OPTIMAL WEIGHTED PARTIAL DECISION COMBINING FOR FADING CHANNEL DIVERSITY

by

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Abstract

A diversity combining scheme is examined that utilizes a demodulator's hard decisions in conjunction with knowledge of each decision's reliability. A maximum-likelihood bit decision is made, based on these partial decisions from the demodulator and on measurements of the state of the fading channel. The technique is sub-optimal since hard decisions are processed, but it may find application in low cost receiver design. The technique is optimal in the sense that a minimum probability of bit error is achieved, given a set of partial decisions and knowledge of their reliability.

Performance analysis for the case of non-coherent frequency shift keying on a slow Rayleigh fading channel with additive white Gaussian noise includes the derivation of a tight upper bound on the probability of bit error, and estimates of the asymptotic performance relative to standard diversity schemes such as majority-voting, selection diversity, square-law, and maximal ratio combining. These results are supported by simulation results for bit and packet error rates in an example system. With five independent bit repeats and a BER of $10^{-3}$, the receiver is about 3 dB more efficient than majority-voting, and about 1 dB more efficient than selection diversity. The gain in efficiency, relative to the standard partial decision combination schemes, increases with the number of repeats.

The degradation in performance in a practical receiver implementation is addressed, and it is demonstrated that near ideal performance may be obtained with only a few reliability weights quantized to a small number of levels. Furthermore, this performance is maintained over a wide range of average signal to noise ratio without having to adapt the reliability weights. When the reliability estimate is corrupted by additive white Gaussian noise, it is demonstrated that simple low-
pass filtering of the signal strength estimate is sufficient to obtain near ideal performance. The performance is degraded in the presence of cochannel interference, but for a moderate level of interference the performance is demonstrated to be superior to majority-voting or selection diversity.

Other results include a method to estimate the optimal quantization thresholds, and a method to obtain the probability of error of selection diversity receivers employing signal to noise ratio measurement quantization. The selection diversity analysis is applicable to the more general case of Rician fading.
# Table of Contents

Abstract ..................................................... ii
List of Figures ................................................ vi
List of Tables ................................................ viii
Acknowledgements .......................................... ix

1 Introduction ................................................. 1
  1.1 Background ............................................. 1
  1.2 Optimal Diversity Combining ......................... 4
  1.3 Thesis Organization .................................. 6

2 Idealized WPD Receiver ................................... 8
  2.1 WPD Receiver Concept ................................ 8
  2.2 Optimal Weighting of Partial Decisions ............ 9
  2.3 Discussion ............................................ 13
  2.4 Analysis of WPD Receiver Performance ............... 15
    2.4.1 Methods for an Analytical Solution ............. 15
    2.4.2 Bounds on the WPD Bit Error Rate .............. 17
  2.5 Comparison of WPD and Traditional Diversity Schemes 20
    2.5.1 Asymptotic BER Characteristics ................. 23
    2.5.2 Efficient Use of Branch Information .......... 24
    2.5.3 Optimal Number of Diversity Branches .......... 27
  2.6 Simulation ........................................... 33
    2.6.1 Simulation Model ................................ 33
    2.6.2 Simulation Description ......................... 39
    2.6.3 Simulation Parameters ......................... 41
    2.6.4 Simulation Results ............................. 43
  2.7 Error Rate Approximations .......................... 47
# TABLE OF CONTENTS

3 Some Issues in Practical WPD Receiver Design .................................................. 53
   3.1 SNR and Weight Quantization ................................................................. 53
      3.1.1 BER Evaluation Method ............................................................... 54
      3.1.2 Threshold Optimization ............................................................... 57
         Capacity Method ................................................................................. 57
         Chernoff Bound Method .................................................................... 62
      3.1.3 Performance With Quantized SNR ..................................................... 64
      3.1.4 Performance with Quantized Weights .............................................. 73
   3.2 SNR Estimation ......................................................................................... 76
      3.2.1 Receiver Model .............................................................................. 76
      3.2.2 Simulation Method, AWGN ............................................................. 80
      3.2.3 Signal Strength Estimation ............................................................... 83
      3.2.4 Simulation Method, Cochannel Interference ...................................... 88
      3.2.5 Simulation Results ..................................................................... 93

4 Conclusions ....................................................................................................... 105

References ............................................................................................................ 108

Appendix A ........................................................................................................... 112

Appendix B .......................................................................................................... 115

Appendix C .......................................................................................................... 122

Appendix D .......................................................................................................... 125
List of Figures

1.1 Example of Rayleigh Fading .................................. 3
2.1 Weighted Partial Decision Diversity Receiver ................. 12
2.2 Square-Law NCFSK Diversity Receiver .......................... 21
2.3 Maximal-Ratio NCFSK Diversity Receiver ....................... 22
2.4 Bit Error Rate vs. Number of Diversity Branches. ............ 26
2.5 Chernoff Bound Exponent Functions ............................ 30
2.6 BER vs. Number of Diversity Branches, Fixed $E_b/N_0$ .... 32
2.7 AMPS Land to Mobile Control Channel Format .................. 34
2.8 Normalized Autocorrelation Function of the Received Envelope 42
2.9 Bit Error Rates for 3 and 5 Repeats ........................... 45
2.10 Packet Error Rates for 5 Repeats .............................. 46
2.11 WPD Bit Error Rate Bound and Simulation Results .......... 48
2.12 Uncorrected Packet Error Rate ............................... 51
2.13 Single Bit Error Corrected Packet Error Rate ................ 52
3.1 SNR Quantization ............................................. 54
3.2 Discrete Channel Model Paths Corresponding to a Repeated '1' 55
3.3 Discrete Memoryless Channel Model ............................ 58
3.4 Discrete Channel Model Capacity ................................ 61
3.5 Comparison of Single Threshold Values ........................ 65
3.6 Comparison of Error Rates using a Single Threshold ........... 66
3.7 BER with One Threshold ..................................... 67
3.8 BER with Three Thresholds ................................... 67
3.9 Convergence of WPD BER ...................................... 69
3.10 Optimized and Fixed Threshold Performance .................. 70
3.11 Convergence of Selection Diversity BER ....................... 72
3.12 Weight Quantization Characteristic ............................ 73
LIST OF FIGURES

3.13 Quantized Weight BER. .......................... 75
3.14 NCFSK Receiver, Bandpass Filter Implementation ............. 78
3.15 Bandpass Filter NCFSK Receiver, Block Diagram ............... 78
3.16 Frequency Response of Signal Strength Estimator Filter ........ 87
3.17 Tracking Behaviour of the Signal Strength Estimator .......... 87
3.18 Phasor Representation of the IF Filter Envelope Detector Output .... 89
3.19 WPD Bit Error Rate with SNR Estimation .................. 94
3.20 WPD Bit Error Rate with SNR Estimation, SIR = 15 dB .......... 96
3.21 WPD Packet Error Rate with SNR Estimation ................ 98
3.22 WPD Packet Error Rate with SNR Estimation, SIR = 15 dB .... 99
3.23 Selection Diversity BER with SNR Estimation ............... 101
3.24 Selection Diversity BER with SNR Estimation, SIR = 15 dB ... 102
3.25 Selection Diversity PER with SNR Estimation ............. 103
3.26 Selection Diversity PER with SNR Estimation, SIR = 15 dB .... 104

A.1 A Linear Bound on $x^{1/2}$. .......................... 113

A.2 Ratios of Two Bounds on the Moment Generating Function to the Numerically Computed Result. .................. 114

D.1 Quantized SNR Probability Density Function .................. 126

D.2 Selection Diversity Quantized SNR Cumulative Distribution Func-
tion ........................................ 127
List of Tables

2.1  Asymptotic Loss of Square-Law, Selection, and WPD Combining Relative to Maximal-Ratio Combining .......................... 25

2.2  Estimated and Exact Values for the Optimum Number of Branches . 31
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Chapter 1

Introduction

1.1 Background

Methods for achieving reliable and efficient mobile digital communications have received considerable attention in the technical literature. Much of this theoretical and empirical knowledge is applied in practical engineering designs to arrive at a best system design. This best solution is a compromise between many factors, including various performance criteria, hardware cost, reliability, design effort, and compatibility with present and future systems. As part of the effort to optimize the system design, error rate performance is often sacrificed by implementing theoretically sub-optimal demodulators. This thesis introduces and analyzes a sub-optimum digital receiver that may be added to the designer's repertoire of techniques for mobile digital communications.

In urban or suburban environments, digital communications over VHF/UHF mobile radio channels is severely impaired by multipath induced fading [1]. Local buildings and other objects act as scatterers that can provide multiple propagation
paths to the receiver. Unfortunately, each path has its associated attenuation and delay so that the resulting sum of these signals at the receiver will exhibit the effects of destructive and constructive interference. As the mobile receiver moves through this interference pattern the magnitude and phase of the resultant RF signal will fluctuate. For urban areas the received signal energy is often modeled as the resultant of the superposition of a large number of scattered signals arriving from equally likely directions, with no line of sight (LOS) component [2]. In this model the received signal amplitude is Rayleigh distributed. A more general model incorporates a static LOS component in addition to an infinite number of scatterers and leads to a received signal amplitude that is Rician distributed [3]. The Rician distribution reduces to a Rayleigh distribution as the LOS component is decreased to zero.

An additional variation in the received signal strength is due to changes in the nature of the propagation paths. Significant variations in the large scale geography, e.g. hills, can shadow the receiver from the transmitter. With this shadowing effect the logarithm of the received signal amplitude is usually modeled by a normal distribution and the rate of change of signal strength is slow relative to Rayleigh fading [4].

We will concentrate on the performance of receivers over small regions in urban centers. The appropriate model therefore is a Rayleigh distributed received signal amplitude. The typical behaviour of a Rayleigh fading channel over time is shown

---

1 Even if the receiver does not move it is possible to have a time variation of the received signal due to moving reflectors. For example, moving trucks or buses may reflect signals towards a stationary receiver [5].
in Fig. 1.1. Fades occur at a rate that increases [4] with the Doppler frequency

\[ f_D = \frac{v}{\lambda} \]  \hspace{1cm} (1.1)

where \( v \) is the vehicle velocity and \( \lambda \) is the signal wavelength \(^2\). Thus the fade rate increases with the vehicle speed and signal frequency.

Diversity techniques are often used to combat fading by combining information received from different branches. A branch is the virtual transmitter to receiver propagation path formed by the superposition of the multitude of individual paths. If the branches fade independently it is unlikely that they will all experience a deep fade simultaneously, and it is thus hoped that a more reliable decision may be made.

\(^2\) The Doppler frequency is the maximum Doppler frequency shift experienced by any individual ray as it arrives at the vehicle.
Diversity schemes may be classified according to the source of the branch information. For example, *space*, *time*, and *frequency* diversity obtain multiple versions of the signal via spatially separated antennas, repeated signal elements, and transmission over multiple frequency bands respectively. Ideally, the fading on the branches would be statistically independent, which would be true for sufficiently separated antennas, time samples, or frequency bands, so that the correlation between branches is negligible. The effectiveness of a diversity system is reduced when the branches are correlated, but substantial gains are still achievable with considerable correlation. For example, at a target BER of $10^{-3}$ a two branch *square-law* receiver with a power correlation coefficient of 0.6 achieves 85% of the dB gain provided by two completely independent branches.\(^3\)

Effective use of the branch information is determined by the combining and detection methods used. Traditional methods of diversity combining, *maximal-ratio*, *equal gain*, *selection*, and *square-law* combining are discussed and analyzed in [1,6,7,8,9,10].

### 1.2 Optimal Diversity Combining

In general there are different *optimal* combining methods, each corresponding to the particular noise statistics, the amount of channel state information available, and the modulation/demodulation methods employed. For fading in additive white Gaussian noise (AWGN) the best possible combination of the branch signals is obtained with *maximal-ratio* combining. This optimal linear combining method

\(^3\)This dB gain is with respect to single branch reception of non-coherent frequency shift keying. These results are obtained using Figure 2 of [7] and (2.28).
achieves the highest possible instantaneous SNR and also the lowest BER. However, it requires the perfect measurement and correction of relative phase errors of each branch, so that the RF signal may be added coherently, and further requires that each branch be weighted by the ratio of its received signal amplitude to noise power. In the absence of amplitude and phase measurements for the respective branches, the appropriate optimum receiver for non coherent frequency shift keying (NCFSK) in slow fading\(^4\) uses a square-law combiner [9]. The relative performance of these schemes will be discussed later.

Another possible approach is to utilize a set of hard decisions derived from the branches\(^5\). A commonly used method for time diversity is majority voting, where each bit is repeated an odd number, \(L\), of times. The receiver makes a decision on each bit repeat, and then decides in favor of the majority of these \(L\) partial decisions. If channel state measurement information is available, then another possible approach is to use selection diversity, where the only decision retained is that corresponding to the branch with the largest signal to noise ratio. Another option is erasure, where the partial decisions from the poorest SNR branches are ignored, and the remaining partial decisions are considered equally important [4,11]. However, neither of the latter two schemes use all of the available partial decisions and one might expect some improvement if all of the partial decisions were combined intelligently. This thesis introduces and analyzes the optimal combination scheme for partial decisions on a fading channel. The method consists of combining the partial decisions according to their reliability to yield a maximum likelihood

\(^4\) The term slow fading refers to the case in which the received amplitude and phase are almost constant over the bit period. The condition of slow fading is assumed throughout this thesis, unless stated otherwise.

\(^5\) We shall refer to a hard decision made on a particular branch as a partial decision.
While the proposed technique may not be expected to perform as well as maximal-ratio or square-law combining the technique may be expected to yield an improvement over commonly used methods of partial decision combining.

1.3 Thesis Organization

The remainder of this thesis is organized as follows, with the principal results of each chapter listed below.

Chapter 2 introduces and analyzes the optimal weighted partial decision (WPD) receiver with the assumption that the knowledge of the channel state is perfect. Specific results obtained for NCFSK are;

- Some useful bounds on the bit error rate are developed and used to compare the WPD receiver with various traditional combining schemes.
- The optimal number of diversity branches given a fixed energy per bit is estimated.
- Simulation results are presented to provide examples of the WPD receiver performance and to illustrate the accuracy with which the BER is predicted by an upper bound derived earlier.

Chapter 3 demonstrates the robustness of the WPD receiver when various degradations are encountered in a practical receiver design. Results include;
• An efficient method for optimizing the WPD receiver thresholds when the SNR is quantized.

• Examples of the degradation in BER performance when the SNR and weights are quantized.

• An expression for the BER of selection diversity when the SNR is quantized.

• Examples of the performance degradation expected when the channel state estimation is corrupted by AWGN or cochannel interference.

Chapter 4 contains a summary of the main results presented in the thesis, as well as a brief discussion on some related topics.
Chapter 2

Idealized WPD Receiver

2.1 WPD Receiver Concept

As discussed in the introduction, the basic concept of the WPD receiver is to weight all of the partial decisions according to their reliability. While we have not as yet mathematically defined reliability, the intuitive basis for the combination scheme is straightforward. For example, on a Rayleigh fading channel the probability that a bit decision will be wrong during a deep fade will be virtually 1/2. Clearly there is little value in utilizing this particular decision. From an information theoretic view we can model the fading channel as a time varying binary symmetric channel (BSC), and if we consider a sub-channel formed by considering only the deeply faded signal, then the information carrying capacity of this sub-channel is virtually zero. On the other hand, consider a situation in which the received signal is quite high, say 20 dB above the noise. Then the probability that a bit decision is correct will be nearly one, and the decision should be considered completely reliable. This

\footnote{This division of the fading channel into several sub-channels of varying capacity will be made mathematically explicit in section 3.1.2.1.}
case corresponds to a sub-channel with a capacity of virtually one information bit per channel bit, so that no redundancy is required to arrive at an error free decision. For the continuum of channel conditions encountered on a typical fading channel our intuitive model of the WPD receiver would then weight bit decisions by appropriate confidence, or reliability, factors.

At the outset of this thesis work the expected gain due to a reliability weighting of the partial decisions was investigated using some ad-hoc weighting schemes. Some of the possible candidates for the weighting functions included;

- a weight function linear with the BSC crossover probability,
- a weight function equal to the received SNR,
- a weight function equal to the capacity of the BSC.

The performance of the capacity weighting scheme was estimated via simulation and the results indicated that the basic concept of reliability weighting showed some promise. Rather than use an ad-hoc scheme, the question of how to optimally weight the partial decisions is discussed in the next section.

### 2.2 Optimal Weighting of Partial Decisions

Consider a set of $L$ partial decisions, $\{d_i\}_{i=1}^L$, made on a message bit, $m \in \{0, 1\}$, where

$$d_i = \begin{cases} 
1 & \text{if the } i^{th} \text{ decision is a '1'} \\
-1 & \text{otherwise.}
\end{cases}$$

(2.1)
The maximum a posteriori probability (MAP) decision rule is to choose \( m = 1 \) if

\[
Pr \left( \{ d_i \}_{i=1}^L | m = 1 \right) Pr(m = 1) > Pr \left( \{ d_i \}_{i=1}^L | m = 0 \right) Pr(m = 0)
\]  
(2.2)

where \( Pr(m = k) \) \( k = 0, 1 \) is the probability of sending \( m = k \). This MAP decision rule is optimum in the sense of minimizing the probability of bit error. If the messages are equally likely, then the MAP decision rule corresponds to the maximum likelihood (ML) decision rule,

\[
Pr \left( \{ d_i \}_{i=1}^L | m = 1 \right) > Pr \left( \{ d_i \}_{i=1}^L | m = 0 \right).
\]  
(2.3)

It is shown in [12] that if the \( L \) partial decisions, \( \{ d_i \}_{i=1}^L \), are independent, then the ML decision statistic is the weighted sum

\[
D_L = \sum_{i=1}^L w_i d_i,
\]  
(2.4)

where the weights, \( w_i \), are given by

\[
w_i = c \ln \left( \frac{1 - p_i}{p_i} \right), \quad c \text{ any positive constant}
\]  
(2.5)

with \( p_i \) being the probability that the \( i^{th} \) decision is incorrect. The optimal decision rule is to decide '0' was sent if \( D_L < 0 \). If \( D_L \geq 0 \) then '1' is chosen.

The optimal decision statistic (2.4) and weights (2.5) were derived in [12] for application in the processing of multiple hard limited samples, i.e. partial decisions, per bit. The principle is that if a signalling waveform is not rectangular, then the partial decisions should not be processed with equal weights. For example, partial decisions made on portions of the signal waveform that are of low amplitude, e.g. the beginning of a raised cosine pulse, are intuitively less reliable than partial decisions based on the high amplitude portions of the signal. Knowledge of the
signalling waveform and the pdf of the noise can then be used to calculate the optimal weights in (2.5).

In our application we wish to know what the optimal decision rule is for partial decisions made on a number of bit repeats undergoing fading. This problem is closely related to that of [12]. We assume that the noise samples corrupting the individual bit transmissions are statistically independent. Each partial decision made by the demodulator will have a particular probability of error depending on the channel conditions for that bit repeat. Then for a set of $L$ bit repeats, the $L$ bit decisions constitute a set of partial decisions on a particular message bit, as in [12], except that the probability of error for each partial decision is no longer a quantity that is known \textit{a priori} from the signalling waveform and the pdf of the noise. Rather, the probability of error, $p_i$, for each partial decision, $d_i$, is to be determined from the state of the fading channel. Thus, for a particular set of $L$ partial decisions, $\{d_i\}_{i=1}^L$, and the set of corresponding demodulator bit error probabilities, $\{p_i\}_{i=1}^L$, determined from the fading channel measurements, the maximum likelihood decision statistic and weights are (2.4) and (2.5).

Our problem then is to measure the state of the fading channel so that an accurate evaluation of the demodulator error probability, $p_i$, and in turn the weight, $w_i$, may be made. Since the BER is directly related to the received SNR, it would be natural to measure the received signal level during the fading process. The SNR may then be estimated from the ratio of the measured received signal power to the noise power\footnote{If the receiver noise is dominated by internally generated front-end thermal noise, then a laboratory bench measurement of the intermediate frequency (IF) filter output power with no input signal will be a good estimate of the received noise power in actual use. As will be discussed later, external noise will adversely affect the weight determination.}.

11
CHAPTER 2. IDEALIZED WPD RECEIVER

The WPD receiver structure is shown in figure 2.1. Rather than explicitly calculate the weights from the received SNR the receiver could make use of a lookup table indexed by the received signal level. A sufficiently fast received signal level measurement is readily available in some receivers by taking the output of the already existing average received power indicator prior to low pass filtering [13]. Practical issues involving the accuracy required of the weight determination procedure will be discussed in Chapter 3.
2.3 Discussion

Majority-voting, selection of the strongest signal, and erasure of the weaker signals were discussed earlier as combination methods that utilize partial decisions. It is interesting to view them as special cases of reliability weighting. Majority-voting simply assigns equal weights to all of the partial decisions. With unit weights, (2.4) implements majority-voting as an up/down counter. Selection diversity assigns a non-zero weight only to the branch with the strongest instantaneous SNR. Thus the single most reliable decision is used. Finally, the erasure method ignores those branches with a sufficiently low SNR and treats all the others as equally reliable. Since the traditional methods of majority-voting, selection, and erasure are in fact non-optimal weighting schemes, we can therefore predict equal or improved performance using the optimum WPD receiver.

For the special case of two branches, the combination of two weighted partial decisions will result in the decision being made in favor of the partial decision with the higher weight. Since this higher weight corresponds to the branch with the higher SNR, we see that, for $L = 2$, a WPD combiner is equivalent to selection diversity. This equivalence does not require that the WPD receiver use optimal weights. Any reliability weighting scheme that has weights monotonically increasing with SNR will be equivalent to selection diversity, for $L = 2$.

In most applications the vehicle speed, or fading rate, is not constant. The choice of a method for combatting the effects of fading should be influenced by the distribution of fading rates that will be encountered. We can make some observations on the performance of these various systems as the fade rate is varied.
At high Doppler frequencies the received signal level will be almost independent from bit repeat to bit repeat. This condition is convenient mathematically and is usually assumed for the performance evaluation of diversity schemes. Expressions for the BER are sometimes obtainable in this case and these will be reviewed in the next section.

As the fade rate is lowered the bit repeats will no longer be independent and the effective gain of the diversity system is decreased. Analytical solutions for this case of correlated branches are rare [7,8]. As the fade rate is decreased still further, so that for a set of repeats the signal level is constant, the BER will be determined by averaging the error rate obtained for this fixed SNR over the fading SNR distribution. Note that as this very slow fading rate is approached the weights for a set of bits received in the WPD combiner will be practically equal. Thus the WPD receiver performance will be equivalent to majority-voting at these very slow fade rates. Again, this result is independent of the optimality of the weights used. An alternative view of the majority-voting scheme at very slow fade rates is that it is optimal. Corresponding observations may be made for selection diversity at very slow fade rates. In this case the receiver will choose one of $L$ equally reliable decisions. Clearly this is not the best use of equally reliable decisions since many equally good decisions have been ignored.
2.4 Analysis of WPD Receiver Performance

This section discusses the estimation of the WPD receiver BER when NCFSK is used on a Rayleigh fading channel corrupted by additive white Gaussian noise.

2.4.1 Methods for an Analytical Solution

An *exact* expression for the WPD receiver BER has not yet been obtained. Below we briefly outline some attempts to determine the exact BER, but stop short of a complicated series solution in favour of a tight upper bound derived in the next section.

For a Rayleigh distributed received amplitude the SNR at the receiver will be exponentially distributed with mean $\gamma_0$, as [11]

$$f_\Gamma(\gamma) = \frac{1}{\gamma_0} e^{-\gamma/\gamma_0} \quad , \gamma \geq 0. \quad (2.6)$$

The receiver SNR is defined as $\Gamma = E_S/N_0$, where $E_S$ is the received signal energy in a bit period, and $N_0/2$ is the two-sided noise power spectral density. An ideal binary NCFSK receiver, with SNR $\gamma$, will make an error with probability [9]

$$p_e(\gamma) = \frac{1}{2} e^{-\gamma/2}. \quad (2.7)$$

Assume that a '1' is transmitted a number, $L$, of times. Then for each possible value of SNR, $\gamma$, the weighted partial decision variable, $X \triangleq w_i d_i$, has two values, i.e.,

$$X(\gamma) = \begin{cases} 
  w_i(\gamma) & , \text{with pdf value of } f_{\gamma_e}(\gamma) \\
  -w_i(\gamma) & , \text{with pdf value of } f_{\gamma_e}(\gamma)
\end{cases} \quad (2.8)$$
where

\[ f_{\gamma_e} = f_\Gamma(\gamma) [1 - p_e(\gamma)] \] (2.9)
\[ f_{\gamma_e} = f_\Gamma(\gamma)p_e(\gamma). \] (2.10)

Then \( f_X(x) \) is obtained from a transformation of (2.9) and (2.10) using (2.8), i.e. [14]

\[ f_X(x) = \begin{cases} f_{\gamma_e} [\gamma(x)] |\gamma'(x)|, & x < 0 \\ f_{\gamma_e} [\gamma(x)] |\gamma'(x)|, & x > 0 \end{cases} \] (2.11)

where \( \gamma(x) \) is the inverse function of (2.8). Performing the transformation in (2.11) yields,

\[ f_X(x) = \begin{cases} \frac{2^{1+2/\gamma_0}}{\gamma_0} \frac{e^{(1+2/\gamma_0)x}}{(1 + e^x)^{2+2/\gamma_0}}, & x < 0 \\ \frac{2^{1+2/\gamma_0}}{\gamma_0} \frac{e^{-2x/\gamma_0}}{(1 + e^{-x})^{2+2/\gamma_0}}, & x > 0 \end{cases} \] (2.12)

The probability of bit error is

\[ P_{e,wpd} = Pr(D_L < 0) = \int_{-\infty}^{0} f_{D_L}(y) \, dy \] (2.13)

where \( f_{D_L}(y) \) is the pdf of the final weighted decision variable, \( D_L \), as defined in (2.4). If the bit repeats are separated by at least the reciprocal of twice the Doppler frequency, the weighted partial decision variables may be considered to be independent [1] and,

\[ f_{D_L}(x) = \frac{f_X(x) * f_X(x) * \cdots * f_X(x)}{L} \] (2.14)

where * denotes convolution. However, computation of \( P_{e,wpd} \) via direct convolution is unwieldy for \( L > 2 \), and one may attempt to use the moment generating function of \( X \),

\[ G_X(s) = E[e^{-sy}] = \int_{-\infty}^{\infty} e^{-sy} f_X(y) \, dy. \] (2.15)
For $L$ independent repeats, the moment generating function of the final weighted decision variable, $D_L$, is $G_X(s)$. Taking the inverse transform of $G_X(s)$ and substituting this in (2.13) yields

$$P_{e,WPD} = \frac{1}{2\pi j} \int_{-\infty}^{0} \int_{-j\infty}^{j\infty} e^{st} G_X(s) \, ds \, dx \, .$$  (2.16)

It can be easily shown that

$$G_X(s) = \frac{1+2/\gamma_0}{\gamma_0} \left[ B_{\frac{1}{2}}(1+2/\gamma_0+s,1-s) + B_{\frac{1}{2}}(2/\gamma_0-s,2+s) \right]$$  (2.17)

where $B_{\alpha}(a,b)$ is the incomplete Beta function [15,16]. The direct evaluation of (2.16) appears intractable due to the nature of the incomplete Beta function. Alternatively, one may return to the derivation of the optimal decision rule [12] and attempt to find the pdf of the product of the $L$ random variables

$$\left( \frac{1-p_i}{p_i} \right)^{d_i} \, .$$  (2.18)

using the Mellin transform [17,18,19]. Unfortunately, the expression for the Mellin transform is also in the form of an incomplete Beta function. A series approximation approach could be taken by expressing the incomplete Beta function (2.17) as a hypergeometric series, [15], but the large number of terms arising from raising a truncated hypergeometric series to the $L^{th}$ power would best be handled by an algebraic manipulation program such as $REDUCE$, or $MACSYMA$ [20,21]. Rather than use a complicated series approximation for the WPD BER we will develop a tight upper bound.

### 2.4.2 Bounds on the WPD Bit Error Rate

We will find two upper bounds on the WPD bit error rate. The first, based on the Chernoff bound, is an easy to evaluate expression, but unfortunately rather
loose. The second bound is a somewhat more awkward expression to evaluate, but it will be shown later to be very tight and can thus provide an accurate means of computing the WPD bit error rate analytically.

The procedure to find a Chernoff bound is to first find $s = s^*$ that minimizes the moment generating function defined by (2.15). Then

$$P_{e,wpd} \leq [G_X(s^*)]^L.$$  \hfill (2.19)

In our case

$$G_X(s) = \int_{-\infty}^{\infty} e^{-sv} f_X(y) dy$$

$$= \frac{2^{1+2/\gamma_0}}{\gamma_0} \int_0^{\infty} \frac{e^{-(1+2/\gamma_0-s)v} + e^{-(2/\gamma_0+s)v}}{(1 + e^{-v})^{2+2/\gamma_0}} dy.$$ \hfill (2.20)

Inspection of (2.20) reveals that the integral exists only for $-2/\gamma_0 < s < 1 + 2/\gamma_0$, and also that the integral is symmetric with respect to $s = 1/2$. Thus the minimum value of (2.20) occurs at $s^* = 1/2$. Substituting $s^*$ and $v = e^{-v}$ into (2.20) we obtain

$$G_X(s^*) = \frac{2^{1+2/\gamma_0}}{\gamma_0} \int_0^{1} \frac{dv}{v^{1/2-2/\gamma_0}(1 + v)^{2+2/\gamma_0}}.$$ \hfill (2.21)

Unfortunately, the integral in (2.21) is not expressible in terms of elementary functions. In Appendix A it is shown that a good upper bound on (2.21) is given by

$$P_{e,wpd} < \left[ \frac{4}{4 + \gamma_0} \left( \frac{4 + 5\gamma_0}{4 + 3\gamma_0} \right)^{1/2} \right]^L.$$ \hfill (2.22)

We may find a much tighter upper bound on the WPD bit error rate as follows. Since the weights given by (2.5) are optimum, the use of any other weighting scheme will result in some performance loss. The amount of the loss incurred will depend
on the sensitivity of the WPD receiver to the weight accuracy. It is demonstrated later that the WPD receiver is quite insensitive to the weight accuracy, and we can thus hope to find a good upper bound on the WPD bit error rate if we can find a good approximation to the optimal weighting function and if the approximation leads to a tractable formulation.

We find a reasonable approximation to the optimal weighting function by considering its behaviour as the SNR is varied. The optimum weighting function for NCFSK is obtained by substituting (2.7) into (2.5),

\[ w = \ln(2e^{\gamma/2} - 1). \tag{2.23} \]

For large signal to noise ratios the optimal weights approach

\[ w = \ln 2 + \gamma/2 \approx \gamma/2. \tag{2.24} \]

The behaviour of the optimal weights at small signal to noise ratios may be found by first re-writing (2.23) as

\[ w = \ln[1 + (2e^{\gamma/2} - 2)]. \tag{2.25} \]

Using series expansions for \( \ln(1 + x) \) and \( e^{\gamma/2} \) in (2.25), and retaining only the terms up to the first powers of \( x \) it is easily shown that, for small SNR,

\[ w_i \approx \gamma. \tag{2.26} \]

We see that for both high and low SNR the optimal weighting function is proportional to the SNR, \( \gamma \). It seems reasonable then to approximate the optimal weighting function by a linear function of SNR and then attempt to obtain an analytical solution. The tightness of the bound can then be checked by comparing its prediction of BER with that obtained by simulation.
Recall that a scaling of the weighting function will not affect its performance, so we will take \( w_i = \gamma_i \) and attempt to carry out the exact BER analysis. In Appendix B it is shown that the solution is

\[
P_{e,wpd} \leq P_{e,wpd} |_{w=\gamma} = \frac{1}{(2 + \gamma_0)^L} \sum_{k=1}^{L} \left\{ \left( \frac{2L - k - 1}{L - 1} \right) \left[ (-1)^{L-2L} + 2^L \left( \frac{2 + \gamma_0}{4 + \gamma_0} \right)^{L-k} \right] \right. \\
+ \sum_{j=1}^{L-1} \left( -1 \right)^{j-1} \binom{L}{j} \sum_{i=0}^{L-k-j} \binom{L-k+j-i-1}{i} \binom{L-j+i-1}{j-1} \left( \frac{2 + \gamma_0}{4 + \gamma_0} \right)^{L-k-i} \right\}
\]  

Equation (2.27) is valid for \( L \geq 2 \), and it is exact for \( L = 2 \). In Section 2.7 the (2.27) will be demonstrated to yield an accurate estimate of the WPD bit error rate.

2.5 Comparison of WPD and Traditional Diversity Schemes

In this section we shall make some general comparisons for NCFSK diversity reception using the WPD receiver versus the traditional partial decision combining schemes of majority-voting and selection diversity. These schemes will also be compared with the analog square-law and maximal-ratio receivers. For reference, the block diagrams of the WPD, square-law, and maximal-ratio receivers are shown in Figures 2.1, 2.2, and 2.3 respectively. Block diagrams for the selection diversity and majority-voting receivers are not shown separately since they may be considered as special cases of the WPD receiver.

First, we note that the BER of an ideal NCFSK receiver over the Rayleigh
fading channel is

\[ P = E[p_e(\gamma)] = \int_0^\infty p_e(\gamma)f_r(\gamma)\,d\gamma = \frac{1}{2 + \gamma_0}. \]  

(2.28)

The BER for majority-voting on an odd number, \( L \), of independent bit receptions is simply the probability that more than half of the partial decisions are in error. With an even number of repeats, there is the possibility of a tie. We have,

\[
P_{e,mv} = \begin{cases} 
\sum_{k=\frac{L}{2}+1}^{L} \binom{L}{k} P^k(1-P)^{L-k}, & L \text{ odd.} \\
\frac{1}{3} \left( \frac{L}{L/2} \right) P^{L/2}(1-P)^{L/2} + \sum_{k=\frac{L}{2}+1}^{L} \binom{L}{k} P^k(1-P)^{L-k}, & L \text{ even.}
\end{cases}
\]

(2.29)
The BER for selection diversity is [7,9]

\[ P_{e_{,sel}} = \frac{2^{L-1}L!}{\prod_{k=1}^{L} (\gamma_0 + 2k)} = \frac{L!}{2 \prod_{k=1}^{L} (k + \gamma_0/2)} \]  

(2.30)

The analog techniques of square-law and maximal-ratio combining have bit error rates [7,9]

\[ P_{e_{,sq}} = P_e \sum_{k=0}^{L-1} \binom{L+k-1}{k} (1 - P)^{L-k} \]  

(2.31)

\[ P_{e_{,mr}} = 2^{L-1} P^L \]  

(2.32)
CHAPTER 2. IDEALIZED WPD RECEIVER

2.5.1 Asymptotic BER Characteristics

We consider the behaviour of these combination techniques for large signal to noise ratios. In this case the dependence of BER on $\gamma_0$ is

$$P_{e,mv} \approx \begin{cases} \left( \frac{L}{L+\frac{1}{2}} \right)^{L+1} \approx \left( \frac{L}{L+1} \right)^{\frac{1}{2}L}, & \gamma_0 \gg L, \quad L \text{ odd} \\ \frac{1}{2} \left( \frac{L}{L/2} \right)^{L} \approx \frac{1}{2} \left( \frac{L!}{(L/2)!} \right)^{\frac{1}{2}}, & \gamma_0 \gg L, \quad L \text{ even} \end{cases}$$

(2.33)

$$P_{e,sel} \approx \frac{2^{L-1} L!}{\gamma_0^L}, \quad \gamma_0 \gg L$$

(2.34)

$$P_{e,aeq} \approx \frac{(2L - 1)!}{L!(L - 1)!} \frac{1}{\gamma_0^L}, \quad \gamma_0 \gg L$$

(2.35)

$$P_{e,mar} \approx \frac{2^{L-1}}{\gamma_0^L}, \quad \gamma_0 \gg L$$

(2.36)

Thus with selection, square-law and maximal-ratio diversity the BER decreases inversely with the $L^{th}$ power of $\gamma_0$, while for majority-voting with $L$ odd the exponent of $\gamma_0$ is $-1/2 (L + 1)$. For majority-voting with $L$ even, the exponent of $\gamma_0$ is $-L/2$. We expect that the WPD receiver BER should also decrease inversely with the $L^{th}$ power of $\gamma_0$. This is confirmed by evaluating (2.22) or (2.27) for large $\gamma_0$. Equation (2.22) gives simply

$$P_{e,wpd} < \left( \frac{4\sqrt{5/3}}{\gamma_0} \right)^L.$$

(2.37)

At high signal to noise ratios the asymptotic increase in SNR required to achieve the same error rate for most of these systems may be evaluated. The loss incurred by using square-law or selection diversity receivers instead of maximal-ratio com-

---

3 The form of $P_{e,aeq}$ in (2.35) is obtained from the leading term in an alternate expression for (2.31) derived in [7].
CHAPTER 2. IDEALIZED WPD RECEIVER

Combination is obtained by equating (2.35) and (2.34) to (2.36), which yields,

\[
\frac{\gamma_{\text{sql}}}{\gamma_{\text{mr}}} = \left[ \frac{(2L-1)!}{L!(L-1)!2^{L-1}} \right]^\frac{1}{t}
\]  \hspace{1cm} \text{(2.38)}

and, for selection diversity,

\[
\frac{\gamma_{\text{sel}}}{\gamma_{\text{mr}}} = [L]^\frac{1}{t}.
\]  \hspace{1cm} \text{(2.39)}

We may also evaluate (2.27) as \( \gamma_0 \to \infty \), to obtain a slightly pessimistic estimate of the loss of the WPD receiver relative to maximal-ratio combination. This loss estimate is

\[
\frac{\gamma_{\text{wpd}}}{\gamma_{\text{mr}}} = \left[ \frac{1}{2^{2L-1}} \sum_{k=1}^{L} \left( \frac{2L-k-1}{L-1} \right) \left( -1 \right)^{L-2k} + 2^k \right]^{\frac{1}{L}} \hspace{1cm} \text{(2.40)}
\]

\[+ \sum_{j=1}^{L-1} \left( -1 \right)^{L-j} \binom{L}{j} \sum_{i=0}^{L-k-j-1} \binom{L-j+i-1}{i} \right] \right]^{\frac{1}{L}} \]

The results of evaluating (2.39), (2.38) and (2.40) for several \( L \) are summarized in Table 2.1. It can be seen that the WPD receiver suffers less than 1.1 dB loss relative to the square-law receiver for \( L \leq 15 \). Selection diversity suffers a greater loss as \( L \) is increased, and is 5.6 dB worse than square-law combining with \( L = 15 \).

2.5.2 Efficient Use of Branch Information

The BER of the various combination schemes may also be compared as a function of the number of branches to see how effectively the branch information is used.

The bound of (2.27) and the bit error rates as given by (2.29) through (2.32)
CHAPTER 2. IDEALIZED WPD RECEIVER

<table>
<thead>
<tr>
<th>Number of Branches $L$</th>
<th>Loss [dB]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Square-Law</td>
</tr>
<tr>
<td>2</td>
<td>0.9</td>
</tr>
<tr>
<td>3</td>
<td>1.3</td>
</tr>
<tr>
<td>4</td>
<td>1.6</td>
</tr>
<tr>
<td>5</td>
<td>1.8</td>
</tr>
<tr>
<td>6</td>
<td>1.9</td>
</tr>
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<td>7</td>
<td>2.0</td>
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<td>8</td>
<td>2.1</td>
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<tr>
<td>9</td>
<td>2.2</td>
</tr>
<tr>
<td>10</td>
<td>2.26</td>
</tr>
<tr>
<td>11</td>
<td>2.31</td>
</tr>
<tr>
<td>12</td>
<td>2.35</td>
</tr>
<tr>
<td>13</td>
<td>2.39</td>
</tr>
<tr>
<td>14</td>
<td>2.42</td>
</tr>
<tr>
<td>15</td>
<td>2.45</td>
</tr>
</tbody>
</table>

Table 2.1: Asymptotic Loss of Square-Law, Selection, and WPD Combining Relative to Maximal-Ratio Combining

are plotted in Figure 2.4.

It can be seen that significant gains for the WPD receiver are indicated by (2.27) as $L$ is increased. Selection diversity will eventually offer no further improvement as $L$ increases, while the BER for majority-voting and the WPD receiver continue to decrease. The BER of majority-voting decreases in a stepwise fashion with $L$. Voting on an even number of branches offers exactly zero improvement over using one less branch. The WPD receiver makes the most efficient use of the branch

---

4 The formulas of (2.27), (2.29), and (2.31), are awkward to compute directly for large $L$ since the factorial terms in the binomial coefficient become very large. One may avoid numerical overflow by computing the binomial coefficient as

$$\frac{M!}{N!(M-N)!} = \exp\{\ln[M!] - \ln[N!] - \ln[(M-N)!]\} = \exp\{\ln[\prod_{i=1}^{M} i] - \ln[\prod_{j=1}^{N} j] - \ln[\prod_{k=1}^{M-N} k]\} = \exp\{\sum_{i=1}^{M} \ln[i] - \sum_{j=1}^{N} \ln[j] - \sum_{k=1}^{M-N} \ln[k]\}. $$

5 This arises from the properties of the binomial distribution, and a proof may be found in [23], for example.
information of any of the partial decision schemes. As one might expect, the analog combination schemes are more efficient than the partial decision combiners. In particular, it can be shown [22] that a $L$ branch square-law combiner achieves the same probability of error as a $2L - 1$ branch majority-voting combiner.
2.5.3 Optimal Number of Diversity Branches

In the previous section we considered the improvement in BER obtained by the various combiners when an increasing number of diversity branches is used. This compared their performance as more and more energy was invested in the transmission of a single message bit. Consider instead that the total amount of energy transmitted per message bit is fixed. Then, as an additional branch is added the average power per repetition will be decreased (assuming a fixed repetition duration), but the additional branch increases the likelihood of avoiding fades. For the partial decision combiners we note that on one hand, each partial decision will become less reliable as more branches are added, but on the other hand we have more decisions on which to decide the final outcome. These two tendencies have opposite effects on the BER and one would expect that an optimum number of branches exists to achieve a best compromise. The square-law combiner can also be shown [11] to have an optimum number of branches, since it employs non-coherent combination of the branch signals. This is in contrast to maximal-ratio combining which makes the best possible use of the branch signals via coherent combining. It can be shown [9] that in maximal-ratio combining an optimum number of repeats does not exist since additional branches never decrease performance.

Previous results exist for the optimal number of diversity branches for selection diversity [24] and square-law receivers [11,25]. We will incorporate these results in the discussion to follow and we will estimate the optimal number of diversity branches for the WPD and majority-voting receivers.

For a fixed transmitted energy per message bit and $L$ branches with independent
Rayleigh fading, the ratio of the average received energy of a message bit to the noise density is
\[
\frac{E_b}{N_0} = L \gamma_0. \tag{2.41}
\]

Recall that the Chernoff bound of (2.19) is normally in the form
\[
Pr(\text{bit error}) \leq G_X^L(s^*). \tag{2.42}
\]

This may be re-written as
\[
Pr(\text{bit error}) \leq e^{-L \ln \left( \frac{1}{G_X(s^*)} \right)} \tag{2.43}
\]
\[
\leq e^{- \frac{E_b}{N_0} L \ln \left( \frac{1}{G_X(s^*)} \right)} \tag{2.44}
\]
\[
\leq e^{- \frac{E_b}{N_0} \frac{1}{\gamma_0} \ln \left( \frac{1}{G_X(s^*)} \right)} \tag{2.45}
\]
\[
\leq e^{- \frac{E_b}{N_0} g(\gamma_0)} \tag{2.46}
\]

where we have defined
\[
g(\gamma_0) = \frac{1}{\gamma_0} \ln \left( \frac{1}{G_X(s^*)} \right). \tag{2.47}
\]

We will refer to (2.47) as the Chernoff bound exponent function \(^6\). The form of the Chernoff bound in (2.46) shows that for a given \(E_b/N_0\), the lowest BER indicated by the bound occurs for the maximum of the Chernoff bound exponent function. Let \(\gamma_0^*\) denote the SNR at which this maximum is attained, then our estimate of the optimal number of branches, \(L^*\), is obtained from substituting \(\gamma_0^*\) into (2.41);
\[
L^* \approx \frac{1}{\gamma_0^* N_0}. \tag{2.48}
\]

\(^6\) In [11] a similar form of the Chernoff bound is referred to as an efficiency function and is used to estimate the optimum number of branches for the square-law receiver.
For the WPD receiver the Chernoff bound is given by (2.22) and we have

\[ g_{wpd}(\gamma_0) = \frac{1}{\gamma_0} \ln \left[ \frac{(4 + \gamma_0)}{4} \frac{\sqrt{4 + 3\gamma_0}}{\sqrt{4 + 5\gamma_0}} \right]. \tag{2.49} \]

For majority-voting, we can obtain a Chernoff bound by considering the majority-voting receiver to act as an up/down counter. It counts up with probability \( P \), and down with probability \( 1 - P \). The moment generating function is then

\[ G_{mv}(s) = E[e^{st}] = Pe^{s} + (1 - P)e^{-s}. \tag{2.50} \]

The value of \( s \) that minimizes (2.50) is easily found and yields

\[ G_{mv}(s^*) = 2\sqrt{P(1 - P)} = \frac{2}{2 + \gamma_0} \sqrt{1 + \gamma_0} \tag{2.51} \]

and

\[ g_{mv}(\gamma_0) = \frac{1}{\gamma_0} \ln \left[ \frac{2 + \gamma_0}{2\sqrt{1 + \gamma_0}} \right]. \tag{2.52} \]

For the square-law receiver, we have from [11] that

\[ G_{sq}(s^*) = 4P(1 - P) = \frac{4(1 + \gamma_0)}{(2 + \gamma_0)^2} \tag{2.53} \]

so that

\[ g_{sq}(\gamma_0) = \frac{1}{\gamma_0} \ln \left[ \frac{(2 + \gamma_0)^2}{4(1 + \gamma_0)} \right] \tag{2.54} \]

\[ = 2g_{mv}(\gamma_0). \tag{2.55} \]

The exponent functions, (2.49) and (2.54), are plotted in Figure 2.5. Note that both are maximum at \( \gamma_0 \approx 5 \text{ dB} \approx 3 \). Thus our estimate of the optimum number of branches for these schemes is

\[ L^* \approx \frac{1}{3} \frac{E_b}{N_0}. \tag{2.56} \]
Figure 2.5: Chernoff Bound Exponent Functions
CHAPTER 2. IDEALIZED WPD RECEIVER

Table 2.2: Estimated and Exact Values for the Optimum Number of Branches

<table>
<thead>
<tr>
<th>$E_b/N_0$ [dB]</th>
<th>Majority-Voting</th>
<th>Selection</th>
<th>Square-Law</th>
<th>WPD (Bound)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Est.</td>
<td>Exact</td>
<td>Est.</td>
<td>Exact</td>
</tr>
<tr>
<td>10</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>2</td>
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<tr>
<td>12</td>
<td>5</td>
<td>3</td>
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<td>3</td>
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<tr>
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<td>13</td>
<td>11</td>
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<td>4</td>
</tr>
<tr>
<td>18</td>
<td>21</td>
<td>19</td>
<td>6</td>
<td>6</td>
</tr>
</tbody>
</table>

For selection diversity it has been shown [24] that the optimum number of branches is well estimated by

$$L^* \approx \sqrt{E_b/2N_0}.$$  \hspace{1cm} (2.57)

To see how accurate these guidelines are in predicting the optimum number of branches one can search for the optimum $L$ with a few values of $E_b/N_0$. The results of this search using the exact BER expressions for the majority-voting, selection, and square-law receivers, and using the bound of (2.27) for the WPD receiver are summarized in Table 2.2. It can be seen that the predicted optimum numbers of branches are reasonably close to the exact result, with the exception of majority-voting at lower SNR. The loss in using the predicted optimum number of branches is very small as demonstrated by the plot of BER as a function of $L$ in Figure 2.6. Note that as $E_b/N_0$ is decreased, there will be a point for each of the imperfect combination schemes (i.e. all but maximal-ratio) where the the loss associated with imperfect combination overcomes any gain inherent in having multiple samples of the fading signal. In other words, the optimal number of branches for the imperfect combination schemes will be one as $E_b/N_0$ becomes small. This is demonstrated in Figure 2.6 for majority-voting with $E_b/N_0 = 10$ dB.
CHAPTER 2. IDEALIZED WPD RECEIVER

Figure 2.6: BER vs. Number of Diversity Branches, Fixed $E_b/N_0$. 
Finally, we emphasize that the maximal-ratio receiver has no optimum number of branches since it perfectly combines all of the branch information. It can be shown [9] that as the number of branches approaches infinity, with a fixed energy per bit of $E_b/N_0$, the effects of Rayleigh fading are completely mitigated, i.e.

$$\lim_{L \to \infty} P_{e,mr} = \frac{1}{2} e^{-\frac{E_b}{2 N_0}}.$$  

(2.58)

Thus the BER for maximal-ratio combining with an infinite number of Rayleigh faded branches is the same as the BER for an unfaded signal with the same bit energy.

2.6 Simulation

To obtain an estimate of the WPD receiver performance in an example system a Monte Carlo type simulation program was developed. This section outlines the simulation method and presents some results for the bit and packet error rates.

The simulation results to be presented are for a simple model of the Advanced Mobile Phone Service (AMPS) land to mobile control channel [26,27]. This cellular system was selected since it is a well known modern system, and since it uses majority voting to help combat fading.

2.6.1 Simulation Model

The AMPS system employs FSK modulation and detection of a 10 kbps Manchester encoded data stream. In the land to mobile control channel five repeats of a forty bit packet are sent, with an effective forty bit gap between block repeats to decrease
Figure 2.7: AMPS Land to Mobile Control Channel Format. (Figure taken from [39].)

their dependence, as shown in Figure 2.7. Upon reception of the packet repeats the mobile decoder performs majority voting on the individual bits and then corrects one bit error per packet. The error correction utilizes a (40,28) BCH code, which is a shortened version of the (63,51) BCH code\textsuperscript{7}.

To efficiently simulate the WPD receiver we shall use the ideal NCFSK bit error probability model given in (2.7). This will not accurately predict the error rate to be expected in a practical receiver for several reasons, which will be outlined below. However, it provides a simple and reasonable approximation to the expected error

\textsuperscript{7}The (40,28) code has a minimum distance of 5, which implies a capability for double error correction. However the decoder is configured to correct only one error, to achieve a lower probability of undetected packet error.
rate, and in fact is used in the performance specifications for the AMPS system [13].

The idealized NCFSK model implicitly assumes that a receiver with matched filters is used. The receiver block diagram is equivalent to the square-law receiver in Figure 2.2 with $L = 1$. Some of the factors contributing to a deviation of the actual cellular system performance from our simulation results are:

**Demodulator Implementation Method.** Practical NCFSK demodulators in cellular receivers usually employ a limiter/discriminator followed by an integrate and dump circuit. Performance analysis of limiter/discriminator based receivers attempts to incorporate the effect of *click* noise [28,29,30]. The generation of a click, or *spike*, may be understood by representing the received signal plus noise on a phasor diagram. The discriminator's function is to differentiate the received phase, and a click will be generated if the resultant signal plus noise phasor rapidly encircles the origin. For example, if the noise happens to cause the received phase to completely encircle the origin in the positive direction, then the phase change will be $2\pi$ and a positive spike will be generated of area $2\pi$. At high SNR, the probability of the noise overcoming the signal and resulting in an encirclement of the origin is very low. As the SNR is lowered the signal approaches the noise level and the likelihood of click occurrences increases.

Other receiver implementations might conceivably utilize matched filters or bandpass filters instead of the limiter/discriminator scheme. In this case there would inevitably be some cross-talk between the filters, as well as loss of signal power due to bandlimiting by the receiver intermediate frequency
(IF) filter. The effects of these factors on the BER is analyzed in [31].

Intersymbol Interference. The transmitted pulses are filtered prior to transmission in order to control adjacent channel interference and are also shaped by the receiver IF filter. The resulting intersymbol interference (ISI) may increase the BER [32,33,34]. As the IF bandwidth is decreased there will be a higher level of ISI which will tend to decrease performance. However, this reduction in IF bandwidth also admits less noise to the receiver. Thus, for a given bit rate and frequency deviation there will be an optimal IF filter bandwidth for a specified receiver structure. Several investigations into the relationship between IF filter bandwidth, frequency deviation, and the resulting BER have been made, including [32,33,34].

Manchester Encoded Data. The cellular signalling format uses Manchester encoded bits as inputs to the FM modulator. This facilitates bit synchronization, since each bit has a transition. It also has no DC component, so that implementation via an integrator followed by a phase modulator with limited range is practical. Of course the multiplication of the data sequence by a square wave increases the occupied bandwidth of the signal [35], but in cellular systems relatively wide channels, 30 kHz, are available and spectral efficiency is instead obtained via frequency re-use. The ISI will be increased from the use of Manchester encoding but this is traded off for the convenient synchronization and implementation features. The performance of Manchester encoded data and non-return to zero (NRZ) data through limiter discriminator systems is investigated in [33,34].

Interference. Cellular systems operate in co-channel and adjacent channel in-
interference generated by neighbouring cells. The performance of cellular signalling is very dependent on the cell frequency plan, cell site antenna design, and control of transmitter power [36,37]. If all of the transceivers increase their power levels in an effort to overcome the AWGN, the level of interference will also increase and may dominate the system performance.

A form of interference generated by neighbouring vehicles is ignition noise, which is impulsive in nature. The contribution of ignition noise towards the BER may be significant in some systems [38]. The contribution of ignition noise towards the BER in the AMPS system is considered to be low [39], since the arrival rate of the impulses is low compared to the bit rate.

The effects of interference will be discussed further in Section 3.2.

Miscellaneous Propagation Phenomena. As discussed in Section 1.1, we have assumed that the propagation is described by Clarke's multi-scatterer model, so that the received amplitude is Rayleigh distributed. There will be, of course, a variety of fading environments encountered in practice, including the presence of a direct LOS component and shadowing of the signal. Also, fading has been assumed to be flat across the frequency band. For UHF frequencies in dense urban areas the coherence bandwidth is of the order of 40 kHz [4]\(^8\). Fading on frequencies separated less than 40 kHz will then be largely correlated, so that the assumption of flat fading in the AMPS 30 kHz bandwidth should be reasonable.

We have also assumed the condition of slow fading, i.e. that the signal amplitude and phase are effectively constant during a bit. The time varying phase

\(^{8}\) Here the coherence bandwidth is taken as the bandwidth within which fading has a 0.9 or greater correlation.
leads to a small but irreducible BER as the SNR is increased [1]. This is referred to as random FM, since the variation of received phase over the bit period is equivalent to an undesired random frequency modulation.

**Miscellaneous Receiver Implementation Losses.** In addition to the items mentioned above there will be additional degradation due to various receiver implementation losses such as imperfect bit synchronization, inaccuracies in channel measurement information, and quantization of variables in a digital implementation. Some of these factors will be addressed in Chapter 3.

In summary, there are many practical aspects of receiver implementation that will alter the BER performance from the ideal. The exact amount of degradation will depend on the particular receiver implementation, but we note from the AMPS specification [13] that it is entirely feasible to have less than 3 dB degradation from the ideal NCFSK bit error rate. This includes all of the effects listed above except for the interference and propagation phenomena. In Chapter 3 a more comprehensive simulation is developed that relaxes some of our assumptions of ideality. Another benefit of the more detailed simulation is that it simulates the random variables internal to the demodulator, so that bit and packet error rates for the square-law receiver may be obtained in addition to the majority-voting, selection, and WPD receivers.

\(^{9}\text{At 850 MHz with a 10 kbps bit rate and a 14 kHz peak to peak frequency deviation the BER limit is of the order of } 10^{-5} \text{ for vehicle speeds below 70 mph [4].} \)
CHAPTER 2. IDEALIZED WPD RECEIVER

2.6.2 Simulation Description

The computer simulation program utilizes routines developed in [40] to generate an exponentially distributed random variable that represents the squared envelope of a sine wave under-going Rayleigh fading. The fading routines are based on the fading simulation program discussed in [41]. Briefly, two pseudo-random Gaussian number sequences are generated that represent the Fourier transform of the in-phase and quadrature components of a complex Gaussian random process. The autocorrelation function of the complex valued received amplitude for a particular vehicle speed, derived in [2], yields the power spectral density function of the received signal as

$$S(f) = \frac{1}{\pi f_D} \frac{1}{\sqrt{1 - f^2/f_D^2}} \quad |f| \leq f_D. \quad (2.59)$$

The Gaussian random number sequences are weighted to achieve this spectral shape, and then inverse fast Fourier transformed to yield a sequence of time domain samples of the complex amplitudes. Finally, summing the squares of the in-phase and quadrature components yields the desired exponentially distributed sequence\(^{10}\).

For each bit the fading sequence value is used to represent the received SNR, and is then used in (2.7) to yield \(p_i\). The weight \(w_i\) is then obtained from (2.5). A bit error is injected if the output of a uniformly distributed pseudo-random number generator, with range \((0,1)\), is less than \(p_i\).

The program maintains various counters and registers to simulate and monitor

\(^{10}\)The phase of the received signal could be obtained from the ratio of the quadrature to in-phase components, but this is not required here since we are simulating a non-coherent receiver.
CHAPTER 2. IDEALIZED WPD RECEIVER

the WPD, majority-voting, and selection diversity receivers. The various bit and packet error events are counted by \( M \) separate sets of error counters that maintain data from \( M \) different fading sequences. This provides \( M \) independent trials, from which we may compute \( M \) sample means that estimate any particular error rate. If we have counted a sufficient number of error events then the sample means should be approximately normally distributed according to the central limit theorem. We can then compute a confidence interval using the \( t \)-distribution [42]. Denoting the \( m^{th} \) sample mean for an actual error rate \( p \) as \( \hat{p}_m \), then our final estimate \( \hat{p} \) is the mean of \( \{\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_M\} \), and an \( (1 - \alpha) \cdot 100\% \) confidence interval about \( \hat{p} \) is

\[
\hat{p} \pm t_{\frac{\alpha}{2}} \frac{s}{\sqrt{M}}
\]

where \( t_{\frac{\alpha}{2}} \) is the argument of the \( t \) distribution that leaves an area of \( \alpha/2 \) in the tail of the distribution, and \( s^2 \) is the sample variance.

The program tests periodically to see if a specified confidence interval has been attained and stops execution if this criterion has been met. Since some of the simulations at higher signal to noise ratios will run for a considerable time the program periodically sends intermediate results to a file for preliminary viewing. The program also periodically updates a simulation state file that contains the present values of the random number generator seeds and all of the error event counters. This simulation state file is used to gracefully continue the simulation in the event of a computer system crash, rather than having to re-start from scratch.
2.6.3 Simulation Parameters

All simulation results were obtained using a Doppler frequency, $f_D$, of 49 Hz and a bit rate of 8192 bits per second (bps). This bit rate was chosen since the fading simulation is constrained to use bit rates that are a power of two\textsuperscript{11}, and 8192 bps is the closest power of two bit rate to the 10 kbps data rate used by AMPS. The fade rate was chosen using the guideline that the bit repeats will be approximately independent for separations, $\tau$, of at least the reciprocal of twice the Doppler frequency [1]. For cellular signalling with bit repeats separated by 80 bits at 10 kbps, or 8 msec, the Doppler frequency should be greater than

$$f_D > \frac{1}{2\tau} = \frac{1}{2 \cdot 8 \text{ msec}} \approx 60 \text{ Hz.}$$

(2.61)

At 850 MHz this Doppler frequency is equivalent to a vehicle speed of 47 mph. For our simulation we must scale the Doppler frequency since we are simulating 8192 bps, rather than 10 kbps. The equivalent $f_D$ for 8192 bps is then 49 Hz. This Doppler frequency is the minimum according to our guideline to achieve independent bit repeats, and we will use it to see if there is any appreciable deviation of the simulation results from that predicted by (2.29) and (2.30), and to compare the WPD simulation results with the bound of (2.27).

The degree of correlation between the fading envelopes of two bit repeats can be found from the normalized autocorrelation function of the received envelope,

$$\rho(\tau) \approx J_0^2(2\pi f_D \tau)$$

(2.62)

where $J_0(x)$ is the zero order Bessel function of the first kind and $\tau$ is the time separation. Equation (2.62) is obtained from combining equations (16) and (17).

\textsuperscript{11}This constraint is due to the fact that the fading envelope generator uses a FFT to produce one second of fading envelope data.
Figure 2.8: Normalized Autocorrelation Function of the Received Envelope

in [2] and is plotted in Figure 2.8. For our bit separation of at $\tau = 8$ msec with 10 kbps, and $f_D = 60$ Hz (2.62) yields a correlation coefficient of 0.1. Given this small correlation and the previous observation that square-law combining suffers only minor degradation with correlated branches [7], it seems reasonable to expect that the simulation results for the other combining methods will be fairly close to their predicted error rates. This will be confirmed in the next section.
2.6.4 Simulation Results

The BER performance of the majority-voting, selection, and WPD receivers as determined by simulation\footnote{The BER curve for the selection and square-law receivers with $L = 5$ was obtained via the more general simulation method explained in Chapter 3. This was done since these receivers were not incorporated into the simple simulation method used in this chapter. All of the relevant simulation parameters such as the packet format, Doppler frequency, etc. are identical in both of the simulation runs. The error bars for all simulation results in this thesis indicate 95\% confidence intervals.} is shown in Figure 2.9. Also shown are the bit error rates predicted for the standard partial decision combining schemes, as given by (2.29), and (2.30). The BER obtained from the simulations for majority-voting, selection diversity, and square-law combining are only marginally higher than predicted. This confirms that at $f_D\tau = 0.48$ the bit repeats are nearly independent, and we should not expect any significant decrease in BER if the bit separation is further increased. The results for three and five repeats are shown. As mentioned earlier, an $L$ branch square-law combiner achieves the same error rate as a $2L-1$ branch majority-voting receiver [22], so that square-law combining with $L = 3$ is equivalent to majority-voting with $L = 5$.

With a nominal AMPS signal to noise ratio of 15 dB, the AMPS format of majority-voting on 5 repeats achieves a BER of $2.5 \times 10^{-4}$. For this target BER, the WPD receiver is over 3.5 dB more efficient. As shown earlier, this gain will increase indefinitely in an ideal system as the SNR is increased. Relative to selection diversity, the WPD receiver shows a gain of more than 1 dB. As the SNR is increased the asymptotic gain of the WPD receiver relative to selection diversity will be about 1.4 dB as shown in Table 2.1. With three repeats, the WPD receiver is approximately 4 dB more efficient than majority-voting and only slightly better.
than selection diversity. This is consistent with our earlier observation that WPD and selection diversity are equivalent for $L = 2$, and the gain offered by the WPD method increases with $L$. Finally, we note that for 3 or 5 repeats, the WPD receiver is less than 1 dB poorer than square-law combining.

The packet error rates (PER) for five repeats are shown in Figure 2.10. For a PER of $10^{-3}$, the WPD strategy is over 3 dB more efficient than majority-voting. Here, even with no error correction, the WPD and selection diversity receivers outperform majority-voting with single bit error correction, and the WPD receiver has more than a 1 dB advantage over selection diversity.

As discussed in Section 2.3, a more complete performance comparison could average the error rate over the Doppler frequency distribution. We have demonstrated that with a few independent bit repeats the performance gain over majority-voting is significant, and the gain relative to selection diversity is more modest. At very low Doppler frequencies, the error rates of the WPD and majority-voting receivers will be equivalent, while selection diversity will offer no gain relative to a single bit transmission. Thus, if there is a wide range of expected Doppler frequencies, using the WPD receiver will provide better performance than either of the majority voting or selection diversity schemes.
Figure 2.9: Bit Error Rates for 3 and 5 Repeats
Figure 2.10: Packet Error Rates for 5 Repeats
2.7 Error Rate Approximations

In this section we demonstrate that the upper bound given by (2.27) is quite tight. We also demonstrate that some simple bounds on the packet error rate are in good agreement with the simulation results.

Three estimates of the WPD bit error rate are shown in Figure 2.11. The fading simulation results from the previous section have been re-plotted, along with the bound (2.27), and another simulation that used completely independent bit repeats. This latter simulation generated independent samples of an exponentially distributed random variable by summing the squares of two independent Gaussian random number generators. As discussed in the previous section, the fading simulation with $f_D \cdot \tau = 0.48$ yielded bit error rates that are slightly higher than predicted for the standard diversity schemes with completely independent bit repeats. Thus we can expect the fading simulation to yield a slightly higher BER for the WPD receiver as well. The independent bit repeat simulation plotted in Figure 2.11 confirms this. Note that the BER predicted by the upper bound of (2.27) is practically identical to the independent bit repeat simulation results and can thus be used to accurately estimate the WPD bit error rate. This does not show that 2.27 is accurate at higher signal to noise ratios or for large values of L, but, at the typical values of SNR and L used here the accuracy is very good. There is no reason to expect a dramatic loss of accuracy as the SNR or L is increased.

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13 The Gaussian random number generation program is documented in [43].

14 The independent bit repeat simulation plot in Figure 2.11 extends only to 12 dB. The 95% confidence intervals are indicated by error bars, which extend only about one line thickness, at worst.
Figure 2.11: WPD Bit Error Rate Bound and Simulation Results
On the contrary, there will be a fixed dB loss as the SNR approaches infinity since both the bound and the actual WPD error rate have the same asymptotic slope. For five repeats, this fixed dB loss appears to be negligible.

Next, we demonstrate the effectiveness of approximating the packet error rate by some simple bounds. The Rayleigh fading channel is usually considered to be bursty, i.e. bit errors tend to cluster together rather than being randomly spaced. Since a packet in error may often contain a number of errors, relatively few packets will absorb a specified number of bit errors, compared to the number of packets that would be required to absorb the same number of independently occurring bit errors. For a bursty channel then, it would seem reasonable to form an upper bound on the uncorrected PER by assuming that bit errors occur independently, resulting in more packet errors for a given BER, i.e.

\[ P( \text{packet error} ) = P(> 0 \text{ bit errors in packet} ) \leq 1 - (1 - P_e)^N \]  
\[ (2.63) \]

where \( P_e = P( \text{bit error after } L \text{ repeats}) \), and \( N \) is the packet length.

One may form a bound on the \( t \)-error corrected PER by considering that, at worst, the errors that occur are grouped into exactly \( t + 1 \) errors per packet. Over a long observation period of say \( M \) bits, there will be \( MP_e \) bits in error. At worst, this will result in \( MP_e/(t + 1) \) packet errors. The bound on the PER is then

\[ P(> t \text{ bit errors in packet} ) \leq \frac{MP_e/(t + 1)}{M/N} = \frac{NP_e}{t + 1}. \]  
\[ (2.64) \]

Equations (2.63) and (2.64) are plotted in Figures 2.12 and 2.13 along with the simulation data. For the standard diversity schemes the value of \( P_e \) used is taken from the analytical BER expressions. For the WPD receiver \( P_e \) is determined from the bound, (2.27).
With no error correction, Figure 2.12 shows that, for packet error rates below $10^{-1}$, both the independent bit error model (2.63) or the bound (2.64) are within 1 dB of the simulation results. As the SNR is increased, the accuracy of both bounds improves. The accuracy of the PER prediction by (2.63) or (2.64) corresponds to single bit errors being the dominant error mechanism. This is due in part to diversity decreasing the effective fade duration [9], so that the probability of single bit errors can dominate even at moderate Doppler frequencies.

If the channel, including the effect of diversity, is such that the sum of the probabilities of $t + 2, t + 3, \ldots, N$ errors per packet are significant compared to the probability of $t + 1$ errors per packet, then the bound of (2.64) will not be very tight. The looser bound of (2.64) with one bit error correction is shown in Figure 2.13. Below a PER of about $10^{-1}$ the bound is about 1dB off the square-law, WPD, and selection diversity curves, while for majority-voting it is about 2 dB off. One would expect that the PER bound (2.64) will be looser as $t$ is increased.
CHAPTER 2. IDEALIZED WPD RECEIVER

Figure 2.12: Uncorrected Packet Error Rate
Figure 2.13: Single Bit Error Corrected Packet Error Rate
Chapter 3

Some Issues in Practical WPD Receiver Design

In this chapter we demonstrate the robustness of the WPD receiver when several of the assumptions discussed in the previous chapter are removed. The performance of the WPD receiver is assessed when various parameters are quantized, and when the signal strength measurement is imperfect. Also, a relatively efficient method is presented to determine the optimal quantization thresholds. Finally, it is demonstrated that a good signal strength estimate is obtainable by simple low pass filtering of the IF filter output envelope.

3.1 SNR and Weight Quantization

Let the received SNR be quantized into $N + 1$ regions by $N$ thresholds, $\{T_k\}_{k=1}^{N}$, as shown in Figure 3.1. These regions will give rise to $N + 1$ different reliability weights. We will consider the effect of the SNR quantization shown in Figure 3.1 and the effect of the quantization of the corresponding weights separately. That is, we first assume that the weights are represented exactly, then consider quantization of these weights.
3.1.1 BER Evaluation Method

The probability of bit error for the WPD receiver when a perfect SNR measurement is quantized may be evaluated as follows. The random variables resulting from the thresholding of the SNR can be represented by a discrete channel model. If we assume that a '1' is sent $L$ times, and assume that the repetitions are independent, then all of the possible combinations of the receiver random variables may be represented by the tree shown in Figure 3.2. At each repeat, the receiver makes a partial decision from one of the $N + 1$ reliability regions, resulting in $2(N + 1)$ possible values. The probability that the SNR falls in the $k^{th}$ SNR region is represented by the branch labeled $P_k$, and stemming from this branch are the probabilities of then making a correct decision or an error, $P_{c|k}$ and $P_{e|k}$ respectively. With $L$ repeats,
there is a total of $M = [2(N + 1)]^L$ sequences of random variables represented by the $M$ final nodes in Figure 3.2.

Let $w^{(k)}$ denote the optimal weight for the $k^{th}$ SNR region. Since a bit received in the $k^{th}$ SNR region has a probability of error equal to $P_{e|k}$, we have from (2.5) that

$$w^{(k)} = \ln \left( \frac{1 - P_{e|k}}{P_{e|k}} \right)$$

(3.1)
CHAPTER 3. SOME ISSUES IN PRACTICAL WPD RECEIVER DESIGN 56

where

\[ P_{e|k} = \frac{1}{P_k} \int_{T_k}^{T_{k-1}} p_e(\gamma) f_T(\gamma) \, d\gamma \]  
\[ = \frac{1}{P_k} \frac{1}{2 + \gamma_0} \left[ e^{-\left(1 + \frac{1}{\gamma_0}\right) T_{k-1}} - e^{-\left(1 + \frac{1}{\gamma_0}\right) T_k} \right] \]  
\[ (3.2) \]
\[ (3.3) \]

and

\[ P_k = \int_{T_{k-1}}^{T_k} f_T(\gamma) \, d\gamma = e^{\frac{-T_{k-1}}{\gamma_0}} - e^{\frac{-T_k}{\gamma_0}}. \]  
\[ (3.4) \]

Define a set of variables, \( U = \{u_j\}_{j=1}^M \), where the subscript \( j \) indicates one of the \( M \) possible paths in the tree of Figure 3.2. The value of \( u_j \) represents the value of the final decision variable for that particular path, and is computed by summing \( d_i w_i^{(k)} \) for the \( j^{th} \) path. The probability of \( u_j \) occurring, \( p(u_j) \), is given by the product of the probabilities of the branches associated with the \( j^{th} \) path. The BER is evaluated by summing the probabilities of all the paths that lead to an error,

\[ P_e = \sum_{u_j < 0} p(u_j) + \frac{1}{2} \sum_{u_j = 0} p(u_j), \]  
\[ (3.5) \]

where the second term accounts for the fact that half of any ties will result in an error.

This procedure will later be used to compute the probability of bit error for the WPD receiver with unquantized and quantized weights, but before doing so the issue of how to select the thresholds is discussed.
3.1.2 Threshold Optimization

This section discusses various methods that may be used to select the SNR quantization thresholds shown in Figure 3.1. We seek a set of thresholds \( \{T_k\}_{k=1}^N \) that minimizes the probability of bit error.

A direct, but inefficient, approach would be to optimize the BER by using a non-linear function optimization routine and the BER evaluation algorithm described in Section 3.1.1. In an alternate method, which we shall refer to as the capacity method, thresholds are chosen so as to maximize the discrete memoryless channel (DMC) capacity. This method has been used previously in different contexts in [44,23].

We will apply the direct optimization and capacity methods here, and we will also introduce a method that chooses thresholds that minimize a Chernoff bound.

3.1.2.1 Capacity Method

Recall that Figure 3.2 summarizes the possible events that will occur in a WPD receiver for \( L \) independent repeats of the message bit \( m = 1 \). The DMC model for one repeat may be obtained from Figure 3.2 by re-drawing the paths arising from one repeat to also show the paths for \( m = 0 \), and collapsing the transition probabilities \( P_{c|k} P_k \) and \( P_{c|k} P_k \) to form \( P_e^k \) and \( P_{e|k} \) respectively. The result is shown in Figure 3.3. We will show that the capacity of such a channel is simply the weighted average of the capacities of the sub-channels formed by the SNR quantization regions.
Figure 3.3: Discrete Memoryless Channel Model
CHAPTER 3. SOME ISSUES IN PRACTICAL WPD RECEIVER DESIGN 59

The capacity of the channel in Figure 3.3 is

\[ C = \max_{\{P(x = i)\}} [I(X;Y)] \] (3.6)

where \(\{P(x = i)\}\) is the set of input probability assignments, and \(I(X;Y)\) is the average mutual information between the sets of inputs and outputs, \(X\) and \(Y\). For a symmetric DMC\(^1\), as we have here, capacity is attained for equally likely inputs \([45]\). Furthermore, with the maximizing input probability assignment the capacity will be equal to the mutual information between any one of the inputs\(^2\) and the set of outputs, i.e. [45, pg. 91]

\[
C = I(x = i; Y) = \sum_j P(j | i) \log_2 \frac{P(j | i)}{P(j)} \text{ information bits/channel bits}.
\] (3.7)

From Figure 3.3 we note that

\[
P(j = 1) = 1/2P_e 1 + 1/2P_e 1 = 1/2[P_e + P_e] = 1/2P_1 \tag{3.8}
\]

\[
P(j = 2) = 1/2[P_e + P_e] = 1/2P_1
\]

or in general,

\[
P(j = 2k - 1) = P(j = 2k) = 1/2P_k \quad , k = 1, 2, \ldots, N + 1. \tag{3.9}
\]

The capacity may be written as

\[
C = \sum_{j=1}^{2(N+1)} P(j | i) \log_2 \frac{P(j | i)}{P(j)} \tag{3.10}
\]

\[
= \sum_{k=1}^{N+1} \left[ P(2k - 1 | i) \log_2 \frac{P(2k - 1 | i)}{P(2k - 1)} + P(2k | i) \log_2 \frac{P(2k | i)}{P(2k)} \right].
\]

\(^1\)For a formal definition of symmetry in a DMC, see [45, pg.94].

\(^2\)Provided that the input chosen has a non-zero probability assignment.
Choosing \( i = 1 \), and noting from Figure 3.4 that \( P(2k - 1 \mid i = 1) = P_{ck} \), and \( P(2k \mid i = 1) = P_{ck} \), we have

\[
C = \sum_{k=1}^{N+1} \left[ P_{ck} \log_2 \frac{2P_{ck}}{P_k} + P_{ck} \log_2 \frac{2P_{ck}}{P_k} \right]
\]

\[
= \sum_{k=1}^{N+1} P_k \left[ P_{c|k} \log_2 2P_{c|k} + P_{c|k} \log_2 2P_{c|k} \right]
\]

\[
= \sum_{k=1}^{N+1} P_k \left[ \left( P_{c|k} + P_{e|k} \right) \log_2 2 \left( 1 - P_{e|k} \right) \log_2 \left( 1 - P_{e|k} \right) + P_{e|k} \log_2 P_{e|k} \right]
\]

\[
= \sum_{k=1}^{N+1} P_k \left[ 1 - h(P_{e|k}) \right] \quad (3.11)
\]

\[
= \sum_{k=1}^{N+1} P_k C_{\text{BSC}}(P_{e|k}) . \quad (3.12)
\]

In (3.11) \( h(\cdot) \) is the binary entropy function and \( C_{\text{BSC}}(\cdot) \) is the capacity of a BSC. The final equation (3.12) shows that the capacity of the channel is the weighted average of the capacities of the \( N + 1 \) sub-channels formed by thresholding. Note that the form of this result does not depend on the actual SNR distribution\(^3\). These results are consistent with the intuitive argument presented in Section 2.1 that views independent samples from a fading channel as coming from sub-channels of different capacities.

The DMC capacity is plotted in Figure 3.4 for a few values of \( N \). The capacity rapidly approaches its limiting value as \( N \) is increased. This behaviour indicates that most of the gain is achievable, in principle, with a small number of thresholds.

\(^3\)For example, the capacity formula will be similar when the fading is Rician distributed, except for the actual values of the \( P_k \) and \( P_{e|k} \).
Selecting the thresholds \( \{T_k\}_{k=1}^N \) by maximizing the capacity (3.12) may be efficiently done numerically, using a non-linear function optimization program\(^4\). One can take advantage of some reasonably efficient optimization routines by providing the function (3.12) and its first partial derivatives with respect to \( \{T_k\}_{k=1}^N \). Section 3.1.3 discusses how well the capacity method predicts the optimal thresholds.

---

4The non-linear function optimization program used here was *LINOPT*, which is documented in [46]. Alternatively, one might consider finding the thresholds by setting the partial derivatives of (3.12) to zero, and attempt to solve the resulting set of \( N \) simultaneous non-linear equations. This approach would have involved the development of routines to solve the simultaneous non-linear equations. Instead, the approach used employs an available routine for optimizing a single non-linear function of several variables that is efficient and reliable.
3.1.2.2 Chernoff Bound Method

In this section we introduce the Chernoff bound as a performance criterion for optimization of WPD receiver thresholds. The method is not restricted to the modulation, noise, path gain, and quantization characteristics used here.\(^6\)

We begin by deriving the Chernoff bound for the WPD receiver. We represent the probability of occurrence of discrete random variables in the WPD receiver by a channel transition matrix, and require that the transition matrix be symmetric. This is satisfied, by definition, in channels that are symmetric with respect to the input symbols.

The general form of a binary input, Q-ary output, channel transition matrix is

\[
T = \begin{bmatrix}
    p_{11} & p_{12} & \cdots & p_{1Q} \\
    p_{21} & p_{22} & \cdots & p_{2Q}
\end{bmatrix}.
\]

If \(T\) is symmetric, then from the definition \([45]\) it may be partitioned as

\[
\begin{bmatrix}
    \begin{pmatrix} p_{11} & p_{21} \\ p_{21} & p_{11} \end{pmatrix} & \begin{pmatrix} p_{12} & p_{22} \\ p_{22} & p_{12} \end{pmatrix} & \cdots & \begin{pmatrix} p_{1k} & p_{2k} \\ p_{2k} & p_{1k} \end{pmatrix} & \cdots & \begin{pmatrix} p_{1(M-1)} & p_{2(M-1)} \\ p_{2(M-1)} & p_{1(M-1)} \end{pmatrix} & \begin{pmatrix} p_{1M} \\ p_{1M} \end{pmatrix}
\end{bmatrix}.
\]

The second subscript indicates the partition number, \(k = 1, 2, \ldots, M\), which correspond to a reliability class. Note that the final partition in (3.14) is not a \(2 \times 2\) matrix. This would arise if the output \(j = Q\) was equally likely given that either a '1' or a '0' was sent. This corresponds to an erasure, since the occurrence of this output offers no indication as to what was sent.

\(^6\)For example, the Chernoff bound approach may also be applied to binary antipodal signalling on a non-fading channel with AWGN. This proposal was subsequently implemented in \([47]\).
It is shown in Appendix C that if the transition matrix does not contain any erasure partitions\(^6\) then the Chernoff bound on the probability of bit error is

\[
p_e \leq \left[ G_x(s^*) \right]^L = \left[ 2 \sum_{k=1}^M \sqrt{P_{1k}P_{2k}} \right]^L
\]

(3.15)

where \(P_{1k}\) and \(P_{2k}\) are the transition probabilities of the \(k^{th}\) partition of \(T\). The value of (3.15) is that it concisely gives the Chernoff bound in terms of the transition matrix elements. Thus it may be applied, for example, whether the \(Q\) outputs are derived from a fading or static channel, or whether the \(Q\) outputs are derived from a signal corrupted by Gaussian or non-Gaussian noise.

Our strategy here is to choose the receiver thresholds \(\{T_k\}_{k=1}^N\) so that (3.15) is minimized, or equivalently,

\[
\min \sum_{k=1}^{N+1} \sqrt{P_{1k}P_{2k}}.
\]

(3.16)

Equation (3.16) may be efficiently solved numerically by a non-linear function optimization program. It is straightforward to substitute the equations for \(P_{ek}\) and \(P_{ck}\) into (3.16), and to compute the first partial derivatives of (3.16) with respect to \(\{T_k\}_{k=1}^N\) to provide to the numerical optimization program\(^7\). The performance of the Chernoff bound method for choosing the thresholds will be discussed next.

---

\(^6\)Appendix C also presents a slight modification of (3.15) for the case when an erasure partition is present. This would be useful, for example, in optimizing the WPD receiver thresholds for binary antipodal signalling with an even number of thresholds.

\(^7\)The numerical optimization routine used here was the same as used for the capacity method, namely \(LINOPT\)[46].
CHAPTER 3. SOME ISSUES IN PRACTICAL WPD RECEIVER DESIGN

3.1.3 Performance With Quantized SNR

In this section we utilize the algorithm of Section 3.1.1 to compare the BER using optimal thresholds with the error rates using the thresholds found by the capacity and Chernoff bound methods.

Figure 3.5 shows the value of the threshold in a single threshold receiver as determined by direct optimization\(^8\), the capacity method, and the Chernoff bound method. The optimum threshold increases slightly with the number of repeats, and generally the Chernoff bound threshold is quite close, especially as the number of thresholds is increased. The capacity threshold appears to differ significantly from the optimum as the SNR is increased. However, one must evaluate the resulting BER using the various thresholds to make a meaningful comparison. A plot of BER vs. SNR is shown in Figure 3.6 for 3, 5, and 7 repeats. It can be seen that the capacity method performs quite well, but not as well as the Chernoff bound method which gives results very close to the optimum.

The previous comparisons have been for a receiver employing a single threshold. Comparison of Figures 3.7 and 3.8 reveals that as the number of thresholds is increased, the sensitivity of BER to the accuracy of the thresholds decreases. This is what we would expect as the number of thresholds is increased, since even randomly selected thresholds would eventually quantize the received SNR sufficiently that the BER would approach that of an unquantized channel.

\(^8\) Direct minimization of the BER evaluation algorithm was done using the routine \textit{NLPQL}, which is documented in [46]. The thresholds were verified to be optimum within 0.1 dB by perturbing the thresholds obtained from the numerical optimization, and checking that the resulting BER was higher.
Figure 3.5: Comparison of Single Threshold Values
Figure 3.6: Comparison of Error Rates using a Single Threshold
CHAPTER 3. SOME ISSUES IN PRACTICAL WPD RECEIVER DESIGN

Figure 3.7: BER with One Threshold

Figure 3.8: BER with Three Thresholds
CHAPTER 3. SOME ISSUES IN PRACTICAL WPD RECEIVER DESIGN 68

From the previous discussion and BER plots, it is apparent that as the number of thresholds is increased, we obtain diminishing returns in terms of increased performance. This is summarized in Figure 3.9, where the BER for an increasing number of optimal thresholds is shown, along with the bound on the WPD error rate (2.27) which is an accurate estimate of the BER for an infinite number of thresholds. It can be seen that the BER rapidly approaches its limit, and most of the gain attainable is reached with a few thresholds. This is in agreement with the discussion in Section 3.1.2.1 on the capacity of the quantized channel.

Note that the thresholds used in the preceding BER curves were optimized for each value of average SNR. The question arises as to how much degradation is incurred when optimization is not carried out for each SNR. Figure 3.10 shows the BER for both a continuously optimized threshold and a threshold optimized only at an SNR of 15 dB. The degradation is very slow as the SNR deviates from the 15 dB optimization point, even for this most sensitive case of a single threshold. Thus, even if the distribution of the average SNR is quite wide, only a slight performance degradation should result from fixing the single threshold at its nominal design level. The degradation is even less when a few more thresholds are used, as will be demonstrated in the next section.

As discussed in the introduction, the logarithm of the received signal amplitude averaged over a sufficient number of wavelengths so that the effect of Rayleigh fading is removed, is well modeled by a normal distribution. The standard deviation of this shadowing distribution was measured to be 8 to 12 dB in some large cities [1]. Thus the range of the average SNR may in fact be quite wide.
Figure 3.9: Convergence of WPD BER
Figure 3.10: Optimized and Fixed Threshold Performance
CHAPTER 3. SOME ISSUES IN PRACTICAL WPD RECEIVER DESIGN

It is interesting to compare the performance of the WPD and selection diversity receivers with SNR quantization. It is possible to obtain an expression for the BER of selection diversity in this case. It is shown in Appendix D that with $N$ SNR thresholds, the selection diversity BER is

$$P_{e, sel} = \sum_{k=1}^{N+1} \left[ (A_k + P_k)^L - A_k^L \right] P_{e|k}$$  \hspace{1cm} (3.17)

where

$$A_k = \sum_{i=1}^{k-1} P_i$$  \hspace{1cm} (3.18)

and $P_k, P_{e|k}$ are defined by (3.4) and (3.3). Equation (3.17) is plotted in Figure 3.11 for a few values of $N$, along with (2.30) which is the selection diversity BER using an unquantized SNR distribution. The thresholds used are optimum, and were found by direct minimization of (3.17) using the same routine, NLPQL [46], as was used for the WPD BER algorithm.

Only a few thresholds are needed to achieve almost all of the attainable gain, although the convergence to the $N \to \infty$ result is not as rapid as for the WPD receiver (see Figure 3.9).
Figure 3.11: Convergence of Selection Diversity BER
3.1.4 Performance with Quantized Weights

In this section we demonstrate the degradation incurred when the WPD receiver weights are quantized. Let there be $B$ bits available for weight quantization, providing $2^B$ quantization levels. Scaling of the optimal weight values does not affect the resulting performance, so that one may normalize the weights prior to quantization. Since the largest weight, $w_{\text{max}}$, has the greatest influence on the final decision variable, we choose to scale the weights such that $w_{\text{max}}$ is quantized to $2^B$. The quantization characteristic is shown in Figure 3.12. There may be superior weight quantization schemes, but our intention is simply to demonstrate that a small number of quantization levels can be sufficient.
CHAPTER 3. SOME ISSUES IN PRACTICAL WPD RECEIVER DESIGN 74

The BER of the WPD receiver with quantized weights is easily evaluated using the algorithm of Section 3.1.1. Figure 3.13 shows the results of quantization with a few values of $B$, and also the BER for unquantized weights. Note that the quantized weights are based on fixed SNR thresholds that are optimal at an average SNR of 15 dB. Also shown is the best performance possible, achieved using unquantized weights that are optimized at each average SNR.

In the example shown in Figure 3.13, we see that four bits for quantization is sufficient to obtain almost exactly the same performance as unquantized weights. With three bits for quantization the degradation is less than 0.5 dB with respect to the unquantized weights, for a BER greater than $10^{-7}$.

Note that the loss due to the use of fixed weights, instead of optimizing the thresholds and weights at each average SNR, is very small. Comparison of the three threshold example of Figure 3.13 with the one threshold case of Figure 3.10 shows that with three thresholds the degradation incurred due to the use of fixed weights has decreased. This is consistent with the earlier discussion on the reduced sensitivity of the BER to weight accuracy as the number of thresholds is increased.

The examples presented in this and the previous section demonstrate that with a reasonably small number of bits for weight quantization and a few fixed weights, the BER performance is effectively the same as the ideal case with no SNR or weight quantization, over a very wide range of average SNR. It is reasonable to expect that the number of weights and quantization levels required to attain near ideal performance will not increase dramatically as the number of repeats is increased.
Figure 3.13: Quantized Weight BER.
3.2 SNR Estimation

In this section we investigate the degradation in WPD receiver performance due to imperfect knowledge of the channel state. The effects of an imperfect signal strength measurement due to noise and cochannel interference are shown. Simple low-pass filtering of the signal strength estimate is shown to improve the performance of both the WPD and selection diversity schemes, if cochannel interference is not dominant.

The performance of the various receivers was estimated using computer simulations. By using a suitable receiver model the random variables internal to the receiver need be computed only once per bit. This approach yields an efficient simulation procedure relative to a multiple sample per bit model. The receiver model, simulation methods, and simulation results are discussed in the following sections.

3.2.1 Receiver Model

The development of the receiver model was motivated by the fact that practical received signal strength indicator circuits estimate the amount of power passing through the IF filter. We will assume a dual bandpass filter implementation of a NCFSK receiver, as shown in Figure 3.14. A pair of idealized bandpass filters corresponding to each of the two signal frequencies are followed by envelope detectors and bit rate sampling circuits. The envelope samples may be compared to form a partial decision, or squared and summed over a number of repeats to implement a square-law receiver. The estimate of the received signal strength is formed by
sampling the envelope of the IF filter output.

Figure 3.15 shows an equivalent receiver block diagram to that of Figure 3.14. We assume that the IF and detection filters have ideal 'brickwall' passbands, and that the detection filters have a bandwidth, $B/2$, half that of the IF filter.

The assumptions made for the bandpass filter model are the same as those for the matched filter model in Section 2.6.1. Briefly, it is assumed that the signal energy of a cosine burst influences only the output of its respective filter, and that all of the transmitted signal energy is received. Intersymbol interference and imperfect symbol synchronization effects are neglected. Also, we make the usual assumption of slow fading.

The assumption of no filter crosstalk and reception of all the signal energy will only be true if almost all of the spectral energy of the frequency burst is contained within the detection filter bandwidth. The usual rule of thumb is that the occupied bandwidth of a pulse is the reciprocal of the pulse duration. Since we wish to estimate the performance degradation due to the use of an imperfect signal strength measurement, and not the degradation due to a band-limited signal, we will assume that the conditions of no filter crosstalk or signal loss are valid. This will be approximately true for $R \leq B/2$. 

Figure 3.14: NCFSK Receiver, Bandpass Filter Implementation

Figure 3.15: Bandpass Filter NCFSK Receiver, Block Diagram
The relationship between the BER performance of the bandpass and matched filter receivers is evident from the following expressions,

\[ P_{e,bp} = \frac{1}{2} e^{-P_s/2N} \]  
(3.19)

\[ P_{e, mf} = \frac{1}{2} e^{-E_b/2N_0} \]  
(3.20)

where \( P_s \) is the signal power and \( N \) is the total noise power at the output of one of the ideal bandpass filters [9]. The two BER expressions are identical in form since in each case, the receiver decides in favour of the larger of a Rayleigh versus a Rician random variable. In each case, the filter with no signal input outputs a zero mean Gaussian random process which is envelope detected and sampled, resulting in a Rayleigh random variable. Also, in each receiver, the filter with the signal input outputs a non-zero mean Gaussian random process which is envelope detected and sampled, resulting in a Rician random variable. The relationship between the two exponents in (3.20) and (3.19) is simply,

\[ \frac{P_s}{N} = \frac{E_b/T}{N_0B/2} = \frac{R}{B/2} \frac{E_b}{N_0} \]  
(3.21)

where \( T \) is the bit duration. Equation (3.21) implies that the two receivers would have the same error rate if \( R = B/2 \). However, as discussed above, this would only be approximately true since at \( R = B/2 \) there would be some crosstalk and signal loss.

The random variables computed by the simulation program during each bit are derived in the next section.

\(^{10}\)This is due to the fact that a Gaussian random process through a linear filter results in a Gaussian random process [11].
3.2.2 Simulation Method, AWGN

Assume that a cosine burst of frequency $f_1$ is transmitted, and that the channel fading attenuates the signal energy to $E_s$. The received signal is

$$x(t) = \sqrt{\frac{2E_s}{T}} \cos(2\pi f_1 t + \theta) + w(t), \quad 0 \leq t < T,$$

(3.22)

where $\theta$ is a random phase angle and $w(t)$ is AWGN of power spectral density $N_0/2$. The random phase $\theta$ is uniformly distributed over $[0, 2\pi)$ since Rayleigh fading is assumed. The outputs of the bandpass filters are

$$y_1(t) = \sqrt{\frac{2E_s}{T}} \cos(2\pi f_1 t + \theta) + n_{c1}(t) \cos(2\pi f_1 t) - n_{s1} \sin(2\pi f_1 t)$$

(3.23)

and

$$y_2(t) = n_{c2}(t) \cos(2\pi f_2 t) - n_{s2} \sin(2\pi f_2 t)$$

(3.24)

where we have used the narrowband representation of the noise at the filter outputs, i.e. $n_{c1}, n_{s1}, n_{c2},$ and $n_{s2}$ are all independent Gaussian random processes with zero mean and variance $N = N_0 B/2$ [48].

Expanding (3.23) yields

$$y_1(t) = \sqrt{\frac{2E_s}{T}} \cos \theta \cos(2\pi f_1 t) - \sqrt{\frac{2E_s}{T}} \sin \theta \sin(2\pi f_1 t) + n_{c1}(t) \cos(2\pi f_1 t) - n_{s1} \sin(2\pi f_1 t)$$

(3.25)

$$= \left[ \sqrt{\frac{2E_s}{T}} \cos \theta + n_{c1}(t) \right] \cos(2\pi f_1 t) - \left[ \sqrt{\frac{2E_s}{T}} \sin \theta + n_{s1}(t) \right] \sin(2\pi f_1 t).$$

(3.26)

The output of an envelope detector is the magnitude of the vector sum of the in-phase and quadrature components. Thus, from (3.26) the samples of the envelope
detectors at time $t = T$ are

$$R_1(t) |_{t=T} = \left\{ \left[ \sqrt{\frac{2E_s}{T}} \cos \theta + n_{c1}(T) \right]^2 + \left[ \sqrt{\frac{2E_s}{T}} \sin \theta + n_{s1}(T) \right]^2 \right\}^{1/2}$$ \hspace{1cm} (3.27)

and

$$R_2(t) |_{t=T} = \left[ n_{c2}^2(T) + n_{s2}^2(T) \right]^{1/2}.$$ \hspace{1cm} (3.28)

Using appropriate transformations of these random variables it can be shown that $R_1(T)$ is Rician distributed, and $R_2(T)$ is Rayleigh distributed [11].

The output of the IF filter is

$$y_3(t) = y_1(t) + y_2(t)$$

$$= \left[ \sqrt{\frac{2E_s}{T}} \cos \theta + n_{c1}(t) \right] \cos(2\pi f_1 t) - \left[ \sqrt{\frac{2E_s}{T}} \sin \theta + n_{s1}(t) \right] \sin(2\pi f_1 t)$$

$$+ n_{c2}(t) \cos(2\pi f_2 t) - n_{s2} \sin(2\pi f_2 t)$$ \hspace{1cm} (3.29)

$$= y_{c1}(t) \cos(2\pi f_1 t) - y_{s1}(t) \sin(2\pi f_1 t)$$

$$+ n_{c2}(t) \cos(2\pi f_2 t) - n_{s2}(t) \sin(2\pi f_2 t).$$ \hspace{1cm} (3.30)

In the last equation we have introduced $y_{c1}(t)$ and $y_{s1}(t)$ to denote the in-phase and quadrature components, with respect to $f_1$, of the received signal plus noise. Writing $f_1 = f_{IF} - \Delta f$, $f_2 = f_{IF} + \Delta f$, and expanding then collecting all the terms that change slowly with respect to the carrier frequency, we obtain

$$y_3(t) = [y_{c1}(t) \cos 2\pi \Delta f t + y_{s1}(t) \sin 2\pi \Delta f t$$

$$+ n_{c2}(t) \cos 2\pi \Delta f t - n_{s2}(t) \sin 2\pi \Delta f t] \cos 2\pi f_{IF} t$$

$$+ [y_{c1}(t) \sin 2\pi \Delta f t - y_{s1}(t) \cos 2\pi \Delta f t$$

$$- n_{c2}(t) \sin 2\pi \Delta f t - n_{s2}(t) \cos 2\pi \Delta f t] \sin 2\pi f_{IF} t.$$ \hspace{1cm} (3.31)
The output of the IF filter's envelope detector, $R_3(t)$, is then the magnitude of the vector sum of the in-phase and quadrature terms in square brackets in (3.31).

For signal frequencies that are centered on the bandpass filters, i.e. $\Delta f = B/4$, as shown in Figure 3.15, and choosing $R = B/2$, the IF envelope is

$$R_3(t) |_{t=T} = \left\{ [y_{e1}(T) + n_{c2}(T)]^2 + [y_{s1}(T) + n_{s2}(T)]^2 \right\}^{1/2}. \quad (3.32)$$

The evaluation of $y_{e1}(T)$ and $y_{s1}(T)$ may be simplified if we ignore the random phase variable $\theta$. That is we set $\theta = 0$ for each bit, which will not influence the pdf at the output of the envelope detector since the envelope detector output is independent of the carrier phase, assuming that $f_1 \gg 1/T$. This results in

$$y_{e1}(T) = \sqrt{\frac{2E_s}{T}} + n_{c1}(T) \quad (3.33)$$

$$y_{s1}(T) = n_{s1}(T) \quad (3.34)$$

Thus, for each bit, the simulation requires the Rayleigh distributed received amplitude, $\sqrt{2E_s/T}$, and four independent Gaussian random variables, $n_{c1}(T)$, $n_{s1}(T)$, $n_{c2}(T)$, $n_{s2}(T)$. From these $R_1(T), R_2(T)$, and $R_3(T)$ are computed via (3.27), (3.28), and (3.32). These envelope detector outputs can be compared to determine a partial decision, or squared and summed over a number of repeats to simulate a square law receiver. For the partial decision receivers, $R_3(T)$ can be used to provide an estimate of the received signal strength, as discussed in the next section.
CHAPTER 3. SOME ISSUES IN PRACTICAL WPD RECEIVER DESIGN

3.2.3 Signal Strength Estimation

The accuracy of using the envelope of the IF filter output as a signal strength estimate is determined by several factors. Since \( R_s(T) \) is the sampled envelope of the sum of the signal component plus noise, the signal strength estimate will be poor at low SNR. Also, since the envelope detector output is a non-linear function of the input signal, it will have a non-zero mean even with no signal input\(^\text{11}\). Also, any additional interference will contribute to the magnitude of the received signal strength estimate, so that the estimate will be high if the signal to interference ratio (SIR) is low.

Once the signal strength estimate has been obtained, the receiver could use a lookup table to obtain the corresponding weight. From the results of the previous section on quantization we know that only a few fixed weights quantized to a small number of levels are required. The calculation of the weights would normally be done during the receiver design so that the weight accuracy is dependent on a priori knowledge of the characteristics of the expected noise. If the received noise varies from its design model, due to variability in manufacturing, installation, or interference, then we can expect some degradation in performance. We shall consider separately the effects of AWGN and cochannel interference on the WPD receiver performance.

When the signal strength estimate is derived from the envelope of the fading signal plus AWGN, we note that the estimator need only track frequencies up to the

\(^{\text{11}}\text{It is possible to compensate for the non-linear transfer characteristic of the envelope detector by using a non-linear warping function for correction. We shall not consider such compensation here, but will simply observe the resulting performance without correction of the non-linearity.}\)
order of the Doppler frequency. A low-pass filter following the envelope detector could be used to remove a considerable portion of the IF bandwidth noise, since the IF bandwidth is considerably greater than the Doppler frequency. We model this filtering by digitally filtering the sampled output of the envelope detector, since the simulation only has access to the receiver random variables once per bit. The effectiveness of the filtering in a real implementation will be better than that shown here, since the simulation processes aliased samples of the noise. This is because the sampling frequency, \( R \), is less than twice the noise bandwidth at the output of the envelope detector.

The filter bandwidth must be chosen wide enough so that it can reliably track the signal fades, and the phase characteristic should not delay the tracked signal significantly. As a rule of thumb, we note that the the spectrum of an envelope detected Rayleigh faded signal extends to \( 2f_D \) [2], so that the filter bandwidth should be at least twice the maximum expected Doppler frequency. In practice, we would be using a non-ideal filter, and the bandwidth of the filter should be chosen even greater than \( 2f_D \) to accommodate the fading signal, and to minimize undesirable delay introduced by the filter. One might consider compensating for the filter delay by delaying the weighting of the received data, but this adds to the complexity of the receiver. We will use a very simple low-pass filter with no delay compensation to see how well a basic implementation performs.

The digital filter used is a simple, first order, infinite impulse response filter,

\[
y(n) = ay(n - 1) + x(n) \tag{3.35}
\]

and was chosen since it represents the least sophisticated filtering strategy that one might use in practice. The input \( x(n) \) is taken to be the logarithm of the
sampled envelope detector output, since logarithmic signal strength estimators are common in FM receiver integrated circuits\textsuperscript{12}. We would expect that due to the logarithmic function, the bandwidth of the signal component would be wider than at the output of the envelope detector. This can be seen by considering that during a typical deep fade, which is short in duration, the argument of the log function will be quite small. This results in a large negative output of a short duration and thus of high frequency content. We can therefore expect that the low-pass filter cutoff frequency will have to be wider than the simple $2f_D$ guideline for an envelope detector.

The Fourier transform of the low-pass filter impulse response is

$$H(e^{j\omega}) = H(z)|_{z=e^{j\omega}} = \frac{1}{1 - a e^{-j\omega}}. \quad (3.36)$$

The value of the constant, $a$, that yields a specified 3 dB cutoff frequency, $f_{co}$, is

$$a = [2 - \cos(2\pi f_{co}/T)] - \sqrt{(2 - \cos(2\pi f_{co}/T))^2 - 1}. \quad (3.37)$$

For land mobile radio communications at UHF, we expect a maximum Doppler frequency of about 100 Hz. This corresponds to 130 kmh at 850 MHz. Our guideline for the low-pass filter cutoff frequency would have us set $f_{co} > 2f_D = 200$ Hz, assuming that we had an ideal filter. For a trial, we will choose $f_{co} = 4f_D = 400$ Hz since we are using a logarithmic signal strength estimator, and since we are using a low pass filter with non-ideal gain and phase characteristics. The value of $a$ for this cutoff frequency is 0.74 at our simulation bit-rate of 8192 bps. The frequency response shown in Figure 3.16 shows the poor low-pass characteristic of this filter. The tracking performance of the filter can be seen in Figure 3.17, where

\textsuperscript{12} For example, the Plessey SL6652 IC.
a sample of a Rayleigh faded signal is plotted versus time, along with the low-pass filter output. There is a delay of the order of 5 bits and the amplitude fluctuations of the signal are not precisely followed. Despite these shortcomings, we will use this filter to observe whether it can improve the WPD receiver performance.
CHAPTER 3. SOME ISSUES IN PRACTICAL WPD RECEIVER DESIGN

Figure 3.16: Frequency Response of Signal Strength Estimator Filter.

\[ y(n) = 0.74 y(n-1) + x(n) \]
Cutoff frequency = 400 Hz
Bit rate = 8192 bps

Figure 3.17: Tracking Behaviour of the Signal Strength Estimator
3.2.4 Simulation Method, Cochannel Interference

We now consider the case where the signal strength estimate is derived from the envelope of the faded signal plus cochannel FSK interference and AWGN.

The cochannel interference is assumed to be due to a number of interferers, and the received level of each is taken to be Rayleigh distributed. These conditions have been used to model the cochannel interference level for the AMPS cellular system [49]. If any number of cochannel interferers are Rayleigh faded, then the received level of interference is also Rayleigh [49]. Unfortunately, the spectrum of the interference will be same as that of the signal, so that low-pass filtering of the IF filter envelope detector output will not be of benefit in an interference dominated system. We will develop a simulation procedure that uses a pessimistic model for the signal strength estimate, by assuming that the envelope detector output is not filtered. As shown in Section 3.2.2 the IF filter envelope detector output can be written as the vector sum of the in-phase and quadrature components of the signal and narrowband noise components, and we extend this to include additional components arising from the interference. The Rayleigh distributed interference level is equivalent to sampling the magnitude of a complex Gaussian random process. Thus we may represent the signal, cochannel interference, and the AWGN by the phasor diagram in Figure 3.18, where the in-phase and quadrature components of each vector is a Gaussian random variable.
Figure 3.18: Phasor Representation of the IF Filter Envelope Detector Output. The IF filter envelope detector output sample, $R_3(T)$, is the vector sum of the faded signal and the narrowband representations of the AWGN and cochannel interference. The signal and AWGN components are discussed in Section 3.2.2. The fading cochannel interference has narrowband in-phase and quadrature components, $n_{ci}(t)$ and $n_{si}(t)$, that are independent Gaussian random processes.
CHAPTER 3. SOME ISSUES IN PRACTICAL WPD RECEIVER DESIGN

The faded signal level and the AWGN noise components are generated as described in Section 3.2.2. If the bit repeats are separated by several fades then the envelope detector samples will be independent, and the fading interference levels for a set of bit repeats will be accurately modeled using independent complex Gaussian noise samples.

The SIR is $E_s/E_I$, where $E_s$ and $E_I$ are the average bit period energies of the signal and interference, respectively. The narrowband model of the interference is

$$i(t) = n_{ci}(t) \cos(2\pi f_{IF}t) - n_{si}(t) \sin(2\pi f_{IF}t).$$  \hspace{1cm} (3.38)

The variances of the Gaussian components, $n_{ci}(t)$ and $n_{si}(t)$ are found from

$$E_I = E \left[ \int_0^T i^2(t) \, dt \right] = \frac{T}{2} E[n_{ci}^2(t)] + \frac{T}{2} E[n_{si}^2(t)] \hspace{1cm} (3.39)$$

so that

$$E[n_{ci}^2(t)] = E[n_{si}^2(t)] = \frac{E_I}{T}. \hspace{1cm} (3.40)$$

Next, we consider the modeling of the demodulator performance in cochannel FSK interference. With fading interferers, there will be a varying number of interferers that contribute to a bit error. In [52] the performance of an ideal NCFSK demodulator is derived for a non-fading signal in the presence of any number of equal energy, non-fading, cochannel interferers. The exact BER formulas are complex to evaluate and as such are impractical for our simulation. However, as the number of cochannel interferers is increased the BER approaches that obtained by assuming that the cochannel interference is equivalent to narrowband Gaussian noise of the same energy. This assumption is pessimistic when there are a small number of interferers, and the static SIR and SNR values are high [52,53]. Also, for high SIR and SNR, the assumption of equal strength interferers is pessimistic.
[54]. At lower values of SNR and SIR, the assumption of narrowband Gaussian noise is slightly optimistic if there are a very small number of interferers, e.g. one or two [53,52]. However, in our case the instantaneous interference level is the superposition of a number of fading signals, and it is more probable that a number of interferers contribute to any significant interference level. Also, since there are a number of interferers, it is likely that some interference energy will be present in both of the detection filters. These considerations indicate that errors due exclusively to a single tone are not likely to be the most common. Thus, with more than one interferer usually causing errors, the modeling of the interference by an equivalent amount of narrowband Gaussian noise may be reasonably good.

Our modeling strategy, then, is to simulate the effect of the fading interference on the demodulator by using the instantaneous level of interference to adjust the variance of additional AWGN.

The interference energy for a particular bit is

$$\int_0^T i^2(t) \, dt = \frac{T}{2} \left[ n_{c1}^2(T) + n_{s1}^2(T) \right], \tag{3.41}$$

where $n_{c1}(T)$ and $n_{s1}(T)$ were shown earlier to have variance $E_I/T$. The division of AWGN between the detection filters does not affect the error rate with equally likely data bits in an ideal NCFSK receiver [52]. For our simulation we must divide the noise equally between the filters since we are only simulating the reception of a ‘1’. The variances of the Gaussian components in the narrowband representation of the equivalent interference noise for the filter at $f_1$ are found from

$$\frac{1}{2} \int_0^T i^2(t) \, dt = E \left[ \int_0^T (n_{c1}(t) \cos(2\pi f_1 t) - n_{s1}(t) \sin(2\pi f_1 t))^2 \, dt \right]. \tag{3.42}$$

The variances of $n_{c1}$, $n_{s1}$, will be equal, and will also be equal to the variances of
the equivalent noise components for the filter at $f_2$, $n_{ci2}$, $n_{si2}$, so that using (3.42) and (3.41) one obtains

$$E[n_{ci1}^2(T)] = E[n_{si1}^2(T)] = E[n_{ci2}^2(T)] = E[n_{si2}^2(T)] = \frac{1}{4} \left[ n_{ci}^2(T) + n_{si}^2(T) \right].$$

(3.43)

The simulation results for both no cochannel interference and an unfiltered signal strength estimate with cochannel interference are discussed in the next section.
3.2.5 Simulation Results

The BER of the WPD receiver using five repeats with no cochannel interference is shown in Figure 3.19. Also shown are the BER curves for the majority-voting and square-law receivers as determined by simulation and as predicted by (2.29) and (2.31). The Doppler frequency, bit-rate and packet format are the same as used for the simulation discussed in Section 2.6.4, i.e. \( f_D = 49 \) Hz, \( R = 8192 \) bps, and the packets are forty bits long with single packet length gaps between repeats. Four curves are shown for the WPD receiver which correspond to the following methods of obtaining the SNR estimate:

- **Exact.** Exact SNR measurement.
- **Signal plus noise.** The unfiltered envelope detector output is used to estimate the received signal strength.
- **Filtered signal plus noise.** The logarithm of the envelope detector output is low-pass filtered and used to estimate the received signal strength.
- **Filtered signal, no noise.** The logarithm of the signal without noise is low-pass filtered and used to estimate the received signal strength. This will reveal the degradation of the BER due to the distortion of the signal by the filter.

Inspection of Figure 3.19 shows the separate contributions to the BER degradation by the noise and low-pass filter characteristics. At a target error rate of \( 10^{-3} \) the WPD receiver suffers a one dB loss using the signal plus noise estimate instead of the ideal SNR estimate. With the low-pass filter, the filtered signal plus noise curve is very close to the filtered signal with no noise curve. This indicates that the low-pass filter has removed almost all of the effect of the noise. However, as the SNR is increased the filtered signal plus noise BER curve shows an increasing
Figure 3.19: WPD Bit Error Rate with SNR Estimation
loss relative to the ideal SNR measurement. This is due to the distortion of the received signal level by the filter. At high SNR, errors occur mainly during deep fades, which are of short duration, and the delay introduced by our simple filter becomes significant. At very high SNR, the filter distortion will limit the BER, and better performance would be obtained without low-pass filtering. Despite this imperfect filtering, the low-pass filter does improve the BER over the range of SNR shown. The filter design involves a trade-off between good noise attenuation and the effects of filter distortion at high SNR. A compromise filter design can be sought to obtain a BER characteristic close to that of the ideal SNR measurement, over a certain operating range of average SNR.

Figure 3.20 shows the performance of the WPD receiver when interference is present. We have taken the SIR to be 15 dB as an approximation to the cochannel interference expected in cellular systems. In the AMPS system the design objective is to achieve a SIR of 17 dB or greater over 90% of the coverage area in order to maintain good voice quality [36]. We have used a pessimistic value of SIR for our example.

Returning to Figure 3.20, we see that compared to the case of no interference shown in Figure 3.19, the interference significantly affects the BER. Two curves for the WPD receiver are shown, with the labels describing the SNR estimation method as follows:

Signal. The SNR estimate is formed from the ratio of a perfect signal strength measurement to the value of the known noise PSD, $N_0$.

Signal plus noise and interference. The SNR estimate is formed from the ratio of the unfiltered envelope detector output to the known noise PSD, $N_0$. 
CHAPTER 3. SOME ISSUES IN PRACTICAL WPD RECEIVER DESIGN

Figure 3.20: WPD Bit Error Rate with SNR Estimation, SIR=15 dB
For a target BER of $10^{-3}$, comparing the signal plus noise and interference curve in Figure 3.20 with the signal plus noise curve in Figure 3.19, i.e. no lowpass filtering of the envelope detector output, we see that despite the interference the WPD receiver has slightly increased its gain over majority-voting. Unfortunately, if the interference level is increased, then the signal strength estimate may become so inaccurate that the WPD receiver may perform worse than majority-voting. However, as discussed earlier, the SIR value used here is a pessimistic approximation to that expected. If lowpass filtering of the envelope detector output is used, then we can expect the WPD performance to approach the signal curve in Figure 3.20 for low SNR, where the performance is dominated by AWGN. As the SNR is increased, the signal plus noise and interference curve will represent the expected performance since filtering the envelope detector output will not mitigate the effects of cochannel interference. However, for the range of SNR shown in Figure 3.20 we note that the BER curves are not yet approaching their limiting value set by the interference. Thus, there is considerable influence due to AWGN in this region, and lowpass filtering should increase the gains over that reported here. Without filtering, the gain of the WPD receiver over majority-voting is over 2 dB at a BER of $10^{-3}$. Also, as the SNR is increased, the WPD receiver may achieve a lower BER limit than majority-voting, depending on the level of interference. Similar gains in performance relative to majority-voting using the PER as a criterion may be observed in Figures 3.21 and 3.22, for the cases of no interference and SIR $= 15$ dB respectively.
Figure 3.21: WPD Packet Error Rate with SNR Estimation
Figure 3.22: WPD Packet Error Rate with SNR Estimation, SIR = 15 dB
It is interesting to compare the performance of selection diversity to the WPD receiver when both utilize the same signal strength estimation schemes. Figures 3.23 through 3.26 show the performance of selection diversity with and without interference. Below an SNR of about 12 dB in AWGN, the effect of the noise on the unfiltered signal strength estimate has degraded the selection diversity receiver BER so that it is higher than majority-voting. With interference, the selection diversity BER relative to majority-voting is further increased. The filtered signal strength estimate will improve the BER and PER for lower SNR, as in the WPD receiver. However, the selection diversity receiver appears to be more sensitive to SNR estimation error than the WPD receiver. This can be seen by comparing Figures 3.23 and 3.19. The loss in efficiency at a BER of $10^{-3}$ by using an unfiltered signal strength estimate is about 50% greater for the selection diversity receiver. The greater sensitivity of the selection diversity receiver is also evident when a filtered signal strength estimate is used, or whether BER or PER is used as the performance measure. Compared to selection diversity, the WPD receiver with an unfiltered signal strength estimate offers an improvement of about 3 dB, at a BER of $10^{-3}$ and an SIR of 15 dB. The improvement relative to selection diversity and majority-voting will increase with the number of repeats.
Figure 3.23: Selection Diversity BER with SNR Estimation
Figure 3.24: Selection Diversity BER with SNR Estimation, SIR = 15 dB.
Figure 3.25: Selection Diversity PER with SNR Estimation
Figure 3.26: Selection Diversity PER with SNR Estimation, SIR = 15 dB.
Chapter 4

Conclusions

In this thesis we have examined the technique of weighted partial decision combining for diversity improvement on fading channels. The motivation for using partial decisions rather than a completely soft decision process is that it may be preferable in some applications due to constraints on receiver cost.

In Chapter 2, the WPD receiver performance for the case of NCFSK demodulation on a slow Rayleigh fading channel was analyzed. A tight upper bound on the BER was derived, as were estimates of the asymptotic performance of the scheme relative to majority-voting, selection diversity, square-law and maximal ratio combining. The WPD receiver performance was shown to be equal to, or better than, majority voting or selection diversity. For low Doppler frequencies the WPD receiver is equivalent to majority-voting, while selection diversity is less efficient, offering no gain relative to a single transmission. With five independent bit repeats and a target BER of $10^{-3}$, the WPD receiver is about 3 dB more efficient than majority voting, and about 1 dB more efficient than selection diversity. This gain increases with the number of repeats. The WPD receiver is less than
1.1 dB poorer than the square-law receiver, up to about 15 repeats, while at this number of repeats selection diversity is about 6 dB poorer than the square-law receiver. Simulation results for packet error rates show similar gains over the standard partial decision combiners. A simple guideline was developed to estimate the optimal number of independent branches, given a fixed energy per bit. The optimal number of branches for the majority-voting, WPD, and square-law receivers is approximately one third of the value of the available energy per bit.

The performance degradation due to various receiver implementation effects was considered in Chapter 3. It was demonstrated that near optimal WPD performance is achievable with only a few reliability weights quantized to a small number of levels, and without threshold optimization as the average SNR varies. This level of performance can be nearly maintained despite an AWGN corrupted received signal strength estimate, by using a simple low pass filter to smooth the IF filter envelope detector output. It was also demonstrated that the selection diversity receiver is slightly more sensitive to signal strength estimator errors. When the signal strength estimate is corrupted by cochannel interference, the performance degradation may be considerable, but for a moderate level of interference the WPD receiver maintains some performance advantage over majority-voting and selection diversity.

The results indicate that near ideal WPD receiver performance in AWGN can be obtained with a simple implementation. Since many land-mobile radio receivers already employ a received signal strength measurement circuit, implementation of the WPD receiver requires only that the signal strength measurement be low-pass filtered, sampled, and used to index a small lookup table of reliability weights which
are summed to form the final decision variable. The WPD receiver may be appropriate when the receiver design is constrained to use a hard decision demodulator, and the system is power limited with a modest level of cochannel interference. In such cases, one might consider the WPD receiver when a fixed number of bit repeats are used, or when a variable number of transmissions are employed with memory ARQ (automatic repeat request).

Also developed in Chapter 3 was a method to estimate the optimal quantization thresholds for the WPD receiver. The method minimizes a Chernoff bound on the probability of bit error, and is applicable to other signalling, channel fading, and noise characteristics. Also, an expression for the probability of bit error for a selection diversity receiver with quantization was derived. The analysis is applicable for the more general case of a Rician faded signal level.

Finally, it is emphasized that this thesis has concentrated on simple repetition diversity, because of its practicality in low cost receivers. However, it is possible to extend the decision rule to apply to coded diversity reception on slowly fading channels, so that the receiver forms a maximum-likelihood packet decision. This area is presently under study.
References


REFERENCES


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Appendix A

In this appendix it is shown that

\[ G_x(s^*) = \frac{2^{2+2/\gamma_0}}{\gamma_0} \int_0^1 \frac{dv}{v^{1/2-2/\gamma_0}(1+v)^{2+2/\gamma_0}} < \frac{4}{4 + \gamma_0} \left( \frac{4 + 5\gamma_0}{4 + 3\gamma_0} \right)^{1/2} \]  \hspace{1cm} (A.1)  

so that we can obtain (2.22),

\[ P_{e.wp} < \left[ \frac{4}{4 + \gamma_0} \left( \frac{4 + 5\gamma_0}{4 + 3\gamma_0} \right)^{1/2} \right]^L. \]

Let

\[ I = \int_0^1 \frac{dv}{v^{1/2-2/\gamma_0}(1+v)^{2+2/\gamma_0}} \]  \hspace{1cm} (A.2)  

\[ = \int_0^1 \frac{1}{v^{5/2}} \left( \frac{v}{1+v} \right)^{2+2/\gamma_0} dv. \]  \hspace{1cm} (A.3)  

By making the substitution \( t = v/(1+v) \) we can obtain

\[ I = \int_0^{1/2} t^{-1/2+2/\gamma_0}(1-t)^{1/2} dt = B_{1/2}(1/2 + 2/\gamma_0, 3/2). \]  \hspace{1cm} (A.4)  

One might evaluate \( I \) using tables or numerical integration routines for the incomplete Beta function, but we can find a good upper bound on \( B_{1/2}(1/2 + 2/\gamma_0, 3/2) \) that is more readily evaluated. Substituting \( x = 1 - t \), gives

\[ I = \int_{1/2}^1 x^{1/2}(1-x)^{-1/2+2/\gamma_0} dx. \]  \hspace{1cm} (A.5)
We may upper bound $I$ by bounding the function $x^{1/2}$ over the region $x = (1/2, 1)$ by a linear approximation to $x^{1/2}$, as shown in Figure A.1.

Thus

$$ I < I_1(x_0) = \int_{1/2}^{1} (mx + b)(1 - x)^{-1/2 + 2/\gamma_0} dx. \quad (A.6) $$

The line $y_1 = mx + b$ is chosen to be tangent to $y = x^{1/2}$ at the point $x_0$, and $m$ and $b$ are easily found in terms of $x_0$. If one minimizes (A.6) with respect to the tangent point $x_0$, it is found that the minimum value of $I_1(x_0)$ occurs at $x_0^* = (4 + 5\gamma_0)/(8 + 6\gamma_0)$. The resulting bound on $I$ is

$$ I < I_1(x_0^*) = \frac{1}{(1/2 + 2/\gamma_0)^2} \left[ \frac{4 + 5\gamma_0}{8 + 6\gamma_0} \right]^{1/2} \quad (A.7) $$

which, after multiplication by $2^{2+2/\gamma_0}/\gamma_0$, is equivalent to (A.1).

The ratio $I_1/I$, is plotted as a function of SNR in Figure A.2, where $I$ was
Figure A.2: Ratios of Two Bounds on the Moment Generating Function to the Numerically Computed Result.

evaluated numerically. Also shown is the ratio $I_2/I$, where $I_2 = 1 + \pi/2$ was found in [51] and is asymptotic to $I$ as $\gamma_0 \to \infty$.

Over the range of SNR from -5 dB to 30 dB, the bound $I_1$ is less than 0.5% greater than the exact value $I$. Although the bound $I_2$ is asymptotic to $I$, it is not tighter than $I_1$ unless the SNR is greater than about 37 dB. For more typical signal to noise ratios $I_1$ is much tighter.

\footnote{The FORTRAN routines BETA and RIBETA [50] were used to evaluate the incomplete Beta function.}
Appendix B

In this appendix we derive equation (2.27), which gives the WPD receiver BER when weights proportional to the received SNR are used. We assume that an ideal NCFSK receiver is operating in AWGN with a Rayleigh faded signal amplitude.

Assume that a '1' is transmitted \( L \) times, and that \( L \) partial decisions, \( \{d_i\}_{i=1}^L \), are made, where \( d_i \) is defined in (2.1). With the weights equal to the received SNR, it will be convenient to define the weighted partial decision variable as \( Y = -\gamma d_i \). Note that a positive value of \( Y \) corresponds to a partial decision error. Let the sum of \( L \) independent samples of \( Y \) be denoted by \( Y_L \), and denote the pdf of \( Y_L \) by \( f_{Y_L}(y) \). The probability of bit error is

\[
P_{e,wpd | \gamma = \gamma} = Pr(Y_L > 0) = \int_0^\infty f_{Y_L}(y) \, dy. \tag{B.1}
\]

For each possible value of SNR, \( \gamma \), the weighted partial decision variable, \( Y \), has two values,

\[
Y(\gamma) = \begin{cases} 
-\gamma & \text{with pdf value of } f_{\gamma}\gamma(\gamma) \\
\gamma & \text{with pdf value of } f_{\gamma}\mu(\gamma) 
\end{cases} \tag{B.2}
\]

where

\[
f_{\gamma}\gamma(\gamma) = f_\Gamma(\gamma) [1 - p_e(\gamma)] \tag{B.3}
\]

\[
f_{\gamma}\mu(\gamma) = f_\Gamma(\gamma) p_e(\gamma). \tag{B.4}
\]
The pdf of \( Y \) is

\[
f_Y(y) = \begin{cases} 
  f_r(-y) \left[1 - p_e(-y)\right], & y < 0 \\
  f_r(y)p_e(y), & y > 0
\end{cases}
\] (B.5)

The moment generating function of \( Y \) is

\[
G_Y(s) = E\left[e^{-sv}\right] = \int_{-\infty}^{\infty} e^{-sv} f_Y(y) dy 
\] (B.6)

\[
= \int_0^{\infty} f_r(y) \left[(1 - p_e(y)) e^{sv} + p_e(y)e^{-sv}\right] dy 
\] (B.7)

\[
= \frac{1}{\gamma_0} \int_0^{\infty} e^{-v/\gamma_0} \left[(1 - 1/2e^{-v/2}) e^{sv} + 1/2e^{-v/2}e^{-sv}\right] dy 
\] (B.8)

\[
= \frac{1}{2\gamma_0} \left[\frac{1}{s + 1/2 + 1/\gamma_0} + \frac{2}{-s + 1/\gamma_0} - \frac{1}{-s + 1/2 + 1/\gamma_0}\right]. 
\] (B.9)

This can be put in the form

\[
G_Y(s) = \frac{1}{2\gamma_0} \frac{R(s)}{s + 1/2 + 1/\gamma_0} 
\] (B.10)

where

\[
R(s) = \frac{-2/\gamma_0 s + 1/2(1/2 + 1/\gamma_0)^2}{(-s + 1/\gamma_0)(-s + 1/2 + 1/\gamma_0)}. 
\] (B.11)

Note that \( R(s) \) incorporates all of the right half plane (RHP) poles of \( G_Y(s) \). The moment generating function of \( Y_L \) is

\[
G_{Y_L} = G_Y^L(s), 
\] (B.12)

which can be written as

\[
G_{Y_L} = L(s) + R(s) 
\] (B.13)

where \( L(s) \) is the partial fraction expansion of \( G_{Y_L}(s) \) corresponding to its left half plane (LHP) poles, and \( R(s) \) is the partial fraction expansion corresponding to its RHP poles. Note that to obtain \( f_{Y_L} \) for \( y \geq 0 \) one needs to find the inverse
Laplace transform of $\mathcal{L}(s)$. The probability of error could then be obtained via the integration in (B.1). We may avoid the Laplace transform inversion and subsequent integration by noting that

$$P_{e, wpd} |_{w=\gamma} = \int_0^\infty f_{Y_L}(y) \, dy \quad \text{(B.14)}$$

$$= \lim_{s \to 0} \int_0^\infty e^{-sv} f_{Y_L}(y) \, dy \quad \text{(B.15)}$$

$$= \lim_{s \to 0} \mathcal{L}(s). \quad \text{(B.16)}$$

Thus, the probability of error may be found by evaluating the LHP portion of $G_{Y_L}(s)$ at $s = 0$.

In our case, using (B.10) in (B.12) gives

$$G_{Y_L}(s) = \frac{1}{(2\gamma_0)^L} \frac{R^L(s)}{(s + 1/2 + 1/\gamma_0)^L}. \quad \text{(B.17)}$$

The LHP partial fraction expansion of $G_{Y_L}(s)$ is

$$\mathcal{L}(s) = \frac{1}{(2\gamma_0)^L} \sum_{k=1}^{L} \frac{a_k}{(s + 1/2 + 1/\gamma_0)^k} \quad \text{(B.18)}$$

where

$$a_k = \frac{1}{(L - k)!} \left. \frac{d^{L-k}}{ds^{L-k}} \left[ R^L(s) \right] \right|_{s = -(1/2+1/\gamma_0)}. \quad \text{(B.19)}$$

In order to evaluate the $a_k$ we will find it convenient to split up $R(s)$ into its partial fractions,

$$R(s) = \frac{-2/\gamma_0 s + (1/2 + 1/\gamma_0)(1 + 2/\gamma_0)}{(-s + 1/\gamma_0)(-s + 1/2 + 1/\gamma_0)} \quad \frac{1 + 4/\gamma_0}{-s + 1/\gamma_0} - \frac{1 + 2/\gamma_0}{-s + 1/2 + 1/\gamma_0}. \quad \text{(B.20)}$$

We now have

$$R^L(s) = \sum_{j=0}^{L} \binom{L}{j} \left( \frac{1 + 4/\gamma_0}{-s + 1/\gamma_0} \right)^j \left( \frac{-(-1+2/\gamma_0)}{-s + 1/2 + 1/\gamma_0} \right)^{L-j}. \quad \text{(B.21)}$$
Consider the $n^{th}$ derivative of $R^L(s)$,

$$\frac{d^n}{ds^n}[R^L(s)] = \sum_{j=0}^{L} \binom{L}{j} b_{jn}$$

(B.22)

where we have defined $b_{jn}$ as

$$b_{jn} = \frac{d^n}{ds^n} \left[ \left( \frac{1+4/\gamma_0}{-s+1/\gamma_0} \right)^j \left( -\frac{(1+2/\gamma_0)}{-s+1/2+1/\gamma_0} \right)^{L-j} \right].$$

(B.23)

For $j = 0$ we have

$$b_{0n} = \frac{d^n}{ds^n} \left[ \left( -\frac{(1+2/\gamma_0)}{-s+1/2+1/\gamma_0} \right)^L \right]$$

(B.24)

$$= (-1)^L (1+2/\gamma_0)^L \frac{(L+n-1)!}{(L-1)!} \frac{1}{(-s+1/2+1/\gamma_0)^{L+n}}$$

(B.25)

and for $j = L$,

$$b_{Ln} = \frac{d^n}{ds^n} \left[ \left( \frac{1+4/\gamma_0}{-s+1/\gamma_0} \right)^L \right]$$

(B.26)

$$= (1+4/\gamma_0)^L \frac{(L+n-1)!}{(L-1)!} \frac{1}{(-s+1/\gamma_0)^{L+n}}.$$  

(B.27)

For the terms, $b_{j1}, b_{j2}, \ldots, b_{j(L-1)}$, the form of $b_{jn}$ is

$$b_{Ln} = \frac{d^n}{ds^n}[u \, v]$$

(B.28)

where

$$u = \left( \frac{1+4/\gamma_0}{-s+1/\gamma_0} \right)^j$$

(B.29)

and

$$v = \left( \frac{-\left(1+2/\gamma_0\right)}{-s+1/2+1/\gamma_0} \right)^{L-j}$$

(B.30)

Using Leibnitz's theorem for the differentiation of a product of two functions [15],

$$\frac{d^n}{dx^n}[u \, v] = u \frac{d^n v}{dx^n} + \binom{n}{1} \frac{d u}{dx} \frac{d^{n-1} v}{dx^{n-1}} + \binom{n}{2} \frac{d^2 u}{dx^2} \frac{d^{n-2} v}{dx^{n-2}} + \cdots + \frac{d^n u}{dx^n} v$$

$$= \sum_{i=0}^{n} \binom{n}{i} \frac{d^{n-i} u}{dx^{n-i}} \frac{d^i v}{dx^i}.$$  

(B.31)
we have, for $n = 1, 2, \ldots, L - 1$,
\[
b_{jn} = \sum_{i=0}^{n} \binom{n}{i} \frac{d^{n-i}}{ds^{n-i}} \left[ \frac{(1 + 4/\gamma_0)^j}{-(s + 1/2 + 1/\gamma_0)} \right] \frac{d^i}{ds^i} \left[ \frac{-(1 + 2/\gamma_0)}{-(s + 1/2 + 1/\gamma_0)} \right] \]
(B.32)
\[
= \sum_{i=0}^{n} \binom{n}{i} (1 + 4/\gamma_0)^j \frac{(j + n - i - 1)!}{(j - 1)!} \frac{1}{-(s + 1/\gamma_0)^{j+n-i}} \cdot (-1)^{L-j} (1 + 2/\gamma_0)^{L-j} \frac{(L - j + n - i - 1)!}{(L - j - 1)!} \frac{1}{-(s + 1/2 + 1/\gamma_0)^{L-j+n-i}}. \quad \text{(B.33)}
\]

We require the $(L-k)^{th}$ derivative of $R^L(s)$ evaluated at the LHP pole, which gives
\[
\frac{d^n}{ds^n} \left[ R^L(s) \right]_{s=-\frac{1}{2}+\frac{1}{\gamma_0}} = \sum_{j=0}^{L} b_{j(L-k)} \bigg|_{s=-\frac{1}{2}+\frac{1}{\gamma_0}} \quad \text{(B.34)}
\]
\[
= b_{0(L-k)} \bigg|_{s=-\frac{1}{2}+\frac{1}{\gamma_0}} + b_{L(L-k)} \bigg|_{s=-\frac{1}{2}+\frac{1}{\gamma_0}} + \sum_{j=1}^{L-1} \binom{L}{j} b_{j(L-k)} \bigg|_{s=-\frac{1}{2}+\frac{1}{\gamma_0}}. \quad \text{(B.35)}
\]

Using (B.16) with (B.18), and (B.19), gives
\[
P_{e,wpd} \big|_{w=\gamma} = \frac{1}{(2\gamma_0)^L} \sum_{k=1}^{L} \frac{1}{(1/2 + 1/\gamma_0)^k} \frac{1}{(L-k)!} \left\{ \frac{d^{L-k}}{ds^{L-k}} \left[ R^L(s) \right]_{s=-\frac{1}{2}+\frac{1}{\gamma_0}} \right\}. \quad \text{(B.36)}
\]

Using (B.36), (B.35), (B.33), (B.25), and (B.27), yields
\[
P_{e,wpd} \big|_{w=\gamma} = \frac{1}{(2\gamma_0)^L} \sum_{k=1}^{L} \frac{1}{(1/2 + 1/\gamma_0)^k} \frac{1}{(L-k)!} \left\{ \frac{d^{L-k}}{ds^{L-k}} \left[ R^L(s) \right]_{s=-\frac{1}{2}+\frac{1}{\gamma_0}} \right\}.
\]
\[
= \frac{1}{(2\gamma_0)^L} \sum_{k=1}^{L} \frac{1}{(1/2 + 1/\gamma_0)^k} \frac{1}{(L-k)!} \left\{ \frac{(2L-k-1)!}{(L-1)!} \left[ \frac{(-1)^L}{(1 + 2/\gamma_0)^L-k} + \frac{(1 + 4/\gamma_0)^L}{(1/2 + 2/\gamma_0)^{2L-k}} \right] \right. \\
+ \sum_{j=1}^{L-1} (-1)^{L-j} \binom{L}{j} \sum_{i=0}^{L-k} \binom{L-k}{i} \frac{(L-k+j-i-1)!}{(j-i)!} \frac{(L-j+i-1)!}{(L-j-i)!} \\
\left. \cdot \frac{(1 + 4/\gamma_0)^j}{(1 + 2/\gamma_0)^{L-k+j-i}} \frac{1}{(1 + 2/\gamma_0)^i} \right\}. \quad \text{(B.37)}
\]
After some algebra, the preceding equation can be simplified to

\[
P_{e \text{wpd} | w=\gamma} = \frac{1}{(2 + \gamma_0)^L} \sum_{k=1}^{L} \left( \frac{2L - k - 1}{L - 1} \right) \left[ (-1)^L 2^{k - L} + 2^L \frac{2 + \gamma_0}{4 + \gamma_0} \right]^{L - k}
\]

\[
+ \sum_{j=1}^{L-1} (-1)^{L-j} \left( \begin{array}{c} L \\ j \end{array} \right) \sum_{i=0}^{L-k} 2^{j-i} \left( \begin{array}{c} L - k + j - i - 1 \\ j - 1 \end{array} \right) \left( L - j + i - 1 \right) \left( \frac{2 + \gamma_0}{4 + \gamma_0} \right)^{L - k - i}
\]

which is (2.27).

As a check on the above derivation of (2.27), one can use the form of \( G_Y(s) \) given by (B.9) and write

\[
G_Y(s) = \frac{1}{2\gamma_0} \left[ A(s) + B(s) \right]
\]

where

\[
A(s) = \frac{1}{s + 1/2 + 1/\gamma_0}
\]

and

\[
B(s) = \frac{2}{-s + 1/\gamma_0} - \frac{1}{-s + 1/2 + 1/\gamma_0}.
\]

Then

\[
G_{Y_L}(s) = \frac{1}{(2\gamma_0)^L} \sum_{k=0}^{L} \left( \begin{array}{c} L \\ k \end{array} \right) A^k(s) B^{L-k}(s).
\]

By finding the LHP portion of this form of \( G_{Y_L}(s) \), and setting \( s \) to 0, one can obtain an alternate form of (2.27),
\[ P_{e,wpd \mid u=\gamma} = \frac{1}{(2 + \gamma_0)^L} \]

\[ + \frac{1}{(2 + \gamma_0)^L} \sum_{k=1}^{L-1} \binom{L}{k} \sum_{j=1}^{k} \left( \binom{L - j - 1}{k - j - 1} \left[ (-1)^{L-k} 2^{j-L} + 2^{L-K} \frac{2 + \gamma_0}{4 + \gamma_0} \right]^{L-j} \right) \]

\[ + \sum_{m=1}^{L-k-1} (-1)^{L-k-m} \binom{L-k}{m} \sum_{i=0}^{k-j} 2^{2m+k-i-L} \]

\[ \cdot \left( \binom{m + k - j - i - 1}{m - 1} \right) \left( \binom{L - k - m + i - 1}{i} \right) \left( \frac{2 + \gamma_0}{4 + \gamma_0} \right)^{k-j+m-i} \]  \quad \text{(B.43)}

Equation (B.43) and (2.27) were checked to give identical numerical results over a broad range of SNR and for several values of \( L \). Note that the \( k^{th} \) term in (B.43) is the probability that an error is caused by \( L - k \) incorrect partial decisions outweighing \( k \) correct partial decisions.
Appendix C

In this appendix the Chernoff bound on the probability of bit error for a binary WPD receiver employing quantization is shown to be given by (3.15),

\[ p_e \leq [G_X(s^*)]^L = \left[ 2 \sum_{k=1}^{M} \sqrt{P_{1k}P_{2k}} \right]^L \]

where \( P_{1k} \) and \( P_{2k} \) are the transition probabilities in the \( k^{th} \) partition of a symmetric channel transition matrix that has no erasure partitions. A Chernoff bound for the case of an erasure partition will also be given.

As discussed in section 3.1.2.2, a symmetric channel transition matrix, \( T \), may be partitioned as

\[
\begin{pmatrix}
(P_{11} & P_{21}) & (P_{12} & P_{22}) & \cdots & (P_{1k} & P_{2k}) & \cdots & (P_{1(M-1)} & P_{2(M-1)}) & (P_{1M})
\end{pmatrix}.
\]

Let \( j = 1, 2, \ldots, Q \), index the columns of \( T \) and let \( i = (0, 1) \) index the rows, with \( i = 0 \) corresponding to the message bit \( m = 0 \). Assume that '1' is transmitted and that \( L \) independent outputs \( \{j_i\}_{i=1}^L \) are obtained from the channel. A maximum-likelihood receiver will decide that a '1' was sent if

\[ P \left[ \{j_i\}_{i=1}^L \mid i = 1 \right] > P \left[ \{j_i\}_{i=1}^L \mid i = 0 \right]. \tag{C.1} \]

Since we have assumed that the samples are independent, then the ML decision
rule reduces to
\[ \prod_{i=1}^{L} P(j_i | 1) > \prod_{i=1}^{L} P(j_i | 0) \]  
(C.2)
\[ \prod_{i=1}^{L} \frac{P(j_i | 1)}{P(j_i | 0)} > 1 \]  
(C.3)
\[ \sum_{i=1}^{L} \ln \left( \frac{P(j_i | 1)}{P(j_i | 0)} \right) > 0. \]  
(C.4)

We seek to bound the probability of error by a Chernoff bound, i.e.
\[ P \left( \sum_{i=1}^{L} \ln \left( \frac{P(j_i | 1)}{P(j_i | 0)} \right) \leq 0 \right) \leq G_X(s^*) \]  
(C.5)

where \( G_X(s^*) \) is the minimized moment generating function for the random variable \( X \), where \( X \) takes on the values
\[ x_j = \ln \left( \frac{P(j_i | 1)}{P(j_i | 0)} \right), j = 1, 2, \ldots, Q. \]  
(C.6)

Since we have assumed that a '1' has been transmitted, \( x_j \) occurs with probability \( P(j_i | 1) \). To find \( s^* \) we write
\[ G_X(s) = E \left[ e^{-sx} \right] = \sum_{j=1}^{Q} P(j_i | 1)e^{-sx_j} \]
\[ = \sum_{j=1}^{Q} P(j_i | 1)e^{-s \ln \left( \frac{P(j_i)}{P(0)} \right)} = \sum_{j=1}^{Q} P(j_i | 1)^{1-s} P(j_i | 0)^s. \]  
(C.7)

This sum over the \( Q \) columns of \( T \) may be re-written as a sum over the \( M \) partitions of \( T \), resulting in
\[ G_X(s) = \sum_{k=1}^{M-1} \left[ P_{2k}^{1-s} P_{1k}^s + P_{1k}^{1-s} P_{2k}^s \right] + P_{1M} \]  
(C.8)

Equation (C.8) is symmetric with respect to \( s = 1/2 \), and since the moment generating function is a convex function of \( s \) [45], \( G_X(s) \) is minimized by \( s = s^* = 1/2 \).
Thus

\[ G_X(s^*) = \sum_{k=1}^{M-1} 2P_{1k}^{1/2}P_{2k}^{1/2} + P_{1M} \]  \hspace{1cm} (C.9)

and

\[
p_e \leq |G_X(s^*)|^L \leq \left[ 2 \sum_{k=1}^{M-1} \sqrt{P_{1k}P_{2k} + P_{1M}} \right]^L . \]  \hspace{1cm} (C.10)

Equation (C.10) is in a form for the case of an erasure partition. If the erasure partition is absent the channel transition matrix will consist of \( M \) 2 \times 2 partitions, and (C.10) simplifies to

\[
p_e \leq |G_X(s^*)|^L = \left[ 2 \sum_{k=1}^{M} \sqrt{P_{1k}P_{2k}} \right]^L
\]

which is (3.15).
Appendix D

Here we derive an expression for the probability of error of a selection diversity receiver employing a quantized SNR measurement.

Let the pdf of the received SNR be denoted by $f_r(\gamma)$. The received SNR is quantized using $N$ thresholds $\{T_k\}_{k=1}^N$, as shown in Figure 3.1. Let the pdf of the quantized SNR be denoted as $f_X(x)$, where $x \in \{x_1, x_2, \ldots, x_{N+1}\}$. A representative $f_X(x)$ is shown in Figure D.1.

A selection diversity receiver operating on $L$ independent samples of the quantized SNR will choose the sample with SNR, $x^*$, that is the largest of those available. The probability that $x^*$ is equal to, or less than, some value $y$ is equal to the probability that all of the samples are at the same level, $y$, or less. Since we have assumed that each branch fades independently the cumulative distribution function of $X^*$ is

$$F_{X^*}(y) = P(X_1 \leq y, X_2 \leq y, \ldots, X_L \leq y) \quad (D.1)$$

$$= \prod_{i=1}^L P(X_i \leq y) \quad (D.2)$$

$$= \prod_{i=1}^L F_{X_i}(y) \quad (D.3)$$

$$= [F_X(y)]^L \quad (D.4)$$
where from (D.3) to (D.4) we have assumed that the $X_i$ are identically distributed. A representative $F_{x^*}(y)$ is shown in Figure D.2.

The pdf of $X^*$, $f_{x^*}(y)$, is the derivative of $F_{x^*}(y)$. When the selected SNR is in the $k^{th}$ SNR region, the receiver will make an error with probability

$$P_{e|k} = \frac{1}{P_k} \int_{T_{k-1}}^{T_k} p_e(\gamma) f_\Gamma(\gamma) \, d\gamma \quad (D.5)$$

where $P_k$ is the probability that the SNR is in region $k$ in Figure 3.1, and $p_e(\gamma)$ is the probability of receiver error at SNR $\gamma$. The probability of error is then the sum of the conditional probabilities given by (D.5), weighted by the pdf of $X^*$, i.e.

$$P_{e,sel} = P_1L P_{e|1} + [(P_1 + P_2)^L - P_1^L] P_{e|2} + \cdots$$

$$= \sum_{k=1}^{N+1} \left[ \left( \sum_{i=1}^{k} P_i \right)^L - \left( \sum_{i=1}^{k-1} P_i \right)^L \right] P_{e|k} \quad (D.6)$$

$$= \sum_{k=1}^{N+1} \left[ (A_k + P_k)^L - (A_k)^L \right] P_{e|k} \quad (D.7)$$

Figure D.1: Quantized SNR Probability Density Function
Figure D.2: Selection Diversity Quantized SNR Cumulative Distribution Function

where we have defined

\[ A_k = \sum_{i=1}^{k-1} P_i. \]  (D.8)

Equations (D.7) and (D.8) are identical to (3.17) and (3.18) in Section 3.1.3. Note that equation (D.7) is not restricted to any particular SNR distribution,\(^1\) so that it may be used, for example, with a Rician SNR distribution. As the number of SNR quantization regions is increased, the BER given by (D.7) will approach the BER for a selection diversity receiver without SNR quantization. As demonstrated in Section 3.1.3, with 3 repeats, about 4 thresholds are required to closely approximate the unquantized selection diversity BER.

\(^1\)If the \(P_k\) and \(P_{cl,k}\) cannot be obtained in closed form, then one may evaluate these probabilities by numerical integration.