter the market, since their revenues from covering a long option would be less than the amount that they would have to pay to acquire the option. This was justified since we assumed that CLO was downward sloping. If CLO were not downward sloping everywhere, then clearly, there may be investors at some higher bases who would also take up covered positions. In short, by assuming that the schedules were monotonic, we ensured that the set of bases corresponding to any trading behaviour would be convex. A lack of monotonicity in the schedules could conceivably destory the convexity of these groups, but neither will it change the manner in which

[47] We assumed this to be constant, but if we do allow the tax rates to be heterogeneous, then we have to add that axis to the diagram that we used. In particular, we should add a third orthogonal axis, and the schedules HSO, etc., will be two-dimensional surfaces.

the price is determined, nor will it change the existence of such trading behaviour. The assumption of monotonicity serves the same role as the assumption of a constant tax rate for the economy. Using the assumption permits easier visualisation of the problem at hand without cluttering up the analysis.

We are now in a position to examine the effect of relaxing our earlier assumption that the shadow cost, ρ, of being able to circumvent the constraint of having to hold on to the inventory, was zero. If it were not zero, then we would have to lower HSO and CSO, while at the same time raise WSO and CLO. Recall that ρ was proportional to how far away the basis of the inventory was from the current stock price, S_0. Investors whose inventories have a basis of S_0, will have $\rho = 0$, whereas others may not. The introduction of a non-zero shadow cost will hence cause the curves HSO and CSO to pivot downwards with the fulcrum at η and ν respectively. Similarly, WSO and CLO will pivot upwards with μ and ξ as fulcrums respectively. From figure 3, we infer that this will reduce the volume of trading in naked options and increase the trading in covered options.

We could include the schedules relevant to the group of investors who wish to *add* options to their existing portfolios into figure 3, provided we have a reasonable and tractable way of computing these schedules. Observe that the addition of such investors will provide us with $\mathcal{F}_H$, $\mathcal{F}_W$, $\mathcal{F}_{CSO}$, and $\mathcal{F}_{CLO}$. We also will essentially increase the number of participants in the model. The knowledge of the distribution of bases and investor preferences would, however, still be required to determine the option price. In addition, the inclusion of this group of investors does not appear to have any unambiguous implications of the price, in the absence of further assumptions. Consequently, the benefits of adding the second group to this framework, (or of adding what we have done here to a reasonable model containing only the other type of investors) is not clear at this juncture.

A final comment that must be made is on the issue of an apparent indeterminacy of the equilibrium price. If, for example, it turns out that

$$\text{Min}\left(HSO(S_0,\tau_g,\tau_o), CSO(S_0,\tau_g,\tau_o)\right) > \text{Max}\left(WSO(S_0,\tau_g,\tau_o), CLO(S_0,\tau_g,\tau_o)\right) \quad ,$$

then the equilibrium price could lie anywhere between these two points when the number of investors in $\mathcal{A}$ is equal to the number of investors in $\mathcal{B}$. In this case, it appears as if nobody will create synthetic options and there will be no covered option positions. However, since investors in $\mathcal{C}$ can enter into a covered option position with *any* option on this stock, they'll choose the one which gives them the highest profit. Hence, if P was too "high" relative to the prices of other options, we would find more investors with long stock inventories writing covered options and less taking covered long positions[48]. This will drive P downward. Consequently, the prices of options on the

[48] They will go to other options on the same stock

same stock will be jointly determined by the movement of investors in C across options. If we view such a joint determination of option prices as a "general equilibrium", then our current study is in the flavour of a "partial equilibrium".

2.3 The impact of tax rates

The effect of a change in τ_o, the ordinary income tax rate, on the schedules HSO, WSO, CSO, and CLO is not unambiguous. To get a feel for this, notice that the four schedules have a common form with respect to τ_o. Consider equation (2), as representative of that form. In equation (2), an increase in τ_o results in an increase in C^u and C^d. On the other hand, ϱ and R^* decrease. In other words, the denominator in equation (2) unambiguously decreases as a result of an increase in τ_o, but the direction in which the numerator moves is not known. Intuitively speaking, a change in the ordinary income tax rate affects the final payoffs *as well as* the composition of the hedge portfolio. In addition, these changes are in opposing directions — consequently, the direction in which schedule will change will depend largely on which of these two changes is the larger.

On the other hand, a change in τ_g, the capital gains tax rate does not affect the final payoffs from the option. Consequently, the direction in which the schedules HSO, WSO, CSO, and CLO will move in response to a change in τ_g, can be predicted. The intuition behind the directions of movements is rather simple. Recall that, in the derivation of the schedules, we only apportioned the capital gains taxes paid to the replicating portfolio. All capital losses, it was assumed, would have been optimally realised on the stock were it not being used for replication. Hence, these could be viewed as "leakages" from the replication process. They tend to *raise* the costs of replication, and *lower* the revenues from replication. Now, if the capital gains tax rate were to rise, the magnitude of these leakages would also rise. Consequently, we would expect that, the cost of creating a long synthetic option and the cost of covering a short option, would rise along with τ_g, whereas, the revenue from creating a short synthetic option and from covering a long option, would fall as τ_g rises. As expected, this is borne out by the following lemmas.

Lemma 7 *A rise in the capital gains tax rate* ceteris paribus, *is accompanied by a rise in the cost of creating a long synthetic option and in the cost of covering a short option. In addition, the revenue from creating a short synthetic option and from covering a long option declines from this rise in the capital gains tax rate.*

Proof Refer to Appendix 1

The implications of a rise in the capital gains tax rate on the equilibrium trading behaviour is easy to visualise if we look at figure 3. Notice that as a result of a rise in τ_g, HSO and CSO move upwards whereas, WSO and CLO move downward. The result of a rise in HSO is that fewer investors will find their cost of creating a synthetic option to be lower than the market price of the option. This will result in a higher demand for naked option positions. If CSO moves upwards, the number of investors who find their cost of covering a short option to be less than the market price for the option, will go down. This will result in a reduced supply of options from investors who take up covered positions. In a very similar manner, the downward shift of WSO and CLO will result in an increased supply of naked option positions and a reduced demand for options from investors who take up covered positions. In effect, the volume of trading in covered options is expected to decline, whereas, the volume of trading in naked options will rise. There is no unambiguous prediction on the direction in which the equilibrium price will move as a result of a rise in the capital gains tax rate. The supply *and* demand for naked and covered options are affected and the direction of movement of the equilibrium price will depend largely on the parameters of the option and on the distribution of individuals along S_B in our economy. We summarise these observations in the following lemma.

Lemma 8 *A rise in the capital gains tax rate* ceteris paribus, *is accompanied by an increased supply and demand for naked option positions* and *a decreased supply and demand for covered option portfolios.*

Proof Immediate from Figure 3.

2.4 Deviations of the equilibrium price from the Black-Scholes price

Though we cannot predict whether the equilibrium price will be above or below the Black-Scholes price, under the assumptions of our model, we can make predictions about the "direction" in which this difference will move under certain conditions. To start off, let

$$\delta = P - \mathbf{BS} \quad ,$$

where P is the equilibrium price of the call option, and **BS** is its Black-Scholes price. Clearly, δ is a function of *all* the variables that we have talked of so far, including stock, option and macro-economic parameters. It turns out that we can say something about the relative magnitude of δ under the following mutually exclusive scenarios.

Scenario 1 The stock price is S_0 today, *and*, it has been at about this level for a while. In other words, the current stock price is not at the end of a long climb or of an extended decline, and there have been times in the recent past when the market price for the stock was above S_0 and other times when it was below S_0.

Scenario 2 The current stock price is S_0 (the same level as in Scenario 1), but this price is the result of an extended climb in the stock price. In other words, S_0 is the highest price the stock has reached in recent history.

The **BS** price of a call option, in both these scenarios, would be the same since the history of the stock price is irrelevant in the computation of that price, and since everything else is the same. However, the equilibrium price in our model will not be the same in the two scenarios above. Consequently, δ would also be different. Denote by P_1 and P_2, the equilibrium prices under the two scenarios and let

$$\delta_1 = P_1 - \mathbf{BS} \quad ,$$

and

$$\delta_2 = P_2 - \mathbf{BS} \quad .$$

We shall show that we can make predictions on the relative magnitudes of P_1 and P_2 and consequently on δ_1 and δ_2. This will yield an empirically testable hypothesis from our model. In order to compare P_1 and P_2, we shall determine the excess demand in Scenario 2, *if P_1 were the market price*. If at this price P_1, demand exceeds supply, then P_2 must be higher than P_1. On the other hand, if supply exceeds demand, then P_2 must be lower than P_1.

The major difference between the two scenarios that is relevant to us is in the distribution of the bases of stock inventories in the economy, namely, $\mathcal{A}(x)$, $\mathcal{B}(x)$, and $\mathcal{C}(x)$. Recall that $\mathcal{D}$ is the group of investors with inventories of long stocks with basis S_B below S_0 and that $\mathcal{E}$ is the group of investors with inventories of short stocks with basis S_B above S_0. In addition, $\mathcal{F}$ was the set of investors with no such inventories. Label the groups as $\mathcal{D}_1$, $\mathcal{E}_1$, $\mathcal{F}_1$, $\mathcal{D}_2$, $\mathcal{E}_2$, and $\mathcal{F}_2$, to correspond to the two scenarios.

There is nothing to suggest that the composition of $\mathcal{D}_1$ and $\mathcal{D}_2$ should be different, since in both

cases the stock prices are the same and, in addition, investors had opportunities to buy stocks at prices below S_0. On the other hand, the composition of $\mathcal{E}_1$ and $\mathcal{E}_2$ could be substantially different. Under scenario 2, the stock price has not exceeded S_0 in the past, and hence, $\mathcal{E}_2$ would be devoid of any members. This is, of course, not the case with $\mathcal{E}_1$ since the stock price has been above S_0. However, since the total number of investors in the economy can be assumed to be the same[47], under the two scenarios, $\mathcal{F}_2$, must contain more investors than $\mathcal{F}_1$. To summarise,

$$
\begin{aligned}
|\mathcal{D}_1| &= |\mathcal{D}_2| \quad , \\
|\mathcal{E}_1| &\geq |\mathcal{E}_2| \quad , \\
|\mathcal{F}_1| &\leq |\mathcal{F}_2| \quad ,
\end{aligned}
$$

where the cardinality of each set is a measure of the number of investors in that set. Finally, assume that investor preferences have not been affected by the history of the stock price in the sense that if an investor *desires* the payoffs from an option under scenario 1, he will continue to do so under scenario 2.

The investors in $\mathcal{A}$ who desire the payoffs from long options and would have been in $\mathcal{E}_1$ or $\mathcal{F}_1$ under scenario 1, will be in $\mathcal{F}_2$ under scenario 2. These individuals will continue to demand naked options[48]. However, investors in $\mathcal{B}$ who would have been in $\mathcal{E}_1$, *and* would have replicated short option positions under scenario 1, will be in $\mathcal{F}_2$ under scenario 2 and since $P_1 > \mathcal{F}_W$, they would prefer to write options rather than replicate *were the price set at* P_1. This implies that there will be an *increase* in the supply of naked options from this group under scenario 2 (compared to scenario 1), *if* the equilibrium price were not changed from P_1. The individuals who were writing options (under scenario 1) at the price of P_1, will continue to do so, and there will be no impact on the supply of naked options from these investors. Finally, investors who are in $\mathcal{C}$, and would have been in $\mathcal{E}_1$ (under scenario 1), and who would have *bought* covered options at the price P_1, will now (under scenario 2) be in $\mathcal{F}_2$, and will not find it profitable to do so anymore since to preclude arbitrage, P_1 must be greater than $\mathcal{F}_{CLO}$. Hence, at P_1, under scenario 2, there will be a lower demand for covered options than under scenario 1.

In short, at a price of P_1, there will be greater supply (and lower demand) of options under scenario 2 than under scenario 1. However, P_1 was the price in scenario 1 that equated supplies and demands. Hence, under scenario 2, at the price P_1, the supply of call-options will exceed the demand for call options. This implies that the equilibrium price under scenario 2 must be lower than P_1. In other words,

$$P_2 \leq P_1 \quad .$$

[47] Failing which, we are comparing oranges to apples.

[48] Recall that the equilibrium price is not greater than $\mathcal{F}_H$.

In a very similar manner we can argue that the equilibrium price would be *higher*, had the stock price reached S_0 at the end of a decline[49]. In addition, observe that these conclusions are independent of the option parameters such as maturity and strike price, the macroeconomic parameters such as interest rates and stock price distribution parameters such as u and d, as long as the parameters are the same under both scenarios. Finally, observe that the Black-Scholes price, **BS**, would be identical in both scenarios. Hence, δ, which is the difference between the market price and the Black-Scholes price, will be lower[50] under scenario 2. This gives us the following empirical implication.

Hypothesis 1 *The* difference *between the market price of a call option and its Black-Scholes price* (as measured by δ) *will be lower at the end of a long climb in the price of the underlying stock. Conversely, the difference will be higher at the end of a extended decline in the price of the underlying stock.*

Another conjecture that we can make comes from the demand for and supply of options directly. The demand and supply of options on a stock need not be identical for different option parameters. For example, the demand for options with high exercise prices may be lower or higher than the demand for options with lower exercise prices on the same stock. In short, the demand and supply may be functions of the exercise price. Depending on the nature of these functions, δ may be an increasing or decreasing function of the strike price. If, for example, the difference between the demand for and supply of options, measured by the difference in the size of the groups $\mathcal{A}$ and $\mathcal{B}$, is an increasing function of the exercise price, so will δ. On the other hand, if the difference between demand and supply is a decreasing function of the exercise price, so will δ. Since the determination of these supplies and demands is exogeneous to our model, the function relating this difference to the exercise price will have to be assumed[51]. We would expect, however, that the slope of this function would be of the same sign for options on stocks that are positively correlated and of opposite signs for options on stocks that are negatively correlated. The actual sign of the slope could, in any event, be different at different points in time for options on the same stock.

This is one possible explanation for the findings of the Black[1975] and Macbeth-Merville[1979] studies. Black[1975] found that δ was a decreasing function of the strike price whereas, Macbeth-Merville[1979] reported δ to be an increasing function of the strike price.

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[49] We would now consider the set $\mathcal{D}$ to have changed.

[50] If it were negative under scenario 1, it will be more so under scenario 2. If it were positive in scenario 1, however, it could conceivably change sign and become negative in scenario 2.

[51] It may, however, be possible to relate this function to the price history of the stock via an appropriate model.

The next chapter contains a few numerical examples to illustrate these concepts. One of the observations we make there is that δ could be an increasing **or** decreasing function of the strike price *even* when the distribution of individuals along S_B remains the same. A change in the parameters of the option and of the stock, is sufficient to achieve this. However, there is nothing systematic in this behaviour that is apparent.

Chapter 3

A few numerical examples

3.1 Structuring the problem

This section contains a few numerical examples to illustrate the various points made in the study. For computational ease, the population density functions $\mathcal{A}(\bullet)$, $\mathcal{B}(\bullet)$, and $\mathcal{C}(\bullet)$ were assumed to be of the form shown in Figure 4.

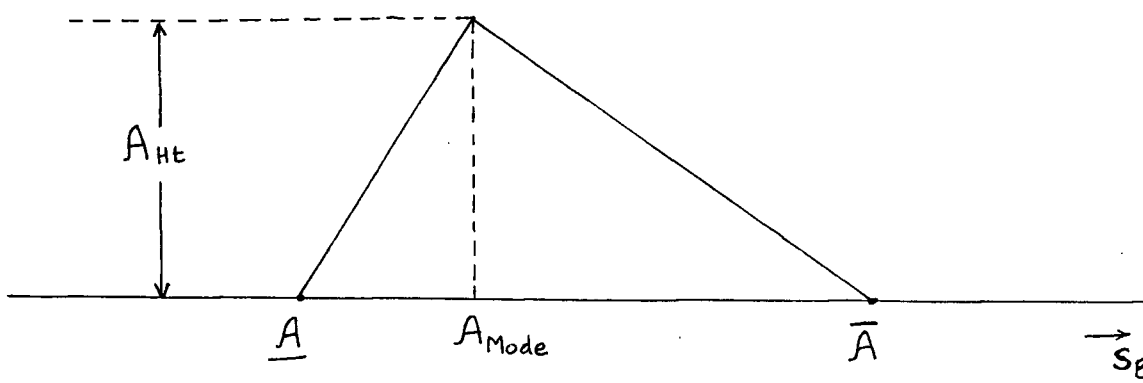

Figure 4

The three functions, $\mathcal{A}(\bullet)$, $\mathcal{B}(\bullet)$, and $\mathcal{C}(\bullet)$ are hence specified completely by the twelve parameters

$$\underline{\mathcal{A}},\ \overline{\mathcal{A}},\ \mathcal{A}_{\text{Mode}},\ \mathcal{A}_{\text{Ht}},\ \underline{\mathcal{B}},\ \overline{\mathcal{B}},\ \mathcal{B}_{\text{Mode}},\ \mathcal{B}_{\text{Ht}},\ \underline{\mathcal{C}},\ \overline{\mathcal{C}},\ \mathcal{C}_{\text{Mode}},\ \text{and}\ \mathcal{C}_{\text{Ht}}\ .$$

The number of investors in $\mathcal{A}$ with inventories of stocks that have a basis between two points x and y can now be computed as the area under the above function between x and y. In a similar manner, we can specify the number of investors in $\mathcal{B}$ or $\mathcal{C}$ between any two bases.

We also assume that

$$\mathcal{A} \cap \mathcal{F} \ = \ \mathcal{B} \cap \mathcal{F} \ = \ \emptyset \ .$$

Recall that $\mathcal{A}_{\mathcal{F}}$ and $\mathcal{B}_{\mathcal{F}}$ were the cardinalities of $\mathcal{A} \cap \mathcal{F}$ and $\mathcal{B} \cap \mathcal{F}$ respectively. Hence, this translates into $\mathcal{A}_{\mathcal{F}} = \mathcal{B}_{\mathcal{F}} = 0$. This is really a stronger assumption than necessary. The assumption that $\mathcal{A}_{\mathcal{F}} = \mathcal{B}_{\mathcal{F}} \neq 0$, will yield the same equilibrium price. This is so since, for all equilibrium prices, these investors would find it preferable to buy/sell the option, rather than create synthetic ones[52]. Hence, their demands and supplies exactly cancel each other out.

All the examples that we present have options with $N = 10$. We can interpret this as the number of *months* to maturity of the option. Under this interpretation, R is the *monthly* interest rate.

3.2 Example 1

Consider an option with the following parameters :

$$\begin{array}{lll} u = 1.3 & S_0 = 20 & \tau_o = 0.3 \\ d = 0.9 & N = 10 & \tau_g = 0.05 \end{array} \qquad R = 1.01$$

---

[52] This comes about from an earlier assumption that P lies in $(\mathcal{F}_W, \mathcal{F}_H)$.

	$\mathcal{A}$	$\mathcal{B}$	$\mathcal{C}$
Lower Support	10	10	10
Upper Support	30	30	30
Mode	15	15	15
Max. Height	10	10	80

The equilibria corresponding to different exercise prices can be viewed in Table 3 (Appendix 2). Table 4 contains the equilibria that result, if we raise the *modes* of the three functions $\mathcal{A}(\bullet)$, $\mathcal{B}(\bullet)$, and $\mathcal{C}(\bullet)$ to 20, and finally, Table 5 contains the equilibria corresponding the case where the three modes are at 25.

The case where the modes were at 15 corresponds to a situation where the stock price has had a history of a climbing price, whereas the last case corresponds to a history of a declining price. The second case is the result of a price history that was basically flat. Notice, from tables 3, 4 and 5, that the equilibrium price is a non-decreasing function of the modes[53]. Finally, observe that δ is *not* a monotonic function of the strike price. It is quite easy, however, to find cases where it *is* a monotonic function of the strike price. In addition, the monotonicity could be positive or negative. The findings of the Black [1975] and the Macbeth-Merville [1979] studies *may* have isolated these chance occurrences, but it appears unlikely. The odds are that there is another explanation for these observations. We shall consider one possibility in this study, but first we shall examine the impact of an increase in the demands for or supply of naked options on the equilibrium price. We shall also examine the impact of a change in the capital gains tax rate, τ_g, on the equilibrium price.

3.2.1 A change in investor demands & supplies of naked options

An exogeneous increase/decrease in the demand for naked call options can be simulated by increasing/decreasing the height of $\mathcal{A}(\bullet)$. The impact of such an increase is presented in Tables 6, 7, and 8 when the modes of the distribution are at 15, 20 and 25. Table 6 corresponds to the case where the exercise price of the option is at \$15, and all other parameters are the same as before. The Black-Scholes price for the option in this case works out to be \$7.50. Table 7 and 8, on the other hand, correspond to options whose exercise prices are at \$20 and \$25 respectively. When the exercise price is \$20, the Black-Scholes price is \$5.02, whereas, with an exercise price of \$25, the Black-Scholes price drops to \$3.48.

[53] It also turns out for this case, that the prices of options with lower exercise price are more sensitive to changes in the modes of the three distributions of bases.

Observe that the price of the option rises with an increase in the demand for naked long positions. This is not unusual, since the price of any asset that is in finite supply, must show such monotonicity. For the sake of completeness, notice from Tables 9, 10, and 11, that the price decreases[54] with increases in the supply of naked option positions. We achieve this by increasing the height of $\mathcal{B}(\bullet)$. In table 9, the strike price is at \$15, whereas, in tables 10 and 11, the strike price is at \$20 and \$25, respectively.

3.2.2 The impact of the capital gains tax rate

Recall that a change in the capital gains tax rate τ_g, will raise the schedules HSO and CSO and lower the schedules WSO and CLO, at any price. However, we had argued that no definite predictions can be made about the direction in which the equilibrium price would move. We could, however, draw inferences about the volume of trade in covered options. In tables 12 through 20, we can view the effect that a change in the capital gains tax rate has on the price and on the supplies and demands. Tables 12, 13, and 14 contain the results for an option with an exercise price of \$15, when the modes of the distributions $\mathcal{A}(\bullet)$, $\mathcal{B}(\bullet)$, and $\mathcal{A}(\bullet)$ are at 15, 20, and 25 respectively. Tables 15, 16, and 17 display the results for an option with an exercise price of \$20 and finally, tables 18, 19, and 20 display the results for an option with an exercise price of \$25.

Observe that the trading in covered options declines as τ_g rises, and that the price is not monotonically related to the capital gains tax rate. This lack of monotonicity is most evident in Tables 12, 13 and 14. However, as one can observe in tables 15, 16 and 17, monotonicity may be evident under certain parametric conditions. Finally, in tables 18, 19 and 20, the price does not appear to be significantly affected by changes in the capital gains tax rate.

A "high" mode in our examples corresponds to a scenario where the stock price has had a history of a declining price, whereas a "low" mode characterises the case where the stock price has risen over time. Since the Black-Scholes price will not be altered by the location of the mode, we expect (from Hypothesis 1) that the equilibrium price should rise with an increase in the modes of $\mathcal{A}(\bullet)$, $\mathcal{B}(\bullet)$ and $\mathcal{C}(\bullet)$. Confirmation of this is obtained in tables 6 through 11. Observe also that the change in the price is not necessarily a monotonic function of the level of the modes either.

[54] If it did not do so, we would need to worry about the stability of the price equilibrium.

3.3 Deviations from the Black-Scholes price at different exercise prices

The difference between our equilibrium price and the Black-Scholes price is not necessarily monotonic in the strike price of the option. In table 3, δ, the difference between the equilibrium price and the Black-Scholes price, is seen to depict this lack of monotonicity. Further, from tables 4 and 5, we can observe that even the sensitivity of δ, to an increase in the mode, is not monotonic in the strike price. We shall construct a few examples here to illustrate this lack of monotonicity.

3.3.1 Example 2

Let

$$\begin{array}{lll} u = 1.1 & S_0 = 20 & \tau_o = 0.2 \\ d = 0.9 & N = 10 & \tau_g = 0.1 \end{array} \qquad R = 1.01$$

	$\mathcal{A}$	$\mathcal{B}$	$\mathcal{C}$
Lower Support	6	6	5
Upper Support	30	29	30
Mode	9	10	10
Max. Height	150	10	100

In figure 5 in Appendix 4 we have plotted the equilibrium price and the Black-Scholes price as functions of the strike price. The solid line is the equilibrium price of our model, whereas, the broken line is the Black-Scholes price. The difference, δ, between our equilibrium price and the Black-Scholes price, can be viewed in figure 6. Observe that though there is a general trend for the bias to be downward sloping, there are local deviations from such a trend.

3.3.2 Example 3

Consider another example where we alter the population density functions

$$\begin{array}{lll} u = 1.1 & S_0 = 20 & \tau_o = 0.2 \\ d = 0.9 & N = 10 & \tau_g = 0.1 \end{array} \qquad R = 1.01$$

	$\mathcal{A}$	$\mathcal{B}$	$\mathcal{C}$
Lower Support	6	6	5
Upper Support	30	29	30
Mode	26	26	26
Max. Height	10	50	100

Figure 7 in Appendix 4 we have plotted the equilibrium price and the Black-Scholes price as functions of the strike price. The solid line is the equilibrium price of our model, whereas, the broken line is the Black-Scholes price. The difference, δ, between our equilibrium price and the Black-Scholes price, can be viewed in figure 8. Once again, observe that though there is a general trend for the bias to be upward sloping, there are local deviations from such a trend.

3.3.3 Example 4

The following example depicts substantial swings in δ. We obtain this example by raising the variance of the stock price in example 2

$$\begin{array}{lll} u = 1.5 & S_0 = 20 & \tau_o = 0.2 \\ d = 0.7 & N = 10 & \tau_g = 0.1 \end{array} \qquad R = 1.01$$

	$\mathcal{A}$	$\mathcal{B}$	$\mathcal{C}$
Lower Support	6	6	5
Upper Support	30	29	30
Mode	9	10	10
Max. Height	150	10	100

Figure 9 in Appendix 4 contains the plot of our equilibrium price and the Black-Scholes price as functions of the strike price. Once again, the solid line is the equilibrium price of our model, whereas, the broken line is the Black-Scholes price. The difference, δ, between our equilibrium price and the Black-Scholes price, can be viewed in figure 10. No distinct trend is visible in this case for the strike-price bias.

3.3.4 Example 5

As a final example, consider the following variation of example 3

$$\begin{array}{lll} u = 1.3 & S_0 = 20 & \tau_o = 0.2 \\ d = 0.9 & N = 10 & \tau_g = 0.1 \end{array} \quad R = 1.01$$

	$\mathcal{A}$	$\mathcal{B}$	$\mathcal{C}$
Lower Support	6	6	5
Upper Support	30	29	30
Mode	26	26	26
Max. Height	10	150	100

Figure 11 in Appendix 4 contains the plot of our equilibrium price and the Black-Scholes price as functions of the strike price. As before, the solid line is the equilibrium price of our model, whereas, the broken line is the Black-Scholes price. The difference, δ, between our equilibrium price and the Black-Scholes price, can be viewed in figure 12.

3.3.5 A different explanation for the strike-price bias

It is clearly possible to construct examples of options where the difference, δ, is an increasing or a decreasing function of the strike price. However, such cases are special and it is unlikely that our world restricts itself to such options. In this section, we shall construct a more realistic example where δ is monotonic in the exercise price.

The studies of Black [1975] and Macbeth-Merville [1979] found that δ was monotonic in the strike price, though the direction of monotonicity was reversed in the two studies. Not only did they find that δ was monotonic in the exercise price, but they also estimated that the function $\delta(E)$ changed signs at around the current stock price. Though the *location* of $\delta(E)$ is a very different question from the sign of its slope, separating the two issues in an empirical test is quite another matter. One can be reasonably sure of the slope of the function, but since the Black-Scholes price is subject to estimation errors, it is difficult to be sure that we have not either over or underestimated it. Hence, even if the equilibrium price were above (or below) the Black-Scholes for the entire range of strike prices in the market, an over-estimate (or under-estimate), of the Black-Scholes price could easily lead us to believe that the function $\delta(E)$ is positive for some exercise prices and negative for others. In any case, estimation errors aside, it is possible to construct scenarios where $\delta(E)$ is

monotonic and changes signs at any point.

The supply and demand for naked option positions is not likely to be the same for all options in the economy, even if they are on the same underlying asset. In fact, in a market where investors are contemplating on switching from stocks to options, it is quite likely that the demands and supplies of naked option positions[55] are themselves monotonic functions of the exercise price. The direction of monotonicity could also be positive or negative, depending on a variety of factors exogeneous to our model. For example, if the demand for naked options increases with the strike price, we would expect that the function $\delta(E)$ is increasing in the strike price. On the other hand, if the supply of naked options increases (or the demand decreases) with the strike price, we would expect that $\delta(E)$ is decreasing in the strike price. We use this idea to construct examples of situations where $\delta(E)$ displays monotonicity *and* "proper" location with respect to the current stock price. Consider the following parameters :

$$\begin{array}{lll} u = 1.1 & S_0 = 20 & \tau_o = 0.2 \\ d = 0.95 & N = 10 & \tau_g = 0.1 \end{array} \qquad R = 1.01$$

	$\mathcal{A}$	$\mathcal{B}$	$\mathcal{C}$
Lower Support	5	5	5
Upper Support	30	30	30
Mode	10	10	10
Max. Height	$\mathcal{A}_{\mathrm{Ht}}$	$\mathcal{B}_{\mathrm{Ht}}$	100

Table 21 has been generated by assuming the following :

$$\begin{aligned} \mathcal{A}_{\mathrm{Ht}} &= 100 \\ \mathcal{B}_{\mathrm{Ht}} &= 4\,E \end{aligned}$$

where

$$E = \text{Exercise Price}$$

Observe that $\delta(E)$ is decreasing in the exercise price and in addition, it is positive for options that are "in the money" and negative for those that are "out of the money". This is consistent with the findings of Black [1975].

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[55] We cannot say this for covered positions, since such demands and supplies come about from endowments and not directly from preferences.

Table 22 has, on the other hand has been generated by assuming that

$$\mathcal{A}_{\mathrm{Ht}} = \begin{cases} \left(\frac{4}{3}\right)(E+10)\,, & \text{if } E < 20; \\ 2\,E\,. & \text{otherwise.} \end{cases}$$
$$\mathcal{B}_{\mathrm{Ht}} = 100$$

In this case, we find that $\delta(E)$ is increasing in the exercise price and in addition is negative for *in the money* options and positive for *out of the money* options. This is consistent with the findings of Macbeth-Merville [1979].

Other scenarios can be constructed by appropriately specifying the form of the demands and supplies in the economy. However, the objective of this exercise was to illustrate that the findings of the two mentioned studies is consistent with investor rationality and market efficiency, given a rich enough structure of demands and supplies. The heterogeneity of personal tax rates is a factor which further adds to the number of possible scenarios that can result in reality.

3.4 Conclusions and future research

In this paper, we demonstrated that investor preferences and endowments play a role in the determination of option prices. This result is obtained by assuming that capital markets are not complete for the costless transfer of basis liability between investors. In other words, it is not possible for one investor who has a stock with a basis of S_B, to transfer the stock to another investor, without being affected by taxes[56]. Consequently, each investor will perceive the value of a hypothetical security that allows such basis transfers, to be different. The price of options in equilibrium is in part influenced by the assessments of different investors of the value of such a security. Consequently, the option price can be viewed as being comprised of two components. The first component, which we shall call the arbitrage component, is the one that has been tackled in great detail by earlier models. The magnitude of this component is clearly dependent upon the distributional characteristics of the stock, and on other factors such as interest rates, etc. However, given the existence of the second component, no model which *does not* acknowledge its existence will be able to explain the observed option prices. This component of the option price is some "average" of the various assessments of the value of the hypothetical security that permits basis transfers. In the process of reaching equilibrium, some investors who feel that the returns from the option are excessively high, will make some finite "arbitrage" profits. In equilibrium, however, all

[56] If this were possible, then no two investors who are subject to different rates of taxation will have different stock bases in equilibrium.

investors are in agreement that the price is such that the riskless bond dominates any combination of the option and the hedge that yields a riskless payoff. The actual location of the equilibrium price, hence, depends to a large degree on the initial preferences, endowments and beliefs of investors[57]. Those who felt that a riskless combination of options and stocks would dominate the riskless bond, before equilibrium was reached, end up taking covered option positions in equilibrium.

In this study, we assumed that all covers and replication strategies involved the use of stocks and bonds only. It is easy to notice that an appropriate portfolio consisting of puts, stocks and bonds can also be used to replicate the payoffs from a call-option. In addition, such a portfolio will not have to be rebalanced regularly and consequently, we may be able to obtain a model for the joint determination of call and put prices, given a distribution of investors with inventories of stocks at different bases. We also assumed that the stock price is not affected by the existence of options. Considering the fact that any covered position in options also involves some stock, it is an easy exercise to realise that the demands and supplies of stocks will be affected by the location of the option price. We should hence, strive to obtain models that explain the joint determination of option and stock prices in equilibrium. This may explain some of the anamolies that we observe in both the stock and option markets.

[57] We assumed that beliefs were homogeneous.

Appendix 1

Proofs

We start this appendix by proving that the equilibrium price must satisfy

$$P \geq \text{Max}\left(0, (S_0 - E)\right) \quad .$$

The proof that is provided demonstrates that if the above relation were not true, then arbitrage is possible. To start off, assume that $P < 0$. We shall construct a portfolio that requires no investment today, and will yield a positive payoff in all possible future states. Consider the following portfolio :

1) Buy a call option and pay P dollars today. You receive money since $P < 0$.

2) Invest the $|P|$ dollars in a riskless bond.

At maturity, you could let the option expire, and obtain a capital gains tax shelter of $P\tau_o$. Since P is negative, this would be a capital gains tax liability. Alternatively, you could exercise the option, and buy the stock for E. The value of the option in this eventuality is $S_N - E$, where S_N is the terminal price of the stock. If you can sell the stock and realise a capital loss shelter (basis of stock is $P + E$), you may do so. Choose to exercise, only if it benefits you to do so. The payoff from this option hence is *not less than*

$$\text{Max}\left(P\tau_o, (S_N - E)\right) \quad .$$

With the call strategy that we outlined above, you will receive $|P|$ dollars today, since P is negative, and in the future you pay $|P|\tau_o$ dollars and in addition, you may also *receive* $S_N - E$ dollars (whenever $S_N - E \geq P\tau_o$). Hence, the *minimum* amount that you pay out in the future is $|P\tau_o|$ dollars.

You will receive $|P|(R^*)^N$ dollars in the future from the investment in the bond, where N is the time to maturity of the option. Now,

$$\begin{aligned}
(R^*)^N &\geq R^* && \text{since } R^* \geq 1 \\
&= R - (R-1)\tau_o \\
&= R(1-\tau_o) + \tau_o \\
&> \tau_o && \text{since} R(1-\tau_o) > 0
\end{aligned}$$

Since $R^* > \tau_o$, you will receive more money from your lending, in the future, than the capital gains taxes that you have to pay on your call option, no matter what tax bracket you are in. Since your net cash flow today is zero, this is a money making machine and cannot exist in equilibrium. Hence, in equilibrium,

$$P \geq 0 \quad .$$

If $P > 0$, then the call strategy is one which yields *less* than the riskless rate of return when the option expires unexercised, and *more* than the riskless rate when the final stock price, S_N, exceeds $(E + PR^*)$.

To show that P exceeds $(S_0 - E)$, assume the converse

$$P + E < S_0$$

and consider the following portfolio strategy :

1) Buy a call option today. You pay P dollars for this today

2) Sell a stock short. You will receive S_0 dollars today.

3) Since by assumption[58], $S_0 > P$, you will be left with some cash. Invest this in a riskless bond.

The value of this portfolio is zero today by construction. We shall demonstrate that the future cash flows from the portfolio, V_f will be positive in all possible future states, under the assumption that S_0 exceeds $P + E$.

[58] The strike price E is usually non-negative

At maturity of the option, if the final stock price, S_N is less than the strike price, E, let the option expire. This will give you a tax shelter of $P\tau_o$ dollars. Liquidate your stock position if it means that you will get some capital gains tax shelter. Otherwise, defer the capital gains realisation. The value of the stock position is hence *at least* S_N. The net cash flow to you is hence *at least*

$$P\tau_o - S_N \quad .$$

If, at maturity, $S_N > E$, then exercise the option. Since this is like purchasing a stock, you will pay capital gains taxes on your short position. The amount of the taxes paid will be $\tau_o(S_0 - P - E)$. This is because you sold the stock for S_0 and bought one for $P + E$. The net cash flow to you is hence[59]

$$-E - \tau_o(S_0 - P - E) \quad .$$

The cash flow from the investment in the bond is always

$$(S_0 - P)\,(R^*)^N \quad .$$

Putting these terms together, we obtain the net cash flow to you, from the portfolio, at maturity of the option as

$$V_f = \begin{cases} P\tau_o - S_N + (S_0 - P)(R^*)^N & \text{if } S_N < E\ , \\ -E - \tau_o(S_0 - P - E) + (S_0 - P)(R^*)^N & \text{otherwise}\ . \end{cases}$$

Consider the cash flow when $S_N < E$:

$$\begin{aligned} P\tau_o - S_N + (S_0 - P)(R^*)^N &\geq P\tau_o - S_N + (S_0 - P)R^* && \text{since } R^* \geq 1 \\ &= \left(S_0 - P - \frac{S_N}{R^*}\right) R^* + P\tau_o \\ &> \left(S_0 - P - \frac{S_N}{R^*}\right) R^* \\ &> \left(S_0 - P - \frac{S_N}{R^*}\right) \\ &> \left(S_0 - P - \frac{E}{R^*}\right) && \text{since } S_N < E \\ &> (S_0 - P - E) \\ &> 0 && \text{since } S_0 > P + E \end{aligned}$$

[59] If you can keep the long stock and the short stock separate for tax purposes, your payoff will be $-E$.

Also, use the definition of R^* to rewrite the cash flows from the portfolio when $S_N > E$ as :

$$\begin{aligned}
(S_0 - P)(R^*)^N - E - \tau_o(S_0 - P - E) &\geq (S_0 - P)R^* - E - \tau_o(S_0 - P - E) \\
&= R(1-\tau_o)(S_0 - P) - E(1-\tau_o) \\
&= (1-\tau_o)(RS_0 - RP - E) \\
&= R(1-\tau_o)(S_0 - P - E) + E(1-\tau_o)(R-1) \\
&\geq R(1-\tau_o)(S_0 - P - E) \\
&> 0 \qquad\qquad \text{since } S_0 > P + E
\end{aligned}$$

Hence, the cash flows from this portfolio is always positive if $S_0 > P + E$. In addition, the required investment today is zero. This is not consistent with equilibrium. Hence,

$$P > S_0 - E$$

Lemma 1 *The hedge portfolio that replicates a long option* **or** *covers a short option position will be long in stocks.*

Proof Notice from Table 1 that the payoffs from the long option are positively correlated with the stock price and that the payoffs from the short option are negatively correlated with the stock price. Hence, the portfolio that replicates a long option and the portfolio that covers a short option must be positively correlated with the stock. Assume that the hedge portfolio is short in stocks. We shall show that this leads to a contradiction. For notational ease, let S represent the stock price at the start of the maturity period. In other words, replace S_{N-1} by S.

The payoffs from every short stock in the hedge portfolio when the stock price moves from S to uS and the hedge is liquidated is given by

$$H^u = -uS - \tau_g(S_B - uS)I(S_B > uS)$$

where we have only apportioned the capital gains to the hedge and assumed that capital losses would have been optimally realised even if the stock had not been used for replication. Similarly, if the stock price had moved to dS, the payoff from each short stock in the hedge portfolio would have been

$$H^d = -dS - \tau_g(S_B - dS)I(S_B > dS) \quad .$$

If the hedge portfolio is to be positively correlated with the stock and if $u > d$, then

$$H^u \geq H^d \quad .$$

$\boxed{\text{Case 1: } S_B \leq dS < uS}$

$$\begin{aligned} H^u \geq H^d \Rightarrow \quad & -uS \geq -dS \\ \Rightarrow \quad & u \geq d \\ \Rightarrow \quad & \textbf{Contradiction} \end{aligned}$$

$\boxed{\text{Case 2: } dS < S_B < uS}$

$$\begin{aligned} H^u \geq H^d \Rightarrow \quad & -uS \geq -dS - \tau_g(S_B - dS) \\ \Rightarrow \quad & (u-d)S \leq \tau_g(S_B - dS) \\ \Rightarrow \quad & (u-d)S \leq \tau_g(uS - dS) \qquad \text{since } S_B < uS \\ \Rightarrow \quad & (u-d)(1-\tau_g) \leq 0 \end{aligned}$$

This is a **contradiction,** since by assumption $\tau_g < 1$.

$\boxed{\text{Case 3: } dS < uS \leq S_B}$

$$\begin{aligned} H^u \geq H^d \Rightarrow \quad & -uS - \tau_g(S_B - uS) \geq -ds - \tau_g(S_B - dS) \\ \Rightarrow \quad & (u-d)(1-\tau_g) \leq 0 \end{aligned}$$

Once again, this is a **contradiction.** Hence, the portfolio cannot be short in stocks.

■

Lemma 2 *The hedge portfolio that replicates a short option* **or** *covers a long option position will be short in stocks.*

Proof Notice from Table 1 that the payoffs from the short option are negatively correlated with the stock price and that the payoffs from the long option are positively correlated with the stock price. Hence, the portfolio that replicates a short option and the portfolio that covers a long option must be negatively correlated with the stock. Assume that the hedge portfolio is long in

stocks. We shall show that this leads to a contradiction. For notational ease, let S represent the stock price at the start of the maturity period. In other words, replace S_{N-1} by S.

The payoffs from every long stock in the hedge portfolio when the stock price moves from S to uS and the hedge is liquidated is given by

$$H^u = uS - \tau_g(uS - S_B)I(S_B < uS)$$

where we have only apportioned the capital gains to the hedge and assumed that capital losses would have been optimally realised even if the stock had not been used for replication. Similarly, if the stock price had moved to dS, the payoff from each short stock in the hedge portfolio would have been

$$H^d = dS - \tau_g(dS - S_B)I(S_B < dS) \quad .$$

If the hedge portfolio is to be negatively correlated with the stock and if $u > d$, then

$$H^u \leq H^d \quad .$$

$\boxed{\text{Case 1: } S_B \geq uS > uS}$

$$\begin{aligned} H^u \leq H^d \Rightarrow \quad & uS \leq dS \\ \Rightarrow \quad & u \leq d \\ \Rightarrow \quad & \textbf{Contradiction} \end{aligned}$$

$\boxed{\text{Case 2: } dS < S_B < uS}$

$$\begin{aligned} H^u \geq H^d \Rightarrow \quad & uS(1-\tau_g) \leq dS - \tau_g S_B \\ \Rightarrow \quad & uS(1-\tau_g) \leq dS - \tau_g dS \quad \text{since } S_B > dS \\ \Rightarrow \quad & (u-d)(1-\tau_g) \leq 0 \end{aligned}$$

This is a **contradiction,** since by assumption $\tau_g < 1$.

$\boxed{\text{Case 3: } uS > dS \geq S_B}$

$$\begin{aligned} H^u \geq H^d \Rightarrow \quad & uS(1-\tau_g) \leq dS(1-\tau_g) \\ \Rightarrow \quad & u \leq dS(1-\tau_g) \end{aligned}$$

Once again, this is a **contradiction.** Hence, the portfolio cannot be long in stocks. ■

Lemma 3 *If $\varrho \in (0,1)$, then HSO is a non-increasing function of S_B.*

Proof Observe that in equation (3), $C^u \geq C^d$. This can be verified by looking at Table 1 where a higher final stock price implies higher option payoffs. Hence, if the result of a rise in S_B is a rise in ϱ, then, Φ_{N-1} will also rise. This will result in a higher value of HSO. Hence,

$$\frac{\partial \varrho}{\partial S_B} \leq 0 \Rightarrow \frac{\partial HSO(S_B, \tau_o, \tau_g)}{\partial S_B} \leq 0$$

It is easy to verify from equation (3) that as S_B rises, $\theta(u)$ and $\theta(d)$ **either** rise **or** do not change. Let $\theta(u)$ rise by x_u and $\theta(d)$ rise by x_d. Hence,

$$x_u\,,\ x_d \geq 0$$

Let ϱ change from ϱ_{old} to ϱ_{new} as a result of the rise in S_B. We shall show that $\varrho_{old} \geq \varrho_{new}$. Now,

$$\varrho_{new} = \frac{R^* - \theta(d) - x_d}{\theta(u) - \theta(d) + x_u - x_d} \leq \frac{R^* - \theta(d) - x_d}{\theta(u) - \theta(d) - x_d}$$

But we know that

$$\forall\, x > 0\,, \qquad \frac{a - x}{b - x} < \frac{a}{b} \qquad \text{if} \quad 0 < a < b \quad .$$

Hence,

$$\frac{R^* - \theta(d) - x_d}{\theta(u) - \theta(d) - x_d} \leq \varrho_{old}$$

This implies that

$$\varrho_{new} \leq \varrho_{old}$$

■

Lemma 4 *If $\omega \in (0,1)$, then WSO is a non-increasing function of S_B.*

Proof Notice from Table 1 and equation (3) that the payoffs to the option,

$$C^u \leq C^d < 0 \quad .$$

Consequently, a reduction in ω implies that Φ_{N-1} (which was negative) will become *less negative*. In other words, the absolute value of Φ_{N-1} will fall. This will result in a reduced WSO. Hence,

$$\frac{\partial \omega}{\partial S_B} \leq 0 \quad \Rightarrow \quad \frac{\partial\, WSO(S_B, \tau_o, \tau_g)}{\partial S_B} \leq 0 \quad .$$

Now, notice in equation 3 that

$$\frac{\partial \lambda(u)}{\partial S_B}, \frac{\partial \lambda(d)}{\partial S_B} \geq 0 \quad .$$

Let ω change from ω_{old} to ω_{new} as the consequence of a rise in S_B, and let the corresponding change in $\lambda(u)$ and $\lambda(d)$ be x_u and x_d respectively. Hence,

$$x_u\,,\ x_d \geq 0$$

and

$$\begin{aligned} \omega_{new} &= \frac{R^* - \lambda(d) - x_d}{\lambda(u) - \lambda(d) + x_u - x_d} \\ &\leq \frac{R^* - \lambda(d) - x_d}{\lambda(u) - \lambda(d) - x_d} \end{aligned}$$

and since $\omega \in (0,1)$,

$$\leq \omega_{old} \quad .$$

■

Lemma 5 *If $\varsigma \in (0,1)$, then CSO is a non-increasing function of S_B.*

Proof For convenience, let us label the four indicator functions in equation (4) as follows

$$\begin{aligned} I(S_B < uS_{N-1} < E) &= I_1 \\ I(S_B < dS_{N-1} < E) &= I_2 \\ I(uS_{N-1} > E) &= I_3 \\ I(E < uS_{N-1} < S_B) &= I_4 \\ I(dS_{N-1} > E) &= I_5 \\ I(E < dS_{N-1} < S_B) &= I_6 \end{aligned}$$

Equation 4 can then be written as

$$\left\{\begin{aligned}
\Phi_{N-1} &= \frac{1}{R^*}\{\varsigma[C^u + F^u] + (1-\varsigma)[C^d + F^d]\} \\
\varsigma &= \frac{R^* - \Gamma(d)}{\Gamma(u) - \Gamma(d)} \\
\Gamma(u) &= u - \tau_g\left(u - \frac{S_B}{S_{N-1}}\right) I_1 \\
\Gamma(d) &= d - \tau_g\left(d - \frac{S_B}{S_{N-1}}\right) I_2 \\
F^u &= \tau_g\Big([P + E - S_B]I_3 + [S_B - uS_{N-1}]I_4\Big) \\
F^d &= \tau_g\Big([P + E - S_B]I_5 + [S_B - dS_{N-1}]I_6\Big)
\end{aligned}\right\}$$

$\boxed{\text{Case 1: } uS_{N-1} < E}$

Since $d < u$, it must be that $uS_{N-1} < dS_{N-1}$. Hence,

$$I_3 = I_4 = I_5 = I_6 = F^u = F^d = 0$$

The proof in this case is very identical to that in Lemma 3, and will not be explicitly outlined.

$\boxed{\text{Case 2: } uS_{N-1} > E \text{ and } dS_{N-1} < E}$

There are four possible subcases under this case as follows:

a) $S_B \leq dS_{N-1}$

b) $dS_{N-1} < S_B < E < uS_{N-1}$

c) $dS_{N-1} < E < S_B < uS_{N-1}$

d) $dS_{N-1} < E < uS_{N-1} < S_B$

We shall consider each one of these sub-cases independently.

a) In this sub-case,

$$I_1 = I_4 = I_5 = I_6 = 0$$

and

$$I_2 = I_3 = 1$$

Hence,

$$\begin{aligned}
\Gamma(u) &= u \\
\Gamma(d) &= d - \tau_g\left(d - \frac{S_B}{S_{N-1}}\right) \\
F^u &= \tau_g(P + E - S_B) \\
F^d &= 0
\end{aligned}$$

Notice that C^u exceeds C^d. Hence,

$$S_B \uparrow \;\Rightarrow\; F^u \downarrow, \Gamma(d) \uparrow \;\Rightarrow\; \varsigma \downarrow$$

And again,

$$\varsigma \downarrow, F^u \downarrow \;\Rightarrow\; \Phi_{N-1} \downarrow \;\Rightarrow\; CSO \downarrow$$

b) In this sub-case,

$$I_1 = I_2 = I_4 = I_5 = I_6 = 0$$

and $I_3 = 1$. Hence,

$$\begin{aligned}
\Gamma(u) &= u \\
\Gamma(d) &= d \\
F^u &= \tau_g(P + E - S_B) \\
F^d &= 0
\end{aligned}$$

Now,

$$S_B \uparrow \;\Rightarrow\; F^u \downarrow \;\Rightarrow\; \Phi_{N-1} \downarrow \;\Rightarrow\; CSO \downarrow$$

c) In this sub-case,

$$I_1 = I_2 = I_4 = I_5 = I_6 = 0$$

and $I_3 = 1$. This is identical to b) above.

d) In this sub-case,

$$I_1 = I_2 = I_5 = I_6 = 0$$

and

$$I_3 = I_4 = 1$$

Hence,

$$\begin{aligned} \Gamma(u) &= u \\ \Gamma(d) &= d \\ F^u &= \tau_g(P + E - uS_{N-1}) \\ F^d &= 0 \end{aligned}$$

Here CSO is independent of S_B.

$\boxed{\text{Case 3: } dS_{N-1} > E}$

There are four possible sub-cases under this case as follows:

a) $S_B \leq E \leq dS_{N-1} < uS_{N-1}$

b) $E \leq S_B \leq dS_{N-1} < uS_{N-1}$

c) $E \leq dS_{N-1} < S_B < uS_{N-1}$

d) $E \leq dS_{N-1} < uS_{N-1} \leq S_B$

We shall consider each one of these sub-cases independently.

a) In this sub-case,

$$I_1 = I_2 = I_4 = I_6 = 0$$

and

$$I_3 = I_5 = 1$$

Hence,

$$\begin{aligned}\Gamma(u) &= u\\ \Gamma(d) &= d\\ F^u &= \tau_g(P+E-S_B)\\ F^d &= \tau_g(P+E-S_B)\end{aligned}$$

Therefore,

$$S_B \downarrow \Rightarrow \quad F^u \downarrow, F^d \downarrow \Rightarrow \quad \Phi_{N-1} \downarrow \Rightarrow \quad CSO \downarrow$$

b) In this sub-case,

$$I_1 = I_2 = I_4 = I_6 = 0$$

and

$$I_3 = I_5 = 1$$

Hence,

$$\begin{aligned}\Gamma(u) &= u\\ \Gamma(d) &= d\\ F^u &= \tau_g(P+E-S_B)\\ F^d &= \tau_g(P+E-S_B)\end{aligned}$$

This is identical to a) above.

c) In this sub-case,

$$I_1 = I_2 = I_4 = 0$$

and

$$I_3 = I_5 = I_6 = 1$$

Hence,

$$\begin{aligned}\Gamma(u) &= u\\ \Gamma(d) &= d\\ F^u &= \tau_g(P+E-S_B)\\ F^d &= \tau_g(P+E-dS_{N-1})\end{aligned}$$

In here, ς is independent of S_B and

$$\begin{aligned}\varsigma F^u + (1-\varsigma)F^d &= \tau_g(P + E - S_B) + (1-\varsigma)\tau_g(S_B - dS_{N-1}) \\ &= \tau_g(P + E - (1-\varsigma)dS_{N-1}) - \varsigma\tau_g S_B\end{aligned}$$

As S_B rises, this term above falls and as a result Φ_{N-1} falls and consequently CSO drops.

d) In this sub-case,

$$I_3 = I_5 = I_4 = I_6 = 1$$

and

$$I_1 = I_2 = 0$$

Hence,

$$\begin{aligned}\Gamma(u) &= u \\ \Gamma(d) &= d \\ F^u &= \tau_g(P + E - uS_{N-1}) \\ F^d &= \tau_g(P + E - dS_{N-1})\end{aligned}$$

All the terms are independent of S_B. Hence, as S_B changes, Φ_{N-1} does not change in this region.

Finally observe that CSO is continuous in S_B and that

$$\begin{aligned}\lim_{S_B \to |dS_{N-1}|_-} \left\{CSO\right\} &= \lim_{S_B \to |dS_{N-1}|_+} \left\{CSO\right\} \\ \lim_{S_B \to |uS_{N-1}|_-} \left\{CSO\right\} &= \lim_{S_B \to |uS_{N-1}|_+} \left\{CSO\right\}\end{aligned}$$

This completes the proof. ■

Lemma 6 *If $\psi \in (0,1)$, then CLO is a non-increasing function of S_B.*

Proof The proof here is very similar to that in Lemma 4. Once again, a drop in ψ implies a reduction in $|\Phi_{N-1}|$ and consequently, in CLO. A rise in S_B results in a rise (or no change) in $\Lambda(u)$ and $\Lambda(d)$, which in turn results in a decline in ψ. ■

Lemma 7 *A rise in the capital gains tax rate ceteris paribus is accompanied by a rise in the cost of creating a long synthetic option and in the cost of covering a short option. In addition, the revenue from creating a short synthetic option and from covering a long option declines from this rise in the capital gains tax rate.*

Proof Consider HSO first. Observe that in equation 2, that $C^u \geq C^d$. This comes about from Table 1 where a higher final stock price implies higher option payoffs. Hence, if a rise in τ_g is accompanied by a rise in ϱ, then Φ_{N-1} will also rise and so will HSO. In other words,

$$\frac{\partial \varrho}{\partial \tau_g} \geq 0 \quad \Longrightarrow \quad \frac{\partial (HSO)}{\partial \tau_g} \geq 0 \quad .$$

Now,

$$\frac{\partial \varrho}{\partial \tau_g} = \frac{[\theta(u) - \theta(d)] \left(d - \frac{S_B}{S_{N-1}}\right) \mathcal{I}(d) \; - \; [R^* - \theta(d)] \left\{ \left(d - \frac{S_B}{S_{N-1}}\right) \mathcal{I}(d) - \left(u - \frac{S_B}{S_{N-1}}\right) \mathcal{I}(u) \right\}}{[\theta(u) - \theta(d)]^2}$$

where

$$\begin{aligned} \mathcal{I}(d) &= \mathcal{I}(S_B < dS_{N-1} < P + E) \\ \mathcal{I}(u) &= \mathcal{I}(S_B < uS_{N-1} < P + E) \end{aligned}$$

The sign of $\partial \varrho / \partial \tau_g$ is the same as the sign of the numerator of the above expression. We shall show that it is non-negative always.

$\boxed{\text{Case 1: } \mathcal{I}(u) = \mathcal{I}(d) = 0}$

In this case,

$$\frac{\partial \varrho}{\partial \tau_g} = 0$$

$\boxed{\text{Case 2: } \mathcal{I}(u) = \mathcal{I}(d) = 1}$

From the definition of the indicator function, we learn that

$$u - \frac{S_B}{S_{N-1}} > d - \frac{S_B}{S_{N-1}} > 0$$

The numerator hence reduces to

$$(u - d) \left\{ \left(d - \frac{S_B}{S_{N-1}}\right) + (R^* - d) \right\} > (u - d) \left(d - \frac{S_B}{S_{N-1}}\right) > 0$$

Hence

$$\frac{\partial \varrho}{\partial \tau_g} > 0$$

$\boxed{\text{Case 3: } I(u) = 1 \; ; \; I(d) = 0}$

The numerator reduces to

$$(R^* - d)\left(u - \frac{S_B}{S_{N-1}}\right)$$

and this is positive. Hence the derivative is positive.

$\boxed{\text{Case 4: } I(u) = 0 \; ; \; I(d) = 1}$

The numerator reduces to

$$\left(d - \frac{S_B}{S_{N-1}}\right)(u - R^*)$$

and this is positive. Hence

$$\frac{\partial \varrho}{\partial \tau_g} \geq 0$$

Consequently, HSO rises with τ_g.

Consider CSO now. A glance at equation 4 tells us that following the same logic as with HSO, we can show that

$$\frac{\partial \varsigma}{\partial \tau_g} \geq 0 \quad .$$

In addition,

$$C^u \geq C^d \geq 0 \quad .$$

Hence, if

$$F^u \geq F^d \geq 0$$

and

$$\frac{\partial F^u}{\partial \tau_g} \geq \frac{\partial F^d}{\partial \tau_g} \geq 0 \quad ,$$

then we can infer that CSO also rises with τ_g

Now first of all recall that the basis of investors who are concerned with covering a short option position, must be *below* the current market price of the stock, S_0, since these are the investors who

will have inventories of long stocks. In addition, we showed in an earlier section that to eliminate arbitrage possibilities,

$$P + E > S_0 \quad .$$

Hence, in F^u and F^d, it must be that

$$P + E > S_B \quad .$$

In addition, the indicator function of the second term in F^u and F^d ensures that the term is always non-negative. Also notice that since $u > d$, $F^u \geq F^d$. In other words,

$$F^u \geq F^d \geq 0 \quad .$$

Finally observe that the derivative of F^u and F^d with respect to τ_g is non-negative by the same argument that was used in showing that they themselves are non-negative. This implies that CSO rises with τ_g.

Coming to WSO, we notice that C^u and C^d are negative and that $C^u < C^d$. Consequently, if (in equation 3) ω were to decrease with a rise in τ_g, then Φ_{N-1} (which was negative) will become *less negative*. In other words, the absolute value of Φ_{N-1} will decline. This will reduce WSO. In other words,

$$\frac{\partial \omega}{\partial \tau_g} \geq 0 \quad \Longrightarrow \quad \frac{\partial (WSO)}{\partial \tau_g} \geq 0 \quad .$$

Now,

$$\frac{\partial \omega}{\partial \tau_g} = \frac{[\lambda(u) - \lambda(d)]\left(d - \frac{S_B}{S_{N-1}}\right) I(d) - [R^* - \lambda(d)]\left\{\left(d - \frac{S_B}{S_{N-1}}\right) I(d) - \left(u - \frac{S_B}{S_{N-1}}\right) I(u)\right\}}{[\lambda(u) - \lambda(d)]^2}$$

where

$$\begin{aligned} I(d) &= I(dS_{N-1} < S_B \text{ and } dS_{N-1} < E) \\ I(u) &= I(uS_{N-1} < S_B \text{ and } uS_{N-1} < E) \end{aligned}$$

The sign of $\partial\omega/\partial\tau_g$ is the same as the sign of the numerator of the above expression. We shall show that it is non-positive always.

$\boxed{\text{Case 1: } I(u) = I(d) = 0}$

In this case,

$$\frac{\partial \omega}{\partial \tau_g} = 0$$

$\boxed{\text{Case 2: } I(u) = I(d) = 1}$

In this case, the option is not being exercised and the payoffs C^u and C^d equal $P\tau_o$. Hence, a change in ω will not affect Φ_{N-1}.

$\boxed{\text{Case 3: } I(u) = 0\ ;\ I(d) = 1}$

The numerator reduces to

$$\left(d - \frac{S_B}{S_{N-1}}\right)(u - R^*)$$

and this is non-positive, since by assumption $u > R^*$ and in this case,

$$d - \frac{S_B}{S_{N-1}} \leq 0 \quad .$$

The only combination that we have not considered is when $I(u) = 1$ and $I(d) = 0$. This is not possible since by assumption $u > d$ and this ensures that

$$I(u) = 1 \implies I(d) = 1 \quad .$$

Hence

$$\frac{\partial \omega}{\partial \tau_g} \leq 0$$

Consequently, WSO declines with an increase in τ_g.

In the case of CLO, the proof is very similar to the case of WSO. This can be verified by comparing equations 3 and 5. The only difference is that in equation 5, when the two indicator functions are unity, it is not necessary that C^u and C^d are the same. Hence, we shall only consider this one case to show that as τ_g rises, ψ declines. All the other cases are proved in a fashion **identical** to WSO and will not be repeated here. When the two indicator functions both take on a value of unity, the sign of $\partial\psi/\partial\tau_g$ is the same as the sign of

$$[\Lambda(u) - \Lambda(d)]\left(d - \frac{S_B}{S_{N-1}}\right) - [R^* - \Lambda(d)]\left\{\left(d - \frac{S_B}{S_{N-1}}\right) - \left(u - \frac{S_B}{S_{N-1}}\right)\right\} \quad .$$

This reduces to

$$(u - d)\left(R^* - \frac{S_B}{S_{N-1}}\right) \quad .$$

which in turn is negative since $u > R^*$ and since in this case

$$\left(u - \frac{S_B}{S_{N-1}}\right) \leq 0 \quad .$$

Hence $\partial\psi/\partial\tau_g$ is non-positive and consequently, CLO decreases with an increase in τ_g. ■

Appendix 2

Tables

	$S_N < E$	$E < S_N < P + E$	$S_N > P + E$
Payoffs to long option	$P\tau_o$	$(S_N - E)(1 - \tau_o) + P\tau_o$	$(S_N - E)$
Payoffs to short option	$-P\tau_o$	$(E - S_N)(1 - \tau_o) - P\tau_o$	$(E - S_N)(1 - \tau_o) - P\tau_o$

Payoffs from options

Table 1

	$\mathcal{A}$	$\mathcal{B}$	$\mathcal{C}$
$HSO \geq P$	Buy option	•	•
$HSO < P$	Long synthetic option	•	•
$WSO > P$	•	Short synthetic option	•
$WSO \leq P$	•	Write option	•
$CLO > P$	•	•	Buy option and hedge
$CLO \leq P$	•	•	•
$CSO > P$	•	•	•
$CSO \leq P$	•	•	Write option and hedge

Optimal Trading Rules

Table 2

Strike Pr.	14.00	16.00	18.00	20.00	22.00	24.00
Eqlbm. Pr.	8.59	7.91	6.84	5.75	5.04	4.42
B-S Price	8.12	7.00	6.01	5.02	4.26	3.74
δ	0.47	0.91	0.83	0.74	0.78	0.68
D_{naked}	100.00	100.00	100.00	100.00	100.00	100.00
D_{Cover}	266.67	123.93	210.66	266.67	0.00	0.00
S_{naked}	100.00	100.00	100.00	100.00	100.00	100.00
S_{Cover}	266.67	123.93	210.66	266.67	0.00	0.00
ξ	8.99	7.94	6.90	5.85	5.04	4.42
μ	7.88	6.88	5.91	4.93	4.22	3.69
η	9.05	7.98	6.92	5.86	5.05	4.42
ν	8.49	7.85	6.75	5.65	5.13	4.52

Modes at 15 (End of a bull market).

Table 3

Strike Pr.	14.00	16.00	18.00	20.00	22.00	24.00
Equlbm Pr.	8.84	7.91	6.86	5.81	5.04	4.42
B-S Price	8.12	7.00	6.01	5.02	4.26	3.74
δ	0.72	0.91	0.85	0.80	0.78	0.68
D_{naked}	100.00	100.00	100.00	100.00	100.00	100.00
D_{Cover}	400.00	165.09	260.36	342.50	0.00	0.00
S_{naked}	100.00	100.00	100.00	100.00	100.00	100.00
S_{Cover}	400.00	165.09	260.36	342.50	0.00	0.00
ξ	9.05	7.94	6.90	5.86	5.04	4.42
μ	7.95	6.88	5.91	4.94	4.21	3.69
η	9.10	7.98	6.93	5.88	5.05	4.42
ν	8.51	7.86	6.76	5.66	5.13	4.52

Modes at 20 (Stable stock price history).

Table 4

Strike Pr.	14.00	16.00	18.00	20.00	22.00	24.00
Eqlbm Pr.	9.01	7.92	6.87	5.84	5.05	4.42
B-S Price	8.12	7.00	6.01	5.02	4.26	3.74
δ	0.89	0.92	0.87	0.82	0.79	0.68
D_{naked}	100.00	100.00	100.00	100.00	100.00	100.00
D_{Cover}	266.67	118.64	189.74	244.82	0.00	0.00
S_{naked}	100.00	100.00	100.00	100.00	100.00	100.00
S_{Cover}	266.67	118.64	189.74	244.82	0.00	0.00
ξ	9.08	7.94	6.90	5.87	5.04	4.42
μ	8.00	6.88	5.92	4.95	4.22	3.69
η	9.14	7.98	6.93	5.88	5.05	4.42
ν	8.53	7.86	6.76	5.67	5.13	4.52

Modes at 25. (End of a bear market)

Table 5

Modes	$\mathcal{A}_{Ht}$	Eqlbm. Pr.	D_{naked}	D_{Cover}	S_{naked}	S_{Cover}
15.00	10.00	8.44	100.00	83.47	100.00	83.47
15.00	20.00	8.46	200.00	25.99	100.00	125.99
15.00	30.00	8.49	300.00	0.00	100.00	200.00
15.00	40.00	8.53	391.41	0.00	100.00	291.41
15.00	50.00	8.55	438.64	0.00	100.00	338.64
20.00	10.00	8.44	100.00	115.51	100.00	115.51
20.00	20.00	8.46	200.00	49.75	100.00	149.75
20.00	30.00	8.49	300.00	0.00	100.00	200.00
20.00	40.00	8.53	380.85	0.00	100.00	280.85
20.00	50.00	8.55	408.43	0.00	100.00	308.43
25.00	10.00	8.45	100.00	81.60	100.00	81.60
25.00	20.00	8.46	200.00	6.52	100.00	106.52
25.00	30.00	8.53	288.51	0.00	100.00	188.51
25.00	40.00	8.57	320.18	0.00	100.00	220.18
25.00	50.00	8.63	347.92	0.00	100.00	247.92

Price Sensitivity to Demand with $E = 15$. Black-Scholes price = \$ 7.50

Table 6

Modes	$\mathcal{A}_{Ht}$	Eqlbm. Pr.	D_{naked}	D_{Cover}	S_{naked}	S_{Cover}
15.00	10.00	5.75	100.00	266.67	100.00	266.67
15.00	20.00	5.79	200.00	259.51	100.00	359.51
15.00	30.00	5.82	300.00	223.68	100.00	423.68
15.00	40.00	5.84	400.00	167.08	100.00	467.08
15.00	50.00	5.86	500.00	96.21	100.00	496.21
20.00	10.00	5.81	100.00	342.50	100.00	342.50
20.00	20.00	5.84	200.00	264.26	100.00	364.26
20.00	30.00	5.85	300.00	177.72	100.00	377.72
20.00	40.00	5.87	400.00	86.60	100.00	386.60
20.00	50.00	5.89	495.15	0.00	100.00	395.15
25.00	10.00	5.84	100.00	244.82	100.00	244.82
25.00	20.00	5.86	200.00	151.89	100.00	251.89
25.00	30.00	5.87	300.00	58.14	100.00	258.14
25.00	40.00	5.90	365.40	0.00	100.00	265.40
25.00	50.00	5.94	366.67	0.00	100.00	266.67

Price sensitivity to Demand with $E = 20$. Black-Scholes price = \$ 5.02

Table 7

Modes	$\mathcal{A}_{Ht}$	Eqlbm. Pr.	D_{naked}	D_{Cover}	S_{naked}	S_{Cover}
15.00	10.00	4.11	100.00	0.00	100.00	0.00
15.00	20.00	4.13	100.00	0.00	100.00	0.00
15.00	30.00	4.15	100.00	0.00	100.00	0.00
15.00	40.00	4.26	133.33	0.00	100.00	33.33
15.00	50.00	4.28	166.67	0.00	100.00	66.67
20.00	10.00	4.11	100.00	0.00	100.00	0.00
20.00	20.00	4.25	100.00	0.00	100.00	0.00
20.00	30.00	4.27	150.00	0.00	100.00	50.00
20.00	40.00	4.28	200.00	0.00	100.00	100.00
20.00	50.00	4.30	250.00	0.00	100.00	150.00
25.00	10.00	4.11	100.00	0.00	100.00	0.00
25.00	20.00	4.27	133.33	0.00	100.00	33.33
25.00	30.00	4.30	200.00	0.00	100.00	100.00
25.00	40.00	4.33	266.67	0.00	100.00	166.67
25.00	50.00	4.39	333.33	0.00	100.00	233.33

Price sensitivity to Demand with $E = 25$. Black-Scholes price = \$ 3.48

Table 8

Modes	β_{Ht}	Eqlbm. Pr.	D_{naked}	D_{Cover}	S_{naked}	S_{Cover}
15.00	10.00	8.44	100.00	83.47	100.00	83.47
15.00	20.00	8.42	100.00	131.84	200.00	31.84
15.00	30.00	8.37	100.00	200.00	300.00	0.0
15.00	40.00	6.94	100.00	266.67	366.67	0.0
15.00	50.00	6.89	100.00	266.67	366.67	0.0
20.00	10.00	8.44	100.00	115.51	100.00	115.51
20.00	20.00	8.43	100.00	170.62	200.00	70.62
20.00	30.00	8.40	100.00	220.94	300.00	20.94
20.00	40.00	8.37	100.00	300.00	400.00	0.0
20.00	50.00	7.48	100.00	400.00	500.00	0.0
25.00	10.00	8.45	100.00	81.60	100.00	81.60
25.00	20.00	8.43	100.00	146.40	200.00	46.40
25.00	30.00	8.40	100.00	211.04	300.00	11.04
25.00	40.00	8.38	100.00	300.00	400.00	0.0
25.00	50.00	8.35	100.00	400.00	500.00	0.0

Price sensitivity to Supply with $E = 15$. Black-Scholes price = \$ 7.50

Table 9

Modes	β_{Ht}	Eqlbm. Pr.	D_{naked}	D_{Cover}	S_{naked}	S_{Cover}
15.00	10.00	5.75	100.00	266.67	100.00	266.67
15.00	20.00	5.71	100.00	266.67	200.00	166.67
15.00	30.00	5.67	100.00	266.67	300.00	66.67
15.00	40.00	4.60	100.00	266.67	366.67	0.0
15.00	50.00	4.60	100.00	266.67	366.67	0.0
20.00	10.00	5.81	100.00	342.50	100.00	342.50
20.00	20.00	5.78	100.00	398.09	200.00	298.09
20.00	30.00	5.72	100.00	400.00	300.00	200.00
20.00	40.00	5.67	100.00	400.00	400.00	100.00
20.00	50.00	5.51	100.00	400.00	500.00	0.0
25.00	10.00	5.84	100.00	244.82	100.00	244.82
25.00	20.00	5.83	100.00	337.08	200.00	237.08
25.00	30.00	5.81	100.00	427.13	300.00	227.13
25.00	40.00	5.79	100.00	509.58	400.00	209.58
25.00	50.00	5.72	100.00	533.33	500.00	133.33

Price sensitivity to Supply with $E = 20$. Black-Scholes price = \$ 5.02

Table 10

Modes	β_{Ht}	Eqlbm. Pr.	D_{naked}	D_{Cover}	S_{naked}	S_{Cover}
15.00	10.00	4.11	100.00	0.0	100.00	0.0
15.00	20.00	4.10	100.00	100.00	200.00	0.0
15.00	30.00	4.08	100.00	200.00	300.00	0.0
15.00	40.00	3.17	100.00	266.67	366.67	0.0
15.00	50.00	3.15	100.00	266.67	366.67	0.0
20.00	10.00	4.11	100.00	0.0	100.00	0.0
20.00	20.00	4.10	100.00	100.00	200.00	0.0
20.00	30.00	4.09	100.00	200.00	300.00	0.0
20.00	40.00	4.08	100.00	300.00	400.00	0.0
20.00	50.00	3.70	100.00	400.00	500.00	0.0
25.00	10.00	4.11	100.00	0.0	100.00	0.0
25.00	20.00	4.10	100.00	100.00	200.00	0.0
25.00	30.00	4.09	100.00	200.00	300.00	0.0
25.00	40.00	4.08	100.00	300.00	400.00	0.0
25.00	50.00	4.08	100.00	400.00	500.00	0.0

Price sensitivity to Supply with $E = 25$. Black-Scholes price = \$ 3.48

Table 11

Modes	τ_g	τ_o	Eqlbm Pr.	δ	D_{naked}	D_{Cover}	S_{naked}	S_{Cover}
15.00	0.00	0.30	8.30	0.81	100.00	266.67	100.00	533.33
15.00	0.03	0.30	8.42	0.92	100.00	216.34	100.00	216.34
15.00	0.06	0.30	8.45	0.95	100.00	43.51	100.00	43.51
15.00	0.09	0.30	8.44	0.94	100.00	0.00	100.00	0.00
15.00	0.12	0.30	8.46	0.96	100.00	0.00	100.00	0.00
15.00	0.15	0.30	8.54	1.04	100.00	0.00	100.00	0.00
15.00	0.18	0.30	8.49	1.00	100.00	0.00	100.00	0.00
15.00	0.21	0.30	8.43	0.93	100.00	0.00	100.00	0.00
15.00	0.24	0.30	8.26	0.77	100.00	0.00	100.00	0.00
15.00	0.27	0.30	8.22	0.72	100.00	0.00	100.00	0.00
15.00	0.30	0.30	8.20	0.70	100.00	0.00	100.00	0.00

Impact of the long-term capital gains tax rates. $E = 15$, Modes at 15. **BS** = \$ 7.50.

Table 12

Modes	τ_g	τ_o	Eqlbm Pr.	δ	D_{naked}	D_{Cover}	S_{naked}	S_{Cover}
20.00	0.00	0.30	8.40	0.90	100.00	400.00	100.00	400.00
20.00	0.03	0.30	8.44	0.94	100.00	274.59	100.00	274.59
20.00	0.06	0.30	8.45	0.95	100.00	61.52	100.00	61.52
20.00	0.09	0.30	8.44	0.94	100.00	0.00	100.00	0.00
20.00	0.12	0.30	8.46	0.96	100.00	0.00	100.00	0.00
20.00	0.15	0.30	8.55	1.05	100.00	0.00	100.00	0.00
20.00	0.18	0.30	8.50	1.00	100.00	0.00	100.00	0.00
20.00	0.21	0.30	8.49	1.00	100.00	0.00	100.00	0.00
20.00	0.24	0.30	8.54	1.04	100.00	0.00	100.00	0.00
20.00	0.27	0.30	8.56	1.07	100.00	0.00	100.00	0.00
20.00	0.30	0.30	8.52	1.02	100.00	0.00	100.00	0.00

Impact of the long-term capital gains tax rates. $E = 15$, Modes at 20. **BS** = \$ 7.50.

Table 13

Modes	τ_g	τ_o	Eqlbm Pr.	δ	D_{naked}	D_{Cover}	S_{naked}	S_{Cover}
25.00	0.00	0.30	8.51	1.01	100.00	533.33	100.00	266.67
25.00	0.03	0.30	8.45	0.95	100.00	205.66	100.00	205.66
25.00	0.06	0.30	8.45	0.95	100.00	41.73	100.00	41.73
25.00	0.09	0.30	8.44	0.94	100.00	0.00	100.00	0.00
25.00	0.12	0.30	8.46	0.96	100.00	0.00	100.00	0.00
25.00	0.15	0.30	8.55	1.05	100.00	0.00	100.00	0.00
25.00	0.18	0.30	8.56	1.06	100.00	0.00	100.00	0.00
25.00	0.21	0.30	8.57	1.07	100.00	0.00	100.00	0.00
25.00	0.24	0.30	8.57	1.07	100.00	0.00	100.00	0.00
25.00	0.27	0.30	8.58	1.08	100.00	0.00	100.00	0.00
25.00	0.30	0.30	8.55	1.05	100.00	0.00	100.00	0.00

Impact of the long-term capital gains tax rates. $E = 15$, Modes at 25. **BS** = \$ 7.50.

Table 14

Modes	τ_g	τ_o	Eqlbm Pr.	δ	D_{naked}	D_{Cover}	S_{naked}	S_{Cover}
15.00	0.00	0.30	5.47	0.46	100.00	266.67	100.00	533.33
15.00	0.03	0.30	5.64	0.62	100.00	266.67	100.00	266.67
15.00	0.06	0.30	5.80	0.78	100.00	233.35	100.00	233.35
15.00	0.09	0.30	5.85	0.83	100.00	86.70	100.00	86.70
15.00	0.12	0.30	5.87	0.86	100.00	0.00	100.00	0.00
15.00	0.15	0.30	5.88	0.86	100.00	0.00	100.00	0.00
15.00	0.18	0.30	5.89	0.87	100.00	0.00	100.00	0.00
15.00	0.21	0.30	5.89	0.88	100.00	0.00	100.00	0.00
15.00	0.24	0.30	5.91	0.90	100.00	0.00	100.00	0.00
15.00	0.27	0.30	5.93	0.92	100.00	0.00	100.00	0.00
15.00	0.30	0.30	5.94	0.92	100.00	0.00	100.00	0.00

Impact of the long-term capital gains tax rates. $E = 20$, Modes at 15. **BS** = \$ 5.02.

Table 15

Modes	τ_g	τ_o	Eqlbm Pr.	δ	D_{naked}	D_{Cover}	S_{naked}	S_{Cover}
20.00	0.00	0.30	5.62	0.61	100.00	400.00	100.00	400.00
20.00	0.03	0.30	5.76	0.74	100.00	400.00	100.00	400.00
20.00	0.06	0.30	5.82	0.81	100.00	278.10	100.00	278.10
20.00	0.09	0.30	5.85	0.84	100.00	116.83	100.00	116.83
20.00	0.12	0.30	5.87	0.85	100.00	0.00	100.00	0.00
20.00	0.15	0.30	5.91	0.89	100.00	0.00	100.00	0.00
20.00	0.18	0.30	5.89	0.87	100.00	0.00	100.00	0.00
20.00	0.21	0.30	5.92	0.90	100.00	0.00	100.00	0.00
20.00	0.24	0.30	5.93	0.91	100.00	0.00	100.00	0.00
20.00	0.27	0.30	5.93	0.92	100.00	0.00	100.00	0.00
20.00	0.30	0.30	5.94	0.92	100.00	0.00	100.00	0.00

Impact of the long-term capital gains tax rates. $E = 20$, Modes at 20. **BS** = \$ 5.02.

Table 16

Modes	τ_g	τ_o	Eqlbm Pr.	δ	D_{naked}	D_{Cover}	S_{naked}	S_{Cover}
25.00	0.00	0.30	5.88	0.86	100.00	533.33	100.00	266.67
25.00	0.03	0.30	5.86	0.84	100.00	266.67	100.00	266.67
25.00	0.06	0.30	5.84	0.82	100.00	199.20	100.00	199.20
25.00	0.09	0.30	5.86	0.84	100.00	80.77	100.00	80.77
25.00	0.12	0.30	5.87	0.85	100.00	0.00	100.00	0.00
25.00	0.15	0.30	5.91	0.89	100.00	0.00	100.00	0.00
25.00	0.18	0.30	5.89	0.87	100.00	0.00	100.00	0.00
25.00	0.21	0.30	5.92	0.90	100.00	0.00	100.00	0.00
25.00	0.24	0.30	5.93	0.92	100.00	0.00	100.00	0.00
25.00	0.27	0.30	5.93	0.92	100.00	0.00	100.00	0.00
25.00	0.30	0.30	5.94	0.92	100.00	0.00	100.00	0.00

Impact of the long-term capital gains tax rates. $E = 20$, Modes at 25. **BS** = \$ 5.02.

Table 17

Modes	τ_g	τ_o	Eqlbm Pr.	δ	D_{naked}	D_{Cover}	S_{naked}	S_{Cover}
15.00	0.00	0.30	4.10	0.62	100.00	266.67	100.00	0.00
15.00	0.03	0.30	4.10	0.62	100.00	0.00	100.00	0.00
15.00	0.06	0.30	4.11	0.62	100.00	0.00	100.00	0.00
15.00	0.09	0.30	4.11	0.63	100.00	0.00	100.00	0.00
15.00	0.12	0.30	4.11	0.63	100.00	0.00	100.00	0.00
15.00	0.15	0.30	4.11	0.63	100.00	0.00	100.00	0.00
15.00	0.18	0.30	4.11	0.63	100.00	0.00	100.00	0.00
15.00	0.21	0.30	4.12	0.64	100.00	0.00	100.00	0.00
15.00	0.24	0.30	4.11	0.63	100.00	0.00	100.00	0.00
15.00	0.27	0.30	4.12	0.64	100.00	0.00	100.00	0.00
15.00	0.30	0.30	4.13	0.65	100.00	0.00	100.00	0.00

Impact of the long-term capital gains tax rates. $E = 25$, Modes at 15. **BS** = \$ 3.48.

Table 18

Modes	τ_g	τ_o	Eqlbm Pr.	δ	D_{naked}	D_{Cover}	S_{naked}	S_{Cover}
20.00	0.00	0.30	4.10	0.62	100.00	400.00	100.00	0.00
20.00	0.03	0.30	4.10	0.62	100.00	0.00	100.00	0.00
20.00	0.06	0.30	4.11	0.62	100.00	0.00	100.00	0.00
20.00	0.09	0.30	4.11	0.63	100.00	0.00	100.00	0.00
20.00	0.12	0.30	4.11	0.63	100.00	0.00	100.00	0.00
20.00	0.15	0.30	4.11	0.63	100.00	0.00	100.00	0.00
20.00	0.18	0.30	4.11	0.63	100.00	0.00	100.00	0.00
20.00	0.21	0.30	4.12	0.63	100.00	0.00	100.00	0.00
20.00	0.24	0.30	4.12	0.64	100.00	0.00	100.00	0.00
20.00	0.27	0.30	4.13	0.64	100.00	0.00	100.00	0.00
20.00	0.30	0.30	4.13	0.64	100.00	0.00	100.00	0.00

Impact of the long-term capital gains tax rates. $E = 25$, Modes at 20. **BS** = \$ 3.48.

Table 19

Modes	τ_g	τ_o	Eqlbm Pr.	δ	D_{naked}	D_{Cover}	S_{naked}	S_{Cover}
25.00	0.00	0.30	4.10	0.62	66.67	0.00	100.00	266.67
25.00	0.03	0.30	4.10	0.62	100.00	0.00	100.00	0.00
25.00	0.06	0.30	4.11	0.62	100.00	0.00	100.00	0.00
25.00	0.09	0.30	4.11	0.63	100.00	0.00	100.00	0.00
25.00	0.12	0.30	4.11	0.63	100.00	0.00	100.00	0.00
25.00	0.15	0.30	4.11	0.63	100.00	0.00	100.00	0.00
25.00	0.18	0.30	4.12	0.63	100.00	0.00	100.00	0.00
25.00	0.21	0.30	4.12	0.64	100.00	0.00	100.00	0.00
25.00	0.24	0.30	4.12	0.64	100.00	0.00	100.00	0.00
25.00	0.27	0.30	4.13	0.64	100.00	0.00	100.00	0.00
25.00	0.30	0.30	4.13	0.65	100.00	0.00	100.00	0.00

Impact of the long-term capital gains tax rates. $E = 25$, Modes at 25. **BS** = \$ 3.48.

Table 20

Ex. Pr.	Eqlbm Pr.	Blk-Schl	δ	D_{naked}	D_{Cover}	S_{naked}	S_{Cover}
10.00	11.12	10.95	0.17	1001.70	0.00	875.00	126.70
11.00	10.19	10.04	0.15	1041.53	0.00	925.00	116.53
12.00	9.27	9.14	0.13	1078.07	0.00	975.00	103.07
13.00	8.34	8.24	0.11	1118.65	0.00	1025.00	93.65
14.00	7.43	7.34	0.09	1134.34	0.00	1075.00	59.34
15.00	6.60	6.48	0.12	1215.24	0.00	1125.00	90.24
16.00	5.71	5.62	0.09	1250.00	0.00	1175.00	75.00
17.00	4.96	4.86	0.11	1243.20	0.00	1225.00	18.20
18.00	4.13	4.10	0.03	1250.00	25.00	1275.00	0.00
19.00	3.45	3.43	0.03	1250.00	75.00	1325.00	0.00
20.00	2.87	2.87	0.00	1250.00	125.00	1375.00	0.00
21.00	2.30	2.31	-0.01	1250.00	175.00	1425.00	0.00
22.00	1.83	1.86	-0.02	1250.00	225.00	1475.00	0.00
23.00	1.34	1.53	-0.19	1250.00	250.00	1500.00	0.00
24.00	1.04	1.19	-0.16	1250.00	250.00	1500.00	0.00
25.00	0.75	0.88	-0.13	1250.00	250.00	1500.00	0.00
26.00	0.62	0.72	-0.11	1250.00	250.00	1500.00	0.00
27.00	0.49	0.57	-0.09	1250.00	250.00	1500.00	0.00
28.00	0.35	0.42	-0.07	1250.00	250.00	1500.00	0.00
29.00	0.24	0.29	-0.05	1250.00	250.00	1500.00	0.00
30.00	0.22	0.24	-0.02	1250.00	250.00	1500.00	0.00

An Example of a decreasing strike-price bias.

Table 21

Ex. Pr.	Eqlbm Pr.	Blk-Schl	δ	D_{naked}	D_{Cover}	S_{naked}	S_{Cover}
10.00	10.56	10.95	-0.38	375.00	250.00	625.00	0.00
11.00	9.66	10.04	-0.38	375.00	250.00	625.00	0.00
12.00	8.74	9.14	-0.39	375.00	250.00	625.00	0.00
13.00	8.22	8.24	-0.01	375.00	250.00	625.00	0.00
14.00	7.32	7.34	-0.02	375.00	250.00	625.00	0.00
15.00	6.47	6.48	-0.01	375.00	250.00	625.00	0.00
16.00	5.52	5.62	-0.09	375.00	250.00	625.00	0.00
17.00	4.68	4.86	-0.18	450.00	175.00	625.00	0.00
18.00	4.00	4.10	-0.10	466.67	158.33	625.00	0.00
19.00	3.39	3.43	-0.04	483.33	141.67	625.00	0.00
20.00	2.87	2.87	0.00	500.00	125.00	625.00	0.00
21.00	2.35	2.31	0.04	525.00	100.00	625.00	0.00
22.00	1.91	1.86	0.05	550.00	75.00	625.00	0.00
23.00	1.56	1.53	0.03	554.37	70.44	625.00	0.00
24.00	1.22	1.19	0.02	625.00	0.00	625.00	0.00
25.00	0.95	0.88	0.07	625.00	0.00	625.00	0.00
26.00	0.78	0.72	0.06	625.00	0.00	625.00	0.00
27.00	0.61	0.57	0.04	625.00	0.00	625.00	0.00
28.00	0.44	0.42	0.02	625.00	0.00	625.00	0.00
29.00	0.33	0.29	0.04	625.00	0.00	625.00	0.00
30.00	0.27	0.24	0.03	625.00	0.00	625.00	0.00

An Example of an increasing strike-price bias.

Table 22

Appendix 3

Plots related to the numerical examples

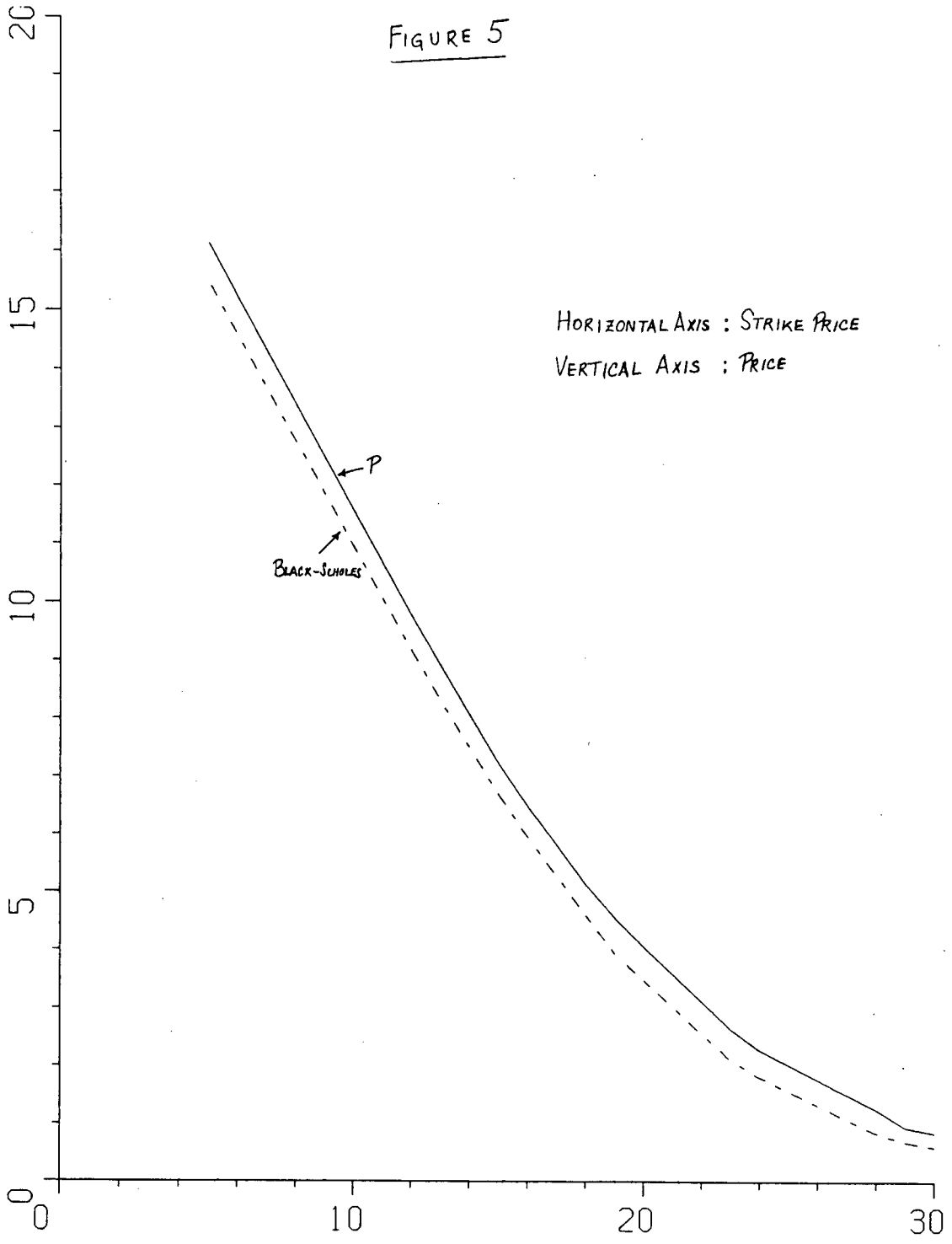


FIGURE 5
HORIZONTAL AXIS : STRIKE PRICE
VERTICAL AXIS : PRICE
P
BLACK-SCHOLES
0
5
10
15
20
0
10
20
30

FIGURE 6

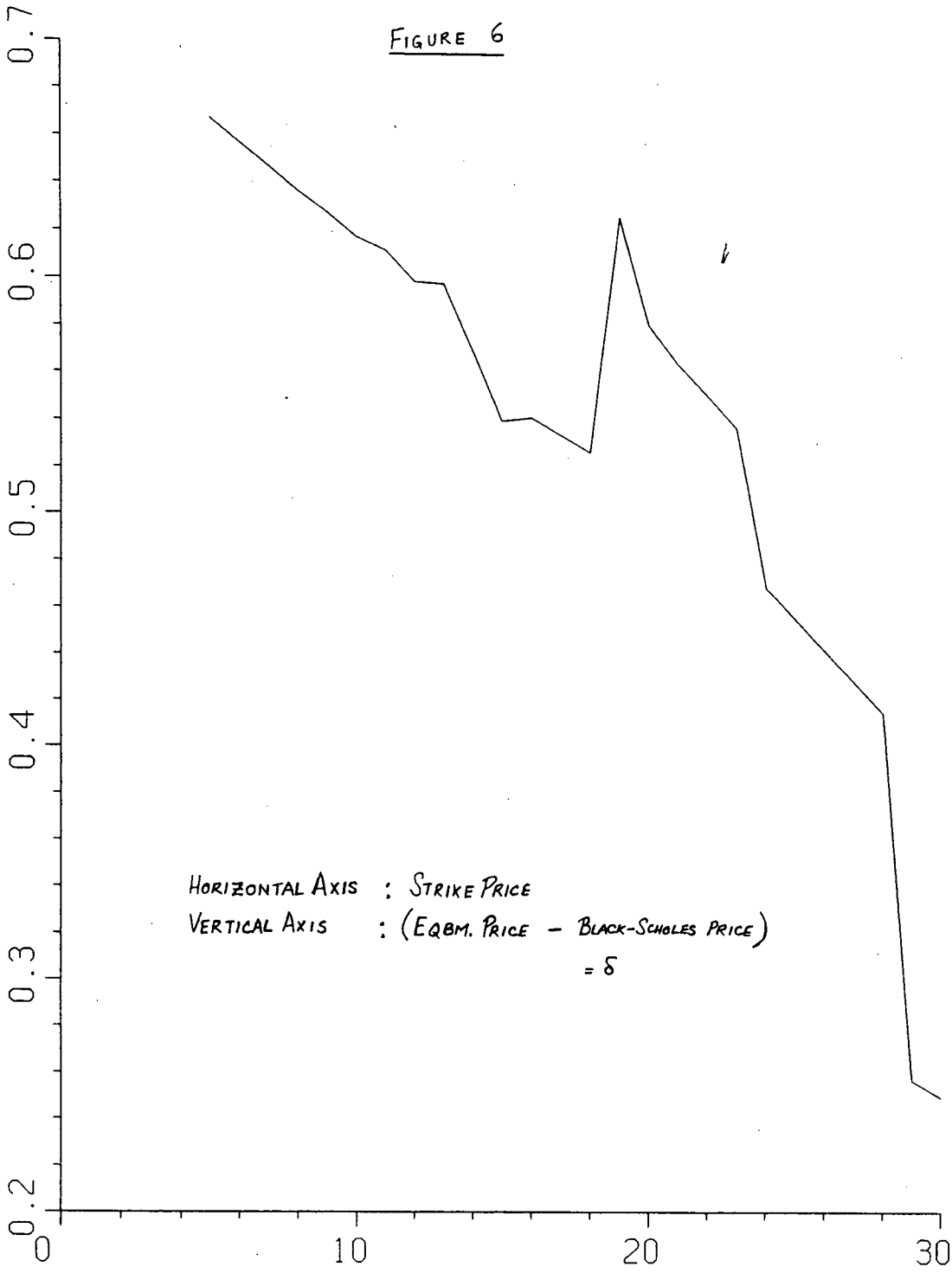

0.7
0.6
0.5
0.4
0.3
0.2
0
10
20
30
HORIZONTAL AXIS : STRIKE PRICE
VERTICAL AXIS : (EQBM. PRICE − BLACK-SCHOLES PRICE)
= δ

FIGURE 7

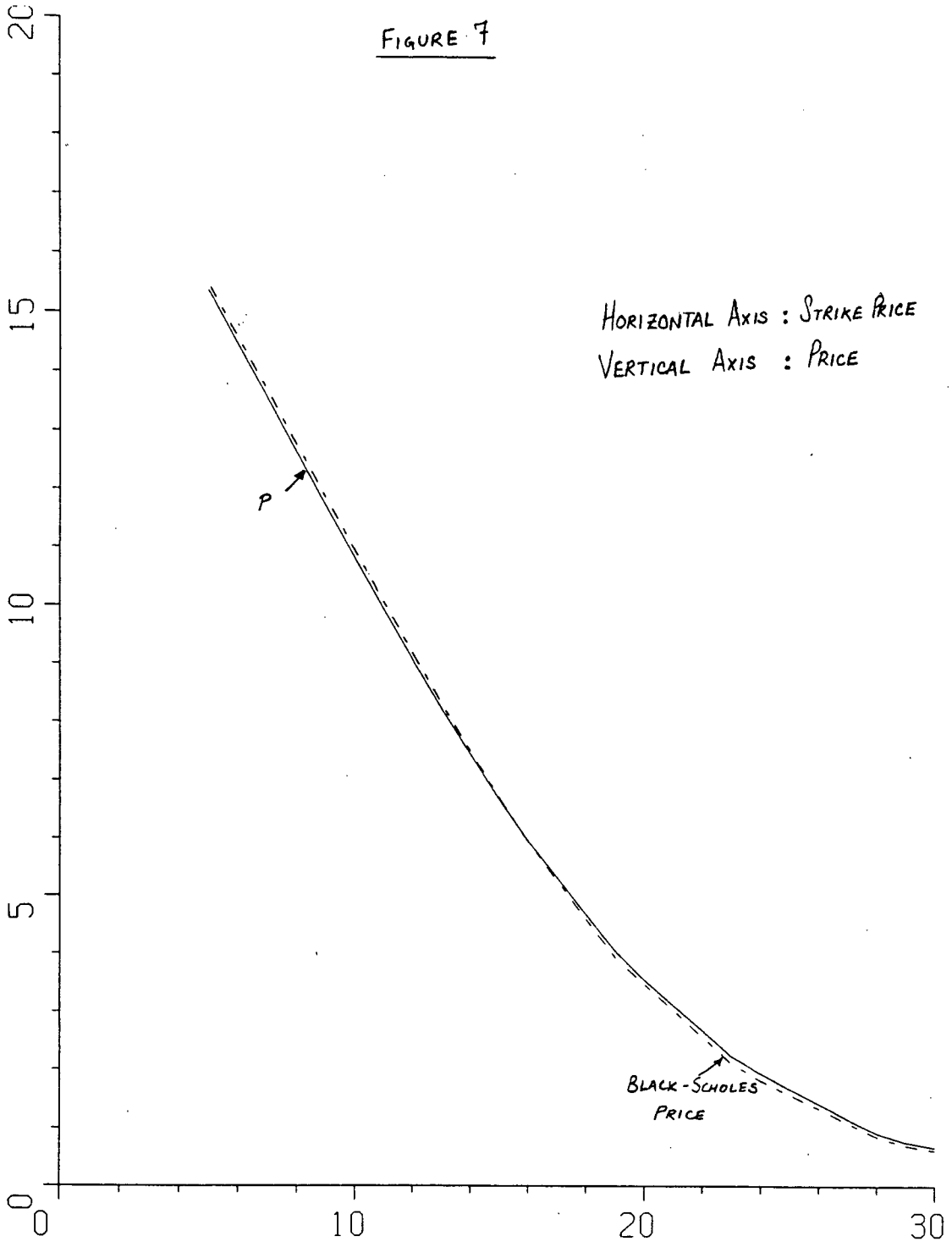
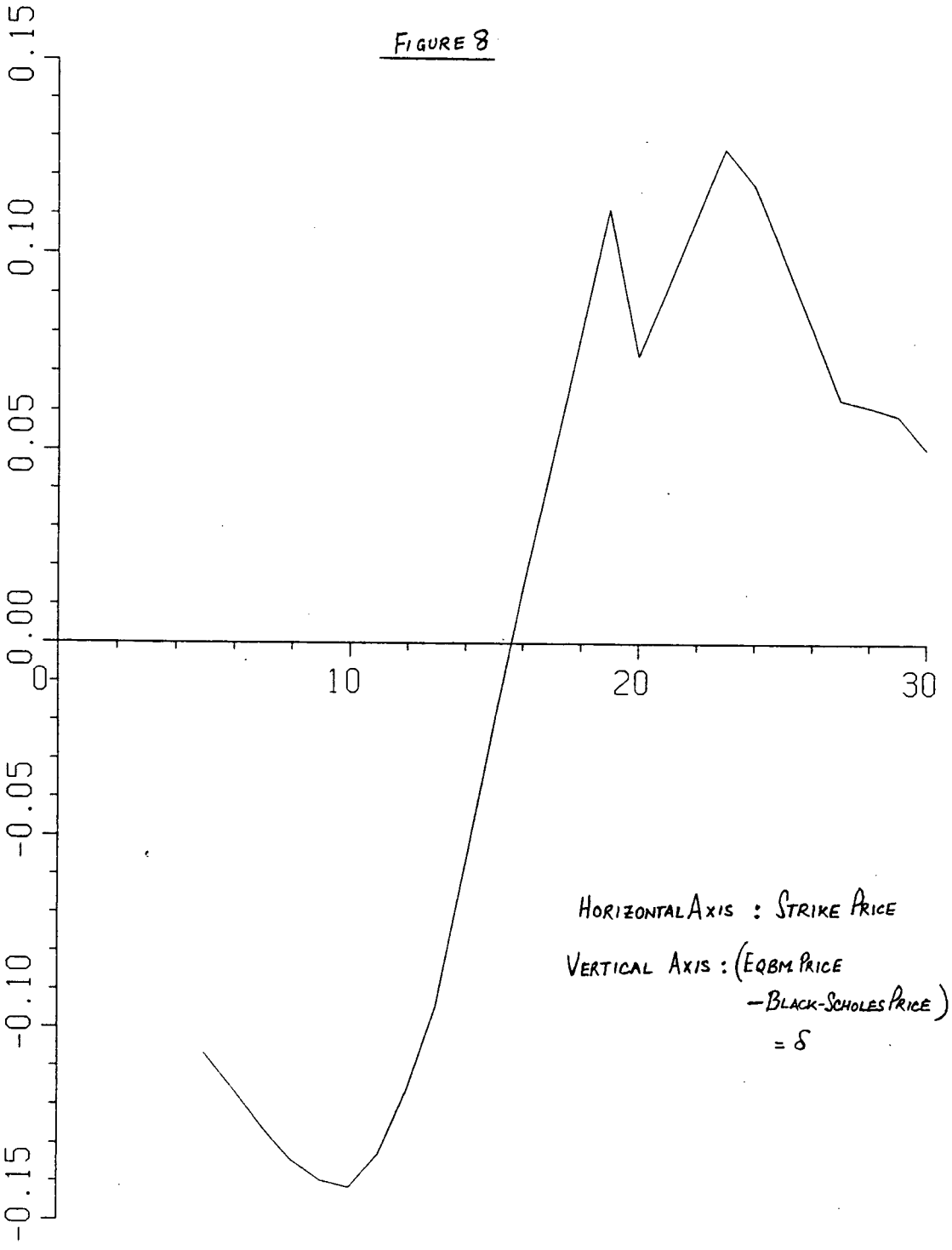

HORIZONTAL AXIS : STRIKE PRICE
VERTICAL AXIS : PRICE
P
BLACK-SCHOLES PRICE
0
5
10
15
20
0
10
20
30

FIGURE 8


0.15
0.10
0.05
0.00
-0.05
-0.10
-0.15
0
10
20
30
HORIZONTAL AXIS : STRIKE PRICE
VERTICAL AXIS : (EQBM PRICE
- BLACK-SCHOLES PRICE)
= δ

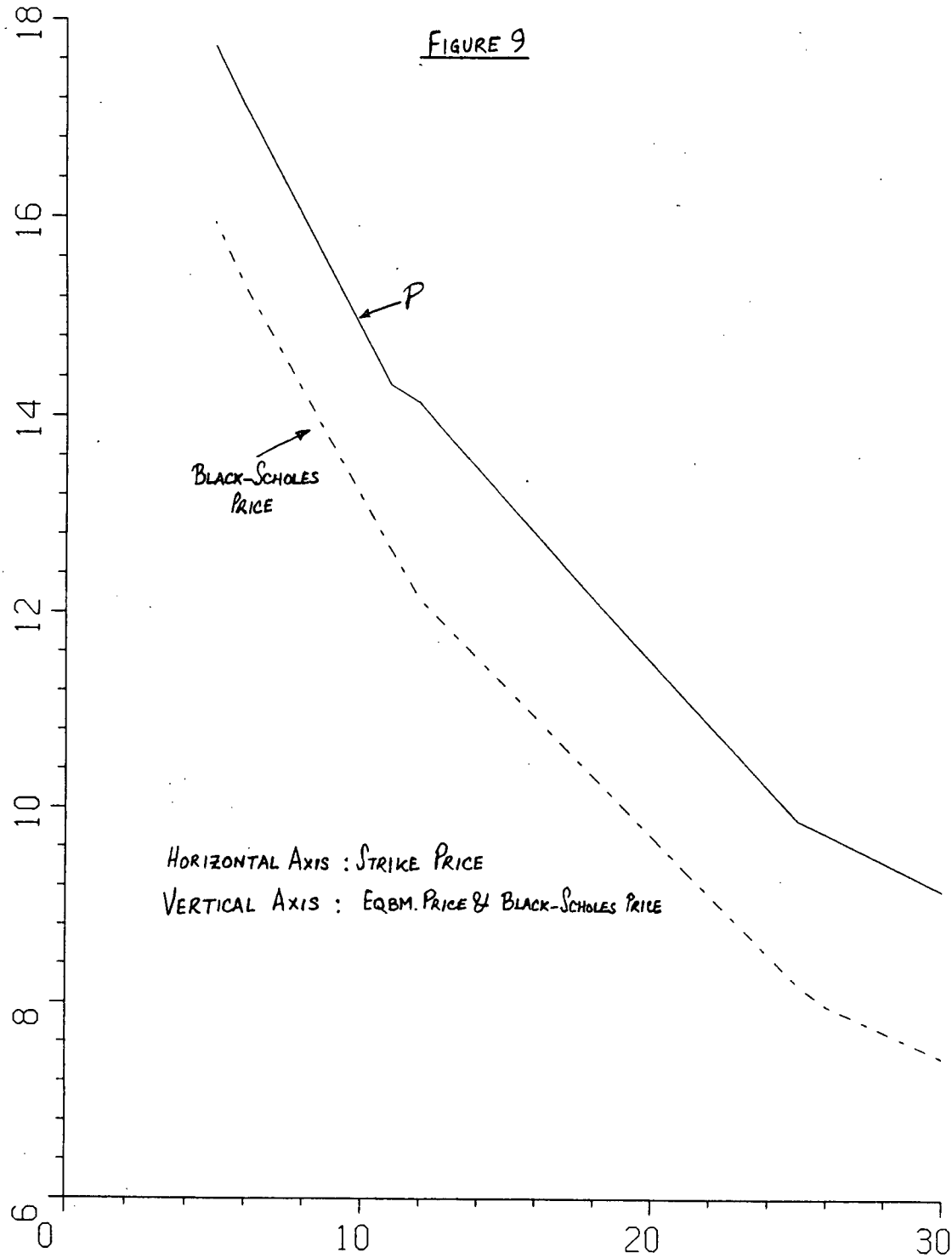


FIGURE 9
P
BLACK-SCHOLES PRICE
HORIZONTAL AXIS : STRIKE PRICE
VERTICAL AXIS : EQBM. PRICE & BLACK-SCHOLES PRICE
18
16
14
12
10
8
6
0
10
20
30

FIGURE 10

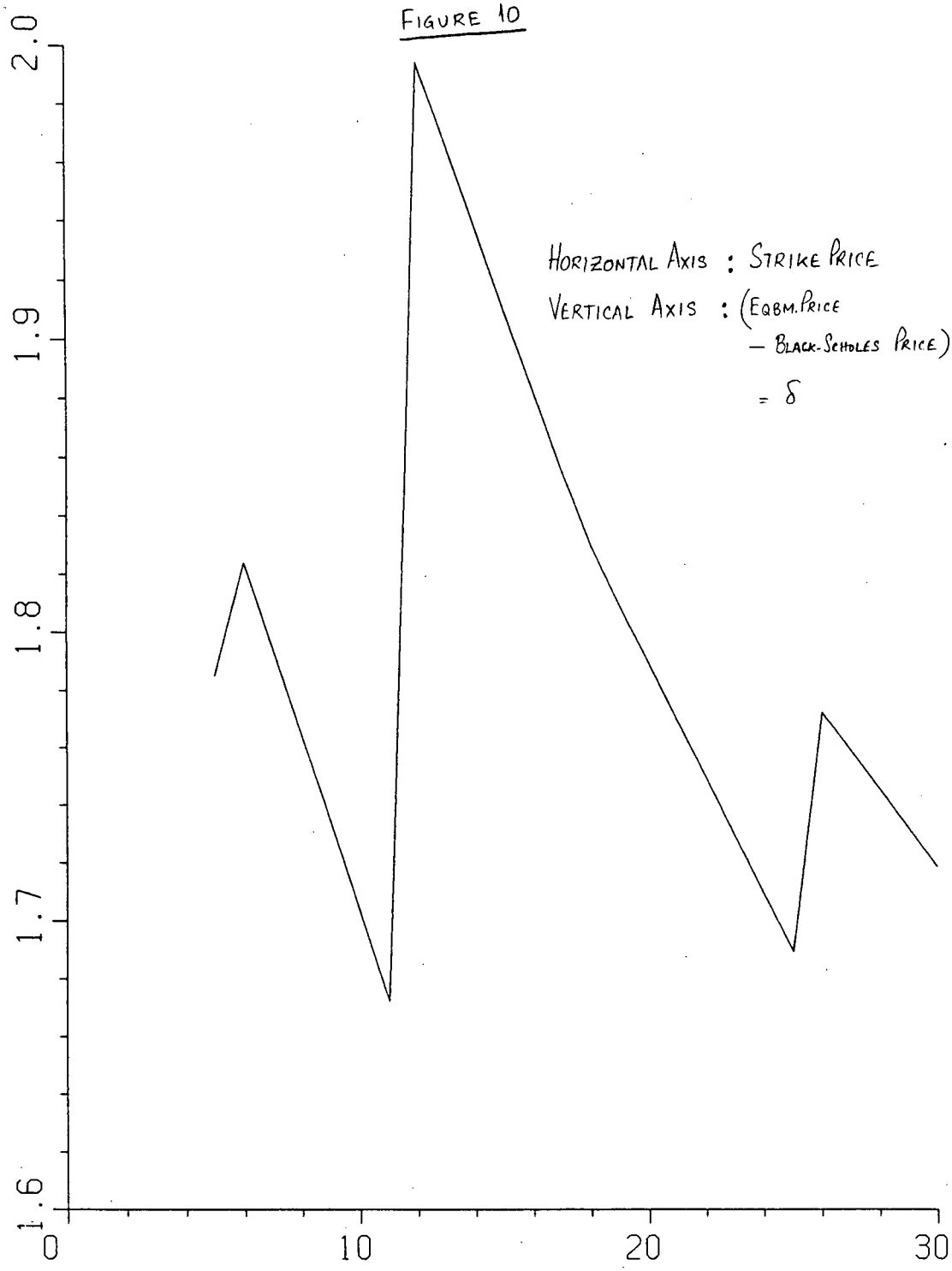
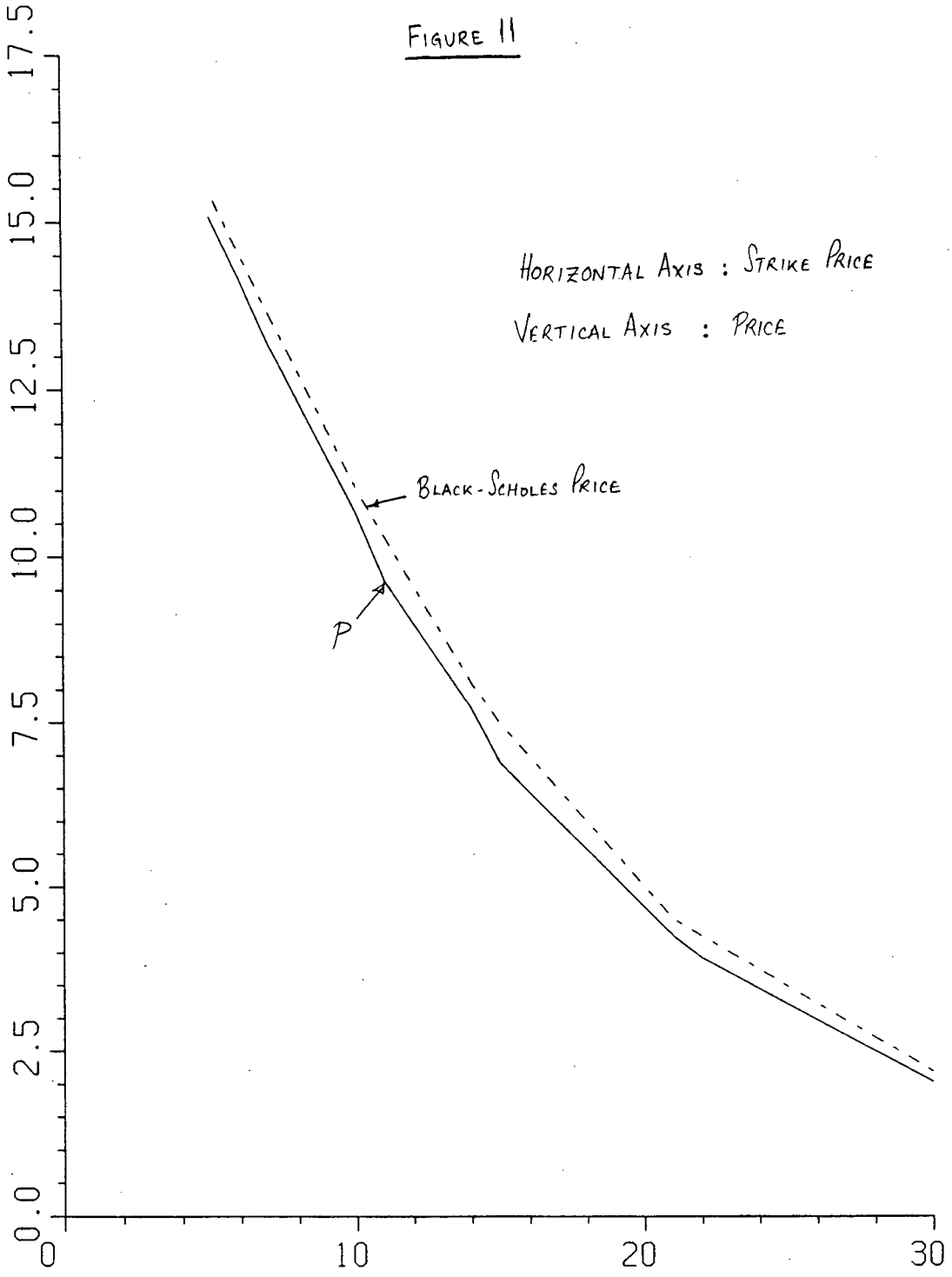

2.0
1.9
1.8
1.7
1.6
0
10
20
30
HORIZONTAL AXIS : STRIKE PRICE
VERTICAL AXIS : (EQBM. PRICE
— BLACK-SCHOLES PRICE)
= δ

FIGURE 11


17.5
15.0
12.5
10.0
7.5
5.0
2.5
0.0
0
10
20
30
HORIZONTAL AXIS : STRIKE PRICE
VERTICAL AXIS : PRICE
BLACK-SCHOLES PRICE
P

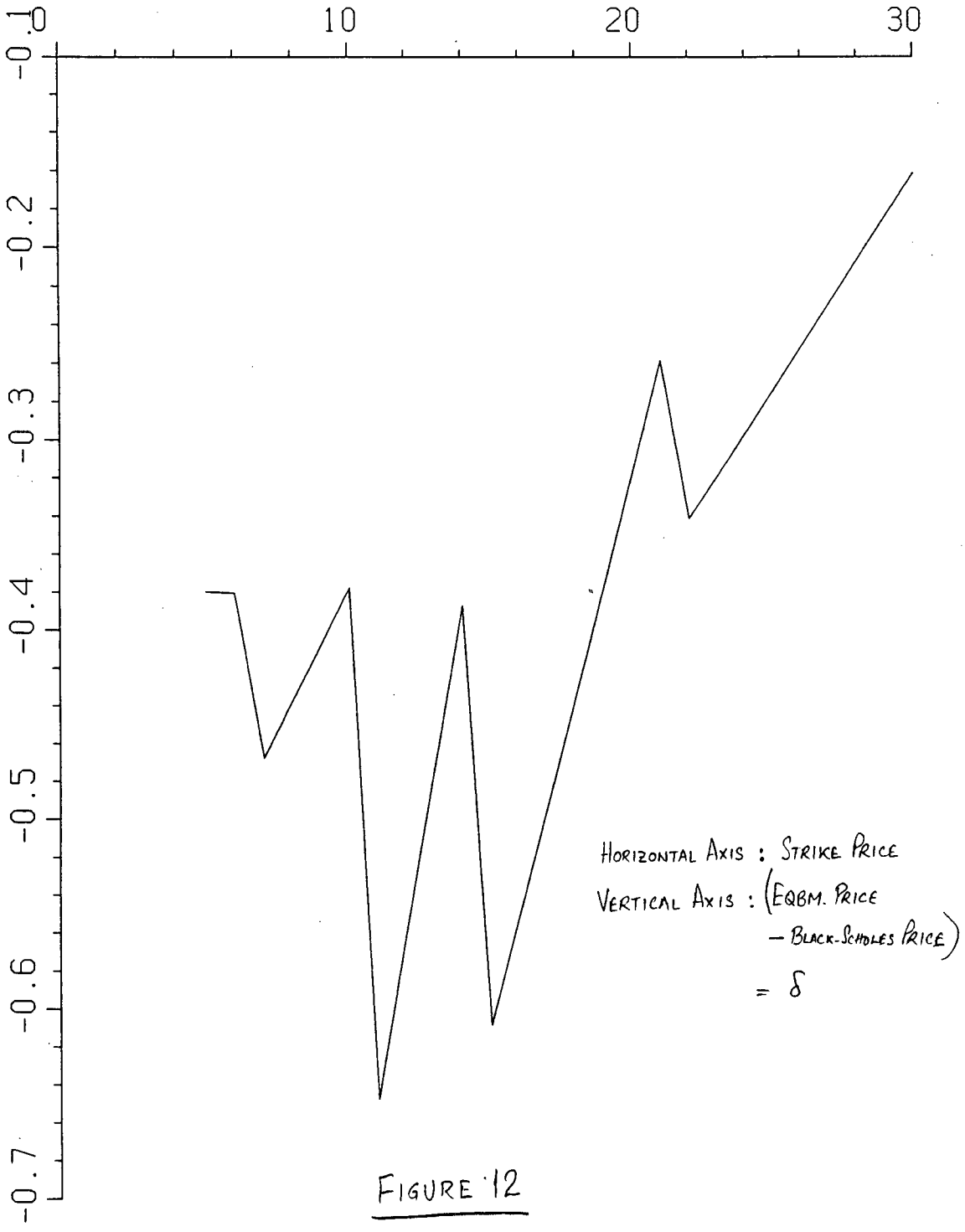

0
10
20
30
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.7
HORIZONTAL AXIS : STRIKE PRICE
VERTICAL AXIS : (EQBM. PRICE
– BLACK-SCHOLES PRICE)
= δ

FIGURE 12

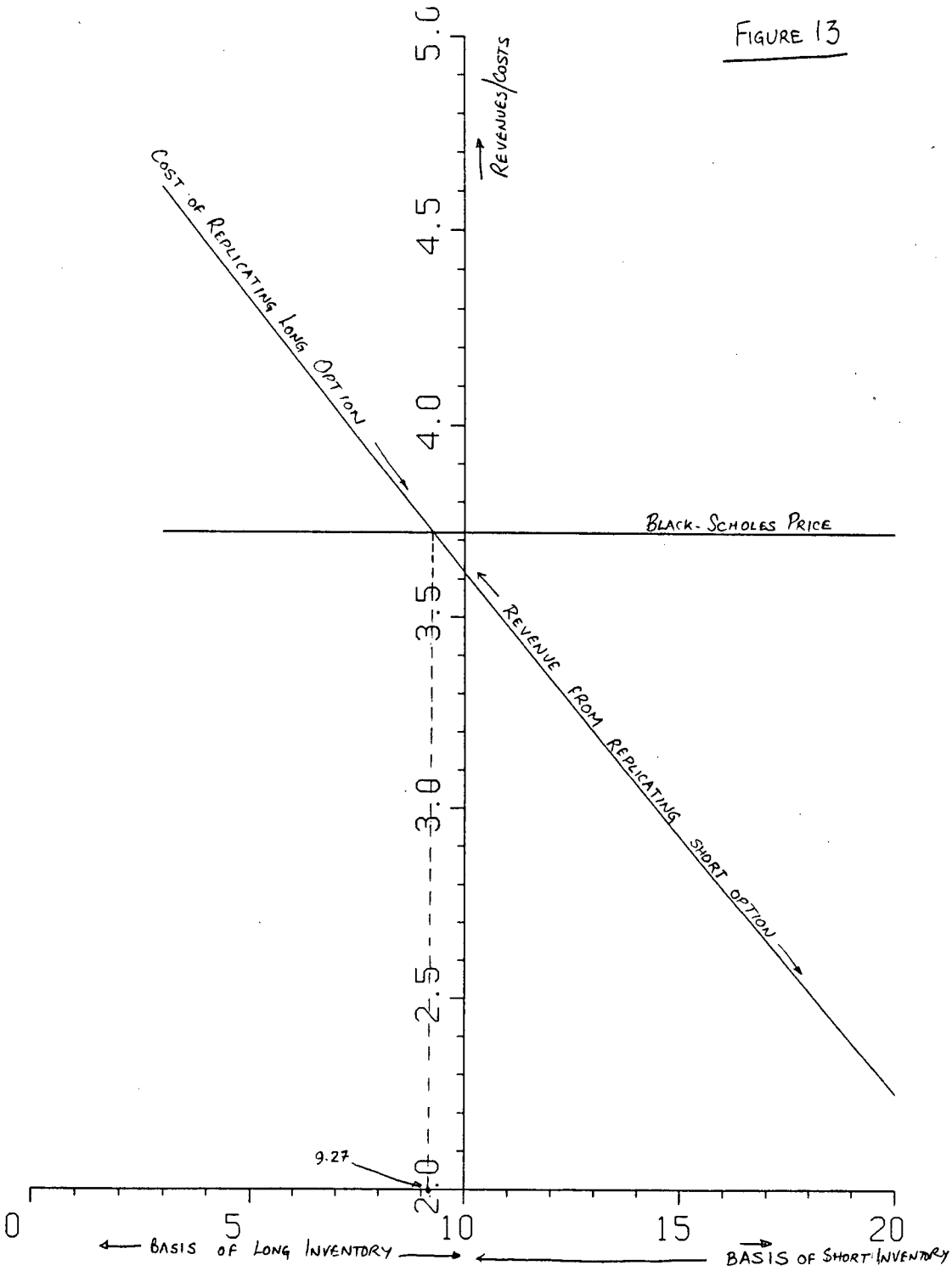


FIGURE 13
REVENUES/COSTS
5.0
4.5
4.0
3.5
3.0
2.5
2.0
COST OF REPLICATING LONG OPTION
BLACK-SCHOLES PRICE
REVENUE FROM REPLICATING SHORT OPTION
9.27
0
5
10
15
20
BASIS OF LONG INVENTORY
BASIS OF SHORT INVENTORY

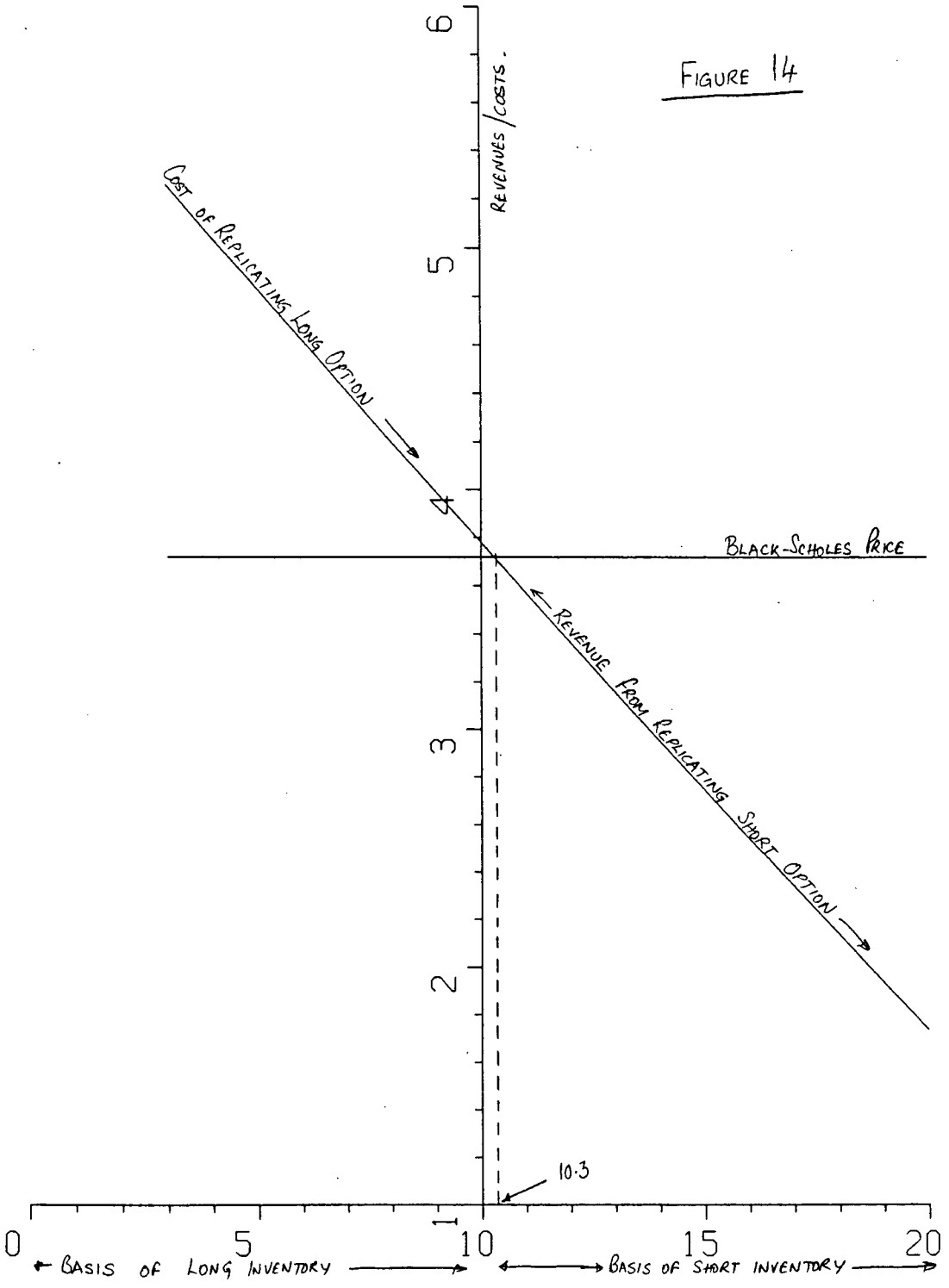


Figure 14
Revenues / Costs.
6
5
4
3
2
1
Cost of Replicating Long Option
Black-Scholes Price
Revenue from Replicating Short Option
10·3
0
5
10
15
20
← Basis of Long Inventory →
← Basis of Short Inventory →

Appendix 4

Notation used in **"The impact of taxes on the pricing of call options"**

$\mathcal{A}$ = The set of investors who desire the payoffs generated by holding a particular call option.

$\mathcal{B}$ = The set of investors who desire the payoffs generated by writing a particular call option.

$\mathcal{C}$ = Those who do not belong to $\mathcal{A}$ or $\mathcal{B}$.

$\mathcal{D}$ = The set of investors who are holding onto stocks that they bought sometime ago at prices below the current stock price. They desist from selling them for fear of realising capital gains taxes at the capital gains tax rate which is assumed to be less than the ordinary income tax rate.

$\mathcal{E}$ = The set of investors who have a short position on stocks that they sold at prices in excess of the current market price. They desist from closing out their position for fear of realising capital gains taxes.

$\mathcal{F}$ = The set of investors who do not belong to $\mathcal{D}$ or $\mathcal{E}$.

R = The pre-tax rate of return on a riskless bond.

R^* = The after-tax rate of return on a riskless bond.

τ_o = The ordinary income tax rate.

τ_g = The capital gains tax rate.

S_B = The basis of an investor's stock holding for computation of capital gains taxes.

S = The stock price.

u = The magnitude of an "up" move in the stock price.

d = The magnitude of an "down" move in the stock price.

E = The exercise price of a call option.

N = The number of periods to maturity of the call option.

S_N = The stock price at maturity of the option.

S_0 = The stock price at the commencement of the option.

S_{N-1} = The stock price at the start of the maturity period of the option.

C^u = The payoff to the option at termination if the stock price moves from S_{N-1} to uS_{N-1}.

C^d = The payoff to the option at termination if the stock price moves from S_{N-1} to dS_{N-1}.

α = The value of the stock position in a replicating portfolio at the start of the period when the portfolio is formed.

y = The value of the bond position in a replicating portfolio at the start of the period when the portfolio is formed.

Φ_{N-1} = The total value of a replicating portfolio at the start of the period when the portfolio is formed.

$I(\bullet)$ = An indicator function that takes on a value of unity when its argument is **true** and a value of zero otherwise.

HSO = The cost of replicating a long option position for an investor who has a basis of S_B, and marginal tax rates of τ_o and τ_g.

WSO = The revenue from replicating a short option position for an investor who has a basis of S_B, and marginal tax rates of τ_o and τ_g.

CSO = The cost of covering a short option position for an investor who has a basis of S_B, and marginal tax rates of τ_o and τ_g.

CLO = The revenue from covering a long option position for an investor who has a basis of S_B, and marginal tax rates of τ_o and τ_g.

P = The equilibrium price of the option.

BS = The Black-Scholes price of the option.

η = The cost of replicating a long option position for an investor who has a basis of S_0, and marginal tax rates of τ_o and τ_g.

δ = The difference between the equilibrium price and the Black-Scholes price of the option.

μ = The cost of replicating a short option position for an investor who has a basis of S_0, and marginal tax rates of τ_o and τ_g.

ν = The cost of covering a short option position for an investor who has a basis of S_0, and marginal tax rates of τ_o and τ_g.

ξ = The revenue from covering a long option position for an investor who has a ba-

sis of S_0, and marginal tax rates of τ_o and τ_g.

$\mathcal{F}_H$ = The cost of replicating a long option position for an investor in $\mathcal{F}$.

$\mathcal{F}_W$ = The revenue from replicating a short option position for an investor in $\mathcal{F}$.

$\mathcal{F}_{CSO}$ = The cost of covering a short option position for an investor in $\mathcal{F}$.

$\mathcal{F}_{CLO}$ = The revenue from covering a long option position for an investor in $\mathcal{F}$.

$\mathcal{A}(x)$ = The density function of the distribution of stock bases of investors in $\mathcal{A}$.

$\mathcal{B}(x)$ = The density function of the distribution of stock bases of investors in $\mathcal{B}$.

$\mathcal{C}(x)$ = The density function of the distribution of stock bases of investors in $\mathcal{C}$.

ϱ = This is defined in Equation 2.

$\theta(\bullet)$ = This is defined in Equation 2.

ω = This is defined in Equation 3.

$\lambda(\bullet)$ = This is defined in Equation 3.

ς = This is defined in Equation 4.

$\Gamma(\bullet)$ = This is defined in Equation 4.

F^u = This is defined in Equation 4.

F^d = This is defined in Equation 4.

ψ = This is defined in Equation 5.

$\Lambda(\bullet)$ = This is defined in Equation 5.

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