ON THE DYNAMICS OF SPACECRAFT WITH A SLEWING APPENDAGE

by

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ABSTRACT

A relatively general formulation for studying librational dynamics of a rigid satellite with a rigid appendage free to undergo any arbitrary slewing motion is developed. The governing nonlinear, nonautonomous and coupled equations are solved numerically. A parametric study suggests that the system can become unstable under critical combinations of inertia, geometric, orbital and slewing time history parameters. The fundamental information is relevant to the design of satellites and the Orbiter based experiments, and construction of the proposed space station, which would involve complex slewing motion of structural components.
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LIST OF SYMBOLS

\( \bar{a} \) position vector for mass element of appendage, Figure 2.3

\( \bar{c}, \bar{c}_o \) position vectors, Figure 2.3

\( c_\epsilon \) nondimensional gravitational constant, \( 3/(1+\epsilon \cos \theta) \)

\( c_\mu \) gravitational constant, \( 3\mu/R_c^3 \)

\( \bar{d}_a \) position vector, \( -\bar{c} -\bar{c}_o + \bar{h} + \bar{f}_3 \), Appendix B

\( \bar{d}_s \) position vector, \( -\bar{c} -\bar{c}_o \), Appendix B

\( \bar{d}_m, \bar{d}_s \) differential elements for appendage and central body, respectively, Figure 2.3

\( \bar{f} \) position vector, Figure 2.3

\( \bar{f}_3 \) projection of \( \bar{f} \) onto body axes \( x,y,z \); \( [M_a]\bar{f} \)

\( f_\epsilon \) nondimensional eccentricity function, \( -2\epsilon \sin \theta/(1+\epsilon \cos \theta) \)

\( \bar{g} \) position vector, \( -\bar{c} -\bar{c}_o + \bar{h} + \bar{f} \), Figure 2.3

\( \bar{h} \) position vector, Figure 2.3

\( i,j,k \) unit vectors along \( x,y,z \) axes, respectively

\( \bar{i}_a, \bar{j}_a, \bar{k}_a \) unit vectors along \( x_a, y_a, z_a \) axes, respectively

\( \bar{i}_s, \bar{j}_s, \bar{k}_s \) unit vectors along \( x_s, y_s, z_s \) axes, respectively

\( \{l\} \) direction cosines for unit vector along \( \bar{R}_c \) with respect to body axes \( x,y,z \)

\( m, m_a, m_s \) satellite, appendage and central body masses, respectively

\( \{q\} \) generalized coordinate vector for librational degrees of freedom

\( q_i \) generalized coordinate for \( i^{th} \) librational degree of freedom

\( \bar{r} \) position vector for a mass element \( \bar{d}m \)

\( \bar{r}_a \) position vector for mass element \( \bar{d}m_a, -\bar{c} -\bar{c}_o + \bar{h} + \bar{f} + \bar{a} \), Figure 2.3
\( \vec{r}_s \) position vector for mass element \( dm_s \), 
\( \vec{s} \) position vector for mass element of central body, Figure 2.3

t time

\( x,y,z \) satellite body coordinates with origin at \( C \), Figure 2.2

\( x_a,Y_a,z_a \) appendage body coordinates with origin at \( O_a \), Figure 2.2

\( x_s,Y_s,z_s \) central body coordinates with origin at \( O_s \), Figure 2.2

C instantaneous center of mass of satellite

\( C_0 \) center of mass of satellite before appendage slew motion

\( D_a,D_s \) domains associated with appendage and central body of satellite, Figure 2.2

E center of force

\([E]\) unit matrix

\([H]\) angular momentum with respect to body axes \( x,y,z \) due to slewing appendage

\([I]\) satellite inertia diadic

\([I_a]\) appendage inertia diadic

\( I_{arb} \) arbitrary inertia used to nondimensionalize the equations of motion

\([I_s]\) central body inertia diadic

\([M_a]\) transformation matrix defining rotation between \( x_a,Y_a,z_a \) and \( x,y,z \) axes

\( O_a,O_s \) centers of mass for appendage and central body, respectively, Figure 2.3

\( P \) hinge location on central body, Figure 2.3

\( \{Q\} \) generalized force vector for librational degrees of freedom

\( Q_i \) generalized force for \( i^{th} \) librational degree of freedom
<table>
<thead>
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<th>Symbol</th>
<th>Definition</th>
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<tr>
<td>$\vec{R}$</td>
<td>position vector from the center of force to a mass element dm</td>
</tr>
<tr>
<td>$R_C, \vec{R}_C$</td>
<td>scalar distance and position vector, respectively, from the center of force to the satellite's instantaneous center of mass</td>
</tr>
<tr>
<td>$T$</td>
<td>kinetic energy</td>
</tr>
<tr>
<td>$T_e$</td>
<td>effective kinetic energy contributing to the librational equations of motion</td>
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<tr>
<td>$U$</td>
<td>potential energy</td>
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<tr>
<td>$U_e$</td>
<td>effective potential energy contributing to the librational equations of motion</td>
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<tr>
<td>$X, Y, Z$</td>
<td>inertial coordinate system with origin at $E_0$</td>
</tr>
<tr>
<td>$X_0, Y_0, Z_0$</td>
<td>orbital coordinate system with origin at $C_0$</td>
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<tr>
<td>$a, \beta$</td>
<td>appendage slew motion coordinates, pitch and roll, respectively</td>
</tr>
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<td>initial appendage position coordinates, pitch and roll, respectively</td>
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<td>$a_f, \beta_f$</td>
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<td>$\xi_s$</td>
<td>inertia ratio for central body, $I_{s,z}/I_{s,x}$</td>
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<td>$\eta$</td>
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<tr>
<td>$\theta$</td>
<td>true anomaly</td>
</tr>
<tr>
<td>$\mu$</td>
<td>universal gravitational constant</td>
</tr>
<tr>
<td>$\vec{\xi}$</td>
<td>vector of normalized coordinates specifying hinge location on central body</td>
</tr>
<tr>
<td>$\rho$</td>
<td>mass density ratio of appendage to central body</td>
</tr>
<tr>
<td>$\tau_a, \tau_\beta$</td>
<td>slew motion durations for appendage pitch and roll, respectively</td>
</tr>
<tr>
<td>$\psi, \phi, \lambda$</td>
<td>librational coordinates defining pitch, roll and yaw, respectively</td>
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\( \vec{\omega}, \{\omega\} \)  satellite angular velocity vector

\( \vec{\omega}_a, \{\omega_a\} \) appendage angular velocity vector relative to central body

\( \Delta \alpha, \Delta \beta \) specified changes in coordinates due to appendage slew motion, pitch and roll, respectively

Dots and primes represent differentiation with respect to \( t \) and \( \theta \), respectively, unless otherwise defined; superscript * implies nondimensional parameter unless otherwise indicated.
1. INTRODUCTION

1.1 Preliminary Remarks

The Space Shuttle is operational and has proved its versatility in undertaking diverse missions. The next logical step would be to use it in the construction of operational bases for scientific exploration, satellite launch and maintenance, manufacture of products in the favourable microgravity environment, and the earth oriented applied technology programs. To this end, the U.S. has committed itself to the construction of a space station by the early nineties using the Shuttle based Space Transportation System for ferrying material and construction crew. The program presents a host of possibilities involving relative slewing motion of structural members, subassemblies and platforms. Even the steady state operational phase can involve slewing motion of antennas, scientific instruments, telescopes and other systems as in the case of the proposed Galileo spacecraft. Transfer of load over the space station using a Mobile Remote Manipulator System (MRMS) may require complex slewing time histories and can create challenging transient dynamics, stability and control problems.

In its utmost generality the problem can be quite complex: slewing motion of one or more translating, flexible systems supported by a flexible platform. To get some appreciation as to the fundamental dynamics as affected by the system inertia distribution, flexibility, orbital
parameters, slewing time histories and generalized forces, it is planned to approach the problem in an increasing order of complexity. As a first step, this thesis considers the slewing dynamics of a rigid appendage, of arbitrary inertia, with reference to a rigid platform in a general orbit.

1.2 A Brief Review of the Literature

Literature pertaining to librational dynamics of rigid and flexible spacecraft is indeed enormous and any attempt at a comprehensive review of this vast body of literature is considered unwarranted and beyond the scope of the present thesis. Fortunately, there are several papers that cover specific aspects of satellite dynamics quite thoroughly¹⁻⁷.

Of considerable interest is the relative motion of appendages involving deployment dynamics where a spacecraft extends flexible appendages outwardly from its central body. The complexity of the problem has often led to analyses involving simplifying assumptions. Lang and Honeycutt⁸ as well as Cloutier⁹ represented an appendage by point masses, while several other authors had treated the flexible members as rigid bodies¹⁰⁻¹². Cherchas and his associates¹³⁻¹⁴ as well as Dow et al.¹⁵ considered uniform flexible appendages with specific configurations and fixed deployment velocities. Jankovic¹⁶ investigated dynamics of the CTS solar panels during deployment and correlated measured tip acceleration with the analytical prediction. Ibrahim and Misra¹⁷ studied the librational dynamics of a body deploying
two plate type flexible members normal to the orbiting plane. The effect of deployment velocities and plate properties on the librational response was investigated. A general formulation for satellites with flexible deploying beam type appendages has been presented by Lips and Modi\textsuperscript{18}. Interaction between the libration dynamics, flexibility, and deployment was studied and it was noted that certain combinations of system parameters gave rise to instability. More recently, a rather general formulation representing the class of spacecraft with deploying flexible, extensible beam/plate type appendages has been presented by Ibrahim and Modi\textsuperscript{19}. This is of particular interest as the formulation takes into account the shifting center of mass and the changing central rigid body inertia, which are factors that also occur in the class of spacecraft with a slewing appendage. The control of a satellite with deploying appendages indeed represents an intriguing problem. The challenge has been well met with an analysis proffered by Sellappan and Bainum\textsuperscript{20} on the stochastic optimal control of a spinning spacecraft with movable telescoping appendages.

Also of relevant interest is the dynamics of slewing spacecraft. Several proposed missions require rapid large angle maneuvers and a high degree of pointing accuracy. This dictates a need for controlled satellite slewing maneuvers. Barba and Aubrun\textsuperscript{21} presented a mathematical description of the problem of controlling slewing motions of spacecraft by means of gyroscopic devices and gave geometrical
interpretations of the energy and momentum transfer through the "momentum sphere" and the "energy ellipsoid". A number of investigators\textsuperscript{22-25}, among them Kranton\textsuperscript{26}, and Skaar and Kraige\textsuperscript{27}, have addressed the problem of control via gyroscopic means, attempting to find control laws that minimize either maneuver time or energy consumption. Chen and Kane\textsuperscript{28} approached the slewing problem from a different angle, focusing attention not on control torques, but on the internal angular momentum of a gyrostat. Exploiting the use of active control, Dwyer and Batten\textsuperscript{29} opted for thrusters as well as reaction wheels. On the other hand, Junkins et al.\textsuperscript{30} have relied on the interaction between electromagnets and the Earth's magnetic field while Lunscher and Modi\textsuperscript{31} utilized solar pressure to execute slewing maneuvers. Junkins and Turner\textsuperscript{32}, and Vadali and Junkins\textsuperscript{33} introduced optimal large angle maneuvers of rigid spacecraft, while Juang et al.\textsuperscript{34} obtained a closed form solution for feedback control. Frauenholz\textsuperscript{35} used a maneuver reconstruction process to improve slew performance, and Dwyer and Fadali\textsuperscript{36} developed single step optimization strategies suitable for simultaneous slew trajectory planning and tracking. Control laws involving nonlinear feedback were presented by Carrington and Junkins\textsuperscript{37-39}, and Dwyer\textsuperscript{40}. The slewing dynamics of flexible spacecraft have been analyzed by several investigators\textsuperscript{41-55}, with contributions by Breakwell\textsuperscript{54} and Meirovitch\textsuperscript{55} of considerable significance.
The area of instrument or appendage pointing systems is quite relevant to this investigation. Of particular interest is the study done by Chrétien et al.\textsuperscript{56} in which the attitude control of a satellite with a rotating solar array was considered. Other pointing systems analyzed were: an auxiliary instrument station, by Joshi\textsuperscript{57}; the Voyageur high gain antenna, by Jahanshahi\textsuperscript{58}; and the Space Telescope pointing control unit, by Dougherty et al.\textsuperscript{59} Also, Broquet et al.\textsuperscript{60} studied antenna pointing systems for large communications satellites, and Yuan and Stieber\textsuperscript{61} examined the robust beam-pointing and attitude control of a flexible spacecraft. The pointing performance of a dual-spin spacecraft was evaluated by Hayati and Jahanshahi\textsuperscript{62} while Bell and Lin\textsuperscript{63} presented a study of pointing requirements and control approaches for an Earth Observation System. Alberts et al.\textsuperscript{64} have developed a scheme for controlling the motion of a lightweight flexible arm. Laurenson\textsuperscript{65} presented a modal analysis of rotating flexible structures while Swigert\textsuperscript{66} investigated the effect of torque shapes on the response of a flexible structure.

1.3 Purpose and Scope of the Investigation

From the review of the relevant literature, it is apparent that the dynamics of satellites with slewing appendages has considerable ground to cover.

The purpose of this investigation is to attempt a preliminary analysis of the problem in the hope that it will
provide insight into the dynamical behaviour of satellites with slewing appendages. To gain better appreciation of the system behaviour, the model is purposely kept simple: spacecraft with a rigid central body and rigid appendage operating in absence of any environmental forces.

The study is initiated with the formulation of the equations of motion for a general spacecraft model consisting of two connected, rigid bodies of arbitrary geometry and inertia distribution, where one body exhibits specified relative rotation about the other.

This is followed by the response analysis of a specific configuration, a cylindrical central body with a cylindrical appendage, as affected by the appendage slew motion, satellite inertia and geometry, initial conditions and orbit eccentricity. Even for this relatively simple situation, numerical integration of the highly nonlinear, nonautonomous and coupled system of equations presents a challenging task. Emphasis throughout is on methodology and physical appreciation of system behaviour which may help investigation of more sophisticated models. The thesis ends with some useful concluding remarks and thoughts on possible directions for future studies.
2. FORMULATION OF THE PROBLEM

The study is initiated with the consideration of a general spacecraft model consisting of two connected, rigid bodies of arbitrary geometry and inertia distribution, where one body executes specified relative rotation about the other. The equations of motion are developed for the general model.

Next, a particular case of cylindrical geometry for the central body and the appendage is considered. A set of satellite parameters specify possible variations of this configuration. Variations in the appendage slewing motion are expressed in terms of time history parameters.

2.1 Spacecraft Model

Consider a satellite comprised of two rigid bodies of arbitrary geometry and inertia distribution, connected through an ideal (frictionless) joint (Figure 2.1.). The appendage is capable of executing relative rotation about the central body causing a shift in the satellite center of mass. The satellite is free to negotiate an arbitrary, specified trajectory. The position vector $\vec{R}_C$ and true anomaly $\theta$ define the location of the instantaneous center of mass $C$ of the spacecraft with respect to the inertial coordinate system $X,Y,Z$ having its origin at $E$, the center of force. $C_0$ represents location of $C$ in absence of any appendage slew motion. An orthogonal, orbiting reference frame $X_0,Y_0,Z_0$ with its origin at $C_0$ is so oriented that $Z_0$
Figure 2.1 Geometry of orbiting spacecraft with a slewing appendage.
and $X_0$ are along the local vertical and horizontal, respectively, while $Y_0$ is aligned with the orbit normal. The satellite body coordinate system $x,y,z$ with unit vectors $\hat{i},\hat{j},\hat{k}$ has its origin at $C$ and coincides with the orbital frame $X_0,Y_0,Z_0$ in the absence of any librations and relative appendage rotation. Orientation of the satellite axes $x,y,z$ at any instant $t$ relative to orbital frame $X_0,Y_0,Z_0$ is described by a set of modified Eulerian rotations $q_1,q_2,q_3$.

The spacecraft is schematically illustrated in Figure 2.2. The domain, mass, and center of mass for the central body are referred to, respectively, as $D_s, m_s, O_s$. Corresponding parameters for the appendage are specified as, $D_a, m_a, O_a$, respectively. The total mass $m$ of the satellite is thus given by

$$m = m_s + m_a.$$

Located at $O_s$ is a central body coordinate frame $x_s,y_s,z_s$ with unit vectors $\hat{i}_s,\hat{j}_s,\hat{k}_s$. Similarly located at $O_a$ is an appendage body coordinate system $x_a,y_a,z_a$ with unit vectors $\hat{i}_a,\hat{j}_a,\hat{k}_a$.

The satellite body frame $x,y,z$ for the entire system (central body and appendage) is taken to be parallel to the central body frame $x_s,y_s,z_s$.

The orientation of the axes $x_a,y_a,z_a$ relative to the frame $x,y,z$ is specified by a transformation matrix $[M_a]$, which is determined by the order and number of rotations.
Figure 2.2 Domains associated with central and appendage components of spacecraft and corresponding reference coordinate systems.
used to arrive at frame $x_a', y_a', z_a$ from satellite body axes $x, y, z$.

Let $\bar{s}$ and $\bar{a}$ define the positions of the differential mass elements $dm_s$ and $dm_a$ in domains $D_s$ and $D_a$, respectively (Figure 2.3).

$$\bar{s} = x_s \hat{i} + y_s \hat{j} + z_s \hat{k}$$

$$\bar{a} = x_a \hat{i} + y_a \hat{j} + z_a \hat{k}$$

The vector $\bar{c}$ determines the position of the instantaneous center of mass $C$ relative to point $C_0$. The vector $\bar{h}$ locates hinge point $P$ with respect to $O_s$, and $\bar{f}$ positions $O_a$ with respect to $P$. Due to rotation of the appendage relative to the central body, $\bar{c}$ and the satellite inertia diadic will be functions of time.

2.2 Development of the General Equations of Motion

2.2.1 Kinetic energy

The kinetic energy of the system can be written as

$$T = \frac{1}{2} \int_{m} \dot{R} \cdot \ddot{R} \ dm ,$$

where $m$ is the total mass of the spacecraft and $\bar{R}$ defines the position of a differential element of mass $dm$ with respect to $E$. If $\bar{R}$ is the position vector of the element
Figure 2.3 Schematic diagram showing position vectors to mass elements in different domains.
mass with respect to the body coordinate system \(x, y, z\), then

\[
\vec{R} = \vec{R}_c + \vec{r},
\]

where \(\vec{r}\) for different domains is defined as

\[
\vec{r} = \begin{cases} 
\vec{r}_s = -\vec{c} - \vec{c}_o + \vec{s}, & \text{D}_s \text{ domain;} \\
\vec{r}_a = -\vec{c} - \vec{c}_o + \vec{h} + \vec{f} + \vec{a}, & \text{D}_a \text{ domain.}
\end{cases}
\]

Differentiating \(\vec{R}\) with respect to time

\[
\dot{\vec{R}} = \dot{\vec{R}}_c + \frac{\partial \vec{r}}{\partial t} + (\vec{\omega} \times \vec{r}) \quad \ldots \quad (2.2)
\]

where \(\vec{\omega}\) is the librational angular velocity of the satellite; \(\dot{\vec{R}}_c\) is the orbital (tangential and radial) velocity of the satellite; and \(\frac{\partial \vec{r}}{\partial t}\) represents the time rate of change of \(\vec{r}\) in the reference coordinate system \(x, y, z\).

Substituting equation (2.2) into equation (2.1)

\[
T = \frac{1}{2} \int m \dot{\vec{R}}_c \cdot \dot{\vec{R}}_c + \frac{\partial \vec{r}}{\partial t} \cdot \frac{\partial \vec{r}}{\partial t} + (\vec{\omega} \times \vec{r}) \cdot (\vec{\omega} \times \vec{r})
\]

\[
+ 2\dot{\vec{R}}_c \cdot \frac{\partial \vec{r}}{\partial t} + 2\dot{\vec{R}}_c \cdot (\vec{\omega} \times \vec{r}) + 2\frac{\partial \vec{r}}{\partial t} \cdot (\vec{\omega} \times \vec{r}) \, dm
\]

It may be noted that
\[
\frac{\partial \tilde{r}}{\partial t} = \frac{\partial \tilde{r}_S}{\partial t} = -\dot{\tilde{c}} \quad \text{in the } D_S \text{ domain and}
\]
\[
= \frac{\partial \tilde{r}_a}{\partial t} = -\dot{\tilde{c}} + \tilde{\omega}_a (\tilde{r} + \tilde{a}) \quad \text{in the } D_a \text{ domain},
\]

where \( \tilde{\omega}_a \) represents the relative angular velocity of the appendage with respect to the central body and \( \dot{\tilde{c}} \) is the velocity of the instantaneous center of mass \( C \) in the reference coordinate system \( x,y,z \).

Note, the kinetic energy terms without generalized coordinates or velocities do not contribute to Lagrange's equations of motion for the librational degrees of freedom due to differentiation with \( q \) and \( \dot{q} \) involved. Hence, the effective component of the kinetic energy contributing in formulation of the equations of motion, \( T_e \), can be written as,

\[
T_e = \frac{1}{2} \int_m (\tilde{\omega} \times \tilde{r}) \cdot (\tilde{\omega} \times \tilde{r}) \, dm + 2\tilde{R}_C \cdot (\tilde{\omega} \times \tilde{r}) + 2\frac{\partial \tilde{r}}{\partial t} \cdot (\tilde{\omega} \times \tilde{r}) \, dm.
\]

Since \( \tilde{R}_C \) is a constant over the spacecraft,

\[
\int_m \tilde{R}_C \cdot (\tilde{\omega} \times \tilde{r}) \, dm = \tilde{R}_C \cdot \tilde{\omega} \int_m \tilde{r} \, dm.
\]

But \( \int_m \tilde{r} \, dm = 0 \), so the kinetic energy expression takes the form,

\[
T_e = \frac{1}{2} \int_m (\tilde{\omega} \times \tilde{r}) \cdot (\tilde{\omega} \times \tilde{r}) \, dm + \int_m \frac{\partial \tilde{r}}{\partial t} \cdot (\tilde{\omega} \times \tilde{r}) \, dm.
\]
Clearly, the first part of the kinetic energy,

$$\frac{1}{2} \int m (\bar{\omega}_i \bar{r}) \cdot (\bar{\omega}_i \bar{r}) \, dm,$$

can be written as

$$\frac{1}{2} \{\omega\}^T [I] \{\omega\};$$

and the second term of the kinetic energy,

$$\int m \frac{\partial \bar{r}}{\partial t} \cdot (\bar{\omega}_i \bar{r}) \, dm,$$

can be expressed as (Appendix A)

$$\{\omega\}^T \{H\}.$$

So the kinetic energy of the system can be expressed in a compact form as

$$T_e = \frac{1}{2} \{\omega\}^T [I] \{\omega\} + \{\omega\}^T \{H\}, \quad \ldots (2.3)$$

where:

$$\{\omega\} \quad \text{angular velocity vector;}$$

$$[I] \quad \text{time dependent satellite inertia matrix;}$$
time dependent angular momentum vector with respect to satellite body frame due to rotation of the appendage relative to the central body;

\( \frac{1}{2}\{\omega\}^T[I]{\omega} \) kinetic energy due to pure librational rotation of the satellite;

\( \{\omega\}^T[H] \) kinetic energy due to coupling between librational rotation of the satellite and rotation of the appendage relative to the central body.

As the appendage undergoes relative rotation about the central body, the satellite's moments and products of inertia change. This change is primarily due to reorientation of the appendage with respect to the central body and to a lesser degree the relocation or shifting of the satellite's instantaneous center of mass. Computation of the time dependent satellite inertia matrix [I] is achieved through use of the parallel-axis theorems for moments and products of inertia (Appendix B).

2.2.2 Potential energy

The gravitational potential energy of the system can be written as

\[
U = -\mu \int_m \frac{dm}{|R|}
\]  

\(\text{(2.4)}\)
After binomial expansion and truncation of the series, the potential energy can be written as

$$U = -\mu \left( m_s + m_a \right) / R_c - \mu / (2R_c^3)^* \left[ I_{xx}(1-3l_x^2) + I_{yy}(1-3l_y^2) + I_{zz}(1-3l_z^2) + 6(I_{xy}l_xl_y + I_{yz}l_yl_z + I_{zx}l_zl_x) \right]$$

$$= -\mu (m_s + m_a) / R_c - \mu / (2R_c^3) \text{tr}[I] + 3\mu / (2R_c^3) \{l\}^T[I]\{l\},$$

where:

- $\mu$ universal gravitational constant;
- $R_c$ distance from the center of force to the satellite center of mass;
- $\{l\}$ vector of direction cosines for unit vector along $\overline{R}_c$ with respect to body axes $x, y, z$, respectively, $\{l_x, l_y, l_z\}$;
- $I_{xx}, I_{yy}, I_{zz}$ components of satellite inertia matrix $[I]$.

Here the first term represents potential energy due to the satellite treated as a point mass and the rest of the expression is the contribution due to its finite size and rotation. The first two terms will disappear upon differentiation of the potential energy for the librational degrees of freedom in Lagrange's equations of motion. Hence, the effective potential energy so far as the derivation of
the equations of motion are concerned, can be finally written as

\[ U_e = c \mu / 2 [l][I][l] \]  \hspace{1cm} \text{...(2.5)}

where \( c \mu = 3\mu / R_c^3 \).

2.3 Governing Equations of Motion

The governing equations of motion can now be obtained using the Lagrange principle,

\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = Q_i \]  \hspace{1cm} \text{...(2.6)}

where \( q_i \) and \( Q_i \), respectively, represent the generalized coordinate and force for the \( i^{th} \) librational degree of freedom. The development of the equations of motion is given in detail in Appendix C. The nondimensional equations of motion, in vector-matrix form, appear as follows

\[
[M]^T [I^*] [M] \dot{q} + [\dot{q}][([M_v] - [M_q])] [I^*] [M] \dot{q} \\
+ [M]^T [I^*] [\dot{q}] [M_v] \{\dot{q}\} + [\dot{q}][([M_v] - [M_q]) ([I^*] [N] \\
+ \{H^*}\] + [([M]^T [I^*] - [N_q]) [I^*]) [M] + [M]^T [I^*] ([N_v] \\
+ \{H^*\} + [I^*] \{N\} f_\epsilon\} + c_\epsilon [L_q] [I^*] \{l\} = \{Q^*\}.
\]

It should be emphasized that the nondimensional system parameters (matrices \([I^*]\), \([i^*]\), and vectors \(\{H^*\}, \{\dot{H}^*\}\)) are
functions of the satellite configuration and the appendage slew motion. Hence, the equations of motion can be easily obtained for any satellite configuration that can be described by the general spacecraft model of section 2.1. Furthermore, it should be noted that the equations of motion are valid for any sequence of modified Eulerian rotations chosen to describe librational motion for the satellite.

2.4 Librational Terms for the Equations of Motion

The orientation of body axes x, y, z relative to the orbital coordinate frame $X_0, Y_0, Z_0$ at any instant $t$ can be described by a set of modified Eulerian rotations as follows (Figure 2.4):

- $\psi$ (pitch) about $Y_0$ giving $x', y', z'$ ;
- $\phi$ (roll) about $x'$ giving $x'', y'', z''$ ;
- $\lambda$ (yaw) about $z''$ giving $x, y, z$ .

The librational terms for this particular sequence of modified Eulerian rotations are listed in Appendix D.

2.5 Specific Satellite Configuration Selected for Study

2.5.1 Cylindrical central body with a cylindrical appendage

Consider the central body as a cylinder with axis $z_s$ aligned with its axis of symmetry. Similarly, let the appendage be a cylinder with axis $z_a$ aligned with its axis
Figure 2.4 Eulerian rotations showing pitch ($\psi$), roll ($\phi$), and yaw ($\lambda$) librations.
of symmetry (Figure 2.5).

As a result of this configuration, the transverse moments of inertia for each body are equal due to the axisymmetric character of a cylinder, and the products of inertia vanish.

Let the hinge location on the appendage be restricted to the center of the axial end of the appendage cylinder and let the placement of the hinge on the central cylinder be arbitrary.

Let the density ratio be known.

2.5.2 Governing parameters for the two cylinder configuration

The inertia properties of a cylinder can be represented by the ratio of its axial to transverse moment of inertia. A ratio close to zero represents a slender rod, while that approaching two describes a squat, flat dish (Figure 2.6).

The inertia parameters are selected as follows:

\[
\begin{align*}
\xi_s & \quad \text{inertia ratio of central cylinder, } I_{s,z}/I_{s,x} ; \\
\xi_a & \quad \text{inertia ratio of appendage cylinder, } I_{a,z}/I_{a,x} ; \\
\eta & \quad \text{mass ratio of appendage to central body, } m_a/m_s .
\end{align*}
\]

The parameters \( \xi_s, \xi_a, \eta \) suffice to describe all possible inertial combinations for a two cylinder satellite.

To specify the location of the hinge on the central cylinder, consider a normalized coordinate system \( \xi_x, \xi_y, \xi_z \).
Figure 2.5 Specific satellite configuration - cylindrical central body with a cylindrical appendage.
Figure 2.6 Variation of the inertia ratio $\zeta$ with satellite geometry.
located at $O_s$ with $\xi_x$, $\xi_y$, normalized using the central cylinder radius and $\xi_z$ with reference to half the length of the central cylinder. The geometric parameter

$$\bar{\xi} = (\xi_x, \xi_y, \xi_z)^T$$

specifies the hinge position $P$ with respect to $O_s$, the center of mass for the central cylinder.

Thus, satellite parameters $\xi_s$, $\xi_a$, $\eta$, $\bar{\xi}$ specify any possible orientation of a cylindrical satellite with a cylindrical appendage.

### 2.5.3 Appendage rotations

The orientation of a line in space can be determined by specifying two angles. The line of direction for the appendage relative to the central body can therefore be specified by two angles relative to the central body.

Consider then, located at the appendage center of mass, the appendage coordinate system $x'_a, y'_a, z'_a$ parallel to the spacecraft body axes $x, y, z$ (Figure 2.7). The orientation of the appendage coordinate axes $x'_a, y'_a, z'_a$ relative to the body axes $x, y, z$ is specified by the following set of rotations:

- $\alpha$ (appendage pitch) about $y'_a$ giving $x''_a, y'_a, z'_a$;
- $\beta$ (appendage roll) about $x''_a$ giving $x'_a, y'_a, z'_a$. 
Figure 2.7 Appendage slew motion relative to central body.
The resulting transformation matrix \([M_a]\) and appendage relative angular velocity vector \(\vec{\omega}_a\) and their respective derivatives are listed in Appendix E.

2.5.4 Appendage time history parameters

The path traversed by the appendage as it moves from one position to another can be specified by a time history curve. Two time history functions are required to completely specify the appendage's slew motion composed of pitch \(\alpha\) and roll \(\beta\). If the shape of the function is known (i.e., exponential, ramp, sinusoidal, cubic, etc.) then the time history is completely specified once the initial position, the final position, and the duration of the slew maneuver are given. The parameters for the time history curves are defined as follows:

\[
\begin{align*}
\alpha_i, \beta_i & \quad \text{initial position in pitch and roll, respectively;} \\
\alpha_f, \beta_f & \quad \text{final position in pitch and roll, respectively;} \\
\tau_\alpha, \tau_\beta & \quad \text{duration of slew motion in pitch and roll, respectively.}
\end{align*}
\]

2.6 System Parameters

In section 2.3, the nondimensional equations of motion were formulated based on the general spacecraft model described in section 2.1. The system parameters \([I^*]\),
The equations of motion for any satellite configuration are easily obtained by evaluating the system parameters for that particular satellite.

In this section, the system parameters are evaluated for the two cylindrical element satellite configuration with the pitch and roll appendage slew sequence.

2.6.1 Instantaneous center of mass

The rotation of the appendage relative to the central body causes a shifting of the satellite's center of mass. Determination of the position, velocity, and acceleration of the satellite's center of mass is necessary for calculation of its inertia matrix $[I^*]$, angular momentum vector $\{H^*\}$, and their respective derivatives $[\dot{I}^*]$ and $\{\dot{H}^*\}$. The position, velocity, and acceleration of the shifting center of mass for the two cylinder satellite with the pitch, roll appendage rotation sequence are presented in Appendix F.

2.6.2 Evaluation of system parameters

The system parameters $[I^*]$, $\{H^*\}$, $[\dot{I}^*]$, $\{\dot{H}^*\}$ are functions of the satellite configuration and the appendage time history.

The desired inertial parameters $\xi_5$, $\xi_a$ are obtained by setting $I_{arb} = I_{S,x}$. Then from section 2.3 (Appendix C):
\[ [I^*] = [I] / I_{s,x} ; \]
\[ \{H^*\} = \{H\} / (I_{s,x} \dot{\theta}) ; \]
\[ [\dot{i}^*] = [\dot{i}] / (I_{s,x} \dot{\theta}) ; \]
\[ \{\ddot{H}^*\} = \{\ddot{H}\} / (I_{s,x} \dot{\theta}^2) . \]

The elements of \([I^*], \{H^*\}, [\dot{i}^*], \{\ddot{H}^*\}\) for the specific satellite configuration under consideration with the appendage pitch, roll slew sequence are respectively listed in Appendices G, H, I, J.
3. RESULTS AND DISCUSSION

3.1 Computational Considerations

The governing nonlinear, nonautonomous and coupled equations of motion are programmed for numerical integration using an AMDAHL 470-V8 digital computer. The integration routine (DGEAR - I.M.S.L.) is based on the implicit Adams' method with built-in error control\textsuperscript{67-68}.

The dynamics of the system is represented by three second-order differential equations corresponding to three librational degrees of freedom. Using the conventional procedure, the equations are transferred into six first-order equations where all degrees of freedom are solved for simultaneously as an initial value problem. For the procedure to succeed, it is necessary to use the latest available data while updating derivatives. The program is coded in FORTRAN using double-precision variables throughout.

3.2 Effect of Appendage Time History

Several appendage time history functions were considered to evaluate their effect on system response. The angular displacement, velocity and acceleration for the selected appendage time histories are shown in Figures 3.1 and 3.2. The equations for the various time histories are listed in Appendix K.
Figure 3.1 Appendage time history with finite initial velocity: (a) exponential displacement; (b) velocity; (c) acceleration.
Appendage Time
History Parameters

$\alpha_i = -90^\circ$
$\alpha_f = 0^\circ$
$T_\alpha = .001$ orbit
$\beta_i = \beta_f = \tau_\beta = 0$

Figure 3.1 (cont.) Appendage time histories with finite initial velocity: (d) sinusoidal or ramp displacement; (e) velocity; (f) acceleration.
Figure 3.2 Appendage time history with zero initial velocity: (a) exponential displacement; (b) velocity; (c) acceleration.

Appendage Time History Parameters

\[
\begin{align*}
\alpha_i &= -90^\circ \\
\alpha_f &= 0^\circ \\
\tau_a &= 0.001 \text{ orbit} \\
\beta_i = \beta_f = \tau_\beta &= 0
\end{align*}
\]
Appendage Time
History Parameters

- $\alpha_i = -90^\circ$
- $\alpha_f = 0^\circ$
- $\tau_\alpha = .001$ orbit
- $\beta_i = \beta_f = \tau_\beta = 0$

Figure 3.2 (cont.)  Appendage time histories with zero initial velocity: (d) sinusoidal, ramp, or cubic displacement; (e) velocity; (f) acceleration.
Figure 3.3 shows the system response to the time histories of Figure 3.1 with various durations of slew motions. It can be seen that slew motions with a finite initial velocity can readily cause a satellite to become unstable. Clearly, the usefulness of such slew motions is limited as restrictions must be imposed on the duration of the slewing maneuvers to ensure acceptable response.

Figure 3.4 shows the system response to the time histories of Figure 3.2 and illustrates the limiting effect a smooth starting slew motion has on the amplitude of libration. Note, even changing the slew period by two orders of magnitude seems to have virtually no effect on the response. Thus it can be concluded that time histories with zero initial velocity are preferable to slew motions that "jump start".

Figure 3.5(a) shows the system response to the smooth time histories of Figure 3.2 for appendage slew motion duration of 0.001 orbit while Figures 3.5(b),(c) present the responses to the same time histories but with extended slew motion durations of 0.1 and 0.3 orbits, respectively. It is of interest to note that the smooth time histories do not affect the satellite response substantially. Note, the system response to the smooth time histories remain the same for slew motions of short duration (approximately 0.1 orbit) and begin to deviate as \( \tau_a \) becomes relatively large. Thus for smooth slew motions of short duration, time history variations have negligible effect on system response. The
Figure 3.3 System response showing effect of appendage time histories with finite initial velocity: (a) exponential; (b) sinusoidal; (c) ramp.
Satellite Parameters

\[
\begin{align*}
\zeta_s &= 0.06 \\
\zeta_a &= 0.06 \\
\eta &= 0.06 \\
\xi' &= (0 \ 0 \ 1)
\end{align*}
\]

Appendage Time History Parameters

\[
\begin{align*}
\alpha_i &= -90^\circ \\
\alpha_f &= 0^\circ \\
\beta_i &= \beta_f = \tau_\beta = 0
\end{align*}
\]

Initial Conditions

\[
\begin{align*}
\psi(0) &= \psi'(0) = 0 \\
\phi(0) &= \phi'(0) = 0 \\
\lambda(0) &= \lambda'(0) = 0
\end{align*}
\]

Eccentricity

\[
\varepsilon = 0
\]

Figure 3.4 System response showing effect of appendage time histories with zero initial velocity: (a) sinusoidal; (b) ramp; (c) cubic.
Figure 3.5 System response showing the effect of duration of slewing maneuver with zero initial velocity: (a) 0.001 orbit; (b) 0.1 orbit; (c) 0.3 orbit.
system response is mainly governed by a disturbance torque during slew motion. For slew motions of longer duration, the gravitational field has more "time" to exercise its influence, hence the increased deviation in system responses to various time histories with lengthy durations.

A close up view of the system response during slew motion illustrated in Figure 3.5(a) is given in Figure 3.6. It can be seen that all smooth slew motions of short duration, regardless of the time history chosen, result in the same conditions of libration at the end of the slew motion. Thus despite the variation in librations during slewing maneuvers, the system response after the slew motion is identical.

As the smooth time histories do not affect the steady state response except for a long slow period, from now on the attention is focused on the cubic function.

Figure 3.7 shows the effect of appendage slew motion in pitch, with a specific cubic time history, on system response. Effect of slew duration ($\tau_a$) as well as initial and final appendage orientations are considered. As can be expected, effect of the gravitational moment during slew motion is more pronounced for longer $\tau_a$ (Figure 3.7a). Furthermore, it can be concluded that longer the duration to traverse a specific path, smaller the resulting amplitude of libration.

Figure 3.7(b) illustrates the effect of the magnitude of the slew motion on system response. It shows that the
Figure 3.6 System response during slew motion for various time histories: (a) pitch response; (b) pitch rate response.
### Satellite Parameters
- \( \xi_s = 0.06 \)
- \( \xi_a = 0.06 \)
- \( \eta = 0.06 \)
- \( \xi^T = (0 \\ 0 \\ 1) \)

### Appendage Parameters
- \( C_a = 0.06 \)
- \( \gamma = 0.06 \)

### Cubic Time History Parameters
- \( \beta_i = \beta_f = \tau_f = 0 \)

### Initial Conditions
- \( \psi(0) = \psi'(0) = 0 \)
- \( \phi(0) = \phi'(0) = 0 \)
- \( \lambda(0) = \lambda'(0) = 0 \)

### Eccentricity
- \( \varepsilon = 0 \)

---

**Figure 3.7** System response showing effect of appendage pitch: (a) duration; (b) initial position; (c) final position.
longer the path that must be traversed within a specific duration of time, the greater the resulting amplitude of libration.

Influence of the appendage's relative orientation on system response is vividly illustrated in Figure 3.7(c). The orientation of the appendage relative to the central body determines the satellite inertia diadic, and hence the orientation of the axis of minimum moment of inertia with respect to the satellite body axes. The equilibrium state for the satellite is determined by the orientation of the axis of minimum moment of inertia - the gravitational moment tends to align the minimum moment of inertia axis towards the center of force. Therefore, a position of the appendage relative to the central body determines the equilibrium state of the satellite.

Influence of the satellite's orbital velocity on the response during the symmetric pitch slew motion is shown in Figure 3.8. It is interesting to note that the symmetric pitch slew motions of the appendage do not result in symmetric response - there is a phase shift. Note also a change in the satellite's equilibrium state as affected by the appendage's relative orientation.

Figure 3.9 shows the response due to appendage roll slew motion. Now all the librational degrees of freedom are excited. Strong coupling exists between roll and yaw, and the satellite yaw is apparently unlimited. Although libration in yaw gives no indication as to the stability of
Satellite Parameters

\[ \xi_s = 0.06 \]
\[ \xi_a = 0.06 \]
\[ \eta = 0.06 \]
\[ \bar{r}_x = (0 \ 0 \ 1) \]

Appendage Parameters

Cubic Time History Parameters

\[ \tau_a = 0.001 \text{ orbit} \]
\[ \beta_i = \beta_f = \tau_B = 0 \]

Initial Conditions

\[ \psi(0) = \psi'(0) = 0 \]
\[ \phi(0) = \phi'(0) = 0 \]
\[ \lambda(0) = \lambda'(0) = 0 \]

Eccentricity

\[ \varepsilon = 0 \]

Figure 3.8 System response showing effect of appendage pitch: (a) initial position; (b) final position.
### Satellite Parameters

- $\zeta_s = 0.06$
- $\zeta_a = 0.06$
- $\eta = 0.06$
- $\xi^* = (0 \ 0 \ 1)$

### Appendage Parameters

- Cubic Time
- History Parameters
- $\tau_\beta = 0.001$ orbit
- $\alpha_i = \alpha_i = \tau_\alpha = 0$

### Initial Conditions

- $\psi(0) = \psi'(0) = 0$
- $\phi(0) = \phi'(0) = 0$
- $\lambda(0) = \lambda'(0) = 0$
- Eccentricity $\varepsilon = 0$

### Figure 3.9

System response due to appendage slew motion in roll: (a) pitch response; (b) roll response; (c) yaw response.
a satellite, it does affect its pointing accuracy (except when an instrument is aligned with the spin axis).

Of course, in general, the appendage is free to undergo both roll and pitch motion simultaneously to attain an arbitrary orientation. Response for such general slew motion is presented in Figures 3.10 and 3.11. As can be expected, now all the librational degrees of freedom are excited, with yaw particularly sensitive to the appendage roll.

Figure 3.12 shows the effect of different slew trajectories leading to the same appendage orientation. Three cases are considered. Solid lines represent the response of a satellite with the appendage executing pitch and roll slew motion simultaneously from $\theta=0$ to $\theta=0.001$ orbit. This is contrasted against the librational response with the appendage first slewing in pitch over 0.0005 orbit, and then executing roll slew motion to give total $r=0.001$ orbit. Finally, in the third case the sequence of slew motion is reversed, i.e., the roll followed by the pitch over the same duration. It is significant that different slew trajectories leading to the same orientation of the appendage have little effect on the satellite response. The librational dynamics is mainly governed by the final satellite inertia diadic and the disturbance torque from the appendage slew motion. The deviation in satellite response is, of course, attributed to the variation in the slew time histories of the appendage. For slew motions of longer duration, the effect of the specified appendage path on the
### Satellite Parameters
- $\xi_s = 0.06$
- $\xi_o = 0.06$
- $\eta = 0.06$
- $\xi^T = (0 0 1)$

### Appendage Parameters
- Cubic Time
- History Parameters
  - $\alpha_i = \beta_i = 0^\circ$
  - $\tau_a = .001$ orbit
  - $\tau_\beta = .001$ orbit

### Initial Conditions
- $\psi(0) = \psi'(0) = 0$
- $\phi(0) = \phi'(0) = 0$
- $\lambda(0) = \lambda'(0) = 0$
- Eccentricity $\varepsilon = 0$

### Figure 3.10
System response due to appendage pitch and roll slew motions:
(a) pitch response; (b) roll response; (c) yaw response.
Figure 3.11 System response due to appendage pitch and roll slew motions: (a) pitch response; (b) roll response; (c) yaw response.
### Satellite Parameters
- $\xi_s = 0.06$
- $\xi_a = 0.06$
- $\eta = 0.06$
- $\xi^T = (0 \ 0 \ 1)$

### Appendage Parameters
- Initial Conditions
  - $\psi(0) = \psi'(0) = 0$
  - $\phi(0) = \phi'(0) = 0$
  - $\lambda(0) = \lambda'(0) = 0$
- Eccentricity $\epsilon = 0$

### Initial Conditions
- $\alpha_i = \beta_i = 0^\circ$
- $\alpha_f = \beta_f = 30^\circ$
- $\tau_{\text{total}} = .001 \text{ orbit}$

### Figure 3.12
System response showing effect of path of appendage during slew motion:
- (a) pitch response
- (b) roll response
- (c) yaw response

\[ \psi^\circ \]
\[ \phi^\circ \]
\[ \lambda^\circ \]
satellite response will become more pronounced because of the gravitational moment, as mentioned before.

3.3 Effect of Satellite Inertia and Geometry

Of critical importance in design would be the influence of inertia parameters associated with the central body and the appendage. These are $\xi_a$, $\xi_s$ (ratio of axial to transverse inertia of the appendage and the central body, respectively) and $\eta$, their mass ratio. Figure 3.13 indicates influence of these parameters on the librational dynamics. The trends are consistent with what one would expect based on physical considerations. As the central body becomes more slender (dumbbell satellite) the librational amplitude diminishes markedly (Figure 3.13a). On the other hand, slewing motion of a slender appendage leads to a larger amplitude libration (Figure 3.13b). Of course, as the relative mass of the appendage increases, the pitch librations grow. Clearly, the inertia ratio of the central body is the most critical parameter in assessing the satellite stability, because the gravity gradient stabilizing moment is governed by the slenderness of the satellite body. The inertia ratio of the appendage ($\xi_a$) and appendage/central body mass ratio ($\eta$) play lesser roles as their destabilizing effect on satellite response can be compensated through careful selection of the appendage time histories.
Satellite Parameters
\[ \bar{\xi} = (0, 0, 1) \]

App. Cubic Time History Parameters
\[ \begin{align*}
\alpha_1 &= -90^\circ \\
\alpha_f &= 0^\circ \\
\tau_a &= 0.001 \text{ orbit} \\
\beta_i &= \beta_f = \tau_\beta = 0
\end{align*} \]

Initial Conditions
\[ \begin{align*}
\psi(0) &= \psi'(0) = 0 \\
\phi(0) &= \phi'(0) = 0 \\
\lambda(0) &= \lambda'(0) = 0 \\
\varepsilon &= 0
\end{align*} \]

Eccentricity
\[ \begin{align*}
\xi_a &= 0.06 \\
\eta &= 0.06 \\
\xi_a &= 0.01 \\
\xi_a &= 0.06 \\
\xi_a &= 0.30 \\
\xi_a &= 0.80 \\
\xi_a &= 1.60
\end{align*} \]

Figure 3.13 System response showing effect of: (a) central body inertia ratio; (b) appendage inertia ratio; (c) appendage/central body mass ratio.
Figure 3.14 exhibits the effect of hinge location, i.e., the appendage placement on the central body, on the response with appendage pitch slew motion. Figure 3.14(a) illustrates influence of the hinge location in the orbital plane on the satellite's equilibrium state—a consequence of the satellite inertia diadic which, in turn, determines the orientation of the axis of minimum moment of inertia. It suggests that location of the hinge must be given careful consideration in order to obtain a desired orientation for the axis of minimum moment of inertia, and thus achieve a desirable equilibrium state for the satellite. Figures 3.14(b),(c) show the pitch and roll responses of the satellite due to variation of hinge location across the orbital plane. Note that the out of plane hinge location induces the out of plane motion (Figure 3.14c) as well as the planar motion (Figure 3.14b) due to coupling.

Figures 3.15,3.16 show, respectively, response of a satellite with prescribed appendage roll slew motion as affected by hinge location within the orbital plane (Figure 3.15) and across the orbital plane (Figure 3.16). Clearly, appendage placement on the central body influences the system response, which is particularly significant in the degree of freedom for the out of plane hinge location.

Next, the attention was directed at determination of a combination of system parameters which may lead to unstable librational response. However, the system proved to be rather stable. The reluctance of a satellite to tumble when
Satellite Parameters
\[ \xi_s = 0.06 \]
\[ \xi_a = 0.06 \]
\[ \eta = 0.06 \]

Appendage Parameters
\[ \alpha_i = -90^\circ \]
\[ \alpha_{i'} = 0^\circ \]
\[ \tau_{\alpha} = 0.001 \text{ orbit} \]
\[ \beta_i = \beta_{i'} = \tau_{\beta} = 0 \]

History Parameters

Initial Conditions
\[ \psi(0) = \psi'(0) = 0 \]
\[ \phi(0) = \phi'(0) = 0 \]
\[ \lambda(0) = \lambda'(0) = 0 \]

Eccentricity
\[ \epsilon = 0 \]

Figure 3.14 System response due to appendage pitch slew motion as affected by hinge location:
(a) in the orbital plane;
(b), (c) across the orbital plane.
<table>
<thead>
<tr>
<th>Satellite Parameters</th>
<th>App. Cubic Time History Parameters</th>
<th>Initial Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\zeta_s = 0.06$</td>
<td>$\beta_i = -90^\circ$</td>
<td>$\psi(0) = \psi'(0) = 0$</td>
</tr>
<tr>
<td>$\zeta_a = 0.06$</td>
<td>$\beta_i = 0^\circ$</td>
<td>$\phi(0) = \phi'(0) = 0$</td>
</tr>
<tr>
<td>$\eta = 0.06$</td>
<td>$\tau_{\beta} = .001$ orbit</td>
<td>$\lambda(0) = \lambda'(0) = 0$</td>
</tr>
<tr>
<td>$r = (\ldots)$</td>
<td>$\alpha_i = \alpha_i = \tau_{\alpha} = 0$</td>
<td>Eccentricity $\varepsilon = 0$</td>
</tr>
</tbody>
</table>

Figure 3.15  System response due to appendage roll slew motion as affected by hinge location in the orbital plane: (a) pitch response; (b) roll response.
Figure 3.16 System response due to appendage roll slew motion as affected by hinge location across the orbital plane: (a) pitch response; (b) roll response.
subjected to a smooth slew disturbance is evident in Figure 3.17. Even when faced with an adverse combination of inertia and geometry, the satellite response continues to be stable. This suggests that a conventional satellite would indeed be unlikely to tumble from the action of a smooth slewing maneuver.

3.4 Effect of Impulsive Disturbance

Figure 3.18 shows the response of a satellite with prescribed appendage pitch slew motion as affected by impulsive disturbance in the orbital plane (Figure 3.18a) and across the orbital plane (Figures 3.18b,c). Figure 3.18(a) illustrates the sensitivity of pitch response to inplane impulsive disturbances. This suggests that a reduction in pitch amplitude can be obtained by judiciously slewing the appendage at an appropriate positive or negative initial velocity. Figure 3.18(b) reflects the insensitivity of pitch response to out of plane impulsive disturbances. This is somewhat surprising as there does not seem to be any influence from the out of plane response, suggesting lower order of magnitude of the coupling terms.

Figures 3.19 and 3.20 show, respectively, response of a satellite to appendage roll slew motion with an impulsive disturbance in the orbital plane and across the orbital plane. As before, Figure 3.19 suggests that a reduction in pitch amplitude can be obtained by judiciously slewing the appendage with a desirable impulse. It also shows that
### Satellite Parameters
- $\xi_s = 1.90$
- $\xi_a = 0.001$
- $\eta = 0.25$
- $\vec{c} = (0 \ 1 \ 1)$

### Appendage Parameters
- Initial Conditions
  - $\psi(0) = \psi'(0) = 0$
  - $\phi(0) = \phi'(0) = 0$
  - $\lambda(0) = \lambda'(0) = 0$

### Initial Conditions
- $\alpha_i = \beta_i = 0^\circ$
- $\alpha_f = \beta_f = 60^\circ$
- $\tau_a = \tau_B = .001$ orbit

### Eccentricity
- $\epsilon = 0$

---

**Figure 3.17** System response for an adverse combination of satellite inertia and geometry: (a) pitch response; (b) roll response; (c) yaw response.
Figure 3.18 System response due to appendage pitch slew motion as affected by an impulsive disturbance: (a) in the orbital plane; (b), (c) across the orbital plane.
Figure 3.19 System response due to appendage roll slew motion as affected by an impulsive disturbance in the orbital plane: (a) pitch response; (b) roll response.
Figure 3.20 System response due to appendage roll slew motion as affected by an impulsive disturbance across the orbital plane: (a) pitch response; (b) roll response.
inplane initial disturbances have negligible effect on the roll response. On the other hand, the transverse impulsive disturbances do affect, slightly, the roll response (Figure 3.20). Thus, relatively, only a small reduction in roll amplitude can be achieved with an inplane or out of plane impulsive disturbance. In fact, the presence of initial librational velocities during appendage roll slew motions can increase the roll amplitude.

3.5 Effect of Orbit Eccentricity

Eccentricity of the orbit introduces a forcing function on the system, generally leading to a worsening of the librational response.

The effect of orbit eccentricity on the response of a satellite performing appendage roll slew motion at the perigee of orbit is shown in Figure 3.21. It can be seen that the librational response of the satellite during slew motion is adversely affected by orbit eccentricity. This is significant as it shows that the presence of eccentricity may induce outright tumbling of a satellite performing slew motion at the perigee of the orbit.
Satellite Parameters

$\xi_s = 0.06$
$\xi_o = 0.06$
$\eta = 0.06$
$\xi^T = (0 \ 0 \ 1)$

Appendage Cubic Time

History Parameters

$\beta_i = -90^\circ$
$\beta_f = 0^\circ$
$\tau_\beta = .001 \text{ orbit}$
$\alpha_i = \alpha_f = \tau_\alpha = 0$

Initial Conditions

$\psi(0) = \psi'(0) = 0$
$\phi(0) = \phi'(0) = 0$
$\lambda(0) = \lambda'(0) = 0$

Figure 3.21 System response due to appendage roll slew motion as affected by orbit eccentricity:
(a) pitch response; (b) roll response.
4. CONCLUDING REMARKS

4.1 Summary of Conclusions

A relatively general formulation presented here for studying librational dynamics of a rigid satellite with a rigid slewing appendage represents a first step towards an analysis of more complex systems. The analysis provides a useful insight into interactions between the inertia parameters, orbit geometry, slew time histories and initial conditions. The salient features of the analysis and the conclusions based on them may be summarized as follows:

(i) Appendage time histories of long duration with smooth starting and stopping motions lead to a substantially low librational response.

(ii) The appendage should be as light and as flat as possible to minimize disturbance torque during appendage slew motions.

(iii) The hinge location and orientation of the appendage influences the satellite inertia diadic and hence defines the equilibrium state for the satellite about which librational response occurs.

(iv) Pitch slew motion does not excite out of plane motion for any arbitrary hinge location in the orbital plane. However, during out of plane location of the hinge, the same slew motion excites all the librational degrees of freedom. For a roll slew
motion, irrespective of the hinge location, pitch, roll and yaw are always excited. This is consistent with the earlier studies with reference to librational dynamics of rigid satellites.

(v) Stability (no tumbling) in pitch and roll can be maintained with gravity gradient stabilization provided a suitable combination of satellite and appendage time history parameters are chosen.

(vi) Gravity gradient stabilized satellites with slewing appendages have poor pointing accuracy. Out of plane satellite motion results in libration in yaw which causes the appendage to revolve about the satellite spin axis. Gravity gradient stabilization does little to reduce yaw libration of cylindrical satellites. As well, gravity gradient stabilization is unable to control large libration caused by fast appendage slew motions, nor does it eliminate the offset error caused by a nonzero equilibrium state.

(vii) It appears that, in general, the eccentricity affects the librational response adversely during slewing maneuvers.

4.2 Recommendations for Future Work

This preliminary investigation has provided some insight into the dynamical behaviour of spacecraft with a slewing appendage. However, it is only a first step towards
analysis of more realistic and hence complex models. A few suggestions concerning direction for future investigation are recorded below:

(i) Most satellites are provided with some form of damping mechanism, in addition to the internal damping, to minimize librational response. It would be of considerable interest to evaluate the performance of the satellite due to the presence of damping.

(ii) A high degree of pointing accuracy is required for many satellite missions. An active control should be developed to meet all pointing accuracy criteria, in the presence of slewing maneuvers.

(iii) A stability study is important and should be undertaken. To this end, the concept integral manifold in the phase space may prove useful.

(iv) Many satellite missions require near earth orbits, where free molecular reaction forces (aerodynamic drag) may have significant effect on the satellite performance.

(v) The U.S. has committed itself to a space station by early 1990's. The proposed baseline configuration of the space station involves a vast array of highly flexible beams (keel), power booms, solar panels, antennas, etc., extending to tens of meters. MRMS to be used for construction of the space station will
be, as the name suggests, a mobile unit with its own flexible appendages. Thus, an analysis of a system with slewing flexible appendages traversing a flexible, orbiting structure will be needed in the dynamical analysis and control of the proposed space station.


APPENDIX A - ANGULAR MOMENTUM VECTOR \{H\}

\[ f_m(\bar{\omega}\bar{x}\bar{r}) \cdot \frac{\partial \bar{r}}{\partial t} \, dm = f_{m_s}(\bar{\omega}\bar{x}\bar{r}_s) \cdot \frac{\partial \bar{r}_s}{\partial t} \, dm_s \]

\[ + f_{m_a}(\bar{\omega}\bar{x}\bar{r}_a) \cdot \frac{\partial \bar{r}_a}{\partial t} \, dm_a \quad ....\quad (A.1) \]

Recalling from section 2.2.1 that:

\[ \bar{r}_s = -\bar{c} -\bar{c}_o +\bar{s} ; \]
\[ \bar{r}_a = -\bar{c} -\bar{c}_o +\bar{h} +\bar{f} +\bar{a} ; \]
\[ \frac{\partial \bar{r}_s}{\partial t} = -\dot{\bar{c}} ; \]
\[ \frac{\partial \bar{r}_a}{\partial t} = -\dot{\bar{c}} +\bar{a}\chi(\bar{f} +\bar{a}) ; \]

the integral in domain \( D_s \) becomes

\[ f_{m_s}(\bar{\omega}\bar{x}\bar{r}_s) \cdot \frac{\partial \bar{r}_s}{\partial t} \, dm_s = f_{m_s}\bar{\omega}\chi(-\bar{c} -\bar{c}_o +\bar{s}) \cdot -\dot{\bar{c}} \, dm_s \]
\[ = m_s\bar{\omega}\chi(\bar{c} +\bar{c}_o) \cdot \dot{\bar{c}} \quad ....\quad (A.2) \]

since \( f_{m_s}\bar{s} \, dm_s = 0 \).

Evaluation of the integral in the domain \( D_a \) is as follows:

\[ f_{m_a}(\bar{\omega}\bar{x}\bar{r}_a) \cdot \frac{\partial \bar{r}_a}{\partial t} \, dm_a = f_{m_a}\bar{\omega}\chi(-\bar{c} -\bar{c}_o +\bar{h} +\bar{f} +\bar{a}) \cdot (-\dot{\bar{c}} +\bar{a}\chi(\bar{f} +\bar{a})) \, dm_a \]

With \( \bar{g} = -\bar{c} -\bar{c}_o +\bar{h} +\bar{f} \),

\[ f_{m_a}(\bar{\omega}\bar{x}\bar{r}_a) \cdot \frac{\partial \bar{r}_a}{\partial t} \, dm_a = f_{m_a}\bar{\omega}\chi(\bar{g} +\bar{a}) \cdot (-\dot{\bar{c}} +\bar{a}\chi(\bar{f} +\bar{a})) \, dm_a \]
\[ = f_{m_a}\bar{\omega}\chi(\bar{g} +\bar{a}) \cdot \dot{\bar{c}} +[\bar{\omega}\chi(\bar{g} +\bar{a}) \cdot \bar{a}\chi(\bar{f} +\bar{a})] \, dm_a \]

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\[
= \int_{m_a} -\ddot{\omega} \cdot (\dddot{\omega} \cdot \dddot{g} + \dddot{\omega} \cdot \dddot{a}) + (\dddot{\omega} \cdot \dddot{a}) ((\dddot{g} + \dddot{a}) \cdot (\dddot{f} + \dddot{a}))
- (\dddot{\omega} \cdot (\dddot{f} + \dddot{a})) ((\dddot{g} + \dddot{a}) \cdot \dddot{a}) \, dm_a
\]
\[
= \int_{m_a} -\ddot{c} \cdot (\dddot{\omega} \cdot \dddot{g} - \dddot{\omega} \cdot \dddot{a})
+ (\dddot{\omega} \cdot \dddot{a}) \, (\dddot{g} \cdot \dddot{f} + \dddot{a} \cdot \dddot{f} + \dddot{g} + \dddot{a}\cdot \dddot{a})
- (\dddot{\omega} \cdot \dddot{f} + \dddot{a}) \, (\dddot{\omega} \cdot \dddot{g} + \dddot{a} \cdot \dddot{g})
\]
\[
= \int_{m_a} -\ddot{c} \cdot (\dddot{\omega} \cdot \dddot{g} - \dddot{\omega} \cdot \dddot{a})
+ (\dddot{\omega} \cdot \dddot{a}) \, (\dddot{g} \cdot \dddot{f} + \dddot{a} \cdot \dddot{f} + \dddot{g} + \dddot{a}\cdot \dddot{a})
- (\dddot{\omega} \cdot \dddot{f})(\dddot{\omega} \cdot \dddot{a}) - (\dddot{\omega} \cdot \dddot{a})(\dddot{\omega} \cdot \dddot{a}) \, dm_a .
\]

However, since \( \int_{m_a} \ddot{a} \, dm_a = 0 \), the integral becomes:

\[
= m_a (-\ddot{\omega} \cdot \dddot{g}) - 0 + m_a (\dddot{\omega} \cdot \dddot{a}) (\dddot{g} \cdot \dddot{f}) + 0 + 0 +
\int_{m_a} (\dddot{\omega} \cdot \dddot{a}) (\dddot{a} \cdot \dddot{a}) \, dm_a - m_a (\dddot{\omega} \cdot \dddot{f})(\dddot{\omega} \cdot \dddot{g}) - 0 - 0
- \int_{m_a} (\dddot{\omega} \cdot \dddot{a}) (\dddot{a} \cdot \dddot{a}) \, dm_a
\]
\[
= m_a [-\ddot{c} \cdot (\dddot{\omega} \cdot \dddot{g} + (\dddot{\omega} \cdot \dddot{a})(\dddot{g} \cdot \dddot{f} - (\dddot{\omega} \cdot \dddot{f})(\dddot{\omega} \cdot \dddot{g}])
+ \int_{m_a} (\dddot{\omega} \cdot \dddot{a}) (\dddot{a} \cdot \dddot{a}) - (\dddot{\omega} \cdot \dddot{a})(\dddot{\omega} \cdot \dddot{a}) \, dm_a
\]
\[
= m_a [-\ddot{c} \cdot (\dddot{\omega} \cdot \dddot{g} + (\dddot{\omega} \cdot \dddot{a})(\dddot{g} \cdot \dddot{f} - (\dddot{\omega} \cdot \dddot{f})(\dddot{\omega} \cdot \dddot{g})]
+ \int_{m_a} (\dddot{\omega} \cdot \dddot{a}) \cdot (\dddot{a} \cdot \dddot{a}) \, dm_a
\]
\[
= m_a (\dddot{\omega} \cdot \dddot{g}) \cdot (\dddot{\omega} \cdot \dddot{a} \cdot \dddot{c}) + \int_{m_a} (\dddot{\omega} \cdot \dddot{a}) \cdot (\dddot{\omega} \cdot \dddot{a}) \, dm_a .
\]

The integral on the right hand side in the above expression can be evaluated as follows:

\[
\int_{m_a} (\dddot{\omega} \cdot \dddot{a}) \cdot (\dddot{\omega} \cdot \dddot{a}) \, dm_a = \int_{m_a} \dddot{\omega} \cdot (\dddot{a} \cdot (\dddot{\omega} \cdot \dddot{a})) \, dm_a
= \dddot{\omega} \cdot \int_{m_a} \dddot{a} \cdot (\dddot{\omega} \cdot \dddot{a}) \, dm_a
= \dddot{\omega} \cdot [I_a] \dddot{\omega}
\]

where \([I_a] = \) appendage body inertia diadic with respect to frame \( x_a', y_a', z_a \).
The integral in domain $D_a$ thus becomes:

$$\int_{m_a} (\omega x_r) \cdot \frac{\partial r_a}{\partial t} \, dm_a = m_a (\omega x_g) \cdot (\omega_a x_f - \ddot{c}) + \omega \cdot [I_a] \ddot{\omega}_a. \quad \cdots (A.3)$$

Substituting from equations (A.2) and (A.3) into (A.1):

$$\int_{m} (\omega x_r) \cdot \frac{\partial r}{\partial t} \, dm = m_s (\omega x (-\ddot{c} - \ddot{c}_o) \cdot \ddot{c} + m_a (\omega x_g) \cdot (\omega_a x_f - \ddot{c}) + \omega \cdot [I_a] \ddot{\omega}_a$$

$$= m_s \ddot{\omega} \cdot (-\ddot{c} - \ddot{c}_o) \cdot \ddot{c} + m_a \ddot{\omega} \cdot \ddot{g}_x (\omega_a x_f - \ddot{c}) + \omega \cdot [I_a] \ddot{\omega}_a$$

$$\omega \cdot (m_s (-\ddot{c} - \ddot{c}_o) \cdot \ddot{c} + m_a \ddot{g}_x (\omega_a x_f - \ddot{c}) + [I_a] \ddot{\omega}_a)$$

$$= \omega \cdot \ddot{H}$$

where $\ddot{H}$ represents the angular momentum vector. Let $\{H\} =$ projection of $\ddot{H}$ onto body axes $x,y,z$. The angular momentum vector with respect to frame $x,y,z$ now appears as:

$$\{H\} = m_s (-\ddot{c} - \ddot{c}_o) x - \ddot{c} + m_a (-\ddot{c} - \ddot{c}_o + \ddot{h} + [M_a] \ddot{f}) \cdot x ([M_a] (\omega_a x_f) - \ddot{c})$$

$$+ [M_a] [I_a] \ddot{\omega}_a$$

where $\ddot{f}$, $\ddot{\omega}_a$ and $[I_a]$ are with respect to the appendage body axes $x_a, y_a, z_a$.

From Figure 2.3, it is apparent that vector $(-\ddot{c} - \ddot{c}_o)$ locates the center of central body mass $O_s$ with respect to the instantaneous center of mass $C$, and that vector $(-\ddot{c} - \ddot{c}_o + \ddot{h} + [M_a] \ddot{f})$ locates the center of appendage mass $O_a$ with respect to the instantaneous center of mass $C$.

Let

$$\ddot{d}_s = -\ddot{c} - \ddot{c}_o,$$

$$\ddot{d}_a = -\ddot{c} - \ddot{c}_o + \ddot{h} + [M_a] \ddot{f}.\)
Note that \( \dot{d}_s = -\dot{c} \).

The angular momentum vector with respect to body axes \( x, y, z \) can now be written as

\[
\{H\} = m_s \ddot{d}_s \times \dot{d}_s + m_a \ddot{d}_a \chi ([M_a] (\bar{\omega}_a x f) + \dot{d}_s) + [M_a][I_a] \bar{\omega}_a.
\]
APPENDIX B - SATELLITE INERTIA MATRIX [I]

The satellite inertia matrix [I] is computed using the parallel-axis theorems for moments and products of inertia,

\[
[I] = [I_s] + m_s [\bar{d}_s \cdot \bar{d}_s [E] - \bar{d}_s \bar{d}_s^T] + [M_a][I_a][M_a]^T + m_a [\bar{d}_a \cdot \bar{d}_a [E] - \bar{d}_a \bar{d}_a^T],
\]

where:

- \([I]\) satellite inertia diadic with respect to satellite body axes \(x,y,z\);
- \([I_s]\) central body inertia diadic with respect to coordinate axes \(x_s,y_s,z_s\);
- \([I_a]\) appendage inertia diadic with respect to coordinate axes \(x_a,y_a,z_a\);
- \([M_a]\) transformation matrix relating body axes \(x,y,z\) to coordinate axes \(x_a,y_a,z_a\);
- \([E]\) unit matrix;
- \(\bar{d}_s\) position vector from \(C\) to \(O_s\), \(-\vec{c} - \vec{c}_o\), Figure 2.3;
- \(\bar{d}_a\) position vector from \(C\) to \(O_a\), \(-\vec{c} - \vec{c}_o + \bar{h} + [M_a] \bar{f}\), Figure 2.3;
- \(m_s\) mass of central body;
- \(m_a\) mass of appendage.
APPENDIX C - EQUATIONS OF MOTION

From sections 2.2.1 and 2.2.2, the kinetic and potential energies for the system are given by

\[ T_e = \frac{1}{2} \{\omega\}^T \{I\} \{\omega\} + \{\omega\}^T \{H\} ; \]

\[ U_e = c_{\mu} / 2 \{l\}^T \{I\} \{l\} ; \]

where \( c_{\mu} = 3\mu / R_c^3 \).

The satellite angular velocity vector can be written as

\[ \{\omega\} = [M] \{\dot{q}\} + \{N\} \dot{\theta} ; \]

and clearly,

\[ \{\omega\}^T = \{\dot{q}\}^T [M]^T + \dot{\theta} \{N\}^T \]

where \([M]\) and \([N]\) represent respective projections of \(\{\dot{q}\}\) and \(\dot{\theta}\) on the satellite body axes \(x,y,z\).
The governing equations of motion can now be obtained using the Lagrange equation

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}} \right) - \frac{\partial T}{\partial q} + \frac{\partial U}{\partial q} = \{Q\},
\]

where \{q\} and \{Q\} represent the vectors of generalized coordinates and forces, respectively.

Substituting for the kinetic and potential energies:

\[
\begin{align*}
\frac{d}{dt} & \left( \frac{1}{2} \frac{\partial}{\partial \dot{q}} \{\omega\}^T [I] \{\omega\} + \frac{1}{2} \{\omega\}^T [I] \frac{\partial}{\partial \dot{q}} \{\omega\} + \frac{\partial}{\partial \dot{q}} \{\omega\}^T [H] \right) \\
& - ( \frac{\partial}{\partial \dot{q}} \{\omega\}^T [I] \{\omega\} + \frac{1}{2} \{\omega\}^T [I] \frac{\partial}{\partial \dot{q}} \{\omega\} + \frac{\partial}{\partial \dot{q}} \{\omega\}^T [H] ) \\
& + c_\mu / 2 \frac{\partial}{\partial \dot{q}} \{\dot{l}\}^T [I] \{\dot{l}\} + c_\mu / 2 \{\dot{l}\}^T [I] \frac{\partial}{\partial \dot{q}} \{\dot{l}\} = \{Q\},
\end{align*}
\]

i.e.,

\[
\begin{align*}
\frac{d}{dt} & \left( \frac{\partial}{\partial \dot{q}} \{\omega\}^T [I] \{\omega\} + \frac{\partial}{\partial \dot{q}} \{\omega\}^T [H] \right) \\
& - ( \frac{\partial}{\partial \dot{q}} \{\omega\}^T [I] \{\omega\} + \frac{\partial}{\partial \dot{q}} \{\omega\}^T [H] ) \\
& + c_\mu \frac{\partial}{\partial \dot{q}} \{\dot{l}\}^T [I] \{\dot{l}\} = \{Q\},
\end{align*}
\]

or

\[
\begin{align*}
\frac{d}{dt} & ( \frac{\partial}{\partial \dot{q}} \{\omega\}^T \{\omega\} + \{H\} ) \\
& - \frac{\partial}{\partial \dot{q}} \{\omega\}^T \{\omega\} + \{H\} ) \\
& + c_\mu \frac{\partial}{\partial \dot{q}} \{\dot{l}\}^T [I] \{\dot{l}\} = \{Q\}.
\end{align*}
\]

But \( \frac{\partial}{\partial \dot{q}} \{\omega\}^T = [M]^T \) and \( \frac{\partial}{\partial \dot{q}} \{\omega\}^T \) can be written as

\[
\frac{\partial}{\partial \dot{q}} \{\omega\}^T = [q] [M_q] + \dot{\theta}[N_q];
\]
where:

\[
[q] = \begin{bmatrix}
{0}^T & {0}^T & {0}^T \\
{0}^T & {q}^T & {0}^T \\
{0}^T & {0}^T & {q}^T
\end{bmatrix}
\]

\[
[M_q] = \frac{\partial}{\partial [\dot{q}]} [M]^T ;
\]

\[
[N_q] = \frac{\partial}{\partial [\dot{q}]} [N]^T .
\]

Similarly, let

\[
[L_q] = \frac{\partial}{\partial [\dot{q}]} [/]^T .
\]

Substituting for \( \frac{\partial}{\partial [\dot{q}]} [\omega]^T \), the equations of motion become:

\[
\frac{d}{dt} ([M]^T ([I] [\omega] + [H]) ) - \frac{\partial}{\partial [\dot{q}]} [\omega]^T ([I] [\omega] + [H]) + c_\mu \frac{\partial}{\partial [\dot{q}]} [/]^T [I] [/] = \{Q\}
\]

\[
[[\dot{M}]]^T ([I] [\omega] + [H]) + [M]^T ([I] [\omega] + [I] [\dot{\omega}] + [\dot{H}]) - \frac{\partial}{\partial [\dot{q}]} [\omega]^T ([I] [\omega] + [H]) + c_\mu \frac{\partial}{\partial [\dot{q}]} [/]^T [I] [/] = \{Q\}
\]

\[
\]

At this point, \( [\omega] \) must be evaluated:

\[
[\omega] = [M] [\dot{q}] + [N] \dot{\theta} ;
\]
\{\omega\} = [\dot{M}]\{\dot{q}\} + [M]\{\ddot{q}\} + \{\ddot{N}\}\dot{\theta} + \{\dot{N}\}\ddot{\theta}.

If $[M]$ is written in the form

\[
[M] = \begin{bmatrix}
m_x & o_x & a_x \\
m_y & o_y & a_y \\
m_z & o_z & a_z
\end{bmatrix},
\]

then

\[
[M] = \begin{bmatrix}
\dot{m}_x & \dot{o}_x & \dot{a}_x \\
\dot{m}_y & \dot{o}_y & \dot{a}_y \\
\dot{m}_z & \dot{o}_z & \dot{a}_z
\end{bmatrix}.
\]

Note that $\dot{m}_x$ can be written as

\[
\dot{m}_x = \{\dot{q}\}^T \bar{m}_{xv},
\]

where $\bar{m}_{xv}$ represents vector of coefficients for $\{\dot{q}\}^T$ within $\dot{m}_x$. Applying similar procedure to the remaining elements of $[\dot{M}]$,

\[
[M] = [\dot{q}] [M_v],
\]
where
\[
[M_v] = \begin{bmatrix}
\bar{m}_{xv} & \bar{o}_{xv} & \bar{a}_{xv} \\
\bar{m}_{yv} & \bar{o}_{yv} & \bar{a}_{yv} \\
\bar{m}_{zv} & \bar{o}_{zv} & \bar{a}_{zv}
\end{bmatrix}.
\]

Clearly, \([\dot{M}]^T\) can be written as
\[
[\dot{M}]^T = [\dot{q}] [M_{vt}],
\]
where
\[
[M_{vt}] = \begin{bmatrix}
\bar{m}_{xv} & \bar{m}_{yv} & \bar{m}_{zv} \\
\bar{o}_{xv} & \bar{o}_{yv} & \bar{o}_{zv} \\
\bar{a}_{xv} & \bar{a}_{yv} & \bar{a}_{zv}
\end{bmatrix}.
\]

Similarly, if \(\{N\}\) is written in the form
\[
\{N\} = \begin{bmatrix}
{n_x} \\
{n_y} \\
{n_z}
\end{bmatrix},
\]
then
\[
\{\dot{N}\} = \begin{pmatrix}
\dot{n}_x \\
\dot{n}_y \\
\dot{n}_z 
\end{pmatrix}.
\]

Noting that \(\dot{n}_x\) can be written as
\[
\dot{n}_x = \bar{n}_{xv} \{q\},
\]
where \(\bar{n}_{xv} \) represents vector of coefficients for \(\{q\}\) within \(\dot{n}_x\), \(\{\dot{N}\}\) can be written as
\[
\{\dot{N}\} = [N_v]\{q\},
\]
where
\[
[N_v] = \begin{pmatrix}
\bar{n}_{xv} \\
\bar{n}_{yv} \\
\bar{n}_{zv}
\end{pmatrix}.
\]

Now \(\{\dot{\omega}\}\) can be written as
\[
\{\dot{\omega}\} = [\dot{q}] [M_v] \{\dot{q}\} + [M] \{\dot{\omega}\} + [N_v] \{\dot{\omega}\} + \{N\} \ddot{\theta}.
\]

Substituting for \([\dot{M}]^T\), \(\frac{\partial}{\partial \{q\}}\{\omega\}^T\), \(\{\dot{\omega}\}\), and \(\frac{\partial}{\partial \{q\}}\{l\}^T\), the equations of motion become:
\[ ([\dot{q}][M_{vt}] - [\dot{q}][M_q] - \dot{\theta}[N_q]) \{ [I]\{\omega\} + \{H\} \} + [M]^T \{ [I]\{\omega\} + \{\dot{H}\} \} + [M]^T [I] \{ \dot{[q]}[M_v]\{[q] + [M] [\dot{q}] + [N_v] [\dot{q}] \dot{\theta} + \{N\dot{\theta}\} \} + c_\mu[L_q][I] \{ \} = \{Q\} \). 

On substituting for \{\omega\}, the equation takes the form

\[ ([\dot{q}][M_{vt}] - [\dot{q}][M_q] - \dot{\theta}[N_q]) \{ [I]\{[M] [\dot{q}] + \{N\dot{\theta}\} + \{H\} \} \]
\[ + [M]^T [I] \{ [M] [\dot{q}] + \{N\dot{\theta}\} + \{H\} \} + [M]^T [I] \{ [q] [M_v] \{\dot{q}\} \}
\[ + [M] \{ \dot{q} \} + [N_v] \{ \dot{q} \} \dot{\theta} + \{N\dot{\theta}\} + c_\mu [L_q] \{ I \} \{ \} = \{Q\} , \]

\[ [\dot{q}][M_{vt}] - [\dot{q}][M_q] - \dot{\theta}[N_q] \{ [I]\{[M] [\dot{q}] \} \{M_{vt} - [M_q] \} (I) \{N\} \dot{\theta} + \{H\} \}
\[ + [M]^T [I] \{ [M] [\dot{q}] + [M]^T ( I) \{N\} \dot{\theta} + \{\dot{H}\} \} + [M]^T [I] \{ M_v \} \{q\} \dot{\theta}
\[ + [M]^T [I] \{ N \} \dot{\theta} + c_\mu [L_q] \{ I \} \{ \} = \{Q\} . \]

Rearranging the terms gives

\[ [M]^T [I] \{ M \} \{\dot{q}\} + [\dot{q}][M_{vt}] - [M_q] \} \{ I \} \{ M \} \{\dot{q}\}
\[ + [M]^T [I] \{ q \} \{ M_v \} \{\dot{q}\} + [\dot{q}][M_{vt}] - [M_q] \} \{ I \} \{ N \} \dot{\theta} + \{H\} \}
\[ + ([M]^T [I] \{ M \} - \dot{\theta}[N_q] \} \{ I \} \{ M \} + [M]^T [I] \{ N_v \} \dot{\theta} \} \{\dot{q}\}
\[ - \dot{\theta}[N_q] \} \{ I \} \{ N \} \dot{\theta} + \{H\} \} + [M]^T \{ [I] \{ N \} \dot{\theta} + [I] \{ N \} \dot{\theta} + \{\dot{H}\} \}
\[ + c_\mu [L_q] \{ I \} \{ \} = \{Q\} . \]

The equations of motion can be nondimensionalized through multiplication by
where $I_{arb}$ is an arbitrary inertia chosen to nondimensionalize the equations of motion.

Noting that \( \{q\} = \{q\} , \)
\( \{\ddot{q}\} = \{\dot{q}\} \dot{\theta}^2 + \{\dot{q}\} \ddot{\theta} , \)

the nondimensionalization of \( \{q\} \) and \( \{\ddot{q}\} \) results in

\[
\frac{\{\dot{q}\}}{\dot{\theta}} = \{\dot{q}\} , \\
\frac{\{\ddot{q}\}}{\dot{\theta}^2} = \{\dddot{q}\} + \{\dot{q}\} f_{\epsilon} ,
\]

where:
\[
f_{\epsilon} = \frac{\dddot{\theta}/\dot{\theta}^2}{1 + \epsilon \cos \theta} = -\frac{2 \epsilon \sin \theta}{1 + \epsilon \cos \theta} ;
\]
\( \epsilon = \text{orbit eccentricity}. \)

The equations of motion are nondimensionalized by multiplication with \( 1/(I_{arb} \dot{\theta}^2) \), giving

\[
[M]^T [I^*] [M] \{\dddot{q}\} + [\dot{q}]([M_v t] - [M_q]) [I^*] [M] \{\dot{q}\} + [M]^T [I^*] [\dot{q}] [M_v] \{\dot{q}\} + [\dot{q}]([M_v t] - [M_q])([I^*] [N] + [H^*]) + ([M]^T [i^*] [M] - [N_q] [I^*] [M] + [M]^T [I^*] [N_v]) \{\dot{q}\} - [N_q]([I^*] [N] + [H^*]) + [M]^T ([i^*] [N] + [I^*] [N] f_{\epsilon} + [H^*]) + c_{\epsilon} [L_q] [I^*] \{/\} = \{Q^*\} ,
\]

where: \([I^*] = [I]/I_{arb}\)
\(\{\mathbf{H}^*\} = \{\mathbf{H}\}/(I_{arb} \dot{\theta});\)
\(\{\mathbf{i}^*\} = \{\mathbf{i}\}/(I_{arb} \dot{\theta});\)
\(\{\mathbf{H}^*\} = \{\mathbf{H}\}/(I_{arb} \dot{\theta}^2);\)
\(\{\mathbf{Q}^*\} = \{\mathbf{Q}\}/(I_{arb} \dot{\theta}^2);\)
\(c_\epsilon = c_\mu/\dot{\theta}^2 = 3\mu/(R_c^3 \dot{\theta}^2) = \frac{3}{1+\epsilon \cos \theta}.\)

Rearranging the terms finally results in the nondimensional governing equations of motion appearing as

\[
\begin{align*}
[M]^T [\mathbf{I}^*] [M] \{\ddot{\mathbf{q}}\} + [\dot{\mathbf{q}}]([M_{vt}] -[M_{q}]) [\mathbf{I}^*] [M] \{\mathbf{q}\} \\
+ [M]^T [\mathbf{I}^*] [\dot{\mathbf{q}}] [M_{v}] \{\mathbf{q}\} + [\dot{\mathbf{q}}]([M_{vt}] -[M_{q}]) ([\mathbf{I}^*] \{N\} \\
+ \{\mathbf{H}^*\}) + ([M]^T [\mathbf{I}^*] -[N_{q}] [\mathbf{I}^*]) [M] +[M]^T [\mathbf{I}^*] ([N_{v}] \\
+[M] f_{\epsilon}) \{\dot{\mathbf{q}}\} -[N_{q}] ([\mathbf{I}^*] \{N\} + \{\mathbf{H}^*\}) +[M]^T ([\mathbf{I}^*] \{N\} \\
+ \{\mathbf{H}^*\} +[\mathbf{I}^*] \{N\} f_{\epsilon}) + c_\epsilon [L_{q}] [\mathbf{I}^*] \{l\} = \{\mathbf{Q}^*\} .
\end{align*}
\]
APPENDIX D - LIBRATIONAL TERMS

The librational terms are evaluated for the rotation sequence of \( \psi \) (pitch), \( \phi \) (roll), \( \lambda \) (yaw).

\[ \{\omega\} = [M] \{\dot{q}\} + \{N\} \dot{\theta} \]

where:

\[ \{\dot{q}\} = (\dot{\psi} \; \dot{\phi} \; \dot{\lambda})^T ; \]

\[ [M] = \begin{bmatrix} m_x & o_x & a_x \\ m_y & o_y & a_y \\ m_z & o_z & a_z \end{bmatrix} = \begin{bmatrix} c\phi s\lambda & c\lambda & 0 \\ c\phi c\lambda & -s\lambda & 0 \\ -s\phi & 0 & 1 \end{bmatrix} ; \]

\[ \{N\} = \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \begin{bmatrix} c\phi s\lambda \\ c\phi c\lambda \\ -s\phi \end{bmatrix} ; \]

\[ [M]^T = \begin{bmatrix} c\phi s\lambda & c\phi c\lambda & -s\phi \\ c\lambda & -s\lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} ; \]
\[ [M_q] = \begin{bmatrix}
\frac{\partial [M]^T}{\partial \psi} \\
\frac{\partial [M]^T}{\partial \phi} \\
\frac{\partial [M]^T}{\partial \lambda}
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
-s\phi \lambda & -s\phi \lambda & -c \phi \\
-c \phi \lambda & -c \phi \lambda & 0 \\
-s \lambda & -c \lambda & 0 \\
0 & 0 & 0
\end{bmatrix} \]
\[
\{ N \}^T = \begin{pmatrix} c \phi s \lambda & c \phi c \lambda & -s \phi \end{pmatrix} ;
\]

\[
\{ N \} = \begin{bmatrix} \frac{\partial \{ N \}^T}{\partial \psi} \\ \frac{\partial \{ N \}^T}{\partial \phi} \\ \frac{\partial \{ N \}^T}{\partial \lambda} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -s \phi s \lambda & -s \phi c \lambda & -c \phi \\ c \phi c \lambda & -c \phi s \lambda & 0 \end{bmatrix} ;
\]

\[
\{ \ell \} = \begin{bmatrix} -s \psi c \lambda + c \psi s \phi s \lambda \\ s \psi s \lambda + c \psi s \phi c \lambda \\ c \psi c \phi \end{bmatrix} ;
\]

\[
\{ L \} = \begin{bmatrix} \frac{\partial \{ \ell \}^T}{\partial \psi} \\ \frac{\partial \{ \ell \}^T}{\partial \phi} \\ \frac{\partial \{ \ell \}^T}{\partial \lambda} \end{bmatrix} = \begin{bmatrix} -c \psi c \lambda & -s \psi s \phi s \lambda & c \psi s \lambda & -s \psi s \phi c \lambda & -s \psi c \phi \\ c \psi c \phi s \lambda & c \psi c \phi c \lambda & -c \psi s \phi \end{bmatrix} ;
\]

\[
\left[ \begin{array}{c} s \psi s \lambda + c \psi s \phi c \lambda \\ s \psi c \phi s \lambda \\ s \psi c \phi c \lambda \\ s \psi c \phi c \lambda \\ s \psi c \phi c \lambda \\ s \psi c \phi c \lambda \end{array} \right] = \begin{bmatrix} s \psi c \phi c \lambda & s \psi c \phi c \lambda & s \psi c \phi c \lambda & s \psi c \phi c \lambda & s \psi c \phi c \lambda & s \psi c \phi c \lambda \end{bmatrix} ;
\]
\[
\dot{[M]} = \frac{d}{dt}[M] = \{q\} [M_v];
\]

\[
[M_v] = \begin{bmatrix}
\ddot{m}_{xv} & \ddot{o}_{xv} & \ddot{a}_{xv} \\
\ddot{m}_{yv} & \ddot{o}_{yv} & \ddot{a}_{yv} \\
\ddot{m}_{zv} & \ddot{o}_{zv} & \ddot{a}_{zv}
\end{bmatrix}.
\]

The elements of \([M_v]\) are obtained as follows:

\[
\dot{m}_x = \frac{d}{dt}(m_x) = \dot{\psi}(0) + \dot{\phi}(-s\phi\lambda) + \dot{\lambda}(c\phi\lambda),
\]

therefore \[
\ddot{m}_{xv} = \begin{bmatrix} 0 \\ -s\phi\lambda \\ c\phi\lambda \end{bmatrix}.
\]

Similarly, for the remaining elements of \([M_v]\):

\[
\ddot{m}_{yv} = \begin{bmatrix} 0 \\ -s\phi\lambda \\ c\phi\lambda \end{bmatrix}; \quad \ddot{m}_{zv} = \begin{bmatrix} 0 \\ -c\phi \\ 0 \end{bmatrix};
\]
\[ \ddot{\alpha}_{xv} = \ddot{\alpha}_{yv} = \ddot{\alpha}_{zv} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \]

\[ \ddot{\omega}_{xv} = \ddot{\omega}_{yv} = \ddot{\omega}_{zv} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \]
\[
[M_v] = \begin{bmatrix}
0 & 0 & 0 \\
-s\phi s\lambda & 0 & 0 \\
c\phi c\lambda & -s\lambda & 0 \\
0 & 0 & 0 \\
-s\phi c\lambda & 0 & 0 \\
-c\phi s\lambda & -c\lambda & 0 \\
0 & 0 & 0 \\
-c\phi & 0 & 0 \\
0 & 0 & 0 
\end{bmatrix}
\]
\[
[M_{vt}] = \begin{bmatrix}
0 & 0 & 0 \\
-s\phi s\lambda & -s\phi c\lambda & -c\phi \\
c\phi c\lambda & -c\phi s\lambda & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
-s\lambda & -c\lambda & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
\[
\{ N \} = \frac{d}{dt}\{ N \} ;
\]

\[
\dot{n}_x = \frac{d}{dt}(n_x),
\]

\[
= \dot{\psi}(0) + \dot{\phi}(-s\phi s\lambda) + \dot{\lambda}(c\phi c\lambda),
\]

therefore \[
\bar{n}_{xv} = \begin{bmatrix} 0 \\ -s\phi s\lambda \\ c\phi c\lambda \end{bmatrix}
\]

Similarly, \[
\bar{n}_{yv} = \begin{bmatrix} 0 \\ -s\phi c\lambda \\ -c\phi s\lambda \end{bmatrix}, \quad \bar{n}_{zv} = \begin{bmatrix} 0 \\ -c\phi \end{bmatrix};
\]

\[
[N_v] = \begin{bmatrix} \bar{n}_{xv}^T \\ \bar{n}_{yv}^T \\ \bar{n}_{zv}^T \end{bmatrix} = \begin{bmatrix} 0 & -s\phi s\lambda & c\phi c\lambda \\ 0 & -s\phi c\lambda & -c\phi s\lambda \\ 0 & -c\phi & 0 \end{bmatrix}.
\]
APPENDIX E - TRANSFORMATION TERMS FOR SLEW MANEUVERS

Transformation matrix \([M_a]\) and its derivatives \([\dot{M}_a], [\ddot{M}_a]\).

\[
\begin{bmatrix}
  m_{xa} & 0_{xa} & a_{xa} \\
  m_{ya} & 0_{ya} & a_{ya} \\
  m_{za} & 0_{za} & a_{za}
\end{bmatrix},
\]

\[
\begin{bmatrix}
  \cos \alpha & \sin \alpha \sin \beta & \sin \alpha \cos \beta \\
  0 & \cos \beta & -\sin \beta \\
  -\sin \alpha & \cos \alpha \sin \beta & \cos \alpha \cos \beta
\end{bmatrix}.
\]

\[
[\dot{M}_a] = \frac{[\dot{M}_a]}{\hat{\theta}} =
\begin{bmatrix}
  \ddot{m}_{xa} & \dot{0}_{xa} & \dot{a}_{xa} \\
  \ddot{m}_{ya} & \dot{0}_{ya} & \dot{a}_{ya} \\
  \ddot{m}_{za} & \dot{0}_{za} & \dot{a}_{za}
\end{bmatrix},
\]

where:

\[
\begin{align*}
\ddot{m}_{xa} &= \dot{a} m_{za} \\
\ddot{m}_{ya} &= 0 \\
\ddot{m}_{za} &= -\dot{a} m_{xa} \\
\dot{0}_{xa} &= \dot{a} o_{za} + \dot{\beta} a_{xa}.
\end{align*}
\]
\[ \ddot{\varphi}_y = \dot{\beta} \dot{a}_y; \]
\[ \ddot{\varphi}_z = -\dot{\varphi}_x + \dot{\beta} a_z; \]
\[ \dddot{a}_x = \dot{\varphi}_a - \dot{\beta} \dot{a}_x; \]
\[ \dddot{a}_y = -\dot{\beta} \dot{a}_y; \]
\[ \dddot{a}_z = -a \dot{a}_x - \dot{\beta} \dot{a}_z. \]

\[
\begin{bmatrix}
\ddot{m}_x \\
\ddot{m}_y \\
\ddot{m}_z
\end{bmatrix} = \frac{[\ddot{M}_a]}{\dot{\varphi}^2} = \begin{bmatrix}
\ddot{a}_x \\
\ddot{a}_y \\
\ddot{a}_z
\end{bmatrix}
\begin{bmatrix}
\dot{m}_x & \ddot{m}_x & \dddot{m}_x \\
\dot{m}_y & \ddot{m}_y & \dddot{m}_y \\
\dot{m}_z & \ddot{m}_z & \dddot{m}_z
\end{bmatrix},
\]

where:

\[
\begin{align*}
\ddot{m}_x &= (\dot{a} + \dot{\varphi}_e) m_z + \dot{a} \ddot{m}_z; \\
\ddot{m}_y &= 0; \\
\ddot{m}_z &= -(\dot{a} + \dot{\varphi}_e) m_x - \dot{a} \ddot{m}_x; \\
\dot{m}_x &= (\ddot{a} + \ddot{\varphi}_e) o_z + \ddot{a} \dddot{m}_z + (\ddot{\beta} + \dot{\beta} \varphi_e) a_x + \dot{\beta} \dddot{a}_x; \\
\dot{m}_y &= (\ddot{\beta} + \dot{\beta} \varphi_e) a_y + \ddot{\beta} \dddot{a}_y; \\
\dot{m}_z &= -(\ddot{a} + \ddot{\varphi}_e) o_x - \ddot{a} \dddot{m}_x + (\ddot{\beta} + \dot{\beta} \varphi_e) a_z + \dot{\beta} \dddot{a}_z; \\
\ddot{a}_x &= (\ddot{\varphi}_a) a_z + \ddot{a} \dddot{a}_x - (\ddot{\beta} + \dot{\beta} \varphi_e) o_x - \dddot{\varphi}_a; \\
\ddot{a}_y &= -(\ddot{\beta} + \dot{\beta} \varphi_e) o_y - \ddot{\beta} \dddot{a}_y; \\
\ddot{a}_z &= -(\ddot{\varphi}_a) a_z - \ddot{a} \dddot{a}_z -(\ddot{\beta} + \dot{\beta} \varphi_e) o_z - \dddot{\varphi}_a;
\end{align*}
\]

and \( \varphi_e \) appears as a consequence of nondimensionalization as

\[
f_e = \ddot{\varphi}_e / \dot{\varphi}^2 = -2\epsilon \sin\theta, \quad \text{where} \quad \epsilon = \text{orbit eccentricity.}
\]
The appendage relative angular velocity vector $\tilde{\omega}_a^*$ and its derivative $\dot{\tilde{\omega}}_a^*$ can be written as:

$$\tilde{\omega}_a^* = \tilde{\omega}_a / \dot{\theta},$$

$$\dot{\tilde{\omega}}_a^* = \ddot{\tilde{\omega}}_a / \dot{\theta}^2,$$

where

$$= \begin{pmatrix} \omega_a^* & \dot{i}_a \\ \omega_y^* & \dot{j}_a \\ \omega_z^* & \dot{k}_a \end{pmatrix} = \begin{pmatrix} \beta & \dot{i}_a \\ \dot{\omega}_y & \dot{j}_a \\ \dot{\omega}_z & \dot{k}_a \end{pmatrix};$$

$$= \begin{pmatrix} \dot{\omega}_a^* & \dot{i}_a \\ \dot{\omega}_y^* & \dot{j}_a \\ \dot{\omega}_z^* & \dot{k}_a \end{pmatrix},$$

$$= \begin{pmatrix} \ddot{\beta} + \ddot{\beta} f_e & \dot{i}_a \\ (\ddot{a} + \dot{a} f_e) o_y + \dot{\omega}_y^* & \dot{j}_a \\ (\ddot{a} + \dot{a} f_e) a_y + \dot{\omega}_z^* & \dot{k}_a \end{pmatrix}. $$
APPENDIX F - CENTER OF MASS TERMS

Determination of position, velocity and acceleration of the center of mass

First, the dimensions of the two cylinder body satellite configuration are established. The radii and lengths of the two cylinders are determined in terms of the satellite parameters $\xi_s$, $\xi_a$, $\eta$. The hinge location on the central cylinder is specified by $\vec{\xi} = (\xi_x, \xi_y, \xi_z)^T$, thus giving $\vec{h}$. The hinge location on the appendage is, by constraint, located at the center of the axial end, which gives $\vec{f}$. The vector $(\vec{h} + \vec{f})$ locates $O_a$ with respect to $O_s$ (Figure 2.3). The position, velocity, and acceleration of the instantaneous center of mass is then determined.

Dimensions of the central and appendage cylinders

Let: $w_s = \text{radius of central cylinder}$;
$l_s = \text{length of central cylinder}$;
$V_s = \text{volume of central cylinder}$;
$w_a = \text{radius of appendage cylinder}$;
$l_a = \text{length of appendage cylinder}$;
$V_a = \text{volume of appendage cylinder}$;
$\rho = \text{density ratio of appendage to central body}$.

From section 2.5.1 the central body and the appendage are taken to be symmetric about their respective axis $z_s$ and $z_a$. 99
The moments of inertia are:

\[ I_{s,z} = \frac{1}{2} m_s w_s^2 \]

axial moment of inertia for central cylinder;

\[ I_{s,x} = \frac{1}{12} m_s (3w_s^2 + l_s^2) \]

transverse moment of inertia for central cylinder;

\[ I_{a,z} = \frac{1}{2} m_a w_a^2 \]

axial moment of inertia for appendage cylinder;

\[ I_{a,x} = \frac{1}{12} m_a (3w_a^2 + l_a^2) \]

transverse moment of inertia for appendage cylinder.

The volumes of the cylinders are given by:

\[ V_s = \pi w_s^2 l_s \]

\[ V_a = \pi w_a^2 l_a \]

....(F.1)

....(F.2)

From section 2.5.2, the inertia parameters are given as:

\[ \xi_s = I_{s,z} / I_{s,x} \]

....(F.3)

\[ \xi_a = I_{a,z} / I_{a,x} \]

....(F.4)

\[ \eta = m_a / m_s \]

....(F.5)

Equations (F.3), (F.4) and (F.5) after appropriate substitutions can be rewritten as:
$\zeta_s = \frac{6 \omega_s^2}{3\omega_s^2 + l_s^2}$; ....(F.6)

$\zeta_a = \frac{6 \omega_a^2}{3\omega_a^2 + l_a^2}$; ....(F.7)

$\eta = \frac{\rho V_a}{V_s}$. ....(F.8)

Equation (F.1) can be rearranged as

$$l_s = \frac{V_s}{(\pi \omega_s^2)},$$ ....(F.9)

and with substitution from (F.8), (F.2) can be rewritten

$$l_a = \eta \frac{V_s}{(\rho \pi \omega_a^2)}.$$ ....(F.10)

Substitution of (F.9) into (F.6) and (F.10) into (F.7) gives:

$$\omega_s = \left(\frac{V_s^2 \zeta_s}{3\pi^2 (2 - \zeta_s)}\right)^{\frac{1}{6}};$$ ....(F.11)

$$\omega_a = \left(\frac{V_s^2 \eta^2 \zeta_a}{3\pi^2 \rho^2 (2 - \zeta_a)}\right)^{\frac{1}{6}}. ....(F.12)$$

The effect of $V_s$ is to impose a constraint on the size of the satellite configuration. The numerical value of $V_s$ would
play a significant role in an aerodynamic study where the surface area of a satellite affects its response due to drag. As this study does not consider aerodynamic effect, the numerical value of $V_s$ becomes arbitrary and is therefore conveniently set equal to unity. Furthermore, since this study considers bodies of known density (section 2.5.1), the density ratio $\rho$ can also be set equal to unity. For $V_s = 1$ and $\rho = 1$, and with substitutions of equations (F.11) and (F.12) into (F.9) and (F.10), respectively, equations (F.9) to (F.12) can be rewritten as:

$$w_s = \left( \frac{\xi_s}{3 \pi^2 \left( 2 - \xi_s \right)} \right)^{\frac{1}{6}}; \quad \ldots \text{(F.13)}$$

$$l_s = \left( \frac{3 \left( 2 - \xi_s \right)}{\pi \xi_s} \right)^{\frac{1}{3}}; \quad \ldots \text{(F.14)}$$

$$w_a = \eta^\frac{1}{3} \left( \frac{\xi_a}{3 \pi^2 \left( 2 - \xi_a \right)} \right)^{\frac{1}{6}}; \quad \ldots \text{(F.15)}$$

$$l_a = \eta^\frac{1}{3} \left( \frac{3 \left( 2 - \xi_a \right)}{\pi \xi_a} \right)^{\frac{1}{3}}. \quad \ldots \text{(F.16)}$$
Determination of position vector \( \vec{h} + \vec{f}_3 \)

\( \vec{h} \) locates the hinge position \( P \) with respect to coordinate system \( x_s, y_s, z_s \) (Figure 2.5),

\[
\vec{h} = \begin{pmatrix}
    h_x \\ h_y \\ h_z
\end{pmatrix} = \begin{pmatrix}
    \hat{x}_s \\ \hat{y}_s \\ \hat{z}_s
\end{pmatrix},
\]

where:

\[
\begin{align*}
    h_x &= \ell_x w_s; \\
    h_y &= \ell_y w_s; \\
    h_z &= \ell_z l_s/2.
\end{align*}
\]

\( \vec{f} \) locates \( O_a \) with respect to \( P \) along the appendage coordinate axes \( x_a, y_a, z_a \) (Figure 2.5). Since the hinge location on the appendage cylinder is confined to the center of the axial end,

\[
\vec{f} = \begin{pmatrix}
    f_x \\ f_y \\ f_z
\end{pmatrix} = \begin{pmatrix}
    \hat{x}_a \\ \hat{y}_a \\ \hat{z}_a
\end{pmatrix},
\]
where:
\[ f_x = 0; \]
\[ f_y = 0; \]
\[ f_z = \frac{l_a}{2}. \]

Recalling that \( x_s,y_s,z_s \) is parallel to \( x,y,z \) at any instant, a projection of \( \bar{f} \) on the body axes \( x,y,z \) is the same as a projection on the axes \( x_s,y_s,z_s \),

\[ \bar{f}_3 = [M_a] \bar{f} = \begin{pmatrix} f_{3x} \\ f_{3y} \\ f_{3z} \end{pmatrix} \]

where:
\[ f_{3x} = (\frac{l_a}{2}) a_{xa}; \]
\[ f_{3y} = (\frac{l_a}{2}) a_{ya}; \]
\[ f_{3z} = (\frac{l_a}{2}) a_{za}. \]

Note that \( \bar{h} + \bar{f}_3 \) locates \( O_a \) with respect to \( O_s \),

\[ \bar{h} + \bar{f}_3 = \begin{pmatrix} (\xi_x w_s + (\frac{l_a}{2}) a_{xa}) & \hat{i} \\ (\xi_y w_s + (\frac{l_a}{2}) a_{ya}) & \hat{j} \\ (\xi_z l_s/2 + (\frac{l_a}{2}) a_{za}) & \hat{k} \end{pmatrix}. \]
Position, velocity and acceleration of instantaneous center of mass

Let the position vector locating $O_s$ with respect to $C$ along the body axes $x,y,z$ be denoted by $\vec{d}_s$. Similarly, let the position vector locating $O_a$ with respect to $C$ along the body axes $x,y,z$, be denoted by $\vec{d}_a$. These are the same position vectors as mentioned earlier in Appendix B.

From Figure 2.3:

$$\vec{d}_s = -\vec{c} - \vec{c}_o;$$

$$\vec{d}_a = -\vec{c} - \vec{c}_o + \vec{h} + \vec{f}_3.$$ 

The position of $C$, measured from $O_s$, is given by

$$(\vec{c}_o + \vec{c}) (m_s + m_a) = 0 * m_s + (\vec{h} + \vec{f}_3) m_a,$$

$$\vec{c}_o + \vec{c} = (m_a / (m_s + m_a)) (\vec{h} + \vec{f}_3),$$

$$= \frac{n}{n+1} (\vec{h} + \vec{f}_3).$$

Therefore:

$$\vec{d}_s = \frac{-n}{n+1} (\vec{h} + \vec{f}_3);$$

$$\vec{d}_a = \frac{1}{n+1} (\vec{h} + \vec{f}_3).$$
So \( \dd_s \) and \( \dd_a \) appear as:

\[
\dd_s = \begin{pmatrix}
  d_{sx} \\
  d_{sy} \\
  d_{sz}
\end{pmatrix}
\]

where:

\[
\begin{align*}
  d_{sx} &= \frac{-\eta}{\eta+1} \left( \xi_x w_s + \left( \frac{l_a}{2} \right) a_{xa} \right); \\
  d_{sy} &= \frac{-\eta}{\eta+1} \left( \xi_y w_s + \left( \frac{l_a}{2} \right) a_{ya} \right); \\
  d_{sz} &= \frac{-\eta}{\eta+1} \left( \xi_z l_s/2 + \left( \frac{l_a}{2} \right) a_{za} \right);
\end{align*}
\]

\[
\dd_a = \begin{pmatrix}
  d_{ax} \\
  d_{ay} \\
  d_{az}
\end{pmatrix}
\]

where:

\[
\begin{align*}
  d_{ax} &= \frac{1}{\eta+1} \left( \xi_x w_s + \left( \frac{l_a}{2} \right) a_{xa} \right); \\
  d_{ay} &= \frac{1}{\eta+1} \left( \xi_y w_s + \left( \frac{l_a}{2} \right) a_{ya} \right); \\
  d_{az} &= \frac{1}{\eta+1} \left( \xi_z l_s/2 + \left( \frac{l_a}{2} \right) a_{za} \right).
\end{align*}
\]
Let the velocity of \( O_s \) and \( O_a \) with respect to \( C \) be denoted by \( \vec{v}_s \) and \( \vec{v}_a \), respectively. The velocity vectors are time derivatives of the position vectors, divided by the true anomaly rate \( \dot{\theta} \). Hence,

\[
\vec{v}_s = \frac{\dot{d}_s}{\dot{\theta}} = \begin{pmatrix}
    v_{sx} \hat{i} \\
    v_{sy} \hat{j} \\
    v_{sz} \hat{k}
\end{pmatrix},
\]

where:

\[
v_{sx} = -\frac{\eta}{\eta + 1} \left( \frac{l}{a/2} \right) \ddot{a}_xa;
\]

\[
v_{sy} = -\frac{\eta}{\eta + 1} \left( \frac{l}{a/2} \right) \ddot{a}_ya;
\]

\[
v_{sz} = -\frac{\eta}{\eta + 1} \left( \frac{l}{a/2} \right) \ddot{a}_za;
\]

\[
\vec{v}_a = \frac{\dot{d}_a}{\dot{\theta}} = \begin{pmatrix}
    v_{ax} \hat{i} \\
    v_{ay} \hat{j} \\
    v_{az} \hat{k}
\end{pmatrix},
\]

where:

\[
v_{ax} = \frac{1}{\eta + 1} \left( \frac{l}{a/2} \right) \ddot{a}_xa;
\]

\[
v_{ay} = \frac{1}{\eta + 1} \left( \frac{l}{a/2} \right) \ddot{a}_ya;
\]

\[
v_{az} = \frac{1}{\eta + 1} \left( \frac{l}{a/2} \right) \ddot{a}_za.
\]
Let the acceleration of $O_S$ with respect to $C$ be denoted by $\ddot{a}_s$. It is a second time derivative of the position vector $\dddot{d}_s$, divided by $\dot{\theta}^2$,

$$\dddot{a}_s = \dddot{d}_s / \dot{\theta}^2 = \left\{ \begin{array}{c}
    a_{sx} \hat{i} \\
    a_{sy} \hat{j} \\
    a_{sz} \hat{k}
\end{array} \right\},$$

where:

$$a_{sx} = \frac{-\eta}{n+1} \left( l / 2 \right) \dddot{a}_x;$$

$$a_{sy} = \frac{-\eta}{n+1} \left( l / 2 \right) \dddot{a}_y;$$

$$a_{sz} = \frac{-\eta}{n+1} \left( l / 2 \right) \dddot{a}_z.$$
APPENDIX G - NONDIMENSIONAL SATELLITE INERTIA MATRIX

[I] was given in Appendix B.

\[ [I^*] = [I] / I_{s,x}, \]
\[ = [I_s] / I_{s,x} + (m_s / I_{s,x}) \left( \overline{d_s} \cdot \overline{d_s} [E] - \overline{d_s} \overline{d_s}^T \right) \]
\[ + [M_a] [ [I_a] / I_{s,x} ] [M_a]^T \]
\[ + (m_a / I_{s,x}) \left( \overline{d_a} \cdot \overline{d_a} [E] - \overline{d_a} \overline{d_a}^T \right), \]

where:

\[ [I_s^*] = [I_s] / I_{s,x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \xi_s \end{bmatrix} ; \]
\[ [I_a^*] = [I_a] / I_{s,x} = \begin{bmatrix} \eta k_2 & 0 & 0 \\ 0 & \eta k_2 & 0 \\ 0 & 0 & \xi_a \eta k_2 \end{bmatrix} ; \]
\[ k_s = \frac{m_s}{I_s,x} = 12 / k_1 ; \]

\[ k_a = \frac{m_a}{I_s,x} = 12 \eta / k_1 ; \]

\[ k_1 = 3 w_s^2 + l_s^2 ; \]

\[ k_2 = \frac{(3 w_a^2 + l_a^2)}{k_1} . \]
APPENDIX H - NONDIMENSIONAL ANGULAR MOMENTUM VECTOR

{H} was given in Appendix A.

\[ \{H^*\} = \{H\} / (I_{s,x} \dot{\theta}) , \]
\[ = (m_s / I_{s,x}) \bar{d}_s \times (\bar{d}_s / \dot{\theta}) \]
\[ + (m_a / I_{s,x}) \bar{d}_a \times \]
\[ ( [M_a] (\bar{\omega}_a / \dot{\theta}) \times [M_a] \bar{f} + (\ddot{d}_s / \dot{\theta}) ) \]
\[ + [M_a] ([I_a] / I_{s,x}) (\bar{\omega}_a / \dot{\theta}) , \]
\[ = k_s \bar{d}_s \times \bar{v}_s \]
\[ + k_a \bar{d}_a \times ( [M_a] \bar{\omega}_a \times [M_a] \cdot \bar{f} + \bar{v}_s ) \]
\[ + [M_a] [I_a^*] \bar{\omega}_a^* , \]

where:
\[ [I_a^*] = [I_a] / I_{s,x} \quad \text{(Appendix G)}; \]
\[ \bar{\omega}_a^* = \bar{\omega}_a / \dot{\theta} \quad \text{(Appendix E)}; \]
\[ \bar{v}_s = \dot{\bar{d}}_s / \dot{\theta} \quad \text{(Appendix F)}; \]
\[ k_s = m_s / I_{s,x} \quad \text{(Appendix G)}; \]
\[ k_a = m_a / I_{s,x} \quad \text{(Appendix G)}. \]
Recalling from section 2.6.2,

\[ [i^*] = \frac{[i]}{(I_s, x^2 \dot{\theta})} ; \]

\[
= k_s \left[ 2 \ddot{d}_s \cdot \vec{v}_s [E] - \vec{v}_s \ddot{d}_s^T - \ddot{d}_s \vec{v}_s^T \right]
\]

\[
+ [\ddot{M}_a] [I_a^*] [M_a]^T + [M_a] [I_a^*] [\ddot{M}_a]^T
\]

\[
+ k_a \left[ 2 \ddot{d}_a \cdot \vec{v}_a [E] - \vec{v}_a \ddot{d}_a^T - \ddot{d}_a \vec{v}_a^T \right].
\]
APPENDIX J - NONDIMENSIONAL DERIVATIVE OF ANGULAR MOMENTUM VECTOR

Recalling from section 2.6.2,

\[ \{ \dot{H}^* \} = \{ \dot{H} \} / (I_a S \dot{\theta}^2) ; \]
\[ = k_s ( \bar{v}_s \times \vec{v}_s + \bar{d}_s \times a_s ) \]
\[ + k_a \bar{v}_a \times ( [M_m] \bar{\omega}_a \times [M_m] \bar{f} + \bar{v}_s ) \]
\[ + k_a \bar{d}_a \times \{ ( [\tilde{M}_m] \bar{\omega}_a^* + [M_m] \bar{\omega}_a^* ) \times [M_m] \bar{f} \}
\[ + [M_m] \bar{\omega}_a^* \times ([\tilde{M}_m] \bar{f} + [M_m] \bar{f}/\dot{\theta} ) + \bar{a}_s \} \]
\[ + [\tilde{M}_m] [I_1^*] \bar{\omega}_a^* + [M_m] [I_1^*] \bar{\omega}_a^* . \]

Noting that \( \ddot{f} = \bar{v}_s \times \vec{v}_s = 0 \), \( \{ \dot{H}^* \} \) reduces to

\[ \{ \dot{H}^* \} = k_s ( \bar{d}_s \times a_s ) \]
\[ + k_a \bar{v}_a \times ( [M_m] \bar{\omega}_a \times [M_m] \bar{f} + \bar{v}_s ) \]
\[ + k_a \bar{d}_a \times \{ ( [\tilde{M}_m] \bar{\omega}_a^* + [M_m] \bar{\omega}_a^* ) \times [M_m] \bar{f} \}
\[ + [M_m] \bar{\omega}_a^* \times ([\tilde{M}_m] \bar{f} + \bar{a}_s ) \}
\[ + [\tilde{M}_m] [I_1^*] \bar{\omega}_a^* + [M_m] [I_1^*] \bar{\omega}_a^* . \]
The appendage pitch during a slewing maneuver is given by

\[ a = a_i + \Delta a , \]

where \( \Delta a \) represents the change in appendage pitch. \( \Delta a \) is generated through specification of the appendage pitch parameters \( a_i, a_f, \tau_a \).

Equations of time histories with finite initial velocities

Exponential time history:

\[ \Delta a = (a_f - a_i) \left( 1 - \exp\left( -20 \frac{\theta}{\tau_a} \right) \right) . \]

Sinusoidal time history:

\[ \Delta a = (a_f - a_i) \sin\left( \frac{\pi \theta}{2 \tau_a} \right) . \]

Ramp time history is composed of a linear function (during first 95% of \( \tau_a \)), followed by a quadratic function (during last 5% of \( \tau_a \)):

\[ \Delta a = (a_f - a_i) \left( 2(0.95)\tau_a A + B \right) \theta , \text{ for } \theta \leq 0.95 \tau_a \]
\[ \Delta a = (a_f - a_i) \left( A \theta^2 + B \theta + C \right) , \text{ for } \theta \geq 0.95 \tau_a \]
where 
\[ A = \frac{1}{(\tau_a^2 (0.95^2 - 1))} \]
\[ B = -2 / (\tau_a (0.95^2 - 1)) \]
\[ C = 0.95^2 / (0.95^2 - 1) \]

**Equations of time histories with zero initial velocities**

**Cubic time history:**

\[ \Delta a = (a_f - a_i) \left( (3 \theta^2 / \tau_a^2) - (2 \theta^3 / \tau_a^3) \right) \]

**Exponential time history:**

\[ \Delta a = (a_f - a_i) \left( 1 - \exp(-20 \theta^2 / \tau_a^2) \right) \]

**Sinusoidal time history:**

\[ \Delta a = (a_f - a_i) \left( 1 + \sin(\pi \theta / \tau_a - \pi / 2) \right) / 2 \]

**Ramp time history** is composed of a quadratic function (during first 5% of \( \tau_a \)), a linear function (during next 90% of \( \tau_a \)), followed by a quadratic function (during last 5% of \( \tau_a \)):

\[ \Delta a = (a_f - a_i) D \theta^2, \quad \text{for } \theta \leq 0.05\tau_a; \]
\[ \Delta a = (a_f - a_i) (E \theta + F), \quad \text{for } 0.05\tau_a \leq \theta \leq 0.95\tau_a; \]
\[ \Delta a = (a_f - a_i) (A \theta^2, + B \theta + C), \quad \text{for } \theta \geq 0.95\tau_a. \]
Here: \[ A = \frac{1}{(\tau_a^2 (0.95-1) (1-0.05+0.95))}; \]
\[ B = \frac{-2}{(\tau_a (0.95-1) (1-0.05+0.95))}; \]
\[ C = 1 + \frac{1}{((0.95-1) (1-0.05+0.95))}; \]
\[ D = \frac{1}{(\tau_a^2 0.05 (1-0.05+0.95))}; \]
\[ E = \left( A 0.95^2 \tau_a^2 + B 0.95 \tau_a + C - D 0.05^2 \tau_a^2 \right) \]
\[ \left/ (0.95 \tau_a - 0.05 \tau_a) \right; \]
\[ F = \left( D \tau_a^3 (0.95)0.05^2 - 0.05 \tau_a (A 0.95^2 \tau_a^2 \right)
\[ + B 0.95 \tau_a + C)) \left/ (0.95 \tau_a - 0.05 \tau_a \right). \]

The appendage roll during a slewing maneuver is given by

\[ \beta = \beta_i + \Delta \beta, \]

where \( \Delta \beta \) represents the change in appendage roll. \( \Delta \beta \) is generated through use of the same equations that gave \( \Delta a \) but with appendage roll parameters \( \beta_i, \beta_f, \tau_\beta \) appropriately replacing \( a_i, a_f, \tau_a \).