A PARTIAL LOGIC OF DESCRIPTIONS

by

PETER J. APOSTOLI

Hnrs. B. A. The University of British Columbia, November 1984

A THESIS SUBMITTED IN PATIAL FULFILLMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF ARTS

in

THE FACULTY OF GRADUATE STUDIES

Department of Philosophy

We accept this thesis as conforming

to the required standard

THE UNIVERSITY OF BRITISH COLUMBIA

November 1986

Department of Philosophy

The University of British Columbia
1956 Main Mall
Vancouver, Canada
V6T 1Y3

Date Oct. 14 1986

Abstract

Let a "partial logic" for a first order predicate language $\mathbb{L}$ be a formal proof-theory $\mathbb{PT}$ for sentences of $\mathbb{L}$ together with a model theoretic semantics for $\mathbb{PT}$ which can be considered a generalization of classical first-order Tarskian semantics in the following sense: if $\mathbb{M}$ is a model for $\mathbb{PT}$ then $\mathbb{M}$ is a **partial** function from the set of sentences of $\mathbb{L}$ into the set $\{T, F\}$ of classical truth values such that 1) every atomic sentence of $\mathbb{L}$ receives exactly one truth value, and 2) if $\mathbb{M}$ agrees with a given Tarskian model $\mathbb{TM}$ on the assignment of truth values to the atomic sentences of $\mathbb{L}$, then $\mathbb{M}$ agrees with $\mathbb{TM}$ everywhere $\mathbb{M}$ is defined. In this paper we utilize formal techniques developed by P. C. Gilmore for intensional set theories without excluded middle to present a sound and complete partial logic Pld for the first order predicate calculus with definite descriptions. Pld utilizes truth value gaps to systematically treat symbolic sentences that contain "improper" description terms, and can be seen as an acceptable formalization of the Strawsonian view that the semantic-well-formedness of a grammatically subject-predicate sentence of English presupposes the propriety of any definite description occurring as subject term therein.

Contents

Acknowledgements

Many thanks are due to the Department of Philosophy for the support I received during my Masters work. I also want to thank the people in the Department of Computer Science (in particular, Paul Gilmore and Akira Kanda) for their generosity. Special thanks are due to Dick Robinson and Howard Jackson for their speed-reading abilities.

Section 0: Introduction

As is well known, there has been considerable debate among language philosophers concerning the proper treatment of sentences of natural language which contain improper definite descriptions. In particular, there has been heated debate over the question of whether or not grammatically subject-predicate sentences which contain improper definite descriptions as subject terms merit truth values. In contrast, logicians, although differing in their particular logistic methods for treating symbolic sentences containing "improper" description terms, have (historically) been virtually unaminous in their religious belief that *all* sentences of a formal description calculus should be evaluated. Indeed, Russell and Frege, although on opposte sides of the natural language debate, both offer proof-theoretic treatments of description terms which require (i.e., are sound and complete with respect to) semantics that close truth value gaps for all symbolic sentences.

The purpose of this paper is to challenge the assumption that any acceptable description calculus should maintain the law of excluded middle. Utilizing formal techniques originally developed for intensional set theories without excluded middle by P.C. Gilmore in the papers "Combining Unrestricted Abstraction with Universal Quantification" [1] and "Natural Deduction Based Set Theories: A New Resolution of Old Paradoxes" [2], I present a sound and complete "partial logic of descriptions" (abbreviated as 'Pld') which systematically employs truth value gaps in the treatment of certain symbolic sentences containing improper description terms. The semantic treatment such sentences receive in Pld may be seen as a formalization of the Strawsonian [1] view that the truth evaluation of an English subject-predicate sentence which contains a definite description as subject term **presupposes** the propriety of that description, since the symbolic translation of such a sentence is evaluated relative to a given model of Pld only **after** the propriety of every description term occurring in the symbolic sentence has been established relative to the given model.

By a "partial logic" for a first order predicate language $\mathbb{L}$ we intend a formal proof-theory $\mathbb{PT}$

for sentences of $\mathbb{L}$ together with a model theoretic semantics for $\mathbb{PT}$ which can be considered a **generalization** of classical first-order Tarskian semantics in the following sense: if $\mathbb{M}$ is a model for $\mathbb{PT}$ then $\mathbb{M}$ is a **partial** function from the set of sentences of $\mathbb{L}$ into the set $\{T, F\}$ of classical truth values such that 1) every atomic sentence of $\mathbb{L}$ recieves exactly one truth value, and 2) if $\mathbb{M}$ agrees with a given Tarskian model $\mathbb{TM}$ on the assignment of truth values to the atomic sentences of $\mathbb{L}$, then $\mathbb{M}$ agrees with $\mathbb{TM}$ everywhere $\mathbb{M}$ is defined.

Partial logics are not to be confused with many-valued logics, say, three-valued systems in which the third value is intended to represent a classical-truth-value gap, since models for such systems are total assignment functions. (Intuitively, where the recursive definition of truth for such a three-valued system assigns the third value to a sentence S on the basis of a **succeeding** computation, we treat such sentences S by computations which **fail.**)

So a partial logic is simply a logical system which allows truth value gaps, and Pld is simply a partial logic for the first order identity calculus with descriptions in which certain symbolic sentences which contain description terms that lack descriptums relative to a given model $\mathbb{M}$ do not receive a truth value relative to $\mathbb{M}$. We may characterize virtually all other proof-theoretic treatments of descriptions (or at least the historically important ones) as offering **totalizing** solutions to the problems presented by improper description terms in the sense that they are sound with respect to total models only, that is, models in which excluded middle holds.

In this section we give a brief outline of the three historically important proof-theoretic solutions (i.e., those proposed by Hilbert and Bernays [1], Frege [1], [2], and Whitehead and Russell [1]) to the problem posed by improper description terms occurring in symbolic sentences and appraise the relative merits of Pld visa-vis these totalizing solutions. Then in sections 1. - 5. we set out the syntax and semantics for Pld and prove that Pld is both sound and complete. In section 6 we discuss what we call the "Frege-Strawson Doctrine" (F-S D), the view that grammatically singular subject-predicate sentences of a natural language that contain improper definite descriptions as subject terms have no truth value, and evaluate to what extent Pld can be said to be a formalization of that doctrine. We will see that Pld treats in a natural way the thorny problem of integrating the F-S D with a general theory of truth for complex symbolic sentences.

Any proof-theoretic treatment of descriptions has to treat in some acceptable way the problem that the classically valid inference forms of excluded middle and existential generalization / universal specification are not (in general) intuitively sound in languages containing syntactically well-formed but semantically vacuous singular terms. The failure of existential generalization / universal specification in languages containing syntactically well-formed but semantically vacuous singular terms is self explanatory. The argument that the occurrence of improper description terms in symbolic sentences threatens the validity of excluded middle in formal systems is essentially the same as the argument that vacuous singuar terms occurring as subject terms in grammatically subject-predicate sentences of a natural language threaten the legitimacy of excluded middle for natural languages:

Let the "naive" view of the semantics of grammatically singular subject-predicate sentences of a natural language be the view that the utterer of a such a sentence S utters a sentence that is true, respectively, false relative to a context of utterance $\mathbb{C}$ just in case the grammatical predicate of S "applies" relative to $\mathbb{C}$ to the object "picked out" relative to $\mathbb{C}$ by the subject term of **S.** Note that this account has the same general form as has the account usually given of the truth evaluation relative to a model $\mathbb{M}$ of an atomic sentence $\ulcorner \mathbf{P}(t) \urcorner$ of a formal language (for simplicity, we assume $\ulcorner \mathbf{P}(t) \urcorner$ consists of the unary predicate letter '**P**' followed by singular term 't', although this restriction is not essential to the point I wish to make): $\ulcorner \mathbf{P}(t) \urcorner$ is true, respectively, false, relative to $\mathbb{M}$ just in case the object $\mathbb{M}$ assigns to 't' belongs, respectively, does not belong, to the set of objects that $\mathbb{M}$ assigns to '**P**'.

Now, it seems a harmless rephrasing of the naive view to say that the utterer of a grammatically singular subject-predicate sentence S of a natural language utters a sentence that is true, false relative to $\mathbb{C}$ just in case the subject term of S "picks out" an object relative to $\mathbb{C}$ **and** the grammatical predicate of S applies, does not apply, relative to $\mathbb{C}$, to that object. In other words, it seems that the naive view of how singular subject-predicate sentence of a natural language come to have truth values (relative to a context of utterance) **presupposes** (Frege [1], Stawson [1]) that the subject terms of those sentences have referents (relative to that context of utterance).

Similarily, we can rephrase the above account of the evaluation of the symbolic sentence $\ulcorner \mathbf{P}(t) \urcorner$

relative to $\mathbb{M}$ as follows: $\ulcorner \mathbf{P}(t) \urcorner$ is true, false, relative to $\mathbb{M}$ just in case $\mathbb{M}$ assigns some object to '*t*' **and** that object belongs, respectively, does not belong, to the set of objects that $\mathbb{M}$ assigns to '**P**'. So it appears that the standard method of evaluating the truth of symbolic sentences **A** relative to a model $\mathbb{M}$ presupposes that the singular terms (or at least those of the primitive basis of the language) occurring in **A** have denotations relative to $\mathbb{M}$. Of course, as Russell points out in "On Denoting" [1], we cannot simply assign an arbitrary truth value (say, False) to sentences which contain improper definite descriptions as subject terms and maintain classical negation without breaching the law of noncontradiction.

Now Russell [1], explicitly states that Theory of Definite Description is intended to solve the problem of the "apparent" failure of excluded middle in English posed by improper definite descriptions occurring as subject terms of grammaticaly subject-predicate sentences of English. Indeed, the Russellian Theory of Descriptions does not maintain the validity of existential generalization / universal specification as applied to descriptions. In contrast, Frege, who first articulated the view that (at least subject-predicate) sentences of natural language that contain vacuous singular subject terms do not have truth values, seems concerned with the *general* problem of the failure of certain intuitively acceptable modes of inference in languages that contain syntactically well formed but semantically vacuous singular terms. Frege [1, pp. 70] writes

> it is a defect of languages that expressions are possible within them, which, in their grammatical form, seemingly determined to designate an object, nevertheless do not fulfill this condition in special cases . . . . it is customary in logic texts to warn against the ambiguity of expressions as a source of fallicies . . . . it is a least appropriate to issue a warning against apparent names that have no designation ;

Although Frege does not explicitly mention the intuitive modes of reasoning he feels that "logically perfect" languages should support, Frege's famous dictum, intended as definitive of the notion of a formal language,

> that in a logically perfect language (logical symbolism) every expression constructed as a name in a grammatically correct manner out of already introduced symbols, in fact designate an object .

certainly ensures that any reasonable formal semantic theory that satisfies it will maintain both excluded middle and unrestricted generalization / specification as valid forms of inference.

The three proof-theoretic treatment of descriptions we are concerned with comparing to Pld - Hibert and Bernay's [1] treatment of descriptive terms in a first-order arithmetic language, Frege's methods [1], [2], and the Whitehead-Russell [1] method presented in P.M.- can best be compared in terms of the above dictum: while the Hilbert - Bernays and the Fregean treatments are based upon different methods of satisfying this same dictum, the Russell's Theory of Descriptions is based upon a *rejection* of the dictum.

For the purposes of the following discussion we introduce the strict-substitution operator $[t / x]$, which, when applied to a formula or term E, yields the result $[t / x]E$ of substituting t for all free occurrences of x in E. Renaming of variables bound in the formula or term is automatically effected so that any free occurrence of a variable in t is a free occurrence in the result of the application.

1) Russell's method: The proof-theoretic treatment of description terms that Russell presents in P.M. is based upon the semantic intuition motivating the Russelian Theory of Descriptions, that the propriety of an English definite description (that is, the condition that there exists a unique object satisfying the defining formula of the definite description) is part of the semantic content or *meaning* of the sentences in which the description occurs. Accordingly, a grammaticaly subject-predicate sentence S of English containing a definite description D.D as subject term is treated as semantically equivalent to, and an abreviation of, (the material mode equivalent of) the quantificational statement that there exists a unique object x satisfying the defining formula of D.D. and x bears the grammatical predicate of S.

In P.M., Russell provides a proof-theoretic formalization of the Theory of Descriptions in terms of the following contextual definiton schema:

$$[(\iota x)\Phi \ / y]F \equiv_{\mathrm{df}} (\exists_y)[(\forall_x)(\Phi \leftrightarrow x = y) \wedge F]$$

where Φ is an arbitrary symbolic formula containing at most free occurrence of the variable x, and F is an arbitrary symbolic formula that contains a free occurrence of y, where y is different from x. This schema may be thought of as a two-way rewrite rule; for any instance I of the schemata, any occurrence (in a symbolic sentence occurring in a derivation) of the symbolic sentence that is the *definiendum*, respectively, *definiens* of $\mathbb{I}$ may be replaced by the symbolic sentence that is the *definiens*, respectively, *definiendum* of $\mathbb{I}$. The important point here is that the description operator is not part of the primitve basis of the Russellian language; rather, it is a symbol that is defined only within arbitrary sentential contexts.

Let us consider the "Russellian" description calculus obtained from the classical first order identity calculus by the addition of the above contextual definition to the proof theory for the identity calculus. Now, there are two ways in which we could set up a semantics for this Russellian proof theory of descriptions. We could leave the descriptor operator, and hence all description terms, out of the primitive basis of the language, in which case our semantics would simply be the classical semantics for the identity calculus, and no symbolic sentence (of the classical identity calculus with descriptions) that contains description terms would be evaluated relative to any model of our semantics. On the other hand, we could the add description terms to the elementary syntax of our language and add a semantic rule analgous to the Russellian contextual definition to our semantics so that a sentence of the form $(\exists_y)[(\forall_x)(\Phi \leftrightarrow x = y) \wedge F]$ is true relative to a given model iff $[(\iota x)\Phi / y]F$ is (where these two sentences satisfy the same restrictions as those placed on applications of the Russellian contextual definition above). The important point is that, in either case, this proof-theoretic treatment of description terms requires a semantics in which definite descriptions (i.e., description terms) are not assigned denotations **at all**. That is, if $\mathbb{M}$ is a model of a Russellian proof-theory of descriptions, then a symbolic sentence $[(\iota x)\Phi \; / y]F$ that contains a description term $(\iota x)\Phi$ is **not** evaluated on the basis of an assignment of an object from the domain of $\mathbb{M}$ to $(\iota x)\Phi$. So a Russellian semantics for formal description calculi represents a rejection of Frege's dictum that every singular term, in particular, every description term, of a formal language should be assigned a denotation (relative to a given model, we would say) since within a Russellian semantics, description terms do not function to denote objects.

As mentioned, the main drawback of the Russellian proof-theoretic treatment of descriptions is that existential generalization / universal specification are not in general sound (i.e., validity preserving) when applied to description terms. For suppose $(\forall_y)F$ is a classically valid symbolic sentence, for some symbolic formula F containing free occurrence of y. Consider a symbolic formula Φ containing at most free occurrence of a variable x such that $\neg(\exists_y)(\forall_x)(\Phi \leftrightarrow x = y)$ is classically valid (for example, the formula $\neg(x = x)$ will do). Then clearly the symbolic sentence $\neg(\exists_y)[(\forall_x)(\Phi \leftrightarrow x = y) \wedge F]$ is classically valid. Then, if our Russellian description calculus is complete with respect to classical semantics, we are able to derive $\neg(\exists_y)[(\forall_x)(\Phi \leftrightarrow x = y) \wedge F]$. The above definition, applied "left to right", allows us to re-write this sentence as $\neg[(\iota x)\Phi \;/\; y]F$. Then by an application of unrestricted existential generalization applied to the constant term $(\iota x)\Phi$, we may derive $(\exists_y)\neg F$. But since $(\forall_y)F$ is a classically valid, so is $\neg(\exists_y)\neg F$. Hence, $(\exists_y)\neg F$ is not classically valid. So $(\exists_y)\neg F(y)$ is derivable but not classically valid, i.e., our Russellian description calculus with unrestricted existential generalization of constant singular terms is not sound. Indeed, this example shows that such calculus (if complete) is not syntactically consistent, since in this example both of $(\exists_y)\neg F$, $\neg(\exists_y)\neg F$ are derivable.

So any sound and complete Russellian calculus of descriptions must restrict the application of the deductive rule existential generalization so that only "provably proper" description terms, that is constant description terms $(\iota x)\Phi$ such that, for some variable y, the sentence $(\exists_y)(\forall_x)(\Phi \leftrightarrow x = y)$ is provable, may be generalized. It is easily shown that a similar restriction must be put on universal specification. One way of restricting existential generalization in this way is simply to require that any application of existential generalization in which the generalized singular term is a description term must be preceded by a subordinate proof of the propriety of the description term.

2) <u>Frege's methods</u>: Frege [1], [2], suggests two formal treatments of descriptions both of which involve satisfying his dictum, that every well-formed singular term of the language have a denotation, by making the arbitrary convention that otherwise improper description terms are assigned some object from the range of the individual variables. There are several ways that this convention can be implemented proof-theoretically, depending on the type of objects that belong to the domain of discourse of the model(s) of the system. In [2], Frege presents a typeless system in which both classes as well as their elements belong to the range of the individual variables, so he is able to make the convention that any constant description term whose descriptive formula is not uniquely satisfied by some object in the range of the individual variables be assigned as descriptum

the class of those objects satisfying the defining formula. So, relative to a model $\mathbb{M}$ of this type of system, different constant description terms whose defining formulas are not uniquely satisfied by some object from the domain of $\mathbb{M}$ may have different descriptums.

Frege suggests a slightly simpler treatment in [1]. Here, a single object from the range of the individual variables of the system is chosen as the common descriptum of all description terms whose defining formula is not uniquely satisfied. For example, if the system is a first order arithmetic, zero might be chosen. Alternatively, if the range of the individual variables contains classes of individuals, the null set might be chosen as the common descriptum of all otherwise improper description terms (this is Quine's [1] choice in Mathematical Logic).

Now, the literature contains two interesting proof-theoretic implementations of Frege's second method (we ignore the first method, since for our purposes, it is merely a more complicated method of achieving the goal of the second, viz., the assignment of arbitrary denotations to otherwise improper description terms).

Carnap [1] suggests a proof-theoretic treatment of descriptions in which a particular individual constant of the language, say a_0, is chosen to denote the common descriptum of all constant description terms whose defining formula is not uniquely satisfied. More precisely, if $\mathbb{M}$ is a model for this particular proof-theoretic treatment of descriptions, then $\mathbb{M}$ assigns some object **o** from the domain of $\mathbb{M}$ to the individual constant a_0 and for any constant description term $(\iota x)\Phi$ of the language, if Φ is not uniquely satisfied relative to $\mathbb{M}$ by some object in the domain of $\mathbb{M}$, then $\mathbb{M}$ assigns **o** to $(\iota x)\Phi$.

In this formal treatment of descriptions a sentence of the form $[(\iota x)\Phi\ /\ y]F$, where Φ is an arbitrary symbolic formula containing at most free occurrence of x and F is a symbolic formula containing free occurrence of y distinct from x, is treated as logically equivalent to the following sentence $(\exists_y)[(\forall_x)(\Phi \leftrightarrow x = y) \wedge F] \vee (\neg(\exists_y)[(\forall_x)(\Phi \leftrightarrow x = y)] \wedge [a_0 / y]F)$. Then the proof theory for this treatment of descriptions is that of the first order identity calculus together with the following axiom schema (or the corresponding contextual definition):

$$[(\iota x)\Phi\ /\ y]F \leftrightarrow (\exists_y)[(\forall_x)(\Phi \leftrightarrow x = y) \wedge F] \vee (\neg(\exists_y)[(\forall_x)(\Phi \leftrightarrow x = y)] \wedge [a_0 / y]F).$$

and the semantics for this treatment of descriptions is that of the first order identity calculus together with a semantic rule that makes $[(\iota x)\Phi \ / y]F$ semantically equivalent to $(\exists_y)[(\forall_x)(\Phi \leftrightarrow x = y) \wedge F] \vee (\neg(\exists_y)[(\forall_x)(\Phi \leftrightarrow x = y)] \wedge [a_0 / y]F)$.

The authors Kalish, Montague and Mar [1] present a natural deduction based treatment of descriptions based on Frege's second method. Here, the term chosen to denote the common descriptum of all otherwise improper constant description terms is the 'absurd' constant description term $(\iota x)\neg(x = x)$. As in Carnap's treatment, if $\mathbb{M}$ is a model for this particular proof-theoretic treatment of descriptions, then $\mathbb{M}$ assigns some object $\mathbf{o}$ from the domain of $\mathbb{M}$ to the the description term $(\iota x)\neg(x = x)$ and for any constant description term $(\iota x)\Phi$ of the language, if Φ is not uniquely satisfied relative to $\mathbb{M}$ by some object in the domain of $\mathbb{M}$, then $\mathbb{M}$ assigns $\mathbf{o}$ to $(\iota x)\Phi$.

Kalish, Montague and Mar present two inference rules for introducing description terms into their natural deductions derivations. P.D. (Proper Description) allows the symbolic formula $[(\iota x)\Phi \ / x]\Phi$ to be derived from a formula of the form $(\exists_y)(\forall_x)(\Phi \leftrightarrow x = y)$, where Φ is an arbitrary formula containing no free occurrence of y. I.D. (Improper Description) allows the formula $(\iota x)\Phi = (\iota x)\neg(x = x)$ to be derived from a formula of the form $\neg(\exists_y)(\forall_x)(\Phi \leftrightarrow x = y)$, where Φ is an arbitrary symbolic formula containing no free occurrence of y.

As mentioned above, the main advantage of the Fregean treatments over the Russellian is that they maintain the soundness of the inference rules existential generalization and universal specification. It is easy to show that they also preserve excluded middle (as does the Russellian). The main disadvantage of the Fregean treatments over the Russellian is that the meanings that certain symbolic sentences containing description terms receive in these treatments diverge from their intuitive meanings. For example, when we consider the following two sentences

1) $(\forall_x)(\Phi \leftrightarrow x = \mathbf{b})$

2) $\mathbf{b} = (\iota x)\Phi$

(where $\mathbf{b}$ is an individual constant) as translations of English sentences, we expect them to be semantically equivalent - indeed, it is easily verified that they are semantically equivalent within a

Russellian semantics. But it is easy to see that there are Fregean models in which 1) is false but 2) is true: let $\mathbb{M}$ be any Fregean model such that there is no object **o** in the domain of $\mathbb{M}$ which uniquely satisfies Φ. Further, assume that $\mathbb{M}$ assigns the "default object" (i.e., the element of the domain of $\mathbb{M}$ which has been chosen as the common descriptum in $\mathbb{M}$ of all constant description terms whose defining formulas are not uniquely satisfied relative to $\mathbb{M}$ by some object in the domain of $\mathbb{M}$) to the individual constant **b**. Then 2) is true relative to $\mathbb{M}$ but 1) is, by hypothesis, false.

3) <u>The method of Hilbert and Bernays</u>: In [1], Hibert and Bernays present an arithmetic calculus (that is, a proof-theory with the natural numbers as the intended model) based upon a proof-theoretic implementation of Frege's dictum (that a formal language must insure that every syntactically well-formed singular term have a denotation) which is an alternative to the Fregean convention of assigning a "default object" to otherwise improper description terms. Rather than defining "term / formula of the language" in a context-free grammar (per the classical approach to defining the elementary syntax of logical languages) this treatment defines the elementary syntax and the notion of proof by simultaneous induction such that a string of the form $(\iota x)\Phi$, and hence any "sentence" in which it occurs, is allowed into the language only if there is a proof that there is exactly one natural number satisfying the defining formula Φ. Hence, whereas Frege satisfies by **semantic** means his dictum that all singular terms defined in the elementary syntax of a logical language have denotations (by assigning a "default object" to otherwise improper description terms), Hilbert and Bernays satisfies Frege's dictum by modifying the elementary **syntax** of classical arithmetic with descriptions so that a string of the form $(\iota x)\Phi$ is considered well formed only if (there is a proof that) there is exactly one natural number satisfying Φ.

Hilbert and Bernays' single deductive rule for introducing description terms into derivations provides the original motivation for Gilmore's natural deduction rule for descriptions presented in [2] and (hence) both the proof theoretic and the semantic rules for treating description terms in Pld. We might think of the Hilbert and Bernays' deductive rule for descriptions, and indeed Pld's, as a "one-way" implementation of the Russellian contextual definition that allows us to introduce description terms into arbitrary sentential contexts but not eliminate them from sentential contexts.

Roughly speaking, the Hilbert and Bernays deductive rule for description terms (as well as Gilmore's and Pld's deductive rules for descriptions) allows the symbolic sentence that is the *definiendum* of an instance of the Russellian contextual definition to be concluded from a derivation line consisting of the symbolic sentence which is the *definiens* of that instance; there is no corresponding rule for introducing description terms into derivations on the basis of a derivation line consisting of the **denial** (negation) of the symbolic sentence that is the *definiens* of an instance of the Russellian contextual definition (as there is in the Russellian and Fregean proof theoretic treatments of descriptions). Hence, only provably proper description terms (which are the only description terms in Hilbert and Bernays' language) can be introduced into symbolic sentences in a derivation.

If we forget, for the moment, that the language of the Hilbert and Bernays arithmetic calculus is only a proper subset of what we might call "classical arithmetic with descriptions", it would seem that this property of a proof-theory alone, that the only symbolic sentences that can be introduced into a derivations are those in which the only occurring description terms are provably proper, will insure that excluded middle fails for that proof-theory. For surely no sentence of the form $((\iota\boldsymbol{x})\neg(\boldsymbol{x} = \boldsymbol{x}) = \mathbf{b}) \vee \neg((\iota\boldsymbol{x})\neg(\boldsymbol{x} = \boldsymbol{x}) = \mathbf{b})$ will be derivable in such a system. Indeed, this sentence of the classical identity calculus with descriptions is not a theorem of Pld. But of course, this sentence is not even in Hilbert and Bernays' language. It is easy to show that excluded middle holds for every sentence of classical arithmetic with descriptions that is in Hilbert and Bernays' language, so their proof-theoretic treatment of descriptions can be classified, along with Frege's and Russell's, as a totalizing treatment in the sense that it is sound only with respect to models that are **total** Tarskian functions from the set of sentences of the language into {T , F}.

One obvious drawback of the Hilbert and Bernays proof-theoretic treatment of descriptions is that this system's language does not have a decidable elementary syntax. Indeed, it is easily seen that the set of well-formed strings of that language is recursively enumerable but not recursive.

Carnap [1] points out another drawback of the general approach of satisfying Frege's dictum by restricting the elementary syntax of the classical description calculus. While this approach may be convenient in an arithmetic calculus, he suggests, its extension to general proof-theories in which "factual" symbolic sentences may figure as premises to deductions has the consequence that the set of well formed strings of the language, and hence the formal notion of *grammaticality*, will depend on "the contingency of facts". We might put Carnap's point here slightly more generally as follows: "The approach of satisfying Frege's dictum by restricting the elementary syntax of the classical description calculus lends itself to an applied system like arithmetic with descriptions since the set of constant description terms of classical description calculus that are "provably proper in arithmetic", and hence are well-formed expressions for the language thus restricted, although undecidable, is at least *determinate* (at least for a Platonist). However, if this approach is extended to a general description calculus in which we can axiomatize arbitrary extralogical first order theories, the language thus restricted becomes indeterminate since the set of constant description terms of classical description calculus that are "provably proper" now depends on the particular extralogical theory whose axioms are figuring as premises in derivations of the general calculus".

Now it seems to me that the soundness of Hilbert and Bernays proof-theoretic treatment of descriptions in arithmetic illustrates that Frege's dictum places a requirement on the notion of a formal language that is stronger than is necessary to achieve Frege's goal of eliminating the possibility of falacious inferences within a formal language. For the type of fallacious inferences that are due to the occurrence of sentences containing vacuous singular terms in arguments of natural language can be avoided in languages that do not meet the condition that every well formed singular term is assigned a denotation. As the Hilbert-Bernays treatment shows, to avoid making fallacious inferences in a formal language, we need only avoid **reasoning** over sentences of the language that contain vacuous singular terms. In other words, to avoid fallacious inferences in a formal language with descriptions, we need only exclude improper description terms from occurring in sentences in our **derivations**, rather than from the language altogether.

So Hilbert and Bernay's practice of excluding from their language those sentences that contain classical description terms that are (arithmetically) improper is not necessary to satisfy the goal of Frege's dictum (if not the letter). Of course, any proof-theory of descriptions that uses the Hilbert-Bernays technique (of allowing a description term to be introduced into a sentence in a derivation only after its propriety has been proven) but does **not** restrict its elementary syntax in the manner of Hilbert and Bernays will not be complete with respect to any **classical** semantics for that language. For, as we have seen, the classically valid sentence $((\iota x)\neg(x = x) = \mathbf{b}) \vee \neg((\iota x)\neg(x = x) = \mathbf{b})$ will not be derivable in such a system.

Now, consider the semantics for such a system obtained from Tarskian first order semantics by adding to the Tarskian definition of 'truth relative to a model' a rule that is the semantic version of our "one-way" deductive rule for description terms, so that the *definiendum* of an instance $\mathbb{I}$ of the Russellian contextual definition is true relative to a model $\mathbb{M}$ of this semantics if the *definiens* of $\mathbb{I}$ is true relative to $\mathbb{M}$ but not conversely. This semantics is found to be a generalization of Tarskian semantics in which excluded middle holds for all sentences of the classical identity calculus but fails in a model $\mathbb{M}$ for (certain) symbolic sentences, and only symbolic sentences, that contain constant description terms that are improper relative to $\mathbb{M}$. Then this semantics together with our Hibert-Bernaysian proof-theory constitutes a sound and complete partial logic of descriptions based upon a decidable elementary syntax. Essentially, we maintain both the semantic completeness of our proof-theory and the decidability of our elementary syntax by making the nonrecursive procedure of determining the propriety of a given description term part of the **semantics** of our system (where it belongs), rather than polluting our elementary syntax with it, as Hilbert and Bernays would have us do. This is the motivation for Pld.

Section 1: The System Pld.

1.1 Elementary syntax

The language for Pld has as its primitive basis the sets of symbols given in 1.1.1. - 1.1.9. below:

.1. An enumerable set *Var* of **individual variables**: 'u', 'v', 'w', 'x', 'y', with or without numeric subscripts. An individual variable is a **term**. An individual variable has an **occurrence** in itself. An occurence of an individual variable in itself is a **free occurrence** in itself.

.2. An enumerably infinite set *Par* of **individual parameters**: 'l', 'm', 'n','p', 'q', with or without numeric subscripts. An individual parameter is a **term**. An individual parameter has an **occurrence** in itself.

.3. A finite set *Cnst* of **individual constants**: 'a', 'b', 'c', 'd', 'e' , with or without numeric subscripts An individual constant is a **term** and has an **occurrence** in itself.

.4. A finite set *Fun* of **unary functors**: '$\mathscr{f}$', '$\mathscr{g}$', '$\mathscr{h}$', '$\mathscr{i}$', with or without numeric subscripts.

.5. For each n, a denumerable set set $Pred^n$ of n-ary **descriptive predicate constants**: 'P^n', 'Q^n', 'R^n', 'S^n', with or without numeric subscripts.

.6. A set {'='}containing a single binary **logical predicate constant**.

.7. A set {'$\neg$' , '$\rightarrow$', '$\wedge$'} of **sentential connectives.**

.8. A set {'( )' , 'ι'} of **variable binding operators.** 'ι' is called the **descriptive operator**.

.9. A set {'(' , ')' , '.'} of **punctuation marks.**

For the purposes of discussion we will assume as available the connective and quantifier sets of the full predicate calculus on the understanding that expressions over the larger language are defined in terms of the above primitive basis in the usual way. For each set S of symbols given in 1.1.1. - 1.1.5., above, we let the **bold-face** version of the member symbols be metalinguistic variables ranging over S. Each symbol which is a member of one of the sets given in 1.1.6. - 1.1.9. occurs in the metalanguage as a metalinguistic constant denoting itself in the object language. Let $\boldsymbol{r}, \boldsymbol{s}, \boldsymbol{t}$, with or without numeric subscripts, be metalinguistic variables ranging over terms. Metalinguistic concatenation denotes object language concatenation. Then,

.1.0. **Elementary formulas** are of the form $\boldsymbol{P}^n(\boldsymbol{t}_1, \ldots, \boldsymbol{t}_n)$ and $(\boldsymbol{t}_1 = \boldsymbol{t}_2)$. A (free) occurrence of a variable in one of $\boldsymbol{t}_1, \ldots, \boldsymbol{t}_n$ is a **(free) occurrence** in $\boldsymbol{P}^n(\boldsymbol{t}_1, \ldots, \boldsymbol{t}_n)$. A (free) occurrence of a variable in $\boldsymbol{t}_1$ or in $\boldsymbol{t}_2$ is a **(free) occurrence** in $(\boldsymbol{t}_1 = \boldsymbol{t}_2)$. Elementary formulas are **formulas**. An elementary formula of the form $\boldsymbol{P}^n(\boldsymbol{t}_1, \ldots, \boldsymbol{t}_n)$, $\boldsymbol{t}_1 = \boldsymbol{t}_2$ is an **atomic formula** if the $\boldsymbol{t}_1, \boldsymbol{t}_2, \ldots, \boldsymbol{t}_n$ are basic terms.

Let '$\boldsymbol{afm}$', respectively, '$\boldsymbol{F}$', '$\boldsymbol{G}$', '$\boldsymbol{H}$', 'Φ', 'Ψ', 'Ξ', with or without numeric subscripts, be metalinguistic variables over atomic formulas, respectively, formulas.

.1.1. If $\boldsymbol{F}$ and $\boldsymbol{G}$ are formulas, then the following expressions are **formulas:** $\neg\boldsymbol{F}$, $(\boldsymbol{x})(\boldsymbol{F})$, $(\boldsymbol{F} \rightarrow \boldsymbol{G})$, $(\boldsymbol{F} \wedge \boldsymbol{G})$. A (free) occurrence of a variable in $\boldsymbol{F}$ is a **(free) occurrence** in $\neg\boldsymbol{F}$. A (free) occurrence of a variable in $\boldsymbol{F}$ or $\boldsymbol{G}$ is a **(free) occurrence** in both of $(\boldsymbol{F} \rightarrow \boldsymbol{G})$, $(\boldsymbol{F} \wedge \boldsymbol{G})$. A free occurence of a variable other than $\boldsymbol{x}$ in $\boldsymbol{F}$ is a **free occurrence** in $(\boldsymbol{x})(\boldsymbol{F})$; no other variable has a free occurence in $(\boldsymbol{x})(\boldsymbol{F})$.

.1.2. If $\boldsymbol{t}$ is a term and $\not{f}$ is a unary functor, then the result $\not{f}(\boldsymbol{t})$ of applying $\not{f}$ to $\boldsymbol{t}$ is a term. If $\boldsymbol{F}$ is a formula, then $\iota\boldsymbol{x}.\boldsymbol{F}$ is a **term**, and a **description term**. A (free) occurrence of a term $\boldsymbol{r}$ in $\boldsymbol{t}$ is a **(free) occurrence** in $\not{f}(\boldsymbol{t})$. An occurrence of a term $\boldsymbol{r}$ in $\boldsymbol{F}$ is an **occurrence** in $\iota\boldsymbol{x}.\boldsymbol{F}$. A free occurrence of a variable other than $\boldsymbol{x}$ in $\boldsymbol{F}$ is a **free occurrence** in $\iota\boldsymbol{x}.\boldsymbol{F}$; no other variable has a free occurrence in $\iota\boldsymbol{x}.\boldsymbol{F}$. A term in which no variable has a free occurrence is a **constant term.**

.1.3. A term which contains no occurrence of a description term is a **basic term.**

.1.4. Let $\mathbb{P} : \mathbb{N} \rightarrow Par$ be a bijective enumeration of Par. Let π be any subset of Par. Then $\delta(\pi)$ is the set of constant basic terms in which the only occurring parameters are from $Par \cup \{\mathbb{P}(1)\}$. So $\delta(Par)$ is the set of constant basic terms of Pld.

.1.4. A **sentence** is a formula in which no variable has a free occurrence. Let π be any subset of *Par*. Then $\Sigma(\pi)$ is the set of sentences in which the only occurring parameters are from π. So $\Sigma(Par)$ is the set of all sentences of Pld.

Throughout the remainder of this paper, we will use the symbol '$\cong$' to denote the relation of syntactic identity holding between terms, formulas and signed sentences of Pld. In other words, the expression '$\mathbb{E}_1 \cong \mathbb{E}_2$' shall mean that the expression denoted by the metalinguistic variable '$\mathbb{E}_1$' is syntactically the same expression as that denoted by '$\mathbb{E}_2$'. We do this to avoid confusion with the object language symbol for identity '='. Further, we will use both of '$\cong$', '=' to denote the relation of set theoretic identity holding between sets of signed sentences; '=' will be used to denote this metatheoretic relation only when there is no danger of confusing it with the object language symbol '='. In particular, '$\cong$' will be used to denote the identity relation holding between those sets of signed sentences which are denoted in this text by the display of their member signed sentences.

The following expressions, with or without numeric subscripts, will be used as metalinguistic variables over atomic sentences, respectively, sentences: ***'asnt'***, respectively, ***'snt'***, '**A**', '**B**', '**C**', '**D**'. The logical syntax and the semantics for Pld is presented in the manner of Gilmore [1], [2] using the notions of *signed sentence* and *sequent* of signed sentences. If **A** is a sentence, then both of $\pm$**A** are **signed sentences**. A **sequent** is any finite set of signed sentences. Semantically, the signature $\pm$ indicates, respectively, the assignment of truth or falsehood to a sentence relative to a model. In the logical syntax, the signature $\pm$ indicates that the sentence so signed occurs to the right, respectively, left, of the Gentzen arrow. Also used is the simultaneous strict-substitution operator $[t_1, \ldots, t_n / x_1, \ldots, x_n]$, which, when applied to a formula or term, substitutes t_i $(1 \leq i \leq n)$ for all free occurrences of x_i in the formula or term, such that none of the x_i occurring free in any of the $t_1, \ldots, t_n$, are substituted for. Renaming of variables bound in the formula or term is automatically effected so that any free occurrence of a variable in t_i is a free occurrence in the result of the application. The substitution operator $[t_1, \ldots$

$., t_n / x_1, \ldots, x_n]$ will be abbrieviated as $[\underline{t} / \underline{x}]$, where $\underline{t} \cong t_1, \ldots, t_n$ and $\underline{x} \cong x_1, \ldots, x_n$. We will also apply the simultaneous strict-substitution operator $[t_1, \ldots, t_n / x_1, \ldots, x_n]$ to sequents such that $[t_1, \ldots, t_n / x_1, \ldots, x_n]Seq$ denotes the sequent obtained from the sequent *Seq* by replacing every signed sentence $\pm snt$ in *Seq* by respectively $\pm[t_1, \ldots, t_n / x_1, \ldots, x_n]snt$.

1.2. **Logical syntax**

We present a set of axioms and first order rules of deduction much as in Gilmore [1], [2]. Rule 1.2.2.6. below is significanly modified over the rule 5.3.2. for descriptions presented in Gilmore [2], where description terms are defined on the basis of set abstraction terms from the first order set theory. 1.2.2.6. represents a weakening of the restriction 5.3.2. places on the "existance" and "uniqueness" premises of an application of that rule such that the "input" sentences to an application of 1.2.2.6. do not have to be themselves derivable sentences, but rather merely positively signed members of a derivable sequent. Indeed, as we demonstrate in section 6, this modification is necessary to the completeness of the set of rules given below relative to the semantics we will define.

.1. Let t be in $\delta(Par)$. Then the set of axioms is the set of all sequents of the form $\{-asnt, +asnt\}$ or of the form $\{+(t = t)\}$.

.2. Let t, r, s be constant, possibly non-basic, terms. The rules of deduction are given by the following schemata:

.1.± (thinning)

$$\frac{Seq}{Seq \cup \{\pm \mathrm{A}\}}$$

.2.±

$$\frac{Seq \cup \{\pm \mathrm{A}\}}{Seq \cup \{\mp \neg \mathrm{A}\}}$$

.3. ±

$$\frac{Seq1 \cup \{+\mathrm{A}\} \qquad Seq2 \cup \{+\mathrm{B}\}}{Seq1 \cup Seq2 \cup \{+(\mathrm{A} \wedge \mathrm{B})\}} \qquad\qquad \frac{Seq \cup \{-\mathrm{A}\ ,\ -\mathrm{B}\}}{Seq \cup \{-(\mathrm{A} \wedge \mathrm{B})\}}$$

.4.±

$$\frac{Seq \cup \{-\mathrm{A}\ ,\ +\mathrm{B}\}}{Seq \cup \{+(\mathrm{A} \rightarrow \mathrm{B})\}} \qquad\qquad \frac{Seq1 \cup \{+\mathrm{A}\} \qquad Seq2 \cup \{-\mathrm{B}\}}{Seq1 \cup Seq2 \cup \{-(\mathrm{A} \rightarrow \mathrm{B})\}}$$

.5.±

$$\frac{Seq \cup \{+[p\ /\ x]F\}}{Seq \cup \{+(x)F\}} \qquad\qquad \frac{Seq \cup \{-[t\ /\ x]F\}}{Seq \cup \{-(x)F\}}$$

where, for .5.+, p is a parameter which does not occur in the conclusion $Seq \cup \{+(x)F\}$.

.6.

$$\frac{Seq1 \cup \{+[t\ /\ x]\Phi\} \quad Seq2 \cup \{+(x)(v)((\Phi \wedge [v\ /\ x]\Phi) \rightarrow x = v)\} \quad Seq3 \cup \{\pm[t\ /\ u]F\}}{Seq1 \cup Seq2 \cup Seq3 \cup \{\pm[\iota x.\Phi\ /\ u]F\}}$$

where, v different from x.

.7.

$$\frac{Seq1 \cup \{+[r\ /\ u]F\} \qquad Seq2 \cup \{-[s\ /\ u]F\}}{Seq1 \cup Seq2 \cup \{-(r = s)\}}$$

.8. (cut)

$$\frac{Seq1 \cup \{+\mathrm{A}\} \qquad Seq2 \cup \{-\mathrm{A}\}}{Seq1 \cup Seq2}$$

An instance of one of the above rule schemata will be called an **application** of the rule. For a given application $\mathbb{A}$ of one of the above rules, the sequent(s) above, respectively, below, the vinculum are called the **premise(s)**, respectively, **conclusion, to** $\mathbb{A}$. Let the signed sentential variables (i.e., '$\pm\mathbf{A}$', '$\pm\mathbf{B}$', '$\pm[t\ /\ x]\Phi$ ', '$\pm(x)F$', '$\mp(\mathrm{A} \wedge \mathrm{B})$', etc.) explicitly given in the premises, respectively, conclusion, of a given rule schema $\mathbb{R}$ be said to denote the **input sentence(s)**, respectively, **output sentence,** to a given application of $\mathbb{R}$.

We have, then, a three premise rule for introducing sentences containing definite descriptions (i.e., description terms) into derivations. Notice that .6. is unique among the above rules in that it does not have a dual rule treating the case where the input sentences to an application have negative signatures. This asymmetric treatment of descriptions is motivated by the intuition that we cannot, in genaral, validly introduce a sentence containing a description into a derivation on *either* side of the turnstile (i.i., Gentzen arrow) without first securing the existence of a descriptum for the description to denote. This corresponds to the semantical intuition behind the Frege-Strawson doctrine that a sentence of natural language which is grammatically of subject-predicate form and which contains a vacuous singular subject term cannot have a truth value associated with it. We can say, then, that any description which is introduced into a Pld derivation by 1.2.2.6. has large scope, in the Russellian sense, over the sentence in which it occurs.

.3. An axiom from 1.2.1. is a **derivation tree.** If Σ is a derivation tree whose endsequent *Seq* is the premise of an application $\mathbb{A}$ of one of 1.2.2.1.±, 1.2.2.2.±, 1.2.2.3.-, 1.2.2.4.+, 1.2.2.5.±, then the tree obtained from Σ by appending the conclusion of $\mathbb{A}$ to *Seq* is a **derivation tree.** If Σ, Σ' are derivation trees whose endsequents respectively *Seq*, *Seq'* are premises of an application $\mathbb{A}$ of one of 1.2.2.3.+, 1.2.2.4.-, 1.2.2.7., 1.2.2.8. then the tree obtained from Σ and Σ' by appending the conclusion of $\mathbb{A}$ to *Seq* and *Seq'* is a **derivation tree.** If Σ, Σ', Σ'' are derivation trees whose endsequents respectively *Seq*, *Seq'*, *Seq"* are premises of an application $\mathbb{A}$ of 1.2.2.6. then the tree obtained from Σ and $\Sigma'$and Σ'' by appending the conclusion of $\mathbb{A}$ to *Seq* and *Seq'* and *Seq"* is a **derivation tree**.

.4. A sequent *Seq* of signed sentences is **derivable** iff it is the endsequent of a derivation tree. A sentence *snt* is **derivable** iff the sequent $\{+snt\}$ is derivable.

.5. A sentence *snt* is **grounded** iff the sequent.$\{+snt , -snt\}$ is derivable. Let *Grd* denote the set of grounded sentences.

.6. The depth $D(\Sigma)$ of a derivation tree Σ is zero if Σ ia an axiom; otherwise $D(\Sigma) = 1 + D(\Sigma')$

where Σ' is Σ's deepest proper subtree.

1.3 Semantics

A **term model** for a formal system T is an interpretation for T whose domain of discourse is the set of constant terms defined in the elementary syntax for T. In this subsection we define a class of term models, called **bases**, for Pld.

.1. Let $\pi \subseteq Par$ be any set of parameters. A **base *bse* with domain $\delta(\pi)$** is any set *bse* of signed atomic sentences from $\Sigma(\pi)$ which satisfies the following three conditions:

1) For every atomic sentence ***asnt*** $\in \Sigma(\pi)$, exactly one of $\pm$***asnt*** is a member of *bse*.
2) For all terms $\boldsymbol{t}$ in $\delta(\pi)$, $+\boldsymbol{t} = \boldsymbol{t}$ is in *bse*.
3) For all terms $\boldsymbol{r}, \boldsymbol{s}$ in $\delta(\pi)$, if both of $+[\boldsymbol{r} / x]\boldsymbol{afm}$, $-[\boldsymbol{s} / x]\boldsymbol{afm}$ are in *bse*, then so is $-\boldsymbol{r} = \boldsymbol{s}$.

For the remainder of this paper, we use simply the term 'base' to mean 'base with domain $\delta(Par)$' unless otherwise indicated.

.2. Let *Set* be a set of signed sentences. The **semantic successor *Sc(Set)* of *Set*** is the smallest set satisfying the following semantic rules:

.1. If respectively $\pm$**A** is in *Set*, then respectively $\mp\neg$**A** is in *Sc(Set)*.

.2. If each of +**A**, +**B**, respectively, one of -**A**, -**B**, is in *Set*, then respectively $\pm$(**A** $\wedge$ **B**) is in *Sc(Set)*.

.3. If one of -**A**, +**B**, respectively, each of +**A**, -**B**, is in *Set*, then respectively $\pm$(**A** $\rightarrow$ **B**) is in *Sc(Set)*.

.4. If *Set* contains $+[\boldsymbol{t} / x]\boldsymbol{F}$ for all $\boldsymbol{t}$ in $\delta(Par)$, respectively, $-[\boldsymbol{t} / x]\boldsymbol{F}$ for some $\boldsymbol{t}$ in $\delta(Par)$, then respectively $\pm(x)\boldsymbol{F}$ is in *Sc(Set)*.

.5. If each of $+[\boldsymbol{t} / x]\boldsymbol{\Phi}$, $+(x)(v)((\boldsymbol{\Phi} \wedge [v / x]\boldsymbol{\Phi}) \rightarrow x = v)$, and one of, respectively, $\pm[\boldsymbol{t} / u]\boldsymbol{F}$, is in *Set*, then respectively $\pm[\iota x.\boldsymbol{\Phi} / u]\boldsymbol{F}$ is in *Sc(Set)*, where $\boldsymbol{t}$ is a constant, possibly non-basic, term.

In the semantic rule 1.3.2.5. for descriptions, let '$+[t\,/\,x]\Phi$', '$+(x)(v)((\Phi \wedge [v\,/\,x]\Phi) \rightarrow x = v)$' be said to denote the signed sentence which is the **existence**, respectively, **uniqueness, presupposition** of an instance of 1.3.2.5.. '$\pm[t\,/\,u]F$' will be said to denote the signed sentence which is the **major statement** of an instance of 1.3.2.5.. We have then, a rule which is the obvious semantic analog to the derivational rule 1.2.2.7. for descriptions. Note the assymetric treatment descriptions receive semantically: 1.3.2.5. has no dual rule treating the cases where one of the presuppositions of an instance of the rule is false, i.e., of negative signature, in *bse*.

.3. Let ***Set*** be a set of signed sentences. We define by transfinite induction on μ a sequence $\langle Set_\mu \rangle$ as follows:

.0. $Set_0 = Set$;

.1. $Set_{\mu+1} = Set_\mu \cup Sc(Set_\mu)$ for successor ordinals μ ;

.2. $Set_\mu = \bigcup Set_\beta, 0 \leq \beta < \mu$ for limit ordinals μ.

.4. The **semantic closure** $\mathbb{Cl}$**(*bse*) of *bse*** is the union set of the bse_μ for ordinals μ less than the first nondenumerable ordinal, i.e., $Cl(bse) = bse_{\in_0}$.

.5. Let *bse* be a base. A sequent ***Seq*** is **valid for *bse*** iff $[\underline{t}\,/\,\underline{p}]Seq \cap Cl(bse) \neq \varnothing$, where $\underline{p}$ is an n-ary vector on *Par* which contains all of the parameters which occur in signed sentences of ***Seq*** and $\underline{t}$ is any n-ary vector on $\delta(Par)$.

.6. A sequent ***Seq*** is **valid** iff ***Seq*** is valid for all bases. A sentence ***snt*** is **valid** iff $\{+snt\}$ is a valid sequent.

<u>Lemma 1.3.7.</u>: $Sc(Cl(bse)) \subseteq Cl(bse)$.

<u>Proof of 1.3.7.</u>: Suppose otherwise, i.e., suppose $Sc(Cl(bse)) \supset Cl(bse)$. Then there is a sequence $bse_0, \ldots, bse_{\in_0}, Sc(Cl(bse))$ of sets of signed sentences each of which properly includes its immediate predecessor. But this sequence is order type $\in_0 + 1$, and so is of nondenumerable cardnality. Hence, there are nondenumerably many distinct sentences of Pld. But

Pld has only denumerably many sentences. Contradiction. □

We should note that it is possible to define the semantic closure $Cl(bse)$ of a base bse by *finite* induction only as bse_{ω}. That is, one can show by induction on i, $0 \leq i < \omega$, that $Sc(bse_{\omega}) \supset bse_{\omega}$. However, we maintain definition 1.3.4. of $Cl(bse)$ as $bse_{\epsilon 0}$ since that definition makes the proof of 1.3.7. much easier than would otherwise be the case.

Note that since the semantic rule 1.2.3.4. limits the range of the universal quantifier to the set $\delta(Par)$ of constant basic terms, and since no description terms occur in any sentence of any base, the bases we are considering all have $\delta(Par)$ as their domain of individuals. Hence, description terms play no *semantic* role in the class of term models defined in 1.3.. Rather, the role that description terms play in Pld is of a purely syntactic nature: descriptions function in the logical syntax (specifically, through the rule of universal specification, .5.-) to abbreviate the results of certain formal deductions and hence to facilitate certain further deductions, deductions which can be achieved without the use of descriptions. Hence, although, Pld is nominalistically interpreted, in the sense that its models are syntactic, description terms do not function as individuals in those models. The fact that description terms do not belong to the domain of individuals in our term models is expressed by lemma 1.3.10. below which "says" that a description term will "bear a range of properties" relative to a given model of Pld only if there is a basic term which bears the same range of properties. This fact proves crucial to the proof (lemma 1.3.9.) that the class of term models we have defined "obey the law of noncontradiction" in the sense that no sentence is assigned both the value true and the value false relative to any given model.

Lemma 1.3.8.: Let bse be any base and let $\boldsymbol{r}, \boldsymbol{s}$ be any terms in $\delta(Par)$ such that the signed sentence $+\boldsymbol{r} = \boldsymbol{s}$ is in $Cl(bse)$. Then, for all formulas $\boldsymbol{F}$, if $\pm[\boldsymbol{s} / \boldsymbol{x}]\boldsymbol{F}$ is in bse_{μ}, so is respectively $\pm[\boldsymbol{r} / \boldsymbol{x}]\boldsymbol{F}$.

Proof of 1.3.8.: Suppose $+\boldsymbol{r} = \boldsymbol{s}$ is in $Cl(bse)$. Then, since $+\boldsymbol{r} = \boldsymbol{s}$ is atomic, $+\boldsymbol{r} = \boldsymbol{s}$ is in bse. We show by transfinite induction on μ in definition 1.3.3. that for all μ, if $\pm[\boldsymbol{s} / \boldsymbol{x}]\boldsymbol{F}$ is in bse_{μ}, so is respectively $\pm[\boldsymbol{r} / \boldsymbol{x}]\boldsymbol{F}$. So let $\pm[\boldsymbol{s} / \boldsymbol{x}]\boldsymbol{F}$ be any signed sentence in bse_{μ}. Clearly, we may assume that x occurs free in F, since otherwise the claim holds vacuously.

Base step: $\mu = 0$. By clause 1) of definition 1.3.1., since bse contains $+r = s$, bse does not contain $-r = s$. Hence, by the contrapositive of clause 3) of 1.3.1., for every atomic formula afm, if $\pm[s / x]afm$ is in bse_μ, then so is respectively $\pm[r / x]afm$. But bse contains atomic formulas *only*, so the claim holds.

Induction step: Assume that for all $\beta < \mu$, for all formulas G, if $\pm[s / x]G$ is in bse_β, so is respectively $\pm[r / x]G$. We want to show that respectively $\pm[r / x]F$ is in bse_μ.

For limit ordinals μ: Since $\pm[s / x]F$ is in bse_μ, and $bse_\mu = \bigcup bse_\beta$, $0 \leq \beta < \mu$, respectively $\pm[s / x]F$ is in bse_β for some $\beta < \mu$. By the hypothesis of induction, then, respectively $\pm[r / x]F$ is in $bse_\beta \subseteq bse_\mu$.

Successor ordinals μ: By the hypothesis of induction, we may assume that $\pm[s / x]F$ is not in bse_β for any $\beta < \mu$, since otherwise respectively $\pm[r / x]F$ is in $bse_\beta \subseteq bse_\mu$. Then, since $bse_\mu = bse_{\mu-1} \cup Sc(bse_{\mu-1})$, $\pm[s / x]F$ is in $Sc(bse_{\mu-1})$. There are five main cases:

i) $\pm[s / x]F$ is in $Sc(bse_{\mu-1})$ by virtue of an instance of the semantic rule 1.3.2.1. and is respectively of the form $\pm[s / x]\neg G \cong \pm\neg[s / x]G$ for some formula G. Then respectively $\pm[s / x]G$ is in $bse_{\mu-1}$. So, by the hypothesis of induction, respectively $\pm[r / x]G$ is in $bse_{\mu-1}$. Then, by 1.3.2.1., respectively $\mp\neg[r / x]G \cong \pm[r / x]\neg G \cong \pm[r / x]F$ is in $Sc(bse_\mu) \subseteq bse_\mu$.

ii) $\pm[s / x]F$ is in $Sc(bse_{\mu-1})$ by virtue of an instance of the semantic rule 1.3.2.2. and is respectively of the form $\pm[s / x](G \wedge H) \cong \pm([s / x]G \wedge [s / x]H)$. Then each of $+[s / x]G$, $+[s / x]H$, respectively, one of $-[s / x]G$, $-[s / x]H$, is in $bse_{\mu-1}$. So, by the hypothesis of induction each of $+[r / x]G$, $+[r / x]H$, respectively, one of $-[r / x]G$, $-[r / x]H$, is in $bse_{\mu-1}$. By 1.3.2.2., then, respectively $\pm([r / x]G \wedge [r / x]H) \cong \pm[r / x](G \wedge H) \cong \pm[r / x]F$ is in $Sc(bse_\mu) \subseteq bse_\mu$.

iii) $\pm[s / x]F$ is in $Sc(bse_{\mu-1})$ by virtue of an instance of the semantic rule 1.3.2.3.. This case is similar to ii).

iv) $\pm[s / x]F$ is in $Sc(bse_{\mu-1})$ by virtue of an instance of the semantic rule 1.3.2.4. and is respectively of the form $\pm[s / x](y)G \cong \pm(y)[s / x]G$ for some formula G. Then $bse_{\mu-1}$ contains $+[t / y][s / x]G \cong +[s / x][t / y]G$ for all t in $\delta(Par)$, respectively, $-[t / y][s / x]G \cong -[s / x][t / y]G$ for some t in $\delta(Par)$. So, by the hypothesis of induction, $bse_{\mu-1}$ contains $+[r / x][t / y]G \cong +[t / y][r / x]G$ for all t in $\delta(Par)$, respectively, $-[r / x][t / y]G \cong -[t / y][r / x]G$ for some t in $\delta(Par)$. Hence, by 1.3.2.4., $Sc(bse_{\mu}) \subseteq bse_{\mu}$ contains respectively $\pm(y)[r / x]G \cong \pm[r / x](y)G \cong \pm[r / x]F$.

v) $\pm[s / x]F$ is in $Sc(bse_{\mu-1})$ by virtue of an instance $\mathbb{I}$ of the semantic rule 1.3.2.5. for descriptions and is respectively of the form $\pm[s / x][\iota y.\Phi / u]G$ for some constant description term $\iota y.\Phi$ and formula G. Since s is a constant term, we may rewrite the sentence $[s / x][\iota y.\Phi / u]G$ as $[\iota y.[s / x]\Psi / u][s / x]G$ for some formula Ψ such that $\Phi \cong [s / x]\Psi$. Then, since $\pm[\iota y.[s / x]\Psi / u][s / x]G$ is in $Sc(bse_{\mu-1})$ by virtue $\mathbb{I}$, $bse_{\mu-1}$ contains both of the presuppositions $+[t / y][s / x]\Psi$, $+(y)(v)(([s / x]\Psi \wedge [v / y][s / x]\Psi) \rightarrow y = v)$, as well as the major statement respectively $\pm[t / u][s / x]G$ for some constant term t. Since both of t, s are constant terms, the presuppositions may be written as: $+[s / x][t / y]\Psi$, $+[s / x](y)(v)((\Psi \wedge [v / y]\Psi) \rightarrow y = v)$ and the major statement may be respectively rewritten as $\pm[s / x][t / u]G$. Then, by the hypothesis of induction, $bse_{\mu-1}$ contains both of $+[r / x][t / y]\Psi$, $+[r / x](y)(v)((\Psi \wedge [v / y]\Psi) \rightarrow y = v)$, respectively $\pm[r / x][t / u]G$. Again, these signed sentences may be written as: $+[t / y][r / x]\Psi$, $+(y)(v)(([r / x]\Psi \wedge [v / y][r / x]\Psi) \rightarrow y = v)$, respectively $\pm[t / u][r / x]G$. Since $bse_{\mu-1}$ contains all of $+[t / y][r / x]\Psi$, $+(y)(v)(([r / x]\Psi \wedge [v / y][r / x]\Psi) \rightarrow y = v)$, respectively $\pm[t / u][r / x]G$, by 1.3.2.5., $Sc(bse_{\mu}) \subseteq bse_{\mu}$ contains respectively $\pm[\iota y.[r / x]\Psi / u][r / x]G \cong \pm[r / x]F$. □

Lemma 1.3.9.: Let *bse* be any base, *snt* any sentence. Then, for all ordinals μ, not both of $\pm$*snt* are in bse_{μ}.

Lemma 1.3.10.: Let *bse* be any base. Then, for all ordinals μ, for evey constant description term

$\iota x.\Phi$, there is a basic term $t_{\iota x.\Phi}$ in $\delta(Par)$ such that for all formulas F, if $\pm[\iota x.\Phi / u]F$ is a signed sentence in bse_μ, then respectively $\pm[t_{\iota x.\Phi} / u]F$ is in bse_μ. $t_{\iota x.\Phi}$ is called a **descriptum for $\iota x.\Phi$ in bse.**

Proof of 1.3.9. and 1.3.10.: We prove 1.3.9., 1.3.10 by simultaneous transfinite induction on μ in definition 1.3.3. showing that all ordinals μ satisfy both of 1.3.9., 1.3.10.. Let bse be any base.

Base step: $\mu = 0$. Since $bse_0 = bse$ is a base, by condition 1) of definition 1.3.1., not both of $\pm snt$ are in bse_0. So 1.3.9. holds. Since $bse_0 = bse$ contains signed atomic sentences only, and no atomic sentence contains an occurrence of any description term, 1.3.10. holds vacuously.

Induction step: $\mu > 0$. Assume that for all $\beta < \mu$, bse_β satisfies both of 1.3.9., 1.3.10.. That is, assume that a) for all $\beta < \mu$, not both of $\pm snt$ are in bse_β, and b) for all $\beta < \mu$, for evey constant description term $\iota x.\Phi$, there is a basic term $t_{\iota x.\Phi}$ in $\delta(Par)$ such that for all formulas F, if $\pm[\iota x.\Phi / u]F$ is a signed sentence in bse_β, then respectively $\pm[t_{\iota x.\Phi} / u]F$ is in bse_β. We need to show that bse_μ satisfies both of 1.3.9., 1.3.10..

For limit ordinals μ: Since $bse_\mu = \bigcup bse_\beta$, $0 \leq \beta < \mu$, both of 1.3.9., .1.3.10., hold trivially for bse_μ by the hypothesis of induction.

For successor ordinals μ: We show first that bse_μ satisfies 1.3.10.. Let $\iota x.\Phi$ be any description term. We want to show that there is a $t_{\iota x.\Phi}$ in $\delta(Par)$ such that for all formulas F, if $\pm[\iota x.\Phi / u]F$ is a signed sentence in bse_μ, then respectively $\pm[t_{\iota x.\Phi} / u]F$ is in bse_μ. Clearly, we may assume that $\iota x.\Phi$ occurs in a signed sentence $ssnt$ belonging to bse_μ, since otherwise the claim holds vacuously. There are two main cases:

a) For no $\gamma < \mu$, constant term t, variable v, does bse_γ contain both of the signed sentences $+[t / x]\Phi$, $+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)$. Since $\iota x.\Phi$ occurs in some signed sentence $ssnt$ belonging to bse_μ and $\langle bse_\mu \rangle$ is well ordered under $\subseteq$, there is a least $\gamma \leq \mu$ such that $\iota x.\Phi$ occurs in some signed sentence $ssnt$ belonging to bse_γ. Since bse_0 contains signed atomic sentences only, $\gamma \neq 0$. If γ is a nonzero limlt ordinal, then there is a $\beta < \gamma$ such that $ssnt \in bse_\beta$. Hence, γ is a successor ordinal. Let $t_{\iota x.\Phi}$ be any term in $\delta(Par)$. Now, we show by induction on α that for all α, $\gamma \leq \alpha \leq \mu$, for all formulas F, if $\pm[\iota x.\Phi / u]F$ is in bse_α, then respectively $\pm[t_{\iota x.\Phi} / u]F$ is in bse_α. Clearly, our desired result that for all formulas F, if $\pm[\iota x.\Phi / u]F$ is a signed sentence in bse_μ, then respectively $\pm[t_{\iota x.\Phi} / u]F$ is in bse_μ, will follow from this result. So assume that the signed sentence $\pm[\iota x.\Phi / u]F$ is in bse_α

<u>Base step</u>: $\alpha = \gamma$. Now, since γ is a successor ordinal, by definition 1.3.3., $bse_\gamma = bse_{\gamma-1} \cup Sc(bse_{\gamma-1})$. Since $\iota x.\Phi$ does not occur in any signed sentence in $bse_{\gamma-1}$ it follows that $\pm[\iota x.\Phi / u]F$ is in bse_γ by virtue of being in $Sc(bse_{\gamma-1})$. Since $\iota x.\Phi$ is a constant term and $bse_{\gamma-1}$ does not contain the presuppositions to any instance of semantic rule 1.3.2.5., $\pm[\iota x.\Phi / u]F$ is obtained in $Sc(bse_{\gamma-1})$ by an instance $\mathbb{I}$ of one of the semantic rule 1.3.2.2., 1.3.2.3.. We will assume that $\mathbb{I}$ is an instance $\mathbb{I}$ of 1.3.2.2., since the other case is similar. Then $\pm[\iota x.\Phi / u]F$ is of the form $-(\mathrm{A} \wedge [\iota x.\Phi / u]H)$ for some sentence A and formula H containing free occurrence of u.,where $bse_{\gamma-1}$ contains the the signed sentence -A. Then by an instance of 1.3.2.2., $Sc(bse_{\gamma-1})$ contains the signed sentence $-(\mathrm{A} \wedge [t_{\iota x.\Phi} / u]H) \cong -[t_{\iota x.\Phi} / u]F$. So the claim holds.

<u>Induction step</u>: $\alpha > \gamma$. Assume c) that that for all β, $\gamma \leq \beta < \alpha$, for all formulas F, if $\pm[\iota x.\Phi / u]F$ is in bse_β, then respectively $\pm[t_{\iota x.\Phi} / u]F$ is in bse_β. We want to show that for all formulas F, if $\pm[\iota x.\Phi / u]F$ is in bse_α, then respectively $\pm[t_{\iota x.\Phi} / u]F$ is in bse_α. So let F be any formula such that $\pm[\iota x.\Phi / u]F$ is a signed sentence in bse_α We show that respectively $\pm[t_{\iota x.\Phi} / u]F$ is in bse_α. <u>For limit ordinals α</u>: Since $bse_\alpha = \bigcup bse_\beta$, $0 \leq \beta < \alpha$, the claim holds trivially by hypothesis of induction c). <u>For successor ordinals α:</u> We may assume that for no $\beta < \alpha$ is $\pm[\iota x.\Phi / u]F$ in bse_α, since otherwise, by the hypothesis of induction, the claim holds trivially. Then $\pm[\iota x.\Phi / u]F$ is in $Sc(bse_{\alpha-1})$. Since $\alpha - 1 < \mu$ and no for no $\beta < \mu$ does bse_β contain the presuppositions to any instance of 1.3.2.5., there are just four cases:

i) $\pm[\iota x.\Phi / u]F$ is obtained in $Sc(bse_{\alpha-1})$ by an application of semantic rule 1.3.2.1. and is respectively of the form $\pm[\iota x.\Phi / u]\neg G$, for some formula G. Then $bse_{\alpha-1}$ contains repectively

$\mp[\iota x.\Phi / u]G$ and so, by the hypothesis of induction c), contains $\mp[t_{\iota x.\Phi} / u]G$. Hence, by semantic rule 1.3.2.1., respectively $\pm\ [t_{\iota x.\Phi} / u]\neg G \cong \pm[t_{\iota x.\Phi} / u]F$ is in $Sc(bse_{\alpha-1}) \subseteq bse_{\alpha}$.

ii) $\pm[\iota x.\Phi / u]F$ is obtained in $Sc(bse_{\alpha-1})$ by an application of semantic rule 1.3.2.2. and is respectively of the form $\pm[\iota x.\Phi / u](G \wedge H)$, for some formulas G, H. Then $bse_{\alpha-1}$ contains both of $+[\iota x.\Phi / u]G$, $+[\iota x.\Phi / u]H$, respectively one of $-[\iota x.\Phi / u]G$, $-[\iota x.\Phi / u]H$. By hypothesis of induction c), $bse_{\alpha-1}$ contains both of $+[t_{\iota x.\Phi} / u]G$, $+[t_{\iota x.\Phi}\ u]H$, respectively one of $-[t_{\iota x.\Phi} / u]G$, $-[t_{\iota x.\Phi} / u]H$. Hence, by semantic rule 1.3.2.2., respectively $\pm[t_{\iota x.\Phi} / u](G \wedge H) \cong \pm[t_{\iota x.\Phi} / u]F$ is in $Sc(bse_{\alpha-1}) \subseteq bse_{\alpha}$.

iii) $\pm[\iota x.\Phi / u]F$ is obtained in $Sc(bse_{\alpha-1})$ by an application of semantic rule 1.3.2.3. and is respectively of the form $\pm[\iota x.\Phi / u](G \rightarrow H)$. This case is similar to ii).

iv) $\pm[\iota x.\Phi / u]F$ is obtained in $Sc(bse_{\alpha-1})$ by an application of semantic rule 1.3.2.4. and is respectively of the form $\pm[\iota x.\Phi / u](v)G$, for some formula G. Then $bse_{\alpha-1}$ contains $+[\iota x.\Phi / u][s / v]G$ for all s in $\delta(Par)$, respectively, $-[\iota x.\Phi / u][s / v]G$ for some s in $\delta(Par)$. So, by hypothesis of induction c), $bse_{\alpha-1}$ contains $+[t_{\iota x.\Phi} / u][s / v]G$ for all s in $\delta(Par)$, respectively, $-[t_{\iota x.\Phi} / u][s / v]G$ for some s in $\delta(Par)$. Since both of s, $t_{\iota x.\Phi}$ are constant terms, $\pm[t_{\iota x.\Phi} / u][s / v]G \cong \pm[s / v][t_{\iota x.\Phi} / u]G$. So $bse_{\alpha-1}$ contains $+[s / v][t_{\iota x.\Phi} / u]G$ for all s in $\delta(Par)$, respectively, $-[s / v][t_{\iota x.\Phi} / u]G$ for s in $\delta(Par)$. Hence, by semantic rule 1.3.2.4., respectively $\pm(v)[t_{\iota x.\Phi} / u]G \cong \pm[t_{\iota x.\Phi} / u](v)G \cong \pm[t_{\iota x.\Phi} / u]F$ is in $Sc(bse_{\alpha-1}) \subseteq bse_{\alpha}$.

In all cases then, respectively $\pm[t_{\iota x.\Phi} / u]F$ is in bse_{α}. This completes the induction step for case a).

b) For some $\gamma < \mu$, constant term t, variable v, bse_{γ} contains both of the signed sentences $+[t / x]\Phi$, $+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)$. Then, since $\langle bse_{\mu}\rangle$ is well ordered under $\subseteq$, there is a least $\gamma < \mu$ such that for some $\gamma < \mu$, constant term t, variable v, bse_{γ} contains both of the signed sentences $+[t / x]\Phi$, $+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)$. Since $\gamma < \mu$, by hypothesis of induction b), we may assume that this t belongs to $\delta(Par)$. So let $t_{\iota x.\Phi}$ be t. Now, we show by induction on α that for all $\alpha, \gamma \leq \alpha \leq \mu$, for all formulas F, if $\pm[\iota x.\Phi / u]F$ is in bse_{α}, then respectively $\pm[t_{\iota x.\Phi} / u]F$ is in bse_{α}. Clearly, our desired result that for all formulas F, if $\pm[\iota x.\Phi / u]F$ is a signed sentence in bse_{μ}, then respectively $\pm[t_{\iota x.\Phi} / u]F$ is in bse_{μ}, will follow from this result. So assume that the signed sentence $\pm[\iota x.\Phi / u]F$ is in bse_{α}

Base step: $\alpha = \gamma$. Since for no $\beta < \gamma$ does bse_{β} contain the pressupositions to any instance of the semantic rule for descriptions 1.3.2.5., it follows by the argument given in case a) above that for all formulas F, if $\pm[\iota x.\Phi / u]F$ is in bse_{γ}, then respectively $\pm[t_{\iota x.\Phi} / u]F$ is in bse_{γ}.

Induction step: $\alpha > \gamma$. Assume c) that that for all $\beta, \gamma \leq \beta < \alpha$, for all formulas F, if $\pm[\iota x.\Phi / u]F$ is in bse_{β}, then respectively $\pm[t_{\iota x.\Phi} / u]F$ is in bse_{β}. We want to show that for all formulas F, if $\pm[\iota x.\Phi / u]F$ is in bse_{α}, then respectively $\pm[t_{\iota x.\Phi} / u]F$ is in bse_{α}. So let F be any formula such that $\pm[\iota x.\Phi / u]F$ is a signed sentence in bse_{α} We show that respectively $\pm[t_{\iota x.\Phi} / u]F$ is in bse_{α}. For limit ordinals α: Since $bse_{\alpha} = \bigcup bse_{\beta}, 0 \leq \beta < \alpha$, the claim holds trivially by hypothesis of induction c). For successor ordinals α: We may assume that for no $\beta < \alpha$ is $\pm[\iota x.\Phi / u]F$ in bse_{α}, since otherwise, by the hypothesis of induction, the claim holds trivially. Then $\pm[\iota x.\Phi / u]F$ is in $Sc(bse_{\alpha-1})$. There are five subcases: subcases in which $\pm[\iota x.\Phi / u]F$ is in $Sc(bse_{\alpha-1})$ by virtue of an instance $\mathbb{I}$ of one of semantic rules 1.3.2.1, 1.3.2.2., 1.3.2.3., 1.3.2.4., are treated as subcases i) - iv) of case a) above. So we consider only the subcase where $\mathbb{I}$ is an instance of the semantic rule for descriptions 1.3.2.5. Then there are two sub-subcases:

1) It is not the case that, for some constant term r and variable v, both of the following hold:
i) the presuppositions to $\mathbb{I}$ are of the form $+[r / x]\Phi$, $+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)$.
ii) the major statement to $\mathbb{I}$ is respectively of the form $\pm[r / u]F$.
Then F is of the form $[\iota z.\Psi / y]G$, for some constant description term $\iota z.\Psi$ and formula G, and the presuppositions, respectively, major statement to $\mathbb{I}$ are of the form $+[s / z]\Psi$, $+(z)(v)((\Psi \wedge [v / z]\Psi) \rightarrow z = v)$, respectively, $\pm[\iota x.\Phi / u][s / y]G \cong \pm[s / y][\iota x.\Phi / u]G$, for some constant term s and variable v. So $bse_{\alpha-1}$ contains all of $+[s / z]\Psi$, $+(z)(v)((\Psi \wedge [v / z]\Psi) \rightarrow z = v)$, respectively $\pm[\iota x.\Phi / u][s / y]G$. Then by hypothesis of induction c), $bse_{\alpha-1}$ contains

all of $+[s / z]\Psi$, $+(z)(v)((\Psi \wedge [v / z]\Psi) \rightarrow z = v)$, respectively $\pm[t_{\iota x.\Phi} / u][s / y]G \cong \pm[s / y][t_{\iota x.\Phi} / u]G$. So, by semantic rule 1.3.2.5., $Sc(bse_{\alpha-1}) \subseteq bse_{\alpha}$ contains respectively $\pm[\iota z.\Psi / y][t_{\iota x.\Phi} / u]G \cong \pm[t_{\iota x.\Phi} / u][\iota z.\Psi / y]G \cong \pm[t_{\iota x.\Phi} / u]F$. So the claim holds.

2) The presuppositions to $\mathbb{I}$ are of the form $+[r / x]\Phi$, $+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)$, and the major statement to $\mathbb{I}$ is respectively of the form $\pm[r / u]F$, for some constant term r and variable v. So $bse_{\alpha-1}$ contains all of $+[r / x]\Phi$, $+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)$, respectively $\pm[r / u]F$. Since $\alpha - 1 < \mu$, by the hypothesis of induction b), we may assume that r is in $\delta(Par)$. But we established that the signed sentence $+[t_{\iota x.\Phi} / x]\Phi$ belongs to bse_{γ}. Since $\gamma < \alpha$, $bse_{\gamma} \subseteq bse_{\alpha-1}$. So all of $+[t_{\iota x.\Phi} / x]\Phi$, $+[r / x]\Phi$, $+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)$, are in $bse_{\alpha-1}$. Since $+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)$ is in $bse_{\alpha-1}$, it is easily verified that by semantic rule 1.3.2.4. there is a $\beta < \alpha - 1$ such that $+(([t_{\iota x.\Phi} / x]\Phi \wedge [s / x]\Phi) \rightarrow t_{\iota x.\Phi} = s)$ is in bse_{β}. Then, it is easily verified that by semantic rule 1.3.2.3. there is a $\zeta < \beta$ such that one of $-([t_{\iota x.\Phi} / x]\Phi \wedge [s / x]\Phi)$, $+(t_{\iota x.\Phi} = s)$ is in bse_{ζ}. Assume that $-([t_{\iota x.\Phi} / x]\Phi \wedge [s / x]\Phi) \in bse_{\zeta}$. Then it is easily verified that by semantic rule 1.3.2.2. there is a $\lambda < \zeta$ such that one of $-[t_{\iota x.\Phi} / x]\Phi$, $-[s / x]\Phi$ is in bse_{λ}. Since $\lambda < \zeta < \beta < \alpha - 1$, $\lambda < \alpha - 1$, and hence $bse_{\lambda} \subseteq bse_{\alpha-1}$. So both of $+[t_{\iota x.\Phi} / x]\Phi$, $+[s / x]\Phi$ and one of $-[t_{\iota x.\Phi} / x]\Phi$, $-[s / x]\Phi$ is in $bse_{\alpha-1}$. So there is a sentence ***snt*** such that both of $\pm$***snt*** are in $bse_{\alpha-1}$. Since $\alpha - 1 < \mu$, this contradicts hypothesis of induction a). So the signed sentence $+(t_{\iota x.\Phi} = s)$ is in $bse_{\zeta} \subseteq bse_{\alpha-1}$. So by lemma 1.3.8., since $\pm[s / u]F$ is in $bse_{\alpha-1}$, respectively $\pm[t_{\iota x.\Phi} / u]F$ is in $bse_{\alpha-1} \subseteq bse_{\alpha}$. So the claim holds.

In all cases, then, respectively $\pm[t_{\iota x.\Phi} / u]F$ is in bse_{α} This completes the inner induction step of the proof that bse_{μ} satisfies lemma 1.3.10.. Thus, the outer induction step of the proof that for all ordinals μ, bse_{μ} satisfies lemma 1.3.10., is now complete. We now complete the induction step of the proof that bse_{μ} satisfies lemma 1.3.9.: Let ***snt*** be any sentence. Assume, contrary to that which is to be shown, that bse_{μ} contains both of $\pm$***snt***. Since the sequence $\langle bse_{\mu} \rangle$ is well ordered under $\subseteq$, there are least β, γ, $0 < \beta$, $\gamma \leq \mu$, such that $+$***snt*** $\in bse_{\beta}$ and $-$***snt*** $\in bse_{\gamma}$. Now, by definition 1.3.3., $bse_{\beta} = bse_{\beta} \cup Sc(bse_{\beta-1})$ and $bse_{\gamma} = bse_{\gamma-1} \cup Sc(bse_{\gamma-1})$. Since for no ζ, α such that $\zeta < \beta$ and $\alpha < \gamma$ is $+$***snt*** $\in bse_{\zeta}$, $-$***snt*** $\in bse_{\alpha}$ it follows that $+$***snt*** $\in Sc(bse_{\beta-1})$ and

-$snt \in Sc(bse_{\gamma-1})$. There are two cases:

1) snt contains no occurrence of any description term. Then there are just four subcases:

i) snt is of the form $\neg\mathbf{A}$ for some sentence $\mathbf{A}$. Then $+\neg\mathbf{A}$ is in $Sc(bse_{\beta-1})$ and $-\neg\mathbf{A}$ is in $Sc(bse_{\gamma-1})$ by virtue of semantic rule 1.3.2.1.. and so $-\mathbf{A} \in bse_{\beta-1}$ and $+\mathbf{A} \in bse_{\gamma-1}$. Assume, without loss of generality, that $\gamma \leq \beta$. So $bse_{\gamma-1} \subseteq bse_{\beta-1}$ and hence both of $\pm\mathbf{A}$ are in $bse_{\beta-1}$. Since $\beta \leq \mu$, $\beta - 1 < \mu$. So there is a $\beta < \mu$ such that both of $\pm\mathbf{A}$ are in bse_{β}. This contradicts hypothesis of induction a).

ii) snt is of the form $(\mathbf{A} \wedge \mathbf{B})$ for some sentences $\mathbf{A}$, $\mathbf{B}$. Then $+(\mathbf{A} \wedge \mathbf{B})$ is in $Sc(bse_{\beta-1})$ and $-(\mathbf{A} \wedge \mathbf{B})$ is in $Sc(bse_{\gamma-1})$ by virtue of semantic rule 1.3.2.2. and so both of $+\mathbf{A}$, $+\mathbf{B}$ are in $bse_{\beta-1}$ and one of $-\mathbf{A}$, $-\mathbf{B}$ is in $bse_{\gamma-1}$. Assume, without loss of generality, that $\gamma \leq \beta$. Then both of $+\mathbf{A}$, $+\mathbf{B}$ and one of $-\mathbf{A}$, $-\mathbf{B}$ are in $bse_{\beta-1}$. Then either both of $\pm\mathbf{A}$ or both of $\pm\mathbf{B}$ are in $bse_{\beta-1}$. Since $\beta \leq \mu$, $\beta - 1 < \mu$. So there is a $\beta < \mu$ and a sentence snt such that both of $\pm snt$ are in bse_{β}. This contradicts the hypothesis of induction a).

iii) snt is of the form $(\mathbf{A} \rightarrow \mathbf{B})$ for some sentences $\mathbf{A}$, $\mathbf{B}$. This case is similar to case ii) above.

iv) snt is of the form $(x)F$ for some formula F. Then $+(x)F$ is in $Sc(bse_{\beta-1})$ and $-(x)F$ is in $Sc(bse_{\gamma-1})$ by virtue of semantic rule 1.3.2.4. and so $+[t / x]F$ is in $bse_{\beta-1}$ for all terms $t \in \delta(Par)$ and $-[r / x]F$ is in $bse_{\gamma-1}$ for some term $r \in \delta(Par)$. Assume, without loss of generality, that $\gamma \leq \beta$. Then both of $+[r / x]F$, $-[r / x]F$ are in $bse_{\beta-1}$. Since $\beta \leq \mu$, $\beta - 1 < \mu$. So there is a $\beta < \mu$ and a sentence $[r / x]F$ such that both of $\pm[r / x]F$ are in bse_{β}. This contradicts hypothesis of induction a).

2) snt contains an occurrence of some description term and so is of the form $[\iota x.\Phi / u]F$ for some description term $\iota x.\Phi$ and formula F containing free occurrence of u.. In case neither $+snt \in Sc(bse_{\beta-1})$ nor $-snt \in Sc(bse_{\gamma-1})$ by virtue of the semantic rule for descriptions 1.3.2.5., this case reduces to case 1) above. So assume, without loss of generality, that $+snt \in Sc(bse_{\beta-1})$ by

virtue of an instance $\mathbb{B}$ 1.3.2.5.. Then there are two subcases:

i) $-snt \in Sc(bse_{\gamma-1})$ by virtue of an instance $\mathbb{G}$ of 1.3.2.5.. Since $+[\iota x.\Phi / u]F \in Sc(bse_{\beta-1})$ by virtue of $\mathbb{B}$, $bse_{\beta-1}$ contains the two presuppositions $+[s / x]\Phi$, $+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)$, as well as the major statement $+[s / u]F$, to $\mathbb{B}$, for some constant term s and variable v. And since $-[\iota x.\Phi / u]F \in Sc(bse_{\gamma-1})$ by virtue $\mathbb{G}$, $bse_{\gamma-1}$ contains the existential presupposition $+[r / x]\Phi$ and the major statement $-[r / u]F$ to $\mathbb{G}$ for some constant terms r. Since $\gamma - 1 < \mu$, both of s, r may by hypothesis of induction b) be assumed to belong to $\delta(Par)$. Then, since $\gamma - 1 < \mu$, it follows from hypothesis of induction a), by an argument similar to the one given in the outer induction step for the proof of lemma 1.3.10., that the signed sentence $+(s = r)$ belongs to bse. So, by lemma 1.3.8., since both of $+(s = r)$, $-[r / x]\Phi$ belong to $bse_{\gamma-1}$, it follows that the signed sentence $-[s / x]\Phi$ belongs to $bse_{\gamma-1}$. So both of $\pm[s / x]\Phi$ belong to $bse_{\gamma-1}$. This contradicts hypothesis of induction a).

ii) $-snt \in Sc(bse_{\gamma-1})$ by virtue of an instance $\mathbb{G}$ of some semantic rule other than 1.3.2.5.. Since $+[\iota x.\Phi / u]F \in Sc(bse_{\beta-1})$ by virtue of $\mathbb{B}$, $bse_{\beta-1}$ contains the two presuppositions $+[s / x]\Phi$, $+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)$, as well as the major statement $+[s / u]F$, to $\mathbb{B}$, for some constant term s and variable v. Let us suppose that $\mathbb{G}$ is an instance of 1.3.2.2.; the other cases are similar to this one. Then F is of the form $(G \wedge H)$ for some formulas G, H. Since $-[\iota x.\Phi / u](G \wedge H) \cong -([\iota x.\Phi / u]G \wedge [\iota x.\Phi / u]H) \in Sc(bse_{\gamma-1})$ by virtue of $\mathbb{G}$, $bse_{\gamma-1}$ contains one of the signed sentences $-[\iota x.\Phi / u]G$, $-[\iota x.\Phi / u]H$. Now, since $\gamma - 1 < \mu$, it follows from hypothesese of induction a) and b), by the argument given in the outer induction step for the proof of lemma 1.3.10., that there are $\zeta < \gamma - 1$ and $t_{\iota x.\Phi} \in \delta(Par)$ such that bse_ζ contains the signed sentence $+[t_{\iota x.\Phi} / x]\Phi$ and $bse_{\gamma-1}$ contains one of the signed sentences $-[t_{\iota x.\Phi} / u]G$, $-[t_{\iota x.\Phi} / u]H$. Assume, without loss of generality, that $\gamma \leq \beta$. So $bse_\zeta \subseteq bse_{\gamma-1} \subseteq bse_{\beta-1}$ and hence all of $+[t_{\iota x.\Phi} / x]\Phi$, $+[s / x]\Phi$, $+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)$, $+[s / u]F \cong +([s / u]G \wedge [s / u]H)$, and one of $-[t_{\iota x.\Phi} / u]G$, $-[t_{\iota x.\Phi} / u]H$ are in $bse_{\beta-1}$. Since $\beta - 1 < \mu$ and all of $+[t_{\iota x.\Phi} / x]\Phi$, $+[s / x]\Phi$, $+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)$ are in $bse_{\beta-1}$, it follows from hypothesis of induction a) that the signed sentence $+(t_{\iota x.\Phi} = s)$ belongs to bse. So, by lemma 1.3.8., since

$+(t_{\iota x.\Phi} = s)$ and one of $-[t_{\iota x.\Phi} / u]G$, $-[t_{\iota x.\Phi} / u]H$ both belong to $bse_{\beta-1}$, it follows that one of $-[s / u]G$, $-[s / u]H$ belongs to $bse_{\beta-1}$.

Now, formulas G, H are respectively of the form $[\underline{\iota z.\Psi} / \underline{x}]G'$, $[\underline{\iota y.\Xi} / \underline{x}]H'$, for some formulas G', H' containing no occurrences of any constant description terms, where the $\underline{\iota z.\Psi} \cong \iota z_1.\Psi_1, \ldots, \iota z_n.\Psi_n$, $\underline{\iota y.\Xi} \cong \iota y_1.\Xi_1, \ldots, \iota y_m.\Xi_m$ are all of the constant description terms occurring in G, H. So $bse_{\beta-1}$ contains $+[s / u]F \cong +([s / u]G \wedge [s / u]H) \cong +([s / u][\underline{\iota z.\Psi} / \underline{x}]G' \wedge [s / u][\underline{\iota y.\Xi} / \underline{x}]H')$ and one of $-[s / u][\underline{\iota z.\Psi} / \underline{x}]G'$, $-[s / u][\underline{\iota y.\Xi} / \underline{x}]H'$. Since $\beta - 1 < \mu$, by hypothesis of induction b), there are $\underline{t}_{\iota z.\Psi} \cong t_{\iota z1.\Psi 1}, \ldots, t_{\iota zn.\Psi n}$, $\underline{t}_{\iota z.\Xi} \cong t_{\iota z1.\Xi 1}, \ldots, t_{\iota zm.\Xi m}$ in $\delta(Par)$ such that $bse_{\beta-1}$ contains $+([s / u][\underline{t}_{\iota z.\Psi} / \underline{x}]G' \wedge [s / u][\underline{t}_{\iota z.\Xi} / \underline{x}]H')$ and one of $-[s / u][\underline{t}_{\iota z.\Psi} / \underline{x}]G'$, $-[s / u][\underline{t}_{\iota z.\Xi} / \underline{x}]H'$. Now, since neither of G', H' contain an occurrence of any constant description term, $([s / u][\underline{t}_{\iota z.\Psi} / \underline{x}]G' \wedge [s / u][\underline{t}_{\iota z.\Xi} / \underline{x}]H')$ contains no occurrence of any constant description term. Since $bse_{\beta-1}$ contains $+([s / u][\underline{t}_{\iota z.\Psi} / \underline{x}]G' \wedge [s / u][\underline{t}_{\iota z.\Xi} / \underline{x}]H')$, there is a least $\zeta \leq \beta - 1$ such that bse_ζ contains $+([s / u][\underline{t}_{\iota z.\Psi} / \underline{x}]G' \wedge [s / u][\underline{t}_{\iota z.\Xi} / \underline{x}]H')$. So $+([s / u][\underline{t}_{\iota z.\Psi} / \underline{x}]G' \wedge [s / u][\underline{t}_{\iota z.\Xi} / \underline{x}]H')$ is in $Sc(bse_{\zeta-1})$. Moreover, since $([s / u][\underline{t}_{\iota z.\Psi} / \underline{x}]G' \wedge [s / u][\underline{t}_{\iota z.\Xi} / \underline{x}]H')$ contains no occurrence of any constant description term, $+([s / u][\underline{t}_{\iota z.\Psi} / \underline{x}]G' \wedge [s / u][\underline{t}_{\iota z.\Xi} / \underline{x}]H')$ is in $Sc(bse_{\zeta-1})$ by virtue of semantic rule 1.3.2.2.. So $bse_{\zeta-1}$ contains both of $+[s / u][\underline{t}_{\iota z.\Psi} / \underline{x}]G'$, $+[s / u][\underline{t}_{\iota z.\Xi} / \underline{x}]H'$. Since $bse_{\zeta-1} \subseteq bse_{\beta-1}$, it follows that $bse_{\beta-1}$ contains both of $+[s / u][\underline{t}_{\iota z.\Psi} / \underline{x}]G'$, $+[s / u][\underline{t}_{\iota z.\Xi} / \underline{x}]H'$ and one of $-[s / u][\underline{t}_{\iota z.\Psi} / \underline{x}]G'$, $-[s / u][\underline{t}_{\iota z.\Xi} / \underline{x}]H'$. So there is a $\beta < \mu$ and a sentence ***snt*** such that both of $\pm$***snt*** are in bse_β. This contradicts hypothesis of induction a).

In all cases then, the assumption that both of $\pm$*snt* are in bse_μ leads to a contradiction. This completes the induction step of the proof that for all ordinals μ, bse_μ satisfies lemma 1.3.9.. Thus, the proofs of lemmata 1.3.9. and 1.3.10. are now complteted. □

<u>Lemma 1.3.11.</u>: Let *bse* be any base and let $\iota x.\Phi$ be any constant description term such that for some constant term r and variable v, $Cl(bse)$ contains both of the signed sentences $+[r / x]\Phi$, $+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)$. Let $t_{\iota x.\Phi} \in \delta(Par)$ be a descriptum of $\iota x.\Phi$ in *bse*. Then,

for all formulas F, if $\pm[t_{\iota x.\Phi} / u]F$ is a signed sentence in $Cl(bse)$, then so is respectively $\pm[\iota x.\Phi / u]F$.

Proof of 1.3.11.: Let bse, $\iota x.\Phi$ and $t_{\iota x.\Phi}$ be as above. Assume that, for some formula F, $\pm[t_{\iota x.\Phi} / u]F$ is a signed sentence in $Cl(bse)$. We want to show that respectively $\pm[\iota x.\Phi / u]F$ is in in $Cl(bse)$. Since $Cl(bse) = \bigcup bse_\mu, 0 \leq \mu < \in_0$, respectively $\pm[t_{\iota x.\Phi} / y]F$ is in bse_ζ for some $\zeta < \in_0$. Since $Cl(bse)$ contains both of the signed sentences $+[r / x]\Phi$, $+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)$, there are $\beta, \gamma < \in_0$ such that $+[r / x]\Phi \in bse_\beta$ and $+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v) \in bse_\gamma$. Assume, without loss of generality, that $\beta < \gamma$. Then $bse_\beta \subseteq bse_\gamma$ and so bse_γ contains both of the signed sentences $+[r / x]\Phi$, $+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)$. So, by an instance of 1.3.2.5. whose major statement is $+[r / x]\Phi$, $Sc(bse_\gamma) \subseteq bse_{\gamma+1}$ contains the signed sentence $+[\iota x.\Phi / x]\Phi$. Then by lemma 1.3.10., $bse_{\gamma+1}$ contains the signed sentence $+[t_{\iota x.\Phi} / x]\Phi$. There are two cases:

i) $\zeta \leq \gamma$. Then $bse_{\gamma+1}$ contains both of $+[t_{\iota x.\Phi} / x]\Phi$, $+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)$, as well as respectively $\pm[t_{\iota x.\Phi} / y]F$. Hence, by 1.3.2.5., $Sc(bse_{\gamma+1}) \subseteq Cl(bse)$ contains respectively $\pm[\iota x.\Phi / y]F$.

ii) $\zeta > \gamma$. Then bse_ζ contains both of $+[t_{\iota x.\Phi} / x]\Phi$, $+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)$, as well as respectively $\pm[t_{\iota x.\Phi} / y]F$. Hence, by 1.3.2.5., $Sc(bse_\zeta) \subseteq Cl(bse)$ contains respectively $\pm[\iota x.\Phi / y]F$. □

The significance of lemma 1.3.10. is that it guarantees that for every base bse and every constant description term t, there is a basic terms representing or "covering" t in bse. This fact is crucial to the soundness proof of the next section.

Section 2: A Soundness Result for Pld.

We say that Pld is **semantically sound** iff every derivable sequent is valid. In this section we show that Pld is (semantically) sound.

Lemma 2.0.: Let *bse* be any base, *s* any constant term, $\iota x.\Phi$ any constant description term, and F any formula containing free occurrence of the variable u. Then, for all ordinals μ, if bse_μ contains both of the signed sentences respectively $\pm[s / u]F$, $\mp[\iota x.\Phi / u]F$, then, for some constant term t and variable v, bse_μ contains both of the signed sentences $+[t / x]\Phi$, $+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)$.

Proof of 2.0.: Let *bse* be any base, *s* any constant term, $\iota x.\Phi$ any constant description term, and F any formula containing free occurrence of the variable u. Assume that bse_μ contains both of respectively $\pm[s / u]F$, $\mp[\iota x.\Phi / u]F$, and, contrary to that which is to be shown, for no constant term t and variable v does bse_μ contain both of the signed sentences $+[t / x]\Phi$, $+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)$. By definition 1.3.3., $bse_\mu = \bigcup bse_\beta$, $0 \leq \beta < \mu$, so, since $\mp[\iota x.\Phi / u]F$ belongs to bse_μ, there is a $\beta \leq \mu$ such that $\mp[\iota x.\Phi / u]F \in bse_\beta$. Now, since $\iota x.\Phi$ occurs in the signed sentence $\mp[\iota x.\Phi / u]F \in bse_\beta$ and $\langle bse_\mu \rangle$ is well ordered under $\subseteq$, there is a least $\zeta \leq \beta$ such that $\iota x.\Phi$ occurs in some signed sentence *ssnt* in bse_ζ. We now establish the following claim:

Claim: Let F be any formula containing free occurrence of a variable u and s be any constant term such that bse_μ contains both of the signed sentences respectively $\pm[s / u]F$, $\mp[\iota x.\Phi / u]F$. Then, there is $\beta \leq \mu$ such that for no $\gamma < \beta$ does $\iota x.\Phi$ occur in any signed sentence in bse_γ and for some sentence A and formula G containing free occurrence of u, one of the following two conditions holds:

i) there is a sentence of the form $(A \wedge [s / u]G)$, $([s / u]G \wedge A)$ such that bse_β contains the signed sentence $-(A \wedge [\iota x.\Phi / u]G)$, $-([s / u]G \wedge A)$ as well as the signed sentence $+(A \wedge [s / u]G)$,

$+([s / u]G \wedge \mathrm{A})$

ii) there is a sentence of the form $(\mathrm{A} \rightarrow [s / u]G)$, $([s / u]G \rightarrow \mathrm{A})$ such that bse_β contains the signed sentence $+(\mathrm{A} \rightarrow [\iota x.\Phi / u]G)$, $+([\iota x.\Phi / u]G \rightarrow \mathrm{A})$ as well as the signed sentence $-(\mathrm{A} \rightarrow [s / u]G)$, $-([s / u]G \rightarrow \mathrm{A})$.

<u>Proof of claim</u>: Let F be any formula containing free occurrence of u and s be any constant term such that bse_μ contains both of the signed sentences respectively $\pm[s / u]F$, $\mp[\iota x.\Phi / u]F$. We show by induction on μ for $\mu \geq \zeta$ that there is $\beta \leq \mu$ such that for no $\gamma < \beta$ does $\iota x.\Phi$ occur in any signed sentence in bse_γ and for some sentence A and formula G containing free occurrence of u, one of conditions i), ii) hold.

<u>Base step</u>: $\mu = \zeta$. Clearly, μ is a successor ordinal. So $bse_\mu = bse_{\mu-1} \cup \mathcal{S}c(bse_{\mu-1})$. Since $\mp[\iota x.\Phi / u]F \in bse_\mu$ and for no $\beta < \mu = \zeta$ does $\mp[\iota x.\Phi / u]F$ occur in bse_β, it follows that $\mp[\iota x.\Phi / u]F \in \mathcal{S}c(bse_{\mu-1})$ by virtue of an instance $\mathbb{I}$ of some semantic rule. Since $\iota x.\Phi$ is a constant term and for no $\beta < \mu = \zeta$ does $\iota x.\Phi$ occur in any signed sentence in bse_β, it follows that $\mathbb{I}$ is not an instance of either of the semantic rules 1.3.2.1., 1.3.2.4.. By assumption, for no constant term t and variable v does $bse_\mu \supset bse_{\mu-1}$, and hence $bse_{\mu-1}$, contain both of the signed sentences $+[t / x]\Phi$, $+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)$, so $\mathbb{I}$ is not an instance of the semantic rule for descriptions 1.3.2.5.. Then there are just two cases:

a) $\mathbb{I}$ is an instance of the semantic rule 1.3.2.2.-.. Then, for some sentence A containing no occurrence of $\iota x.\Phi$ and some formula G containing free occurrence of u, $\mp[\iota x.\Phi / u]F$ is of the form of one of $-(\mathrm{A} \wedge [\iota x.\Phi / u]G)$, $-([\iota x.\Phi / u]G \wedge \mathrm{A})$, where $bse_{\mu-1}$ contains -A. Then $\pm[s / u]F$ is of the form $+(\mathrm{A} \wedge [s / u]G)$, $+([s / u]G \wedge \mathrm{A})$. So μ is such for no $\beta < \mu$ does $\iota x.\Phi$ occur in any signed sentence in bse_β and bse_μ contains $-(\mathrm{A} \wedge [\iota x.\Phi / u]G)$, $-([\iota x.\Phi / u]G \wedge \mathrm{A})$ as well as $-(\mathrm{A} \wedge [\iota x.\Phi / u]G)$, $-([\iota x.\Phi / u]G \wedge \mathrm{A})$. So condition i) holds.

b) $\mathbb{I}$ is an instance of the semantic rule 1.3.2.3.+.. Then, for some sentence A containing no occurrence of $\iota x.\Phi$ and some formula G containing free occurrence of u, $\mp[\iota x.\Phi / u]F$ is of the form of one of $+(A \rightarrow [\iota x.\Phi\ /\ u]G)$, $+([\iota x.\Phi / u]G \rightarrow A)$, where $bse_{\mu-1}$ contains $-A$, $+A$. Then $\pm[s / u]F$ is of the form $-(A \rightarrow [s\ /\ u]G)$, $-([s / u]G \rightarrow A)$. So μ is such for no $\beta < \mu$ does $\iota x.\Phi$ occur in any signed sentence in bse_β and bse_μ contains $+(A \rightarrow [\iota x.\Phi\ /\ u]G)$, $+([\iota x.\Phi / u]G \rightarrow A)$ as well as $-(A \rightarrow [s\ /\ u]G)$, $-([s / u]G \rightarrow A)$. So condition ii) holds.

In both cases, then, one of conditions i), ii) holds.

Induction step: $\mu > \zeta$. Assume that for all γ, $\zeta \leq \gamma < \mu$, for all formulas H containing free occurrence of u and all constant terms t such that bse_γ contains both of the signed sentences respectively $\pm[t / u]H$, $\mp[\iota x.\Phi / u]H$ there is $\beta \leq \gamma$ such that for no $\gamma < \beta$ does $\iota x.\Phi$ occur in any signed sentence in bse_γ and for some sentence A and formula G containing free occurrence of u, one of conditions i), ii) hold.

For limit ordinals μ: The claim holds trivially by the hypothesis of induction.

For successor ordinals μ: Now, bse_μ contains both of respectively $\pm[s / u]F$, $\mp[\iota x.\Phi / u]F$. Since by assumption s is a constant term, by lemma 1.3.10, we may assume that s is in $\delta(Par)$. Let $\iota z.\Psi$ be any constant description term occurring in $F$and let F' be the result of uniformly replacing every occurrence of $\iota z.\Psi$ in F by a descriptum $t_{\iota z.\Psi}$for $\iota z.\Psi$ in bse whose existance is guaranteed by lemma 1.3.10. By lemma 1.3.10, bse_μ contains both of respectively $\pm[s / u]F'$, $\mp[\iota x.\Phi / u]F'$. Since $bse_\mu = \bigcup bse_\beta$, $0 \leq \beta < \mu$, and $\langle bse_\mu \rangle$ is well ordered under $\subseteq$, there are least γ, $\beta \leq \mu$ such that $\pm[s / u]F' \in bse_\gamma$and respectively $\mp[\iota x.\Phi / u]F' \in bse_\beta$. Clearly, β is a successor ordinal, and so $bse_\beta = bse_{\beta-1} \cup Sc(bse_{\beta-1})$. So $\mp[\iota x.\Phi / u]F' \in Sc(bse_{\beta-1})$ by virtue of an instance $\mathbb{G}$ of some semantic rule. Since F' contains no occurrence of any constant description term, and for no constant term t and variable v does $bse_\mu \supset bse_{\beta-1}$, and therefore $bse_{\beta-1}$, contain both of the signed sentences $+[t / x]\Phi$, $+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)$, it follows that $\mathbb{G}$ is not an application of the rule for descriptions 1.3.2.5.. So there are just four cases:

1) $\mathbb{G}$ is an instance of 1.3.2.1.. Then F' is of the form $\neg H$ for some formula H and so $\mp[\iota x.\Phi /$

$u]F' \cong \mp[\iota x.\Phi / u]\neg H$. So $bse_{\beta-1}$ contains respectively $\pm[\iota x.\Phi / u]H$. Now, bse_γcontains respectively $\pm[s / u]F' \cong \pm[s / u]\neg H$. Since $[s / u]\neg H$ is not atomic, $\gamma \neq 0$. Clearly, then, $\gamma$is a successor ordinal, and so $bse_\gamma = bse_{\gamma-1} \cup Sc(bse_{\gamma-1})$. Since for no $\kappa < \gamma$does bse_κcontain $\pm[s / u]\neg H$, it follows that $\pm[s / u]\neg H \in Sc(bse_{\gamma-1})$ by virtue of an instance $\mathbb{B}$ of some semantic rule. Since $\pm[s / u]F' \cong \pm[s / u]\neg H$ contains no occurrence of any constant description term, it follows that $\mathbb{B}$ is not an application of the rule for descriptions 1.3.2.5.. Then $\mathbb{B}$ is an instance of 1.3.2.1.. So $bse_{\gamma-1}$ contains respectively $\mp[s / u]H$. Assume, without loss of generality, that $\beta \geq \gamma$. Then $bse_{\gamma-1} \subseteq bse_{\beta-1}$ and so $bse_{\beta-1}$ contains both of respectively $\pm[\iota x.\Phi / u]H$, $\mp[s / u]H$. Clearly, since u has free occurrence in $F' \cong \neg H$, it follows that u has free occurrence in H. Since $\beta \leq \mu$, $\beta - 1 < \mu$, so by the hypothesis of induction, there is $\eta \leq \beta - 1$ such that for no $\gamma < \eta$ does $\iota x.\Phi$ occur in any signed sentence in $bse_\gamma$and for some sentence A and formula G containing free occurrence of u, one of conditions i), ii) hold. Since $\beta \leq \mu$, $\eta \leq \mu$. Hence, there is a $\eta \leq \mu$ such that for no $\gamma < \eta$ does $\iota x.\Phi$ occur in any signed sentence in $bse_\gamma$and for some sentence A and formula G containing free occurrence of u, one of conditions i), ii) hold. So the claim holds.

1) $\mathbb{G}$ is an instance of 1.3.2.2.. Then F' is of the form $(G \wedge H)$ for some formulas G, H and so $\mp[\iota x.\Phi / u]F' \cong \mp[\iota x.\Phi / u](G \wedge H) \cong \mp([\iota x.\Phi / u]G \wedge [\iota x.\Phi / u]H)$. So $bse_{\gamma-1}$ contains one of $-[\iota x.\Phi / u]G$, $-[\iota x.\Phi / u]H$ respectively both of $+[\iota x.\Phi / u]G$, $+[\iota x.\Phi / u]H$. Now, bse_βcontains respectively $\pm[s / u]F' \cong \pm([s / u]G \wedge [s / u]H)$. Since $([s / u]G \wedge [s / u]H)$ is not atomic, $\beta \neq 0$. Clearly, then, $\beta$is a successor ordinal, and so $bse_\beta = bse_{\beta-1} \cup Sc(bse_{\beta-1})$. So $\pm([s / u]G \wedge [s / u]H) \in Sc(bse_{\beta-1})$ by virtue of an instance $\mathbb{B}$ of some semantic rule. Since $\pm[s / u]F' \cong \pm([s / u]G \wedge [s / u]H)$ contains no occurrence of any constant description term, it follows that $\mathbb{B}$ is not an application of the rule for descriptions 1.3.2.5.. Then $\mathbb{B}$ is an instance of 1.3.2.1.. So $bse_{\beta-1}$ contains both of $+[s / u]G$, $+[s / u]H$ respectively one of $-[s / u]G$, $-[s / u]H$. Assume, without loss of generality, that $\beta \geq \gamma$. Then $bse_{\beta-1}$contains both of $+[s / u]G$, $+[s / u]H$ and one of $-[\iota x.\Phi / u]G$, $-[\iota x.\Phi / u]H$ respectively one of $-[s / u]G$, $-[s / u]H$ and both of $+[\iota x.\Phi / u]G$, $+[\iota x.\Phi / u]H$. Now, we show that there is a formula K containing free

occurrence of u such that $bse_{\beta-1}$ contains both of $\pm[t / u]K$, $\mp[\iota x.\Phi / u]K$; clearly, this result follows if both of G, H contain free occurrence of u. So assume that it is not the case that both of G, H contain free occurrence of u. Since u has free occurrence in $F' \cong (G \wedge H)$, u has free occurrence in one G, H. So assume, without loss of generality, that G contains free occurrence of u but H does not. In this case, $[s / u]H \cong [\iota x.\Phi / u]H$. It follows that $bse_{\beta-1}$ does not contain $-[\iota x.\Phi / u]H \cong -[s / u]H$, for suppose it does: Then, contrary to lemma 1.3.9., $bse_{\beta-1}$ contains both of $+[s / u]H$, $-[s / u]H$. Since $bse_{\beta-1}$ does not contain $-[\iota x.\Phi / u]H \cong -[s / u]H$, it follows that $bse_{\beta-1}$ contains both of $+[s / u]G$, $-[\iota x.\Phi / u]G$ respectively both of $-[s / u]G$, $+[\iota x.\Phi / u]G$. So there is a formula K containing free occurrence of u such that $bse_{\beta-1}$ contains both of $\pm[t / u]K$, $\mp[\iota x.\Phi / u]K$. Since $\beta \leq \mu$, $\beta - 1 < \mu$, so by the hypothesis of induction, there is a $\eta \leq \beta - 1$ such that for no $\gamma < \eta$ does $\iota x.\Phi$ occur in any signed sentence in bse_γ and for some sentence A and formula G containing free occurrence of u, one of conditions i), ii) hold. Since $\beta \leq \mu$, $\eta \leq \mu$. Hence, there is a $\eta \leq \mu$ such that for no $\gamma < \eta$ does $\iota x.\Phi$ occur in any signed sentence in bse_γ and for some sentence A and formula G containing free occurrence of u, one of conditions i), ii) hold. So the claim holds.

3) $\mathbb{G}$ is an instance of 1.3.2.3.. This case is similar to case 2) above

4) $\mathbb{G}$ is an instance of 1.3.2.4.. Then F' is of the form $(z)H$ for some formula H and variable z and so $\mp[\iota x.\Phi / u]F' \cong \mp[\iota x.\Phi / u](z)H \cong \mp(z)[\iota x.\Phi / u]H$. So $bse_{\gamma-1}$ contains $-[t / z][\iota x.\Phi / u]H$ for some $t \in \delta(Par)$ respectively $+[t / z][\iota x.\Phi / u]H$ for all $t \in \delta(Par)$ Now, bse_β contains respectively $\pm[s / u]F' \cong \pm[s / u](z)H \cong \pm(z)[s / u]H$. Since $(z)[s / u]H$ is not atomic, $\beta \neq 0$. Clearly, then, β is a successor ordinal, and so $bse_\beta = bse_{\beta-1} \cup Sc(bse_{\beta-1})$. So $\pm(z)[s / u]H \in Sc(bse_{\beta-1})$ by virtue of an instance $\mathbb{B}$ of some semantic rule. Since $\pm[s / u]F' \cong \pm(z)[s / u]H$ contains no occurrence of any constant description term, it follows that $\mathbb{B}$ is not an application of the rule for descriptions 1.3.2.5.. Then $\mathbb{B}$ is an instance of 1.3.2.4.. So $bse_{\beta-1}$ contains $+[t / z][s / u]H$ for all $t \in \delta(Par)$ respectively $-[t / z][s / u]H$ for some $t \in \delta(Par)$. Clearly, since u has free occurrence in $F' \cong (z)H$, u has free occurrence in H. Assume, without loss of generality, that $\beta \geq \gamma$. Then $bse_{\beta-1}$ contains $-[t / z][\iota x.\Phi / u]H$ for some $t \in \delta(Par)$ and $+[t /$

$z][s / u]H$ for all $t \in \delta(Par)$ respectively $+[t / z][\iota x.\Phi / u]H$ for all $t \in \delta(Par)$ and $-[t / z][s / u]H$ for some $t \in \delta(Par)$. Clearly, then, since $[t / z][\iota x.\Phi / u]H \cong [\iota x.\Phi / u][t / z]H$ and $[t / z][s / u]H \cong [s / u][t / z]H$, it follows that $[t / z]H$ is a formula containing free occurrence of u such that $bse_{\beta-1}$ contains both of $\pm[t / u][t / z]H$, $\mp[\iota x.\Phi / u]$ $[t / z]H$. Since $\beta \leq \mu$, $\beta - 1 < \mu$, so by the hypothesis of induction, there is a $\eta \leq \beta - 1$ such that for no $\gamma < \eta$ does $\iota x.\Phi$ occur in any signed sentence in bse_γ and for some sentence A and formula G containing free occurrence of u, one of conditions i), ii) hold. Since $\beta \leq \mu$, $\eta \leq \mu$. Hence, there is a $\eta \leq \mu$ such that for no $\gamma < \eta$ does $\iota x.\Phi$ occur in any signed sentence in bse_γ and for some sentence A and formula G containing free occurrence of u, one of conditions i), ii) hold. So the claim holds.

In all cases then, there is a $\beta \leq \mu$ such that for no $\gamma < \beta$ does $\iota x.\Phi$ occur in any signed sentence in bse_γ and for some sentence A and formula G containing free occurrence of u, one of the following two conditions holds. This completes the induction step of the proof of the claim. We now return to the main proof of lemma 2.0..

Since bse_μ contains both of respectively $\pm[s / u]F$, $\mp[\iota x.\Phi / u]F$, by the claim, there is a $\beta \leq \mu$ such that for no $\gamma < \beta$ does $\iota x.\Phi$ occur in any signed sentence in bse_γ and for some sentence A and formula G containing free occurrence of u, one of conditions i), ii) holds. Assume, without loss of generality, that condition i) holds. Then, there is a sentence of the form (A $\wedge$ $[s / u]G$), ($[s / u]G$ $\wedge$ A) such that bse_β contains the signed sentence -(A $\wedge$ $[\iota x.\Phi / u]G$), -($[\iota x.\Phi / u]G$ $\wedge$ A) as well as the signed sentence +(A $\wedge$ $[s / u]G$), +($[s / u]G$ $\wedge$ A). Since by assumption s is a constant term, by lemma 1.3.10, we may assume that s is in $\delta(Par)$. Let $\iota z.\Psi$ be any constant description term occurring in A or in G, and let respectively A', G' be the result of uniformly replacing every occurrence of $\iota z.\Psi$ in respectively A', G' by a descriptum $t_{\iota z.\Psi}$ for $\iota z.\Psi$ in bse whose existance is guaranteed by lemma 1.3.10. By lemma 1.3.10, bse_β contains both of -(A' $\wedge$ $[\iota x.\Phi / u]G'$), -($[\iota x.\Phi / u]G'$ $\wedge$ A') as well as the signed sentence +(A' $\wedge$ $[s / u]G'$), +($[s / u]G'$ $\wedge$ A'). Clearly, β is a successor ordinal, and so $bse_\beta = bse_{\beta-1} \cup Sc(bse_{\beta-1})$. Since for no $\gamma < \beta$ does $\iota x.\Phi$ occur in any signed sentence in bse_γ, it follows that -(A' $\wedge$ $[\iota x.\Phi / u]G'$), -($[\iota x.\Phi / u]G'$ $\wedge$ A') does not belong to $bse_{\beta-1}$. So -(A' $\wedge$ $[\iota x.\Phi / u]G'$), -($[\iota x.\Phi / u]G'$ $\wedge$

A') belongs to $Sc(bse_{\beta-1})$ by virtue of an instance $\mathbb{B}$ of some semantic rule. Since neither **A** nor G contains an occurrence of any constant description term, and for no constant term t and variable v does $bse_{\mu} \supset bse_{\beta-1}$, and therefore $bse_{\beta-1}$, contain both of the signed sentences $+[t / x]\Phi$, $+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)$, it follows that $\mathbb{B}$ is not an instance of the semantic rule for descriptions 1.3.2.5.. Hence, $\mathbb{B}$ is an instance of 1.3.2.2., and so $bse_{\beta-1}$ contains -**A'**. Since bse_{β} contains the signed sentence $+(\mathbf{A}' \wedge [s / u]G')$, $+([s / u]G' \wedge \mathbf{A}')$, there is a least $\gamma \leq \beta$ such that bse_{γ} contains the signed sentence $+(\mathbf{A}' \wedge [s / u]G')$, $+([s / u]G' \wedge \mathbf{A}')$. Clearly, γ is a successor ordinal, and so $bse_{\gamma} = bse_{\gamma-1} \cup Sc(bse_{\gamma-1})$. Since $+(\mathbf{A}' \wedge [s / u]G')$, $+([s / u]G' \wedge$ **A'**) does not belong to $bse_{\gamma-1}$, $+(\mathbf{A}' \wedge [s / u]G')$, $+([s / u]G' \wedge \mathbf{A}')$ belongs to $Sc(bse_{\beta-1})$ by virtue of an instance $\mathbb{G}$ of some semantic rule. Now, $+(\mathbf{A}' \wedge [s / u]G')$, $+([s / u]G' \wedge \mathbf{A}')$ contains no occurremce of any constant description term, $\mathbb{B}$ is not an instance of 1.3.2.5.. Hence, $\mathbb{G}$ is an instance of 1.3.2.2., and so $bse_{\gamma-1}$ contains both of +**A'**, $+[s / u]G'$. Assume, without loss of generality, that $\gamma \geq \beta$. Hence, $bse_{\beta-1} \subseteq bse_{\gamma-1}$. So $bse_{\gamma-1}$ contains both of ±**A'**. This contradicts lemma 1.3.9.. Thus lemma 2.0 is established. □

Theorem 2.1.: Let ***Seq*** be any sequent. If ***Seq*** is derivable, then ***Seq*** is valid.

Proof of 2.1: Let ***Seq*** be any derivable sequent. Let $\underline{p} \cong p_1 , \ldots , p_n$ be an n-ary vector on *Par*containing all of the parameters which occur in some member signed sentence of ***Seq***. We need to show that for all bases *bse* and all n-ary vectors $\underline{t} \cong t_1 , \ldots , t_n$ on $\delta(Par)$, $[\underline{t} / \underline{p}]Seq \cap Cl(bse) \neq \varnothing$, from which it follows by definition 1.3.5. and 1.3.6. that ***Seq*** is valid. So let *bse* be any base and $\underline{t}$ any n-ary vector on *Par*. Since ***Seq*** is derivable, by definition 1.2.4., there is a derivation tree Π such that ***Seq*** is the endsequent of Π. We show by strong induction on the depth $D(\Pi)$ that $[\underline{t} / \underline{p}]Seq \cap Cl(bse) \neq \varnothing$.

Base step: $D(\Pi) = 0$. Then by 1.2.6. ***Seq*** is an axiom and so by 1.2.1. there are two cases:

i) ***Seq*** is of the form $\{-\boldsymbol{asnt} , +\boldsymbol{asnt}\}$ for some atomic ***asnt***. Since the $\underline{t}$ are all members of $\delta(Par)$, $[\underline{t} / \underline{p}]\boldsymbol{asnt}$ is an atomic sentence. So by condition 1) of1.3.1., one of $\pm[\underline{t} / \underline{p}]\boldsymbol{asnt}$ belong to *bse*. Hence $\{-[\underline{t} / \underline{p}]\boldsymbol{asnt} , +[\underline{t} / \underline{p}]\boldsymbol{asnt}\} \cong [\underline{t} / \underline{p}]\{-\boldsymbol{asnt} , +\boldsymbol{asnt}\} \cong [\underline{t} / \underline{p}]Seq \cap Cl(bse) \neq$

$\varnothing$.

ii) *Seq* is the form $\{+(t = t)\}$ for some $t \in \delta(Par)$. Clearly, $[\underline{t} / \underline{p}]t \in \delta(Par)$, so by condition 2) of1.3.1., the signed atomic sentence $+([\underline{t} / \underline{p}]t = [\underline{t} / \underline{p}]t)$ belongs to *bse*. So $\{+([\underline{t} / \underline{p}]t = [\underline{t} / \underline{p}]t)\} \cong [\underline{t} / \underline{p}]\{+(t = t)\} \cong [\underline{t} / \underline{p}]Seq \cap Cl(bse) \neq \varnothing$.

In both cases, then, $[\underline{t} / \underline{p}]Seq \cap Cl(bse) \neq \varnothing$.

<u>Induction step</u>: Assume valid all sequents which are derivable as endsequents of derivation trees Σ such that $D(\Sigma) < D(\Pi)$. We show that $[\underline{t} / \underline{p}]Seq \cap Cl(bse) \neq \varnothing$ by considering eight cases corresponding to the eight deductive rules given in 1.2.2. applications of which *Seq* may be the conclusion in Π:

i) *Seq* is the conclusion of an application $\mathbb{A}$ of the thinning rule 1.2.2.1.±. Then *Seq* is respectively of the form $Seq' \cup \{\pm snt\}$ for some sequent *Seq'* and sentence *snt*, where *Seq'* is the premise to $\mathbb{A}$. Then *Seq'* is the endsequent of Π's deepest proper subtree Σ. Since $D(\Sigma) < D(\Pi)$, by the hypothesis of induction, *Seq'* is valid. So, by definition 1.3.6., $[\underline{t} / \underline{p}]Seq' \cap Cl(bse) \neq \varnothing$. Trivially, then, $([\underline{t} / \underline{p}]Seq' \cup [\underline{t} / \underline{p}]\{\pm snt\}) \cong [\underline{t} / \underline{p}](Seq' \cup \{\pm snt\}) \cap Cl(bse) \neq \varnothing$. So $[\underline{t} / \underline{p}]Seq \cap Cl(bse) \neq \varnothing$.

ii) *Seq* is the conclusion of an application $\mathbb{A}$ of 1.2.2.2.±. Then *Seq* is respectively of the form $Seq' \cup \{\mp\neg A\}$ for some sequent *Seq'* and sentence A, where respectively $Seq' \cup \{\pm A\}$ is the premise to $\mathbb{A}$. Then $Seq' \cup \{\pm A\}$ is the endsequent of Π's deepest proper subtree Σ. Since $D(\Sigma) < D(\Pi)$, by the hypothesis of induction, $Seq' \cup \{\pm A\}$ is valid. So, by definition 1.3.6., $[\underline{t} / \underline{p}](Seq' \cup \{\pm A\}) \cong ([\underline{t} / \underline{p}]Seq' \cup [\underline{t} / \underline{p}]\{\pm A\}) \cap Cl(bse) \neq \varnothing$. There are two cases:

a) $[\underline{t} / \underline{p}]Seq' \cap Cl(bse) \neq \varnothing$. Trivially, then, respectively $([\underline{t} / \underline{p}]Seq' \cup [\underline{t} / \underline{p}]\{\mp\neg A\}) \cong [\underline{t} / \underline{p}](Seq' \cup \{\mp\neg A\}) \cap Cl(bse) \neq \varnothing$. So $[\underline{t} / \underline{p}]Seq \cap Cl(bse) \neq \varnothing$.

b) $[\underline{t} / \underline{p}]Seq' \cap Cl(bse) = \varnothing$. Then, since respectively $([\underline{t} / \underline{p}]Seq' \cup [\underline{t} / \underline{p}]\{\pm A\}) \cap Cl(bse) \neq \varnothing$, it follows that respectively $\pm[\underline{t} / \underline{p}]A \in Cl(bse)$. So, by semantic rule 1.3.2.2.±, respectively $\mp\neg[\underline{t}/\underline{p}]A \cong \mp[\underline{t}/\underline{p}]\neg A \in Sc(Cl(bse))$. By lemma 1.3.7., $Sc(Cl(bse)) \subseteq Cl(bse)$, so $\mp[\underline{t}/\underline{p}]\neg A \in Cl(bse)$). Hence, respectively $([\underline{t}/\underline{p}]Seq' \cup \{\mp[\underline{t}/\underline{p}]\neg A\}) \cong [\underline{t}/\underline{p}](Seq' \cup \{\mp\neg A\}) \cap Cl(bse) \neq \varnothing$. So $[\underline{t}/\underline{p}]Seq \cap Cl(bse) \neq \varnothing$.

iii) ***Seq*** is the conclusion of an application $\mathbb{A}$ of 1.2.2.3.±.. We separate the two cases:

iii.1) ***Seq*** is the conclusion of an application $\mathbb{A}$ of 1.2.2.3.+.. Then ***Seq*** is of the form $Seq' \cup \{+(A \wedge B)\}$ for some sequent ***Seq'*** and sentences **A**, **B**. Then there are sequents ***Seq1***, ***Seq2*** such that $Seq' = Seq1 \cup Seq2$ and $Seq1 \cup \{+A\}$, $Seq2 \cup \{+B\}$ are the premises to $\mathbb{A}$. Then both of $Seq1 \cup \{+A\}$, $Seq2 \cup \{+B\}$ are endsequents of proper subtrees respectively Σ_1, Σ_2 of Π. Since $D(\Sigma_1), D(\Sigma_2) < D(\Pi)$, by the hypothesis of induction, both of $Seq1 \cup \{+A\}$, $Seq2 \cup \{+B\}$ are valid. So, by definition 1.3.6., $[\underline{t}/\underline{p}](Seq1 \cup \{+A\}) \cong ([\underline{t}/\underline{p}]Seq2 \cup \{+[\underline{t}/\underline{p}]A\}) \cap Cl(bse) \neq \varnothing$ and $[\underline{t}/\underline{p}](Seq2 \cup \{+B\}) \cong ([\underline{t}/\underline{p}]Seq2 \cup \{+[\underline{t}/\underline{p}]B\}) \cap Cl(bse) \neq \varnothing$. There are two cases:

iii.1.a) $([\underline{t}/\underline{p}]Seq1 \cup [\underline{t}/\underline{p}]Seq2) \cap Cl(bse) \neq \varnothing$. Since $Seq' = Seq1 \cup Seq2$, it follows that $[\underline{t}/\underline{p}]Seq' \cong ([\underline{t}/\underline{p}]Seq1 \cup [\underline{t}/\underline{p}]Seq2)$. So $[\underline{t}/\underline{p}]Seq' \cap Cl(bse) \neq \varnothing$. Trivially, then, $([\underline{t}/\underline{p}]Seq' \cup [\underline{t}/\underline{p}]\{+(A \wedge B)\}) \cong [\underline{t}/\underline{p}](Seq' \cup \{+(A \wedge B)\}) \cap Cl(bse) \neq \varnothing$. So $[\underline{t}/\underline{p}]Seq \cap Cl(bse) \neq \varnothing$.

iii.1.b) $([\underline{t}/\underline{p}]Seq1 \cup [\underline{t}/\underline{p}]Seq2) \cap Cl(bse) = \varnothing$. Then, since $([\underline{t}/\underline{p}]Seq1 \cup \{+[\underline{t}/\underline{p}]A\}) \cap Cl(bse) \neq \varnothing$ and $([\underline{t}/\underline{p}]Seq2 \cup \{+[\underline{t}/\underline{p}]B\}) \cap Cl(bse) \neq \varnothing$., it follows that both of $+[\underline{t}/\underline{p}]A$, $+[\underline{t}/\underline{p}]B \in Cl(bse)$. So, by semantic rule 1.3.2.3.+, $+([\underline{t}/\underline{p}]A \wedge [\underline{t}/\underline{p}]B) \cong +[\underline{t}/\underline{p}](A \wedge B) \in Sc(Cl(bse))$. By lemma 1.3.7., $Sc(Cl(bse)) \subseteq Cl(bse)$, so $+[\underline{t}/\underline{p}](A \wedge B) \in Cl(bse)$). Hence, $[\underline{t}/\underline{p}]Seq' \cup \{+[\underline{t}/\underline{p}](A \wedge B)\} \cong [\underline{t}/\underline{p}](Seq' \cup \{+(A \wedge B)\}) \cap Cl(bse) \neq \varnothing$. So $[\underline{t}/\underline{p}]Seq \cap Cl(bse) \neq \varnothing$.

iii.2) ***Seq*** is the conclusion of an application $\mathbb{A}$ of 1.2.2.3.-.. Then ***Seq*** is of the form $Seq' \cup \{-(A \wedge B)\}$ for some sequent ***Seq'*** and sentences **A**, **B**, where $Seq' \cup \{-A, -B)\}$ is the premise to $\mathbb{A}$. Then $Seq' \cup \{-A, -B)\}$ is endsequent of Π's deepest proper subtree Σ. Since

$\mathcal{D}(\Sigma) < \mathcal{D}(\Pi)$, by the hypothesis of induction, $Seq' \cup \{-A, -B)\}$ is valid. So, by definition 1.3.6., $[\underline{t} / \underline{p}](Seq' \cup \{-A, -B)\}) \cong ([\underline{t} / \underline{p}]Seq' \cup [\underline{t} / \underline{p}]\{-A, -B)\}) \cap Cl(bse) \neq \emptyset$. There are two cases:

iii.2.a) $([\underline{t} / \underline{p}]Seq' \cap Cl(bse) \neq \emptyset$. Trivially, then, $([\underline{t} / \underline{p}]Seq' \cup [\underline{t} / \underline{p}]\{-(A \wedge B)\}) \cong [\underline{t} / \underline{p}](Seq' \cup \{-(A \wedge B)\}) \cap Cl(bse) \neq \emptyset$. So $[\underline{t} / \underline{p}]Seq \cap Cl(bse) \neq \emptyset$.

iii.2.b) $([\underline{t} / \underline{p}]Seq' \cap Cl(bse) = \emptyset$. Then, since $([\underline{t} / \underline{p}]Seq' \cup [\underline{t} / \underline{p}]\{-A, -B)\}) \cap Cl(bse) \neq \emptyset$, it follows that $[\underline{t} / \underline{p}]\{-A, -B)\}) \cap Cl(bse) \neq \emptyset$ and hence that one of $-[\underline{t} / \underline{p}]A$, $-[\underline{t} / \underline{p}]B$ belongs to $Cl(bse)$. So, by semantic rule 1.3.2.2.-, $-([\underline{t} / \underline{p}]A \wedge [\underline{t} / \underline{p}]B) \cong -[\underline{t} / \underline{p}](A \wedge B) \in Sc(Cl(bse))$. By lemma 1.3.7., $Sc(Cl(bse)) \subseteq Cl(bse)$, so $-[\underline{t} / \underline{p}](A \wedge B) \in Cl(bse))$. Hence, $[\underline{t} / \underline{p}]Seq' \cup \{-[\underline{t} / \underline{p}](A \wedge B)\} \cong [\underline{t} / \underline{p}](Seq' \cup \{-(A \wedge B)\}) \cap Cl(bse) \neq \emptyset$. So $[\underline{t} / \underline{p}]Seq \cap Cl(bse) \neq \emptyset$.

iv) Seq is the conclusion of an application of respectively 1.2.2.4.±.. This case is similar to case iii) above.

v) Seq is the conclusion of an application $\mathbb{A}$ of 1.2.2.5.±.. We separate the two cses:

v.1) Seq is the conclusion of an application $\mathbb{A}$ of 1.2.2.5.+.. Then Seq is of the form $Seq' \cup \{+(x)F\}$ for some sequent Seq' and formula F, where for some $q \in Par$ not occurring in any signed sentence of $Seq' \cup \{+(x)F\}$, $Seq' \cup \{+[q / x]F\}$ is the premise to $\mathbb{A}$. Then $Seq' \cup \{+[q / x]F\}$ is endsequent of Π's deepest proper subtree Σ. Since $\mathcal{D}(\Sigma) < \mathcal{D}(\Pi)$, by the hypothesis of induction, $Seq' \cup \{+[q / x]F\}$ is valid. Now, we want to show that $Seq' \cup \{+(x)F\}$ is valid, that is, we want to show that for all n-ary vectors $\underline{t}$ on $\delta(Par)$, $[\underline{t} / \underline{p}](Seq' \cup \{+(x)F\}) \cap Cl(bse) \neq \emptyset$, where $\underline{p}$ is an n-ary vector on Par containing all of the parameters which occur in some member of $Seq' \cup \{+(x)F\}$. Let $\underline{q} \cong q_1, \ldots, q_i \cong q, \ldots, q_m$ be an m-ary vector on Par containing all of the parameters which occur in the members of $Seq' \cup \{+[q_i / x]F\}$ such that q is the i'th term of $\underline{q}$. Clearly $\underline{q}$ contains all of the parameters which occur in some member of $Seq' \cup \{+(x)F\}$. Let $\underline{s} \cong s_1, \ldots, s_i, \ldots, s_m$ be any m-ary vector on, and let r be any term in, $\delta(Par)$. Let $|r / s_i|\underline{s} \cong s_1, \ldots, r, \ldots, s_m$ be the m-ary vector which is obtained from $\underline{s}$ be replacing the i'th term s_i in $\underline{s}$ by r. Since $Seq' \cup \{+[q_i / x]F\}$ is valid, it follows that $[\underline{s} / \underline{q}](Seq' \cup \{+[q_i / x]F\}) \cong ([\underline{s} / \underline{q}]Seq' \cup [\underline{s} / \underline{q}]\{+[q_i / x]F\}) \cap Cl(bse) \neq \emptyset$. There are two cases:

v.1.a) $[\underline{s} / \underline{q}]Seq' \cap Cl(bse) \neq \emptyset$. Trivially, then, $([\underline{s} / \underline{q}]Seq' \cup [\underline{s} / \underline{q}]\{+(x)F\}) \cong [\underline{s} / \underline{q}](Seq' \cup \{+(x)F\}) \cap Cl(bse) \neq \emptyset$. So $[\underline{s} / \underline{q}]Seq \cap Cl(bse) \neq \emptyset$.

v.1.b) $[\underline{s} / \underline{q}]Seq' \cap Cl(bse) = \emptyset$. Since q_i does not occur in any member of Seq', it follows that $[\underline{s} / \underline{q}]Seq' \cong [|r / s_i|\underline{s} / \underline{q}]Seq'$. So, $[|r / s_i|\underline{s} / \underline{q}]Seq' \cap Cl(bse) = \emptyset$. But since $Seq' \cup \{+[q_i / x]F\}$ is valid, it follows that $[|r / s_i|\underline{s} / \underline{q}](Seq' \cup \{+[q_i / x]F\}) \cong ([|r / s_i|\underline{s} / \underline{q}]Seq' \cup [|r / s_i|\underline{s} / \underline{q}]\{+[q_i / x]F\}) \cap Cl(bse) \neq \emptyset$. So $[|r / s_i|\underline{s} / \underline{q}]\{+[q_i / x]F\} \cap Cl(bse) \neq \emptyset$, in other words, $+[|r / s_i|\underline{s} / \underline{q}][q_i / x]F \in Cl(bse)$. Now, since all of the terms in $|r / s_i|\underline{s}$ are constant terms, $[|r / s_i|\underline{s} / \underline{q}][q_i / x]F \cong [r / x][|r / s_i|\underline{s} / \underline{q}]F$. Since q_i does not occur in F, $[r / x][|r / s_i|\underline{s} / \underline{q}]F \cong [r / x][\underline{s} / \underline{q}]F$. So $+[r / x][\underline{s} / \underline{q}]F \in Cl(bse)$. Since r is an arbitrary term in $\delta(Par)$, for all $r \in \delta(Par)$, $+[r / x][\underline{s} / \underline{q}]F \in Cl(bse)$. Hence, by semantic rule 1.3.2.4.+., $+(x)[\underline{s} / \underline{q}]F \cong +[\underline{s} / \underline{q}](x)F \in Sc(Cl(bse))$. By lemma 1.3.7., $Sc(Cl(bse)) \subseteq Cl(bse)$, so $+[\underline{s} / \underline{q}](x)F \in Cl(bse))$. Hence, $[\underline{s} / \underline{q}]Seq' \cup \{+[\underline{s} / \underline{q}](x)F\} \cong [\underline{s} / \underline{q}](Seq' \cup \{+(x)F\}) \cap Cl(bse) \neq \emptyset$. So $[\underline{t} / \underline{p}]Seq \cap Cl(bse) \neq \emptyset$.

v.2) Seq is the conclusion of an application $\mathbb{A}$ of 1.2.2.5.-. Then Seq is of the form $Seq' \cup \{-(x)F\}$ for some sequent Seq' and formula F, where for some constant term t, $Seq' \cup \{-[t / x]F\}$ is the premise to $\mathbb{A}$. Then $Seq' \cup \{-[t / x]F\}$ is endsequent of Π's deepest proper subtree Σ. Since $D(\Sigma) < D(\Pi)$, by the hypothesis of induction, $Seq' \cup \{-[t / x]F\}$ is valid. So $[\underline{t} / \underline{p}](Seq' \cup \{-[t / x]F\}) \cong ([\underline{t} / \underline{p}]Seq' \cup [\underline{t} / \underline{p}]\{-[t / x]F\}) \cap Cl(bse) \neq \emptyset$. There are two main cases:

v.2.a) $[\underline{t} / \underline{p}]Seq' \cap Cl(bse) \neq \emptyset$. Trivially, then, $([\underline{t} / \underline{p}]Seq' \cup [\underline{t} / \underline{p}]\{-(x)F\}) \cong [\underline{t} / \underline{p}](Seq' \cup \{-(x)F\}) \cap Cl(bse) \neq \emptyset$. So $[\underline{t} / \underline{p}]Seq \cap Cl(bse) \neq \emptyset$.

v.2.b) $[\underline{t}/\underline{p}]Seq' \cap Cl(bse) = \varnothing$. Since $([\underline{t}/\underline{p}]Seq' \cup [\underline{t}/\underline{p}]\{-[t/x]F\}) \cap Cl(bse) \neq \varnothing$, it follows that $[\underline{t}/\underline{p}]\{-[t/x]F\}) \cap Cl(bse) \neq \varnothing$. So $-[\underline{t}/\underline{p}][t/x]F \cong -[[\underline{t}/\underline{p}]t/x][\underline{t}/\underline{p}]F \in Cl(bse))$. There are two subcases:

v.2.b.α) $[\underline{t}/\underline{p}]t \in \delta(Par)$. Then, by semantic rule 1.3.2.4.-., $-(x)[\underline{t}/\underline{p}]F \cong -[\underline{t}/\underline{p}](x)F \in Sc(Cl(bse))$. By lemma 1.3.7., $Sc(Cl(bse)) \subseteq Cl(bse)$, so $-[\underline{t}/\underline{p}](x)F \in Cl(bse))$. Hence, $[\underline{t}/\underline{p}]Seq' \cup \{-[\underline{t}/\underline{p}](x)F\} \cong [\underline{t}/\underline{p}](Seq' \cup \{-(x)F\}) \cap Cl(bse) \neq \varnothing$. So $[\underline{t}/\underline{p}]Seq \cap Cl(bse) \neq \varnothing$.

v.2.b.β) $[\underline{t}/\underline{p}]t \notin \delta(Par)$. Then $[\underline{t}/\underline{p}]t$ is a constant description term, so by lemma 1.3.9., there is a descriptum $r_{[\underline{t}/\underline{p}]t} \in \delta(Par)$ for $[\underline{t}/\underline{p}]t$ in bse such that $-[r_{[\underline{t}/\underline{p}]t}/x][\underline{t}/\underline{p}]F \in Cl(bse))$. So this subcase reduces to v.2.b.α) above.

vi) Seq is the conclusion of an application $\mathbb{A}$ of 1.2.2.6.. Then Seq is respectively of the form $Seq1 \cup Seq2 \cup Seq3 \cup \{\pm[\iota x.\Phi/u]F\}$ for some sequents $Seq1, Seq2, Seq3$, description term $\iota x.\Phi$ and formula F, where for some constant term t, the premises to $\mathbb{A}$ are $Seq1 \cup \{+[t/x]\Phi\}$, $Seq2 \cup \{+(x)(v)((\Phi \wedge [v/x]\Phi) \rightarrow x = v)\}$ and respectively $Seq3 \cup \{\pm[t/u]F\}$. Since these premises to $\mathbb{A}$ are endsequents of proper subtrees of Π, by the hypothesis of induction, they are all valid. So $[\underline{t}/\underline{p}](Seq1 \cup \{+[t/x]\Phi\}) \cong ([\underline{t}/\underline{p}]Seq1 \cup [\underline{t}/\underline{p}]\{+[t/x]\Phi\}) \cap Cl(bse) \neq \varnothing$ and $[\underline{t}/\underline{p}](Seq2 \cup \{+(x)(v)((\Phi \wedge [v/x]\Phi) \rightarrow x = v)\}) \cong ([\underline{t}/\underline{p}]Seq2 \cup [\underline{t}/\underline{p}]\{+(x)(v)((\Phi \wedge [v/x]\Phi) \rightarrow x = v)\}) \cap Cl(bse) \neq \varnothing$ and respectively $[\underline{t}/\underline{p}](Seq3 \cup \{\pm[t/u]F\}) \cong ([\underline{t}/\underline{p}]Seq3 \cup [\underline{t}/\underline{p}]\{\pm[t/u]F\}) \cap Cl(bse) \neq \varnothing$. There are two main cases:

vi.a) $[\underline{t}/\underline{p}]Seq1 \cup [\underline{t}/\underline{p}]Seq2 \cup [\underline{t}/\underline{p}]Seq3 \cap Cl(bse) \neq \varnothing$. Trivially, then, $[\underline{t}/\underline{p}]Seq1 \cup [\underline{t}/\underline{p}]Seq2 \cup [\underline{t}/\underline{p}]Seq3 \cup \{\pm[\iota x.\Phi/u]F\} \cong [\underline{t}/\underline{p}](Seq1 \cup Seq2 \cup Seq3 \cup \{\pm[\iota x.\Phi/u]F\}) \cap Cl(bse) \neq \varnothing$. So $[\underline{t}/\underline{p}]Seq \cap Cl(bse) \neq \varnothing$.

vi.b) $([\underline{t}/\underline{p}]Seq1 \cup [\underline{t}/\underline{p}]Seq2 \cup [\underline{t}/\underline{p}]Seq3) \cap Cl(bse) = \varnothing$. So, $[\underline{t}/\underline{p}]\{+[t/x]\Phi\} \cap Cl(bse) \neq \varnothing$ and $[\underline{t}/\underline{p}]\{+(x)(v)((\Phi \wedge [v/x]\Phi) \rightarrow x = v)\}) \cap Cl(bse) \neq \varnothing$ and respectively $[\underline{t}/\underline{p}]\{\pm[t/u]F\}) \cap Cl(bse) \neq \varnothing$. So all of $+[\underline{t}/\underline{p}][t/x]\Phi \cong +[[\underline{t}/\underline{p}]t/x][\underline{t}/\underline{p}]\Phi$, $+[\underline{t}/$

$\underline{p}](x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v) \cong +(x)(v)(([\underline{t} / \underline{p}]\Phi \wedge [v / x][\underline{t} / \underline{p}]\Phi) \rightarrow x = v)$, respectively $\pm[\underline{t} / \underline{p}][t / u]F\} \cong \pm[[\underline{t} / \underline{p}]t / u][\underline{t} / \underline{p}]F$ belong to $Cl(bse)$). Then, since $[\underline{t} / \underline{p}]t$ is a constant term, by semantic rule 1.3.2.5., respectively $\pm[\iota x.[\underline{t} / \underline{p}]\Phi / u][\underline{t} / \underline{p}]F \cong \pm[\underline{t} / \underline{p}][\iota x.\Phi / u]F \in Sc(Cl(bse))$. By lemma 1.3.7., $Sc(Cl(bse)) \subseteq Cl(bse)$, so respectively $\pm[\underline{t} / \underline{p}][\iota x.\Phi / u]F \in Cl(bse)$). Hence, $[\underline{t} / \underline{p}]Seq1 \cup [\underline{t} / \underline{p}]Seq2 \cup [\underline{t} / \underline{p}]Seq3 \cup [\underline{t} / \underline{p}]\ \{\pm[\iota x.\Phi / u]F\} \cong [\underline{t} / \underline{p}](Seq1 \cup Seq2 \cup Seq3 \cup \{\pm[\iota x.\Phi / u]F\}) \cap Cl(bse) \neq \varnothing$. So $[\underline{t} / \underline{p}]Seq \cap Cl(bse) \neq \varnothing$.

vii) Seq is the conclusion of an application $\mathbb{A}$ of 1.2.2.7.. Then Seq is respectively of the form $Seq1 \cup Seq2 \cup \{-(r = s)\}$ for some sequents $Seq1, Seq2$, and constant terms r, s, where for some formula F, the premises to $\mathbb{A}$ are $Seq1 \cup \{+[r / u]F\}, Seq2 \cup \{-[s / u]F\}$. In case u does not occur free in F, $[r / u]F \cong [s / u]F$. Then, by the arguement given in case x below, $([\underline{t} / \underline{p}]Seq1 \cup [\underline{t} / \underline{p}]Seq2) \cap Cl(bse) \neq \varnothing$, and hence $([\underline{t} / \underline{p}]Seq1 \cup [\underline{t} / \underline{p}]Seq2 \cup [\underline{t} / \underline{p}]\{-(r = s)\}) \cong [\underline{t} / \underline{p}](Seq1 \cup Seq2 \cup \{-(r = s)\}) \cap Cl(bse) \neq \varnothing$. So $[\underline{t} / \underline{p}]Seq \cap Cl(bse) \neq \varnothing$. So assume that u has free occurrence in F. Since the premises to $\mathbb{A}$ are endsequents of proper subtrees of Π, by the hypothesis of induction, they are all valid. So $[\underline{t} / \underline{p}](Seq1 \cup \{+[r / u]F\}) \cong ([\underline{t} / \underline{p}]Seq1 \cup [\underline{t} / \underline{p}]\{+[r / u]F\}) \cap Cl(bse) \neq \varnothing$ and $[\underline{t} / \underline{p}](Seq2 \cup \{-[s / u]F\}) \cong ([\underline{t} / \underline{p}]Seq2 \cup [\underline{t} / \underline{p}]\{-[s / u]F\}) \cap Cl(bse) \neq \varnothing$. There are two main subcases:

vii.a) $([\underline{t} / \underline{p}]Seq1 \cup [\underline{t} / \underline{p}]Seq2) \cap Cl(bse) \neq \varnothing$. Trivially, then, $([\underline{t} / \underline{p}]Seq1 \cup [\underline{t} / \underline{p}]Seq2 \cup [\underline{t} / \underline{p}]\{-(r = s)\}) = [\underline{t} / \underline{p}](Seq' \cup \{-(r = s)\}) \cap Cl(bse) \neq \varnothing$. So $[\underline{t} / \underline{p}]Seq \cap Cl(bse) \neq \varnothing$.

vii.b) $([\underline{t} / \underline{p}]Seq1 \cup [\underline{t} / \underline{p}]Seq2) \cap Cl(bse) = \varnothing$. Since $([\underline{t} / \underline{p}]Seq1 \cup [\underline{t} / \underline{p}]\{+[r / u]F\}) \cap Cl(bse) \neq \varnothing$ and $([\underline{t} / \underline{p}]Seq2 \cup [\underline{t} / \underline{p}]\{-[s / u]F\}) \cap Cl(bse) \neq \varnothing$, it follows that $[\underline{t} / \underline{p}]\{+[r / u]F\} \cap Cl(bse) \neq \varnothing$ and $[\underline{t} / \underline{p}]\{-[s / u]F\} \cap Cl(bse) \neq \varnothing$ and hence that both $+[\underline{t} / \underline{p}][r / u]F \cong +[[\underline{t} / \underline{p}]r / u][\underline{t} / \underline{p}]F$, $-[\underline{t} / \underline{p}][s / u]F \cong -[[\underline{t} / \underline{p}]s / u]\ \underline{t} / \underline{p}]F$ belong to $Cl(bse)$. By definition 1.3.3., $Cl(bse) = \bigcup bse_{\mu}$, $0 \leq \mu < \epsilon_0$, so, since both of $+[[\underline{t} / \underline{p}]r / u][\underline{t} / \underline{p}]F$, $-[[\underline{t} / \underline{p}]s / u][\underline{t} / \underline{p}]F$ belong to $Cl(bse)$, there are $\beta, \gamma \leq \epsilon_0$ such that $+[[\underline{t} / \underline{p}]r / u][\underline{t} / \underline{p}]F \in bse_{\beta}$ and $-[[\underline{t} / \underline{p}]s / u][\underline{t} / \underline{p}]F \in bse_{\gamma}$. Assume, without loss of generality, that $\beta \geq \gamma$. Then $bse_{\gamma} \subseteq bse_{\beta}$ and so both of $+[[\underline{t} / \underline{p}]r / u][\underline{t} / \underline{p}]F$, $-[[\underline{t} / \underline{p}]s / u][\underline{t} / \underline{p}]F$ belong to bse_{β} . There are two main sub-subcases:

vii.b.i) both of $[\underline{t}/\underline{p}]r$, $[\underline{t}/\underline{p}]s$ are basic terms. Then, since $[\underline{t}/\underline{p}]r$, $[\underline{t}/\underline{p}]s$ are constant terms, $[\underline{t}/\underline{p}]r, [\underline{t}/\underline{p}]s \in \delta(Par)$. Then, by the contrapositive of lemma 1.3.8., since both of $+[[\underline{t}/\underline{p}]r / u][\underline{t}/\underline{p}]F$, $-[[\underline{t}/\underline{p}]s / u][\underline{t}/\underline{p}]F$ belong to bse_β, the signed sentence $+([\underline{t}/\underline{p}]r = [\underline{t}/\underline{p}]s)$ does not belong to $Cl(bse)$, thus, $+([\underline{t}/\underline{p}]r = [\underline{t}/\underline{p}]s) \notin bse$. So, by condition 1) of definition 1.3.1., $-([\underline{t}/\underline{p}]r = [\underline{t}/\underline{p}]s) \cong -[\underline{t}/\underline{p}](r = s) \in bse \subseteq Cl(bse)$. Hence, $[\underline{t}/\underline{p}]Seq' \cup \{-[\underline{t}/\underline{p}](r = s)\} \cong [\underline{t}/\underline{p}](Seq' \cup \{-(r = s)\}) \cap Cl(bse) \neq \varnothing$. So $[\underline{t}/\underline{p}]Seq \cap Cl(bse) \neq \varnothing$.

vii.b.ii) not both of $[\underline{t}/\underline{p}]r$, $[\underline{t}/\underline{p}]s$ are basic terms, i.e., at least one of $[\underline{t}/\underline{p}]r$, $[\underline{t}/\underline{p}]s$ is a constant description term. Assume without loss of generality that $[\underline{t}/\underline{p}]s$ is a constant description term $\iota x.\Phi$. Then there are two sub-sub-subcases:

vii.b.ii.1) $[\underline{t}/\underline{p}]r \in \delta(Par)$, i.e., $[\underline{t}/\underline{p}]r$ is a basic term. Since $-[\iota x.\Phi / u][\underline{t}/\underline{p}]F$ belongs to bse_β, by lemma 1.3.10., there is a descriptum $t_{\iota x.\Phi} \in \delta(Par)$ for $\iota x.\Phi$ in bse such that $-[t_{\iota x.\Phi} / u][\underline{t}/\underline{p}]F$ belongs to bse_β. Since both of $+[[\underline{t}/\underline{p}]r / u][\underline{t}/\underline{p}]F$, $-[t_{\iota x.\Phi} / u][\underline{t}/\underline{p}]F$ belong to bse_β, by the same argument as that given in b.i) above, the signed sentence $-([\underline{t}/\underline{p}]r = t_{\iota x.\Phi})$ belongs to $bse \subseteq Cl(bse)$. Now, since u occurs free in F, u occurs free in $[\underline{t}/\underline{p}]F$. So, since both of $+[[\underline{t}/\underline{p}]r / u][\underline{t}/\underline{p}]F$, $-[\iota x.\Phi / u][\underline{t}/\underline{p}]F$ belong to bse_β, by lemma 2.0., for some constant term t and variable v, $bse_\beta \subseteq Cl(bse)$ contains both of the signed sentences $+[t/x]\Phi$, $+(x)(v)((\Phi \wedge [v/x]\Phi) \rightarrow x = v)$. Then, by lemma 1.2.11, since all of $-([\underline{t}/\underline{p}]r = t_{\iota x.\Phi})$, $+[t/x]\Phi$, $+(x)(v)((\Phi \wedge [v/x]\Phi) \rightarrow x = v)$ belong to $Cl(bse)$ so does the signed sentence $-([\underline{t}/\underline{p}]r = \iota x.\Phi) \cong -([\underline{t}/\underline{p}]r = [\underline{t}/\underline{p}]s) \cong -[\underline{t}/\underline{p}](r = s)$. Hence, $[\underline{t}/\underline{p}]Seq' \cup \{-[\underline{t}/\underline{p}](r = s)\} \cong [\underline{t}/\underline{p}](Seq' \cup \{-(r = s)\}) \cap Cl(bse) \neq \varnothing$. So $[\underline{t}/\underline{p}]Seq \cap Cl(bse) \neq \varnothing$.

vii.b.ii.2) $[\underline{t}/\underline{p}]r \notin \delta(Par)$, i.e., $[\underline{t}/\underline{p}]r$ is a constant description term, say $\iota z.\Psi$ for some formula Ψ. Since $+[\iota z.\Psi / u][\underline{t}/\underline{p}]F$ belongs to bse_β, by lemma 1.3.10., there is a descriptum $t_{\iota z.\Psi} \in \delta(Par)$ for $\iota z.\Psi$ in bse such that $+[t_{\iota z.\Psi} / u][\underline{t}/\underline{p}]F$ belongs to bse_β. By the argument given in b.ii.1) above, there is a descriptum $t_{\iota x.\Phi} \in \delta(Par)$ for $\iota x.\Phi$ in bse such that $-[t_{\iota x.\Phi} /$

$u][\underline{t} / \underline{p}]F$ belongs to bse_β. Since both of $+[t_{\iota z.\Psi} / u][\underline{t} / \underline{p}]F$, $-[t_{\iota x.\Phi} / u][\underline{t} / \underline{p}]F$ belong to bse_β, by the same argument as that given in vii.b.i) above, the signed sentence $-(t_{\iota z.\Psi} = t_{\iota x.\Phi})$ belongs to $bse \subseteq Cl(bse)$. By the argument of vii.b.ii.1) above, it follows that $-(t_{\iota z.\Psi} = \iota x.\Phi)$ belongs to $bse \subseteq Cl(bse)$. So, since both of $+[\iota z.\Psi / u][\underline{t} / \underline{p}]$, $-[\iota x.\Phi / u][\underline{t} / \underline{p}]F$ belong to bse_β, by lemma 2.0., for some constant term t and variable v, $bse_\beta \subseteq Cl(bse)$ contains both of the signed sentences $+[t / z]\Psi$, $+(z)(v)((\Psi \wedge [v / z]\Psi) \rightarrow z = v)$. Then, by lemma 1.2.11, since all of $-(t_{\iota z.\Psi} = \iota x.\Phi)$, $+[t / z]\Psi$, $+(z)(v)((\Psi \wedge [v / z]\Psi) \rightarrow z = v)$ belong to $Cl(bse)$ so does the signed sentence $-(\iota z.\Psi = \iota x.\Phi) \cong -([\underline{t} / \underline{p}]r = [\underline{t} / \underline{p}]s) \cong -[\underline{t} / \underline{p}](r = s)$. Hence, $[\underline{t} / \underline{p}]Seq' \cup \{-[\underline{t} / \underline{p}](r = s)\} \cong [\underline{t} / \underline{p}](Seq' \cup \{-(r = s)\}) \cap Cl(bse) \neq \varnothing$. So $[\underline{t} / \underline{p}]Seq \cap Cl(bse) \neq \varnothing$.

viii) ***Seq*** is the conclusion of an application A of the cut rule 1.2.2.8.. Then, trivially by lemma 1.3.9., $[\underline{t} / \underline{p}]Seq \cap Cl(bse) \neq \varnothing$.

In all cases, then, $[\underline{t} / \underline{p}]Seq \cap Cl(bse) \neq \varnothing$. Hence, theorem 2.1. and the soundness of Pld is established. □

<u>Corollary to Soundness</u> ("Syntactic Consistency"): For no sentence ***snt*** are both of $\{+snt\}$, $\{-snt\}$ derivable sequents.

<u>Proof</u>: Assume there is a sentence ***snt*** such that both of $\{+snt\}$, $\{-snt\}$ are derivable. Let *bse* be any base, $\underline{p}$ any *n*-ary vector on *Par* containing all of the parameters occurring in ***snt*** and $\underline{t}$ any *n*-ary vector on $\delta(Par)$ and. By soundness theorem 2.1., both of $\{+snt\}$, $\{-snt\}$ are valid, so by definitions 1.3.5. and 1.3.6., $[\underline{t} / \underline{p}]\{+snt\} \cap Cl(bse) \neq \varnothing$ and $[\underline{t} / \underline{p}]\{-snt\} \cap Cl(bse) \neq \varnothing$. So both of $\pm[\underline{t} / \underline{p}]snt$ belong to $Cl(bse)$. Since $Cl(bse) = \bigcup bse_\mu$, $0 \leq \mu < \epsilon_0$, there are $\beta, \gamma <$ $< \epsilon_0$ such that $+[\underline{t} / \underline{p}]snt \in bse_\beta$ and $-[\underline{t} / \underline{p}]snt \in bse_\gamma$. Assume, without loss of generality, that $\beta < \gamma$. Then $bse_\gamma \subseteq bse_\beta$ and so both of $\pm[\underline{t} / \underline{p}]snt$ belong to bse_β. This contradicts lemma 1.3.9.. □

Section 3: An Alternative Logical Syntax for Pld.

For the purposes of the semantic completeness proof of section 4, we present a set of deductive rules which is equivalent to the set presented in 1.2.2.; section 4 establishes the semantic completeness of the new theory Pld2. The logical syntax of Pld2 is obtained from the set of deductive rules of 1.2.2. by replacing the three premise description rule 1.2.2.6.,

$$.6.\quad \frac{\mathit{Seq1} \cup \{+[t/x]\Phi\} \quad \mathit{Seq2} \cup \{+(x)(v)((\Phi \wedge [v/x]\Phi) \rightarrow x = v)\} \quad \mathit{Seq3} \cup \{\pm[t/u]F\}}{\mathit{Seq1} \cup \mathit{Seq2} \cup \mathit{Seq2} \cup \{\pm[\iota x.\Phi/u]F\}}$$

by the following two binary rules 1.2.2.6.+ and 1.2.2.6.-:

$$.6.+.\quad \frac{\mathit{Seq1} \cup \{+[t/x](\Phi \wedge F)\} \qquad \mathit{Seq2} \cup \{+(x)(v)((\Phi \wedge [v/x]\Phi) \rightarrow x = v)\}}{\mathit{Seq1} \cup \mathit{Seq2} \cup \{+[\iota x.\Phi/x]F\}}$$

$$.6.\text{-}.\quad \frac{\mathit{Seq1} \cup \{+[t/x](\Phi \wedge \neg F)\} \qquad \mathit{Seq2} \cup \{+(x)(v)((\Phi \wedge [v/x]\Phi) \rightarrow x = v)\}}{\mathit{Seq1} \cup \mathit{Seq2} \cup \{-[\iota x.\Phi/x]F\}}$$

A **Pld2 derivation tree** is defined in the obvious way. We say that a sequent *Seq* is **2-derivable** iff it is the endsequent of a Pld2 derivation tree. We need to show that a sequent *Seq* is 2-derivable iff it is derivable. The claim that every derivable sequent is 2-derivable is trivial. However, the proof that Pld is semantically complete requires the converse claim, that every 2-derivable sequent is derivable, which we now establish. We first prove a couple of simple lemmata concerning the "old" notion of derivability:

Lemma 3.1.: Let *Seq* be any sequent, **A** any sentence. If $\mathit{Seq} \cup \{+\neg \mathbf{A}\}$ is derivable, then so is $\mathit{Seq} \cup \{-\mathbf{A}\}$.

Proof of 3.1.: The proof of 3.1. is by induction on the depth $D(\Pi)$ of an arbitrary derivation tree

Π of $Seq \cup \{+\neg A\}$ in the manner of the proof of lemma 3.2. below.

Lemma 3.2.: Let Seq be any sequent, A, B any sentences. If $Seq \cup \{+(A \wedge B)\}$ is derivable, then there are sequents $Seq1, Seq2$ such that $Seq1 \cup Seq2 = Seq$ and both of $Seq1 \cup \{+A\}$, $Seq2 \cup \{+B\}$ are derivable.

Proof of 3.2.: Assume that $Seq \cup \{+(A \wedge B)\}$ is derivable, say, as endsequent of a derivation tree Π. We show by induction on the depth $D(\Pi)$ of Π that there are sequents $Seq1, Seq2$ such that $Seq1 \cup Seq2 = Seq$ and both of $Seq1 \cup \{+A\}$, $Seq2 \cup \{+B\}$ are derivable. Clearly, we may omit the basis step, since by assumption Π contains nonatomic sentences and so is not an axiom. So assume the claim holds for all sequents which are derivable as endsequents of derivation trees Σ such that $D(\Sigma) < D(\Pi)$. We show the claim holds for $Seq \cup \{+(A \wedge B)\}$ by case analysis. Since the proof is trivial and somewhat tedious, we consider only three cases:

1) $Seq \cup \{+(A \wedge B)\} \cong Seq' \cup \{+(x)F\,,\, +(A \wedge B)\}$ is the conclusion of an application $\mathbb{A}$ in Π of the quantifier rule 1.2.2.5+. Then the premise to $\mathbb{A}$ is $Seq' \cup \{+[p\,/\,x]F\,,\, +(A \wedge B)\}$ where p is a parameter that does not occur in $Seq \cup \{+(A \wedge B)\}$. Now, $Seq' \cup \{+[p\,/\,x]F\,,\, +(A \wedge B)\}$ is the endsequent of the deepest proper subtree Σ of Π. Since $D(\Sigma) < D(\Pi)$, by the hypothesis of induction, there are $Seq'1$ and $Seq'2$ such that $Seq'1 \cup Seq'2 \cong Seq' \cup \{+[p\,/\,x]F\}$ and both of $Seq'1 \cup \{+A\}, Seq'2 \cup \{+B\}$ are derivable. So $\{+[p\,/\,x]F\}$ belongs to one of $Seq'1, Seq'2$. Without loss of generality, assume that $\{+[p\,/\,x]F\} \in Seq'1$. There are two cases:

i) $\{+[p\,/\,x]F\} \notin Seq'2$. Since p does not occur in $Seq \cup \{+(A \wedge B)\} \cong Seq' \cup \{+(x)F\,,\, +(A \wedge B)\}, p$ does not occur in $(Seq'1 - \{+[p\,/\,x]F\}) \cup \{+(x)F\,,\, +A\}$. So the following is a legitimate application of rule 1.2.2.5+.:

$$\frac{(Seq'1 - \{+[p\,/\,x]F\}) \cup \{+[p\,/\,x]F\,,\, +A\}}{(Seq'1 - \{+[p\,/\,x]F\}) \cup \{+(x)F\,,\, +A\}}$$

Since $Seq'1 \cup \{+A\} \cong (Seq'1 - \{+[p\,/\,x]F\}) \cup \{+[p\,/\,x]F\,,\, +A\}$ is derivable, it follows that

$(Seq'1 - \{+[p/x]F\}) \cup \{+(x)F, +A\}$ is derivable. Hence, both of $(Seq'1 - \{+[p/x]F\}) \cup \{+(x)F, +A\}$, $Seq'2 \cup \{+B\}$ are derivable. Since $Seq'1 \cup Seq'2 \cong Seq' \cup \{+[p/x]F\}$ and $\{+[p/x]F\} \notin Seq'2$, $(Seq'1 - \{+[p/x]F\}) \cup \{+(x)F\} \cup Seq'2 \cong Seq' \cup \{+(x)F\}$. So, the claim holds.

ii) $\{+[p/x]F\} \in Seq'2$. Since p does not occur in $Seq \cup \{+(A \wedge B)\}$, p does not occur in $(Seq'2 - \{+[p/x]F\}) \cup \{+(x)F, +B\}$. So the following is a legitimate application of rule 1.2.2.5+.:

$$\frac{(Seq'2 - \{+[p/x]F\}) \cup \{+[p/x]F, +B\}}{(Seq'2 - \{+[p/x]F\}) \cup \{+(x)F, +B\}}$$

Since $Seq'2 \cup \{+B\} \cong (Seq'2 - \{+[p/x]F\}) \cup \{+[p/x]F, +B\}$ is derivable, it follows that $(Seq'2 - \{+[p/x]F\}) \cup \{+(x)F, +B\}$ is derivable. By case a), we know that $(Seq'1 - \{+[p/x]F\}) \cup \{+(x)F, +A\}$ is derivable. Since $Seq'1 \cup Seq'2 \cong Seq' \cup \{+[p/x]F\}$, $(Seq'1 - \{+[p/x]F\}) \cup \{+(x)F\} \cup (Seq'2 - \{+[p/x]F\}) \cup \{+(x)F\} \cong Seq' \cup \{+(x)F\}$. So the claim holds. **QED CASE 1).**

2) $Seq \cup \{+(A \wedge B)\} \cong Seq' \cup \{+(C \wedge D), +(A \wedge B)\}$ is the conclusion of an application $\mathbb{A}$ in Π of rule 1.2.2.3+., for some sentences **C**, **D** and sequent *Seq'*. Then $\mathbb{A}$ is of the following form:

$$\frac{Seq'1 \cup \{+C\} \qquad Seq'2 \cup \{+D\}}{Seq' \cup \{+(C \wedge D), +(A \wedge B)\}}$$

where $Seq'1 \cup Seq'2 \cong Seq' \cup \{+(A \wedge B)\}$. There are three cases:

i) $\{+(A \wedge B)\} \in Seq'1$ and $\{+(A \wedge B)\} \notin Seq'2$. Then $Seq'1 \cong (Seq'1 - \{+(A \wedge B)\}) \cup \{+(A \wedge B)\}$. Since $(Seq'1 - \{+(A \wedge B)\}) \cup \{+(A \wedge B), +C\}$ is derivable as endsequent of a proper subtree of Π, by the hypothesis of induction, there are *Seq3* and *Seq4* so that $Seq3 \cup Seq4 \cong (Seq'1 - \{+(A \wedge B)\}) \cup \{+C\}$ and both of $Seq3 \cup \{+A\}$, $Seq4 \cup \{+B\}$ are derivable. Assume, without loss of generality, that $+C \in$ *Seq3*. There are two cases:

a) $+C \notin$ *Seq4*. The following is a legitimate application of 1.2.2.3.:

$$\frac{(Seq3 - \{+C\}) \cup \{+C\} \cup \{+A\} \qquad Seq'2 \cup \{+D\}}{Seq'2 \cup (Seq3 - \{+C\}) \cup \{+(C \wedge D)\} \cup \{+A\}}$$

Since both of $Seq'2 \cup \{+D\}$, $Seq3 \cup \{+A\} \cong (Seq3 - \{+C\}) \cup \{+C\} \cup \{+A\}$ are derivable, it follows that $Seq'2 \cup (Seq3 - \{+C\}) \cup \{+(C \wedge D)\} \cup \{+A\}$ is derivable. Further, we know that $Seq4 \cup \{+B\}$ is derivable. Now, since $Seq'1 \cup Seq'2 \cong Seq' \cup \{+(A \wedge B)\}$ and $Seq3 \cup Seq4 \cong (Seq'1 - \{+(A \wedge B)\}) \cup \{+C\}$, it follows from the facts that $\{+(A \wedge B)\} \notin Seq'2$ and $+C \notin Seq4$ that $Seq'2 \cup (Seq3 - \{+C\}) \cup \{+(C \wedge D)\} \cup Seq4 \cong Seq' \cup \{+(C \wedge D)\}$. So the claim holds.

b) $+C \in$ *Seq4*. The following is a legitimate application of 1.2.2.3.:

$$\frac{(Seq4 - \{+C\}) \cup \{+C\} \cup \{+B\} \qquad Seq'2 \cup \{+D\}}{Seq'2 \cup (Seq4 - \{+C\}) \cup \{+(C \wedge D)\} \cup \{+B\}}$$

Since both of $Seq'2 \cup \{+D\}$, $Seq4 \cup \{+B\} \cong (Seq4 - \{+C\}) \cup \{+C\} \cup \{+B\}$ are derivable, it follows that $Seq'2 \cup (Seq4 - \{+C\}) \cup \{+(C \wedge D)\} \cup \{+B\}$ is derivable. Further, by case a) we know that $Seq'2 \cup (Seq3 - \{+C\}) \cup \{+(C \wedge D)\} \cup \{+A\}$ is derivable. Now, since $Seq'1 \cup Seq'2 \cong Seq' \cup \{+(A \wedge B)\}$ and $Seq3 \cup Seq4 \cong (Seq'1 - \{+(A \wedge B)\}) \cup \{+C\}$, it follows from the fact that $\{+(A \wedge B)\} \notin Seq'2$ that $Seq'2 \cup (Seq4 - \{+C\}) \cup \{+(C \wedge D)\} \cup Seq'2 \cup (Seq3 - \{+C\}) \cup \{+(C \wedge D)\} \cong Seq' \cup \{+(C \wedge D)\}$. So the claim holds.

ii) $\{+(A \wedge B)\} \notin Seq'1$ and $\{+(A \wedge B)\} \in Seq'2$. This case is similar to case i) above.

iii) $\{+(A \wedge B)\} \in Seq'1$ and $\{+(A \wedge B)\} \in Seq'2$. Then, by similar argument to case i), there are *Seq3* and *Seq4* so that $Seq3 \cup Seq4 \cong (Seq'1 - \{+(A \wedge B)\}) \cup \{+C\}$ and both of $Seq3 \cup \{+A\}$, $Seq4 \cup \{+B\}$ are derivable and there are *Seq5* and *Seq6* so that $Seq5 \cup$

$Seq6 \cong (Seq'2 - \{+(\mathrm{A} \wedge \mathrm{B})\}) \cup \{+\mathrm{D}\}$ and both of $Seq5 \cup \{+\mathrm{A}\}, Seq6 \cup \{+\mathrm{B}\}$ are derivable. The following is a legitimate application of 1.2.2.3.:

$$\frac{(Seq3 - \{+\mathrm{C}\}) \cup \{+\mathrm{A}\ , +\mathrm{C}\} \qquad (Seq5 - \{+\mathrm{D}\}) \cup \{+\mathrm{A}\ , +\mathrm{D}\}}{(Seq3 - \{+\mathrm{C}\}) \cup (Seq5 - \{+\mathrm{D}\}) \cup \{+(\mathrm{C} \wedge \mathrm{D})\ , +\mathrm{A}\}}$$

In case $+\mathrm{C} \in Seq3$, $Seq3 \cup \{+\mathrm{A}\} \cong (Seq3 - \{+\mathrm{C}\}) \cup \{+\mathrm{A}\ , +\mathrm{C}\}$. In case $+\mathrm{C} \notin Seq3$, since $Seq3 \cup \{+\mathrm{A}\}$ is derivable, $(Seq3 - \{+\mathrm{C}\}) \cup \{+\mathrm{A}\ , +\mathrm{C}\}$ is derivable by an application of thinning. So in both cases, $(Seq3 - \{+\mathrm{C}\}) \cup \{+\mathrm{A}\ , +\mathrm{C}\}$ is derivable. By similar reasoning, $(Seq5 - \{+\mathrm{D}\} \cup \{+\mathrm{A}\ , +\mathrm{D}\}$ is derivable. Hence, $(Seq3 - \{+\mathrm{C}\}) \cup (Seq5 - \{+\mathrm{D}\}) \cup \{+(\mathrm{C} \wedge \mathrm{D})\ , +\mathrm{A}\}$ is derivable. A similar argument establishes that $(Seq4 - \{+\mathrm{C}\}) \cup (Seq6 - \{+\mathrm{D}\}) \cup \{+(\mathrm{C} \wedge \mathrm{D})\ , +\mathrm{B}\}$ is derivable also. Since $Seq3 \cup Seq4 \cong (Seq'1 - \{+(\mathrm{A} \wedge \mathrm{B})\}) \cup \{+\mathrm{C}\}$ and $Seq5 \cup Seq6 \cong (Seq'2 - \{+(\mathrm{A} \wedge \mathrm{B})\}) \cup \{+\mathrm{D}\}$, it follows from the fact that $Seq'1 \cup Seq'2 \cong Seq' \cup \{+(\mathrm{A} \wedge \mathrm{B})\}$ that $(Seq3 - \{+\mathrm{C}\}) \cup (Seq5 - \{+\mathrm{D}\}) \cup \{+(\mathrm{C} \wedge \mathrm{D})\} \cup (Seq4 - \{+\mathrm{C}\}) \cup (Seq6 - \{+\mathrm{D}\}) \cup \{+(\mathrm{C} \wedge \mathrm{D})\} \cong Seq' \cup \{+(\mathrm{C} \wedge \mathrm{D})\}$. So the claim holds. QED CASE 2).

3) $Seq \cup \{+(\mathrm{A} \wedge \mathrm{B})\} \cong Seq' \cup \{\pm[\iota x.\Phi / u]F\ , +(\mathrm{A} \wedge \mathrm{B})\}$ is the conclusion of an application $\mathbb{A}$ in Π of rule 1.2.2.6., for some description term $\iota x.\Phi$ and formula F. Then $\mathbb{A}$ is respectively of the following form:

$$\frac{Seq1 \cup \{+[t/x]\Phi\} \qquad Seq2 \cup \{+(x)(v)((\Phi \wedge [v/x]\Phi) \rightarrow x = v)\} \qquad Seq3 \cup \{\pm[t/u]F\}}{Seq' \cup \{\pm[\iota x.\Phi / u]F\ , +(\mathrm{A} \wedge \mathrm{B})\}}$$

where $Seq1 \cup Seq2 \cup Seq3 \cong Seq' \cup \{+(\mathrm{A} \wedge \mathrm{B})\}$. There are four main cases:

i) $+(\mathrm{A} \wedge \mathrm{B}) \in Seq1$ and $+(\mathrm{A} \wedge \mathrm{B}) \notin Seq2 \cup Seq3$. Then, since $Seq1 \cup \{+[t/x]\Phi\} \cong (Seq1 - \{+(\mathrm{A} \wedge \mathrm{B})\}) \cup \{+(\mathrm{A} \wedge \mathrm{B})\ , +[t/x]\Phi\}$ is derivable as the endsequent of a proper subtree of Π, by the hypothesis of induction, there are $Seq4$, $Seq5$ such that $Seq4 \cup Seq5 \cong (Seq1 - \{+(\mathrm{A} \wedge \mathrm{B})\}) \cup \{+[t/x]\Phi\}$ and both of $Seq4 \cup \{+\mathrm{A}\}, Seq5 \cup \{+\mathrm{B}\}$ are derivable.

Assume without loss of generality that $+[t / x]\Phi \in \textit{Seq4}$. There are two cases:

a) $+[t / x]\Phi \notin \textit{Seq5}$. The following is a legitimate application of 1.2.2.6.:

$$\frac{(\textit{Seq4} - \{+[t/x]\Phi\}) \cup \{+\mathrm{A}\ ,+[t/x]\Phi\} \quad \textit{Seq2} \cup \{+(x)(v)((\Phi \wedge [v/x]\Phi) \rightarrow x = v)\} \quad \textit{Seq3} \cup \{\pm[t/u]F\}}{(\textit{Seq4} - \{+[t/x]\Phi\}) \cup \textit{Seq2} \cup \textit{Seq3} \cup \{\pm[\iota x.\Phi / u]F\ , +\mathrm{A}\}}$$

Since all of $\textit{Seq4} \cup \{+\mathrm{A}\} \cong (\textit{Seq4} - \{+[t/x]\Phi\}) \cup \{+\mathrm{A}\ , +[t/x]\Phi\}$, $\textit{Seq2} \cup \{+(x)(v)((\Phi \wedge [v/x]\Phi) \rightarrow x = v)\}$, $\textit{Seq3} \cup \{\pm[t/u]F\}$ are derivable, it follows that $(\textit{Seq4} - \{+[t/x]\Phi\}) \cup \textit{Seq2} \cup \textit{Seq3} \cup \{\pm[\iota x.\Phi / u]F\ , +\mathrm{A}\}$ is derivable. So both of $(\textit{Seq4} - \{+[t/x]\Phi\}) \cup \textit{Seq2} \cup \textit{Seq3} \cup \{\pm[\iota x.\Phi / u]F\ , +\mathrm{A}\}, \textit{Seq5} \cup \{+\mathrm{B}\}$ are derivable. Now, since $\textit{Seq1} \cup \textit{Seq2} \cup \textit{Seq3} \cong \textit{Seq}' \cup \{+(\mathrm{A} \wedge \mathrm{B})\}$ and $\textit{Seq4} \cup \textit{Seq5} \cong (\textit{Seq1} - \{+(\mathrm{A} \wedge \mathrm{B})\}) \cup \{+[t/x]\Phi\}$, it follows by the facts that $+(\mathrm{A} \wedge \mathrm{B}) \notin \textit{Seq2} \cup \textit{Seq3}$ and $+[t/x]\Phi \notin \textit{Seq5}$ that $(\textit{Seq4} - \{+[t/x]\Phi\}) \cup \textit{Seq2} \cup \textit{Seq3} \cup \{\pm[\iota x.\Phi / u]F\} \cup \textit{Seq5} \cong \textit{Seq}' \cup \{\pm[\iota x.\Phi / u]F\}$. So the claim holds.

b) $+[t/x]\Phi \in \textit{Seq5}$. Then $\textit{Seq5} \cup \{+\mathrm{B}\} \cong (\textit{Seq5} - \{+[t/x]\Phi\}) \cup \{+[t/x]\Phi\ ,+\mathrm{B}\}$. The following is a legitimate application of 1.2.2.6.:

$$\frac{(\textit{Seq5} - \{+[t/x]\Phi\}) \cup \{+\mathrm{B}\ ,+[t/x]\Phi\} \quad \textit{Seq2} \cup \{+(x)(v)((\Phi \wedge [v/x]\Phi) \rightarrow x = v)\} \quad \textit{Seq3} \cup \{\pm[t/u]F\}}{(\textit{Seq5} - \{+[t/x]\Phi\}) \cup \textit{Seq2} \cup \textit{Seq3} \cup \{\pm[\iota x.\Phi / u]F\ , +\mathrm{B}\}}$$

Since all of $\textit{Seq5} \cup \{+\mathrm{B}\} \cong (\textit{Seq5} - \{+[t/x]\Phi\}) \cup \{+\mathrm{B}\ ,+[t/x]\Phi\}, \textit{Seq2} \cup \{+(x)(v)((\Phi \wedge [v/x]\Phi) \rightarrow x = v)\}, \textit{Seq3} \cup \{\pm[t/u]F\}$, are derivable, it follows that $(\textit{Seq5} - \{+[t/x]\Phi\}) \cup \textit{Seq2} \cup \textit{Seq3} \cup \{\pm[\iota x.\Phi / u]F\ , +\mathrm{B}\}$ is derivable. But by case a), we know that $(\textit{Seq4} - \{+[t/x]\Phi\}) \cup \textit{Seq2} \cup \textit{Seq3} \cup \{\pm[\iota x.\Phi / u]F\ , +\mathrm{A}\}$ is also derivable. Now, since $\textit{Seq1} \cup \textit{Seq2} \cup \textit{Seq3} \cong \textit{Seq}' \cup \{+(\mathrm{A} \wedge \mathrm{B})\}$ and $\textit{Seq4} \cup \textit{Seq5} \cong (\textit{Seq1} - \{+(\mathrm{A} \wedge \mathrm{B})\}) \cup \{+[t/x]\Phi\}$, it follows by the fact that $+(\mathrm{A} \wedge \mathrm{B}) \notin \textit{Seq2} \cup \textit{Seq3}$ that $(\textit{Seq5} - \{+[t/x]\Phi\}) \cup \textit{Seq2} \cup \textit{Seq3} \cup \{\pm[\iota x.\Phi / u]F\} \cup (\textit{Seq4} - \{+[t/x]\Phi\}) \cup \textit{Seq2} \cup \textit{Seq3} \cup \{\pm[\iota x.\Phi / u]F\ , +\mathrm{A}\} \cong \textit{Seq}' \cup \{\pm[\iota x.\Phi / u]F\}$. So the claim holds.

ii) $+(A \wedge B) \in$ *Seq1* and $+(A \wedge B) \in$ *Seq2* and $+(A \wedge B) \notin$ *Seq3*. By case i), there are *Seq4*, *Seq5* such that *Seq4* $\cup$ *Seq5* $\cong$ (*Seq1* $- \{+(A \wedge B)\}) \cup \{+[t/x]\Phi\}$ and both of *Seq4* $\cup \{+A\}$, *Seq5* $\cup \{+B\}$ are derivable. By a similar argunent, since $+(A \wedge B) \in$ *Seq2*, we know that there are *Seq6*, *Seq7* such that *Seq6* $\cup$ *Seq7* $\cong$ (*Seq2* $- \{+(A \wedge B)\}) \cup \{+(x)(v)((\Phi \wedge [v/x]\Phi) \rightarrow x = v)\}$ and both of *Seq6* $\cup \{+A\}$, *Seq7* $\cup \{+B\}$ are derivable. Let *Seq6** $\cong$ (*Seq6* $- \{+(x)(v)((\Phi \wedge [v/x]\Phi) \rightarrow x = v)\}) \cup \{+(x)(v)((\Phi \wedge [v/x]\Phi) \rightarrow x = v)\}$. The following is a legitimate application of 1.2.2.6.:

$$\frac{(Seq4 - \{+[t/x]\Phi\}) \cup \{+A\ , +[t/x]\Phi\} \qquad Seq6^{*} \cup \{+A\} \qquad Seq3 \cup \{\pm[t/u]F\}}{(Seq4 - \{+[t/x]\Phi\}) \cup (Seq6 - \{+(x)(v)((\Phi \wedge [v/x]\Phi) \rightarrow x = v)\}) \cup Seq3 \cup \{\pm[\iota x.\Phi/u]F\ , +A\}}$$

In case $+[t/x]\Phi \in$ *Seq4*, *Seq4* $\cup \{+A\} \cong$ (*Seq4* $- \{+[t/x]\Phi\}) \cup \{+A\ ,+[t/x]\Phi\}$; in case $+[t/x]\Phi \notin$ *Seq4*, (*Seq4* $- \{+[t/x]\Phi\}) \cup \{+A\ ,+[t/x]\Phi\}$ is derivable from *Seq4* $\cup \{+A\}$ by an application of thinning. In both cases, then, (*Seq4* $- \{+[t/x]\Phi\}) \cup \{+A\ ,+[t/x]\Phi\}$ is derivable. A similar argument establishes that *Seq6** $\cup \{+A\}$ is derivable. Since *Seq3* $\cup \{\pm[t/u]F\}$ is derivable, it follows that (*Seq4* $- \{+[t/x]\Phi\}) \cup$ (*Seq6* $- \{+(x)(v)((\Phi \wedge [v/x]\Phi) \rightarrow x = v)\}) \cup$ *Seq3* $\cup \{\pm[\iota x.\Phi/u]F\ , +A\}$ is derivable. A similar argument establishes that (*Seq5* $- \{+[t/x]\Phi\}) \cup$ (*Seq7* $- \{+(x)(v)((\Phi \wedge [v/x]\Phi) \rightarrow x = v)\}) \cup$ *Seq3* $\cup \{\pm[\iota x.\Phi/u]F\ , +B\}$ is derivable as well. Now, since *Seq1* $\cup$ *Seq2* $\cup$ *Seq3* $\cong$ *Seq'* $\cup \{+(A \wedge B)\}$ and *Seq4* $\cup$ *Seq5* $\cong$ (*Seq1* $- \{+(A \wedge B)\}) \cup \{+[t/x]\Phi\}$ and *Seq6* $\cup$ *Seq7* $\cong$ (*Seq2* $- \{+(A \wedge B)\}) \cup \{+(x)(v)((\Phi \wedge [v/x]\Phi) \rightarrow x = v)\}$, it follows by the fact that $+(A \wedge B) \notin$ *Seq3* that (*Seq4* $- \{+[t/x]\Phi\}) \cup$ (*Seq6* $- \{+(x)(v)((\Phi \wedge [v/x]\Phi) \rightarrow x = v)\}) \cup$ *Seq3* $\cup \{\pm[\iota x.\Phi/u]F\} \cup$ (*Seq5* $- \{+[t/x]\Phi\}) \cup$ (*Seq7* $- \{+(x)(v)((\Phi \wedge [v/x]\Phi) \rightarrow x = v)\}) \cup$ *Seq3* $\cup \{\pm[\iota x.\Phi/u]F\} \cong$ *Seq'* $\cup \{\pm[\iota x.\Phi/u]F\}$. So the claim holds.

iii) $+(A \wedge B) \in$ *Seq1* and $+(A \wedge B) \notin$ *Seq2* and $+(A \wedge B) \in$ *Seq3*. This case is similar to case ii) above.

iv) $+(A \wedge B) \in$ *Seq1* and $+(A \wedge B) \in$ *Seq2* and $+(A \wedge B) \in$ *Seq3*. This case is similar to case ii) above. **QED CASE 3).**

This completes our case analysis. By the principle of induction, lemma 3.2. is established. □

We may now establish the desired result that every 2-derivable sequent is derivable:

Lemma 3.3.: Let ***Seq*** be any sequent. If ***Seq*** is 2-derivable, then ***Seq*** is derivable.

Proof of 3.3.: Assume that ***Seq*** is 2-derivable, say, as endsequent of a Pld2 derivation tree Π. We show by induction on the depth $\mathbb{D}(\Pi)$ of Π that ***Seq*** is derivable.

Base step: $\mathbb{D}(\Pi) = 0$, i.e., Π is an axiom. Then Π is derivable.

Induction step: Assume derivable every sequent that is 2-derivable as endsequent of a Pld2 derivation tree Σ such that $\mathbb{D}(\Sigma) < \mathbb{D}(\Pi)$. Clearly, if ***Seq*** is the conclusion of an application of one of .1.2.2.1.±, 1.2.2.2.±, 1.2.2.3.±, 1.2.2.4.±, 1.2.2.5.±, 1.2.2.7.±, 1.2.2.8.±, then by the hypothesis of induction, ***Seq*** is derivable. So assume that ***Seq*** is the conclusion of an application $\mathbb{A}$ of either 1.2.2.6.+ or 1.2.2.6.-. We consider these two cases:

i) ***Seq*** is the conclusion of an application $\mathbb{A}$ in Π of 1.2.2.6.+. Then, $\boldsymbol{Seq} \cong \boldsymbol{Seq1} \cup \boldsymbol{Seq2} \cup \{+[\iota x.\Phi / x]F\}$ for some sequents ***Seq1***, ***Seq2***, term t, formula F, and description term $\iota x.\Phi$ and $\mathbb{A}$ is of the following form:

$$\frac{\boldsymbol{Seq1} \cup \{+[t/x](\Phi \wedge F)\} \qquad \boldsymbol{Seq2} \cup \{+(x)(v)((\Phi \wedge [v/x]\Phi) \rightarrow x = v)\}}{\boldsymbol{Seq1} \cup \boldsymbol{Seq2} \cup \{+[\iota x.\Phi / x]F\}}$$

Since $\boldsymbol{Seq1} \cup \{+[t/x](\Phi \wedge F)\}$ and $\boldsymbol{Seq2} \cup \{+(x)(v)((\Phi \wedge [v/x]\Phi) \rightarrow x = v)\}$ are 2-derivable as endsequents of proper subtrees of Π, by the hypothesis of induction, both of ***Seq1*** $\cup \{+[t/x](\Phi \wedge F)\}$, $\boldsymbol{Seq2} \cup \{+(x)(v)((\Phi \wedge [v/x]\Phi) \rightarrow x = v)\}$ are derivable. Since ***Seq1*** $\cup \{+[t/x](\Phi \wedge F)\} \cong \boldsymbol{Seq1} \cup \{+([t/x]\Phi \wedge [t/x]F)\}$ is derivable, by lemma 3.2., there are sequents ***Seq3***, ***Seq4*** such that $\boldsymbol{Seq3} \cup \boldsymbol{Seq4} \cong \boldsymbol{Seq1}$ and both of $\boldsymbol{Seq3} \cup \{+[t/x]\Phi\}$, $\boldsymbol{Seq4}\ \{+[t/x]F\}$ are derivable. The following is a legitimate application of 1.2.2.6.:

$$\frac{\textit{Seq3} \cup \{+[t/x]\Phi\} \quad \textit{Seq2} \cup \{+(x)(v)((\Phi \wedge [v/x]\Phi) \rightarrow x = v)\} \quad \textit{Seq4} \cup \{+[t/x]F\}}{\textit{Seq3} \cup \textit{Seq2} \cup \textit{Seq4} \cup \{+[\iota x.\Phi/x]F\}}$$

So it follows that $\textit{Seq3} \cup \textit{Seq2} \cup \textit{Seq4} \cup \{+[\iota x.\Phi/x]F\} \cong \textit{Seq1} \cup \textit{Seq2} \cup \{+[\iota x.\Phi/x]F\} \cong \textit{Seq}$ is derivable. So the claim holds.

ii) $\textit{Seq}$ is the conclusion of an application $\mathbb{A}$ in Π of 1.2.2.6.-. Then, $\textit{Seq} \cong \textit{Seq1} \cup \textit{Seq2} \cup \{-[\iota x.\Phi/x]F\}$ for some sequents $\textit{Seq1}, \textit{Seq2}$, term t, formula F, and description term $\iota x.\Phi$ and $\mathbb{A}$ is of the following form:

$$\frac{\textit{Seq1} \cup \{+[t/x](\Phi \wedge \neg F)\} \qquad \textit{Seq2} \cup \{+(x)(v)((\Phi \wedge [v/x]\Phi) \rightarrow x = v)\}}{\textit{Seq1} \cup \textit{Seq2} \cup \{-[\iota x.\Phi/x]F\}}$$

Since $\textit{Seq1} \cup \{+[t/x](\Phi \wedge \neg F)\}$ and $\textit{Seq2} \cup \{+(x)(v)((\Phi \wedge [v/x]\Phi) \rightarrow x = v)\}$ are 2-derivable as endsequents of proper subtrees of Π, by the hypothesis of induction, both of $\textit{Seq1} \cup \{+[t/x](\Phi \wedge \neg F)\}$, $\textit{Seq2} \cup \{+(x)(v)((\Phi \wedge [v/x]\Phi) \rightarrow x = v)\}$ are derivable. Since $\textit{Seq1} \cup \{+[t/x](\Phi \wedge \neg F)\} \cong \textit{Seq1} \cup \{+([t/x]\Phi \wedge [t/x]\neg F)\}$ is derivable, by lemma 3.2., there are sequents $\textit{Seq3}, \textit{Seq4}$ such that $\textit{Seq3} \cup \textit{Seq4} \cong \textit{Seq1}$ and both of $\textit{Seq3} \cup \{+([t/x]\Phi\}$, $\textit{Seq4} \cup \{+([t/x]\neg F)\}$ are derivable. Since $\textit{Seq4} \cup \{+([t/x]\neg F)\}$ is derivable, by lemma 3.1., $\textit{Seq4} \cup \{-([t/x]F)\}$ is derivable. The following is a legitimate application of 1.2.2.6.:

$$\frac{\textit{Seq3} \cup \{+[t/x]\Phi\} \quad \textit{Seq2} \cup \{+(x)(v)((\Phi \wedge [v/x]\Phi) \rightarrow x = v)\} \quad \textit{Seq4} \cup \{-[t/x]F\}}{\textit{Seq3} \cup \textit{Seq2} \cup \textit{Seq4} \cup \{-[\iota x.\Phi/x]F\}}$$

So it follows that $\textit{Seq3} \cup \textit{Seq2} \cup \textit{Seq4} \cup \{-[\iota x.\Phi/x]F\} \cong \textit{Seq1} \cup \textit{Seq2} \cup \{-[\iota x.\Phi/x]F\} \cong \textit{Seq}$ is derivable. So the claim holds.

In both cases, $\textit{Seq}$ is derivable. So by the principle of induction, lemma 3.3. is established. $\square$

3.4.: A derivation tree Π is a **strict derivation tree** iff Π contains no applications of the cut rule 1.2.2.8.± and no applications $\mathbb{A}$ of the description rule 1.2.2.6. such that $\mathbb{A}$ has a nonelementary output sentence. A sequent ***Seq*** is **strictly derivable** iff it is derivable as endsequent of a strict derivation tree. A Pld2 derivation tree Π is a **strict Pld2 derivation tree** iff Π contains no applications of the cut rule 1.2.2.8.± and no applications $\mathbb{A}$ of either of the description rules 1.2.2.6.+, 1.2.2.6.-, such that $\mathbb{A}$ has a nonelementary output sentence. A sequent ***Seq*** is **strictly 2-derivable** iff it is 2-derivable as endsequent of a strict Pld2 derivation tree.

Lemma 3.5.: Let ***Seq*** be any sequent. If ***Seq*** is strictly 2-derivable, then ***Seq*** is strictly derivable.

Proof of 3.5.: The proof of lemma 3.5. is obtained from the proof of lemma 3.3. by uniformly substituting the expression "strict derivation tree" for "derivation tree", "strict Pld2 derivation tree" for "Pld2 derivation tree", "strictly derivable" for "derivable", "strictly 2-derivable" for "2-derivable", and "***afm***" for "***F***" throughout that proof.

This concludes section 3. We are now prepared to establish the semantic completeness of Pld by establishing the semantic completeness of Pld2.

Section 4: Semantic Completeness for Pld.

We say that Pld is **semantically complete** iff every valid sequent is derivable. In this section we show that Pld is semantically complete.

4.1 Preliminaries.

For the purposes of the completeness proof, we generalize the notion of 2-derivability (strict 2-derivability) defined in section 3 so that we may consider the 2-derivability (strict 2-derivability) of arbitrary sets of signed sentences. A possibly infinite set ***Set*** of signed sentences is said to be **2-derivable** (strictly 2-derivable) iff there is a sequent $\boldsymbol{Seq} \subseteq \boldsymbol{Set}$ such that ***Seq*** is 2-derivable (strictly 2-derivable). In other words, a set of signed sentences is 2-derivable (strictly 2-derivable) iff some finite subset of it is 2-derivable (strictly 2-derivable). Observe that a sequent is 2-derivable (strictly 2-derivable) in the extended sense defined here iff it is 2-derivable (strictly 2-derivable) in the sense of section 3.

Let bijective numerations $\mathbb{P}$: $\mathbb{N} \to Par$, $\mathbb{T}$: $\mathbb{N} \to \delta(Par)$, $\mathbb{SS}$: $\mathbb{N} \to \{\pm \boldsymbol{snt} : \boldsymbol{snt} \in \Sigma(Par)\}$, of the set of respectively parameters, constant basic terms, signed sentences, of Pld be given. Let $\mathbb{Ind}$: $\{\pm \boldsymbol{snt} : \boldsymbol{snt} \in \Sigma(Par)\} \to \mathbb{N}$ be the functional inverse of $\mathbb{SS}$, i.e., $\mathbb{Ind}(\mathbb{SS}(i)) = i$; i will be called the **index** of the signed sentence $\mathbb{SS}(i)$. If ***Set*** is a set of signed sentences, let $\mathcal{P}(\boldsymbol{Set})$ be the set of parameters which occur in at least one member of ***Set***. Then,

.1. Let ***Set*** be a set of signed sentences and $\pm$***snt*** any member of ***Set***. Then, ***Set*** is **downward closed** iff the following conditions hold:

1) For no atomic sentence ***asnt*** does ***Set*** contain both of $\pm$***asnt***.

2) For no ***t*** in $\delta(Par)$ does ***Set*** contain the signed sentence $+(\boldsymbol{t} = \boldsymbol{t})$

3) For no ***r***, ***s*** in $\delta(Par)$ does ***Set*** contain both of $+(\boldsymbol{r} = \boldsymbol{s})$, $-(\boldsymbol{s} = \boldsymbol{r})$.

4) For no r, s in $\delta(Par)$ and atomic formula afm does Set contain all of $-(r = s)$, $-[r / v]afm$, $+[s / v]afm$.

5) If $\pm snt$ is respectively of the form $\pm\neg\mathbf{A}$ for some sentence $\mathbf{A}$, then Set contains respectively $\mp\mathbf{A}$.

6) If $\pm snt$ is respectively of the form $\pm(\mathbf{A} \wedge \mathbf{B})$ for some sentences $\mathbf{A}$, $\mathbf{B}$, then Set contains one of $+\mathbf{A}$, $+\mathbf{B}$, respectively, both of $-\mathbf{A}$, $-\mathbf{B}$.

7) If $\pm snt$ is respectively of the form $\pm(\mathbf{A} \rightarrow \mathbf{B})$ for some sentences $\mathbf{A}$, $\mathbf{B}$, then Set contains both of $-\mathbf{A}$, $+\mathbf{B}$, respectively, one of $+\mathbf{A}$, $-\mathbf{B}$.

8) If $\pm snt$ is respectively of the form $\pm(x)F$ for some formula F, then Set contains $+[p / x]F$ for some p in $\mathcal{P}(Set)$, respectively, $-[t / x]F$ for every t in $\delta(\mathcal{P}(Set))$.

9) If $\pm snt$ is a signed nonatomic elementary sentence of the form $+[\iota x.\Phi / u]afm$ for some constant description term $\iota x.\Phi$ and atomic formula afm, then Set satisfies one of the following conditions:

i) Set contains the signed sentence $+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)$ for some variable v.

ii) Set contains the signed sentence $+[t / x](\Phi \wedge [x / u]afm)$ for every $t \in \delta(\mathcal{P}(Set))$.

10) If $\pm snt$ is a signed nonatomic elementary sentence of the form $-[\iota x.\Phi / u]afm$ for some constant description term $\iota x.\Phi$ and atomic formula afm, then Set satisfies one of the following conditions:

i) Set contains the signed sentence $+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)$ for some variable $v \neq x$.

ii) Set contains the signed sentence $+[t / x](\Phi \wedge \neg[x / u]afm)$ for every $t \in \delta(\mathcal{P}(Set))$.

4.2. Statement and Proof of the Completeness Theorem.

Theorem 4.2.1.: Let Seq be a sequent of signed sentences. If Seq is valid, then Seq is derivable.

We establish 4.2.1. by proving the much stronger result that if ***Seq*** is valid then ***Seq*** is strictly derivable. This stronger result follows by lemma 3.5. from the following theorem, which we now prove:

Theorem 4.2.2: Let *Seq* be any sequent. If *Seq* is valid then *Seq* is strictly 2-derivable.

Proof of 4.2.2.: We will show that any sequent that is not strictly 2-derivable is not valid, that is, is invalidated by some base, by establishing the following two lemmata:

Lemma 4.2.3.: Let *Seq* be any sequent. If *Seq* is not strictly 2-derivable, then *Seq* has a downward closed extension Set_{ω}.

Lemma 4.2.4.: Every downward closed set ***Set*** of signed sentences determines a base bse_{Set} such that $Set \cap Cl(bse_{Set}) = \emptyset$.

Note that the significance of the notion of *downward closed* is revealed by lemmas 4.2.3. and 4.2.4.: every downward closed sequent is invalidated by some base. Clearly, our desired result that any sequent which is not strictly 2-derivable is invalidated by some base is an immediate consequence of lemmas 4.2.3. and 4.2.4..

Proof of 4.2.3.: We establish lemma 4.2.3. by defining a process which constructs downward closed extensions for arbitrary sequents which are not strictly 2-derivable. We say that a signed sentence $\pm snt$ is **reducible** iff ***snt*** is neither elementary nor, for some formula ***F***, of the form $(x)F$. Let $S \subseteq \mathbb{N}$ and ***Set*** be a set of signed sentences. Let $\pm snt$ be a signed nonatomic sentence in ***Set*** such that $\mathbb{Ind}(\pm snt) \notin S$ and for all $j < i$, if $j \in S$, then $\mathbb{SS}(j) \notin Set$. In other words, let $\pm snt$ be the first signed nonatomic sentence in ***Set*** whose index is not in S. Then ***Set'*** is said to be an **S-reduction of *Set*** iff $\pm snt$ is reducible and ***Set'*** is obtained from ***Set*** by adding new signed sentences to ***Set'*** according to the following rules:

1) If $\pm snt$ is respectively of the form $\pm\neg$A for some sentence A, then respectively $\mp$A is added to ***Set*** ;

2) If $\pm$***snt*** is of the form +(**A** $\wedge$ **B**) for some sentences **A**, **B**, then one of +**A**, +**B** is added to *Set*;

3) If $\pm$***snt*** is of the form -(**A** $\wedge$ **B**) for some sentences **A**, **B**, then both of -**A**, -**B** are added to *Set*;

4) If $\pm$***snt*** is of the form +(**A** $\rightarrow$ **B**) for some sentences **A**, **B**, then both of -**A**, +**B** are added to *Set* ;

5) If $\pm$***snt*** is of the form -(**A** $\rightarrow$ **B**) for some sentences **A**, **B**, then one of +**A**, -**B** is added to *Set*.;

4.2.3.1: **A reduction sequence** $\langle Set_i \rangle$ **of a sequent** $\mathbb{Seq}$ **with respect to an enumeration** $\mathbb{SS}$ is a sequence of sets Set_i of signed sentences defined by simultaneous recursion with sequences $\langle S_i \rangle$, $\langle D1_i \rangle$ of index sets S_i, $D1_i$ as follows: Let $\pm snt_i$ be the first signed nonatomic sentence in Set_i whose index is not in S_i. Let $\mathbb{R}(i) \subseteq S_i$ be the set of indices in S_i of signed sentences ***ssnt*** such that ***ssnt*** is either of the form -(*x*)***F*** for some formula ***F*** or of the form $\pm[\iota x.\Phi / u]afm$ for some description term $\iota x.\Phi$ and atomic formula ***afm***. Then,

0) $Set_0 = Seq$. $S_0 = \varnothing$. $D1_i = \varnothing$.

1) If $\pm snt_i$ is reducible, then Set_{i+1} is a S_i -reduction of Set_i. $S_{i+1} = S_i \cup \{\mathrm{Ind}(\pm snt_i)\}$. $D1_{i+1} = D1_i$.

2) If $\pm snt_i$ is of the form +(*x*)***F*** for some formula ***F***, then Set_{i+1} is obtained from Set_i by adding to Set_i the new signed sentence $+[p / x]F$, where ***p*** is the first parameter that does not occur in any signed sentence of Set_i. In other words, $Set_{i+1} = Set_i \cup \{+[p / x]F\}$. $S_{i+1} = (S_i \cup \{\mathrm{Ind}(\pm snt_i)\}) - \mathbb{R}(i)$. $D1_{i+1} = D1_i$.

3) If $\pm snt_i$ is of the form -(*x*)***F*** for some formula ***F***, then Set_{i+1} is obtained from Set_i by adding to Set_i all signed sentences of the form $-[t / x]F$, where ***t*** is in $\delta(\wp(Set_i))$. In other words, $Set_{i+1} = Set_i \cup \{+[t / x]F : t \in \delta(\wp(Set_i))\}$. $S_{i+1} = S_i \cup \{\mathrm{Ind}(\pm snt_i)\}$. $D1_{i+1} = D1_i$.

4) If $\pm$***snt*** is a signed nonatomic elementary sentence of the form $+[\iota x.\Phi / u]afm$ for some description term $\iota x.\Phi$ and atomic formula ***afm***, then Set_{i+1}, S_{i+1}, $D1_{i+1}$ are obtained from Set_i, S_i, $D1_i$ by one of the following rules:

a) If $\mathbb{I}\mathrm{nd}(\pm snt_i) \notin \mathrm{D1}_i$, then Set_{i+1} is obtained from Set_i by adding to Set_i the new signed sentence $+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)$ for some variable $v \neq x$. In other words, $Set_{i+1} = Set_i \cup \{+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)\}$. $\mathrm{S}_{i+1} = \mathrm{S}_i \cup \{\mathbb{I}\mathrm{nd}(\pm snt_i)\}$. $\mathrm{D1}_{i+1} = \mathrm{D1}_i$.

b) Set_{i+1} is obtained from Set_i by adding to Set_i all signed sentences of the form $+[t / x](\Phi \wedge [x / u]afm)$, where $t \in \delta(\mathcal{P}(Set_i))$. In other words, $Set_{i+1} = Set_i \cup \{+[t / x](\Phi \wedge [x / u]afm): t \in \delta(\mathcal{P}(Set_i))\}$. $\mathrm{S}_{i+1} = \mathrm{S}_i \cup \{\mathbb{I}\mathrm{nd}(\pm snt_i)\}$. $\mathrm{D1}_{i+1} = \mathrm{D1}_i \cup \{\mathbb{I}\mathrm{nd}(\pm snt_i)\}$.

5) If $\pm snt$ is a signed nonatomic elementary sentence of the form $-[\iota x.\Phi / u]afm$ for some description term $\iota x.\Phi$ and atomic formula afm, then Set_{i+1}, S_{i+1}, $\mathrm{D1}_{i+1}$ are obtained from Set_i, S_i, $\mathrm{D1}_i$ by one of the following rules:

a) If $\mathbb{I}\mathrm{nd}(\pm snt_i) \notin \mathrm{D1}_i$ then Set_{i+1} is obtained from Set_i by adding to Set_i the signed sentence $+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)$ for some variable $v \neq x$. In other words, $Set_{i+1} = Set_i \cup \{+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)\}$. $\mathrm{S}_{i+1} = \mathrm{S}_i \cup \{\mathbb{I}\mathrm{nd}(\pm snt_i)\}$. $\mathrm{D1}_{i+1} = \mathrm{D1}_i$.

b) Set_{i+1} is obtained from Set_i by adding to Set_i all signed sentences of the form $+[t / x](\Phi \wedge \neg[x / u]afm)$, where $t \in \delta(\mathcal{P}(Set_i))$. In other words, $Set_{i+1} = Set_i \cup \{+[t / x](\Phi \wedge \neg[x / u]afm) : t \in \delta(\mathcal{P}(Set_i))\}$. $\mathrm{S}_{i+1} = \mathrm{S}_i \cup \{\mathbb{I}\mathrm{nd}(\pm snt_i)\}$. $\mathrm{D1}_{i+1} = \mathrm{D1}_i \cup \{\mathbb{I}\mathrm{nd}(\pm snt_i)\}$.

6) If there is no such signed sentence $\pm snt_i$, that is, if Set_i contains no signed nonatomic sentence whose index is not in S_i, then $Set_{i+1} = Set_i$. $\mathrm{S}_{i+1} = \mathrm{S}_i$. $\mathrm{D1}_{i+1} = \mathrm{D1}_i$.

Observe that definition 4.2.3.1. is well-defined in the sense that the "new" parameter p required by clause 2) will always exist, since for all i, $\mathcal{P}(Set_i)$ is finite, as we now show: Since Seq is a finite set of signed sentences, $\mathcal{P}(Set_0)$ is finite. If $\mathcal{P}(Set_i)$ is finite, so is $\mathcal{P}(Set_{i+1})$, since the only clause of 4.2.3.1. which adds new parameters to any $\mathcal{P}(Set_i)$ (i.e., 2) adds only finitely many such parameters (just one, in fact).

We observe that 4.2.3.1. is well-defined also in the sense that clauses 1) - 6) inclusive of that definition are mutually exclusive as well as collectively exhaustive. It is important to note that rules a), b) of clauses 4) and 5) of 4.2.3.1. are not to be thought of as mutually exclusive and exhaustive and therefore *determinate* subcases of 4). Rather, they are *alternative possible ways* in which a Set_{i+1} may be obtained from Set_i in a reduction sequence $\langle Set_i \rangle$.

We will see that for arbitrary sequents *Seq* that are not strictly 2-derivable there exists an infinite reduction sequence $\langle Set_i \rangle$ the union set of which proves to be the downward closed extension of *Seq* whose existence is claimed in lemma 4.2.3.. The significant observation here is that any reduction sequence $\langle Set_i \rangle$ has the property that for each Set_i in $\langle Set_i \rangle$, if Set_i is valid in a base *bse* with domain $\delta(\mathcal{P}(Set_i))$, then Set_{i+1} is valid in *bse*. Hence, if a base *bse* invalidates any Set_i in any reduction sequent $\langle Set_i \rangle$ of a sequent *Seq*, it follows by induction that *bse* invalidates *Seq*. Indeed, the claim that the union set of a reduction sequent $\langle Set_i \rangle$ is downward closed implies that $\langle Set_i \rangle$ has the property observed above and that *there are* Set_i in $\langle Set_i \rangle$ and *bse* such that *bse* invalidates Set_i.

Lemma 4.2.3.2: Let *Seq* be any sequent. If *Seq* is not strictly 2-derivable, then *Seq* has an infinite reduction sequence $\langle Set_i \rangle$ such that no Set_i in $\langle Set_i \rangle$ is strictly 2-derivable.

Proof of 4.2.3.2.: Assume *Seq* is not strictly 2-derivable. We show by induction on i that for every i, there is a Set_i such that Set_i is not strictly 2-derivable.

Base step: $i = 0$. By assumption $Set_0 = Seq$ is not strictly 2-derivable.

Induction step: $i > 0$. Assume Set_i is not strictly 2-derivable. We show that there is a Set_{i+1} which is not strictly 2-derivable by exhaustive case analysis. There are two main cases:

1) There is no signed nonatomic sentence $\pm snt_i$ such that $\pm snt_i \in Set_i$ and $\mathbb{I}\mathrm{nd}(\pm snt_i) \notin S_i$. In this case, by clause 6) of definition 4.2.3.1., $Set_{i+1} = Set_i$. By the hypothesis of induction, Set_{i+1} is not strictly 2-derivable, so the claim holds.

2) There is a signed nonatomic sentence $\pm snt_i$ such that $\pm snt_i \in Set_i$ and $\mathbb{I}\mathrm{nd}(\pm snt_i) \notin S_i$. There are nine subcases:

i) $\pm snt_i$ is respectively of the form $\pm\neg A$ for some sentence A. Then by clause 1) of 4.2.3.1. Set_{i+1} exists and is a S_i-reduction of Set_i of type 1). In this case, $Set_{i+1} \cong Set_i \cup \{\mp A\}$, respectively. We show that if Set_{i+1} is strictly 2-derivable, then so is Set_i, from which it follows by the hypothesis of induction that Set_{i+1} is not strictly 2-derivable. Assume that Set_{i+1} is strictly 2-derivable. Then some finite subset Seq' of Set_{i+1} is strictly 2-derivable. In case $Seq' \subseteq Set_i$, Set_i is strictly 2-derivable by virtue of its strictly 2-derivable finite subset Seq'. In case $Seq' \not\subseteq Set_i$, respectively $\mp A \in Seq'$. The following is a legitimate application of rule 1.2.2.2..:

$$\frac{(Seq' - \{\mp A\}) \cup \{\mp A\}}{(Seq' - \{\mp A\}) \cup \{\pm\neg A\}}$$

So, since $Seq' \cong (Seq' - \{\mp A\}) \cup \{\mp A\}$ is strictly 2-derivable, so is $(Seq' - \{\mp A\}) \cup \{\pm\neg A\}$. Since $Seq' \subseteq Set_{i+1} \cong Set_i \cup \{\mp A\}$, $(Seq' - \{\mp A\}) \cup \{\pm\neg A\} \subseteq Set_i \cup \{\pm\neg A\} \cong Set_i$. Since Set_i has a finite subset $(Seq' - \{\pm A\}) \cup \{\pm\neg A\}$ which is strictly 2-derivable, Set_i is strictly 2-derivable. So the claim holds.

ii) $\pm snt_i$ is of the form $+(A \wedge B)$ for some sentences A, B. In this case (since $+(A \wedge B) \in Set_i$), $Set_i \cong Set_i \cup \{+(A \wedge B)\}$. Then by clause 1) of 4.2.3.1., two Set_{i+1} exist as S_i-reductions of Set_i of type 2): $Set_i \cup \{+A\}$ and $Set_i \cup \{+B\}$. We show that if both Set_{i+1} are strictly 2-derivable, then so is Set_i, from which it follows by the hypothesis of induction that at least one of the Set_{i+1} is not strictly 2-derivable. So assume that both of $Set_i \cup \{+A\}$, $Set_i \cup \{+B\}$ are strictly 2-derivable. Then there are finite $Seq1 \subseteq Set_i \cup \{+A\}$, $Seq2 \subseteq Set_i \cup \{+B\}$ such that both of $Seq1$, $Seq2$ are strictly 2-derivable. In case either of $Seq1$, $Seq2$ are subsets of Set_i, then clearly Set_i has a finite subset which is strictly 2-derivable, and so is strictly 2-derivable. So assume neither of $Seq1$, $Seq2$ are subsets of Set_i. Then $+A \in Seq1$ and $+B \in Seq2$. The following is a legitimate application of rule 1.2.2.3.+.:

$$\frac{(Seq1 - \{+A\}) \cup \{+A\} \qquad (Seq2 - \{+B\}) \cup \{+B\}}{(Seq1 - \{+A\}) \cup (Seq2 - \{+B\}) \cup \{+(A \wedge B)\}}$$

Since both of $Seq1 \cong (Seq1 - \{+A\}) \cup \{+A\}$, $Seq2 \cong (Seq2 - \{+B\}) \cup \{+B\}$ are strictly 2-derivable, it follows that $(Seq1 - \{+A\}) \cup (Seq2 - \{+B\}) \cup \{+(A \wedge B)\}$ is strictly

2-derivable. Now, since $Seq1 \subseteq Set_i \cup \{+A\}$ and $Seq2 \subseteq Set_i \cup \{+B\}$, it follows that $(Seq1 - \{+A\}) \cup (Seq2 - \{+B\}) \cup \{+(A \wedge B)\} \subseteq Set_i \cup \{+(A \wedge B)\} \cong Set_i$. Since Set_i has a strictly 2-derivable finite subset $(Seq1 - \{+A\}) \cup (Seq2 - \{+B\}) \cup \{+(A \wedge B)\}$, Set_i is strictly 2-derivable. So the claim holds.

iii) $\pm snt_i$ is of the form $-(A \wedge B)$ for some sentences A, B. Then, by clause 1) of 4.2.3.1., Set_{i+1} exists and is an S_i-reduction of Set_i of type 3). In this case, $Set_i \cong Set_i \cup \{-(A \wedge B)\}$ and $Set_{i+1} \cong Set_i \cup \{-A , -B\}$. We show that if Set_{i+1} is strictly 2-derivable, then so is Set_i, from which it follows by the hypothesis of induction that Set_{i+1} is not strictly 2-derivable. So assume that $Set_i \cup \{-A , -B\}$, is strictly 2-derivable. Then there is finite $Seq' \subseteq Set_i \cup \{-A , -B\}$ such that Seq' is strictly 2-derivable. In case $Seq' \subseteq Set_i$, Set_i is strictly 2-derivable by virtue of its strictly 2-derivable finite subset Seq'. In case $Seq' \nsubseteq Set_i$, one of $-A$, $-B \in Set_i$. Without loss of generality, assume that $-A \in Set_i$. There are two cases:

a) $-B \in Seq'$. Then $Seq' \cong (Seq' - \{-A , -B\}) \cup \{-A , -B\}$. Consider the following legitimate application of rule 1.2.2.3.-:

$$\frac{(Seq' - \{-A , -B\}) \cup \{-A , -B\}}{(Seq' - \{-A , -B\}) \cup \{-(A \wedge B)\}}$$

Since $(Seq' - \{-A , -B\}) \cup \{-A , -B\} \cong Seq'$ is strictly 2-derivable, it follows that $(Seq' - \{-A , -B\}) \cup \{-(A \wedge B)\}$ is strictly 2-derivable.

b) $-B \notin Seq'$. Then $Seq' \cong (Seq' - \{-A , -B\}) \cup \{-A\}$. The the following is a legitimate application of the thinning rule followed by a legitimate application of rule 1.2.2.3.-:

$$\cfrac{\cfrac{(Seq' - \{-A , -B\}) \cup \{-A\}}{(Seq' - \{-A , -B\}) \cup \{-A , -B\}}}{(Seq' - \{-A , -B\}) \cup \{-(A \wedge B)\}}$$

Since $(Seq' - \{-A, -B\}) \cup \{-A\} \cong Seq'$ is strictly 2-derivable, it follows that $(Seq' - \{-A, -B\}) \cup \{-(A \wedge B)\}$ is strictly 2-derivable.

In both cases, then, $(Seq' - \{-A, -B\}) \cup \{-(A \wedge B)\}$ is strictly 2-derivable. Since $Seq' \subseteq Set_{i+1} \cong Set_i \cup \{-A, -B\}$, it follows that $(Seq' - \{-A, -B\}) \cup \{-(A \wedge B)\} \subseteq Set_i \cup \{-(A \wedge B)\} = Set_i$. Hence, Set_i has a strictly 2-derivable finite subset and so Set_i is strictly 2-derivable. So the claim holds.

iv) $\pm snt_i$ is the form $+(A \rightarrow B)$ for some sentences **A**, **B**. This case is similar to subcase iii).

v) $\pm snt_i$ is of the form $-(A \rightarrow B)$ for some sentences **A**, **B**. This case is similar to subcase ii).

vi) $\pm snt_i$ is of the form $+(x)F$ for some formula F. In this case, $Set_i \cong Set_i \cup \{+(x)F\}$. Then, by clause 2) of 4.2.3.1., Set_{i+1} exists and $Seq_{i+1} \cong Set_i \cup \{+[p / x]F\}$ where p is the first parameter that does not occur in any member of Set_i. We show that if Set_{i+1} is strictly 2-derivable, then so is Set_i, from which it follows by the hypothesis of induction that Set_{i+1} is not strictly 2-derivable. So assume that $Seq_{i+1} \cong Set_i \cup \{+[p / x]F\}$ is strictly 2-derivable. Then there is finite $Seq' \subseteq Set_i \cup \{+[p / x]F\}$ such that Seq' is strictly 2-derivable. In case $Seq' \subseteq Set_i$, Set_i is strictly 2-derivable by virtue of its strictly 2-derivable finite subset Seq'. In case $Seq' \not\subseteq Set_i$, $+[p / x]F \in Seq'$. Since p does not occur in any member of Set_i, the following is a legitimate application of rule 1.2.2.5.+:

$$\frac{(Seq' - \{+[p / x]F\}) \cup \{+[p / x]F\}}{(Seq' - \{+[p / x]F\}) \cup \{+(x)F\}}$$

Since $Seq' \cong (Seq' - \{+[p / x]F\}) \cup \{+[p / x]F\}$ is strictly 2-derivable, it follows that $(Seq' - \{+[p / x]F\}) \cup \{+(x)F\}$ is strictly 2-derivable. Since $Seq' \subseteq Seq_{i+1} \cong Set_i \cup \{+[p / x]F\}$, it follows that $(Seq' - \{+[p / x]F\}) \cup \{+(x)F\} \subseteq Set_i \cup \{+(x)F\} \cong Set_i$. Hence, Set_i has a strictly 2-derivable finite subset and so Set_i is strictly 2-derivable.

vii) $\pm snt_i$ is of the form $-(x)F$ for some fomula F. In this case, $Set_i \cong Set_i \cup \{-(x)F\}$. Then, by clause 3) of 4.2.3.1., Set_{i+1} exists and $Seq_{i+1} \cong Set_i \cup \{-[t/x]F : t \in \delta(\mathcal{P}(Set_i))\}$. We show that if Set_{i+1} is strictly 2-derivable, then so is Set_i, from which it follows by the hypothesis of induction that Set_{i+1} is not strictly 2-derivable. So assume that $Seq_{i+1} \cong Set_i \cup \{-[t/x]F : t \in \delta(\mathcal{P}(Set_i))\}$ is strictly 2-derivable. Then there is finite $Seq' \subseteq Set_i \cup \{-[t/x]F : t \in \delta(\mathcal{P}(Set_i))\}$ such that Seq' is strictly 2-derivable. In case $Seq' \subseteq Set_i$, Set_i is strictly 2-derivable by virtue of its strictly 2-derivable finite subset Seq'. In case $Seq' \nsubseteq Set_i$ there is a nonempty finite set $\{-[t_1/x]F, \ldots, -[t_n/x]F\} \subseteq \{-[t/x]F : t \in \delta(\mathcal{P}(Set_i))\}$ such that $\{-[t_1/x]F, \ldots, -[t_n/x]F\} \subseteq Seq'$ where the $-[t_1/x]F, \ldots, -[t_n/x]F$ are all the signed sentences in $\{-[t/x]F : t \in \delta(\mathcal{P}(Set_i))\}$ which are members of Seq'. The following is a finite sequence of legitimate applications of rule 1.2.2.5.-:

$$\dfrac{\dfrac{\dfrac{(Seq' - \{-[t_1/x]F, \ldots, -[t_n/x]F\}) \cup \{-[t_1/x]F, \ldots, -[t_n/x]F\}}{(Seq' - \{-[t_1/x]F, \ldots, -[t_n/x]F\}) \cup \{-[t_2/x]F, \ldots, -[t_n/x]F\} \cup \{-(x)F\}}}{(Seq' - \{-[t_1/x]F, \ldots, -[t_n/x]F\}) \cup \{-[t_3/x]F, \ldots, -[t_n/x]F\} \cup \{-(x)F\}}}{}$$

$$\vdots$$

$$\dfrac{(Seq' - \{-[t_1/x]F, \ldots, -[t_n/x]F\}) \cup \{-[t_n/x]F\} \cup -(x)F\}}{(Seq' - \{-[t_1/x]F, \ldots, -[t_n/x]F\}) \cup \{-(x)F\}}$$

Since $Seq' \cong (Seq' - \{-[t_1/x]F, \ldots, -[t_n/x]F\}) \cup \{-[t_1/x]F, \ldots, -[t_n/x]F\}$ is strictly 2-derivable, it follows that $(Seq' - \{-[t_1/x]F, \ldots, -[t_n/x]F\}) \cup \{-(x)F\}$ is strictly 2-derivable. Since $Seq' \subseteq Seq_{i+1} \cong Set_i \cup \{-[t/x]F : t \in \delta(\mathcal{P}(Set_i))\}$ and the $-[t_1/x]F, \ldots, -[t_n/x]F$ are *all* the signed sentences in $\{-[t/x]F : t \in \delta(\mathcal{P}(Set_i))\}$ which are members of Seq' it follows that $(Seq' - \{-[t_1/x]F, \ldots, -[t_n/x]F\}) \cup \{-(x)F\} \subseteq Set_i \cup \{+(x)F\} \cong Set_i$. Hence, Set_i has a strictly 2-derivable finite subset and so Set_i is strictly 2-derivable. So the claim holds.

viii) $\pm snt_i$ is a signed elementary sentence of the form $+[\iota x.\Phi / u]afm$ for some description term $\iota x.\Phi$ and atomic formula afm. In this case, $Set_i \cong Set_i \cup \{+[\iota x.\Phi / u]afm\}$, for some set of signed sentences Set. Then by rules a), b) of clause 4) of 4.2.3.1., two Set_{i+1} exist: a) $Set_i \cup \{+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)\}$ and b) $Set_i \cup \{+[t / x](\Phi \wedge [x / u]afm) : t \in \delta(\mathcal{P}(Set_i))\}$. We show that if both Set_{i+1} are strictly 2-derivable, then so is Set_i, from which it follows by the hypothesis of induction that at least one of the Set_{i+1} is not strictly 2-derivable. So assume that both of Set_{i+1}a), Set_{i+1}b) are strictly 2-derivable. Then there are finite $Seq1 \subseteq Set_{i+1}$a), $Seq2 \subseteq Set_{i+1}$b) such that both of $Seq1$, $Seq2$, are strictly 2-derivable. In case one of $Seq1$, $Seq2$, is a subset of Set_i, then clearly Set_i is strictly 2-derivable. So assume neither of $Seq1$, $Seq2$, are subsets of Set_i. Then $+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v) \in Seq1$ and there is nonempty finite set $\{+[t_1 / x](\Phi \wedge [x / u]afm), \ldots, +[t_n / x](\Phi \wedge [x / u]afm)\} \subseteq \{+[t / x](\Phi \wedge [x / u]afm) : t \in \delta(\mathcal{P}(Set_i))\}$ such that $\{+[t_1 / x](\Phi \wedge [x / u]afm), \ldots, +[t_n / x](\Phi \wedge [x / u]afm)\} \subseteq Seq2$ where the $+[t_1 / x](\Phi \wedge [x / u]afm), \ldots, +[t_n / x](\Phi \wedge [x / u]afm)$ are the only signed sentences in $\{+[t / x](\Phi \wedge [x / u]afm) : t \in \delta(\mathcal{P}(Set_i))\}$ which are members of $Seq2$. Let $Seq1^* \cong Seq1 - \{+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)\}$, and let $Seq2^* \cong Seq2 - \{+[t_1 / x](\Phi \wedge [x / u]afm), \ldots, +[t_n / x](\Phi \wedge [x / u]afm)\}$. Then the following is a finite sequence of legitimate applications of 1.2.2.6.+:

$$\frac{Seq2^* \cup \{+[t_1 / x](\Phi \wedge [x / u]afm), \ldots, +[t_n / x](\Phi \wedge [x / u]afm)\} \qquad Seq1}{Seq2^* \cup Seq1^* \cup \{+[t_2 / x](\Phi \wedge [x / u]afm), \ldots, +[t_n / x](\Phi \wedge [x / u]afm)\} \cup \{+[\iota x.\Phi / u]afm\} \qquad Seq1}$$

$$\frac{}{Seq2^* \cup Seq1^* \cup \{+[t_3 / x](\Phi \wedge [x / u]afm), \ldots, +[t_n / x](\Phi \wedge [x / u]afm)\} \cup \{+[\iota x.\Phi / u]afm\} \qquad Seq1}$$

$$\vdots$$

$$\frac{Seq2^* \cup Seq1^* \cup \{+[t_n / x](\Phi \wedge [x / u]afm)\} \cup \{+[\iota x.\Phi / u]afm\} \qquad Seq1}{Seq2^* \cup Seq1^* \cup \{+[\iota x.\Phi / u]afm\}}$$

Since both of $Seq1$, $Seq2 \cong Seq2^* \cup \{+[t_1 / x](\Phi \wedge [x / u]afm), \ldots, +[t_n / x](\Phi \wedge [x / u]afm)\}$ are strictly 2-derivable, it follows that $Seq2^* \cup Seq1^* \cup \{+[\iota x.\Phi / u]afm\}$ is strictly 2-derivable. Now, since $Seq1 \subseteq Set_i \cup \{+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)\}$ and $Seq2 \subseteq Set_i \cup \{+[t / x](\Phi \wedge [x / u]afm) : t \in \delta(\mathcal{P}(Set_i))\}$, it follows by the facts that $Seq1^* \cong$

$Seq1 - \{+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)\}$ and $Seq2^* \cong Seq2 - \{+[t_1 / x](\Phi \wedge [x / u]afm), \ldots, +[t_n / x](\Phi \wedge [x / u]afm)\}$, where the $+[t_1 / x](\Phi \wedge [x / u]afm), \ldots, +[t_n / x](\Phi \wedge [x / u]afm)$ are the only members of $\{+[t / x](\Phi \wedge [x / u]afm) : t \in \delta(\wp(Set_i))\}$ which belong to $Seq2$, that $Seq2^* \cup Seq1^* \subseteq Set_i$. Hence, $Seq2^* \cup Seq1^* \cup \{+[\iota x.\Phi / u]afm\} \subseteq Set_i \cup \{+[\iota x.\Phi / u]afm\} \cong Set_i$. Hence, Set_i has a strictly 2-derivable finite subset and so Set_i is strictly 2-derivable. So the claim holds.

ix) $\pm snt_i$ is a signed elementary sentence of the form $-[\iota x.\Phi / u]afm$ for some description term $\iota x.\Phi$ and atomic formula afm. In this case, $Set_i \cong Set_i \cup \{-[\iota x.\Phi / u]afm\}$. Then by rules a), b) of clause 5) of 4.2.3.1., two Set_{i+1} exist: a) $Set_i \cup \{+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)\}$ and b) $Set_i \cup \{+[t / x](\Phi \wedge \neg[x / u]afm) : t \in \delta(\wp(Set_i))\}$. We show that if both Set_{i+1} are strictly 2-derivable, then so is Set_i, from which it follows by the hypothesis of induction that at least one of the Set_{i+1} is not strictly 2-derivable. So assume that both of Set_{i+1}a), Set_{i+1}b) are strictly 2-derivable. Then there are finite $Seq1 \subseteq Set_{i+1}$a), $Seq2 \subseteq Set_{i+1}$b) such that both of $Seq1$, $Seq2$, are strictly 2-derivable. In case one of $Seq1$, $Seq2$, is a subset of Set_i then clearly Set_i is strictly 2-derivable. So assume neither of $Seq1$, $Seq2$, are subsets of Set_i. Then $+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v) \in Seq1$ and there is a nonempty finite set $\{+[t_1 / x](\Phi \wedge \neg[x / u]afm), \ldots, +[t_n / x](\Phi \wedge \neg[x / u]afm)\} \subseteq \{+[t / x](\Phi \wedge \neg[x / u]afm) : t \in \delta(\wp(Set_i))\}$ such that $\{+[t_1 / x](\Phi \wedge \neg[x / u]afm), \ldots, +[t_n / x](\Phi \wedge \neg[x / u]afm)\} \subseteq Seq2$, where the $+[t_1 / x](\Phi \wedge \neg[x / u]afm), \ldots, +[t_n / x](\Phi \wedge \neg[x / u]afm)$ are the only signed sentences in $\{+[t / x](\Phi \wedge \neg[x / u]afm) : t \in \delta(\wp(Set_i))\}$ which are members of $Seq2$. Let $Seq1^* \cong Seq1 - \{+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)\}$, and let $Seq2^* \cong Seq2 - \{+[t_1 / x](\Phi \wedge \neg[x / u]afm), \ldots, +[t_n / x](\Phi \wedge \neg[x / u]afm)\}$. Then the following is a finite sequence of legitimate applications of 1.2.2.6.-:

$$\frac{Seq2^* \cup \{+[t_1 / x](\Phi \wedge \neg[x / u]afm), \ldots, +[t_n / x](\Phi \wedge \neg[x / u]afm)\} \qquad Seq1}{\dfrac{Seq2^* \cup Seq1^* \cup \{+[t_2 / x](\Phi \wedge \neg[x / u]afm), \ldots, +[t_n / x](\Phi \wedge \neg[x / u]afm)\} \cup \{-[\iota x.\Phi / u]afm\} \quad Seq1}{Seq2^* \cup Seq1^* \cup \{+[t_3 / x](\Phi \wedge \neg[x / u]afm), \ldots, +[t_n / x](\Phi \wedge \neg[x / u]afm)\} \cup \{-[\iota x.\Phi / u]afm\} \quad Seq1}}$$

.

.

$$\frac{Seq2^* \cup Seq1^* \cup \{+[t_n / x](\Phi \wedge \neg[x / u]afm)\} \cup \{-[\iota x.\Phi / u]afm\}}{Seq2^* \cup Seq1^* \cup \{-[\iota x.\Phi / u]afm\}} \quad Seq1$$

Since both of ***Seq1***, ***Seq2*** $\cong$ ***Seq2**** $\cup \{+[t_1 / x](\Phi \wedge \neg[x / u]afm), \ldots, +[t_n / x](\Phi \wedge \neg [x / u]afm)\}$ are strictly 2-derivable, it follows that ***Seq2**** $\cup$ ***Seq1**** $\cup \{-[\iota x.\Phi / u]afm\}$ is strictly 2-derivable. Now, since ***Seq1*** $\subseteq$ $Set_i \cup \{+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)\}$ and ***Seq2*** $\subseteq Set_i \cup \{+[t / x](\Phi \wedge \neg[x / u]afm) : t \in \delta(\wp(Set_i))\}$, it follows by the facts that ***Seq1**** $\cong$ ***Seq1*** $- \{+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)\}$ and ***Seq2**** $\cong$ ***Seq2*** $- \{+[t_1 / x](\Phi \wedge \neg[x / u]afm), \ldots, +[t_n / x](\Phi \wedge \neg[x / u]afm)\}$, where the $+[t_1 / x](\Phi \wedge \neg[x / u]afm), \ldots, +[t_n / x](\Phi \wedge \neg[x / u]afm)$ are the only members of $\{+[t / x](\Phi \wedge \neg[x / u]afm) : t \in \delta(\wp(Set_i))\}$ which belong to ***Seq2***, that ***Seq2**** $\cup$ ***Seq1**** $\subseteq Set_i$. Hence, ***Seq2**** $\cup$ ***Seq1**** $\cup \{-[\iota x.\Phi / u]afm\} \subseteq Set_i \cup \{-[\iota x.\Phi / u]afm\} \cong Set_i$. Hence, Set_i has a strictly 2-derivable finite subset and so Set_i is strictly 2-derivable. So the claim holds.

This conclusion our case analysis. By the principle of induction, lemma 4.2.3.2. is established. □

<u>Lemma 4.2.3.3.</u>: Let $\langle Set_i \rangle$ be any infinite reduction sequence of sequent ***Seq***, and let $\langle S_i \rangle$ be the sequence of index sets defined with $\langle Set_i \rangle$ in 4.2.3.1.. Then, for any Set_i in $\langle Set_i \rangle$ and any signed nonatomic sentence $\pm snt \in Set_i$ such that $\mathbb{Ind}(\pm snt) \notin S_i$, there is a $j \geq i$ such that $\pm snt$ is the *first* signed nonatomic sentence in Set_j such that $\mathbb{Ind}(\pm snt) \notin S_j$.

<u>Proof of 4.2.3.3</u>: Let Set_i be any term of $\langle Set_i \rangle$ and let $\pm snt$ be any signed nonatomic sentence in Set_i such that $\mathbb{Ind}(\pm snt) \notin S_i$. Then for all $j \leq i$, $\pm snt \in Set_j$. We show by induction on the number $N(i, \mathbb{Ind}(\pm snt))$ of signed sentences $ssnt$ of the form $+(x)F$ such that $\mathbb{Ind}(ssnt) < \mathbb{Ind}(\pm snt)$ and $\mathbb{Ind}(ssnt) \notin S_i$ that there is a j such that $\pm snt$ is the first signed nonatomic sentence in Set_j such that $\mathbb{Ind}(\pm snt) \notin S_j$.

<u>Base step</u>: $N(i, \mathbb{Ind}(\pm snt)) = 0$. Suppose the claim does not hold, that is, suppose that for all $j \geq i$, there is a signed nonatomic sentence $ssnt \in Set_j$ such that $\mathbb{Ind}(ssnt) < \mathbb{Ind}(\pm snt)$ and $\mathbb{Ind}(ssnt) \notin S_i$. For any i, let $ssnt_i$ be the first signed nonatomic sentence in Set_i such that

$\mathbb{I}\mathrm{nd}(ssnt_i) \notin S_i$. Then, for all $j \geq i$, $\mathbb{I}\mathrm{nd}(ssnt_j) < \mathbb{I}\mathrm{nd}(\pm snt)$. So there is an infinite sequence of natural numbers $\mathbb{I}\mathrm{nd}(ssnt_i)$, $\mathbb{I}\mathrm{nd}(ssnt_{i+1}), \ldots, \mathbb{I}\mathrm{nd}(ssnt_j), \ldots$ such that for all $j \geq i$, $\mathbb{I}\mathrm{nd}(ssnt_j) < \mathbb{I}\mathrm{nd}(\pm snt)$ and $\mathbb{I}\mathrm{nd}(ssnt_{j+1}) \notin S_j$ and $\mathbb{I}\mathrm{nd}(ssnt_{j+1}) \in S_{j+1}$. We now show by induction on j that for all $j \geq i$, $S_j \subseteq S_{j+1}$: For $j = i$, since $\mathbb{I}\mathrm{nd}(ssnt_j) < \mathbb{I}\mathrm{nd}(\pm snt)$ and $\mathbb{I}\mathrm{nd}(ssnt_j) \notin S_j$ and $N(i, \mathbb{I}\mathrm{nd}(\pm snt)) = 0$, it follows that $ssnt_j$ is not, for some formula F, of the form $+(x)F$. Hence S_{j+i} is obtained from S_j in $\langle S_i \rangle$ by a clause of 4.2.3.1. other than clause 2). Since clause 2) is the only clause of 4.2.3.1. that deletes indices from an S_i in $\langle S_i \rangle$, it follows that $S_j \subseteq S_{j+1}$. For $j \geq i$, assume that for all k, $i \leq k < j$, $S_k \subseteq S_{k+1}$. Since $N(i, \mathbb{I}\mathrm{nd}(\pm snt)) = 0$, it follows that $N(j, \mathbb{I}\mathrm{nd}(\pm snt)) = 0$. So, snt_j is not of the form $+(x)F$, and so, since S_{j+1} is obtained from S_j by a clause of 4.2.3.1. other than 2), $S_j \subseteq S_{j+1}$. **QED.** We now continue with the main proof of the base step: Since for all $j \geq i$, $\mathbb{I}\mathrm{nd}(ssnt_{j+1}) \notin S_j$ and $\mathbb{I}\mathrm{nd}(ssnt_{j+1}) \in S_{j+1}$, it follows by the fact that for all $j \geq i$, $S_j \subseteq S_{j+1}$ that for all $j \geq i$, for all $k < j$, $\mathbb{I}\mathrm{nd}(ssnt_k) \neq \mathbb{I}\mathrm{nd}(ssnt_j)$. So each $\mathbb{I}\mathrm{nd}(ssnt_j)$ in the sequence $\mathbb{I}\mathrm{nd}(ssnt_i)$, $\mathbb{I}\mathrm{nd}(ssnt_{i+1}), \ldots, \mathbb{I}\mathrm{nd}(ssnt_j), \ldots$ is distinct. Hence, there are infinitely many natural numbers n such that $n < \mathbb{I}\mathrm{nd}(\pm snt)$. But for any given natural number there are only finitely many natural numbers less than it. Contradiction. So the claim holds.

<u>Induction step</u>: $N(i, \mathbb{I}\mathrm{nd}(\pm snt)) > 0$. Assume as the hypothesis of induction that for all Set_i in $\langle Set_i \rangle$ and any signed nonatomic sentence $ssnt \in Set_i$ such that $\mathbb{I}\mathrm{nd}(ssnt) \notin S_i$, if $N(i, \mathbb{I}\mathrm{nd}(ssnt)) < N(i, \mathbb{I}\mathrm{nd}(\pm snt))$, then there is a $j \geq i$ such that $ssnt$ is the first signed nonatomic sentence in Set_j such that $\mathbb{I}\mathrm{nd}(ssnt) \notin S_j$. We show that there is a $j \geq i$ such that $\pm snt$ is the first signed nonatomic sentence in Set_j such that $\mathbb{I}\mathrm{nd}(ssnt) \notin S_j$. For suppose the opposite. Suppose, that is, that for all $j \geq i$, there is a signed nonatomic sentence $ssnt \in Set_j$ such that $\mathbb{I}\mathrm{nd}(ssnt) < \mathbb{I}\mathrm{nd}(\pm snt)$ and $\mathbb{I}\mathrm{nd}(ssnt) \notin S_i$. For any i, let $ssnt_i$ be the first signed nonatomic sentence in Set_i such that $\mathbb{I}\mathrm{nd}(ssnt_i) \notin S_i$. Then, for all $j \geq i$, $\mathbb{I}\mathrm{nd}(ssnt_j) < \mathbb{I}\mathrm{nd}(\pm snt)$. Now, for all $j \geq i$, S_{j+1} is obtained from S_j in $\langle S_i \rangle$ by adding a $\mathbb{I}\mathrm{nd}(ssnt_j)$ to S_j (per one of clauses 1), 2), 3), 4), 5) of 4.2.3.1.). So there is an infinite sequence of natural numbers $\mathbb{I}\mathrm{nd}(ssnt_i)$, $\mathbb{I}\mathrm{nd}(ssnt_{i+1}), \ldots, \mathbb{I}\mathrm{nd}(ssnt_j), \ldots$ such that for all $j \geq i$, $\mathbb{I}\mathrm{nd}(ssnt_j) < \mathbb{I}\mathrm{nd}(\pm snt)$ and

$\mathbb{I}\mathrm{nd}(ssnt_{j+1}) \notin S_j$ and $\mathbb{I}\mathrm{nd}(ssnt_{j+1}) \in S_{j+1}$. In case for all $j \geq i$, $ssnt_j$ is not, for some formula F, of the form $+(x)F$, it follows that for all $j \geq i$, $S_j \subseteq S_{j+1}$. In this case, then, for all $j \geq i$, for all $k < j$, $\mathbb{I}\mathrm{nd}(ssnt_k) \neq \mathbb{I}\mathrm{nd}(ssnt_j)$. Hence, each $\mathbb{I}\mathrm{nd}(ssnt_j)$ in the sequence $\mathbb{I}\mathrm{nd}(ssnt_i)$, $\mathbb{I}\mathrm{nd}(ssnt_{i+1})$, . . . , $\mathbb{I}\mathrm{nd}(ssnt_j)$, . . . is distinct, which contradicts the fact that there are only finitely many distinct natural numbers n such that $n < \mathbb{I}\mathrm{nd}(\pm snt)$. So let $v \geq i$ be a number such that $ssnt_v$ is the signed sentence in the sequence $\mathbb{I}\mathrm{nd}(ssnt_i)$, $\mathbb{I}\mathrm{nd}(ssnt_{i+1})$, . . . , $\mathbb{I}\mathrm{nd}(ssnt_j)$, . . . of the form $+(x)F$ such that for no j, $i \leq j < v$, is $ssnt_j$ of the form, for some formula G, $+(x)G$. It follows that for all j, $i \leq j < v$, $S_j \subseteq S_{j+1}$, and so $S_i \subseteq S_v$. Hence, $\mathrm{N}(i, \mathbb{I}\mathrm{nd}(\pm snt)) \geq \mathrm{N}(v, \mathbb{I}\mathrm{nd}(\pm snt))$. Now we know that $\mathbb{I}\mathrm{nd}(ssnt_v) < \mathbb{I}\mathrm{nd}(\pm snt)$ and $\mathbb{I}\mathrm{nd}(ssnt_v) \notin S_v$. By clause 2) of 4.2.3.1., $S_{v+1} = (S_v - \mathbb{R}(v)) \cup \{\mathbb{I}\mathrm{nd}(ssnt_v)\}$. Since $\mathbb{R}(v)$ contains no indices of signed sentences of the form $+(x)F$, it follows that $\mathrm{N}(v + 1, \mathbb{I}\mathrm{nd}(\pm snt)) = \mathrm{N}(v, \mathbb{I}\mathrm{nd}(\pm snt)) - 1$. Hence, $\mathrm{N}(v + 1, \mathbb{I}\mathrm{nd}(\pm snt)) < \mathrm{N}(i, \mathbb{I}\mathrm{nd}(\pm snt))$. So by the hypothesis of induction, there is a $j \geq v + 1$ such that $\pm snt$ is the first signed nonatomic sentence in Set_{v+1} such that $\mathbb{I}\mathrm{nd}(ssnt) \notin S_{v+1}$. Since $v + 1 > i$, it follows that there is a $j \geq i$ such that $\pm snt$ is the first signed nonatomic sentence in Set_j such that $\mathbb{I}\mathrm{nd}(ssnt) \notin S_j$. So the claim holds.

This completes the proof of lemma 4.2.3.3. □

Lemma 4.2.3.4: Let Seq be any sequent which is not strictly 2-derivable. Let $\langle Set_i \rangle$ be one of the infinite reduction sequences of Seq whose existence is asserted by lemma 4.2.3.2. and let Set_ω be the union set of $\langle Set_i \rangle$, i.e., $Set_\omega = \bigcup Set_i$,$0 \leq i < \omega$. Then, Set_ω is downward closed.

Proof of 4.2.3.4.: We show that Set_ω satisfies all of conditions 1) - 10) inclusive of definition 4.1.1.. Let $\pm snt$ be any member of Set_ω. Then,

1) For no atomic sentence $asnt$ does Set_ω contain both of $\pm asnt$.

Proof: Suppose that Set_ω contain both of $\pm asnt$ for some atomic sentence $asnt$. Since $Set_\omega = \bigcup Set_i$,$0 \leq i < \omega$, there are $i, j < \omega$ such that $+asnt \in Set_i$ and $-asnt \in Set_j$. Asssume, without loss of generality, that $i \geq j$. Then both of $\pm \boldsymbol{asnt} \in \boldsymbol{Set}_i$ and so $\{+\boldsymbol{asnt}, -\boldsymbol{asnt}\} \subseteq \boldsymbol{Set}_i$. Since $\{+\boldsymbol{asnt}, -\boldsymbol{asnt}\}$ is an axiom, it is strictly 2-derivable. Since $\boldsymbol{Set}_i$ has a strictly 2-derivable finite subset, by definition $\boldsymbol{Set}_i$ is strictly 2-derivable. This contradicts the claim of lemma 4.2.3.2. that no $\boldsymbol{Set}_i$ in $\langle \boldsymbol{Set}_i \rangle$ is strictly 2-derivable. So $\boldsymbol{Set}_\omega$ satisfies condition 1) of 4.1.1.. □

2) For no $\boldsymbol{t}$ in $\delta(Par)$ does $\boldsymbol{Set}_\omega$ contain the signed sentence $+(\boldsymbol{t} = \boldsymbol{t})$.

Proof: Suppose that $\boldsymbol{Set}_\omega$ contains the signed sentence $+(\boldsymbol{t} = \boldsymbol{t})$ for some basic constant term $\boldsymbol{t}$ in $\delta(Par)$. Since $\boldsymbol{Set}_\omega = \bigcup \boldsymbol{Set}_i$,$0 \leq i < \omega$, there is $i < \omega$ such that $+(\boldsymbol{t} = \boldsymbol{t}) \in \boldsymbol{Set}_i$. Since $\{+(\boldsymbol{t} = \boldsymbol{t})\}$ is an axiom, it is strictly 2-derivable. Since $\boldsymbol{Set}_i$ has a strictly 2-derivable finite subset, by definition $\boldsymbol{Set}_i$ is strictly 2-derivable. This contradicts the claim of lemma 4.2.3.2. that no $\boldsymbol{Set}_i$ in $\langle \boldsymbol{Set}_i \rangle$ is strictly 2-derivable. So $\boldsymbol{Set}_\omega$ satisfies condition 2) of 4.1.1.. □

3) For no $\boldsymbol{r}, \boldsymbol{s}$ in $\delta(Par)$ does $\boldsymbol{Set}$ contain both of $+(\boldsymbol{r} = \boldsymbol{s})$, $-(\boldsymbol{s} = \boldsymbol{r})$.

Proof: Suppose there are $\boldsymbol{r}, \boldsymbol{s}$ in $\delta(Par)$ such that $\boldsymbol{Set}_\omega$ contains both of $+(\boldsymbol{r} = \boldsymbol{s})$, $-(\boldsymbol{s} = \boldsymbol{r})$. Since $\boldsymbol{Set}_\omega = \bigcup \boldsymbol{Set}_i$,$0 \leq i < \omega$, there is $i, j < \omega$ such that $+(\boldsymbol{r} = \boldsymbol{s}) \in \boldsymbol{Set}_i$ and $-(\boldsymbol{s} = \boldsymbol{r}) \in \boldsymbol{Set}_j$. Assume without loss of generality that $i \geq j$. Then $+(\boldsymbol{r} = \boldsymbol{s}), -(\boldsymbol{s} = \boldsymbol{r}) \in \boldsymbol{Set}_i$. Since the sequents $\{+(\boldsymbol{s} = \boldsymbol{s})\}$, $\{-(\boldsymbol{r} = \boldsymbol{s}), +(\boldsymbol{r} = \boldsymbol{s})\}$ are both axioms, the following is a legitimate Pld2 strict derivation of the sequent $\{+(\boldsymbol{r} = \boldsymbol{s}), -(\boldsymbol{s} = \boldsymbol{r})\}$:

$$\frac{\{+(\boldsymbol{s} = \boldsymbol{s})\} \qquad \{+(\boldsymbol{r} = \boldsymbol{s}), -(\boldsymbol{r} = \boldsymbol{s})\}}{\{+(\boldsymbol{r} = \boldsymbol{s}), -(\boldsymbol{s} = \boldsymbol{r})\}}$$

Thus $\{+(\boldsymbol{r} = \boldsymbol{s}), -(\boldsymbol{s} = \boldsymbol{r})\}$ is a strictly 2-derivable sequent. But by lemma 4.2.3.2. no $\boldsymbol{Set}_i, \boldsymbol{Set}_i$ in $\langle \boldsymbol{Set}_i \rangle$ is strictly 2-derivable. So $\{+(\boldsymbol{r} = \boldsymbol{s}), -(\boldsymbol{s} = \boldsymbol{r})\}$ is not a subset of $\boldsymbol{Set}_i$ for any i. So $\boldsymbol{Set}_\omega$ satisfies condition 3) of 4.1.1.. □

4) For no $\boldsymbol{r}, \boldsymbol{s}$ in $\delta(Par)$ and atomic formula $\boldsymbol{afm}$ does $\boldsymbol{Set}$ contain all of $-(\boldsymbol{r} = \boldsymbol{s})$, $-[\boldsymbol{r} / \boldsymbol{v}]\boldsymbol{afm}$, $+[\boldsymbol{s} / \boldsymbol{v}]\boldsymbol{afm}$.

Proof: Suppose that for some $\boldsymbol{r}, \boldsymbol{s}$ in $\delta(Par)$ and atomic formula $\boldsymbol{afm}$, $\boldsymbol{Set}$ contain all of $-(\boldsymbol{r} = \boldsymbol{s})$,

-$[r / v]afm$, +$[s / v]afm$. Since $Set_\omega = \bigcup Set_i$, $0 \leq i < \omega$, there are $i, j, k < \omega$ such that -$(r = s)$ $\in Set_i$, -$[r / v]afm \in Set_j$, and +$[s / v]afm \in Set_k$. Assume, without loss of generality, that $i \geq j \geq k$. Then all of -$(r = s)$, -$[r / v]afm$, +$[s / v]afm \in Set_i$ and so $\{$-$(r = s)$, -$[r / v]afm$, +$[s / v]afm\} \subseteq Set_i$. We show that $\{$-$(r = s)$, -$[r / v]afm$, +$[s / v]afm\}$ is a strictly 2-derivable sequent. The following is a legitimate application of the identity rule 1.2.2.7.:

$$\frac{\{-[r / v]afm\ ,\ +[r / v]afm\} \qquad\qquad \{-[s / v]afm\ ,\ +[s / v]afm\}}{\{-(r = s)\ ,\ -[r / v]afm\ ,\ +[s / v]afm\}}$$

Since both of $\{$-$[r / v]afm$, +$[r / v]afm\}$, $\{$-$[s / v]afm$, +$[s / v]afm\}$ are axioms, it follows that $\{$-$(r = s)$, -$[r / v]afm$, +$[s / v]afm\}$ is a strictly 2-derivable sequent. Since Set_i has a strictly 2-derivable finite subset, by definition Set_i is strictly 2-derivable. This contradicts the claim of lemma 4.2.3.2. that no Set_i in $\langle Set_i\rangle$ is strictly 2-derivable. So Set_ω satisfies condition 4) of 4.1.1.. □

Since $\pm snt \in Set_\omega = \bigcup Set_i$,$0 \leq i < \omega$, and $\langle Set_i\rangle$ is well-ordered under $\subseteq$, it follows that $\pm snt \in Set_i$ for some least i, $0 \leq i < \omega$. We now show that Set_ω satisfies conditons 5) - 10) of definition 4.1.1.:

5) If $\pm snt$ is respectively of the form $\pm\neg$A, for some sentence A, then Set_ω contains respectively $\mp$A.

Proof: Assume that $\pm snt$ is respectively of the form $\pm\neg$A. Since for all $j < i$, $\pm\neg$A $\notin Set_j$, it follows by definition 4.2.3.1. of $\langle Set_i\rangle$ that respectively Ind($\pm\neg$A) $\notin$ S_i . Since $\pm\neg$A $\in Set_i$ and Ind($\pm\neg$A) $\notin$ S_i , by lemma 4.2.3.3., there is a $j \geq i$ such that $\pm\neg$A is the first signed sentence in Set_j such that respectively Ind($\pm\neg$A) $\notin$ S_j . Hence, by clause 1) of 4.2.3.1., Set_{j+1} is an S_i -reduction of Set_j of type 1) and so $Set_{j+1} \cong Set_j \cup \{\mp\text{A}\}$, respectively. Since $Set_{j+1} \subseteq Set_\omega$, it follows that respectively $\mp$A $\in Set_\omega$. So Set_ω satisfies condition 5) of 4.1.1. □

6) If $\pm snt$ is respectively of the form $\pm$(A $\wedge$ B), for some sentences A, B, then Set_ω contains one of +A, +B, respectively, both of -A, -B.

Proof: Assume that $\pm snt$ is respectively of the form $\pm(\mathbf{A} \wedge \mathbf{B})$. Since for all $j < i$, $\pm(\mathbf{A} \wedge \mathbf{B}) \notin Set_j$, it follows by definition 4.2.3.1. of $\langle Set_i \rangle$ that respectively $\mathbb{Ind}(\pm(\mathbf{A} \wedge \mathbf{B})) \notin S_i$. Since $\pm(\mathbf{A} \wedge \mathbf{B}) \in Set_i$ and $\mathbb{Ind}(\pm(\mathbf{A} \wedge \mathbf{B})) \notin S_i$, by lemma 4.2.3.3., there is a $j \geq i$ such that $\pm(\mathbf{A} \wedge \mathbf{B})$ is the first signed sentence in Set_j such that respectively $\mathbb{Ind}(\pm(\mathbf{A} \wedge \mathbf{B})) \notin S_j$. Hence, by clause 1) of 4.2.3.1., Set_{j+1} is an S_j-reduction of Set_j of type 2) and so Set_{j+1} contains one of, respectively, both of, **A**, **B**. Since $Set_{j+1} \subseteq Set_\omega$, it follows that Set_ω contains one of, respectively, both of, **A**, **B**. So Set_ω satisfies condition 6) of 4.1.1.. □

7) If $\pm snt$ is respectively of the form $\pm(\mathbf{A} \rightarrow \mathbf{B})$, for some sentences **A**, **B**, then Set_ω contains both of -**A**, +**B**, respectively, one of +**A**, -**B**.

Proof: This case is similar to condition 6) above. □

8) If, for some formula F, $\pm snt$ is respectively of the form $\pm(x)F$, then Set_ω contains $+[p / x]F$ for some p in $\mathcal{P}(Set_\omega)$, respectively, $-[t / x]F$ for every t in $\delta(\mathcal{P}(Set_\omega))$.

Proof: There are two cases:

i) $\pm snt$ is of the form $+(x)F$. Since for all $j < i$, $+(x)F \notin Set_j$, it follows by definition 4.2.3.1. of $\langle Set_i \rangle$ that $\mathbb{Ind}(+(x)F) \notin S_i$. Since $+(x)F \in Set_i$ and $\mathbb{Ind}(+(x)F) \notin S_i$, by lemma 4.2.3.3., there is a $j \geq i$ such that $+(x)F$ is the first signed sentence in Set_j such that $\mathbb{Ind}(+(x)F) \notin S_j$. Hence, by clause 2) of 4.2.3.1., $Set_{j+1} \cong Set_j \cup \{+[p / x]F\}$, where p is in $\mathcal{P}(Set_j) \subseteq \mathcal{P}(Set_\omega)$. Since $Set_{j+1} \subseteq Set_\omega$, it follows that $+[p / x]F \in Set_\omega$. So Set_ω satisfies condition 8).

ii) $\pm snt$ is of the form $-(x)F$. We want to show that $-[t / x]F \in Set_\omega$ for every t in $\delta(\mathcal{P}(Set_\omega))$. So let s be any term in $\delta(\mathcal{P}(Set_\omega))$. Since the sequence $\langle \mathcal{P}(Set_i) \rangle$ is well ordered under $\subseteq$, there is a least i such that $s \in \delta(\mathcal{P}(Set_i)$; let that least i be i'. It follows that $\mathcal{P}(Set_{i'-1}) \subseteq \mathcal{P}(Set_{i'})$ but $\mathcal{P}(Set_{i'}) \nsubseteq \mathcal{P}(Set_{i'-1})$. Since for all i, clause 2) is the only clause of 4.2.3.1. that adds new parameters to a $\mathcal{P}(Set_i)$, it follows that $Set_{i'}$ is obtained from $Set_{i'-1}$ in $\langle Set_i \rangle$ by clause 2). Hence, $S_{i'} = \varnothing$. There are two subcases:

a) $i' < i$. Since for all $j < i$, $-(x)F \notin Set_j$, it follows by definition 4.2.3.1. of $\langle Set_i \rangle$ that $\mathrm{Ind}(-(x)F) \notin S_i$. Then, since $-(x)F \in Set_i$ and $\mathrm{Ind}(-(x)F) \notin S_i$, by lemma 4.2.3.3., there is a $j \geq i$ such that $-(x)F$ is the first signed sentence in Set_j such that $\mathrm{Ind}(+(x)F) \notin S_j$. Then, by clause 3) of 4.2.3.1., $Set_{j+1} \cong Set_j \cup \{+[t / x]F : t \in \delta(\mathcal{P}(Set_j))\}$. Since $j \geq i > i'$, $\mathcal{P}(Set_{i'}) \subseteq \mathcal{P}(Set_j)$, and so $\delta(\mathcal{P}(Set_{i'})) \subseteq \delta(\mathcal{P}(Set_j))$. Hence, $-[s / x]F \in Set_{j+1} \subseteq Set_\omega$. So Set_ω satisfies condition 8).

b) $i \leq i'$. Since $Set_i \subseteq Set_{i'}$, $-(x)F \in Set_{i'}$. Then by lemma 4.2.3.3., since $\mathrm{Ind}(-(x)F) \notin S_{i'}$, there is a $j \geq i'$ such that $-(x)F$ is the first signed sentence in Set_j such that $\mathrm{Ind}(+(x)F) \notin S_j$. Then, by clause 3) of 4.2.3.1., $Set_{j+1} \cong Set_j \cup \{+[t / x]F : t \in \delta(\mathcal{P}(Set_j))\}$. Since $j \geq i'$, $\mathcal{P}(Set_{i'}) \subseteq \mathcal{P}(Set_j)$, and so $\delta(\mathcal{P}(Set_{i'})) \subseteq \delta(\mathcal{P}(Set_j))$. Hence, $-[s / x]F \in Set_{j+1} \subseteq Set_\omega$. So Set_ω satisfies condition 8).

In all cases, then, Set_ω satisfies condition 8). □

9) If $\pm snt$ is a signed nonatomic elementary sentence of the form $+[\iota x.\Phi / u]afm$ for some constant description term $\iota x.\Phi$ and atomic formula afm, then Set_ω satisfies one of the following conditions:

i) Set_ω contains the signed sentence $+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)$ for some variable v.

ii) Set_ω contains the signed sentence $+[t / x](\Phi \wedge [x / u]afm)$ for every $t \in \delta(\mathcal{P}(Set))$.

<u>Proof:</u> Assume that $\pm snt$ is of the form $+[\iota x.\Phi / u]afm$. Since for all $j < i$, $+[\iota x.\Phi / u]afm \notin Set_j$, it follows by definition 4.2.3.1. of $\langle Seq_i \rangle$ that $\mathrm{Ind}(+[\iota x.\Phi / u]afm) \notin S_i$. Since $+[\iota x.\Phi / u]afm \in Set_i$ and $\mathrm{Ind}(+[\iota x.\Phi / u]afm) \notin S_i$, by lemma 4.2.3.3., there is a $j \geq i$ such that $+[\iota x.\Phi / u]afm$ is the first signed sentence in Set_j such that $\mathrm{Ind}(+[\iota x.\Phi / u]afm) \notin S_j$. Let j' be the least such j, i.e., let j' be such that $+[\iota x.\Phi / u]afm$ is the first signed sentence in $Set_{j'}$ such that $\mathrm{Ind}(+[\iota x.\Phi / u]afm) \notin S_{j'}$ and for all $k < j'$, it is not the case that $+[\iota x.\Phi / u]afm$ is the first signed sentence in Set_k such that $\mathrm{Ind}(+[\iota x.\Phi / u]afm) \notin S_k$. We observe that by definiton 4.2.3.1., for all i and signed sentences $ssnt$, $\mathrm{Ind}(ssnt) \in \mathrm{D1}_i$ only if there is a $j < i$ such

that $ssnt$ is the first signed sentence in Set_j such that $\mathbb{I}\mathrm{nd}(ssnt) \notin S_j$. Hence, $\mathbb{I}\mathrm{nd}(+[\iota x.\Phi / u]afm) \notin D1_{j'}$. Hence, by clause 4) of 4.2.3.1., there are two cases:

i) $Set_{j'+1}$ is obtained from $Set_{j'}$ by rule a) of clause 4). Then, $Set_{j'+1} \cong Set_{j'} \cup \{+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)\}$ for some variable v. Since $Set_{j'+1} \subseteq Set_\omega$, it follows that $+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v) \in Set_\omega$. So Set_ω satisfies i) of condition 9).

ii) $Set_{j'+1}$ is obtained from $Set_{j'}$ by rule b) of clause 4). Then, $D1_{j'+1} = D1_{j'} \cup \{\mathbb{I}\mathrm{nd}(+[\iota x.\Phi / u]afm)\}$. We want to show that $+[t / x](\Phi \wedge [x / u]afm) \in Set_\omega$ for every t in $\delta(\mathcal{P}(Set_\omega))$. So let s be any term in $\delta(\mathcal{P}(Set_\omega))$. Since the sequence $\langle \mathcal{P}(Set_i) \rangle$ is well ordered under $\subseteq$, there is a least i such that $s \in \delta(\mathcal{P}(Set_i))$; let that least i be i'. It follows that $\mathcal{P}(Set_{i'-1}) \subseteq \mathcal{P}(Set_{i'})$ but $\mathcal{P}(Set_{i'}) \nsubseteq \mathcal{P}(Set_{i'-1})$. Since for all i, clause 2) is the only clause of 4.2.3.1. that adds new parameters to a $\mathcal{P}(Set_i)$, it follows that $Set_{i'}$ is obtained from $Set_{i'-1}$ in $\langle Set_i \rangle$ by clause 2). Hence, $S_{i'} = \varnothing$. There are two subcases:

a) $j' + 1 > i'$. We have that $+[\iota x.\Phi / u]afm$ is the first signed sentence in $Set_{j'}$ such that $\mathbb{I}\mathrm{nd}(+[\iota x.\Phi / u]afm) \notin S_{j'}$ and that $Set_{j'+1}$ is obtained from $Set_{j'}$ by rule b) of clause 4). Hence, $Set_{j'+1} \cong Set_{j'} \cup \{+[t / x](\Phi \wedge [x / u]afm) : t \in \delta(\mathcal{P}(Set_{j'}))\}$. Clearly, since $j' \geq i'$ and $s \in \delta(\mathcal{P}(Set_{i'}))$, $s \in \delta(\mathcal{P}(Set_{j'}))$. So $+[s / x](\Phi \wedge [x / u]afm) \in Set_{j'+1} \subseteq Set_\omega$. So Set_ω satisfies ii) of condition 9).

b) $i' \geq j' + 1$. Since $j' \geq i$, it follows that $i' > i$, so we know that $+[\iota x.\Phi / u]afm \in Set_{i'}$. Since $S_{i'} = \varnothing$, $\mathbb{I}\mathrm{nd}(+[\iota x.\Phi / u]afm) \notin S_{i'}$, so by lemma 4.2.3.3. there is a $j \geq i'$ such that $+[\iota x.\Phi / u]afm$ is the first signed sentence in Set_j such that $\mathbb{I}\mathrm{nd}(+[\iota x.\Phi / u]afm) \notin S_j$. Now, since $\mathbb{I}\mathrm{nd}(+[\iota x.\Phi / u]afm) \in D1_{j'+1}$ and $j \geq i' \geq j' + 1$, it follows by definition 4.2.3.1. that $\mathbb{I}\mathrm{nd}(+[\iota x.\Phi / u]afm) \in D1_j$. Hence, Set_{j+1} is not obtained from Set_j by rule a) of clause 4) of 4.2.3.1., but rather is obtained from Set_j by rule b) of clause 4). Hence, $Set_{j+1} \cong Set_j \cup \{+[t / x](\Phi \wedge [x / u]afm) : t \in \delta(\mathcal{P}(Set_j))\}$. Clearly, since $j \geq i'$ and $s \in \delta(\mathcal{P}(Set_{i'}))$, $s \in \delta(\mathcal{P}(Set_j))$. So $+[s / x](\Phi \wedge [x / u]afm) \in Set_{j+1} \subseteq Set_\omega$. So Set_ω satisfies ii) of condition 9).

In all cases, then, Set_ω satisfies condition 9) of 4.1.1.. □

10) If $\pm snt$ is a signed nonatomic elementary sentence of the form $-[\iota x.\Phi / u]afm$ for some constant description term $\iota x.\Phi$ and atomic formula afm, then Set_ω satisfies one of the following conditions:

i) Set_ω contains the signed sentence $+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)$ for some variable v.

ii) Set_ω contains the signed sentence $+[t / x](\Phi \wedge \neg[x / u]afm)$ for every $t \in \delta(\wp(Set))$.

Proof: The proof of 10) is similar to the proof of 9) above. □

This completes the proof of lemma 4.2.3.4.. Lemma 4.2.3. is established by lemmas 4.2.3.2. and 4.2.3.4. as follows: Let Seq be any sequent which is not strictly 2-derivable. By 4.2.3.2., Seq has an infinite reduction sequence $\langle Set_i \rangle$ such that $Seq = Set_0$ and no Set_i in $\langle Set_i \rangle$ is strictly 2-derivable. Then by 4.2.3.4., $Set_\omega = \bigcup Set_i$,$0 \leq i < \omega$, is downward closed. Since $Seq \subseteq Set_\omega$, it follows that Seq has a downward closed extension Set_ω. We now restate and prove lemma 4.2.4.:

Lemma 4.2.4.: Every downward closed set Set of signed sentences determines a base $bse(Set)$ such that $Set \cap Cl(bse(Set)) = \emptyset$.

Proof of 4.2.4.: Let Set be a downward closed set of signed sentences. Let F be any formula. We recursively define a downward closed extension Set^* of Set as follows:

0) $Set_0 = Set$;

1) Set_{i+1} is obtained from Set_i by adding to Set_i all signed sentences of the form $\pm[p / x]F$ such that respectively $\pm[\mathbb{P}(1) / x]F$ is in Set_i , where p is the first (relative to the enumeration $\mathbb{P}$) parameter that does not occur in any member of Set_i . In other words $Set_{i+1} = Set_i \cup \{\pm[\mathbb{P}(i) / x]F : \pm[\mathbb{P}(1) / x]F \in Set_i$ and $\mathbb{P}(i) \in Par - \wp(Set_i)$ and for all

$j < i$, $\mathbb{P}(j) \in \wp(Set_i)\}$.

2) $Set^* = \bigcup Set_i$, $0 \leq i < \omega$.

Lemma 4.2.4.1.: Let Set be a downward closed set of signed sentences, and let Set^* be as defined above. Then Set^* is downward closed.

Proof of 4.2.4.1.: We show by induction on i in the recursive definition of Set^* that every Set_i is downward closed. Let $\pm snt$ be any member of Set^*. Then,

Base step: $i = 0$. By assumption, $Set_0 = Set$ is downward closed. So the claim holds.

Induction step: $i > 0$. Assume that for all $j < i$, Set_j is downward closed. We show that Set_i satisfies all of conditions 1) -10) inclusive of definition 4.1.1:

1) For no atomic sentence $asnt$ does Set_i contain both of $\pm asnt$.

Proof: Assume that Set_i contains both of $\pm asnt$ for some atomic sentence $asnt$. Now, $asnt$ can be written as $[p / x]afm$ for some atomic formula afm containing no occurrence of p, where p is the first parameter in $Par - \wp(Set_{i-1})$. Clearly, then, since both of $\pm[p / x]afm$ are in Set_i and afm contains no occurrence of p, it follows by clause 1) of the recursive definition of Set^* that Set_{i-1} contains both of $\pm[\mathbb{P}(1) / x]afm$. This contradicts the hypothesis of induction.

2) For no t in $\delta\ (\wp(Set_i))$ does Set_i contain the signed sentence $+(t = t)$.

Proof: Assume that Set_i contains the signed sentence $+(t = t)$ for some t in $\delta(\wp(Set_i))$. Now the signed sentence $+(t = t)$ can be written as $+[p\ / x](s = s)$ for some basic term s containing no occurrence of p, where p is the first parameter in $Par - \wp(Set_{i-1})$. Since $+[p / x](s = s) \in Set_i$ and s contains no occurrence of p, it follows by clause 1) of the recursive definition of Set^* that $+[\mathbb{P}(1) / x](s = s) \in Set_{i-1}$. But clearly, the signed sentence $+[\mathbb{P}(1) / x](s = s)$ is of the form $+(r = r)$ for some r in $\delta\ (\wp(Set_i))$. So for some r in $\delta\ (\wp(Set_i))$, Set_{i-1} contains the signed sentence $+(r = r)$. This contradicts the hypothesis of induction.

3) For no r, s in $\delta(\mathcal{P}(Set_i))$ does Set_i contain both of $+(r = s)$, $-(s = r)$.

Proof: The proof that Set_i satisfies condition 3) is similar to the proof above that Set_i satisfies condition 1).

4) For no r, s in $\delta(\mathcal{P}(Set_i))$ atomic formula afm , does Set_i contain all of $-(r = s)$, $-[r / v]afm$, $+[s / v]afm$.

Proof: Assume that for some r, s in $\delta(\mathcal{P}(Set_i))$ and atomic formula afm, Set_i contains all of $-(r = s)$, $-[r / v]afm$, $+[s / v]afm$. Now $-(r = s)$, $-[r / v]afm$, $+[s / v]afm$ can be respectively writtten as $-[p / x](t_1 = t_2)$, $-[p / x]$ $[[p / x]t_1 / v]afm'$, $+[p / x][[p / x]t_2 / v]afm'$ for some basic terms t_1 , t_2 and atomic formula afm' such that none of t_1 , t_2 , afm' contain an occurrence of p, where p is the first parameter in $Par - \mathcal{P}(Set_{i-1})$. Since all of $-[p / x](t_1 = t_2)$, $-[p / x][[p / x]t_1 / v]afm'$, $+[p / x][[p / x]t_1 / v]afm'$ are in Set_i , and none of t_1 , t_2, afm' contain an occurrence of p, it follows all of $-[\mathbb{P}(1) / x](t_1 = t_2) \cong -([\mathbb{P}(1) / x]t_1 = [\mathbb{P}(1) / x]t_2)$, $-[\mathbb{P}(1) / x][[\mathbb{P}(1) / x]t_1 / v]afm'$, $+[\mathbb{P}(1) / x][[\mathbb{P}(1) / x]t_1 / v]afm'$ are in Set_{i-1} . This contradicts the hypothesis of induction.

5) If $\pm snt$ is respectively of the form $\pm\neg$A, then Set_i contains respectively $\mp$A.

Proof: Assume that $\pm snt$ is respectively of the form $\pm\neg$A. Now $\neg$A is of the form $[p / x]\neg F$ for some formula F containing no occurrence of p, where p is the first parameter in $Par - \mathcal{P}(Set_{i-1})$. Since respectively $\pm[p / x]\neg F$ is in Set_i and F contains no occurrence of p, it follows that the signed sentence respectively $\pm[\mathbb{P}(1) / x]\neg F$ is in Set_{i-1} . Set_{i-1} is downward closed by the hypothesis of induction, so by condition 5) of 4.1.1., Set_{i-1} contains respectively $\mp[\mathbb{P}(1) / x]F$. Then by clause 1) of the recursive definition of Set^*, Set_i contains respectively $\mp[p / x]F \cong \mp$A. So the claim holds.

6) If $\pm snt$ is respectively of the form $\pm$(A $\wedge$ B), then Set contains one of +A, +B, respectively, both of -A, -B.

Proof: Assume that $\pm snt$ is respectively of the form $\pm(\mathbf{A} \wedge \mathbf{B})$ for some signed sentences **A**, **B**. Now **A**, **B** are respectively of the form $[p / x]F$, $[p / x]G$, for some formulas F, G containing no occurrence of p, where p is the first parameter in $Par - \mathscr{P}(Set_{i-1})$. Since respectively $\pm(\mathbf{A} \wedge \mathbf{B}) \cong \pm([p / x]F \wedge [p / x]G)$ is in Set_i and neither F nor G contains an occurrence of p, it follows by clause 1) of the recursive definition of Set^* that Set_{i-1} contains respectively $\pm([\mathbb{P}(1) / x]F \wedge [\mathbb{P}(1) / x]G)$. By the hypothesis of induction, Set_{i-1} is downward closed, so by condition 6) of 4.1.1., Set_{i-1} contains one of $+[\mathbb{P}(1) / x]F$, $+[\mathbb{P}(1) / x]G$, respectively, both of $-[\mathbb{P}(1) / x]F$, $-[\mathbb{P}(1) / x]G$. Then by clause 1) of the recursive definition of Set^*, Set_i contains one of $+[p / x]F$, $+[p / x]G$, respectively, both of $-[p / x]F$, $-[p / x]G$. So Set_i contains one of +**A**, +**B**, respectively, both of -**A**, -**B**. So the claim holds.

7) If $\pm snt$ is respectively of the form $\pm(\mathbf{A} \rightarrow \mathbf{B})$, then Set_i contains both of -**A**, +**B**, respectively, one of +**A**, -**B**.

Proof: The proof that Set_i satisfies condition 7) is similar to the proof above that Set_i satisfies condition 6).

8) If $\pm snt$ is respectively of the form $\pm(x)F$, then Set_i contains $+[p / x]F$ for some p in $\mathscr{P}(Set_i)$, respectively, $-[t / x]F$ for every t in $\delta(\mathscr{P}(Set_i))$.

Proof: Assume that $\pm snt$ is respectively of the form $\pm(x)F$, for some formula F. Now, F is of the form $[p / y]G$, for some formula G containing no occurrence of p, where p is the first parameter in $Par - \mathscr{P}(Set_{i-1})$. We treate the two cases seperately:

i) $\pm snt$ is of the form $+(x)F$. Since $+(x)F \cong +(x)[p / y]G \cong +[p / y](x)G$ is in Set_i and G contains no occurrence of p, it follows by clause 1) of the recursive definition of Set^* that Set_{i-1} contains $+[\mathbb{P}(1) / y](x)G \cong +(x)[\mathbb{P}(1) / y]G$. By the hypothesis of induction, Set_{i-1} is downward closed, so by condition 7) of 4.1.1., Set_{i-1} contains $+[q / x][\mathbb{P}(1) / y]G \cong +[\mathbb{P}(1) / y][q / x]G$ for some q in $\mathscr{P}(Set_{i-1})$. Then by clause 1) of the recursive definition of Set^*, Set_i contains $+[p / y][q / x]G \cong +[q / x][p / y]G \cong +[q / x]F$ for some q in $\mathscr{P}(Set_{i-1}) \subseteq \mathscr{P}(Set_i)$. So the claim holds.

ii) $\pm snt$ is of the form $-(x)F$. Since $-(x)F \cong -(x)[p / y]G \cong -[p / y](x)G$ is in Set_i and G contains no occurrence of p, it follows by clause 1) of the recursive definition of Set^* that Set_{i-1} contains $-[\mathbb{P}(1) / y](x)G \cong +(x)[\mathbb{P}(1) / y]G$. By the hypothesis of induction, Set_{i-1} is downward closed, so by condition 8) of 4.1.1., Set_{i-1} contains $-[t / x][\mathbb{P}(1) / y]G$. Let r be any term in $\delta(\mathcal{P}(Set_i))$. Now, r is of the form $[p / y]s$ for some variable y and term s such that s contains no occurrence of p. Then $[\mathbb{P}(1) / y]s$ is a term in $\delta(\mathcal{P}(Set_{i-1}))$. So Set_{i-1} contains $-[[\mathbb{P}(1) / y]s / x][\mathbb{P}(1) / y]G$. Hence, by clause 1) of the recursive definition of Set^*, since neither G nor s contain an occurrence of p , Set_i contains $-[[p / y]s / x][p / y]G \cong -[[p / y]s / x]F \cong -[r / x]F$, which is what we want to show.

In both cases, then, the claim holds.

9) If $\pm snt$ is a signed nonatomic elementary sentence of the form $+[\iota x.\Phi / u]afm$ for some constant description term $\iota x.\Phi$ and atomic formula afm, then Set_i satisfies one of the following conditions:

i) Set_i contains the signed sentence $+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)$ for some variable v.

ii) Set_i contains the signed sentence $+[t / x](\Phi \wedge [x / u]afm)$ for every $t \in \delta(\mathcal{P}(Set_i))$.

<u>Proof</u>: Assume that $\pm snt$ is a signed nonatomic elementary sentence of the form $+[\iota x.\Phi / u]afm$ for some constant description term $\iota x.\Phi$ and atomic formula afm. Now, the formulas Φ, afm are respectively of the form $[p / y]\Psi$, $[p / y]afm'$ for some variable y and formulas Ψ, afm' such that Ψ and afm' contain no occurrence of p, where p is the first parameter in $Par - \mathcal{P}(Set_{i-1})$. Since $+[\iota x.\Phi / u]afm \cong +[\iota x.[p / y]\Psi / u]\ [p / y]afm' \cong +[p / y][\iota x.\Psi / u]afm'$ is in Set_i and neither Φ nor afm contain an occurrence of p, it follows by clause 1) of the recursive definition of Set^* that Set_{i-1} contains $+[\mathbb{P}(1) / y][\iota x.\Psi / u]afm' \cong +[\iota x.[\mathbb{P}(1) / y]\Psi / u][\mathbb{P}(1) / y]afm'$. By the hypothesis of induction, Set_{i-1} is downward closed, so by condition 9) of 4.1.1., there are two cases:

i) Set_{i-1} contains the signed sentence $+(x)(v)(([\mathbb{P}(1) / y]\Psi \wedge [v / x]\ [\mathbb{P}(1) / y]\Psi) \rightarrow x = v)$.

Now, the signed sentence $+(x)(v)(([\mathbb{P}(1) / y]\Psi \wedge [v / x]\,[\mathbb{P}(1) / y]\Psi) \rightarrow x = v)$ may be written as $+[\mathbb{P}(1) / y](x)(v)((\Psi \wedge [v / x]\Psi) \rightarrow x = v)$. So by clause 1) of the recursive definition of ***Set****, since Ψ contains no occurrence of ***p***, Set_i contains $+[p / y](x)(v)((\Psi \wedge [v / x]\Psi) \rightarrow x = v) \cong +(x)(v)(([p / y]\Psi \wedge [v / x]\,[p / y]\Psi) \rightarrow x = v) \cong +(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)$. So the claim holds.

ii) For every $t \in \delta(\mathcal{S}(Set_{i-1}))$, Set_{i-1} contains the signed sentence $+[t / x]([\mathbb{P}(1) / y]\Psi \wedge [\mathbb{P}(1) / y]afm')$. Now let ***r*** be any term in $\delta(\mathcal{S}(Set_i))$. Now, ***r*** is of the form $[p / y]s$ for some term ***s*** such that ***s*** contains no occurrence of ***p***. Then $[\mathbb{P}(1) / y]s$ is a term in $\delta(\mathcal{S}(Set_{i-1}))$. So Set_{i-1} contains the signed sentence $+[[\mathbb{P}(1) / y]s / x]([\mathbb{P}(1) / y]\Psi \wedge [\mathbb{P}(1) / y]afm') \cong +[\mathbb{P}(1) / y]\,[s / x](\Psi \wedge afm')$.Since none of s, Ψ, afm' contain an occurrence of ***p***, by clause 1) of the recursive definition of ***Set****, Set_i contains $+[p / y][s / x](\Psi \wedge afm') \cong +[[p / y]s / x]([p / y]\Psi \wedge [p / y]afm') \cong +[r / x](\Phi \wedge afm)$, which is what we want to show.

In both cases, then, Set_i satisfies condition 9).

10) If $\pm snt$ is a signed nonatomic elementary sentence of the form $-[\iota x.\Phi / u]afm$ for some constant description term $\iota x.\Phi$ and atomic formula ***afm***, then Set_i satisfies one of the following conditions:

i) Set_i contains the signed sentence $+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)$ for some variable ***v***.

ii) Set_i contains the signed sentence $+[t / x](\Phi \wedge \neg[x / u]afm)$ for every $t \in \delta(\mathcal{S}(Set_i))$.

<u>Proof</u>: The proof that Set_i satisfies condition 10) is similar to the proof above that Set_i satisfies condition 9).

That ***Set**** is downward closed follows immediately from the facts that, by definition, $Set^* = \bigcup Set_i$, $0 \leq i < \omega$ and that for all i, $0 \leq i < \omega$, Set_i is downward closed. This completes the proof of lemma 4.2.4.1. □

<u>Lemma 4.2.4.2.</u>: Let ***Set*** be a downward closed set of signed sentences, and let ***Set**** be as defined above. Let $\pm snt$ be any signed sentence in ***Set****. Then, if $\pm snt$ is of the form $-(x)F$ for

some formula F containing free occurrence of x, then $\mathcal{S}^{\delta}(Set^*) = Par$.

Proof of 4.2.4.2.: Let $\pm snt$ be any signed sentence in Set^* such that $\pm snt$ is of the form $-(x)F$ for some formula F containing free occurrence of x. Since by lemma 4.2.4.1. Set^* is downward closed, by condition 8) of definition 4.1.1., Set^* contains the signed sentence $-[t / x]F$ for every t in $\delta(\mathcal{S}^{\delta}(Set^*)$. Since by definition 1.1.1.4., $\mathbb{P}(1) \in \delta(\mathcal{S}^{\delta}(Set^*)$, it follows that Set^* contains the signed sentence $-[\mathbb{P}(1) / x]F$. Then it is easily verified by induction on i in the definition of Set^* that Set^* contains the signed sentence $-[\mathbb{P}(i) / x]F$ for every $i < \omega$. Since F contains free occurrence of x, it follows that $\mathcal{S}^{\delta}(Set^*) = Par$. □

Lemma 4.2.4.3.: Let Set be a downward closed set of signed sentences, and let Set^* be as defined above. Let $\pm snt$ be any signed sentence in Set^*. Then, if $\pm snt$ is of the form $\pm[\iota x.\Phi / u]afm$ for some constant description term $\iota x.\Phi$ and atomic formula afm containing free occurence of u, and for no variable v does Set^* contain the signed sentence $+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)$, then $\mathcal{S}^{\delta}(Set^*) = Par$.

Proof of 4.2.4.3.: Let $\pm snt$ be any signed sentence in Set^* of the form $\pm[\iota x.\Phi / u]afm$ for some constant description term $\iota x.\Phi$ and atomic formula afm containing free occurence of u such that for no variable v does Set^* contain the signed sentence $+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)$. Since, by lemma 4.2.4.1., Set^* is downward closed and i) of condition 9), respectively, 10) of 4.1.1. is not satisfied, by ii) of condition 9), respectively, 10) of 4.1.1., Set^* contains the signed sentence $+[t / x](\Phi \wedge [x / u]afm)$, respectively $+[t / x](\Phi \wedge \neg[x / u]afm)$, for every t in $\delta(\mathcal{S}^{\delta}(Se^*)$. Since, by definition 1.1.1.4., $\mathbb{P}(1) \in \delta(\mathcal{S}^{\delta}(Set)$, it follows that Set^* contains the signed sentence $+[\mathbb{P}(1) / x](\Phi \wedge [x / u]afm)$, respectively, $+[\mathbb{P}(1) / x](\Phi \wedge \neg[x / u]afm)$. Then it is easily verified by induction on i in the definition of Set^* that Set^* contains the signed sentence $+[\mathbb{P}(i) / x](\Phi \wedge [x / u]afm)$, respectively, $+[\mathbb{P}(i) / x](\Phi \wedge \neg[x / u]afm)$ for every $i < \omega$. Since afm contains free occurrence of u, it follows that $\mathcal{S}^{\delta}(Set^*) = Par$. □

Now, let Set be a downward closed set of signed sentences, and let Set^* be as defined above. Let B1 be the set of atomic sentences $\mp asnt$ such that respectively $\pm asnt$ is in Set^*. We extend B1 to a base $bse(Set)$ with domain $\delta(\mathcal{S}^{\delta}(Set^*))$ by the following operations:

a) Let t be any term in $\delta(Par)$. Then B2 is obtained from B1 by adding to B1 all signed atomic sentences of the form $+(t = t)$.

b) Let r, s be any terms in $\delta(Par)$ such that neither of $\pm(r = s)$ is a member of B2. Then B3 is obtained from B2 by adding $-(r = s)$ to B2.

c) Let ***asnt*** be any atomic sentence such that neither of ±***asnt*** is in B3, and let $r, s \in \delta(Par)$. Then B4 = *bse*(*Set*) is obtained from B3 by adding to B3 one and only one of ±***asnt*** with the following proviso: if ***asnt*** is of the form $[r / x]$***afm*** for some atomic formula ***afm*** and $+(r = s) \in$ B3, then if $+[r / x]$***afm*** is added to B3, then so is $+[s / x]$***afm***.

Lemma 4.2.4.4.: Let *Set* be a downward closed set of signed sentences, and let *bse*(*Set*) be as defined above. Then *bse*(*Set*) is a base with domain $\delta(Par)$.

Proof of 4.2.4.4.: We show that *bse*(*Set*) satisfies all of conditions 1), 2), 3) of definition 1.3.1. :

1) For every atomic sentence ***asnt***, exactly one of ±***asnt*** is in *bse*(*Set*).

Proof: By operation c) above, *bse*(*Set*) contains at least one of ±***asnt*** for every atomic sentence ***asnt***. To show that *bse*(*Set*) contains at most one of ±***asnt*** for every atomic sentence ***asnt***, we show that B1 has this property and that operations a), b), c) preserve the property. Since by hypothesis *Set** is downward, by condition 1) of definition 4.1.1.1., *Set** contains at most one of ±***asnt*** for every atomic sentence ***asnt***. It follows by the definition of B1 that B1 contains at most one of ±***asnt*** for every atomic sentence ***asnt***. Now we show that operation a) preserves this property: Assume that B2 contains both of ±***asnt*** for some atomic sentence ***asnt***. In case ***asnt*** is of the form $(t = t)$, since operation a) adds to B1 signed sentences with positive signature only, B1 contains the signed sentence $-(t = t)$. Then by the definition of B1, *Set** contains the signed sentence $+(t = t)$. But since *Set** is downward closed, by condition 2) of 4.1.1., *Set** contains no such signed sentence. Contradiction. In case ***asnt*** is not of the form $(t = t)$ for some $t \in \delta(\mathcal{S}(Set^*))$, since operation a) adds to B1 signed sentences of this form only, B1 contains both of ±***asnt*** . Now we show that operation b) preserves the desired property: Assume that B3 contains both of $\pm$***asnt*** for some atomic sentence ***asnt*** and that for no atomic sentence ***asnt*** does B2 contain both of $\pm$***asnt***. Then we know that at least one of $\pm$***asnt*** is added to B2 by operation b) and since the only type of signed sentence operation b) adds to B2 are identity statements of negative signature, it follows that ***asnt*** is of the form $(r = s)$ for some $r, s \in \delta(Par)$ and that by operation b) adds the signed sentence $-(r = s)$ to B2. Then, by operation b), B2 does not contain $+(r = s)$, i.e., +***asnt*** $\notin$ B2. Contradiction. Finally, it is trivial to show that operation c) preserves the desired property.

2) For all terms $t \in \delta(Par)$, $+(t = t)$ is in *bse*(*Set*).

Proof: By operation b), B2 $\subseteq$ *bse*(*Set*) satisfies this condition.

3) For all terms $r, s \in \delta(Par)$ and atomic formulas ***afm***, if both of $+[r / x]$***afm***, $-[s / x]$***afm*** are signed sentences in *bse*(*Set*), then so is $-(r = s)$.

Proof: Let r, s be any terms in $\delta(Par)$ and let ***afm*** be any atomic formula. We want to show that if both of $+[r / x]$***afm***, $-[s / x]$***afm*** are signed sentences in *bse*(*Set*), then so is the signed sentence $-(r = s)$. So we show the contrapositive, that if $-(r = s) \notin$ *bse*(*Set*), then it is not the case that both of $+[r / x]$***afm***, $-[s / x]$***afm*** $\in$ *bse*(*Set*). So assume that $(r = s) \notin$ *bse*(*Set*). We show first that B2 does not contain both of $+[r / x]$***afm***, $-[s / x]$***afm***: We may assume that x has free occurrence in ***afm***, since otherwise the claim holds trivially by the fact that *bse*(*Set*) satisfies condition 1). Since by condition 1) above *bse*(*Set*) contains at least one of $\pm$***asnt*** for every atomic sentence ***asnt***, it follows that $+(r = s) \in$ *bse*(*Set*). Since neither operation b) nor c) adds signed sentences of the form $+(r = s)$ (i.e., positive identity statements) to B2 or B3, it follows that $+(r = s) \in$ B2. In case r is syntactically identical to s, so that the signed sentence $+(r = s)$ is of the form $+(r = r)$, it is trivially the case that if the signed sentence $+[r / x]$***afm*** is in B2, then so is $+[s / x]$***afm***. In this case, then, since by condition 1) *bse*(*Set*) contains at most one of $\pm$***asnt*** for every atomic sentence ***asnt***, it is not the case that both of $+[r / x]$***afm***, $-[s / x]$***afm*** are in B2. In case r is not syntactically identical to s, since operation a) adds to B1 signed sentences of the form $+(t = t)$ only, it follows that $+(r = s) \in$ B1. Since $+(r = s) \in$ B1, by the definition of B1, $-(r = s) \in$

*Set**. Since *Set** is downward closed and $-(r=s) \in$ *Set**, by condition 4) of 4.1.1., *Set** does not contain both of $-[r / x]afm$, $+[s / x]afm$. So, by the definition of B1, B1 does not contain both of $+[r / x]afm$, $-[s / x]afm$. It follows that B2 does not contain both of $+[r / x]afm$, $-[s / x]afm$, for suppose that B2 does contain both of $+[r / x]afm$, $-[s / x]afm$. Since B1 does not contain both of $+[r / x]afm$, $-[s / x]afm$ and operation a) adds to B1 signed sentences with positive signature only, it follows that B1 contains $-[s / x]afm$ and operation a) adds $+[r / x]afm$ to B1. Since operation a) adds to B1 signed sentences of the form $+(t=t)$ only and x has free occurrence in afm, $+[r / x]afm$ is of the form $+(r=r)$. Then $-[s / x]afm \in$ B1 is of one of three forms: $-(s=s)$, $-(s=r)$, or $-(r=s)$. Since *Set** is downward closed, by condition 2) of 4.1.1., *Set** does not contain $+(s=s)$ and hence, by definition, B1 does not contain $-(s=s)$. Since $+(r=s) \in$ B1 and *bse*(*Set*) satisfies condition 1), $-(r=s) \notin$ B1. Hence, $-[s / x]afm \in$ B1 is of the form $-(s=r)$. So B1 contians both of $+(r=s)$, $-(s=r)$, and hence *Set** contains both of $-(r=s)$, $+(s=r)$. But since *Set** is downward closed, by condition 3) of 4.1.1., *Set** does not contain both of $-(r=s)$, $+(s=r)$. Contradiction. So we have shown that B2 does not contain both of $+[r / x]afm$, $-[s / x]afm$: But, trivially, operation c) preserves this property, so B4 = *bse*(*Set*) does not contain both of $+[r / x]afm$, $-[s / x]afm$: □

<u>Lemma 4.2.4.5.</u>: Let *Set* be a downward closed set of signed sentences, and let *Set**, *bse*(*Set*) be as defined above. Then, $Set^* \cap Cl(bse(Set)) = \varnothing$.

<u>Proof of 4.2.4.5.</u>: We show by transfinite induction on μ in definition 1.3.3. that for all μ, $Set^* \cap bse(Set)_\mu = \varnothing$. So assume that there is an ordinal μ such that $Set^* \cap bse(Set)_\mu \neq \varnothing$. Since $\langle bse(Set)_\mu \rangle$ is well ordered under $\subseteq$, there is a least ordinal μ such that $Set^* \cap bse(Set)_\mu \neq \varnothing$. So let $\pm snt \in Set^* \cap bse(Set)_\mu$. There are three main cases:

i) $\mu = 0$: Since $\pm snt \in bse(Set)_0$, snt is atomic. So, since $\pm snt \in Set^*$, by the definition of B1, respectively $\mp snt \in$ B1. Since B1 $\subseteq bse(Set) = bse(Set)_0$, it follows that both of $\pm snt \in bse(Set)$. This contradicts lemma 4.2.4.4..

ii) μ is a nonzero limit ordinal: Since $\pm snt \in bse(Set)_\mu$ and $bse(Set)_\mu = \bigcup bse(Set)_\beta$, $0 \leq \beta$

$< \omega$, there is a $\beta < \omega$ such that respectively $\pm snt \in bse(Set)_\beta$. Contradiction.

iii) μ is a successor ordinal: There are five subcases:

1) $\pm snt$ is respectively of the form $\pm\neg$A, for some sentence **A**. Since $\pm\neg$A $\in$ *Set** and, by lemma 4.2.4.1., *Set** is downward closed, by condition 5) of 4.1.1., *Set** contains the signed sentence respectively $\mp$A. Since $\pm\neg$A $\in bse(Set)_\mu$ and $\langle bse(Set)_\mu \rangle$ is well ordered under $\subseteq$, there is a least ordinal $\beta \leq \mu$ such that respectively $\pm\neg$A $\in bse(Set)_\beta$. So, since $\pm\neg$A $\in bse(Set)_\beta$ but for all $\gamma < \beta$, respectively $\pm\neg$A $\notin bse(Set)_\gamma$, it follows that $\pm\neg$A $\in Sc(bse(Set)_{\beta-1})$ by virtue of the semantic rule 1.3.2.1.. Hence $bse(Set)_{\beta-1}$ contains the signed sentence respectively $\mp$A. Hence $\mp$A $\in$ *Set** $\cap\, bse(Set)_{\beta-1}$. But since $\beta \leq \mu$, $\beta - 1 < \mu$. So there is a $\beta < \mu$ such that *Set** $\cap\, bse(Set)_\beta \neq \varnothing$. Contradiction.

2) $\pm snt$ is respectively of the form $\pm$(**A** $\wedge$ **B**) for some signed sentences **A**, **B**. Since $\pm$(**A** $\wedge$ **B**) $\in$ *Set** and, by lemma 4.2.4.1., *Set** is downward closed, by condition 6) of 4.1.1., *Set** contains one of +**A**, +**B**, respectively, both of -**A**, -**B**. Since $\pm$(**A** $\wedge$ **B**) $\in bse(Set)_\mu$ and $\langle bse(Set)_\mu \rangle$ is well ordered under $\subseteq$, there is a least ordinal $\beta \leq \mu$ such that respectively $\pm$(**A** $\wedge$ **B**) $\in bse(Set)_\beta$. So, since $\pm$(**A** $\wedge$ **B**) $\in bse(Set)_\beta$ but for all $\gamma < \beta$, respectively $\pm$(**A** $\wedge$ **B**) $\notin bse(Set)_\gamma$, it follows that $\pm$(**A** $\wedge$ **B**) $\in Sc(bse(Set)_{\beta-1})$ by virtue of the semantic rule 1.3.2.2.. Hence $bse(Set)_{\beta-1}$ contains both of +**A**, +**B**, respectively, one of -**A**, -**B**. Hence *Set** $\cap$ $bse(Set)_{\beta-1}$ contains one of +**A**, +**B**, respectively, one of -**A**, -**B**. But since $\beta \leq \mu$, $\beta - 1 < \mu$. So there is a $\beta < \mu$ such that *Set** $\cap\, bse(Set)_\beta \neq \varnothing$. Contradiction.

3) $\pm snt$ is respectively of the form $\pm$(**A** $\rightarrow$ **B**) for some signed sentences **A**, **B**. This subcase is similar to subcase 2) above.

4) $\pm snt$ is respectively of the form $\pm(x)F$ for some formula F. Since $\pm(x)F \in bse(Set)_\mu$ and $\langle bse(Set)_\mu \rangle$ is well ordered under $\subseteq$, there is a least ordinal $\beta \leq \mu$ such that respectively $\pm(x)F \in bse(Set)_\beta$. So, since $\pm(x)F \in bse(Set)_\beta$ but for all $\gamma < \beta$, respectively $\pm(x)F \notin bse(Set)_\gamma$, it follows that $\pm(x)F \in Sc(bse(Set)_{\beta-1})$ by virtue of the semantic rule 1.3.2.4.. We treate the two

cases seperately:

a) $\pm snt$ is of the form $+(x)F$. Hence $bse(Set)_{\beta-1}$ contains the signed sentence $+[t / x]F$ for every t in $\delta(Par)$. Since $+(x)F \in Set^*$ and, by lemma 4.2.4.1., Set^* is downward closed, by condition 8) of 4.1.1., Set^* contains the signed sentence $+[p / x]F$ for some p in $\mathcal{P}(Set^*)$. Since $p \in Par, p \in \delta(Par)$. So $bse(Set)_{\beta-1}$ contains the signed sentence $+[p / x]F$. Hence $Set^* \cap bse(Set)_{\beta-1}$ contains the signed sentence $+[p / x]F$. But since $\beta \leq \mu$, $\beta - 1 < \mu$. So there is a $\beta < \mu$ such that $Set^* \cap bse(Set)_{\beta} \neq \emptyset$. Contradiction.

b) $\pm snt$ is of the form $-(x)F$. Hence $bse(Set)_{\beta-1}$ contains the signed sentence $-[t / x]F$ for some t in $\delta(Par)$. Since $-(x)F \in Set^*$ and, by lemma 4.2.4.1., Set^* is downward closed, by condition 8) of 4.1.1., Set^* contains the signed sentence $-[t / x]F$ for every t in $\delta(\mathcal{P}(Set^*))$. In case x does not occur free in F, $[t / x]F \cong F$, and so $Set^* \cap bse(Set)_{\beta-1}$ contains the signed sentence $-F$. In case x occurs free in F, by lemma 4.2.4.2., $\mathcal{P}(Set^*) = Par$. Then, in this case, Set^* contains the signed sentence $-[t / x]F$ for every t in $\delta(Par)$. Hence $Set^* \cap bse(Set)_{\beta-1}$ contains the signed sentence $-[t / x]F$ for some t in $\delta(Par)$. But since $\beta \leq \mu$, $\beta - 1 < \mu$. So in either case there is a $\beta < \mu$ such that $Set^* \cap bse(Set)_{\beta} \neq \emptyset$. Contradiction.

5) $\pm snt$ is a signed elementary sentence of the form $+[\iota x.\Phi / u]afm$ for some constant description term $\iota x.\Phi$ and atomic formula afm: If $+[\iota x.\Phi / u]afm$ is atomic, $+[\iota x.\Phi / u]afm \in bse(Set)_0$ and this case reduces to case i). So we may assume that u occurs free in afm. There are two subcases:

a) Set^* contains the signed sentence $+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)$ for some variable v. Since $+[\iota x.\Phi / u]afm \in bse(Set)_{\mu}$ and $\langle bse(Set)_{\mu} \rangle$ is well ordered under $\subseteq$, there is a least ordinal β, $0 < \beta \leq \mu$, such that $+[\iota x.\Phi / u]afm \in bse(Set)_{\beta}$. So, since $+[\iota x.\Phi / u]afm \in bse(Set)_{\beta}$ for $\beta > 0$, but for all $\gamma < \beta$, $+[\iota x.\Phi / u]afm \notin bse(Set)_{\gamma}$, it follows from the fact that $+[\iota x.\Phi / u]afm$ is a signed *elementary* sentence (and as such cannot be the output of an instance of any semantic rule other than 1.3.2.5.) that $+[\iota x.\Phi / u]afm \in Sc(bse(Set)_{\beta-1})$ by virtue of an instance $\mathbb{I}$ of the semantic rule for descriptions 1.3.2.5.. Hence $bse(Set)_{\beta-1}$ contains the uniqueness presupposition $+(x)(z)((\Phi \wedge [z/x]\Phi) \rightarrow x = z)$ to $\mathbb{I}$ for some variable z. Now it is clear that by semantic rule 1.3.2.4., if $bse(Set)_{\beta-1}$ contains the signed sentence $+(x)(z)((\Phi \wedge [z / x]\Phi) \rightarrow x = z)$ for *some* variable z, then $bse(Set)_{\beta-1}$ contains the signed sentence $+(x)(z)((\Phi \wedge [z / x]\Phi) \rightarrow x = z)$ for *all* variables z. So $Set^* \cap bse(Set)_{\beta-1}$ contains the signed sentence $+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)$ for some variable v. But since $\beta \leq \mu$, $\beta - 1 < \mu$. So there is a $\beta < \mu$ such that $Set^* \cap bse(Set)_\beta \neq \emptyset$. Contradiction.

b) For no variable v does Set^* contain the signed sentence $+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)$. Since $+[\iota x.\Phi / u]afm \in Set^*$ and, by lemma 4.2.4.1., Set^* is downward closed, by condition 9) of 4.1.1., Set^* contains the signed sentence $+[t / x](\Phi \wedge [x / u]afm) \cong +([t / x]\Phi \wedge [t / u]afm)$ for every $t \in \delta(\mathcal{P}(Set^*))$. Then by condition 6) of 4.1.1., for every $t \in \delta(\mathcal{P}(Set^*))$, Set^* contains one of the signed sentences $+[t / x]\Phi$, $+[t / u]afm$. Since $+[\iota x.\Phi / u]afm \in Set^*$ and u occurs free in afm and for no variable v does Set^* contain the signed sentence $+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)$, it follows by lemma 4.2.4.3. that $\delta(\mathcal{P}(Set^*)) = Par$. So, for every $t \in \delta(Par)$, Set^* contains one of the signed sentences $+[t / x]\Phi$, $+[t / u]afm$.

Since $+[\iota x.\Phi / u]afm \in bse(Set)_\mu$ and $\langle bse(Set)_\mu \rangle$ is well ordered under $\subseteq$, there is a least ordinal β, $0 < \beta \leq \mu$, such that $+[\iota x.\Phi / u]afm \in bse(Set)_\beta$. So, since $+[\iota x.\Phi / u]afm \in bse(Set)_\beta$ but for all $\gamma < \beta$, $+[\iota x.\Phi / u]afm \notin bse(Set)_\gamma$, it follows from the fact that $+[\iota x.\Phi / u]afm$ is a signed *elementary* sentence (and as such cannot be the output of an instance of any semantic rule other than 1.3.2.5.) that $+[\iota x.\Phi /u]afm \in Sc(bse(Set)_{\beta-1})$ by virtue of an instance $\mathbb{I}$ of the semantic rule for descriptions 1.3.2.5.. Hence $bse(Set)_{\beta-1}$ contains the existence presupposition $+[s / x]\Phi$ as well as the major statement $+[s / u]afm$ to $\mathbb{I}$ for some constant, possibly nonbasic, term s. In case s is not in $\delta(Par)$, by lemma 1.3.10., there is a descriptum t_s for s in $bse(Set)$ such that both of $+[t_s / x]\Phi$, $+[t_s / u]afm$ are in $bse(Set)_{\beta-1}$. So $bse(Set)_{\beta-1}$ contains both of $+[s / x]\Phi$, $+[s / u]afm$ for some $s \in \delta(Par)$. So, since for every $t \in \delta(Par)$, Set^* contains one of the signed sentences $+[t / x]\Phi$, $+[t / u]afm$ and there is an $s \in \delta(Par)$ such that $bse(Set)_{\beta-1}$ contains both of $+[s / x]\Phi$, $+[s / u]afm$, it follows that there is an $s \in \delta(Par)$ such that $Set^* \cap bse(Set)_{\beta-1}$ contains one of $+[s / x]\Phi$, $+[s / u]afm$. But since $\beta \leq \mu$, $\beta - 1 <$

μ. So there is a $\beta < \mu$ such that $\boldsymbol{Set^*} \cap bse(Set)_\beta \neq \varnothing$. Contradiction.

6) $\pm\boldsymbol{snt}$ is a signed elementary sentence of the form $-[\iota x.\Phi / u]\boldsymbol{afm}$ for some constant description term $\iota x.\Phi$ and atomic formula $\boldsymbol{afm}$: If $-[\iota x.\Phi / u]\boldsymbol{afm}$ is atomic, $-[\iota x.\Phi / u]\boldsymbol{afm} \in bse(Set)_0$ and this case reduces to case i). So we may assume that $\boldsymbol{u}$ occurs free in $\boldsymbol{afm}$. There are two subcases:

a) $\boldsymbol{Set^*}$ contains the signed sentence $+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)$ for some variable $\boldsymbol{v}$. Since $-[\iota x.\Phi / u]\boldsymbol{afm} \in bse(Set)_\mu$ and $\langle bse(Set)_\mu \rangle$ is well ordered under $\subseteq$, there is a least ordinal β, $0 < \beta \leq \mu$, such that $-[\iota x.\Phi / u]\boldsymbol{afm} \in bse(Set)_\beta$. So, since $-[\iota x.\Phi / u]\boldsymbol{afm} \in bse(Set)_\beta$ for $\beta > 0$, but for all $\gamma < \beta$, $-[\iota x.\Phi / u]\boldsymbol{afm} \notin bse(Set)_\gamma$, it follows from the fact that $-[\iota x.\Phi / u]\boldsymbol{afm}$ is a signed *elementary* sentence that $-[\iota x.\Phi / u]\boldsymbol{afm} \in Sc(bse(Set)_{\beta-1})$ by virtue of an instance $\mathbb{I}$ of the semantic rule for descriptions 1.3.2.5.. Hence $bse(Set)_{\beta-1}$ contains the uniqueness presupposition $+(x)(z)((\Phi \wedge [z / x]\Phi) \rightarrow x = z)$ to $\mathbb{I}$ for some variable z. Now it is clear that by semantic rule 1.3.2.4., if $bse(Set)_{\beta-1}$ contains the signed sentence $+(x)(z)((\Phi \wedge [z / x]\Phi) \rightarrow x = z)$ for *some* variable z, then $bse(Set)_{\beta-1}$ contains the signed sentence $+(x)(z)((\Phi \wedge [z / x]\Phi) \rightarrow x = z)$ for *all* variables z. So $\boldsymbol{Set^*} \cap bse(Set)_{\beta-1}$ contains the signed sentence $+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)$ for some variable $\boldsymbol{v}$. But since $\beta \leq \mu$, $\beta - 1 < \mu$. So there is a $\beta < \mu$ such that $\boldsymbol{Set^*} \cap bse(Set)_\beta \neq \varnothing$. Contradiction.

b) For no variable $\boldsymbol{v}$ does $\boldsymbol{Set^*}$ contain the signed sentence $+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)$. Since $-[\iota x.\Phi / u]\boldsymbol{afm} \in \boldsymbol{Set^*}$ and, by lemma 4.2.4.1., $\boldsymbol{Set^*}$ is downward closed, by condition 9) of 4.1.1., $\boldsymbol{Set^*}$ contains the signed sentence $+[t / x](\Phi \wedge \neg[x / u]\boldsymbol{afm}) \cong +([t / x]\Phi \wedge \neg[t / u]\boldsymbol{afm})$ for every $t \in \delta(\mathscr{S}(\boldsymbol{Set^*}))$. Then by condition 6) of 4.1.1., for every $t \in \delta(\mathscr{S}(\boldsymbol{Set^*}))$, $\boldsymbol{Set^*}$ contains one of the signed sentences $+[t / x]\Phi$, $+\neg[t / u]\boldsymbol{afm}$. Since $+[\iota x.\Phi / u]\boldsymbol{afm} \in \boldsymbol{Set^*}$ and $\boldsymbol{u}$ occurs free in $\boldsymbol{afm}$ and for no variable $\boldsymbol{v}$ does $\boldsymbol{Set^*}$ contain the signed sentence $+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)$, it follows by lemma 4.2.4.3. that $\delta(\mathscr{S}(\boldsymbol{Set^*})) = \boldsymbol{Par}$. So, for every $t \in \delta(\boldsymbol{Par})$, $\boldsymbol{Set^*}$ contains one of the signed sentences $+[t / x]\Phi$, $+\neg[t / u]\boldsymbol{afm}$. If $\boldsymbol{Set^*}$ contains the signed sentence $+\neg[t / u]\boldsymbol{afm}$, then by condition 5) of

4.1.1., *Set** contains the signed sentence -$[t / u]afm$. So, for every $t \in \delta(Par)$, *Set** contains one of the signed sentences +$[t / x]\Phi$, -$[t / u]afm$. Since +$[\iota x.\Phi / u]afm \in bse(Set)_\mu$ and $\langle bse(Set)_\mu \rangle$ is well ordered under $\subseteq$, there is a least ordinal β, $0 < \beta \leq \mu$, such that -$[\iota x.\Phi / u]afm \in bse(Set)_\beta$. So, since +$[\iota x.\Phi / u]afm \in bse(Set)_\beta$ for $\beta > 0$, but for all $\gamma < \beta$, -$[\iota x.\Phi / u]afm \notin bse(Set)_\gamma$, it follows from the fact that -$[\iota x.\Phi / u]afm$ is a signed *elementary* sentence that -$[\iota x.\Phi /u]afm \in Sc(bse(Set)_{\beta-1})$ by virtue of an instance $\mathbb{I}$ of the semantic rule for descriptions 1.3.2.5.. Hence $bse(Set)_{\beta-1}$ contains the existence presupposition +$[s / x]\Phi$ as well as the major statement -$[s / u]afm$ to $\mathbb{I}$ for some constant, possibly nonbasic, term s. In case s is not in $\delta(Par)$, by lemma 1.3.10., there is a descriptum t_s for s in $bse(Set)$ such that both of +$[t_s / x]\Phi$, -$[t_s / u]afm$ are in $bse(Set)_{\beta-1}$. So $bse(Set)_{\beta-1}$ contains both of +$[s / x]\Phi$, -$[s / u]afm$ for some $s \in \delta(Par)$. So, since for every $t \in \delta(Par)$, *Set** contains one of the signed sentences +$[t / x]\Phi$, -$[t / u]afm$ and there is an $s \in \delta(Par)$ such that $bse(Set)_{\beta-1}$ contains both of +$[s / x]\Phi$, -$[s / u]afm$, it follows that there is an $s \in \delta(Par)$ such that $Set^* \cap bse(Set)_{\beta-1}$ contains one of +$[s / x]\Phi$, -$[s / u]afm$. But since $\beta \leq \mu$, $\beta - 1 < \mu$. So there is a $\beta < \mu$ such that $Set^* \cap bse(Set)_\beta \neq \varnothing$. Contradiction.

In all cases, then, the assumption that there is an ordinal μ such that $Set^* \cap bse(Set)_\mu \neq \varnothing$ leads to a contradiction. Hence, for all μ, $Set^* \cap bse(Set)_\mu = \varnothing$. Then it follows by the fact that, by definition, $Cl(bse(Set)) = \bigcup bse(Set)_\mu$, $0 \leq \mu < \epsilon_0$, that $Set^* \cap Cl(bse(Set)) = \varnothing$. □

Lemma 4.2.4., the claim that every downward closed set *Set* of signed sentences determines a base *bse*(*Set*) such that $Set \cap Cl(bse(Set)) = \varnothing$, now follows from lemmata 4.2.4.1., 4.2.4.4., 4.2.4.5. established above: Let *Set* be any downward closed set of signed sentences. Then by 4.2.4.1., *Set* has a downward closed extension *Set**. By lemma 4.2.4.4., there exists a base *bse*(*Set*), which, by lemma 4.2.4.5., is such that $Set^* \cap Cl(bse(Set)) = \varnothing$. Since $Set \subseteq Set^*$, it follows that $Set \cap Cl(bse(Set)) = \varnothing$. □

Theorem 4.2.2., the claim that every valid sequent is strictly 2-derivable, now follows from lemmata 4.2.3. and 4.2.4. Hence, lemma 4.2.1. and the completeness of Pld has been established.

☐

Section 5: Corollaries to Soundness and Completeness.

In this section we state and prove some normal form results concerning Pld derivations which fall out of theorems 2.1. and 4.2.2. and then give a characterization of the set of sentences of Pld for which excluded middle holds.

Corallary 5.2 (Normal Form Lemma): Every derivable sequent is strictly derivable.

Proof of 5.2: Let ***Seq*** be any derivable sequent. Then, by soundness theorem 2.1., ***Seq*** is valid. Then by lemma 4.2.2., ***Seq*** is strictly 2-derivable. So by lemma 3.5., ***Seq*** is strictly derivable. □

Corollary 5.3: Every derivable sequent can be derived as endsequent of a derivation tree containing no application of the cut rule 1.2.2.8..

Proof of 5.3: By corollary 5.2. and definition of **strictly derivable.** □

Corollary 5.4: Every derivable sequent can be derived as endsequent of a derivation tree all of whose applications of the description rule 1.2.2.6. have elementary output sentences.

Proof of 5.4: By corollary 5.2. and definition of **strictly derivable.** □

Notice that corollary 5.3. allows us to give a proof of the syntactic consistency of Pld that is an alternative to the proof given in section 2. We restate and reprove the syntactic consistency of Pld:

Corollary (Syntactic Consistency): For no sentence ***snt*** are both of {+***snt***}, {-***snt***} derivable sequents.

Proof: Assume there is a sentence ***snt*** such that both of {+***snt***}, {-***snt***} are derivable. Since the null sequent can be derived by appending an application of the cut rule 1.2.2.8. to the derivations of $\{+snt\}$, $\{-snt\}$, it follows that the null sequent is derivable. Then by corollary 5.3., the null sequent is derivable as endsequent of a derivation tree containing no applications of 1.2.2.8. But since all Pld axioms are non-null sequents and every Pld deduction rule $\mathbb{R}$ except for 1.2.2.8. is such that if all of an applications of $\mathbb{R}$'s premises are non-null then that applications conclusion is non-null, it follows by a simple induction on the depth of derivation trees that the null sequent is not derivable without application of 1.2.2.8.. Contradiction □

We say that **excluded middle holds for a sentence** ***snt*** of Pld iff for all bases *bse*, one of $\pm$*snt* belongs to *Cl*(*bse*). Then, it follows from soundness and completeness by definition 1.2.5. that the set of sentences of Pld for which excluded middle holds is precisely the set *Grd* of grounded sentences. Since *Grd* is r.e., we have an r.e. characterization of set of sentences of Pld for which excluded middle holds.

Section 6: Some Remarks on the Frege-Strawson Doctrine.

The "Frege-Strawson doctrine" (F-S D) has thus far been characterized as the claim that a sentence of a natural language that is grammatically of subject-predicate form and that contains a vacuous singular subject term cannot have a (classical) truth value associated with it. We have suggested that Pld formalizes the semantic intuition behind F-S D as applied to improper definite description in terms of a theory of truth which allows truth value gaps. In this section we evaluate the extent to which Pld can be said to be a formalization of the F-S D and then consider the problem of integrating F-S D with a general theory of truth for compound sentences of natural language. Indeed, we find that Pld suggests a natural such extension of the F-S D.

A definite description of English can be informally characterized as a noun phrase of the form $\ulcorner$the $\Phi\urcorner$ where 'Φ' is an English noun phrase, qualified or unqualified, in the singular. A proper definite description of English can be informally characterized as a definite description of the form $\ulcorner$the $\Phi\urcorner$ such that there is exactly one Φ. The formal analog to an English definite description within Pld syntax is, of course, the description term. Let us say that a Pld constant description term of the form $\iota x.\Phi$ is **proper relative to a base *bse*** iff for some constant term t and variable v, $Cl(bse)$ contains both of the signed sentences $+[t / x]\Phi$, $+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)$. We say that $\iota x.\Phi$ is **improper relative to a base *bse*** iff $\iota x.\Phi$ is not proper relative to a *bse*.

When we consider that the formal analog within predicate logic to a singular subject-predicate sentence of natural language is the atomic (or, in the terminology of Pld, the 'elementary') symbolic sentence, that is, a sentence consisting of an n-ary predicate letter followed by n-many singular terms, it is easy to see that Pld offers a precise formalization of the F-S D as applied to definite descriptions. For, if A is an elementary sentence, then A lacks a truth value relative to a model *bse* if and only if A contains a description terms that is improper relative to *bse*, as we now show:

<u>Theorem 6</u>: Let *bse* be any base, A any elementary sentence. Then, one of $\pm$A belongs to $Cl(bse)$

iff every constant description term $\iota x.\Phi$ occurring in A is proper relative to *bse.*

Proof of 6.:

$\Rightarrow$ Assume that $\pm$A belongs to *Cl*(*bse*). Let $\iota x.\Phi$ be any constant description term occurring in A; we may assume that there is one, since otherwise it is trivially the case that every constant description term $\iota x.\Phi$ occurring in A is proper relative to *bse*. So A is of the form $[\iota x.\Phi / u]F$ for some elementary formula F containing free occurrence of a variable u. Then, since $\pm[\iota x.\Phi / u]F$ belongs to *Cl*(*bse*) and *Cl*(*bse*) $= \bigcup bse_\mu$, $0 \leq \mu < \in_0$, there is a $\mu < \in_0$ such that $\pm[\iota x.\Phi / u]F$ belongs to bse_μ. Let respectively Φ', F' be the result of uniformily replacing every occurrence in respectively Φ, F of any constant description term $\iota z.\Psi \not\equiv \iota x.\Phi$ occurring in $[\iota x.\Phi / u]F$ by a descriptum $t_{\iota z.\Psi} \in \delta(Par)$ for $\iota z.\Psi$ in *bse* whose existence is guaranteed by lemma 1.3.10.. By lemma 1.3.10., respectively $\pm[\iota x.\Phi' / u]F'$ belongs to bse_μ. Since $bse_\mu = \bigcup bse_\beta$, $0 \leq \beta < \mu$, and $\langle bse_\mu \rangle$ is well ordered under $\subseteq$, there is a least $\beta < \mu$ such that $\pm[\iota x.\Phi' / u]F'$ belongs to bse_β. Since $[\iota x.\Phi' / u]F'$ is not atomic, $\beta \neq 0$. Clearly, then, β is a successor ordinal. So $bse_\beta = bse_{\beta-1} \cup Sc(bse_{\beta-1})$. Then $\pm[\iota x.\Phi' / u]F'$ belongs to $Sc(bse_\beta)$ by virtue an instance $\mathbb{I}$ of some semantic rule. Since $[\iota x.\Phi' / u]F'$ is an elementary sentence, $\mathbb{I}$ is an instance of the semantic rule for descriptions 1.3.2.5. Since $\iota x.\Phi$ is the only constant description term occurring in $[\iota x.\Phi' / u]F'$, there are presuppositions to $\mathbb{I}$ of the form $+[t / x]\Phi$, $+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)$ for some constant term t and variable v. Hence, there are constant term t and variable v such that $bse_{\beta-1} \subseteq Cl(bse)$ contains both of the signed sentences $+[t / x]\Phi$, $+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)$. So $\iota x.\Phi$ is proper relative to *bse*.

$\Leftarrow$ Assume that every constant description term $\iota x.\Phi$ occurring in A is proper relative to *bse*. We show that one of $\pm$A belongs to *Cl*(*bse*) by induction on the number #(A) of description terms occurring in A.

Base step: #(A) = 0. Since A contains no description terms, A is a sentence of classical logic. Now it is easy to show that for all sentences **B**, if **B** contains no occurrence of any description term, then the sequent {-**B** , +**B**} is derivable, by induction on the syntactic complexity (defined in the usual way for sentences of classical logic) of **B**; indeed this is precisely the proof of the redundancy of Gentzen's original set of axioms containing the sequent S $\rightarrow$ S for every sentence S of first order classical logic. So, by semantic completeness, $\{-A, +A\} \cap Cl(bse) \neq 0$. Hence, one of $\pm$A belongs to $Cl(bse)$.

Induction step: #(A) > 0. Assume that one of $\pm$**B** belongs to $Cl(bse)$ for all sentences **B** of Pld such that #(**B**) < #(A). Let $\iota x.\Phi$ be any constant description term occurring in A; since #(A) > 0 we know there is one. So A is of the form $[\iota x.\Phi / u]F$ for some elementary formula F containing free occurrence of a variable u. Since one of $\pm[\iota x.\Phi / u]F$ belongs to $Cl(bse)$ and $Cl(bse) = \bigcup bse_\mu$, $0 \leq \mu < \epsilon_0$, there is a $\mu < \epsilon_0$ such that $\pm[\iota x.\Phi / u]F$ belongs to bse_μ. Then by lemma 1.3.10., there is a descriptum $t_{\iota x.\Phi} \in \delta(Par)$ for $\iota x.\Phi$ in bse such that respectively $\pm[t_{\iota x.\Phi} / u]F$ belongs to bse_μ. Clearly, $\#([t_{\iota x.\Phi} / u]F) < \#(A)$. So by the hypothesis of induction, one of, not respectively, $\pm[t_{\iota x.\Phi} / u]F$ belongs to $Cl(bse)$. By assumption $\iota x.\Phi$ is proper relative to bse, so there are constant term t and variable v such that $Cl(bse)$ contains both of the signed sentences $+[t / x]\Phi$, $+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)$. Then $Cl(bse)$ contains both of $+[t / x]\Phi$, $+(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)$ and one of $\pm[t_{\iota x.\Phi} / u]F$. Then by lemma 1.3.11., $Cl(bse)$ contains one of $\pm[\iota x.\Phi / u]F$. So the claim holds. □

As explained in the introductory section 0, the F - S D, as a doctrine concerning the primitive linguistic operations of reference and predication, is *prima facie* plausible to one who holds what we have called the "naive" view of the semantics of grammatically singular subject-predicate sentences. However, when we consider the difficulty of formulating a *compositional* semantic theory for sentences of natural language that satisfies the F-S D, that doctrine loses much of its initial attractiveness. For, if the semantic value of grammatically compound sentences is determined by the semantic value of its syntactic constituents, in particular, if the truth value of a compound sentence is a function of those of its subsentences, then *every* sentence is such that its truth evaluation depends ultimately on the truth evaluation of subject-predicate sentences. So, on any compositional semantic theory, the F-S D has ramifications for the evaluation of *all* types of sentences of natural language.

In particular, a compositional semantic theory for natural language that satisfies the F-S D must decide how the "undefinedness" of a subsentence of a given compound sentence effects the truth value the given sentence. Now, the simple-minded solution of straight forwardly generalizing the F-S D to all sentences of natural language, so that (on this generalized view) *any* sentence that contains a vacuous sigular term must lack a truth value, has obvious counter-intuitive consequences. For on this generalization of the F-S D, the English sentence 'Either grass is green or the present king of France is bald.' lacks a truth value. But clearly this consequence (of the generalized F-S D) threatens the monotonicity of many intuitively monotonic natural language inferences, since from the truth of a sentence **S** we may infer the truth of the sentence ⌜**S** or **P**⌝, where **P** is any English sentence.

More poignant examples of counter-intuitive consequences of the generalized F-S D can be found: Let S be any English sentence containing no occurrence of a vacuous singular term. Then, the English sentence ⌜Either S or not-S or the present king of France is bald.⌝, though seemingly a *logical* truth, lacks a truth value on this view. So does the the following English sentence which may be considered axiomatic of the Russellian Theory of Descriptions: 'If there is exactly one present King of France and he is bald, then the present king of France is bald'.

So there is a class of intuitively true (false) English sentences containing improper definite descriptions that present (at least *prima facie*) counterexamples to the acceptablity of the simple-minded generalization of the F-S D. Now, it is reasonable to require of any acceptable (compositional) semantic theory for natural language that satisfies the F-S D that it respect our intuitions concerning the truth value of the sentences in this "target" class. And certainly this requirement is all the more pressing on semantic theories for artificial logical languages designed to formalize the intuitive logical relationships holding between sentences of natural language. Then it is a virtue of Pld that it does respect the truth value of (the symbolic sentences corresponding to) the sentence in this class, as we now show.

We may characterize the target class of intuitively true (or false) English sentences containing improper definite descriptions whose truth value we are interested in preserving as being the union of the following three classes: 1) the class of English sentences whose truth (falsehood) seems to be guaranteed by the truth (falsehood) of their contingently true (false) subsentences, 2) the class of

English sentences S whose truth (falsehood) seems to be guaranteed by the (logical) truth (falsehood) of subsentences of S that correspond to theorems of the first order identity calculus without descriptions, and 3) the class of English sentences that correspond to theorems of a Russellian first order description calculus (which are not theorems of the conservative extension of first order identity calculus obtained by adding description terms to the elementary syntax of that theory).

The English sentence 'Either grass is green or the present king of France is bald.' is a representative of the class of English sentences containing improper definite descriptions whose truth seems to be guaranteed by the truth of their contingently true subsentences. Now, the formal analog to the notion of the contingent truth of an English sentence is the defined notion of "truth (validity)-relative-to a-base" of a symbolic sentence of Pld. Clearly, the unrestricted monotonicity of the semantic rules 1.3.2.2., 1.3.2.3 governing the binary sentential connective of Pld insures that if **A** is true relative to a given base *bse*, then so is (the unabreviated form of) the sentence **A** $\vee$ **B** where **B** is any symbolic sentence, in particular, one that contains a constant description term that is improper relative to *bse*.

The English sentence, ⌜Either **S** or not-**S** or the present king of France is bald.⌝, where **S** is any English sentence containing no occurrence of a vacuous singular term, is a representative the class of logically true English sentences containing improper definite descriptions whose truth seems to be guaranteed by the truth of their logically true subsentences. The formal analog within Pld of the intuitive notion of the logical truth of an English sentence is, of course, the defined notion of validity, or truth relative to all bases, of a symbolic sentence. Again, relative to a base *bse*, the formal analog within Pld of the notion of a vacuous singular term of English is the defined notion of a description term that is improper relative to *bse*. Now, it is easily shown (by an argument similar to the proof of lemma 6 from right to left) that, for any base *bse*, if **A** is a sentence of Pld in which the only occurring description terms are proper relative to *bse*, then (the unabreviated form of) the sentence **A** $\vee \neg$**A** is true relative to *bse*. Hence, it follows that the sentence (**A** $\vee \neg$**A**) $\vee$ **B** is true relative to *bse* for every sentence **B**. So we can say that if **A** is a sentence of Pld in which the only occurring description terms are proper relative to all bases *bse*, then $(\mathbf{A} \vee \neg\mathbf{A}) \vee \mathbf{B}$ is a logical truth of Pld, even if **B** contains an description term that is improper to some or all bases.

At this point one might object that since we have argued for the acceptability of Pld as a formal semantic theory satisfying the F-S D on the grounds that Pld insures the truth (falsehood) of symbolic sentences that correspond to intuitively true English sentences, we should bite the bullet and admit that the failure of excluded middle in Pld is a failure of Pld, period, since most people accept ⌜S or not-S⌝, for any sentence S, as a logical truth of English. However, one would here be objecting to the F-S D itself, rather than our particular formalization of it, since it is the basic intuition behind the F-S D that English sentences of the form ⌜S or not-S⌝ are **not**, in general, logically true.

The English sentence, 'If there is exactly one present king of France and he is bald, then the present king of France is bald', is a representative the class of intuitively true English sentences that correspond to theorems of a Russellian first order description calculus that are not theorems of the first order identity calculus (nor may be obtained from such theorems by the uniform weak substitution of description terms for terms of the first order identity calculus). Now, it is not the case that all symbolic sentences of Pld of the form $(\exists_x)(\forall_y)((\Phi \leftrightarrow y = x) \wedge F) \rightarrow [\iota x.\Phi / x]F$ are valid for arbitirary formulas Φ, F, since sequents of the form $\{-[t / x]\Phi,\ -(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v),\ -[t / u]F,\ +[\iota x.\Phi / u]F\}$ are not in general valid for arbitirary formulas Φ, F. However, if we restrict Φ and F so that all of the signed sentences $[t / x]\Phi$, $(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v)$, $[t / u]F$ are grounded, equivalently, obey excluded middle, then it is trivially the case that, by the semantic rule for descriptions 1.3.2.5., $\{-[t / x]\Phi,\ -(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v),\ -[t / u]F,\ +[\iota x.\Phi / u]F\}$ is a valid sequent. So, for every base *bse*, if neither of Φ, F contain description terms that are improper relative to *bse*, then the sequent $\{-[t / x]\Phi,\ -(x)(v)((\Phi \wedge [v / x]\Phi) \rightarrow x = v),\ -[t / u]F,\ +[\iota x.\Phi / u]F\}$ is valid relative to *bse*. In particular, if neither of Φ, F contain any description terms, the sequent in question is valid. So, if we construe the English predicates '*x* is a present King of France', '*x* is bald' as primitive, that is, corresponding to predicate letters of Pld, then the symbolic sentence of Pld corresponding to the English 'If there is exactly one present king of France and he is bald, then the present king of

France is bald' is valid.

We should note that sequents of the form $\{-[t/x]\Phi,\ -(x)(v)((\Phi \wedge [v/x]\Phi) \rightarrow x = v),\ -[t/u]F,\ +[\iota x.\Phi/u]F\}$, where all of the signed sentences $[t/x]\Phi$, $(x)(v)((\Phi \wedge [v/x]\Phi) \rightarrow x = v)$, $[t/u]F$ are grounded but neither of $[t/x]\Phi$, $(x)(v)((\Phi \wedge [v/x]\Phi) \rightarrow x = v)$ are valid/derivable, are examples of sequents that are valid relative to the semantics of Pld, but that are not derivable within a Pld version of logical syntax suggested for descriptions in Gilmore [3]. This is because the rule for descriptions given in [3] require the "presuppositions" to the application of the rule to be themselves derivable, rather than merely members of a derivable sequent. A simplified version of the rule given in [3], modified in the manner of Pld, is as follows:

$$\frac{\{+[t/x]\Phi\} \qquad \{+(x)(v)((\Phi \wedge [v/x]\Phi) \rightarrow x = v)\} \qquad Seq \cup \{\pm[t/u]F\}}{Seq \cup \{\pm[\iota x.\Phi/u]F\}}$$

It is clear that a logical syntax $\mathbb{LS}$ treating description terms with this rule only allows a sequent of the form $Seq \cup \{\pm[\iota x.\Phi/u]F\}$ to be derivable only if either there are term t and variable v such that both of the signed sentences $+[t/x]\Phi$, $+(x)(v)((\Phi \wedge [v/x]\Phi) \rightarrow x = v)$ are derivable or Seq is itself derivable (in which case $\pm[\iota x.\Phi/u]F$ is obtained in the derivation of $Seq \cup \{\pm[\iota x.\Phi/u]F\}$ by thinning). Now, consider a sequent $Seq \cong \{-[t/x]\Phi,\ -(x)(v)((\Phi \wedge [v/x]\Phi) \rightarrow x = v),\ -[t/u]F,\ +[\iota x.\Phi/u]F\}$, where all of the signed sentences $[t/x]\Phi$, $(x)(v)((\Phi \wedge [v/x]\Phi) \rightarrow x = v)$, $[t/u]F$ are grounded but neither of $[t/x]\Phi$, $(x)(v)((\Phi \wedge [v/x]\Phi) \rightarrow x = v)$ are derivable, which is such that $\{-[t/x]\Phi,\ -(x)(v)((\Phi \wedge [v/x]\Phi) \rightarrow x = v),\ -[t/u]F\}$ is not a derivable sequent. Clearly such a sequent exists. Since neither of $[t/x]\Phi$, $(x)(v)((\Phi \wedge [v/x]\Phi) \rightarrow x = v)$ are derivable sentences and $\{-[t/x]\Phi,\ -(x)(v)((\Phi \wedge [v/x]\Phi) \rightarrow x = v),\ -[t/u]F\}$ is not a derivable sequent, it follows that Seq is not derivable in $\mathbb{LS}$. But, since all of the signed sentences $[t/x]\Phi$, $(x)(v)((\Phi \wedge [v/x]\Phi) \rightarrow x = v)$, $[t/u]F$ are grounded, Seq is valid in Pld.

Bibliography

Carnap, R.

[1] *Meaning and Necessity*, Chicago, 1947

Frege, G.

[1] Über Sinn und Bedeutung, *Zeitschrift fur Philosophie und Kritik*, vol. 100 (1892), pp. 25 - 50. English translation in *Translations from the Philisophical Writings of Gottlob Frege*, edited by Geach and Black, Oxford, 1960.

[2] *Grundegesetze der Arithmetik*, vol. 1, Jena, 1893. English translation in *The Basic Laws of Arithmetic*, translated and edited with an introduction by M.Furth, Berkeley and Los Angeles, 1964.

Gilmore, P.C.

[1] Combining Unrestricted Abstraction with Universal Quantification, *To H. B. Curry: Essays on Combinatorial Logic, Lambda Calculus and Formalism*, J.P. Seldin and J.R. Hendley (eds.), Academic Press, i980, pp. 99 - 124.

[2] Natural Deduction Based Set Theories: A New Resolution of the Old Paradoxes, *Journal of Symbolic Logic*, June 1986.

Hilbert, D., and Bernays, P.

[1] *Grundlagen der Mathematik*, vol. 1, Berlin, 1934.

Kalish, D., Montague, R., Mar, G.,

[1] *Logic: Techniques of Formal Reasoning*, New York, 1980.

Quine, W.V.

[1] *Mathematical Logic*, New York, 1940.

Russell, B.

[1] On Denoting, *Mind*, vol. 14 (1905), pp. 479 - 93.

Strawson

[1] On Referring, *Mind*, vol. 59 (1950), pp. 320 - 344.

Whitehead, A. N., and Russell, B.

[1] *Principia Mathematica*, vol. 1, London, 1910; 2nd edition, 1925.