LINEAR EQUIVALENTS OF NONLINEAR SYSTEMS

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B.Sc., The University of British Columbia, 1983

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE

in

THE FACULTY OF GRADUATE STUDIES

Department of Mathematics

We accept this thesis as conforming
to the required standard

THE UNIVERSITY OF BRITISH COLUMBIA

April 1987

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Abstract

Consider the following nonlinear system

\[ \dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i, \]
\[ y = h(x), \]

where \( x \in \mathbb{R}^n \), \( f, g_1, \ldots, g_m \) are \( C^\infty \) function in \( \mathbb{R}^n \) and \( h \) is a \( C^\infty \) function in \( \mathbb{R}^p \), all defined on a neighborhood of 0. The problem of finding a necessary and sufficient condition such that system (1) can be transformed to a linear controllable system by a state coordinate change and feedback has been studied quite well. In this thesis, we first discuss a few different approaches to this problem and eventually we will show that the slightly different versions of the necessary and sufficient condition discovered are equivalent. Next we consider system (1) with all \( u_i = 0 \) together with system (2), and study the dual problem of transforming it to a linear observable system by a state and output coordinate change. Finally, we consider briefly system (1) and (2) with nonzero \( u_i \) and study the problem of transforming it to a linear system that is both completely controllable and observable. Examples are given and applications to local stabilization and estimation are discussed.
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Acknowledgement

I wish to thank Dr. U. Haussmann for his helpful comments, criticisms, and suggestions on this thesis, and for his patience in reviewing the draft. I am also grateful to the Mathematics Department for providing the financial support during the preparation of this thesis.
CHAPTER 1
Introduction and background

§1.1 Introduction

Consider the linear controllable system of the form

\[ \dot{x} = Ax + Bu \]  \hspace{1cm} (1.1)

where \( A \) and \( B \) are \( n \times n \) and \( n \times m \) constant matrices respectively, \( x \in \mathbb{R}^n, u \in \mathbb{R}^m \), and satisfying the controllability rank condition

\[ \text{rank}(B, AB, \ldots, A^{n-1} B) = n. \]

We will denote this system by the pair \((A, B)\). In the early seventies, some interest was shown in the problem of transforming (1.1) to another linear controllable system \((\hat{A}, \hat{B})\) by transformations of the form

(i) a nonsingular linear change of state coordinates:

\[ x = P\xi \]

(ii) a nonsingular linear change of input coordinates:

\[ u = Qv \]

(iii) a linear state feedback:

\[ u = Wx \]

so that system \((A, B)\) is transformed to system \((\hat{A}, \hat{B})\) in the following way:
((A, B) \rightarrow (\hat{A}, \hat{B}) = (P^{-1}(A + BW)P, P^{-1}BQ))

Brunovsky gave a necessary and sufficient condition under which such a transformation is possible. The condition is that

\( r_i = \hat{r}_i, \text{ for } i = 0, \ldots, n - 1, \)

where

\( r_0 = \text{rank } B \)

\( r_i = \text{rank } (B, AB, \ldots, A^iB) - \text{rank } (B, AB, \ldots, A^{i-1}B), \)

for \( 1 \leq i \leq n - 1, \) and \( \hat{r}_i \) are similarly defined on the system \((\hat{A}, \hat{B})\).

He called the system \((A, B)\) feedback equivalent or F-equivalent to the system \((\hat{A}, \hat{B})\) if such a transformation exists, and showed that systems that are F-equivalent to each other are F-equivalent to a certain linear controllable system of the form

\[
\dot{\xi} = \tilde{A}\xi + \tilde{B}v
\]

(1.2)

where

\[
\tilde{A} = \begin{pmatrix}
A_1 & 0 & \cdots & 0 \\
& O & A_2 & \cdots & 0 \\
& & \ddots & \ddots & \vdots \\
O & O & \cdots & A_{r_0}
\end{pmatrix}, \quad 
\tilde{B} = \begin{pmatrix}
b_1 & O & \cdots & 0 & O \\
& O & b_2 & \cdots & O & O \\
& & \ddots & \ddots & \vdots & \ddots \\
& & & O & O & \cdots & b_{r_0} & O
\end{pmatrix}
\]

\[
A_i = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
& \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix}, \quad 
\begin{pmatrix}
b_i
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
\vdots \\
0 \\
1
\end{pmatrix}, \quad i = 1, \ldots, r_0,
\]

where \(A_i\) and \(b_i\) have dimensions \(\kappa_i \times \kappa_i\) and \(\kappa_i \times 1\) respectively. Moreover, the integers \(\kappa_1, \ldots, \kappa_m\) can be reordered so as to satisfy

\( 0 \leq \kappa_i \leq n, \quad \kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_{r_0} \geq 1, \)
\[ \kappa_i = 0, \text{ for } i > r_0, \text{ and } \sum_{i=1}^{m} \kappa_i = n. \]

and are generally called the controllability indices. They are related to the \( r_j \)'s in the following way:

\[ \kappa_i = \text{the number of } r_j \text{'s that are } \geq i. \]

System (1.2) is generally said to be in Brunovsky canonical form and will be referred to as system (BCF) for brevity. Since then much work has been done to generalize the result to nonlinear systems, specifically, to find a necessary and sufficient condition under which a nonlinear control system can be transformed locally to the linear controllable system (BCF). The particular class of nonlinear control system considered by many has the form

\[ \dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i, \quad (1.3) \]

where \( x \in \mathbb{R}^n, f, g_1, \ldots, g_m \) are \( C^\infty \) vector fields on some neighborhood of 0 in \( \mathbb{R}^n \), and \( f(0) = 0 \). Such systems have both practical and theoretical significances as many physical systems, especially those in engineering, can be described in this form.

**Remark.** There is no loss of generality in assuming that \( f(0) = 0 \), since if \( f(x_0) = 0 \), then letting \( \bar{x} = x - x_0 \), we have

\[ f(x) = f(x + x_0) = \bar{f}(\bar{x}), \text{ and } g_i(x) = \bar{g}_i(\bar{x}). \]

In the new variable \( \bar{x} \), system (1.3) is written as

\[ \dot{\bar{x}} = \dot{x} = \bar{f}(\bar{x}) + \sum_{i=1}^{m} \bar{g}_i(\bar{x})u_i, \]

where now \( \bar{f}(0) = 0 \).
The class of transformations considered by most people is a modification of the one considered by Brunovsky, specifically, it consists of

(C1) a nonlinear change of state coordinates:

\[ x = \tilde{x}(\xi) \]

such that the map \( \tilde{x} : V \subset \mathbb{R}^n \to U \subset \mathbb{R}^n \) is a local diffeomorphism at 0 mapping the origin to the origin, i.e., \( \tilde{x}(0) = 0 \),

(C2) a linear change of input coordinates:

\[ u_i = \sum_{j=1}^{m} q_{ij}(x)v_j, \quad i = 1, \ldots, m \]

where \( q_{ij} \) are \( C^\infty \) real-valued functions on \( U \) such that the \( m \times m \) matrix \( Q(x) = (q_{ij}(x)) \) is nonsingular on a neighborhood of 0 so that \( g_i \to \sum_{j=1}^{m} g_j q_{ji} \).

(C3) a nonlinear state feedback:

\[ u_i = w_i(x), \quad i = 1, \ldots, m \]

where \( w_i(x) \) are \( C^\infty \) real-valued functions defined on \( U \), and \( w_i(0) = 0 \) for \( i = 1, \ldots, m \), so that \( f \to f + \sum_{i=1}^{m} g_i w_i \).

The family of all such transformations forms a group which will be called \( G_c \). We call two systems \( G_c \)-related if one can be transformed to the other by a transformation in \( G_c \). It can be seen that this relation is an equivalence relation, i.e., it is reflexive, symmetric, and transitive. Hence, we will also call such systems \( G_c \)-equivalent for convenience.

Of particular interest is the characterization of systems (1.3) that are locally \( G_c \)-equivalent to the linear controllable system (BCF). In [2], Brockett obtained some results for the single input case. Jakubczyk and Respondek [9],
and Hunt, Su, and Meyer [7] got similar results for the multi-input case. In [8], Hunt and Su used the global inverse function theorem to get conditions under which a global transformation exists for the single input case.

As a dual to the above linearization problem of a nonlinear control system, the linearization of a nonlinear system with outputs has also been considered by some researchers. Such a system has the form

\[ \dot{x} = f(x) \]
\[ y = h(x) \]

where \( x \in \mathbb{R}^n \), \( y \in \mathbb{R}^p \), \( f(x) \) is a \( C^\infty \) vector field on some neighborhood of 0 in \( \mathbb{R}^n \), and \( h(x) \) is a \( C^\infty \) map from some neighborhood of 0 in \( \mathbb{R}^n \) to \( \mathbb{R}^p \) such that \( h(0) = 0 \). \( x \) and \( y \) are generally called the state and output of the system respectively.

Now the group of transformations consists of:

(O1) a nonlinear change of state coordinates as described earlier in (C1).

(O2) a nonlinear change of output coordinates:

\[ y = \tilde{y}(\phi) \]

such that the map \( \tilde{y} : W \subset \mathbb{R}^p \rightarrow \tilde{y}(W) \subset \mathbb{R}^p \) is a local diffeomorphism on \( W \) mapping the origin to the origin, i.e., \( \tilde{y}(0) = 0 \), and \( W \) is some neighborhood of 0 in \( \mathbb{R}^p \).

We will call this group of transformations \( G_o \) and we similarly define \( G_o \)-related or \( G_o \)-equivalent systems. Analogous to the previous problem, we are interested in the characterization of systems (1.4) that are locally \( G_o \)-equivalent to a system in observable form given below.

\[ \dot{\xi} = A\xi + \alpha(\phi) \quad (1.5) \]
\[ \phi = C\xi \]
where $\xi \in \mathbb{R}^n, \phi \in \mathbb{R}^p$, $\alpha$ is a vector-valued function depending only on the output $\phi$, and $(C, A)$ is an observable pair, i.e.,

$$\text{rank} \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} = n.$$ 

Incidentally, if (1.4) is linear and observable, i.e., if it has the form

$$\begin{align*}
\dot{x} &= Ax \\
y &= Cx
\end{align*}$$

(1.6)

and $(C, A)$ is an observable pair, then it is possible to transform (1.6) to (1.5) where $A = \tilde{A}$ as before and $C = \tilde{C}$ where

$$
\tilde{C} = \begin{pmatrix}
c_1 & O & \cdots & O \\
O & c_2 & \cdots & O \\
\vdots & \vdots & \ddots & \vdots \\
O & O & \cdots & c_{d_0} \\
O & O & \cdots & O
\end{pmatrix}, \quad c_i = (1 \ 0 \ \cdots \ 0), \quad i = 1, \ldots, d_0,
$$

$A_i$ and $c_i$ have dimensions $\mu_i \times \mu_i$ and $1 \times \mu_i$ respectively. Moreover, the integers $\mu_1, \ldots, \mu_p$ can be reordered so as to satisfy

$$0 \leq \mu_i \leq n, \quad \mu_1 \geq \mu_2 \geq \cdots \geq \mu_{d_0} \geq 1,$$

$$\mu_i = 0, \quad \text{for } i > d_0, \quad \text{and } \sum_{i=1}^p \mu_i = n,$$

and are generally called the observability indices. It can be checked that

$$\mu_i = \text{the number of } d_j \text{'s that are } \geq i,$$

where the $d_i$'s are defined as follows.

$$
d_0 = \text{rank } C,
$$

$$
d_i = \text{rank } \begin{pmatrix} C \\ CA \\ \vdots \\ CA^i \end{pmatrix} - \text{rank } \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{i-1} \end{pmatrix}, \quad \text{for } 1 \leq i \leq n - 1.
$$
Indeed, if \((C, A)\) is an observable pair, then \((A^T, C^T)\) is an controllable pair. That means there exist matrices \(P, W, Q\), such that

\[
\tilde{A} = P^{-1}(A^T + C^T W)P, \quad \text{and} \quad \tilde{B} = P^{-1}C^T Q,
\]

or

\[
A = (P^T)^{-1} \tilde{A}^T P^T - W^T C, \quad \text{and} \quad \tilde{B}^T = Q^T C (P^T)^{-1}.
\]

Let \(x = S \xi\), and \(y = H \phi\) be the state and output coordinate change respectively. If

\[
S = (P^T)^{-1}, \text{and} \quad H = (Q^T)^{-1},
\]

then from (1.6) it follows that

\[
\dot{\xi} = S^{-1} A S \xi \\
= P^T ((P^T)^{-1} \tilde{A}^T P^T - W^T C)(P^T)^{-1} \xi \\
= (\tilde{A}^T - S^{-1} W^T C S) \xi \\
= (\tilde{A}^T - S^{-1} W^T (Q^T)^{-1} Q^T C S) \xi.
\]

Also

\[
y = H \phi = (Q^T)^{-1} \phi = CS \xi,
\]

so

\[
\phi = Q^T C S \xi = \tilde{B}^T \xi.
\]

Hence,

\[
\dot{\xi} = \tilde{A}^T \xi + D \phi,
\]

where \(D = - S^{-1} W^T (Q^T)^{-1}\). It is obvious that a permutation of the states will transform the system in coordinate \(\xi\) and \(\phi\) into system (1.5). The relationship between the integers \(\mu_i\) and \(d_j\) is clear in view of our previous discussion on the control system (1.1).
System (1.5) will be referred to as the dual Brunovsky observer form or system (DBOF) for brevity if \((C, A) = (\bar{C}, \bar{A})\). We are particularly interested in finding necessary and sufficient conditions that system (1.4) can be transformed to this form. In [13], Krener and Isidori gave a necessary and sufficient condition such that this is possible for the single output case. Bestle and Zeitz [1] also studied this problem for the single output case and without the output change in coordinates, \((O2)\). They gave the necessary condition that the map for the state coordinate change has to satisfy: it must solve a system of partial differential equations. They did not examine the solvability of these equations; however, they showed how to design a nonlinear observer based on the linear system in canonical form if a transformation exists. Krener and Respondek [14] also studied this problem for the multi-output case.

In this thesis, we will study the above problems in some details. In the next section, we will give some definitions and background material developed in differential geometry. In chapter 2, we will derive a necessary and sufficient condition that system (1.3) be \(Gc\)-equivalent to system (BCF) and show that it is equivalent to those in the literature. Thereafter, an example is given in section 2.2. In section 2.3, we will determine a control law that will stabilize the nonlinear system for small deviation from the equilibrium, and an application will be discussed in section 2.4. We will derive a necessary and sufficient condition that system (1.4) be \(Go\)-equivalent to system (DBOF) in section 3.1, followed by an example in section 3.2. In section 3.3, we will develop the theory for the local estimation of states and then we give the design of a local asymptotic observer in section 3.4. In chapter 4, we will find a necessary and
sufficient condition that the system

\[ \dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i \]
\[ y = h(x) \]  \hspace{1cm} (1.7)

be Gc-equivalent to a linear system that is both controllable and observable of the form

\[ \dot{\xi} = \tilde{A}\xi + \tilde{B}v \]
\[ \phi = \tilde{C}\xi \]  \hspace{1cm} (1.8)

where Gc is the group of transformations of the type (C1), (C2), (C3), and (O2). An example will be given and a regulator problem will be discussed. We will sometimes use the word 'linearizable' in place of 'Gc-equivalent' or 'Go-equivalent' when the meaning is clear.

§1.2 Preliminaries

In this section, we will introduce some definitions developed in differential geometry that are relevant to our later discussions. Specifically, Brockett's survey [3] on nonlinear systems and differential geometry, Hermann and Krener's paper [6] on nonlinear controllability and observability, and, in particular, Isidori's book [9] on nonlinear control systems provide some background material for this section.

For nonlinear system analysis, we need to define some terminologies. Given a $C^\infty$ manifold $M$, let $C(M)$ be the set of $C^\infty$ real-valued functions on $M$, $V(M)$ be the set of $C^\infty$ vector fields on $M$, and $V^*(M)$ be the set of covariant vector fields or covector fields on $M$. We define a $k$ dimensional $C^\infty$ distribution $\Delta$ on $M$ as a mapping assigning to each point $p$ of $M$ a $k$ dimensional tangent space $T_pM$ to $M$ at $p$; it is a submodule of $V(M)$ over the ring $C(M)$. 9
Furthermore, for each \( p \in M \), there exists a neighborhood \( U \) of \( p \) and \( k \) \( C^\infty \) vector fields \( v_1, \ldots, v_k \) such that

\[
\Delta(q) = \text{span}\{v_i(q) : i = 1, \ldots, k\}, \quad \forall q \in U.
\]

Following the notations of Isidori [9], if

\[
W = \{w_i \in V(M) : i \in I\}
\]

for some index set \( I \), we denote the distribution generated by elements of \( W \) to be

\[
\text{sp\{w}_i : i \in I\},
\]

which is the set of all linear combinations of vector fields in \( W \) with coefficients in \( C(M) \), and will be denoted by \( \tilde{W} \). Pointwise, for each \( p \in M \),

\[
\tilde{W}(p) : p \mapsto \text{span}\{w_i(p) : i \in I\}
\]

where \( \text{span}\{w_i(p) : i \in I\} \) denotes all \( \mathbb{R} \) - linear combinations of \( w_i(p) \).

Likewise, we define a \( k \) dimensional \( C^\infty \) codistribution \( \Delta^* \) on \( M \) as a mapping assigning to each point \( p \) of \( M \) a \( k \) dimensional cotangent space \( T^*_p M \) to \( M \) at \( p \), it is a submodule of \( V^*(M) \) over the ring \( C(M) \). For each \( p \in M \), there exists a neighborhood \( U \) of \( p \) and \( k \) \( C^\infty \) covector fields \( v^*_1, \ldots, v^*_k \) such that

\[
\Delta^*(q) = \text{span}\{v^*_i(q) : i = 1, \ldots, k\}, \quad \forall q \in U.
\]

If \( W^* = \{v^*_i \in V^*(M) : i \in I\} \), then we define \( \tilde{W}^* \) similarly.

**Definition:** A distribution \( \Delta \) (or codistribution) is said to be nonsingular on \( U \) if

\[
\text{dim} \ \Delta(p) = \text{constant}, \quad \forall p \in U,
\]

and is said to be nonsingular if it is nonsingular on \( M \). If a point \( p \) has such a neighborhood \( U \) on which the distribution is nonsingular, then \( p \) is called a regular point.
Definition: Given a distribution $\Delta$, we can define a codistribution, the annihilator of $\Delta$, by

$$\Delta^\perp: p \mapsto \{v^* \in T^*_p M : \langle v^*, v \rangle = 0, \forall v \in \Delta(p)\}.$$  

Similarly, given a codistribution $\Delta^*$, we can define a distribution, the annihilator of $\Delta^*$, by

$$\Delta^{\ast \perp}: p \mapsto \{v \in T_p M : \langle v^*, v \rangle = 0, \forall v^* \in \Delta^*(p)\}.$$  

Moreover, we have the properties

$$\dim \Delta + \dim \Delta^\perp = \dim M,$$

and

$$\dim \Delta^* + \dim \Delta^{\ast \perp} = \dim M.$$  

We will assume $M = \mathbb{R}^n$ in the sequel. We define the Lie bracket $[\ , \ ]$ of two vector fields $g_1, g_2 \in \mathcal{V}(\mathbb{R}^n)$ as

$$[g_1, g_2] = \frac{\partial g_2}{\partial x} g_1 - \frac{\partial g_1}{\partial x} g_2,$$

which is a column vector valued function.

The Lie bracket operation satisfies the following properties:

(i) it is skew symmetric, i.e., $[g_1, g_2] = -[g_2, g_1]$,

(ii) it is bilinear over $\mathbb{R}$, i.e., if $a_1, a_2$ are real numbers, $g_1, g_2, g_3 \in \mathcal{V}(\mathbb{R}^n)$, then

$$[a_1 g_1 + a_2 g_2, g_3] = a_1 [g_1, g_3] + a_2 [g_2, g_3],$$

(iii) it satisfies the Jacobi identity

$$[g_1, [g_2, g_3]] + [g_3, [g_1, g_2]] + [g_2, [g_3, g_1]] = 0.$$
If \( \alpha, \beta \in C(R^n) \), \( f, g \in V(R^n) \), then

\[
[\alpha f, \beta g] = \alpha \cdot \beta \cdot [f, g] + L_f(\beta) \cdot \alpha \cdot g - L_g(\alpha) \cdot \beta \cdot f,
\]

where \( \cdot \) indicates ordinary product.

The Lie derivative of a vector field \( g \) with respect to \( f \) is defined as

\[
L_f(g) = [f, g], \quad \text{or} \quad \text{ad}_f g,
\]

and inductively, we define

\[
\text{ad}^k_f g = [f, \text{ad}^{k-1}_f g], \quad \text{for} \quad k \geq 1, \quad \text{with} \quad \text{ad}^0_f g = g.
\]

We see that \( \text{ad}^k_f g \) is again a vector field on \( R^n \) represented as a column vector.

Note also that the Jacobi Identity can be equivalently written as

\[
L_{g_1} [g_2, g_3] = [L_{g_1} (g_2), g_3] + [g_2, L_{g_1} (g_3)].
\]

If \( h \in C(R^n) \), then \( dh \) is a covector field on \( R^n \) defined by

\[
dh = (\frac{\partial h}{\partial x_1}, \ldots, \frac{\partial h}{\partial x_n}),
\]

which is a row vector valued function on \( R^n \), and the Lie derivative of \( h \) with respect to a vector field \( f \) in \( V(R^n) \) is

\[
L_f(h) = \langle dh, f \rangle = \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} f_i,
\]

where \( f_i \) is the \( i \)th component of \( f \). Again \( L_f(h) \) is in \( C(R^n) \). If \( w \) is a covector field, then the Lie derivative of \( w \) with respect to \( f \) is

\[
L_f(w) = (\frac{\partial w^T}{\partial x} f)^T + w \frac{\partial f}{\partial x}.
\]
In particular, if \( w \) is an exact covector field, i.e., \( w = dh \), then

\[
L_f(dh) = f^T \frac{\partial^2 h}{\partial x^2} + \frac{\partial h}{\partial x} \frac{\partial f}{\partial x},
\]

which is a row vector valued function. Inductively, we define

\[
L_f^k(dh) = L_f L_f^{k-1}(dh), \quad \text{for} \quad k \geq 1, \quad \text{with} \quad L_f^0(dh) = dh.
\]

\( L_f^k(dh) \) is again in \( \mathcal{V}^*(R^n) \). We also have the relation

\[
L_f(dh) = dL_f(h),
\]

i.e., the operators \( L_f \) and \( d \) commute, and the Leibnitz type formula

\[
L_f(\langle dh, g \rangle) = \langle L_f(dh), g \rangle + \langle dh, [f, g] \rangle,
\]

where \( h \in C(R^n) \), and \( f, g \in V(R^n) \). If \( \alpha, \beta \in C(R^n), f \in V(R^n), \) and \( w \in V^*(R^n) \), then

\[
L_{\alpha f}(\beta w) = \alpha \cdot \beta \cdot L_f(w) + \alpha \cdot L_f(\beta) \cdot w + \beta \cdot \langle w, f \rangle \cdot d\alpha,
\]

and if \( S \) is a set of vector fields or covector fields on \( R^n f \in V(R^n) \), then

\[
L_f S = \{ L_f \tau : \tau \in S \}.
\]

**Definition**: Let \( V = \{ f_1(x), \ldots, f_m(x) \} \) be a collection of \( C^\infty \) vector fields on \( R^n \), \( V \) is called involutive if there exist \( \gamma_{ijk} \in C(R^n) \) such that

\[
[f_i, f_j] = \sum_{k=i}^{m} \gamma_{ijk} \cdot f_k, \quad 1 \leq i, j \leq m.
\]

**Definition**: A distribution \( \Delta \) is involutive if \( [f, g] \in \Delta \) whenever \( f, g \in \Delta \).

**Remark.** It is easy to see that if a finite collection of vector fields \( V \) is involutive, then the distribution \( \tilde{V} \) generated by \( V \) is an involutive distribution.
CHAPTER 2
Linearization of nonlinear control systems

§2.1 Necessary and sufficient condition

In this section we will find necessary and sufficient conditions for the nonlinear system

\[ \dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i \]  

(2.1)

to be locally \(C^\infty\)-equivalent to the linear controllable system (BCF) around 0. We will examine previous work by Jakubczyk and Respondek [10], Isidori [9], and Hunt, Su, and Meyer [7]. They gave close but seemingly different conditions. We will show their proofs or the ideas behind them when necessary. Finally, we will prove that their results are equivalent.

First, we introduce some notation. Call \((w,Q)\) a feedback pair, where \(w\) is a \(m\)-vector of \(C^\infty\) real-valued functions and \(Q\) is an \(m \times m\) matrix of \(C^\infty\) real-valued functions, all defined on a neighborhood \(U\) of 0, if they satisfy the properties:

(i) \(w(0) = 0,\)

(ii) \(Q(x)\) is nonsingular for all \(x \in U.\)

We define \(S_0 = \{g_1, \ldots, g_m\},\) and \(S_k = S_{k-1} \cup [f, S_{k-1}],\) for \(k \geq 1.\)

Similarly, we define \(\hat{S}_0 = \{\hat{g}_1, \ldots, \hat{g}_m\},\) and \(\hat{S}_k = \hat{S}_{k-1} \cup [\hat{f}, \hat{S}_{k-1}],\) where

\[ \hat{g}_j = \sum_{i=1}^{m} g_i q_{ij} \]

\[ \hat{f} = f + \sum_{i=1}^{m} g_i w_i, \]

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and \( (w, (g_{ij})) \) is a feedback pair.

For simplicity, assume \( g_1, \ldots, g_m \) are linearly independent on a neighborhood of 0.

**Proposition 2.1** The following statements are equivalent:

(A) System (2.1) is locally \( G_c \)-equivalent to the linear controllable system (BCF) around 0.

(B) There exists a feedback pair \((w, Q)\), and integers \( \kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_m \geq 1 \) with \( \sum_{i=1}^{m} \kappa_i = n \), such that the following conditions are satisfied on a neighborhood of 0:

(i) \([\text{ad}^k \tilde{g}_j, \text{ad}^l \tilde{g}_i] = 0\), for \( 1 \leq i, j \leq m, k = 0, \ldots, \kappa_j - 1, l = 0, \ldots, \kappa_i - 1 \),

(ii) \( \text{ad}^k \tilde{g}_j = 0 \), for \( j = i, \ldots, m \),

(iii) \( \text{dim span}\{\text{ad}^k \tilde{g}_j : j = 1, \ldots, m, k = 0, \ldots, \kappa_j - 1\} = n \).

(C) cf. [10] The following conditions are satisfied on a neighborhood of 0:

(i) The distribution \( \mathcal{S}_j \) is involutive for \( j = 0, \ldots, n - 1 \),

(ii) \( \mathcal{S}_j \) is nonsingular for \( j = 0, \ldots, n - 1 \),

(iii) \( \text{dim } \mathcal{S}_{n-1} = n \).

**Remark.** We will prove that (A) implies (B), (B) implies (C), and (C) implies (A). A direct proof that (B) implies (A) will be shown in the course of proving proposition 4.1.

First, call \((\tilde{x}(\xi), w, Q)\) the linearizing triple if the map \( \tilde{x}(\xi) \), corresponding to the state coordinate change, and the feedback \( u = w(x) + Q(x)v \), transform system (2.1) into system (BCF).

**Proof:** \((A) \Rightarrow (B)\). Let \((\tilde{x}(\xi), w, Q)\) be the linearizing triple with controllability indices \( \kappa_1, \ldots, \kappa_m \) satisfying \( \kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_m \geq 1 \) and \( \sum_{i=1}^{m} \kappa_i = n \).
Differentiate $\tilde{x}(\xi)$ with respect to $t$,

$$
\dot{x} = \frac{\partial \tilde{x}}{\partial \xi} \frac{d\xi}{dt} = f(x) + G(x)u
$$

$$
\frac{\partial \tilde{x}}{\partial \xi}(A\xi + Bv) = f(x) + G(x)(w(x) + Q(x)v)
$$

$$
= f(x) + G(x)w(x) + G(x)Q(x)v
$$

$$
= \tilde{f}(x) + \tilde{G}(x)v,
$$

where

$$
\tilde{f}(x) = f(x) + G(x)w(x)
$$

$$
\tilde{G}(x) = G(x)Q(x).
$$

If (2.2) holds for arbitrary $v$, then it is necessary that

$$
\frac{\partial \tilde{x}}{\partial \xi} \tilde{A} = \tilde{f}(\tilde{x}(\xi)),
$$

(2.3)

$$
\frac{\partial \tilde{x}}{\partial \xi} \tilde{B} = \tilde{G}(\tilde{x}(\xi)).
$$

(2.4)

Let $\sigma_0 = 0$ and $\sigma_j = \sum_{i=1}^{j} \kappa_i$, for $j \geq 1$, then

$$
\xi = \begin{pmatrix} 
\xi_1 \\
\xi_2 \\
\vdots \\
\xi_{\sigma_1} \\
\xi_{\sigma_1+1} \\
\vdots \\
\xi_{\sigma_m} 
\end{pmatrix},
$$

or, if we let $\xi_{ik} = \xi_{\sigma_{i-1} + k}$, for $i = 1, \ldots, m$, $k = 1, \ldots, \kappa_i$, then

$$
\xi = \begin{pmatrix} 
\xi_{11} \\
\xi_{12} \\
\vdots \\
\xi_{1\kappa_1} \\
\xi_{21} \\
\vdots \\
\xi_{m\kappa_m} 
\end{pmatrix}, \quad \tilde{A}\xi = \begin{pmatrix} 
\xi_{12} \\
\vdots \\
\xi_{1\kappa_1} \\
0 \\
\xi_{22} \\
\vdots \\
\xi_{m\kappa_m} 
\end{pmatrix}, \quad \text{and } \tilde{B} = (e_{1\kappa_1}, \ldots, e_{m\kappa_m}),
$$
where $e_{ik}$ is a standard unit vector with a 1 in the $\sigma_{i-1} + k$ th position. By (2.4)

$$\frac{\partial \tilde{z}}{\partial \xi} \tilde{B} = G(\tilde{z}(\xi))Q(\tilde{z}(\xi))$$

$$= (G(\tilde{z}(\xi))q_1(\tilde{z}(\xi)), \ldots, G(\tilde{z}(\xi))q_m(\tilde{z}(\xi))).$$

Also

$$\frac{\partial \tilde{B}}{\partial \xi} = \left( \frac{\partial \tilde{z}}{\partial \xi} e_{1\kappa_1}, \ldots, \frac{\partial \tilde{z}}{\partial \xi} e_{m\kappa_m} \right)$$

$$= \left( \frac{\partial \tilde{z}}{\partial \xi_{1\kappa_1}}, \ldots, \frac{\partial \tilde{z}}{\partial \xi_{m\kappa_m}} \right),$$

so

$$\frac{\partial \tilde{z}}{\partial \xi_{i\kappa_i}} = G(\tilde{z}(\xi))q_i(\tilde{z}(\xi)) \triangleq \hat{g}_i(\tilde{z}(\xi)), \quad i = 1, \ldots, m. \quad (2.5)$$

Differentiating (2.3) partially with respect to $\xi_{ik+1}$, we have

$$\frac{\partial}{\partial \xi_{ik+1}} \left( \frac{\partial \tilde{z}}{\partial \xi} \tilde{A} \right) = \frac{\partial}{\partial \xi_{ik+1}} (f(\tilde{z}(\xi)))$$

$$= \frac{\partial f}{\partial x} \frac{\partial \tilde{z}}{\partial \xi_{ik+1}}. \quad (2.6)$$

Also

$$\frac{\partial}{\partial \xi_{ik+1}} \left( \frac{\partial \tilde{z}}{\partial \xi} \tilde{A} \right) = \left( \frac{\partial}{\partial \xi_{ik+1}} \left( \frac{\partial \tilde{z}}{\partial \xi} \right) \right) \tilde{A} \xi + \frac{\partial \tilde{z}}{\partial \xi} \left( \frac{\partial}{\partial \xi_{ik+1}} (\tilde{A} \xi) \right)$$

$$= \left( \frac{\partial}{\partial \xi} \left( \frac{\partial \tilde{z}}{\partial \xi_{ik+1}} \right) \right) \tilde{A} \xi + \frac{\partial \tilde{z}}{\partial \xi} \left( \frac{\partial}{\partial \xi_{ik+1}} (\tilde{A} \xi) \right).$$

Since

$$\frac{\partial}{\partial \xi_{ik+1}} (\tilde{A} \xi) = \begin{cases} e_{ik}, & \text{if } k = 1, \ldots, \kappa_i - 1; \\ 0, & \text{if } k = 0, \end{cases}$$

then

$$\frac{\partial \tilde{z}}{\partial \xi} \left( \frac{\partial}{\partial \xi_{ik+1}} (\tilde{A} \xi) \right) = \begin{cases} \frac{\partial \tilde{z}}{\partial \xi_{ik}}, & \text{if } k = 1, \ldots, \kappa_i - 1; \\ 0, & \text{if } k = 0. \end{cases}$$

Hence

$$\frac{\partial}{\partial \xi_{ik+1}} \left( \frac{\partial \tilde{z}}{\partial \xi} \tilde{A} \xi \right) = \begin{cases} \left( \frac{\partial}{\partial \xi} \left( \frac{\partial \tilde{z}}{\partial \xi_{ik+1}} \right) \right) \tilde{A} \xi + \frac{\partial \tilde{z}}{\partial \xi_{ik}}, & \text{if } k = 1, \ldots, \kappa_i - 1; \\ \left( \frac{\partial}{\partial \xi} \left( \frac{\partial \tilde{z}}{\partial \xi_{ik+1}} \right) \right) \tilde{A} \xi, & \text{if } k = 0. \end{cases}$$
Rewriting the above equation, we have

\[
\frac{\partial \hat{x}}{\partial \xi_{ik}} = \frac{\partial}{\partial \xi_{ik+1}} (\frac{\partial \hat{x}}{\partial \xi} \hat{A} \xi) - (\frac{\partial}{\partial \xi} (\frac{\partial \hat{x}}{\partial \xi_{ik+1}})) \hat{A} \xi
\]

\[
= \frac{\partial \hat{f}}{\partial x} \frac{\partial \hat{x}}{\partial \xi_{ik+1}} - (\frac{\partial}{\partial \xi} (\frac{\partial \hat{x}}{\partial \xi_{ik+1}})) \hat{A} \xi, \quad \text{if } k = 1, \ldots, \kappa_i - 1, \quad (2.7)
\]

and

\[
0 = \frac{\partial \hat{f}}{\partial x} \frac{\partial \hat{x}}{\partial \xi_{ik+1}} - (\frac{\partial}{\partial \xi} (\frac{\partial \hat{x}}{\partial \xi_{ik+1}})) \hat{A} \xi, \quad \text{if } k = 0. \quad (2.8)
\]

Let \( k = \kappa_i - 1 \) in (2.7), then

\[
\frac{\partial \hat{x}}{\partial \xi_{i\kappa_i-1}} = \frac{\partial \hat{f}}{\partial x} \frac{\partial \hat{x}}{\partial \xi_{i\kappa_i}} - (\frac{\partial}{\partial \xi} (\frac{\partial \hat{x}}{\partial \xi_{i\kappa_i}})) \hat{A} \xi.
\]

Since from (2.5)

\[
\frac{\partial \hat{x}}{\partial \xi_{i\kappa_i}} = \hat{g}_i(\hat{x}(\xi)),
\]

then

\[
\frac{\partial \hat{x}}{\partial \xi_{i\kappa_i-1}} = \frac{\partial \hat{f}}{\partial x} \hat{g}_i - \frac{\partial \hat{g}_i}{\partial \xi} \frac{\partial \hat{x}}{\partial \xi} \hat{A} \xi
\]

\[
= \frac{\partial \hat{f}}{\partial x} \hat{g}_i - \frac{\partial \hat{g}_i}{\partial \xi} \hat{f}
\]

\[
= -\text{ad}_f \hat{g}_i. \quad (2.9)
\]

Letting \( k = \kappa_i - 2 \) in (2.7) and using (2.9), we find

\[
\frac{\partial \hat{x}}{\partial \xi_{i\kappa_i-2}} = \frac{\partial \hat{f}}{\partial x} (-\text{ad}_f \hat{g}_i) - \frac{\partial}{\partial x} (-\text{ad}_f \hat{g}_i) \hat{f}
\]

\[
= \text{ad}_f^2 \hat{g}_i.
\]

If we repeat this process for \( k = \kappa_i - 3, \ldots, 1 \), we find

\[
\frac{\partial \hat{x}}{\partial \xi_{i\kappa_i-k}} = (-1)^k \text{ad}_f^k \hat{g}_i, \quad \text{for } k = 0, \ldots, \kappa_i - 1. \quad (2.10)
\]

Let \( k = \kappa_i - 1 \) in (2.10), then

\[
\frac{\partial \hat{x}}{\partial \xi_{i1}} = (-1)^{\kappa_i-1} \text{ad}_f^{\kappa_i-1} \hat{g}_i, \quad \text{for } k = 0, \ldots, \kappa_i - 1.
\]
From (2.8),
\[ 0 = \frac{\partial \hat{f}}{\partial x} \frac{\partial \hat{x}}{\partial \xi_1} - \frac{\partial}{\partial x} \left( \frac{\partial \hat{x}}{\partial \xi_i} \right) \hat{f} = -[\hat{f}, (-1)^{\kappa - 1} \text{ad}_{\hat{f}}^{-1} \hat{g}_i], \]
or
\[ \text{ad}_{\hat{f}}^{\kappa} \hat{g}_i = 0. \quad (2.11) \]

Since the map \( \hat{x}(\xi) \) is assumed to be smooth, the mixed partials must agree; this implies (i). (ii) holds in view of (2.11), and (iii) is necessary since the vector fields \{\text{ad}^i \hat{g}_j : j = 1, \ldots, m, i = 0, \ldots, \kappa_j - 1\} are the columns of the matrix \( \frac{\partial \hat{x}}{\partial \xi} \) and it is required that this matrix be nonsingular on a neighborhood of 0.

To show (B) \( \Rightarrow \) (C), we require the following lemma.

**Lemma 2.2** Suppose either \( \tilde{S}_k \) or \( \bar{S}_k \) is involutive for \( 0 \leq k \leq n - 1 \), then \( \tilde{S}_k = \bar{S}_k \) for \( 0 \leq k \leq n \). In addition, if either \( \tilde{S}_k \) or \( \bar{S}_k \) is nonsingular on a neighborhood of 0, then both are.

**Proof:** Suppose \( \tilde{S}_k \) is involutive for \( 0 \leq k \leq n - 1 \). Let
\[ G = (g_1, \ldots, g_m), \quad \hat{G} = (\hat{g}_1, \ldots, \hat{g}_m), \quad Q = (q_{ij}), \]
then
\[ \hat{G} = GQ, \quad \text{i.e.,} \quad \hat{g}_j = \sum_{i=1}^{m} g_i q_{ij}. \]
This implies \( \tilde{S}_0 \subset \bar{S}_0 \). Since \( Q \) is nonsingular on a neighborhood of 0, we can write \( G = \hat{G} \hat{Q} \), where \( \hat{Q} = Q^{-1} \). It is obvious that the reverse inclusion holds.

By induction, suppose \( \tilde{S}_k = \bar{S}_k \) for \( 0 \leq k \leq l < n \). If \( \hat{v} \in \tilde{S}_{l+1} \), then
\[ \hat{v} = \sum_{i=1}^{m} \alpha_i \cdot \text{ad}_{\hat{f}}^{l+1} \hat{g}_i + \hat{g}^l, \]

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where $\tilde{g}^l$ is a vector field in $\tilde{S}_i$ and $\alpha_i$ is a real-valued $C^\infty$ function defined on a neighborhood of 0. Also

$$ad^{l+1}_j \tilde{g}_i = [f + \sum_{j=1}^{m} g_j w_j, ad^l_j \tilde{g}_i]$$

$$= [f, ad^l_j \tilde{g}_i] + \sum_{j=1}^{m} [g_j w_j, ad^l_j \tilde{g}_i].$$

Since $ad^l_j \tilde{g}_i \in \tilde{S}_i$, and $\tilde{S}_i = \tilde{S}_i$, we have

$$[f, ad^l_j \tilde{g}_i] \in \tilde{S}_{i+1}.$$

Since $g_j \in \tilde{S}_i = \tilde{S}_i, \ j = 1, \ldots, m$, and $\tilde{S}_i$ is involutive, therefore,

$$[g_i w_i, ad^l_j \tilde{g}_i] \in \tilde{S}_i = \tilde{S}_i,$$

so

$$ad^{l+1}_j \tilde{g}_i \in \tilde{S}_{i+1}, \ i = 1, \ldots, m.$$

This shows that $\tilde{\theta} \in \tilde{S}_{i+1}$, and $\tilde{S}_{i+1} \subset \tilde{S}_{i+1}$.

To show $\tilde{S}_{i+1} \subset \tilde{S}_{i+1}$, let $v \in \tilde{S}_{i+1}$, then

$$v = \sum_{i=1}^{m} g_i \cdot ad^{l+1}_j g_i + \tilde{g}^l,$$

where $\tilde{g}^l \in \tilde{S}_i$, and $\gamma_i \in C(U)$, for some neighborhood $U$ of 0. But

$$ad^{l+1}_j g_i = [f, ad^l_j g_i]$$

$$= [f - \sum_{j=1}^{m} g_j w_j, ad^l_j g_i]$$

$$= [f, ad^l_j g_i] - \sum_{j=1}^{m} [g_j w_j, ad^l_j g_i].$$

Similar argument as before will show that

$$ad^{l+1}_j g_i \in \tilde{S}_{i+1}, \ i = 1, \ldots, m,$$
so \( v \in \tilde{S}_{l+1} \), and this shows that \( \tilde{S}_{l+1} \subset \tilde{S}_{l+1} \). Thus \( \tilde{S}_{l+1} = \tilde{S}_{l+1} \) and \( \tilde{S}_k = \tilde{S}_k \) for \( 0 \leq k \leq n \) by induction. If \( \tilde{S}_k \) is involutive for \( 0 \leq k \leq n \), we can show the same result by a similar proof. QED. 

Now we continue the proof that (B) implies (C) in proposition 2.1. Conditions (B-i) and (B-ii) implies that the distribution \( \tilde{S}_j \) is involutive for \( j = 0, \ldots, \kappa_1 - 1 \) because the Lie bracket of any two vector fields in \( \tilde{S}_j \) is zero. Hence by lemma 2.2, \( \tilde{S}_j = \tilde{S}_j \) and is involutive for \( j = 0, \ldots, \kappa_1 \). Furthermore, condition (B-iii) says that the \( n \) vector fields in \( \tilde{S}_{\kappa_1 - 1} \) are linearly independent on a neighborhood of 0; hence, \( \tilde{S}_j \) and \( S_j \) are nonsingular on a neighborhood of 0. This proves conditions (C-i) and (C-ii). Condition (C-iii) holds since \( \tilde{S}_{\kappa_1 - 1} = \tilde{S}_{\kappa_1 - 1} \) and the latter has dimension \( n \). Clearly \( \dim \tilde{S}_{n-1} = n \) since \( \tilde{S}_i \subset \tilde{S}_{i+1} \) for all \( i \geq 0 \) and \( \kappa_1 \leq n \).

(C) \( \Rightarrow \) (A). The proof was provided by Jakubczyk and Respondek [10]. We made few modifications except for some changes of notation and some added details. The following lemma was proved in [10].

**Lemma 2.3** Let \( \tilde{S}_0 \subset \tilde{S}_1 \subset \cdots \subset \tilde{S}_k \) denote a sequence of involutive \( C^\infty \) distribution on a \( n \)-manifold \( M \) having dimensions \( s_0 \leq s_1 \leq \cdots \leq s_k \) respectively. Then around any point \( x_0 \in M \) there exists a coordinate system \((x,U)\) such that the integral manifolds of \( \tilde{S}_j \) around 0 are of the form

\[
M_j = \{ p \in U : x_i(p) = c_i, \quad i = s_j + 1, \ldots, n \}, \quad \text{for} \quad j = 0, \ldots, \hat{k}
\]

where \( c_i \) are constants.

Hypotheses (C-i) and (C-ii) imply those of lemma 2.3. Hence there exists a coordinate system such that the integral manifolds of \( \tilde{S}_j \) are as described in the lemma. Here \( \hat{k} \) is the smallest integer such that \( \dim \tilde{S}_{\hat{k}} = n \) on a neighborhood of 0.
In the new coordinates, vector fields in $S_j$ have the last $(n - s_j)$ components equal to zero. For convenience, we use the same variable $x$ for the new coordinates and the same $f$ expressed in these new coordinates.

We define $r_0 = s_0$, and $r_i = s_i - s_{i-1}$, for $i \geq 1$, and let

$$ f = \begin{pmatrix} f^0 \\ f^1 \\ \vdots \\ f^k \end{pmatrix}, \quad x = \begin{pmatrix} x^0 \\ x^1 \\ \vdots \\ x^k \end{pmatrix}, $$

where

$$ f^i = \begin{pmatrix} f_{s_{i-1} + 1} \\ \vdots \\ f_{s_i} \end{pmatrix}, $$

and has $r_i$ components. In the new coordinates, system (2.1) has the form

$$ \begin{align*}
\dot{x}^0 &= f^0(x) + \sum_{i=1}^{m} g_i^0(x)u_i \\
\dot{x}^1 &= f^1(x) \\
&\vdots \\
\dot{x}^k &= f^k(x)
\end{align*} \tag{2.12} $$

where $f^i(0) = 0$, for $i = 0, \ldots, k$.

We will show that

$$ \frac{\partial f^j}{\partial x^i} = 0, \quad \text{for} \quad i = 0, \ldots, j - 2, \tag{2.13} $$

and

$$ \text{rank} \frac{\partial f^j}{\partial x^{j-1}} = r_j, \quad \text{for} \quad j = 1, \ldots, k. \tag{2.14} $$

Since $[f, g] = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g$, if we let

$$ [f, g] = \begin{pmatrix} [f, g]^0 \\ \vdots \\ [f, g]^k \end{pmatrix}, $$

we have
then

\[ [f, g]^j = \frac{\partial g^j}{\partial x} f - \frac{\partial f^j}{\partial x} g = \sum_{i=0}^{k} \left( \frac{\partial g^j}{\partial x^i} f^i - \frac{\partial f^j}{\partial x^i} g^i \right). \]

For \( j \geq 2 \), if \( g \in \bar{S}_{j-2} \), then

\[
g = \begin{pmatrix}
g^0 \\
\vdots \\
g^{j-2} \\
0 \\
\vdots \\
0
\end{pmatrix},
\]

and \([f, g] \in \bar{S}_{j-1}\), so

\[ 0 = [f, g]^j = \sum_{i=0}^{k} \frac{\partial g^j}{\partial x^i} f^i - \sum_{i=0}^{j-2} \frac{\partial f^j}{\partial x^i} g^i. \]

Since \( g^i = 0 \), the first sum is zero, hence

\[ \frac{\partial f^j}{\partial x^i} = 0, \quad i = 0, \ldots, j-2. \]

For \( j \geq 1 \), if \( g \in \bar{S}_{j-1} \), then \( g^i = 0 \), for \( i \geq j \), and \([f, g] \in \bar{S}_j\), so

\[ [f, g]^j = -\sum_{i=1}^{j-1} \frac{\partial f^j}{\partial x^i} g^i = -\frac{\partial f^j}{\partial x^{j-1}} g^{j-1}. \]

This implies rank \( \frac{\partial f^j}{\partial x^{j-1}} = r_j \), since there are \( r_j \) vector fields in \( \bar{S}_j \) that are linearly independent of vector fields in \( \bar{S}_{j-1} \).

We will show that system (2.12) can be transformed into the form

\begin{equation}
\begin{align*}
\dot{x}^0 &= v \\
\dot{x}^1 &= \dot{x}^1 \\
&\vdots \\
\dot{x}^k &= \dot{x}^k
\end{align*}
\end{equation}
where the variables in $\tilde{x}^i$ are the first $r_i$ variables in $x^{i-1}$. We see that system (2.15) is the permuted form of system (BCF).

First, consider the transformation

$$
\tilde{x}^{k-1} = \begin{pmatrix}
  f^k \\
x_2^{k-1}
\end{pmatrix}
$$

$$
\tilde{x}^i = x^i, \quad i \neq k - 1,
$$

(2.16)

where $x_2^{k-1}$ is a $(r_{k-1} - r_k)$ vector consisting of variables in $x^{k-1}$ such that, on a neighborhood of 0,

$$
\text{rank} \frac{\partial \tilde{x}^{k-1}}{\partial x^{k-1}} = r_{k-1}.
$$

This is possible by property (2.14). Moreover, the transformation $x \mapsto \tilde{x}$ is a local diffeomorphism around 0 since the Jacobian matrix $\frac{\partial \tilde{x}}{\partial x}$ is nonsingular around 0, Furthermore,

$$
\begin{align*}
\tilde{x}^{k-1} &= \frac{\partial \tilde{x}^{k-1}}{\partial x^{k-1}} \tilde{x}^{k-1} \\
&= \frac{\partial \tilde{x}^{k-1}}{\partial x^{k-1}} f^{k-1} \\
&\triangleq \hat{f}^{k-1},
\end{align*}
$$

and

$$
\frac{\partial \hat{f}^{k-1}}{\partial \tilde{x}^{k-2}} = \frac{\partial \tilde{x}^{k-1}}{\partial x^{k-1}} \frac{\partial f^{k-1}}{\partial \tilde{x}^{k-2}},
$$

because $\tilde{x}^{k-1}$ given by (2.16) is independent of $x^{k-2}$.

Since $\frac{\partial \tilde{x}^{k-1}}{\partial x^{k-1}}$ is a $(r_{k-1} \times r_{k-1})$ nonsingular matrix and $\frac{\partial f^{k-1}}{\partial \tilde{x}^{k-2}}$ has rank $r_{k-1}$, therefore

$$
\text{rank} \frac{\partial \hat{f}^{k-1}}{\partial \tilde{x}^{k-2}} = r_{k-1}.
$$

Moreover, using (2.13), we find that

$$
\frac{\partial \hat{f}^{k-1}}{\partial \tilde{x}^i} = 0, \quad i = 0, \ldots, \hat{k} - 3.
$$
Thus properties (2.13) and (2.14) are preserved. Replacing $\tilde{x}$ by $x$ again, the system has the form

$$\begin{align*}
\dot{x}^0 &= f^0 + G^0 u \\
\vdots \\
\dot{x}^{k-1} &= f^{k-1} \\
\dot{x}^k &= x_{i-1}^{k-1}
\end{align*}$$

where $x_{i-1}^{k-1}$ are the first $r_k$ components of $x^{k-1}$, and $G^0$ is a $m \times m$ matrix whose $i$th column is $g_i^0$.

If we repeat this process, then after another $k - 2$ steps, the transformed system is

$$\begin{align*}
\dot{x}^0 &= f^0 + G^0 u \\
\dot{x}^1 &= x_{i-1}^0 \\
\vdots \\
\dot{x}^k &= x_{i-1}^{k-1}
\end{align*}$$

where $x_{i-1}^i$ are the first $r_{i+1}$ components of $x^{i}$ for $i = 0, \ldots, k - 1$.

Now let $u = (G^0)^{-1}(v - f^0)$, the first equation becomes $\dot{x}^0 = v$. A further state coordinate change $x \mapsto Px$, where $P$ is an appropriate permutation matrix, will transform this last system into the system (BCF) described earlier. This completes the proof of proposition 2.1. QED. $\blacksquare$

**Remark.** Condition (B-i) can be simplified in the following lemma.

**Lemma 2.4** Under hypothesis (B-ii), i.e.,

$$\text{ad}^k_{\hat{g}_i} \hat{g}_i = 0, \ i = 1, \ldots, m.$$ 

The following statements are equivalent:

(i) $[\text{ad}^k_{\hat{g}_j} \hat{g}_j, \text{ad}^l_{\hat{g}_i} \hat{g}_i] = 0, \ 1 \leq i, j \leq m, k = 0, \ldots, \kappa_j - 1, l = 0, \ldots, \kappa_i - 1.$

(i') $[\text{ad}^k_{\hat{g}_j} \hat{g}_j, \hat{g}_i] = 0, \ 1 \leq i, j \leq m, k = 0, \ldots, \kappa_j - 1.$
**Proof:** (i) implies (i') is clear as condition (i') is a special case of (i) by setting $l = 0$. For the converse, we will prove (i) by induction on $l$. (i) is true for $l = 0$ by (i'). Assume (i) holds for some $l \leq \kappa_i - 2$ and for $k = 0, \ldots, \kappa_j - 1$. Then using the Jacobi identity,

$$[\text{ad}^k \hat{g}_j, \text{ad}^{l+1} \hat{g}_i] = [\hat{f}, [\text{ad}^k \hat{g}_j, \text{ad}^l \hat{g}_i]] - [\text{ad}^{k+1} \hat{g}_j, \text{ad}^l \hat{g}_i].$$

By assumption, the first term is zero, and if $k + 1 \leq \kappa_j - 1$ or $k + 1 = \kappa_j$, then the second term is zero by the inductive hypothesis or by condition (ii) respectively. Hence (i) holds for $l + 1$ and therefore holds for $l = 0, \ldots, \kappa_i - 1$ by induction. QED.

Consequently, condition (B) can be replaced by condition (B') consisting of (i') in lemma 2.4, (B-ii), and (B-iii).

**Corollary 2.5** Conditions (A), (B'), and (C) are equivalent.

**Remark.** Since $\tilde{S}_j = S_j$ for $j \geq 0$, then

$$r_i = \text{dim } \tilde{S}_i - \text{dim } \tilde{S}_{i-1} = \text{dim } \tilde{S}_i - \text{dim } \tilde{S}_{i-1}, \text{ for } i \geq 1$$

and $r_0 = \text{dim } \tilde{S}_0 = \text{dim } \tilde{S}_0$.

It can be checked that the sets of integers $\{r_i\}_{i=0}^{n-1}$ and $\{\kappa_j\}_{j=1}^{m}$ are related as follows:

$$r_i = \text{the number of } \kappa_j\text{'s that are } \geq i + 1$$

$$\kappa_j = \text{the number of } r_i\text{'s that are } \geq j.$$ 

(2.17)

To show this, construct the array of vector fields in $S_{\kappa_{1-1}}$ as shown below:

\[
\begin{pmatrix}
X_{1}^{0} & X_{2}^{0} & X_{3}^{0} & \cdots & X_{m}^{0} \\
X_{1}^{1} & X_{2}^{1} & \vdots & \cdots & \vdots \\
\vdots & \vdots & \vdots & \cdots & X_{m}^{\kappa_{m}-1} \\
\vdots & \vdots & X_{3}^{\kappa_{3}-1} & \cdots & Y \\
X_{1}^{\kappa_{1}-1} & X_{2}^{\kappa_{2}-1} & Y & \cdots & Y.
\end{pmatrix}
\]

(2.18)

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where $X^j_i$ denotes $\text{ad}^j_i g_i$ and $Y$ corresponds to vector fields that are linearly dependent on those to the left or above.

The number of entries of $X^j_i$ in the $i$th row is $r_{i-1}$ for $i \geq 1$, and the number of $X^j_i$ in the $j$th column is $\kappa_j$. It is easy to check that the $r_i$'s and $\kappa_j$'s are related as claimed. Moreover, the $r_i$'s satisfy:

$$r_0 \geq r_1 \geq \cdots \geq r_{\kappa_1 - 1} \geq 1, \quad r_i = 0, \quad \text{for} \quad i \geq \kappa_1, \quad \text{and} \quad \sum_{i=0}^{\kappa_1 - 1} r_i = n.$$

Incidentally, Isidori, and Hunt, Su, and Meyer gave similar but weaker conditions. We will state and explain them below. First, Isidori's conditions [9, p.237] are as follows:

(D)

(i) the distribution $\bar{S}_i$ is involutive for all $i \geq 0$ such that $m_{k-i-1} \neq 0$, where the integers $m_0, \ldots, m_k$ are defined by

$$m_0 = r_k$$
$$m_0 + m_1 = r_{k-1}$$
$$\vdots$$
$$m_0 + m_1 + \cdots + m_k = r_0.$$

(ii) $\bar{S}_i$ is nonsingular on a neighborhood of $0$ for all $i \geq 0$.

(iii) $\text{dim} \bar{S}_k(0) = n$.

Next, Hunt, Su, and Meyer's conditions [7] are as follows:

(E)

(i) the distribution $\bar{S}_{\kappa_i-2}$ is involutive for $i = 1, \ldots, m$.

(ii) $\bar{S}_{\kappa_i-2} = \text{sp}\{v : v \in S_{\kappa_i-2} \cap S\}$, for $i = 1, \ldots, m$, where $S = \{\text{ad}^j_i g_i : i = 1, \ldots, m, \quad j = 0, \ldots, \kappa_i - 1\}$.

(iii) $\bar{S} = \text{sp}\{v \in S\}$ spans an $n$ dimensional space on a neighborhood of $0$. 

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Remark. A reordering of the vector fields $g_1,\ldots,g_m$ may be required in condition (E).

It can be seen that conditions (ii) and (iii) in (D) and (C) are the same and are equivalent to those in (E). We will show that (D-i) and (E-i) are also equivalent.

The integers $m_i$ defined in (D) can be expressed as

$$m_i = r_k - r_{k-i+1}, \quad \text{for} \quad i = 1,\ldots,k,$$

or

$$m_{k-l-1} = r_{l+1} - r_{l+2}, \quad \text{for} \quad l = -1,\ldots,k-2,$$

and $m_0 = r_k$.

So $m_{k-l-1} \neq 0$ iff $r_{l+1} - r_{l+2} > 0$.

If $\kappa_i$ and $r_j$ are as defined in (2.17) and the vector fields are arranged as in (2.18), we see that

$$\kappa_{r_{l+2}} > \kappa_i, \quad \text{for} \quad i = r_{l+2} + 1,\ldots,r_{l+1}.$$

In this case

$$\kappa_i = l + 2, \quad \text{for} \quad i = r_{l+2} + 1,\ldots,r_{l+1}$$

or

$$l = \kappa_i - 2, \quad \text{for} \quad i = r_{l+2} + 1,\ldots,r_{l+1},$$

so the distribution $\bar{S}_i$ that are required to be involutive in (D-i) have the form $\bar{S}_{\kappa_i-2}$ for $i = 1,\ldots,m$, which is condition (E-i). We note that in (E-i) the same distribution will be enumerated twice if two $\kappa_i$ are the same. Hence, Isidori’s conditions are equivalent to those of Hunt, Su, and Meyer.

The equivalence of condition (C-i) and (D-i) is proved in the following lemma.
Lemma 2.6 If \( \tilde{S}_i \) is nonsingular on a neighborhood \( U \) of 0 for \( i = 0, \ldots, \kappa_1 - 1 \), then the following conditions are equivalent:

(C-i) \( \tilde{S}_i \) is involutive for \( i = 0, \ldots, \kappa_1 - 2 \).

(D-i) \( \tilde{S}_{\kappa_1 - 2} \) is involutive for \( i = 1, \ldots, m \).

Proof: Clearly, (C-i) implies (D-i). To prove the converse, it suffices to consider the case with two distinct \( \kappa_i \). Hence assume that

\[ \kappa_1 = \cdots = \kappa_l > \kappa_{l+1} = \cdots = \kappa_m, \]

for some \( l \) between 1 and \( m \).

With no loss of generality, assume that the vector fields are arranged as in (2.18), i.e.

\[
\begin{pmatrix}
X_1^0 & \cdots & X_l^0 & X_{l+1}^0 & \cdots & X_m^0 \\
X_1^1 & \cdots & X_l^1 & \vdots & \cdots & \vdots \\
\vdots & \vdots & \vdots & X_{l+1}^{\kappa_m - 1} & \cdots & X_m^{\kappa_m - 1} \\
X_1^{\kappa_1 - 1} & \cdots & X_l^{\kappa_1 - 1} & Y & \cdots & Y
\end{pmatrix}
\]

By the nonsingularity assumption of \( \tilde{S}_i \) on \( U \) for \( i = 0, \ldots, \kappa_1 - 1 \), the \( Y \)'s correspond to the vector fields that are linearly dependent on \( U \) on those to the left and above. We can assume that \( \kappa_m > 2 \), since if \( \kappa_m \leq 2 \), then the second part of the following proof is adequate. So condition (D-i) says that \( \tilde{S}_{\kappa_m - 1} \) and \( \tilde{S}_{\kappa_1 - 1} \) are involutive. Suppose \( \tilde{S}_{\kappa_m - 3} \) is not involutive, then there exist vector fields, \( v_i, v_j \in \tilde{S}_{\kappa_m - 3} \) such that

\[ [v_i, v_j] \notin \tilde{S}_{\kappa_m - 3}. \]

Since \( \tilde{S}_{\kappa_m - 2} \) is involutive, then

\[ [v_i, v_j] = \sum_{k=1}^{m} \alpha_k \cdot \text{ad}_{g_k}^{\kappa_m - 2} g_k + \bar{g}_{\kappa_m - 3}, \tag{2.19} \]

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where \( \alpha_k \in C(U) \) and not all \( \alpha_k \) equal to zero, and as before \( \tilde{g}^i \in \tilde{S}_i \) for all \( i \geq 0 \). Using the Jacobi identity,

\[
[f, [v_i, v_j]] = [[f, v_i], v_j] + [v_i, [f, v_j]]
\]

where

\[
[f, v_i], \quad [f, v_j] \in \tilde{S}_{\kappa_m-2}, \text{ since } v_i, v_j \in \tilde{S}_{\kappa_m-3}
\]

we have

\[
[[f, v_i], v_j], \quad [v_i, [f, v_j]] \in \tilde{S}_{\kappa_m-2},
\]

by involutivity of \( \tilde{S}_{\kappa_m-2} \). Hence,

\[
[f, [v_i, v_j]] \in \tilde{S}_{\kappa_m-2}. \tag{2.20}
\]

But from (2.19)

\[
[f, [v_i, v_j]] = [f, \sum_{k=1}^{m} \alpha_k \cdot \operatorname{ad}^\kappa_{f^{-2}} g_k + \tilde{g}^\kappa_{m-3}]
\]

\[
= \sum_{k=1}^{m} \alpha_k \cdot \operatorname{ad}^\kappa_{f^{-1}} g_k + \sum_{k=1}^{m} \operatorname{ad}^\kappa_{f^{-2}} g_k \mathcal{L}_f(\alpha_k) + [f, \tilde{g}^\kappa_{m-3}]
\]

\[
= \sum_{k=1}^{m} \alpha_k \cdot \operatorname{ad}^\kappa_{f^{-1}} g_k + \tilde{g}^\kappa_{m-2}.
\]

Since at least one \( \alpha_k \) is not zero and \( \operatorname{ad}^\kappa_{f^{-1}} g_1, \ldots, \operatorname{ad}^\kappa_{f^{-1}} g_m \) are linearly independent of vector fields in \( \tilde{S}_{\kappa_m-2} \). Therefore

\[
[f, [v_i, v_j]] \notin \tilde{S}_{\kappa_m-2}.
\]

This contradicts (2.20). Hence \( \tilde{S}_{\kappa_m-3} \) is involutive. By induction \( \tilde{S}_i \) is involutive for all \( i \leq \kappa_m - 3 \).

Now suppose \( \kappa_1 - \kappa_m \geq 2 \) so that \( \kappa_1 - 3 \geq \kappa_m - 1 \), otherwise if \( \kappa_1 - \kappa_m = 1 \), then \( \kappa_1 - 2 = \kappa_m - 1 \), and \( \tilde{S}_{\kappa_1-2} \) is involutive by hypothesis; hence, no proof is needed.
Suppose $\tilde{S}_{\kappa_1-3}$ is not involutive, then there exist vector fields $v_i, v_j \in \tilde{S}_{\kappa_1-3}$ such that not both are in $\tilde{S}_{\kappa_m-2}$ and

$$[v_i, v_j] \notin \tilde{S}_{\kappa_1-3}.$$ 

But $\tilde{S}_{\kappa_1-2}$ is involutive by hypothesis, so

$$[v_i, v_j] = \sum_{k=1}^{l} \gamma_k \cdot \text{ad}^{\kappa_1-2}_f g_k + \tilde{g}^{\kappa_1-3},$$

where $\gamma_k \in C(U)$ and not all $\gamma_k$ equal to zero. Taking Lie derivative of $[v_i, v_j]$ with respect to $f$ as before and using the fact that $\text{ad}^{\kappa_1-1}_f g_1, \ldots, \text{ad}^{\kappa_1-1}_f g_l$ are linearly independent of vector fields in $\tilde{S}_{\kappa_1-2}$, we find that this leads to a contradiction. Hence, $\tilde{S}_{\kappa_1-3}$ is involutive, and, if we repeat the same argument for $\tilde{S}_{\kappa_1-4}$ and so on, we find that $\tilde{S}_i$ is involutive for $i = \kappa_m - 1, \ldots, \kappa_1 - 3$. Hence $\tilde{S}_i$ is involutive for $i = 0, \ldots, \kappa_1 - 2$. QED.

Remark. Looking at the proof of lemma 2.2, one might think that condition (D-i) can be replaced by the condition that $\tilde{S}_{\kappa_1-2}$ be involutive. But this is not true since involutivity of $\tilde{S}_{\kappa_m-1}$ does not imply the involutivity of $\tilde{S}_{\kappa_m-2}$. To show this, consider a contrapositive proof as before and assume that $\tilde{S}_{\kappa_m-2}$ is not involutive given that $\tilde{S}_{\kappa_m-1}$ is. So there exist $v_i, v_j \in \tilde{S}_{\kappa_m-2}$ such that

$$[v_i, v_j] = \sum_{k=1}^{m} \alpha_k \cdot \text{ad}^{\kappa_m-1}_f g_k + \tilde{g}^{\kappa_m-2},$$

and not all $\alpha_k$ equal to zero. But if $\alpha_k = 0$ for $k = 1, \ldots, l$, then

$$[f, [v_i, v_j]] = \sum_{k=l+1}^{m} \alpha_k \cdot \text{ad}^{\kappa_m}_f g_k + \tilde{g}^{\kappa_m-1}.$$ 

Since

$$\text{ad}^{\kappa_m}_f g_k \in \text{sp}\{\text{ad}^{\kappa_m}_f g_j : j = 1, \ldots, l\} + \tilde{S}_{\kappa_m-1}, \quad \text{for} \quad k = l + 1, \ldots, m,$$

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so if \( sp\{ad_f^m g_k : k = l + 1, \ldots, m\} \subseteq \tilde{S}_{\kappa m - 1} \),
then \([f, [u_i, v_j]] \in \tilde{S}_{\kappa m - 1} \), and this will not lead to a contradiction.

In summary, we have shown that (B), (B') (C), (D), and (E) are equivalent conditions such that the nonlinear system (2.1) can be linearized to a system in Brunovsky canonical form.

Incidentally, the way that (D) and (E) are proved is based on the study of the map \( \xi = \tilde{\xi}(x) \) or simply \( \xi(x) \) so that in the \( \xi \) coordinates

\[
\dot{\xi} = \tilde{A}\xi + \tilde{B}v.
\]

Since this provides a different proof of conditions (C), (D), or (E), we will derive the necessary conditions that \( \xi(x) \) has to satisfy.

First, differentiating \( \xi(x) \) with respect to \( t \), we have

\[
\frac{\partial \xi}{\partial x} \dot{z} = \tilde{A}\xi + \tilde{B}v
\]
or

\[
\frac{\partial \xi}{\partial x}(f + \sum_{i=1}^{m} g_i u_i) = \tilde{A}\xi + \tilde{B}v. \tag{2.21}
\]

Recalling the form of \( \tilde{A} \) and \( \tilde{B} \), and since \( \xi(x) \) does not depend on \( u_i, i = 1, \ldots, m \), therefore, we have

\[
L_f(\xi) = \xi_{l+1}, \quad l = 1, \ldots, \sigma_1 - 1, \sigma_1 + 1, \ldots, \sigma_2 - 1, \sigma_2 + 1, \ldots, n - 1 \tag{2.22}
\]

and

\[
\langle d\xi_{\sigma_j}, f \rangle + \sum_{i=1}^{m} u_i \langle d\xi_{\sigma_j}, g_i \rangle = v_j, \quad \text{for} \quad j = 1, \ldots, m \tag{2.23}
\]

\[
\langle d\xi_i, g_i \rangle = 0, \quad \text{for} \quad i = 1, \ldots, m, \tag{2.24}
\]

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and for \( l = 1, \ldots, \sigma_1 - 1, \sigma_1 + 1, \ldots, \sigma_2 - 1, \sigma_2 + 1, \ldots, n - 1 \). For convenience, we use double subscripts notation. For \( i = 1, \ldots, m \), let

\[
\xi_{i\kappa_i} = \xi_{\sigma_i}
\]

\[
\xi_{i\kappa_i - 1} = \xi_{\sigma_i - 1}
\]

\[
\vdots
\]

\[
\xi_{i1} = \xi_{\sigma_i - 1 + 1},
\]

then (2.22), (2.23), and (2.24) become

\[
L_{f}^{k-1}(\xi_{j1}) = \xi_{jk}, \quad j = 1, \ldots, m, k = 2, \ldots, \kappa_j
\]

(2.25)

\[
\langle d\xi_{jk}, f \rangle + \sum_{i=1}^{m} u_i \langle d\xi_{jk}, g_i \rangle = v_j, \quad 1 \leq i, j \leq m
\]

(2.26)

\[
\langle d\xi_{jk}, g_i \rangle = 0, \quad 1 \leq i, j \leq m, k = 1, \ldots, \kappa_j - 1.
\]

(2.27)

Substituting (2.25) into (2.26) and (2.27), we have

\[
\langle dL_{f}^{k-1}(\xi_{j1}), f \rangle + \sum_{i=1}^{m} u_i \langle dL_{f}^{k-1}(\xi_{j1}), g_i \rangle = v_j, \quad 1 \leq i, j \leq m
\]

(2.28)

and

\[
\langle dL_{f}^{k}(\xi_{jk}), g_i \rangle = 0, \quad 1 \leq i, j \leq m, k = 0, \ldots, \kappa_j - 2.
\]

(2.29)

Therefore, we need to find \( m \) functions \( \xi_{11}, \xi_{21}, \ldots, \xi_{m1} \) such that (2.29) is satisfied. Moreover, the matrix

\[
M^0 = (m_{ij}^0), \text{where } m_{ij}^0 = \langle dL_{f}^{k-1}(\xi_{i1}), g_j \rangle,
\]

is required to be nonsingular on a neighborhood of 0 in order that \( u_1, \ldots, u_m \), be solvable from (2.28). These conditions are stated as (a) in the following lemma.
Lemma 2.7  The following statements are equivalent:

(a)

(i) 
\[ \langle dL_f^k(\xi_i), g_j \rangle = 0, \quad 1 \leq i, j \leq m, \quad k = 0, \ldots, \kappa_i - 2. \]

(ii) the \( m \times m \) matrix \( M^0 = (m^0_{ij}) \) is nonsingular on a neighborhood \( U \) of 0, where
\[ m^0_{ij} = \langle dL_f^{\kappa_i-1}(\xi_i), g_j \rangle. \]

(b)

(i) 
\[ \langle dL_f^k(\xi_i), \text{ad}_f^l g_j \rangle = 0, \quad \text{for} \quad 1 \leq i, j \leq m, \]
and for \( l = 0, \ldots, \kappa_i - 2, \quad k = 0, \ldots, \kappa_i - 2 - l. \)

(ii) For \( l = 0, \ldots, \kappa_1 - 1 \), the \( r_l \times m \) matrix \( M^l = (m^l_{ij}) \) has rank \( r_l \) on \( U \), where
\[ m^l_{ij} = \langle dL_f^{\kappa_i-l-1}(\xi_i), \text{ad}_f^l g_j \rangle. \]
Furthermore, \( M^l = (-1)^l M^0 \), for \( l = 0, \ldots, \kappa_1 - 1 \), where \( M^0_l \) is the first \( r_l \) rows of \( M^0 \).

Proof: Clearly, (b) \( \Rightarrow \) (a). The converse can be proved by induction. By (a-i), (b-i) is true for \( l = 0 \). Assume (b-i) holds for \( l \leq p < \kappa_i - 2, \quad k \leq \kappa_i - 2 - p \) for some \( p \), and for \( i = 1, \ldots, m \), then
\[ \langle dL_f^k(\xi_i), \text{ad}_f^{p+1} g_j \rangle = L_f \langle dL_f^k(\xi_i), \text{ad}_f^p g_j \rangle - \langle dL_f^{k+1}(\xi_i), \text{ad}_f^p g_j \rangle \]
\[ = 0, \quad \text{for} \quad k \leq \kappa_i - 2 - (p + 1). \]

This follows from the induction hypothesis since \( k + 1 \leq \kappa_i - 2 - p \). Hence (b-i) holds for \( l \leq \kappa_i - 2 \) by induction. Again, (b-ii) is proved by induction. Clearly
(b–ii) is true for \( l = 0 \) by (a–ii). Assume (b–ii) holds for \( l \leq p < \kappa_1 - 1 \) for some \( p \), i.e.

\[
M^l = (-1)^l M^0_l, \quad \text{for} \quad l \leq p < \kappa_1 - 1.
\]

Then for \( i \leq r_{p+1} \)

\[
m^p_{ij} = \langle dL^\kappa_i \omega^{p+1} (\xi_i), \text{ad}^p_j g_j \rangle
\]

\[= L_f \langle dL^\kappa_i \omega^{p-2} (\xi_i), \text{ad}^p_j g_j \rangle - \langle dL^\kappa_i \omega^{p-1} (\xi_i), \text{ad}^p_j g_j \rangle.\]

By property (b–i), the first term is zero, since \( p \leq \kappa_i - 2 \), therefore

\[
m^p_{ij} = -\langle dL^\kappa_i \omega^{p-1} (\xi_i), \text{ad}^p_j g_j \rangle
\]

\[= -m^p_{ij}, \quad \text{for} \quad 1 \leq i \leq r_{p+1}\]

hence

\[
M^{p+1} = (-1)^p M^0_{p+1} = (-1)^{p+1} M^0_{p+1}.
\]

This proves (b–ii), that is, \( M^l = (-1)^l M^0_l \), for \( l = 0, \ldots, \kappa_1 - 1 \). Obviously \( M^l \) has full rank on \( U \) if \( M^0 \) is nonsingular on \( U \). QED.

The necessity of conditions (C), (D), or (E) can be proved as follows:

Let

\[
Y_l = \begin{pmatrix}
\langle dL^\kappa_i \omega^{l-1} (\xi_{11}) \\
\vdots \\
\langle dL^\kappa_i \omega^{l-1} (\xi_{il}) 
\end{pmatrix}, \quad \text{for} \quad l = 0, \ldots, \kappa_1 - 1.
\]

Remark. \( Y_l \) is an \( r_l \times n \) matrix where \( r_l \) is the number of \( \kappa_i \geq l + 1 \). The rows in \( Y_1, \ldots, Y_{\kappa_1 - 1} \) are the rows of the matrix \( \frac{\partial \xi}{\partial x} \), so they are linearly independent on \( U \).

Let

\[
L^k_j G = (\text{ad}^k_j g_1, \ldots, \text{ad}^k_j g_m), \quad \text{for} \quad k = 0, \ldots, \kappa_1 - 1.
\]
Then conditions (b-i) and (b-ii) in lemma 2.7 can be written as

\[
\begin{pmatrix}
Y_0 \\
Y_1 \\
\vdots \\
Y_{\kappa_1-1}
\end{pmatrix}
\begin{pmatrix}
G, L_f G, \cdots, L_f^{\kappa_1-1} G
\end{pmatrix}
= \begin{pmatrix}
M^0 & \times & \times & \times \\
O & -M_i^0 & \times & \times \\
\vdots & \vdots & \vdots & \vdots \\
O & O & \cdots & (-1)^{\kappa_1-1} M_{\kappa_1-1}^0
\end{pmatrix}
\] (2.30)

where \( M_i^0 \) is an \( r_i \times m \) matrix consisting of the first \( r_i \) rows of \( M^0 \). In particular, \( \text{rank} \ M_i^0 = r_i \) on \( U \). Let

\[
\tilde{G}_j = (G, L_f G, \cdots, L_f^j G),
\]

\[
\tilde{Y}_j = \begin{pmatrix}
Y_0 \\
\vdots \\
Y_j
\end{pmatrix}, \quad \hat{Y}_j = \begin{pmatrix}
Y_{j+1} \\
\vdots \\
Y_{\kappa_1-1}
\end{pmatrix}
\]

and

\[
\tilde{M}_j = \begin{pmatrix}
M^0 & \times \\
\vdots & \vdots \\
O & \cdots
\end{pmatrix}
\] (2.31)

Then from (2.30)

\[
\tilde{Y}_j \tilde{G}_j = \tilde{M}_j,
\] (2.31)

and

\[
\hat{Y}_j \tilde{G}_j = 0,
\] (2.32)

for \( j = 0, \ldots, \kappa_1 - 1 \). Since the number of rows in \( \hat{Y}_j \) is \( \sum_{i=0}^j r_i = s_j \), and they are linearly independent covector fields on \( U \), so on \( U \)

\[
\text{rank} \ \hat{Y}_j = s_j
\]

and

\[
\text{rank} \ \tilde{Y}_j = n - s_j.
\]

Since

\[
\text{rank} \ \tilde{M}_j = \text{rank} \ \hat{Y}_j = s_j,
\]
then (2.31) implies

\[ \text{rank } \tilde{G}_j \geq s_j, \]

and (2.32) implies

\[ \text{rank } \tilde{G}_j \leq n - (n - s_j) = s_j. \]

Hence,

\[ \text{rank } \tilde{G}_j = s_j, \quad \text{for } j = 0, \ldots, \kappa_1 - 1. \]

Since

\[ \tilde{S}_j = \text{sp}\{ g : g \text{ belongs to a column of } \tilde{G}_j \}, \]

then \( \tilde{S}_j \) is nonsingular on \( U \) with dimension equal to \( s_j \). In particular,

\[ \dim \tilde{S}_{\kappa_1 - 1} = s_{\kappa_1 - 1} = \sum_{i=0}^{\kappa_1 - 1} r_i = n. \]

This proves the necessity of conditions (ii) and (iii) in (C), (D), and (E). To prove the involutivity requirement, we use the following lemma [9, p.21].

**Lemma 2.8** A nonsingular distribution \( \mathcal{S} \) of dimension \( k \) is involutive iff \( \mathcal{S}^\perp \) is locally spanned by \( n - k \) exact one forms.

From (2.32), for \( j = 0, \ldots, \kappa_1 - 1, \), \( \tilde{S}_j \) annihilates \( n - s_j \) linearly independent covector fields which are exact one forms. Hence, by lemma 2.8, it is necessary that \( \tilde{S}_j \) be involutive for \( j = 0, \ldots, \kappa_1 - 1 \). This completes the proof of the necessity of the condition (C), which are, by previous proof, equivalent to (B), (D), and (E).

Likewise, the sufficiency of the hypothesis (C) can be proved in a way different from that by Jakubczyk and Respondek. The idea is to show that there exists \( m \) functions \( \xi_1, \ldots, \xi_m \), such that the \( n \) covector fields \( dL_{\gamma}^{k}(\xi_i) \), for \( i = 1, \ldots, m, k = 0, \ldots, \kappa_i - 1 \) are linearly independent on a neighborhood of 0,
and satisfy condition (a) or (b) in lemma 2.7. Then the feedback pair \((w, Q)\) are constructed so that in the coordinates \(\xi_{ik}\) defined by

\[
\xi_{ik} = L_j^{-1}(\xi_i), \quad \text{for} \quad i = 1, \ldots, m, \quad k = 1, \ldots, \kappa_i,
\]

the system is in Brunovsky canonical form. The details of this proof are given in Isidori's book [9, p.237]. We observed that Hunt, Su, and Meyer [7] give a way to construct the functions \(\xi_1, \ldots, \xi_m\).

**Remark.** Frequently we need to solve a system of partial differential equations; the condition for the solvability of such a system is stated in the following proposition (Spivak [15], p.254).

**Proposition 2.9** Let \(U \times V \subset \mathbb{R}^m \times \mathbb{R}^n\) be open, where \(U\) is a neighborhood of \(0 \in \mathbb{R}^m\), and let \(f^j : U \times V \rightarrow \mathbb{R}^n\) be \(C^\infty\) functions, \(j = 1, \ldots, m\). Then for every \(x_0 \in V\), there exists a unique solution \(x : W \rightarrow V\) defined in a neighborhood \(W\) of \(0 \in \mathbb{R}^m\), satisfying

\[
x(0) = x_0
\]

\[
\frac{\partial x}{\partial y_j} = f^j(y, x(y)), \quad j = 1, \ldots, m, \quad \forall y \in W
\]

iff there is a neighborhood of \((0, x_0) \in U \times V\) on which

\[
\frac{\partial f^i}{\partial y_i} + \frac{\partial f^j}{\partial x} f^i = \frac{\partial f^i}{\partial y_j} + \frac{\partial f^i}{\partial x} f^j, \quad 1 \leq i, j \leq m.
\]

**Remark.** (2.34) is the condition obtained by setting the mixed partials of \(x\) equal and is called the integrability condition. As a special case, if \(f^j\) is independent of \(y\) for all \(j\), then (2.34) becomes

\[
\frac{\partial f^j}{\partial x} f^i = \frac{\partial f^i}{\partial x} f^j, \quad 1 \leq i, j \leq m,
\]

\([f^j, f^i] = 0, \quad 1 \leq i, j \leq m\).
We will outline the procedure for solving system (2.33). First solve the ordinary differential equation

\[ \frac{dx^1(y_1)}{dy_1} = f^1(y_1, 0, \ldots, 0, x^1(y_1, 0, \ldots, 0)) \]
\[ x^1(0) = x_0. \]

Next solve the ordinary differential equation

\[ \frac{dx^2(y_1, y_2)}{dy_2} = f^2(y_1, y_2, 0, \ldots, 0, x^2(y_1, y_2, 0, \ldots, 0)) \]
\[ x^2(y_1, 0) = x^1(y_1), \]

and so on. The last ordinary differential equation to be solved is

\[ \frac{dx^m(y_1, \ldots, y_m)}{dy_m} = f^m(y_1, \ldots, y_m, x^m(y_1, \ldots, y_m)) \]
\[ x^m(y_1, \ldots, y_{m-1}, 0) = x^{m-1}(y_1, \ldots, y_{m-1}). \]
§2.2 Example

In this section we will exhibit an example concerning the result of the previous section. In particular, we will use condition \((B')\) to solve for the feedback pair \((w, Q)\) after we check that hypothesis \((C)\) is satisfied. Then the method outlined after proposition 2.9 is applied to solve a system of partial differential equations for the map \(\hat{x}(\xi)\). We also use the fact that if \(g_1(x), \ldots, g_m(x)\), are linearly independent vector fields on some open set \(U\), and if

\[
\sum_{i=1}^{m} c_i(x)g_i(x) = 0, \quad \forall x \in U,
\]

then

\[
c_i(x) = 0, \quad \forall x \in U, \quad i = 1, \ldots, m.
\]

To show this, suppose there is a \(x_1 \in U\) such that \(c_i(x_1) \neq 0\) for some \(i\), then

\[
\sum_{i=1}^{m} c_i(x_1)g_i(x_1) = 0
\]

would contradict the linearly independence of \(g_1(x_1), \ldots, g_m(x_1)\).

Example:

Consider system (2.1) with \(n = 3, m = 2\), and

\[
f = \begin{pmatrix} \sqrt{1-x_1^2} (x_2 - \ln(x_3 + 1)) \\ \sin x_1 \\ 0 \end{pmatrix},
\]

\[
g_1 = \begin{pmatrix} 0 \\ \cos x_1 \\ 0 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 \\ 0 \\ (x_3 + 1)e^{x_2} \end{pmatrix}.
\]
For simplicity, we first employ the following transformation. Let
\[
u = \begin{pmatrix} -\tan x_1 \\ \sec x_1 \\ 0 \end{pmatrix} + \begin{pmatrix} \sec x_1 & 0 \\ 0 & e^{-x_2/(x_3 + 1)} \end{pmatrix} v, \tag{2.2.2}
\]
so that system (2.1) becomes
\[
\dot{x} = \tilde{f}(x) + \tilde{g}_1(x)v_1 + \tilde{g}_2(x)v_2
\]
where
\[
\tilde{f} = \begin{pmatrix} \sqrt{1 - x_1^2}(x_2 - \ln(x_3 + 1)) \\ 0 \\ 0 \end{pmatrix}, \quad \tilde{g}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \tilde{g}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{2.2.3}
\]
For convenience, we drop the \(^\sim\) sign and replace \(v\) by \(u\). We find, after some computation,
\[
\text{ad}_f g_1 = \begin{pmatrix} \sqrt{1 - x_1^2} \\ 0 \\ 0 \end{pmatrix}, \quad \text{ad}_f g_2 = \begin{pmatrix} \frac{-\sqrt{1 - x_1^2}}{x_3 + 1} \\ 0 \\ 0 \end{pmatrix} = -\frac{1}{x_3 + 1} \text{ad}_f g_1 \tag{2.2.4}
\]
so \(g_1, g_2, \text{ad}_f g_1\) are linearly independent on \(U = \{x : |x_1| < 1, x_3 > -1\}\). Also \([g_1, g_2] = 0\). Hence \(\tilde{S}_0\) is involutive and hypothesis (C) is satisfied with \(r_0 = 2, r_1 = 1, \kappa_1 = 2, \kappa_2 = 1\).

Next we have to find linearly independent vector fields \(\hat{g}_1, \hat{g}_2, \text{ad}_f \hat{g}_1\) such that the Lie bracket between any two is zero and
\[
\text{ad}_f \hat{g}_2 = 0, \quad \text{ad}_f^2 \hat{g}_1 = 0,
\]
where \(\hat{f} = f + g_1 w_1 + g_2 w_2\), and \(\hat{g}_i = g_1 q_{1i} + g_2 q_{2i}, \quad i = 1, 2\), cf. (B'). The aim is to find \(w\) and \(Q\) so that we can then find the linearizing transformation \(\hat{x}(\xi)\). We can also assume that \(q_{21} = 0\) since
\[
\text{ad}_f \hat{g}_1 = [\hat{f}, \hat{g}_1] = (\text{ad}_f g_1) q_{11} + (\text{ad}_f g_2) q_{21} + \tilde{g}^0
\]
where \( g^0 \in \mathcal{S}_0 \), and since \( \text{ad}_fg_2 \in sp\{g_1, g_2, \text{ad}_fg_1\} \). Now solve

\[
0 = [\hat{g}_1, \hat{g}_2]
\]

\[
= [g_1 q_{11}, g_1 q_{12} + g_2 q_{22}]
\]

\[
= g_1 \left( \frac{\partial g_{12}}{\partial x} g_1 \right) q_{11} - g_1 \left( \frac{\partial q_{11}}{\partial x} g_1 \right) q_{12}
+ g_2 \left( \frac{\partial q_{22}}{\partial x} g_1 \right) q_{11} - g_1 \left( \frac{\partial q_{11}}{\partial x} g_2 \right) q_{22}.
\]

Since \( g_1, g_2 \) are linearly independent, so setting their coefficients equal to zero, we have

\[
\frac{\partial q_{22}}{\partial x} g_1 = 0,
\]

and

\[
\left( \frac{\partial q_{12}}{\partial x} g_1 \right) q_{11} - \left( \frac{\partial q_{11}}{\partial x} g_1 \right) q_{12} - \left( \frac{\partial q_{11}}{\partial x} g_2 \right) q_{22} = 0,
\]

or

\[
\frac{\partial q_{22}}{\partial x_2} = 0,
\]

and

\[
\frac{\partial q_{12}}{\partial x_2} q_{11} - \frac{\partial q_{11}}{\partial x_2} q_{12} - \frac{\partial q_{11}}{\partial x_2} q_{22} = 0. \tag{2.2.5}
\]

Assuming \( w_1 = w_2 = 0 \), then \( \hat{f} = f \). Next solve

\[
0 = [\hat{f}, \hat{g}_2]
\]

\[
= [f, g_1 q_{12} + g_2 q_{22}]
\]

\[
= (\text{ad}_fg_1) q_{12} + g_1 \left( \frac{\partial q_{12}}{\partial x} f \right)
+ (\text{ad}_fg_2) q_{22} + g_2 \left( \frac{\partial q_{22}}{\partial x} f \right).
\]

Using (2.2.4), and setting coefficients of \( \text{ad}_fg_1, g_1, g_2 \) equal to zero, we have

\[
q_{12} - q_{22} / (x_3 - 1) = 0, \quad \frac{\partial q_{12}}{\partial x} f = 0, \text{ and } \frac{\partial q_{22}}{\partial x} f = 0,
\]

or

\[
q_{22} = q_{12} (x_3 + 1), \quad \frac{\partial q_{12}}{\partial x_1} = 0, \text{ and } \frac{\partial q_{22}}{\partial x_1} = 0.
\]

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Next we solve

\[ 0 = [\dot{g}_1, \text{ad}_f \dot{g}_2] \]
\[ = [g_1 q_{11}, (\text{ad}_f g_1) q_{11} - g_1 (\frac{\partial q_{11}}{\partial x} f)] \]
\[ = [g_1, (\text{ad}_f g_1)] q_{11}^2 + (\text{ad}_f g_1)(\frac{\partial q_{11}}{\partial x} g_1) q_{11} \]
\[ - g_1(\frac{\partial q_{11}}{\partial x} (\text{ad}_f g_1)) q_{11} - g_1(\frac{\partial}{\partial x} (\frac{\partial q_{11}}{\partial x} f) g_1) q_{11} \]
\[ + g_1 \left( \frac{\partial q_{11}}{\partial x} g_1 \right) (\frac{\partial q_{11}}{\partial x} f) . \]

Setting coefficients equal to zero as before, we have

\[ \frac{\partial q_{11}}{\partial x} g_1 = 0, \text{ and } -\frac{\partial q_{11}}{\partial x} \text{ad}_f g_1 q_{11} - \left( \frac{\partial}{\partial x} \left( \frac{\partial q_{11}}{\partial x} f \right) g_1 \right) q_{11} = 0, \]

or \[ \frac{\partial q_{11}}{\partial x_2} = 0, \text{ and } -2 \frac{\partial q_{11}}{\partial x_1} \sqrt{1 - x_1^2} = 0. \]

The last equality leads to \[ \frac{\partial q_{11}}{\partial x_1} = 0. \]

Similarly, we solve \[ 0 = [\dot{g}_2, \text{ad}_f \dot{g}_1] \] to get \[ \frac{\partial q_{12}}{\partial x_3} = 0, \text{ and } \frac{\partial q_{12}}{\partial x_2} = 0. \]

So \[ q_{11} = \text{constant}. \]

Hence let \[ q_{11} = 1. \]

We also check that condition (B'-ii) is satisfied, i.e.

\[ \text{ad}_f \dot{g}_1 = \text{ad}_f g_1 = 0, \]

and

\[ \text{ad}_f \dot{g}_2 = 0. \]

Let \[ q_{12} = 1, \] then \[ q_{22} = x_3 + 1, \]

and

\[ Q = \begin{pmatrix} 1 & 1 \\ 0 & x_3 + 1 \end{pmatrix}, \]

which is nonsingular on \( U \). Combining with the transformation (2.2.2), we have

\[ u = \begin{pmatrix} -\tan x_1 \\ 0 \end{pmatrix} + \begin{pmatrix} \sec x_1 \\ 0 \end{pmatrix} e^{-x_2/(x_3 + 1)} \begin{pmatrix} 1 & 1 \\ 0 & x_3 + 1 \end{pmatrix} v \]
\[ = w(x) + Q(x)v \]

where \( w(x) = \begin{pmatrix} -\tan x_1 \\ 0 \end{pmatrix} \), and \( Q(x) = \begin{pmatrix} \sec x_1 & \sec x_1 \\ 0 & e^{-x_2} \end{pmatrix} \).

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Now we solve the system of partial differential equations (2.10) for the map \( \tilde{x}(\xi) \). For convenience, we drop the \(^-\) sign and use single subscript for \( \xi \). So the system of equations to be solved are

\[
\frac{\partial x}{\partial \xi_1} = \frac{\partial x}{\partial \xi_3} = \hat{g}_2 = \begin{pmatrix} 0 \\ 1 \\ x_3 + 1 \end{pmatrix},
\]

\[
\frac{\partial x}{\partial \xi_2} = \frac{\partial x}{\partial \xi_2} = \hat{g}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},
\]

\[
\frac{\partial x}{\partial \xi_1} = \frac{\partial x}{\partial \xi_1} = -\text{ad}_f \hat{g}_1 = \begin{pmatrix} \sqrt{1 - x_1^2} \\ 0 \\ 0 \end{pmatrix}.
\]

Using the method outlined after proposition (2.9), we first solve

\[
\frac{dx(\xi_1)}{d\xi_1} = \begin{pmatrix} \sqrt{1 - x_1^2} \\ 0 \\ 0 \end{pmatrix}, \quad x(0) = 0.
\]

We get

\[
x(\xi_1) = \begin{pmatrix} \sin \xi_1 \\ 0 \\ 0 \end{pmatrix}.
\]

Next we solve

\[
\frac{dx(\xi_1, \xi_2)}{d\xi_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad x(\xi_1, 0) = x(\xi_1).
\]

We have

\[
x(\xi_1, \xi_2) = \begin{pmatrix} \sin \xi_1 \\ \xi_2 \\ 0 \end{pmatrix}.
\]

Finally we solve

\[
\frac{dx(\xi_1, \xi_2, \xi_3)}{d\xi_3} = \begin{pmatrix} 0 \\ 1 \\ x_3 + 1 \end{pmatrix}, \quad x(\xi_1, \xi_2, 0) = x(\xi_1, \xi_2).
\]

We get the map

\[
x_1 = \sin \xi_1
\]

\[
x_2 = \xi_3 + \xi_2
\]

\[
x_3 = e^{\xi_3} - 1.
\]
§2.3 Local stabilization of a nonlinear system

For the nonlinear system (2.1), it may not be easy to determine if an arbitrary point \( x \in \mathbb{R}^n \) can be transferred to a given point under an admissible control, but under the assumption that the nonlinear system with \( f(x_0) = 0 \) is linearizable around \( x_0 \) to a linear controllable system, it can be seen that points in some neighborhood of \( x_0 \) can be transferred to \( x_0 \) in finite time under some control. We will elaborate on this point in this section. First we introduce some definitions.

Let \( \Omega \) be the set of all admissible control, which will be assumed to be unconstrained, so \( \Omega = \mathbb{R}^m \).

Let \( u[t_0, t_1] \in \Omega \) denote the control acting on the system from time \( t_0 \) to time \( t_1 \).

Let \( \psi(t; x_0, t_0, u) \) denote the trajectory of the system at time \( t \) that originated from \( x_0 \) at time \( t_0 \) under the control \( u[t_0, t] \).

Let \( C(x_0) = \{ x : \exists u \in \Omega \text{ such that } \psi(t_1; x, t_0, u) = x_0, \text{for some } t_1 \geq t_0 \} \) i.e., \( C(x_0) \) is the set of points that can be transferred to \( x_0 \) at some finite time \( t_1 \).

If \( U \) is a neighborhood of \( x_0 \), we define

\[
C_U(x_0) = \{ x \in C(x_0) : \psi(t; x, t_0, u) \in U, t \in [t_0, t_1], u \in \Omega \}
\]

**Definition**: System (2.1) is said to be controllable at \( x_0 \) if \( C(x_0) = \mathbb{R}^n \), and it is said to be controllable if this is true for all \( x_0 \in \mathbb{R}^n \).

**Definition**: System (2.1) is said to be locally controllable at \( x_0 \) if for every neighborhood \( U \) of \( x_0 \), \( C_U(x_0) \) is also a neighborhood of \( x_0 \), and it is said to be
locally controllable if this is true for all \( x_0 \in \mathbb{R}^n \).

**Definition** : System (2.1) is said to be weakly controllable at \( x_0 \) if there exists a neighborhood \( U \) of \( x_0 \) such that \( C(x_0) = U \), and it is said to be weakly controllable if this is true for all \( x_0 \in \mathbb{R}^n \).

**Definition** : System (2.1) is said to be locally weakly controllable at \( x_0 \) if there exists a neighborhood \( U \) of \( x_0 \) such that for every neighborhood \( V \) of \( x_0 \) contained in \( U \), \( C_V(x_0) \) is also a neighborhood of \( x_0 \), and it is said to be locally weakly controllable if this is true for all \( x_0 \in \mathbb{R}^n \).

**Remark.** These definitions resemble those given by Hermann and Krener [6]. The difference is that our set \( C(x_0) \) denotes a set of points that can be transferred to \( x_0 \), whereas the set \( A(x_0) \) defined in [6] denotes a set of points that can be transferred from \( x_0 \). Moreover, the definition for weak controllability has been modified so that it is analogous to that for weak observability given in the next chapter. Furthermore, it can be seen that the following implications hold.

\[
\begin{align*}
\text{locally controllable} & \quad \implies \quad \text{controllable} \\
\downarrow & \quad \downarrow \\
\text{locally weakly controllable} & \quad \implies \quad \text{weakly controllable}
\end{align*}
\]

We will review some results from linear system theory. Consider the linear time-invariant system

\[
\dot{x} = Ax + Bu. \tag{1.1}
\]

**Lemma 2.10** System (1.1) is controllable iff the controllability rank condition holds.
The above lemma is well known and will not be proved here. As noted by Hermann and Krener in [6], the four concepts of controllability are equivalent for linear system. But this is not the subject of our discussion, and we will limit ourselves to showing the equivalence of controllable and locally controllable system as in the next lemma.

**Lemma 2.11** System (1.1) is locally controllable iff the controllability rank condition holds.

The proof is briefly shown below.

**Proof:** Necessity of the rank condition is clear as local controllability implies controllability.

For sufficiency. Let $\Phi(t, t_0)$ be the state transition matrix of the system (1.1), i.e.,

$$\frac{d\Phi(t, t_0)}{dt} = A\Phi(t, t_0)$$

$$\Phi(t_0, t_0) = I.$$

Let

$$M(t, t_0) = \int_{t_0}^{t} \Phi(t, s)BB^T\Phi^T(t, s)ds.$$ 

It is known that $M(t, t_0)$ is nonsingular for $t > t_0$ iff the controllability rank condition holds.

Let

$$u(t) = -P(t, t_1, t_0)x_0$$

where

$$P(t; t_1, t_0) = B^T\Phi(t_1, t)M^{-1}(t_1, t_0)\Phi(t_1, t_0).$$
Then
\[
x(t) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,s)Bu(s)ds
\]
\[
= \Phi(t,t_0)x_0 - \int_{t_0}^{t} \Phi(t,s)BB^T \Phi^T(t,s)M^{-1}(t_1,t_0)\Phi(t_1,t_0)x_0ds
\]
\[
= K(t; t_1, t_0)x_0
\]
where
\[
K(t; t_1, t_0) = \Phi(t,t_0) - \int_{t_0}^{t} \Phi(t,s)BB^T \Phi^T(t,s)M^{-1}(t_1,t_0)\Phi(t_1,t_0)ds.
\]
It can be checked that
\[
x(t_1) = K(t_1; t_1, t_0)x_0 = 0.
\]
So the above control will transfer any point \(x_0\) to the origin. Moreover,
\[
|x(t)| \leq \sup_{t_0 \leq t \leq t_1} |x(t)| = \sup_{t_0 \leq t \leq t_1} |K(t; t_1, t_0)||x_0|
\]
\[
= \|K(t_1, t_0)||x_0||.
\]
So given any \(\epsilon\) neighborhood of 0, choose \(\delta < \frac{\epsilon}{\|K(t_1, t_0)||x_0||}\), then
\[
\|x_0\| < \delta \text{ implies } |x(t)| < \epsilon \text{ for } t \in [t_0, t_1].
\]

Therefore system (1.1) is locally controllable at 0 and, hence, at all \(x \in \mathbb{R}^n\) since the system is linear and time–invariant. QED. \(\Box\)

Corollary 2.12 If the controllability rank condition holds then system (1.1) is locally weakly controllable.

Proposition 2.13 If the nonlinear system (2.1) is locally linearizable to a linear controllable system at \(x_0\), then system (2.1) is locally weakly controllable at \(x_0\).

Remark. Proposition 2.13 is intuitively clear in view of lemma 2.11 or corollary 2.12. Nevertheless, we will prove it below.
Proof: Let \((\tilde{x}, w, Q)\) be the linearizing triple that transforms (2.1) to (1.1). In particular,

\[
\tilde{x}: \quad V \subset \mathbb{R}^n \rightarrow \tilde{x}(V) = U
\]

is a diffeomorphism from \(V\) onto \(U\) such that \(x_0 \in \tilde{x}(V)\). Let \(\xi_0 = \tilde{x}^{-1}(x_0)\), then, since \(\tilde{x}\) is continuous, for every neighborhood \(U_1\) of \(x_0\) contained in \(U\), \(\tilde{x}^{-1}(U_1) = V_1\) is a neighborhood of \(\xi_0\) such that \(\tilde{x}(V_1) = U_1\). But in the \(\xi\) coordinate around \(\xi_0\), the system is linear and controllable, so there exists a neighborhood \(V_2\) of \(\xi_0\) contained in \(V_1\), and a control

\[
v(t) = -P(t; t_1, t_0)\tilde{\xi}
\]

such that any point \(\xi \in V_2\) at time \(t_0\) can be transferred to \(\xi_0\) at time \(t_1\) with the trajectory \(\gamma(t; \xi, t_0, v) \in V_1\) for \(t \in [t_0, t_1]\). This implies \(C_{V_1}(\xi_0)\) is a neighborhood of \(\xi_0\).

But for any \(v\) and any \(\xi \in V_1\), if \(u = w + Qv\), and \(\bar{x} = \tilde{x}(\xi)\), then the trajectory for the nonlinear system is

\[
\psi(t; \bar{x}, t_0, u) = \tilde{x}(\gamma(t; \xi, t_0, v)), \quad \text{for} \quad t \in [t_0, t_1],
\]

since \((\tilde{x}(\xi), w, Q)\) is the linearizing triple. So \(\psi(t; \bar{x}, t_0, u) \in \tilde{x}(V_1) = U_1\). In particular, \(\psi(t_1; \bar{x}, t_0, u) = x_0\), since \(\gamma(t_1; \xi, t_0, v) = \xi_0\) and \(\tilde{x}(\xi_0) = x_0\). Moreover, since \(\tilde{x}\) is a local diffeomorphism, \(\tilde{x}(V_2)\) is a neighborhood of \(x_0\). Hence, we conclude that every point in \(\tilde{x}(V_2)\) can be transferred to \(x_0\) by some feedback control \(u\) with the trajectory lying inside \(U_1\). This proves that \(C_{U_1}(x_0)\) is a neighborhood of \(x_0\). QED. \(\Box\)
§2.4 Local asymptotic stabilization problem

In this section we will consider the particular case that \( x_0 = 0 \), and \( f(0) = 0 \), i.e., the origin is an equilibrium point. From the result of the previous section, we see that if the nonlinear system (2.1) is linearizable around 0 to a linear controllable system and if the admissible control set is \( \mathbb{R}^m \), then there exists a control such that any point in a sufficiently small neighborhood of 0 can be transferred to the origin in a finite time with the trajectory lying inside some given neighborhood of 0. But such a control is usually time dependent and is undesirable in practice. Furthermore, as far as stability of the system is concerned, it suffices to have a control that can transfer some initial state to a neighborhood of 0, which is sufficiently small. For these reasons, we consider the following problem.

Local asymptotic stabilization problem:

For the system (2.1), find a control law \( u \), and neighborhoods \( U, U_1 \) (if possible) of 0 with \( U_1 \subset U \) such that any initial state \( x_0 \in U_1 \) can be transferred to the origin asymptotically under this control law, i.e., \( x_0 \to 0 \) as \( t \to \infty \). Moreover, the trajectory \( \psi(t; x_0, t_0, u) \in U \), for all \( t \geq t_0 \).

Consider again the linear controllable system

\[
\dot{\xi} = A\xi + Bv. \tag{1.1}
\]
If we feedback the state linearly into the input, i.e., if

\[ v = -P \xi \]

where \( P \) is a constant matrix, then the closed loop system becomes

\[ \dot{\xi} = (A - BP) \xi. \]

It is known from linear system theory that we can find a constant matrix \( P \) such that the characteristic values of \((A - BP)\) can be arbitrarily assigned in the complex plane. Since the solution of the closed loop system is

\[ \xi(t) = e^{(A-BP)t} \xi_0 \]

then

\[ \|\xi\| \leq ke^{\lambda t} \|\xi_0\|, \]

where \( k \) is some positive constant and

\[ \lambda = \max\{Re \lambda_1, \ldots, Re \lambda_n\}, \]

where \( \lambda_1, \ldots, \lambda_n \) are the characteristic values of \((A - BP)\).

As noted above, we can choose \( P \) so that \( \lambda \) is negative. Thus, given any \( \epsilon \) neighborhood \( V \) of 0, if \( \|\xi_0\| < \frac{\epsilon}{k} \), then \( \xi(t) \in V \), for all \( t \geq 0 \). Moreover, \( \xi(t) \) approaches 0 as \( t \) approaches \( \infty \).

If \((\hat{x}, w, Q)\) is the linearizing triple transforming system (2.1) to (1.1), and if \( V \) is sufficiently small so that the map \( \hat{x} \) is a diffeomorphism on \( V \), then clearly \( \hat{x}(\xi(t)) \) for \( t \geq 0 \), is the trajectory for the nonlinear system under the control

\[ u = w(x) - Q(x)P\xi \]

\[ = w(x) - Q(x)P\hat{x}^{-1}(x) \]
with the property that

\[ \dot{x}(\xi(t)) \in \bar{x}(V), \text{ for } t \geq 0, \text{ and } \dot{x}(\xi(t)) \to 0 \text{ as } t \to \infty. \]

In particular, if the linear controllable system (1.1) is in Brunovsky canonical form, the constant matrix \( P \) can be easily determined so that the characteristic values of the closed loop system are as desired.

In summary, the control law

\[ u = w(x) - Q(x)P\dot{x}^{-1}(x) \]

will be called the local asymptotic control law since the linearization is valid only locally. Moreover, system (2.1) subjected to small deviation of the state from the equilibrium point 0 can be stabilized asymptotically.
CHAPTER 3
Linearization of nonlinear systems with outputs

§3.1 Necessary and sufficient condition

In this section, we will consider the dual problem of the one considered in the previous chapter; that is, we want to find necessary and sufficient conditions that the nonlinear uncontrolled system with output of the form

\[
\begin{align*}
\dot{x} &= f(x) \\
y &= h(x)
\end{align*}
\]  

(3.1)

where \(h(0) = 0\), is locally Go-equivalent to the linear observable system in dual Brunovsky observer form

\[
\begin{align*}
\dot{\xi} &= \tilde{A}\xi + \alpha(\phi) \\
\phi &= \tilde{C}\xi
\end{align*}
\]  

(3.2)

As mentioned in the introduction, Krener and Isidori [11] gave a necessary and sufficient condition such that this is possible by a state coordinate change for the single output case. Isidori also gave a detailed proof in [9, p.248]. Bestle and Zeitz [1] studied the same problem and allowed \(f\) and \(h\) to be time dependent. They derived the necessary condition by partially differentiating the map \(\tilde{x}(\xi)\) with respect to the coordinate \(\xi\) just as we did in the previous chapter. They obtained a system of partial differential equations which, using our notation and for the time independent case, is equivalent to system (2.10), namely

\[
\frac{\partial \tilde{x}}{\partial \xi_k} = (-1)^{n-k}\text{ad}_{f}^{n-k}g, \quad \text{for} \quad k = 1, \ldots, n.
\]  

(3.3)
Clearly if $g$ is known, then all $n$ partial derivatives can be computed. They derived $g$ by partially differentiating the output and found that
\[ g(x) = \Phi^{-1}(x)(1,0,\ldots,0)^T, \]
where
\[ \Phi(x) = \begin{pmatrix} \frac{\partial h}{\partial x} \\ N_1(\frac{\partial h}{\partial x}) \\ \vdots \\ N^{n-1}(\frac{\partial h}{\partial x}) \end{pmatrix}, \]
and the operator is defined by
\[ N(\frac{\partial h}{\partial x}) = \frac{\partial}{\partial t}(\frac{\partial h}{\partial x}) + f^T \frac{\partial^2 h}{\partial x^2} + \frac{\partial h}{\partial x} \frac{\partial f}{\partial x}, \]
which, for the time independent case, is just the Lie derivative of a covector field $dh$ by $f$. However, they did not examined the solvability of system (3.3). Krener and Respondek [14] also studied this problem for the multi-output case. It can be seen that their result (theorem 5.1) is equivalent to what we will derive in this section, although their theorem is stated in a more abstract way. Our derivation in the sequel was inspired by the papers mentioned above.

For simplicity, assume $\frac{\partial h}{\partial x}$ has rank $p$ where $p$ is the dimension of $h$. First, we introduce the following notation.

Let
\[ E_j = \left\{ L_f^i(h) : i = 1,\ldots,p, l = 0,\ldots, j \right\}, \quad j = 0,\ldots,n-1. \]
We define
\[ \bar{E}_j = sp\{ \tau : \tau \in E_j \} \]
that is, $\bar{E}_j$ is a $C^\infty$ codistribution on $R^n$ generated by $E_j$. 

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Let
\[ e_j(x) = \dim \bar{E}_j(x), \quad j = 0, \ldots, n - 1, \]
and define
\[ d_0 = e_0 \]
\[ d_j = e_j - e_{j-1}, \quad \text{for} \quad j = 1, \ldots, n - 1. \]
We also call \((\hat{x}, \hat{y})\) the linearizing pair if \(\hat{x}\) and \(\hat{y}\) are the maps that correspond to the state and output coordinate change respectively.

**Proposition 3.1** System \((3.1)\) is locally Go-equivalent to system \((DBOF)\) at 0 iff the following conditions are satisfied on a neighborhood \(U\) of 0.

(i) there exists linearly independent vector fields \(g_1, \ldots, g_p\) such that for \(i = 1, \ldots, p, j = 1, \ldots, p,\)
\( (a) \quad \langle L^k_f (dh_i), g_j \rangle = 0, \quad \text{for} \quad k = 0, \ldots, \mu_j - 2 \)
\( (b) \quad \text{the} \ p \times p \text{ matrix } N = (n_{ij}) \text{ is nonsingular, where} \)
\[ n_{ij} = \langle L^k_f (dh_i), g_j \rangle \]
\( (c) \quad [\text{ad}^l_f g_i, \text{ad}^k_f g_j] = 0, \quad 1 \leq i, j \leq p, l = 0, \ldots, \mu_i - 1, k = 0, \ldots, \mu_j - 1, \)
where \(\mu_j = \text{the number of } d_i \text{ such that } d_i \geq j.\)

(ii) \(\dim \bar{E}_j = e_j = \text{constant}, \quad j = 0, \ldots, n - 1.\)

(iii) \(\dim \bar{E}_{n-1} = n.\)

**Proof:** For necessity, assume the maps \(x = \hat{x}(\xi)\) and \(y = \hat{y}(\phi)\) exist so that \((3.1)\) is transformed to \((3.2)\) with observability indices \(\mu_1, \ldots, \mu_p\) satisfying
\[ \mu_1 \geq \mu_2 \geq \cdots \geq \mu_p \geq 1 \quad \text{and} \quad \sum_{i=1}^{p} \mu_i = n. \]
Differentiate \(\hat{x}(\xi)\) with respect to \(t\), we have
\[ \dot{x} = \frac{\partial \hat{x}}{\partial \xi} \xi = f(x). \]
So
\[ \frac{\partial}{\partial \xi}(\tilde{A}\xi + \alpha(\phi)) = f(\tilde{x}(\xi)). \] (3.4)

Using double subscripts for \( \xi \) as in the previous chapter and differentiating (3.4) partially with respect to \( \xi_{ik+1} \), we have

\[ \frac{\partial}{\partial \xi_{ik+1}}(\frac{\partial}{\partial \xi}(\tilde{A}\xi + \alpha(\phi))) = \frac{\partial}{\partial \xi_{ik+1}}(f(\tilde{x}(\xi))) = \frac{\partial f}{\partial \tilde{x}} \frac{\partial \tilde{x}}{\partial \xi_{ik+1}}. \] (3.5)

Also
\[ \frac{\partial}{\partial \xi_{ik+1}}(\frac{\partial}{\partial \xi}(\tilde{A}\xi + \alpha(\phi))) = \frac{\partial}{\partial \xi_{ik+1}}(\frac{\partial}{\partial \xi}(\tilde{A}\xi + \alpha(\phi))) + \frac{\partial}{\partial \xi}(\frac{\partial}{\partial \xi_{ik+1}}(\tilde{A}\xi + \alpha(\phi))) = (\frac{\partial}{\partial \xi}(\frac{\partial}{\partial \xi_{ik+1}})(\tilde{A}\xi + \alpha(\phi))) + \frac{\partial}{\partial \xi}(\frac{\partial}{\partial \xi_{ik+1}}(\tilde{A}\xi + \alpha(\phi))). \]

As in the proof of proposition 2.1, we have

\[ \frac{\partial}{\partial \xi}(\frac{\partial}{\partial \xi_{ik+1}}(\tilde{A}\xi)) = \begin{cases} \frac{\partial}{\partial \xi}, & \text{if } k = 1, \ldots, \mu_i - 1; \\ 0, & \text{if } k = 0. \end{cases} \] (3.6)

Since
\[ \phi = \tilde{C}\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_p \end{pmatrix}, \]
then
\[ \frac{\partial \phi}{\partial \xi_{ik+1}} = \begin{cases} 0, & \text{if } k = 1, \ldots, \mu_i - 1; \\ e_i, & \text{if } k = 0, \end{cases} \]

where \( e_i \) denotes a standard unit vector in \( \mathbb{R}^p \).

Hence
\[ \frac{\partial \alpha(\phi)}{\partial \xi_{ik+1}} = \begin{cases} 0, & \text{if } k = 1, \ldots, \mu_i - 1; \\ \frac{\partial \alpha}{\partial \phi_i}, & \text{if } k = 0. \end{cases} \] (3.7)
Substituting (3.6) and (3.7) into (3.5), we have

\[
\frac{\partial z}{\partial \xi_{ik}} = \frac{\partial f}{\partial x} \frac{\partial z}{\partial \xi_{ik+1}} - \left( \frac{\partial}{\partial \xi} \left( \frac{\partial z}{\partial \xi_{ik+1}} \right) \right)(\tilde{A} \xi + \alpha(\phi)) \quad \text{if } k = 1, \ldots, \mu_i - 1
\]  

(3.8)

and

\[
\frac{\partial z}{\partial \xi_k} \frac{\partial \alpha}{\partial \phi_i} = \frac{\partial f}{\partial x} \frac{\partial z}{\partial \xi_{ik+1}} - \left( \frac{\partial}{\partial \xi} \left( \frac{\partial z}{\partial \xi_{ik+1}} \right) \right)(\tilde{A} \xi + \alpha(\phi)), \quad \text{if } k = 0.
\]

(3.9)

Let \( k = \mu_i - 1 \) in (3.8), then

\[
\frac{\partial z}{\partial \xi_{ik}} = \frac{\partial f}{\partial x} \frac{\partial z}{\partial \xi_{ik+1}} - \frac{\partial}{\partial \xi} \left( \frac{\partial z}{\partial \xi_{ik+1}} \right)(\tilde{A} \xi + \alpha(\phi)).
\]

If we let

\[
\frac{\partial z}{\partial \xi_{ik}} = g_i(\tilde{x}(\xi)),
\]

then, as before, we find

\[
\frac{\partial z}{\partial \xi_{ik}} = (-1)^k \text{adj}^k g_i, \quad \text{for } k = 0, \ldots, \mu_i - 1,
\]

(3.10)

and

\[
\frac{\partial z}{\partial \xi_k} \frac{\partial \alpha}{\partial \phi_i} = (-1)^\mu \text{adj}^\mu g_i.
\]

(3.11)

(3.10) can be written as

\[
\frac{\partial z}{\partial \xi_{ik}} = (-1)^\mu - k \text{adj}^{\mu - k} g_i, \quad \text{for } k = 1, \ldots, \mu_i.
\]

(3.12)

From the output \( y = h(x) \) and the map \( y = \tilde{y}(\phi) \), we have

\[
\tilde{y}(\phi(\xi)) = h(\tilde{x}(\xi)).
\]

(3.13)

Differentiating (3.13) partially with respect to \( \xi \), we have

\[
\frac{\partial \tilde{y}}{\partial \phi} \frac{\partial \phi}{\partial \xi} = \frac{\partial h}{\partial x} \frac{\partial \tilde{x}}{\partial x}.
\]
or
\[
\frac{\partial \tilde{y}}{\partial \phi_i} \tilde{C} = \frac{\partial h}{\partial x} \frac{\partial \tilde{x}}{\partial \xi}.
\] (3.14)

Since
\[
\frac{\partial \tilde{y}}{\partial \phi_i} \tilde{C} = \left( \frac{\partial \tilde{y}}{\partial \phi_1} O \mid \frac{\partial \tilde{y}}{\partial \phi_2} O \mid \cdots \mid \frac{\partial \tilde{y}}{\partial \phi_p} O \right)
\] (3.15)

where \( O \) consists of zero entries and the \( i \)th block of the matrix has dimension \( p \times \mu_i - 1 \), then from (3.14) and (3.15), for \( i = 1, \ldots, p \),

\[
\frac{\partial \tilde{y}}{\partial \phi_i} = \frac{\partial h}{\partial x} \frac{\partial \tilde{x}}{\partial \xi_{i1}}, \quad \text{if } k = 2, \ldots, \mu_i
\]

or
\[
\frac{\partial \tilde{y}}{\partial \phi_i} \delta^k_j = \frac{\partial h}{\partial x} \frac{\partial \tilde{x}}{\partial \xi_{ik}}, \quad \text{for } k = 1, \ldots, \mu_i,
\] (3.16)

where
\[
\delta^k_j = \begin{cases} 
1 & \text{if } k = j, \\
0 & \text{if } k \neq j.
\end{cases}
\]

Substituting (3.12) into (3.16), we get
\[
\frac{\partial \tilde{y}}{\partial \phi_i} \delta^k_j = \frac{\partial h}{\partial x} \left( (-1)^{\mu_i - k} \text{ad}_f^{\mu_i - k} g_i \right), \quad \text{for } k = 1, \ldots, \mu_i.
\] (3.17)

So for \( 1 \leq i, j \leq p \),
\[
\frac{\partial \tilde{y}}{\partial \phi_i} \delta^k_j = (-1)^{\mu_i - k} \langle dh_j, \text{ad}_f^{\mu_i - k} g_i \rangle, \quad \text{for } k = 1, \ldots, \mu_i.
\] (3.18)

Since the Jacobian matrix \( \frac{\partial \tilde{y}}{\partial \phi_i} (\phi) \) is required to be nonsingular on a neighborhood of \( 0 \in \mathbb{R}^p \), hence it is necessary that
\[
\langle dh_j, \text{ad}_f^{\mu_i - k} g_i \rangle = 0, \quad \text{for } 1 \leq i, j \leq p, \quad k = 2, \ldots, \mu_i,
\]
or
\[
\langle dh_j, \text{ad}_f^k g_i \rangle = 0, \quad \text{for } 1 \leq i, j \leq p, \quad k = 0, \ldots, \mu_i - 2,
\] (3.19)
and the \( p \times p \) matrix \( N^0 = (n^0_{ji}) \) is nonsingular on a neighborhood of 0, where
\[
n^0_{ji} = \langle dh_j, \text{ad}_{f}^{-1} g_i \rangle. \tag{3.20}
\]

The above conditions are stated as \((a')\) in the following lemma which can be considered as dual to lemma 2.7.

**Lemma 3.2** The following statements are equivalent:

\((a')\)

(i) 
\[
\langle dh_j, \text{ad}_f^k g_i \rangle = 0, \quad 1 \leq i, j \leq p, \quad k = 0, \ldots, \mu_i - 2.
\]

(ii) the \( p \times p \) matrix \( N^0 = (n^0_{ji}) \) is nonsingular on a neighborhood \( U \) of 0, where
\[
n^0_{ji} = \langle dh_j, \text{ad}_{f}^{-1} g_i \rangle.
\]

\((b')\)

(i) 
\[
\langle L^l_f(dh_j), \text{ad}_f^k g_i \rangle = 0, \quad \text{for} \quad 1 \leq i, j \leq p,
\]

and for \( l = 0, \ldots, \mu_i - 2, \quad k = 0, \ldots, \mu_i - 2 - l.\)

(ii) For \( l = 0, \ldots, \mu_1 - 1 \), the \( p \times d_l \) matrix \( N^l = (n^l_{ji}) \) has rank \( d_l \) on \( U \), where
\[
n^l_{ji} = \langle L^l_f(dh_j), \text{ad}_f^{\mu_i - l - 1} g_i \rangle.
\]

Furthermore, \( N^l = (-1)^l N^0_l \), for \( l = 0, \ldots, \mu_1 - 1 \), where \( N^0_l \) is the first \( d_l \) columns of \( N^0 \).

**Proof:** The equivalence of \((a')\) and \((b')\) is quite clear if we compare lemma 2.7 and lemma 3.2. We see that the roles of covector fields and vector fields are
interchanged. In particular, we notice the following correspondence:

\[ L^k_f(d\xi_i) \leftrightarrow \text{ad}^k_f g_i \]
\[ \text{ad}^l_f g_j \leftrightarrow L^l_f(dh_j) \]

To relate condition \((a')\) to the one stated in proposition 3.1, we use the following lemma.

**Lemma 3.3** \((a')\) or \((b')\) is equivalent to \((c')\) given below.

\((c')\)

(i) \( \langle L^l_f(dh_j), g_i \rangle = 0, \quad 1 \leq i, j \leq p, \quad l = 0, \ldots, \mu_i - 2. \)

(ii) the \( p \times p \) matrix \( N = (n_{ji}) \) is nonsingular on a neighborhood of 0, where

\[ n_{ji} = \langle L^\mu_{-1} f(dh_j), g_i \rangle. \]

**Proof:** We show that \((b')\) and \((c')\) are equivalent. Clearly, \((b'-i) \Rightarrow (c'-i)\) since the latter is a special case of the former by letting \( k = 0 \).

Next we show that \((c'-i) \Rightarrow (b'-i)\). For \( i = 1, \ldots, p \), let \( l = \mu_i - 2 \) in \((c'-i)\), we have

\[ 0 = \langle L^\mu_{-2} f(dh_j), g_i \rangle = L_f \langle L^\mu_{-3} (dh_j), g_i \rangle - \langle L^\mu_{-3} (dh_j), \text{ad}_f g_i \rangle. \]

The first term is zero by \((c'-i)\), Repeating this process, we get the last equation

\[ (-1)^{\mu_i-2} \langle dh_j, \text{ad}^\mu_{-2} f g_i \rangle = 0. \]

We see that

\[ \langle L^l_f(dh_j), \text{ad}^k_f g_i \rangle = 0, \quad \text{for} \quad l = 0, \ldots, \mu_i - 2, \quad k = \mu_i - 2 - l. \]
If we repeat with \( l = \mu_i - 3, \ldots, 0 \), we get condition (b'-i). To show the equivalence of (b'-ii) and (c'-ii), we note that
\[
n_{ji}^0 = \langle dh_j, \text{ad}_{f}^{\mu_i-1} g_i \rangle
\]
\[
= L_f(dh_j, \text{ad}_{f}^{\mu_i-2} g_i) - \langle L_f(dh_j), \text{ad}_{f}^{\mu_i-2} g_i \rangle.
\]
The first term is zero by (b'-i). Repeating this process, we have
\[
n_{ji}^0 = (-1)^{\mu_i} \langle (L_f)^{\mu_i-1} (dh_j), g_i \rangle, \quad \text{for} \quad 1 \leq i, j \leq p.
\]
Thus, the columns in \( N \) and \( N^0 \) are the same except for a possible sign change. This shows that (c'-ii) is equivalent to (a'-ii) which is equivalent to (b'-ii).

QED.

Returning now to the proof of necessity in proposition 3.1, let
\[
Z_l = (\text{ad}_{f}^{\mu_i-l-1} g_1, \ldots, \text{ad}_{f}^{\mu_i-l-1} g_j), \quad \text{for} \quad l = 0, \ldots, \mu_1 - 1,
\]
where
\[
j = \text{the number of } \mu_i \text{ such that } \mu_i - l - 1 \geq 0. \tag{3.21}
\]
Obviously, this number depends on \( l \), so we will denote it as \( d_l \), and, for the moment, ignore the previous definition; that is, \( d_j = e_j - e_{j-1} \). We will show that this is indeed true.

The columns of \( Z_l \) are linearly independent since they are the columns of the Jacobian matrix \( \frac{\partial z_l}{\partial \xi} \) by (3.12).

Hence
\[
\text{rank } Z_l = d_l.
\]
Note also that the following relation between \( \mu_i \) and \( d_j \) holds.
\[
\mu_i = \text{the number of } d_j \text{ such that } d_j \geq i.
\]
Let
\[
L^k_f(dh) = \begin{pmatrix}
L^k_f(dh_1) \\
\vdots \\
L^k_f(dh_p)
\end{pmatrix},
\]

then condition (b') in lemma 3.2 can be written as
\[
\begin{pmatrix}
dh \\
L_f(dh) \\
\vdots \\
L^\mu_1(dh)
\end{pmatrix}
(Z_0, Z_1, \ldots, Z_{\mu_1-1}) = \begin{pmatrix}
N^0 & O & \cdots & O \\
x & -N^0_1 & \cdots & O \\
\vdots & \vdots & \ddots & \vdots \\
x & x & \cdots & (-1)^{\mu_1-1} N^0_{\mu_1-1}
\end{pmatrix},
\]

where \(N^0_i\) is as defined in lemma 3.2.

If we let
\[
\tilde{Z}_j = (Z_0, \ldots, Z_j), \quad \hat{Z}_j = (Z_{j+1}, \ldots, Z_{\mu_1-1}),
\]
and
\[
\tilde{E}_j = \begin{pmatrix}
dh \\
\vdots \\
L^\mu_1(dh)
\end{pmatrix}, \quad \tilde{N}_j = \begin{pmatrix}
N^0 & \cdots & O \\
x & \ddots & \vdots \\
x & \cdots & (-1)^j N^0_j
\end{pmatrix},
\]

then (3.22) can be written as
\[
\tilde{E}_j \tilde{Z}_j = \tilde{N}_j, \quad \text{for } j = 0, \ldots, \mu_1 - 1,
\]
and
\[
\tilde{E}_j \hat{Z}_j = 0, \quad \text{for } j = 0, \ldots, \mu_1 - 2.
\]

Since
\[
\text{rank } \tilde{Z}_j = \text{rank } \tilde{N}_j = \sum_{i=0}^{j} d_i,
\]
then by (3.23)
\[
\text{rank } \tilde{E}_j \geq \sum_{i=0}^{j} d_i.
\]
Since
\[
\text{rank } \tilde{Z}_j = n - \text{rank } \tilde{Z}_j
\]
\[
= n - \sum_{i=0}^j d_i,
\]
then by (3.24)
\[
\text{rank } \tilde{E}_j \leq \sum_{i=0}^j d_i.
\]
Hence, we have
\[
\text{rank } \tilde{E}_j = \sum_{i=0}^j d_i.
\]
Since the codistribution
\[
\tilde{E}_j = sp\{\tau : \tau \in E_j\}
\]
\[
= sp\{\tau : \tau \text{ belongs to a row of } \tilde{E}_j\}
\]
then
\[
\text{rank } \tilde{E}_j = \dim \tilde{E}_j = e_j.
\]
Therefore, we have
\[
e_j = \sum_{i=0}^j d_i, \quad \text{for } j = 0, \ldots, \mu_1 - 1.
\]
Clearly, \(d_j = e_j - e_{j-1}\) as claimed. This also shows that the dimension \(d_j\) or \(e_j\) is constant on a neighborhood of 0. Furthermore
\[
\dim E_{\mu_1-1} = \sum_{i=0}^{\mu_1-1} d_i = \sum_{i=1}^{\mu} \mu_i = n.
\]
This proves the necessity of (ii) and (iii) in proposition (3.1).

From equation (3.10), we get the integrability condition \((i-c)\) by setting the mixed partials of the map \(\tilde{x}(\xi)\) equal. Since \((i-a)\) and \((i-b)\) in prop. 3.1 are just \((c'-i)\) and \((c'-ii)\) respectively in lemma 3.3, the necessity of the hypotheses in prop. 3.1 is proved.
For sufficiency, assume that the hypotheses (i), (ii), and (iii) in proposition 3.1 are satisfied. Hypothesis (iii) implies that there is a least integer \( \hat{k} \) such that

\[
\dim \mathcal{E}_k(x) = n, \quad \forall x \in U
\]

and that

\[
\sum_{i=0}^{\hat{k}} d_i = \sum_{i=1}^{\hat{k}} (e_i - e_{i-1}) + e_0 = e_{\hat{k}} = n.
\]

Moreover, the \( \mu_j \)'s defined in hypothesis (i) imply the relation (3.21), and

\[
\mu_1 \geq \mu_2 \geq \cdots \geq \mu_p \geq 1, \quad \sum_{i=1}^{p} \mu_i = \sum_{i=0}^{\hat{k}} d_i = n.
\]

Clearly, \( \hat{k} = \mu_1 - 1 \). Now let the map \( \tilde{x}(\xi) \) be defined by

\[
\frac{\partial \tilde{x}}{\partial \xi_{ik}} = (-1)^{\mu_i-k} \text{ad}_{g_i}^{\mu_i-k} \quad \text{for} \quad k = 1, \ldots, \mu_i, \quad (3.25)
\]

and \( \tilde{x}(0) = 0 \). (3.25) is solvable for \( \tilde{x}(\xi) \) since the integrability condition (i-c) is satisfied.

Let

\[
\xi = \tilde{x}(x) = \tilde{x}^{-1}(x),
\]

then

\[
\dot{\xi} = \frac{\partial \tilde{x}}{\partial x} \dot{x} = \frac{\partial \tilde{x}}{\partial x} f(\tilde{x}(\xi)) \triangleq \tilde{f}(\xi).
\]

From (3.25)

\[
\frac{\partial \tilde{\xi}}{\partial x} \frac{\partial \tilde{x}}{\partial \xi_{ik}} = \frac{\partial \tilde{x}}{\partial x} (-1)^{\mu_i-k} \text{ad}_{g_i}^{\mu_i-k} \quad \text{for} \quad k = 1, \ldots, \mu_i,
\]

which implies

\[
\frac{\partial}{\partial \xi_{ik}} = \frac{\partial \tilde{x}}{\partial x} (-1)^{\mu_i-k} \text{ad}_{g_i}^{\mu_i-k} \quad \text{for} \quad k = 1, \ldots, \mu_i.
\]
where \( \frac{\partial}{\partial \xi_{ik}} \) in local coordinates is the unit vector \( e_{ik} \). Hence, for \( k = 1, \ldots, \mu_i - 1 \), and using the fact that the Lie bracket of vector fields is invariant with respect to a change of state coordinates, we have

\[
\frac{\partial}{\partial \xi_{ik}} = \frac{\partial}{\partial x} (\text{ad} f^{-k} [f(x), \text{ad} f^{k-1} g_i(x)])
\]

\[
= (-1)^{\mu_i-k} \left[ \frac{\partial}{\partial x} f(\tilde{x}(\xi)), \frac{\partial}{\partial x} \text{ad} f^{\mu_i-k-1} g_i(\tilde{x}(\xi)) \right]
\]

\[
= (-1)^{\mu_i-k} [\tilde{f}(\xi), (-1)^{\mu_i-k-1} \frac{\partial}{\partial \xi_{ik+1}}]
\]

This implies

\[
\frac{\partial \tilde{f}(\xi)}{\partial \xi_{ik+1}} = e_{ik}, \quad \text{for} \quad i = 1, \ldots, p, \quad k = 1, \ldots, \mu_i - 1,
\]

or, if we use double subscripts for the components of \( \tilde{f}(\xi) \), then for \( i = 1, \ldots, p, \quad k = 1, \ldots, \mu_i - 1 \),

\[
\frac{\partial \tilde{f}_{ij}(\xi)}{\partial \xi_{ik+1}} = \begin{cases} 
0 & \text{if } j \neq i \text{ or } j = i, l \neq k, \\
1 & \text{if } j = i, \text{ and } l = k.
\end{cases}
\]

This shows that \( \tilde{f}_{ik}(\xi) \) has the form

\[
\tilde{f}_{ik}(\xi) = \xi_{i+1} + \alpha_{ik}(\xi_1, \ldots, \xi_{p1}), \quad \text{for} \quad i = 1, \ldots, p, \quad k = 1, \ldots, \mu_i - 1,
\]

\[
\tilde{f}_{i\mu}(\xi) = \alpha_{i\mu}(\xi_1, \ldots, \xi_{p1}).
\]  

(3.26)

To get the map for the output coordinate change, consider the output relation

\[
y = h(x).
\]  

(3.27)

Substituting the map \( \tilde{x} \) into (3.27), we have

\[
y = \tilde{y}(\xi) = h(\tilde{x}(\xi)).
\]
Differentiating \( y \) partially with respect to \( \xi_{ik} \), we have

\[
\frac{\partial \tilde{y}}{\partial \xi_{ik}} = \frac{\partial h}{\partial x} \frac{\partial \tilde{x}}{\partial \xi_{ik}}.
\]

Using (3.25), then for \( j = 1, \ldots, p, \)

\[
\frac{\partial \tilde{y}_j}{\partial \xi_{ik}} = \langle dh_j, (-1)^{i-1} \text{ad}_{f_j}^{i-1} g_i \rangle = \begin{cases} 
0 & \text{if } k = 2, \ldots, \mu_i, \\
\langle dh_j, (-1)^{i-1} \text{ad}_{f_j}^{i-1} g_i \rangle & \text{if } k = 1.
\end{cases}
\]

So \( \tilde{y} \) is independent of \( \xi_{ik} \) for \( i = 1, \ldots, p, k = 2, \ldots, \mu_i \).

Letting \( \phi_i = \xi_{i1}, \) then

\[
\frac{\partial \tilde{y}_j}{\partial \phi_i} = \frac{\partial \tilde{y}_j}{\partial \xi_{i1}} = \langle dh_j, (-1)^{i-1} \text{ad}_{f_j}^{i-1} g_i \rangle, \quad \text{for } i \leq i, j \leq p.
\]

We notice that

\[
\frac{\partial \tilde{y}_i}{\partial \phi_i} = (-1)^{i-1} n_{ji}^0 = n_{ji},
\]

where \( n_{ji} \) is defined in lemma (3.3), i.e. \( \frac{\partial \tilde{y}}{\partial \phi} = N \). Since the matrix \( N = (n_{ji}) \) is nonsingular on some neighborhood of 0 by hypothesis, and \( \tilde{x}(0) = 0, \phi(0) = 0, \) we see that the Jacobian matrix \( \frac{\partial \tilde{y}}{\partial \phi} (\phi) \) is nonsingular for \( \phi \) in some neighborhood of \( 0 \in \mathbb{R}^p \).

Now we show that \( \tilde{x}(\xi) \) is a local diffeomorphism around 0; that is, \( \frac{\partial \tilde{x}}{\partial \xi} \) is nonsingular on some neighborhood of 0. We have shown that hypotheses (i-a) and (i-b) are equivalent to (b') in lemma 3.2, which can be expressed as equations (3.23) and (3.24).

Letting \( j = \mu_1 - 1 \) in (3.23), we have

\[
\tilde{E}_{\mu_1 - 1} \tilde{Z}_{\mu_1 - 1} = \tilde{N}_{\mu_1 - 1}.
\]

Since

\[
\text{rank } \tilde{E}_{\mu_1 - 1} (x) = e_{\mu_1 - 1} (x) = n, \quad \forall x \in U,
\]

\[
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\]
and

$$\text{rank } \tilde{N}_{\mu_1-1}(x) = \sum_{i=0}^{\mu_1-1} d_i(x) = n, \quad \forall x \in U,$$

then

$$\text{rank } \tilde{Z}_{\mu_1-1}(x) = n, \quad \forall x \in U.$$ 

But $\tilde{Z}_{\mu_1-1}$ is a $n \times n$ matrix consisting of the columns of $\frac{\partial \tilde{x}}{\partial \xi}$, so $\frac{\partial \tilde{x}}{\partial \xi}$ is nonsingular for $\forall x \in U$.

Replacing $\xi_i$ by $\phi_i$ in (3.26), we see that (3.26) is the required form to which system (3.1) is supposed to be transformed. We note that $\alpha(\phi)$ can be computed from equation (3.4) or by solving (3.11). QED. \(\Box\)

Remark. If $f(0) = 0$, then necessarily $\alpha(0) = 0$ as seen from equation (3.4).
§3.2 Example

We will give an example which satisfies the hypothesis of proposition 3.1 and we will then find the transformations which map the nonlinear system (3.1) to the dual Brunovsky observer form. For convenience, we will drop the \( \sim \) sign for the maps and write \( x = x(\xi) \) and \( y = y(\phi) \).

Consider
\[
\dot{x} = f(x) \\
y = h(x)
\]
where
\[
f(x) = \begin{pmatrix} 
\sin x_2 (1 + x_1) \\
\sec x_2 (1 + x_1) \\
\cos x_4 \\
x_3 
\end{pmatrix}, \quad h(x) = \begin{pmatrix} 
x_1 \\
e^{x_4} - 1
\end{pmatrix}.
\]

We find that
\[
dh = \begin{pmatrix} 
1 & 0 & 0 & 0 \\
0 & 0 & 0 & e^{x_4} \\
0 & 0 & 0 & e^{x_4} \\
0 & 0 & 0 & x_3 e^{x_4} 
\end{pmatrix},
\]
\[
L_f(dh) = \begin{pmatrix} 
\sin x_2 (1 + x_1) \cos x_2 & 0 & 0 & 0 \\
0 & 0 & 0 & e^{x_4} \\
0 & 0 & 0 & x_3 e^{x_4}
\end{pmatrix},
\]
so
\[
\text{rank} \begin{pmatrix} 
dh \\
L_f(dh)
\end{pmatrix} = 4
\]
on \( U \), where \( U = \{ x : x_1 > -1, -\pi/2 < x_2 < \pi/2 \} \). So \( d_0 = 2, d_1 = 2, \mu_1 = \mu_2 = 2 \), and (ii) and (iii) in proposition 3.1 are satisfied.

Let
\[
g_1(x) = \begin{pmatrix} 
0 \\
1 \\
0 \\
0
\end{pmatrix} a_1(x), \quad g_2(x) = \begin{pmatrix} 
0 \\
0 \\
1 \\
0
\end{pmatrix} a_2(x),
\]

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where \( a_1(x) \), and \( a_2(x) \) \( \in C^\infty(U) \), and not equal to zero on \( U \). We see that (i-a) and (i-b) in proposition 3.1 are satisfied. In particular, if \( a_1(x) = \sec x_2, a_2(x) = 1 \), then (i-c) is also satisfied. So

\[
g_1(x) = \begin{pmatrix} 0 & \sec x_2 \\ \sec x_2 & 0 \end{pmatrix}, \quad g_2(x) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},
\]

and

\[
ad_fg_1 = \begin{pmatrix} 1 + x_1 \\ 0 \\ 0 \end{pmatrix}, \quad ad_fg_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},
\]

Solving system (3.10) with \( x(0) = 0 \), we find that

\[
x_1 = e^{\xi_1} - 1, \quad (3.2.1)
\]
\[
x_2 = \arcsin \xi_2, \quad (3.2.2)
\]
\[
x_3 = \xi_4, \quad (3.2.3)
\]
\[
x_4 = \xi_3. \quad (3.2.4)
\]

Substituting this map into \( h(x) \), we find

\[
y(\phi) = \begin{pmatrix} e^{\phi_1} - 1 \\ e^{\phi_2} - 1 \end{pmatrix},
\]

where \( \phi_1 = \xi_1 \), and \( \phi_2 = \xi_3 \). Now we find \( \alpha(\phi) \) from the maps \( x(\xi) \) and \( y(\phi) \) by differentiating \( x \) and equating to \( f \). From (3.2.1)

\[
\dot{x}_1 = e^{\xi_1} \xi_1
\]
\[
\sin x_2(1 + x_1) = (1 + x_1)(\xi_2 + \alpha_1)
\]
\[
\sin x_2(1 + x_1) = (1 + x_1)(\sin x_2 + \alpha_1).
\]

From (3.2.2)

\[
\dot{x}_2 = \sec x_2 \xi_2
\]
\[
\sec x_2(1 + x_1) = \sec x_2(\alpha_2).
\]
From (3.2.3)
\begin{align*}
\dot{x}_3 &= \dot{\xi}_4 \\
\cos x_4 &= \alpha_4.
\end{align*}

Finally, from (3.2.4)
\begin{align*}
\dot{x}_4 &= \dot{\xi}_3 \\
x_3 &= \xi_4 + \alpha_3.
\end{align*}

Hence \(\alpha_1 = 0, \alpha_2 = 1 + x_1 = e^{\phi_1}, \alpha_4 = \cos \xi_3 = \cos \phi_2\), and \(\alpha_3 = 0\).
§3.3 Local estimation of a nonlinear system

Consider the more general nonlinear system (1.7) with inputs and outputs. Clearly system (3.1) is a special case of (1.7) with \( g_i(x) = 0, i = 1, \ldots, m \). Quite often the system state \( x \) is required to feedback to the system for stabilization purpose; but in many systems, the states may not be accessible and measured. In such cases, it may be necessary to estimate the states from knowledge of the past output. This is called an observation problem (or reconstruction problem by Kalman [12]). We will not discuss in depth the observability problem of a more general nonlinear system, but rather, we will restrict to those that are locally linearizable to a linear observable system.

Just as a nonlinear system (2.1) can be locally stabilized under the assumption that it can be locally linearized to a linear controllable system, we expect that the states of the nonlinear system (3.1) can be locally estimated from knowledge of the outputs if it can be locally linearized to a linear observable system. This is the subject of this and the following section.

Analogous to the controllability concept of the previous chapter, we will give some definitions on observability, which can be considered as dual to those on controllability. These definitions were given by Hermann and Krener [6].

Just as before, we denote by \( \psi(t; x_0, t_0, u) \) the state of the nonlinear system at \( t \) that originated from \( x_0 \) at \( t_0 \) for a control \( u|_{t_0, t} \), and by \( y(\psi(t; x_0, t_0, u)) \) the corresponding output at \( t \).

Let \( I(x_0) = \{ x : y(\psi(t; x, t_0, u)) = y(\psi(t; x_0, t_0, u)) \} \),
for all \( t, \bar{t} \), with \( \bar{t} \leq t \leq t_0 \), and all \( u|_{\bar{t}, t_0} \in \Omega \}, \) i.e., \( I(x_0) \) is the set of points
that are indistinguishable from $x_0$ at time $t_0$ from knowledge of the past outputs under all admissible controls.

If $U$ is a neighborhood of $x_0$, then we define

$$I_U(x_0) = \{ x : x \in I(x_0), \psi(t; x, t_0, u) \in U, \text{for all } \bar{t} \leq t \leq t_0, \text{ and all } u[\bar{t}, t_0] \in \Omega \}$$

**Definition:** System (1.7) is said to be observable at $x_0$ if $I(x_0) = x_0$, and it is said to be observable if this is true for all $x_0 \in \mathbb{R}^n$.

**Definition:** System (1.7) is said to be locally observable at $x_0$ if for every neighborhood $U$ of $x_0$, $I_U(x_0) = x_0$, and it is said to be locally observable if this is true for all $x_0 \in \mathbb{R}^n$.

**Definition:** System (1.7) is said to be weakly observable at $x_0$ if there exists a neighborhood $U$ of $x_0$ such that $I(x_0) \cap U = x_0$, and it is said to be weakly observable if this is true for all $x_0 \in \mathbb{R}^n$.

**Definition:** System (1.7) is said to be locally weakly observable at $x_0$ if there exists a neighborhood $U$ of $x_0$ such that for every neighborhood $V$ of $x_0$ contained in $U$, $I_V(x_0) = x_0$, and it is said to be locally weakly observable if this is true for all $x_0 \in \mathbb{R}^n$.

It can be seen that the following implication holds.

\[
\begin{align*}
\text{locally observable} & \quad \Rightarrow \quad \text{observable} \\
\downarrow & \\
\text{locally weakly observable} & \quad \Rightarrow \quad \text{weakly observable}
\end{align*}
\]

We will also review some linear system theory. Consider system (1.5)

\[
\begin{align*}
\dot{\xi} &= A\xi + \alpha(\phi) \\
\phi &= C\xi
\end{align*}
\]  

(1.5)
where $\alpha(0) = 0$. This system can be considered as a linear system driven by an input.

Let $\gamma(t; \xi_0, t_0)$ be the state of system (1.5) at time $t$ that originated from $\xi_0$ at $t_0$, and let $\phi(\gamma(t; \xi_0, t_0))$ be the corresponding output.

The state and output equations of system (1.5) can be written as the following integral equations.

\[
\xi(t) = \Phi(t, t_0) \xi_0 + \int_{t_0}^{t} \Phi(t, s) \alpha(\phi) ds, \tag{3.28}
\]

\[
\phi(t) = C\Phi(t, t_0) \xi_0 + C \int_{t_0}^{t} \Phi(t, s) \alpha(\phi) ds. \tag{3.29}
\]

Clearly, if $\xi_0 = 0$, then, since $\alpha(0) = 0$, $\phi(t) = 0$, for all $t$. Hence, for system (1.5), we can state our definition of unobservability as follows:

**Definition**: System (1.5) is unobservable at $\xi_0$

if $\xi_0$ is indistinguishable from 0 for all past outputs

iff $\phi(\gamma(t; \xi_0, t_0)) = 0$, for $t \leq t_0$.

Recall from linear system theory the following lemma.

**Lemma 3.4** System (1.5) is observable iff the observability rank condition holds.

**Lemma 3.5** System (1.5) is locally observable iff the observability rank condition holds.

We will sketch the proof below.

**Proof**: Consider the integral equations (3.28) and (3.29) for $t \leq t_0$. From (3.29), we can write

\[
z(t) = C\Phi(t, t_0) \xi_0 \tag{3.30}
\]

where $z(t) = \phi(t) - C \int_{t_0}^{t} \Phi(t, s) \alpha(\phi) ds$. 

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Multiply both sides of (3.30) by $\Phi^T(t, t_0)C^T$ and integrate to obtain

$$\int_{t_0}^{t} \Phi^T(s, t_0)C^T z(s) ds = W(t, t_0) \xi_0,$$

where $W(t, t_0) = \int_{t_0}^{t} \Phi^T(s, t_0)C^T C\Phi(s, t_0) ds$. From a known theorem, $W(t, t_0)$ is nonsingular for $t < t_0$ iff the observability rank condition holds. In fact (Kalman [12])

$$\dim \text{ range } W(t, t_0) = \text{rank } \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}$$

for all $t < t_0$. So if the observability rank condition holds, then $\phi(t) = 0$, for $t < t_0$ would imply $z(t) = 0$, for $t < t_0$, hence $\xi_0 = 0$. So the only point that is indistinguishable from zero is zero, i.e., $I(\xi_0) = \xi_0$. In particular, for any $\epsilon$ neighborhood $U$ of $\xi_0$, and for all $\bar{t} < t_0$ such that $\gamma(t; \xi_0, t_0) \in U$ for $\bar{t} \leq t \leq t_0$, $I_U(\xi_0) = \xi_0$. Conversely, if system (1.5) is locally observable, then it is observable. By lemma 3.4, the observability rank condition holds. Indeed, since $z(t)$ is known from the output $\phi$, we can determine $\xi_0$ from the past output in the above equality for any $\xi_0$ only if $W(t, t_0)$ is nonsingular for $t < t_0$; otherwise if $W(t, t_0)$ is singular for some $t < t_0$, then it is singular for all $t < t_0$ and hence any point $\xi_0 \in \text{Ker } W(t, t_0)$ is unobservable.

**Corollary 3.6** If the observability rank condition holds then system (1.5) is locally weakly observable.

**Proposition 3.7** If the nonlinear system (3.1) is locally linearizable to the linear observable system (1.5) around $x_0$, then the nonlinear system is locally weakly observable at $x_0$.

**Proof:** Let $(\bar{x}, \bar{y})$ be the linearizing pair given as

- $\bar{x} : V \subset R^n \rightarrow \tilde{z}(V) = U$
- $\xi \rightarrow x$

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and
\[
\tilde{y} : W \subset \mathbb{R}^p \rightarrow \tilde{y}(W)
\]

Since \( \tilde{x} \) is continuous, for every neighborhood \( U_1 \) of \( x_0 \) in \( \tilde{U} \), \( \tilde{x}^{-1}(U_1) = V_1 \) is a neighborhood of \( \xi_0 \) such that \( \tilde{x}(\xi_0) = x_0 \). But in the \((\xi, \phi)\) coordinate around \( \xi_0 \), the system is locally weakly observable. This implies that for all \( t_0 \), there exists a \( \bar{t} < t_0 \) such that if

\[
\phi(\gamma(t; \xi; t_0)) = \phi(\gamma(t; \xi_0; t_0)), \quad \forall t \in [\bar{t}, t_0]
\]

and \( \gamma(t; \xi; t_0) \in V_1 \) for all \( t \in [\bar{t}, t_0] \), then \( \xi = \xi_0 \). This implies \( I_{V_1}(\xi_0) = \xi_0 \).

Since \( \tilde{x} \) and \( \tilde{y} \) are local diffeomorphism and hence one to one, and \( \tilde{x}(V_1) = U_1 \), the result follows by mapping trajectory and output in the \((\xi, \phi)\) coordinates into trajectory and output in the \((x, y)\) coordinate, and we have \( I_{U_1}(x_0) = x_0 \).

QED. \&
§3.4 Local asymptotic estimation problem

In this section we will discuss a model which will give an estimate of the state of a nonlinear system. For a linear system of the form

\[ \dot{\xi} = A\xi + Bv \]

\[ \phi = C\xi \]  

(3.31)

the system described by

\[ \dot{\xi} = F\dot{\xi} + K\phi + Hv \]

(3.32)

is called an observer or asymptotic state estimator of system (3.31) if for any \( \dot{\xi}(t_0), \dot{\xi}(t) \) approaches \( \xi(t) \) as \( t \) approaches \( \infty \).

Linear system theory says that (3.32) is an observer for (3.31) iff the observability rank condition holds, and

\[ F = A - KC, \quad H = B. \]

Thus the observer has the form

\[ \dot{\xi} = (A - KC)\dot{\xi} + K\phi + Bv \]

and the estimation error \( e = \xi - \hat{\xi} \) is described by the dynamical equation

\[ \dot{e} = \dot{\xi} - \dot{\hat{\xi}} = (A - KC)e. \]

\( K \) can be chosen so that the characteristic values of \( (A - KC) \) lie in the left half complex plane iff \( (C, A) \) is an observable pair. In such case, \( e(t) \to 0 \) as \( t \to \infty \).
For the nonlinear system (3.1), we can state the local asymptotic estimation problem as follows:

Local asymptotic estimation problem:

For the system (3.1), determine a local asymptotic observer and a neighborhood $U$ of 0 (if possible) such that any state in $U$ can be estimated asymptotically, that is, if $\hat{x}$ is the estimate, then the estimation error $e(t) = (x(t) - \hat{x}(t)) \to 0$ as $t \to \infty$.

We will consider only those nonlinear system that can be transformed to a linear observable system around 0. Our observer design below is based on the linear system and is a slight modification of that given by Bestle and Zeitz [1]. We made the modification since we have allowed change in the output coordinates and our system is multi-output. We observe also that Krener and Isidori [11], and Krener and Respondek [14] also mention a similar design.

From the form of the linear observer it is reasonable to consider the nonlinear observer of the form

$$\dot{x} = f(\hat{x}) + K(\phi - \hat{\phi})$$

$$= f(\hat{x}) + \hat{K}(C\dot{\xi} - C\hat{\xi}),$$

(3.33)

where $\hat{\xi}$ is the estimate of the state of the linear system, $\hat{x} = \hat{x}(\hat{\xi})$ is the estimate of the actual state of the nonlinear system, and $\hat{K}$ is a $n \times p$ matrix which is to be determined so that $\hat{x}(t) \to x(t)$ as $t \to \infty$. From equation (3.33), we have

$$\dot{x} = \frac{\partial \hat{x}}{\partial \xi} \hat{\xi} = f(\hat{x}) + \hat{K}(C\xi - C\hat{\xi})$$

so that

$$\hat{\xi} = (\frac{\partial \hat{x}}{\partial \xi})^{-1} (f(\hat{x}) + \hat{K}(C\xi - C\hat{\xi})).$$
Using (3.4)

\[ \dot{\xi} = A\dot{\xi} + \alpha(\dot{\phi}) + \tilde{K}C(\xi - \dot{\xi}), \]

where \( \tilde{K} = (\frac{\partial^2 \tilde{u}}{\partial \xi^2})^{-1} \tilde{K} \).

Let \( e \) be the observer error in the new coordinates, i.e., \( e = \xi - \dot{\xi} \). So

\[
\dot{e} = \dot{\xi} - \dot{\xi} = A\xi + \alpha(\phi) - (A\dot{\xi} + \alpha(\phi) + \tilde{K}C(\xi - \dot{\xi})) = (A - \tilde{K}C)e + \alpha(\phi) - \alpha(\phi) .
\]

Using the first order approximation for \( \alpha(\phi) - \alpha(\phi) \), we have

\[
\alpha(\phi) - \alpha(\phi) = \frac{\partial \alpha}{\partial \phi} \mid_{\phi} \frac{\partial \phi}{\partial \xi}(\xi - \dot{\xi}) + o(\xi - \dot{\xi}) = \frac{\partial \alpha}{\partial \phi} \mid_{\phi} Ce + o(e) = \frac{\partial \alpha}{\partial \phi} Ce + o(e)
\]

where \( \frac{\partial \alpha}{\partial \phi} \) is \( \frac{\partial \alpha}{\partial \phi} \) evaluated at \( \phi \).

Hence,

\[
\dot{e} = (A - (\tilde{K} - \frac{\partial \hat{\alpha}}{\partial \phi})C)e + o(e) = (A - LC)e + o(e), \tag{3.34}
\]

where \( L = \tilde{K} - \frac{\partial \hat{\alpha}}{\partial \phi} \).

The asymptotics of system (3.34) depend on the characteristic values of \( (A - LC) \). So if \( (C, A) \) is an observable pair, then we can choose \( L \) such that these characteristic values lie in the left half plane. In such case, \( e(t) \rightarrow 0 \), as \( t \rightarrow \infty \).

Now \( \tilde{K} \) in equation (3.33) can be determined as follows:

\[
\hat{K} = \frac{\partial \hat{\alpha}}{\partial \xi} \tilde{K} = \frac{\partial \hat{\alpha}}{\partial \xi} (L + \frac{\partial \hat{\alpha}}{\partial \phi})
\]
where $\frac{\partial \hat{x}}{\partial \xi}$ is $\frac{\partial \check{x}}{\partial \xi}$ evaluated at $\check{x}$. Since the estimation error of the state of the nonlinear system is

$$\hat{x} - x = \check{x}(\xi) - \check{x}(\xi),$$

and $\check{x}$ is continuous, it is clear that $\hat{x} \to x$, as $t \to \infty$.

Hence, the complete observer is

$$\hat{x} = \frac{\partial \hat{x}}{\partial \xi} (L + \frac{\partial \check{x}}{\partial \phi}) C(\xi - \check{\xi})$$

$$= f(\check{x}) + \frac{\partial \hat{x}}{\partial \xi} (L + \frac{\partial \check{x}}{\partial \phi})(\tilde{y}^{-1}(y) - \tilde{y}^{-1}(\check{y})).$$

This observer will be called the local asymptotic observer since the linearization is only valid locally around 0. Note that if the pair $(C, A)$ is in dual Brunovsky canonical form, i.e., $(C, A) = (\tilde{C}, \tilde{A})$, then the determination of the matrix $L$ for given characteristic values of $(A - LC)$ is relatively easy.
CHAPTER 4
Linearization of nonlinear control systems with outputs

§4.1 Necessary and sufficient condition

We now consider the problem of finding the necessary and sufficient condition such that the nonlinear system of the form

\[ \dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i \]

\[ y = h(x) \]

(4.1)
can be transformed locally to a linear system that is both completely controllable and observable of the form

\[ \dot{\xi} = A\xi + Bv \]

\[ \phi = C\xi \]

(4.2)

In particular, we are interested in the form such that \((A, B)\) is in Brunovsky canonical form and \((C, A)\) is in dual Brunovsky canonical form. The class of transformations consists of the type \((C1), (C2), (C3),\) and \((O2)\) discussed in the introduction. It is clear that we require that \(f(0) = 0\), and \(h(0) = 0\) if the state and output coordinate transformations map the origin to origin.

For simplicity, we also assume \(g_1, \ldots, g_m\) are linearly independent and rank \(\frac{\partial h}{\partial x} = p = m\) on some neighborhood of 0. As a consequence of the previous discussions, we have the following result:

**Proposition 4.1** System (4.1) is locally Gco-equivalent to system (4.2) at 0 iff there exists a feedback pair \((w, Q)\) such that the following conditions are satisfied on a neighborhood \(U\) of 0.
(i) For \( i = 1, \ldots, m, \quad j = 1, \ldots, m, \)

(a) \((L_f^k(dh_i), \hat{g}_j) = 0, \quad \text{for} \quad k = 0, \ldots, \kappa_j - 2,\)

(b) the \( m \times m \) matrix \( N = (n_{ij}) \) is nonsingular, where

\[
n_{ij} = (L_f^j)^{-1} (dh_i), \hat{g}_j),
\]

(c) \([\text{ad}^k_f \hat{g}_j, \hat{g}_i] = 0, \quad 1 \leq i, j \leq m, \quad k = 0, \ldots, \kappa_j - 1,\)

(d) \(\text{ad}^\kappa_f \hat{g}_i = 0, \quad 1 \leq i \leq m,\)

where

\[
\kappa_j = \text{the number of } r_i \text{ such that } r_i \geq j.
\]

(ii) \( \dim \tilde{E}_j = \dim \tilde{S}_j = \text{constant}, \quad j = 0, \ldots, n - 1.\)

(iii) \( \dim \tilde{E}_{n-1} = \dim \tilde{S}_{n-1} = n \)

where

\[
\tilde{E}_j = \{L_f^l(dh_i) : i = 1, \ldots, m, l = 0, \ldots, j\}, \quad j = 0, \ldots, n - 1
\]

and

\[
\tilde{E}_j = \text{sp}\{\tau : \tau \in \tilde{E}_j\}
\]

\( \tilde{S}_j, \tilde{S}_j, \hat{f}, \hat{g} \) and \( r_i \) are as defined in chapter 2.

**Proof:** Necessity of the hypotheses is clear if we replace \( u \) by \( w + Qv \) and carry out the steps used to derive the necessary condition in prop. 2.1 and prop. 3.1 with \( f \) replaced by \( \hat{f} \) and \( g_i \) replaced by \( \hat{g}_i \). The necessity of (ii) is also obvious since the controllability indices and observability indices defined by the pair \((\hat{A}, \hat{B})\) and \((\hat{C}, \hat{A})\) respectively must agree.

For sufficiency, we proceed in a similar way as that used to prove proposition 3.1. Let the map \( \tilde{x}(\xi) \) be defined by

\[
\frac{\partial \tilde{x}}{\partial \xi_{ik}} = (-1)^{\kappa_i - k} \text{ad}^\kappa_f^{-k} \hat{g}_i, \quad \text{for} \quad k = 1, \ldots, \kappa_i, \quad (4.3)
\]

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and \(\tilde{x}(0) = 0\). (4.3) is solvable for \(\tilde{x}(\xi)\) by conditions (i-c) and (i-d).

Let
\[
\xi = \tilde{\xi}(x) = \tilde{x}^{-1}(x),
\]
then
\[
\dot{\xi} = \frac{\partial \tilde{\xi}}{\partial x} \dot{x} = \frac{\partial \tilde{\xi}}{\partial x} (f(x) + G(x)w) = \frac{\partial \tilde{\xi}}{\partial x} (f(x) + G(x)(w + Qv)) = \frac{\partial \tilde{\xi}}{\partial x} (\hat{f}(x) + \hat{G}(x)v) = \frac{\partial \tilde{\xi}}{\partial x} \hat{f}(x) + \frac{\partial \tilde{\xi}}{\partial x} (\sum_{i=1}^{m} \hat{g}_i(x)v_i)
\]
\[
\dot{\xi} = \hat{f}(\xi) + \sum_{i=1}^{m} \hat{g}_i(\xi)v_i,
\]
where \(\hat{f}(\xi) = \frac{\partial \tilde{\xi}}{\partial x} \hat{f}(\tilde{x}(\xi))\), \(\hat{g}_i(\xi) = \frac{\partial \tilde{\xi}}{\partial x} \hat{g}_i(\tilde{x}(\xi))\), and \(\hat{f}(x)\) and \(\hat{g}_i(x)\) are as defined before. Multiplying both sides of (4.3) by \(\frac{\partial \tilde{\xi}}{\partial x}\), we have
\[
\frac{\partial \tilde{\xi}}{\partial x} \frac{\partial \tilde{x}}{\partial \xi_{ik}} = \frac{\partial \tilde{\xi}}{\partial x} (-1)^{\kappa - k} \text{ad}_{\hat{f}}^{\kappa - k} \hat{g}_i, \quad \text{for} \quad k = 1, \ldots, \kappa_i. (4.4)
\]

Proceeding as in the proof of proposition 3.1, we have
\[
\hat{f}_{ik}(\xi) = \xi_{ik+1} + \alpha_{ik}(\xi_1, \ldots, \xi_{m_1}), \quad \text{for} \quad i = 1, \ldots, m, \quad k = 1, \ldots, \kappa_i - 1,
\]
\[
\hat{f}_{i\kappa_i}(\xi) = \alpha_{i\kappa_i}(\xi_1, \ldots, \xi_{m_1}). (4.5)
\]

Letting \(k = \kappa_i\) in equation (4.4), we have
\[
e_{i\kappa_i} = \hat{g}_i(\xi), \quad \text{for} \quad i = 1, \ldots, m.
\]
Moreover, for $i = 1, \ldots, m$,

$$
\frac{\partial f(\xi)}{\partial \xi_1} = \left[ \frac{\partial}{\partial \xi_1}, f(\xi) \right] = \left[ \frac{\partial \tilde{x}}{\partial x}, \frac{\partial \tilde{z}}{\partial \xi_1}, \frac{\partial \tilde{\xi}}{\partial x} \right] \hat{f}(\xi(x))
$$

$$
= \frac{\partial \tilde{\xi}}{\partial x} \left[ \frac{\partial \tilde{x}}{\partial \xi_1}, \frac{\partial \tilde{z}}{\partial x}, f(x) \right]
$$

$$
= \frac{\partial \tilde{\xi}}{\partial x} \left[ (-1)^{\kappa_i-1} \text{ad}_f^{\kappa_i-1} \hat{g}_i(x), \hat{f}(x) \right]
$$

$$
= \frac{\partial \tilde{\xi}}{\partial x} \left[ (-1)^{\kappa_i} [\hat{f}(x), \text{ad}_f^{\kappa_i-1} \hat{g}_i(x)] \right]
$$

$$
= \frac{\partial \tilde{\xi}}{\partial x} \left[ (-1)^{\kappa_i} \text{ad}_f^{\kappa_i} \hat{g}_i(x) \right]
$$

$$
= 0,
$$

where the last equality follows from condition (i-d). This shows that $\alpha_{ik}$ can only be constants. But by a remark following the proof of proposition 3.1, $\alpha_{ik}(0) = 0$. Hence $\alpha_{ik} = 0$, for $i = 1, \ldots, m$, $k = 1, \ldots, \kappa_i$.

Clearly in the $\xi$ coordinates, the system has the form

$$
\dot{\xi} = \tilde{A}\xi + \tilde{B}v.
$$

The rest of the proof is the same as that given in proposition 3.1. QED. \( \Box \)

Remark. From the proof of the sufficiency of the hypotheses in proposition 4.1. We have also shown a direct proof that hypothesis (B) in proposition 2.1 is sufficient that system (2.1) can be transformed to the system in Brunovsky canonical form. Note also that we have shown that $\tilde{S}_j = \hat{S}_j$ in lemma 2.2 under the hypotheses of (i-c) and (i-d) so we can replace $\hat{S}_j$ by $\tilde{S}_j$ in hypotheses (ii) and (iii). Moreover, it is clear that $\tilde{S}_j$ is required to be involutive for $j = 0, \ldots, n - 1$, and $\dim \tilde{S}_{n-1} = n$ on a neighborhood of 0. However, it is not yet clear that $\tilde{E}_j = \hat{E}_j$ under the other hypotheses in the proposition. But under certain condition, we can show that this is true.

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Let
\[ E = \{ L^k_f(dh_i) : i = 1, \ldots, m, k = 0, \ldots, \kappa_i - 1 \} \]
be the set of \( n \) linearly independent covector fields. This is possible by hypotheses (ii) and (iii), perhaps after rearranging the entries in the vector valued function \( h(x) \). As before we can form the array (2.18) of covector fields with \( X^j_i \) denoting \( L^j_f(dh_i) \) and the \( Y \)'s being defined similarly.

**Lemma 4.2** Under the hypotheses of proposition 4.1 with
\[ n_{ij} = \delta^j_i, \quad (4.6) \]
then
\[ \langle L^k_f(dh_i), \hat{g}_j \rangle = 0, \quad \text{for} \quad k = 0, \ldots, \kappa_i - 2, \quad j = 1, \ldots, m. \quad (4.7) \]

**Proof:** Obviously (4.7) holds for \( j = i \) by hypothesis (i-a). For \( j \neq i \), by hypothesis (4.6)
\[ \langle L^\kappa_f^{-1}(dh_i), \hat{g}_j \rangle = 0. \quad (4.8) \]
Hence
\[ \langle L^k_f(dh_i), \hat{g}_j \rangle = 0, \quad \text{for} \quad k = 0, \ldots, \kappa_j - 1, \quad (4.9) \]
and we have
\[ \langle L^k_f(dh_i), \text{ad}^l_f \hat{g}_j \rangle = 0, \quad \text{for} \quad k + l \leq \kappa_j - 1, \quad (4.10) \]
as seen by the proof of lemma 2.7 or lemma 3.3. Now
\[ \langle L^\kappa_f^{-1}(dh_i), \hat{g}_j \rangle = L_f \langle L^\kappa_f^{-1}(dh_i), \hat{g}_j \rangle - \langle L^\kappa_f^{-1}(dh_i), \text{ad}^l_f \hat{g}_j \rangle. \]
The first term of the right hand side of the above equation is zero by (4.10). Hence
\[ \langle L^\kappa_f^{-1}(dh_i), \hat{g}_j \rangle = -\langle L^\kappa_f^{-1}(dh_i), \text{ad}^l_f \hat{g}_j \rangle. \]
Repeatedly applying the Leibnitz type formula and (4.10), we find that

\[ \langle L_f^{k+1}(dh_i), g_j \rangle = (\langle L_f^k(dh_i), \hat{g}_j \rangle \hat{g}_j \rangle = 0, \]

where the last equality follows from hypothesis (i-d). Now

\[ \langle L_f^k(dh_i), \hat{g}_j \rangle = 0, \quad \text{for} \quad k = 0, \ldots, \kappa_j. \]

This leads to

\[ \langle L_f^k(dh_i), \text{ad}_{f}^{l} \hat{g}_j \rangle = 0, \quad \text{for} \quad k + l \leq \kappa_j. \]

Clearly, continuing in a similar manner we have

\[ \langle L_f^k(dh_i), \hat{g}_j \rangle = 0, \quad \text{for} \quad k \geq 0. \quad (4.11) \]

In particular, (4.11) holds for \( k = 0, \ldots, \kappa_i - 2 \). QED. \( \Box \)

**Lemma 4.3** Under the hypotheses of proposition 4.1 with condition (4.6),

\[ L_f^k(dh_i) = L_f^k(dh_i), \quad i = 1, \ldots, m, \quad k = 0, \ldots, \kappa_i - 1. \quad (4.12) \]

**Proof:** By induction, for \( i = 1, \ldots, m \), (4.12) holds for \( k = 0 \). If \( \kappa_i = 1 \), then no proof is needed. If \( \kappa_i \geq 2 \), then assume (4.12) holds for some \( k \leq \kappa_i - 2 \). Now

\[ L_f^{k+1}(dh_i) = L_f(L_f^k(dh_i)) \]

\[ = L_f + \sum_{j=1}^{m} g_J w_J (L_f^k(dh_i)) \]

\[ = L_f(L_f^k(dh_i)) + \sum_{j=1}^{m} L_{g_j w_j} (L_f^k(dh_i)) \]

\[ = L_f^{k+1}(dh_i) + \sum_{j=1}^{m} (L_{g_j}(L_f^k(dh_i))) w_j + (\langle L_f^k(dh_i), g_j \rangle) dw_j \]

\[ = L_f^{k+1}(dh_i) + \sum_{j=1}^{m} (d(\langle L_f^k(dh_i), g_j \rangle)) w_j + (\langle L_f^k(dh_i), g_j \rangle) dw_j. \]
We have used the fact that

\[ L_g(dh) = dL_g(h), \]

where \( g \in V(R^n) \) and \( h \in C(R^n) \). By lemma 4.2, the second and third terms are equal to zero since

\[ sp\{g_1, \ldots, g_m\} = sp\{\hat{g}_1, \ldots, \hat{g}_m\}. \]

Hence

\[ L^k_j(dh_i) = L^k_j(dh_i). \]

This proves (4.12) for \( k \leq \kappa_i - 1 \). QED. \\

**Proposition 4.4** Under the hypotheses of proposition 4.1 with condition (4.6), then

\[ E_j = E_j, \quad \text{for} \quad j = 0, \ldots, n - 1, \]

iff

\[ d(\hat{Q}_i w) \in sp\{r : r \in \hat{E}_j\}, \quad i = 1, \ldots, m, \quad (4.13) \]

where \( \hat{Q}_i \) is the \( i \)th row of \( \hat{Q} = Q^{-1} \).

**Proof:** Let

\[ E = \{L^k_j(dh_i) : i = 1, \ldots, m, k = 0, \ldots, \kappa_i - 1\}. \]

Clearly

\[ \hat{E}_j \subseteq E_j, \quad \text{for} \quad j = 0, \ldots, n - 1, \]

since by (4.12) and the definition of \( \hat{E} \),

\[ L^k_j(dh_i) \in sp\{r : r \in \hat{E} \cap \hat{E}_j\} = sp\{r : r \in E \cap E_j\} \]
for \( j \geq 0, i = 1, \ldots, m \). Since \( \hat{G} = GQ \), then \( G = \hat{G} \hat{Q} \) where \( \hat{Q} = Q^{-1} \), i.e.

\[
g_j = \sum_{i=1}^{m} \hat{g}_i \hat{q}_{ij},
\]

where \( \hat{g}_{ij} \) is the \( ij \)th entry of \( \hat{Q} \).

So, for \( i = 1, \ldots, m \),

\[
L_f^{\kappa_i} (dh_i) = L_f (L_f^{-1}^{\kappa_i} (dh_i))
\]

\[
= L_f - \sum_{j=1}^{m} g_j w_j (L_f^{-1}^{\kappa_i} (dh_i))
\]

\[
= L_f (L_f^{-1}^{\kappa_i} (dh_i)) - \sum_{j=1}^{m} L_g w_j (L_f^{-1}^{\kappa_i} (dh_i))
\]

\[
= L_f^{\kappa_i} (dh_i) - \sum_{j=1}^{m} ((L_f^{-1}^{\kappa_i} (dh_i), g_j)) w_j + ((L_f^{-1}^{\kappa_i} (dh_i), g_j)) d w_j
\]

\[
= L_f^{\kappa_i} (dh_i) - \sum_{j=1}^{m} \sum_{l=1}^{m} (d((L_f^{\kappa_i} (dh_i), \hat{g}_i \hat{q}_{lj})) w_j
\]

\[
+ ((L_f^{\kappa_i} (dh_i), \hat{g}_i \hat{q}_{lj})) d w_j
\]

\[
= L_f^{\kappa_i} (dh_i) - \sum_{j=1}^{m} (d(\hat{q}_{ij}) w_j + (\hat{q}_{ij}) d w_j)
\]

\[
= L_f^{\kappa_i} (dh_i) - \sum_{j=1}^{m} d(\hat{q}_{ij} w_j)
\]

\[
= L_f^{\kappa_i} (dh_i) - d(\hat{Q} w).
\]

Hence

\[
L_f^{\kappa_i} (dh_i) \in \bar{E}_{\kappa_i} \quad \text{iff} \quad d(\hat{Q} w) \in \bar{E}_{\kappa_i}.
\]

Noting the fact that if

\[
L_f^{\kappa_i} (dh_i) \in \text{sp}\{L_f^{\kappa_i} (dh_1), \ldots, L_f^{\kappa_i} (dh_{i-1})\} + \bar{E}_{k-1},
\]

then

\[
L_f^{\kappa_i+1} (dh_i) \in \text{sp}\{L_f^{\kappa_i+1} (dh_1), \ldots, L_f^{\kappa_i+1} (dh_{i-1})\} + \bar{E}_{k}.
\]

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Therefore

$$\tilde{E}_j \subset \tilde{E}_j, \quad j = 0, \ldots, n - 1 \quad \text{iff} \quad d(\tilde{Q}_i w) \in \tilde{E}_{\kappa_i}, \quad i = 1, \ldots, m,$$

and the result follows. QED. ||

**Remark.** Note that condition (4.6) is the one that requires

$$\frac{\partial \tilde{u}}{\partial \phi} = I,$$

that is, no change in output coordinates other than a rearrangement of the entries of $h(x)$.

In summary, if the class of transformations consists only of (C1), (C2), (C3), and a permutation of the output coordinates, then it is necessary that $n_{ij} = \delta_{i}^{j}$ in hypothesis (i-b) of proposition 4.1. Since condition (4.13) is not known to be necessary, all we can say about the dimensions of $E_j$ and $\tilde{S}_j$ is that, on a neighborhood of 0

(ii') $\dim E_j \geq \dim \tilde{S}_j, \quad j = 0, \ldots, n - 1,$

(iii') $\dim E_{n-1} = \dim \tilde{S}_{n-1} = n.$

**Remark.** We see that conditions (ii'), (iii'), and the involutivity requirement of the distribution $\tilde{S}_j$ for $j = 0, \ldots, n - 1$ can be checked from the known data of the system (4.1). Also the conditions that $\dim \tilde{S}_{n-1} = n$ and $\dim E_{n-1} = n$ are just the nonlinear analogues of the controllability rank condition and the observability rank condition respectively discussed earlier for the linear system.
§4.2 Example

Consider system (4.1) with \( f, g_1, g_2 \) as in the example of chapter 2; that is,

\[
f(x) = \begin{pmatrix} \sqrt{1-x_1^2}(x_2 - \ln(x_3 + 1)) \\ \sin x_1 \\ 0 \end{pmatrix},
\]
\[
g_1(x) = \begin{pmatrix} 0 \\ \cos x_1 \\ 0 \end{pmatrix}, \quad g_2(x) = \begin{pmatrix} 0 \\ 0 \\ (x_3 + 1)e^{x_2} \end{pmatrix},
\]

and

\[
h(x) = \begin{pmatrix} x_1 \\ \ln(1 + x_3) \end{pmatrix}.
\]

Recall we found that

\[
w(x) = \begin{pmatrix} -\tan x_1 \\ 0 \end{pmatrix}, \quad Q(x) = \begin{pmatrix} \sec x_1 & \sec x_1 \\ 0 & e^{-x_2} \end{pmatrix}
\]
\[
\hat{f}(x) = \begin{pmatrix} \sqrt{1-x_1^2}(x_2 - \ln(x_3 + 1)) \\ 0 \\ 0 \end{pmatrix},
\]
\[
\hat{g}_1(x) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \hat{g}_2(x) = \begin{pmatrix} 0 \\ 1 \\ (x_3 + 1) \end{pmatrix},
\]

and

\[
dh = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \frac{1}{x_3+1} \end{pmatrix}
\]
\[
L_f(dh) = \begin{pmatrix} \frac{-x_1}{\sqrt{1-x_1^2}}(x_2 - \ln(1 + x_3)) & \sqrt{1-x_1^2} \\ 0 & 0 & \frac{-x_1}{x_3+1} \end{pmatrix}.
\]

So

\[
\langle dh, \hat{g}_1 \rangle = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \langle dh, \hat{g}_2 \rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \langle L_f(dh), \hat{g}_1 \rangle = \begin{pmatrix} \sqrt{1-x_1^2} \\ 0 \end{pmatrix}.
\]

We see that the hypotheses in prop. 4.1 are satisfied on \( U \) with \( \kappa_1 = 2 \), and \( \kappa_2 = 1 \), where \( U = \{ x : |x| < 1, x_3 > -1 \} \).

It can be checked that the state and output coordinate changes are given by

\[
x = \begin{pmatrix} \sin \xi_1 \\ \xi_2 + \xi_3 \\ e^{\xi_3} - 1 \end{pmatrix}, \quad y = \begin{pmatrix} \sin \phi_1 \\ \phi_2 \end{pmatrix}.
\]
§4.3 Local asymptotic estimation and control

Nonlinear systems of the form (4.1) that can be transformed to a linear system (4.2) which is completely controllable and observable may be rare, but from the discussions of the previous chapters, for such systems we expect that the nonlinear system can be locally estimated and stabilized. Indeed from the result of section 2.4, the local asymptotic control law is

\[ u = w(\hat{x}) - Q(\hat{x})P\hat{x}^{-1}(\hat{x}) \]  

(4.14)

where \( P \) is chosen that all the characteristic values of \((A - BP)\) have negative real parts and the estimated state is used for the synthesis of the control since the state of the nonlinear system is unavailable by assumption. And from the result of section 3.4 and the form of observer for the linear system with controls, it is clear that the nonlinear observer for the system (4.1) should have the form

\[ \dot{x} = f(\hat{x}) + G(\hat{x})u + \hat{K}(\phi - \hat{\phi}), \]  

(4.15)

where \( u = w(\hat{x}) + Q(\hat{x})v, \hat{x} = \hat{\hat{x}}(\hat{\xi}), \) and \((\hat{x}, w, Q)\) is the linearizing triple. Hence

\[ \dot{\hat{x}} = \hat{f}(\hat{x}) + \hat{G}(\hat{x})v + \hat{K}(\phi - \hat{\phi}), \]  

(4.16)

where \( \hat{f}(\hat{x}) = f(\hat{x}) + G(\hat{x})w(\hat{x}) \) and \( \hat{G}(\hat{x}) = G(\hat{x})Q(\hat{x}) \). Furthermore, by the derivation in section 3.4 and the proof of proposition 4.1, we have

\[ \dot{\xi} = \frac{\partial \hat{x}}{\partial \xi} \dot{\hat{\xi}} = \hat{f}(\hat{x}) + \hat{G}(\hat{x})v + \hat{K}(C\xi - C\hat{\xi}), \]

so that

\[ \dot{\xi} = (\frac{\partial \hat{x}}{\partial \xi})^{-1}(\hat{f}(\hat{x}) + \hat{G}(\hat{x})v + \hat{K}(C\xi - C\hat{\xi})) \]

\[ = A\hat{\xi} + Bv + \hat{K}C(\xi - \hat{\xi}), \]
where $\tilde{K} = (\frac{\partial \tilde{x}}{\partial \xi})^{-1} \tilde{K}$. Hence the observer error in the new coordinates satisfies

$$\dot{e} = \dot{\xi} - \dot{x} = A\xi + Bv - (A\dot{\xi} + Bv + \tilde{K}C(\xi - \dot{\xi})) = (A - \tilde{K}C)e.$$

Since the pair $(C, A)$ is observable, we can choose $\tilde{K}$ such that the characteristic polynomial of $(A - \tilde{K}C)$ is as desired. In particular, if all the characteristic values of $(A - \tilde{K}C)$ have negative real parts, then $e(t) \to 0$ as $t \to \infty$. In such case, we will call (4.15) or (4.16) the local asymptotic observer for the system (4.1). For the system (4.1) we will define the local regulator as the system comprising of the local asymptotic control law (4.14) and the local asymptotic observer (4.15). From a result called the separation principle in linear system theory (Kailath [11]), we know that a regulator for a linear system is stable if the linear system is both completely controllable and observable. Since state and output of a nonlinear system can be mapped homeomorphically onto those of a linear system if the nonlinear system is linearizable, we readily see that the above controller and observer design serves the purpose of locally stabilizing the nonlinear system. The nonlinear system (4.1) and the local regulator are depicted in figure 1.
Fig. 1. Local regulator
Bibliography


