OPTIMAL PUBLIC POLICIES IN SMALL OPEN ECONOMIES

By

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ABSTRACT

Until recently, proofs establishing the existence of gains from trade have used the assumption that the government can alter the distribution of income by a set of lump sum transfers, i.e., the government has at its disposal a set of household specific transfer instruments. However, recent work has been devoted to situations where these transfer instruments are inadmissible. Dixit and Norman (1980: 79-80) demonstrate that a government that can alter all domestic commodity taxes can ensure that no individual is made worse off by moving from autarky to free trade. It turns out, however, that this Dixit and Norman proof of the gains from trade shows only that the autarky equilibrium can be replicated under free trade and not that positive gains will occur.

One of the purposes of this thesis is to investigate the problem of the gains from trade when a variety of tax and transfer instruments are available.

It is fruitful to regard the problem of the gains from trade as a policy reform question: can the government in the home country find a small (differential) perturbation in the country's initial international trade prohibitive tariffs which, accompanied with a suitable (differential) perturbation in the country's commodity tax structure, results in a strict Pareto improvement? In order to answer the question, a model for the production side of an economy is presented in Chapter 2. It is established that, under some very weak conditions, there are (differential) tariff perturbations that improve the country's initial net balance of trade. In Chapter 4, it is shown that these
productivity gains can be distributed to the consumers in the economy in a strict Pareto improving way by suitably adjusting the country's initial commodity tax rates. The principal tool for establishing these results is a duality theorem: Motzkin's Theorem.

Chapter 3 develops two approximative formulae for measuring the productivity gain accruing from a change of tariffs.

Some examples of strict Pareto improving perturbations in commodity taxes and tariffs are given in Chapter 7. These include proportional and uniform reductions of tariffs as well as a change toward uniformity in the country's initial tariff structure.

Next, the government is assumed to be able to adjust only the home country's initial vectors of tariffs and lump sum transfers but not the vector of commodity taxes. Conditions for strict Pareto improving tariff and transfer perturbations to exist are developed.

In Chapter 9 it is shown that neither the existence of strict gains from trade under commodity taxation or under lump sum compensation necessarily implies the other.

Examples of strict Pareto improving changes in tariffs, taxes and transfers are given in Chapter 10. These include proportional reductions of tariffs and/or taxes and movements toward uniformity in the tax rates for domestic and tradeable commodities. The role of normality of commodities in consumption in policy recommendation results is also discussed.

Chapter 11 develops sufficient conditions for a perturbation in the home country's tax structure, which causes international trade, to be strict Pareto improving.
In Chapter 12 the goal of the government is to choose a policy that reduces the level of economic inequality associated with the initial observed equilibrium in the economy. It is shown that inequality reducing perturbations in commodity taxes and tariffs exist, if the preferences and initial commodity endowments of the consumers satisfy certain conditions.
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1. INTRODUCTION

The question of whether there are gains from trade is an interesting one. Until recently, proofs establishing the existence of gains from trade have used the assumption that the government can alter the distribution of income by a set of lump sum transfers; i.e., the government has at its disposal a set of household specific transfer instruments. However, recent work has been devoted to situations where these transfer instruments are inadmissible. Dixit and Norman (1980: 79-80) demonstrate that a government that can alter all domestic commodity taxes can ensure that no individual is made worse off by moving from autarky to free trade. However, Kemp and Wan (1983) provide an example of an economy which shows that the availability of commodity tax instruments alone (without the use of lump sum transfers) is not sufficient to ensure that a Pareto improvement will occur moving from autarky to free trade. Thus the Dixit and Norman proof of the gains from trade shows only that the autarky equilibrium may be replicated under free trade and not that positive gains will occur.

One of the purposes of this thesis is to investigate the problem of the gains from trade when a variety of tax and transfer instruments are available.

It is fruitful to regard the problem of the gains from trade as a policy reform question. Suppose that the initial autarky equilibrium in the home country is a consequence of the government tariff policy; i.e., the initial tariffs on internationally tradeable commodities, in a
country open for international trade, are chosen to be such that the resulting producer prices faced by the domestic production sector coincide with the autarky equilibrium prices for tradeables. Can the government find a small (differential) perturbation in the initial international trade prohibitive tariffs which, accompanied with a suitable (differential) perturbation in the home country's commodity tax structure, results in a strict Pareto improvement, i.e., in a strict welfare improvement for all households in the economy? If this is possible, the government should adopt the policy of (infinitesimally) changing the country's initial tax and tariff structure—a policy which also gainfully opens the country for international trade.

The problem of the gains from trade thus becomes the following: under what conditions do strict Pareto improving perturbations of commodity taxes and tariffs exist? In order to answer this question, a model for the production side of an economy is presented in Chapter 2. The model is written in duality terms, assuming that there are K constant returns to scale production industries in the home country. It is established in Theorem 2.1 that, under some very weak conditions on the production sectors' technologies, there exist perturbations in the home country's initial tariff structure that improve the country's initial net balance of trade, i.e., the net amount of tradeables revenue generated by the K industries. In Chapter 4, it is shown that these productivity gains can be distributed to the consumers in the economy in a strict Pareto improving way by suitably adjusting the country's initial commodity tax rates. The principal tool for establishing these
results is a duality theorem: Motzkin's Theorem. The general
equilibrium model employed in this thesis is similar to the model used
in Diewert (1983b).

Chapter 3 addresses the problem of measuring the gain accruing
from a productivity (i.e., net balance of trade) improving change of
tariffs. It is shown that approximative formulae for the gain can be
found by applying Diewert's measurement of deadweight loss methodology.
The proposed formulae are based on observable data and approximate the
productivity gain to the second order.

Chapter 5 discusses the gains from trade problem in more detail.

In Chapter 6 it is assumed that only a subset of the country's
initial commodity taxes and tariffs can be perturbed. Chapter 7 gives
some examples of strict Pareto improving perturbations in tariffs and
commodity taxes. These include proportional and uniform reductions of
the home country's initial tariffs. It is also shown that a change
toward uniformity in the country's initial tariff structure can be
strict Pareto improving.

In Chapter 8 it is assumed that the government can adjust the
country's initial vector of tariffs and the initial vector of household
specific transfers. The commodity tax rates in the economy are assumed
to be fixed. The problem considered is: under what conditions are
there strict Pareto improving (differential) perturbations in the
country's initial tariffs and transfers? As in Chapter 2, the
conditions for strict Pareto improving tariff and transfer changes to
exist are developed by first considering the existence of productivity
improving tariff perturbations.
The problem of the gains from trade is readressed in Chapter 9: the sufficient conditions for the existence of strict gains from trade under commodity taxation and under lump sum compensation are compared. It is shown that neither the existence of strict gains under commodity taxation or under lump sum compensation necessarily implies the other.

In Chapter 10 some examples of strict Pareto improving changes in tariffs, taxes and lump sum transfers are given. These include proportional reductions of tariffs and taxes, proportional reductions of either domestic or tradeables commodity taxes, and movements toward uniformity in the tax rates for domestic and tradeable commodities. The role of normality of commodities in consumption in policy recommendation results is also discussed.

Chapter 11 considers the welfare effects of a change in the home country's commodity tax structure (without a change in tariffs or lump sum transfers). If the initial equilibrium of the economy is an autarky equilibrium, the conditions given in Theorem 11.1 can be interpreted to be sufficient for a perturbation of taxes, which causes international trade, to be strict Pareto (hence, welfare) improving.

In Chapter 12 the main policy goal of the government is assumed to be the reduction of economic inequality in the society. The goal of the analysis in this chapter is to operationalize the concept of economic inequality in such a way that practical policy questions can be answered. It is established that economic inequality reducing commodity tax and tariff perturbations exist if the consumers' preferences and initial commodity endowments satisfy certain conditions. For example,
there must exist a commodity with respect to which the preferences of the "rich" and the "poor" in the economy significantly differ. In this case, a proportional reduction of tariffs can be made inequality reducing by suitably perturbing the home country's commodity tax structure.
2. PRODUCTIVITY IMPROVING CHANGES IN TARIFFS

2.1 Equilibrium for the Production Side of an Economy

The production side of an economy is assumed to consist of $K$ constant returns to scale sectors, indexed $k=1,...,K$. There are $N+M$ commodities, $N$ of which are domestic (not internationally tradeable). The prices of the domestic goods are denoted by $p \in \mathbb{R}^N_+$. The tradeables prices $w \in \mathbb{R}^M_+$ are internationally given. Hence, the country in question is assumed to be small.

The technology of the $k$th industry (or producer) is represented by its unit production possibility set $C^k$, $k=1,...,K$. Thus, if $(y^k, f^k) \in C^k$, the vector $y^k = (y^k_1, ..., y^k_N)$ and the vector $f^k = (f^k_1, ..., f^k_M)$ of domestic (net) supplies and the net exports are producible by sector $k$ when it operates at unit scale. If $y^k_n < 0$, $n \in [1,...,N]$, the $n$th domestic good is used as an input in sector $k$ and if $f^k_m < 0$, $m \in [1,...,M]$, the $m$th tradeable good is an import for sector $k$, $k=1,...,K$.

The unit production sets $C^k$ are defined with respect to some always used input (or with respect to some always produced output). For example, if for sector $k$ an always used input exists (e.g., land), some amount of this input is chosen as the unit level, and the set $C^k$ then consists of all vectors $(y^k, f^k)$ that the sector can produce using one unit of land. Each set $C^k$, $k=1,...,K$, is assumed to be nonempty, closed and bounded from above.

Define $z^k \in \mathbb{R}_+$ as the amount of the always used input (or the always produced output) in sector $k$, $k=1,...,K$. This variable gives the
scale of sector $k$. Using the scale $z^k$, the total production possibility set of sector $k$ is defined as $T^k = \{z^k c^k: z^k \geq 0\}$. Each $T^k, k=1,...,K,$ is a nonempty, closed cone.

According to duality theory, since the $K$ production sectors are assumed to behave competitively, the unit production possibility sets $C^k, k=1,...,K,$ can equivalently be represented using the production sectors' unit profit functions $\pi^k$ defined for $k=1,...,K$ by

\[(2.1) \quad \pi^k(p, w + \tau) = \max_{y^k, f^k} \{p^T y^k + (w + \tau)^T f^k: (y^k, f^k) \in C^k\},\]

where $\tau \in \mathbb{R}^M$ is a vector of trade taxes and/or subsidies. The vector $(w + \tau)$ in (2.1) is thus the vector of prices for tradeable commodities faced by domestic producers. If each internationally traded good is either used as an input by all sectors or produced as an output by all sectors, the components of the vector $\tau = (\tau_1, ..., \tau_M)^T$ may be interpreted as follows: if $f_m^k > 0$ and $\tau_m > 0 (< 0)$, net exports of the $m$th internationally traded good by sector $k$ are subsidized (taxed); if $f_m^k < 0$ and $\tau_m > 0 (< 0)$, net imports of good $m$ into sector $k$ are taxed (subsidized). In what follows, $\tau$ is called the tariff vector. The unit profit functions $\pi^k, k=1,...,K,$ are well-defined since, by assumption, each set $C^k$ is closed and bounded from above. Furthermore, each unit profit function is convex and linearly homogenous in the prices $(p, w + \tau)$.

Assuming that the unit profit functions $\pi^k, k=1,...,K,$ are twice continuously differentiable, the sectoral price-dependent input output coefficients $y^k$ and $f^k, k=1,...,K,$ can be determined using Hotelling's Lemma:
(2.2) \hspace{1cm} y^k = \nabla_p \pi^k(p, w + \tau), \hspace{1cm} k = 1, \ldots, K; \hspace{1cm} Y \equiv [y^1, \ldots, y^K] \nabla_w \pi^k(p, w + \tau), \hspace{1cm} k = 1, \ldots, K; \hspace{1cm} F \equiv [f^1, \ldots, f^K].

The vectors \( \nabla_p \pi^k(p, w + \tau) \) and \( \nabla_w \pi^k(p, w + \tau) \) denote the first order partial derivatives of the functions \( \pi^k, k = 1, \ldots, K, \) with respect to the vectors \( p \) and \( w, \) respectively. Linear homogeneity of the unit profit functions and Euler's Theorem imply

(2.4) \hspace{1cm} \pi^k(p, w + \tau) = p^T \nabla_p \pi^k(p, w + \tau) + (w + \tau)^T \nabla_w \pi^k(p, w + \tau) = p^T y^k - (w + \tau)^T f^k, \hspace{1cm} k = 1, \ldots, K.

If the scale \( z^k \) is positive \((z^k > 0)\), the vectors \( y^k z^k \) and \( f^k z^k \) give sector \( k \)'s total domestic and net export supplies. Hence, the total profit earned by sector \( k \) is

(2.5) \hspace{1cm} \pi^k_z = p^T y^k z^k + (w + \tau)^T f^k z^k, \hspace{1cm} k = 1, \ldots, K.

Then, using the matrices \( Y \) and \( F \) defined in (2.2)-(2.3), the (row) vector of the industry total profits can be written as \( p^T Y z + (w + \tau)^T F z. \)

The producers' aggregate symmetric substitution matrix \( S \) is defined by

(2.6) \hspace{1cm} S = \begin{bmatrix} S_{pp} & S_{pw} \\ S_{wp} & S_{ww} \end{bmatrix} = \begin{bmatrix} \frac{K}{2} \sum_{k=1}^{K} \frac{\nabla_p \pi^k(p, w + \tau) z^k}{\nabla_p \pi^k(p, w + \tau) z^k} & \frac{K}{2} \sum_{k=1}^{K} \frac{\nabla_w \pi^k(p, w + \tau) z^k}{\nabla_w \pi^k(p, w + \tau) z^k} \\ \frac{K}{2} \sum_{k=1}^{K} \frac{\nabla_p \pi^k(p, w + \tau) z^k}{\nabla_p \pi^k(p, w + \tau) z^k} & \frac{K}{2} \sum_{k=1}^{K} \frac{\nabla_w \pi^k(p, w + \tau) z^k}{\nabla_w \pi^k(p, w + \tau) z^k} \end{bmatrix}.
In (2.6), the matrix block \( S_{pp} \) gives the responses of the domestic total net supplies to changes in domestic prices \( p \), \( S_{pw} \) gives the responses of the domestic total net supplies to changes in tradeables prices \( w + \tau \), and \( S_{ww} \) gives the responses of the total net export supplies to changes in prices \( (w, + \tau) \). The matrix \( S \) is positive semidefinite, since the unit profit functions \( \pi^k \), \( k = 1, \ldots, K \), are convex in prices \( (p, w + \tau) \). Linear homogeneity of the unit profit functions and Euler's Theorem imply

\[
\begin{align*}
\text{(2.7)} \quad [p^T, (w + \tau)^T] S &= 0^T_{N+M}.
\end{align*}
\]

This means that the producer substitution matrix \( S \) has at least one zero eigenvector which is the vector of producer prices \( (p, w + \tau) \).  

The equilibrium conditions for the production side of the economy, assuming that each sector \( k, k = 1, \ldots, K \), is operating at a positive scale, are:

\[
\begin{align*}
\text{(2.8)} \quad \sum_{k=1}^{K} \pi^k(p^*, w + \tau^*)z^k &= y^*, \\
\text{(2.9)} \quad \pi^k(p^*, w + \tau^*) &= 0, \quad k = 1, \ldots, K, \\
\text{(2.10)} \quad \sum_{k=1}^{K} w \pi^k(p^*, w + \tau^*)z^k &= b^*.
\end{align*}
\]

According to (2.8)-(2.9), at an equilibrium (indexed with an asterisk), net supply of domestic commodities equals an exogenously given \( y^* \) \( (y^* \) can be, for example, the consumers' net demand vector or a vector of domestic goods
endowments) and all industries make zero pure profits.\(^9\) Equation (2.10) defines \(b^*\), the net amount of foreign exchange earned by the domestic producers.

The \(N + K + 1\) equations (2.8)-(2.10) endogenously determine the equilibrium vector of domestic prices \(p^*\), the equilibrium industry scales \(z^*\), and the equilibrium net balance of payments \(b^*\). The exogenous variables in the model are the net output of domestic commodities, \(y^*\), the constant international prices \(w\) and the tariffs \(\tau^*\). It is assumed that there exists an initial equilibrium where (2.8)-(2.10) are satisfied, and the vectors of domestic prices and industry scales are strictly positive, i.e., \(p^* \in \mathbb{R}^N_{++}, z^* \in \mathbb{R}^K_{++}\), given \(y^*\) and the prices of tradeable commodities \((w + \tau^*)\).

For the subsequent analysis it is required that the endogenous \(p^*\), \(z^*\) and \(b^*\) be regarded as (once continuously differentiable) implicit functions of the prices \((w + \tau^*)\). The conditions that guarantee the existence of these implicit functions can be derived by totally differentiating the model (2.8)-(2.10) at the initial equilibrium:\(^\text{10}\)

\[
(2.11) \quad B_p \Delta p^* + B_z \Delta z^* + B_b \Delta b^* = B_\tau \Delta \tau^*,
\]

where \(B_p = \begin{bmatrix} S_{pp} \\ Y^T \\ w^T S_{wp} \end{bmatrix}, \ B_z = \begin{bmatrix} Y \\ 0 \end{bmatrix}, \ B_b = \begin{bmatrix} O_N \\ O_K \end{bmatrix}\) and \(B_\tau = \begin{bmatrix} -S_{pw} \\ -F^T \\ -w^T S_{ww} \end{bmatrix}\).

According to the Implicit Function Theorem, the functions \(p^* (w + \tau^*), z^* (w + \tau^*)\) and \(b^* (w + \tau^*)\) around the initial equilibrium exist if the
matrix \([B_p, B_z, B_b]\) is invertible. Under this supposition, the directional derivatives of the functions \(p^*(w + \tau^*), z^*(w + \tau^*)\) and \(b^*(w + \tau^*)\) evaluated at the initial equilibrium are determined by the matrix \([B_p, B_z, B_b]^{-1} B\tau^*\).

Diewert and Woodland (1977: Appendix) show that necessary and sufficient conditions for the matrix \([B_p, B_z, B_b]\) to be invertible are:

\[
\begin{align*}
(2.12) & \quad \text{rank } Y = K (\leq N) \\
(2.13) & \quad \text{rank } (S_{pp} + YY^T) = N.
\end{align*}
\]

It is assumed henceforth that (2.12)-(2.13) are satisfied at the initial equilibrium. Economic interpretations for assumptions (2.12) and (2.13) are discussed in Sections 2.2 and 2.3.

2.2 Continuity of the Producers' Total Net Supply Functions

In order to develop an interpretation for assumption (2.12), the economy's GNP function\(^{11}\) \(G(w + \tau, y^*)\) must be first defined:

\[
(2.14) \quad G(w + \tau, y^*) = \max \left\{ \sum_{k=1}^{K} (w + \tau)^T f^k z^k : \begin{array}{c}
(y^k, f^k) \in C^k, z^k \geq 0 \\
\sum_{k=1}^{K} y^k z^k \geq y^*; k = 1, \ldots, K
\end{array} \right\}.
\]

In (2.14), the producers' net revenues from the sales of tradeables are maximized with respect to the constraint that the production sectors, in
the aggregate, supply a predetermined amount $y^*$ of domestic commodities. (If domestic good $n, n \in [1, \ldots, N]$, is an input, the producers' total demand for this factor must not exceed the given endowment $-y^*_n \geq 0$.)

Using the Karlin (1959: p. 201) — Uzawa (1958: p. 34) Saddle Point Theorem, the concave programming problem (2.14) can be written in an equivalent dual form:

(2.15) $G(w + \tau, y^*) = \min_{p \geq 0, N} \left\{ -p^Ty^* : -\pi^k(p, w + \tau) \geq 0, k = 1, \ldots, K \right\}.$

In (2.15), the producers' costs from using the domestic commodity vector $y^*$ as a net input are minimized with respect to the constraint that no production sector earns positive (pure) profits.\textsuperscript{12}

In order to illustrate problem (2.15), consider an economy where $N = 2$ and $K = 3$. In this particular case, $N$ is less than $K$ and assumption (2.12) is violated. Figure 1 is drawn by slightly modifying the factor price diagram presented in Woodland (1982: p. 48).

The shaded area in Fig. 1a) is the feasible solution set for problem (2.15). For the fixed $y^*$ depicted in the figure,\textsuperscript{13} the cost minimizing vector of domestic commodity prices is represented by the point $p^0$, where the unit profit level curves of the production sectors (at fixed $(w + \tau^*)$) intersect.\textsuperscript{14} Clearly, there exist industry scales $z^1 > 0, z^2 > 0, z^3 > 0$ such that $y^* = -\frac{\pi^1}{p}z^1 - \frac{\pi^2}{p}z^2 - \frac{\pi^3}{p}z^3$, i.e., in Fig. 1a) all three industries operate at a positive scale.

However, if the tariffs $\tau^*$ are perturbed, the producers' zero unit profit curves shift, as depicted in Fig. 2.1b). As a result, one of the
Figure 1 - Relative Numbers of Domestic Commodities and Production Industries: Continuity of Total Industry Net Supplies.
three industries is likely to cease production at a positive scale: in Fig. 1b), sector 1 does not operate at the new cost minimizing prices $p^1$, since its unit profits at these prices are negative. Hence, it can be seen that if the number of production sectors $K$ exceeds the number of domestic commodities $N$, as in the example above, the producers' total net supply functions $y^k z^k$ and $f^k z^k$, $k=1,...,K$, are likely to be discontinuous — an outcome to be avoided if differential analysis is to be applied.\(^{15}\) The assumption that $K \leq N$ can thus be regarded as a continuity constraint on the sectoral total net supply functions.

Continuity of the sectoral total net supply functions also depends on the rank of the matrix $Y$. To see this, consider Figure 2.

Figure 2 is drawn assuming that there are two production industries and two domestic commodities with prices $p = (p^1, p^2)$ in the economy. In Fig. 2a) the gradients of the sectoral unit profit functions $\pi^k$, $k = 1,2$, with respect to the domestic goods prices are linearly dependent, i.e., the rank of $(y^1, y^2)$ is 1 ($< K = 2$). At the prices $p^0$ both sectors are operating at a positive scale. Suppose now that the prices $(w + \tau^*)$ change. The change in the tradeables producer prices causes a shift in the sectoral unit profit level curves; a possible outcome is depicted in Fig. 2b). After the change in $(w + \tau^*)$ only sector 2 can earn zero pure profits and hence, only sector 2 will stay operative. Sector 1 will close down, causing a discontinuity in its total net supplies.

If, however, the gradients of the unit profit functions $\pi^1$ and $\pi^2$ with respect to the domestic commodity prices are linearly independent, i.e.,
Figure 2 - Rank of the Matrix Y and Continuity of Total Industry Net Supplies.
the rank of \((y^1, y^2)\) is 2 (\(=K\)), the sectoral zero unit profit curves will intersect both before and after a small perturbation in the tradeables producer prices \((w + \tau^*)\). In this case, both sectors will stay operative despite the tradeables price change. Assuming that the unit profit functions \(\pi^k\), \(k = 1, 2\), are twice continuously differentiable, the sectoral total net supply functions \(y^k z^k\) and \(f^k z^k\), \(k = 1, \ldots, K\), are continuous.

2.3 Local Controllability of the Production Sector and Changes in the Home Country's Net Balance of Trade

The goal of this section is to provide interpretations for assumption (2.13) and for a restriction imposed later in the analysis on the net balance of trade function \(b^*(w + \tau^*)\). It will be required that

\[
\begin{align*}
\nabla_\tau b(w + \tau^*) &\neq 0^T_M.
\end{align*}
\]

It will be seen that both assumptions (2.13) and (2.16) are related to local controllability of production in the home country, where the concept of local controllability is that defined by Guesnerie (1977) and Weymark (1979).

Let us start by considering the assumption that the matrix \(S_{pp} + YY^T\) is positive definite,\(^{16}\) i.e., assumption (2.13). As shown in Section 2.3, this supposition is sufficient (together with assumption (2.12)) for the implicit functions \(p^*(w + \tau^*), z^*(w + \tau^*)\) and \(b^*(w + \tau^*)\) to be well defined. Hence, from the mathematical point of view, assumption
(2.13) is needed to guarantee the existence of the inverse demand functions $p^*(w + \tau^*)$, the industry scale functions $z^*(w + \tau^*)$, and the net balance of trade function $b^*(w + \tau^*)$ around the initial equilibrium (which solves (2.8)-(2.10)).

To give an intuitive meaning for assumption (2.13), it is necessary to briefly explain the agenda for the following sections. The production side of an economy, described by the model (2.8)-(2.10), is analyzed. The authority choosing the exogenous vector of tariffs $\tau^*$ is called the government of the home country. The government is assumed to have a policy goal: to improve the country's initial net balance of trade $b^*$ by suitably changing the initial equilibrium tariffs $\tau^*$, while maintaining the aggregate domestic net supply at its initial level $y^*$. In other words, the government is assumed to search for a perturbation of the initial tariffs $\tau^*$ such that, after the country's production sector has adjusted to the change in the relative producer prices $(w + \tau^*)$, a higher level of net export revenue is attained without sacrificing any of the initial domestic net supply $y^*$. It is evident that the induced change in the relative producer prices $(w + \tau^*)$ will generally change both the producers' domestic and tradeables net supplies. Hence, to achieve its policy goal, the government must be able to influence domestic goods production in the home country in such a way that, in the aggregate, the change in the domestic net supply is zero even though the sectoral net supplies do not generally stay at their initial levels. Consider the Guesnerie-Weymark definition of local controllability of production:
**Definition 2.1:** (Guesnerie (1977), Weymark (1979))

The government is said to have local control of production in an economy characterized by the equations (2.8)-(2.10), if the rank of the producer substitution matrix S is maximal \((= N + M - 1)\) so that it is possible to induce a differential change in supplies in any direction on the economy's production possibility frontier by a suitable differential change of producer prices.

Returning to the government's policy problem, it seems clear that if the government has local control of production at the initial equilibrium, it can induce the particular kind of change in the industry net exports and in the domestic net supplies (if such a change exists) that will leave the aggregate \(y^*\) constant while, at the same time, the initial net balance of trade \(b^*\) is being improved. It can also be seen that when the producer substitution matrix S is of maximal rank \((= N + M - 1)\), assumption (2.13) is satisfied.\(^{18}\) There seems thus to be a connection between assumption (2.13) and local controllability of production in the country.

Inspection of assumption (2.13) shows, however, that the producer substitution matrix S need not be of maximal rank for assumption (2.13) to be satisfied, i.e., local controllability of production, in the sense of Guesnerie and Weymark, is not necessary for (2.13) to hold. A weaker concept of controllability of production is in order:

**Definition 2.2:**

Domestic goods production in an economy described by the model (2.8)-(2.10) is said to be locally controllable around the initial equilibrium (which
satisfies (2.8)-(2.10)), if there exist once continuously differentiable functions $p^*(y^*, w + \tau^*)$ and $z^*(y^*, w + \tau^*)$ such that (2.8)-(2.10) is satisfied when the initial vector of tariffs is $\tau^*$.

Here, local controllability of domestic goods production is defined to mean that there exist functions $p^*(y^*, w + \tau^*)$ and $z^*(y^*, w + \tau^*)$ (around the initial equilibrium) which can be used to solve the appropriate perturbations in the equilibrium domestic prices $p^*$ and industry scales $z^*$, once a suitable (net balance of trade improving) change in the tariffs $\tau^*$ has been established. (The change in the vector $y^*$ is zero if the initial equilibrium net supply of domestic commodities is maintained).

**Lemma 2.1:**
Suppose that assumptions (2.12)-(2.13) are satisfied. Then, the government has local control of domestic goods production in the home country in the sense of Definition 2.2.

**Proof:**
Consider equations (2.8)-(2.9) which endogenously determine domestic prices $p^*$ and industry scales $z^*$, given tariffs $\tau^*$ and a vector of domestic net supplies $y^*$. Differentiating (2.8)-(2.9) around the initial values of $p$, $z$, $\tau$, and $y$:

$$B_p \Delta p^* + B_z \Delta z^* = B_y \Delta y^*,$$

where $B_p = \begin{bmatrix} S_{pp} & \cdot \\ y^T \end{bmatrix}$, $B_z = \begin{bmatrix} Y \\ 0_{KxK} \end{bmatrix}$ and $B_y = \begin{bmatrix} I_N \\ 0_{KxN} \end{bmatrix}$.
It follows that the domestic prices $p^*$ and industry scales $z^*$ can be regarded as (once continuously differentiable) implicit functions of the domestic net supplies $y^*$ and tariffs $\tau^*$, if the inverse matrix $[B_p, B_z]^{-1}$ exists. According to Diewert and Woodland (1977: Appendix), the matrix $[B_p, B_z]$ is invertible if and only if (2.12)-(2.13) are satisfied, which has been assumed. Hence, the implicit functions $p^*(y^*, w + \tau^*)$ and $z^*(y^*, w + \tau^*)$ exist. QED

Using Lemma 2.1, assumption (2.13) can be given an interpretation as a sufficient condition for local controllability of domestic goods production.\textsuperscript{19}

Let us now turn to consider tradeables production and the net balance of trade function $b^*(w + \tau^*)$. As shown in Section 2.1, assumptions (2.12)-(2.13) are sufficient for this function to exist. Diewert (1983a) shows that assumptions (2.12)-(2.13) are also sufficient for the economy's GNP function $G(w + \tau, y^*)$ to be twice continuously differentiable. Woodland (1982: p. 59) proves that the producers' aggregate net supply of tradeable commodities, $f(w + \tau^*, y^*)$, can be obtained as a vector of partial derivatives of the GNP function:

$$f(w + \tau^*, y^*) = \sum_{k=1}^{K} \nabla_w k(p^*, w + \tau^*) z^k = \nabla_w G(w + \tau^*, y^*).$$

Then, applying the results of Diewert (1983a: pp. 189-190), under assumptions (2.12)-(2.13),
(2.19) \[ \frac{\partial^2}{\partial w \partial \tau^*} G(w + \tau^*, y^*) = \left[ - S_{wp} D_{11} - FD^T_{21}, I_M \right] S \begin{bmatrix} -D_{11} S_{wp} - D_{12} F^T \end{bmatrix}, \]

where the matrices \( D_{11} \) and \( D_{12} \) are blocks in the symmetric inverse matrix

(2.20) \[ D = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} = \begin{bmatrix} S_{pp} & Y \\ Y^T & 0_{KxK} \end{bmatrix}^{-1} \]

and \( I_M \) is an \((M \times M)\) identity matrix. The matrix \( \frac{\partial^2}{\partial w \partial \tau^*} G(w + \tau^*, y^*) \) gives the (net) export supply responses to changes in prices \((w + \tau^*)\) holding domestic net outputs constant at \( y^* \). The matrix \( \frac{\partial^2}{\partial w \partial \tau^*} G(w + \tau^*, y^*) \) is positive semidefinite since the GNP function \( G(w + \tau^*, y^*) \) is convex in prices \((w + \tau^*)\). Because the GNP function is also linearly homogenous in \((w + \tau^*)\), the matrix \( \frac{\partial^2}{\partial w \partial \tau^*} G(w + \tau^*, y^*) \) has at least one zero eigenvector: the vector of tradeables producer prices \((w + \tau^*)\).

The following lemma connects the net balance of trade function \( b^*(w + \tau^*) \) and the matrix \( \frac{\partial^2}{\partial w \partial \tau^*} G(w + \tau^*, y^*) \).

**Lemma 2.2:**

The gradient of the net balance of trade function \( b^*(w + \tau^*) \) with respect to the tariff vector \( \tau^* \) is:

(2.21) \[
\nabla_{\tau} b^*(w + \tau^*) = - \tau^{*T} \left[ S_{ww} - (S_{wp}, F)D(S_{wp}, F)^T \right] = - \tau^{*T} \frac{\partial^2}{\partial w \partial \tau^*} G(w + \tau^*, y^*),
\]

where the matrix \( D \) is defined in (2.20).
Proof:

After differentiating (2.8)-(2.10) at the initial equilibrium, the gradient \( \nabla_{\tau} b^*(w + \tau^*) \) can be solved from:

\[
\begin{bmatrix}
S_{pp} & Y & 0_N \\
Y^T & 0_K & 0_K \\
-\tau^{*T}S_{wp} & -\tau^{*T}F & -1
\end{bmatrix}
\begin{bmatrix}
\nabla_{\tau} p^* \\
\nabla_{\tau} z^* \\
\nabla_{\tau} b^*
\end{bmatrix}
= \begin{bmatrix}
-S_{pw} \\
-F \\
\tau^{*T}S_{ww}
\end{bmatrix}.
\]

Since the inverse matrix \( D \) exists, (2.2) can be solved for \( \nabla_{\tau} b^*(w + \tau^*) \):

\[
(2.23) \quad \nabla_{\tau} b^*(w + \tau^*) = -\tau^{*T}S_{ww} + \tau^{*T}(S_{wp}D_{11} + FD_{21})S_{pw} + \tau^{*T}(S_{wp}D_{12} + FD_{22})F^T
\]

\[
= -\tau^{*T}[S_{ww} - (S_{wp}, F)D(S_{wp}, F)^T]
\]

Consider now the quadratic form \([- S_{wp}D_{11} - FD_{12}^T, I_M] S [- S_{wp}D_{11} - FD_{12}^T, -I_M]^T\).
Hence, $V_\tau^* b^*(w + \tau^*) = -\tau^* \nabla^2_{ww} G(w + \tau^*, y^*)$ using (2.19). QED

Using Lemma 2.2, assumption (2.16) can be given two interpretations. On one hand, if the government wishes to induce an improvement in the home country's initial net balance of trade by changing the tariffs $\tau^*$, it seems natural to require that the gradient $V_\tau^* b^*$ is nonzero. In fact, in Section 2.4 it will be shown that (2.16) is a necessary condition for strict improvements in $b^*$ to exist. On the other hand, assumption (2.16) is related to local controllability of the tradeables production in the country (keeping domestic goods production fixed at $y^*$): if the rank of the matrix $\nabla^2_{ww} G(w + \tau^*, y^*)$ is maximal ($= M - 1$), then, according to Definition 2.1, net export production in the home country is locally controllable, i.e., any differential change in net export supplies on the economy's production possibility frontier for tradeables (at fixed $y^*$) can be induced by a suitable change in tariffs $\tau^*$. This means
that if strict productivity improving directions of net export production change exist, they can be attained by perturbing the initial tariffs $\tau^*$ appropriately. Are there such directions of change in net tradeables production? It will be shown in section 2.4 that a necessary condition for strict productivity improvements to exist is that the initial vector of tariffs $\tau^*$ is nonzero and nonproportional to the international prices $w$, i.e., the relative prices for tradeables at home and abroad do not coincide. But if the initial tariffs $\tau^*$ are nonzero and nonproportional to the international tradeables prices $w$, and if the matrix $\nabla^2_{ww} G(w + \tau^*, y^*)$ is of maximal rank, the gradient $\nabla_{\tau^*} b^*(w + \tau^*)$ must be nonzero, i.e., (2.16) is satisfied: (2.16) can be regarded as a combined assumption concerning the existence of productivity improving directions of change in the home country's net export supply and the economy's ability to attain them through differential changes in the relative producer prices $(w + \tau^*)$.

It should be noted, however, that local controllability of tradeables production, in the sense of Guesnerie and Weymark, is not necessary for (2.16) to be satisfied: the gradient $\nabla_{\tau^*} b^*(w + \tau^*)$ can be nonzero even though the rank of the matrix $\nabla^2_{ww} G(w + \tau^*, y^*)$ is less than $M - 1$ (as long as the tariff vector $\tau^*$ is not a zero eigenvector of the matrix $\nabla^2_{ww} G(w + \tau^*, y^*)$). In this case, the tradeables production possibility frontier in the home country (given a fixed $y^*$) is ridged and/or kinked, but, around the initial equilibrium, at least some directions of change in the economy's net export supply (that can be attained by a differential perturbation of the tariffs $\tau^*$) exist. For the condition (2.16) to be
satisfied, these directions of net export supply change must also be such that the directional derivative of the net balance of trade function is nonzero in the corresponding direction of change in the tariffs $\tau^*$.  

2.4 Existence of Productivity Improving Changes in Tariffs

In the previous section, the government's policy goal was defined as follows: find a (differential) change in the tariffs $\tau^*$ such that the home country's initial net balance of trade $b^*$ is improved, while keeping the domestic goods net supply at its initial level $y^*$. In this section, the goal of the analysis is to derive sufficient conditions that make this policy goal feasible, i.e., the aim is to develop minimal sufficient conditions for:

\begin{equation}
\text{(2.25)} \quad \Delta p^*, \Delta z^*, \Delta b^*, \Delta \tau^* \text{ such that (2.11) is satisfied and } \Delta b^* > 0.23,24
\end{equation}

A change of $\tau^*$ for which (2.25) holds is called a productivity improving change of tariffs $\tau^*$.

Theorem 2.1:

Suppose i) rank $Y = K \leq N$, ii) rank $(S_{pp} + YY^T) = N$, iii) $\tau^T \nu_{ww}^2$ $G(w + \tau^*, y^*) \neq 0_{M}^T$. Then, there exists a productivity improving change in tariffs $\tau^*$.

The proof of Theorem 2.1 makes use of two lemmas, which are established in Appendix 1.
Lemma 2.3:
Any vector \( \lambda \in \mathbb{R}^{N+K+1} \) satisfying the equations \( \lambda^T [B_p, B_z] = 0^T_{N+K} \) must be of the form

\[
(2.26) \quad \lambda^T = k[(p^* + \delta)^T, \gamma^T, 1], \quad k \in \mathbb{R},
\]

where

\[
(2.27) \quad \delta^T = \tau^* T [S_{wp} D_{11} + P D_{12}], \quad \gamma^T = \tau^* T [S_{wp} D_{12} + P D_{22}].
\]

Lemma 2.4:
For the vector \( \lambda \) solved in Lemma 2.3,

\[
(2.28) \quad \lambda^T B_p = k \tau^* T \gamma^T_{ww} G(w + \tau^*, y^*), \quad k \in \mathbb{R}.
\]

Proof of Theorem 2.1:
A sufficient condition for a productivity improving change in tariffs \( \tau^* \) to exist is (2.25). An equivalent form for this condition can be derived using a theorem of alternative, Motzkin's Theorem:

\[
(2.29) \quad \text{there does not exist a vector } \lambda \in \mathbb{R}^{N+K+1} \text{ such that } \lambda^T [B_p, B_z, -B_\tau] = 0^T_{N+K+M}, \lambda^T B_b < 0.
\]

On the contrary, suppose such a \( \lambda \) exists. By Lemma 2.3, a vector \( \lambda \) that solves the equations \( \lambda^T [B_p, B_z] = 0^T_{N+K} \) must be of the form \( \lambda^T = k[(p^* + \delta)^T, \gamma^T, 1], \quad k \in \mathbb{R} \). For such a \( \lambda \), \( \lambda^T B_b = -k \). Thus, for \( \lambda^T B_b \) to be negative, \( k > 0 \) (and \( k \) may be chosen to be one). By Lemma 2.4,
\[ \lambda^T_{B_\tau} = \tau^T y^2 w g(w + \tau^*, y^*). \] By assumption iii), \[ \lambda^T_{B_\tau} \neq 0^T_M, \] a contradiction. QED

The following two propositions give examples of productivity improving tariff changes.

**Proposition 2.1:**

If the assumptions of Theorem 2.1 are satisfied, a proportional reduction of tariffs \( \tau^* \) will increase the amount of foreign exchange produced by the domestic production sector without diminishing the net supply of domestic commodities.

**Proof:**

A sufficient condition for a proportional reduction in tariffs \( \tau^* \) to be productivity improving is:

\[(2.30) \quad \text{there exist } \Delta p^*, \Delta z^*, \Delta b^*, \Delta \tau^* \text{ such that } (2.11) \text{ holds, } \Delta b^* > 0 \text{ and } \Delta \tau^* = -\tau \tau^*, r \geq 0.\]

Applying Motzkin's Theorem, an equivalent condition is:

\[(2.31) \quad \text{there does not exist a vector } \lambda \in \mathbb{R}^{N+K+1} \text{ such that } \lambda^T [B_p, B_z] = 0^T_{N+K}, \lambda^T_{B_b} < 0, \lambda^T_{B_\tau} \tau^* \leq 0.\]

Proceeding as in the proof of Theorem 2.1, the equations \( \lambda^T [B_p, B_z] = 0^T_{N+K} \) are solved for \( \lambda \). For this \( \lambda \), \[ \lambda^T_{B_\tau} \tau^* = k \tau^T y^2 w g(w + \tau^*, y^*), \]

where the proportionality factor \( k \) must be positive since \( \lambda^T_{B_b} < 0 \). Thus,
k may be set equal to one. By assumption, the tariff vector \( \tau^* \) is not a zero eigenvector of the positive semidefinite matrix \( \nabla^2_{\omega \omega} G(\omega + \tau^*, y^*) \). Hence, \( \lambda^T B \tau^* = \tau^T \nabla^2_{\omega \omega} G(\omega + \tau^*, y^*) \tau^* > 0 \). QED

**Proposition 2.2:**

Suppose that \( \tau^* \geq 0 \) (imports are taxed and exports are subsidized) and the assumptions of Theorem 2.1 are satisfied. Then, there exists at least one tradeable such that lowering the tariff on that good leads to a productivity improvement.

**Proof:**

According to Lemma 2.2, \( \nabla_{\tau} b^*(\omega + \tau^*) = -\tau^T \nabla^2_{\omega \omega} G(\omega + \tau^*, y^*) \). Hence, \( \tau^T \nabla^2_{\omega \omega} G(\omega + \tau^*, y^*) \tau^* = -\nabla_{\tau} b(\omega + \tau^*) \tau^* > 0 \) (the matrix \( \nabla^2_{\omega \omega} G(\omega + \tau^*, y^*) \) is positive semidefinite and, by assumption, \( \tau^* \) is not a zero eigenvector of the matrix \( \nabla^2_{\omega \omega} G(\omega + \tau^*, y^*) \)). If the vector \( \tau^* \) is nonnegative, at least one of the components in the vector \( \tau^T \nabla^2_{\omega \omega} G(\omega + \tau^*, y^*) \) must be positive. Pick any tariff \( \tau_m^* \), \( m \in [1,...,M] \), corresponding to the positive number \( (\partial b^*/\partial \tau_m^*) \). Then, lowering the tariff \( \tau_m^* \) will increase the amount of foreign exchange earned by the production sectors, given that a fixed \( y^* \) is being supplied. QED

A policy implication of Proposition 2.1 is that small, competitive countries could improve their productivity performance by proportionally reducing their trade barriers. If all tariffs \( \tau^* \) are nonpositive (imports
are subsidized and exports are taxed), a result parallel to Proposition 2.2 holds: there exists at least one tradeable commodity \( m \), \( m \in [1, \ldots, M] \), such that an increase in \( \tau_m^* \) is productivity improving.\(^{27}\)

Assumption iii) in Theorem 2.1 can be written in terms of the aggregate producer substitution matrix \( S \). Using (2.19), the following expression is found:

\[
(2.32) \quad \tau^* \nabla^2_{ww} G(w + \tau^*, y^*) = \tau^* [-S_{wp} D_{11} - F D_{12}^T, I_M] S [-D_{11} S_{pw} - D_{12} F^T, I_M].
\]

The equations (2.32) can also be written as

\[
(2.33) \quad \tau^* \nabla^2_{ww} G(w + \tau^*, y^*) = [-\delta^T, \tau^T] S [-D_{11} S_{pw} - D_{12} F^T, I_M],
\]

where the vector \( \delta \) is defined by

\[
(2.34) \quad \delta^T = \tau^*[S_{wp} D_{11} + F D_{12}^T].
\]

Then, since the matrix \( S \) is positive semidefinite,

\[
(2.35) \quad \tau^* \nabla^2_{ww} G(w + \tau^*, y^*) \tau^* = [-\delta^T, \tau^T] S [-\delta, \tau^T] \geq 0.
\]
Assumption iii) can now be replaced by:

(2.36) the vector \([- \delta^T, \tau^T]\) is not proportional to any zero eigenvector of the producer substitution matrix \(S\).

If the only zero eigenvector of the matrix \(S\) is the vector of domestic producer prices \((p^*, w + \tau^*)\), (2.36) is equivalent to either

(2.37) tariffs \(\tau^* (\neq 0_M)\) are not proportional to the international prices \(w\)

or

(2.38) the vector of producer prices for domestic commodities \(p^*\) is not proportional to the vector \(\delta\) defined in (2.34).

Lemma 2.5 in Appendix 1 establishes the equivalence of (2.37) and (2.38). Diewert (1983b: p. 273) shows that the vector \((p^* + \delta)\) is the appropriate productive efficiency vector of shadow prices (for domestic commodities) for choosing government projects when distortionary tariffs and taxes are present in the economy. Using (2.37) and (2.38), it can be added that the domestic producer prices \(p^*\) should be used as the shadow prices for cost benefit analysis only if i) the initial vector of tariffs \(\tau^*\) is zero, i.e. \(\tau^* = 0_M\), or if ii) \(\tau^*\) is proportional to the international prices \(w\).

Under the rather strong maximal rank supposition about the producer substitution matrix \(S\), the seemingly complicated assumption iii) in Theorem
2.1 simplifies to an easily understood nonproportionality condition. Furthermore, if the producer substitution matrix $S$ is of maximal rank, the domestic goods producer substitution matrix $S_{pp}$ is of full rank ($= N$), which means that assumption ii) of Theorem 2.1 is satisfied.

**Proposition 2.3:**
Suppose i) rank $Y = K \leq N$, ii) rank $S = N + M - 1$, iii) the initial vector of tariffs $\tau^* (\neq 0_M)$ is not proportional to the international prices $w$. Then, there exists a productivity improving change in $\tau^*$; the change in $\tau^*$ may be chosen to be a proportional reduction.

If there is only one aggregate production sector in the home country, even the rank assumption on the domestic goods net supply matrix $Y$ in Proposition 2.3 may be erased. The resulting extremely simple version of Theorem 2.1 reveals the basic economic conditions which are sufficient for productivity improving tariff changes to exist: there must be substitution in production, and the relative producer prices for tradeable commodities in the home country must differ from the international prices $w$. The additional suppositions in Theorem 2.1 are needed to cover the more general cases where the economy's total production possibility set may be kinked and the number of production sectors exceeds one.

The central role of the substitutability and nonproportionality assumptions is emphasized in Theorem 2.2 and in its corollary below: they establish that strict productivity improving tariff changes, i.e., tariff changes that cause a strict increase in the home country's initial net
balance of trade \( b^* \), cannot exist if assumption iii) of Theorem 2.1 is violated.

Let us first consider the most general necessary conditions for nonexistence of strict productivity improving tariff perturbations. These conditions can be regarded equivalently as necessary conditions for productivity optimality of the initial equilibrium.

**Theorem 2.2:**

A necessary condition for a strict productivity improving tariff change to not exist is:

\[
(2.39) \quad \text{there is a vector } \lambda \in \mathbb{R}^{N+K+1} \text{ such that } \lambda^T [B_p', B_z', -B_x] = 0^T_{N+K+M}, \\
\lambda^T B_b < 0.
\]

**Proof:**

A necessary condition for productivity optimality of the initial equilibrium is:

\[
(2.40) \quad \text{there do not exist } \Delta p^*, \Delta z^*, \Delta b^*, \Delta \tau^* \text{ such that (2.11) is satisfied and } \Delta b^* > 0.
\]

Using Motzkin's Theorem and the proof of Theorem 2.1, it can be seen that (2.39) and (2.40) are equivalent. QED

**Corollary 2.1:**

Suppose i) rank \( Y = K \leq N \), ii) rank \((S_{pp} + YY^T) = N\), iii) \( \tau^T \frac{v^2}{w_w} \)
\[ G(w + \tau^*, y^*) = 0^T_M. \] Then, the initial equilibrium satisfies the necessary condition for productivity optimality given in Theorem 2.2, and no strict productivity improving directions of change in tariffs \( \tau^* \) exist.

**Proof:**

Assumptions i)-ii) of Corollary 2.1 and Lemma 2.3 imply that a vector \( \lambda \) which solves the equations \( \lambda^T [B_{\lambda}, B_z] = 0^T_{N+K} \) must be of the form

\[ \lambda^T = k[(p^* + \delta)^T, y^T, 1], \]

where \( k \in \mathbb{R} \). Choose \( k = 1 \) so that

\[ \lambda^T B_{\lambda} = -1 < 0. \]

By Lemma 2.4, \( \lambda^T B_{\lambda} = \tau^* \nabla^2_{w,w} G(w + \tau^*, y^*). \)

Hence, by assumption iii), \( \lambda^T B_{\lambda} = 0^T_M \) and (2.39) is satisfied. QED

Corollary 2.1 implies that (2.16) is a necessary condition for strict productivity improvements in tariffs \( \tau^* \) to exist.\(^{28}\) In other words, if the gradient of the net balance of trade function \( b^*, \nabla_{\tau} b^*(w + \tau^*) \), is a zero \( M \)-vector (and (2.12) and (2.13) are satisfied), the initial equilibrium satisfies the necessary condition (2.39) for productivity optimality. However, (2.39) is not sufficient for the initial equilibrium to be a productivity maximum, i.e., such that the maximum for the net balance of trade function \( b^*(w + \tau^*) \) is attained.
3. HOW LARGE IS THE PRODUCTIVITY GAIN?

It was shown in the previous chapter that, under certain rather weak conditions, productivity improving changes in tariffs exist. How large are these productivity gains, i.e., the improvements in the home country's initial net balance of trade $b^*$? Approximating formulae for the increase in $b^*$ can be developed using the Allais-Diewert method of measuring the deadweight loss due to inefficient schemes of taxation.

Before considering the approximations for the productivity gain, it is useful to give a brief review of the derivation of the Allais-Diewert deadweight loss measure.\(^1\)

Diewert (1983c) defines the maximal amount of foreign exchange available to the production sector, under the condition that the production sector in the aggregate supplies at least the initial equilibrium amount $y^*$ of domestic commodities, as the solution $b^o$ to

\[
(3.1) \quad b^o = \max \left\{ \sum_{k=1}^{K} w^T f^k z^k : \sum_{k=1}^{K} y^k z^k \geq y^*, \quad k=1, \ldots, K \right\}.
\]

The value $b^o$ that solves (3.1) is at least as large as the initial observed net balance of trade $b^* \left( = \sum_{k=1}^{K} w^T f^k z^k \right)$.\(^2\) The Allais-Diewert production loss pertaining to the initial equilibrium is then defined as
(3.2) \( A_L \equiv b^0 - b^* \geq 0 \).

Using the Karlin (1959: p. 201) - Uzawa (1958: p. 34) Saddle Point Theorem, (3.1) can be written in an equivalent dual form:

\[
(3.3) \quad b^0 = \max_{z \geq 0_K} \min_{p \geq 0_N} \left\{ \sum_{k=1}^{K} \pi^k(p,w)z^k - p^T y^* : k=1, \ldots, K \right\},
\]

where the vector of Lagrange multipliers \( p \) is the (producer) shadow price vector for domestic commodities when no tariff distortions are present in the economy. The first order necessary conditions for (3.3) are:

\[
(3.4) \quad \pi^k(p,w) = 0, \quad k=1, \ldots, K,
\]

\[
(3.5) \quad \sum_{k=1}^{K} \pi^k(p,w)z^k = y^*.
\]

It is assumed that a unique solution \( z \in \mathbb{R}_+^K, p \in \mathbb{R}_+^N \) for (3.4) - (3.5) exists, and that the industry scales \( z \) and domestic prices \( p \) can be regarded as implicit functions of the exogenous \( (w, y^*) \) around the solution for (3.4) - (3.5).³

Let us now compare the optimal reference equilibrium satisfying (3.4) - (3.5) to the initial distorted equilibrium characterized by the
equations (2.8) - (2.9). The optimal equilibrium can be mapped to the distorted equilibrium by defining a \( \xi \)-equilibrium for which

\[
(3.6) \quad \pi^k(p(\xi), w + \tau^* \xi) = 0, \quad k=1, \ldots, K,
\]

\[
(3.7) \quad \sum_{k=1}^{K} \pi^k(p(\xi), w + \tau^*) z^k(\xi) = y^*.
\]

If \( \xi = 0 \), (3.6) - (3.7) defines the same optimal equilibrium as the equations (3.4) - (3.5), and if \( \xi = 1 \), (3.6) - (3.7) defines the initial observed equilibrium that solves (2.8) - (2.9).

For each equilibrium of the economy indexed by \( \xi \) (\( 0 \leq \xi \leq 1 \)), Diewert defines the Allais objective function \( A(\xi) \) for the economy as the net amount of foreign exchange that the production sector (in the aggregate) produces at the equilibrium indexed by \( \xi \):

\[
(3.8) \quad A(\xi) = \sum_{k=1}^{K} w^T \pi^k(p(\xi), w + \tau^*) z^k(\xi).
\]

Using (3.8), the Allais-Diewert loss measure \( A_L \) defined in (3.2) can be written as

\[
(3.9) \quad A_L = A(0) - A(1).
\]
Next, the value $A(1)$ is approximated using the second order Taylor Series expansion of $A$ around $\xi = 0$, $A(0) + A'(0)(1 - 0) + 1/2 A''(0)(1 - 0)^2$. Using this formula, the Allais-Diewert loss $A_L$ becomes approximately $-A'(0) - 1/2 A''(0)$. It can be shown that the derivative of the Allais objective function at $\xi = 0$, $A'(0)$, is zero and hence,

$$A_L = -1/2 A''(0),$$

where

$$A''(0) = [p'(0)^T, \tau^*]^T S^o \begin{bmatrix} p'(0) \\ \tau^* \end{bmatrix} \geq 0.$$

In (3.11), $p'(0)$ denotes the derivatives of the domestic commodity prices $p$ with respect to $\xi$, evaluated at $\xi = 0$. These derivatives can be calculated by differentiating the equations (3.4) - (3.5) at the solution of (3.4) - (3.5) and by setting $\xi = 0$. The producer substitution matrix $S^o$ in (3.11) is evaluated at the optimal undistorted equilibrium; positive semidefiniteness of $S^o$ implies that $-A''(0) \geq 0$.

Let us now return to the problem of measuring the productivity gain accruing from a productivity improving perturbation of tariffs $\tau^*$. Suppose that the initial tariff distorted equilibrium is indexed by 1, the new (after the tariff change) equilibrium by 2, and the optimal undistorted equilibrium by 0. It seems natural to measure the productivity gain, $A_G$, between the equilibria 1 and 2 by the difference
(3.12) \( A_G = b^2 - b^1 \geq 0, \)

where \( b^1 \) is the observed net balance of trade in equilibrium 1 (formerly denoted by \( b^* \)) and \( b^2 \) is the net balance of trade after the change in tariffs \( \tau^1 \) (formerly denoted by \( \tau^* \)). If the perturbation in the initial tariffs \( \tau^1 \) is strict productivity improving, the gain \( A_G \) is nonnegative.\(^6\)

Adding and subtracting the maximal net balance of trade \( b^0 \) on the right hand side of (3.12), the gain \( A_G \) can be written as

\[
(3.13) \quad A_G = (b^0 - b^1) - (b^0 - b^2) = A_G^{0+1} - A_G^{0+2}.
\]

In (3.13), \( A_G^{0+1} \) denotes the Allais – Diewert productivity loss between the optimal and actual observed equilibria, and \( A_G^{0+2} \) is the Allais – Diewert loss between the optimal and new (after the change in tariffs) equilibria. The productivity gain \( A_G \) has thus been expressed as a difference of productivity losses between the three equilibria 0, 1 and 2.

Applying Diewert's second order approximation rule (3.10) – (3.11) to (3.13), the approximate gain accruing from a strict productivity improving change in tariffs \( \tau^1 \) is:

\[
(3.14) \quad A_G = \frac{1}{2} [p'(0)^T, \tau^1 T] S^o [p'(0)^T, \tau^1 T]^T - \\
\frac{1}{2} [p'(0)^T, \tau^2 T] S^o [p'(0)^T, \tau^2 T]^T,
\]
where $\tau^2$ denotes the vector of tariffs at the equilibrium 2.

In order to calculate the gain approximation (3.14), the government must have information about the aggregate producer substitution matrix $S^0$, the domestic price derivatives $p'(0)$ (which depend on the net supply matrices $Y^0$ and $F^0$ at the optimal equilibrium), and the policy variables $\tau^1$ and $\tau^2$. In practice, unfortunately, it may be impossible to form estimates for the unobserved matrices $S^0$, $Y^0$, and $F^0$. Further approximations to (3.14) are thus called for. One possibility is, if the optimal and observed equilibria are not too far from each other, to replace the matrices $S^0$, $F^0$ and $Y^0$ in (3.14) by the observed matrices $S^1$, $F^1$ and $Y^1$. This approximation, however, involves an error, the size of which is not known.

Is there then any way of measuring the productivity gain as accurately as in (3.14), using only the observed information at equilibrium 1? It turns out that this is indeed possible, if the following version of the Quadratic Approximation Lemma is employed.

**Lemma 3.1:** (The Quadratic Approximation Lemma; Diewert (1976: p. 118)).

For a twice continuously differentiable function $f(z)$, $z \in \mathbb{R}^N$,

\[(3.15) \quad f(z^1) - f(z^0) = \nabla f(z^0)^T (z^1 - z^0) + \frac{1}{2} (z^1 - z^0)^T \nabla^2 f(z^0) (z^1 - z^0)\]

if and only if

\[(3.16) \quad f(z) = a_0 + a^T z + \frac{1}{2} z^T A z,\]
where $A^T = A$, or if and only if

$$f(z^1) - f(z^0) = \frac{1}{2} [\nabla f(z^0) + \nabla f(z^1)]^T (z^1 - z^0).$$

Lemma 3.1 establishes two exact expressions for the change in the value of a quadratic function $f(z)$: (3.15) uses the second order Taylor Series expression, while (3.17) uses the inner product of the average gradient $1/2[\nabla f(z^0) + \nabla f(z^1)]$ with the difference $(z^1 - z^0)$.

If the function $f(z)$ is not quadratic, i.e., of the form (3.16), (3.15) gives a second order approximation to the change $f(z^1) - f(z^0)$. The formula (3.17) also provides an approximation to $f(z^1) - f(z^0)$ which, by rewriting (3.17) in the form

$$f(z^1) - f(z^0) \approx \frac{1}{2} [\nabla f(z^0)^T (z^1 - z^0)] + \frac{1}{2} [\nabla f(z^1)^T (z^1 - z^0)],$$

can be given a new interpretation: the term $\nabla f(z^0)^T (z^1 - z^0)$ in (3.18) yields a first order approximation to the change $f(z^1) - f(z^0)$ around the point $z^0$, whereas the term $\nabla f(z^1)^T (z^1 - z^0)$ provides a first order approximation to $f(z^1) - f(z^0)$ around the point $z^1$. Thus, in (3.18), the change in the value of the function $f$ is approximated using the average of two first order approximations. Lemma 3.1 shows that, for a quadratic function, (3.18) provides exactly as good an approximation to the change $f(z^1) - f(z^0)$ as the second order expression (3.15); both are in fact exact. If the function $f$ is not quadratic, the approximation (3.18) is approximately as accurate as the second order formula (3.15).
The above result can be used to derive another approximation formula for the productivity gain measure $A_G$ defined in (3.12). Using (3.13), $A_G$ can be regarded as a difference of two loss measures $A_L^{0+1}$ and $A_L^{0+2}$ which themselves are defined as changes in the Allais objective function $A(\xi)$. Let us approximate the loss $A_L^{0+1}$, not to the second order around the optimal equilibrium, but using the average of the two first order losses around the equilibria 0 and 1. To the first order, around the equilibrium 1,

$$(3.19) \quad A_L^{0+1} = A(0) - A(1) \approx A'(1),$$

and to the first order, around the equilibrium 0,

$$(3.20) \quad A_L^{0+1} = A(0) - A(1) \approx -A'(0).$$

Hence, by taking the average of (3.19) and (3.20),

$$(3.21) \quad A_L^{0+1} \approx \frac{1}{2} [A'(1) - A'(0)].$$

Diewert (1983c: p. 170) shows that $A'(0) = 0$, which implies

$$(3.22) \quad A_L^{0+1} \approx \frac{1}{2} A'(1).$$
Similarly, for the measure $A_L^{0+2}$,

(3.23)  \[ A_L^{0+2} = \frac{1}{2} A'(2). \]

The average loss measures (3.22) and (3.23) are approximately as accurate as the corresponding second order measures derived using the formulae (3.10) - (3.11).

Applying (3.22) - (3.23), the productivity gain measure $A_G$ defined in (3.13) can be approximated by

(3.24)  \[ A_G = \frac{1}{2} [A'(1) - A'(2)]. \]

Using the results in Diewert (1983c), the derivatives of the Allais objective function, $A'(1)$ and $A'(2)$, can be shown to equal

(3.25)  \[ A'(1) = [p'(1)^T, \tau_1^1T] S^1 [p'(1)^T, \tau_1^1T]^T \]

and

(3.26)  \[ A'(2) = [p'(2)^T, \tau_2^2T] S^2 [p'(2)^T, \tau_2^2T]^T. \]

Formulae (3.24) - (3.26) provide the second approximation for the productivity gain $A_G$ in this section. The measure (3.24) depends on the observed producer substitution matrix $S^1$ and the observed net
output matrices $Y^1$ and $F^1$. It also depends on the corresponding unobserved matrices in the new (after the tariff change) equilibrium. If, however, the perturbation in tariffs $\tau^1$ is small, it is reasonable to assume that the changes in the producer substitution matrix $S^1$ are also small. Hence, the matrix $S^2$ can be approximated by the matrix $S^1$. After this adjustment, the productivity gain measure (3.24) becomes

$$A_G = \frac{1}{2} \left[ p'(1)^T, \tau^1 \right] S^1 \left[ p'(1)^T, \tau^1 \right]^T$$

$$- \frac{1}{2} \left[ p'(2)^T, \tau^2 \right] S^1 \left[ p'(2)^T, \tau^2 \right]^T. 10$$

The approximation error involved in substituting the matrix $S^1$ for the matrix $S^2$ in the second term of (3.27) is likely to be considerably smaller than the error in (3.14), if, in (3.14), the matrix $S^1$ is used instead of the matrix $S^0$.

If the net output matrices after the tariff change, $Y^2$ and $F^2$, can be assumed to be close to the initial observed matrices $Y^1$ and $F^1$, the derivatives $p'(2)$ in (3.27) can be replaced by the initial equilibrium price derivatives $p'(1)$. In this case, the gain measure (3.27) simplifies to

$$A_G = \frac{1}{2} \left[ (\tau^1 - \tau^2)^T S^1_{wp} p'(1) + p'(1)^T S^1_{pw} (\tau^1 - \tau^2) \right. $$

$$\left. + \tau^1 S^1_{ww} (\tau^1 - \tau^2) \right].$$
This approximation depends only on observable variables.\textsuperscript{11}

It can be seen from (3.28) that the productivity gain $A_G$ increases proportionally with the production substitution terms $S_{pw}^1$, $S_{wp}^1$ and $S_{ww}^1$, i.e., if all the matrices $S_{pw}^1$, $S_{wp}^1$ and $S_{ww}^1$ are multiplied by a scalar $\alpha > 0$, the gain $A_G$ is replaced by $\alpha A_G$: the more substitution there is in the domestic production sector, the larger the gains from productivity improving tariff policies are likely to be.

In the case of proportional changes in tariffs $\tau^1$, it is possible to calculate the derivative of the approximate gain (3.28) with respect to a proportionality factor. Assume, for example, that $\tau^2 = k\tau^1$, $k \in (0,1)$. Using (3.28),

\begin{equation}
(3.29) \quad A_G = \frac{1}{2} \left[ (1 - k) \tau^1 S_{wp}^1 p'(1) + (1 - k) p'(1) S_{pw}^1 \tau^1 + \tau^1 S_{ww}^1 \tau^1 - k^2 \tau^1 S_{ww}^1 \tau^1 \right].
\end{equation}

Hence,

\begin{equation}
(3.30) \quad \frac{dA_G}{dk} = \frac{1}{2} \left[ - \tau^1 S_{wp}^1 p'(1) - p'(1)^T S_{pw}^1 \tau^1 - 2k \tau^1 S_{ww}^1 \tau^1 \right].
\end{equation}

If the producer substitution matrix for domestic and tradeable commodities is zero ($S_{pw} = 0_{N \times M}$), the derivative (3.30) becomes

\begin{equation}
(3.31) \quad \frac{dA_G}{dk} = -2k \tau^1 S_{ww}^1 \tau^1 \leq 0.
\end{equation}
The weak inequality in (3.31) follows from the positive semidefiniteness of the matrix $S_{ww}^{-1}$. From (3.31) it can be seen that as the proportionality factor $k$ decreases (i.e., as the proportional reduction of tariffs $\tau^1$, which is strict productivity improving by Proposition 2.1, becomes larger), the productivity gain $A_G$ increases.\textsuperscript{12}

Finally, it should be noted that the productivity gain measure $A_G$ defined in (3.12) is only a partial, production side, measure of gain as opposed to a general equilibrium gain formula. Nonetheless, it is valuable as a lower bound for the total general equilibrium gain accruing from a productivity improving change of tariffs. If a more accurate approximation of the general equilibrium gain is needed, the method of measuring the productivity gain presented in this section could be adapted to the general equilibrium context by employing the Debreu-Diewert measure of deadweight loss defined in Diewert (1984).
4. STRICT PARETO IMPROVING CHANGES IN COMMODITY TAXES AND TARIFFS

4.1 A General Equilibrium Model

There are $H$ consumers (households), indexed by $h = 1, \ldots, H$, in the economy. The preferences of the consumers are represented by their expenditure functions $m^h(u^h, q, v)$, which are defined as the minimum net cost (factor supplies are indexed negatively) of achieving a given utility level $u^h$. In addition to $u^h$ the expenditure functions $m^h$, $h = 1, \ldots, H$, depend on the $N$-dimensional vector of domestic consumer prices $q = (p + t) \in \mathbb{R}^N_+$, where $p$ is the vector of domestic producer prices and $t$ is a vector of taxes or subsidies on domestic commodities. The expenditure functions $m^h$, $h = 1, \ldots, H$, also depend on a vector of consumer prices for internationally tradeable goods equal to $v = (w + \tau + s) \in \mathbb{R}^M_+$, where $w$ is the world price vector, $\tau$ is the tariff vector and $s$ is a vector of taxes or subsidies on internationally tradeable commodities.

The expenditure functions $m^h$, $h = 1, \ldots, H$, are assumed to be twice continuously differentiable. They are also concave and linearly homogenous in prices $(q, v)$. Using Shephard's Lemma, the consumers' Hicksian (compensated) net demand functions for domestic and tradeable commodities can be derived as first order partial derivatives of the expenditure functions $m^h$:

\begin{equation}
\chi^h(u^h, q, v) = \nabla_q m^h(u^h, q, v), \quad h = 1, \ldots, H,
\end{equation}
and

\[(4.2) \quad e^h(u^h, q, v) \equiv v^m(u^h, q, v), \quad h=1, \ldots, H.\]

The consumer net demand matrices for domestic and tradeable goods are defined using (4.1) and (4.2):

\[(4.3) \quad X = [x^1, \ldots, x^H]; \quad E = [e^1, \ldots, e^H].\]

The aggregate consumer substitution matrix \( \Sigma \) is defined by

\[(4.4) \quad \Sigma = \begin{bmatrix}
\Sigma_{qq} & \Sigma_{qv} \\
\Sigma_{vq} & \Sigma_{vv}
\end{bmatrix}^H \Sigma = \begin{bmatrix}
\Sigma_{qq} & \Sigma_{qv} \\
\Sigma_{vq} & \Sigma_{vv}
\end{bmatrix}^H = \begin{bmatrix}
\Sigma_{qq} & \Sigma_{qv} \\
\Sigma_{vq} & \Sigma_{vv}
\end{bmatrix} = \begin{bmatrix}
\Sigma_{qq} & \Sigma_{qv} \\
\Sigma_{vq} & \Sigma_{vv}
\end{bmatrix}^H.

In (4.4), the matrix block \( \Sigma_{qv} \), for example, gives the aggregate net demand responses to changes in tradeables consumer prices for domestic commodities. The other blocks of the substitution matrix \( \Sigma \) have analogous interpretations.

The matrix \( \Sigma \) is symmetric, negative semidefinite (since the expenditure functions \( m^h \), \( h=1, \ldots, H \), are concave in prices \( q, v \)) and satisfies

\[(4.5) \quad [q^T, v^T] \Sigma = 0^T_{N+M}.\]
This follows because the expenditure functions \( m^h, h=1,\ldots,H \), are linearly homogenous in prices \((q, v)\). Equations (4.5) imply that the rank of the matrix \( \Sigma \) can be at most \( N + M - 1 \).

It may be that some of the \( N \) domestic commodities are producer durables or supplied independently of prices by the consumers. Suppose the \( n^{th} \) domestic good is such a commodity. Then, the \( n^{th} \) row and column of the matrix \( E_{qq} \) consist of zeros. Similarly, if an internationally traded good \( m, m \in \{1,\ldots,M\} \), does not enter into the preferences of any consumer, or if it passes through the domestic production sector before being available to consumers, the column \( m \) of the matrix \( E_{qv} \) is a column of zeroes, and the \( m^{th} \) row of the matrix \( E \) is a row of zeroes.

Since the expenditure functions \( m^h(u^h, q, v), h=1,\ldots,H \), are nondecreasing in \( u^h, h=1,\ldots,H \), the utility of each household can be measured in terms of income, holding the consumer prices \((q, v)\) fixed at some given level. If the consumer prices are fixed at their initial equilibrium levels \((q^*, v^*)\), the equations \( u^h = m^h(u^h, q^*, v^*) \), \( h=1,\ldots,H \), define the consumers' utility levels \( u^h \) in the neighbourhood of the initial equilibrium. This money metric scaling of utilities implies the following restrictions:

\[
(4.6) \quad \forall_{u^h} m^h(u^*^h, q^*, v^*) = 1, \ h=1,\ldots,H,
\]

and
\[(4.7) \quad v^2_{juh} m^h(u^*, q^*, v^*) = 0, \ h=1, \ldots, H.\]

In addition, Diewert (1978: p. 146) shows that, if the money metric scaling of utilities is applied, the matrices \(\Sigma_{\text{qu}}\) and \(\Sigma_{\text{vu}}\) defined below can be interpreted as the income derivative matrices of the consumers' ordinary demand functions:

\[(4.8) \quad \Sigma_{\text{qu}} \equiv [m_{1\text{qu}}, \ldots, m_{H\text{qu}}],\]

where \(m^h_{\text{qu}} = v^2_{qu} m^h(u^*, q^*, v^*), \ h=1, \ldots, H,\)

and

\[(4.9) \quad \Sigma_{\text{vu}} \equiv [m_{1\text{vu}}, \ldots, m_{H\text{vu}}],\]

where \(m^h_{\text{vu}} = v^2_{vu} m^h(u^*, q^*, v^*), \ h=1, \ldots, H.\)

The government in the economy imposes tariffs on tradeables and taxes both domestic and tradeable commodities. Furthermore, the government may give the households (positive or negative) lump sum transfers. The vector of lump sum transfers is denoted by \(g \in \mathbb{R}^H.\)

With its income, the government buys domestic and tradeable commodities in amounts \(x^0 \in \mathbb{R}^N_+\) and \(e^0 \in \mathbb{R}^M_+.\) These commodities are used to produce public goods and services for the private sector. The vectors \(x^0\) and
\( e^0 \) are assumed to stay constant throughout the analysis so that no public goods explicitly appear in the model.

There are \( H + N + K + 1 \) equations that characterize an equilibrium (indexed with an asterisk) when the demand side of the economy is taken into account:

\[
\begin{align*}
\text{(4.10)} & \quad m^h(u^h, q^*, v^*) = g^h, \ h=1, \ldots, H, \\
\text{(4.11)} & \quad \pi^k(p^*, w + \tau^*) = 0, \ k=1, \ldots, K, \\
\text{(4.12)} & \quad \sum_{h=1}^{H} \sum_{q} m^h(u^h, q^*, v^*) + x^0 = \sum_{k=1}^{K} \sum_{p} \pi^k(p^*, w + \tau^*) z^k, \\
\text{(4.13)} & \quad w^T \sum_{h=1}^{H} \sum_{v} m^h(u^h, q^*, v^*) + w^T e^0 = \sum_{k=1}^{K} w^T \sum_{w} \pi^k(p^*, w + \tau^*) z^k - b^*. 
\end{align*}
\]

According to (4.10) - (4.13), the consumers (households) equate their expenditures on domestic and tradeable commodities minus their factor incomes to their lump sum revenues (which may be zero)\(^1\), the \( K \) constant returns to scale production sectors earn zero (pure) profits,\(^2\) consumer aggregate net demand for domestic commodities equals their aggregate net supply, and the balance of payments net surplus equals \( b^* \).
If (4.10) - (4.13) are satisfied, by Walras' Law, the government budget constraint

\[(4.14) \quad t^* x^0 + (r^* + s^*) \Sigma e^0 + t^* \Sigma x^h + s^* \Sigma e^h \]

also holds. With a simple manipulation (4.14) can be rewritten as:

\[(4.15) \quad t^* x^0 + s^* \Sigma e^h + r^* [\Sigma e^h - \Sigma f^k z^k] \]

\[= q^* x^0 + v^* e^0 + \sum_{h=1}^H g^h + b^* \]

This form of the budget constraint implies that the government expenditures on domestic and tradeable commodities plus the lump sum transfers forwarded to the consumers plus the balance of payments net surplus must equal the government tax and tariff income.

The exogenous variables in (4.10) - (4.13) are the international prices w which also provide a price normalization in the model, and the government policy instruments g*, t*, s* and t*. The endogenous
variables, which are determined in (4.10) - (4.13) as implicit functions of the exogenous variables, are $u^*$ (household utility levels), $p^*$ (domestic producer prices), $z^*$ (industry scales) and $b^*$ (balance of trade net surplus).³

It is assumed that an initial equilibrium satisfying (4.10) - (4.13) and $(p^*, z^*) > O_{N+K}$ exists. Total differentiation of (4.10) - (4.13) at the initial values of the variables of the model yields:

\begin{equation}
(4.16) \quad A\Delta u^* = B_p \Delta p^* + B_z \Delta z^* + B_b \Delta b^* + B_t \Delta t^* + B_s \Delta s^* + B_g \Delta g^*,
\end{equation}

where the matrices $A$, $B_p$, ..., $B_g$ are defined as follows:⁴

\[
A = \begin{bmatrix}
I_H \\
\Sigma_{qu} \\
0_{KxH} \\
-w^T \Sigma_{vu}
\end{bmatrix},
B_p = \begin{bmatrix}
-X^T \\
-\Sigma_{qq} + S_{pp} \\
Y^T \\
-w^T \Sigma_{vq} + w^T S_{wp}
\end{bmatrix},
B_z = \begin{bmatrix}
0_{HxK} \\
Y \\
o_{KxH} \\
-w^T F
\end{bmatrix},
B_b = \begin{bmatrix}
0_H \\
0_N \\
0_K \\
-1
\end{bmatrix},
\]

\[
B_t = \begin{bmatrix}
-E^T \\
-\Sigma_{qv} + S_{pw} \\
p^T \\
-w^T \Sigma_{vv} + w^T S_{ww}
\end{bmatrix}.
\]
Using the Implicit Function Theorem, the derivatives of the endogenous \( u, p, z \) and \( b \) with respect to the exogenous \( \tau, t, s \), and \( g \) are determined by the matrix \([A, -B, -B, -B, -B, -B, -B, -B, -B]^{-1} [B, B, B, B, B, B, B, B, B] \).

4.2 Existence of Strict Pareto Improving Changes in Commodity Taxes and Tariffs

A strict Pareto improving change in the commodity tax rates \((t, s)\) and tariffs \(\tau\) is one which leads to a utility increase for each consumer (household) in the economy. The goal of this section is to develop sufficient conditions for such a tax and tariff perturbation to exist when, in addition, the tax/tariff change is required to lead to an increase in the home country's initial net balance of trade. More precisely, the problem is the following: under what conditions, starting from an initial equilibrium which satisfies (4.10) - (4.13), do there exist \( \Delta u^*, \Delta p^*, \Delta z^*, \Delta b^*, \Delta t^*, \Delta s^*, \Delta \tau, \Delta g^* \) such that (4.16) holds and \( \Delta u^* \gg 0_H, \Delta b^* > 0, \Delta g^* = 0_H \)?

One of the sufficient conditions given in Theorem 4.1 below is a restriction on the consumers' preferences and on their initial commodity
endowments. It is required that

(4.17) there is no solution \( a > 0 \) to \( a^T [X^T, E^T] = 0^T_{N+M} \).

This supposition is satisfied if there is some domestic good \( n \), \( n \in [1, \ldots, N] \), which is in net demand or in net supply by every household, i.e., \( x^h_n > 0 \) or \( < 0 \) for all \( h=1, \ldots, H \). Alternatively, it is sufficient to have an internationally traded commodity \( m \), \( m \in [1, \ldots, M] \), in net demand or in net supply by every consumer; in this case, \( e^h_m > 0 \) or \( < 0 \) for all \( h=1, \ldots, H \). Goods that are demanded or supplied by all consumers in the economy are often called Diamond-Mirrlees goods.

Weymark (1979: pp. 176-177) shows that the existence of a Diamond-Mirrlees good (or a composite Diamond-Mirrlees commodity) in the economy guarantees that some Pareto improving directions of consumer price changes (i.e., commodity tax changes) exist. Without this assumption, the household preferences might be conflicting to such a degree that no Pareto improvement through a perturbation of the economy's initial commodity tax structure is possible. Assumption (4.17) thus implies sufficient homogeneity of consumer preferences without which Theorem 4.1 cannot be established.\(^6\)

**Theorem 4.1:**

Suppose (i) rank \( Y = K \leq N \), (ii) rank \([S_{pp} + YY^T]\) = \( N \), (iii) \( \tau^T \gamma_w \frac{V^2}{w} G(w + \tau^*, y^*) \neq 0^T_M \) and (iv) there is no solution \( a > 0 \).
to $a^T[x^T, e^T] = 0^T_{N+M}$. Then, a (differential) strict Pareto and productivity improving change in the economy's initial tariff and commodity tax structures exists, holding the initial vector of lump sum transfers constant. This change in taxes and tariffs also improves the home country's net balance of trade.

Proof:

Applying Motzkin’s Theorem, a sufficient condition for a strict Pareto and productivity improving tax and tariff change to exist is:

(4.18) there is no vector $\lambda^T = [\lambda^T_1, \lambda^T_2, \lambda^T_3, \lambda^T_4] \in R^{H+N+K+1}$ such that $\lambda^T[B_p, B_z, B_t, B_s] = 0^T_{N+K+M+N+M}$, $\lambda^T[A, -B_a] > 0^T_{H+1}$.

Consider the equations $\lambda^T[B_p, B_z, B_t] = 0^T_{N+K+N}$. Subtract the $N$ equations corresponding to the matrix $B_t$ from the first $N$ equations corresponding to the matrix $B_p$. This implies

(4.19) $[\lambda^T_2, \lambda^T_3, \lambda^T_4] \begin{bmatrix} S_{pp} & Y \\ Y^T & 0_{KxK} \\ w^T_{Sp} & w^T_F \end{bmatrix} = 0^T_{N+K}$.

Equations (4.19) have already been solved in Lemma 2.3. Hence,
if \( \lambda^4 = k \in \mathbb{R} \), \( \lambda^2 = k(p^* + \delta)^T \) and \( \lambda^3 = k\gamma^T \), where the vectors \( \delta \) and \( \gamma \) are those defined in (2.27). The inequalities \( \lambda^T[A, -B_b] > 0_{H+1}^T \) imply that \( \lambda^4 = k \geq 0 \). In order to determine if (4.18) can be satisfied, two cases need to be considered.

(i) \( \lambda^4 = k = 0 \):

If \( k = 0 \), then also \( \lambda^2 = 0_N^T \) and \( \lambda^3 = 0_K^T \). In order to satisfy (4.18), the following must hold:

\[
\begin{align*}
\lambda^TA &= \lambda^1 > 0_H^T, \\
\lambda^TB &= \lambda^1X^T = 0_N^T, \\
\lambda^BS &= \lambda^1E^T = 0_M^T.
\end{align*}
\]

By assumption, there is no solution to (4.20) and hence no solution to (4.18).

(ii) \( \lambda^4 = k > 0 \):

Set \( k = 1 \). Let us consider the equations \( \lambda^T[B_b, B_s] = 0_{N+M}^T \). The goal here is to show that these equations cannot be satisfied if 
\( \lambda^T = [\lambda^1, (p^* + \delta)^T, \gamma^T, 1] \) (the vector \( \lambda^1 \) may be solved from the equations \( \lambda^T[B_b, B_s] = 0_{N+M}^T \), but knowledge of \( \lambda^1 \) is not required for the proof of the theorem). Using the above defined vector \( \lambda \),

\[
\begin{align*}
\lambda^TB &= -\lambda^1E^T - (p^* + \delta)^T \Sigma_{qv} + (p^* + \delta)^T S_{pw} + \gamma^T F \Sigma_{vv} \\
&\quad + w^T S_{ww} \\
&= (p^* + \delta)^T S_{pw} + \gamma^T F + w^T S_{ww}.
\end{align*}
\]
since $\lambda^T B = 0^T_M$. Applying Lemma 2.4, it can be seen that

$$
(4.22) \quad \lambda^T B = -\tau^* T^2 \left( y^* + \tau^*, y^* \right) > 0^T_M. \quad \text{QED}
$$

Theorem 4.1 shows that a strict productivity improving change in the initial equilibrium tariffs $\tau^*$ can be converted to a strict Pareto improving change of the tariffs $\tau^*$ and commodity tax rates $(t^*, s^*)$, without a change in the consumers' initial lump sum incomes. The first three assumptions needed to establish the result have been encountered in Theorem 2.1. They imply that a strict productivity improving tariff change, starting from the initial equilibrium, exists. In order to distribute these productivity gains to the households in a strict Pareto improving way, the fourth assumption which involves the consumer preferences and endowments, must be satisfied.

The role of assumptions (i) - (iv) is further clarified, if Theorem 4.1 is compared to some results of Diamond and Mirrlees (1971). In their classic paper, Diamond and Mirrlees consider, among other things, the existence of Pareto improving changes in commodity taxes in a closed economy. They show that if (i) all production in the economy is under direct government control, (ii) the initial equilibrium production choice lies inside the home country's production possibility set, and (iii) a Diamond-Mirrlees good exists, then a Pareto improving change in the economy's initial commodity tax rates is possible. They argue further that if, alternatively, production in the economy takes
place in a private production sector, Pareto-improving tax changes starting from the initial equilibrium still exist, if the producer and consumer prices in the economy can be perturbed independently from each other and the above mentioned assumptions (ii) - (iii) are satisfied.  

Consider now the existence result in Theorem 4.1. Let us define international trade as an additional production technology made available for the home country. This artificial technology can be expressed as the set

\[ \bar{T} = \left\{ (\sum_{k=1}^{K} y^k, \sum_{k=1}^{K} f^k) : \sum_{k=1}^{K} y^k \leq 0^N, w^T \sum_{k=1}^{K} f^k \leq 0 \right\}. \]

The economy's total production possibility set is then generated by the sum of the domestic production technologies \( T^k, k=1,...,K \) and the set \( \bar{T} \). In Figure 3, which is drawn assuming that there are two internationally tradeable commodities \( (M = 2) \) and one consumer \( (H = 1) \) in the home country, the curve \( PP' \) gives the domestic production possibility frontier for tradeables keeping domestic goods net supply constant. The frontier \( PP' \) is generated by the domestic technology sets \( T^k, k=1,...,K \). The line denoted by \( w \) defines the production possibility frontier for the total technology \( \sum_k T^k + \bar{T} \).

Suppose first that all production in the country is under direct public control, and that the line denoted by \( w \) in Fig. 3 is the relevant production possibility frontier for tradeables. Suppose further, that
the initial equilibrium in the economy corresponds to the points A and B in Figure 3. Since the initial production choice A lies inside the feasible production possibility set, using the Diamond-Mirrlees argument, the only condition needed for a Pareto improving change in the commodity tax rates \((t^*, s^*)\) and tariffs \(\tau^*\) to exist, is that there is a Diamond-Mirrlees commodity in the economy. Assumption (iv) in Theorem 4.1 is the weakest sufficient version of this assumption.

Suppose then that the domestic production sector in the home country consists of \(K\) constant returns to scale industries. In this case, the production possibility frontier \(PP'\) generated by the sum of the sectoral technologies \(T_k, k=1,...,K\), becomes a constraint: the government can choose a strict Pareto improving change in the initial commodity tax rates and tariffs only if the change is such that the new production choice, established after the perturbation in the taxes and tariffs, lies on the frontier \(PP'\). Assumptions (i) - (iii) in Theorem 4.1 guarantee that a net balance of trade improving tariff change, that satisfies this condition, exists. This change moves the economy's production choice from A, in Fig. 3, toward the point C along the curve \(PP'\). Since the consumer preferences satisfy Assumption (iv) in Theorem 4.1, and since, in Theorem 4.1, the consumer and producer prices of commodities can be perturbed independently of each other, the government can adjust the initial commodity tax rates \((t^*, s^*)\) simultaneously with a change in tariffs \(\tau^*\) so that a strict Pareto improvement is attained. Theorem 4.1 can thus be regarded as a generalized open economy version of the results of Diamond and Mirrlees.
Figure 3 - Strict Pareto Improving Perturbations in Tariffs and Commodity Taxes.
Finally, one more comment should be made about what was not assumed in Theorem 4.1: nothing was said about the initial values of the commodity tax rates $t^*$ and $s^*$. It was only assumed that the government is able to adjust all of these tax rates if need be. Would Theorem 4.1 still hold, if the initial tax rates ($t^*$, $s^*$) happened to be Diamond-Mirrlees optimal, i.e., they maximize some social welfare function $W(u)$ with respect to the constraints of the general equilibrium model (4.10) – (4.13)? It turns out that only the properties of the initial tariff vector $\tau^*$ matter. If, at the initial equilibrium, the tariffs $\tau^*$ and the commodity tax rates ($\tau^*$, $s^*$) are Diamond-Mirrlees optimal, then no strict Pareto and productivity improving changes in them exist. But if the vector of tariffs $\tau^*$ is arbitrary, Diamond-Mirrlees optimality of the commodity tax rates does not change the conclusion of Theorem 4.1. Rather, if the commodity taxes are not optimal at the initial equilibrium, there exist strict Pareto improving tax and tariff changes with $\Delta \tau^* = 0_M$ i.e., the initial tariffs $\tau^*$ need not be perturbed at all. Considering practical policies, this means that as long as the initial commodity tax rates ($t^*$, $s^*$) in the home country are not Diamond-Mirrlees optimal, the government can attain strict Pareto (hence, welfare) improvements by changing only the commodity taxes. After these improvement possibilities have been exhausted, the more complex policies involving changes in tariffs are needed.
4.3 Necessary Conditions for Pareto Optimality: Nonexistence of Strict
Pareto and Productivity Improving Tax and Tariff Changes

Having established sufficient conditions implying the existence of strict Pareto and productivity improving tariff and commodity tax perturbations, it is natural to enquire when these policy changes do not exist. A result in this vein was already established in Section 2.4 as Theorem 2.2, where it was shown that strict productivity improving changes in the initial equilibrium tariffs $\tau^*$ are not possible if the gradient of the net balance of trade function $b^*(w + \tau^*)$ with respect to the tariffs $\tau^*$ is zero. Theorem 2.2 can also be interpreted to give a set of necessary conditions that the initial equilibrium of the economy must satisfy, if it is a local productivity optimum. The practical significance of this result lies in preventing a search for differential improvements in the government tariff policy when none exist.

In this section, the most general necessary conditions for Pareto and productivity optimality of the initial equilibrium are established. Thereafter, some special cases are considered.

Theorem 4.2:

A necessary condition for strict Pareto and productivity improving commodity tax and tariff changes to not exist, i.e., a necessary condition for Pareto and productivity optimality of the initial equilibrium, is:
(4.24) there is a vector $\lambda \in \mathbb{R}^{H+N+K+1}$ such that $\lambda^T [A, -B_b] > 0^T_{H+1}$,

and $\lambda^T [B_p, B_z, B_t, B_s, B_\tau] = 0^T_{N+K+N+M+M^*}$.

**Proof:**

A necessary condition for the nonexistence of strict Pareto and productivity improvements is:

\begin{equation}
\text{(4.25) there do not exist } \Delta u^*, \Delta p^*, \Delta z^*, \Delta b^*, \Delta t^*, \Delta s^*, \Delta \tau^* \text{ such that (4.16) holds and } \Delta u^* \gg 0_H, \Delta b^* > 0. \tag{4.25}
\end{equation}

Using Motzkin's Theorem and the proof of Theorem 4.1, it can be shown that (4.25) and (4.24) are equivalent. QED

Although Theorem 4.2 advises the government not to search for strict Pareto and productivity improvements when (4.24) is satisfied, it is rather difficult to see from the statement of the result, how it is related to Theorems 2.1 and 4.1. In order to interpret Theorem 4.2, a special kind of initial equilibrium for the economy is introduced.

An equilibrium is called $\beta$-optimal with respect to the commodity tax rates $(t, s)$ and tariffs $\tau$, if it solves the nonlinear programming problem

\begin{equation}
\text{(4.26) } \max_{u, p, z, b, t, s, \tau} \{ \beta^T u : (4.10) - (4.13) \text{ hold, } g = \text{constant} \}.
\end{equation}
In (4.26), the government is assumed to choose the tax and tariff rates 
\( (t, s, \tau) \) to maximize a social welfare function of the form 
\( W(u) = \beta^T u \), 
\( \beta > 0 \). The welfare weights \( \beta \) can be regarded as the gradient vector 
\( \nabla_u W(u) \) of some general social welfare function \( W(u) \). (The function 
\( \beta^T u \) is thus a local linearization of \( W(u) \)). The constraints in (4.26) 
restrict the \( \beta \)-optimum to be a competitive equilibrium with a fixed 
(possibly zero) vector of lump sum transfers. Essentially, (4.26) 
corresponds to the (closed economy) Diamond, Mirrlees (1971) – Diewert 
(1978) optimal tax problem; the only difference is that, in (4.26), the 
trade tariffs \( \tau \) are also assumed to be set so as to maximize social 
welfare.\(^{11}\)

If the initial equilibrium of the economy is a \( \beta \)-optimum with 
respect to the commodity tax rates \( (t^*, s^*) \) and tariffs \( \tau^* \), then 
it must necessarily satisfy:

\[
(4.27) \quad \text{there do not exist } \Delta u^*, \Delta p^*, \Delta z^*, \Delta b^*, \Delta \tau^*, \Delta s^*, \Delta \tau^* \text{ such that (4.16) holds, } \beta^T \Delta u^* > 0, \Delta b^* > 0 \text{ and } \Delta g^* = 0_H^*.
\]

If (4.27) is violated, the initial equilibrium cannot be a welfare 
maximum. Applying Motzkin's Theorem, (4.27) can be written as

\[
(4.28) \quad \text{there exists a vector } \lambda \in \mathbb{R}^{H+N+K+1} \text{ such that } \lambda^T A = \beta^T (> 0^T_H), 
\]

\[
-\lambda^T B_b > 0 \text{ and } \lambda^T [B_p, B_z, B_t, B_s, B_\tau] = O^T_{N+K+N+M+M^*}.
\]
Comparing (4.28) and (4.24), it can be seen that, if the initial equilibrium is a Pareto and productivity optimum, it is also a welfare maximum for the social welfare function \((\lambda^T A)u\), where the vector \(\lambda\) is defined by (4.28).

**Proposition 4.1:**

If, at the initial equilibrium of the economy, no strict Pareto and productivity improving commodity tax and tariff changes exist, the equilibrium is a welfare maximum for the social welfare function \((\lambda^T A)u\), where the vector \(\lambda\) solves \(\lambda^T [B_p, B_z, B_t, B_s, B_r] = \lambda^T B_p > 0\).

Proposition 4.1 is a Negishi (1960) type result showing that a Pareto and productivity optimum is also a welfare maximum with respect to some welfare function of the form \(W(u) = \beta^T u\). In particular, if, initially, the government is not assumed to choose the tax and tariff rates in the country so as to maximize social welfare, and if (4.28) is satisfied, the observed equilibrium is revealed to be a welfare maximum with respect to a welfare function which may or may not have socially acceptable welfare weights.

It was assumed in the optimization problem (4.26) that the government can adjust the tariffs \(\tau\) in any way deemed optimal. But if all or some of the tariffs \(\tau^*\) are fixed, it would be useful to know, if any particular values of the initial vector of tariffs \(\tau^*\) satisfy (4.24) under the supposition that only the domestic commodity tax rates
t* and s* can be chosen optimally.

**Proposition 4.2:**

Let the initial equilibrium be $\beta$-optimal with respect to the commodity tax rates $(t^*, s^*)$, and suppose that the assumptions (i) - (ii) and (iv) of Theorem 4.1 are satisfied. Then, if $\tau^T \nabla_{ww}^2 G(w + \tau^*, y^*) = 0^T_M$, the initial equilibrium satisfies the necessary condition for Pareto and productivity optimality given in Theorem 4.2, and no strict Pareto and productivity improving directions of change in the tariffs $\tau^*$ and commodity taxes $(t^*, s^*)$ exist.

**Proof:**

$\beta$-optimality of the initial equilibrium implies that there exists a vector $\lambda \in \mathbb{R}^{H+N+K+1}$ such that 

$$
\lambda^T [A, -B_b] > 0^T_{H+1}, \lambda^T [B_p, B_z, B_t, B_s] = 0^T_{N+K+N+M}.
$$

Applying the proof of Theorem 2.1, for a vector $\lambda$ that solves the equations $\lambda^T [B_p, B_z] = 0^T_{N+K}$, the vector $\lambda^T B_\tau$ equals $\tau^T \nabla_{ww}^2 G(w + \tau^*, y^*)$. By assumption, $\tau^T \nabla_{ww}^2 G(w + \tau^*, y^*) = 0^T_M$.

Then, in addition to the optimality conditions given above, also (4.24) is satisfied. QED

Proposition 4.2 implies that the gradient vector of the net balance of trade function, $\nabla_{\tau} b^*(w + \tau^*)$, which equals $-\tau^T \nabla_{ww}^2 G(w + \tau^*, y^*)$, must be nonzero for strict Pareto and productivity improvements in taxes $(t^*, s^*)$ and tariffs $\tau^*$ to exist.
The assumption in Proposition 4.2 that the vector
\[ \tau^T \gamma^2_{ww} G(w + \tau^*, y^*) \] equals zero can be replaced by a condition involving the aggregate producer substitution matrix S:

\[ (4.29) \quad [-\delta^T, \tau^*] S = 0^T_{N+M}, \]

where the vector \( \delta \) is defined in (2.27). Then, if the matrix \( S \) is of maximal rank ( = \( N + M - 1 \)), a sufficient condition implying (4.29) is that the relative producer prices for tradeables in the home country and abroad coincide. It follows that if, initially, \( \tau^* = 0_M \), or if \( \tau^* \) is some multiple of the international prices \( w \), no strict Pareto and productivity improving commodity tax and tariff perturbations, starting from the initial equilibrium, are possible; i.e., the initial equilibrium satisfies the necessary condition (4.24) for Pareto and productivity optimality. Furthermore, it can be argued, using the programming problem (3.3), that when \( \tau^* = 0_M \) or when \( \tau^* \) is some multiple of the international prices \( w \), the initial equilibrium must in fact be a productivity maximum. Hence, for a small country that cannot influence international commodity prices, zero tariffs are Pareto and productivity optimal even though lump sum transfers are not a feasible government policy instrument as long as the domestic commodity tax rates can be chosen optimally, the producer substitution matrix \( S \) is of maximal rank, and assumptions (i) and (iv) of Theorem 4.1 are satisfied.

The above statement can also be interpreted as an efficiency of total production result. Assuming that the producer substitution matrix
S is of maximal rank, and assumptions (i) and (iv) of Theorem 4.1 hold, starting from an initial equilibrium that satisfies (4.10) - (4.13) with 
\[ \nabla_T b^*(w + \tau^*) = \tau^* T^2 \nabla^2_{w} G(w + \tau^*, y) \neq 0 \]
there exists a path of strict Pareto and productivity improvements leading to a Pareto and productivity optimum, where the relative producer prices \( w \) and \( (w + \tau^*) \) have been equalized. This Pareto and productivity optimum is efficient with respect to the total production technology \( \sum T^k + \bar{T} \) defined in (4.23). (In Fig. 3, the Pareto and productivity optimum corresponds to the first best equilibrium \((C,D)\).) Using Diamond-Mirrlees (1971) terminology, it can be said that total production efficiency is desirable (with respect to the technology \( \sum T^k + \bar{T} \)) in a small country, if the producer substitution matrix \( S \) is of maximal rank, assumptions (i) and (iv) of Theorem 4.1 are satisfied, and the commodity tax rates \((\tau^*, s^*)\) can be chosen optimally to maximize social welfare.

4.4 **Strict Pareto and Productivity Improving Changes in Commodity Taxes and Tariffs when no Domestic Goods Exist**

In the formulation of Theorem 4.1, it was implicitly assumed that some domestic commodities exist, i.e., \( N > 0 \). It is tempting to simply note that if all goods in the home country are internationally tradeable, assumptions (i) - (ii) of the theorem can be erased and the
result is restored\textsuperscript{14}. A closer inspection of the theorem shows, however, that although the sufficient conditions for strict Pareto and productivity improvements to exist, when \( N \) equals zero, are very similar to those given in Theorem 4.1, the interpretation of the general equilibrium model (4.10) - (4.13) changes if \( N = 0 \).

If the number of production sectors in the model (4.10) - (4.13) equals the number of domestic commodities i.e., \( N = K (> 0) \), the sectoral production technologies and the tradeables prices \((w + \tau^*)\) determine the equilibrium prices \( p^* \) for domestic goods in (4.11).\textsuperscript{15} (If the number of the production industries is less than the number of domestic goods \((K < N)\), the domestic market equilibrium conditions (4.12) also affect the prices \( p^* \).) If there are no domestic commodities in the economy, the tradeables prices \((w + \tau^*)\) determining the industry (pure) profits \( \pi^k, k=1,\ldots,K \), instead of the prices \( p^* \) in (4.11), i.e., the production industries do not generally earn zero profits when \( N = 0 \). The existence of these possibly positive profits creates \( K \) artificial domestic factors, to which the profits are imputed. Hence, in the end, \( N \), the number of domestic commodities, must be positive.

If the only domestic commodities in the economy are the \( K \) factors to which the sectoral positive profits are being imputed, the domestic net supply matrix \( Y \) becomes an \((K \times K)\)-identity matrix: each sector supplies one unit of its "ownership", as the newly created domestic commodities might be called. The aggregate producers' zero profit condition can then be written as \( p^*T + (w + \tau^*)^T F = 0_K^T \), where \( p^* \),
the price vector for the artificial factors of production, gives the sectors' pure profits. The producer substitution matrix $S$ is of the form

$$ (4.30) \quad S = \begin{bmatrix} O_{K \times K} & O_{K \times M} \\ O_{M \times K} & S_{ww} \end{bmatrix}. $$

The consumers hold endowments of industry ownership shares, denoted by $x^h$, $h=1,\ldots,H$ ($\sum_{h=1}^{H} x^h_k = 1$, for all $k=1,\ldots,K$).

**Theorem 4.3:**

Let the only domestic commodities in the home country be the ownership shares in the production sectors $k$, $k=1,\ldots,K$. Then, if (i) there is no solution $a > 0_H$ to $a^T [X^T, E^T] = 0^T_{K+M}$, and (ii) $\tau^T S_{ww} \neq 0^T_M$, there exists a strict Pareto and productivity improving change in the initial equilibrium tariffs $\tau^*$ and commodity tax rates $(\tau^*, s^*)$.

**Proof:**

If $Y = I_{K \times K}$, the rank of the matrix $Y$ is $K (= N)$, and the matrix $Y Y^T$ is positive definite. Hence, assumptions (i) - (ii) of Theorem 4.1 are satisfied. Since $\tau^T S_{ww} \neq 0^T_M$, assumption (iv) of Theorem 4.1 is also satisfied. (Note that $\nabla^2_{ww} G(w + \tau^*, y^*) = S_{ww}$, if $S$ is of the form (4.30).) QED
Assumption (ii) in Theorem 4.3 has a similar interpretation as assumption (iv) has in Theorem (4.1); if only the production side of the economy were considered, then proceeding as in Theorem 2.1, it would be possible to show that this supposition implies the existence of strict productivity improving changes in tariffs $t^*$. These productivity gains can be distributed to the households in a strict Pareto improving way, if the consumer preferences are sufficiently homogenous, i.e., assumption (i) of Theorem 4.3 is satisfied. If the only domestic commodities are the ownership shares in the production industries, this condition is easily met: it is enough to have a production sector $k, k \in [1,\ldots,K]$, such that $x^h_k > 0$ for all $h=1,\ldots,H$. Then, lowering the initial domestic tax rate $t^*_k$ is a strict Pareto improving tax change direction.
5. EXISTENCE OF STRICT GAINS FROM TRADE WHEN LUMP SUM TRANSFERS ARE NOT A FEASIBLE GOVERNMENT POLICY INSTRUMENT

It has been shown that, under certain rather weak assumptions about the initial equilibrium, the government can cause a strict productivity and Pareto improvement by adjusting the initial commodity tax rates \((t^*, s^*)\) and tariffs \(\tau^*\) appropriately. This general result has a perhaps surprising application: now, it is easy to prove the existence of positive gains from trade, even if no lump sum transfers may be used to redistribute consumer income.

Suppose that the initial equilibrium of the economy is an autarky equilibrium, i.e., there is no international trade. In order to describe autarky using the open economy model (4.10) - (4.13), it is assumed that the lack of international trade is caused by the government tariff policy. To this end, the initial equilibrium tariffs \(\tau^*\) are defined as

\[
(5.1) \quad \tau^* = \omega^a - w,
\]

where \(\omega^a\) is the autarky equilibrium vector of tradeables prices\(^1\) and \(w\) is the observed international price vector. It follows from (5.1) that \((w + \tau^*) = \omega^a\). This means that if the tariffs \(\tau^*\), defined by (5.1), are used in the open economy model (4.10) - (4.13), the model characterizes an autarky equilibrium.\(^2\)
In order to apply Theorem 4.1, it is assumed that the conditions (i) - (iv) of the theorem are satisfied. Then, there exists a strict Pareto and productivity improving change in the initial tariffs \( \tau^* \) and commodity tax rates \((t^*, s^*)\). It follows that, if the government allows international trade by perturbing the international trade prohibitive tariffs \( \tau^* \), it can also change the initial commodity tax rates \((t^*, s^*)\) in such a way that all households in the economy strictly benefit: strict gains from trade exist.\(^3\)

It should be noted, however, that a perturbation of the tariffs \( \tau^* \) is not always necessary for strict gains from trade to exist: if the autarky commodity tax rates \((t^*, s^*)\) are not \( \beta \)-optimal,\(^4\) strict Pareto improvements can be found by changing only the tax rates \((t^*, s^*)\). At the new (perturbed) levels of the commodity tax rates the initial tariffs \( \tau^* \) are not international trade prohibitive; trade will thus be opened up, causing a productivity and welfare improvement.\(^5\)

Proposition 4.2 gives necessary conditions for Pareto and productivity optimality of the initial equilibrium, when only commodity tax and tariff rates are used as government policy instruments. In the present context, Proposition 4.2 may be interpreted to give necessary conditions for the nonexistence of strict gains from trade. For example, if the initial commodity tax rates \((t^*, s^*)\) are \( \beta \)-optimal, and if the initial trade prohibitive tariffs \( \tau^* \) equal zero (or if they are proportional to the international prices \( w \)) so that the gradient of the net balance of trade function \( b^*(w + \tau^*) \) is a zero \( M \)-vector, the potential productivity gains accruing from the participation in
international trade have already been exhausted in autarky, and no further strict gains from trade are possible.

The initial autarky equilibrium can also satisfy the condition
\[ \nabla_\tau b^* (w + \tau^*) = 0^T_M \] when \( \tau^* \neq 0^M_M \) and \( \tau^* \) is not proportional to the international prices \( w \). In this case, since the home country's production possibility frontier is ridged and/or kinked, no differential strict Pareto and productivity improving perturbations in the tariffs \( \tau^* \) and commodity tax rates \( (t^*, s^*) \) exist.

Consider Figure 4. There are two tradeable commodities in the economies depicted in Figures 4a) and 4b). The production possibility frontier for these commodities, given a fixed net supply of domestic goods \( y^* \), is represented by the curve \( PP' \). The initial autarky equilibrium in Fig. 4a) is denoted by A: the producers face the tradeables prices \( w^a (= w + \tau^*) \), while the consumer prices for tradeables are \( v^a (= w^a + s^*) \). At these prices the consumer attains the utility level \( u^A \).

The point B in Fig. 4a) shows the economy's production choice under free undistorted trade \( (\tau^* = 0^M_M, s^* = 0^M_M) \). The line \( w \) denotes the exogenous international prices. Let us denote the consumer indifference curve tangent to the line \( w \) (not shown in Fig. 4a)) by \( u^C \). (The point C denotes the tangency point between the line \( w \) and the curve \( u^C \).)

The first best autarky equilibrium in Fig. 4a) lies at \( A' \) where the consumer and producer prices for tradeables coincide \( (w^a = v^a) \).
Figure 4 - Existence of Strict Gains from Trade
Because of a government tax revenue requirement and the lack of nondistortionary tax instruments, the actual initial autarky equilibrium at A is only a second best equilibrium, i.e., $u^A < u^{A'}$. But if the conditions of Theorem 4.1 are satisfied, the government can move the equilibrium from A, i.e., from autarky, toward the first best (free trade) equilibrium at B and C in such a way that the consumer in the economy strictly benefits. The shift is accomplished by suitably perturbing the initial commodity tax rates $(t^*, s^*)$ and tariffs $\tau^*$. If, for example, the tariffs $\tau^*$ are reduced to zero,\(^6\) production will shift to B, while the consumer will reach the indifference curve tangent to the consumer price line $v(= w + s^a)$.

The consumer's utility level on this indifference curve can be higher or lower (in the Fig. 4a) it is lower) than the first best autarky utility level $u^{A'}$, but it is not lower than $u^A$ and not higher than $u^C$.

If, starting from the equilibrium with zero tariffs, the commodity taxes can be changed further, the gains from trade (the difference between $u^A$ and the indifference curve tangent to the line $v$) can be made even larger.

It is easy to see in Fig. 4a) that, if the international prices $w$ and the autarky producer prices $(w + \tau^*)$ coincide, strict gains from trade can be found only by changing the commodity tax rates $(t^*, s^*)$, if they are not $\beta$-optimal at the initial equilibrium (starting from an equilibrium, where production takes place at B, and the consumer attains the utility level corresponding to the consumer prices $v$, the consumer's welfare can be improved, if the tax rates $s^*$ (and $t^*$) are perturbed optimally; in the Figure 4a) the consumer price line $v$ could
be rotated toward the line w as far as the government revenue constraint allows).

Figure 4b) illustrates the case where the autarky equilibrium at A satisfies the necessary conditions for Pareto and productivity optimality given in Proposition 4.2 even though the trade prohibitive tariffs $\tau^*$ are nonzero and nonproportional to the international prices w: the gradient of the net balance of trade function $b^*(w + \tau^*)$ at A is a zero 2-vector since, at A, the matrix $\gamma^2_{ww} G(w + \tau^*, y^*)$ is a zero (2 x 2)-matrix. It follows that no strict (differential) Pareto and productivity improving tax and tariff changes starting from A are possible: strict gains from international trade do not exist.

The existence of the gains from trade is further discussed in Chapter 9. In that chapter, lump sum transfers are assumed to be admissible, and the conditions under which strict gains from trade exist when either commodity taxation or lump sum compensation is used to distribute the gains to the consumers, are compared.
6. EXISTENCE OF STRICT PRODUCTIVITY AND PARETO IMPROVEMENTS WHEN ONLY A LIMITED SET OF COMMODITY TAXES AND TARIFFS CAN BE PERTURBED

Suppose that, instead of being able to adjust all the \( N + M \) initial commodity tax rates \( t^* \) and \( s^* \) at will, the government is constrained to perturb only the \( N \) domestic tax rates \( t^* \). Are strict Pareto and productivity improving changes in tariffs and taxes still possible? In particular, are the \( N \) domestic tax rates sufficient to distribute the gains from international trade to the households in the economy in a strict Pareto improving way?

**Theorem 6.1:**

Suppose that (i) \( \text{rank } Y = K \leq N \), (ii) \( S_{pp} + YY^T \) is positive definite, (iii) there is no solution to \( a > 0 \) to \( a^T X^T = 0_N^T \), (iv) \( \tau^T v^2 w G(w + \tau^*, y^*) \neq 0_M^T \), and (v) \( g^* = 0_H \). Then, there exists a strict Pareto and productivity improving change in tariffs \( \tau^* \) and commodity tax rates \( t^* \).

**Proof:**

It is sufficient to show that there does not exist a vector \( \lambda \in \mathbb{R}^{H+N+K+1} \) such that \( \lambda^T[A, -B] > 0_{H+1}^T \), \( \lambda^T[B_p, B_z, B_t] = 0_{N+K+N}^T \), \( \lambda^T_B = 0_M^T \). Following the proof of Theorem 4.1, the vector \( \lambda \) that solves the \( N + K \) equations \( \lambda^T[B_p, B_z] = 0_{N+K}^T \) must be of the form \( k[(p^* + \delta)^T, y^T, 1], k \in \mathbb{R} \), in its three last components. Assumption
(iii) implies that if \( k = 0 \), no vector \( \lambda \) satisfying the equations and inequalities given above exists.

Suppose \( k = 1 \):

Consider the equations \( \lambda_B^T \tau = 0^T_M \). Using the definition of \( B_\tau \) in (4.16), these equations can be written as

\[
\lambda_B^T \tau = -\lambda_T^{T} - (p^* + \delta)^T \Sigma_v^T - \omega v^T \epsilon v + (p^* + \delta)^T S_{pw} + \gamma_T T + w v^T S_{ww}.
\]

By assumption, \( \lambda_B^T \tau = 0^T_N \). This implies, using (4.16), that

\[
\lambda^T \tau q - (p^* + \delta) v^T \epsilon v - \omega v^T \epsilon v = 0,
\]

where \( q^* = (p^* + t^*) \).

Using the homogeneity of the expenditure functions, (6.2) becomes

\[
\lambda^T \tau q^* v^* - (p^* + \delta) v^T \epsilon v q^* - \omega v^T \epsilon v q^* = 0,
\]

where \( v^* = (w + s^* + r^*) \).

Since \( g^* = 0 \), (6.3) yields \( \lambda^T \tau v^* = 0 \). It follows, using (6.1), that

\[
\lambda^T \tau v^* = [(p^* + \delta)^T S_{pw} + \gamma_T T S_{ww}] v^* = (p^* + \delta)^T S_{pw} v^* = -v^T w^2 g(w + r^*, y^*) v^*.
\]
By assumption, \( \tau^T \gamma^2 \frac{G(w + \tau^*, y^*)}{\omega} \neq 0 \), hence, since \( \nu^* >> \omega \), 
\( \lambda^T B \nu^* \neq 0 \) and the vector \( \lambda^T B \tau \) cannot be zero. QED

Theorem 6.1 shows that strict Pareto and productivity improving
tariff and tax perturbations are still possible even though the
government is constrained to adjust only the \( N \) domestic commodity tax
rates \( \tau^* \) in addition to the tariffs \( \tau^* \). Theorem 6.1 also implies
that the separation of the consumer and producer sectors is not
necessary for strict Pareto and productivity improvements to exist in an
open economy.¹ (In Theorem 4.1, the assumption that all commodity tax
rates \( (\tau^*, s^*) \) can be freely adjusted means that that the consumer
and producer prices can be perturbed independently from each other.)

The assumptions of Theorem 6.1 are not much stricter than the
assumptions of Theorem 4.1. The only additional restrictive supposition
is that the (possibly composite) Diamond-Mirrlees good, that is supposed
to exist, must be a domestic good. The assumption that there are no
lump sum transfers at the initial equilibrium is not necessary for the
result to hold. Inspection of the proof for Theorem 6.1 shows that the
transfer vector \( g^* \) must only be such that 
\( \lambda^T_1 g^* = -\tau^* \gamma^2 \frac{G(w + \tau^*, y^*)}{\omega} \nu^* \),
where the vector \( \lambda^T_1 \) solves the equations
\( \lambda^T_1 X^T + (p^* + \delta)^T \Sigma_{qq} + \omega^T \Sigma_{vq} = 0^T \), i.e., the equations \( \lambda^T B \tau = 0^T \). Nonexistence of
transfers at the initial equilibrium is sufficient for the inequality
\( \lambda^T_1 g^* = -\tau^* \gamma^2 \frac{G(w + \tau^*, y^*)}{\omega} \nu^* \) to be satisfied.

¹
Theorem 6.1 proves that productivity gains from international trade can be distributed to the households in a strict Pareto improving way also when the set of controllable tax instruments is restricted to \( t^* \), the set of domestic commodity tax rates. Yet, existence of fixed tax distortions \( s^* \) in the home country gives rise to some complications. First, in contrast to the results of the previous sections, zero tariffs (free trade) are not generally optimal for a small country, if nonoptimal commodity taxes that cannot be adjusted are present in the system. This can be seen as follows: write equation (6.1) in the form

\[
(6.4) \quad \lambda^T B = -\lambda^T E + t^* T \Sigma_{qv} + (\tau^* + s^*) T \Sigma_{vw} - \tau^*[S_{np,11} + FD^T] \Sigma_{qv} \\
+ \tau^* T \Sigma_{ww} G(w + \tau^*, y^*).
\]

If the initial tariffs \( \tau^* \) are zero, but \( (\tau^*, s^*) \neq 0_{N+M} \), equation (6.4) does not reduce to a zero identity; in other words, even though the tariffs \( \tau^* \) at the initial equilibrium were zero, some strict Pareto and productivity improving changes in \( \tau^* \) and commodity tax rates \( t^* \) would still exist (assuming that the other conditions of Theorem 6.1 are satisfied.) \(^2\)

Yet, there is a case where free trade, i.e., zero tariffs, is Pareto and productivity optimal for a small country, irrespective of the fixed initial commodity tax rates \( s^* \). This occurs when there is no substitution in consumption, i.e., when \( \Sigma = O(N+M)X(N+M) \). \(^3\) It is well known that taxes on goods in fixed demand are nondistortionary; hence,
free trade under these circumstances is optimal for a small country.

The above result that free trade is not generally Pareto and productivity optimal for a small country if some distortionary taxes in the country exist, can be rephrased to express **nondesirability of total production efficiency under the presence of market distortions.** In this form, the statement is an application of Guesnerie's (1977) earlier propositions to an open economy.

Consider then the existence of strict Pareto and productivity improvements in tariffs and commodity taxes when **only one initial tariff** \( \tau^*_m \), \( m \in [1,\ldots,M] \), **can be varied.** In this case, Theorems 2.1 and 4.1 can still be applied, if all commodity tax rates \((t^*, s^*)\) or at least the rates \( t^* \) are adjustable: a sufficient condition for a strict Pareto and productivity improvement in \((t^*, s^*)\) and \( \tau^*_m \) to exist, supposing that assumptions (i), (ii) and (iv) of Theorem 4.1 are satisfied, is that the derivative \( \nabla_{\tau^*_m} b^*(w + \tau^*) = \tau^*_m \nabla^2_{ww} G(w + \tau^*, y^*)_{\tau^*_m} \) is nonzero. (The notation \( \nabla^2_{ww} G(w + \tau^*, y^*)_{\tau^*_m} \) refers to the \( m^{th} \) column of the matrix \( \nabla^2_{ww} G(w + \tau^*, y^*)_m \).

If **only one initial domestic commodity tax rate** \( t^*_n \), \( n \in [1,\ldots,N] \), in addition to the tariffs \( \tau^* \) **is variable, strict Pareto improvements in** \( \tau^* \) and \( t^*_n \) **are not possible, but the government can still generate a strict welfare (and productivity) improvement in an exogenously given social welfare function** \( W(u) = \beta^T u \).
Theorem 6.2:

Suppose that (i) rank \( Y = K \leq N \), (ii) \( S_{pp} + YY^T \) is positive definite, and (iii) the vector \( \lambda \in \mathbb{R}^{H+N+K+1} \) defined by

\[
\lambda^T = [\beta^T, O_{N+K+1}^T][A, -B_p, -B_z', - (B_t')_n]^{-1}
\]

does not solve the equations \( \lambda^T B_T = O_N^T \). Then, there exists a strict welfare and productivity improving change in the tariffs \( \tau^* \) and the tax rate \( \tau_n^*, n \in [1,...,N] \).

Proof:

A sufficient condition for a strict welfare and productivity improving change in the tariffs \( \tau^* \) and the tax rate \( \tau_n^*, n \in [1,...,N] \), to exist is:

\[
(6.6) \text{there exist } \Delta u^*, \Delta p^*, \Delta z^*, \Delta b^*, \Delta \tau_n^*, \Delta \tau^* \text{ such that (4.16) is satisfied and } \beta^T \Delta u^* > O_H^T, \Delta b^* > 0.
\]

Equivalently, by Motzkin's Theorem:

\[
(6.7) \text{there must not exist a vector } \lambda \in \mathbb{R}^{H+N+K+1} \text{ such that } \lambda^T A = \beta^T, \lambda^T [B_p, B_z, (B_t)_n] = O_{N+K+1}^T, \lambda^T B_T = O_M^T, - \lambda^T B_b > 0.
\]
By the Implicit Function Theorem, the inverse matrix

\[ [A, - B_p, - B_z, - B_t]^{-1} \]

exists, and hence (6.5) defines a unique \( \lambda \) for the vector of welfare weights \( \beta^* \). By assumption, for this \( \lambda \), the vector \( \lambda^T B_t \) is not zero; thus (6.7) holds. QED

Why is it that only a strict welfare (not a strict Pareto) improving change in tariffs \( \tau^* \) and the tax rate \( t^*_n, n \in [1, \ldots, N] \), can be found? The intuitive explanation is that, to produce Pareto improvements, the government needs a sufficiently large number of free tax instruments; for welfare improvements, only one adjustable tax rate can be sufficient. The technical reason for the nonexistence of strict Pareto improvements above is that the vector \( \lambda \), which can be solved from (6.5), cannot be determined from the equations \( \lambda^T [B_p, B_z, (B^*_n)] = 0^{T}_{N+K+1} \) that would have to be used in order to show the existence of strict Pareto improvements in \( \tau^* \) and \( t^*_n, n \in [1, \ldots, N] \).

If a tradeable commodity tax rate \( s^*_m, m \in [1, \ldots, M] \), were the free tax instrument, a theorem analogous to Theorem 6.2 could be established. In this case, in (6.5), the column \( (-B_t)_{-n} \) would be replaced by the column \( (-B^*_s)_{-m} \), but the other assumptions of the theorem would stay the same.
7. SOME PIECENELM POLICY RESULTS WHEN NO LUMP SUM TRANSFERS ARE USED

AS GOVERNMENT POLICY INSTRUMENTS

The previous sections have been entirely concerned with the existence of strict Pareto and productivity improving government policy perturbations. Thus far, very little has been said about the specific nature of the Pareto and productivity improving policy changes.

It was established earlier that, for example, a proportional reduction of the initial equilibrium tariffs $\tau^*$ could be strict productivity improving. In this section, it will be shown that such a reduction is also a strict Pareto improvement, if the initial commodity tax rates ($t^*$, $s^*$) are adjusted accordingly. Some other examples of strict Pareto improving tariff and tax perturbation policies are also given.

Consider first a reduction of the initial tariffs $\tau^*$ when $\tau^* \geq 0_M$, and an increase of $\tau^*$ when $\tau^* \leq 0_M$.\footnote{Theorem 7.1:}

Let assumptions (i) - (iv) of Theorem 4.1 be satisfied so that a strict Pareto and productivity improving change in the initial tariffs $\tau^*$ and commodity taxes ($t^*$, $s^*$) exists. Then, (i) if the tariffs $\tau^*$ are nonnegative, i.e., $\tau^* \geq 0_M$, the change in $\tau^*$ may be taken to be a reduction ($\Delta \tau^* \leq 0_M$), and (ii) if the tariffs $\tau^*$ are nonpositive, i.e., $\tau^* \leq 0_M$, the change in $\tau^*$ may be taken to be an increase ($\Delta \tau^* \geq 0_M$).
Proof:

Consider the case (i) where $\tau^* \geq 0$. A sufficient condition for a strict Pareto and productivity improvement in tariffs and commodity taxes to exist is:

\begin{equation}
\text{(7.1)} \quad \text{there is no vector } \lambda \in \mathbb{R}^{H+N+K+1} \text{ such that } \lambda^T [A, -B_b] > 0^T_{H+1}, \lambda^T [B_p, B_z, B_t, B_s] = 0^T_{N+K+N+M}, \lambda^T B_t > 0^T_M.
\end{equation}

The condition (7.1) is derived using Motzkin's Theorem; the constraint $\Delta \tau^* \leq 0_M$ results in the inequality $\lambda^T B_t \geq 0^T_M$ in (7.1). Consider the equations

\begin{equation}
(7.2) \quad \lambda^T B_t (w + \tau^*) = \lambda^T \begin{bmatrix} -E^T \\ -\Sigma^T_{qv} + S^T_{pw} \\ F^T \\ -w^T \Sigma_{vv} + w^T S_{ww} \end{bmatrix} (w + \tau^*)
\end{equation}

\begin{equation}
= \lambda^T \begin{bmatrix} x^T q^* + E^T s^* - g^* \\ \Sigma^T_{qq} + \Sigma^T_{qv} s^* - S^T_{pp} p^* \\ -Y^T p^* \\ w^T \Sigma_{vq}^* - w^T S_{wp} p^* + w^T \Sigma_{vv} s^* \end{bmatrix}.
\end{equation}
The homogeneity of the expenditure and unit profit functions, and the equations \( q^T X + v^T E = g^T \) and \( p^T Y + (\omega + \tau^*)^T F = 0^T \) yield the latter form of \( \lambda^T B_t (\omega + \tau^*) \). Since \( \lambda^T [B_t, B_s] = 0^T_{N+M} \) by assumption,

\[
(7.3) \quad \lambda^T B_t (\omega + \tau^*) = -\lambda_1^T g^* - \lambda_2^T S_{pp}^* - \lambda_3^T v^T p^* - w^T S_{wp}^* \\
= -\lambda_1^T g^* - \lambda_1^T k^T p^* - \lambda_2^T q^T p^* - w^T q^T v^* ,
\]

using \( \lambda^T B_p = 0^T_N \). Further, because \( \lambda^T B_t = 0^T_N \),

\[
(7.4) \quad \lambda^T B_t (\omega + \tau^*) = -\lambda_1^T g^* = -\lambda_1^T [X^T q^* + E^T v^*] = \lambda_1^T X^T q^* - \lambda_1^T E^T v^* .
\]

The equations \( \lambda^T [B_t, B_s] = 0^T_{N+M} \) imply

\[
(7.5) \quad \lambda^T B_t (\omega + \tau^*) = (p^* + \delta^T) T \Sigma_{qq} q^* + w^T \Sigma_{qq} q^* + (p^* + \delta^T) T \Sigma_{qv} v^* \\
+ w^T \Sigma_{qv} v^* = 0 ,
\]

using the homogeneity of the expenditure functions \( m^h, h=1,\ldots,H \). In (7.5) the fact that the vector \( \lambda \) must be of the form \( [\lambda_1^T, (p^* + \delta^T), \gamma^T, 1] \) has also been employed. By assumption, the vector \( (\omega + \tau^*) \) is strictly positive (\( \gg 0_M \)) and the vector \( \lambda^T B_t \) is nonzero (since \( \tau^T \Sigma_{ww} G(\omega + \tau^*, y^*) \neq 0^T_M \); see the proof of Theorem 4.1). Thus, the vector
\lambda^T B_\tau must contain at least one negative (and positive) element, i.e., (7.1) is violated.

Consider now the case where \( \tau^* \leq \mathcal{O}_M \). The vector \( \tau^* \) is such that \((w + \tau^*) > \mathcal{O}_M \). A sufficient condition for a strict Pareto and productivity improving increase in \( \tau^* \) to exist is:

\[ (7.6) \text{ there is no vector } \lambda \in \mathbb{R}^{H+N+K+1} \text{ such that } \]

\[ \lambda^T [A, -B_d] > 0^T_{H+1}, \lambda^T [B_p, B_z, B_t, B_s] = 0^T_{N+K+M}, \lambda^T B_\tau \leq 0^T_M \]

Using the same reasoning as above, it can be seen that for any \( \lambda \) satisfying (7.6) (except the inequalities \( \lambda^T B_\tau \leq 0^T_M \)), the inequalities \( \lambda^T B_\tau \leq 0^T_M \) must be violated. Hence (7.6) holds, and a strict Pareto and productivity improving perturbation in the tariffs \( \tau^* \) and tax rates \( (t^*, s^*) \) exists. QED

Theorem 7.1 shows that a movement toward free trade, i.e., zero tariffs, is strict Pareto and productivity improving for a small country, if all the initial commodity tax rates \( (t^*, s^*) \) can be adjusted suitably and the other conditions of the theorem are satisfied. However, a closer inspection of the proof of Theorem 7.1 shows that also an increase of positive tariffs and a reduction of negative tariffs can be strict Pareto and productivity improving. How can this seemingly counterintuitive result be explained? The economic...
justification seems to be that a movement toward equalized relative producer prices for tradeables at home and abroad, i.e., toward equalized relative prices \((w + \tau^*)\) and \(w\), is strict Pareto and productivity improving under the conditions of Theorem 4.1, whether the change is accomplished by reducing (increasing) the tariffs \(\tau^* > 0\) (\(\tau^* < 0\)) or by increasing (reducing) them.

It may sometimes be desirable to choose a particular kind of reduction in the home country's trade barrier. For example, a proportional or uniform reduction of tariffs may be considered politically fair. It turns out that a proportional reduction of tariffs is strict Pareto improving under the conditions of Theorem 4.1 but that a uniform reduction of tariffs requires somewhat stronger assumptions; the responses of the domestic production sectors' net export supply functions to changes in tradeables producer prices must be such that a decrease in each tariff \(\tau^*_m\), \(m \in \{1, \ldots, M\}\), leads to an increase in the home country's net balance of trade.

**Theorem 7.2:**

Let assumptions (i) - (ii) and (iv) of Theorem 4.1 be satisfied. Assume further that \(v^T \cdot b^*(w + \tau^*) = -\tau^T v^2_{ww} G(w + \tau^*, y^*) < 0\). Then, a strict Pareto improving reduction of tariffs \(\tau^*\), accompanied by a change in the initial commodity tax rates \((t^*, s^*)\), exists, and the change in the tariffs \(\tau^*\) can be chosen to be a uniform reduction, i.e., \(\Delta \tau^*_m = -h\), \(h > 0\), \(m \in \{1, \ldots, M\}\).
Proof:

It is sufficient to show that there is no vector $\lambda \in \mathbb{R}^{H+N+K+1}$ such that $\lambda^T[A, -B] > 0^T_{H+1}$, $\lambda^T[B_p, B_z, B_L, B_s] = 0^T_{N+K+N+M}$, $\lambda^TB_T H_M \geq 0$, where $h_M$ is an $M$-vector consisting of numbers $h(>0)$. As shown in the proof of Theorem 4.1, for a vector $\lambda$ satisfying $\lambda^T[B_p, B_z, B_L, B_s] = 0^T_{N+K+N+M}$, the following must hold: $\lambda^TB \tau = -\tau^T \nabla^2_{w'w} G(w + \tau, y)$. Then, the inequalities $\lambda^TB_T H_M \geq 0$ can be written as $-\tau^T \nabla^2_{w'w} G(w + \tau, y^*) h_M \geq 0$. But, by assumption, $\nabla^*_{w'w} (w + \tau^*) < 0^T_M$ and $h \gg 0^T_M$. Hence, $\nabla^*_{w'w} (w + \tau^*) h_M < 0$. QED

The assumptions in Theorem 7.2 are also sufficient for a third kind of reduction in tariffs $\tau^*$ to be strict Pareto and productivity improving. Suppose that $\tau^* \gg 0^T_M$, i.e., net exports are subsidized and net imports are taxed. In this case, the government may be interested in bringing the tariff rates $\tau^*$ closer to each other; in other words, a change toward uniformity in tariff rates may be desirable.

**Proposition 7.1:**

Let the initial vector of tariffs $\tau^*$ be positive, i.e., $\tau^* \gg 0^T_M$. Suppose that the assumptions of Theorem 7.2 are satisfied. Then, there exists a strict Pareto and productivity improving change in the initial tariffs $\tau^*$ and commodity tax rates $(t^*, s^*)$, and the change of tariffs $\tau^*$ can be chosen to be a reduction toward a (nonnegative) uniform tariff structure.
Proof:

Define $\hat{\tau}_M \equiv (\hat{\tau}, \ldots, \hat{\tau})^T (\geq 0_M)$ as the uniform set of tariffs, toward which the initial tariffs $\tau^*$ are perturbed. The direction of change in $\tau^*$ is $\Delta \tau^* = -(\tau^* - \hat{\tau}) < 0$. Using the proof of Theorem 7.2, for a strict Pareto and productivity improvement to exist when $\Delta \tau^* = -k(\tau^* - \hat{\tau})$, $k > 0$, it is sufficient to show that there is no vector $\lambda \in \mathbb{R}^{H+N+K+1}$ such that $\lambda^T [A, -B_B] > 0_{H+1}^T$, $\lambda^T [B_p, B_z, B_t, B_s] = 0_{N+K+N+M}^T$, $\lambda^T B_{\tau} (\tau^* - \hat{\tau}) \geq 0$. By assumption, $\lambda^T B_{\tau} = \nabla_{\tau} b^*(\omega + \tau^*) < 0_M^T$ and thus, $\lambda^T B_{\tau} (\tau^* - \hat{\tau}) < 0_M$ when $(\tau^* - \hat{\tau}) > 0_M$. QED

It should be noted that, if the initial vector of tariffs $\tau^*$ is negative, i.e., net export are taxed and net imports are subsidized, under the conditions of Proposition 7.1, there exists a strict Pareto and productivity improving decrease in tariffs $\tau^*$ toward a uniform tariff structure.\(^5\)

Proposition 7.1 is closely related to a result established by Hatta (1977b). Hatta showed that, under certain conditions, in a one consumer one producer economy, a reduction of the highest (ad valorem) tariff rate to the level of the next highest tariff rate improves the welfare of the consumer. To derive his result, Hatta assumed that lump sum transfers are admissible, and that no distortionary commodity taxes exist in the home country. In Proposition 7.1, in contrast, no lump sum transfers are assumed to be available, distortionary (specific) commodity taxes are present, and the numbers of consumers and
production industries are not restricted to one. The assumption that
the gradient of the net balance of trade function \( b^*(w + \tau^*) \) with
respect to tariffs \( \tau^* \) is negative at the initial equilibrium appears
to be a generalization of a supposition used by Hatta: the good with
the highest tariff rate must be a substitute in production to the other
tradeable commodities. (If \( \tau^* \gg 0 \), for the vector \( \tau^* V^2 \)
\( G(w + \tau^*, y^*) \) to be positive, positive terms in the matrix \( V^2 \)
\( G(w + \tau^*, y^*) \), i.e., substitution in production, must dominate.)

Is there anything more to be said about the changes in the
initial tax rates \( t^* \) and \( s^* \)? For example, is it possible to lower
also them, when tariffs \( \tau^* \geq 0 \) are being reduced? Let us suppose
that all initial commodity tax rates in the economy are positive, i.e.,
\( t^* > 0_N \), \( s^* > 0_M \). (This means that the government is taxing
commodities bought by the consumers, whereas factors of production sold
by the households are being subsidized.) Then, assuming that the
initial equilibrium is not a \( \beta \)-optimum with respect to the tax and
tariff rates \( (t^*, s^*, \tau^*) \) (in which case, no strict Pareto and
productivity improvement in them could exist), a strict Pareto and
productivity improvement can be attained by simultaneously reducing \( t^* \),
\( s^* \) and \( \tau^* \).

Theorem 7.2:
Suppose that the initial commodity tax rates \( (t^*, s^*) \) are
positive, i.e., \( t^* > 0_N \), \( s^* > 0_M \), and the tariffs \( \tau^* \) satisfy
\( r^* \geq 0 \). Assume in addition that \((r^*, s^*, \tau^*)\) do not solve the problem

\[
\text{(7.7) } \max_{u, p, z, b, t, s, \tau} \left\{ g^T u : (4.10) - (4.13) \text{ hold, } g^* = 0 \right\},
\]

for any \( \beta > 0 \). Then, there exists a strict Pareto and productivity improving simultaneous reduction in \( t^*, s^* \) and \( \tau^* \) (i.e., \( \Delta t^* \leq 0 \), \( \Delta s^* \leq 0 \), \( \Delta \tau^* \leq 0 \)).

**Proof:**

A sufficient condition for a strict Pareto productivity improving simultaneous reduction in \( t^*, s^* \) and \( \tau^* \) to exist is:

\[
\text{(7.8) there is no vector } \lambda \in \mathbb{R}^{H+N+K+1} \text{ such that } \lambda^T [A, -B_b] > 0^T_{H+1},
\]

\[
\lambda^T [B_p, B_z] = 0^T_{N+K}, \lambda^T [B_t, B_s, B_{\tau}] \geq 0^T_{N+M+M^*}.
\]

Since the initial equilibrium is not a \( \beta \)-optimum, for any vector \( \lambda \) satisfying \( \lambda^T [A, -B_b] > 0^T_{H+1} \) and \( \lambda^T [B_p, B_z] = 0^T_{N+K} \), the vector

\[
\lambda^T [B_t, B_s, B_{\tau}] \text{ is not a zero } (N+M+M)-\text{vector. It follows from the inequalities } \lambda^T A > 0^T_H \text{ that } \lambda \neq 0^T_{H+N+K+1}. \text{ Then, for any } \lambda \in \mathbb{R}^{H+N+K+1} (\neq 0),
\]
(7.9) \[
\lambda^T [B_L, B_S, B_T] \begin{bmatrix}
    t^* \\
    s \\
    w + \tau^*
\end{bmatrix} = \lambda^T \begin{bmatrix}
    -X^T \\
    -\Sigma q q^* - \Sigma v v^* - \Sigma v w^* + S_p w \\
    0_{KxN} \\
    O_{KxM} \\
    F^T \\
    -w^T \Sigma v q - w^T \Sigma v v^* - w^T \Sigma v w^* + w^T S_{wp} \\
\end{bmatrix}
\]

\[
= \lambda^T \begin{bmatrix}
    X_p^* - g^* \\
    \Sigma q p^* - S_{pp} p^* \\
    T^* \\
    -Y_p \\
    w^T \Sigma v q^* - w^T S_{wp} p^* \\
\end{bmatrix} = -\lambda^T g^* = 0
\]

using the equations \( \lambda^T B_p = O_N^T \) when \( g^* = O_H \).

Since, by assumption, \( t^* > O_N, s^* > O_M \) and \( (w + \tau^*) \gg O_M \), the vector \( \lambda^T [B_L, B_S, B_T] \) must contain negative elements (it is not zero because the initial equilibrium is not a \( \beta \)-optimum). Thus, (7.8) is satisfied.

QED

Corollary 7.2.1:

If \( t^* > O_N, s^* > O_M, \tau^* \gg O_M \) and rank \([B_p, B_z, B_t, B_s, B_t] = H+N+K+1 \), a strict simultaneous Pareto and productivity improving reduction in \( t^*, s^* \) and \( \tau^* \) exists.
Proof:

If rank $[B_p, B_z, B_t, B_s, B_t] = H+N+K+1$, the only solution to the equations $\lambda^T [B_p, B_z, B_t, B_s, B_t] = 0_{N+K+N+M+M}$ is $\lambda = 0_{H+N+K+1}$. Then, $\lambda^T A = 0_H$ and $\lambda^T B = 0$, which contradict the conditions $\lambda^T A > 0_H$ and $\lambda^T B < 0$ in (7.8). QED

If the initial commodity tax rates $(t^*, s^*)$ are negative $(t^* < 0_N, s^* < 0_M)$ so that commodities bought by the consumers are subsidized, and factors of production sold by the households are taxed, it can be shown, using the same analysis as in the proof of Theorem 7.2, that a strict Pareto and productivity improving increase in $t^*, s^*$ and $\tau^* (\leq 0_N)$ is possible. This policy change translates to reduced subsidies on consumer goods, reduced taxes on factors of production (e.g., labor), smaller import subsidies and lower export taxes, if such exist.

The conclusion of Theorem 7.2 may not seem so surprising; it is only claimed that a simultaneous reduction of (positive) distortionary taxes and tariffs is desirable. Yet, it is not self-evident that a strict Pareto and productivity improvement can be achieved through a change in $(t^*, s^*)$ and $\tau^*$, without a change in the initial equilibrium vector of lump sum transfers (in Theorem 7.2, $g^* = 0_H$ and $Ag^* = 0_H$). Actually, according to Theorem 7.2, if the assumptions of Theorem 7.2 are satisfied, there exists an entire path of
simultaneous strict Pareto and productivity improving reductions in the positive \((t^*, s^*)\) and \(\tau^*\) which, in the limit, lead to a nondistortionary government which behaves like an additional private production sector in the economy. One should, however, be cautious in interpreting this result as a practical policy recommendation: the general equilibrium model \((4.10) - (4.13)\), which is used to derive Theorem 7.2, characterizes a static, perfectly competitive economy, where no externalities in consumption or production are present. Furthermore, the outcome of a strict Pareto and productivity improvement may be unacceptable from an economic equality point of view.

If the government is restricted to adjust a subset of the commodity tax rates \((t^*, s^*)\), results analogous to Theorem 7.2 can still be established. For example, the following statements can be proved using similar techniques as in the proof of Theorem 7.2:

(i) If \(t^* > 0\), \(s^* = 0\), and the initial tax rates \(t\) and the tariffs \(\tau\) are not \(\beta\)-optimal, there exists a simultaneous strict Pareto and productivity improving reduction of \(t^*\) and \(\tau^*\).

(ii) If \(t^* = 0\), \(s^* > 0\), and the initial tax rates \(s\) and the tariffs \(\tau\) are not \(\beta\)-optimal, there exists a simultaneous strict Pareto and productivity improving reduction of \(s^*\) and \(\tau^*\).

(iii) If \(t^*_n > 0\), \(n \in [1, \ldots, N]\), \(t^*_{n} = 0_{n-1}\), \(10\), \(s^* = 0\), \(g^* = 0\) and the
initial tax rate $t^*_n$ and the tariffs $\tau^*$ are not $\beta$-optimal, there exists a simultaneous strict Pareto and productivity improving reduction of $t^*_n$ and $\tau^*$. 
8. PARETO IMPROVING POLICY PERTURBATIONS WITH LUMP SUM TRANSFERS

8.1 A Second Model for the Production Side of An Economy

Consider a model of an economy's production side consisting of the domestic supply equals demand equations (4.12), the zero profit relations (2.9), and the net balance of trade equation (4.13). These N+K+I relationships endogenously determine the equilibrium vector of domestic prices p*, the equilibrium vector of industry scales z*, and the home country's equilibrium net balance of trade b*, given the exogenous tariffs τ*, the vector of international prices w, a fixed vector of domestic commodity tax rates (t*, s*), and a fixed vector of household utilities u*.

The new model of the economy's domestic production sector differs from the model (2.8) - (2.10) in that the production sector is not required to supply a fixed vector y* of domestic commodities but, instead, the consumers are assumed to be kept at fixed utility levels u* h, h=1, ..., H, which equal the observed initial equilibrium levels of consumer welfare.

It is assumed that a solution to the equations (4.12), (2.9) and (4.13) exists. It is also assumed that the domestic p* and the industry scales z* that solve (4.12), (2.9) and (4.13) are strictly positive. Differentiating the equations (4.12), (2.9) and (4.13) around the initial solution p* >> 0_N, z* >> 0_K, b* ε R:

\[ B_p \Delta p^* + B_z \Delta z^* + B_b \Delta b^* = B_\tau \Delta \tau^* \]
where \( B_p = \begin{bmatrix} S_{pp} - \Sigma_{qq} \\ Y^T \\ w^T(S_{wp} - \Sigma_{vq}) \end{bmatrix}, B_z = \begin{bmatrix} Y \\ 0_{K \times K} \\ w^T_F \end{bmatrix}, B_b = \begin{bmatrix} 0_N \\ 0_K \\ -1 \end{bmatrix} \)

\[ B_\tau = \begin{bmatrix} -S_{pw} + \Sigma_{qv} \\ -F^T \\ -w^T(S_{ww} - \Sigma_{vv}) \end{bmatrix} \]

Applying the Implicit Function Theorem, the endogenous \( p^*, z^* \) and \( b^* \) can be regarded as implicit functions of the exogenous tradeables prices \( (w + \tau^*) \) (at fixed \( u^*, t^* \) and \( s^* \)), if the matrix \([B_p, B_z, B_b]\) is invertible. Under this supposition, the directional derivatives of the functions \( p^*(w + \tau^*), z^*(w + \tau^*) \) and \( b^*(w + \tau^*) \), evaluated at the initial solution to (4.12), (2.9) and (4.13), are given by the matrix \([B_p, B_z, B_b]^{-1} B_\tau^*\).

Using the results of Diewert and Woodland (1977: Appendix), it can be seen that the necessary and sufficient conditions for the matrix \([B_p, B_z, B_b]\) to be invertible are (2.12) and

\[(8.2) \quad \text{rank} (S_{pp} - \Sigma_{qq} + YY^T) = N.\]

Henceforth, it is assumed that (2.12) and (8.2) are satisfied at the initial solution to (4.12), (2.9) and (4.13).

Assumption (8.2) has a similar economic interpretation as assumption (2.13). Consider the following version of Definition 2.2:
**Definition 8.1:**

Domestic goods production is said to be locally controllable around the initial solution to (4.12), (2.9) and (4.13), if there exist continuously differentiable functions \( p^*(u^*, w + \tau^*, t^*, s^*) \) and \( z^*(u^*, w + \tau^*, t^*, s^*) \) around the initial solution to (4.12), (2.9) and (4.13) which satisfy (4.12), (2.9), and (4.13) when the exogenous variables \( u, \tau, t \) and \( s \) assume their initial equilibrium values.

The idea behind Definition 8.1 is simple. It is assumed that the policy goal of the government is to find such a perturbation of the initial tariffs \( \tau^* \) that the amount of net foreign exchange earned by the production industries, in the aggregate, is increased, while the consumers in the economy are kept at their initial equilibrium utility levels \( u^* \). In order to achieve its target, the government must be able to influence domestic goods production in the country in such a way that as the tariffs \( \tau^* \) (and hence, the consumer prices for tradeables \( (w + \tau^* + s^*) \)) are perturbed from their initial levels, the induced change in the consumers' welfare is zero. Local controllability of domestic goods production in the sense of Definition 8.1 is sufficient to this end: the functions \( p^*(u^*, w + \tau^*, t^*, s^*) \) and \( z^*(u^*, w + \tau^*, t^*, s^*) \) define the appropriate changes in domestic goods prices and industry scales corresponding to given changes in \( u^*, \tau^*, t^* \) and \( s^* \).

As in Lemma 2.1, it can be shown that Assumptions (2.12) and (8.2) are sufficient for local controllability of domestic goods production in the sense of Definition 8.1. If the consumer
substitution matrix for domestic goods $\Sigma_{qq}$ is a zero $(N \times N)$-matrix, condition (8.2) coincides with condition (2.13), and controllability of domestic goods production in the sense of Definition 8.1 coincides with the controllability concept of Definition 2.2. If the matrix $\Sigma_{qq}$ is nonzero, however, condition (8.2) is less restrictive than condition (2.13).

As before, the gradient vector of the net balance of trade function $b^*(w + \tau^*)$ will play an important role in the subsequent analysis. Since the model consisting of the equations (4.12), (2.9) and (4.13) differs from the model (2.8) - (2.10), a new expression for the vector $\nabla_{\tau} b^*(w + \tau^*)$ must be found. The gradient $\nabla_{\tau} b^*(w + \tau^*)$ is given by the last row of the matrix $[B_p, B_z, B_b]^{-1} B_{\tau}$:

\[
(8.3) \quad \nabla_{\tau} b^*(w + \tau^*) = w^T \left[ [S_{ww} - \Sigma_{vv}] - [S_{wp} - \Sigma_{vq}, F] \tilde{D} \right. \\
\left. [S_{wp} - \Sigma_{qv}, F]^T \right],
\]

where the symmetric inverse matrix $\tilde{D}$ is defined by

\[
(8.4) \quad \tilde{D} = \begin{bmatrix} \tilde{D}_{11} & \tilde{D}_{12} \\ \tilde{D}_{21} & \tilde{D}_{22} \end{bmatrix} = \begin{bmatrix} S_{pp} - \Sigma_{qq} & Y \\ Y^T & 0 \end{bmatrix}_K^{-1}.
\]
Assumptions (2.12) and (8.2) guarantee that the inverse matrix \( \bar{D} \) exists.

In order to develop an interpretation for the matrix on the right hand side of (8.3), a new function, called the home country's constant utility tax adjusted balance of trade function \( B \), is defined:

\[
B(w, w + \tau, w + \tau + s, t) \equiv \max_{z, Y, F, X, E} \left\{ -w^T e^0 + \sum_{k=1}^{K} (w + \tau)^T f_k^z k \right\}
\]

\[
- \sum_{h=1}^{H} (w + \tau + s)^T e^h - \sum_{h=1}^{H} t x^h + \sum_{k=1}^{K} y^h z^k
\]

\[
\sum_{h=1}^{H} x^h - x^0 \geq 0, (y^h, f^h) \in C^h, (x^h, e^h) \in \mathcal{N}^h(u^*)
\]

The function \( B \) gives the maximal net amount of foreign exchange that the private production industries can earn, net of government tax revenue, when the consumers are kept at their initial equilibrium utility levels \( u^* \); the sets \( \mathcal{M}^h(u^*) \) in (8.5) represent the net consumption vectors \( (x^h, e^h), h=1, \ldots, H \), that can achieve utility level \( u^* \), \( h=1, \ldots, H \).

The function \( B \) is convex and linearly homogenous in its arguments. Using the Karlin (1959: p. 201) - Uzawa (1958: p. 34) Saddle Point Theorem, the convex programming problem (8.5) can be written in a dual form which gives an alternative expression for the function \( B \):
\[(8.6) \quad B(w, w + \tau, w + \tau + s, t) = \max z \geq 0_K, p \geq 0_N \min \{ -w^T e^0 + \sum_{k=1}^{K} \pi^k (p, w + \tau) z^k - \sum_{h=1}^{H} m^h (u^h, p + t, w + \tau + s) \} - p^T x^0 \].

In (8.6), the functions $\pi^k$, $k=1, \ldots, K$, and $m^h$, $h=1, \ldots, H$, are the previously defined sectoral unit profit and household expenditure functions, respectively. Problem (8.6) shows that $B$ is equal to the net value of private production valued at the prices $(p, w + \tau)$ minus the net value of public production valued at the prices $(p, w)$ minus the net household expenditures valued at the prices $(p + t, w + \tau + s)$.

If $\tau = \tau^*$, $s = s^*$ and $t = t^*$, the first order conditions for problem (8.6) become the zero profit and domestic supply equals demand equations (2.9) and (4.12) which are assumed to be satisfied at the economy's initial equilibrium. Hence, if conditions (2.12) and (8.2) are satisfied, the tax and tariff variables are fixed at their initial values $(t^*, s^*, \tau^*)$, and the household utilities are fixed at $u^*$, the equations (2.9) and (4.12) can be used to determine the industry scales $z^*$ and domestic prices $p^*$ as implicit functions of the international prices $w$. This means that $B$ becomes a function of $w$:

\[(8.7) \quad B(w) = B(w, w + \tau^*, w + \tau^* + s^*, t^*). \]
The gradient of the function $\tilde{B}$ can be calculated by differentiating the objective function in (8.6):

\begin{equation}
\nabla_w \tilde{B}(w) = -e^0 + \sum_{k=1}^{K} \nabla_{w \pi} (p(w), w + \tau^*) z^k(w) - \sum_{h=1}^{H} \nabla_{w \mu} (u^* h, p(w) + \tau^*, w + \tau^* + s^*).
\end{equation}

Thus, $\nabla_w \tilde{B}(w)$ is the net excess supply vector of internationally traded commodities produced by the constant utility economy.

In order to develop the Hessian matrix of the function $B$ the gradient $\nabla_w \tilde{B}(w)$ is differentiated with respect to the international prices $w$. Using the formulae for the derivatives $\nabla_w p(w)$ and $\nabla_w z(w)$ obtained by differentiating the equations (4.12) and (2.9) with respect to $w$, $p$ and $z$:

\begin{equation}
\nabla^2_{ww} \tilde{B}(w) = S_{ww} - \Sigma_{vv} - [S_{wp} - E_{vq}, F] D [S_{wp} - E_{vq}, F]^T.
\end{equation}

Hence, using (8.3),

\begin{equation}
\nabla_{\tau^*} \tilde{b}^*(w + \tau^*) = \omega^T \nabla^2_{ww} \tilde{B}(w).
\end{equation}
Lemma 8.1:

The matrix $V^{2}_{ww} B(w)$ can be written in the form

$$V^{2}_{ww} B(w) = \begin{bmatrix} -(S_{wp} - \Sigma_{vq}) D_{11} - F D_{12}^T, \quad I_M \end{bmatrix}$$

$$\begin{bmatrix} S - \Sigma \end{bmatrix} \begin{bmatrix} \tilde{D}_{11} (S_{pw} - \Sigma_{qv}) - \tilde{D}_{12} F^T \end{bmatrix},$$

where the matrices $\tilde{D}_{11}$ and $\tilde{D}_{12}$ are blocks in the inverse matrix $\tilde{D}$ defined in (8.4).

The proof of Lemma 8.1 is given in Appendix 2. Lemma 8.1 implies that the matrix $V^{2}_{ww} B(w)$ is positive semidefinite, but the zero eigenvectors of the matrix are generally unknown. It can be shown, however, that if the initial equilibrium commodity tax rates are zero, i.e., $t^* = 0_N$ and $s^* = 0_M$, the function $B + w e$, defined using (8.5), must be linearly homogenous in the tradeables prices $(w + \tau^*)$. This means the matrix $V^{2}_{ww} B(w)$ must satisfy the constraint

$$V^{2}_{ww} B(w) (w + \tau^*) = 0_M,$$

if $t^* = 0_N$ and $s^* = 0_M$. 
8.2 Existence of Constant Utility Productivity Improving Changes in Tariffs

The government's policy goal is defined as follows: find such a (differential) change in the home country's initial equilibrium vector of tariffs $\tau^*$ that the country's initial net balance of trade $b^*$ is improved, while the consumers in the economy are kept at their initial equilibrium levels of welfare $u^*_h$, $h=1,...,H$. More precisely, the problem is to determine the minimal sufficient conditions for:

\begin{equation}
(8.12) \text{there exist } \Delta p^*, \Delta z^*, \Delta b^*, \Delta \tau^* \text{ such that (8.1) holds and } \Delta b^* > 0.
\end{equation}

A perturbation of tariffs $\tau^*$ which satisfies (8.12) is called a constant utility productivity improving change of tariffs $\tau^*$.

**Theorem 8.1:**

Suppose (i) rank $Y = K \leq N$, (ii) rank $(S_{pp} - \Sigma_{qq} + YY^T) = N$ and (iii) $w^T \bar{V}_{w\omega} \bar{B}(w) \neq 0_M^T$. Then, there exists a constant utility productivity improving change in tariffs $\tau^*$.

**Proof:**

The proof makes use of two preliminary lemmas, the proofs of which are given in Appendix 2.
Lemma 8.2:

Any vector $\lambda \in \mathbb{R}^{N+K+1}$ satisfying the equations $\lambda^T [B_p, B_z] = 0_{N+K}$ must be of the form

$$(8.13) \quad \lambda^T = k[(p^* + \varepsilon)^T, \theta^T, 1], k \in \mathbb{R},$$

where

$$(8.14) \quad [\varepsilon^T, \theta^T] = [-\tau^* T \Sigma_q - (\tau^* + s^*)^T \Sigma_{vq} + \tau^* T S_{wp}, \tau^* T F] D.$$

Lemma 8.3:

For the vector $\lambda$ solved in Lemma 8.2, the following holds:

$$(8.15) \quad \lambda^T B_\tau = -k \omega^T \Sigma_{ww}^{-1} \bar{B}(\omega), k \in \mathbb{R}.$$

Proof of Theorem 8.1:

A sufficient condition for a constant utility productivity improving change in tariffs $\tau^*$ to exist is $(8.12)$. Using Motzkin's Theorem as in the proof of Theorem 2.1, an equivalent condition can be derived:

$$(8.16) \quad \text{there must not exist a vector } \lambda \in \mathbb{R}^{N+K+1} \text{ such that}$$
\[ \lambda^T[B_p, B_z, -B_T] = 0^T_{N+K+M}, \lambda^T_B < 0. \]

On the contrary, suppose such a \( \lambda \) exists. By Lemma 8.2, a vector \( \lambda \) that solves the equations \( \lambda^T[B_p, B_z] = 0^T_{N+K} \) must be of the form
\[ \lambda^T = k[(p^* + \varepsilon)^T, \theta^T, 1], k \in \mathbb{R}. \]
For such a \( \lambda \), \( \lambda^T_B = -k \). Thus, for \( \lambda^T_B \) to be negative, \( k > 0 \) (and \( k \) may be chosen to be one). Using Lemma 8.3, \( \lambda^T_B = -w^T \nabla^2_B(w). \) By assumption, \( \lambda^T_B \neq 0^T_M \), a contradiction.

QED

An example of constant utility productivity improving tariff changes is provided by Proposition 8.1.

**Proposition 8.1:**

If the assumptions of Theorem 8.1 are satisfied, a change of the tariffs \( \tau^* \) in the direction of the world prices \( w \) will be productivity improving (keeping the households in the economy at their initial equilibrium utility levels \( u^* \)).

**Proof:**

Let \( \Delta \tau^* = rw, r > 0. \) Using the proof of Proposition 2.1, it is sufficient to show that there is no vector \( \lambda \in \mathbb{R}^{N+K+1} \) such that
\[ \lambda^T[B_p, B_z] = 0^T_{N+K}, \lambda^T_B < 0, \lambda^T_B w = 0. \] Using Lemmas 8.2 and 8.3, it can be seen that for the vector \( \lambda \in \mathbb{R}^{N+K+1} \) that satisfies \( \lambda^T[B_p, B_z] = \]
The scalar must be equal \( w^T
\). The matrix \( V_{ww}^2 \) is positive semidefinite and, by assumption, \( w^T V_{ww}^2 (w) \neq 0 \).

Hence, \( w^T V_{ww}^2 (w) w > 0 \). QED

Proposition 8.1 implies that small, competitive countries can improve their productivity performance by shifting their tariff structure toward the international prices \( w \), without sacrificing the welfare of their consumers.

Assumption (iii) in Theorem 8.1 may be written in an alternative form which involves the producer and consumer substitution matrices \( S \) and \( \Sigma \). Using the proof of Lemma 8.3, in Appendix 2, the following formula can be derived:

\[
(8.17) \quad w^T V_{ww}^2 (w) w = w^T \left[ -(S_{wp} - \Sigma_{qv}) D_{11} - F D_{12}^T, I_M \right] [S - \Sigma] \\
\quad - \left[ \tilde{D}_{11} (S_{pw} - \Sigma_{qv}) - \tilde{D}_{12} F^T \right] w \\
\quad - \left[ I_M \right]
\]

\[
= \left[ -(p^* + \varepsilon)^T, w^T \right] (S - \Sigma) \left[ -(p^* + \varepsilon) \right] \geq 0, \text{ } \text{ } \text{ } \text{ } 11
\]

where the vector \( \varepsilon \) is defined in (8.14). Employing (8.17), assumption (iii) in Theorem 8.1 can be replaced by:
(8.18) the vector \[-(p^* + \varepsilon)^T, w^T\] is not proportional to any zero eigenvector of the substitution matrix \((S - I)\).

If there are no distortionary commodity taxes at the initial equilibrium, i.e., \(t^* = 0_N, s^* = 0_M\), and both matrices \(S\) and \(E\) are of maximal rank (= \(N + M - 1\)), (8.18) simplifies to:

\[
(8.19) \text{the initial vector of tariffs } t^* \text{ is nonzero and not proportional to the international prices w}
\]

or

\[
(8.20) \text{the vector of domestic producer prices } p^* \text{ is not proportional to the vector } \varepsilon \text{ defined in (8.14)}.
\]

The equivalence of (8.19) and (8.20) can be derived using a similar calculation as in Lemma 2.5, in Appendix 1. Diewert (1983: p. 289) shows that the vector \((p^* + \varepsilon)\) is the appropriate productive efficiency vector of shadow prices (for domestic goods) for evaluating government projects, when lump sum transfers are available for the policy choosing government. The conditions (8.19) and (8.20) imply that the domestic producer price vector \(p^*\) should be used as the shadow price vector for domestic commodities only if no commodity tax distortions are present in the home country, and the initial tariffs \(t^*\) are zero or proportional
to the international prices $w$. If the initial commodity tax rates $(t^*, s^*)$ differ from zero, the appropriate shadow prices for domestic commodities are $(p^* + e)$.

Using (8.19), Theorem 8.1 may be written in a simplified form which is parallel to Proposition 2.3.

**Proposition 8.2:**

Suppose (i) rank $Y = K \leq N$, (ii) rank $S = \text{rank } \Sigma = N+M-1$, (iii) $(t^*, s^*) = 0_{N+M}$, (iv) the initial vector of tariffs $t^*$ is nonzero and not proportional to the international prices $w$. Then, there exists a constant utility productivity improving change in tariffs $t^*$.

If the consumer substitution matrix $\Sigma$ is a zero $(N + M) \times (N + M)$ matrix, Theorems 8.1 and 2.1 coincide: a constant utility productivity improving change of tariffs is also a productivity improving change of tariffs when domestic goods net supply is kept at its initial equilibrium level. Hence, applying Proposition 2.1, the constant utility productivity improving change in tariffs $t^*$ can be taken to be a proportional reduction of $t^*$, irrespective of the initial domestic commodity tax rates $(t^*, s^*)$.

Consider then the necessary conditions for constant utility productivity optimality of the initial equilibrium. Under these circumstances, no strict constant utility productivity improving changes in tariffs $t^*$ exist.
Theorem 8.2:

A necessary condition for constant utility productivity optimality of the initial equilibrium is:

\[(8.21)\] \[\lambda \in \mathbb{R}^{N+K+1} \text{ such that } \lambda^T[B_p, B_z, -B_{\tau}] = \mathbf{0}^{N+K+M}, \lambda^Tb < 0.\]

Proof:

A necessary condition for a strict constant utility productivity improving change in \(\tau^*\) to not exist is:

\[(8.22)\] \[\text{there do not exist } \Delta p^*, \Delta z^*, \Delta b^*, \Delta \tau^* \text{ such that } (8.1) \text{ is satisfied and } \Delta b^* > 0.\]

Motzkin's Theorem gives the equivalent form (8.21) for (8.22). QED

Corollary 8.2.1:

Suppose (i) rank \(Y = K \leq N\), (ii) rank \((S_{pp} - E_{qq} + YY^T) = N\), (iii) \(\mathbf{w}^T\mathbf{v}^2 B(w) = -\mathbf{0}^T_M\). Then, the initial equilibrium satisfies the necessary condition for constant utility productivity optimality given in Theorem 8.2, and no strict constant utility productivity improving directions of change in tariffs \(\tau^*\) exist.

The proof of Corollary 8.2.1 is analogous to the proof of Corollary 2.1.1.
8.3 Strict Pareto and Productivity Improving Changes in Tariffs and Lump Sum Transfers

In this section, the problem to be considered is that of distributing the gains accruing from a constant utility productivity improving change of tariffs to the consumers. The government in the home country is assumed to have lump sum transfer instruments in its disposal, but the initial equilibrium commodity tax rates \((t^*, s^*)\) are assumed to be kept unchanged. More precisely, the government's policy problem is:

\[
\begin{align*}
\text{(8.23) } & \text{find } \Delta u^*, \Delta p^*, \Delta z^*, \Delta b^*, \Delta t^*, \Delta s^*, \Delta g^*, \text{ such that (4.16) is satisfied and } \Delta u^* \gg 0, \Delta b^* > 0, \Delta t^* = 0_N, \Delta s^* = 0_M.
\end{align*}
\]

Theorem 8.3:

Suppose (i) rank \(Y = K \leq N\), (ii) rank \((S_{pp} - \Sigma_{qq} + YY^T) = N\) and (iii) \(W^T \Sigma_{ww}^{-1} B(w) = 0^T_M\). Then, there exists a strict Pareto and productivity improving change in tariffs \(t^*\) and transfers \(g^*\), without a change in the home country's initial commodity tax structure. Moreover, the change in tariffs and transfers strictly improves the country's initial net balance of trade \(b^*\).
Proof:

Applying Motzkin's Theorem, a sufficient condition for a strict Pareto and productivity improving transfer and tariff change to exist is:

\[(8.27) \quad \text{there is no vector } \lambda \in \mathbb{R}^{H+N+K+1} \text{ such that } \lambda^T[B_p, B_z, B_g] = 0^T_{N+K+H}, \lambda^T[A, -B_b] > 0^T_{H+1}, \lambda^TB_T = 0^T_M, \text{ where the matrices } A \text{ and } [B_p, B_z, B_g, B_b] \text{ are those defined in (4.16)}.\]

Consider the equations \(\lambda^T[B_p, B_z, B_g] = 0^T_{N+K+H}.\) The equations \(\lambda^TB_g = 0^T_H\) imply that the first \(H\) components of the vector \(\lambda\) must be zero, i.e., \(\lambda_1 = 0_H.\) Then, using Lemma 8.2, the other components of the vector \(\lambda\) can be solved. It follows that \(\lambda^T = k[0^T_H, (p^* + \varepsilon)^T, \theta^T, 1], k \in \mathbb{R}, \) where the vectors \(\varepsilon\) and \(\theta\) are defined in (8.14).

The inequality \(\lambda^TB_b > 0\) implies \(k > 0.\) But if \(k = 0,\) then \(\lambda^TA = 0^T_H,\) which contradicts the assumption that \(\lambda^TA > 0^T_H.\) Hence, \(k > 0\) and it may be set to one. Consider now the equations \(\lambda^TB_T = 0^T_M.\) Using Lemma 8.3 and the fact that \(\lambda_1 = 0_H,\) it can be seen that \(\lambda^TB_T = w^TV^2_{ww}B(w).\) (The change of the sign in the previous formula is caused by the differing definitions of the matrices \(B_T\) in Lemma 8.3 and (4.16).) By assumption \(w^TV^2_{ww}B(w) \neq 0^T_M.\) QED
Theorem 8.3 proves that a constant utility productivity improving change of tariffs can be converted to a strict Pareto improving change of tariffs and lump sum transfers, without changing the home country's initial commodity tax structure. The assumptions needed for the result to be established are exactly the same as the conditions for a constant utility productivity improving tariff change to exist. In particular, no homogeneity assumption on consumer preferences is present in Theorem 8.1. This is because a Diamond-Mirrlees commodity always exists in an economy where household specific transfers are admissible.\textsuperscript{14}

8.4 **Necessary Conditions for Pareto Optimality: Nonexistence of Strict Pareto and Productivity Improving Tariff and Transfer Changes**

Consider first the most general necessary conditions for Pareto and productivity optimality of the initial equilibrium when the government is assumed to use lump sum transfers and trade tariffs as its policy instruments.

**Theorem 8.4:**

A necessary condition for strict Pareto and productivity improving tariff and transfer changes to not exist, i.e., a necessary condition for Pareto and productivity optimality of the initial equilibrium is the following:
(8.25) there exists a vector $\lambda \in \mathbb{R}^{H+N+K+1}$ such that $\lambda^T [A, -B_b] > 0^T_{H+1}$, 
$$
\lambda^T [B_p, B_z, B_g] = 0^T_{N+K+H} \text{ and } \lambda^T_T = 0^T_M.
$$

**Proof:**

If the initial equilibrium is a Pareto and productivity optimum, there must not exist $\Delta u^*, \Delta p^*, \Delta z^*, \Delta b^*, \Delta \tau^*, \Delta g^*$ such that (4.16) is satisfied and $\Delta u^* \gg 0_H$ and $\Delta b^* > 0$. Then (8.25) follows by using Motzkin's Theorem as in the proof of Theorem 4.2. QED

Although Theorem 8.4 establishes conditions under which strict differential Pareto and productivity improvements are not possible, the result is rather difficult to interpret. It is helpful to develop some examples of situations where (8.25) is satisfied.

Consider first $\beta$-optima with respect to tariffs and transfers. An equilibrium is said to be **$\beta$-optimal with respect to tariffs $\tau$ and transfers $g$**, if the vectors $\tau$ and $g$ solve the nonlinear programming problem

(8.26) $\max \quad [\beta^T u: (4.10) - (4.13) \text{ hold, } t = \text{ constant, }$ 
\[ u, p, z, b, g, \tau \]
\[ s = \text{ constant}. \]

In (8.26), the vectors $g$ and $\tau$ are chosen so as to maximize the social welfare function $W(u) = \beta^T u$, $\beta > 0_H$, with respect to the constraints
of the general equilibrium model (4.10) - (4.13). The commodity tax rates $t$ and $s$ are assumed to be kept at their previously set levels.

Suppose now that the initial equilibrium of the economy is a $\beta$-optimum with respect to the transfers $g^*$ and tariffs $\tau^*$. Then necessarily,

\begin{equation}
\text{(8.27) there must not exist } \Delta u^*, \Delta \rho^*, \Delta z^*, \Delta b^*, \Delta g^*, \Delta \tau^* \text{ such that (4.16) is satisfied and } \beta^T \Delta u^* > 0, \Delta b^* > 0, \Delta \tau^* = 0_N \text{ and } \\
\Delta s^* = 0_M^*.
\end{equation}

Motzkin's Theorem yields an equivalent form for (8.27):

\begin{equation}
\text{(8.28) there must exist a vector } \lambda \in \mathbb{R}^{H+N+K+1} \text{ such that } \\
\lambda^T A = \beta^T, -\lambda^T B > 0, \lambda^T [B_p, B_z, B_g, B_\tau] = 0^T_{N+K+H+M}.
\end{equation}

Comparing (8.28) and (8.25), it can be inferred that if the initial equilibrium of the economy is a Pareto and productivity optimum, it must also be a welfare maximum with respect to the social welfare function $W(u) = \beta^T u$, where the welfare weights $\beta$ are equal to $(\lambda^T A)$ and the vector $\lambda$ is defined in (8.28). This result corresponds to Proposition 4.1.

Suppose then that only the domestic lump sum transfers $g^*$, but not the tariffs $\tau^*$, can be chosen to maximize social welfare.
Theorem 8.5:

Let the initial equilibrium be a $\beta$-optimum with respect to the transfers $g^*$, and suppose that assumptions (2.12) and (8.2) are satisfied. Then, if $V_{\tau}^*(w + \tau^*) = w^T V_{ww}^2 B(w) = 0^T_M$, no strict Pareto and productivity improving directions of change in tariffs $\tau^*$ and transfers $g^*$ exist.

Proof:

$\beta$-optimality of the initial equilibrium implies that there is a vector $\lambda \in R^{H+N+K+1}$ such that $\lambda^T A = B^T(\beta^T O^T_H), -\lambda^T b > 0, \lambda^T [b_p, b_z, b_g]^T = 0^T_{N+K+H}$. It was shown in the proof of Theorem 8.3 that for a vector $\lambda$, which solves $\lambda^T [b_p, b_z, b_g] = 0^T_{N+K+H}$, the vector $\lambda^T b_{\tau}$ equals $w^T V_{ww}^2 B(w)$. Since, by assumption, $w^T V_{ww}^2 B(w) = 0^T_M$, it follows that $\lambda^T b_{\tau} = 0^T_M$ and (8.25) is satisfied. QED

When is the gradient of the net balance of trade function $b^*(w + \tau^*)$ zero? Using (8.17), the equation $V_{\tau}^*(w + \tau^*) = 0^T_M$ can be replaced by the condition

(8.29) $[-(p^* + \epsilon)^T, w^T] (S - \Sigma) = 0^T_M$.

It follows that $V_{\tau} b^*(w + \tau^*)$ equals zero at least if both aggregate substitution matrices $S$ and $\Sigma$ are zero $(N + M) \times (N + M)$ - matrices, i.e., there is no substitution in consumption or production
at the initial equilibrium. Under these circumstances, no differential change of relative prices in the home country can change the economy's equilibrium vectors of net supply and net demand, and hence no strict Pareto and productivity improvements through (differential) changes in the initial tariffs are possible.\textsuperscript{16}

Using (8.29), it can be seen that \( V_T^* (\omega + \tau^*) \) is zero also if the vector \([- (p^* + \varepsilon)^T, w^T]\) is a zero eigenvector of the matrix \((S - E)\). In general, it is hard to infer when this condition might be satisfied because the zero eigenvectors of the matrix \((S - E)\) are unknown, unless the initial tax rates \((t^*, s^*)\) are zero. Yet, at least one interesting policy related result can be derived. Suppose that, at the initial equilibrium, international trade is free, i.e., \( \tau^* = 0_M \). Using (8.14), it can be inferred that, if either one of the consumer substitution matrices \( \Sigma_{qq} \) and \( \Sigma_{qv} \) differs from zero and \((t^*^T, s^*^T) \neq 0^T_{N + M}\), the vector \( \varepsilon \) must be nonzero. This implies that, at least when the matrices \( S \) and \( E \) are of maximal rank, the vector \([- (p^* + \varepsilon)^T, w^T]\) cannot be a zero eigenvector of the matrix \((S - E)\). Thus, at the initial equilibrium, \( V_T^* (\omega + \tau^*) \neq 0_M \), and some strict Pareto and productivity improving tariff changes exist (assuming that the other conditions of Theorem 8.3 are satisfied). In other words, for a small country, zero tariffs (free trade) are not Pareto and productivity optimal if there are distortionary (nonoptimal) commodity taxes in the economy and the producer and consumer substitution matrices are of maximal rank \((= N + M - 1)^{17} \), even if lump sum transfers could be
employed as government policy instruments. This result can also be regarded as an efficiency of production result: total production efficiency with respect to the technology $\bar{T} + \Sigma T_k$ defined in (4.23) is not desirable in a small country, if there are nonoptimal commodity taxes in the system and the matrices $S$ and $\Sigma$ are of maximal rank, even if lump sum transfers could be chosen optimally to maximize social welfare.

If, however, the initial commodity tax rates $(t^*, s^*)$ are zero, using (8.19), free trade can be said to be Pareto and productivity optimal for a small country, if the lump sum transfers $g^*$ are chosen to maximize social welfare and the substitution matrices $S$ and $\Sigma$ are of maximal rank. Under these circumstances, total production efficiency with respect to the technology $\bar{T} + \Sigma T_k$ is desirable, and the appropriate shadow prices for tradeables for cost benefit analysis are the international prices $w$.

There is also a case, where total production efficiency is desirable in a small country, even though the commodity tax structure in the home country arbitrary. Consider an initial equilibrium where $\Sigma = 0_{(N+M) \times (N+M)}$ i.e., there is no substitution in consumption. Then, the matrix $\nabla^2 \tilde{B}(w)$ coincides with the matrix $\nabla^2 G(w + \tau^*, y^*)$ defined in (2.19). It follows that the vector of producer prices $(p^*, w + \tau^*)$ must be a zero eigenvector of the matrix $\nabla^2 \tilde{B}(w)$. Hence, if $\tau^* = 0_M$, ...
or if \( \tau^* \) is proportional to the international prices \( w \), the gradient
\[
\nabla_b^* (w + \tau^*) \quad \text{(which equals} \quad w^T \nabla_{ww}^2 \bar{B}(w)) \quad \text{must be a zero M-vector,}
\]
irrespective of the initial commodity tax rates \((\tau^*, s^*)\).

The discussion above is illustrated in Figure 5. Figure 5 is drawn assuming that there are two tradeable commodities in the economy; the curve PP' denotes the production possibility frontier for these goods keeping net supplies of other commodities constant. Consider first the Fig. 5a). At the initial equilibrium, production takes place at A and the one consumer in the economy attains utility level \( u^A \) at consumer prices \((w + \tau^* + s^*)\). A strict Pareto and productivity improving change of tariffs and transfers moves the economy's production choice toward the point B, which is the profit maximizing output choice if international trade is free. The consumer is shifted toward the point C, which corresponds to the consumer's first best optimum. Yet, since the commodity tax rates \( s^* \) cannot be perturbed, the first best equilibrium at B and C cannot be reached. This is because the consumer prices \((w + s^*)\) that would be observed if tariffs were reduced to zero, do not generally support the first best indifference curve \( u^C \) at C. Hence, total production efficiency cannot be Pareto and production optimal for the country. Instead, there exists some tariff vector, \( \tau \) denoted by \( \tau \) in Fig. 5a), which is Diamond-Mirrlees optimal given the existing tax distortions \( s^* \). The corresponding second-best equilibrium is denoted by D in Fig. 5a). The consumer faces the prices \((w + s^* + \tau)\) at D. The utility level attained by the consumer at these
Figure 5 - Strict Pareto and Productivity Improvements in Tariffs and Transfers.
prices is at least as high as \( u^A \) but generally less than the first
best level \( u^C \).

In Fig. 5b), the case where \( t^* = 0_N \) and \( s^* = 0_M \) is depicted.
Starting from the initial equilibrium at A, assuming that both matrices
\( S \) and \( \Sigma \) are of maximal rank, it is possible to adjust the initial
tariffs and transfers so that the first best equilibrium at B and C
(total production efficiency) is attained.

In Fig. 5c), the consumer has L-shaped indifference curves, i.e.,
\( \Sigma = 0_{(N+2) \times (N+2)} \). Also under these circumstances, the first equilibrium
at B and C can be reached. This is because, at C, the consumer prices
\( (w + s^*) \) are a support vector of the indifference curve \( u^C \).
9. MORE ON GAINS FROM TRADE

Assuming that the initial vector of tariffs $\tau^*$ equals the international trade prohibitive tariff vector $(w^a - w)$ defined in (5.1), Theorem 8.3 may be applied to show the existence of strict gains from trade when the government uses lump sum transfers as income redistribution instruments. This result should be compared to the discussion in Chapter 4, where it was established that also commodity taxes alone (without lump sum compensation) can be used to distribute the productivity gains accruing from the participation in international trade. What is the connection between these two propositions? Do strict gains from trade always exist when only commodity taxation is admissible, if the conditions implying the existence of strict gains under lump sum compensation are satisfied? Or vice versa: do strict gains from trade under lump sum compensation always exist, if the sufficient conditions implying the existence of strict gains under commodity taxation are satisfied?

Kemp and Wan in their mimeo analyze these questions. They claim that sufficient conditions for strict gains from trade to exist whether or not household specific transfers are admissible are:

(9.1) i. each tradeable good is produced in the home country,

   ii. the production possibility surface of the home country is smooth and strictly concave,

   iii. all consumers in the economy have locally unsatiated preferences in autarky.
The goal of this chapter is to generalize and strengthen this Kemp and Wan result.

Before the generalized proposition can be established, it is necessary to understand why strict gains from trade may not exist if lump sum compensation is not possible (Kemp and Wan provide an example of a situation where this is the case). It can also be shown that, under certain circumstances, strict gains from trade do not exist, unless commodity taxation is a feasible government policy option (an example is offered later in this section). Let us first consider the Kemp-Wan example in more detail.

Kemp and Wan assume that there are two consumers in the home country. The consumers inelastically supply two primary factors. There are also two producers manufacturing two tradeable commodities using fixed coefficients technologies. In the present notation: \( H = 2, \ N = K = 2, \ M = 2, \Sigma_{qq} = 0_{2 \times 2}, \ S = 0_{4 \times 4} \).

Suppose first that lump sum transfers are admissible, but that the commodity tax rates cannot be changed from their initial values \((t^*, s^*)\). Do strict gains from trade exist in these circumstances? To answer the question, Theorem 8.3 is applied. Assumption (i) of the theorem is satisfied, since the producers in the economy supply separate commodities and \( N = K = 2 \). Also assumption (ii) is satisfied, since the matrix \( YY^T \) is positive definite when \( N = K \) and the rank of the matrix \( Y \) equals \( K \). In order to confirm assumption (iii), the matrix \( \bar{w}^2 \bar{B}(w) \) must be calculated. By assumption, \( \Sigma_{qq} = 0_{2 \times 2} \). This implies that \( \Sigma_{qv} = 0_{2 \times 2} \) and it can thus be seen that \( \bar{w}^2 \bar{B}(w) = \Sigma_{vv} \) and \( w^T \Sigma_{vv} \neq 0 \).
Hence, all the assumptions of Theorem 8.3, are satisfied and strict gains from international trade exist.

Consider then the case where only commodity tax rates, but not lump sum transfers, can be perturbed from their initial autarky levels. It is immediately obvious that assumption (iii) of Theorem 4.1 is violated. This is because the producer substitution matrix $S$ is assumed to a zero $(4 \times 4)$-matrix, which implies that $V_{ww}^2 G(w + \tau^*, y^*) = 0_{2 \times 2}$. Hence, according to Proposition 4.2, international trade, caused by a (differential) perturbation of the tariffs $\tau^*$, is not strictly gainful.

How can these differing conclusions be explained? Consider the Figure 6.

Figure 6 represents a two tradeable commodities one consumer economy, where the production possibility frontier for tradeables (keeping domestic goods net supply fixed) is given by the curve PP'. At the autarky equilibrium $A$, the producers face the tradeables prices $(w + \tau^*)$, whereas the consumer prices for tradeable are $(w + \tau^* + s^*)$. At these prices the consumer attains the indifference curve $u^A$.

At the autarky equilibrium point the economy's production possibility frontier for tradeables is kinked. This means that no differential change in the autarky tariffs $\tau^*$ can shift the producers' net supply of tradeables from $A$.

Then, according to Proposition 4.2, no strict increase in the net amount of foreign exchange earned by the production industries is possible. It follows, that the welfare of the
Figure 6 - Existence of Strict Gains from Trade Under Commodity Taxation and Lump Sum Compensation.
consumer cannot be strictly improved by changing the commodity tax rates \( s^* \) (and \( t^* \)).\(^5\) In other words, strict gains from trade under commodity taxation do not exist.

If, however, the initial equilibrium lump sum transfer \( g^* \) for the consumer can be altered, i.e., lump sum transfers are a feasible government policy instrument, the situation of the consumer in Fig. 6 can be improved.\(^6\) Suppose, for example, that the initial tariffs \( t^* \) are perturbed toward the tariffs \( \tau \) depicted in Fig. 6. As a consequence, the producers aggregate net supply of tradeables does not change from its initial level but the consumer is moved along his indifference curve \( u^A \).\(^7\) If, simultaneously with the tariff change, the consumer is given a transfer in the direction of \( \Delta g^* \) in Fig. 6, the consumer is made strictly better off. Hence, international trade, caused by a perturbation in the initial tariffs \( t^* \) is strictly gainful if lump sum compensation is available.

It can be concluded that the Kemp-Wan example is based on the existence of a "substitution gap" between the Theorems 4.1 and 8.3. According to the latter result, substitution in consumption can make a strict Pareto improvement possible even though a strict Pareto and productivity improvement in the sense of Theorem 4.1 does not exist. But the "substitution gap" between Theorems 4.1 and 8.3 can be employed also in another fashion. Suppose that all the conditions of Theorem 4.1 are satisfied; in particular, the producer substitution matrix \( S \) is not a zero \((N + M) \times (N + M) - \) matrix \((S \neq O_{(N+M) \times (N+M)})\), but suppose, in addition, that the consumer substitution matrix \( \Sigma \) is such that
Then, assumption (iii) of Theorem 8.3 is violated and, according to Theorem 8.5, no strict Pareto improving (differential) perturbations of tariffs \( \tau^* \) and transfers \( g^* \) exist. In other words, strict gains from trade under lump sum compensation are not possible although international trade under commodity taxation would be strictly gainful. The substitution effect, which in the Kemp-Wan example gave existence of the strict gains from trade under lump sum compensation, can thus work adversely.

The Kemp-Wan example and the example above demonstrate that neither the existence of strict gains from trade under commodity taxation or the existence of strict gains from trade under lump sum compensation necessarily implies the other. Yet, with the help of Theorems 4.1 and 8.3, a general result can be established.

**Theorem 9.1:**

I. Suppose that the conditions of Theorem 8.3 are satisfied so that strict gains from trade under lump sum compensation exist. Then, if in autarky, (i) rank \((S_{pp} + YY^T) = N\), (ii) there is no solution \( a > 0 \) to \( a^T[X^T, E^T] = 0_{N+H}^T \), and (iii) \( \tau^* v^2_w B(w) = 0^T_M \), strict gains from trade under commodity taxation exist.

II. Suppose that the conditions of Theorem 4.1 are satisfied so that strict gains from trade under commodity taxation exist. Then, if

\[
(9.2) \quad w^T v^2_w B(w) = 0^T_M.
\]
$w^2 w^T \bar{p}(w) \neq 0^T_M$ at the initial autarky equilibrium, strict gains from trade under lump sum compensation exist.

The conditions (i) - (iii) in the first part of Theorem 9.1 are the generalizations of the three Kemp-Wan assumptions (9.1). Kemp and Wan suppose that the consumer preferences are unsatiated around the autarky equilibrium and this assumption is implicitly present also in Theorem 9.1. But the Kemp-Wan requirement that each tradeable good must be produced in autarky in the home country seems to translate to the condition that the same K production sectors that operate in autarky, operate also after international trade has become possible.\textsuperscript{8} The Kemp-Wan assumption that the country's production possibility frontier is globally smooth and strictly concave generalizes to assumption (iii) in Theorem 9.1: kinks and ridges in the production possibility frontier are allowed but, at the autarky equilibrium, the gradient of the net balance of trade function $b^*(w + \tau^*)$ with respect to the trade prohibitive tariffs $\tau^*$ must be nonzero.

Assumption (i) in the part I of Theorem 9.1 is needed when the numbers of production sectors and domestic commodities do not coincide as in the Kemp-Wan case. The role of this assumption is to guarantee local controllability of domestic goods production in the home country in the sense of Definition 2.2. Assumption (ii) in the Part I of Theorem 9.1 ensures that some strict Pareto improving directions of consumer price changes (starting from the autarky equilibrium) exist. This condition is satisfied in the model of Kemp and Wan but they do not
clearly state it as a necessary condition for strict gains from trade when commodity taxations is used to redistribute consumer income.

The second part of Theorem 9.1 emphasizes the fact that strict gains from trade under lump sum compensation need not exist whenever trade under commodity taxation is strictly gainful. It seems, however, that the likelihood of this abnormality is small: for example, the vector $w^T \psi^2 B(w)$ cannot be a zero $M$-vector if both the substitution matrices $S$ and $Z$ are of maximal rank, and the initial equilibrium tariffs $\tau^* (\neq 0_M)$ are not proportional to the international prices $w$.\textsuperscript{9}
10. PROPORTIONAL REDUCTIONS IN DISTORTIONS AND SOME PIECEMEAL POLICY

RESULTS

The goal of the analysis in this chapter is to develop some examples of strict Pareto and productivity improving policy changes when household specific lump sum transfers are an admissible policy instrument. The results found are often similar in nature to those presented in Chapter 7, where lump sum transfers were not allowed, but some differences do occur. First, the existence of arbitrary tax distortions in the economy makes it harder to establish explicit policy recommendations—often to derive a result, it is necessary to assume either that \( t^* = 0_N \), \( s^* = 0_M \) (tax distortions at the initial equilibrium do not exist), or that there is no substitution in consumption between some or all commodities. On the other hand, if lump sum transfer changes are possible, more results, where the changes in the policy instruments are either proportional or shift the policy variables toward uniformity, can be proved.

Let us start by considering a shift toward the international prices \( w \) in the initial equilibrium tariffs \( t^* \)—a policy that was shown to be constant utility productivity improving in Proposition 8.1. If lump sum transfers may be employed, this policy can be converted to a strict Pareto improvement.

**Theorem 10.1:**

Suppose that the assumptions of Theorem 8.3 are satisfied. Then, there exists a strict Pareto and productivity improving change in the
initial equilibrium tariffs and transfers, and the change in tariffs \( \tau^* \) can be chosen to be a movement toward the world price vector \( w \).

**Proof:**

Let \( \Delta \tau^* = rw, \) where \( r > 0 \). A sufficient condition for a strict Pareto and productivity improving change in tariffs \( \tau^* \) and transfers \( g^* \) to exist is:

\[
\text{(10.1) there does not exist a vector } \lambda \in \mathbb{R}^{H+N+K+1} \text{ such that }
\lambda^T[A, -B_b] > 0^T_{H+1}, \lambda^T[B_p, B_z, B_g] = 0^T_{N+K+H} - \lambda^T_B w > 0.
\]

A vector \( \lambda \) satisfying the equations \( \lambda^T[B_p, B_z, B_g] = 0^T_{N+K+H} \) must be of the form \( \lambda^T = k[0^T_H, (p^* + \epsilon)^T, 0^T, 1] \), \( k \in \mathbb{R} \). The inequalities \( \lambda^T[A, -B_b] > 0^T_{H+1} \) imply \( k \geq 0 \). Since \( \lambda^T A > 0^T_H \), \( k \) must be positive. Hence, choose \( k=1 \). Using Lemma 8.3, \( \lambda^T_B w = w^T_{w^2 w^2} B(w) \). (The change in the sign of the vector \( w^T_{w^2 w^2} B(w) \) occurs because the matrices \( B_t \) defined in (8.1) and (4.16) have opposite signs.) By assumption, \( w^T_{w^2 w^2} B(w) \neq 0^T_M \). It follows that, \(-\lambda^T_B w = -w^T_{w^2 w^2} B(w) w < 0 \) since matrix \( w^2 w^2 B(w) \) is positive semidefinite. QED

Theorem 10.1 is a counterpart of Theorem 7.1. According to Theorem 7.1, if the commodity tax rates \((t^*, s^*)\) can be adjusted, a strict Pareto and productivity improving policy is to reduce the initial
(nonnegative) tariffs $\tau^*$. In Theorem 10.1, the change in tariffs $\tau^*$ is not a reduction as such, but it is a reduction in the distortionary gap between the domestic and international producer prices for tradeables. Hence, also it results in a reduction of the home country's trade barrier.

A reduction of the initial equilibrium tariffs $\tau^*$, accompanied by a perturbation of the transfers $g^*$, may be shown to be a strict Pareto and productivity improvement, if either there are no distortionary commodity taxes in the economy, i.e., $t^* = 0_N$ and $s^* = 0_M$, or if the consumer substitution matrix $\Sigma$ is a zero $(N+M) \times (N+M)$-matrix.

Theorem 10.2:

Suppose assumptions (i) - (iii) of Theorem 8.3 are satisfied. Then, if (a) $t^* = 0_N$ and $s^* = 0_M$ or (b) $\Sigma = 0_{(N+M) \times (N+M)}$, there exists a strict Pareto and productivity improving change in tariffs $\tau^*$ and transfers $g^*$. The change in tariffs may be taken to be a proportional reduction.

Proof:

A sufficient condition for the existence of a strict Pareto and productivity improvement in tariffs $\tau^*$ and transfers $g^*$ is (10.1), where the inequalities $-\lambda^T B w > 0$ are replaced by $\lambda^T B \tau^* > 0$.

(a) If $t^* = 0_N$, $s^* = 0_M$, and the vector $\lambda$ is of the form $\lambda^T = [0^T, (p^* + \varepsilon), \theta^T, 1]$, using (4.16) and the homogeneity of the unit profit and expenditure functions, it can be seen that
\[(10.2) \quad \lambda^T B_\tau = \theta^T F^T + \epsilon^T (S_{pw} - \Sigma_{qv}) - \tau^* S_{ww} + (\tau^* + s^*)^T \Sigma_{vv}^*.\]

The vectors \( \epsilon \) and \( \theta \) are defined in (8.14). Using these definitions when \( \tau^* = 0_N \) and \( s^* = 0_M \),

\[(10.3) \quad \lambda^T B_\tau = \tau^* \left[ (-\Sigma_{vq} D_{12} + S_{wp} D_{12} + F D_{22}) F^T + (-\Sigma_{vq} D_{11} + S_{wp} D_{11} + F D_{12}) (S_{wp} - \Sigma_{qv}) - (S_{ww} - \Sigma_{vv}) \right] \]

\[= -\tau^* \left[ -(S_{wp} - \Sigma_{vq}) D_{11} - F D_{12}, I_M \right] (S - \Sigma) \]

\[= -\tau^* \Sigma_{vv}^2 B(w).\]

The last equation above is derived using Lemma 8.1. In order to derive the quadratic form in (10.3), the following properties of the inverse matrix \( \bar{D} \) have been used: \( \bar{D}_{12}^T (S_{pp} - \Sigma_{qq}) \bar{D}_{12} = -\bar{D}_{22}, \bar{D}_{11} \)

\( (S_{pp} - \Sigma_{qq}) \bar{D}_{11} = \bar{D}_{11}, \bar{D}_{11} (S_{pp} - \Sigma_{qq}) \bar{D}_{12} = 0. \)

Since, using (10.3), \( \lambda^T B_\tau = -\tau^* \Sigma_{vv}^2 B(w) \), it follows that \( \lambda^T B_\tau \tau^* = -\tau^* \Sigma_{vv}^2 B(w) \tau^* < 0. \)

(b) If \( \Sigma = 0_{(N+M) \times (N+M)} \), the matrix \( \Sigma_{vv}^2 B(w) \) equals the matrix \( \Sigma_{vv}^2 G(w + \tau^*, y^*) \) for any \( \tau^* \in \mathbb{R}^N \) and \( s^* \in \mathbb{R}^M \). In this case, \( \lambda^T B_\tau \tau^* = -\tau^* \Sigma_{vv}^2 G(w + \tau^*, y^*) \tau^* < 0. \) QED

Let us now assume that although the commodity tax rates in the home country are arbitrary at the initial equilibrium, the government is
able to adjust them in addition to the tariffs $\tau^*$ and transfers $g^*$. It was shown in Theorem 7.2 that, under certain conditions, commodity taxes and tariffs can be simultaneously reduced to produce a strict Pareto and productivity improvement. This result did not require lump sum transfers to be admissible, but if they are, the simultaneous reduction in taxes and tariffs may be chosen to be a proportional reduction.

Theorem 10.3:

Suppose that assumptions (i) - (iii) of Theorem 8.3 are satisfied. Then, there exists a strict Pareto and productivity improving change in $t^*$, $s^*$, $\tau^*$ and $g^*$, and the change in the commodity tax rates and tariffs can be chosen to be a proportional reduction.

Proof:

A condition sufficient to imply the existence of a strict Pareto and productivity improving proportional reduction in commodity taxes and tariffs is:

\[(10.4) \text{ there exists } \Delta u^*, \Delta p^*, \Delta z^*, \Delta b^*, \Delta \tau^*, \Delta s^* \text{ and } \Delta g^* \text{ such that } (4.16) \text{ is satisfied } \Delta u^* \gg 0, \Delta b^* \geq 0, \Delta \tau^* = -r \tau^*, \Delta s^* = -r s^*, r > 0.\]

The Motzkin dual equivalent to (10.4) is:
there is no vector \( \lambda \in \mathbb{R}^{H+N+K+1} \) such that \( \lambda^T [B, B, B] \)

\[
= 0^T_{N+K+H}, \quad \lambda^T [A, -B_b] > 0^T_{H+1}, \quad \lambda^T [B_t, B_s, B_t] \begin{bmatrix} t^* \\ s^* \\ \tau^* \end{bmatrix} > 0.
\]

As shown before, the vector satisfying the equations \( \lambda^T [B, B, B] = 0^T_{N+K+H} \) and \( \lambda^T [A, -B_b] > 0^T_H \) must be of the form \( \lambda^T = [0^T, (p^* + \epsilon)^T, \theta^T, 1] \). For this \( \lambda \), using (4.16),

\[
(10.6) \quad \lambda^T [B_t, B_s, B_t] \begin{bmatrix} t^* \\ s^* \\ \tau^* \end{bmatrix} = \begin{bmatrix} \lambda_2^T, \lambda_3^T, \lambda_4^T \end{bmatrix} \begin{bmatrix} f^T \tau^* \\ \Sigma q \Sigma^* p + \Sigma q v w + S p w \tau^* \\ w T \Sigma q \Sigma^* p + w T \Sigma v v w + w T S w \tau^* \end{bmatrix}
\]

The equations \( \lambda^T B_p = 0^T_N \) and the homogeneity of the unit profit and expenditure functions have been used to derive (10.6). Applying now the definition of the vector \( \lambda \),
(10.7)  
\[ \lambda^T [B_l, B_s, B_r] \begin{bmatrix} t^* \\ s^* \\ \tau^* \end{bmatrix} = (p^* + \varepsilon)^T S_{pp} p^* + (p^* + \varepsilon)^T \Sigma_{qvw} \]
\[ + (p^* + \varepsilon)^T S_{pw} \tau^* + 6T_{pT} (p^* + \varepsilon) \]
\[ + w^T S_{wp} p^* + w^T \Sigma_{vvw} + w^T S_{ww} \tau^*, \]

where also the equation \( \tau^{*T} F = \varepsilon^{TY} \) derived from (B5) in Appendix 2 has been employed. Applying the equations (B5) in Appendix 2,

(10.8)  
\[ \lambda^T [B_l, B_s, B_r] \begin{bmatrix} t^* \\ s^* \\ \tau^* \end{bmatrix} = (p^* + \varepsilon)^T S_{pp} p^* + (p^* + \varepsilon)^T \Sigma_{qvw} \]
\[ + (p^* + \varepsilon)^T S_{pw} \tau^* + (p^* + \varepsilon)^T (\Sigma_{qq} - S_{pp}) \]
\[ (p^* + \varepsilon) + \tau^{*T} S_{wp} (p^* + \varepsilon) + p^{*T} S_{pp} (p^* + \varepsilon) \]
\[ + w^T \Sigma_{vvq} (p^* + \varepsilon) + w^T S_{wp} p^* + w^T \Sigma_{vwv} \]
\[ + w^T S_{ww} \tau^* \]
\[ = [(p^* + \varepsilon)^T, w^T] (\Sigma - S) \begin{bmatrix} p^* + \varepsilon \\ w \end{bmatrix} \]
\[ = -w^T \nabla^2 B(w) w, \]
using Lemma 8.1. Since, by assumption \( w^T \nabla^2_{ww} \tilde{B}(w) \neq 0 \), the scalar 
\(-w^T \nabla^2_{ww} \tilde{B}(w)w\) must be negative. QED

Theorem 10.3 implies that, if the assumptions of the theorem are satisfied, starting from an initial equilibrium with arbitrary commodity tax and tariff structures, there exists a path of tax and tariff reductions which, in the limit, leads to a nondistortionary government financed by lump sum taxation. If the initial equilibrium is an autarky equilibrium, the final equilibrium with \( t^* = 0 \), \( s^* = 0 \), and \( \tau^* = 0 \) coincides with the first best equilibrium depicted in Fig. 3 by the points C and D. In other words, if the assumptions of Theorem 10.3 are satisfied, the domestic economy can be shifted to total production efficiency with respect to the technology \( T + 1 \eta T_k \) defined in (4.23) by proportionally reducing the commodity tax and tariff rates \( t^* \), \( s^* \), and \( \tau^* \).

Proportional reductions in the initial commodity tax rates \( t^* \) and/or \( s^* \) without changes in the tariffs \( \tau^* \) will also usually be strict Pareto and productivity improving. To develop precise results, a new Hessian matrix has to be defined. Consider problem (8.6) which defines the constant utility tax adjusted balance of trade function \( B(w, w + \tau^*, w + \tau^* + s^*, t^*) \). The first order conditions for (8.6) are the zero profit and supply equals demand equations (2.9) and (4.12). These equations are assumed to be satisfied at the initial equilibrium
of the economy and they can thus be used to define the equilibrium domestic price and industry scale vectors $p^*$ and $z^*$ as implicit functions of the variables $w$, $x^*$, $t^*$ and $s^*$. Differentiating the equations (2.9) and (4.12) at the initial equilibrium:

\[
(10.9) \begin{bmatrix}
S_{pp} - \sum_{qq} Y^T \\
Y^T 0_{KxK}
\end{bmatrix} \begin{bmatrix}
\Delta p^* \\
\Delta z^*
\end{bmatrix} = \begin{bmatrix}
\sum_{qv} - S_{pw} \\
-S_T^T
\end{bmatrix} \Delta t^* + \begin{bmatrix}
\sum_{qq} \\
0_{KxN}
\end{bmatrix} \Delta t^* + \begin{bmatrix}
\sum_{qv} \\
0_{KxM}
\end{bmatrix} \Delta s^*.
\]

Equations (10.9) determine the derivatives of the implicit functions $p^*(w + x^*, t^*, s^*)$ and $z^*(w + x^*, t^*, s^*)$ with respect to the exogenous commodity tax rates $(t^*, s^*)$ and the tariffs $t^*$ at the initial equilibrium.

Differentiate the function $B(w, w + x^*, w + t^* + s^*, t^*)$ with respect to $x^*$, $t^*$ and $s^*$. Using (8.6),

\[
(10.10) \nabla_{tt} B = \sum_{k=1}^K \nabla_{tt} k(p^*, w + x^*) z^k - \sum_{h=1}^H \nabla_{tt} h(u^*, q^*, v^*),
\]

\[
(10.11) \nabla_{tt} B = - \sum_{h=1}^H \nabla_{tt} h(u^*, q^*, v^*),
\]
The second order partial derivatives of the function $B$ with respect to the tariffs $t^*$ and tax rates $(t^*, s^*)$ at the initial equilibrium can be calculated by differentiating equations (10.10) - (10.12) and by using (10.9):

\[(10.13) \nabla B = \begin{bmatrix}
\nabla^2_{tt} B & \nabla^2_{tt} B & \nabla^2_{ts} B \\
\nabla^2_{tt} B & \nabla^2_{tt} B & \nabla^2_{ts} B \\
\nabla^2_{st} B & \nabla^2_{st} B & \nabla^2_{ss} B
\end{bmatrix}
= \begin{bmatrix}
S_{ww} - \Sigma_{vv}, -\Sigma_{vq}, -\Sigma_{vv} \\
-\Sigma_{qv}, -\Sigma_{qq}, -\Sigma_{qv} \\
-\Sigma_{vv}, -\Sigma_{vq}, -\Sigma_{vv}
\end{bmatrix}
\]

\[= \begin{bmatrix}
S_{wp} - \Sigma_{vq}, F \\
-\Sigma_{qq}, \Omega_{NxK} \\
-\Sigma_{vq}, \Omega_{MxK}
\end{bmatrix}
\begin{bmatrix}
S_{wp} - \Sigma_{vq}, F \\
-\Sigma_{qq}, \Omega_{NxK} \\
-\Sigma_{vq}, \Omega_{MxK}
\end{bmatrix}^T.
\]

The inverse matrix $D$ in (10.13) is defined in (8.5). Since $B$ is a convex function of its arguments, the matrix $\nabla^2 B$ is positive semidefinite.

Consider now, for example, a proportional reduction in the tradeables taxes $s^*$. One of the sufficient conditions for this policy perturbation to be strict Pareto and productivity improving is that the vector $s^T \nabla^2_{ss} B$ is nonzero, i.e., $s^T \nabla^2_{ss} B \neq 0^T$: the initial vector of
tradeables taxes $s^*$ must not be a zero eigenvector of the matrix $V^2_{ss}B$.

**Theorem 10.4:**

Suppose that (i) rank $Y = K \leq N$ and (ii) rank $(S_{pp} - E_{qq} + Yr^T) = N$ at the initial equilibrium. Then,

(I) If $\tau^* = \mathbf{0}_N$ and $w^T V^2_{ww}B(w) \neq \mathbf{0}_M^T$, there exists a strict Pareto and productivity improving change in the initial commodity taxes $(t^*, s^*)$ and transfers $g^*$. The change in $(t^*, s^*)$ may be chosen to be a proportional reduction.

(II) If $t^* = \mathbf{0}_N$, $\tau^* = \mathbf{0}_M$ and $s^{*T} V^2_{ss}B \neq \mathbf{0}_M^T$ or if $\Sigma_{qv} = \mathbf{0}_{NxM}$ and $s^{*T} V^2_{ss}B \neq \mathbf{0}_M^T$, there exists a strict Pareto and productivity improving change in the initial tax rates $s^*$ and transfers $g^*$. The change in the tradeables taxes $s^*$ may be taken to be a proportional reduction.

(III) If $t^* = \mathbf{0}_M$, $\tau^* = \mathbf{0}_M$ and $t^{*T} V^2_{tt}B \neq \mathbf{0}_N^T$ or $\Sigma_{qv} = \mathbf{0}_{NxM}$ and $t^{*T} V^2_{tt}B = \mathbf{0}_N^T$, there exists a strict Pareto and productivity improving change in the initial tax rates $t^*$ and transfers $g^*$. The change in $t^*$ may be chosen to be a proportional reduction.

The proof of Theorem 10.4 is rather tedious and hence deferred to Appendix 2. The sufficient conditions for proportional reductions in the commodity tax rates $t^*$ and/or $s^*$ to be strict Pareto and productivity improving given in Theorem 10.4 are restrictive: although not much is required concerning the substitution matrices $S$ and $E$ (the
vectors $w^T V^2 B(w)$, $s^T V^2 B$ and $t^T V^2 B$ depend on the matrices $S$ and $\Sigma$), international trade is assumed to be free, i.e., $\tau^* = 0_M$, and some of the tax rates $(t^*, s^*)$ are also assumed to be initially zero. If these conditions are not satisfied, a proportional reduction of $t^*$ and/or $s^*$, even if accompanied with a change in household specific transfers, may cause a welfare and productivity loss.

Theorem 10.5:

Suppose (i) rank $Y = K \leq N$, (ii) rank $(S_{pp} - \Sigma_{qq} + YY^T) = N$, and (iii) the initial equilibrium tariffs $\tau^*$ are such that

$$\nabla_{\tau} b^*(w + \tau^*) (w + \tau^*) < 0.$$ 

Then, there exists a strict Pareto and productivity deproving change of commodity tax rates $(t^*, s^*)$ and lump sum transfers $g^*$, where the change in $(t^*, s^*)$ is a proportional reduction.

Proof:

A sufficient condition for a strict Pareto and productivity deprovement in $(t^*, s^*)$ and $g^*$ to exist, when the change in $(t^*, s^*)$ is a proportional reduction, is:

$$\text{(10.15) there does not exist a vector } \lambda \in \mathbb{R}^{H+N+K+1} \text{ such that}$$
\[
\lambda^T[A, -B_b] < 0^T_{H+1}, \lambda^T[B_p, B_z, B_g] = 0^T_{N+K+H}.
\]

\[
\lambda^T[B_L, B_S]\begin{bmatrix}
t^* \\
s^*
\end{bmatrix} > 0.
\]

A vector satisfying the equations \( \lambda^T[B_p, B_z, B_g] = 0^T_{N+K+H} \) must be of the form \( \lambda^T = k[0^T_H, (p^* + \varepsilon)^T, \theta^T, 1], k \in \mathbb{R} \). The inequalities \( \lambda^T[A, -B_b] < 0^T_H \) imply \( k < 0 \); choose \( k = -1 \). Then, using the proof of Theorem 10.4,

\[
(10.16) \quad \lambda^T[B_L, B_S]\begin{bmatrix}
t^* \\
s^*
\end{bmatrix} = w^T \sigma^2 B(w)(w + \tau^*).
\]

Using (8.10) and (10.14),

\[
(10.17) \quad \lambda^T[B_L, B_S]\begin{bmatrix}
t^* \\
s^*
\end{bmatrix} = \nabla \tau^* (w + \tau^*)(w + \tau^*) < 0. \quad \text{QED}
\]

Theorem 10.5 shows that, under very reasonable conditions, if the initial tariff vector \( \tau^* \) is arbitrary and nonzero, a proportional reduction of the commodity tax rates \( (t^*, s^*) \) may lead to a strict Pareto and productivity deprovement inspite of the fact that lump sum transfers can be used to redistribute consumer income: it is only
required that the producer price weighted net balance of trade derivatives \( \nabla_{m} b^{*}(w + \tau^{*}), m=1, \ldots, M \), sum to a negative number.\(^1\)

If the strong conditions of Theorem 10.4 are slightly modified, it can be shown that movements toward uniformity (at a lower level of taxation) in positive commodity taxes are strict Pareto and productivity improving.

**Proposition 10.1:**

Suppose that Assumptions of Theorem 10.4 are satisfied, except that the conditions on the matrices \( \nabla_{ww} B(w), \nabla_{ss} B \) and \( \nabla_{tt} B \) are replaced by: \( w^{T} \nabla_{ww} B(w) < 0 \), \( s^{T} \nabla_{ss} B < 0 \), \( t^{T} \nabla_{tt} B < 0 \). Then, there exist strict Pareto and productivity improving changes in (I) \((t^{*}, s^{*}) \gg 0_{N+M} \) and \( g^{*} \), (II) \( s^{*} \gg 0_{M} \) and \( g^{*} \), and (III) \( t^{*} \gg 0_{N} \) and \( g^{*} \), where the perturbations in the commodity tax variables are chosen to be movements toward a lower uniform level of taxation.\(^2\)

The proof of Proposition 10.1 is given in Appendix 2.

The sufficient conditions for strict Pareto and productivity improvements to exist given in Proposition 10.1 are closely related to a well known result of Hatta (1977a). Hatta showed that, under certain conditions, the reduction of the highest tax rate (on a domestic good) in a one consumer closed economy is welfare improving. In Proposition 10.1, a similar kind of policy recommendation result is achieved in an open economy without restricting the numbers of consumers, producers and
commodities (except that \( K \leq N \)). The assumptions about the matrices \( \mathbf{V}_{tt}^2 \mathbf{B} \) and \( \mathbf{V}_{ss}^2 \mathbf{B} \) in Proposition 10.1 are generalizations of the Hatta assumption that the good with the highest (domestic) tax rate must be substitutable for all other (domestic) commodities in the economy. This can be seen as follows: suppose, for example, that \( t^* \gg 0 \). Then, using (10.13), the condition \( t^* \mathbf{V}_{tt}^2 \mathbf{B} \mathbf{O}_N^T \) is equivalent to

\[
T^T \left( \mathbf{E} \right) > 0
\]

For the latter vector to be positive, the off-diagonal terms in the matrix \( \mathbf{E} \) must be sufficiently positive, i.e., in a sense, substitution in consumption of the domestic commodities must dominate.

Yet, the connection between Proposition 10.1 and the above mentioned Hatta result is not straightforward since Hatta used ad valorem (not specific) commodity tax rates in his model. Furthermore, one of the assumptions used in his theorem was the Hatta normality condition: the sum of the consumer demand income derivatives weighted by the corresponding producer prices must be positive. \(^3\) No such supposition was needed in Proposition 10.1 or in the policy recommendation results of the previous chapters. In order to resolve the differences between Proposition 10.1 and the results of Hatta, the model employed by Hatta is examined in detail.

Let us first write the Hatta model in the form (4.10) - (4.13). The home country is assumed to be closed with \( N+1 \) domestic commodities. There are \( N \) producers, each one of whom supplies one of the \( N \) commodities, indexed by \( n \in \{1, \ldots, N\} \). The \( [N+1]^{th} \) domestic good is a fixed resource which is used as an input to produce the other domestic commodities. The fixed resource serves as the numeraire commodity in
the model with a price denoted by \( w = 1 \) (hence, the fixed factor is regarded as a "tradeable" good). The government imposes ad valorem taxes \( t \in \mathbb{R}^N \), on the \( N \) domestic commodities and gives lump sum transfers \( g \in \mathbb{R} \) to the single consumer in the country. The initial equilibrium of the Hatta economy is characterized by the equations:

(10.18) \[ m(u^*, 1, q^*) = g^* , \]

(10.19) \[ \pi^*(1, p^*) = p_n^* - a_n = 0, \ n=1,\ldots,N, \]

(10.20) \[ \nabla q^* m(u^*, 1, q^*) = y^* , \]

(10.21) \[ \sum_{n=1}^{N} \nabla \pi_n^*(1, p^*) y^* = \sum_{n=1}^{N} a_n y_n^* = r = \nabla w m(u^*, 1, q^*) , \]

(10.22) \[ q^* = [1 + t^*] p^* . \]

In (10.22), the matrix \([1 + t^*]\) is an \((N \times N)\)-matrix with diagonal elements equal to \((1 + t_n^*)\), \(n=1,\ldots,N\). The other elements of the matrix \([1 + t^*]\) are zero.

According to (10.18), the consumer's expenditures on the \( N \) domestic commodities minus his revenue from selling the fixed factor
equal his lump sum income $g^*$. The producers' unit profit functions 
are of the form (10.19), where the number $a_n$ is the $n$th sector's 
input coefficient for producing one unit of the $n$th output using the 
fixed factor as an input. The equations (10.20) - (10.21) are market 
clearing conditions: consumer demand for the $N$ commodities equals their 
supply $y^*$, and the producer demand for the factor equals its fixed 
supply $r$. The industry output levels $y^*$ serve as the scale variables 
for the $N$ producers. This choice of the production scale is permissible 
because each production sector supplies only one output.

The $1+N+N+1$ equations (10.18) - (10.21) determine endogenously 
the equilibrium utility level of the consumer $u^*$, the equilibrium 
prices $p^*$, the equilibrium output levels $y^*$ and one of the policy 
instruments, given the exogenous variables $a^T = (a_1, \ldots, a_N), r, t^*$ and 
g$. Differentiation of (10.18) - (10.21) at the initial equilibrium 
(which solves (10.18) - (10.21)), yields:

\begin{align}
(10.23) \quad A \Delta u^* &= B_p \Delta p^* + B_y \Delta y^* + B_t \Delta t^* + B_g \Delta g^* \\
\end{align}

where $A = 
\begin{bmatrix}
1 \\
0_N \\
\Sigma_{qu} \\
0 \\
\end{bmatrix}, 
B_p = 
\begin{bmatrix}
-X^T[1 + t^*] \\
I_N \\
-\Sigma_{qq}[1 + t^*] \\
0_N \\
\end{bmatrix}, 
B_y = 
\begin{bmatrix}
0^T_N \\
0_{NxN} \\
I_N \\
-a^T \\
\end{bmatrix}$,
The normality condition given in Hatta (1977a) is $a^T \Sigma_{qu} > 0$. Using the equation (10.19), this condition can be written in the form

\[(10.24) \quad p^T \Sigma_{qu} > 0.\]

It is easy to see that (10.24) is satisfied, if all commodities $n$, $n=1, \ldots, N$, are normal, i.e., if all elements of the vector $\Sigma_{qu}$ are positive.

The following theorem shows the connection between the Hatta normality assumption and the results of the previous chapters.

**Theorem 10.6:**

If the initial equilibrium in the economy that satisfies the equations (10.18) - (10.22) is a $\beta$-optimum with respect to the initial transfer $g^*$, the Hatta normality condition (10.24) is satisfied.
Proof:

Suppose the initial equilibrium is a $\beta$-optimum with respect to the transfer $g^*$. Then, there exists a vector $\lambda^T = [\lambda_{1}^{T}, \lambda_{2}^{T}, \lambda_{3}^{T}, \lambda_{4}^{T}] \in \mathbb{R}^{1+N+N+1}$ such that $\lambda^T A = \beta = 1 > 0$, $\lambda^T [B_p, B_y, B_g] = 0^T_{N+N+1}$. (Since $H = 1$, $\beta$ can be set to equal 1.) The equations $\lambda^T g = 0$ imply that $\lambda_1 = 0$. Hence,

\[
(10.25) \quad [\lambda_{2}^{T}, \lambda_{3}^{T}, \lambda_{4}^{T}] \begin{bmatrix}
I_N & O_{N \times N} \\
-\Sigma_{qq}[1 + t^*] & I_N \\
O_N^T & -a^T
\end{bmatrix} = 0^T_{N+N}.
\]

It follows that $\lambda_3^T = \lambda_4 a^T$ and $\lambda_2^T - \lambda_3^T \Sigma_{qq}[1 + t^*] = 0^T_N$. Set $\lambda_4 = k$, $k \in \mathbb{R}$. Then, $\lambda_3^T = ka^T$, $\lambda_2^T = ka^T \Sigma_{qq}[1 + t^*]$. The equality $\lambda^T A = 1$ becomes $ka^T \Sigma_{qq} = 1$. Hence, $k \neq 0$. In addition, it can be inferred that, in fact, $k > 0$. (This can be seen by replacing the equality (10.21) with a weak inequality ($\leq$). Although, at the initial equilibrium, (10.21) is satisfied as an equality, allowing the possibility of free disposal for the fixed factor $r$ imposes a nonnegativity constraint on the last component of the vector $\lambda$ when the Kuhn-Tucker conditions $\lambda^T A = 1$, $\lambda [B_p, B_y, B_g] = 0^T_{N+N+1}$, $\lambda \in \mathbb{R}^{1+N+N+1}$, are derived for the problem $\max \{u: (10.18) - (10.21) \text{ hold}\}$). Setting $k = 1$, $\lambda^T A = 1 (> 0)$ is equivalent to $a^T \Sigma_{qq} = 1 > 0$. 

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Thus, using (10.19), the Hatta normality condition (10.24) is satisfied. QED

Why is it that normality of goods in demand and $\theta$-optimality of the initial equilibrium are connected? In order to answer the question, consider Figure 7 presented by Hatta (1977a).

There are two goods, $x_1$ and $x_2$, in the economy. The line $PP'$ defines the country's production possibility frontier for $x_1$ and $x_2$ given the amount of the fixed factor $r$. The initial equilibrium for the economy is at $x^0$ where the consumer attains the utility level $u^0$. At $x^0$, the good $x_2$ is taxed at a higher level than the good $x_1$. Suppose the tax rate for $x_2$ is reduced and as a consequence, the equilibrium shifts to $x^1$. The change from $x^0$ to $x^1$ may be decomposed to two parts: first, the consumer is moved along his original indifference curve to $x^*$ and then, up to $x^1$ along his Engel (income consumption) curve. If the consumer's Engel curve cuts the production possibility frontier $PP'$ from the inside to the outside, the move from $x^0$ to $x^1$ must be a welfare improvement.

When does the Engel curve have the form required? Consider the Hatta normality condition. If $p^{\ast T}\Sigma_q u > 0$, an increase in the consumer's transfer income raises the value of his total consumption, i.e., his Engel curve must cut the production possibility frontier from the inside to the outside. Hence, the normality condition in the policy recommendation result of Hatta (1977a) is needed to guarantee that a policy change (here, a reduction of the highest commodity tax rate) is welfare improving.
Figure 7 - Normality of Commodities and the Effects of a Policy Change.
If, on the other hand, the initial equilibrium in the economy is a $B$-optimum with respect to the transfer $g^*$, it can be seen that the consumer's Engel curve must cut the production possibility frontier from the inside to the outside. If this were not the case, the initial equilibrium could not be a welfare-maximum: it would be possible to reduce the consumer's lump sum income and move along his Engel curve to a new equilibrium with a higher level of consumer welfare. The shift would also be productionally feasible because the change would lead to an equilibrium inside the economy's production possibility set. The situation is depicted in Figure 8. The point $x^0$ in Fig. 8 cannot be a $B$-optimum with respect to the transfer $g^*$.\footnote{5}

It can thus be concluded that the Hatta normality condition which is present in many earlier policy recommendation results is satisfied if the initial equilibrium of the Hatta economy is a welfare maximum with respect to the lump sum transfer $g^*$. But, as can be recalled, the earlier policy results given in this chapter were not conditional on the initial equilibrium of the economy being a $B$-optimum with respect to the vector of transfers $g^*$. How are the earlier results then to be understood? How can they imply that some welfare improvements do exist without any normality condition imposed on the domestic or tradeable commodities? The answer is rather simple. If the normality condition in the Hatta economy is violated, the initial equilibrium cannot be a $B$-optimum with respect to the lump sum transfer $g^*$. This means that some strict Pareto improving perturbation only in the transfer $g^*$ exists, without a change in the commodity tax rates $t^*$. However, once
Figure 8 - β-Optimality and the Hatta Normality Condition.
all such Pareto improvement possibilities have been exhausted, the resulting equilibrium is a $\mathcal{B}$-optimum with respect to $g^*$, the Hatta normality condition is satisfied, and then, further Pareto improvements can be attained by perturbing some of the other policy instruments (commodity taxes) together with the lump sum transfer $g^*$, as the earlier theorems demonstrate.

The Hatta model (10.18) - (10.22) is a very much simplified version of the model (4.10) - (4.13). Hence, the normality condition appropriate for the general model is not the simple Hatta assumption (10.24). It was shown in the proof of Theorem 8.1 that the vector $\lambda \in \mathbb{R}^{H+N+K+1}$ associated with a $\mathcal{B}$-optimum with respect to the lump sum transfers $g^*$ in the economy described by (4.10) - (4.13) must be proportional to the vector $\lambda^T = [0^T, (p^* + \varepsilon)^T, \theta^T, 1]$, where the vectors $\varepsilon$ and $\theta$ are defined in (8.14). For this vector $\lambda$, the inequalities $\lambda^T A > 0^T_H$ translate to

\[(10.29) \quad (p^* + \varepsilon)^T \Sigma_{qu} + w^T \Sigma_{vu} > 0^T_H,\]

which is the generalized Hatta normality condition for the model (4.10) - (4.13). If (10.29) is observed to be violated, the initial equilibrium of the economy is not a $\mathcal{B}$-optimum with respect to the transfers $g^*$ and strict Pareto and productivity improving changes in only transfers $g^*$ exist.

The condition (10.29) differs from the Hatta normality condition (10.24) in that the tradeables income derivatives $\Sigma_{vu}$ (weighted by the
international prices \( w \) are added to the formula. Also the domestic income derivatives \( \Sigma_{qu} \) are weighted by the shadow prices \( (p^* + \varepsilon) \) instead of the producer prices \( p^* \). The prices \( (p^* + \varepsilon) \) can be shown to be nonnegative if the initial equilibrium is a \( \beta \)-optimum with respect to the initial transfers \( g^* \). Hence, at a \( \beta \)-optimum, (10.29) is satisfied if all commodities in the economy are normal.

If lump sum transfers are not a feasible government policy instrument, the \((H+N+K+L)\) vector of Lagrange multipliers associated with a \( \beta \)-optimum with respect to the commodity tax rates \((t^*, s^*)\) is proportional to the vector \( \lambda^T = [\lambda_1^T, (p^* + \delta)^T, \gamma^T, 1] \), where \( \lambda \) solves the equations \( \lambda^T[B_p, B_u, B_t, B_s] = 0^T_{N+K+N+M} \). In this case, the generalized normality condition, implied by the inequalities \( \lambda^T A > 0^T_H \), is

\[
(10.30) \quad \lambda_1^T + (p^* + \delta)^T \Sigma_{qu} + \omega^T \Sigma_{vu} > 0^T_H.
\]

If (10.30) is violated, strict Pareto and productivity improving perturbations in only \((t^*, s^*)\) exist.

Let us now return to consider Proposition 10.1 and the Hatta (1977a) result according to which a reduction of the highest domestic tax rate in a closed one consumer economy is welfare improving.
Theorem 10.7: (Hatta (1977): Theorem 1)

Suppose that the model (10.18) - (10.22) is used to characterize an economy where the tax rate \( t_n^* (\gg 0) \), \( n \in [1, \ldots, N] \), on the \( n \)th domestic commodity is the highest of all tax rates \( t^* \). Then, if the \( n \)th domestic commodity is substitutable for all other domestic goods in the sense that

\[
(10.31) \quad p^{*T}(\Sigma_{qq}[p^*])_n > 0,
\]

where \( (\Sigma_{qq}[p^*])_n \) denotes the \( n \)th column of the \((N \times N)\)-matrix \( \Sigma_{qq}[p^*] \), there exists a strict welfare improving change of the tax rate \( t_n^* \), \( n \in [1, \ldots, N] \), and transfer \( g^* \). The change in \( t_n^* \) may be taken to be a reduction toward the level of the next highest domestic tax rate \( t_{n'}^* (\geq 0) \), \( n' \in [1, \ldots, n-1, n+1, \ldots, N] \).

**Proof:**

A sufficient condition for a strict welfare improvement of the above form to exist in the Hatta economy is:

\[
(10.32) \quad \text{there is no vector } \lambda \in \mathbb{R}^{1+N+N+1} \text{ such that } \lambda^T A > 0,
\]

\[
\lambda^T[B_p, B_y, B_g] = \lambda^T_{N+N+1} \Sigma_{qq}[p^*], \quad \lambda^T_{N+N+1} \Sigma_{qq}[p^*] h > 0,
\]

where

\[
h = (t_n^* - t_{n'}^*) > 0.
\]
Using the proof of Theorem 10.6, the vector \( \lambda \) satisfying 
\[
\lambda^T [B_p, B_y, B_g] = 0_{N+K+1}^T \text{ and } \lambda^T A > 0
\]
must be of the form 
\[
\lambda^T = [0, a^T\Sigma_{qq}[1 + t^*], a^T, 1]^T.
\]
Hence, it suffices to show that \( \lambda^T(B_L)^*_n < 0 \). Using (10.23), (10.19) and (10.31),

\( \lambda^T(B_L)^*_n = -a^T(\Sigma_{qq}[p^*])^*_n < 0. \)

QED

In his version of Theorem 10.7, Hatta assumes that the normality condition (10.24) is satisfied. This condition holds, if the initial equilibrium is a \( \beta \)-optimum with respect to the transfer \( g^* \). If this is not the case, then there are some welfare improving changes in only \( g^* (\Delta t^*_n = 0, n=1, \ldots, N) \).

Theorem 10.7 can be regarded as a special case of Proposition 10.1, since a movement toward uniform domestic taxation means reducing the highest domestic tax rate, if all other domestic commodity tax rates equal some common rate \( \bar{t} (\geq 0) \). The apparent dissimilarity of the condition \( t^* T(V^2_{tt}B)_n < 0 \) in Proposition 10.1\(^7\) and the condition \( p^* T(\Sigma_{qq}[p^*])^*_n > 0 \) in Theorem 10.7 can be explained by taking account of the difference between specific and ad valorem taxation: Hatta derives an expression for the consumer welfare change caused by a perturbation in the economy's commodity tax structure, and the sign of the welfare change depends on the signs of the vector \( p^* T \Sigma_{qq}[p^*]; \)\(^8\)
then, using the homogeneity condition $\sum_q \left[ 1 + t^* \right] p^* = 0_N$, Hatta arrives at an alternative form for the vector $p^* \Sigma_q$ which can be interpreted to imply the substitutability of the $n^{th}$ domestic commodity for all other domestic goods, as assumed by Hatta in his theorem.$^9$
11. COMMODITY TAXATION AS A CAUSE OF INTERNATIONAL TRADE

In a paper on the relationship between commodity taxation and international trade Melvin\(^1\) gives an interesting statement: "When trade is caused by a consumption tax, the country imposing the tax may be made worse off by trade so that a prohibitive tariff would be appropriate."

Two questions immediately arise: first, how does a commodity tax "cause" trade, and secondly when is international trade, induced by a change in the country's commodity tax structure, welfare reducing?

In order to answer the first problem, consider an economy at an autarky equilibrium. The autarky (international trade prohibitive) tariffs \(\tau^*\) are those defined in (5.1). The tariffs \(\tau^*\) are defined with respect to the initial equilibrium commodity tax rates and the initial transfers in the home country. Hence, if any of these policy variables are perturbed from their autarky levels, without simultaneously changing the tariffs \(\tau^*\), the relative producer and consumer prices in the economy change, creating a possibility for international trade; adjustments in the home country's autarky commodity tax or transfer vectors cause international trade by indirectly changing the autarky preserving trade barrier \(\tau^*\).

Whether the trade inducing change in the autarky commodity taxes or transfers is strict Pareto and productivity improving depends on the properties of the initial commodity tax rates \((t^*, s^*)\) and transfers \(g^*\). If, for example, \((t^*, s^*)\) are \(\beta\)-optimal (with respect to the constraints: (4.10) - (4.13) hold, \(\tau = \tau^*\) and \(g = g^*\),
no perturbation of only \((t^*, s^*)\) (which would cause international trade) can be strict Pareto or welfare improving. Similarly, if the initial autarky equilibrium is a \(\beta\)-optimum with respect to the transfers \(g^*\) (assuming that the home country's commodity tax structure is fixed), no change of the transfers \(g^*\) alone can be strict Pareto and welfare improving. On the other hand, if the initial commodity tax rates \((t^*, s^*)\) are adjustable but arbitrary in autarky, strict Pareto and productivity improving perturbations in them exist; trade caused by these changes in the home country's commodity tax structure is welfare improving.

Theorem 11.1:

Suppose that in autarky (i) rank \(Y = K \leq N\), (ii) rank \((S_{pp} + YY^T) = N\), (iii) \(t^* \geq 0_N, s^* \geq 0_M, g^* = 0_H\), (iv) \((t^*, s^*)\) are not \(\beta\)-optimal with respect to any social welfare function \(W(u) = \beta^T u, \beta > 0_H\), i.e., any vector \(\lambda \in \mathbb{R}^{H+N+K+1}\) that solves \(\lambda^T[A, -B^*_b] > 0^T_{H+1}\) and \(\lambda^T[B^*_p, B^*_z] = 0^T_{N+K}\) does not satisfy \(\lambda^T[B^*_t, B^*_s] = 0^T_{N+M}\). Then, there exists a strict Pareto and productivity improving reduction of \((t^*, s^*)\), i.e., \(\Delta t^* \leq 0_N, \Delta s^* \leq 0_M\), without a change in the autarky tariffs \(t^*\). If the autarky tax rates \((t^*, s^*)\) are nonpositive, there exists a strict Pareto and productivity improving increase of \((t^*, s^*)\), without a change in the initial tariffs \(t^*\).
Proof:
Suppose that \( t^* \geq 0 \), \( s^* \geq 0 \). It suffices to show that there is no vector \( \lambda \in \mathbb{R}^{H+N+K+1} \) such that \( \lambda^T [A, -B_s] > 0^T_{H+1}, \lambda^T [B^T, B] = 0^T_N \), \( \lambda^T [B_t, B_s] \geq 0^T_{N+M} \). Using (4.16), the homogeneity of the expenditures functions, and assumption (iii),

\[
(11.1) \quad \lambda^T [B_t, B_s] (q^T, v^T) = 0.
\]

By assumption, \( (q,v) \gg 0_{N+M} \) and the vector \( \lambda^T [B_t, B_s] \) is nonzero. It follows that the vector \( \lambda^T [B_t, B_s] \) must contain negative elements, i.e., the inequalities \( \lambda^T [B_t, B_s] \geq 0^T_{N+M} \) must be violated.

If \( t^* \leq 0 \) and \( s^* \leq 0 \), it suffices to show that there is no vector \( \lambda \in \mathbb{R}^{H+N+K+1} \) such that \( \lambda^T [A, -B_s] > 0^T_{H+1}, \lambda^T [B^T, B] = 0^T_N \), \( \lambda^T [B_t, B_s] \leq 0^T_{N+M} \). Using (11.1) and a similar reasoning as above, the inequalities \( \lambda^T [B_t, B_s] \leq 0^T_{N+M} \) must be violated. QED

Theorem 11.1 implies that the government in a small autarkic country, where lump sum transfers are not admissible, can find such a reduction of the initial nonoptimal taxes (subsidies) on the commodities in net demand (supply) by the consumers that the country's initial net balance of trade is strictly improved and every household in the economy is made strictly better of. Equally well, if the country's commodity
tax structure in autarky is such that goods in net demand by the
consumers are subsidized and the factors sold by the households are
taxed, there exists some strict Pareto and productivity improving
reduction in the levels of these subsidies and taxes. 

The example, on which Melvin based his statement cited in the
beginning of this chapter, is in fact closely related to Theorem 11.1.
Melvin considered a world consisting of two countries with identical
production possibility sets. In both countries, government expenditures
are financed using lump sum transfers without commodity taxation (i.e.,
\( t^* = 0_N, s^* = 0_M \)). Because the demand sides of the two economies are
assumed to be similar, the autarky equilibrium prices in the two
countries coincide. It follows that the autarky tariffs \( \tau^* \) in each
country equal zero. However, it can be shown that the zero commodity
tax rates \( (t^*, s^*) \) in the two countries must be \( \beta \)-optimal for the
social welfare function \( W(u) = 1^T_H u \) if \( \tau^* = 0_M \). Hence, assumption (iv)
of Theorem 11.1 is violated. Furthermore, using Proposition 4.2, it can
be seen that when \( \tau^* = 0_M \) and the tax rates \( (t^*, s^*) \) are \( \beta \)-optimal
(for \( \beta = 1_H \)), no strict Pareto and productivity improving directions of
change in the commodity tax rates \( (t^*, s^*) \) and the tariffs \( \tau^* \) exist; in
Melvin's example, the consumers in the country which imposes a nonzero
tax on a tradeable good are actually made worse of.
12. ECONOMIC INEQUALITY AND PUBLIC POLICIES

The analysis in the previous chapters has been entirely concerned with the existence of strict Pareto improvements. A strict Pareto improvement is, of course, also a strict welfare improvement for any increasing social welfare function, but it could be argued, following Blackorby and Donaldson (1977: p. 374), that accepting Pareto improvements as social welfare improvements may "allow too much inequality to creep into a social arrangement." Consider Figure 9.

Suppose that at the initial equilibrium in a two consumer economy, the households 1 and 2 attain the utility levels \((u_1, u_2)\) depicted by the point \(S\) in the figure, and that the utilities \(u_1\) and \(u_2\) are fully comparable. The point \(S'\) gives the symmetric distribution of \((u_1, u_2)\) with respect to the equal utility line \(u_1 = u_2\). Starting from \(S\), a strict Pareto improving change in the government policy instruments shifts the consumers to a new utility (real income) distribution point in the cone ASB. This policy change is also (relative) inequality reducing, if the perturbation causes the consumers to move in the cone ASC, and (relative) inequality increasing, if the households are being shifted in the cone CSB. A policy change that causes a shift in the consumer utilities in the cone S'SA is (relative) inequality reducing, but not Pareto improving.

The primary policy goal of the government, in this chapter, is assumed to be to reduce the initial level of economic inequality in the home country. The aim of this chapter is to operationalize the concept
Figure 9 - Pareto Improving and Inequality Reducing Policy Perturbations.
of economic inequality, i.e., to define it in a way that allows the analysis of practical policy questions: when do inequality reducing commodity tax and tariff perturbations exist and when, if ever, are simple policies (e.g., proportional reductions of tariffs) inequality reducing?

12.1 Existence of Inequality Reducing Policy Perturbations

It will be assumed in this section that each household $h$, $h=1,\ldots,H$, in the economy possesses nonnegative endowments of domestic and internationally tradeable commodities, denoted by $c^h \geq 0^N_h$ and $d^h \geq 0^M_h$ respectively. It is also assumed that if, for example, the $n$th domestic commodity is a labor service sold by the consumer $h$, the element $c^h_n$ in his endowment vector $c^h$ is equal to zero. The reasons for the latter assumption are twofold: first, the government is not likely to obtain correct information about the consumers' labor skills that it needs in order to tax their abilities and secondly, if taxes on the consumers' labor endowments are allowed, it is possible that the individuals in the society are forced to donate their time to the government—an outcome that corresponds to slavery and is obviously incentive in compatible.2

The equations describing the initial equilibrium in the economy are the following:3

$$m^h(u^*_h, q^*, v^*) = g^h + q^*T c^h + v^*T d^h, \ h=1,\ldots,H,$$
Let us denote the matrices of consumer endowments by

\( \mathbf{C} = [c_1^-, \ldots, c_H^+] \); \( \mathbf{D} = [d_1^-, \ldots, d_H^+] \).

Differentiation of the model (12.1) - (12.4) at the initial equilibrium yields:

\( \Delta \mathbf{u}^* = B_p \Delta p^* + B_z \Delta z^* + B_b \Delta b^* + B_t \Delta t^* + B_s \Delta s^* + B_t \Delta t^* \),

where
The matrices $B_z$ and $B_b$ in (12.6) are those defined in (4.16).

It is assumed that the government agrees on the form of a social welfare function $W(u)$; the gradient of this function at the initial equilibrium is denoted by

\[(12.7) \quad \beta \equiv \nabla_u W(u^*) .\]

Define the household average real income $u^*$ at the initial equilibrium as

\[(12.8) \quad u^* = \frac{1}{H} \sum_{h=1}^{H} \Sigma^h (u^*_h, q^*_h, v^*_h).\]
Next, define the Kolm (1969) equal equivalent of the real income vector \( u^* \) as the scalar \( K(u^*) \) which satisfies \( W(K(u^*))L_H = W(u^*_H) \). The scalar \( K(u^*) \) gives the amount of real income which, if attained by each individual in the economy, would give rise to the same level of social welfare as the actual observed vector of real income \( u^* \). Assuming that the social welfare function \( W(u) \) is cardinally scaled, the equation

\[
(12.9) \quad W(u^*) = K(u^*)
\]

is satisfied at the initial equilibrium. Hence, using the Kolm (1969) measure of injustice, a monetary measure of economic inequality at the initial equilibrium, \( I^* \), can be defined:

\[
(12.10) \quad I^* = u^* - W(u^*).
\]

A reduction of economic inequality is defined to mean a reduction in the measure \( I^* \) (\( \Delta I^* < 0 \)). When can such reductions be reached using the initial commodity tax rates \( (t^*, s^*) \) and the tariffs \( \tau^* \) as the government policy instruments? Let us rewrite the equation (12.10) in an equivalent form \(-I^* = W(u^*) - u^*\). Then, it can be seen that a reduction of economic inequality is equivalent to an increase in the
initial value of the social welfare function \( \tilde{W}(u) = W(u) - u \). What are the sufficient conditions for an improvement in \( \tilde{W}(u^*) \) to exist?

The problem is to determine the minimal sufficient conditions for:

\[
\text{(12.11)} \quad \text{there exist } \Delta u^*, \Delta p^*, \Delta z^*, \Delta b^*, \Delta t^*, \Delta s^*, \Delta r^* \text{ such that (12.6) holds, } \beta^T \Delta u^* > 0 \text{ and } \Delta b^* > 0, \text{ where } \beta^T = \nabla_u \tilde{W}(u^*) = (\beta^T - \frac{T}{H})/(\beta^T - \frac{T}{H}).
\]

**Theorem 12.1:**

Suppose that (i) rank \( Y = K \leq N \), (ii) rank \( S_{pp} + YT) = N \), (iii) \( \beta^T[(X^T - C^T), (E^T - D^T)] \neq 0_{N+M} \), (iv) \( \tau^T \nabla^2_{ww} G(w + \tau^*, y^*) \neq 0_{M} \) at the initial equilibrium. Then, there exists a strict welfare and economic equality improving change in the tariffs \( \tau \) and commodity tax rates \( (t^*, s^*) \) (without a change in the lump sum transfers \( g^* \)) which does not reduce the country's initial net balance of trade. The change in tariffs can be chosen to be a proportional reduction.

**Proof:**

Motzkin's Theorem yields an equivalent condition to (12.11):
(12.12) there must not exist a vector \( \lambda \in \mathbb{R}^{H+N+K+1} \) such that
\[
\lambda^T [B_p, B_z, B_t, B_s] = 0^T_{N+K+N+M}, \quad \lambda^T B_t = 0^T_M, \quad \lambda^T A = \beta^T,
\]
\[\lambda^T B_b < 0.\]

The inequality \( \lambda^T B_b < 0 \) implies \( \lambda_4 \geq 0 \). Set \( \lambda_4 = k \geq 0 \). As in the proof of Theorem 2.1, the equations \( \lambda^T [B_p, B_z, B_t, B_s] = 0^T_{N+K+N+M} \) can be used to solve the components \( \lambda_2 \) and \( \lambda_3 \) of the vector \( \lambda \) when \( \lambda_4 = k \):
\[
\lambda_2^T = k(p^* + \delta)^T, \quad \lambda_3^T = k \gamma^T,
\]
where the vectors \( \delta \) and \( \gamma \) are defined in (2.27).

If \( k = 0 \), \( \lambda_2 = 0^N \) and \( \lambda_3 = 0^K \). Then, the equations in (12.12) simplify to

(12.13) \[\lambda_1^T = \beta^T, \quad \lambda_1^T [\lambda^T - C^T] = 0^T_N, \quad \lambda_1^T [E^T - D^T] = 0^T_M.\]

By assumption, there is no solution \( \lambda_1^T = \beta^T \) to (12.13).

Suppose \( k > 0 \). Then, the equations \( \lambda^T B_t = 0^T_M \) are equivalent to \( \tau^T V_{ww}^2 \)
\[G(w + \tau^*, y^*) = 0^T_M, \] where the matrix \( V_{ww}^2 \) \( G(w + \tau^*, y^*) \) is defined in (2.12). By assumption, \( \tau^T V_{ww}^2 \) \( G(w + \tau^*, y^*) \neq 0^T_M \), i.e., (12.12) is satisfied. Proposition 2.1 shows that \( \Delta \tau^* \) may be chosen to be a proportional reduction. QED
Corollary 12.1.1:

If rank $[X^T - C^T, E^T - D^T] = H (\leq N + M)$, condition (iii) in Theorem 12.1 is satisfied.

Proof:

If rank $[X^T - C^T, E^T - D^T] = H$, the only solution to $\lambda^T [X^T - C^T, E^T - D^T] = O^T_{N+M}$ is $\lambda = O_H$. Hence, since $\hat{\beta}^T \neq O_H$, assumption (iii) in Theorem 12.1 must be satisfied. QED

Theorem 12.1 shows that the sufficient conditions that guarantee the existence of improvements in economic equality are very similar to those implying the existence of strict Pareto improvements (when commodity taxes and tariffs are the admissible government policy instruments): assumptions (i), (ii) and (iv) in Theorem 12.1 are sufficient for a strict productivity improving perturbation in the tariffs $\tau^*$ to exist, and assumption (iii) ensures that the gains accruing from a productivity improving perturbation of $\tau^*$ can be distributed to the households in a way that shifts the distribution of real income toward equality. Assumption (iii) can also be given an interesting interpretation. Let us partition the households in the economy into three classes: those with a positive welfare weight $\hat{\beta}^h$, those with a zero welfare weight $\hat{\beta}^0$, and those with a negative welfare weight $\hat{\beta}^h$. The first class of consumers is called "the poor" and the last "the rich." For assumption (iii) in Theorem 12.1 to be satisfied,
it is sufficient to assume that there exists some good (domestic or internationally tradeable) in (net) demand (or supply) by all "the poor" and in (net) supply (or demand) by all "the rich."  In other words, the preferences of "the rich" and "the poor" must significantly differ from each other at least in the case of a commodity. Under this supposition, a proportional reduction of the tariffs $\tau^*$ can be made economic inequality reducing.\(^\text{8}\)

Using Proposition 4.2, it can be seen that strict welfare and economic equality improving directions of change in the tariffs $\tau^*$ and commodity taxes $(t^*, s^*)$ cannot exist, if the initial equilibrium is a $\beta$-optimum with respect to the commodity tax rates $(t^*, s^*)$ and the gradient of the net balance of trade function, $\nabla_T b^*(w + \tau^*)$, equals zero. One can thus conclude that zero tariffs (free trade) are not only Pareto but also "equality optimal" for a small country, if the domestic commodity tax rates are freely adjustable and the producer substitution matrix $S$ is of maximal rank ($= N + M - 1$).

If the government in the home country can adjust the initial vector of lump sum transfers $g^*$, it can be shown that the assumptions of Theorem 8.3, which imply the existence of strict Pareto improving perturbations in the tariffs $\tau^*$ and transfers $g^*$, are also sufficient for a strict welfare and equality improving tariff and transfer change to exist.\(^\text{9}\)
12.2 Existence of Inequality Reducing and Welfare Improving Policy

Perturbations

By comparing the assumptions of Theorem 12.1 to the conditions of Theorem 8.3, which guarantee the existence of strict Pareto improving perturbations in the commodity tax rates \((t^*, s^*)\) and the tariffs \(\tau^*\), the conclusions drawn from Figure 9 can be confirmed: (i) a strict Pareto improvement (if it exists) does not necessarily imply a strict reduction in economic inequality, and (ii) a strict inequality reduction can exist even if a strict Pareto improvement does not. When can the government find changes in \((t^*, s^*, \tau^*)\) that are both Pareto and equality improving? And when are commodity tax and tariff perturbations welfare improving for both social welfare functions \(W(u)\) and \(\overline{W}(u)\)?

Theorem 12.2:

Suppose that (i) \(\text{rank } Y = K \leq N\), (ii) \(\text{rank } (S_{pp}^T + YY) = N\), (iii) \(\tau^* T \nabla_{ww}^2 G(w + \tau^*, y^*) 
eq 0_M^T\) at the initial equilibrium. Then, if \(\hat{\beta}^T \equiv \nabla_u \overline{W}(u^*) = (\beta^T - 1_H^T/H)\) and

\[
(12.14) \quad \text{there is no solution } a > 0_H \text{ to } a^T [X^T - C^T, E^T - D^T] = 0_{N+M}^T,
\]

and
(12.15) there is no solution $r > 0$ to
\[ (r^T + \beta^T) \left[ X^T - C^T, E^T - D^T \right] = o^T_{n+m}, \]

there exists a strict Pareto and equality improving change in the initial commodity tax rates $(t^*, s^*)$ and the tariffs $\tau^*$ (without a change in the initial vector of lump sum transfers $g^*$). The policy perturbation does not reduce the level of the economy's initial net balance of trade $b^*$.

II. If

(12.16) there is no solution $r > 0$ to
\[ (r \hat{\beta}^T + \beta^T) \left[ X^T - C^T, E^T - D^T \right] = o^T_{n+m}, \]

and

(12.17) \[ \hat{\beta}^T \left[ X^T - C^T, E^T - D^T \right] \neq o^T_{n+m}, \]
\[ \beta^T \left[ X^T - C^T, E^T - D^T \right] \neq o^T_{n+m}, \]

there exists a strict welfare and equality improving perturbation of the initial commodity tax rates $(t^*, s^*)$ and the tariffs $\tau^*$ (without a change in the initial vector of lump sum transfers $g^*$). The change does not reduce the level of the economy's initial net balance of trade $b^*$.
Proof:

(I) A sufficient condition for a Pareto and equality improving change in \((t^*, s^*)\) and \(\tau^*\) to exist is:

\[(12.18) \text{ there exist } \Delta u^*, \Delta p^*, \Delta z^*, \Delta b^*, \Delta s^*, \Delta \tau^* \text{ such that (4.16)} \]

is satisfied, \(\Delta b^* \geq 0, \Delta u^* \gg 0\), and \(\beta^T \Delta u^* > 0\).

Using Motzkin's Theorem, an equivalent condition for (12.18) can be derived:

\[(12.19) \text{ there is no vector } \lambda \in \mathbb{R}^{H+N+K+1} \text{ such that } \lambda^T [B_p, B_z, B_t, B_s, B_\tau] = 0, \lambda^T B_b \leq 0, \lambda^T A = v_1^T + v_2^T \beta, (v_1, v_2) > 0_{H+1}.\]

Using \(\lambda^T B_b \leq 0, \lambda_4 \equiv k \geq 0\). Then, as shown in the proof of Theorem 2.1, \(\lambda_2^T = k(p^* + \delta)^T, \lambda_3^T = k \gamma^T\). If \(k = 0\), \(\lambda_2 = 0_N\) and \(\lambda_3 = 0_K\). It follows that \(\lambda^T A = \lambda^T_1\). For (12.19) to be satisfied, there must not exist a solution to

\[(12.20) \lambda^T_1 = v_1^T + v_2^T \beta, - \lambda^T_1 [X - C^T, E^T - D^T] = 0_{N+M}, (v_1, v_2) > 0_{H+1}.\]

Since \((v_1, v_2) > 0_{H+1}, (12.20)\) is equivalent to
(12.21) \[ \lambda_1^T = v_1^T (> 0_H^T) \] or \[ \lambda_1^T = v_1^T + \beta^T, v_1 > 0_H, \] and
\[ -\lambda_1^T[X^T - C^T, E^T - D^T] = 0_{N+M}^T. \]

Using assumptions (12.14) - (12.15) and (12.21), it can be seen that there is no solution to (12.20).

If \( k > 0 \), choose \( k = 1 \). Then, \( \lambda^T B_t = \tau T^T \frac{v_1^2}{w w^T} G(w + \tau^*, y^*) \neq 0_M^T \) by assumption. Hence (12.19) is satisfied.

(II) Now it is required that

(12.22) there exist \( \Delta u^*, \Delta p^*, \Delta z^*, \Delta b^*, \Delta t^*, \Delta s^*, \Delta \tau^* \) such that
(4.16) is satisfied and \( \Delta b^* \geq 0, \beta T^T \Delta u^* > 0, \beta^T \Delta u^* > 0. \)

Equivalently,

(12.23) there must not exist a vector \( \lambda \in \mathbb{R}^{H+N+K+1} \) such that
\[ \lambda^T[B_p, B_z, B_t, B_s, B_t] = 0_{N+K+N+M+M}^T, \lambda^T B_b \leq 0, \lambda^T A = \]
\[ \frac{v_1}{v_2} \beta^T + \frac{v_2}{v_1} \beta^T, (v_1, v_2) > 0_2. \]

Proceeding as above, \( \lambda^T B_b \leq 0 \) implies \( \lambda_4 \equiv k \geq 0. \)

If \( k = 0, \lambda_2 = 0_N \) and \( \lambda_3 = 0_K \). Then, \( \lambda^T A = \lambda_1^T \). For (12.23) to be satisfied, there must not exist a solution to
\( (12.24) \quad \lambda_1^T = \bar{v_1} \beta^T + \bar{v}_2 \beta^T, \quad -\lambda_1^T [X^T - C^T, E^T - D^T] = \mathbf{0}_{N+M}, \quad (\bar{v_1}, \bar{v}_2) > 0_2. \)

Since \((v_1, v_2) > 0_2\), \(12.24\) is equivalent to

\( (12.25) \quad \lambda_1^T = \beta^T + \bar{v}_2 \beta^T, \quad \bar{v}_2 \geq 0 \) or \( \lambda_1^T = \bar{v}_1 \beta^T + \beta^T, \quad \bar{v}_1 \geq 0, \) and

\[-\lambda_1^T [X^T - C^T, E^T - D^T] = \mathbf{0}_{N+M}.\]

Using assumptions \((12.16) - (12.17)\) and \((12.25)\), it can be seen that there is no solution to \((12.24)\).

If \( k > 0 \), choose \( k = 1 \). The equations \( \lambda^T_B = \mathbf{0}_{M} \) simplify to \( \tau^T \frac{v^2}{ww} G(w + \tau^*, y^*) = \mathbf{0}_{M} \). Using assumption (iii), these equations must be violated. QED

**Corollary 12.2.1:**

If rank \([X^T - C^T, E^T - D^T]\) = \( H(\leq N+M) \), conditions \((12.21) - (12.22)\) of Theorem 12.2 are satisfied. If rank \([X^T - C^T, E^T - D^T]\) = \( H(\leq N+M) \) and \( \beta > 0_H \), conditions \((12.16) - (12.17)\) are satisfied.

**Proof:**

If rank \([X^T - C^T, E^T - D^T]\) = \( H(\leq N+M) \), the only solution for the equations \( \lambda^T [X^T - C^T, E^T - D^T] = \mathbf{0}_{N+M} \) is \( \lambda = \mathbf{0}_H \). Hence, \((12.14)\) is
satisfied. Then, since \( \hat{\beta} \neq -r, r > 0_H \), the equations \( (r^T + \hat{\beta}^T) \)
\[
[X^T - C^T, E^T - D^T] = O^T_{N+M}
\]
have no solution \( r > 0_H \). (If \( \sum_{h=1}^{H} \beta^h = 1 \), not all \( \hat{\beta}^h (= \beta^h - 1/H) \) can be negative.)

If \( \text{rank } [X^T - C^T, E^T - D^T] = H \), condition (12.17) must be satisfied. Furthermore, if \( \text{rank } [X^T - C^T, E^T - D^T] = H \) and \( \beta > 0_H \), condition (12.16) must be satisfied (since the vector \( (r \hat{\beta} + \beta) \) cannot be zero).

QED

Assumption (12.14) in Part I of Theorem 12.2 is the generalized Diamond-Mirrlees condition that is needed for a Pareto improving direction of change in the initial commodity tax and tariff to exist. Condition (12.15) guarantees that a Pareto improving direction of change in the initial commodity tax and tariff rates can be made economic inequality reducing. Corollary 12.2.1 implies that the rank condition that, according to Corollary 12.1.1 is sufficient for equality improvements to exist, is also sufficient to ensure the existence of a strict Pareto and equality improving perturbation in \((t^*, s^*, r^*)\).

Condition (12.17) in Part II of Theorem 12.2 corresponds to assumption (iii) in Theorem 12.1. Condition (12.16) imposes a restriction on the welfare weights \( \beta \) (which are determined by the gradient \( \nabla u W(u^*) \)). Corollary 12.2.1 shows that (12.16) is satisfied at least if \( \beta > 0_H \) and the matrix \([X^T - C^T, E^T - D^T]\) is of full (row) rank.12
13. CONCLUSIONS

One of the famous problems in the area of international trade theory has been the question of the gains from trade: is it possible for all the consumers in an autarkic country to benefit if the country is opened up for international trade and if so, under what conditions does this occur?

It is well-known that free international trade enlargens (or at least, does not reduce) the feasible consumption possibility set for the consumers in a previously autarkic country. Hence, if the government in the home country has lump sum transfer instruments at its disposal, it can ensure that every household in the economy will be better off (or at least, not worse of) under free international trade than in autarky.

But what if lump sum redistribution of income is inadmissible? Is it still possible to show that all consumers in an autarkic country can benefit from free international trade? Dixit and Norman (1980) in their textbook provided a result according to which the autarky welfare levels of all households in the home economy can be replicated under free trade if the government can freely adjust the country's commodity tax structure. However, Dixit and Norman did not show that strict gains from trade would occur, i.e., that the shift from autarky to free trade would produce a strict Pareto improvement.

This thesis started as an attempt to find the sufficient conditions for strict gains from trade to exist when only commodity taxation is used to influence the distribution of income. The problem is approached as a policy reform question: when can the government find
strict Pareto improving (differential) perturbations in the country's commodity tax and tariff structure if the initial autarky tariffs are defined to be such that international trade is just being prohibited?

It turns out that under some weak conditions (established in Chapter 2 of the thesis) on the production technologies of the domestic production sectors and on the initial vector of tariffs there exist such changes in the economy's tariff vector that the amount of foreign exchange earned by the domestic producers is increased. As an example, the perturbation of tariffs can be chosen to be a proportional reduction.

In Chapter 4 of the thesis the question of redistributing the increase in the amount of foreign exchange earned by the domestic producers to the consumers is considered. It turns out that a condition on the preferences and initial endowments of the households has to be satisfied; this condition is a generalization of the Diamond-Mirrleess assumptions that a good in net demand or supply by all households exists. It can be shown that if the above mentioned condition on the consumer preferences and the previously developed conditions on the domestic production technologies are satisfied, then strict Pareto improving perturbations in the country's initial equilibrium tariffs and commodity tax rates exist. In particular, if the initial equilibrium is an autarky equilibrium, these conditions are sufficient for strict gains from trade to exist.

In Chapter 3 of the thesis the problem of approximating the size of the productivity gain accruing from a productivity improving tariff
change is analyzed. The measures developed in this chapter also provide production side approximations for the size of the gain from trade when the initially trade prohibitive tariffs are being perturbed.

Having established the existence of strict Pareto improving tariff and commodity tax perturbations, the next question is: what kind of examples of these policy changes can be found? In Chapter 7 of the thesis it is shown that, for example, reductions of positive tariffs and increases of negative tariffs, uniform reductions of tariffs and changes toward uniformity in the country's tariff structure can be strict Pareto improving if the country's initial commodity tax rates can be freely varied.

In Chapter 8, lump sum transfers are assumed to be available and sufficient conditions for strict Pareto improving tariff and transfer perturbations to exist are developed. These production side conditions are also sufficient for strict gains from trade to exist if lump sum transfers can be used to redistribute consumer income.

In Chapter 9, the existence results under different assumptions about the availability of government policy instruments are compared. An interesting proposition derived in this section is that strict gains from trade need not exist under lump sum compensation even if strict gains under commodity taxation would be possible.

Chapter 10 contains some examples of strict Pareto improving commodity tax, tariff and transfer perturbations: these include changes toward international prices in tariffs, simultaneous proportional reductions of commodity taxes and tariffs, and changes toward uniformity in commodity taxes. The results concerning the changes toward
uniformity in commodity taxation generalize the earlier Hatta (1977a)
result according to which a reduction in the highest commodity tax rate
in a closed one consumer economy (where lump sum transfers are
admissible) is welfare improving.

In the last chapter of the thesis the government is assumed to
search for policy reforms that improve economic equality in the
country. In order to consider actual policy perturbations, a reduction
in economic inequality is defined to correspond to an improvement in an
especially defined social welfare function. One of the conditions for
inequality reducing commodity tax and tariff changes to exist turns out
to be a condition on the consumer preferences: there must be a good
with respect to which "the rich" and "the poor" are on the opposite
sides of the (net) market.

In the future, the same methods that, in this thesis, have been
used to analyze policy reforms in a small open economy could be applied
to study multicity price and other policy agreements. One might
also want to develop (production side) approximative measures for the
gains accruing to the countries that agree to implement beneficial
policy reforms. Another intriguing direction of research would be to
combine the theories of optimal tax reforms, imperfect information and
uncertainty. In this case, the goal of the analysis would be to find
policy rules for a government (or a firm) acting under uncertainty about
the other economic agents' goals and characteristics. Of course, one
could also investigate the conditions for gains from the trade to exist
under these circumstances. In a many country case, one could compare
policies (and the gains accruing from them) under imperfect information
to those under certainty and cooperation.
FOOTNOTES

Chapter 2:

1. If some of the production sectors exhibit diminishing returns to scale, new domestic commodities that correspond to the ownership shares of these sectors are added to the model. The new factors absorb the pure profits earned by the diminishing returns to scale sectors.

2. The technologies of the production industries are described using their unit production possibility sets since the technologies of the sectors are assumed to exhibit constant returns to scale. If the total production possibility sets of the industries were used, the sectoral total profit functions would assume only two values, zero and infinity. Each profit function would thus be discontinuous and nondifferentiable.

3. All vectors in this thesis are defined as column vectors; \( x^T \) denotes the transpose (row) vector of \( x \).

4. There are several ways of defining the scale of a production sector. If the sector \( k, k=1,\ldots,K \), produces only one output, the scale of the sector \( k \) is the amount of output produced in that sector in each time period. If joint production is present in sector \( k \), the scale of the sector \( k, k=1,\ldots,K \), can be defined as in Ch. 2 by using an always needed input or alternatively by employing units of value added (Woodland (1982: p. 135)).

5. For unit profit functions and duality, see Diewert and Woodland (1977: pp. 377-378).

6. If some sectors produce internationally traded good \( m, m \in [1,\ldots,M] \), while other sectors utilize good \( m \) as an input, it may be necessary to redefine good \( m \) as two separate commodities: an input good and an output good. It is assumed that each technology set \( C_k, k=1,\ldots,K \) is such that good \( m \) is either produced or used as an input, but not both. Thus after redefinition, each internationally traded good will be either produced (or not utilized at all) by each sector or used as an input by each sector.

7. Notation: \( 0_{N+M} \) denotes an \((N+M)\) - vector of zeroes.

8. If all production sectors have a common always used input (or an always produced output), the matrix \( S \) will have a zero row and column corresponding to this commonly applied input. Then, the rank of the matrix \( S \) is at most \( N+M-2 \).

9. The constant returns to scale assumption together with competitive profit maximization implies that all production sectors earn zero pure profits in equilibrium. If the equilibrium total profits were
positive in some sector, the profits of this sector could be driven to infinitely simply by increasing the industry's scale (since the sector's technology is CRS). Hence, the zero profit conditions (2.9) constrain the equilibrium to the finite scale case.

10. Notation: $\Delta p^*$ denotes an infinitesimal change in $p^*$, usually denoted by $dp^*$. $O_{K\times K}$ is an $(K\times K)$-matrix of zeroes.

11. The properties of the GNP function are given in Woodland (1982: Ch. 3.7).

12. A special case of problem (2.15) is presented in Woodland (1982: Ch. 3). There, it is assumed that each production sector supplies only one tradeable good using domestic commodities as inputs. In this case, assuming that the sector $k$, $k=1,\ldots,K$, produces the $k^{th}$ tradeable, sector $k$'s unit profit function is $\pi^k(p, w + \tau) = (w_k + \tau_k) - c^k(p)$, where $c^k(p)$ is sector $k$'s unit cost for producing the $k^{th}$ tradeable. The problem (2.15) can then be interpreted as minimizing the input cost under the constraint that the producer price for the $k^{th}$ tradeable, $(w_k + \tau_k)$, does not exceed the sector $k$'s unit cost $c^k(p)$.

13. The net input endowment vector $-y^*$ is drawn in the diagram taking $p^0$ as the origin.

14. Convexity of the unit profit functions implies convexity of the unit profit level curves $-\pi^k(p, w + \tau^*) = 0$.

15. All the three production sectors can stay operative inspite of the change in $(w + \tau^*)$ if the change is very special, i.e., it is necessary that all the unit profit level curves corresponding to the new producer prices for tradeables intersect at $p^1$.

16. The matrix $S_{pp} + YY^T$ is positive definite, for example, (i) if $S_{pp} = 0_{NxN}$ and $N = K$ (if rank $Y = K = N$), the matrix $YY^T$ is positive definite), or (ii) if rank $S = N+M-1$, which implies that rank $S_{pp} = N$. Proof for (ii): suppose on the contrary that rank $S_{pp} = N-1$ when rank $S = N+M-1$. Then, there exists $x \in \mathbb{R}^N$, $x \neq 0_N$, such that $x^T S_{pp} = 0^T$. Choose $y^T = (x^T, Q^T_M)$. It follows that $y^T S y = 0$; hence, $y$ is a zero eigenvector of the positive semidefinite $S$. Because both the vectors $(p^*, w + \tau^*)$ and $(x, 0_M)$ are zero eigenvectors of the matrix $S$ (and the vectors are
not linearly dependent since \((w + \tau^*) \in \mathbb{R}^N_{++}\), rank \(S\) is at most \(N+M-2\), which contradicts the assumption that rank \(S = N+M-1\).

17. Note that the vector \(y^*\) can be regarded as a net domestic demand vector.

18. If rank \(S = N+M-1\), then rank \(S_{pp} = N\) and (2.13) is satisfied.

19. Note also that controllability of production in the sense of Guesnerie and Weymark implies controllability of domestic goods production in the sense of Definition 2.2, but not conversely.

20. The vector of domestic goods prices \(p^*\) appropriate in (2.18) is the one that solves problem (2.15). This \(p^*\) is the shadow price vector corresponding to the domestic goods constraint \(y^*\).

21. If \(\nabla_T b^*(w + \tau^*) = 0^T_M\), one cannot determine if the function \(b^*(w + \tau^*)\) is increasing, decreasing or stationary. Yet, if \(\nabla_T b^*(w + \tau^*) = 0^T_M\), \(b^*\) cannot be strictly increasing in its argument at \((w + \tau^*)\).

22. If the matrix \(\nabla^2_{ww} G(w + \tau^*, y^*)\) is a zero \((M \times M)\)-matrix, the condition (2.16) is violated. Under these circumstances, no infinitesimal change in \(\tau^*\) can change the production sectors' aggregate production choice (for tradeables); even if productivity improving directions of change in tradeables supply (given a fixed \(y^*\)) would exist, they could not be attained through differential perturbations in the prices \((w + \tau^*)\).

23. Notation: For a vector \(x \in \mathbb{R}^N\), \(x \gg O_N\) means that each component of \(x\) is positive; \(x \geq O_N\) means that each component of \(x\) is nonnegative; \(x > O_N\) means that \(x \geq O_N\) but \(x \neq O_N\). When \(x \in \mathbb{R}\), \(x \gg O\) is equivalent to \(x > 0\).

24. Note that the existence of a productivity improving tariff change is equivalent to the existence of an Allais production gain. (Diewert (1983c) defines the Allais production loss as the extra amount of foreign exchange the producers could earn by having an optimal internal rearrangement of production instead of one disturbed by the existence of distortionary taxes and tariffs.) Another possible way of defining a productivity improving change in tariffs \(\tau^*\)
is to require that there exist $\Delta p^*, \Delta z^*, \Delta b^*, \Delta \tau^*, \Delta y^*$ such that
$$B_p \Delta p^* + B_z \Delta z^* + B_b \Delta b^* + B_y \Delta y^* = B_\tau \Delta \tau^*$$
and $\Delta b^* > 0$, $\Delta y^* > 0_N$, where $B_y = [-I_N, 0_{N \times K}, 0_N]^T$. This approach corresponds to the Debreu measure of productivity loss defined in Diewert (1983c).

25. Motzkin's Theorem:

Let $Q_1$, $Q_2$, and $Q_3$ be given matrices with $Q_1 \neq 0$. Then, either

(i) $Q_1 \rho > 0$, $Q_2 \rho > 0$, $Q_3 \rho = 0$ has a solution or

(ii) $\alpha_1^T Q_1 + \alpha_2^T Q_2 + \alpha_3^T Q_3 = 0^T$, $\alpha_1 > 0$, $\alpha_2 > 0$, has a solution, but never both.

Motzkin's Theorem and other theorems of alternative are discussed in Mangasarian (1969: pp. 17-37).

In order to show the equivalence of (2.25) and (2.29), choose $\rho^T = [\Delta p^T, \Delta z^T, \Delta b^T, \Delta \tau^T]$, $Q_1 = [0_{l \times N}, 0_{l \times K}, 1, 0_{l \times N}]$, $Q_3 = [B_p, B_z, B_b, -B_\tau]$. Then, there must not exist $\alpha_1 \in R$, $\alpha_3 \in R^{N+K+1}$ such that $\alpha_1 Q_1^T + \alpha_3 Q_3^T = 0_{N+K+1+M}$, $\alpha_1 > 0$. Equivalently, there must not exist $\alpha_1$, $\alpha_3$ such that $\alpha_3^T [B_p, B_z, -B_\tau] = 0_{N+K+1+M}$, $\alpha_3^T B_b = -\alpha_1 < 0$. Choose $\lambda = \alpha_3$ and (2.29) follows.

26. Choose $\rho^T = [\Delta p^T, \Delta z^T, \Delta b^T, \Delta \tau^T, \tau]$, $Q_{11} = [0_{l \times (N+K)}, 1, 0_{l \times N}, 0]$, $Q_{12} = [0_{N+K+1+M}, 1]$, $Q_{31} = [B_p, B_z, B_b, -B_\tau, 0_{(N+K+1) \times 1}], Q_{32} = [0_{M \times (N+K+1)}, I_M, \tau^*]$. Then, there must not exist $\alpha_1 \in R^{N+K+1}$, $\alpha_2 \in R$, $\alpha_3 \in R^M$, $\alpha_4 \in R$ such that $\alpha_2 > 0$, $\alpha_4 > 0$, and $\alpha_1^T [B_p, B_z] = 0_{N+K}$, $\alpha_1^T B_b + \alpha_2 = 0$, $-\alpha_1^T B_\tau + \alpha_3 = 0_M^T$, $\alpha_3^T \tau^* + \alpha_4 = 0$. Equivalently, there must not exist vectors $\alpha_1$, $i=1, \ldots, 4$, such that $\alpha_1^T [B_p, B_z] = 0_{N+K}$, $\alpha_1^T B_b = -\alpha_2 < 0$, $\alpha_1^T B_\tau = \alpha_3^T \tau^* = -\alpha_4 \leq 0$. Hence, there must, not exist $\alpha_1 \in R^{N+K+1}$ such that $\alpha_1^T [B_p, B_z] = 0_{N+K}$, $\alpha_1^T B_b < 0$, $\alpha_1^T B_\tau \tau^* \leq 0$. Take $\lambda^T = \alpha_1^T$. 

27.
27. Note that an increase in a negative $\tau_m^*$, $m \in [1, \ldots, M]$, is a decrease in the magnitude of $\tau_m^*$.

28. $\nabla_{\tau} b^*(w + \tau^*) \neq 0^T_M$ is also one of the sufficient conditions for strict productivity improving tariff changes to exist.

Chapter 3:

1. See Diewert (1983c) for more details.

2. The initial equilibrium solution $(y^*_k, f^*_k, z^*_k)$, $k=1, \ldots, K$, is feasible for the problem (3.1) but not necessarily optimal.

3. The implicit functions exist if (i) rank $Y = K \leq N$ and (ii) $S_{pp} + YY^T$ is positive definite.

4. Alternatively, $A(\xi)$ gives the net value of internationally traded goods produced by the entire production sector, when the goods are valued at the international prices $w$.


6. If the directional derivative of the net balance of trade function in the direction of the tariff change is strictly positive, then for a small finite change of tariffs in this direction, the net balance of trade function must be increasing.

7. The formula (3.23) contains an abuse of notation. The term $A'(2)$ refers to the derivative of the function $A(\xi)$ at $\xi = 1$, where the value $\xi = 1$ corresponds to the new (after the tariff change) equilibrium, previously indexed as the equilibrium 2.

8. See Diewert (1983c): pp. 169-170. Set $\xi = 1$. Using formulae (28) and (30) in Diewert (1983c), the expression (3.25) follows. Note that (28), in Diewert (1983c), contains a typo: the next to last term in (28) should be $\tau_k^T y^k(\xi) z_k^T(\xi)$.

9. The same abuse of notation as encountered in formula (3.23) is present here. See footnote 7.

10. The government might be able to give estimates for the net output matrices $Y^2$ and $F^2$ that determine the price derivatives $p'(2)$.
11. Note that if \( N = K \), \( p'(1)^T = \tau^T p(\gamma 1^T)^{-1} \).

12. Note that if \( S_{pw} = 0_{N \times M} \), also \( A_G = (1 - k^2) \tau^T s_{ww}^1 \tau^1 \geq 0 \).

Chapter 4:

1. Consumer preferences are assumed to be nonsatiated so that (4.10) holds as an equality.

2. If some public goods are inputs into the production process, the possible nonzero pure profits generated by these factors can be imputed to domestic factors of production created for this purpose.

3. The implicit functions \( u^*, p^*, z^* \), and \( b^* \) exist if the matrix \( [A, -B, -B, -B] \) is invertible.

4. Note that in order to derive (4.16), the order of the equations (4.11) and (4.12) has been changed.

5. The Pareto improvement is required to be strict (\( \Delta u^* > 0 \)) in order to avoid problems with actual (finite) changes in utilities, as explained in Diewert (1978: pp. 154-155). Assuming that (i) the implicit functions that determine the endogenous variables as functions of the exogenous variables of the model exist in the neighborhood of the initial equilibrium, and that (ii) the changes in the exogenous variables have been normalized by requiring the sum of the changes squared to be one, the changes \( \Delta u^*, \Delta p^*, \Delta z^* \), and \( \Delta b^* \) may be interpreted as directional derivatives of the implicit functions \( u^*, p^*, z^* \), and \( b^* \) given \( w, \tau^*, t^*, s^* \), and \( g^* \). If for some consumer \( h \), \( h \in \{1, \ldots, H\} \), \( \partial u^h(w^*, \tau^*, t^*, s^*, g^*) / \partial t^*_n = 0, m \in \{1, \ldots, N\} \), the consumer's utility \( u^h \) may actually decrease if the tax rate \( t^*_n \) is perturbed from its initial equilibrium value by a small finite amount. However, if \( \partial u^h / \partial t^*_n > 0 \), the function \( u^h \) is increasing in \( t^*_n \) at \( t^*_n \).

6. A condition sufficient to imply that there is no solution \( a > 0^H \) to \( a^T [X^T, E^T] = 0^T_{N+M} \) is to require that rank \( [X^T, E^T] = H \). This assumption means that the number of domestic commodities (\( N \)) plus the number of tradeable goods (\( M \)) must be equal or greater than the number of households (\( H \)) (i.e., \( H \leq N+M \)). Intuitively, in order to
produce strict Pareto improvements, the government must have a sufficiently large number of free tax instruments (commodity taxes) in its disposal.

7. Choose

\[ p^T = [\Delta p^*, \Delta z^*, \Delta b^*, \Delta t^* T, \Delta s^* T, \Delta r^* T], \]

\[ Q_{11} = [I_H, O_{Hx(N+K+1)+N+M+M}] \]

\[ Q_{12} = [O_{H+K+1}^T, 1, O_{N+M+M}^T], \]

\[ Q_3 = [A, -B_p, -z, -B_b, -B_z, -B_s, -B_r]. \]

There must not exist \( \alpha_1 \in \mathbb{R}^{H+N+K+1}, \alpha_2 \in \mathbb{R}^H, \alpha_3 \in \mathbb{R} \) such that

\[ \alpha_1 A + \alpha_2 = 0^T, \alpha_1 [-B_p, -z, -B_b, -B_z, -B_s, -B_r] = 0^T \]

\[ \alpha_1 B_b + \alpha_3 = 0, \alpha_2 > 0^H, \alpha_3 > 0. \]

Choose \( \lambda^T = -\alpha_1 \) and (4.18) follows.

8. By changing the producer prices, the government can induce a change in production that corresponds to the change in consumer demand caused by a Pareto improving tax change.

9. This is because all the commodity tax rates \((t^*, s^*)\) can be adjusted in Theorem 4.1. If some producer price \((w^*_m + \tau^*_m), m \in [1, \ldots, M]\) is changed, the tax rate \(s^*_m\) can be perturbed so that the effect of the tariff change on the consumers is zero. This kind of separation of production and consumption sectors is not present Theorem 6.1 in Chapter 6.

10. The consumers and producers in the economy choose their net demands and (net) supplies by maximizing utility and profits, respectively. The government can influence these (net) demands and (net) supplies indirectly through changes in relative prices obtained by imposing taxes and tariffs on commodities. The government cannot directly choose the consumers' (net) demand vectors or the producers' (net) supplies.

11. The equilibrium that solves (4.26) is usually a second best equilibrium, since the lump sum transfers \(g\) have not been chosen optimally. Only by accident—if \(g\) happened to be fixed at the optimal level—could the equilibrium be a first best optimum.

12. See Section 2.4.

13. See (2.37) in Section 2.4.

14. If \(N = 0\), controllability of domestic goods production cannot cause problems and assumption (ii) is thus not needed. If \(N = 0\), discontinuity of sectoral net supplies cannot occur if the unit profit functions \(\pi^k(w + \tau^*), k = 1, \ldots, K\), are twice continuously differentiable. Hence, assumption (i) is not needed.
15. This is a version of the McKenzie Factor Price Equalization Theorem; McKenzie (1955).

Chapter 5:

1. The prices $w^a$ are obtained by solving the following autarky general equilibrium model:
   (i) $m^h(u^h, q, v) = g^h, h=1,\ldots,H$
   (ii) $\pi^k(p, w + \tau) = 0, k=1,\ldots,K$
   (iii) $\Sigma_{v} m^h(u^h, q, v) + e^0 = \Sigma_{w} \pi^k(p, w + \tau) z^k$
   (iv) $\Sigma_{q} m^h(u^h, q, v) + x^0 = \Sigma_{p} \pi^k(p, w + \tau) z^k$
   (v) $\tau = 0_M$
   (vi) $w_1 = 1$ (price normalization).

2. Note that the tariffs $\tau^*$ do not generate income to the government. Hence, the autarky government budget constraint is satisfied.

3. Note that in order to reach positive gains from trade the tariffs $\tau^*$ need not be reduced to zero (free trade).

4. $(t^*, s^*)$ are not $\beta$-optimal, if they do not solve the problem
   $\max \{\beta^T u: (4.10) - (4.13) \text{ hold}, \tau^* = (w^a - w), g^* = u, p, z, b, t, s \text{ constant}\}$.

5. Welfare effects of international trade that is caused by changes in the country's commodity tax structure are further discussed in Chapter 11.

6. This change involves a series of infinitesimal changes of $\tau^*$ toward $\tau^* = 0_M$.

7. Finite strict Pareto and productivity improving changes in taxes and tariffs do exist at $A$.

Chapter 6:

1. Note that the vector $s^*$ is kept fixed. Hence, a change of tariffs $\tau^*$ affects both consumers and producers.

2. A necessary condition for $\tau^*$ to be Pareto and productivity optimal is that the vector $\lambda^T B_{\tau}$ is zero; see Proposition 4.2.
3. If \( \Sigma = O_{(N+M) \times (N+M)} \), the equations \( \lambda^T B = O^T_N \) imply \(-\lambda^T X = O^T_N \). There is no positive solution to these equations, but choose \( \lambda_1 = O_{H} \). Then, \( \lambda^T A = (p^* + \delta)^T \Sigma qu + w^T \Sigma vu = p^T \Sigma qu + w^T \Sigma vu \) if \( \tau^* = O_{M} \). It is known that \( p^T \Sigma qu + w^T \Sigma vu = 1^T_H \) (money metric scaling of utilities). Hence, for this \( \lambda \), \( \lambda^T A > 0 \), \( \lambda^T [B_p, B_z, B_t] = 0^T_{N+K+N} \), \( -\lambda^T B > 0 \) and \( \lambda^T B = \tau^T \gamma^2 G(w + \tau^*, y^*) = 0^T_M \) since \( \tau^* = O_{M} \). This is sufficient to imply that no (differential) strict Pareto and productivity improvements exist, starting from the initial equilibrium, if \( \tau^* = O_{M} \) and \( \Sigma = O_{(N+M) \times (N+M)} \). The initial equilibrium thus satisfies a necessary condition for Pareto and productivity optimality. Furthermore, the initial equilibrium must be a productivity maximum since, under free trade, the amount of foreign exchange earned by the production sector is maximal (see problem (3.3)), and if the consumer substitution matrix \( \Sigma \) is a zero matrix, existence of nonzero commodity taxes does not influence the consumer net demand \( y^* \) in (3.3).

4. See Theorem 6.1. A sufficient condition implying assumption (iii) in Theorem 6.1 is that \( \text{rank } X^T = H \). This means that \( H \leq N \), i.e., the number of variable tax instruments must be at least as large as the number of consumers (or consumer groups) in the economy.

5. A sufficient condition for a strict Pareto productivity and improvement in \( \tau^* \) and \( \tau^*_n \), \( n \in [1, \ldots, N] \), to exist is: there is no \( \lambda \in R^{H+N+K+1} \) such that \( \lambda^T [A, -B_b] > 0^T_{H+1}, \lambda^T [B_p, B_z, (B_t)_n] = 0^T_{N+K+N+1}, \lambda^T B = 0^T_M \).

Chapter 7:

1. If a good \( m, m \in [1, \ldots, M] \), is a net export for sector \( k, k=1, \ldots, K \), \( \tau^*_m \geq 0 \) (\( \tau^*_m \leq 0 \)) means that production of the good is subsidized (taxed), whereas if the good is a net import for sector \( k, k=1, \ldots, K \), \( \tau^*_m \geq 0 \) (\( \tau^*_m \leq 0 \)) means that the net import is being taxed (subsidized).

2. Consider an increase in \( \tau^* \) when \( \tau^* \geq O_M \). A sufficient condition for a strict Pareto and productivity improvement to exist is:
there does not exist a vector $\lambda \in R^{H+N+K+1}$ such that $\lambda^T[A, - B_b] > 0^T_{H+1}$, $\lambda^T[B_p, B_z, B_t, B_s] = 0^T_{N+K+N+M}$, $\lambda^T B_t \leq 0^T_{M}$.

Using the same reasoning as in the proof of Theorem 7.1, this condition can be shown to be satisfied.

3. For a proportional reduction in tariffs $\tau^*$ to be strict Pareto and productivity improving, it is sufficient to show that there is no vector $\lambda \in R^{H+N+K+1}$ such that $\lambda^T[A, - B_b] > 0^T_{H+1}, \lambda^T[B_p, B_z, B_t, B_s] = 0^T_{N+K+N+M}$, $\lambda^T B_t \tau^* \geq 0$. For a vector $\lambda$ satisfying $\lambda^T[B_p, B_z, B_t, B_s] = 0^T_{N+K+N+M}$, the scalar $\lambda^T B_t \tau^*$ equals $-\tau^* \nabla^2_{ww} G(w + \tau^*, y^*)$. This number is negative, since the matrix $\nabla^2_{ww} G(w + \tau^*, y^*)$ is positive semidefinite and, by assumption, $\nabla_{\tau} b^*(w + \tau^*) \neq 0^T_{M}$.

4. Note that if $\tau^*_m \leq 0$, $m \in [1, \ldots, M]$, this change amounts to an increase in the absolute value of $\tau^*_m$. The assumption that $\nabla_{\tau} b^*(w + \tau^*)$ is negative can be interpreted as follows: if $\tau^* \geq 0^T_{M}$ (net exports are subsidized and net imports are taxed), substitution in production of tradeables must dominate in the sense that $\tau^* \nabla^2_{ww} G(w + \tau^*, y^*) > 0^T_{M}$. In this case, the uniform reduction of $\tau^*$ is equivalent to a uniform reduction of subsidies for net exports and taxes for net imports. If $\tau^* \leq 0^T_{M}$ (net exports are taxed and net imports subsidized), complementarity in production must dominate in the sense that $\tau^* \nabla^2_{ww} G(w + \tau^*, y^*) < 0^T_{M}$. Then, the uniform reduction of $\tau^*$ amounts to a uniform increase in the export taxes and to a uniform increase in the import subsidies.

5. Use the proof of Proposition 7.1 and note that when $(\tau^* - \hat{\tau}) > 0^T_{M}$, $\nabla_{\tau} b^*(w + \tau^*, y^*) (\tau^* - \hat{\tau}) < 0$ if $\nabla_{\tau} b^*(w + \tau^*, y^*) < 0^T_{M}$. 


6. It can also be shown that a strict Pareto improving increase in positive \( t^*, s^* \) and \( \tau^* \) and a strict Pareto improving decrease in negative \( t^*, s^* \) and \( \tau^* \) exist.

7. In Chapter 8 it will be shown that if lump sum transfers are available, the simultaneous reduction in taxes and tariffs can be chosen to be a proportional reduction.

8. If \( t^* = O_N, s^* = O_M, \tau^* = O_M \) and \( g^* = O_H \), the government budget constraint becomes \( p^T x^0 + w^T e^0 = 0 \), which can only hold if either \( x^0 = O_N, e^0 = O_M \), or if the government behaves like a private producer with the government budget constraint serving the role of the zero profit constraint. This implies that some of the components of \( x^0 \) and \( e^0 \) must be negative, corresponding to inputs bought to produce other domestic or tradeable goods.

9. Government policy choices and economic inequality are discussed in Chapter 12.

10. Notation: \( t_{-n} = [t_1, \ldots, t_{n-1}, t_{n+1}, \ldots, t_N]^T \in \mathbb{R}^{N-1} \).

Chapter 8:

1. Assumption (8.2) is satisfied, if for example, the matrix \( S_{pp} + YY^T \) is positive definite, i.e., (2.13) is satisfied, or if the matrix \( \Sigma_{qq} \) is of full rank \( N \). This occurs when the matrix \( \Sigma \) is of maximal rank (= \( N+M-1 \)).

2. Note that the changes in \( u^*, t^* \) and \( s^* \) are going to be zero.

3. Differentiate (4.12) and (2.9) around the initial values of the variables. The assumptions (2.12) and (8.2) are necessary and sufficient for the implicit functions \( p^*(u^*, w + \tau^*, t^*, s^*) \) and \( z^*(u^*, w + \tau^*, t^*, s^*) \) to exist. Compare this to the proof of Lemma 2.1.

4. Note that local controllability of production in the sense of Definition 2.1 is sufficient to imply local controllability of domestic goods production in the sense of Definition 8.1 (if rank \( S = N+M-1 \), the rank of the matrix \( S_{pp} \) must be \( N \)). However, local controllability of production in the sense of Definition 2.1 is not necessary for local controllability of domestic goods production in the sense of Definition 8.1.
5. If there is no substitution in consumption of domestic goods, the requirement that the consumers are kept at their initial utility levels means that the net supply of domestic commodities must stay at the initial $y^*$. Hence, Definitions 2.2 and 8.1 of local controllability of domestic goods production coincide.

6. The matrix $-\Sigma_{qq}$ is positive semidefinite.

7. See the proof of Lemma 2.2.

8. The sets $C^k$, $k=1,\ldots,K$, and $M^h(u^h)$, $h=1,\ldots,H$, are assumed to be closed and convex; in addition, the Slater constraint qualification condition is assumed to be satisfied. (If the utility functions of the consumers are quasiconcave and continuous from above, the sets $M^h$, $h=1,\ldots,H$, are convex and closed. If there is a feasible solution for (8.5) such that the inequality constraints are strictly satisfied, the Slater condition is satisfied.)

9. Note that the conditions (2.12) and (8.2) are sufficient for the function $B(w)$ to be twice continuously differentiable. They are also sufficient for the function $\nabla_w B$ to be once continuously differentiable.

10. This could be seen also by noting that the function $B(w)$ must be convex in prices $w$, as the function $B$ is convex in its arguments.

11. The weak inequality follows from the positive semidefiniteness of the matrix $(S - \Sigma)$.

12. If both matrices $S$ and $\Sigma$ are of maximal rank and $t^* = 0_N$, $s^* = 0_M$, the matrices $S$ and $\Sigma$ have a common zero eigenvector, i.e., the vector $(p^*, w + \tau^*)$. This vector is also the zero eigenvector of the matrix $(S - \Sigma)$.

13. The equivalence of these concepts is easy to visualize if it is remembered that a zero consumption substitution matrix $\Sigma$ corresponds to L-shaped indifference curves in the two commodity case.

14. Theorem 8.3 could be illustrated using Fig. 3. The Pareto improving change in tariffs and lump sum transfers shifts the economy's production choice toward the point $C$, whereas the consumer is moved toward $D$.

15. See Corollary 8.2.1.
16. Yet, finite Pareto and productivity improving changes in tariffs and transfers may exist.

17. If rank $E = N+M-1$, then rank $E_{qq} = N$. Assuming that $N \geq 1$, $E_{qq}$ must be nonzero. Hence, $E \neq O_N$ if $t^* \neq O_N$.

18. If $t^* = O_N$, $s^* = O_M$, $\tau^* = O_M$ and lump sum transfers are chosen optimally, the initial equilibrium is a productivity maximum, i.e., the first best equilibrium the economy can attain under international trade.

Chapter 9:


2. Suppose $E_{qq} = O_{N\times N}$ and $x^T = (x_1^T, x_2^T)$ is an arbitrary nonzero $(N+M)$-vector. Then, $x^T Ex = x_2^T E_{qq} x_1 + x_1^T E_{vv} x_2 + x_2^T E_{vv} x_2 = 0$, since the matrix $E$ is negative semidefinite. It is also known that $x_2^T E_{vv} x_2 \leq 0$, because $E_{vv}$ is negative semidefinite. If the matrix $E_{qq}$ were nonzero, the vector $x$ could be chosen so as to violate the inequality $x^T Ex \leq 0$. Hence, $E_{qq} = O_{N\times N}$.

3. If $E_{qq} = O_{2\times 2}$, then $E_{vv} v^* = O_{2 \times 2}$. In the Kemp-Wan example, $s^* = O_2$ which implies $E_{vv} (w + \tau^*) = O_{2 \times 2}$. The international trade prohibitive tariffs $\tau^*$ in the Kemp-Wan economy are not proportional to $w$. Then, $w^T E_{vv}$ must be nonzero because the matrix $E_{vv}$ is of maximal rank (=1) in this example.

4. The matrix $V_w f(w + \tau^*, y^*) = \varphi^2_w G(w + \tau^*, y^*)$ is a zero $(2\times 2)$-matrix at $A$.

5. It is assumed that the tax rates $(t^*, s^*)$ are chosen Diamond-Mirrlees optimally with respect to the constraints that $\tau = \tau^* (= w^a - w)$ and $g^* = constant$. Otherwise, a welfare improving change in only $(t^*, s^*)$ would exist.
6. To be exact, one must assume that the initial transfers $g^*$ are not Diamond-Mirrlees optimal with respect to the constraints $\tau = \tau^*$ ($= w^a - w$) and $t = t^*$, $s = s^*$.

7. There must be substitution in consumption, i.e., $\Sigma \neq 0^{(N+2)\times(N+2)}$.

8. See the rank $Y = K (\leq N)$ assumption in Theorem 8.3. If each production sector supplies only one good, as in the Kemp-Wan example, the requirements that each tradeable good is produced in autarky and that each sector operates at a positive scale in autarky are equivalent.

9. For $w^T \Sigma^2 w(b(w)) = 0^T_M$ to be satisfied the vector $[-(p + e)^T, w^T]$ must be a zero eigenvector of the matrix $(S - \Sigma)$ (see 8.18). It follows that, since the matrices $S$ and $-\Sigma$ are positive semidefinite, the vector $[-(p + e)^T, w^T]$ must be a zero eigenvector of $S$ and $\Sigma$. If, however, the vectors $w$ and $\tau^*$ are not proportional, $\tau^* \neq 0^T_M$, and the matrices $S$ and $\Sigma$ have only one zero eigenvector $[(p^T, (w + \tau^*)^T) S = 0^T_{N+M}, [(p + t)^T, (w + \tau^* + s)^T] \Sigma = 0^T_{N+M}]$, then the vector $[-(p + e)^T, w^T]$ cannot be a zero eigenvector of $S$ or $\Sigma$.

Chapter 10:

1. If $\tau^* = 0^T_M$, this condition cannot be satisfied since $V_t b^*(w + \tau^*)w = w^T \Sigma^2 w(b(w))w > 0$ if $V_t b^*(w + \tau^*) \neq 0^T_M$.

2. For example, if $t^* \gg 0^T_N$, there exists a uniform level of domestic commodity taxes $\bar{t} (\geq 0^T_N)$ such that a small perturbation of the initial tax rates in the direction $(t^* - \bar{t})$ is strict Pareto and productivity improving.


4. Note that this corresponds to the earlier idea of a constant utility productivity improvement which can be converted to a Pareto improvement through a perturbation in the initial vector of lump sum transfers.
5. \( x^0 \) is an unstable equilibrium. Hatta shows that the equilibrium in the economy described by his model is stable, if the Hatta normality condition is satisfied.


7. \( t_n^{*T} [v^2]_{tt} B \cdot \alpha_n < 0 \) is one of the sufficient conditions for a reduction in a positive domestic tax rate \( t_n^{*} \), \( n \in [1, \ldots, N] \), to be strict Pareto improving (assuming that the transfers \( g^* \) are adjusted appropriately).

8. See formula (12) in Hatta (1977a).

9. Lemma 2 in Hatta (1977a). Note that, using formula (12) in Hatta (1977a), if only one tax rate \( t_n^{*} \) is reduced, a sufficient condition for this change to be welfare improving is that \( p^{*T} (\Sigma_{qq} [p^{*}]) \cdot \alpha_n > 0 \), which is (10.31). Note that in Hatta's notation \( \Sigma_{qq} = F \).

Chapter 11:


2. See problem (4.26).

3. The proof of Theorem 11.1 could also be used to establish that there exist some strict Pareto and productivity improving increases in the positive commodity tax rates \( (t^*, s^*) \), or strict Pareto and productivity improving reductions in the negative \( (t^*, s^*) \). In the first case, the increased taxes on commodities bought by the consumers are balanced with increased subsidies on their work effort. In the latter case, taxes on factors sold are increased simultaneously with increases in subsidies on commodities demanded by the consumers.

4. Suppose \( W(u) = 1^T_{H} u \). If the vectors \( t^* = 0_{N} \) and \( s^* = 0_{M} \) are optimal when \( t^* = 0_{M} \), there must exist a vector \( \lambda \in \mathbb{R}^{H+N+K+1} \) such that \( \lambda^T_{A} = 1^T_{H'} - \lambda^T_{B_b} > 0 \) and \( \lambda^T_{[B_p, B_z, B_f, B_t]} = 0^T_{N+K+N+M} \) (See (4.28)). A vector satisfying \( \lambda^T_{[B_p, B_z, B_t]} = 0^T_{N+K+N} \) and
\[-\lambda^T B_d > 0\] is of the form \[\lambda^T = [\lambda^T_1, (p^* + \delta)^T, \gamma^T, 1]\] using the proof of Theorem 2.1. If \(\tau^* = 0_M\), then \(\delta = 0_N\) and \(\gamma = 0_K\), using (2.27). Choose \(\lambda_1 = 0_H\). Then, \[\lambda^T = [0^T_H, p^T*, 0^T_K, 1]\] and \(\lambda^T_A = 1^T_H\). (Note that the money metric scaling of utilities implies that \(p^T_{uH} + w^T_{vH} = 1^T_H\).)

Chapter 12:

1. It will be assumed in this chapter that all consumers utilities, which are measured using a money metric, are fully comparable. In the previous chapters, when only Pareto improvements were considered, no comparability assumption was needed. Yet, when the social welfare function \(W(u) = \beta^Tu\), \(\beta > 0_H\), was introduced, it was implicitly assumed that the household utilities are fully comparable.

2. The notion that only the households' non-labor incomes are shifted toward equality using government tax and transfer policies corresponds to Dieuwert's (1984) Capital Income Fair equilibrium concept. This in turn is based on Varian's (1976) idea of Opportunity Fairness.

3. It turns out to be convenient to denote the consumer utilities by \(u^h_h, h = 1, \ldots, H\), instead of \(u^h, h=1,\ldots, H\). Note that the definition of consumer expenditures in (12.1) differs from the definition of \(m^H\) in the equation (4.10). In (12.1), the consumer expenditures are defined as the difference between the value of (gross) purchases of domestic and tradeable commodities minus the value of any labor services supplied. In (4.10), the consumer expenditures were calculated as the difference between the value of the consumers' net consumption and the value of their labor supply.

4. Note that the observed initial equilibrium consumer prices \((q^*, v^*)\) are chosen to serve as the reference prices for the money metric utility functions. Hence, all interpersonal utility comparisons in this section are based on \((q^*, v^*)\). The choice of the social welfare function \(W(u)\) is a value judgement. An economist can only suggest properties of the function \(W(u)\) that the society might think as desirable. The function \(W(u)\) may, for example, be required to be quasiconcave, symmetric, increasing, and cardinally scaled. The last property means that social welfare is increasing along the equally distributed real income line. (This also means that \(W(\lambda^1H) = \lambda\) for all \(\lambda \in \mathbb{R}\).) An example of a
social welfare function that satisfies the above mentioned conditions is the mean of order \( r \) function
\[
W(u) = \left[ \sum_{h=1}^{H} \frac{1}{H} u_h \right]^{1/r}, \quad r < 1, \quad r \neq 0, \quad u \geq 0, \quad u_i \geq 0, \quad h=1, \ldots, H.
\]
\[
H u_{1/H}, \quad r = 0.
\]
For an increasing \( W(u) \), the welfare weights \( \beta \) are positive. If the government wants to assign negative weights to those households whose incomes exceed some "acceptable" upper limit, the government can use the mean-variance social welfare function
\[
W(u) = u - \gamma \left[ \frac{1}{H} (u - \bar{u}_H)^T (u - \bar{u}_H) \right], \quad \bar{u} = \frac{1}{H} \sum_{h=1}^{H} u_h, \quad \gamma > 0.
\]

5. Using Figure 9, a reduction in \( \Gamma^* \) also corresponds to a shift toward the \( u_1 = u_2 \) line starting from the initial equilibrium at \( S \) (if the social welfare function \( W \) is strictly quasi-concave). If \( W \) is utilitarian, the measure \( \Gamma^* \) is identically zero reflecting the governments lack of interest in the distribution of real income in the economy.


7. At least one household must have nonzero (net) demand or supply of the good in question.

8. Note that if \( \beta = 0 \), i.e., \( \beta^h = \frac{1}{H} \) for all \( h = 1, \ldots, H \), strict inequality reductions using commodity tax and tariff perturbations do not exist. In this case, the social welfare function \( W(u) \) is utilitarian, or the real incomes \( u^h \), \( h = 1, \ldots, H \), have been equalized at the initial equilibrium.

9. It is sufficient to show that there is no vector \( \lambda \in \mathbb{R}^{H+N+K+1} \) such that \( \lambda^T [B_p, B_z, B_g, B_t] = 0^T_{N+K+H+M} \), \( \lambda^T A = \beta^T \), \( \lambda^T B_b \leq 0 \). The equations \( \lambda^T B_g = 0^T_H \) imply \( \lambda_1 = 0 \). The inequality \( \lambda^T B_b \leq 0 \) implies \( \lambda_4 = k \geq 0 \). If \( k = 0 \), then \( \lambda^T A = 0^T_H \neq \beta^T \). (For \( k \geq 0 \), the proof of Theorem 8.3 shows that \( \lambda^T_2 = k(p^* + c)^T, \lambda^T_3 = k \delta^T \). If \( k = 1 \), then \( \lambda^T B_t = w^T \gamma^2 \omega^T B(u) \neq 0^T_M \) by assumption.
10. It is necessary for a strict Pareto improving change in commodity taxes and tariffs to exist that the generalized Diamond-Mirrlees condition (4.17) is satisfied. But it can be seen that inspite of the condition (4.17) being satisfied, the vector 
\[ \beta^T[X^T - C^T, E^T - D^T] \] may still be equal to zero, violating the third assumption in Theorem 12.1. (This is because \( \beta \not\in O_H \).) (If \( \beta^T[X^T - C^T, E^T - D^T] = 0^T_{N+M} \), it can be shown that no strict inequality reductions can exist, starting from the initial equilibrium.)

11. There may exist an \( a > 0 \) such that \( a^T[X^T - C^T, E^T - D^T] = 0^T_{N+M} \) but the vector \( \beta^T[X^T - C^T, E^T - D^T] \) is nonzero (\( \beta \not\in O_H \)).

12. If equation (12.16) is written in the form \( \beta^T[X^T - C^T, E^T - D^T] = -r \beta^T[X^T - C^T, E^T - D^T] \), it can be seen that there is no solution to (12.16) if the vectors \( \beta^T[X^T - C^T, E^T - D^T] \) and \( \beta^T[X^T - C^T, E^T - D^T] \) are of the same sign.
REFERENCES


Kemp, M.C. and H.Y. Wan [1983], "Trade Gains Without Lump Sum Compensation?" mimeo, University of New South Wales, Australia.


APPENDIX 1

Proof for Lemma 2.3:

Consider the equations $\lambda^T [B_p, B_z] = O_{N+K}^T$:

$$\begin{bmatrix}
    \lambda_1^T, \lambda_2^T, \lambda_3^T
  \end{bmatrix}
  \begin{bmatrix}
    S_{pp} & Y \\
    Y^T & O_{KxK} \\
    \omega^T S_{wp} & w^T F
  \end{bmatrix}
  = O_{N+K}^T.
$$

Set $\lambda_3 = k \in \mathbb{R}$. Using the equations $S_{pp}^p + S_{pw}^p (w + \tau^*) = O_N$ and $p^T Y + (w + \tau^*)^T F = O_{K}^T ((2.7)$ and $(2.5))$, (A1) can be written as

$$\begin{bmatrix}
    \lambda_1^T, \lambda_2^T
  \end{bmatrix}
  \begin{bmatrix}
    S_{pp} & Y \\
    Y^T & O_{KxK}
  \end{bmatrix}
  = k \begin{bmatrix}
    \tau^T S_{wp} + p^T S_{pp}, \tau^T F + p^T Y
  \end{bmatrix}.$$

Assumptions (i) - (iii) in Theorem 2.1 imply that the matrix on the left hand side of (A2) can be inverted and the inverse is the matrix $D$ defined in (2.20). The properties of this matrix are given in Diewert and Woodland (1977: Appendix). Equations (A2) can be solved for the vector $(\lambda_1^T, \lambda_2^T)$:
(A3) \[ [\lambda_1^T, \lambda_2^T] = k[(\lambda^* + \delta)^T, \gamma^T], \]

where

\[(A4) \quad \delta^T = \tau^*[S_{wp} D_{11} + F D_{12}^T], \gamma^T = \tau^*[S_{wp} D_{12} + F D_{22}]. \]

Equations (A3) are derived using the following properties of the matrix \( D \):

\[ D_{12}^T = D_{21}, [S_{pp} \gamma] D = [I_N, 0_{NxK}]. \quad \text{QED} \]

**Proof of Lemma 2.4:**

Using (2.11),

\[(A5) \quad \lambda^T_{B_{\tau}} = -[(\lambda^* + \delta)^T S_{pw} + \gamma^T F T + w^T S_{ww}]. \]

Applying the definitions of the vectors \( \delta \) and \( \gamma \) in (A4),

\[(A6) \quad \lambda^T_{B_{\tau}} = -k[p^T S_{pw} + \tau^T(S_{wp} D_{11} + F D_{12}^T) S_{pw} + \tau^T(S_{wp} D_{12} + F D_{22}) F^T \]

\[ + w^T S_{ww}] \]

\[ = -k[\tau^T(S_{wp} D_{11} S_{pw} + F D_{12}^T S_{pw} + S_{wp} D_{12} F^T + F D_{22} F^T) - \tau^T S_{ww}] \]
since, by the homogeneity of the unit profit functions, \( p^{TS}_{w}\ +\ w^{TS}_{ww} = -\tau^{TS}_{ww} \). Using the equations \( D_{11} S_{pp} D_{11} = D_{11}, D_{11} S_{pp} D_{12} = 0 \) and \( D_{12} S_{pp} D_{12} = -D_{22} \) given in Diewert and Woodland (1977: Appendix), (A6) may be written as

\[
(A7) \quad \lambda B_{T} = k^{*}[S_{wp} D_{11} - F D_{12}, I_{M}] S [-D_{11} S_{pw} - D_{12} F^{T}] \quad I_{M} = k^{*} T \quad G(w + \tau^{*}, y),
\]

using Lemma 2.2. QED

**Lemma 2.5:**

\[
\tau^{*} = k(w + \tau^{*}), \quad k \neq 0,1, \text{ if and only if } -\delta = k^{*}.
\]

**Proof:**

Suppose \( \tau^{*} = k(w + \tau^{*}), k \neq 0,1 \). Then, \( \tau^{*} = aw \) with \( a = k/(1-k) \).

[If \( k = 1 \), then \( w = 0 \) which violates the assumption that \( w \gg O_{M} \). If \( k = 0 \), then \( \tau^{*} = 0 \) which violates the assumption that \( \tau^{*} \neq 0 \).]

Using the definition of the vector \( \delta \) in (A4), if \( \tau^{*} = aw \),

\[
(A8) \quad \delta^{T} = \tau^{*}[S_{wp} D_{11} + F D_{12}] = aw^{T}[S_{wp} D_{11} + F D_{12}].
\]
The homogeneity of the unit profit functions and the equations

\([S_{pp} Y] D = [I_M, O_{NK}] \) imply

\[
(A9) \quad \delta^T = a\left[-p^T S_{pp} - \tau^T S_{wp}\right] D_{11} + w^T F D_{12}
\]

\[
= -a p^T S_{pp} D_{11} - a \tau^T S_{wp} D_{11} + aw^T F D_{12}
\]

\[
= -a p^T - a \tau^T Y D_{12} - a \tau^T S_{wp} D_{11} + aw^T F D_{12},
\]

which, using the zero profit condition \(p^T Y + (w + \tau)^T F = 0^T\), yields

\[
(A10) \quad \delta^T = -a\left[p^T + \tau^T S_{wp} D_{11} + \tau^T F D_{12}\right] = -a(p^T + \delta)^T.
\]

Then, \(\delta^T = \frac{a p^T}{1 + a} \) \(p^T = kp^T\).

Since all steps of the proof are if and only if statements, equivalence has been shown. QED

Note that proportionality of the vectors \(\tau^*\) and \(w\) implies proportionality of \(\tau^*\) and \((w + \tau^*)\), and hence (2.37) and (2.38) are equivalent.
APPENDIX 2

Proof of Lemma 8.1:

Let us develop the quadratic form (8.11). To simplify notation the matrix \((S - \Sigma)\) is denoted by \[
\begin{bmatrix}
B & A^T \\
A & C
\end{bmatrix}
\]

Then,

\[
\begin{align*}
\nabla^2 \tilde{B}(w) &= \tilde{A} \tilde{D}_{11} \tilde{B} \tilde{D}_{11} A^T + \tilde{F}_T \tilde{D}_{12} \tilde{B} \tilde{D}_{11} A^T \tilde{A} + \tilde{D}_{11} \tilde{B} \tilde{D}_{12} F_T \\
&\quad + \tilde{F}_T \tilde{D}_{12} \tilde{B} \tilde{D}_{12} F_T - \tilde{A} \tilde{D}_{12} F_T - \tilde{D}_{11} \tilde{A} - \tilde{D}_{12} \tilde{F}_T + C \\
&= -\tilde{A} \tilde{D}_{11} \tilde{A} - \tilde{D}_{12} \tilde{F}_T - \tilde{D}_{12} \tilde{A} - \tilde{D}_{22} \tilde{F}_T + C,
\end{align*}
\]

since \(\tilde{D}_{11} \tilde{B} \tilde{D}_{11} = \tilde{D}_{11}, \tilde{D}_{11} \tilde{B} \tilde{D}_{12} = 0, \tilde{D}_{12} \tilde{B} \tilde{D}_{12} = -\tilde{D}_{22}\) (see Diewert and Woodland (1977 : Appendix)). Then,

\[
\begin{align*}
\nabla^2 \tilde{B}(w) &= C - [A, F] A^T - F_T, \\
\end{align*}
\]

which is (8.9). QED

Proof of Lemma 8.2:

Using (8.1) and the equations \(S_{pp} \* p + S_{pw}(w + \tau^*) = 0_N\),

\(p^T_y + (w + \tau^*)_T F = 0_K\),
Assumptions (i) - (ii) in Theorem 8.1 imply that the matrix on left hand side of (B6) can be inverted. Call the inverse \( D = \begin{bmatrix} \tilde{D}_{11} & \tilde{D}_{12} \\ \tilde{D}_{21} & \tilde{D}_{22} \end{bmatrix} \). Then,

\[
(B7) \quad [\lambda_1^T, \lambda_2^T] = k[(p^* + \varepsilon)^T, \theta^T],
\]

where

\[
(B8) \quad [\varepsilon^T, \theta^T] = [-\tau^T \Sigma_{qq} - (\tau^* + s^*)^T \Sigma_{vq} + \tau^T S_{wp}, \tau^T F] D.
\]
Equations (B8) are derived using the equations \((S_{pp} - \Sigma_{qq}) D_{11} + YD_{21} = I, q^T \Sigma_{qq} + v^T \Sigma_{qv} = O^T_N\) and \((S_{pp} - \Sigma_{qq}) D_{12} + YD_{22} = 0_{NxK}^T\)

**Proof of Lemma 8.3:**

Using (8.1) and (B7),

\[(B9) \quad \lambda^T B_T = k[-(p^* + \varepsilon)^T S_{pw} + (p^* + \varepsilon)^T \Sigma_{qv} - \theta^T F - w^T(S_{ww} - \Sigma_{vv})],
\]

\[k \in \mathbb{R}.\]

Using (B5), the homogeneity of the zero profit functions and the zero profit condition \(p^T Y + (w + \tau)^T F = O^T_K\), the vector \([p^* + \varepsilon]^T, \theta^T]\) can be written as

\[(B10) \quad [(p^* + \varepsilon)^T, \theta^T] = -w^T[S_{wp} - \Sigma_{vq}, F] D.\]

Then,
Using (8.10), \( \lambda^T B_T = -kw^T q^2 B(w) \), \( k \in \mathbb{R} \). QED

Proof of Theorem 10.4:

(a) It is sufficient to show that there is no vector \( \lambda \in \mathbb{R}^{H+N+K+1} \) such that \( \lambda^T [A, - B_b] > 0^T_{H+1}, \lambda^T [B_p, B_z, B_g] = 0^T_{N+K+H}, \)

\[
 \lambda^T [B_t, B_s](t^* T, s^T T) > 0. \]

It has been established earlier that a vector \( \lambda \) satisfying \( \lambda^T [B_p, B_z, B_g] = 0^T_{N+K+H} \) must be of the form \( \lambda^T = k [0^T_H, (p^* + \varepsilon)^T, \theta^T, 1], k \in \mathbb{R} \). The inequalities \( \lambda^T [A, - B_b] > 0^T_{H+1} \) imply that \( k > 0 \). Using (4.16), the homogeneity of the unit profit and expenditure functions, and \( \lambda^T B_p = 0^T_N, \)

\[
(\text{B12}) \quad \lambda^T [B_t, B_s](t^* T, s^T T) = \begin{bmatrix} \lambda_2^T, \lambda_3^T, \lambda_4^T \end{bmatrix} \begin{bmatrix} -s_{pw} w + \Sigma_{qv} w \\ -F_w \\ -w^T S_{ww} w + w^T \Sigma_{vv} w \end{bmatrix}. 
\]
Thus, \( \lambda^T[B, B][t^T, s^T]^T = -\lambda^T B w = -w^T v_B^2 w \) using Lemma 8.3. By assumption, \( w^T v_B^2 w \neq 0 \). Hence, \( \lambda^T[B, B][t^T, s^T]^T < 0 \).

(b) It must be shown that there is no \( \lambda \in \mathbb{R}^{H+N+K+1} \) such that \( \lambda^T[A, -B] > 0 \), \( \lambda^T[B, B] = 0 \), \( \lambda^T[B, B] = 0 \), \( \lambda^T[B, B] = 0 \). For \( \lambda^T = [0^T, (p^* + \epsilon)^T, \delta^T, 1] \),

\[
\begin{align*}
(B13) \quad \lambda^T B s^* &= -(p^* + \epsilon)^T v_q \Sigma_v^* s^* - w^T v_q^* s^* \\
&= -(p^* v_q s^*)^T + s^T v_q^* w^* v_q^* s^* - w^T v_q^* s^*,
\end{align*}
\]

using (B8) with \( t^* = 0_N, \tau^* = 0_M \).

Then,

\[
\begin{align*}
(B14) \quad \lambda^T B s^* &= s^T [\Sigma_v^* + v_q^* \Sigma_v^*] s^* \\
&= -s^T v_B^2 s^* < 0,
\end{align*}
\]

since, by assumption, \( s^T v_B^2 s^* \neq 0_M \) and the matrix \( v_B^2 \) is positive semidefinite.

If \( \Sigma_v = 0_{N \times M} \), then \( \lambda^T B s^* = -s^T \Sigma_v^* s^* = w^T v_q^* s^* < 0 \), since, by assumption, \( s^T \Sigma_v \neq 0_M \). (If \( \Sigma_v = 0_{M \times N} \), then \( \Sigma_v (w + s^*) = 0_M \) and thus \( \Sigma_v w = -s^T v_q^* s^* \)).
(c) Now the vector $\lambda^{T}_B t^*$ must be shown to be negative. Using (4.15),

\[
\lambda^{T}_B t^* = -(p^* + \varepsilon)^T \Sigma_{qq} t^* - w^T \Sigma_{vq} t^*
\]

\[
= -p^* T \Sigma_{qq} t^* - w^T \Sigma_{vq} t^* + t^* T \Sigma_{qq} D_{ll} \Sigma_{qq} t^*
\]

\[
= t^* [\Sigma_{qq} + \Sigma_{qq} D_{ll} \Sigma_{qq}] t^*
\]

\[
= -t^* T _{tt}^2 B t^* < 0,
\]

since $t^* T _{tt}^2 B \neq 0^T$ and the matrix $T _{tt}^2 B$ is positive semidefinite.

If $\Sigma_{qq} = 0_{N \times M}$, then

\[
\lambda^{T}_B t^* = -(p^* + \varepsilon)^T \Sigma_{qq} t^*
\]

\[
= -p^* T \Sigma_{qq} t^* + t^* T \Sigma_{qq} D_{ll} \Sigma_{qq} t^*
\]

\[
= t^* [\Sigma_{qq} + \Sigma_{qq} D_{ll} \Sigma_{qq}] t^*
\]

because $\Sigma_{qq} (p^* + t^*) = 0_N$. Hence, $\lambda^{T}_B t^* = -t^* T _{tt}^2 B t^* < 0$, since $t^* T _{tt}^2 B \neq 0$. QED
Proof of Proposition 10.1:

Suppose \( t^* \gg 0_N \). It is sufficient to prove that there is no vector \( \lambda \in \mathbb{R}^{H+N+K+1} \) such that \( \lambda^T [A, -B_b] > 0_{H+1}^T, \lambda^T [B_p, B_z, B_g] = 0_{H+K+1}^T, \lambda^T B_h > 0 \), where the vector \( h \) is defined as the direction of the change in \( t^* : h = (t^* - \bar{t}) > 0_N \), where \( \bar{t} \) is the vector of uniform taxes, \( \bar{t} = (\bar{t}, ..., \bar{t}) \in \mathbb{R}_+^N \). Then, \( \lambda^T B_t h = t^* T \bar{t}^2 B h < 0 \) if, as assumed, \( t^* T \bar{t}^2 B < 0_N^T \) and \( h > 0_N \). The other cases are proved in a similar way.

QED