DYNAMIC REDESIGN OF MODIFIED STRUCTURES

by

P.A. Welch

B.A.Sc., The University of British Columbia, 1985

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF APPLIED SCIENCE

in

THE FACULTY OF GRADUATE STUDIES

DEPARTMENT OF MECHANICAL ENGINEERING

We accept this paper as conforming

to the required standard

The University of British Columbia

August 1987

In presenting this thesis in partial fulfilment of the requirements for an advanced degree at the University of British Columbia, I agree that the Library shall make it freely available for reference and study. I further agree that permission for extensive copying of this thesis for scholarly purposes may be granted by the head of my department or by his or her representatives. It is understood that copying or publication of this thesis for financial gain shall not be allowed without my written permission.

Department of Mechanical Engineering

The University of British Columbia
1956 Main Mall
Vancouver, Canada
V6T 1Y3

Date August 20, 1987

ABSTRACT

A procedure is described for the redesign of undamped unforced linear structural systems to meet specified changes in natural frequency or mode shape. A baseline analysis is conducted using finite elements to obtain a subset of the natural frequencies and mode shapes. A two stage perturbation analysis is then used to obtain the structural changes required to meet the specified changes in natural frequency or mode shape.

TABLE OF CONTENTS

Page

APPENDICES

LIST OF FIGURES

LIST OF TABLES

NOMENCLATURE

$[M]$ = n x n baseline mass matrix

$[K]$ = n x n baseline stiffness matrix

$\{\phi_i\}$ = n x 1 eigenvector corresponding to i-th mode.

ω_i = natural frequency corresponding to i-th mode.

$[\Phi]$ = n x n modal matrix containing the $\{\phi_i\}$'s as columns.

$[\ \omega^2\ ]$ = n x n frequency matrix containing the ω_i^2 as diagonal entries.

$[\ \bar{M}\ ]$ = n x n generalized mass matrix.

$[\ \bar{K}\ ]$ = n x n generalized stiffness matrix.

$[\ \Delta M\ ]$ = n x n change to mass matrix due to change of structural parameters.

$[\ \Delta K\ ]$ = n x n change to stiffness matrix due to change of structural parameters.

$[\ M'\ ]$ = n x n mass matrix of the modified system.

$[\ K'\ ]$ = n x n stiffness matrix of the modified system.

$[\ \Phi'\ ]$ = n x n modal matrix of the modified system.

$[\ \omega^2\ ]$ = n x n frequency matrix of the modified system.

$[\ \Delta\Phi\ ]$ = n x n change to modal matrix due to change of structural parameters.

$[\ \Delta\omega^2\ ]$ = n x n change to frequency matrix due to change of structural parameters.

ϕ_{ki} = k-th degree of freedom of i-th mode.

$[C]$ = n x n matrix where c_{ij} represents the participation of the j-th baseline mode to changes in the i-th modified mode.

α_{ep} = fractional change required in property p of element e to bring about the desired frequency shift.

α_{ep}^{L} = Linear approximation to α_{ep} as given by the linear perturbation equation.

α_{ep}^{G} = Final approximation to α_{ep} as given by the general perturbation equation.

$\{\phi_i^L\}$ = n x 1 Linear approximation to the i'th eigenvector of the changed system.

$[k_e]$ = r x r stiffness matrix of element e.

$[m_e]$ = r x r mass matrix of element e.

$[k_{ep}]$ = r x r matrix by which the change in the elemental stiffness matrix which results from the structural change α_{ep} is given by $[\Delta k_e] = \alpha_{ep} [k_{ep}]$.

$[m_{ep}]$ = r x r matrix by which the change in the elemental mass matrix which results from the structural change α_{ep} is given by $[\Delta m_e] = \alpha_{ep} [m_{ep}]$.

$\{\theta_i\}$ = r x 1 reduced eigenvector for the i'th mode with elements corresponding to the global d.o.f. of the r x r matrices.

CHAPTER 1

INTRODUCTION

In many structural problems the criteria for an acceptable design involve constraints on the free vibration characteristics of a structural system. these constraints may be on one or more of the natural frequencies or the mode shapes. This work will be primarily concerned with structural problems involving frequency constraints only. In most cases the preliminary design does not satisfy all of the required constraints and reanalysis is required. There are two different approaches for this reanalysis procedure. One is to use a direct structural modification procedure in which the new frequencies resulting from a certain structural change are calculated using a method such as modal condensation. This is less time consuming than re-running the finite element program but is still a trial and error approach. The second method, which is employed in this work, is an inverse structural modification procedure in which the structural changes required to bring about a specified frequency shift are calculated. This procedure is based on a perturbation analysis of the basic undamped vibration equations of a discrete system. Consequently only one baseline finite element analysis is required.

A perturbation based inverse modification procedure has previously been implemented as a preliminary design program to interface with the finite element program NASTRAN [19]. This program, called LDRUM - Linear Dynamic Redesign University of Michigan [17], has been rewritten to interface with the finite element program VAST [18] as well as extended to provide improved and varied problem solving. This new program is called INSTRUM - INverse STRUctural Modification program.

1.1 Review of Previous Work

Historically redesign procedures involving only natural frequency were developed first. Since Rayleigh [1] proposed his procedure in 1893, several more methods have surfaced, and only recently have methods been developed that considered mode shape changes as well as frequency changes.

The goal of most frequency objective procedures is to minimize the mass of a structure given a specified frequency objective or to maximize the frequency of a structure for a certain total mass. Constraints are sometimes placed on design variables such as thickness of plates, cross-sectional area of bars, or moment of inertia of beams. Turner proposed one of the first methods to solve this problem [2]. Lagrange multipliers were used to solve the free vibration equations. Taylor solved the problem for an axially vibrating bar by using energy methods [3], while Sheu extended the work of Turner and Taylor to situations where the number of constant stiffness segments was specified, but the boundaries and specific stiffness values of the segments were design variables in the minimum bar weight problem [4]. Pierson reviewed these and other methods in a survey paper [5].

More recently Taylor investigated the problem with only a frequency constraint in terms of a modal correlation [6]. A procedure was developed to scale an existing structural model to meet experimentally measured natural frequencies. The modification scheme is based on the first order terms of a Taylor series expansion about the baseline model. Bellagamba employed an exterior penalty function technique based on the first derivatives of the violated constraints [7]. Additional constraints were imposed on static displacements and element stresses.

The combined natural frequency and mode shape constrained problem has lately been considered using a perturbation based approach. Stetson proposed a first order perturbation method based on the assumption that the new mode shapes could be expressed as linear combinations of the baseline mode shapes [8]. In later works, the technique was formulated in terms of finite elements and applied to several example problems [9-12]. Sanstrom developed first order equations which are similar to Stetson's, but provide a method for specifying mode shape constraints based on physical quantities [13]. Kim formulated the problem using the complete nonlinear perturbation equations [14]. He employed a penalty function method where the objective function was a minimum weight condition and the penalty term was a set of residual force errors. Hoff extended this work successfully in a predictor-corrector algorithm [16]. He used the linear equations of Sanstrom as a first approximation to the solution, updated the mode shapes and used Kim's nonlinear equations as a correction.

1.2 Objectives

The purpose of this work was to develop a general purpose inverse modification program that interfaced fully with the finite element program VAST. The program, called INSTRUM, is designed to return the optimum set of structural changes required to obtain certain desired dynamic characteristics. INSTRUM fits into the dynamic redesign process as shown in Fig. 1.1. One starts with a structural design and makes a finite element model in order to study the dynamic response of the structure. After running VAST for the model's frequencies and mode shapes it is discovered, for example, that one or more of the model's frequencies must be shifted in order to avoid a dangerous forcing frequency. The designer specifies what properties and elements of the model are allowed to

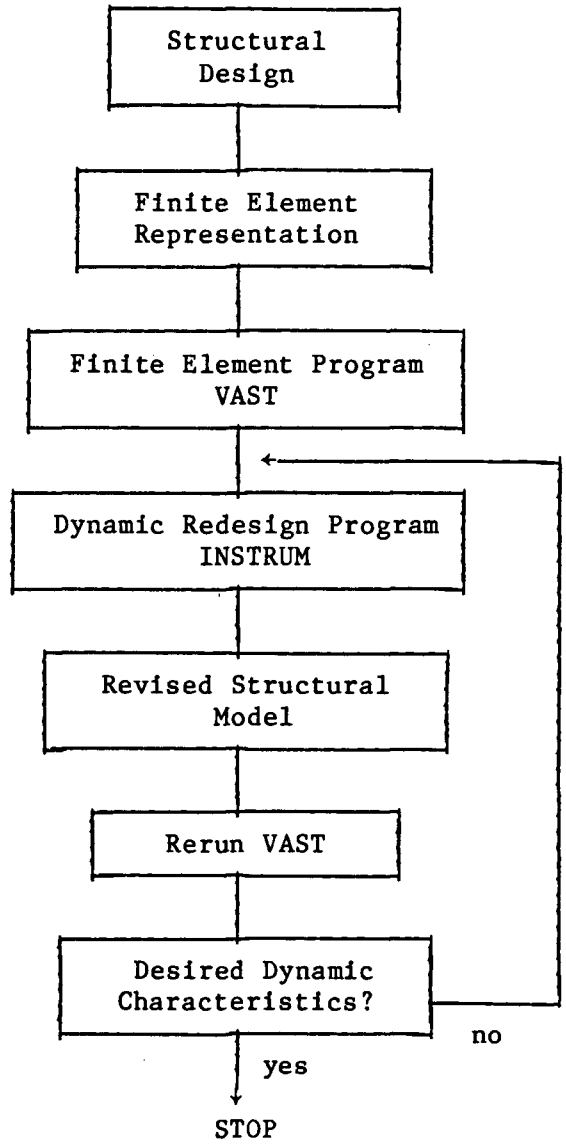


Figure 1.1 Dynamic Redesign Process

change as well as the desired frequency objectives. INSTRUM returns a set of structural changes amounting to a revised structural design. The finite element program VAST can then be run on the new model to check if the frequency objectives have been reached. If the objectives have not been reached then INSTRUM can be run again with these latest finite element results. INSTRUM is based on the preliminary design program LDRUM, written by Hoff, which was designed to interface with the finite element program NASTRAN. LDRUM handled a small library of properties and elements and could only deal with small desired frequency shifts.

This work had three main areas of emphasis. The first was to implement a version of Kim's nonlinear formulation to allow for large desired frequency shifts ("large" in general meaning frequency shifts greater than 10%). The second was to expand the library of properties and elements that could be used in the program. Finally, the third was to gain numerical experience in the performance of INSTRUM and to point out areas which might require further development.

CHAPTER 2

GENERAL INCREMENTAL PERTURBATION METHOD

The following section contains the development of the theory used in the inverse structural modification algorithm. This theory arises from the free vibration equation for an undamped discrete linear system. This equation is perturbed and the results provide linear approximations to the changed frequencies and the changed mode shapes that come about due to changes in the stiffness and mass properties of the system. The next part of the chapter relates these changes in stiffness and mass properties to actual physical structural changes. Then the INSTRUM algorithm is explained showing the different kinds of analysis that can be used for various problems. Finally an alternative formulation of one of the equations is shown which gives insight into the solution behaviour for certain problems.

2.1 Perturbation Theory for Undamped Linear Discrete Systems

2.1.1 Linear Perturbation Equations

The undamped free vibrations of a discrete system are given by

$$[M]\{\ddot{\phi}\} + [K]\{\phi\} = \{0\} \tag{2.1}$$

Eigenvalue analysis produces the eigenvalue problem

$$([K] - \omega^2 [M])\{\phi\} = \{0\} \tag{2.1a}$$

which yields the eigenvalues (or natural frequencies squared)

$$[\ \omega^2\ ] = \begin{bmatrix} \omega_1^2 & & & \\ & \omega_2^2 & & \\ & & \ddots & \\ & & & \omega_n^2 \end{bmatrix} \tag{2.2}$$

and the corresponding eigenvectors (or mode shapes)

$$[\Phi] = [\{\phi\}, \{\phi_2\}, \ldots, \{\phi_n\}] \ . \tag{2.3}$$

Now due to orthogonality of the eigenvectors with respect to [M] and [K]

$$[\Phi]^T\ [M]\ [\Phi] = [\ \bar{M}\ ] \tag{2.4}$$

$$[\Phi]^T\ [K]\ [\Phi] = [\ \bar{K}\ ] \tag{2.5}$$

Then by (2.1a)

$$[\ \bar{K}\ ] = [\ \bar{M}\ ]\ [\ \omega^2\ ] \tag{2.6}$$

Let

[K'], [M'], [Φ'], [ω^2'] represent the characteristics of a modified physical situation. Then in the same way as before

$$[\ \bar{K}'\ ] = [\ \bar{M}'\ ]\ [\ \omega^2{}'\ ] \tag{2.7}$$

$$[\ \bar{K}'\ ] = [\Phi']^T\ [K']\ [\Phi'] \tag{2.8}$$

$$[\ \bar{M}'\ ] = [\Phi']^T\ [M']\ [\Phi'] \tag{2.9}$$

The relationship between the unprimed and the primed quantities can be expressed in terms of perturbations of the original structure.

$$[M'] = [M] + [\Delta M] \tag{2.10}$$

$$[K'] = [K] + [\Delta K] \tag{2.11}$$

$$[\ \omega^2{}'] = [\ \omega^2\ ] + [\ \Delta\omega^2\ ] \tag{2.12}$$

$$[\Phi'] = [\Phi] + [\Delta\Phi] \tag{2.13}$$

The mode shape changes can be represented as linear combinations of the mode shapes obtained from the baseline structure

$$[\Delta\Phi] = [\Phi]\,[C]^T \quad C_{ij} = 0 \text{ for } i = j \tag{2.14}$$

Combining equation (2.14) with (2.13) any direction in the space spanned by the baseline eigenvectors can be realized. Imposing $C_{ij} = 0$ for $i = j$ has the effect of fixing the resulting magnitudes of the changed eigenvectors.

Using relationships (2.8) - (2.11) equation (2.7) can be re-written as

$$[\Phi']^T [\Delta K] [\Phi'] - [\Phi']^T [\Delta M] [\Phi'] [\omega^{2}{}'] = [\Phi']^T [M] [\Phi'] [\omega^{2}{}'] - [\Phi']^T [K] [\Phi'] \tag{2.15}$$

Equation (2.15) is known as the general perturbation equation. At this stage no assumptions have been made and equation (2.15) is an exact equation for a linear undamped discrete system.

Expressions (2.12) and (2.13) are applied to equation (2.15) resulting in

$$[\Phi]^T [\Delta K] [\Phi] + [\Phi]^T [\Delta K] [\Delta\Phi] + [\Delta\Phi]^T [\Delta K] [\Phi] +$$

$$[\Delta\Phi]^T [\Delta K] [\Delta\Phi] - [\Phi]^T [\Delta M] [\Phi] [\ \omega^2\ ] - [\Phi]^T [\Delta M] [\Phi] [\ \Delta\omega^2\ ] -$$

$$[\Phi]^T [\Delta M] [\Delta\Phi] [\ \omega^2\ ] - [\Phi]^T [\Delta M] [\Delta\Phi] [\ \Delta\omega^2\ ] -$$

$$[\Delta\Phi]^T [\Delta M] [\Phi] [\ \omega^2\ ] - [\Delta\Phi]^T [\Delta M] [\Phi] [\ \Delta\omega^2\ ] -$$

$$[\Delta\Phi]^T [\Delta M] [\Delta\Phi] [\ \omega^2\ ] - [\Delta\Phi]^T [\Delta M] [\Delta\Phi] [\ \Delta\omega^2\ ] =$$

$$[\Phi]^T [M] [\Phi] [\ \omega^2\ ] + [\Phi]^T [M] [\Phi] [\ \Delta\omega^2\ ] +$$

$$[\Phi]^T [M] [\Delta\Phi] [\ \omega^2\ ] + [\Phi]^T [M] [\Delta\Phi] [\ \Delta\omega^2\ ] +$$

$$[\Delta\Phi]^T [M] [\Phi] [\ \omega^2\ ] + [\Delta\Phi]^T [M] [\Phi] [\ \Delta\omega^2\ ] +$$

$$[\Delta\Phi]^T [M] [\Delta\Phi] [\ \omega^2\ ] + [\Delta\Phi]^T [M] [\Delta\Phi] [\ \Delta\omega^2\ ] - [\Phi]^T [K] [\Phi] -$$

$$[\Phi]^T [K] [\Delta\Phi] - [\Delta\Phi]^T [K] [\Phi] - [\Delta\Phi]^T [K] [\Delta\Phi] \qquad (2.16)$$

For very small changes it is assumed that the terms of higher order than one in the Δ quantities will be negligible (Appendix A shows an example of the actual magnitudes of these 24 terms for a two element cantilever beam example). Thus assuming small changes and so neglecting all terms not linear in Δ

$$[\Phi]^T [\Delta K] [\Phi] - [\Phi]^T [\Delta M] [\Phi] [\ \omega^2\ ] =$$
$$[\Phi]^T [M] [\Phi] [\ \omega^2\ ] + [\Phi]^T [M] [\Phi] [\ \Delta\omega^2\ ] +$$
$$[\Phi]^T [M] [\Delta\Phi] [\ \omega^2\ ] + [\Delta\Phi]^T [M] [\Phi] [\ \omega^2\ ] -$$
$$[\Phi]^T [K] [\Phi] - [\Phi]^T [K] [\Delta\Phi] - [\Delta\Phi]^T [K] [\Phi] \qquad (2.17)$$

Finally after reducing and using equation (2.14)

$$[\Phi]^T [\Delta K] [\Phi] - [\Phi]^T [\Delta M] [\Phi] [\ \omega^2\ ] =$$

$$[\Phi]^T [M] [\Phi] [\ \Delta\omega^2\ ] + [\Phi]^T [M] [\Phi] [C]^T [\ \omega^2\ ] -$$

$$[\Phi]^T [K] [\Phi] [C]^T \tag{2.18}$$

Carrying out the individual multiplications of equation (2.18)

for $i = j$

$$\{\psi_i\}^T [\Delta K] \{\psi_i\} - \{\psi_i\}^T [\Delta M] \{\psi_i\} \omega_i^2 = M_i \Delta(\omega_i^2) \tag{2.19a}$$

for $i \neq j$

$$\{\psi_j\}^T [\Delta K] \{\psi_i\} - \{\psi_j\}^T [\Delta M] \{\psi_i\} \omega_i^2 = M_j C_{ij} (\omega_i^2 - \omega_j^2) \tag{2.19b}$$

where M_i is the generalized mass corresponding to the i'th mode. The C_{ij}'s are eliminated by using relations obtained from equation (2.14)

$$\Delta\psi_{ki} = C_{i1}\psi_{k1} + C_{i2}\psi_{k2} + \ldots + C_{in}\psi_{kn} = \sum_{j=1}^{n} C_{ij}\, \psi_{kj} \tag{2.20}$$

Solving for C_{ij} in equation (2.19b)

$$C_{ij} = \frac{1}{M_j (\omega_i^2 - \omega_j^2)} \left(\{\psi_j\}^T [\Delta K] \{\psi_i\} - \omega_i^2 \{\psi_j\}^T [\Delta M] \{\psi_i\} \right) \tag{2.21}$$

Applying (2.21) to (2.20) gives an expression which directly relates the physical mode shape changes to the structural changes

$$\Delta\psi_{ki} = \sum_{\substack{j=1 \\ j\neq i}}^{n} \left(\frac{\psi_{kj}}{M_j (\omega_i^2 - \omega_j^2)} \left(\{\psi_j\}^T [\Delta K] \{\psi_i\} - \omega_i^2 \{\psi_j\}^T [\Delta M] \{\psi_i\} \right) \right) \tag{2.22}$$

The expression for $\Delta(\omega_i^2)$, the change in natural frequency squared due to structural changes, follows immediately from (2.19a).

$$\Delta(\omega_i^2) = \frac{1}{M_i} \left(\{\phi_i\}^T [\Delta K] \{\phi_i\} - \{\phi_i\}^T [\Delta M] \{\phi_i\} \omega_i^2 \right) \tag{2.23}$$

Equations (2.22) and (2.23) are the linear perturbation equations originally developed by Sanstrom [13] and are used in the Predictor portion of the INSTRUM algorithm. These equations assume small changes and that the new mode shapes can be expressed as a linear combination of the baseline mode shapes. This last statement is correct only if the complete set of baseline mode shapes is used. In practice this is not possible since only a small subset of the baseline mode shapes is usually available.

2.1.2 Changes in Global Matrices as Structural Changes

The previous section has related the changes in the frequencies and mode shapes to changes in the stiffness and mass matrices of the baseline system. These changes in the stiffness and mass matrices must still be expressed as actual physical structural changes. It is convenient to do this by constraining these structural changes to be changes in the properties of elements from the finite element model. Then the matrices that express the changes in elemental stiffness and mass properties can be individually assembled to form the global changes in stiffness and mass matrices of the baseline system.

The global sized matrices $[\Delta K]$ and $[\Delta M]$ can be divided into elemental components as follows

$$[\Delta K] = \sum_{e=1}^{NE} \sum_{p=1}^{q_e} \alpha_{ep} [k_{ep}] \tag{2.24}$$

$$[\Delta M] = \sum_{e=1}^{NE} \sum_{p=1}^{q_e} \alpha_{ep} [m_{ep}] \tag{2.25}$$

where

e = element number;

NE = total number of elements;

p = number of property being changed;

q_e = number of properties of element e being changed.

α_{ep} = fractional change in property p of element e.

$[k_{ep}]$ = matrix by which the change in the elemental stiffness matrix of element e due to structural change α_{ep} is given by the product $\alpha_{ep} [k_{ep}]$.

$[m_{ep}]$ = matrix by which the change in the elemental mass matrix of element e due to structural change α_{ep} is given by the product $\alpha_{ep} [m_{ep}]$.

Precise definitions of $[k_{ep}]$, $[m_{ep}]$ will be given in Section (3.1).

Substituting (2.24) and (2.25) into (2.22) and (2.23)

$$\Delta\psi_{ki} = \sum_{\substack{j=1 \\ j\neq i}}^{n} \left(\frac{\psi_{ki}}{M_j (\omega_i^2 - \omega_j^2)} \left(\{\psi_j\}^T \left(\sum_{e=1}^{NE} \sum_{p=1}^{q_e} \alpha_{ep} [k_{ep}] \right) \{\psi_i\} - \right. \right.$$

$$\left. \left. \omega_i^2 \{\psi_j\}^T \left(\sum_{e=1}^{NE} \sum_{p=1}^{q_e} \alpha_{ep} [m_{ep}] \right) \{\psi_i\} \right) \right) \tag{2.26a}$$

$$\Delta(\omega_i^2) = \frac{1}{M_i} \left(\{\phi_i\}^T \left(\sum_{e=1}^{NE} \sum_{p=1}^{q_e} \alpha_{ep} [k_{ep}] \right) \{\phi_i\} - \omega_i^2 \{\phi_i\}^T \left(\sum_{e=1}^{NE} \sum_{p=1}^{q_e} \alpha_{ep} [m_{ep}] \right) \{\phi_i\} \right) \tag{2.26b}$$

Equations (2.26a) and (2.26b) are the linear perturbation equations expressing the modal changes as a function of fractional change quantities α_{ep}. For a desired frequency shift the method is to specify the left hand side of equation (2.26b) and solve for these fractional change quantities α_{ep}. This gives initial values for the α_{ep} and comprises the predictor portion of the algorithm in INSTRUM. More accurate values for the α_{ep} are then obtained by using these initial α_{ep} in the corrector portion of the algorithm explained in Section 2.1.4.

2.1.3 Perturbation Influence Terms

A quantity that arises from equation (2.26b) and can be used in choosing the structural changes that have the most effect in changing a certain frequency is the Perturbation Influence Term defined by

$$\{\phi_i\}^T [[k_{ep}] - \omega_i^2 [m_{ep}]] \{\phi_i\} \tag{2.27}$$

This term measures by its relative magnitude the effect that changing property p of element e will have on the i'th frequency. Thus for each property of each element there will be a perturbation influence term for the i'th mode that will show by its relative magnitude in comparison to the other terms how effective

changing that property of that element will be on shifting the i'th frequency. If the term is large and positive it will be effective in increasing the frequency. If the term is large in magnitude but negative it will be effective in decreasing the frequency.

2.1.4 General Perturbation Equation

The corrector portion of the INSTRUM algorithm uses the general perturbation equation already developed as equation (2.15). This equation multiplied out term by term is

$$\{\phi_j'\}^T [\Delta K] \{\phi_i'\} - \{\phi_j'\}^T [\Delta M] \{\phi_i'\} \omega_i^{2\prime} = \{\phi_j'\}^T [M] \{\phi_i'\} \omega_i^{2\prime} - \{\phi_j'\}^T [K] \{\phi_i'\}$$

$$i = 1 \ldots n \qquad (2.28)$$

$$j = 1 \ldots n$$

Applying the definitions for $[\Delta K]$ and $[\Delta M]$ in (2.24) and (2.25)

$$\{\phi_j'\}^T \left(\sum_{e=1}^{NE} \sum_{p=1}^{q_e} \alpha_{ep} [k_{ep}] \right) \{\phi_i'\} - \{\phi_j'\}^T \left(\sum_{e=1}^{NE} \sum_{p=1}^{q_e} \alpha_{ep} [m_{ep}] \right) \{\phi_i'\} \omega_i^{2\prime}$$

$$= \{\phi_j'\}^T [M] \{\phi_i'\} \omega_i^{2\prime} - \{\phi_j'\}^T [K] \{\phi_i'\} \qquad (2.29)$$

$$i = 1 \ldots n$$

$$j = 1 \ldots n$$

These equations define exactly the relationship between the new mode shapes and frequencies and the structural changes required to produce the prescribed

frequency shift. In practice for the type of problems being considered here the changed frequency for a particular mode is known exactly. The modified modes will have been approximated by the linear perturbation equation (2.26a) using the linear approximation to the required structural parameters solved from equation (2.26b). There are $NE*q_e$ unknown structural parameters and in general the system of equations will be underconstrained and some optimizing procedure will be required to choose a suitable solution vector, and the results obtained will depend upon the abilities of the optimization algorithm.

In order to investigate the characteristics of the equations developed in their simplest form consider problems in which only one structural parameter is allowed to change. In this case only one equation from (2.29) for a certain frequency constraint is required. The equation that should be used is that corresponding to the case i = j. In support of this is the fact that if the exact mode shapes are used, the equations corresponding to $i \neq j$ are satisfied identically by virtue of orthogonality and these equations are then independent of frequency. It is reasonable, then, to use only the i = j equations for all problems that involve frequency constraints. Thus the general perturbation equation used in INSTRUM is

$$\{\phi_i'\}^T \left(\sum_{e=1}^{NE} \sum_{p=1}^{q_e} \alpha_{ep} [k_{ep}] \right) \{\phi_i'\} - \{\phi_i'\}^T \left(\sum_{e=1}^{NE} \sum_{p=1}^{q_e} \alpha_{ep} [m_{ep}] \right) \{\phi_i'\} \omega_i^{2\prime}$$

$$= \{\phi_i'\}^T [M] \{\phi_i'\} \omega_i^{2\prime} - \{\phi_i'\}^T [K] \{\phi_i'\} \qquad (2.30)$$

The fractional change quantites α_{ep} found by solving equation (2.30) will be the final answer as given by the corrector portion of the algorithm in INSTRUM.

2.2 INSTRUM Algorithm

The basic procedure is shown schematically in Figure 1.2 and proceeds as follows. Starting with a finite element analysis of the baseline structure the desired frequency objectives, the properties and elements allowed to change, as well as any bounds on these changes must be specified. A functional containing the information from the linear perturbation equations is formed and minimized using an iterative equation solver. This gives a linear approximation to the required structural changes and comprises the predictor portion. The linear approximation to the changed mode shapes is used in the general perturbation equation for each frequency that is being constrained. A second functional containing this information is formed and again minimized using the iterative equation solver. This gives the final required structural changes and comprises the corrector portion.

One complete analysis of the type mentioned above is sufficient for most problems. In addition to this, the option exists in INSTRUM to use the incremental procedure developed by Hoff [16]. A flow chart describing this incremental procedure is shown in Figure 1.3. The procedure uses the structural changes as given by one complete predictor and corrector analysis of INSTRUM to update the elemental and global stiffness and mass matrices. Then Rayleigh's quotient is used on these matrices to approximate the new frequencies resulting from the structural changes. Then the predictor and corrector analyses are performed on this updated system. An option is also included to perform one or more predictor analyses without utilizing the corrector analysis portion.

The perturbation equations are solved subject to the constraints mentioned above by means of a general purpose nonlinear equation solver employing the

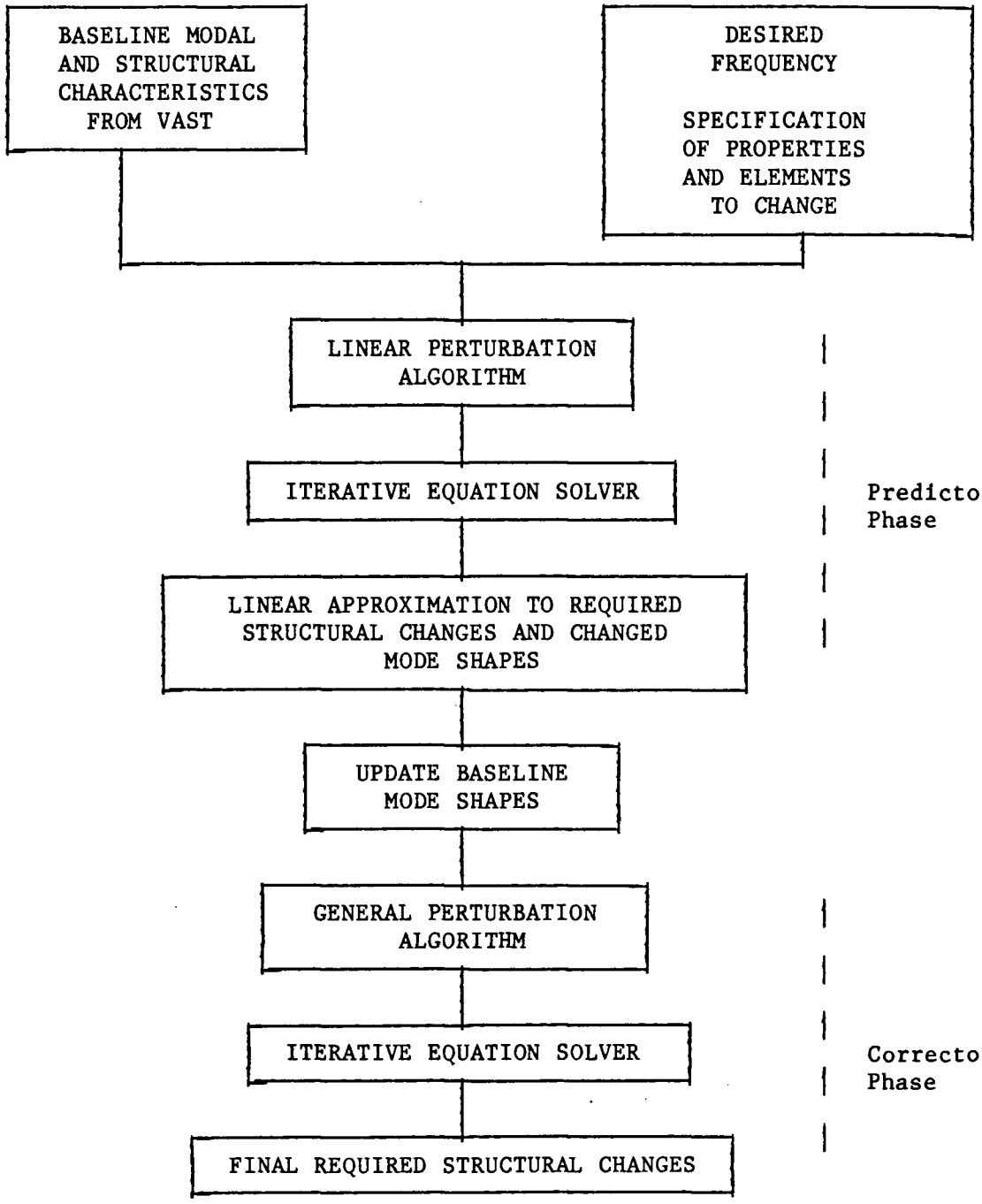


Figure 1.2 Predictor and Corrector Phases in INSTRUM

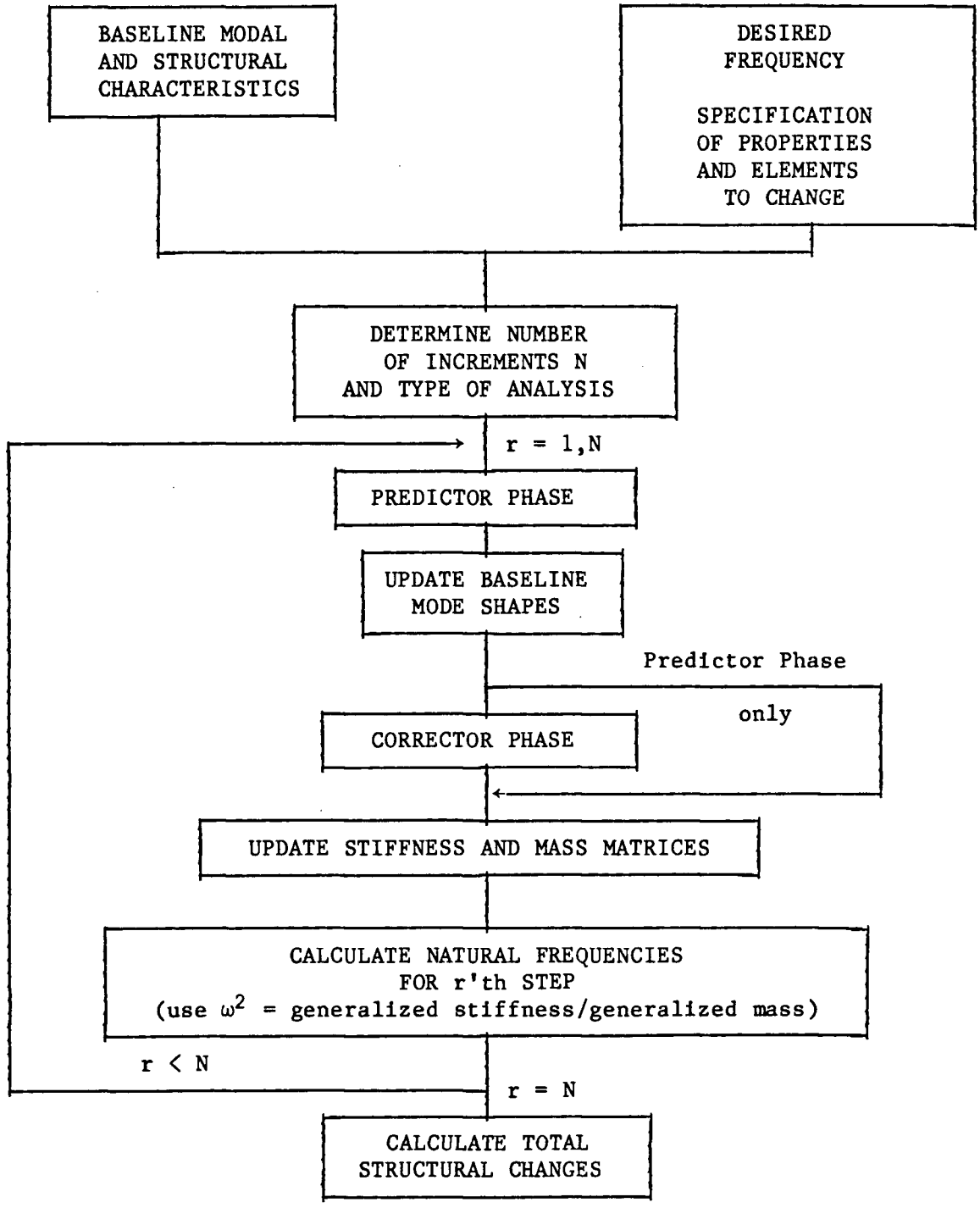


Figure 1.3 Incremental Features of INSTRUM

simplex method with an exterior penalty function. It may be described as minimizing the functional

$$g(\alpha) = \beta_I\, f(\alpha) + \beta_{II}\, \{ \{B\} - [A]\, \{\alpha\}\} + \beta_{III}\, h(\alpha) \qquad (2.31)$$

where

$[A]$ = Linear or general perturbation equations coefficient matrix.

$\{B\}$ = Solution vector of the perturbation equations.

$\{\alpha\}$ = Vector of design variables.

$\{B\} - [A]\{\alpha\}$ = Error in satisfying the linear or the general perturbation equations (depending on if the predictor or corrector analysis is being performed).

$$f(\alpha) = \begin{cases} \sum_{e=1}^{N,E} \sum_{p=1}^{q_p} C_{ep}\, \alpha_{ep} & \text{Minimum Weight Objective} \\ \text{or} & \\ \sum_{e=1}^{N,E} \sum_{p=1}^{q_p} C_{ep}\, \alpha_{ep}^2 & \text{Minimum Change Objective} \end{cases}$$

where

C_{ep} = "Cost" of design variables (e.g. for minimum weight C_{ep} is usually the mass of element e). Can be user defined.

$h(\alpha)$ = Penalty function for exceeding the bounds on design variables.

β_I, β_{II}, β_{III} = User defined constants which can emphasize one particular term of the functional. The default values are 1 and for most problems they should not be changed. The functional is then balanced to satisfy the frequency constraint as the first priority, the optimization criteria and penalty function being secondary priorities.

2.3 Reformulation of General Perturbation Equation

The general perturbation equation as given by the i=j equation of equation (2.28) can be derived using Rayleigh's Quotient instead of the perturbation approach used by Kim [14]. In doing so one can gain insight into the type of accuracy that can be expected and obtain bounds on the solutions for some types of problems.

The eigenvalue problem that results from the equation of motion for an undamped unforced discrete linear system is given by equation (2.1a) as

$$([K] - \omega^2 [M]) \{\phi\} = \{0\} \tag{2.32}$$

The ω_i^2 that satisfy this equation are the squares of the natural frequencies of the system and the corresponding $\{\phi_i\}$ are the mode shapes.

Multiplying equation (2.32) by $\{\phi\}^T$ and rearranging

$$\omega^2 (\phi) = \frac{\{\phi\}^T [K] \{\phi\}}{\{\phi\}^T [M] \{\phi\}} \tag{2.33}$$

Since $[K]$, $[M]$ are symmetric and positive definite the value of $\{\phi\}$ that minimizes equation (2.33) corresponds to the fundamental mode shape of the system $\{\phi_1\}$ [20]. The corresponding value of the quotient $\omega^2(\phi)$ will be the square of the fundamental frequency of the system ω_1^2. The rest of the mode shapes and corresponding frequencies can be found by minimizing the quotient in equation (2.33) subject to the trial functions $\{\phi\}$ being orthogonal (with respect to the global stiffness and mass matrices) to the $\{\phi_i\}$ already solved for. In this way the ω_i^2, $\{\phi_i\}$ can be found successively.

Using the same notation as the perturbation equation derivation let the primed quantities describe the mode shapes, frequencies, and the structural matrices of the changed system. Equation (2.33) then becomes

$$\omega^2 (\phi) = \frac{\{\phi\}^T [K'] \{\phi\}}{\{\phi\}^T [M'] \{\phi\}} \tag{2.34}$$

The changed stiffness and mass matrices can be related to the baseline mass and stiffness matrices using equations (2.10) and (2.11). Substituting these relations into equation (2.34),

$$\frac{\{\phi\}^T [K] \{\phi\} + \{\phi\}^T [\Delta K] \{\phi\}}{\{\phi\}^T [M] \{\phi\} + \{\phi\}^T [\Delta M] \{\phi\}} = \omega^2 \tag{2.35}$$

Thus for the i'th changed frequency and the i'th changed mode shape

$$\frac{\{\phi_i'\}^T [K] \{\phi_i'\} + \{\phi_i'\}^T [\Delta K] \{\phi_i'\}}{\{\phi_i'\}^T [M] \{\phi_i'\} + \{\phi_i'\}^T [\Delta M] \{\phi_i'\}} = \omega_i^{2\prime} \qquad i = 1 \ldots n \tag{2.36}$$

Rearranging

$$\{\phi_i'\}^T [\Delta K] \{\phi_i'\} - \{\phi_i'\}^T [\Delta M] \{\phi_i'\} \omega_i^{2\prime} =$$

$$\{\phi_i'\}^T [M] \{\phi_i'\} \omega_i^{2\prime} - \{\phi_i'\}^T [K] \{\phi_i'\} \qquad i = 1 \ldots n \tag{2.37}$$

This is precisely equation (2.28) for the i=j case. Thus the actual changed mode shapes that satisfy the quotient in equation (2.34) satisfy the general perturbation equation.

This quotient in equation (2.34) will be minimized by the changed mode shape that corresponds to the lowest frequency

$$\frac{\{\phi_1'\}\ [K]\ \{\phi_1'\} + \{\phi_1'\}\ [\Delta K]\ \{\phi_1'\}}{\{\phi_1'\}\ [M]\ \{\phi_1'\} + \{\phi_1'\}\ [\Delta M]\ \{\phi_1'\}} = \omega_1^{2\prime}, \tag{2.38}$$

In practice the changed mode shape is unknown, but the fact that the changed mode shape minimizes the quotient allows bounds to be set on the solutions that will result in certain situations. If it is desired to change the fundamental frequency of a system by making one structural change then there will be one equation and one unknown in the corresponding linear and general perturbation equations. There will be no need to form a functional to be minimized since the linear and general perturbation equations can be solved directly. In these cases certain characteristics of the solution behaviour can be predicted, and these will be discussed fully in Chapter Four.

CHAPTER 3

INSTRUM ELEMENT IMPLEMENTATION

This chapter describes the work involved in interfacing INSTRUM with the finite element program VAST. The $[k_{ep}]$ and $[m_{ep}]$ matricies are defined mathematically and it is shown that there are two distinct types of property changes - a "linear" property change and a "non- linear" property change. Finally, the implementations of the various properties and elements from VAST are explained in detail.

As previously mentioned INSTRUM has been developed from a program written at the University of Michigan called LDRUM. LDRUM was written so that it interfaced with the finite element program NASTRAN. The elemental mass and stiffness matrices, the mode shapes, and the frequencies were all read into LDRUM from NASTRAN data files. In order to obtain the $[k_{ep}]$, $[m_{ep}]$ matrices it was necessary to modify the NASTRAN data file to include dummy elements whose stiffness and mass matrices would correspond to the required $[k_{ep}]$, $[m_{ep}]$ matrices. For example, if the area property of a certain beam element was to be modified, the dummy element in the NASTRAN data file would be a beam element at the same location as the original with 1/1000 of the area and zero moments of inertia. The 1/1000 is used so the extra beam element will not significantly affect the results of the finite element run, and the zero moments of inertia are used to ensure that the resulting stiffness matrix will contain only the terms sensitive to a change in area - precisely the $[k_{ep}]$ matrix corresponding to an area

change. (In this case to obtain the $[k_{ep}]$ matrix the dummy elements stiffness matrix would have to be multiplied by 1000 to account for the difference in area between the actual and dummy elements).

Since INSTRUM was being used to interface with the finite element program VAST, many modifications were made to the original program LDRUM. The files containing the stiffness and mass matrices, the mode shapes, and the frequencies are quite different in VAST than in NASTRAN. Besides this fact it was seen that the method previously employed of determining the $[k_{ep}]$, $[m_{ep}]$ matrices by inserting dummy elements into the finite element data file was inconvenient and limited in the kinds of properties and elements that could be included. It was thus decided to calculate the $[k_{ep}]$, $[m_{ep}]$ matrices while in INSTRUM. The library of properties and elements was increased considerably in this way, while leaving the user free of the inconvenience of having to modify the original finite element data file.

3.1 Linear and Nonlinear Property Changes, Definition of $[k_{ep}]$ Matrices

For the INSTRUM algorithm the changes in elemental matrices that correspond to structural changes are expressed as:

$$[\Delta k_e] = \alpha_{ep} [k_{ep}] \tag{3.1}$$

$$[\Delta m_e] = \alpha_{ep} [m_{ep}] \tag{3.2}$$

where

$[\Delta k_e]$ = the change in the elemental stiffness matrix of element e due to the structural change α_{ep}.

$[\Delta m_e]$ = the change in the elemental mass matrix of element e due to the structural change α_{ep}.

To more precisely see where the $[k_{ep}]$, $[m_{ep}]$ matrices come from consider the following. The changed elemental stiffness matrix arising from a change in the property p from the original value of p_o is

$$[k_e'] = [k_e] + [\Delta k_e] \tag{3.3}$$

$$= [k_e] + \frac{\partial [k_e]}{\partial p}\Big|_{p_o} (p - p_o) + \frac{\partial^2 [k_e]}{\partial p^2}\Big|_{p_o} (p - p_o)^2 \frac{1}{2!} + \ldots \tag{3.4}$$

$$= [k_e] + \frac{\partial [k_e]}{\partial p}\Big|_{p_o} (p_o \alpha_{ep}) + \frac{\partial^2 [k_e]}{\partial p^2}\Big|_{p_o} (p_o \alpha_{ep})^2 \frac{1}{2!} + \ldots \tag{3.5}$$

$$= [k_e] + \alpha_{ep} \left(p_o \frac{\partial [k_e]}{\partial p}\Big|_{p_o}\right) + \alpha_{ep}^2 \left(p_o^2 \frac{\partial^2 [k_e]}{\partial p^2}\Big|_{p_o}\right) \frac{1}{2!} + \ldots \tag{3.6}$$

$$= [k_e] + \alpha_{ep} [k_{ep}] + \alpha_{ep}^2 [k_{ep}^{II}] + \ldots \tag{3.7}$$

Thus if the elemental stiffness matrix $[k_e]$ contains terms in p of no higher order than one then

$$\frac{\partial^n [k_e]}{\partial p^n} \equiv 0 \quad n > 1 \tag{3.8}$$

and

$$[k_e'] \equiv [k_e] + \alpha_{ep} [k_{ep}] \tag{3.9}$$

But if the stiffness matrix $[k_e]$ contains terms in p of higher order than one then

$$[k_e'] \approx [k_e] + \alpha_{ep} [k_{ep}] \tag{3.10}$$

A property change that satisfies equation (3.8) and can be expressed by equation (3.9) will be referred to as a Linear Property Change. A property change that does not satisfy equation (3.8) and which must be expressed by

equation (3.10) will be referred to as a Nonlinear Property Change. A table showing the properties and elements and whether the corresponding changed stiffness matrix can be calculated exactly or only approximately is shown in Table (3.1).

3.2 Element Implementation for Linear Property Changes

As mentioned, the $[k_{ep}]$ matrices are all calculated within INSTRUM. For the linear property changes, i.e. those in which $[k_e'] \equiv [k_e] + \alpha_p [k_{ep}]$, the $[k_{ep}]$ matrices can be found from the original stiffness matrices by picking off the appropriate terms and zeroing the rest of the matrix. An example for a simple 6 degree of freedom beam element is shown in Fig. 3.1.

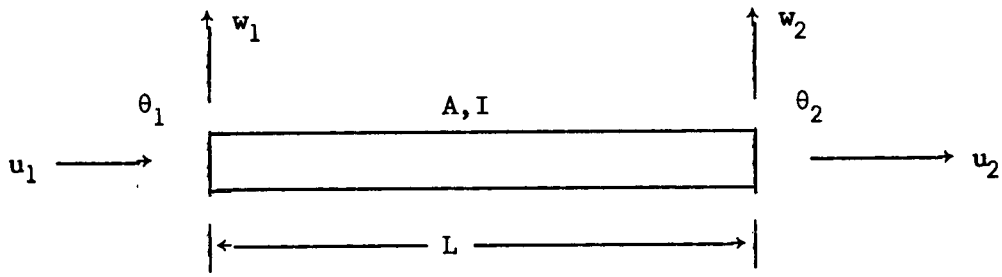
For all the properties implemented the $[m_{ep}]$ matrix is either the elemental mass matrix $[m_e]$, or is identically equal to zero, thus every mass change is a linear property change.

3.3 Element Implementation for Nonlinear Property Changes

There are three elements which can have nonlinear property changes, i.e. changes for which the changed stiffness matrix resulting from some fractional property change can not be calculated exactly. These elements are the triangular plate element (thickness property), the curved beam element (cross-sectional dimensions b,d), and the thick thin shell element (thickness property). Each of these will be considered in turn.

Table 3.1. Vast Elements and Properties Included in INSTRUM

Element	IEC #	Linear Property	Nonlinear Property
Thick Thin Shell	1	ρ, E	t
Beam	3	A, ρ, I1, I2, J, E	
Triangular Plate	4	ρ, E	t
Curved Beam	7	ρ, E	b, d
Rod	8	A, ρ, E	
Triangular Membrane	9	t, ρ, E	
Curved Bar	10	ρ, E	
Stiffened Membrane	11	ρ, E	
8 Node Brick	16	ρ, E	
10 Node Tetrahedron	17	ρ, E	
8 Node Quadralateral	20	t, ρ, E	
Shear Web	23	ρ, E	



$$X = \text{generalized coordinates} = \begin{Bmatrix} u_1 \\ w_1 \\ \theta_1 \\ u_2 \\ w_2 \\ \theta_2 \end{Bmatrix}$$

$$[k_e] = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^9} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}$$

Figure 3.1(a)

$$[k_{ep}]_{p=\text{area}} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Figure 3.1(b) $[k_{ep}]$ matrix for the area property of a 6 d.o.f. beam.

3.3.1 Triangular Plate Element

The thickness change in the triangular plate element is the only nonlinear change considered that can make use of the original stiffness matrix in order to calculate its $[k_{ep}]$. This is because the element formulation considers the bending and membrane actions of the element separately.

The bending terms in a triangular plate element arise from the strain energy calculation [21]

$$U = \frac{1}{2} \{d\}^T \left(\iint_A B^T D B \, dA \right) \{d\} \tag{3.11}$$

where

$$\{d\} = \begin{Bmatrix} w_{xx} \\ w_{yy} \\ 2w_{xy} \end{Bmatrix};$$

$$D = \frac{E t^3}{12 (1 - \nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix};$$

w = transverse deflection of plate;

B = strain - displacement matrix;

t = thickness of plate.

This results in a bending stiffness matrix of the form

$$[k]_{bending} = \frac{E t^3}{12(1 - \nu^2)} \left[\iint_A B^T \underline{D} B \, dA \right] \tag{3.12}$$

where

$$\underline{D} = \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}.$$

Thus all bending terms are proportional to the cube of the thickness of the plate. So for these terms

$$[k_{ep}]_{bending} \approx t \frac{\partial [k]_{bending}}{\partial t} = 3[k]_{bending}. \tag{3.13}$$

The thickness change is a linear property change for the membrane action, and $[k_{ep}]_{membrane} = [k]_{membrane}$. The total $[k_{ep}]$ matrix is obtained from the original stiffness matrix by transforming to local coordinates (where the terms that correspond to bending and those that correspond to membrane action can be distinguished), multiplying the bending terms by three, and transforming back to global coordinates.

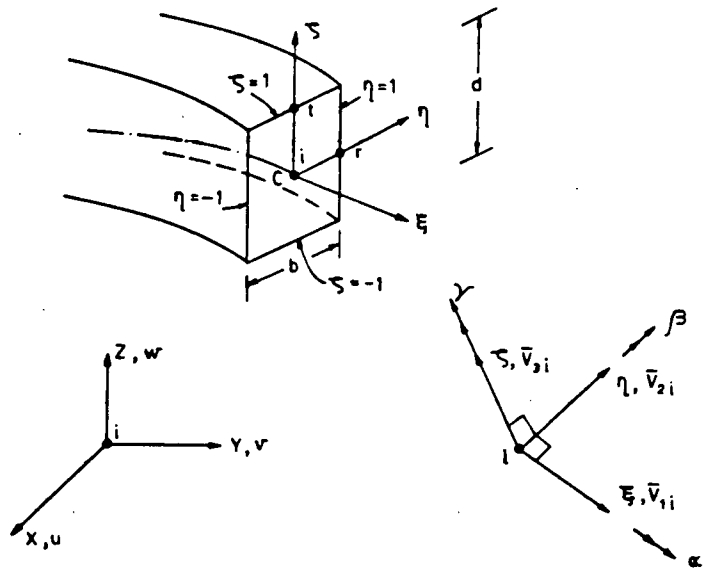
3.3.2 Curved Beam Element

The curved beam element stiffness matrix used in VAST had to be reformulated for use in INSTRUM.

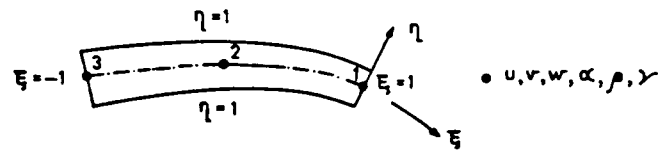
The conventional stiffness matrix formulation as given by Buragohain, Agrawal, and Ayyar proceeds as follows [22].

Using Figure 3.2 for any nodal position i the half width vector $\bar{V}_2$ in the η direction and half depth vector $\bar{V}_3$ in the ζ direction can be expressed as

$$\bar{V}_{2i} = \{x_i y_i z_i\}_r - \{x_i y_i z_i\}_c \tag{3.14}$$



(a) LOCAL AND GLOBAL CO-ORDINATES FOR BEAM



(b) PLAN VIEW OF BEAM ELEMENT

Figure 3.2 Curved Beam Element

$$\bar{V}_{3i} = \{x_i y_i z_i\}_t - \{x_i y_i z_i\}_c \tag{3.15}$$

The element geometry can be defined in terms of mid-surface nodal coordinates and vectors $\bar{V}_{2i}$ and $\bar{V}_{3i}$ as

$$\{xyz\} = \sum_{i=1}^{3} N_i \{x_i y_i z_i\}_c + \sum_{i=1}^{3} N_i \eta \bar{V}_{2i} + \sum_{i=1}^{3} N_i \zeta \bar{V}_{3i} \tag{3.16}$$

The N_i are the quadratic shape functions given by

$$N_1 = \frac{1}{2} \xi (1 + \xi)$$

$$N_2 = \frac{1}{2} (1 - \xi^2) \tag{3.17}$$

$$N_3 = \frac{1}{2} \xi (1 - \xi)$$

Any line through node i, originally normal to the mid-surface and along $\bar{V}_{2i}$ can have three independent translations - u_i, v_i, w_i in the global directions X, Y, Z and two independent rotations - α_i and β_i about the two local axes ξ and η.

The complete 18 d.o.f. element displacement field is given by

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \sum_{i=1}^{3} N_i \begin{Bmatrix} u_i \\ v_i \\ w_i \end{Bmatrix} + \sum_{i=1}^{3} N_i \eta \frac{b_i}{2} [d_i^*] \begin{Bmatrix} \alpha_i \\ \beta_i \\ \gamma_i \end{Bmatrix}$$

$$+ \sum_{i=1}^{3} N_i \zeta \frac{d_i}{2} [d_i^{**}] \begin{Bmatrix} \alpha_i \\ \beta_i \\ \gamma_i \end{Bmatrix} \tag{3.18}$$

Where the $[d_i^*]$ and $[d_i^{**}]$ are matrices of direction cosines given by

$$[d_i^*] = \begin{bmatrix} \ell_3 & 0 & -\ell_1 \\ m_3 & 0 & -m_1 \\ n_3 & 0 & -n_1 \end{bmatrix} \qquad [d_i^{**}] = \begin{bmatrix} -\ell_2 & \ell_1 & 0 \\ -m_2 & m_1 & 0 \\ -n_2 & n_1 & 0 \end{bmatrix}$$

The strains $\{\varepsilon\}$ in global coordinates are expressed in terms of the derivatives in global coordinates as

$$\{\varepsilon\} = = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \alpha_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} = \begin{Bmatrix} u_x \\ u_y \\ u_z \\ v_x \\ v_y \\ v_z \\ w_x \\ w_y \\ w_z \end{Bmatrix}$$

$$= [S] \{u_x u_y u_z v_x \cdots w_z\}^T \tag{3.19}$$

The derivatives with respect to ξ, η, ζ are related to the x, y, z derivatives by means of the Jacobian matrix [J].

$$\{u_\xi \; u_\eta \; u_\zeta\} = [J] \{u_x \; u_y \; u_z\}^T \tag{3.20}$$

thus

$$\{u_x \; u_y \; u_z\} = [J^{-1}] \{u_\xi \; u_\eta \; u_\zeta\}^T \tag{3.21}$$

and

$$\{u_x\ u_y\ u_z\ \ldots\ w_z\} = [AJ]\ \{u_\xi\ u_\eta\ u_\zeta\ \ldots\ w_\zeta\}^T \tag{3.21}$$

where

$$[AJ] = \begin{bmatrix} [J^{-1}] & 0 & 0 \\ 0 & [J^{-1}] & 0 \\ 0 & 0 & [J^{-1}] \end{bmatrix}.$$

Differentiating equation (3.18) gives

$$\{u_\xi\ u_\eta\ u_\zeta\ \ldots\ w_\zeta\} = \sum_{i=1}^{3} [M_i]\ \{u_i\ v_i\ w_i\ \alpha_i\ \beta_i\ \gamma_i\}^T \tag{3.23}$$

where the $[M_i]$ are functions of ξ, η, ζ.

Substitution of equation (3.23) into equation (3.22) and the resultant product into equation (19) results in

$$\{\varepsilon_x\ \varepsilon_y\ \varepsilon_z\ \gamma_{xy}\ \gamma_{yz}\ \gamma_{zx}\} = \sum_{i=1}^{3} [B_i]\ \{u_i\ v_i\ w_i\ \alpha_i\ \beta_i\ \gamma_i\}^T \tag{3.24}$$

where $[B_i] = [S]\ [AJ]\ [M_i]$

Expressing equation (3.24) in another way

$$\{\varepsilon_x \varepsilon_y \varepsilon_z \gamma_{xy} \gamma_{yz} \gamma_{zx}\} = [B]\ \{u_1 v_1 w_1 \alpha_1 \beta_1 \gamma_1 u_2 v_2 w_2 \alpha_2 \beta_2 \gamma_2 u_3 v_3 w_3 \alpha_3 \beta_3 \gamma_3\}^T \tag{3.25}$$

where $[B]$ is given by $[B] = [B_1\ B_2\ B_3]$

The elemental stiffness matrix $[k]$ is given by

$$[k] = \int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1} [B]^T\ [D]\ [B]\ \det [J]\ d\xi\ d\eta\ d\zeta \tag{3.26}$$

where [D] is the elasticity matrix in global coordinates at the particular integration point.

$$= \int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1} [\bar{k}]\, d\xi\, d\eta\, d\zeta \tag{3.27}$$

where $[\bar{k}] = [B]^T [D] [B] \det [J]$.

As mentioned above, for this particular reformulation it is assumed that the b and d dimensions are identical at all three nodes of the element. Then the required derivatives of the stiffness matrix are given by

$$\frac{\partial [k]}{\partial b} = \frac{\partial}{\partial b} \left(\int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1} [\bar{k}]\, d\xi\, d\eta\, d\zeta \right) \tag{3.28}$$

$$= \int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1} \frac{\partial [\bar{k}]}{\partial b}\, d\xi\, d\eta\, d\zeta \tag{3.29}$$

Thus, the terms containing various powers of b,d must be kept separate and so b,d must be treated as variables throughout the multiplications required to obtain $[\bar{k}]$. These multiplications can be kept to a minimum by partitioning the matrices

$$[B]^T [D] [B] \det [J] = [B_1\ B_2\ B_3] [D] \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} \det [J] \tag{3.30}$$

$$= \left[\begin{array}{c|c|c} B_1^T D B_1 & B_1^T D B_2 & B_1^T D B_3 \\ \hline B_2^T D B_1 & B_2^T D B_2 & B_2^T D B_3 \\ \hline B_3^T D B_1 & B_3^T D B_2 & B_3^T D B_3 \end{array}\right] \det [J] \tag{3.31}$$

$$= \left[\begin{array}{c|c|c} \bar{k}_{11} & \bar{k}_{12} & \bar{k}_{13} \\ \hline \bar{k}_{21} & \bar{k}_{22} & \bar{k}_{23} \\ \hline \bar{k}_{31} & \bar{k}_{32} & \bar{k}_{33} \end{array}\right] \tag{3.32}$$

Thus only one $\bar{k}_{ij}$ matrix needs to be calculated in terms of the i,j quantities and the b,d variables. The symbolic Algebra program MAPLE was used to carry out the necessary multiplications [23]. The general form of the $\bar{k}_{ij}$ matrix for the curved beam element is shown in Appendix B. This 6 x 6 matrix is then differentiated with respect to b and d and then by substituting the appropriate values for the i,j quantities the total 18 x 18 matrices $\frac{\partial[\bar{k}]}{\partial b}$, $\frac{\partial[\bar{k}]}{\partial d}$ can be calculated.

The result of the calculations is a very long fortran subroutine that calculates $\frac{\partial \bar{k}_{ij}}{\partial b}$, $\frac{\partial \bar{k}_{ij}}{\partial d}$. This subroutine must be called six times with the i,j values ranging i: i = 1 ... 3, j: j = i ... 3 in order to evaluate the complete $\frac{\partial[\bar{k}]}{\partial b}$, $\frac{\partial[\bar{k}]}{\partial d}$ matrices. Since a numerical integration is required these two matrices are calculated at each gauss point in order to determine the final $\frac{\partial[k]}{\partial b}$, $\frac{\partial[k]}{\partial d}$ matrices. Finally the $[k_{ep}]$ matrices are calculated using the definitions provided by equations (3.6) and (3.7).

3.3.3 Thick Thin Shell Element

The formulation of the thick thin shell element stiffness matrix parallels the curved beam derivation quite closely [24]. There is an additional constraint imposed here in that the element is required to have uniform thickness.

For the thick thin shell element there are 8 nodes and 5 degrees of freedom per node. The $[B_i]$ matrix is thus a 5 x 5 matrix, as is the elasticity matrix [D]. The stiffness matrix again comes from a numerical integration of the form

$$[k] = \int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1} [B]^T [D] [B] \det [J] \, d\xi \, d\eta \, d\zeta \tag{3.37}$$

$$= \int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1} [\bar{k}] \, d\xi \, d\eta \, d\zeta \tag{3.38}$$

In this case the nonlinear property being considered is the thickness property, so the required derivative is

$$\frac{\partial [k]}{\partial t} = \int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1} \frac{\partial [\bar{k}]}{\partial t} \, d\xi \, d\eta \, d\zeta \tag{3.39}$$

where t = thickness of the element.

The total [B] matrix in this case is a 40 x 5 matrix containing 8 5x5 $[B_i]$ matrices corresponding to the 8 nodes.

$$[B] = [B_1 \; B_2 \; B_3 \; B_4 \; B_5 \; B_6 \; B_7 \; B_8] \tag{3.40}$$

The $[B]^T$ [D] [B] det [J] matrix can then be partitioned into the form

$$
\begin{bmatrix}
B_1^T DB_1 & B_1^T DB_2 & B_1^T DB_3 & B_1^T DB_4 & B_1^T DB_5 & B_1^T DB_6 & B_1^T DB_7 & B_1^T DB_8 \\
B_2^T DB_1 & B_2^T DB_2 & B_2^T DB_3 & B_2^T DB_4 & B_2^T DB_5 & B_2^T DB_6 & B_2^T DB_7 & B_2^T DB_8 \\
B_3^T DB_1 & B_3^T DB_2 & B_3^T DB_3 & B_3^T DB_4 & B_3^T DB_5 & B_3^T DB_6 & B_3^T DB_7 & B_3^T DB_8 \\
B_4^T DB_1 & B_4^T DB_2 & B_4^T DB_3 & B_4^T DB_4 & B_4^T DB_5 & B_4^T DB_6 & B_4^T DB_7 & B_4^T DB_8 \\
B_5^T DB_1 & B_5^T DB_2 & B_5^T DB_3 & B_5^T DB_4 & B_5^T DB_5 & B_5^T DB_6 & B_5^T DB_7 & B_5^T DB_8 \\
B_6^T DB_1 & B_6^T DB_2 & B_6^T DB_3 & B_6^T DB_4 & B_6^T DB_5 & B_6^T DB_6 & B_6^T DB_7 & B_6^T DB_8 \\
B_7^T DB_1 & B_7^T DB_2 & B_7^T DB_3 & B_7^T DB_4 & B_7^T DB_5 & B_7^T DB_6 & B_7^T DB_7 & B_7^T DB_8 \\
B_8^T DB_1 & B_8^T DB_2 & B_8^T DB_3 & B_8^T DB_4 & B_8^T DB_5 & B_8^T DB_6 & B_8^T DB_7 & B_8^T DB_8
\end{bmatrix} \det[J]
$$

$$
= \begin{bmatrix}
\bar{k}_{11} & \bar{k}_{12} & \cdots & \bar{k}_{18} \\
\bar{k}_{21} & \bar{k}_{22} & \cdots & \bar{k}_{28} \\
\vdots & \vdots & & \vdots \\
\bar{k}_{81} & \bar{k}_{82} & \cdots & \bar{k}_{88}
\end{bmatrix} \tag{3.42}
$$

As for the case of the curved beam element only one matrix $B_i DB_j \det[J]$ need be multiplied out in terms of variables. The computational savings of performing only one $\bar{k}_{ij}$ calculation is especially obvious in this case. The symbolic

algebra program is again used to perform the multiplications necessary to obtain this $\bar{k}_{ij}$ matrix and to take its derivative. The general form of the $\bar{k}_{ij}$ matrix for the thick thin shell element is shown in Appendix B. Finally, once again the $[k_{ep}]$ matrix is calculated using the definitions provided by equations (3.6) and (3.7).

CHAPTER 4

SOLUTION BEHAVIOUR

This chapter will consider the behaviour and accuracy of INSTRUM solutions for various problems by examining the linear and general perturbation equations that are involved. In particular a discussion is presented of solutions to problems where it is desired to change the fundamental frequency of a system by making one structural change (which must be a linear property change). Under these conditions bounds can be imposed on both the linear and general perturbation equation solutions that will result. It will also be shown that linear perturbation solution accuracy is better if the structural change is to the stiffness properties rather than the mass properties of the system. Finally it will be shown that exact general perturbation equation solutions will result for all problems in which the structural change does not alter the baseline mode shapes.

The following results make use of Rayleighs quotient, which calls for the stiffness and mass matricies involved to be positive definite. Theoretically this is an easy condition to meet, but in practice it is often deemed necessary to transform matricies from local to global coordinates and sometimes this can destroy the positive definiteness of the stiffness and mass matrices. An example from VAST is the thick thin shell element. It has five local degrees of freedom per node and in INSTRUM the stiffness and mass matricies are artificially transformed to six global degrees of freedom per node - destroying positive definiteness. Also the use of lumped mass matrices can lead to a mass matrix which is positive semi-definite if there are zeros on the diagonal, again violating the conditions for Rayleighs quotient. Thus caution must be used when applying the

following results to ensure the problem in question satisfies the conditions for a Rayleigh's quotient result.

4.1 Linear Property Change, One Frequency Constraint

Consider the problem of changing the fundamental frequency of a system by having one structural change that can affect either the mass and/or stiffness of the system. In this case there is only one unknown and one equation for each of the linear and general perturbation equations. Thus the functional is not used and a closed form solution exists.

It will be assumed here that the change in stiffness and the change in mass will be linear property changes, i.e., they can be exactly represented by the linear products $\alpha_{ep}\ [k_{ep}]$, $\alpha_{ep}\ [m_{ep}]$ such that

$$[\Delta K] \equiv \alpha_{ep}\ [k_{ep}] \tag{4.1}$$

$$[\Delta M] \equiv \alpha_{ep}\ [m_{ep}] \tag{4.2}$$

As explained in Chapter 3, this is always true for mass changes but not for every stiffness change.

4.1.1 Linear and General Perturbation Equations

The linear and general perturbation equations for the type of problem mentioned above will be

$$\left(\{\theta_1\}^T\ [k_{ep}]\ \{\theta_1\}^T - \omega_1^2\ \{\theta_1\}^T\ [m_{ep}]\ \{\theta_1\}^T \right) \alpha_{ep}^L = \{\psi_1\}^T\ [M]\ \{\psi_1\}\ (\omega_1^{2\prime} - \omega_1^2) \tag{4.3}$$

$$\left(\{\theta_1^L\}^T\ [k_{ep}]\ \{\theta_1^L\}^T - \omega_1^2\ \{\theta_1^L\}^T\ [m_{ep}]\ \{\theta_1^L\}^T \right) \alpha_{ep}^G = \{\phi_1^L\}^T\ [M]\ \{\phi_1^L\}\ \omega_1^{2\prime} - \{\phi_1^L\}\ [K]\ \{\phi_1^L\} \tag{4.4}$$

respectively.

4.1.2 Linear Equation for Stiffness Change

For a stiffness change it can be shown (see Appendix C) that when constraining the fundamental frequency

$$\alpha_{ep}^{L} \leq \alpha_{ep}^{exact} \tag{4.5}$$

and if $\{\psi_i'\} = \{\psi_i\}$ then when constraining the i'th frequency

$$\alpha_{ep}^{L} = \alpha_{ep}^{exact} \tag{4.6}$$

The result (4.6) comes about since if there is no mass change and no change in mode shape the linear perturbation equation is an exact equation.

4.1.3 Linear Equation for Mass Change

For a mass change it can be shown (see Appendix C) that when constraining the fundamental frequency

$$\alpha_{ep}^{L} < \alpha_{ep}^{exact} \tag{4.7}$$

but that if $\{\psi_i'\} = \{\psi_i\}$ then when constraining the i'th frequency

$$\alpha_{ep}^{L} < \alpha_{ep}^{exact} \tag{4.8}$$

The result (4.8) comes about since even if there is no change in mode shape and $\{\psi_1'\} = \{\psi_1\}$ then equation (4.4) will still not be an exact equation because of the frequency term on the left hand side.

4.1.4 General Equation for Stiffness or Mass Changes

For a structural change involving either a stiffness and/or mass change it can be shown (see Appendix C) that when constraining the fundamental frequency

$$\alpha_{ep}^{G} \leq \alpha_{ep}^{exact} \tag{4.9}$$

and that if $\{\phi_i^L\} = \{\phi_i'\}$ then when constraining the i'th frequency

$$\alpha_{ep}^{G} = \alpha_{ep}^{exact} \tag{4.10}$$

The result (4.10) comes about because if there is no change in mode shape then equation (4.4) is an exact equation no matter if the change is to the stiffness or the mass properties of the system.

CHAPTER 5

INSTRUM EVALUATION

This chapter contains an evaluation of INSTRUM. The examples show the various features of INSTRUM and discuss the effectiveness of the program when presented with different types of problems. To begin an example in which the perturbation influence terms are calculated is shown. Then a series of problems is shown corresponding to the type of problem discussed in chapter 4 where it is desired to change the frequency of a system with one structural change. In these types of problems there is no need to form a functional to be minimized because in both the linear and general equations there will be one equation and one unknown. Problems involving non-linear property changes and higher frequencies are also discussed. Finally a problem with many structural parameters is shown which evaluates the equation solver used in INSTRUM along with its many options.

5.1 Perturbation Influence Terms

The following problem is an example of the usefulness of the perturbation influence terms defined by equation (2.27). INSTRUM can be run with an option to do nothing more than return the perturbation influence terms that correspond to a certain property change and to a certain mode. Each element (or group of elements, if it is desired to change a group of elements uniformly) will have a perturbation influence term which will show by its relative magnitude how effective changing a particular property of that element will be on changing the natural frequency in question.

The structural finite element model, which consists of a frame composed of 5 curved beam elements, is shown in Fig. 5.1 along with the first mode shape. It is desired to find the structural changes that have the greatest effect on increasing or decreasing the first natural frequency.

The perturbation influence terms for this problem are given in Table 5.1 and show the effect of changing 4 properties of each element on the first natural frequency of the frame.

Table 5.1. Perturbation Influence Terms for Curved Beam Frame - First Natural Frequency.

ELEMENT #	P.I.T. b DIMENSION	P.I.T. d DIMENSION	P.I.T. E	P.I.T. DENSITY
1	27,600	8,720	9,450	- 732
2	6,720	- 3,800	5,260	- 9,060
3	- 1,270	- 9,850	4,290	-14,100
4	6,720	- 3,800	5,260	- 9,060
5	27,600	8,720	9,450	- 732

A positive value implies that the structural change would tend to increase the frequency and a negative value implies that the structural change would tend to decrease the frequency. It can be seen that the change that would have the

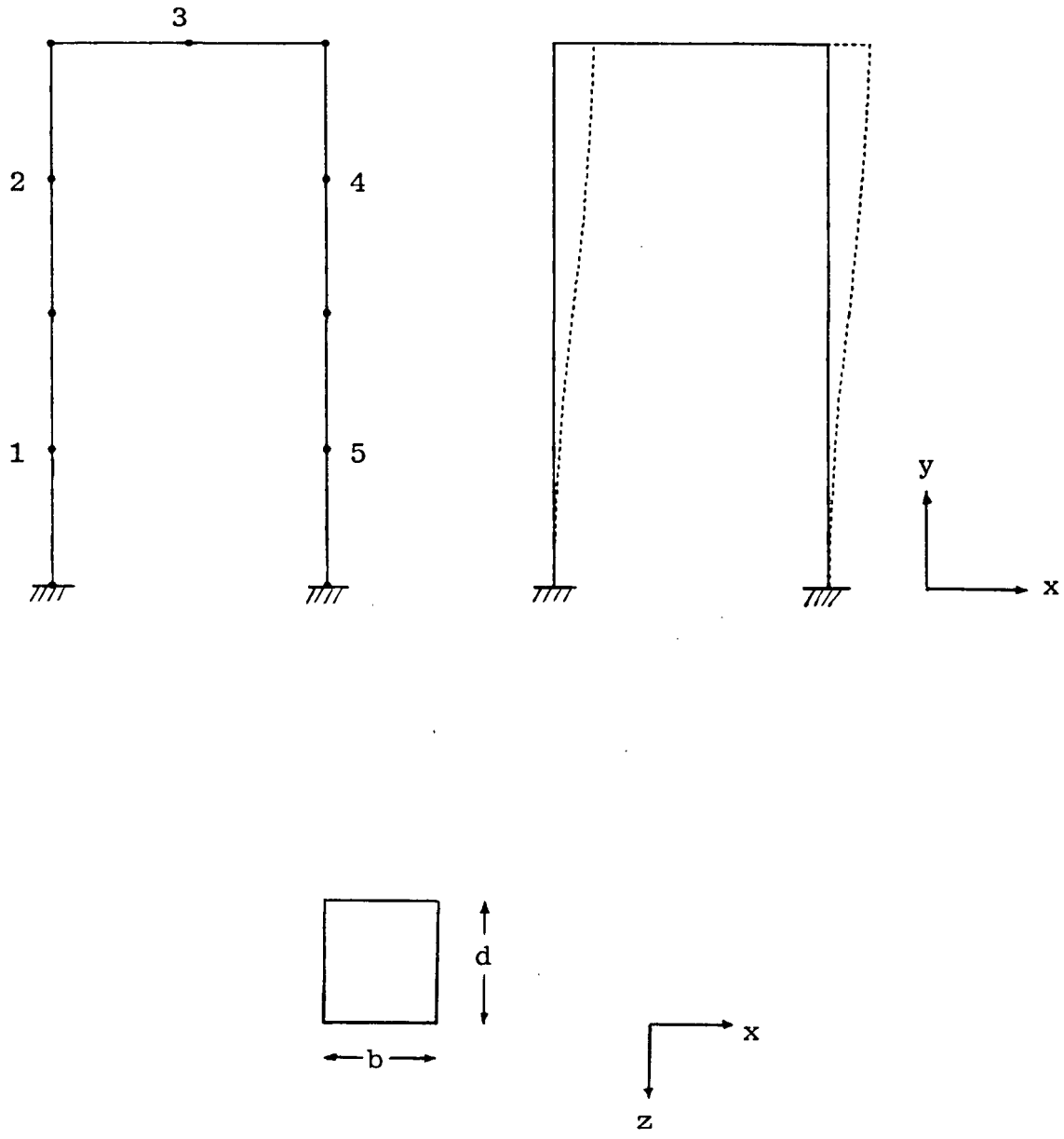


Figure 5.1 Curved Beam Frame Structural Model and First Mode Shape

greatest effect on increasing the first frequency would be to increase the b dimension of either element 1 or 5. The change that would have the greatest effect on decreasing the first frequency would be to increase the density of element 3. This information is shown graphically in Fig. 5.2. Similarly the perturbation influence terms can be calculated for other modes.

5.2 Linear and General Perturbation Equation Accuracy

A series of problems will be presented which show the behaviour of the linear and general perturbation equations for different situations. These problems will all have one thing in common - one desired frequency constraint and one possible structural change. In these types of problems there is no need to form a functional to be minimized because in both the predictor and corrector analyses there will be one equation and one unknown.

Three general types of problems will be considered. One type is linear and general equation accuracy for a stiffness or mass change when the mode shape does not appreciably change. The other two types are when either the mode shape does not change appreciably or when the mode shape changes drastically. It will be seen that the results will differ considerably for each case.

The following results and graphs were produced by running INSTRUM for a range of desired frequency shifts with a certain structural element being allowed to change. The results are all for one complete predictor and corrector analysis. The intermediate predictor results are graphed for comparison with the corrector solutions. In general, the number of modes supplied to INSTRUM from VAST has an effect on the linear approximation to the changed mode shapes and so

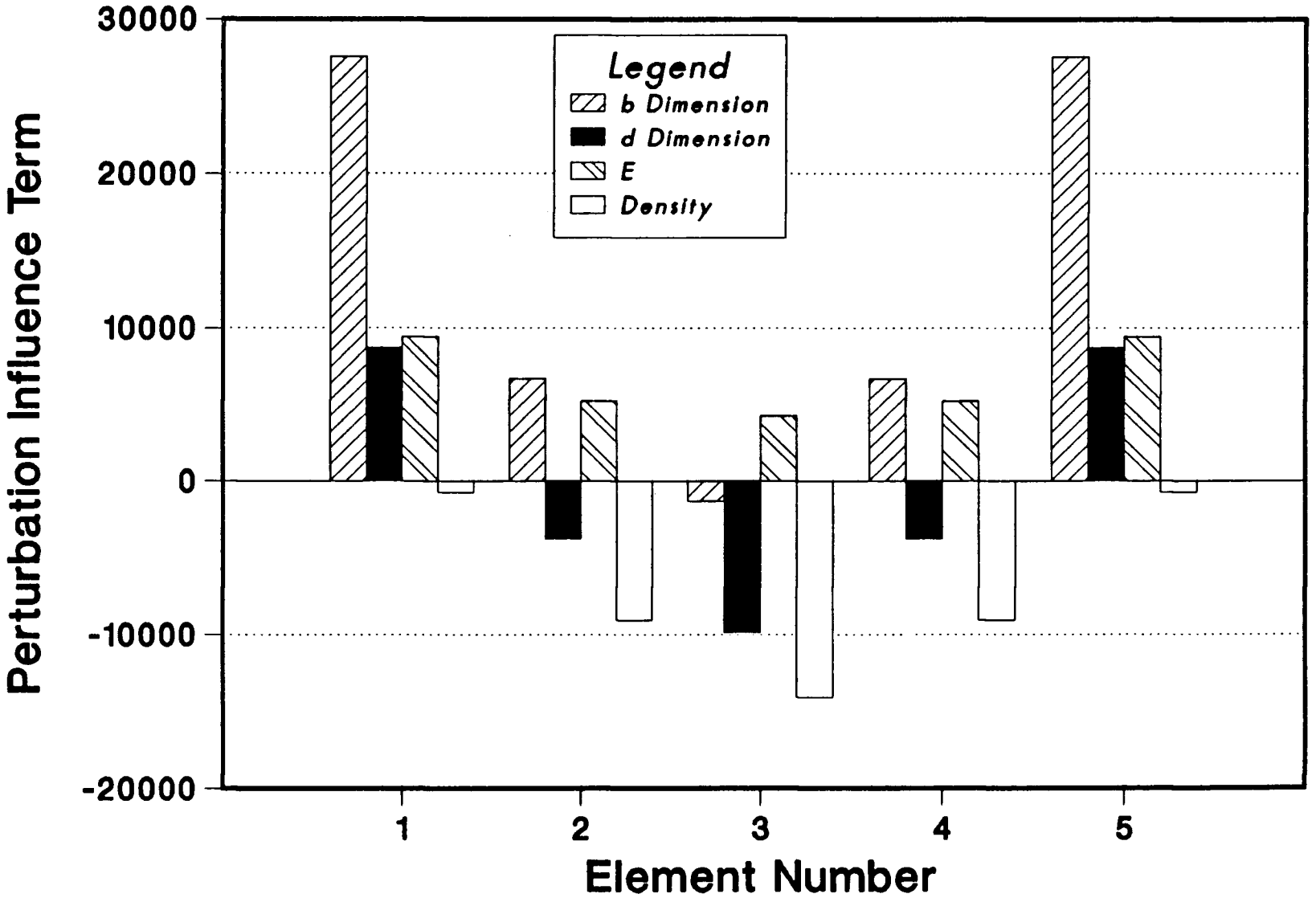


Figure 5.2 Perturbation Influence Terms for Curved Beam Frame - First Frequency

the general equation solution usually depends on the mode shapes supplied. Using the general equation with one mode only does not allow the mode shape to be updated using equation (2.26a) and so the original mode shape is used. The linear equation in all cases uses the original mode shape corresponding to the frequency being changed.

Each graph will correspond to one type of structural change. The predictor analysis results and the corrector analysis results for various numbers of modes are shown along with the exact answers. The exact answers are obtained by re-running VAST with various values of the structural changes. The x axis will be normalized as the ratio of the desired value of a certain frequency to the original value of that frequency. The y axis will then show the fractional change required to bring about the frequency shift. A y value of 0.5 indicates the property in question is to be increased by 50%, while a value of 3 indicates the property is to be increased to 4 times what it was originally. The error in the results should be taken to be the horizontal distance between an INSTRUM result and the exact answer, because this distance indicates the difference in the prescribed frequency and that resulting from implementation of the changes calculated by INSTRUM.

5.2.1 Curved Beam Cantilever Model

A cantilever beam made up of two curved beam elements is shown in Fig. 5.3 along with the first three mode shapes. Figure 5.4 shows the results for changing the density of element 2 on shifting the first natural frequency. Figure 5.5 shows the results for changing E of element 1 and Figure 5.6 shows the results for changing the b dimension for element 1.

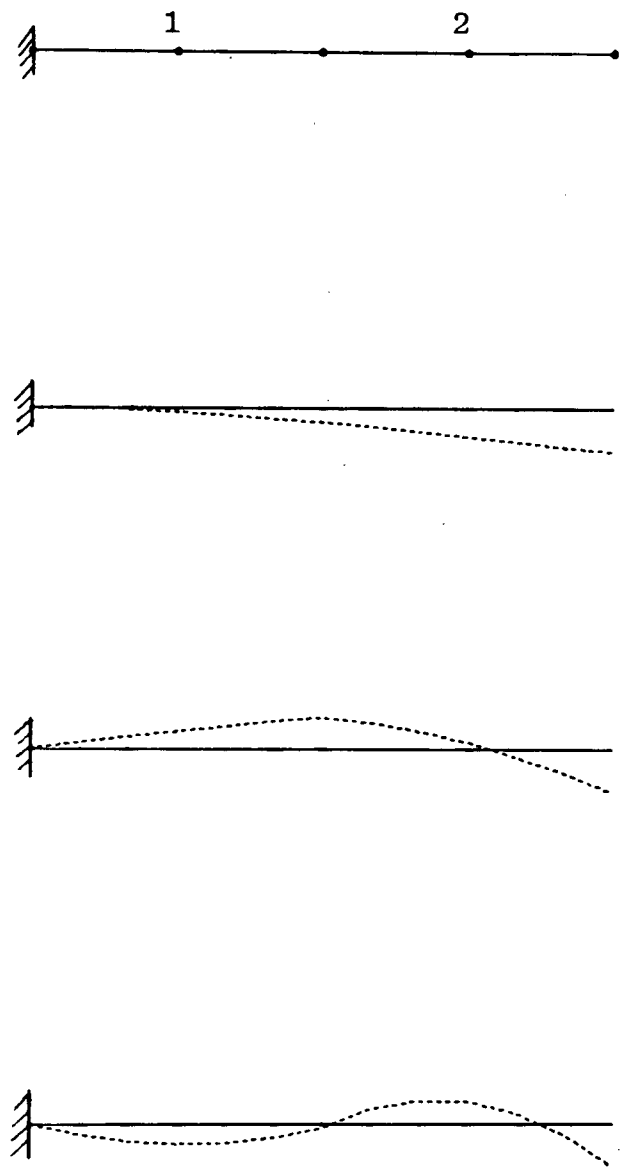


Figure 5.3 Cantilever Beam Model Composed of 2 Curved Beam Elements

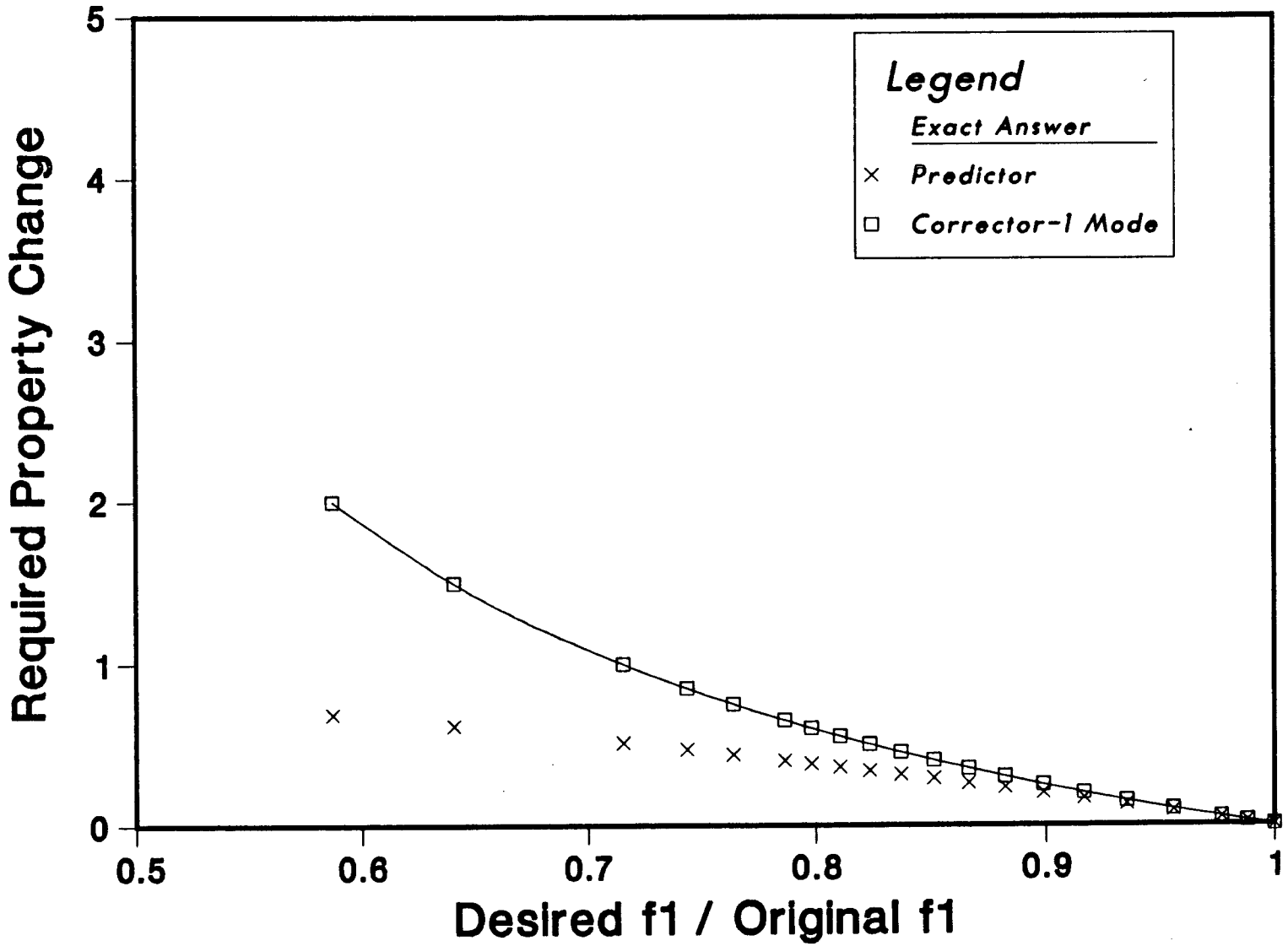


Figure 5.4 Effect of Changing First Frequency by Changing Density of Element 2

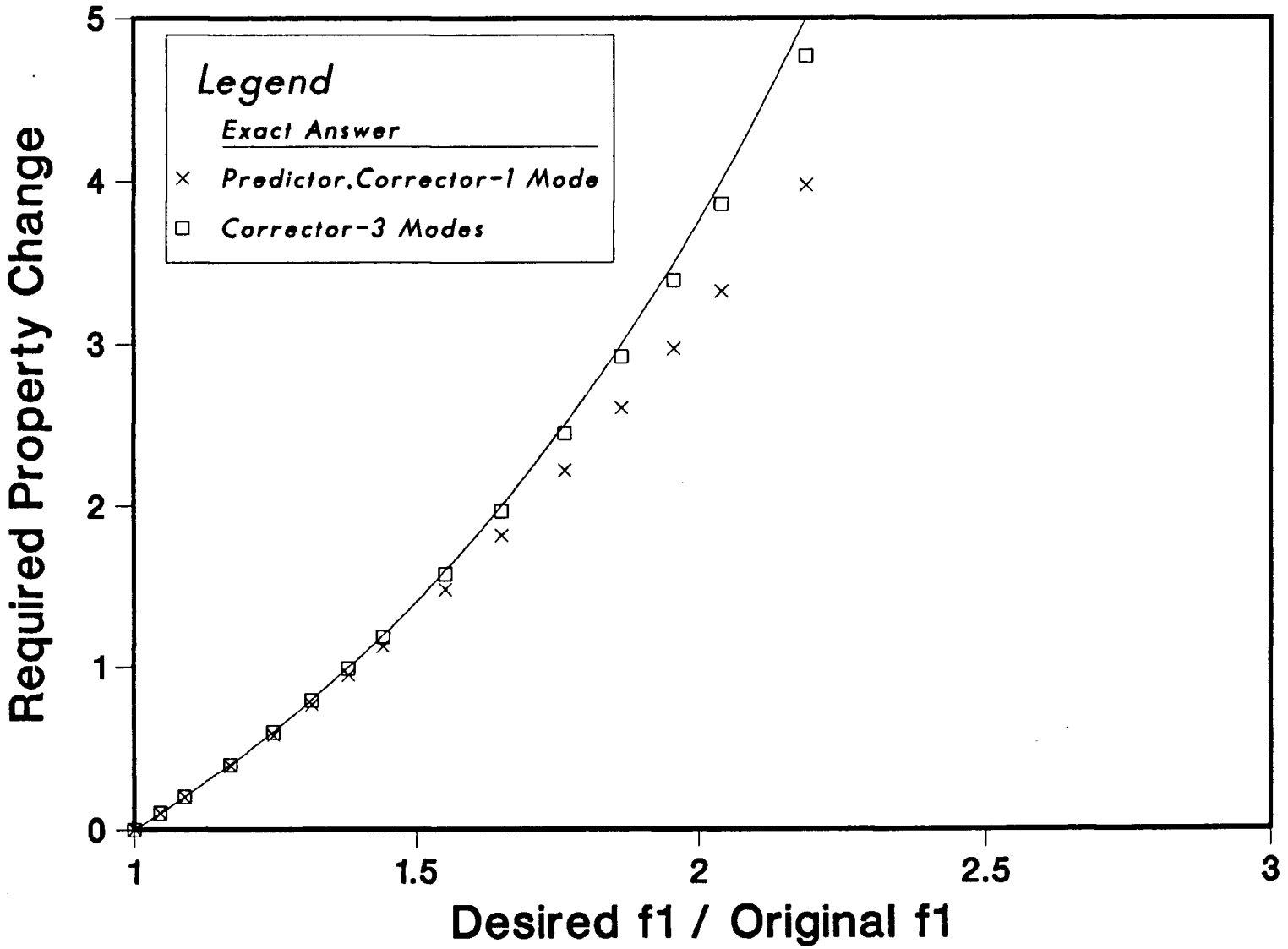


Figure 5.5 Effect of Changing First Frequency by Changing E of Element 1

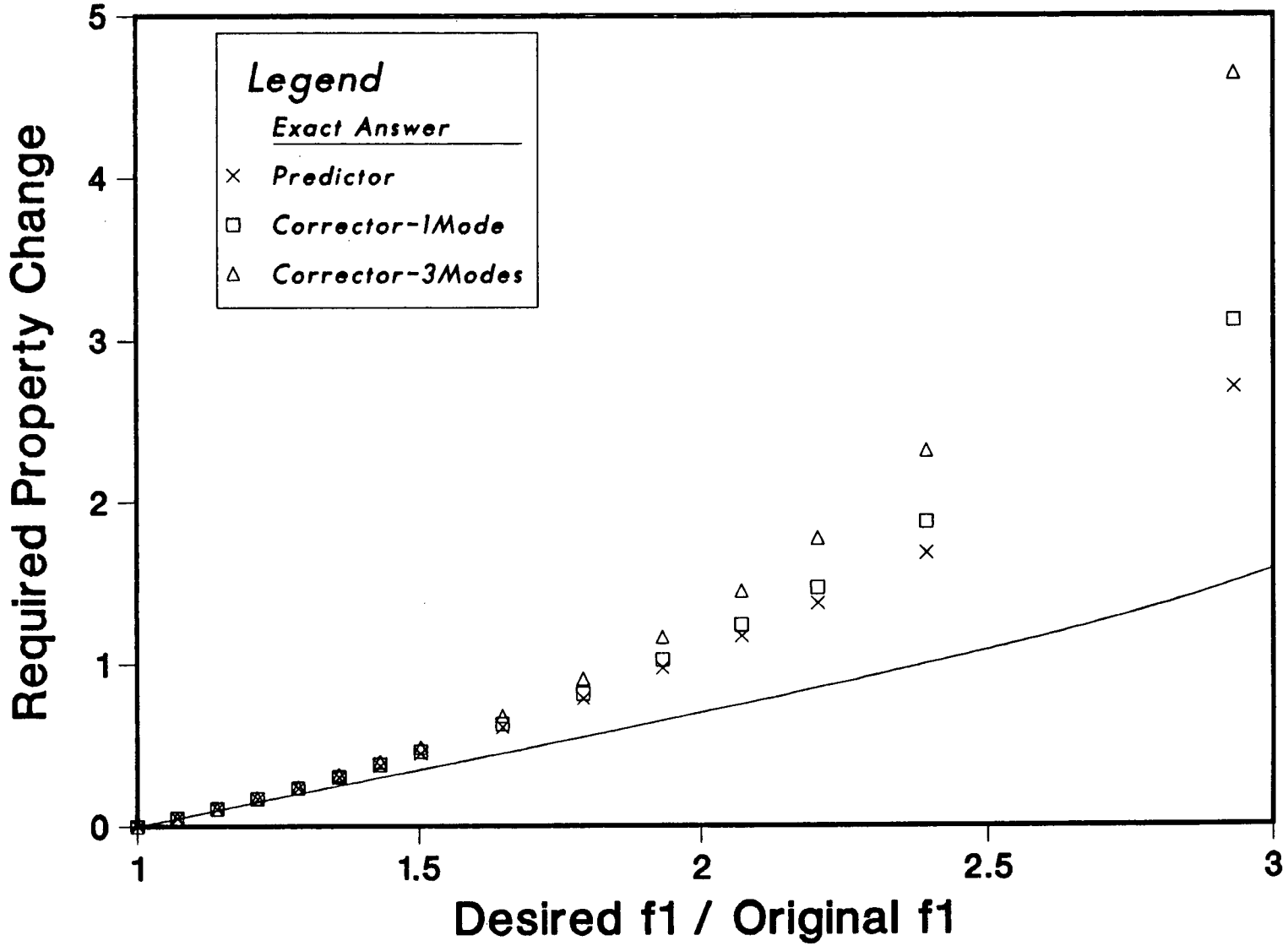


Figure 5.6 Effect of Changing First Frequency by Changing b Dimension of Element 1

Investigating the effect of changing the density of element 2 on the first mode shows that the mode shape does not change in this case. This is reflected in the general equation results in that they are exact when INSTRUM is given either 1 or 3 modes. The linear equation results are poor, but for structural changes that change the mass of the system this is typically the case as explained in Chapter 4.

Investigating the effect of changing E of element 1 on the first mode shows that the first mode shape does not change appreciably for this type of structural change. This is reflected in the relatively high accuracy of the one mode solution. As would be expected the 3 mode solution gives improved accuracy and as shown in Fig. 5.5 provides extremely reliable results for large frequency changes. The predictor and the one mode corrector solution coincide as in this case only a stiffness modification is made and in all such cases the linear perturbation equation and the general perturbation equation are identical.

It may also be noted that for the above two property changes the INSTRUM solutions are less than or equal to the exact solutions in every case. These are two examples of linear property changes, i.e. the $[\Delta K]$ that comes about as a result of a certain fractional change α_{ep} which can be represented identically as

$$[\Delta K] \equiv \alpha_{ep} [k_{ep}] .$$

Changing the b dimension of element 1 comprises a nonlinear change, i.e. a change in which the $[\Delta K]$ that comes about as a result of a certain fractional change α_{ep} can only be approximated as

$$[\Delta K] \approx \alpha_{ep} [k_{ep}] .$$

As can be seen, this has a great effect on the possible accuracy that can be obtained.

No longer are all the INSTRUM results less than or equal to the exact result. In fact all are greater than the exact result. Also no longer does the general equation provide improved accuracy over the linear result. Investigation into this problem shows that the linear approximation to the changed mode shape is quite accurate, and so is not to blame for the inaccuracy.

The general perturbation equation used in INSTRUM for this type of problem is shown in Chapter 4 by equation (4.4). As mentioned in Chapter 4 this equation is exact if the exact changed mode shape is used and if the $[\Delta K]$ given by a structural change is given exactly by α_{ep} $[k_{ep}]$. The general perturbation equation was shown to come from a Rayleighs quotient approach in Chapter 2, and thus the equation is expected to give a good result as long as a good approximation to the changed mode shape is used. The only possible conclusion is that the error results from the fact that this structural change is a nonlinear stiffness property change which cannot be expressed accurately by α_{ep} $[k_{ep}]$.

5.2.2 Battery Pack Solid Element Model

This problem shows the exceptional accuracy that can be obtained for a large problem when the mode shape corresponding to the frequency being shifted does not change as a result of the structural change. The structural model of a battery pack is shown in Fig. 5.7. It consists of 4 solid elements comprising a cylindrical shape (the battery) surrounded by 32 solid elements representing the potting compound. The outside nodes are constricted to be fixed and the battery nodes inside are free to move. There are 68 nodes in all with 408 total degrees of freedom. The first three mode shapes are shown in Fig. 5.8.

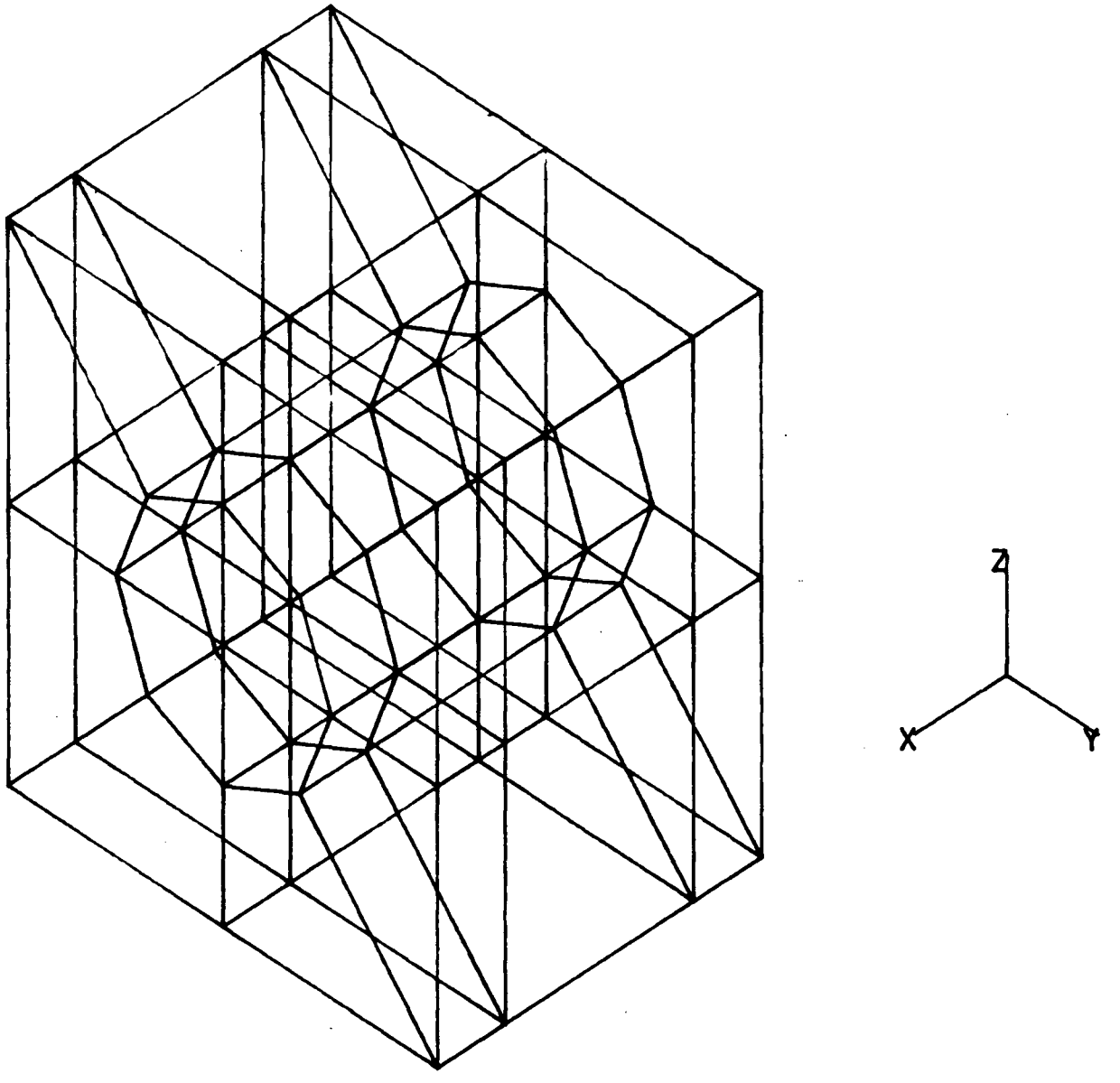


Figure 5.7 Structural Model of Solid Element Battery Pack

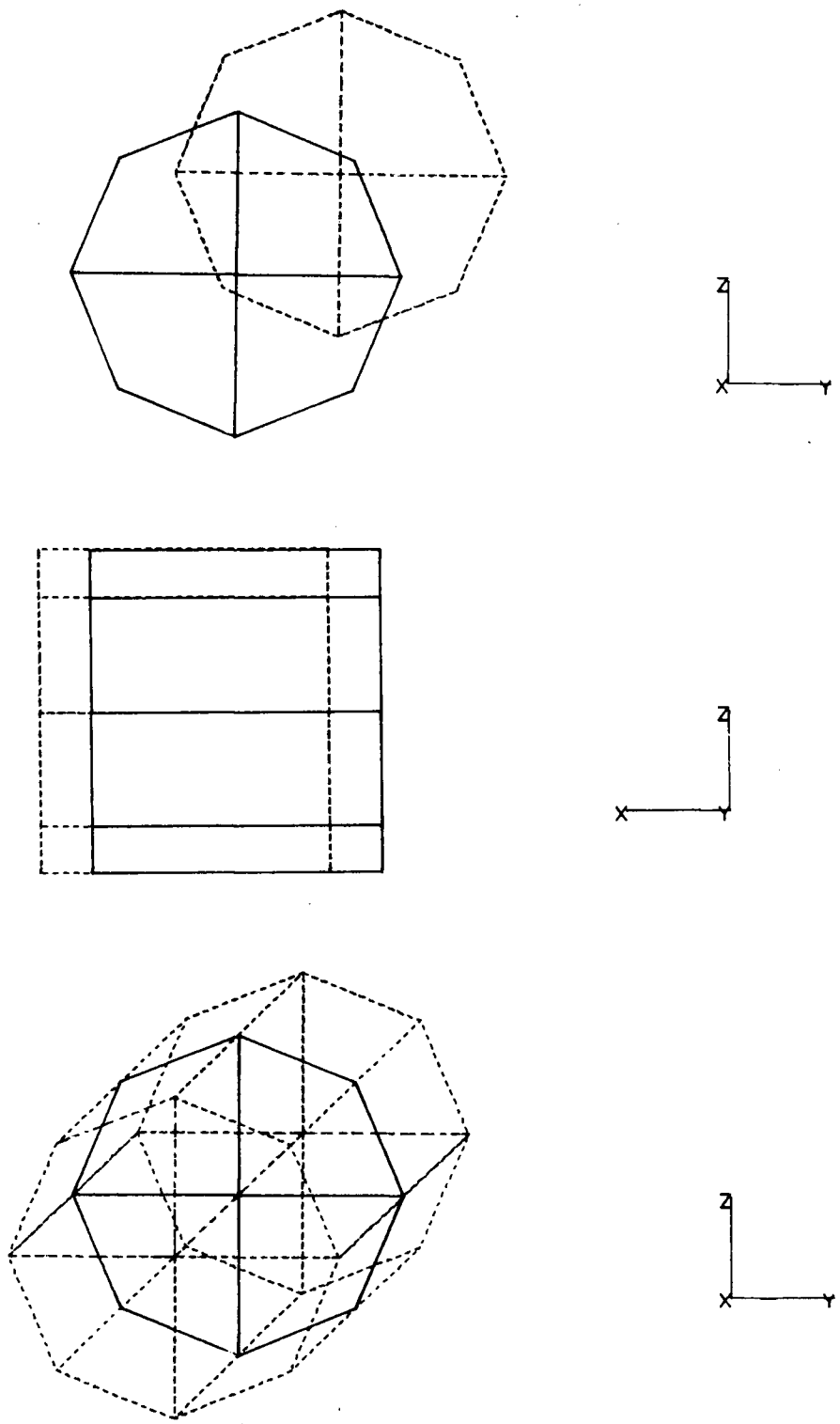


Figure 5.8 Mode Shapes for Solid Element Battery Pack

The INSTRUM problem is to increase the first natural frequency by varying Young's Modulus of the potting compound. The INSTRUM results are shown in Fig. 5.9. Changing E of the potting compound does not change the mode shapes, and since changing E affects only the stiffness properties of the system the linear and general equation solutions all give the exact answer for any sized frequency shift.

5.2.3 Box Model with Superstructure

This problem shows the performance of INSTRUM on a large problem when the structural change affects predominently the mass properties of the system. Figure 5.10 shows a box model made up of 19 quadrilateral membrane elements consisting of 56 nodes and 336 total degrees of freedom. Also shown is the first vertical bending mode of the model. Figure 5.11 shows the INSTRUM results for changing this mode by increasing the thickness of the 5 elements comprising the superstructure (INSTRUM filters out the rigid body modes and so they are not used in the calculation).

The linear equation results are not good but they are not expected to be since the thickness change involves a change in mass properties of the system (see Chapter 4). The general equation results, however, show good accuracy reflecting the improvement that is expected when using the general equation for structural changes that affect the mass properties of a system.

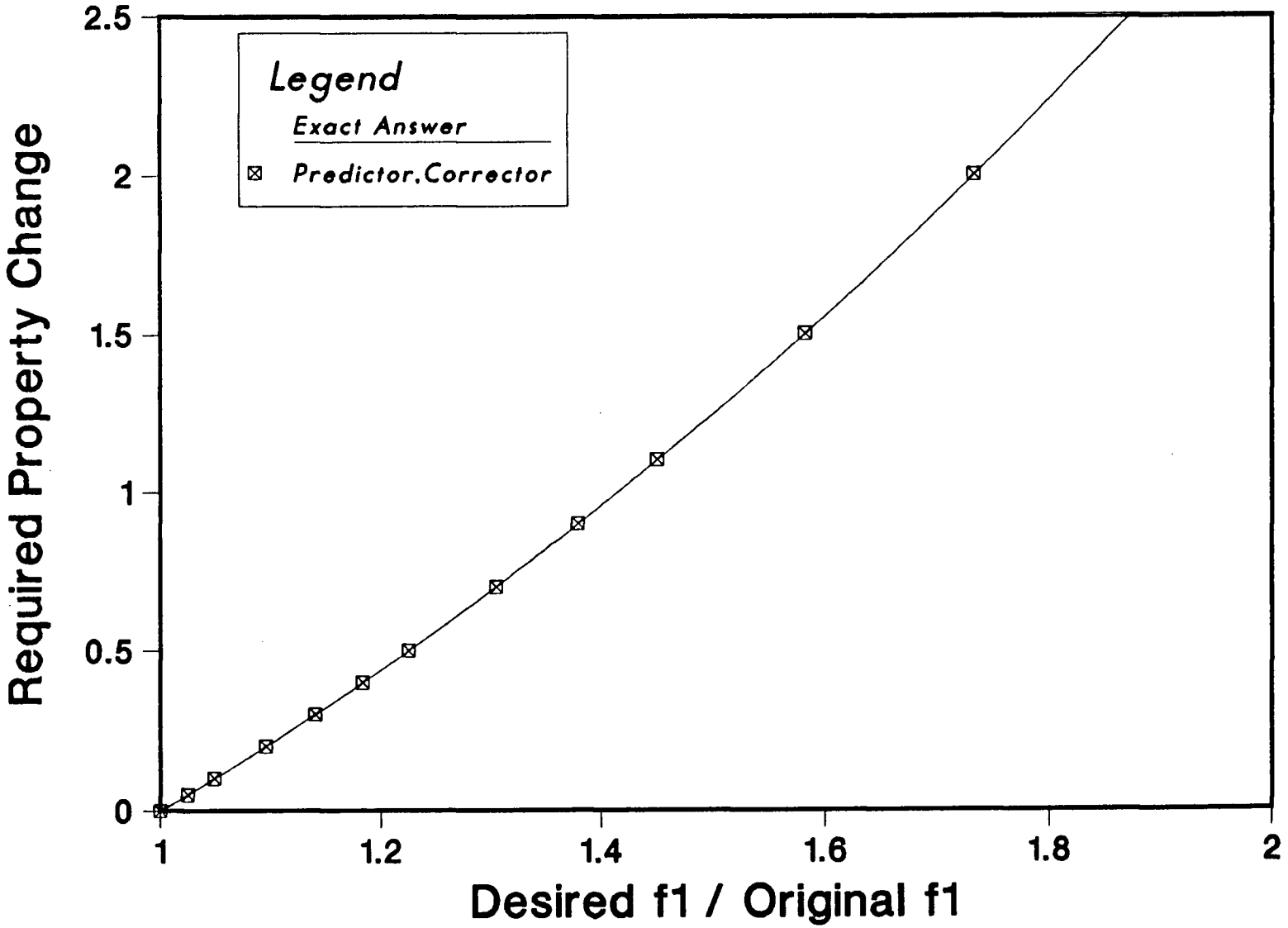


Figure 5.9 Effect of Changing First Frequency by Changing E of Potting Compound

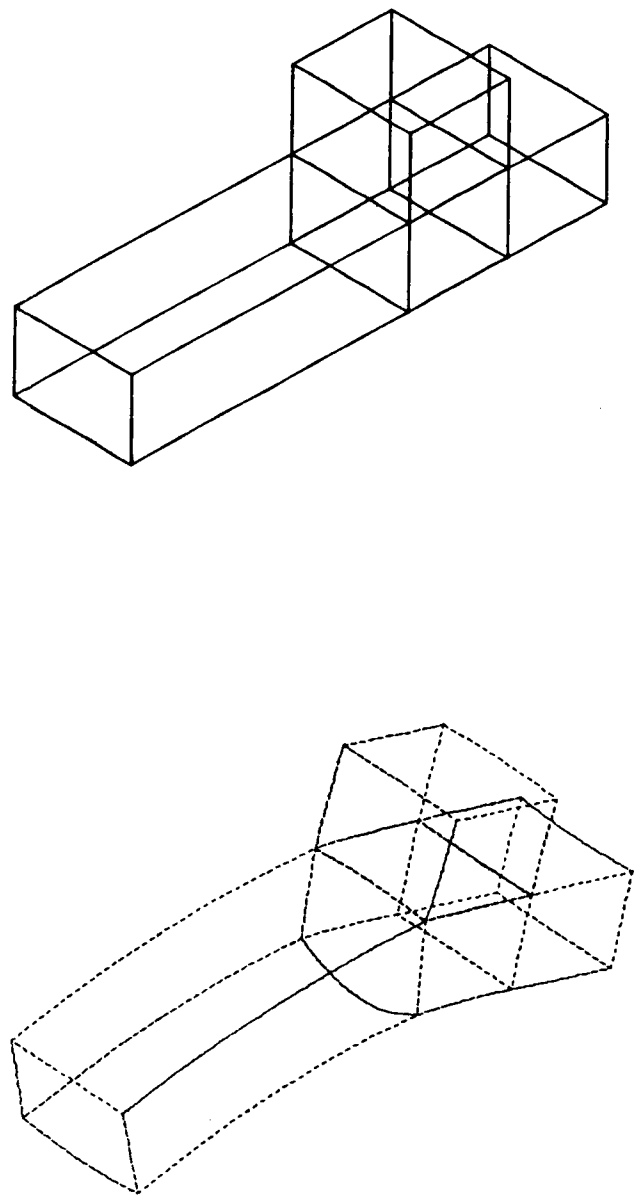


Figure 5.10 Structural Model of Box with Superstructure and Fundamental Vertical Mode

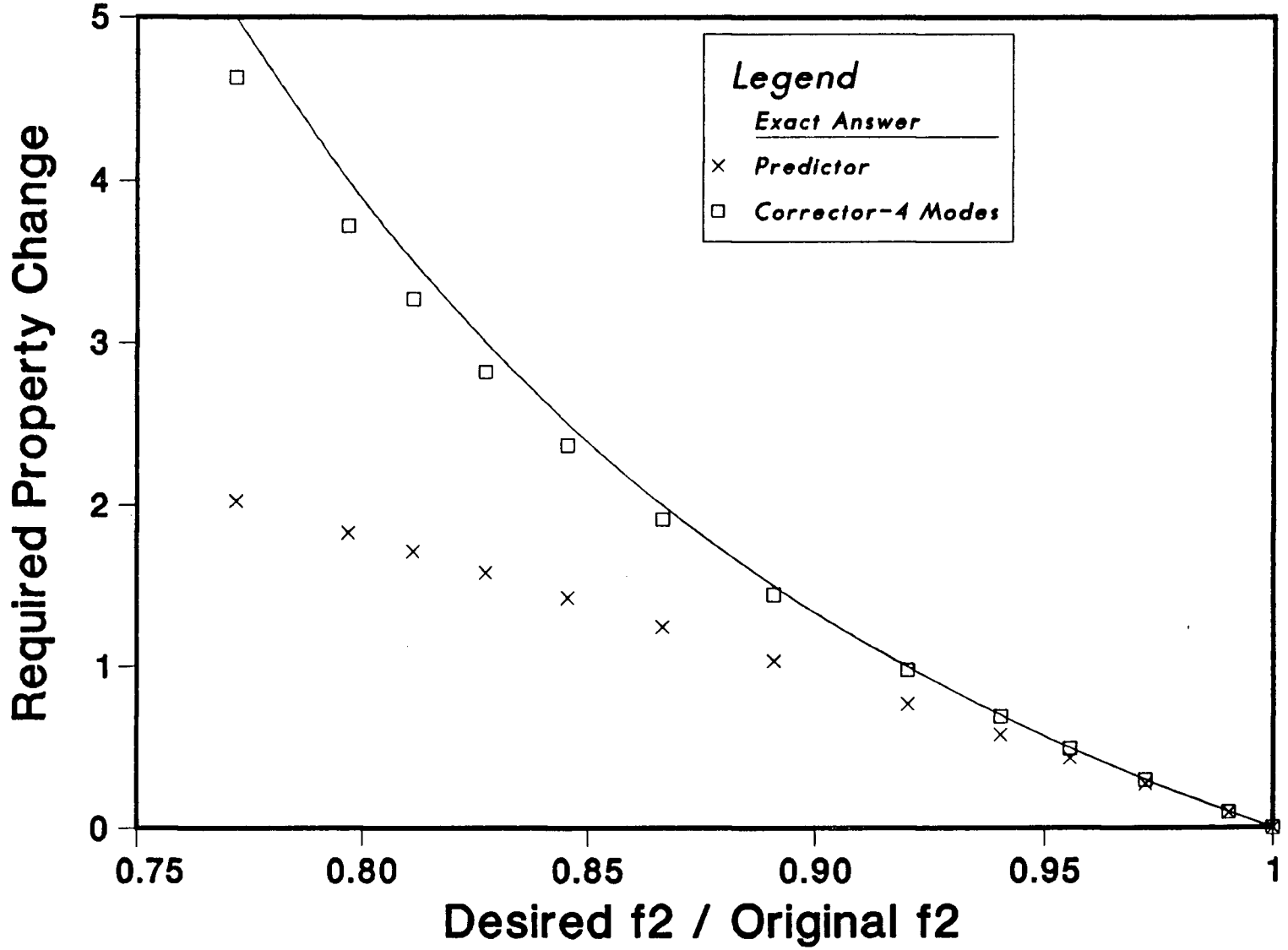


Figure 5.11 Effect of Changing Fundamental Mode by Changing Thickness of Superstructure Elements

5.2.4 Thick Thin Shell Model

This problem shows the behaviour of INSTRUM when a structural change drastically affects the mode shape in question. Figure 5.12 shows the structural model for a curved shell consisting of 4 thick thin shell elements which is simply supported at the corners.

Figure 5.13 shows the INSTRUM results for changing E of all 4 elements uniformly. The mode shapes do not change for this type of structural change and as expected the linear and general results are all exact.

Figure 5.14 shows the INSTRUM results for changing E of element 1 on changing the lowest natural frequency. Investigation shows that the first mode shape changes drastically in this case. The INSTRUM results are inaccurate. This is due to the fact that the mode shape change is such that equation (2.26a) can not approximate it no matter how many mode shapes it is given originally.

5.2.5 Ten Element Cantilever Beam Model

This problem shows the care that must be taken when using INSTRUM to increase the lowest frequency of a system. Consider the problem of increasing the lowest frequency of a cantilever beam by changing the bending moment of inertia of the whole beam. The structural model is composed of 10 general beam elements. From experience it could be assumed that a uniform stiffness change such as this will not affect the mode shape and that only the first mode shape need be used in INSTRUM.

The INSTRUM results are shown in Figure 5.15. Note the exact answer becomes a vertical line at a certain point. This corresponds to the fact that increasing the moment of inertia increases all the bending modes proportionally, but does

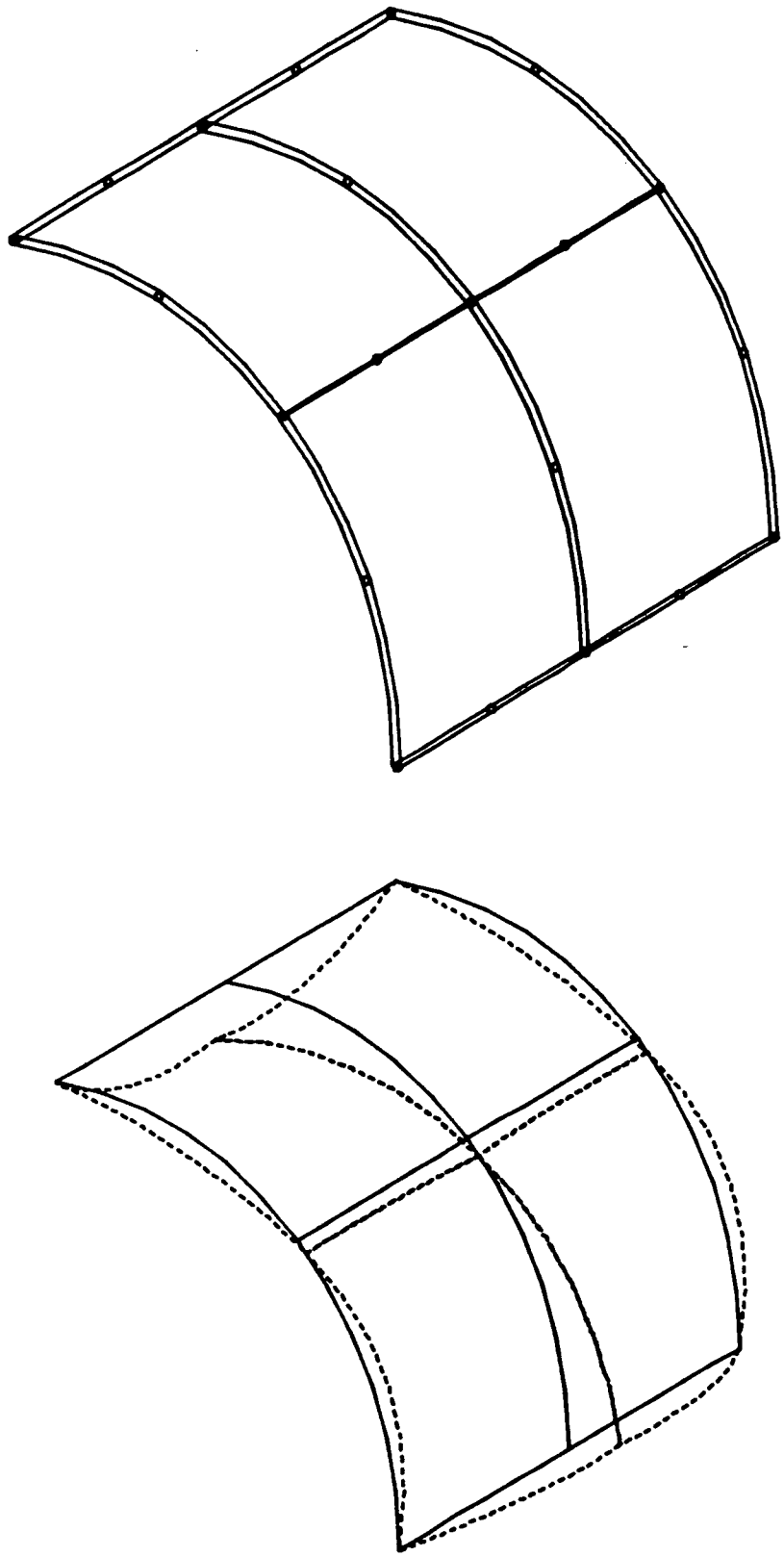


Figure 5.12 Curved Shell Model Composed of Thick Thin Shell Elements and First Mode Shape.

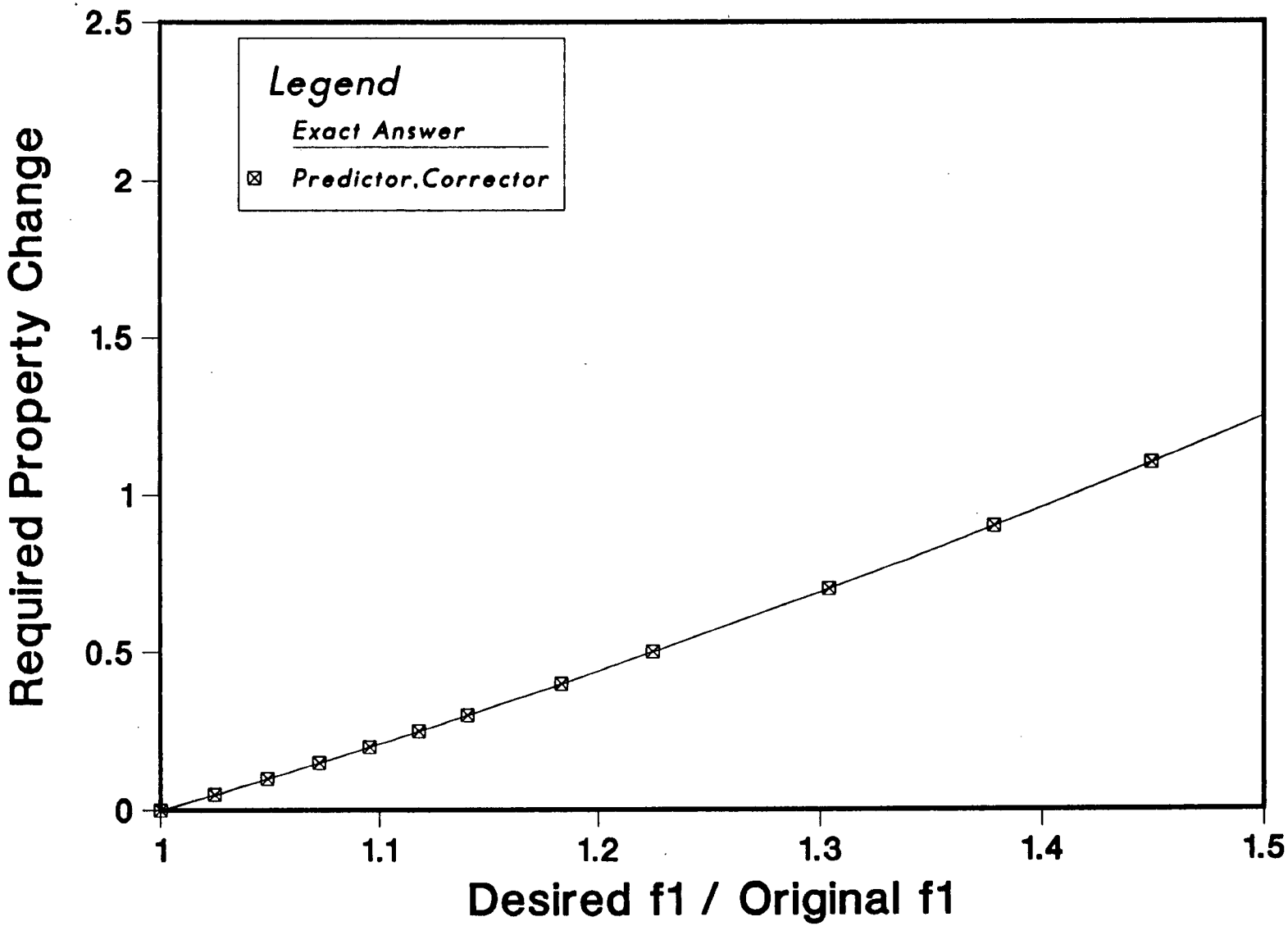


Figure 5.13 Effect of Changing First Frequency by Changing E of all 4 Elements

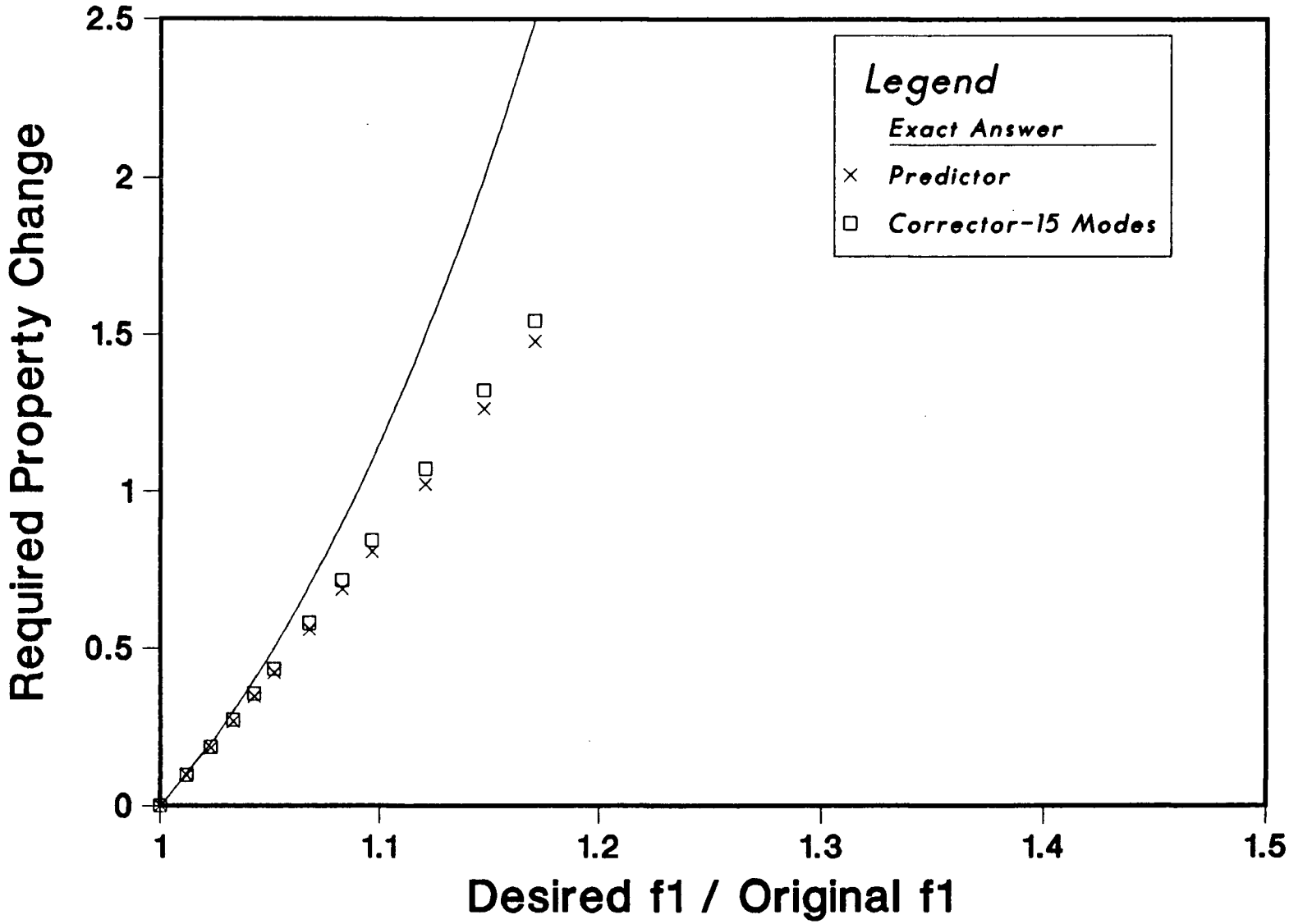


Figure 5.14 Effect of Changing First Frequency by Changing E of Element 1

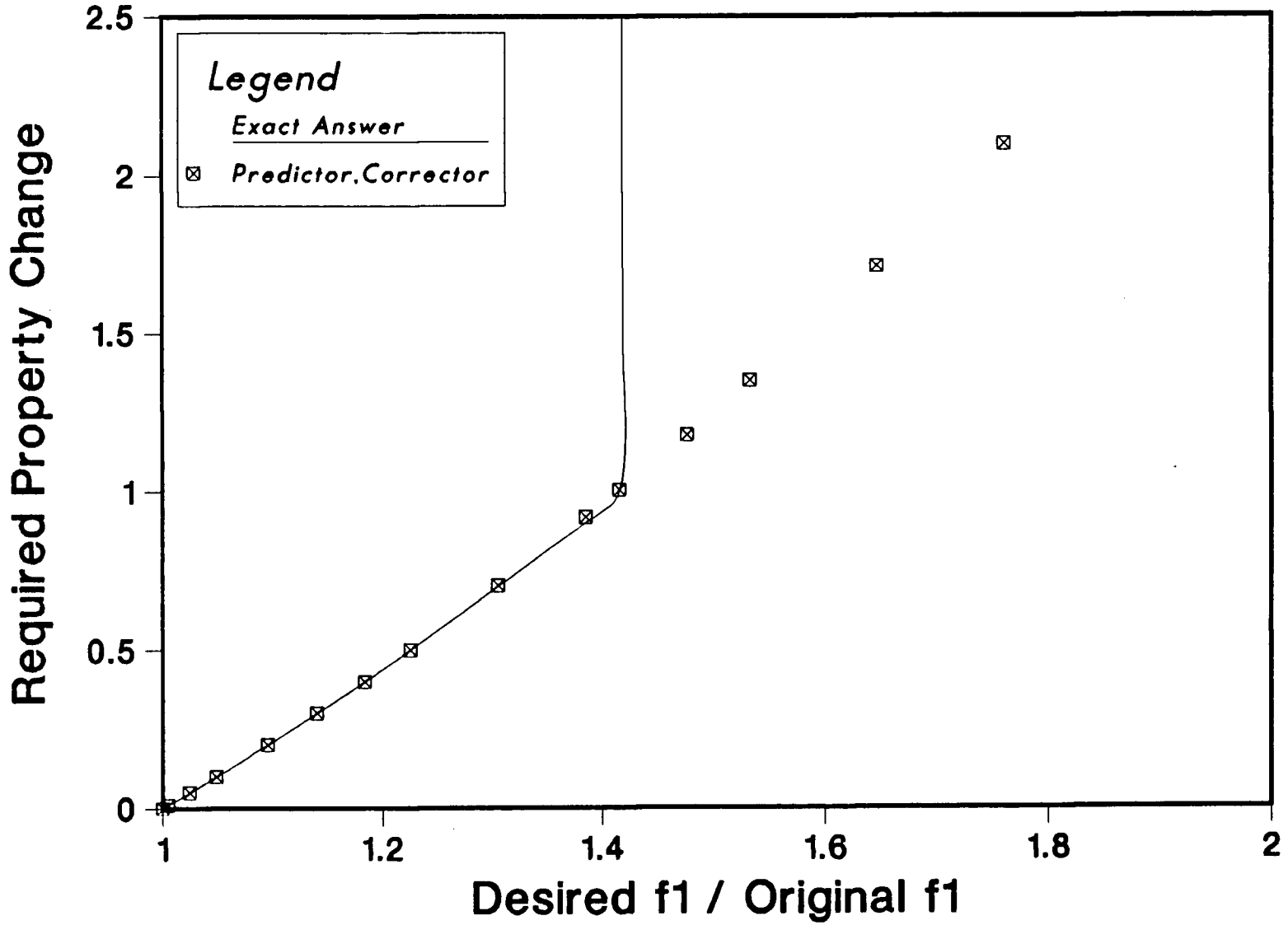


Figure 5.15 Effect of Changing Fundamental Frequency by Changing I of Whole Beam

not affect the axial modes. At a certain point the first axial mode becomes the fundamental mode and no change in inertia will affect it. INSTRUM will not predict this occurrence and it keeps on predicting the structural change required to increase the original bending frequency. Thus for this type of problem care must be exercised to understand the effects of structural changes in consideration of the physical mechanism determining the mode shapes.

5.3 Equation Solver Behaviour

A 10 element cantilever beam problem is used to evaluate the performance of the equation solver in INSTRUM. Recall that the equation solver is used when the number of possible structural changes is different than the number of frequency constraints. The incremental features of INSTRUM will also be applied to this problem and the property change considered will be the density.

INSTRUM has several user input parameters that affect how a solution is determined using the iterative equation solver. As explained in Chapter 2 the user can specify either a minimum weight or minimum change (default) optimization criteria. The user can also emphasize any of the three terms of the functional by manually inputing values of the coefficients of these terms β_I, β_{II}, or β_{III} (default = 1). These terms are the optimization term (minimum weight or minimum change - emphasized by increasing β_I), the modal objective term (either the linear or general equation error - emphasized by increasing β_{II}), and the penalty function term (decides if any bounds on the changes are violated - emphasized by increasing β_{III}).

The problem posed was to change the first frequency from .557 Hz to .3 Hz by changing the density of all 10 elements individually. Only one mode was used in VAST, and in a series of runs the equation solver parameters were varied and the incremental algorithm used. The series of structural changes as returned by INSTRUM were applied to the original finite element model and the finite element program VAST was run on these models to determine the actual frequencies that would result.

Table 5.2 shows the values of f1 obtained from the VAST runs using the INSTRUM results - recall the desired answer is f1 = 0.3 Hz. The results for 1 linear increment are in general poor, but this is to be expected since from Chapter 4 the linear equation tends to give poor results when mass changes are involved. It is interesting to note, though, that 2 linear increments give good results. This is due to the updating procedure explained in Chapter 2 which lessens the error that comes from the frequency term in the linear equation. The general incremental solution is excellent when the minimum change or the minimum weight solver is used and the frequency constraint term is emphasized. Emphasizing the optimization term β_I has the effect of decreasing the accuracy of the answer by shifting the emphasis away from satisfying the perturbation equations.

Table 5.2 Effect of INSTRUM and Solver Parameters on Frequency Prediction.

	Predictor Phase	Corrector Phase	2 Predictor Phase Increments	2 Corrector Phase Increments
Default Values (MC, $\beta \equiv 1$)	.425959	.299982	.309866	.299982
MW Solver $\beta_I = 1000$	.425953	1.27636	.329831	NO SOLUTION could be determined
MC Solver $\beta_I = 100$	.425961	.448279	.372135	.354967
MW Solver $\beta_{II} = 1000$	.425963	.299980	.309867	.299980
MC Solver $\beta_{II} = 100$	.425959	.299982	.309866	.299982

Tables 5.3 and 5.4 show the value of the optimization terms. The weight criteria is defined as the sum of all 10 fractional changes and the change criteria is defined as the sum of the squares of all 10 fractional changes.

Table 5.3 Sum of Structural Changes Showing the Effect of the Minimum Weight Criteria

	Predictor Phase	Corrector Phase	2 Predictor Phase Increments	2 Corrector Phase Increments
Default Values	5.60740	19.3507	17.5198	19.3507
MW Solver $\beta_I = 1000$	3.53060	-8.17230	3.44230	---
MC Solver $\beta_I = 100$	4.66470	2.35730	9.33870	6.20200
MW Solver $\beta_{II} = 1000$	5.29640	18.8919	17.2261	18.8919
MC Solver $\beta_{II} = 100$	5.60740	19.3507	17.5198	19.3507

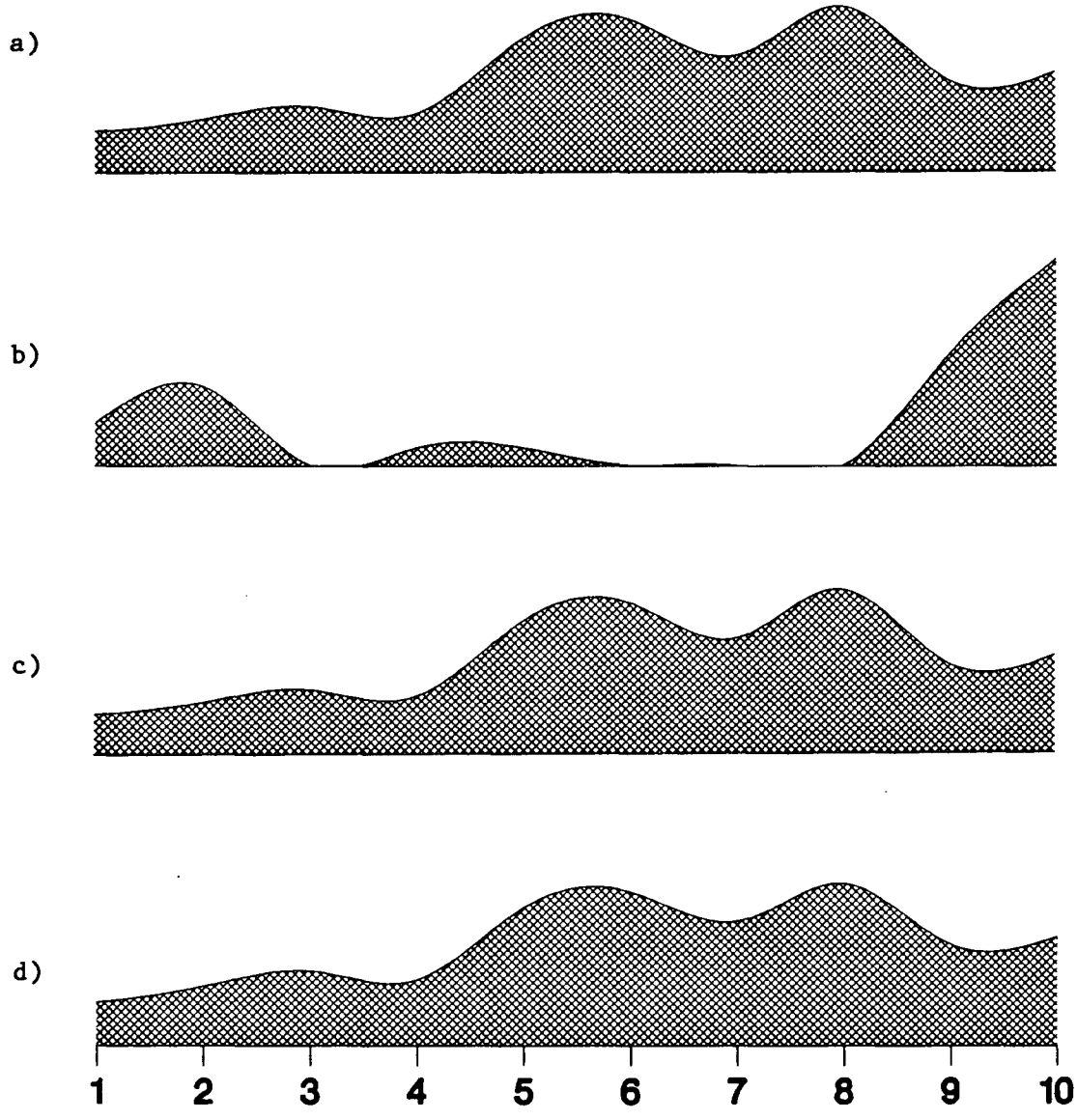
Table 5.4 Sum of Squares of Structural Changes Showing Effect of Minimum Change Criteria

	Predictor Phase	Corrector Phase	2 Predictor Phase Increments	2 Corrector Phase Increments
Default Values	4.19300	49.9738	44.2683	49.9738
MW Solver β_I = 1000	2.88030	6.85850	33.4527	---
MC Solver β_I = 100	3.44360	1.35656	12.6189	1.35656
MW Solver β_{II} = 100	4.19190	49.9711	44.2779	49.9711
MC Solver β_{II} = 100	4.19300	49.9738	44.2683	49.9738

The results show that the minimum change or the minimum weight criteria are best satisfied when β_I is increased, but from Table 5.2 this occurs at the expense of an accurate answer. The one exception to that is the two predictor phase solution for the minimum weight solver, which provides a good minimum weight with a small loss in accuracy. For both the mimimum change and the minimum weight the best optimized solution that also provides a completely

accurate answer is given by the two predictor phase increments. The one and two increment corrector phase solutions are identical in all cases.

The mass distributions that come about from implementing the INSTRUM results are shown in Figure 5.16. The results are in general disappointing. The distribution for the minimum weight solution with 2 predictor phase increments and $\beta_I = 1000$ is of the general form that would be expected for a minimum weight solution of this problem but is physically unrealizable in that most sections are reduced to almost zero.



a) Minimum weight solver, 2 predictor increments

b) Minimum weight solver, β_I = 1000, 2 predictor increments

c) Minimum change solver, 2 predictor increments

d) Minimum change solver, 1 predictor-corrector analysis

Figure 5.16 Mass Distributions for Cantilever Beam

CHAPTER 6

CONCLUSIONS

A perturbation based dynamic redesign method has been developed as the computer program INSTRUM. INSTRUM can be used to solve redesign problems which have been traditionally solved using a trial and error approach. INSTRUM is most effective for problems involving frequency constraints and which have a small (<5) number of structural parameters. Mode shape constraint problems can be solved using INSTRUM but this aspect was not investigated.

There are two distinct types of property changes that can occur and INSTRUM performs differently for each. Problems that involve linear property changes receive the most accurate results from INSTRUM. INSTRUM gives less accurate results for nonlinear property changes and in such cases the predictor analysis solution can be more accurate than the corrector analysis solution. A more accurate means of approximating the change in elemental stiffness that arises from a nonlinear property change is needed.

The following applies to problems involving linear property changes. In general for problems in which the structural change does not affect the mode shapes of the system the INSTRUM corrector phase results will be exact for any desired frequency shift. For problems in which the mode shapes are not affected drastically INSTRUM proves to be accurate for large frequency shifts ("large" being between 10% upwards of 100% or even higher, depending on the problem). For problems in which the mode shapes are affected drastically INSTRUM will not be able to predict the new mode shapes and the answers will not be accurate for large frequency shifts.

The general perturbation equation used in INSTRUM can be derived using a variational approach. This allows bounds to be imposed on the solutions of problems in which it is desired to change the fundamental frequency of a system by making one linear property change.

The equation solver used in INSTRUM is not adequate for problems that involve many more structural change parameters than the number of frequency constraints. The equation solver gave either poorly optimized or physically unrealizable results for a problem with ten structural parameters and one frequency constraint. Changing the values of the coefficients of the three terms of the functional does not improve the result. In particular increasing β_I may make the equation solver satisfy the optimizing criteria without satisfying the frequency constraint.

REFERENCES

1. Rayleigh, B., Theory of Sound, Dover N.Y., 1945, Vol. 1, pp. 113-118, 214-217.

2. Turner, M.J., "Design of Minimum Mass Structures with Specified Natural Frequencies", AIAA Journal, Vol. 5, No. 3, March 1967, pp. 406-412.

3. Taylor, J.E., "Minimum Mass Bar for Axial Vibration at Specified Natural Frequency", AIAA Journal, Vol. 5, No. 10, October 1967, pp. 1911-1913.

4. Sheu, C.Y., "Elastic Minimum-Weight Design for Specified Fundamental Frequency", Int. J. Solids and Structures, Vol. 4, 1968, pp. 953-958.

5. Pierson, B.L., "A Survey of Optimal Structural Design Under Dynamic Constraints", Int. J. Numerical Methods in Engr., Vol. 4, 1972, pp. 491-499.

6. Taylor, J.E., "Scaling a Discrete STructural Model to Match Measured Modal Frequencies", AIAA Journal, Vol. 15, No. 11, November 1977, pp. 1647-1649.

7. Bellagamba, L. and Yang, T.Y., "Minimum-Mass Truss Structures with Constraints on Fundamental Natural Frequency", AIAA Journal, Vol. 19, No. 11, November 1981, pp. 1452-1458.

8. Stetson, K.A., "Perturbation Method of Structural Design Relevant to Holographic Vibration Analysis", AIAA Journal, Vol. 13, No. 4, April 1975, pp. 457-459.

9. Stetson, K.A., and Palma, G.E., "Inversion of First-Order Perturbation Theory and Its Application to Structural Design", AIAA Journal, Vol. 14, No. 4, April 1976, pp. 454-460.

10. Stetson, K.A., Harrison, I.R., and Palma, G.E., "Redesigning Structural Vibration Modes by Inverse Perturbation Subject to Minimal Change Theory", Computer Methods in Applied Mechanics and Engineering, Vol. 16, 1978, pp. 151-175.

11. Stetson, K.A., and Harrison, I.R., "Redesign of Structural Vibration Modes by Finite-Element Inverse Perturbation", J. of Engr. for power, Vol. 103, April 1981, pp. 319-325.

12. Sanstrom, R.E., "Inverse Perturbation Methods for Vibration Analysis", Proceedings, NATO Advance Study Institute on Optimization of Distributed Parameter Structural Systems, University of Iowa, May 20 - June 4, 1980.

13. Sanstrom, R.E., and Anderson, W.J., "Modal Perturbation Methods for Marine Structures", Presented at the Annual Meeting of the Society of Naval Architects and Marine Engineers, Nov. 1982, Transactions of the Society of Naval Architects and Marine Engineers, Vol. 90, 1982.

14. Kim, K.O., Anderson, W.J., and Sandstrom, R.E., "Nonlinear Inverse Perturbation Method in Dynamic Analysis", AIAA Journal, Vol. 21, No. 9, Septebmer 1983.

15. Parsons, M.G., "Optimization Methods for use in Computer-Aided Ship Design", First Ship Technology and Research Symposium, Washington, D.C., August 26-29, 1975.

16. Hoff, C.J., Bernitsas, M.M., Sandstrom, R.E., and Anderson, W.J., "Nonlinear Incremental Inverse Perturbation Method for Structural Redesign", Proceedings, AIAA/ASME/ASCE/AHS24'th Structures, Structural Dynamics and Materials Conference, Lake Tahoe, Nev., May 1983; also AIAA Journal, Vol. 22, No. 9, September 1984, pp. 1304-1309.

17. Hoff, C.J., Linear Dynamic Redesign University of Michigan. User's Manual. Ann Arbor, Michigan, 1984.

18. VAST. Vibration and Strength Analysis Program. Defense Research Establishment Atlantic, Canada.

19. R.H. MacNeal, ed., The NASTRAN Theoretical Manual (Level 16.0), NASA-SP-221 (03), March 1976 (N79-27531, N.T.I.S.).

20. Temple, G., and Bickley, W.G., Rayleighs Principle and its Application to Engineering, Dover Press, 1956, pp. 147-149.

21. Cook, R.D., Concepts and Applications of Finite Element Analysis, Wiley, 1981, pp. 246-248.

22. Buragohain, D.N., Agrawal, S.B., and Ayyar, R.S., "A Matching Super-parametric Beam Element for Shell Beam Systems", Computers & Structures, Vol. 9, 1978, pp. 175-182.

23. Char, B.W., Geddes, K.O., Gonnet, G.H., and Watt, S.M., Maple User's Guide, WATCOM Publications, Waterloo, 1985.

24. NORWOOD, M.E., "Analysis of Shell Structures by Curved Finite Elements", DSS Contract no. 5EY.7040073 Serial 2Ey3-0365, July 1974.

APPENDIX A

PERTURBATION TERMS FOR TWO ELEMENT CANTILEVER BEAM

From equation (2.16) the 24 perturbation terms that result before the nonlinear terms are cancelled are

$$
\begin{aligned}
&[\Phi]^T [\Delta K] [\Phi] + [\Phi]^T [\Delta K] [\Delta\Phi] + [\Delta\Phi]^T [\Delta K] [\Phi] + \\
&[\Delta\Phi]^T [\Delta K] [\Delta\Phi] - [\Phi]^T [\Delta M] [\Phi] [\ \omega^2\ ] - [\Phi]^T [\Delta M] [\Phi] [\ \Delta\omega^2\ ] - \\
&[\Phi]^T [\Delta M] [\Delta\Phi] [\ \omega^2\ ] - [\Phi]^T [\Delta M] [\Delta\Phi] [\ \Delta\omega^2\ ] - \\
&[\Delta\Phi]^T [\Delta M] [\Phi] [\ \omega^2\ ] - [\Delta\Phi]^T [\Delta M] [\Phi] [\ \Delta\omega^2\ ] - \\
&[\Delta\Phi]^T [\Delta M] [\Delta\Phi] [\ \omega^2\ ] - [\Delta\Phi]^T [\Delta M] [\Delta\Phi] [\ \Delta\omega^2\ ] = \\
&[\Phi]^T [M] [\Phi] [\ \omega^2\ ] + [\Phi]^T [M] [\Phi] [\ \Delta\omega^2\ ] + \\
&[\Phi]^T [M] [\Delta\Phi] [\ \omega^2\ ] + [\Phi]^T [M] [\Delta\Phi] [\ \Delta\omega^2\ ] + \\
&[\Delta\Phi]^T [M] [\Phi] [\ \omega^2\ ] + [\Delta\Phi]^T [M] [\Phi] [\ \Delta\omega^2\ ] + \\
&[\Delta\Phi]^T [M] [\Delta\Phi] [\ \omega^2\ ] + [\Delta\Phi]^T [M] [\Delta\Phi] [\ \Delta\omega^2\ ] - [\Phi]^T [K] [\Phi] - \\
&[\Phi]^T [K] [\Delta\Phi] - [\Delta\Phi]^T [K] [\Phi] - [\Delta\Phi]^T [K] [\Delta\Phi] \qquad \text{(A.1)}
\end{aligned}
$$

The purpose of this section is to determine the magnitude of these terms for a simple problem to see if it is justifiable to cancel the terms that are higher order in Δ than one. Note that the terms in equation (A.1) are actually n x n matrices, and so when examining these matrices it is necessary to compare the individual i,j elements.

A cantilever beam composed of 2 general beam elements was the model considered. Using finite element theory this problem was solved for the baseline case, as well as for a range of problems in which the moment of inertia or the

area of the free element was modified. In each case four mode shapes were solved for, and the 24 matricies of equation (A.1) were evaluated.

It is convenient to group terms since many of the terms in the matricies on the left and right hand sides of equation (A.1) cancel. The following 8 groups together comprise 24 terms corresponding to the i'th, j'th elements of the matricies in equation (A.1)

1. $\{\psi_i\}\ ([K] - \omega_j^2\ [M])\ \{\psi_j\}$
2. $\{\psi_i\}^T\ ([K] - \omega_j^2\ [M])\ \{\Delta\psi_j\}$
3. $\{\Delta\psi_i\}^T\ ([K] - \omega_j^2\ [M])\ \{\psi_j\}$
4. $\{\Delta\psi_i\}^T\ ([K] - \omega_j^2\ [M])\ \{\Delta\psi_j\}$
5. $\{\psi_i\}^T\ \Delta([K] - \omega_j^2\ [M])\ \{\psi_j\}$
6. $\{\psi_i\}^T\ \Delta([K] - \omega_j^2\ [M])\ \{\Delta\psi_j\}$
7. $\{\Delta\psi_i\}\ \Delta([K] - \omega_j^2\ [M])\ \{\psi_j\}$
8. $\{\Delta\psi_i\}^T\ \Delta([K] - \omega_j^2\ [M])\ \{\Delta\psi_j\}$

The sum of groups 1 ... 8 should be zero for i,j = 1 ... 4 by equation (A.1). For these 8 groups the linear approximation assumes that groups 4, 6, 7, 8 are all zero, since these groups all contain only terms of higher order in Δ than 1. Group 1 is just the free vibration equation of the baseline system and so is always identically equal to zero. Groups 2 and 3 contain purely linear terms in Δ. Group 5 contains terms linear in Δ except for the $\Delta\omega^2_j\ \{\psi_i\}^T\ [\Delta M]\ \{\psi_j\}$ term.

Figures A.1 to A.4 show the magnitudes of these 8 groups as designated by the hashed bars. The structural modification is changing either the moment of inertia or the area of the free element. Each graph corresponds to a certain structural change, and a certain value of i,j. Graphed beside each of these 8 groups is its corresponding linearized value found by ignoring all terms of higher order in Δ than one. These linearized values are designated by the solid bars.

Figures A.1 and A.2 show the magnitudes of the 8 groups for changing the moment of inertia of element 2. It can be seen from Figure A.1 that for the i=1, j=1 terms the non-linear terms of groups 4,6 and 7 are important. What that corresponds to in the INSTRUM algorithm is the linear portion trying to approximate $\{\phi_1'\}^T [\Delta K] \{\phi_1'\}$ by $\{\phi_1\}^T [\Delta K] \{\phi_1\}$. An interesting result regarding this approximation is seen in Appendix D. The effect is that for structural changes that cause severe mode shape changes the nonlinear terms become the same order as the linear terms and the approximation mentioned above is invalid. For the i=1, j=2 terms the nonlinear terms of groups 4, 6, 8 are important.

Figures A.3 and A.4 show the magnitudes of the 8 groups for changing the area of element 2. Figure A.3 shows how important neglecting the non-linear term in group 5 is. Neglecting that non-linear term is the reason for the result (4.8) and the reason that the linear perturbation equations will not give an exact answer in the case of a mass change even when the mode shapes do not change. The i=1, j=2 graph again reflects the importance of that same non-linear term as well as the nonlinear terms of groups 4, 6 and 7.

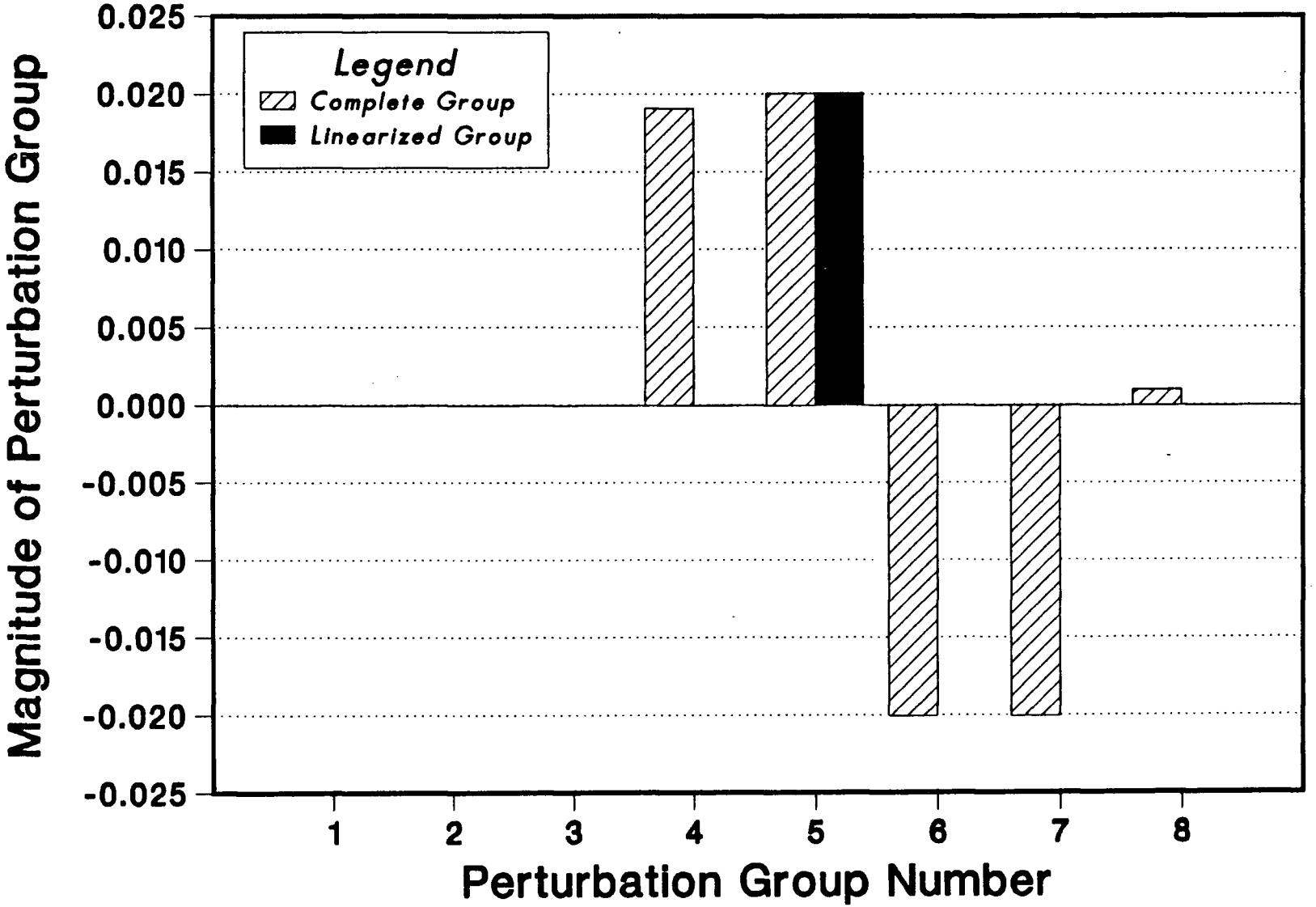


Figure A.1 i=1, j=1 terms for Changing I of Element 2

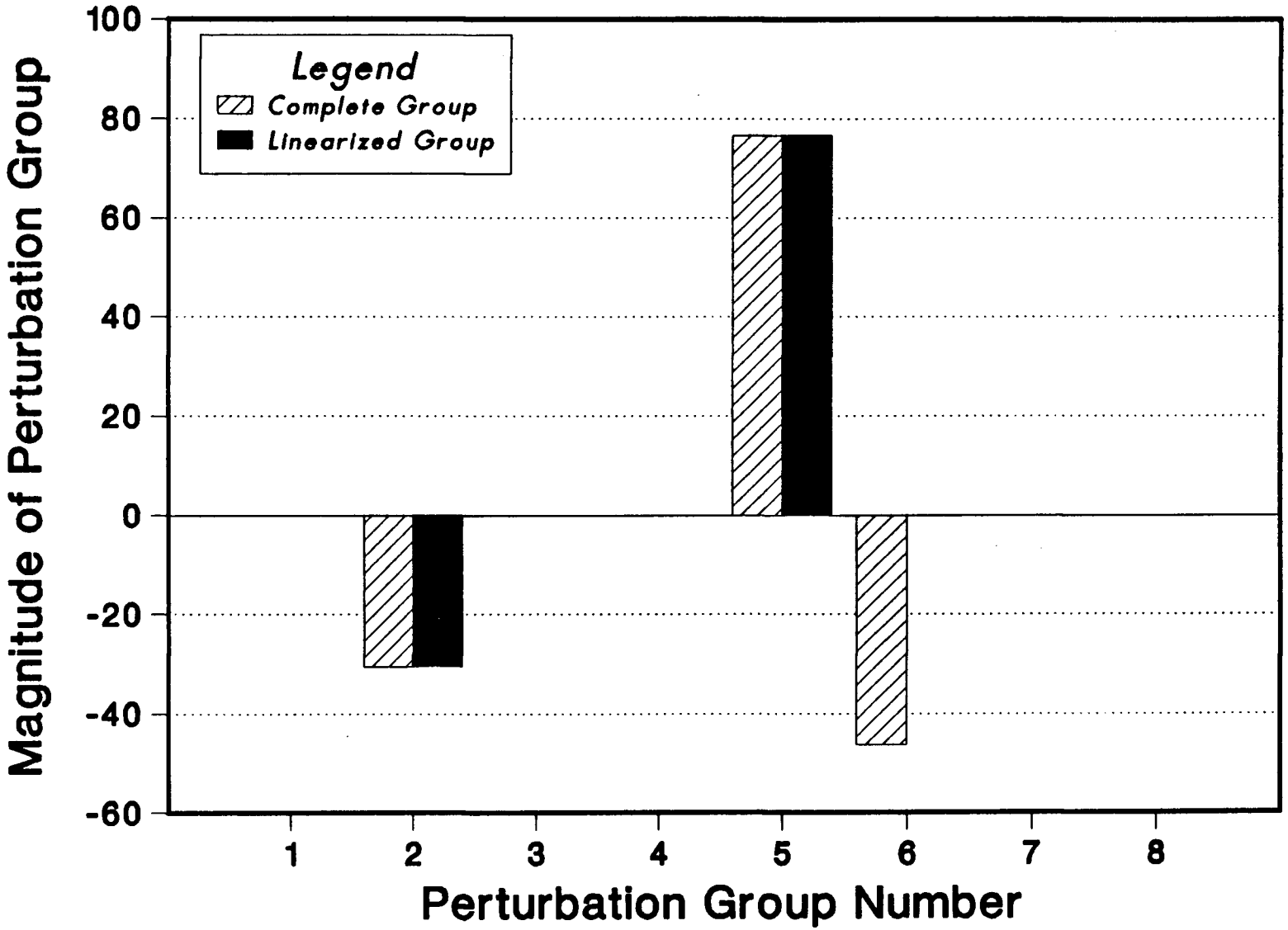


Figure A.2 i=1, j=2 Terms for Changing I of Element 2

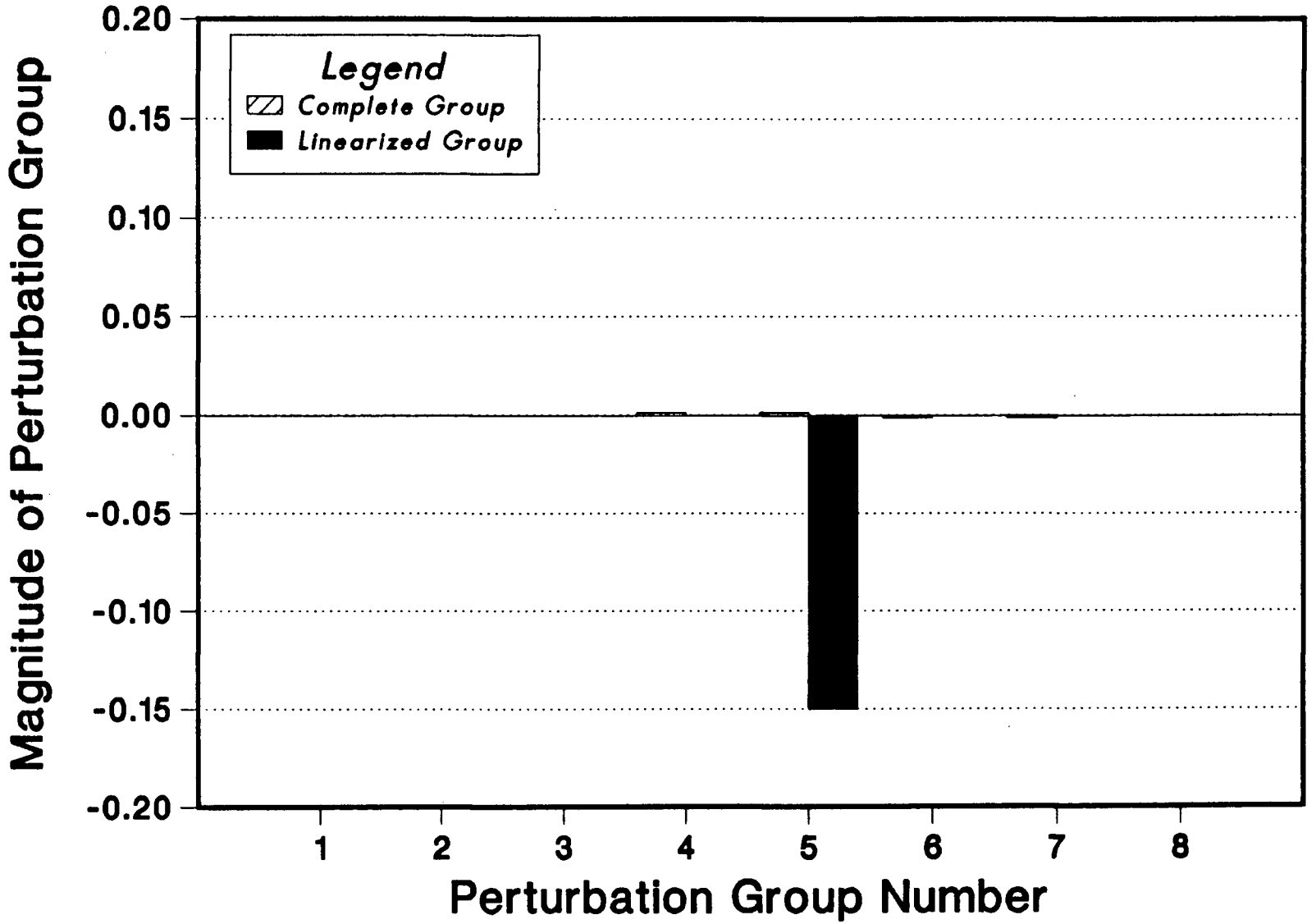


Figure A.3 i=1, j=1 Terms for Changing Area of Element 2

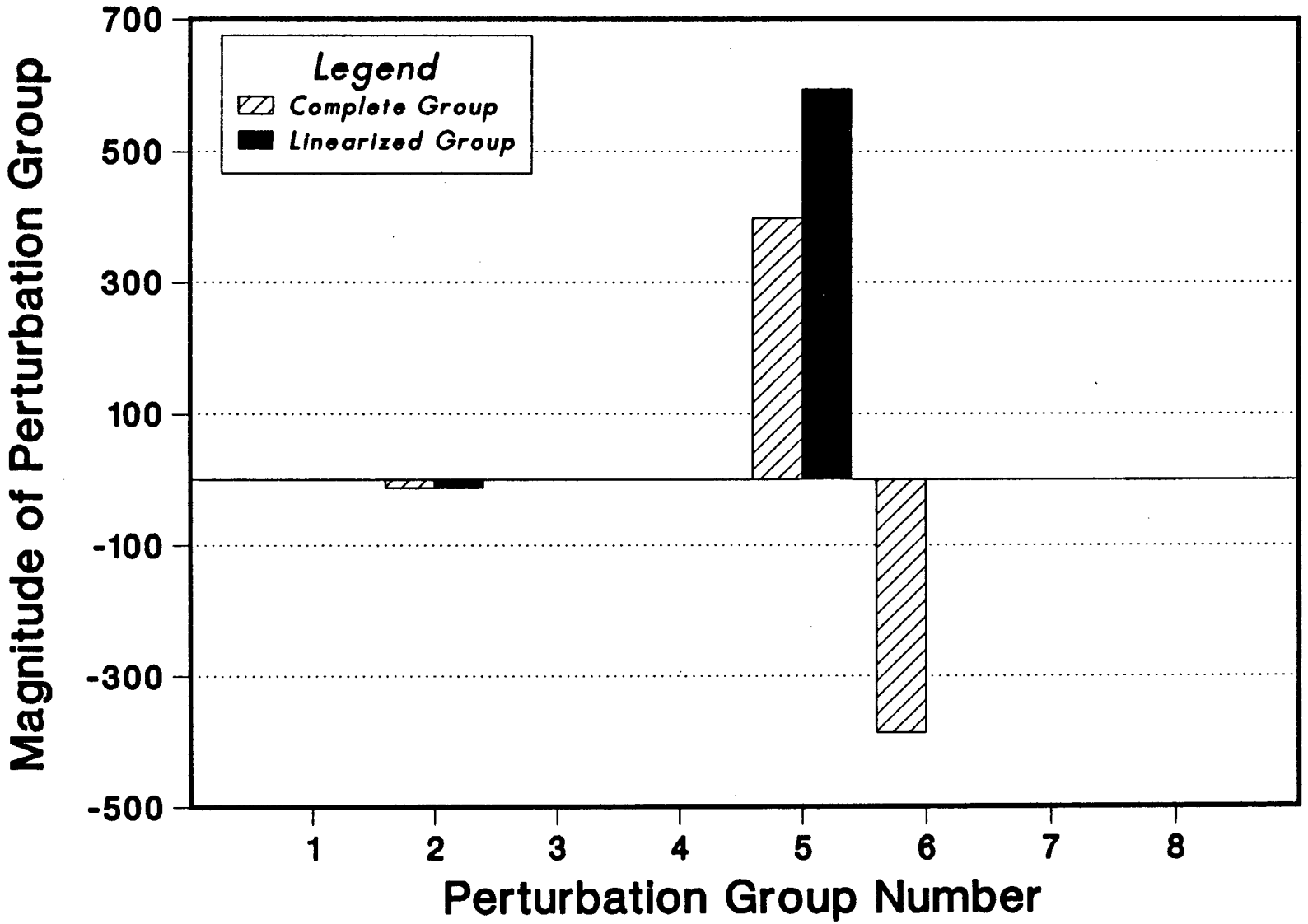


Figure A.4 i=1, j=2 Terms for Changing Area of Element 2

APPENDIX B

FORM OF $\bar{k}_{ij}$ MATRICIES FOR CURVED BEAM AND THICK THIN SHELL ELEMENTS

The general form of the stiffness matricies for the curved beam and the thick thin shell elements are shown here as they were calculated using the symbolic algebra program MAPLE discussed in Chapter 3.

B.1 Curved Beam Element

For the curved beam element the $\bar{k}_{ij}$ matrix turns out to be the following once all the necessary multiplications are carried out:

$$\bar{k}_{ij} = \begin{bmatrix} A1 & A2 & A3 & B1 & B2 & B3 \\ A4 & A5 & A6 & B4 & B5 & B6 \\ A7 & A8 & A9 & B7 & B8 & B9 \\ B10 & B11 & B12 & C1 & C2 & C3 \\ B13 & B14 & B15 & C4 & C5 & C6 \\ B16 & B17 & B18 & C7 & C8 & C9 \end{bmatrix} \tag{B.1}$$

Where the A, B, C terms are of the general form

$$An = \frac{An^1_{ij}\ bd}{D1 + D2\ b + D3\ d} \tag{B.2}$$

$$Bn = \frac{Bn^1_{ij}\ b^2d + Bn^2_{ij}\ bd^2 + Bn^3_{ij}\ bd}{D1 + D2\ b + D3\ d} \tag{B.3}$$

$$Cn = \frac{Cn^1_{ij}\ b^3d + Cn^2_{ij}\ b^2d^2 + Cn^3_{ij}\ b^2d + Cn^4_{ij}\ bd^3 + Cn^5_{ij}\ bd^2 + Cn^6_{ij}\ bd}{D1 + D2\ b + D3\ d} \tag{B.4}$$

where An_{ij}, Bn^1_{ij} ... Bn^3_{ij}, Cn^1_{ij} ... Cn^6_{ij} are functions of ξ, η, ζ as well as i,j. D1, D2, D3 are functions of ξ, η, ζ.

B.2 Thick Thin Shell Element

For the Thick Thin shell element the individual components of the $\bar{k}_{ij}$ matrices are

$$\bar{k}_{ij} = \begin{bmatrix} A1 & A2 & A3 & B1 & B2 \\ A4 & A5 & A6 & B3 & B3 \\ A7 & A8 & A9 & B5 & B6 \\ B7 & B8 & B9 & C1 & C2 \\ B10 & B11 & B12 & C3 & C4 \end{bmatrix} \tag{B.5}$$

where the A, B, C terms are of the following form

$$An = \frac{An^1_{ij}\, t^3 + An^2_{ij}\, t^2 + An^3_{ij}\, t}{D1\, t^2 + D2\, t + D3} \tag{B.6}$$

$$Bn = \frac{Bn^1_{ij}\, t^4 + Bn^2_{ij}\, t^3 + Bn^3_{ij}\, t^2 + Bn^4_{ij}\, t}{D1\, t^2 + D2\, t + D3} \tag{B.7}$$

$$Cn = \frac{Cn^1_{ij}\, t^5 + Cn^2_{ij}\, t^4 + Cn^3_{ij}\, t^3 + Cn^4_{ij}\, t^2 + Cn^5_{ij}\, t}{D1\, t^2 + D2\, t + D3} \tag{B.8}$$

where An^1_{ij} ... An^3_{ij}, Bn^1_{ij} ... Bn^4_{ij}, Cn^1_{ij} ... Cn^5_{ij} are functions of ξ, η, ζ as well as i,j. D1, D2, D3 are functions of ξ, η, ζ.

APPENDIX C

PROOFS OF RESULTS FROM CHAPTER 4

The results stated in Chapter 4 are proved here using a Rayleigh's quotient approach.

C.1 Linear Equation for Stiffness Change

For a structural change that affects only the stiffness properties of the system it can be shown that when constraining the fundamental frequency

$$\alpha_{ep}^{L} \leq \alpha_{ep}^{exact} \tag{C.1}$$

and if $\{\phi_i'\} = \{\phi_i\}$ then when constraining the i'th frequency

$$\alpha_{ep}^{L} = \alpha_{ep}^{exact} \tag{C.2}$$

PROOF

From equation (2.33) for a structural change in stiffness with no change in mass

$$\frac{\{\phi_1'\}^T [K'] \{\phi_1'\}}{\{\phi_1'\}^T [M] \{\phi_1'\}} = \omega_1^{2\prime} \tag{C.3}$$

Since ϕ_1' minimizes the quotient in equation (C.3)

$$\frac{\{\phi_1\}^T [K'] \{\phi_1\}}{\{\phi_1\}^T [M] \{\phi_1\}} \geq \omega_1^{2\prime} \tag{C.4}$$

or

$$\frac{\{\phi_1\}^T [K] \{\phi_1\} + \{\phi_1\}^T [\Delta K] \{\phi_1\}}{\{\phi_1\}^T [M] \{\phi_1\}} \geq \omega_1^2 \tag{C.5}$$

$$\omega_1^2 + \frac{\alpha_{ep}^{exact} \{\theta_1\}^T [k_{ep}] \{\theta_1\}}{\{\phi_1\}^T [M] \{\phi_1\}} \geq \omega_1^{2\prime} \tag{C.6}$$

$$\alpha_{ep}^{exact} \geq \frac{\{\phi_1\}^T [M] \{\phi_1\} (\omega_1^{2\prime} - \omega_1^2)}{\{\theta_1\}^T [k_{ep}] \{\theta_1\}} = \alpha_{ep}^{L} \tag{C.7}$$

$$\alpha_{ep}^{exact} \geq \alpha_{ep}^{L} \tag{C.8}$$

If there is no change in mode shape and $\{\phi_i'\} = \{\phi_i\}$ then the inequality sign in equation (C.4) will be an equal sign. This equation will then be true for the i'th mode so that

$$\alpha_{ep}^{exact} = \alpha_{ep}^{L} \tag{C.9}$$

when constraining the i'th frequency.

C.2 Linear Equation for Mass Change

For a structural change that only affects the mass properties of the system it can be shown that when constraining the fundamental frequency

$$\alpha_{ep}^{L} \leq \alpha_{ep}^{exact} \tag{C.10}$$

but that if $\{\phi_i'\} = \{\phi_i\}$ then when constraining the i'th frequency

$$\alpha_{ep}^{L} < \alpha_{ep}^{exact}. \tag{C.11}$$

PROOF

From equation (2.33) for a structural change in mass with no change in stiffness is

$$\frac{\{\phi_1'\}^T [K] \{\phi_1'\}}{\{\phi_1'\}^T [M'] \{\phi_1'\}} = \omega_1^{2\prime} \tag{C.12}$$

Since ϕ_1' minimizes the quotient in equation (C.12)

$$\frac{\{\phi_1'\}^T [K] \{\phi_1\}}{\{\phi_1\}^T [M'] \{\phi_1\}} \geq \omega_1^{2\prime} \tag{C.13}$$

or

$$\frac{\{\phi_1\}^T [M'] \{\phi_1\}}{\{\phi_1\}^T [K] \{\phi_1\}} \leq \frac{1}{\omega_1^{2\prime}} \tag{C.14}$$

$$\frac{\{\phi_1\}^T [M] \{\phi_1\} + \{\phi_1\}^T [\Delta M] \{\phi_1\}}{\{\phi_1\}^T [K] \{\phi_1\}} \leq \frac{1}{\omega_1^{2\prime}} \tag{C.15}$$

$$\frac{1}{\omega_1^2} + \frac{\alpha_{ep}^{exact} \{\theta_1\}^T [m_{ep}] \{\theta_1\}}{\{\phi_1\}^T [K] \{\phi_1\}} \leq \frac{1}{\omega_1^{2\prime}} \tag{C.16}$$

$$1 + \omega_1^2 \frac{\alpha_{ep}^{exact} \{\theta_1\}^T [m_{ep}] \{\theta_1\}}{\{\phi_1\}^T [K] \{\phi_1\}} \leq \frac{\omega_1^2}{\omega_1^{2\prime}} \tag{C.17}$$

$$1 + \frac{\alpha_{ep}^{exact} \{\theta_1\}^T [m_{ep}] \{\theta_1\}}{\{\phi_1\}^T [M] \{\phi_1\}} \leq \frac{\omega_1^2}{\omega_1^{2\prime}} \tag{C.18}$$

$$\alpha_{ep}^{exact} \leq \frac{\{\phi_1\}^T [M] \{\phi_1\} (\omega_1^2 - \omega_1^{2\prime})}{\{\theta_1\}^T [m_{ep}] \{\theta_1\} \omega_1^{2\prime}} \tag{C.19}$$

$$-\alpha_{ep}^{exact} > \frac{\{\phi_1\}^T [M] \{\phi_1\} (\omega_1^{2\prime} - \omega_1^2)}{\{\theta_1\}^T [m_{ep}] \{\theta_1\} \omega_1^{2\prime}} \tag{C.20}$$

$$\alpha_{ep}^{exact} > \frac{\{\phi_1\}^T [M] \{\phi_1\} (\omega_1^{2\prime} - \omega_1^2)}{-\{\theta_1\}^T [m_{ep}] \{\theta_1\} \omega_1^{2\prime}} \tag{C.21}$$

From equation (C.17) $\omega_1^2 > \omega_1^{2\prime}$ for a positive α and nonzero m_{ep} and so

$$\alpha_{ep}^{exact} > \frac{\{\phi_1\}^T [M] \{\phi_1\} (\omega_1^{2\prime} - \omega_1^2)}{-\{\theta_1\}^T [m_{ep}] \{\theta_1\} \omega_1^2} \tag{C.22}$$

The same inequality holds if α is negative since then $\omega_1^2 < \omega_1^{2\prime}$ and both sides of equation C.22 would be negative with the right hand side being larger in magnitude. Thus, it follows that

$$\alpha_{ep}^{exact} > \alpha_{ep}^{L} \tag{C.23}$$

If there is no change in mode shape and $\{\phi_1'\} = \{\phi_1\}$ then because of the frequency term in equation (C.22) when constraining the i'th frequency the exact answer will still be greater than the linear equation solution.

C.3 General Equation for Stiffness or Mass Changes

For a structural change involving either a stiffness and/or a mass change it can be shown that when constraining the fundamental frequency

$$\alpha_{ep}^{G} < \alpha_{ep}^{exact} \tag{C.24}$$

and that if $\{\phi_i^L\} = \{\phi_i'\}$ then when constraining the i'th frequency

$$\alpha_{ep}^{G} = \alpha_{ep}^{exact} \tag{C.25}$$

PROOF

From Equation (2.33) for a structural change involving both a stiffness and a mass change

$$\frac{\{\phi_1'\}^T [K'] \{\phi_1'\}}{\{\phi_1'\}^T [M'] \{\phi_1'\}} = \omega_1^{2\prime} \tag{C.26}$$

Since ϕ_1' minimizes the quotient in equation (C.26)

$$\frac{\{\phi_1^L\}^T [K'] \{\phi_1^L\}}{\{\phi_1^L\}^T [M'] \{\phi_1^L\}} \geq \omega_1^{2\prime} \tag{C.27}$$

expanding $[K']$ and $[M']$

$$\frac{\{\phi_1^L\}^T [K] \{\phi_1^L\} + \{\phi_1^L\}^T [\Delta K] \{\phi_1^L\}}{\{\phi_1^L\}^T [M] \{\phi_1^L\} + \{\phi_1^L\}^T [\Delta M] \{\phi_1^L\}} \geq \omega_1^2 \tag{C.28}$$

Rearranging

$$\{\phi_1^L\}^T [\Delta K] \{\phi_1^L\} - \omega_1^{2\prime} \{\phi_1^L\} [\Delta M] \{\phi_1^L\} \geq$$

$$\omega_1^{2\prime} \{\phi_1^L\}^T [M] \{\phi_1^L\} - \{\phi_1^L\}^T [K] \{\phi_1^L\} \tag{C.29}$$

Substituting for $[\Delta K]$ and $[\Delta M]$

$$\left(\{\theta_1^L\}^T [k_{ep}] \{\theta_1^L\}^T - \omega_1^2 \{\theta_1^L\}^T [m_{ep}] \{\theta_1^L\}^T \right) \alpha_{ep}^{exact} \geq$$

$$\omega_1^{2\prime} \{\phi_1^L\}^T [M] \{\phi_1^L\} - \{\phi_1^L\}^T [K] \{\phi_1^L\} \tag{C.30}$$

Solving for α_{ep}^{exact}

$$\alpha_{ep}^{exact} > \frac{\omega_1^{2\prime} \{\phi_1^L\}^T [M] \{\phi_1^L\} - \{\phi_1^L\}^T [K] \{\phi_1^L\}}{\{\theta_1^L\}^T [k_{ep}] \{\theta_1^L\} - \omega_1^{2\prime} \{\theta_1^L\}^T [m_{ep}] \{\theta_1^L\}} \tag{C.31}$$

or

$$\alpha_{ep}^{exact} > \alpha_{ep}^{G} \tag{C.32}$$

From equation (C.26) if $\{\phi_i^L\} = \{\phi_i'\}$ the inequality sign in equation (C.27) is replaced by an equal sign and this equation will then be true for the i'th mode. Thus if $\{\phi_i^L\} = \{\phi_i'\}$ then when constraining the i'th frequency

$$\alpha_{ep}^{G} = \alpha_{ep}^{exact} \tag{C.33}$$

APPENDIX D

PROOFS OF WORK INEQUALITIES

D.1 For a Structural Change Involving Stiffness Properties Only

$$\{\phi_1\}^T [\Delta K] \{\phi_1\} > \{\phi_1'\}^T [\Delta K] \{\phi_1'\} \tag{D.1}$$

And for linear property changes

$$\{\theta_1\}^T [k_{ep}] \{\theta_1\} > \{\theta_1'\}^T [k_{ep}] \{\theta_1'\} \tag{D.2}$$

PROOF

Part 1

By Rayleigh's Principle the baseline frequency is given by

$$\omega_1^2 = \frac{\{\phi_1\}^T [K] \{\phi_1\}}{\{\phi_1\}^T [M] \{\phi_1\}} < \frac{\{\phi_1'\}^T [K] \{\phi_1'\}}{\{\phi_1'\}^T [M] \{\phi_1'\}} \tag{D.3}$$

and the changed frequency is given by

$$\omega_1^{2\prime} = \frac{\{\phi_1'\}^T [K'] \{\phi_1'\}}{\{\phi_1'\}^T [M] \{\phi_1'\}} \tag{D.4}$$

It can be assumed that the eigenvectors are normalized with respect to $[M]$ such that

$$\{\phi_1\}^T [M] \{\phi_1\} = 1 \tag{D.5}$$

$$\{\phi_1'\}^T [M] \{\phi_1'\} = 1 \tag{D.6}$$

Then from equation (D.3) using (D.5) and (D.6)

$$\{\phi_1\}^T [K] \{\phi_1\} < \{\phi_1'\}^T [K] \{\phi_1'\} \tag{D.7}$$

From equation (D.4) using (D.5) and (D.6)

$$\{\phi_1'\}^T [K'] \{\phi_1'\} = \omega_1^2' \tag{D.8}$$

expanding

$$\{\phi_1'\}^T [K] \{\phi_1'\} + \{\phi_1'\}^T [\Delta K] \{\phi_1'\} = \omega_1^2' \tag{D.9}$$

or

$$\{\phi_1'\}^T [K] \{\phi_1'\} = \omega_1^2' - \{\phi_1'\}^T [\Delta K] \{\phi_1'\} \tag{D.10}$$

Substitution of equation (D.10) into equation (D.7)

$$\{\phi_1\}^T [K] \{\phi_1\} < \omega_1^2' - \{\phi_1'\}^T [\Delta K] \{\phi_1'\} \tag{D.11}$$

rearranging

$$\{\phi_1'\}^T [\Delta K] \{\phi_1'\} < \omega_1^2' - \{\phi_1\}^T [K] \{\phi_1\} \tag{D.12}$$

using equations (D.3) and (D.5) the result from part 1 is

$$\{\phi_1'\}^T [\Delta K] \{\phi_1'\} < \omega_1^2' - \omega_1^2 \tag{D.13}$$

Part 2

From equation (D.4) using Rayleigh's Principle

$$\omega_1^2' = \frac{\{\phi_1'\}^T [K'] \{\phi_1'\}}{\{\phi_1'\}^T [M] \{\phi_1'\}} < \frac{\{\phi_1\}^T [K'] \{\phi_1\}}{\{\phi_1\}^T [M] \{\phi_1\}} \tag{D.14}$$

Since the eigenvectors are assumed normalized with respect to $[M]$ then

$$\{\phi_1'\}^T [K'] \{\phi_1'\} < \{\phi_1\}^T [K'] \{\phi_1\} \tag{D.15}$$

or

$$\{\phi_1'\}^T [K] \{\phi_1'\} + \{\phi_1'\}^T [\Delta K] \{\phi_1'\} < \{\phi_1\}^T [K] \{\phi_1\} + \{\phi_1\}^T [\Delta K] \{\phi_1\} \tag{D.16}$$

using equation (D.10)

$$\{\phi_1'\}^T [K] \{\phi_1'\} + \omega_1^{2'} - \{\phi_1'\}^T [K] \{\phi_1'\} < \{\phi_1\}^T [K] \{\phi_1\} + \{\phi_1\}^T [\Delta K] \{\phi_1\} \tag{D.17}$$

or

$$\omega_1^{2'} < \{\phi_1\}^T [K] \{\phi_1\} + \{\phi_1\}^T [\Delta K] \{\phi_1\} \tag{D.18}$$

using equations (D.3) and (D.5) the result from part 2 is

$$\omega_1^{2'} - \omega_1^2 < \{\phi_1\}^T [\Delta K] \{\phi_1\} \tag{D.19}$$

Combining equations (D.13) and (D.19)

$$\{\phi_1'\}^T [\Delta K] \{\phi_1'\} < \omega_1^{2'} - \omega_1^2 < \{\phi_1\}^T [\Delta K] \{\phi_1\} \tag{D.20}$$

which implies

$$\{\phi_1'\}^T [\Delta K] \{\phi_1'\} < \{\phi_1\}^T [\Delta K] \{\phi_1\} \tag{D.21}$$

and if there is a linear property change

$$\alpha_{ep}^{exact} \{\theta_1'\}^T [k_{ep}] \{\theta_1'\} < \alpha_{ep}^{exact} \{\theta_1\}^T [k_{ep}] \{\theta_1\} \tag{D.22}$$

or

$$\{\theta_1'\}^T [k_{ep}] \{\theta_1'\} < \{\theta_1\}^T [k_{ep}] \{\theta_1\} \tag{D.23}$$

D.2 <u>For a Structural Change Involving Mass Properties Only</u>

$$\{\phi_1\}^T [\Delta M] \{\phi_1\} < \{\phi_1'\}^T [\Delta M] \{\phi_1'\} \tag{D.24}$$

and

$$\{\theta_1\}^T [m_{ep}] \{\theta_1\} < \{\theta_1'\}^T [m_{ep}] \{\theta_1'\} \tag{D.25}$$

PROOF

Part 1

By Rayleigh's Principle the baseline frequency is given by

$$\omega_1^2 = \frac{\{\phi_1\}^T [K] \{\phi_1\}}{\{\phi_1\}^T [M] \{\phi_1\}} < \frac{\{\phi_1'\}^T [K] \{\phi_1'\}}{\{\phi_1'\}^T [M] \{\phi_1'\}} \tag{D.26}$$

and the changed frequency is given by

$$\omega_1^{2\prime} = \frac{\{\phi_1'\}^T [K] \{\phi_1'\}}{\{\phi_1'\}^T [M'] \{\phi_1'\}} \tag{D.27}$$

It can be assumed that the eigenvectors are normalized with respect to $[K]$ such that

$$\{\phi_1\}^T [K] \{\phi_1\} = 1 \tag{D.28}$$

$$\{\phi_1'\}^T [K] \{\phi_1'\} = 1 \tag{D.29}$$

Then from equation (D.26) using (D.28) and (D.29)

$$\{\phi_1\}^T [M] \{\phi_1\} > \{\phi_1'\}^T [M] \{\phi_1'\} \tag{D.30}$$

From equation (D.27) using (D.29)

$$\{\phi_1'\}^T [M'] \{\phi_1'\} = \frac{1}{\omega_1^{2\prime}} \tag{D.31}$$

expanding

$$\{\phi_1'\}^T [M] \{\phi_1'\} + \{\phi_1'\}^T [\Delta M] \{\phi_1'\} = \frac{1}{\omega_1^{2\prime}} \tag{D.32}$$

or

$$\{\phi_1'\}^T [M] \{\phi_1'\} = \frac{1}{\omega_1^{2\prime}} - \{\phi_1'\}^T [\Delta M] \{\phi_1'\} \tag{D.33}$$

substitution of equation (D.33) into equation (D.30)

$$\{\phi_1\}^T [M] \{\phi_1\} > \frac{1}{\omega_1^{2\prime}} - \{\phi_1'\}^T [\Delta M] \{\phi_1'\} \tag{D.34}$$

rearranging

$$\{\phi_1'\}^T [\Delta M] \{\phi_1'\} > \frac{1}{\omega_1^{2\prime}} - \{\phi_1\}^T [M] \{\phi_1\} \tag{D.35}$$

using equations (D.26) and (D.28) the result from part 1 is

$$\{\phi_1'\}^T [\Delta M] \{\phi_1'\} > \frac{1}{\omega_1^{2\prime}} - \frac{1}{\omega_1^2} \tag{D.36}$$

Part 2

From equation (D.27) using Rayleigh's Principal

$$\omega_1^{2\prime} = \frac{\{\phi_1'\}^T [K] \{\phi_1'\}}{\{\phi_1'\}^T [M'] \{\phi_1'\}} > \frac{\{\phi_1\}^T [K] \{\phi_1\}}{\{\phi_1\}^T [M'] \{\phi_1\}} \tag{D.37}$$

Since the eigenvectors are assumed normalized with respect to $[K]$ then

$$\{\phi_1'\}^T [M'] \{\phi_1'\} > \{\phi_1\}^T [M'] \{\phi_1\} \tag{D.38}$$

or

$$\{\phi_1'\}^T [M] \{\phi_1'\} + \{\phi_1'\}^T [\Delta M] \{\phi_1'\} > \{\phi_1\}^T [M] \{\phi_1\} + \{\phi_1\}^T [\Delta M] \{\phi_1\} \tag{D.39}$$

using equation (D.33)

$$\{\phi_1'\}^T [M] \{\phi_1'\} + \frac{1}{\omega_1^{2\prime}} - \{\phi_1'\}^T [M] \{\phi_1'\} > \{\phi_1\}^T [M] \{\phi_1\} + \{\phi_1\}^T [\Delta M] \{\phi_1\} \tag{D.40}$$

or

$$\frac{1}{\omega_1^{2\,\prime}} > \{\phi_1\}^T [M] \{\phi_1\} + \{\phi_1\}^T [\Delta M] \{\phi_1\} \tag{D.41}$$

using equations (D.26) and (D.28) the result from part 2 is

$$\frac{1}{\omega_1^{2\,\prime}} - \frac{1}{\omega_1^2} > \{\phi_1\}^T [\Delta M] \{\phi_1\} \tag{D.42}$$

Combining equations (D.36) and (D.42)

$$\{\phi_1'\}^T [\Delta M] \{\phi_1'\} > \frac{1}{\omega_1^{2\,\prime}} - \frac{1}{\omega_1^2} > \{\phi_1\}^T [\Delta M] \{\phi_1\} \tag{D.43}$$

which implies

$$\{\phi_1'\}^T [\Delta M] \{\phi_1'\} > \{\phi_1\}^T [\Delta M] \{\phi_1\} \tag{D.44}$$

and since all mass changes are linear property changes

$$\alpha^{exact} \{\theta_1'\}^T [m_{ep}] \{\theta_1'\} > \alpha_{ep}^{exact} \{\theta_1\}^T [m_{ep}] \{\theta_1\} \tag{D.45}$$

or

$$\{\theta_1'\}^T [m_{ep}] \{\theta_1'\} > \{\theta_1\}^T [m_{ep}] \{\theta_1\} \tag{D.46}$$