TWO ESSAYS ON FINANCIAL ECONOMICS:

I. WEIGHTED UTILITY, RISK AVERSION AND PORTFOLIO CHOICE

II. COMPETITIVE BIDDING AND INTEREST RATE FORMATION IN AN INFORMAL FINANCIAL MARKET

by

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Abstract

This thesis consists of two essays. Each essay addresses a research problem involving some aspects of uncertainty and financial economics. Essay I deals with the general question of whether classical results in risk aversion and portfolio choice based on expected utility hypothesis are robust with respect to recent works in nonlinear utility theories generalizing expected utility. We investigate the implications of an axiomatic generalization called weighted utility theory along with the weaker, but unaxiomatized linear Gâteaux utility.

We establish the equivalence among three definitions of individual global risk aversion, i.e., in terms of conditional certainty equivalent, mean preserving spread, and conditional risky-asset demand, without any differentiability assumptions about the preference functional. The only requirement is that the preference ordering be complete, transitive, consistent with first-degree stochastic dominance, and continuous in distribution. The equivalence between the first two definitions is also extended to a comparative context.

We also identify the necessary and sufficient condition for the single risky asset to be a normal good to a weighted utility maximizer with concave lottery-specific utility functions. Unlike its expected utility counterpart, which depends only on the agent's initial wealth and

preferences, this condition also depends on the characteristics of the risky asset.

The second essay examines the role of a sequential competitive bidding process in the endogenous determination of interest rates and the corresponding allocation of loans and savings in a widely observed class of informal financial markets called the 'rotating credit association'. Optimal bidding strategies are obtained for individual agents with concave and time-additive utility functions.

After deriving some comparative statics and efficiency implications of the individual optimal bidding strategy, we impose further restrictions, including risk neutrality, to obtain a tractable form of a Nash equilibrium bidding strategy. This yields, for each agent, an ex post borrowing, as well as lending, interest rate depending on the history of the realized winning bids, including the one for the period in which he won the auction. Weighted by the Nash equilibrium-induced probability of winning in each period, ex ante borrowing and lending interest rates result.

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ESSAY I

WEIGHTED UTILITY, RISK AVERSION AND PORTFOLIO CHOICE

0

INTRODUCTION

0.1 Expected Utility and Finance: History

Given the nature of the topics it studies, finance as a discipline needs tractable, and yet rich enough theories about preferences under uncertainty. Due to its simplicity, expected value was once popular as a criterion for decision making under uncertainty. Investors' risk aversions, evident in the purchase of various types of insurance, the diversification of portfolios, etc., however cast doubts on its theoretical and behavioral validity. In response, two approaches, namely the mean-variance analysis and the expected utility theory, emerged as improved criteria for decision making under uncertainty.

First investigated by Tetens (1789), mean-variance analysis had its impact on finance only after the works of Markowitz (1952a, 1959) and Tobin (1958), and has since laid the foundation for modern portfolio theory. Under certain assumptions, Sharpe (1964), Lintner (1965) and Treynor (1961) derived the capital asset pricing model which establishes the linear relationship between the expected rate of return on a risky asset and its market risk. The simplicity and intuitiveness of mean-variance analysis has led to its widespread acceptance in finance.

Parallel and complementary to mean-variance analysis is the development of expected utility which was first axiomatized by Ramsey (1926), revived by von Neumann and Morgenstern (1947), and refined by Marshak

(1950), Samuelson (1952), Herstein and Milnor (1953), Savage (1954), Anscombe and Aumann (1963), Pratt, Raiffa and Schlaifer (1964), DeGroot (1970), Fishburn (1970), Arrow (1971) and others. Derived from a set of plausible axioms that lead to a simple, tractable representation form, expected utility has provided the foundation for the microeconomics of uncertainty in the last three decades.

Despite its tractability, mean-variance analysis suffers from several theoretical weaknesses. Specifically, Borch (1969) and Feldstein (1969) showed that mean-variance analysis violates stochastic dominance (i.e., a lottery may be preferred to another lottery that always delivers a better outcome with a higher probability) in the sense that, given an indifference curve, one can always construct in the better-than region a lottery which is stochastically dominated by a lottery on the given indifference curve.[1]

It is also well known that, under expected utility theory, mean-variance analysis is valid only if one of the following two assumptions holds: (1) agents have quadratic von Neumann-Morgenstern utility functions; (2) the underlying random variable is normally distributed. Either

[1] Borch (1969) showed that, given two lotteries A and B indifferent in the mean-variance sense, we can always construct another pair of lotteries A' and B' such that $\mu_A = \mu_{A'}$, $\sigma_A = \sigma_{A'}$, $\mu_B = \mu_{B'}$, $\sigma_B = \sigma_{B'}$, but A' stochastically dominates B'. Specifically, let $\mu_A > \mu_B$, $\sigma_A > \sigma_B$, and $p = (\mu_A - \mu_B)^2/[(\mu_A - \mu_B)^2 + (\sigma_A - \sigma_B)^2]$, $\lambda = (\mu_A - \mu_B)/(\sigma_A - \sigma_B)$, $x = \mu_A - \lambda\sigma_A = \mu_B - \lambda\sigma_B$, $y_1 = \mu_A + \sigma_A/\lambda$, $y_2 = \mu_B + \sigma_B/\lambda$. If A' is the lottery of getting x with 1-p chance and getting y_1 with p chance and B' is the lottery of getting x with 1-p chance and getting y_2 with p chance, then A' and B' are such two lotteries.

requirement is not satisfactory. The normal distribution is at times overly restrictive in modelling finance phenomena. The quadratic utility function, on the other hand, is unappealing because it requires a bounded domain and implies increasing absolute risk aversion.

In the area of finance, the following are lines of research that have direct ties with expected utility theory. Pratt (1964) and Arrow (1971) started the literature now known as the theory of risk aversion. Pratt characterized risk aversion by risk premium and, for an expected utility decision maker with von Neumann-Morgenstern utility function u, identified indexes $-u''(x)/u'(x)$ and $-xu''(x)/u'(x)$ as the measure of absolute and relative local risk aversion, respectively (the former is now known as the Arrow-Pratt index). He further characterized comparative risk aversion via risk premium, probability premium, and the Arrow-Pratt index. He also identified the class of utility functions which will exhibit decreasing, constant, or increasing absolute/relative risk aversion, as well as the operations that will preserve such properties.

Arrow independently established justifications for decreasing absolute risk averison and increasing relative risk aversion. He also characterized, in terms of decreasing absolute risk aversion, the normality of risky-asset demand in a one-safe-asset-one-risky-asset world where initial wealth is deterministic. In addition, the safe asset will be a luxury good if the investor's relative risk aversion is increasing in his wealth.

Cass and Stiglitz (1970; 1972) extended Arrow's investigation of wealth effects into the world of many risky assets. They showed that, if the agent's preference exhibits a 'separation property' so that the risky assets as a whole can be viewed as a mutual fund, then Arrow's normality result will continue to hold for these risky assets collectively. They also identified the class of utility functions that have the separation property to either be quadratic or display constant relative risk aversion. Hart (1975) further proved that the separation property is both necessary and sufficient for multiple risky assets to be normal.

Alternatively, by restricting utility functions to those representing the same ordinal preferences, Kihlstrom and Mirman (1974) were able to generalize Arrow and Pratt's characterization of comparative risk aversion to a world of many commodities and thereby investigate its implications for agents' consumption and savings choice. Motivated by Kihlstrom and Mirman's definition of risk aversion, Paroush (1975) further proposed a natural generalization of Arrow-Pratt's risk premium to the world of multiple commodities.

In another direction, Levy and Kroll (1978), Ross (1981), and Kihlstrom, Romer and Williams (1981) investigated the case where the investible initial wealth is random rather than deterministic. It was shown that, unlike the case of deterministic initial wealth, a more risk averse (in the sense of Arrow-Pratt) individual need not be willing to pay a higher premium to insure against a given risk than his less risk averse counterpart. Ross (1981) proposed a stronger condition under which Arrow-Pratt's characterization of comparative risk aversion will carry through even if the initial wealth is random.

While Arrow and Pratt focused their attention on the relation between properties of the utility function and the behavioral implications of risk aversion, Rothschild and Stiglitz (1970; 1971) characterized a decision maker's risk aversion by his response to a particular type of increase in risk, termed 'mean perserving spread'. They showed that the concavity of utility functions and the preference for a distribution over its mean preserving spreads are equivalent in their characterization of risk aversion. Moreover, each of these is more general than the once popular variance criterion in the sense that the former implies, but is not implied by, the latter. Following Rothschild and Stiglitz' mean preserving spread concept, Diamond and Stiglitz (1974) proposed a similar notion called 'mean utility preserving spread' which is useful for characterizing comparative risk aversion for preferences that are linear in probability distribution.

The above investigations along with others not mentioned here constitute the literature on risk aversion which relies critically on the assumption that the agent in question maximizes his expected utility.

0.2 Alternative Preference Theories

While expected utility seems to have served finance well, its empirical validity has been questioned by decision scientists in light of certain widely reported violations of its implications, including the Allais paradox and the concurrence of risk-seeking and risk-averting behavior within an interval of monetary outcomes. Given the prevalence and persistence of these phenomena, the quiet acceptance of expected utility in finance should not be interpreted as unreserved content. Rather, as will be illustrated below, it is mainly due to its tractability and the lack of a viable alternative preference theory. We will, in this section, touch upon several alternative theories that have been reported in the literature and then, in the remainder of this essay, focus on two approaches which hold promise to open some new paths for investigating financial economics beyond expected utility.

Misperception-of-Probability Theories

This line of investigations, started by Edwards (1954), attempts to account for the reported expected utility anomalies by replacing the probability weights p_i in the expected utility expression for a finite lottery, i.e. $\Sigma p_i u(x_i)$, by a nonlinear function $f(p_i)$ (often interpreted as some subjective probabilities) which may not add up to unity.

Subsequent adherents to this view include Handa (1977), who adopted the form $\Sigma f(p_i)x_i$, Karmarkar (1978), who normalized the weights using the form $\Sigma[f(p_i)/\Sigma f(p_i)]u(x_i)$, and Kahneman and Tversky (1979), who edited lotteries before deciding which of their two evaluation equations to be applied.

There are at least three problems with any approach whose representation takes the form $\Sigma f(p_i,x_i)$. First of all, it can only apply to finite lotteries. To see this, suppose the probability distribution is continuous over [a,b]. It is not known how we can calculate $\Sigma f(p_i,x_i)$ or $\int f(p_i,x_i)$.[2] Since finance is frequently concerned with problems involving continuous distributions, this is a serious limitation that casts doubt on their appeal to financial economists as a general decision rule.

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[2] This is because the limit of the value $\Sigma f(p_i,x_i)$ or $\int f(p_i,x_i)$ of any sequence of finite lotteries converging to a continuous probability distribution does not exist.

Another problem with any Edwardsian theory is its inherent tendency to violate stochastic dominance. Kahneman and Tversky (1979) added the detection-of-dominance operation to circumvent this difficulty but paid a steep price in violating transitivity (Chew, 1980; Machina, 1982a).

The third problem with the misperception-of-probability approach lies in its inability to display global risk aversion except when $f(p_i) = p_i$, in which case the expected utility obtains. Global risk aversion, in the sense of preference for a distribution over any of its mean-preserving-spreads (i.e. distributions with equal mean but higher variability) is under expected utility equivalent to pointwise local risk aversion (i.e. aversion towards actuarially fair infinitesimal risks). Since global risk aversion is regarded as an appealing property in finance, financial economists might be reluctant to replace expected utility by theories which are inherently unable to display such a preference property.

General Preference Functionals

Motivated by the paradoxes in his name, Allais is probably the first to consider a preference functional more general than that of expected utility. He argued that a decision maker's preference is represented by a function of the moments of the distribution of some 'cardinal psychological value' functions. Suppose V is the Allais' preference functional. Then

$$V = V(m_1, m_2, \ldots),$$

where

$$m_i = \int s(x)^i dF(x).$$

This model is indeed very general and contains both expected utility and mean-variance analysis as special cases. When the preference depends only on the first moment m_1, expected utility obtains. In Allais' view, expected utility's inability to describe Allais-type choice behavior is because higher moments are unduly ignored. Another special case -- mean-variance analysis -- results when the cardinal psychological value function $s(x)$ is linear in the underlying monetary outcome x, and only the first two moments matter.

It is interesting to note that, if $s(x)$ is nonlinear, then $V = V(m_1,m_2)$ becomes the preference functional for a 'utility mean-variance' analysis -- a case yet to be explored. One possible direction is in developing a 'utility-mean-variance' capital asset pricing model in light of the success and tenacity of the present mean-variance-based capital asset pricing model. Based on Borch (1969)'s illustration, however, it is clear that utility-mean-variance analysis, like mean-variance analysis, will also violate stochastic dominance.

Two other alternative theories to expected utility also belong in the category of general preference functionals - one is Machina's (1982a) 'expected utility analysis' without the independence axiom, the other is weighted utility theory recently proposed in Chew and MacCrimmon (1979a & b), Chew (1980; 1981; 1982; 1983), Fishburn (1983) and Nakamura (1984). The rest of the essay will be devoted to detailed discussions on their assumptions, representation forms, as well as results potentially interesting to finance.[3]

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[3] Other approaches not considered here include the information processing models (e.g. Payne, 1973), Meginniss' (1977) 'entropy' preference, and the theory of regret proposed by Bell (1982) and Loomes and Sugden (1983).

0.3 Organization of the Essay

This essay focuses on comparisons of three preference theories, namely expected utility, weighted utility and linear Gâteaux utility, in terms of their conditions, properties and implications of risk aversion. In Section 1, we summarize their representation functional forms. To avoid mathematic technicalities, they are all stated as hypotheses. We regard the consistency with first-degree stochastic dominance as a property that any useful preference theory must possess. The condition for this property under different theories are given.

In Section 2, we investigate the properties and implications of individual risk aversion, including local risk aversion and global risk aversion. The distinction is important because they do not coincide beyond expected utility. Different notions of risk aversion are defined. Among them are risk aversion in the sense of conditional and unconditional certainty equivalent, risk aversion in terms of mean preserving spread, and pointwise local risk aversion. Risk aversion in terms of conditional certainty equivalent and risk aversion in terms of mean preserving spread are shown to be equivalent regardless of the underlying preference theory. This result only requires the preference to be complete, transitive, consistent with first-degree stochastic dominance and continuous in distribution. On the contrary, some of the risk aversion definitions are only equivalent under expected utility.

Section 3 introduces portfolio choice problem in a world with one safe asset and one risky asset. It is showed that a risk averse agent will never short sell the risky asset as long as its expected return is strictly greater than the safe return. When they are equal, his best strategy is to invest only in the safe asset. If he is a weighted utility maximizer, then he will invest a positive amount in the single risky asset if and only if its expected return is strictly greater than the safe return.

We further show that, if an agent will invest in the risky asset only if its expected return is strictly greater than the safe return, then he must be globally risk averse in the sense of conditional certainty equivalent. This amounts to the equivalence between global conditional-certainty-equivalent risk aversion and global conditional portfolio risk aversion for any preference ordering satisfying completeness, transitivity and first-degree stochastic dominance.

Section 4 characterizes comparative risk aversion across individuals. Again, we prove that comparative risk aversion in terms of conditional certainty equivalent is equivalent to comparative risk aversion in terms of mean preserving spread without depending on specific utility functional forms.

In Section 5, we derive the necessary and sufficient condition for the risky asset to be a normal good for a weighted utility maximizer. This result utilizes the explicit functional form of weighted utility, and is not obtainable under linear Gâteaux utility.

In Section 6, stochastic wealth is introduced to the characterization of decreasing and comparative risk aversion. It appears that weighted utility and linear Gâteaux utility provide more room in allowing additional risks because their utility functions are lottery-specific. The expected utility results in this section are from Ross (1981). The results beyond expected utility were first proved by Machina (1982b) for Fréchet differentiable utility and later extended by Chew (1985) to linear Gâteaux utility. They are reproduced mainly to complete the spectrum of our comparisons.

In this essay, definitions, expressions, lemmas and corollaries are numbered according to the section and the order in which they appear. For the convenience of comparisons across different preference theories, theorems are in general labelled with U, EU, WU or LGU, followed by an Arabic number. Obviously, EU, WU and LGU stand for expected utility, weighted utility, linear Gâteaux utility, respectively. U is used when the result does not depend on a specific preference functional form. The Arabic number indicates the nature of the result. A summary is given below:

Theorem No.	Regarding
1	representation
2	first-degree stochastic dominance (SD)
3	Arrow-Pratt index
4	pointwise local risk aversion (PLRA)
5	global risk aversion (GRA)
6	nonnegative or positive conditional risky-asset demand
7	comparative risk aversion with deterministic wealth (CRA)
8	decreasing risk aversion with deterministic wealth (DRA)
9	comparative risk aversion with stochastic wealth
10	decreasing risk aversion with stochastic initial wealth but deterministic wealth increments
11	decreasing risk aversion with stochastic initial wealth and wealth increments

When there are more than one theorem on the same subject, decimal fractions are used in labelling them. These results are briefly summarized in Section 7 which then concludes this essay by suggesting some potential applications of non-expected utility theories in the area of financial economics.

1

PREFERENCE REPRESENTATION AND STOCHASTIC DOMINANCE

To obtain a preference representation, one can identify a set of normatively appealing axioms about preferences, then construct a preference functional which satisfies these axioms. Alternatively, one can start with a general preference ordering with very little a priori restrictions, and investigate systematically the implications of successively imposed structures. While the historical development of expected utility conformed to the first approach, the latter is useful in providing insights into the meaning of some preference implications which are derived from expected utility but may not be sensitive to the underlying theoretical structures.

Let D denote a space of probability measures on some outcome set without pre-imposed restrictions. The weakest requirement for a preference ordering is the customary completeness and transitivity. We suppose that such a preference ordering is represented by a utility functional V: $D \rightarrow R$ so that, for any F, G ε D, F is weakly preferred to G if and only if $V(F) \geq V(G)$. This rules out any lexico-graphic type preferences.

A preference representation at this level of generality is of little interest. To identify desirable structure to impose on V, define

$$F^{\alpha} \equiv (1-\alpha)F + \alpha G, \qquad \text{where } \alpha \in [0,1]. \tag{1.1}$$

In other words, F^{α} is a probability mixture of F and G. As α increases from 0 to 1, F^{α} goes from F to G. The derivative $\frac{d}{d\alpha} F^{\alpha} = G-F$ is called the 'direction' of F^{α}. The first structure one would consider to impose on V is naturally some sort of smoothness of $V(F^\alpha)$ as F^α shifts from F to G.

Assumption 1.1: $V(F^\alpha)$ is differentiable in α.

So far, we have considered probability measures defined on some outcome space which is very general and may be non-numerical. If we are simply interested in monetary outcomes, we may only consider probability measures defined on some interval J of the real line R, allowing $J \equiv R$ as a special case. We denote by D_J the space of such distributions.

1.1 Linear Gâteaux Utility, Linear Implicit Utility and Weighted Utility

We may further require that $\frac{d}{d\alpha} V(F^\alpha)$ take a specific form. Consider the following:

Assumption 1.2: For every $F \in D_J$, there exists a function $\zeta(\cdot;\cdot)$: $J \times D_J \to R$ such that, for every $F^\alpha = (1-\alpha)F+\alpha G$, $\alpha \in [0,1]$,

$$\frac{d}{d\alpha} V(F^\alpha) = \int \zeta(x;F^\alpha) d[G(x)-F(x)]. \tag{1.2}$$

In functional analysis, $\frac{d}{d\alpha} V(F^\alpha)$ is called the 'Gâteaux differential', and $\zeta(x;F^\alpha)$ the 'Gâteaux derivative' or the 'directional derivative', of $V(F^\alpha)$ at F^α in the direction of G-F (Luenberger, 1969). For a utility functional V satisfying Assumption 1.2, the Gâteaux differential is linear in the direction G-F and may therefore be called a linear Gâteaux preference functional or linear Gâteaux utility.

As it turns out, a subclass of linear Gâteaux preference functionals V can be implicitly defined by the following:

$$\int\phi(x,V(F))dF(x) = 0, \tag{1.3}$$

where ϕ: $J^2 \rightarrow R$ is increasing in x and decreasing in V(F). This class of functionals is called linear implicit utility in Chew (1984). One such example is given by the following:

$$\phi(x,V(F)) \equiv w(x)[v(x)-V(F)]; \tag{1.4}$$

or equivalently,

$$\phi(x,m) \equiv w(x)[v(x)-v(m)], \tag{1.5}$$

where m is the certainty equivalent of distribution F. $\phi(x,V(F))$ is in a sense a 'utility deviation' of an outcome x from the certainty equivalent of F.

Let

$$\phi_2(x,V(F)) \equiv \frac{\partial\phi(x,V(F))}{\partial V(F)}.$$

It can be verified that

$$\frac{\phi(x,V(F))}{-\int\phi_2(x,V(F))dF(x)} \equiv \zeta(x,V(F)): J^2 \rightarrow R \tag{1.6}$$

or

$$\frac{\phi(x,V(F))}{-\int\phi_2(x,V(F))dF(x)} \equiv \zeta(x;F): J \times D_J \rightarrow R \tag{1.7}$$

is the Gâteaux derivative of the linear implicit utility V (defined by (1.3)) at F.

This example turns out to be weighted utility -- a generalization of expected utility. Specifically, weighted utility is a subclass of linear implicit utility with the preference functional V being explicitly given by

$$V(F) = WU(F) = \frac{\int w(x)v(x)dF(x)}{\int w(x)dF(x)}, \tag{1.8}$$

where w(x) is strictly positive and called a 'weight function', v(x) is strictly increasing and called a 'value function'.

With the specific form of V(F), we can obtain its Gâteaux differential as follows:

$$\begin{aligned}
\frac{dV(F^{\alpha})}{d\alpha} &= \frac{dWU(F^{\alpha})}{d\alpha} \\
&= \lim_{\theta\to 0} \frac{1}{\theta}\left[WU(F^{\alpha+\theta})-WU(F^{\alpha})\right] \\
&= \lim_{\theta\to 0} \frac{1}{\theta}\left[\frac{\int vwdF^{\alpha+\theta}}{\int wdF^{\alpha+\theta}} - \frac{\int vwdF^{\alpha}}{\int wdF^{\alpha}}\right] \\
&= \lim_{\theta\to 0} \frac{1}{\theta}\left[\frac{\int vwdF^{\alpha+\theta}\int wdF^{\alpha} - \int vwdF^{\alpha}\int wdF^{\alpha+\theta}}{\int wdF^{\alpha+\theta}\int wdF^{\alpha}}\right] \\
&= \lim_{\theta\to 0} \frac{1}{\theta}\left[\frac{\theta\int vwd[G-F]\int wdF^{\alpha} - \int vwdF^{\alpha}\theta\int wd[G-F]}{\int wdF^{\alpha}\int wdF^{\alpha}+\theta\int wd[G-F]\int wdF^{\alpha}}\right] \\
&= \frac{\int vwd[G-F]\int wdF^{\alpha} - \int vwdF^{\alpha}\int wd[G-F]}{\int wdF^{\alpha}\int wdF^{\alpha}} \\
&= \frac{\int vwd[G-F]-WU(F^{\alpha})\int wd[G-F]}{\int wdF^{\alpha}} \\
&= \frac{\int[v(x)w(x)-WU(F^{\alpha})w(x)]d[G(x)-F(x)]}{\int w(x)dF^{\alpha}(x)} \\
&= \int\zeta(x;F^{\alpha})d[G(x)-F(x)], \qquad (1.9)
\end{aligned}$$

where

$$\zeta(x;F^{\alpha}) = \frac{w(x)[v(x)-WU(F^{\alpha})]}{\int w(x)dF^{\alpha}(x)}\ . \qquad (1.10)$$

Thus, we have verified first of all that the Gâteaux differential of a weighted utility functional is linear in the direction G-F. Secondly, since

$$\int\zeta(x;F)dF(x) = 0,$$

we have shown that weighted utility is indeed a special case of linear implicit utility.

1.2 Fréchet Differentiable Utility

Instead of Assumption 1.2, Machina (1982a) assumes that, in moving from F to G, V is smooth in the sense of Fréchet differentiability, i.e.,

$$V(G) - V(F) = L_F(G-F) + o\|G-F\|, \tag{1.11}$$

where L_F is some linear functional which depends on F. This is equivalent to assuming that, corresponding to each lottery F, there exists a function u(x;F) such that

$$L_F(G-F) = \int u(x;F)d[G-F] = \int u(x;F)dG - \int u(x;F)dF. \tag{1.12}$$

In other words, V(G)-V(F) can be approximated by the difference in the 'expected utilities' of G and F using u(x;F) as a 'local utility function'.

Given (1.11) and (1.12), it can be verified that

$$\frac{d}{d\alpha} V(F^{\alpha})\Big|_{\alpha=0} = \int u(x;F)d[G(x)-F(x)]. \tag{1.13}$$

Thus Machina's Fréchet differentiable utility is also a subclass of linear Gâteaux utility. It is less general than the latter in requiring the existence of an L^1-metric on D_J (induced by the L^1-norm on the linear space spanned by D_J) so that the little o term in (1.11) is well-defined. Machina ensures this by restricting distributions under consideration to those with supports in some compact interval, say [0,M]. We denote by $D_{[0,M]}$ the set of distributions so restricted.

The rest of this essay will develop some risk aversion implications of mainly linear Gâteaux utility and weighted utility. Given the widespread acceptance of expected utility in finance and economics, we will present the results of expected utility as a benchmark for comparisons.

Before we formally introduce the representations of various preference theories, a few clarifications on some assumptions implicitly made and terms and notations not formally defined will avoid confusion. First of all, while Fréchet differentiable utility is defined on $D_{[0,M]}$, expected utility and weighted utility in particular and linear Gâteaux utility in general can be extended to non-numerical outcome space. Since this essay takes a finance perspective and focuses on monetary outcomes, we will only present these utility theories with numerical outcomes. In other words, the von Neumann-Morgenstern utility function of an expected utility decision maker and both the value and the weight functions of a weighted utility decision maker are assumed to be $J \to R$ mappings, where J is a subset of R. We will denote by δ_x the step distribution function with all mass centered at point x. In other words, δ_x stands for the lottery of getting x for sure. Sometimes we need to specify the random variable of a distribution. In such cases, $F_{\tilde{x}}$ is used to denote the distribution of random variable $\tilde{x}$.

Secondly, we use "$\succsim$" to denote a preference ordering on D_J (or $D_{[0,M]}$, depending on the circumstances). For F, G in D_J, "$F \succsim G$" means F is weakly preferred to G. When $F \succsim G$ and $G \succsim F$, we say F and G are indifferent, denoted by "$F \sim G$". If $F \succsim G$ and not $F \sim G$, we say F is strictly preferred to G, denoted by "$F \succ G$".

Thirdly we use the following labels to shorten our statements:

EU	expected utility
WU	weighted utility
LIU	linear implicit utility

FDU Fréchet differentiable utility

LGU linear Gâteaux utility

At times, we will refer to a decision maker by his preference-representing functions. For example, we might refer to an expected utility decision maker with von Neumann-Morgenstern utility function u as 'EU decision maker u', a weighted utility decision maker with value function v and weight function w as 'WU decision maker (v,w)', an FDU decision maker with local utility functions u(x;F) as 'FDU decision maker u(x;F)' and an LGU decision maker whose preference functional V satisfies condition (1.2) as 'LGU decision maker $\zeta(x;F)$'. Or, we may identify a decision maker by his preference functional, i.e. EU for expected utility, WU for weighted utility, and V for FDU and LGU.

Finally, the terms 'decreasing', 'increasing', 'concave', 'convex', etc., are used in the weak sense. When the strict sense applies, it will be obvious by context or so indicated.

1.3 Representation

Hypothesis EU1: There exists a continuous, increasing function $u: J \to R$ such that, for any $F, G \in D_J$,

$$EU(F) \geq EU(G) \iff F \succsim G$$

where

$$EU(F) = \int u(x)dF(x). \tag{1.14}$$

It is known that the utility function u in Hypothesis EU1 exists if and only if the preference ordering $\succsim$ satisfies the following axioms:

Axiom 1 (Completeness): For any F, G ε D_J, either $F \gtrsim G$ or $G \gtrsim F$.

Axiom 2 (Transitivity): For any F, G, H ε D_J, if $F \gtrsim G$ and $G \gtrsim H$, then $F \gtrsim H$.

Axiom 3 (Solvability): For any F, G, H ε D_J, if $F \gtrsim G \gtrsim H$, then there exists a $\beta \in (0,1)$ such that $\beta F+(1-\beta)H \sim G$.

Axiom 4 (Monotonicity): For any F, G ε D_J, if $F \gtrsim G$ and $1 \geqslant \beta > \gamma \geqslant 0$, then $\beta F+(1-\beta)G \gtrsim \gamma F+(1-\gamma)G$.

Axiom 5 (Substitution): For any F, G, H ε D_J, and $p \in [0,1]$, if $F \sim G$, then $pF+(1-p)H \sim pG+(1-p)H$.

Moreover, any u and u* satisfying the relationship

$$u^* = a + bu, \quad b > 0 \tag{1.15}$$

are equivalent representations for a preference ordering.

The first four axioms of expected utility are innocuous and normatively appealing. Axiom 4 is sometimes called 'mixture-monotonicity' (Chew, 1983; 1984; Fishburn, 1983), and is equivalent to a property called 'betweenness' (Chew, 1983).

Definition 1.1: A preference ordering $\gtrsim$ on D_J is said to display the betweenness property if, for any F, G ε D_J satisfying $F \gtrsim G$ and $F^\alpha \equiv (1-\alpha)F+\alpha G$ with $\alpha \in (0,1)$, it is always true that $F \gtrsim F^\alpha \gtrsim G$.

In words, betweenness means that any probability mixture of two lotteries must be intermediate in preference between them. If $F \sim G$, then $F \sim F^\alpha \sim G$. Since $F^\alpha \equiv (1-\alpha)F+\alpha G$ for any $\alpha \in (0,1)$ lies on the line segment connecting F and G, the betweenness property implies that the agent's indifference curves in any simplex of 3-outcome lotteries must be straight lines.

The substitution axiom has been controversial and is the primary cause for the preference functional EU to be linear in distribution. It implies that the agent's indifference curves in the above-mentioned simplex must furthermore parallel each other. Many attempts to generalize expected utility have aimed at relaxing this axiom. We mentioned earlier that Machina assumes Fréchet differentiability to do away with the strong substitution axiom. How weighted utility theory does it will be elaborated shortly.

The function u in (1.14) is usually referred to as the (von Neumann-Morgenstern) utility function. Relationship (1.15) is the affine transformation that defines the uniqueness class of u.

Weighted utility theory is an axiomatic generalization of expected utility advanced by Chew and MacCrimmon (1979a; 1979b), Chew (1980; 1981; 1982; 1983), Fishburn (1983) and Nakamura (1984). Like expected utility, and distinct from other alternative approaches mentioned in Section 0, it is derived from a set of assumptions about the underlying preferences. It retains the completeness, transitivity, solvability and monotonicity axioms of expected utility theory, but weakens its (strong) substitution axiom via the following:

<u>Axiom 5'</u> (Weak Substitution): For any F, $G \in D_J$ such that $F \sim G$ and any $\beta \in (0,1)$, there exists a $\gamma \in (0,1)$ such that, for every $H \in D_J$, $\beta F+(1-\beta)H \sim \gamma G+(1-\gamma)H$.

Axiom 5' differs from Axiom 5 in allowing $\gamma \neq \beta$. γ and β however must still satisfy a relationship called 'ratio consistency':

<u>Definition 1.2</u>: For any F, G, $H \in D_J$, and β_1, β_2, γ_1, $\gamma_2 \in (0,1)$ such that $F \sim G$ and $\beta_i F+(1-\beta_i)H \sim \gamma_i G+(1-\gamma_i)H$ for $i = 1, 2$, if

$$\frac{\gamma_1/(1-\gamma_1)}{\beta_1/(1-\beta_1)} = \frac{\gamma_2/(1-\gamma_2)}{\beta_2/(1-\beta_2)}, \tag{1.16}$$

then, we say the preferences exhibit the ratio consistency property.

A proof of the following lemma appears in Chew (1983, Lemma 2).

Lemma 1.1: Axioms 1, 2, 4 and 5' imply ratio consistency.

In a simplex of lotteries involving three outcomes $\underline{x}$, x, $\bar{x} \in J$, with $\underline{x} < x < \bar{x}$ as illustrated in Figure 1.1, suppose P is the probability mixture of $\delta_{\underline{x}}$ and $\delta_{\bar{x}}$ such that $\delta_x \sim P$. Betweenness and ratio consistency together implies that the indifference curves must be straight lines which 'spoke out' from a point, say A, on the line connecting δ_x and P. (A must be to the right of δ_x or to the left of P, i.e. outside of the simplex, or transitivity will be violated.)

Hypothesis WU1: There exist a strictly increasing function v: $J \to R$ and a strictly positive function w: $J \to R$ such that, for any F, G $\in D_J$,

$$WU(F) \geqslant WU(G) \iff F \succsim G,$$

where

$$WU(F) = \frac{\int v(x)w(x)dF(x)}{\int w(x)dF(x)}. \tag{1.8}$$

The v in (1.8) is referred to as the value function and w the weight function. Suppose another pair of value and weight functions (v*,w*) also represent the same preference ordering. Then v, v*, w and w* must satisfy the following uniqueness-class transformation relationships:

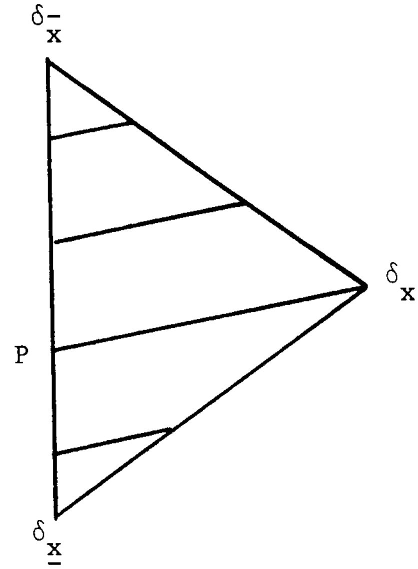
$$v^* = \frac{qv+r}{sv+t} \tag{1.17}$$

$$w^* = (sv+t)w \tag{1.18}$$

where q, t, r and s are constants satisfying $qt > rs$ and $sv+t > 0$.

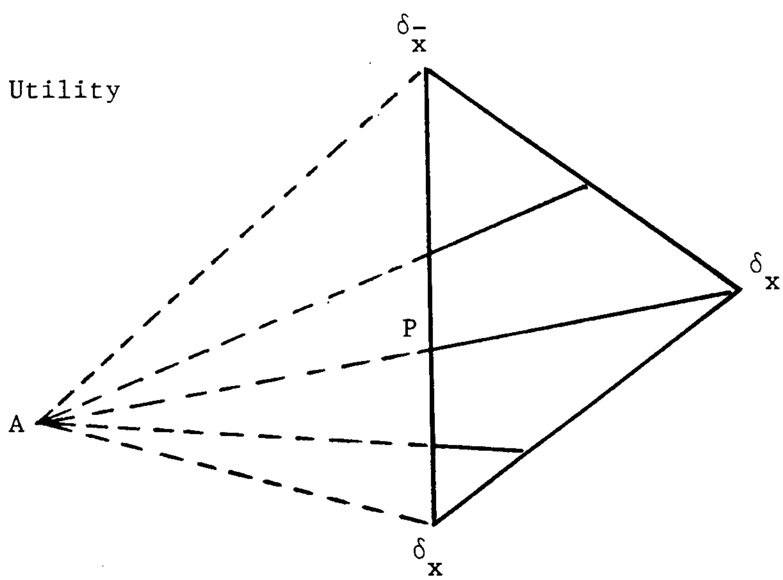
Figure 1.1: Indifference curves in a simplex of lotteries involving three outcomes $\underline{x} < x < \bar{x}$

Expected Utility



$P \equiv (1-p)\delta_{\underline{x}} + p\delta_{\bar{x}} \sim \delta_x$

Weighted Utility



Note from (1.8) that WU(F) is not linear in F and can be rewritten as:

$$WU(F) = \int v(x)dF^w(x), \tag{1.19}$$

where

$$F^w(x) = \frac{\int_{-\infty}^{x} w(t)dF(t)}{\int_{-\infty}^{+\infty} w(t)dF(t)} . \tag{1.20}$$

Clearly, when w is constant, WU will reduce to EU since $F^w(x) = F(x)$.

Because Machina's FDU analysis is not an axiomatic approach, there is not a representation theorem. For ease of comparison, we restate the approach Machina proposed as follows:

Hypothesis FDU1: There exists a Fréchet differentiable preference functional V: $D_{[0,M]} \to R$ such that

$$V(G) - V(F) = \int u(x;F)d[G-F] + o\|G-F\|. \tag{1.21}$$

Machina called the Fréchet derivative $u(x;F)$: $[0,M] \to R$ a local utility function at distribution F.

To obtain testable implications, Machina further assumed the following specific form:

$$V(F) = \int R(t)dF(t) \pm \frac{1}{2} [\int S(t)dF(t)]^2 \tag{1.22}$$

with local utility function

$$u(x;F) = R(x) \pm S(x)\int S(t)dF(t). \tag{1.23}$$

This 'quadratic in probability' functional is known to be incompatible with the betweenness property. In a simplex of 3-outcome lotteries, this means that the agent's indifference curves are in general not straight lines.

Note that the outcome space in Hypothesis FDU1 is $D_{[0,M]}$ while that in both Hypotheses EU1 and WU1 is D_J. When restricted to $D_{[0,M]}$, both EU and WU are also Fréchet differentiable. Furthermore, the functional EU in Hypothesis EU1 has a constant (with respect to distributions) Fréchet derivative u which does not depend on distribution F so that the term $o\|G-F\|$ in expression (1.21) vanishes.

The requirement of a compact outcome space means that Machina's approach might not be extendable to lotteries with non-compact supports. Since LGU contains FDU as a special case and can allow unbounded outcome space,[4] we shall adopt LGU as the most general preference functional to be discussed in this essay. Later in Section 6, we will need to impose more structure on the linear Gâteaux derivative ζ of an LGU functional in order for Machina's results to hold under LGU.

Hypothesis LGU1: There exists a preference functional $V: D_J \to R$ such that

$$\frac{d}{d\alpha} V(F^{\alpha}) = \int \zeta(x;F^{\alpha})d[G(x)-F(x)], \tag{1.2}$$

where $F^{\alpha} \equiv (1-\alpha)F+\alpha G$ and $\zeta(\cdot;\cdot): J \times D_J \to R$.

We will call $\zeta(x;F)$ the lottery specific (w.r.t. F) utility function (LOSUF) of V.

It should be pointed out that both FDU and LGU are so general that one might find them lacking in structural constraints. For instance, it is not known what transformation defines the uniqueness class for FDU or LGU type preferences.

[4] Once we introduce certainty equivalent risk aversion, the support J is required to be bounded from below. See Section 2.

We next examine the conditions needed for each preference functional to display the normatively appealing property called 'stochastic dominance'.

1.4 Stochastic Dominance

It is generally agreed that any preference ordering should be consistent with stochastic dominance defined below:

Definition 1.3: For F, G ε D_J, F is said to stochastically dominate G in the first degree, denoted by $F \geqslant^1 G$, if $F(x) \leqslant G(x)$ for all $x \in J$. If moreover $F(x) < G(x)$ for some $x \in J$, then F is said to strictly stochastically dominate G in the first degree, denoted by $F >^1 G$.

Graphically, stochastic dominance in the first degree means that F and G do not cross and F always lies below (i.e. to the right of) G.

Definition 1.4: A preference ordering $\succsim$ is said to be consistent with stochastic dominance (SD) if $F \succsim G$ whenever $F \geqslant^1 G$.

In other words, if F always delivers a better outcome with a higher probability than G, then F ought to be preferred to G. It is easy to check the following:

Lemma 1.2: If $F \geqslant^1 G$ and $F^\alpha \equiv (1-\alpha)F+\alpha G$, then $F \geqslant^1 F^\alpha \geqslant^1 F^{\alpha'} \geqslant^1 G$ for any α, $\alpha' \in (0,1)$ such that $\alpha \leqslant \alpha'$.

Therefore, regardless of the underlying preference theory, Axiom 4 (Monotonicity) implies that $V(F^\alpha)$ must decrease in α if V is to be consistent with SD.

Theorem U2 (SD): For F, G ε D_J, $F^\alpha \equiv (1-\alpha)F+\alpha G$ where $\alpha \in (0,1)$, and any preference functional $V: D_J \to R$ satisfying Assumption 1.1,

$F \geqslant^1 G$ implies $V(F) \geqslant V(G)$

if and only if

$F \geqslant^1 G$ implies $\frac{d}{d\alpha} V(F^{\alpha}) \leqslant 0$.

When V is an EU functional,

$$
\begin{aligned}
\frac{d}{d\alpha} V(F^{\alpha}) &= \frac{d}{d\alpha} EU(F^{\alpha}) \\
&= \lim_{\theta \to 0} \frac{1}{\theta} \{EU(F^{\alpha+\theta}) - EU(F^{\alpha})\} \\
&= \lim_{\theta \to 0} \frac{1}{\theta} \{\int u(x) d[(1-\alpha-\theta)F+(\alpha+\theta)G] - \int u(x) d[(1-\alpha)F+\alpha G]\} \\
&= \lim_{\theta \to 0} \frac{1}{\theta} \{\int u(x) dF^{\alpha} + \theta \int u(x) d[G-F] - \int u(x) dF^{\alpha}\} \\
&= \int u(x) d[G-F] \\
&= -\int u'(x)[G(x)-F(x)]dx \\
&= \int u'(x)[F(x)-G(x)]dx.
\end{aligned}
$$

Since $F(x)-G(x) \leqslant 0$ by the definition of $\geqslant^1$ and $u'(x) \geqslant 0$ by Hypothesis EU1, it is always true that EU is consistent with SD.

Theorem EU2 (SD): Suppose $u: J \to R$ is continuous and increasing. Then, for any $F, G \in D_J$, $F \geqslant^1 G$ implies $EU(F) \geqslant EU(G)$.

"$u' \geqslant 0$" is commonly referred to as the necessary and sufficient condition for EU to be consistent with SD. Since under EU, u is by construction an increasing function, Theroem EU2 stresses that the preferences of an EU decision maker are consistent with SD.

When V is a WU functional, recall that the Gâteaux differential of $WU(F^{\alpha})$ is

$$\frac{dWU(F^{\alpha})}{d\alpha} = \int \zeta(x;F^{\alpha}) d[G(x)-F(x)] \qquad (1.9)$$

$$= -\int [G(x)-F(x)] d\zeta(x;F^{\alpha}) \qquad \text{(by integration by parts)}$$

$$= -\int \zeta'(x;F^{\alpha})[G(x)-F(x)]dx. \qquad (1.24)$$

If $F \geqslant^1 G$, then $G(x)-F(x) \geqslant 0$ for all $x \in J$ by definition. The necessary and sufficient condition for $\frac{dWU(F^{\alpha})}{d\alpha} \leqslant 0$ is therefore $\zeta'(x;F) \geqslant 0$ for all $x \in J$ at all $F \in D_J$.

Theorem WU2 (SD): Suppose w, v are continuous and bounded; v is strictly increasing and w is strictly positive. Then, for any $F, G \in D_J$,

$F \geqslant^1 G$ implies $WU(F) \geqslant WU(G)$

if and only if

$$\zeta(x;F) = w(x)[v(x)-WU(F)]/\int w dF \qquad (1.25)$$

is an increasing function of x for all $F \in D_J$;

or, equivalently,

$$\phi(x,s) = w(x)[v(x)-v(s)] \qquad (1.26)$$

is an increasing function of x for all $s \in J$.

Confirming Theorem U2, the condition for SD under WU is that the Gâteaux differential of WU(F) be decreasing in the direction G-F. Recall that, under EU where $u'(x) \geqslant 0$ guarantees its consistency with SD, u(x) also has the functional analytical interpretation of a Gâteaux derivative, but does not depend on the distribution F because the EU representation

$$EU(F) = \int u(x)dF(x)$$

is linear in distribution. This observation led Chew and MacCrimmon to name $\zeta(x;F)$ a 'lottery specific utility function (LOSUF)'. We also apply the use of this term to the Gâteaux derivative of an LGU functional. As such, Machina's 'local utility function' is also a LOSUF. We avoid using the term 'local' here since its meaning is different from the 'local' in 'local risk aversion', to be discussed in the next section.

In Theorem WU2, the condition for SD is given in terms of both

$\zeta(x;F)$: $J \times D_J \to R$ and $\phi(x,s)$: $J^2 \to R$. Since $\int w dF$ is constant given F, conditions on $\zeta(x;F)$ and $\phi(x,s)$ are equivalent. As $\phi(x,s)$ is distribution-free, it may at times offer more intuitive interpretations.

The condition for SD in terms of $\phi(x,s)$ can be written as:

$$\phi_1(x,s) = w'(x)[v(x)-v(s)] + w(x)v'(x) \geqslant 0, \tag{1.27}$$

which in turn can be rewritten as:

$$\frac{w'(x)}{w(x)} \quad \begin{matrix} \geqslant \\ \leqslant \end{matrix} \quad -\frac{v'(x)}{v(x)-v(s)} \quad \text{for all } s \begin{matrix} < \\ > \end{matrix} x. \tag{1.28}$$

When $[\ln w(x)]'$ exists, (1.28) is equivalent to:

$$[\ln w(x)]' \quad \begin{matrix} \geqslant \\ \leqslant \end{matrix} \quad -\frac{v'(x)}{v(x)-v(s)} \quad \text{for all } s \begin{matrix} < \\ > \end{matrix} x. \tag{1.29}$$

Let $\bar{v} = \max\{v(x)\}$ and $\underline{v} = \min\{v(x)\}$. Condition (1.29) can then be rewritten as

$$\frac{w'(x)}{w(x)} = [\ln w(x)]' \leqslant \frac{v'(x)}{\bar{v}-v(x)} \qquad \text{if } w'(x) \geqslant 0, \tag{1.30}$$

$$\frac{w'(x)}{w(x)} = [\ln w(x)]' \geqslant -\frac{v'(x)}{v(x)-\underline{v}} \qquad \text{if } w'(x) < 0. \tag{1.31}$$

In other words, SD requires the rate of change of $\ln w(x)$ be bounded from above and from below by the RHS of (1.30) and (1.31), respectively -- i.e., when $w(x)$ is increasing, $\ln w(x)$ cannot increase too fast; when $w(x)$ is decreasing, $\ln w(x)$ cannot decrease too rapidly either.

In the above, we considered multiple distributions. The SD condition is to guarantee that F will be preferred to G as long as $F \geqslant^1 G$. If we are concerned with only one particular distribution, what is the meaning of ζ being increasing?

Note from expression (1.25) that $\zeta(x;F)$ has the interpretation of a "weighted utility-deviation from WU(F)" with $w(x)/\int w dF$ being the weight.

It is therefore not surprising that $\int\zeta(x;F)dF(x) = 0$. The derivative

$$\begin{aligned}\zeta'(x;F) &= \frac{w'(x)[v(x)-WU(F)]+w(x)v'(x)}{\int wdF} \\ &= \frac{w'(x)}{\int wdF}[v(x)-WU(F)] + \frac{w(x)}{\int wdF}v'(x) \qquad (1.32)\end{aligned}$$

is accordingly a "marginal weighted utility-deviation from WU(F)" -- the increase in $\zeta(x;F)$ caused by an infinitesimal increase of x, and is the combined effect of two forces represented by the two terms in (1.32). Given a distribution F, suppose x increases marginally to x^+. WU(F) and $\int wdF$ are constant. w(x) and v(x) may be viewed as unchanged. v'(x) is strictly positive. w'(x) may be positive or negative depending on whether (v,w) is optimistic or pessimistic at x. According to (1.32), when x increases, it causes two effects on ζ. First, it changes the weight w(x). Second, it changes the contingent 'sure utility' v(x). In (1.32), the first term gives the change of ζ resulting from the change in weight, holding the contingent utility-deviation at its initial level. The second term gives the positive effect on ζ caused by the increase in the contingent utility, assuming its weight has not changed. There are four possible cases as listed below:

case	v(x)-WU(F)	$\frac{w'(x)}{\int wdF}$	$\frac{w'(x)}{\int wdF}[v(x)-WU(F)]$	$\frac{w(x)}{wdF}v'(x)$	$\zeta'(x;F)$
(1)	+	+	+	+	+
(2)	+	–	–	+	?
(3)	–	+	–	+	?
(4)	–	–	+	+	+

Obviously, cases (1) and (4) pose no ambiguity. To also have $\zeta'(x;F) \geq 0$ in both case (2) and case (3) requires that the agent be not excessively pessimistic when x is better than his certainty equivalent of the distribution, and not 'overly-optimistic' either when x is below the certainty equivalent.

We will repeatedly see later that, as far as preference properties are concerned, $\zeta(x;F)$ is the WU equivalent of the von Neumann-Morgenstern utility function u(x). If this is to make sense, u(x) must also be able to give a utility-deviation interpretation. This is obviously true in light of the affine EU uniqueness transformation although it is rarely so interpreted in the literature.

For LGU, since the Gâteaux derivative of $V(F^{\alpha})$ at F is $\zeta(x;F)$, the following is true:

Theorem LGU2 (SD): Let $V: D_J \rightarrow R$ be a linear Gâteaux preference functional with LOSUF $\zeta: R \times D_J \rightarrow R$. Then,

$$F \geq^1 G \text{ implies } V(F) \dot{\geq} V(G)$$

if and only if

$$\zeta(x;F) \text{ is increasing in } x \text{ for all } F \in D_J.$$

Given the plausibility of SD, we will consider only the functions that satisfy the required conditions.

2

INDIVIDUAL RISK AVERSION

In financial economics, we often assume that decision makers are risk averse. The notion of risk aversion can however be defined differently based on different concepts. For example, if an agent's certainty equivalent for any lottery is always less than the expected value of that lottery, we may say that he is risk averse in the certainty equivalent sense. Alternatively, if the insurance premium an agent is willing to pay to trade away an arbitrary risk is always greater than the mean of that risk, then we may say that he is risk averse in the sense of insurance premium. We can even define risk aversion in terms of an agent's asset demand or his subjective value of information, etc.

In another direction, it is often of interest to distinguish agents' risk attitudes 'in the small' and 'in the large'. The oberservation that people do hold insurance policies and lottery tickets simultaneously clearly suggests that people have different attitudes towards risks of different 'sizes'. Given a wealth position, an agent's risk aversion towards infinitesimal risks is conventionally termed local risk aversion. In contrast, his risk aversion towards risks in general is called global risk aversion.

Under expected utility, however, these different risk aversion notions are all equivalent to the concavity of the utility function. Since some of these risk aversion notions are not equivalent under WU and LGU, it is necessary that we make distinctions between them. We will begin with local risk aversion.

2.1 Local Risk Aversion

In the literature of risk aversion, local risk aversion refers to risk aversion towards small risks. Suppose $\tilde{\varepsilon}$ is an arbitrary infinitesimal, actuarially fair risk and the decision maker's wealth level is x. If the decision maker always prefers his status quo to taking risk $\tilde{\varepsilon}$ (i.e. $\delta_x \succsim F_{x+\tilde{\varepsilon}}$), we would like to say that his preferences display local risk aversion at x. While the risk being considered here has to be infinitesimal, it need not be actuarially fair. Suppose $E(\tilde{\varepsilon}) \neq 0$ and the agent can pay a premium to insure against this risk. As long as the premium is greater than $E(\tilde{\varepsilon})$, it seems reasonable to say that this agent is averse towards the small risk $\tilde{\varepsilon}$.

To define local risk aversion formally, we first define the term 'insurance premium'.

Definition 2.1: If a decision maker is indifferent between $F_{x+\tilde{\varepsilon}}$ and $\delta_{x+E(\tilde{\varepsilon})-\pi}$, then π is called his (unconditional) insurance premium for risk $\tilde{\varepsilon}$ at x.

We also define 'conditional insurance premium' which will be needed for any nonlinear-in-distribution preference theories such as WU and LGU.

Definition 2.2: If a decision maker is indifferent between $pF_{x+\tilde{\varepsilon}}+(1-p)H$ and $p\delta_{x+E(\tilde{\varepsilon})-\pi}+(1-p)H$, then π is called his conditional insurance premium for risk $\tilde{\varepsilon}$ at x conditional on p and H.

First, note that the risk $\tilde{\varepsilon}$ in Definitions 2.1 and 2.2 is an arbitrary risk which need not be actuarially fair or infinitesimal. Second, the x in Definitions 2.1 and 2.2 can be interpreted as the decision maker's sure wealth position prior to taking the risk $\tilde{\varepsilon}$. In general, π will depend on x, $\tilde{\varepsilon}$, and the individual's attitudes towards risk.

Definition 2.3: If a decision maker's insurance premium for any risk $\tilde{\varepsilon}$, $\pi(x,\tilde{\varepsilon})$, is positive, then his preference is said to display (unconditional) insurance premium risk aversion (IPRA) at x.

Definition 2.4: If a decision maker's conditional insurance premium for any risk $\tilde{\varepsilon}$, $\pi(x,\tilde{\varepsilon}|p,H)$, is positive for any $p \in (0,1]$ and $H \in D_J$ it is conditional upon, then his preference is said to display conditional insurance premium risk aversion (CIPRA) at x.

Local risk aversion is a special case of unconditional IPRA because it restricts the risks under consideration to the infinitesimal ones.

Definition 2.5: If a decision maker's insurance premium for any infinitesimal risk $\tilde{\varepsilon}$, $\pi(x,\tilde{\varepsilon})$, is positive, then his preference is said to display local risk aversion (LRA) at x. If his preference displays LRA at all x, we say that it displays pointwise LRA (PLRA).

Clearly, IPRA implies LRA. We could conceivably define a term called 'conditional PLRA'. This however will not be considered in this essay.

We said previously that π depends on x, $\tilde{\varepsilon}$ and the agent's risk attitudes. Assume that the variance of $\tilde{\varepsilon}$ is σ^2. How can we express π in terms of $\tilde{\varepsilon}$, x and the agent's utility function? The now famous Arrow-Pratt index provides a convenient way.

Definition 2.6: A function $r\colon J \to R$ is an Arrow-Pratt index of a preference ordering if the (unconditional) insurance premium $\pi(x,\tilde{\varepsilon})$ for an infinitesimal risk $\tilde{\varepsilon}$ with variance $\sigma^2 \to 0$ can be written as

$$\pi(x,\tilde{\varepsilon}) = \frac{\sigma^2}{2}\, r(x+E(\tilde{\varepsilon})) + o(\sigma^2). \qquad (2.1)$$

Given Definition 2.5, Corollary 2.1 below is obvious.

Corollary 2.1: A decision maker with Arrow-Pratt index $r(x)$ is LRA at x if and only if $r(x) \geq 0$. He is PLRA if and only if $r(x) \geq 0$ at all $x \in J$.

The following theorems are well known:

Theorem EU3 (Arrow-Pratt Index): The Arrow-Pratt index of an EU decision maker with a continuous, strictly increasing, twice-differentiable von Neumann-Morgenstern utility function $u(x)$ is given by

$$r(x) = -\frac{u''(x)}{u'(x)} . \tag{2.2}$$

Theorem EU4 (PLRA): The preference of an EU decision maker u will display PLRA if and only if u is concave.

For a WU decision maker (v,w), let π be his insurance premium for an infinitesimal, actuarially fair risk $\tilde{\varepsilon}$ with small variance σ^2. It can be shown that, at wealth position x,

$$\pi = \pi(x,\tilde{\varepsilon}) = -\frac{\sigma^2}{2}\left[\frac{v''(x)}{v'(x)} + \frac{2w'(x)}{w(x)}\right] + o(\sigma^2). \tag{2.3}$$

Theorem WU3 (Arrow-Pratt Index): The Arrow-Pratt index of a WU decision maker with properly structured value function v and weight function w is given by

$$r(x) = -\left[\frac{v''(x)}{v'(x)} + \frac{2w'(x)}{w(x)}\right] = -\frac{v''(x)}{v'(x)} - \frac{2w'(x)}{w(x)}. \tag{2.4}$$

It is worth noting that, like its expected utility counterpart, the WU Arrow-Pratt index $r(x)$ in (2.4) is invariant under the uniqueness class transformations (1.17) and (1.18).

Expression (2.4) suggests that a WU decision maker's aversion toward small risks can be seen as coming from two sources represented by two additive terms. The first term $-v''/v'$ can be called the 'value-based risk aversion index' which measures risk aversion attributable to the value function v. The second term $-2w'/w$ can be interpreted as the 'perception-based risk aversion index' or simply the 'optimism (pessimism) index' that reflects certain qualities of the decision maker's perception about the prospects in question (Weber, 1982). To display PLRA, the sum of these two components must be positive at all x. The concavity of v alone is neither necessary nor sufficient for PLRA. When the weight function is constant, EU results, and r(x) reduces to the traditional Arrow-Pratt index.

By Corollary 2.1, the preference of a WU decision maker (v,w) will display LRA at x if and only if r(x) given by (2.4) is positive. Given that

$$r(x) = -\left[\frac{v''(x)}{v'(x)} + \frac{2w'(x)}{w(x)}\right] \tag{2.4}$$

$$= -\{\ln[v'(x)w^2(x)]\}', \tag{2.5}$$

the following is obvious:

<u>Theorem WU4</u> (PLRA): The preference of a WU decision maker (v,w) will display PLRA if and only if $\ln[v'(x)w^2(x)]$ is decreasing in x.

Under EU, the von Neumann-Morgenstern utility function can be recovered from the Arrow-Pratt index r(x) uniquely (up to an affine transformation) via the following:

$$u(x) = \int\exp[-\int r(x)dx]dx. \tag{2.6}$$

Therefore, two EU maximizers who share the same Arrow-Pratt index must have the same utility function.

From (2.5) above, it is clear that, under WU, what we can recapture from the Arrow-Pratt index is $v'(x)w^2(x)$. Therefore, it is possible that two distinct pairs of value and weight functions share the same Arrow-Pratt index and exhibit identical <u>local</u> risk propensities.

The conditions in Theorems EU4 and WU4 for PLRA are both necessary and sufficient. It will be interesting to know what more specific conditions are sufficient for WU to display PLRA. Corollary 2.2 below identifies two such conditions.

Corollary 2.2: The preference of a WU decision maker (v,w) will display PLRA if condition (i) or (ii) below holds:

(i) w is constant and v is concave;

(ii) v is linear and w is decreasing.

Proof: Omitted.

In case (i) where w is constant, WU reduces to EU. Consequently, v being concave is necessary and sufficient for PLRA by Theorem EU4. Under condition (ii) where v is linear, the first term of r(x) in (2.4) vanishes and a decreasing w will result in a positive Arrow-Pratt index.

To characterize LRA for an LGU decision maker V, we must first derive his Arrow-Pratt index. Suppose $\tilde{\varepsilon}$ is an infinitesimal, actuarially fair risk with small variance σ^2. Then, by the definition of insurance premium,

$$V(\delta_{x-\pi}) = V(F_{x+\tilde{\varepsilon}}).$$

Let $F^{\alpha} = (1-\alpha)F_{x+\tilde{\varepsilon}} + \alpha\delta_{x-\pi}$. We have

$$\begin{aligned}
0 = V(\delta_{x-\pi}) - V(F_{x+\tilde{\varepsilon}}) &= \int_0^1 \frac{dV(F^{\alpha})}{d\alpha}\, d\alpha \\
&= \int_0^1 \{\int \zeta(s;F^{\alpha})d[\delta_{x-\pi} - F_{x+\tilde{\varepsilon}}]\}d\alpha \qquad \text{(from (1.2))} \\
&= \int \{\int_0^1 \zeta(s;F^{\alpha})d\alpha\}d[\zeta_{x-\pi} - F_{x+\tilde{\varepsilon}}] \\
&= \int \zeta(s;F^{\alpha'})d[\delta_{x-\pi} - F_{x+\tilde{\varepsilon}}] \qquad \text{for some } \alpha' \varepsilon (0,1).
\end{aligned}$$

Hence,

$$\zeta(x-\pi;F^{\alpha'}) = \int\zeta(x+s;F^{\alpha'})dF_{\tilde{\varepsilon}}.$$

Noting that $\tilde{\varepsilon}$ is a small risk and that $F^{\alpha'} \to \delta_x$ as $\sigma^2 \to 0$, we can take the Taylor's expansion for both sides as follows:

$$\zeta(x-\pi;F^{\alpha'}) \approx \zeta(x-\pi;\delta_x) = \zeta(x;\delta_x) - \pi\,\zeta'(x;\delta_x) + O(\pi^2);$$

and

$$\begin{aligned}\int\zeta(x+s;F^{\alpha'})dF_{\tilde{\varepsilon}} &\approx \int\zeta(x+s;\delta_x)dF_{\tilde{\varepsilon}} \\ &= \int[\zeta(x;\delta_x) + s\zeta'(x;\delta_x) + \frac{s^2}{2}\zeta''(x;\delta_x) + o(s^2)]dF_{\tilde{\varepsilon}} \\ &= \zeta(x;\delta_x) + \frac{\sigma^2}{2}\zeta''(x;\delta_x) + o(\sigma^2).\end{aligned}$$

Therefore,

$$\pi = \pi(x,\tilde{\varepsilon}) = \frac{\sigma^2}{2}\left[-\frac{\zeta''(x;\delta_x)}{\zeta'(x;\delta_x)}\right] + o(\sigma^2) = \frac{\sigma^2}{2}r(x) + o(\sigma^2), \tag{2.7}$$

where

$$r(x) = -\frac{\zeta''(x;\delta_x)}{\zeta'(x;\delta_x)}. \tag{2.8}$$

When $E(\tilde{\varepsilon}) \neq 0$, expression (2.7) becomes

$$\pi = \pi(x,\tilde{\varepsilon}) = \frac{\sigma^2}{2}\left[-\frac{\zeta''(x+E(\tilde{\varepsilon});\delta_{x+E(\tilde{\varepsilon})})}{\zeta'(x+E(\tilde{\varepsilon});\delta_{x+E(\tilde{\varepsilon})})}\right] + o(\sigma^2). \tag{2.9}$$

Hence,

<u>Theorem LGU3</u> (Arrow-Pratt Index): The Arrow-Pratt index of an LGU decision maker V with continuous, strictly increasing, twice-differentiable LOSUF $\zeta(x;F)$ is given by expression (2.8) above.

<u>Theorem LGU4</u> (PLRA): The preference of an LGU decision maker V with LOSUF $\zeta(x;F)$ will display PLRA if $\zeta(x;F)$ is concave in x for all F.

Note that, the conditions of u being concave in Theorem EU4 and $\ln[v'w^2]$ being decreasing in Theorem WU4 are both necessary and sufficient while the concavity of $\zeta(x;F)$ in Theorem LGU4 is only sufficient for PLRA.

2.2 Global Risk Aversion

In contrast to local risk aversion, global risk aversion (GRA) refers to risk aversion in the large. If two distributions F and G share the same mean and G has a higher variability, then we would expect a risk averse agent to prefer F to G. Global risk aversion is such a concept.

Given the prevalence of various forms of lotteries throughout the world, it is perhaps unrealistic to require all agents not to have preference for any actuarially unfair risks -- an implication of global risk aversion. It is, however, desirable for a utility theory, be it linear or nonlinear in distribution, to be able to display some form of global risk aversion when a specific application context calls for it.

After Rothschild and Stiglitz (1970), global risk aversion in finance is better known via 'mean preserving spread'. Instead of jumping into the definition of mean preserving spread, we start with a less general, but simpler and more intuitive concept which we call 'simple mean preserving spread'.

Definition 2.7: For $F \not\equiv G$, G is said to single-cross F at x* from the left if

$$G(x)-F(x) \geqslant 0 \text{ for all } x < x^* \tag{2.10}$$

and

$$G(x)-F(x) \leqslant 0 \text{ for all } x \geqslant x^*. \tag{2.11}$$

When there is no ambiguity about the direction, we say that G and F possess the single crossing property.

Definition 2.8: G is a simple mean preserving spread (simple mps) of F if

(a) G single-crosses F from the left, and

(b) $\int[G(x)-F(x)]dx = 0.$ (2.12)

In Definition 2.8, condition (b) implies that the mean of G and F is identical; condition (a) implies that G has a greater variability than F. For a mean-variance type agent, F clearly dominates G. The single crossing requirement is however not transitive. To see this, suppose F, G and H are three distributions with the same mean. That H single-crosses G and G single-crosses F from the left does not imply that H will single-cross F. The mean preserving spread defined below via second-degree stochastic dominance is less restrictive but transitive.

Definition 2.9: For F, G ε D_J, F is said to stochastically dominate G in the second degree, denoted by $F \geqslant^2 G$, if

$$T(y) = \int_{-\infty}^{y} [G(x)-F(x)]dx \geqslant 0 \text{ for all } y \in J, \tag{2.13}$$

and

$$T(\infty) = \int_J[G(x)-F(x)] = \int_{-\infty}^{+\infty}[G(x)-F(x)]dx = 0. \tag{2.14}$$

Alternatively, G is said to be a mean preserving spread (mps) of F.

When the means of F and G exist, condition (2.14) implies that they are equal. Condition (2.13), in contrast, represents a requirement on their 'squeezed' means -- if we arbitrarily pick a point y and concentrate all the mass over $[y,\infty)$ onto y, then the squeezed mean of G must not be greater than that of F. This can be seen by rewriting condition (2.13) as (2.15) below:

$$T(y) = \left\{\int_{-\infty}^{y}xdF(x)+y[1-F(y)]\right\}-\left\{\int_{-\infty}^{y}xdG(x)+y[1-G(y)]\right\} \geqslant 0 \text{ for all } y. \tag{2.15}$$

Condition (2.13) will obtain if F and G have the single crossing property and satisfy the equal mean condition (2.14). Simple mps is therefore a special case of mps. In fact, Rothschild and Stiglitz (1970) show that an mps of F can be viewed as a result of a sequence of simple mps' of F.

Definition 2.10: A decision maker's preference is said to display mps risk aversion (MRA) at F if he always prefers F to G whenever $F \geq^2 G$. His preference is said to display global MRA (GMRA) if it displays MRA at all F.

Lemma 2.1: If $F \geq^2 G$ and $F^\alpha \equiv (1-\alpha)F+\alpha G$, then $F \geq^2 F^\alpha \geq^2 F^{\alpha'} \geq^2 G$ for any α, $\alpha' \in (0,1)$ such that $\alpha \leq \alpha'$.

Hence, regardless of the underlying preference theory, Axiom 4 (Monotonicity) implies that $V(F^\alpha)$ must decrease in α if V is to display GMRA.

Theorem U5.1 (GMRA): For F, G ε D_J, $F^\alpha \equiv (1-\alpha)F+\alpha G$ where $\alpha \in (0,1)$, and any preference functional $V: D_J \to R$ satisfying Assumption 1.1,

$$F \geq^2 G \text{ implies } V(F) \geq V(G)$$

if and only if

$$F \geq^2 G \text{ implies } \frac{d}{d\alpha} V(F^\alpha) \leq 0.$$

Another way of characterizing risk aversion in the large is via certainty equivalent.

Definition 2.11: If a decision maker is indifferent between F and δ_c, then c is said to be his (unconditional) certainty equivalent (CE) of F.

Definition 2.12: If a decision maker is indifferent between two compound lotteries $pF+(1-p)H$ and $p\delta_c+(1-p)H$, then c is said to be his conditional certainty equivalent (CCE) of F, conditional on probability p and distribution H.

Definition 2.13: If a decision maker always prefers $\delta_{\int x dF}$ to F, then his preference is said to display (unconditional) certainty equivalent risk aversion (CERA) at F. His preference is said to display global CERA (GCERA) if it displays CERA at all F.

Definition 2.14: For any F, H ε D_J and p ε (0,1], if a decision maker always prefers $(1-p)\delta_{\int x dF}+pH$ to $(1-p)F+pH$, then his preference is said to display conditional certainty equivalent risk aversion (CCERA) at F. His preference is said to display global CCERA (GCCERA) if it displays CCERA at all F.

Definition 2.13 (2.14) implies that if a decision maker is GCERA (GCCERA), then his CE (CCE) of any lottery is always smaller than the expected value of that lottery. Under expected utility, the substitution axiom requires that if G ~ F, then for any distribution H and probability p, it must be true that $pG+(1-p)H \sim pF+(1-p)H$. Since $\delta_c \sim F$, the substitution axiom implies that the CE and CCE of any distribution are identical. Therefore, CERA and CCERA are equivalent under expected utility.

Beyond expected utility, CCERA is weaker than the substitution axiom. It simply requires that if $p\delta_c+(1-p)H \sim pF+(1-p)H$, then $c \leqslant \int x dF$ for all p and H. As such, an agent's CE and CCE of a distribution F need not be equal.

A decision maker's CE can be interpreted as the amount he must be paid to give up a lottery with positive expected value. The insurance premium given in Definition 2.3 is a form of CE since, in an insurance context, agents can be viewed as seeking to sell adverse risks -- Given that he has been endowed with a risk of negative expected value, how much would he be willing to pay for trading away this risk? Therefore, we can also use insurance premia to characterize global risk aversion as below (IPRA and CIPRA are defined in Definitions 2.3 and 2.4, respectively):

Definition 2.15: A decision maker's preference is said to display global (unconditional) IPRA (GIPRA) if it displays IPRA at all wealth levels x.

Definition 2.16: A decision maker's preference is said to display global conditional IPRA (GCIPRA) if it displays CIPRA at all wealth levels x.

As CERA and IPRA are equivalent (so are CCERA and CIPRA), we will draw only the one more relevant to the issue under discussion.

Now that we have introduced GCCERA, it is necessary to impose lower boundedness on J. Consider the following property of $\succsim$:

Definition 2.17: A preference ordering $\succsim$ is said to display continuity in distribution (CD) if whenever $F \succ G$ and the sequence $\{G_n\}_{n=0}^{\infty}$ converges to G in distribution, there exists an $N > 0$ such that for every $n > N$, $F \succ G_n$.

Suppose functional V represents preference ordering $\succsim$. When G_n, n = 0, 1, ..., have compactsupports, CD means that $\{V(G_n)\}_{n=0}^{\infty}$ will converge to V(G) if $\{G_n\}_{n=0}^{\infty}$ converges to G. Graphically, CD means that the 'not-worse-than set' is closed, or the 'better-than set' is open. Clearly, CD implies Axiom 3 (Solvability).

In order for GCCERA and CD to be compatible, J must bounded from below. To illustrate, consider $F_q \equiv (1-q)\delta_{x+\theta}+q\delta_{x-[(1/q)-1]\theta}$, where $\theta > 0$, $q \in (0,1]$. SD and GCCERA imply $\delta_{x+\theta} \succ \delta_x \succsim F_q$ for $q \in (0,0.5]$, but F_q converges in distribution to $\delta_{x+\theta}$ as $q \to 0$, contradicting CD. In the remainder of this essay, we will assume that J is bounded from below whenever GCCERA is involved.

So far, we have given four definitions of unconditional global risk aversion, i.e. PLRA, GMRA, GCERA and GIPRA, and two of conditional ones, i.e. GCCERA and GCIPRA. The following is obvious in light of their definitions:

<u>Corollary 2.3</u>: GCCERA $\rightarrow$ GCERA $\rightarrow$ PLRA.

We have also pointed out that GCERA and GIPRA are equivalent (so are GCCERA and GCIPRA). Beyond this, how are they linked together?

It turns out that, GCCERA and GMRA are equivalent regardless of the underlying preference theories. In what follows, we first prove this equivalence for 'elementary lotteries' and then extend it to arbitrary monetary lotteries.

<u>Definition 2.18</u>: A lottery of the form $\Sigma_{i=1}^{N} \frac{1}{N}\delta_{x_i} \equiv \frac{1}{N}\delta_{x_1} + \ldots + \frac{1}{N}\delta_{x_N}$ is called an <u>elementary lottery</u>, denoted by $\underset{\sim}{x} \equiv (x_1, x_2, \ldots, x_N)$.

In other words, an elementary lottery is a lottery which gives N outcomes x_1, ..., x_N with uniform probability 1/N. Note that x_1, ..., x_N need not be distinct. Thus, any lottery involving a finite number of outcomes with rational probabilities can be expressed as an elementary lottery.

The following is due to Hardy, Littlewood and Poyla (1934):

<u>Definition 2.19</u>: For vectors $\underset{\sim}{x}$, $\underset{\sim}{y} \in J^N$, $\underset{\sim}{x}$ is a <u>majorization</u> of $\underset{\sim}{y}$ (or $\underset{\sim}{x}$ <u>majorizes</u> $\underset{\sim}{y}$), denoted by $\underset{\sim}{x} \geqslant^m \underset{\sim}{y}$, if

(a) $$\Sigma_{i=1}^{n} x_i \geqslant \Sigma_{i=1}^{n} y_i \quad \text{for all } 1 \leqslant n \leqslant N \qquad (2.16)$$

and

(b) $\Sigma_{i=1}^{N} x_i = \Sigma_{i=1}^{N} y_i$, (2.17)

where the elements of $\underset{\sim}{x}$ and $\underset{\sim}{y}$ have been arranged in ascending order. When inequality (2.16) holds strictly for at least one n, $\underset{\sim}{x}$ is said to <u>majorize $\underset{\sim}{y}$ strictly</u>.

In Definition 2.19, condition (b) implies that $\underset{\sim}{x}$ and $\underset{\sim}{y}$ share the same mean. There is a sense in condition (a) that $\underset{\sim}{x}$ is more 'centered' towards the mean than $\underset{\sim}{y}$ when $\underset{\sim}{x}$ majorizes $\underset{\sim}{y}$. This sounds similar to the mean preserving spreads given in Definition 2.9. As it turns out, majorization is equivalent to the second-degree stochastic dominance for elementary lotteries.

<u>Lemma 2.2</u>: For elementary lotteries, $\underset{\sim}{x} \geqslant^{m} \underset{\sim}{y}$ if and only if $\underset{\sim}{x} \geqslant^{2} \underset{\sim}{y}$.

Proof: Express $\underset{\sim}{x}$ and $\underset{\sim}{y}$ as $F \equiv \Sigma_{i=1}^{N} \frac{1}{N}\delta_{x_i}$ and $G \equiv \Sigma_{i=1}^{N} \frac{1}{N}\delta_{y_i}$, respectively.

Since

$$\int_{-\infty}^{+\infty} x dF(x) = \Sigma_{i=1}^{N} \frac{1}{N} x_i = \frac{1}{N} \Sigma_{i=1}^{N} x_i ,$$

$$\int_{-\infty}^{+\infty} x dG(x) = \Sigma_{i=1}^{N} \frac{1}{N} y_i = \frac{1}{N} \Sigma_{i=1}^{N} y_i ,$$

condition (2.14) implies and is implied by equality (2.17).

(Sufficiency) We prove (2.16) by induction as follows. First, we show that $x_1 \geqslant y_1$. Suppose the contrary that $x_1 < y_1$. Without loss of generality, also assume $x_n \leqslant y_1 < x_{n+1}$, where n = 1, ..., or N. Consider z = y_1. Then,

$$\begin{aligned} \int_{-\infty}^{z} x dF(x) + z[1-F(z)] &= \frac{1}{N} \Sigma_{i=1}^{n} x_i + \frac{N-n}{N} z = \frac{1}{N} x_1 + \Sigma_{i=2}^{n} x_i + \frac{N-n}{N} z \\ &\leqslant \frac{1}{N} x_1 + \frac{n-1}{N} z + \frac{N-n}{N} z \\ &< \frac{1}{N} y_1 + \frac{N-1}{N} z = \int_{-\infty}^{z} x dG(x) + z[1-G(z)]. \end{aligned}$$

This contradicts (2.15), a condition for $F \geqslant^2 G$. Therefore, $x_1 \geqslant y_1$.

We next assume that, for some $k < N$,

$$\Sigma_{i=1}^{k} x_i \geqslant \Sigma_{i=1}^{k} y_i. \tag{2.18}$$

It remains to be shown that $\Sigma_{i=1}^{k+1} x_i \geqslant \Sigma_{i=1}^{k+1} y_i$. Suppose the contrary that

$$\Sigma_{i=1}^{k+1} x_i < \Sigma_{i=1}^{k+1} y_i. \tag{2.19}$$

Inequalities (2.18) and (2.19) together imply that $x_{k+1} < y_{k+1}$. Assume without loss of generality that $x_{k+1} \leqslant x_n < y_{k+1} \leqslant x_{n+1}$. Let $z = y_{k+1}$.

$$\begin{aligned}
\int_{-\infty}^{z} x dF(x) + z[1-F(z)] &= \frac{1}{N} \Sigma_{i=1}^{n} x_i + \frac{N-n}{N} z \\
&= \frac{1}{N} \Sigma_{i=1}^{k+1} x_i + \frac{1}{N} \Sigma_{i=k+2}^{n} x_i + \frac{N-n}{N} z \\
&\leqslant \frac{1}{N} \Sigma_{i=1}^{k+1} x_i + \frac{N-k-1}{N} z \\
&< \frac{1}{N} \Sigma_{i=1}^{k+1} y_i + \frac{N-k-1}{N} z \\
&= \int_{-\infty}^{z} x dG(x) + z[1-G(z)].
\end{aligned}$$

This again contradicts (2.15). Hence, $\Sigma_{i=1}^{k+1} x_i \geqslant \Sigma_{i=1}^{k+1} y_i$. By induction, it follows that

$$\Sigma_{i=1}^{n} x_i \geqslant \Sigma_{i=1}^{n} y_i \quad \text{for all } 1 \leqslant n \leqslant N. \tag{2.16}$$

(Necessity) For any $z \in J$, suppose without loss of generality that $x_n \leqslant z < x_{n+1}$ and $y_k \leqslant z < y_{k+1}$. There are three cases to consider: (i) $k = n$, (ii) $k < n$, (iii) $k > n$.

Case (i): $k = n$

$$\begin{aligned}
\int_{-\infty}^{z} x dF(x) + z[1-F(z)] &= \frac{1}{N} \Sigma_{i=1}^{n} x_i + \frac{N-n}{N} z \\
&\geqslant \frac{1}{N} \Sigma_{i=1}^{n} y_i + \frac{N-n}{N} z \\
&= \int_{-\infty}^{z} x dG(x) + z[1-G(z)].
\end{aligned}$$

Case (ii): $k < n$

$$\int_{-\infty}^{z} x dF(x) + z[1-F(z)] = \frac{1}{N} \Sigma_{i=1}^{n} x_i + \frac{N-n}{N} z$$

$$\geq \frac{1}{N} \Sigma_{i=1}^{n} y_i + \frac{N-n}{N} z$$

$$= \frac{1}{N} \Sigma_{i=1}^{k} y_i + \frac{1}{N} \Sigma_{i=k+1}^{n} y_i + \frac{N-n}{N} z$$

$$\geq \frac{1}{N} \Sigma_{i=1}^{k} y_i + \frac{N-k}{N} z$$

$$= \int_{-\infty}^{z} x dG(x) + z[1-G(z)].$$

Case (iii): $k > n$

$$\int_{-\infty}^{z} x dF(x) + z[1-F(z)] = \frac{1}{N} \Sigma_{i=1}^{n} x_i + \frac{N-n}{N} z$$

$$\geq \frac{1}{N} \Sigma_{i=1}^{n} y_i + \frac{k-n}{N} z + \frac{N-k}{N} z$$

$$\geq \frac{1}{N} \Sigma_{i=1}^{n} y_i + \frac{1}{N} \Sigma_{i=n+1}^{k} y_i + \frac{N-k}{N} z$$

$$= \frac{1}{N} \Sigma_{i=1}^{k} y_i + \frac{N-k}{N} z$$

$$= \int_{-\infty}^{z} x dG(x) + z[1-G(z)].$$

Q.E.D.

We proved Lemma 2.2 by verifying that the conditions for majorization are equivalent to the conditions for second-degree stochastic dominance. It is easy to check that condition (2.14) is satisfied for elementary lotteries $\underset{\sim}{x}$ and $\underset{\sim}{y}$ if and only if (2.17) holds. The equivalence between (2.15) and (2.16) is not as direct. That (2.16) implies (2.15) is verified via straightforward algebra. That (2.16) is also necessary for (2.15) is proved by induction.

For elementary lotteries, condition (2.15) implies that the 'z-squeezed mean' of $\underset{\sim}{x}$ (i.e., the probability measure on $[z,x_N]$ is squeezed to the point z, where $z \leq x_N$) must not be smaller than the likewise squeezed mean of $\underset{\sim}{y}$. In contrast, condition (2.16) says that the 'n-element partial mean' of $\underset{\sim}{x}$ (i.e., the mean of a reduced vector $(x_1,\dots,x_n)$ with $n \leq N$)

must not be less than that of $\underset{\sim}{y}$. Note that a 'squeezed lottery' is the original lottery with the right tail beyond a fixed point being 'squeezed' to that point while a 'partial lottery' is a truncated lottery of the original one with uniform (conditional) probability 1/n. The following observation should provide more intuition for the equivalence between (2.15) and (2.16).

Corollary 2.4: For elementary lotteries $\underset{\sim}{x}, \underset{\sim}{y} \in J^N$, $\underset{\sim}{x} \geqslant^m \underset{\sim}{y}$ implies

$$\underset{\sim}{x} \geqslant^m \underset{\sim}{y}^1 \geqslant^m \cdots \geqslant^m \underset{\sim}{y}^k \geqslant^m \cdots \geqslant^m \underset{\sim}{y}^{N-1} \equiv \underset{\sim}{y},$$

where

$$\begin{aligned}
\underset{\sim}{x} &\equiv (x_1,x_2,x_3,\ldots,x_n,x_{n+1},x_{n+2},\ldots,x_{N-1},x_N)\\
\underset{\sim}{y}^1 &= (y_1,x_2+(x_1-y_1),x_3,\ldots,x_n,x_{n+1},x_{n+2},\ldots,x_{N-1},x_N)\\
&\cdots\\
\underset{\sim}{y}^n &= (y_1,y_2,y_3,..,y_n,x_{n+1}+\Sigma_{i=1}^n(x_i-y_i),x_{n+2},\ldots,x_{N-1},x_N) \qquad (2.20)\\
&\cdots\\
\underset{\sim}{y}^{N-1} &= (y_1,y_2,y_3,\ldots,y_n,y_{n+1},y_{n+1},\ldots,y_{N-1},x_N+\Sigma_{i=1}^{N-1}(x_i-y_i))\\
&= (y_1,y_2,y_3,\ldots,y_n,y_{n+1},y_{n+1},\ldots,y_{N-1},y_N) \equiv \underset{\sim}{y}.
\end{aligned}$$

Proof: Omitted since it is straightforward.

Corollary 2.4 means that, if $\underset{\sim}{x}$ majorizes $\underset{\sim}{y}$, then $\underset{\sim}{y}$ can be obtained from $\underset{\sim}{x}$ via a finite sequence of mps' or majorizations. From (2.16) and (2.17), we know that $x_1 \geqslant y_1$ and $x_N \leqslant y_N$. Starting from $\underset{\sim}{x}$, $\underset{\sim}{y}^1$ is obtained by pushing x_1 leftwards to y_1 and simultaneously pushing x_2 rightwards by a distance of x_1-y_1. Label this new position as z_2. To obtain $\underset{\sim}{y}^2$ from $\underset{\sim}{y}^1$, again push z_2 leftwards to y_2 (by a distance of $[x_2+(x_1-y_1)]-y_2$) and push x_3 rightwards by the same distance to a position labeled z_3. Continue this process until $\underset{\sim}{y}$ results. Corollary 2.4 tells us that $\underset{\sim}{y}$ will be obtained after N-1 such operations. Since at the <u>ith</u> step (i = 1, 2, ..., or N-1), we push x_i downward and x_{i+1} upward by the same distance, $\underset{\sim}{y}^i$ must be an mps of $\underset{\sim}{y}^{i-1}$. $\underset{\sim}{x} \geqslant^m \underset{\sim}{y}$ implies that, after each iteration, say the <u>ith</u> one, z_{i+1} (the position where x_{i+1} has been pushed to) must be to the right of y_{i+1} so that at the next iteration, the push of z_{i+1} to y_{i+1} is always a leftward one. Only N-1, rather than N, iterations are needed because z_N must coincide with y_N if $\underset{\sim}{x}$ and $\underset{\sim}{y}$ are to have the same mean.

To show that GCCERA and GMRA are equivalent for elementary lotteries, we need Lemma 2.3 below:

<u>Lemma 2.3</u>: Under completeness, transitivity and SD, GCCERA implies that, for every a, ε, θ, p (ε, $\theta > 0$, $p \in (0,1]$), and $H \in D_J$,

$$F \equiv p\{\tfrac{1}{2}\delta_{a-\varepsilon} + \tfrac{1}{2}\delta_{a+\varepsilon}\} + (1-p)H \ \gtrsim\ p\{\tfrac{1}{2}\delta_{a-\varepsilon-\theta} + \tfrac{1}{2}\delta_{a+\varepsilon+\theta}\} + (1-p)H \equiv G.$$

Proof: Let $q = \varepsilon/(\varepsilon+\theta)$. Then,

$$F \ \gtrsim\ p\{\tfrac{q}{2}\delta_{a-\varepsilon-\theta} + \tfrac{1-q}{2}\delta_a + \tfrac{1}{2}\delta_{a+\varepsilon}\} + (1-p)H$$

$$\gtrsim\ p\{\tfrac{q}{2}\delta_{a-\varepsilon-\theta} + (1-q)\delta_a + \tfrac{q}{2}\delta_{a+\varepsilon+\theta}\} + (1-p)H \ \gtrsim\ G.$$

Q.E.D.

<u>Theorem U5.2</u> (GRA): For elementary lotteries and preference ordering $\gtrsim$ satisfying completeness, transitivity and SD, GCCERA <=> GMRA.

Proof: (<=) This is straightforward.

(=>) Suppose $\underset{\sim}{x} \equiv \Sigma_{i=1}^N \frac{1}{N}\delta_{x_i} \geqslant^2 \Sigma_{i=1}^N \frac{1}{N}\delta_{y_i} \equiv \underset{\sim}{y}$. Lemma 2.2 tells us that $\underset{\sim}{x} \geqslant^m \underset{\sim}{y}$, which by Corollary 2.4 implies that $\underset{\sim}{y}$ can be obtained from $\underset{\sim}{x}$ via the sequence $\underset{\sim}{y}^n$ given by (2.20). Since $\underset{\sim}{y}^{n-1}$ and $\underset{\sim}{y}^n$ are the following elementary lotteries:

$$\underset{\sim}{y}^{n-1} \equiv \Sigma_{i=1}^{N} \frac{1}{N}\delta_{y_i^{n-1}} = \frac{N-2}{N}\{\Sigma_{i=1}^{n-1} \frac{1}{N-2}\delta_{y_i} + \Sigma_{i=n+2}^{N} \frac{1}{N-2}\delta_{x_i}\}$$
$$+ \frac{2}{N}\{\frac{1}{2}\delta_{x_n + \Sigma_{i=1}^{n-1}(x_i - y_i)} + \frac{1}{2}\delta_{x_{n+1}}\}$$
$$\underset{\sim}{y}^{n} \equiv \Sigma_{i=1}^{N} \frac{1}{N}\delta_{y_i^{n}} = \frac{N-2}{N}\{\Sigma_{i=1}^{n-1} \frac{1}{N-2}\delta_{y_i} + \Sigma_{i=n+2}^{N} \frac{1}{N-2}\delta_{x_i}\}$$
$$+ \frac{2}{N}\{\frac{1}{2}\delta_{y_n} + \frac{1}{2}\delta_{x_{n+1}+\Sigma_{i=1}^{n}(x_i - y_i)}\},$$

Lemma 2.3 implies that

$$\underset{\sim}{x} \succsim \underset{\sim}{y}^1 \succsim \cdots \succsim \underset{\sim}{y}^n \succsim \cdots \succsim \underset{\sim}{y}^{N-1} \equiv \underset{\sim}{y}.$$

Q.E.D.

The next task is to extend Theorem U5.2 from elementary lotteries to general monetary lotteries. This can be done via CD.

<u>Theorem U5.3</u> (GRA): Under completeness, transitivity, SD and CD, GCCERA <=> GMRA.

Proof: Omitted since it is a special case of Theorem U7.2.

Theorems U5.1 - U5.3 are results on GRA. Because they only require preferences to be complete, transitive, consistent with SD (and in addition be continuous in distribution for Theorem U5.3), and do not depend on specific preference functional forms, we regard them as 'theory-free', and accordingly label them with letter U.

In the literature of risk aversion, it is well known that, under EU, GMRA <=> GCCERA <=> GCERA <=> concavity of u. Machina (1982a) proves that, for the more general Fréchet differentiable utility, GMRA <=> GCCERA <=> the concavity of local utility functions u(x;F). Theorem U5.3 tells us that the equivalence between GMRA and GCCERA is actually more fundamental than believed and does not even depend on Fréchet differentiability. It is true for EU, WU, LIU, FDU, as well as LGU.

When more structures are imposed on the preference functional, stronger implications of global risk aversion are obtainable. The following theorem on EU is well known.

Theorem EU5 (GRA): For an EU decision maker with a continuous, increasing von Neumann-Morgenstern utility function $u(x)$, the following properties are equivalent:

(a) GCCERA;

(b) (Concavity) $u(x)$ is concave;

(c) GCERA;

(d) PLRA.

Given Theorem EU5, we may say that an EU decision maker is GRA if he has a concave utility function, and is consequently GCCERA, GMRA, GCERA and PLRA. Once we depart from EU, we must be specific about the sense of global risk aversion being referred to. For instance, under WU which weakens the (strong) substitution axiom to the weak substitution axiom, a conditional risk aversion definition will imply, but will not be equivalent to, its unconditional counterpart.

Theorem WU5 (GRA): For a WU decision maker (v,w) with LOSUF $\zeta(x;F)$, the following are equivalent:

(a) GCCERA;

(b) (Concavity) $\zeta(x;F)$ is concave in x for all F;

Proof: (a) $\rightarrow$ (b): Suppose there exists H such that $\zeta(x;H)$ is strictly convex in x. Then, for any $x_1 < x_2 < x_3$, there exists q such that

$$0 < \frac{\zeta(x_2;H)-\zeta(x_1;H)}{\zeta(x_3;H)-\zeta(x_1;H)} < q \tag{2.22}$$

$$q < \frac{x_2-x_1}{x_3-x_1} < 1. \tag{2.23}$$

Inequality (2.22) implies that

$$\zeta(x_2;H) < q\zeta(x_3;H)+(1-q)\zeta(x_1;H). \tag{2.24}$$

Define $F \equiv q\delta_{x_3}+(1-q)\delta_{x_1}$. (2.24) becomes

$$\int\zeta(x;H)d\delta_{x_2} < \int\zeta(x;H)dF,$$

or

$$\int\zeta(x;H)d[F-\delta_{x_2}] > 0. \tag{2.25}$$

Let $G = (1-p)F+pH$ and $G' = (1-p)\delta_{x_2}+pH$. Since by expression (1.9),

$$\frac{d}{dp} WU(G)\Big|_{p=1} = \int\zeta(x;H)d[H-F]$$

and

$$\frac{d}{dp} WU(G')\Big|_{p=1} = \int\zeta(x;H)d[H-\delta_{x_2}],$$

inequality (2.25) implies

$$\int\zeta(x;H)d[F-\delta_{x_2}] = \frac{d}{dp}[WU(G')-WU(G)]\Big|_{p=1} > 0.$$

Since $WU(G')\big|_{p=1} = WU(H) = WU(G)\big|_{p=1}$, we have

$$WU((1-p)\delta_{x_2}+pH) < WU((1-p)F+pH) \tag{2.26}$$

for some p sufficiently close to 1. Since $x_2 > qx_3+(1-q)x_1 = \int xdF(x)$ from (2.23), stochastic dominance implies that

$$WU((1-p)\delta_{x_2}+pH) \geqslant WU((1-p)\delta_{\int xdF}+pH). \tag{2.27}$$

(2.26) and (2.27) together imply $WU((1-p)F+pH) > WU((1-p)\delta_{\int xdF}+pH)$, contradicting GCCERA.

(b) $\rightarrow$ (a): Let F, G ε D_J be such that F $\geq^2$ G and define $F^\alpha \equiv (1-\alpha)F+\alpha G$, $\alpha \ \varepsilon \ (0,1)$. Extend expression (1.24) as follows:

$$\frac{dWU(F^\alpha)}{d\alpha} = -\int\zeta'(x;F^\alpha)[G(x)-F(x)]dx \qquad (1.24)$$

$$= -\int\zeta'(x;F^\alpha)d\int_{-\infty}^{x}[G(t)-F(t)]dt$$

$$= -\int\zeta'(x;F^\alpha)dT(x) \qquad \text{(from (2.13))}$$

$$= \int T(x)d\zeta'(x;F^\alpha)$$

$$= \int T(x)\zeta''(x;F^\alpha)dx. \qquad (2.28)$$

Given $T(x) \geq 0$ for all x, $\zeta(x;F)$ being concave in x for all F implies that $\frac{dWU(F^\alpha)}{d\alpha} \leq 0$. By Theorem U5.1, this yields GMRA. Since GMRA <=> GCCERA according to Theorem U5.3, we have GCCERA.

Q.E.D.

Theorems WU5, U5.3 and Corollary 2.3 together give the following relations for WU: $\zeta''(x;F) \leq 0$ at all F <=> GMRA <=> GCCERA => GCERA => PLRA. The following example is provided to demonstrate that, under WU, PLRA does not imply GCCERA in general.

Example 2.1: (PLRA does not imply GCCERA under WU.)

PLRA requires

$$r(x) = -\left[\frac{v''(x)}{v'(x)} + \frac{2w'(x)}{w(x)}\right] \geq 0 \qquad (2.29)$$

for all x. Pick $v(x) = a+bx$, $b > 0$, $x \ \varepsilon \ [0,\infty)$. We have $v''(x) = 0$. Consequently, $r(x) = -\frac{2w'(x)}{w(x)} > 0$ for all x, which implies $w'(x) < 0$ for all x. (Obviously, we are not interested in the case with $w'(x) = 0$.)

Given $v''(x) = 0$ for all x, the GCCERA condition reduces to

$$\zeta''(x;F) = \frac{2w'(x)v'(x) + w''(x)[v(x)-WU(F)]}{\int wdF}. \qquad (2.30)$$

In (2.30), the first term in the numerator of the RHS is always negative. For a strictly convex, decreasing w, we can always construct a distribution F with WU(F) sufficiently small so that the second term $w''(x)[v(x)-WU(F)]$ is sufficiently positive, leading to $\zeta''(x;F) > 0$. Similarly, for a strictly concave, decreasing w, we can always construct a distribution F with WU(F) sufficiently large so that $\zeta''(x;F) > 0$. Hence, PLRA does not imply CCERA in general. ■

Two questions are of interest here: (1) Under what conditions will PLRA imply GCCERA? (2) When will PLRA imply GCERA?

The condition for PLRA is

$$r(x) = -\left[\frac{v''(x)}{v'(x)} + \frac{2w'(x)}{w(x)}\right] \geq 0 \tag{2.29}$$

at all x. The condition for GCCERA is

$$\zeta''(x;F) = \frac{w''(x)[v(x)-WU(F)]+2w'(x)v'(x)+w(x)v''(x)}{\int w dF} \leq 0 \tag{2.31}$$

for all F, which implies, after omitting the argument x:

$$-\left[\frac{v''}{v'} + \frac{2w'}{w}\right] \geq \max\left\{\frac{w''}{w}\frac{v-\underline{v}}{v'},\ -\frac{w''}{w}\frac{\bar{v}-v}{v'}\right\}, \tag{2.32}$$

which can be restated as conditions (2.33) and (2.34) below, noting w, v' > 0:

$$-\left[\frac{v''}{v'} + \frac{2w'}{w}\right] \geq \frac{w''}{w}\frac{v-\underline{v}}{v'} \qquad \text{if } w'' \geq 0, \tag{2.33}$$

$$-\left[\frac{v''}{v'} + \frac{2w'}{w}\right] \geq -\frac{w''}{w}\frac{\bar{v}-v}{v'} \qquad \text{if } w'' < 0. \tag{2.34}$$

The LHS of (2.33) and (2.34) is the WU Arrow-Pratt index. PLRA and GCCERA will be equivalent if condition (2.29) and condition (2.33) or (2.34) coincide. Clearly, a linear weight function will do.

Corollary 2.5: For a WU decision maker (v,w) with linear weight function w, the following are equivalent:

(a) GCCERA;

(b) $\zeta(x;F)$ is concave in x for all F;

(c) GCERA;

(d) PLRA.

Proof: Omitted.

According to this corollary, a decision maker's preference can display GMRA, GCCERA, GCERA and PLRA simultaneously, and yet does not subscribe to EU. This suggests a potential choice of preference model to those who recognize the restrictiveness of EU but are appalled at the complexity of an 'all-out' WU.

We now turn to the second question: When will PLRA and GCERA be equivalent? According to Corollary 2.5, they are equivalent when w is linear. This is only sufficient however. There are other conditions under which PLRA will imply GCERA (but not necessarily GCCERA). For instance, consider a concave v. A sufficient condition for PLRA is w decreasing. Recall that

$$WU(F) = \frac{\int v(x)w(x)dF(x)}{\int w(x)dF(x)} = \int v(x)dF^{w}(x), \tag{1.19}$$

where

$$F^{w}(x) = \frac{\int_{-\infty}^{x} w(t)dF(t)}{\int_{-\infty}^{+\infty} w(t)dF(t)} \,. \tag{1.20}$$

If w is decreasing, F will stochastically dominate F^{w} in the first degree.

Lemma 2.4: Suppose the weight function w is decreasing. Then, for any distribution F, $F \succ^{1} F^{w}$.

Proof: Omitted.

Since (v,w) with v concave will display GCERA when w is constant (a case of EU), (v,w) will be even more inclined to display GCERA when w is strictly decreasing. In such a case, a linear v will suffice for GCERA.

Given the discussion above, Corollary 2.6 is stated without proof.

Corollary 2.6: If w is decreasing and v is concave and at least one of the conditions holds strictly, then PLRA is equivalent to GCERA.

The result of Theorem WU5 can be extended to the more general LGU:

Theorem LGU5 (GRA): For an LGU functional $V: D_J \to R$ with LOSUF $\zeta: J \times D_J \to R$, the following are equivalent:

(a) GCCERA;

(b) (Concavity) $\zeta(x;F)$ is concave in x for all F.

Proof: Omitted since it is similar to the proof of Theorem WU5.

Comparing Theorems EU5, WU5 and LGU5, we observe the following similarities and distinctions. First, the von Neumann-Morgenstern utility function u(x) in condition (b) of Theorem EU5 is replaced by the LOSUF $\zeta(x;F)$ in both Theorem WU5 and Theorem LGU5, confirming that the LOSUF $\zeta(x;F)$ is the non-EU equivalent of the von Neumann-Morgenstern utility function. Second, the GCCERA condition appears in all theorems because it is the strongest form of global risk aversion in the sense that it implies both GCERA and PLRA. Third, unlike Theorem EU5, the GCERA and PLRA conditions are absent in Theorems WU5 and LGU5 because they are implied by, but not equivalent to, the GCCERA and concavity conditions.

In the next section, we will utilize an agent's demand for risky asset to introduce another definition of global risk aversion, called 'portfolio risk aversion'.

3

PORTFOLIO CHOICE PROBLEM

From a finance viewpoint, our ultimate interest in risk aversion lies in its implications for asset demand. We shall present the result in this direction under expected utility and see how other preference functional approaches depart from it. We restrict our investigation to a one-safe-asset-one-risky-asset world.

Definition 3.1: An investment environment which provides only one safe asset with (gross) rate of return r, and one risky asset with (gross) rate of return $\tilde{z}$ is called a simple portfolio set-up.

We will hereafter refer to an asset by its rate of return. The notations to be used in this section are summarized below:

r: gross rate of return on the safe asset;

$\tilde{z}$: gross rate of return on the risky asset;

y_o: positive initial wealth;

x: dollar amount invested in the risky asset;

y_o-x: dollar amount invested in the safe asset;

β: proportion of y_o invested in the risky asset;

$1-\beta$: proportion of y_o invested in the safe asset;

$\tilde{y}$: final wealth.

Definition 3.2: Problem (3.1) below is an investor's (unconditional) simple portfolio choice (SPC) problem:

To find x^* such that, for every $x \neq x^*$, $F_{\tilde{y}^*} \gtrsim F_{\tilde{y}}$, (3.1)

where

$$\tilde{y} = y_o r + x(\tilde{z}-r) \tag{3.2}$$

$$\tilde{y}^* = y_o r + x^*(\tilde{z}-r) \tag{3.3}$$

<u>Definition 3.3</u>: Problem (3.4) below is an investor's <u>conditional simple portfolio choice (CSPC) problem:</u>

To find x^* such that, for every $x \neq x^*$,

$$pF_{\tilde{y}^*}+(1-p)H \succsim pF_{\tilde{y}}+(1-p)H, \tag{3.4}$$

where $\tilde{y}$ and $\tilde{y}^*$ are given by (3.2) and (3.3), respectively, $p \in (0,1]$, and H is a distribution independent of F.

Without a priori restrictions on the preference ordering, it is not guaranteed that the optimal x^* will be unique. It seems desirable to impose the following regularity:

<u>Definition 3.4</u>: In a simple portfolio set-up with safe asset r and risky asset $\tilde{z}$, an investor with initial wealth y_o is said to be an <u>(unconditional) diversifier</u> if his preferences over the set of distributions $\{F_{y_o r+x(\tilde{z}-r)}\}$ are strictly quasi-concave in x.

<u>Definition 3.5</u>: In a simple portfolio set-up with safe asset r and risky asset $\tilde{z}$, an investor with initial wealth y_o is said to be a <u>conditional diversifier</u>, conditional on p and H, if his preferences over the set of distributions $\{pF_{y_o r+x(\tilde{z}-r)}+(1-p)H\}$ are strictly quasi-concave in x.

Again, let us use a simplex of 3-outcome lotteries to illustrate the interpretation of an investor being a diversifier. Recall that, in such a simplex, the indifference curves are parallel straight lines for EU-type preferences, are nonparallel straight lines fanning out from an exterior point for WU-type preferences, and are arbitrary nonintersecting smooth curves for LGU-type preferences. Assume $F_{\tilde{z}} \equiv (1-p)\delta_{\underline{z}}+p\delta_{\bar{z}}$ and $\delta_{y_o\underline{z}}$, $\delta_{y_o r}$, $\delta_{y_o\bar{z}}$ are the three vertices of a simplex, where $\underline{z} < r < \bar{z}$ and $p\bar{z}+(1-p)\underline{z} \geq r$. When $x = 0$, $F_{\tilde{y}} \equiv \delta_{y_o r}$. As x increases, $F_{\tilde{y}}$ will move along a path and $V(F_{\tilde{y}})$ will vary. At $x = y_o$, the path reaches $(1-p)\delta_{y_o\underline{z}}+p\delta_{y_o\bar{z}}$. In general, this path will cross numerous indifference curves. If V is strictly quasi-concave in x, $V(F_{\tilde{y}})$ will either be monotone from $x = 0$ to $x = y_o$ or increase in x up to the optimal x* and then decrease. This will be ensured if all better-than sets in the simplex are convex. Clearly, a GMRA EU or WU investor is generically an unconditional, as well as conditional, diversifer.

Corollary 3.1: A strictly GMRA WU investor is always a conditional diversifier.

Proof: The CSPC problem for a WU investor (v,w) with LOSUF ζ is

$$\max_x \; WU(pF_{y_o r+x(\tilde{z}-r)}+(1-p)H). \tag{3.5}$$

The first and second derivatives of WU above w.r.t. x are given in (3.6) and (3.7) below, respectively:

$$\int\zeta'(y;G)(z-r)dF(z), \tag{3.6}$$

$$\int\zeta''(y;G)(z-r)^2dF(z), \tag{3.7}$$

where $G \equiv pF_{\tilde{y}}+(1-p)H$ and $\tilde{y} = y_o r+x(\tilde{z}-r)$. With $\zeta(y;G)$ strictly concave in y for all $G \in D_J$, (3.7) is always negative, i.e. WU is strictly quasi-concave in x.

Q.E.D.

On the contrary, an LGU functional V(F) does not generally possess such a property. For example, Dekel (1984) showed that, under FDU, the concavity of local utility functions $u(x;F)$ and the quasi-concavity of the preference functional $V(F)$ are jointly sufficient for the demand preferences over assets to be quasi-concave. For LGU, the requirement of an investor being a diversifer means that some degree of arbitrariness in the indifference curves is removed so as to rule out cases where simple portfolio choice decisions might yield undesirable multiple solutions.

We define the x^* that solves an investor's SPC problem (3.1) or CSPC problem (3.4) as his demands for risky asset $\tilde{z}$:

Definition 3.6: Suppose x^* solves the SPC problem (3.1) uniquely for an investor with initial wealth y_o. Then, x^* is called his (unconditional) money risky-asset demand at y_o and $\beta^* \equiv x^*/y_o$ is called his (unconditional) proportional risky-asset demand at y_o.

Definition 3.7: Suppose x^* solves the CSPC problem (3.4) uniquely for an investor with initial wealth y_o. Then, x^* is called his conditional money risky-asset demand at y_o and $\beta^* \equiv x^*/y_o$ is called his conditional proportional risky-asset demand at y_o.

When it is unambiguous, we may omit 'money' and 'proportional' in the above terms. Note that we do not rule out shortsales in both the SPC and the CSPC problems. This however will not be an issue here because it is assumed throughout this essay that $E(\tilde{z}) \geq r$, which implies that x^* is always nonnegative.

Theorem U6 (Nonnegative Conditional Risky-Asset Demand): Suppose x^* solves the CSPC problem (3.4) for any investor whose preferences are complete, transitive and exhibit SD and GCCERA. Then,

(1) x^* $(\beta^*) \geq 0$ if $E(\tilde{z}) \geq r$; moreover, x^* $(\beta^*) = 0$ if $E(\tilde{z}) = r$;

(2) $x^* > 0$ only if $E(\tilde{z}) > r$.

Proof: (1) Suppose $E(\tilde{z}) \geq r$ but $x^* < 0$. Then,

$$E(\tilde{y}^*) = y_o r + x^*[E(\tilde{z})-r] \leq y_o r.$$

SD implies that, for any $p \in [0,1)$ and $H \in D_J$,

$$(1-p)\delta_{E(\tilde{y}^*)}+pH \lesssim (1-p)\delta_{y_o r}+pH.$$

GCCERA implies

$$(1-p)\delta_{E(\tilde{y}^*)}+pH \gtrsim (1-p)F_{\tilde{y}^*}+pH.$$

Hence, by transitivity,

$$(1-p)\delta_{y_o r}+pH \gtrsim (1-p)F_{\tilde{y}^*}+pH.$$

This contradicts the optimality of x^*.

When $E(\tilde{z}) = r$, $E(\tilde{y}) = y_o r + x[E(\tilde{z})-r] = y_o r$ for all x. GCCERA implies

$$(1-p)\delta_{E(\tilde{y})}+pH \equiv (1-p)\delta_{y_o r}+pH \gtrsim (1-p)F_{\tilde{y}}+pH,$$

implying $x^* = 0$.

(2) Suppose $x^* > 0$ but $E(\tilde{z}) < r$. (We need not consider $E(\tilde{z}) = r$ in light of (1) above.) Then, $E(\tilde{y}^*) = y_o r + x^*[E(\tilde{z})-r] < y_o r$. GCCERA, SD and transitivity imply that

$$(1-p)\delta_{y_o r}+pH \gtrsim (1-p)\delta_{E(\tilde{y}^*)}+pH \gtrsim (1-p)F_{\tilde{y}^*}+pH,$$

contradicting optimality of $x^* > 0$.

Q.E.D.

The results of Theorem U6 can be outlined as below:

a. $E(\tilde{z}) > r \rightarrow x^* \geq 0$;

b. $E(\tilde{z}) = r \rightarrow x^* = 0$;

c. $x^* > 0 \rightarrow E(\tilde{z}) > r$.

Note that $x^* > 0$ is sufficient but not necessary for $E(\tilde{z}) > r$ since $x^* > 0$ implies $E(\tilde{z}) > r$, which in turn implies $x^* \geqslant 0$. The equivalence however can be established under EU and WU.

For an EU maximizer, the CSPC problem becomes:

$$\underset{x}{\text{Maximize}} \quad EU[pF_{\tilde{y}}+(1-p)H] = p\int u(y)dF_{\tilde{y}}(y)+(1-p)\int u(s)dH(s) \qquad (3.8)$$

$$\text{s.t.} \quad \tilde{y} = y_o r + x(\tilde{z}-r).$$

The CSPC problem for a WU maximizer with value function v and weight function w is:

$$\underset{x}{\text{Maximize}} \quad WU[pF_{\tilde{y}}+(1-p)H] \qquad (3.5)$$

$$\text{where} \quad \tilde{y} = y_o r + x(\tilde{z}-r).$$

The optimization conditions for (3.8) and (3.5) lead to the following two theorems:

Theorem EU6 (Positive Conditional Risky-Asset Demand): Suppose x^* solves the CSPC problem (3.8) for an EU investor with an increasing, strictly concave utility function $u(y)$. Then

$$x^* \ (\beta^*) > 0 \text{ if and only if } E(\tilde{z}) > r.$$

Proof: Omitted since it is a special case of Theorem WU6. Also, see Arrow (1971).

Theorem WU6 (Positive Conditional Risky-Asset Demand): Suppose x^* solves the CSPC problem (3.5) for a WU investor (v,w) with increasing, strictly concave LOSUF $\zeta(y;F)$. Then,

$$x^* \ (\beta^*) > 0 \text{ if and only if } E(\tilde{z}) > r.$$

Proof: In light of Theorem U6, we need to prove only the sufficiency. Define $G \equiv pF_{\tilde{y}}+(1-p)H$. The FOC and SOC for x^* to solve (3.5) are given by by (3.9) and (3.10) below, respectively:

FOC: $\int \zeta'(y^*;G)(z-r)dF(z) = 0;$ (3.9)

SOC: $\int \zeta''(y^*;G)(z-r)^2 dF(z) \leqslant 0.$ (3.10)

where $\zeta(y;G) = w(y)[v(y)-WU(G)]/\int wdG$

and $y^* = y_o r + x^*(z-r).$

Suppose $E(\tilde{z}) > r$ but $x^* = 0$. (Theorem U6 rules out the possibility of $x^* < 0$.) Then,

$$\int \zeta'(y^*;G)(z-r)dF(z) = \zeta'(y_o r;G)[E(\tilde{z})-r] > 0,$$

contradicting FOC (3.9) and implying $x^* > 0$.

Q.E.D.

Theorem WU6 tells us that, like his expected utility counterpart (cf. Theorem EU6), a GCCERA WU maximizer will invest in the risky asset if and only if the expected return on the risky asset is greater than the sure return on the riskfree asset. Obviously, positive conditional risky-asset demand will imply positive unconditional risky-asset demand. As a matter of fact, the latter requires only GCERA (instead of GCCERA).

The results in Theorems U6, EU6 and WU6 are based on the assumption of GCCERA. If an agent is risk seeking, then it is quite natural for him to have positive risky-asset demand when $E(\tilde{z}) \geqslant r$. This suggests yet another characterization of risk aversion:

Definition 3.8: In a simple portfolio set-up with safe asset r and risky asset $\tilde{z}$, the preference of an investor with initial wealth y_o is said to display (unconditional) portfolio risk aversion (PRA) at y_o if his risky-asset demand at y_o is positive only if $E(\tilde{z}) > r$. His preference is said to display global PRA (GPRA) if it displays PRA at all y_o.

Definition 3.9: In a simple portfolio set-up with safe asset r and risky asset $\tilde{z}$, the preference of an investor with initial wealth y_o is said to display conditional portfolio risk aversion (CPRA) at y_o if his conditional risky-asset demand at y_o is positive only if $E(\tilde{z}) > r$. His preference is said to display global CPRA (GCPRA) if it displays CPRA at all y_o.

GCPRA is stronger than GPRA as GCPRA implies GPRA but the converse is not true. GCPRA is therefore more restrictive than GPRA in the following sense: If for any y_o, an investor is CPRA when p = 1, but not so when p = 0.5, then he is by definition GPRA but not GCPRA. Naturally we expect to find more people displaying GPRA than those displaying GCPRA.

According to Definition 3.9 (3.8), not all investors with positive conditional (unconditional) risky-asset demand are conditional (unconditional) porfolio risk averse, but, if a GCPRA or GPRA agent invests a positive amount in the risky asset, it must be true that $E(\tilde{z}) > r$. In light of Theorem U6, it is clear that any GCCERA investor must also be GCPRA no matter whether his preference functional is EU, WU, FDU or LGU.

Corollary 3.2: Under completeness, transitivity and SD, GCCERA implies GCPRA for a conditional diversifer.

Does GCPRA imply GCCERA in general or under particular preference theories? As it turns out, GCPRA implies GCCERA for any preference ordering satisfying completeness, transitivity, stochastic dominance, and the conditional diversifier assumption. To show this, we need the following lemma:

Lemma 3.1: Suppose x* solves uniquely the CSPC problem (3.4) for a conditional diversifer whose preferences are complete, transitive, consistent with SD and exhibit GCPRA. Then, $x^* = 0$ if $E(\tilde{z}) = r$.

Proof: Suppose $x^* < 0$. (The difinition of GCPRA rules out $x^* > 0$.) Then, $-x^*$ (> 0) is optimal for risky asset $\tilde{z}' \equiv 2r-\tilde{z}$, contradicting the definition of GCPRA.

Q.E.D.

<u>Theorem U5.4</u> (GRA): Under completeness, transitivity and SD, GCCERA is equivalent to GCPRA for a conditional diversifer.

Proof: Given Corollary 2.2, it suffices to prove GCPRA $\rightarrow$ GCCERA. Suppose there exist $p \in (0,1]$, F, $H \in D_J$ such that $pF_{\tilde{y}}+(1-p)H \succ p\delta_{E(\tilde{y})}+(1-p)H$, where $\tilde{y}$ is the r.v. associated with F. For a given y_o, construct $r = E(\tilde{y})/y_o$ and $\tilde{z} = \frac{1}{x}(\tilde{y}-y_o r)+r$. Note that $E(\tilde{z}) = r$. Let x^* be the conditional risky-asset demand in the CSPC problem with the parameters y_o, r, $\tilde{z}$ given above. Then $pF_{y_o r+x^*(\tilde{z}-r)}+(1-p)H \gtrsim pF_{\tilde{y}}+(1-p)H \succ p\delta_{y_o r}+(1-p)H$.

Lemma 3.1 implies $x^* = 0$, giving rise to a contradiction.

Q.E.D.

4

COMPARATIVE RISK AVERSION

In the proceeding sections, we characterized risk aversion in a number of ways. The question to explore next is: What is the meaning of one decision maker being more risk averse than another? What are its behavioral implications?

4.1 Definitions

Since (a) CCE and CE, (b) mps, (c) risky-asset demands, and (d) the concavity of appropriate utility functions have been used to characterize GRA, it is natural to think of them as promising candidates for characterizing comparative risk aversion (CRA). We will consider them one by one.

(a) Certainty Equivalent

Both CCE and CE are concept of a single value, therefore can be easily extended to a comparative risk aversion context. If agent A is more risk averse than agent B in the sense of CCE (CE), we will expect agent A to accept a lower CCE (CE) for any distribution than agent B. Formally, we say that agent A is more GCCERA (GCERA) than agent B if agent A's CCE (CE) of any distribution is smaller than agent B's.

(b) Preference Compensated Spread

Mean preserving spreads do not work quite well in characterizing comparative risk aversion. Recognizing this, Diamond and Stiglitz (1974) proposed a 'mean utility preserving spread' notion (they called it a 'mean utility preserving increase in risk'). Since, as the name says, this notion preserves 'mean utility', it can only be used to characterize comparative risk aversion for EU-type preferences. For more general preferences, more restrictive definitions are needed. The following definition is similar to that in Machina (1982a):

Definition 4.1: Distribution G is said to be a simple compensated spread of distribution F to a decision maker if

(i) G single-crosses F from the left, and

(ii) $F \sim G$.

Compared with Diamond and Stiglitz' mean utility preserving spread (cf. Definition 4.5 below), condition (i) in Definition 4.1 is more restrictive as it only allows distributions which cross once. Condition (ii), however, is more general in not restricting preferences to only expected utility ones. Depending on the preferences subscribed to by the decision maker, there are at least three different cases of simple compensated spread, i.e., EU compensated spread, WU compensated spread, and LGU compensated spread.

Definition 4.2: Distribution G is said to be a simple mean utility preserving spread (simple mups) of distribution F to an expected utility decision maker with von Neumann-Morgenstern utility function u if

(i) G single-crosses F from the left, and

(ii) $EU(F) = EU(G)$.

Definition 4.3: For a WU maximizer with value function v and weight function w, G is said to be a simple weighted utility preserving spread (simple wups) of F if

(a) G single-crosses F from the left; and

(b) WU(G) = WU(F).

Definition 4.4: Distribution G is said to be a simple LGU preserving spread of distribution F to an LGU decision maker V if

(i) G single-crosses F from the left, and

(ii) V(F) = V(G).

Since EU is linear in distribution, the squeezed mean interpretation for second-degree stochastic dominance can be generalized to mean utility via Definition 4.5 below. This will significantly increase the set of permissible distributions.

Definition 4.5: Distribution G is a mean utility preserving spread (mups) of distribution F to an EU decision maker u if

$$\int_{-\infty}^{y} u(x)dF(x)+u(y)[1-F(y)] \geq \int_{-\infty}^{y} u(x)dG(x)+u(y)[1-G(x)] \text{ for all } y, \quad (4.1)$$

and

$$\int_{-\infty}^{\infty} u(x)dF(x) = \int_{-\infty}^{\infty} u(x)dG(x). \quad (4.2)$$

Condition (4.2) says that distributions F and G yield equal expected utility. Condition (4.1) requires that the 'squeezed' expected utility of F over $(-\infty,y]$ be not less than that of G for all y.

To consider a sequence of mups of F, Diamond and Stiglitz parameterized the distribution with a risk factor α. Definition 4.5 can then be restated as Definition 4.5', where a subscript indicates the variable with respect to which a partial derivative is taken.

Definition 4.5' (Diamond and Stiglitz): Given a distribution $F(x,\alpha)$ and a utility function u(x), an increase in α represents a mean utility preserving increase in risk if

$$T(y) = \int_{-\infty}^{y} u'(x)F_{\alpha}(x,\alpha)dx = \int_{-\infty}^{y} F_{\alpha}(x,\alpha)du(x) \geq 0 \text{ for all } y, \quad (4.3)$$

and

$$T(\infty) = \int_{-\infty}^{\infty} u'(x)F_\alpha(x,\alpha)dx = \int_{-\infty}^{\infty} F_\alpha(x,\alpha)du(x) = 0. \tag{4.4}$$

To gain some insight into Definition 4.5', define $F^\alpha \equiv F(x,\alpha) = (1-\alpha)F+\alpha G$, where $\alpha \varepsilon [0,1]$ and G is a mups of F as defined by Definition 4.5. F^α is a mixture of F and G with component of G increasing as α goes from 0 to 1. In other words, $\{F^\alpha : \alpha\varepsilon[0,1]\}$ represents a sequence of mups of F, going from F towards G. Since the same expected utility is preserved from F to G and $F_\alpha(x,\alpha) = \frac{d}{d\alpha}[(1-\alpha)F+\alpha G] = G-F$, conditions (4.3) and (4.4) are equivalent to conditions (4.1) and (4.2), respectively, noting $F(\infty)-G(\infty) = F(-\infty)-G(-\infty) = 0$.

Suppose that agent A is more risk averse than agent B in the sense of compensated spread, and that G is a compensated spread of F to A. It is expected that G will be preferred to F from B's viewpoint because a less risk averse agent should in general demand less compensation for a given increase in risk. Formally, we say that agent A is more MRA than agent B if agent A always prefers F to any of agent B's preference compensated spreads of F.

(c) Risky-Asset Demand

In a simple portfolio set-up, we defined an agent to be CPRA (PRA) if his conditional (unconditional) risky-asset demand is strictly positive only if the expected rate of return on the risky asset is strictly greater than the risk-free rate of return. Suppose both agents A and B are CPRA with respect to $\tilde{z}$ and r. If they have identical initial wealth, then it seems reasonable to expect the more risk averse agent to demand less of the risky asset. Formally, we say that agent A is more GCPRA (GPRA) than agent B if for any r and $\tilde{z}$ such that $E(\tilde{z}) > r$ agent A's demand for $\tilde{z}$ is always less than agent B's.

(d) Concavity of relevant utility functions

Suppose more structures are imposed on a preference functional V so that a GCCREA agent must have a concave utility function (cf. Theorem LGU5). Agent A being more risk averse than agent B suggests that agent A's utility function, if identifiable, is 'more concave' than agent B's.

Definition 4.6: An increasing, continuous function f is said to be at least as concave as (more concave than) another increasing, continuous function g if there exists an increasing concave (strictly concave) function h such that $f(x) \equiv h(g(x))$.

Lemma 4.1 (Pratt): Suppose f and g are two concave, increasing functions. Then,

$$-f''/f' \geqslant (>) -g''/g' \qquad (4.5)$$

if and only if f is at least as concave as (more concave than) g.

In words, Definition 4.6 means that if f is more concave than g, then f can be obtained by 'concavifying' g via an increasing concave function h. For EU maximizers u_A and u_B, u_A being more concave than u_B, by Lemma 4.1, implies that $-u_A''/u_A' \geqslant -u_B''/u_B'$. Since $-u''/u'$ is the Arrow-Pratt index, this means that the more risk averse an EU individual is, the greater his Arrow-Pratt index will be.

Beyond EU, the 'concavity index' will naturally be $-\zeta''(x;F)/\zeta'(x;F)$ since the LOSUF $\zeta(x;F)$ serves as the von Neumann-Morgenstern utility-like function.

4.2 Characterizations

Now that we have clarified the meaning of one agent being more GCCERA, more GMRA, or more GCPRA than another agent, the remaining task in this section is to establish the relations, if any, among them. As it turns out, agent A is more GCCERA than agent B if and only if A is also more GMRA than B regardless of the preference theory they subscribe to. We shall first show that this is true for elementary monetary lotteries and then extend it to general monetary lotteries.

Definition 4.7: G is a simple elementary compensated spread of F to a decision maker if there exist a, ε, p, θ_1, θ_2 (ε, θ_1, $\theta_2 \geqslant 0$, $p \in (0,1]$) and an elementary lottery H such that he is indifferent between

$$F \equiv (1-p)H + p\{\tfrac{1}{2}\delta_{a-\varepsilon} + \tfrac{1}{2}\delta_{a+\varepsilon}\} \tag{4.6}$$

and

$$G \equiv (1-p)H + p\{\tfrac{1}{2}\delta_{a-\varepsilon-\theta_1} + \tfrac{1}{2}\delta_{a+\varepsilon+\theta_2}\}. \tag{4.7}$$

In Definition 4.7, the word 'simple' is used to indicate that G crosses F only once (although the crossing might be an interval). F and G are elementary lotteries because they involve a finite number of outcomes. In Definition 4.7, if $\theta_1 = \theta_2$, G will be an mps of F. For strictly risk averse decision makers, an mps of a distribution F is less desirable than F. In order to make a spread as attractive, the right-tail shift θ_2 must be greater than the left-tail shift θ_1. For a strictly risk seeking individual, the opposite is true (i.e. $\theta_1 > \theta_2$).

Definition 4.8: For elementary lotteries $\underset{\sim}{x}$, $\underset{\sim}{y} \in J^N$, $\underset{\sim}{y}$ is an elementary compensated spread of $\underset{\sim}{x}$ if there exists a nonnegative compensating vector $\underset{\sim}{\eta} \equiv (\eta_1, \eta_2, \ldots, \eta_{N-1})$ such that

$$\underset{\sim}{x} \sim \underset{\sim}{y}^1 \sim \ldots \sim \underset{\sim}{y}^n \sim \ldots \sim \underset{\sim}{y}^{N-1} \equiv \underset{\sim}{y},$$

where

$$\begin{aligned}
\underset{\sim}{x} &\equiv (x_1, x_2, \ldots, x_N) \\
\underset{\sim}{y}^1 &\equiv (y_1, x_2+\eta_1, x_3, \ldots, x_N) \\
&\ldots \\
\underset{\sim}{y}^{n-1} &\equiv (y_1, \ldots, y_{n-1}, x_n+\eta_{n-1}, x_{n+1}, \ldots, x_N) \\
\underset{\sim}{y}^{n} &\equiv (y_1, \ldots, y_{n-1}, y_n, x_{n+1}+\eta_n, x_{n+2}, \ldots x_N) \qquad (4.8) \\
&\ldots \\
\underset{\sim}{y}^{N-1} &\equiv (y_1, \ldots, y_n, \ldots, y_{N-1}, x_N+\eta_{N-1}) = (y_1, \ldots, y_n, \ldots, y_{N-1}, y_N) \equiv \underset{\sim}{y}.
\end{aligned}$$

Note that $\underset{\sim}{y}^{n-1}$ and $\underset{\sim}{y}^{n}$ are respectively G_{n-1} and G_n given below:

$$G_{n-1} = \frac{N-2}{N}\{\Sigma_{i=1}^{n-1} \frac{1}{N-2}\delta_{y_i} + \Sigma_{i=n+2}^{N} \frac{1}{N-2}\delta_{x_i}\} + \frac{2}{N}\{\frac{1}{2}\delta_{x_n+\eta_{n-1}} + \frac{1}{2}\delta_{x_{n+1}}\}$$

$$G_{n} = \frac{N-2}{N}\{\Sigma_{i=1}^{n-1} \frac{1}{N-2}\delta_{y_i} + \Sigma_{i=n+2}^{N} \frac{1}{N-2}\delta_{x_i}\} + \frac{2}{N}\{\frac{1}{2}\delta_{y_n} + \frac{1}{2}\delta_{x_{n+1}+\eta_n}\}.$$

G_n is a simple elementary compensated spread of G_{n-1} if $G_n \sim G_{n-1}$. Definition 4.8 therefore tells us that, if $\underset{\sim}{y}$ is an elementary compensated spread of $\underset{\sim}{x}$, then $\underset{\sim}{y}$ can be obtained from $\underset{\sim}{x}$ via a sequence of simple elementary compensated spreads in the following manner: Starting with $\underset{\sim}{x}$, it must be true that $x_1 \geqslant y_1$. First push x_1 leftwards to y_1 and push x_2 rightwards to, say, z_2 such that the decision maker's preference is preserved. Let us denote the distance of the leftward push at step i by λ_i and the distance of the simultaneous rightward push by η_i. Clearly, $\lambda_1 = x_1 - y_1 \geqslant 0$, and $\eta_1 \geqslant x_1 - y_1$ if the decision maker is GMRA. Next, push z_2 leftwards to y_2, and x_3 rightwards to z_3. This time, $\lambda_2 = z_2 - y_2 =$

$(x_2+\eta_1)-y_2 = \eta_1+(x_2-y_2)$. In general, $\lambda_i = \eta_{i-1}+x_i-y_i$, $\eta_i \geq \lambda_i \geq 0$ if the decision maker is GMRA, and

$$\underset{\sim}{y}^{n-1} = (y_1,\ldots,y_{n-1},x_n+\eta_{n-1},x_{n+1},x_{n+2},\ldots,x_N)$$

$$\underset{\sim}{y}^{n} = (y_1,\ldots,y_{n-1},x_n+\eta_{n-1}-\lambda_n,x_{n+1}+\eta_n,x_{n+2},\ldots,x_N)$$

$$= (y_1,\ldots,y_{n-1},y_n,x_{n+1}+\eta_n,x_{n+2},\ldots,x_N)$$

for $n = 2, 3, \ldots, N-1$. Since $\underset{\sim}{y}^{N-1}$ must be $\underset{\sim}{y}$, we have $x_N+\eta_{N-1} = y_N$, or $\eta_{N-1} = y_N-x_N \geq 0$.

In light of the definition of GMRA, the following corollaries are obvious:

Corollary 4.1: For a risk neutral (in the sense of mps) decision maker, if $\underset{\sim}{y}$ is an elementary compensated spread of $\underset{\sim}{x}$ via the compensating vector $\underset{\sim}{\eta}$, then

$$\eta_k = \Sigma_{i=1}^{k}(x_i-y_i) \qquad \text{for all } 1 \leq k \leq N-1.$$

Corollary 4.2: Suppose $\underset{\sim}{y}$ is an elementary compensated spread of $\underset{\sim}{x}$ for a GMRA decision maker whose preference is complete, transitive and consistent with SD. Then, the compensating vector $\underset{\sim}{\eta}$ satisfies the following:

$$\eta_k \geq \Sigma_{i=1}^{k}(x_i-y_i) \qquad \text{for all } 1 \leq k \leq N-1.$$

Proof: We prove by induction. Suppose $\eta_1 < x_1-y_1$. Then SD and GMRA imply that

$$\underset{\sim}{y}^1 \equiv (y_1,x_2+\eta_1,x_3,\ldots,x_N) \prec (y_1,x_2+(x_1-y_1),x_3,\ldots,x_N)$$

$$\prec (y_1+(x_1-y_1),x_2,x_3,\ldots,x_N) = (x_1,x_2,x_3,\ldots,x_N) \equiv \underset{\sim}{x},$$

contradicting $\underset{\sim}{x} \sim \underset{\sim}{y}^1$. Therefore, it must be true that $\eta_1 \geq x_1-y_1$.

Next, suppose, for some $k < N$, $\eta_k \geq \Sigma_{i=1}^{k}(x_i - y_i)$ but $\eta_{k+1} < \Sigma_{i=1}^{k+1}(x_i - y_i)$. Then,

$$\begin{aligned}
\underset{\sim}{y}^{k+1} &= (y_1, \ldots, y_k, y_{k+1}, x_{k+2} + \eta_{k+1}, x_{k+3}, \ldots, x_N) \\
&\sim (y_1, \ldots, y_k, y_{k+1} + \lambda_{k+1}, x_{k+2}, x_{k+3}, \ldots, x_N) \\
&\prec (y_1, \ldots, y_k, y_{k+1} + \eta_{k+1}, x_{k+2}, x_{k+3}, \ldots, x_N) \qquad \text{(by GMRA and SD)} \\
&\prec (y_1, \ldots, y_k, y_{k+1} + \Sigma_{i=1}^{k+1}(x_i - y_i), x_{k+2}, x_{k+3}, \ldots, x_N) \qquad \text{(by SD)} \\
&\prec (y_1, \ldots, y_k, y_{k+1} + \eta_k + (x_{k+1} - y_{k+1}), x_{k+2}, x_{k+3}, \ldots, x_N) \qquad \text{(by SD)} \\
&= (y_1, \ldots, y_k, x_{k+1} + \eta_k, x_{k+2}, x_{k+3}, \ldots, x_N) = \underset{\sim}{y}^{k}
\end{aligned}$$

giving rise to a contradiction. Thus, $\eta_{k+1} \geq \Sigma_{i=1}^{k+1}(x_i - y_i)$. By induction, we have proved that

$$\eta_k \geq \Sigma_{i=1}^{k}(x_i - y_i) \text{ for all } 1 \leq k \leq N.$$

Q.E.D.

<u>Lemma 4.2</u>: Let $\succsim_A$ and $\succsim_B$ be the respective preference orderings of agents A and B which satisfy completeness, transitivity and SD. If A is more GCCERA than B, then, for any elementary lotteries F and G such that G is a simple elementary compensated spread of F for A, G $\succsim_B$ F.

Proof: Since, for A, G is a simple elementary compensated spread of F, by definition, there must exist constants a, ε, θ_1, θ_2, p (ε, θ_1, $\theta_2 \geq 0$, p ε (0,1]) and an elementary lottery H such that F $\sim_A$ G and

$$F \equiv (1-p)H + p\{\tfrac{1}{2}\ \delta_{a-\varepsilon} + \tfrac{1}{2}\ \delta_{a+\varepsilon}\},$$

$$G \equiv (1-p)H + p\{\tfrac{1}{2}\ \delta_{a-\varepsilon-\theta_1} + \tfrac{1}{2}\ \delta_{a+\varepsilon+\theta_2}\}.$$

Let q_1, q_2 ε [0,1] be such that

$$F \sim_A (1-p) + p\{\frac{q_1}{2}\delta_{a-\varepsilon-\theta_1} + \frac{1-q_1}{2}\delta_a + \frac{1}{2}\delta_{a+\varepsilon}\} \equiv P$$

$$\sim_A (1-p) + p\{\frac{q_1}{2}\delta_{a-\varepsilon-\theta_1} + [1 - \frac{q_1+q_2}{2}]\delta_a + \frac{q_2}{2}\delta_{a+\varepsilon+\theta_2}\} \equiv Q$$

$$\sim_A G.$$

That B is less GCCERA than A implies $G \succsim_B Q \succsim_B P \succsim_B F$.

Q.E.D.

Given F defined by (4.6), Lemma 4.2 implies that, for an identical downward shift θ_1, B will require a lower compensation θ_2 than A will.

Theorem U7.1 (CRA): For elementary lotteries and a pair of preference orderings $\succsim_A$ and $\succsim_B$ satisfying completeness, transitivity and SD, the following conditions are equivalent:

(a) For any elementary lottery F, $CCE_A(F) \leqslant CCE_B(F)$, where $CCE_A(F)$ and $CCE_B(F)$ refer to the CCEs of F (conditional upon any $p \in [0,1]$ and an elementary lottery H) for A and B, respectively.

(b) If $\underset{\sim}{y} \equiv G \equiv \Sigma_{i=1}^{N} \frac{1}{N}\delta_{y_i}$ is an elementary compensated spread of $\underset{\sim}{x} \equiv F \equiv \Sigma_{i=1}^{N} \frac{1}{N}\delta_{x_i}$ to A, then $\underset{\sim}{x}$ is at least as preferable as $\underset{\sim}{y}$ to B.

Proof: (a) → (b): Since $\underset{\sim}{y}$ is an elementary compensated spread of $\underset{\sim}{x}$ to A, $\underset{\sim}{y}$ can be obtained from $\underset{\sim}{x}$ via a sequence of simple elementary compensated spreads $\underset{\sim}{y}^n$ given by (4.8). According to Lemma 4.2,

$$\underset{\sim}{x} \equiv \underset{\sim}{y}^0 \succsim_B \underset{\sim}{y}^1 \succsim_B \cdots \succsim_B \underset{\sim}{y}^n \succsim_B \cdots \succsim_B \underset{\sim}{y}^{N-1} \equiv \underset{\sim}{y}.$$

(b) → (a): This is straightforward recognizing that, for any elementary lotteries F and H, pF+(1-p)H is an elementary compensated spread of $p\delta_{CCE(F)}+(1-p)H$.

Q.E.D.

Given Lemma 4.2, Theorem U7.1 is obvious. Note that, in condition (b), lottery G need not be a simple compensated spread of F.

We are now ready to further extend Theorem U7.1 to general monetary lotteries.

Theorem U7.2 (CRA): The following are equivalent for a pair of preference orderings $\succsim_A$ and $\succsim_B$ which are complete, transitive, continuous in distribution and consistent with SD:

(a) (GCCERA) For any distribution F, $CCE_A(F) \leq CCE_B(F)$, where $CCE_A(F)$ and $CCE_B(F)$ refer to the CCEs of F for A and B, respectively.

(b) (GMRA) If G is a simple compensated spread of an arbitrary distribution F to A, then G is at least as preferable as F to B.

Proof: (a) $\rightarrow$ (b): Suppose G is a simple compensated spread of F to A. Consider $\{i/2^n: i = 1, \ldots, 2^n-1\}$. Let $p^o = F(0)$ and $q^o = G(0)$. Define

$$x_i^n \equiv \begin{cases} \inf\{x \mid F(x)=i/2^n\} & 0 < i/2^n < p^o \\ \sup\{x \mid F(x)=p^o\} & p^o \leq i/2^n < p^o+(1/2^n) \\ \sup\{x \mid F(x)=i/2^n\} & p^o+(1/2^n) \leq i/2^n < 1, \end{cases}$$

$$x_0^n \equiv \begin{cases} 0 & \text{if } p^o \text{ is not in } I_n = \{i/2^n \mid i=1,\ldots,2^n-1\} \\ \inf\{x \mid F(x)=p^o\} & \text{otherwise} \end{cases}$$

and

$$y_i^n \equiv \begin{cases} \inf\{y \mid G(y)=i/2^n\} & 0 < i/2^n < q^o \\ \sup\{y \mid G(y)=q^o\} & p^o \leq i/2^n < q^o+(1/2^n) \\ \sup\{y \mid G(y)=i/2^n\} & q^o+(1/2^n) \leq i/2^n < 1, \end{cases}$$

$$y_0^n \equiv \begin{cases} 0 & \text{if } q^o \text{ is not in } I_n = \{i/2^n \mid i=1,\ldots,2^n-1\} \\ \inf\{y \mid F(y)=q^o\} & \text{otherwise.} \end{cases}$$

Construct $F_n \equiv 2^{-n}\delta_{x_0^n+\varepsilon_n}+\sum_{i=1}^{2^n-1} 2^{-n}\delta_{x_i^n+\varepsilon_n}$ and $G_n \equiv 2^{-n}\delta_{y_0^n+\theta_n}+\sum_{i=1}^{2^n-1} 2^{-n}\delta_{y_i^n+\theta_n}$ such that $G \sim_A G_n \sim_A F_n \sim_A F$. Clearly, F_n and G_n converge in distribution to F and G, respectively. Furthermore, for n sufficiently large, G_n single crosses F_n from the left. By Theorem U7.1, $G_n \succsim_B F_n$.

To prove $G \succsim_B F$, suppose the contrary that $F \succ_B G$. Since $\lim_{n\to\infty} F_n = F$ and $\lim_{n\to\infty} G_n = G$, by CD, there exists a $K > 0$ such that $F_k \succ_B G$ for all $k > K$. Again, by CD, there exists an $M > 0$ such that $F_k \succ_B G_m$ for all $m > M$. Pick $N = \max\{K,M\}$. Then, $F_n \succ_B G_n$ for all $n > N$, giving rise to a contradiction.

(b) → (a): $pF+(1-p)H$ is a simple compensated spread of $p\delta_{CCE_A(F)}+(1-p)H$ to A. Condition (b) implies that

$$pF+(1-p)H \succsim_B p\delta_{CCE(F)}+(1-p)H.$$

Since

$$pF+(1-p)H \sim_B p\delta_{CCE^*(F)}+(1-p)H,$$

by transitivity,

$$p\delta_{CCE^*(F)}+(1-p)H \succsim_B p\delta_{CCE(F)}+(1-p)H.$$

SD implies $CCE_B(F) \geq CCE_A(F)$.

Q.E.D.

We have proved that agent A is more GCCERA than agent B if and only if A is also more GMRA than B regardless of the forms of their preference functionals as long as they are complete, transitive, continuous in distribution, and consistent with stochastic dominance.

We next consider comparative GCPRA. It appears that an investor who is more GCCERA than another investor will also be more GCPRA no matter what utility theory their preferences subscribe to.

Theorem U7.3 (CRA): Suppose the preferences of agents A and B are complete, transitive, continuous in distribution and consistent with SD. Then, A is more GCPRA than B if A is more GCCERA than B.

Proof: In a simple portfolio set-up with $E(\tilde{z}) \geq r$, suppose x_A and $x_B = x_A+\Delta x$ are the respective risky-asset demands of A and B who have identical initial wealth y_o. The optimality of x_B implies that

$$pF_{y_o r+(x_A+\Delta x)(\tilde{z}-r)}+(1-p)H \succsim_B pF_{y_o r+x_A(\tilde{z}-r)}+(1-p)H.$$

It follows that there exists $\theta > 0$ such that

$$pF_{y_o r+(x_A+\Delta x)(\tilde{z}-r)}+(1-p)H \sim_B pF_{y_o r+x_A(\tilde{z}-r)+\theta}+(1-p)H.$$

Suppose $\Delta x < 0$. Then $pF_{y_o r+x_A(\tilde{z}-r)+\theta}+(1-p)H$ will be a compensated spread of $pF_{y_o r+(x_A+\Delta x)(\tilde{z}-r)}+(1-p)H$ to B. Since A is more GCCERA than B, Theorem U7.2, SD and transitivity imply that

$$pF_{y_o r+(x_A+\Delta x)(\tilde{z}-r)}+(1-p)H \succsim_A pF_{y_o r+x_A(\tilde{z}-r)+\theta}+(1-p)H.$$

$$\succ_A pF_{y_o r+x_A(\tilde{z}-r)}+(1-p)H.$$

This contradicts the optimality of x_A to A. Hence, $\Delta x \geq 0$ and $x_B \geq x_A$.

Q.E.D.

Theorems EU7, WU7 and LGU7 below tell us that A being more GCPRA than B also implies A being more GCCERA than B under EU, WU and LGU.

Theorem EU7 (CRA): The following are equivalent for a pair of continuous, increasing von Neumann-Morgenstern utility functions u_A and u_B:

(a) (GCCERA) For any distribution F, $CCE_A(F) \leq CCE_B(F)$, where $CCE_A(F)$

and $CCE_B(F)$ refer to the CCEs of F for u_A and u_B, respectively.

(b) (GCPRA) In a simple portfolio set-up with safe asset r and risky asset $\tilde{z}$, where $E(\tilde{z}) \geq r$, suppose u and u* have identical initial wealth and x_A and x_B are their respective conditional money risky-asset demands. Then $x_A \leq x_B$.

(c) (Concavity) u_A is at least as concave as u_B.

(d) (GCERA) For any distribution F, $CE_A(F) \leq CE_B(F)$, where $CE_A(F)$ and $CE_B(F)$ refer to the CEs of F for u_A and u_B, respectively.

(e) (GPRA) In a simple portfolio set-up with safe asset r and risky asset $\tilde{z}$, where $E(\tilde{z}) \geq r$, suppose u_A and u_B have identical initial wealth and x_A and x_B are their respective unconditional money risky-asset demands. Then $x_A \leq x_B$.

Proof: Omitted since this is well-known.

The following lemma is needed to prove Theorem WU7.

<u>Lemma 4.3</u>: If

(i) G single-crosses F at x* from the left; and

(ii) f(x) and g(x) are two increasing functions, and f(x) is at least as concave in x as g(x),

then

$$\int[G-F]f'(x)dx \geq \frac{f'(x^*)}{g'(x^*)} \int[G-F]g'(x)dx.$$

Proof: Given f(x) at least as concave in x as g(x), for any $x_1, x_2 \in R$ such that $x_1 < x_2$, we have

$$\frac{f'(x_2)}{f'(x_1)} \leq \frac{g'(x_2)}{g'(x_1)} .$$

Applying straight algebra yields the following:

$$\int_{-\infty}^{+\infty} [G-F]f'(x)dx$$

$$= f'(x^*)\int_{-\infty}^{+\infty} [G-F] \frac{f'(x)}{f'(x^*)} dx$$

$$= f'(x^*) \left\{\int_{-\infty}^{x^*}[G-F] \frac{f'(x)}{f'(x^*)} dx + \int_{x^*}^{+\infty}[G-F] \frac{f'(x)}{f'(x^*)} dx\right\}$$

$$\geqslant f'(x^*) \left\{\int_{-\infty}^{x^*}[G-F] \frac{g'(x)}{g'(x^*)} dx + \int_{x^*}^{+\infty}[G-F] \frac{g'(x)}{g'(x^*)} dx\right\}$$

$$= \frac{f'(x^*)}{g'(x^*)} \int_{-\infty}^{+\infty} [G-F]g'(x)dx.$$

Q.E.D.

It should be pointed out that the G and F in Lemma 4.3 need not be distribution functions. Nor do f and g have to be related to G and F in any particular way.

Theorem WU7 (CRA): Under WU, the following are equivalent for two pairs of value and weight functions (v_A,w_A) and (v_B,w_B) with respective LOSUF ζ_A and ζ_B:

(a) (GCCERA) For any $F \in D_J$, $CCE_A(F) \leqslant CCE_B(F)$, where $CCE_A(F)$ and $CCE_B(F)$ refer to the CCE of F for (v_A,w_A) and (v_B,w_B), respectively.

(b) (GCPRA) In a simple portfolio set-up with safe asset r and risky asset $\tilde{z}$, where $E(\tilde{z}) \geqslant r$, let x_A and x_B be the respective conditional risky-asset demands of (v_A,w_A) and (v_B,w_B) who have identical initial wealth. Then $x_A \leqslant x_B$ regardless of the probability and the distribution they are conditional upon.

(c) (Concavity) For any $F \in D_J$, $\zeta_A(x;F)$ is at least as concave in x as $\zeta_B(x;F)$.

In addition, each of the above conditions implies the following:

(d) (GCERA) For any $F \in D_J$, $CE_A(F) \leqslant CE_B(F)$, where $CE_A(F)$ and $CE_B(F)$

refer to the CEs of F for (v_A,w_A) and (v_B,w_B), respectively.

(e) (GPRA) In a simple portfolio set-up with safe asset r and risky asset $\tilde{z}$, where $E(\tilde{z}) \geq r$, let x_A and x_B be the respective risky-asset demands of (v_A,w_A) and (v_B,w_B) who have identical initial wealth. Then $x_A \leq x_B$.

Proof: (a) → (b) follows from Theorem U7.3.

(b) → (c): Suppose there exists $H \in D_J$ such that $\zeta_B(y;H)$ is more concave in y than $\zeta_A(y;H)$. Then, there exist $h_1 < h_2$ and $q \in (0,1)$ such that

$$\frac{\zeta_A'(h_2;H)}{\zeta_A'(h_1;H)} > q > \frac{\zeta_B'(h_2;H)}{\zeta_B'(h_1;H)}$$

so that, for some $\theta > 0$,

$$\frac{\zeta_A(h_2+\theta;H)-\zeta_A(h_2;H)}{\zeta_A(h_1;H)-\zeta_A(h_1-q\theta;H)} > 1 \tag{4.9}$$

$$1 > \frac{\zeta_B(h_2;H)-\zeta_B(h_2-\theta;H)}{\zeta_B(h_1+q\theta;H)-\zeta_B(h_1;H)} . \tag{4.10}$$

Recall that $\int\zeta_A(t;H)d[H-F] = \frac{d}{dp} WU_A[(1-p)F+pH]\Big|_{p=1}$. Inequality (4.9) yields

$$\frac{1}{2}\zeta_A(h_1;H) + \frac{1}{2}\zeta_A(h_2;H) < \frac{1}{2}\zeta_A(h_1-q\theta;H) + \frac{1}{2}\zeta_A(h_2+\theta;H),$$

which is

$$\int\zeta_A(t;H)d[H-(\tfrac{1}{2}\delta_{h_1} + \tfrac{1}{2}\delta_{h_2})] > \int\zeta_A(t;H)d[H-(\tfrac{1}{2}\delta_{h_1-q\theta} + \tfrac{1}{2}\delta_{h_2+\theta})],$$

or

$$\frac{d}{dp}\{WU_A[(1-p)(\tfrac{1}{2}\delta_{h_1} + \tfrac{1}{2}\delta_{h_2})+pH]-WU_A[(1-p)(\tfrac{1}{2}\delta_{h_1-q\theta} + \tfrac{1}{2}\delta_{h_2+\theta})+pH]\}\Big|_{p=1} > 0.$$

Since

$$WU_A[(1-p)(\tfrac{1}{2}\delta_{h_1} + \tfrac{1}{2}\delta_{h_2})+pH] = WU_A[(1-p)(\tfrac{1}{2}\delta_{h_1-q\theta} + \tfrac{1}{2}\delta_{h_2+\theta})+pH]$$

at $p = 1$, it is implied that, for some p sufficiently close to 1,

$$WU_A[(1-p)(\tfrac{1}{2}\delta_{h_1} + \tfrac{1}{2}\delta_{h_2})+pH] < WU_A[(1-p)(\tfrac{1}{2}\delta_{h_1-q\theta} + \tfrac{1}{2}\delta_{h_2+\theta})+pH]. \quad (4.11)$$

Similarly, inequality (4.10) leads to

$$WU_B[(1-p)(\tfrac{1}{2}\delta_{h_1} + \tfrac{1}{2}\delta_{h_2})+pH] > WU_B[(1-p)(\tfrac{1}{2}\delta_{h_1+q\theta} + \tfrac{1}{2}\delta_{h_2-\theta})+pH]. \quad (4.12)$$

Let y_o be the common initial wealth of WU_A and WU_B. Construct a safe asset $r \equiv \frac{1}{y_o}\,\frac{h_1+qh_2}{1+q}$ and a risky asset $\tilde{z} \equiv \frac{1}{2}\delta_{h_1/y_o} + \frac{1}{2}\delta_{h_2/y_o}$. Let x_A and x_B be their respective conditional demands for $\tilde{z}$ (conditional upon p and H). Also let

$$x_1 = y_o[1- \frac{\theta(1+q)}{h_2-h_1}] \quad \text{and} \quad x_2 = y_o[1+ \frac{\theta(1+q)}{h_2-h_1}].$$

Note that $x_1 < y_o < x_2$. It can be verified that (4.11) and (4.12) are equivalent to (4.13) and (4.14) below, respectively:

$$WU_A[(1-p)F_{y_o\tilde{z}}+pH] < WU_A[(1-p)F_{y_or+x_2(\tilde{z}-r)}+pH], \quad (4.13)$$

$$WU_B[(1-p)F_{y_o\tilde{z}}+pH] > WU_B[(1-p)F_{y_or+x_1(\tilde{z}-r)}+pH]. \quad (4.14)$$

Since both WU_A and WU_B are conditional diversifers, the above amounts to $x_B < y_o < x_A$, contradicting condition (b).

(c) → (a): In light of Theorem U7.3, it suffices to prove that (c) implies that WU_A is more GMRA than WU_B. Suppose G is a simple wups of F to WU_A and single-crosses F at x* from the left. As $\zeta_A(x;F)$ is at least as concave in x as $\zeta_B(x;F)$, by Lemma 4.3,

$$\int[G-F]\,\zeta_A'(x;F)dx \geq \frac{\zeta_A'(x^*;F)}{\zeta_B'(x^*;F)} \int[G-F]\,\zeta_B'(x;F)dx. \quad (4.15)$$

Define $F^\alpha = (1-\alpha)F+\alpha G$. It follows that

$$0 = \frac{d}{d\alpha} WU_A(F^\alpha)\Big|_{\alpha=0} = \int\zeta_A(x;F)d[G-F] = -\int[G-F]\,\zeta_A'(x;F)dx$$

$$\leqslant \frac{\zeta_A'(x^*;F)}{\zeta_B'(x^*;F)} \left\{-\int [G-F]\,\zeta_B'(x;F)dx\right\}$$

$$= \frac{\zeta_A'(x^*;F)}{\zeta_B'(x^*;F)} \frac{d}{d\alpha} WU_B(F^\alpha)\Big|_{\alpha=0}.$$

With ζ_A', $\zeta_B' > 0$, this implies $\frac{d}{d\alpha} WU_B(F^\alpha) \geqslant 0$. Hence, $WU_B(G) \geqslant WU_B(F)$.

(a) $\rightarrow$ (d) and (b) $\rightarrow$ (e) follow by definition.

Q.E.D.

Theorem LGU7 (CRA): The following are equivalent for a pair of LGU functionals V_A and V_B with LOSUF $\zeta_A(x;F)$ and $\zeta_B(x;F)$, respectively:

(a) (GCCERA) For any $F \in D_J$, $CCE_A(F) \leqslant CCE_B(F)$, where $CCE_A(F)$ and $CCE_B(F)$ refer to the CCEs of F for A and B, respectively.

(b) (Concavity) For any $F \in D_J$, $\zeta_A(x;F)$ is at least as concave in x as $\zeta_B(x;F)$.

If, in addition, both V_A and V_B are conditional diversifiers, then the above conditions are equivalent to:

(c) (GCPRA) In a simple portfolio set-up with safe asset r and risky asset $\tilde{z}$, where $E(\tilde{z}) \geqslant r$, let x_A and x_B be the respective conditional money risky-asset demand of V_A and V_B who have identical initial wealth. Then $x_A \leqslant x_B$ regardless of the probability and the distribution they are conditional upon.

Each of the above conditions implies (e) and (d) below:

(d) (GCERA) For any $F \in D_J$, $CE_A(F) \leqslant CE_B(F)$, where $CE_A(F)$ and $CE_B(F)$ refer to the CEs of F for A and B, respectively.

(e) (GPRA) In a simple portfolio set-up with safe asset r and risky asset $\tilde{z}$, where $E(\tilde{z}) \geqslant r$, let x_A and x_B be the respective (unconditional) money risky-asset demand of V_A and V_B who have identical initial wealth. Then $x_A \leqslant x_B$.

Proof: Omitted since it is similar to the proof of Theorem WU7. Also, see Theorem 4 in Machina (1982a).

In this section, we first defined the meaning of one individual being more risk averse than another individual in the sense of certainty equivalent, compensated spread, as well as risky-asset demand. We then investigated if the comparative risk aversion in one sense implies the same in another. Our finding can be summarized as follows. First of all, in order to lay the ground for a meaningful study of comparative risk aversion, it is helpful to impose four properties about preference orderings, namely completeness, transitivity, continuity in distribution and consistency with stochastic dominance. Under these fairly basic assumptions, we were able to demonstrate that "A is more GCERA than B" <= "A is more GCCERA than B" <=> "A is more GMRA than B" => "A is more GCPRA than B" => "A is more GPRA than B". This finding is interesting because it contradicts the casual but widely-held belief that the equivalence of comparative GCCERA and GMRA depends on the underlying preference theory. We are thus convinced that GCCERA and GMRA are more fundamental than previously thought in the sense that they are 'theory-independent'.

When we assume linear Gâteaux differentials to obtain LGU, the Gâteaux derivative $\zeta(x;F)$, called LOSUF (abbrevation for lottery-specific utility function), is the utility function whose degree of concavity serves to measure the degree of risk aversion in its conditional sense. As stated in Theorem LGU7, V_A is more GCCERA than V_B iff $\zeta_A(x;F)$ is more concave in x than $\zeta_B(x;F)$ for every F.

When the functional form of ζ is identical at all distributions, or put differently, when ζ is distribution-free, all conditional versions of risk aversion will reduce to their unconditional counterparts. The comparative risk aversion for this case is characterized in Theorem EU7.

Another distinction between Theorem EU7 and Theorem LGU7 is the absence of the conditional diversifer requirement in Theorem EU7. This is because all EU maximizers are generically diversifers.

We have known that WU is a preference theory intermediate between LGU and EU. Comparisions between Theorem LGU7 and Theorem WU7 however reveal few advantages of WU over LGU so far (one being that all WU agents are also inherently diversifers). This does not mean that the additional structures on WU are in vain. We shall find the functional form of WU useful when we wish to obtain more specifics of CRA. For example, suppose two WU agents happen to have the same value function. It can be shown that the agent whose weight function decreases faster will be more GCCERA. Similarly, if they share the same weight function, the agent with more concave LOSUF will be more risk averse.

The appeal of the specific functional form is particularly evident when the problem involves explicit optimization as in the portfolio choice decision (cf. Theorem WU6) or in the study of the normality property of risky-asset demand to be examined in the next section.

5

DECREASING RISK AVERSION AND THE NORMALITY OF RISKY-ASSET DEMAND WITH DETERMINISTIC WEALTH

Section 3 is devoted to studying the behavioral implications of an individual's risk aversion. In Section 4, we compared two individuals and investigated the implications of comparative risk aversion. Now, let us turn back to one single individual but allow his initial wealth to vary. The questions we attempt to answer are: As an individual gets richer, will he be willing to pay a higher or lower insurance premium for a given risk? Will his demand for the risky asset in our one-safe-asset-one-risky-asset world increase or decrease?

These questions can be answered under different assumptions about the riskiness of the agent's wealth. In this section, we continue to assume that the decision maker has deterministic initial wealth. We will allow it to be stochastic in the next section.

This section contains two subsections. In Subsection 5.1, we review the decreasing risk aversion characterization under EU. Subsection 5.2 focuses on the normality of risky-asset demand under WU.

5.1 Decreasing Risk Aversion under Expected Utility

Arrow (1971) convincingly argued for the appeal of decreasing absolute risk aversion which implies that, as an agent gets richer, he should become less risk averse in the sense of demanding cheaper insurance policies and investing more money in the risky asset.

If the same individual with a different level of wealth can be viewed as if he were a different person, then the characterization of comparative risk aversion (CRA) can be restated straightforwardly to characterize decreasing risk aversion (DRA). EU is a preference theory under which this can be done. Theorem EU8 below is a direct translation of Theorem EU7. In accordance with the literature, we replace the certainty equivalent conditions by the insurance premium ones and state the concavity condition in terms of Arrow-Pratt index.

Theorem EU8 (DRA, Arrow-Pratt): The following properties of a continuous, increasing, twice-differentiable von Neumann-Morgenstern utility function $u(y)$ are equivalent:

(a) (Arrow-Pratt Index) $-\dfrac{u''(y_o)}{u'(y_o)} \geq -\dfrac{u''(y_1)}{u'(y_1)}$ for all $y_o \leq y_1$.

(b) (Conditional Insurance Premium) For any $p \in (0,1]$ and $H \in D_J$, suppose $\pi_o = \pi(y_o,\tilde{\varepsilon}|p,H)$ and $\pi_1 = \pi(y_1,\tilde{\varepsilon}|p,H)$ are u's conditional insurance premia for risk $\tilde{\varepsilon}$ at initial wealth levels y_o and y_1, respectively. Then $\pi_o \geq \pi_1$ for all $\tilde{\varepsilon}$ if $y_o \leq y_1$.

(c) (Conditional Risky-Asset Demand) In a simple portfolio set-up with safe asset r and risky asset $\tilde{z}$, where $E(\tilde{z}) \geq r$, suppose x_o and x_1 are u's conditional risky-asset demands at initial wealth levels y_o and y_1, respectively. Then $x_o \leq x_1$ if $y_o \leq y_1$.

(d) (Insurance Premium) Suppose $\pi_o = \pi(y_o,\tilde{\varepsilon})$ and $\pi_1 = \pi(y_1,\tilde{\varepsilon})$ are u's insurance premia for risk $\tilde{\varepsilon}$ at initial wealth levels y_o and y_1, respectively. Then $\pi_o \geq \pi_1$ for all $\tilde{\varepsilon}$ if $y_o \leq y_1$.

(e) (Risky-Asset Demand) In a simple portfolio set-up with safe asset r and risky asset $\tilde{z}$, where $E(\tilde{z}) \geq r$, suppose x_o and x_1 are u's risky-asset demands at initial wealth levels y_o and y_1, respectively. Then $x_o \leq x_1$ if $y_o \leq y_1$.

Theorem EU8 says that, if an EU decision maker's preference exhibits decreasing absolute risk aversion, then his conditional, as well as unconditional, insurance premium for any risk will decrease and his conditional, as well as unconditional, money risky-asset demand will increase as he gets richer. The asset demand implication is often referred to in the literature as the 'normality of risky-asset demand'.

Taking Theorem EU8 as a benchmark, we may consider the following two generalizations. First, if we continue to assume that both the initial wealth and the wealth increment are deterministic but adopt more general preference functionals, how will the characterization be modified? It appears that, beyond EU, the DRA characterizations will not be a straightforward restatement of its CRA counterparts. This will be explained shortly.

The second generalization of Theorem EU8 is to extend the initial wealth or even the wealth increment from deterministic to stochastic. This will be dealt with in Section 6.

5.2 Decreasing Risk Aversion and the Normality of Risky-Asset Demand under Non-Expected Utility

Once the utility function becomes lottery-specific, difficulties arise in directly translating CRA characterizations to DRA ones. To illustrate, consider the unconditional simple portfolio choice problem under WU. In the context of CRA, suppose both investors have the same initial wealth y_o. Let x_A and x_B denote the risky-asset demands of ζ_A and ζ_B, respectively. By optimality, we have

$$\int \zeta_A(y_o r + x_A(z-r); F_{y_o r + x_A(\tilde{z}-r)})dF(z) = 0 \qquad (5.1)$$

and

$$\int \zeta_B(y_o r + x_B(z-r); F_{y_o r + x_B(\tilde{z}-r)})dF(z) = 0. \qquad (5.2)$$

In the context of DRA, assume x_1 is ζ_A's risky-asset demand at y_1 such that

$$\int \zeta_A(y_1 r + x_1(z-r); F_{y_1 r + x_1(\tilde{z}-r)})dF(z) = 0. \qquad (5.3)$$

If investor ζ_A at y_1 can be viewed as investor ζ_B at y_o, we may replace ζ_A by ζ_B and y_1 by y_o in (5.3) to obtain (5.4) below:

$$\int \zeta_B(y_o r + x_1(z-r); F_{y_o r + x_1(\tilde{z}-r)})dF(z) = 0. \qquad (5.4)$$

The CRA characterization in Theorem WU7 can be used to characterize DRA only if the relation between x_B and y_o in (5.2) is identical to that between x_1 and y_o in (5.4). This would be the case if the distributions that ζ_B depends upon in (5.2) (i.e. $F_{y_o r + x_B(\tilde{z}-r)}$) and (5.4) (i.e. $F_{y_o r + x_1(\tilde{z}-r)}$) were identical. Since $x_B \neq x_1$ in general, we conclude that the DRA characterization in terms of asset demand cannot be obtained simply by rephrasing the asset demand condition in Theorem WU7. This argument also applies to the insurance premium condition.

Why does the distribution-dependence of LOSUFs cause problems in characterizing DRA? In the context of CRA, say in terms of insurance premium, we compare the degree of concavity of two individuals' LOSUFs $\zeta_A(y;\delta_{y_o-\pi_A})$ and $\zeta_B(y;\delta_{y_o-\pi_B})$ which are exogenously given and do not shift because the distributions they depend upon remain unchanged. As we turn our interest to DRA, again, we have two LOSUFs, one depends on $F_{y_o+\tilde{\varepsilon}}$, another on $F_{y_1+\tilde{\varepsilon}}$. However, these two LOSUFs are not independent of each other because they originate from the same preference functional. The change of an individual's risk attitudes as his wealth varies therefore depends on the movement along ζ as well as the shift of ζ. This will be explained in greater detail after we derive the necessary and sufficient condition for the normality of risky-asset demand under WU.

One advantage of WU over LGU is its structural specifications which enable us to optimize using calculus and perform comparative statics without utilizing directional or path differentiation. This advantage results in Theorems WU8.1 and WU8.2 below:

<u>Theorem WU8.1</u> (DRA and Conditional Risky-Asset Demand): The following properties of a pair of properly structured value and weight functions (v,w) with increasing, concave LOSUF $\zeta(x;F)$ are equivalent:

(a) $$\rho(y;p,F,H) \equiv -\frac{\dfrac{\zeta''(y;G)}{E_F[\zeta'(G)]} - \dfrac{pw'(y)}{E_G[w]}}{\zeta'(y;G)} \quad \text{(where } G \equiv pF+(1-p)H) \qquad (5.5)$$

is decreasing in y for all $p \in (0,1]$ and F, H ε D_J;

(b) (Conditional Risky-Asset Demand) In a simple portfolio set-up with riskfree asset r and risky asset $\tilde{z} = r+\tilde{\eta}$, where $E(\tilde{\eta}) \geq 0$, let x_o and x_1 be (v,w)'s conditional risky-asset demands at initial wealth levels y_o and y_1, respectively. Then, $x_o \leq x_1$ if $y_o \leq y_1$.

Proof: The CSPC under WU is

$$\underset{x}{\text{Max}}\ V(pF_{y_o r + x\tilde{\eta}} + (1-p)H).$$

The FOC and SOC for x^* to be optimal are as follows:

$$\text{FOC:}\quad \int \zeta'(y^*;G)\eta dF(\eta) = 0 \tag{3.9}$$

$$\text{SOC:}\quad \int \zeta''(y^*;G)\eta^2 dF(\eta) \leq 0, \tag{3.10}$$

where $G \equiv pF_{\tilde{y}^*} + (1-p)H$ and $\tilde{y}^* = y_o r + x^*\tilde{\eta}$.

The rest of the proof is similar to that of Theorem WU8.2.

Q.E.D.

Although Theorem WU8.1 is more general than Theorem WU8.2, we elect to present a complete proof of the latter because it is less complicated notation-wise and sacrifices little substance.

<u>Theorem WU8.2</u> (DRA and Risky-Asset Demand): The following properties of a pair of properly structured value and weight functions (v,w) with increasing, concave LOSUF $\zeta(x;F)$ are equivalent:

$$\text{(a)}\quad \rho(y;F) \equiv -\frac{\dfrac{\zeta''(y;F)}{E[\zeta'(F)]} - \dfrac{w'(y)}{E[w]}}{\zeta'(y;F)} \tag{5.6}$$

or equivalently,

$$\bar{\rho}(y;F) \equiv -\frac{\zeta''(y;F)}{\zeta'(y;F)} + \frac{w'(y)/E[w]}{\zeta'(y;F)/E[\zeta'(F)]} \tag{5.7}$$

is decreasing in y for all F;

(b) (Risky-Asset Demand) In a simple portfolio set-up with safe asset r and risky asset $\tilde{z} = r+\tilde{\eta}$, where $E(\tilde{\eta}) > 0$, let x_o and x_1 be (v,w)'s risky-asset demands at initial wealth levels y_o and y_1, respectively. Then, $x_o \leq x_1$ if $y_o \leq y_1$.

Proof: The SPC problem under WU is

$$\max_{x} \frac{\int v(y)w(y)dF}{\int w(y)dF}, \text{ where } \tilde{y} = y_o r + x\tilde{\eta}.$$

The FOC and SOC for x* to solve the above are as follows:

$$\text{FOC:} \quad \int \zeta'(y^*;F)\eta dF(\eta) = 0 \tag{5.8}$$

$$\text{SOC:} \quad \int \zeta''(y^*;F)\eta^2 dF(\eta) \leq 0. \tag{5.9}$$

Implicitly differentiating the FOC (5.8) w.r.t. x* and y_o yields

$$\frac{dx^*}{dy_o} = \frac{r\int[\zeta''(F) - w'(\frac{\int\zeta'(F)dF}{\int wdF})]\eta dF}{-\int\zeta''(F)\eta^2 dF}, \tag{5.10}$$

where the argument y has been omitted to simplify the expression. By SOC, the denominator of the RHS of (5.10) is positive. Therefore, the sign of $\frac{dx^*}{dy_o}$ is the same as that of

$$\int[\zeta''(F) - w'(\frac{\int\zeta'(F)dF}{\int wdF})]\eta dF$$

which can be rewritten as

$$E[\zeta''(\tilde{y};F)\cdot\tilde{\eta}] - \frac{E[\zeta'(F)]}{E[w]} E[w'(\tilde{y})\cdot\tilde{\eta}]. \tag{5.11}$$

Also, recall that $E(\tilde{z}) \geq r$ implies $x^* \geq 0$ according to Theorem U6.

(a) → (b): We need to consider two cases -- $\eta \geq 0$ and $\eta < 0$.

Case (i): $\eta \geq 0$

$$y = y_o r + x^*\eta \geq y_o r.$$

Expression (5.7) decreasing implies that

$$-\left(\frac{\zeta''(y;F)}{\zeta'(y;F)} - \frac{E[\zeta'(F)]}{E[w]}\frac{w'(y)}{\zeta'(y;F)}\right) \leq \rho(y_o r;F) \equiv \rho_o.$$

Multiply both sides by $-\zeta'(y;F)\eta$:

$$\zeta''(y;F)\eta - w'(y)\eta\frac{E[\zeta'(F)]}{E[w]} \geq -\zeta'(y;F)\eta\rho_o. \tag{5.12}$$

Case (ii): $\eta < 0$

$$y = y_o r + x^*\eta < y_o r.$$

Expression (5.7) decreasing implies that

$$-\left(\frac{\zeta''(y;F)}{\zeta'(y;F)} - \frac{E[\zeta'(F)]}{E[w]}\,\frac{w'(y)}{\zeta'(y;F)}\right) > \rho_o.$$

Multiplying both sides by $-\zeta'(y;F)\eta$ yields (5.12) with stict inequality so that we can take its expectation as follows:

$$E[\zeta''(\tilde{y};F)\tilde{\eta}] - E[w'(\tilde{y})\tilde{\eta}]\,\frac{E[\zeta'(F)]}{E[w]} \geq -\rho_o E[\zeta'(\tilde{y};F)\tilde{\eta}] = 0. \qquad \text{(by FOC)}$$

Hence, $\frac{dx^*}{dy_o} \geq 0$.

(b) $\rightarrow$ (a): To prove necessity, suppose (5.7) is increasing at some initial wealth level $\bar{y}_o$. Then, following steps in the sufficiency proof will lead to $\frac{dx^*}{d\bar{y}_o} < 0$, which contradicts condition (b).

Q.E.D.

The logic in the proof of Theorem WU8.2 is similar to that in Arrow's (Theorem EU8). We first totally differentiate the FOC for the simple portfolio choice problem to obtain the expression of dx^*/dy_o. Then we show that ρ given by (5.6) decreases in y is equivalent to $dx^*/dy_o \geq 0$.

Theorem WU8.2 is a special case of Theorem WU8.1. Nevertheless, the following discussion will focus on Theorem WU8.2 because it is a one-step extension of the now well-known 'normality of risky-asset demand' under EU attributed to Arrow (1971). By 'one-step' we mean that the only generalization from Arrow's result to Theorem WU8.2 is the preference functional. In contrast, Theorem WU8.1 involves two changes -- one is the preference functional, the other is the introduction of another distribution H.

Within the domain of EU, Arrow (1971) showed that, when the initial wealth and the wealth increment are both deterministic, an investor's preference will display decreasing absolute risk aversion if and only if the single risky asset is a normal good to him (i.e. $dx^*/dy_o \geq 0$). In addition, his preference will display increasing relative risk aversion if and only if the safe asset is a superior (or luxury) good (i.e. $d(1-\beta^*)/dy_o \geqslant 0$). The former result is contained in the equivalence of (c) and (e) in Theorem EU8.

It may at first appear somewhat surprising that condition (a) in Theorem WU8.2 is neither the WU Arrow-Pratt index $r(y) = -[\frac{v''(y)}{v'(y)} + \frac{2w'(y)}{w(y)}]$ nor the concavity index $-\zeta''(y:F)/\zeta'(y;F)$. Given that EU is a special case of WU, $\rho(y;F)$ must reduce to the EU Arrow-Pratt index $-v''(y)/v'(y)$ when w is constant. It can be verified that this is indeed so. When w is not constant, neither a decreasing concavity index nor a decreasing Arrow-Pratt index will imply or be implied by a decreasing $\rho(y;F)$. We have partly explained at the start of this subsection why a decreasing concavity index is not the required condition. As to the Arrow-Pratt index, note that $\rho(y;F)$ depends on distribution F while the Arrow-Pratt index r(y) does not. This means whether a risky asset is a normal good to a WU investor depends on not only his risk attitudes but also the attributes of the risky asset. To see how the distribution of $\tilde{z}$ affects one's demand for it, recall that

$$\zeta(y;F) = w(y)[v(y)-WU(F)]/\int wdF. \tag{1.25}$$

In the context of simple portfolio choice, F is the distribution of the final wealth $\tilde{y} = y_o r+x(\tilde{z}-r)$. $\zeta(y;F)$ is a weighted utility-deviation from WU(F) with $w(y)/\int wdF$ being the weight. We may interpret ζ as a weighted regret when $v(y) \leqslant WU(F)$ and a weighted rejoicing when $v(y) > WU(F)$. Let us simply call ζ (or more suitably $-\zeta$) a weighted regret in general. The derivative of ζ given below:

$$\zeta'(y;F) = \frac{w'(y)[v(y)-WU(F)]+w(y)v'(y)}{\int wdF}$$

$$= \frac{w'(y)}{\int wdF}\,[v(y)-WU(F)] + \frac{w(y)}{\int wdF}\,v'(y) \qquad (1.32)$$

is the contingent marginal weighted utility-deviation from WU(F) or the contingent marginal weighted regret for the outcome y. When y increases by \$1, it causes two effects on ζ. The first is a 'weight effect', given by $[w'(y)/E(w)][v(y)-WU(F)]$ -- marginal weight times regret. The second one is a 'utility effect', given by $[w(y)/E(w)][v(y)-WU(F)]' = [w(y)/E(w)]v'(y)$ -- a weighted marginal regret.

The FOC for a WU maximizer's SPC problem is

$$\int \zeta'(y^*;F)\eta dF(\eta) = 0, \qquad (5.8)$$

or equivalently,

$$\int_{-\infty}^{r} \zeta'(y^*;F)\eta dF(\eta) = \int_{r}^{+\infty} \zeta'(y^*;F)\eta dF(\eta). \qquad (5.8')$$

For an additional dollar's investment in $\tilde{z}$, the extra income is $\eta = z-r$ if z realizes. $\zeta'(y^*;F)\eta$ is the marginal utility contingent on the realization of z. The LHS of (5.8') gives the expected marginal disutility from 'bad' outcome states while the RHS gives the expected marginal utility from 'good' outcome states. The FOC means that, at optimality, the expected marginal utility and disutility from investing an additional \$1 in $\tilde{z}$ must balance out so that the agent has no incentive to deviate from his risky-asset demand x*.

Note that in (5.8) $\zeta'(y;F)$ is the only term involving the parameter y_o. Therefore, in order to have the risky-asset demand x* rise when y_o increases, $\zeta'(y;F)$ must behave in a particular manner. Differentiating $\zeta'(y;F)$ w.r.t. y_o yields the following:

$$\frac{\partial}{\partial y_o} \zeta'(y;F) = r\zeta''(y;F) - \frac{w'(y)}{\int w dF} \frac{\partial WU(F)}{\partial y_o}$$

$$= r\left\{\zeta''(y;F) - \frac{w'(y)}{\int w dF} \int \zeta'(F) dF\right\}$$

$$= r\left\{\zeta''(y;F) - \frac{w'(y)}{E[w]} E[\zeta'(F)]\right\}. \tag{5.13}$$

The interpretation of $\rho(y;F)$ defined by (5.6) is better illustrated by expressing it as follows:

$$\rho(y;F) = -\frac{[\frac{\partial}{\partial y_o} \zeta'(y;F)]/rE[\zeta'(F)]}{\zeta'(y;F)} = -\frac{[\frac{\partial}{\partial y_o} \zeta'(y;F)]/\zeta'(y;F)}{rE[\zeta'(F)]}$$

$$= \frac{1}{E[\zeta'(F)]}\left\{-\frac{\zeta''(y;F)}{\zeta'(y;F)} + \frac{\frac{w'(y)}{E[w]} E[\zeta'(F)]}{\zeta'(y;F)}\right\}. \tag{5.6'}$$

$$= -\frac{\frac{\zeta''(y;F)}{E[\zeta'(F)]} - \frac{w'(y)}{E[w]}}{\zeta'(y;F)} \tag{5.6}$$

$rE[\zeta'(F)] = \partial WU(F)/\partial y_o$ is the ex ante expected marginal utility from an extra dollar available for investment. To be consistent with stochastic dominance, it must be positive. Since $rE[\zeta'(F)]$ is constant w.r.t. y, $\rho(y;F)$ will be decreasing in y if and only if

$$-[\frac{\partial}{\partial y_o} \zeta'(y;F)]/\zeta'(y;F) \tag{5.14}$$

is also decreasing in y. Expression (5.14) has the interpretation of a y_o-elasticity, i.e. the proportional change in $\zeta'(y;F)$ induced by \$1 increase in y_o. Note that an increase in y_o will cause an upward shift of WU(F), changing the benchmark based on which the magnitude of regret is measured. Thus, the effect on $\zeta'(y;F)$ is twofold, one resulting from the movement along ζ', the other from the shift of WU(F). These two effects are represented in (5.13) by the two additive terms in the curly bracket.

ρ decreasing in y means that the 'normalized' y_o-elasticity of the marginal weighted regret (normalized by $\partial WU(F)/\partial y_o$) must be decreasing in y. A decreasing y_o-elasticity in turn means that the investor's intensity of regret about not obtaining a marginally better outcome state lessens as he becomes richer. This seems reasonable to be the condition for the normality of risky-asset demand -- as an agent gets richer, he will hold more of the risky asset.

How must the value function v and weight function w behave in order to display decreasing y_o-elasticity in $\zeta'(y;F)$? Before we answer this question, assume that the SPC problem is uniquely solved for a risk averse decision maker, i.e., $\zeta'(y;F) > 0$ and $\zeta''(y;F) < 0$. The following table should be useful.

case	(a) $-\zeta''/\zeta'$	(b) w	(c) w''	(d) w'/ζ'	(e) condition for $(w'/\zeta')' < 0$	(f)*** condition for $\rho' < 0$
(1)	$\downarrow$	$\downarrow$	−	$\downarrow$	−	always
(2)	$\downarrow$	$\downarrow$	+	?	$-\frac{w''}{w'} < -\frac{\zeta''}{\zeta'}$*	$[\frac{\zeta''}{\zeta'}]' > \{\frac{w'/E[w]}{\zeta'(F)/E[\zeta'(F)]}\}'$
(3)	$\downarrow$	$\uparrow$	−	?	$-\frac{w''}{w'} > -\frac{\zeta''}{\zeta'}$*	as above
(4)	$\downarrow$	$\uparrow$	+	$\uparrow$	impossible	as above
(5)	$\uparrow$	$\downarrow$	−	$\downarrow$	−	$-[\frac{\zeta''}{\zeta'}]' < -\{\frac{w'/E[w]}{\zeta'(F)/E[\zeta'(F)]}\}'$
(6)	$\uparrow$	$\downarrow$	+	?	$-\frac{w''}{w'} < -\frac{\zeta''}{\zeta'}$**	as above
(7)	$\uparrow$	$\uparrow$	−	?	$-\frac{w''}{w'} > -\frac{\zeta''}{\zeta'}$**	as above
(8)	$\uparrow$	$\uparrow$	+	$\uparrow$	impossible	impossible

* sufficient but not necessary for $\rho' < 0$
** necessary but not sufficient for $\rho' < 0$
*** independent of (e)

In the above table, column (a) specifies whether ζ is decreasingly or increasingly concave. Since the normality of risky-asset demand is a form of decreasing risk aversion, a decreasingly concave LOSUF is the more natural case. Columns (b) and (c) indicate whether the weight function is increasing or decreasing, concave or convex. When w is decreasing and concave as in cases (1) and (5), the second term of ρ (cf. expression (5.6)) will be decreasing. On the contrary, it will be increasing if w is increasing and convex as in cases (4) and (8). For all other cases, the direction is ambiguous. The condition under which w'/ζ' will be decreasing is given in column (e).

When both $-\zeta''/\zeta'$ and w'/ζ' are decreasing in y, ρ will definitely decrease in y as well. In this category is case (1) as well as cases (2) and (3) when restricted by the additional condition given in column (e). In case (8), the agent's preferences display increasing risk aversion in the sense of risky-asset demand, i.e. he will reduce his investment in $\tilde{z}$ as he becomes wealthier -- a case normatively not very appealing. When $-\zeta''/\zeta'$ decreases but w'/ζ' increases, or $-\zeta''/\zeta'$ increases but w'/ζ' decreases, the decreasing term must dominate in order to have the normality result. It is interesting to note that even if the LOSUF ζ is increasingly concave, normality is still attainable.

An increasingly concave ζ means that the agent will be more GCCERA or GMRA as he gets richer. How can we justify such an agent's demanding more of the risky asset when he has more money to invest? Since this will never occor under EU, let us consider an EU agent as our base case. When the weight function is constant, the second term of ρ vanishes and ρ reduces to

$$\rho_{EU}(y;F) = -\frac{v''(y)/E[v']}{v'(y)}.$$

Since $E[v']$ is constant in y, ρ_{EU} decreasing in y is equivalent to $-v''/v'$ decreasing in y. The latter is of course the EU Arrow-Pratt index. ρ_{EU} is a normalized concavity index whose behavior happens to be consistent with the Arrow-Pratt index.

When w is not constant, it is clear from (5.6) that a concave and/or decreasing w will reinforce the decreasing risk aversion captured by a decreasing normalized concavity index. On the contrary, a convex and/or increasing w will offset all or part of it. Recall that a decreasing w indicates pessimism which is a source of risk aversion. A concave, decreasing w therefore depicts decreasingly pessimistic attitudes. For a WU agent whose utility-based risk aversion is increasing in wealth (i.e. $[-\zeta''/\zeta']' > 0$), if his pessimism decreases sufficiently fast when he becomes richer, he might still increase his holding in the risky asset.

Interestingly, if we define a proportional version of $\rho(y;F)$ as follows:

$$\rho^*(y;F) \equiv y\rho(y;F) = -\frac{\frac{y\zeta''(y;F)}{E[\zeta'(F)]} - \frac{yw'(y)}{E[w]}}{\zeta'(y;F)}, \tag{5.15}$$

it can be shown that $d(1-\beta^*)/dy_o \geqslant 0$ (i.e. the safe asset is a superior good) if and only if $\rho^*(y;F)$ increases in y.

Can we obtain a similar condition for the normality of risky-asset demand under LGU? The presence of the weight function in (5.6) apart from $\zeta(y;F)$ leads us to believe that this is not possible without imposing more structures on the functional V(F) or making further assumptions.

It is worth noting that the insurance premium condition is absent in

Theorems WU8.1 and WU8.2. There appears to be some fundamental distinctions between insurance premium and portfolio choice. First of all, note that the absolute size of the risk in the insurance premium condition remains the same when the agent's initial wealth changes. In contrast, an investor's risky-asset demand is a function of his investible wealth. As he experiences an exogenous increase in his investible funds, he will change his investment in the risky asset. Suppose his initial wealth increases from y_o to $y_o+\Delta y$ and his risky-asset demand changes from x_o to x_1. As long as $x_o \neq x_1$, the absolute size of the risk he is bearing will be altered. We also cannot be sure that the relative size of the risk will not vary.

Secondly, as pointed out earlier, insurance premium is a consequence of a decision maker's perception about the certainty equivalent for a risk. In contrast, risky-asset demand is the result of an optimizing behavior. As such, the derivation of the condition for PRA often makes use of the FOC if it is obtainable. We have applied the same approach in producing Theorems WU8.1 and WU8.2.

A natural way of deriving the condition for DRA in the sense of insurance premium is the comparative static technique. Totally differentiating $WU(F_{y_o+\tilde{\varepsilon}}) = WU(\delta_{y_o-\pi}) = v(y_o-\pi)$ yields

$$\frac{d\pi}{dy_o} = 1 - \frac{\int \zeta'(F_{y_o+\tilde{\varepsilon}})dF_{y_o+\tilde{\varepsilon}}}{v'(y_o-\pi)} \,. \tag{5.16}$$

DRA in the sense of insurance premium calls for $d\pi/dy_o \leqslant 0$, i.e.,

$$\int \zeta'(F_{y_o+\tilde{\varepsilon}})dF_{y_o+\tilde{\varepsilon}} \geqslant v'(y_o-\pi). \tag{5.17}$$

Under EU, (5.17) reduces to

$$\int u' dF_{y_o+\tilde{\varepsilon}} \geq u'(y_o - \pi)$$

which holds for all $\tilde{\varepsilon}$ if and only if $-u''/u'$ is decreasing. When the preference functional is nonlinear in distribution, (5.17) will not be equivalent to $(-\zeta''/\zeta')' \leq 0$ in general.

6

COMPARATIVE AND DECREASING RISK AVERSION INVOLVING STOCHASTIC WEALTH

In Section 5, both the initial wealth and the wealth increment are assumed to be deterministic. When complete insurance is not available or when a safe asset does not exist, this assumption is deemed unrealistic. It is therefore of interest to see how the CRA and DRA characterizations in Theorem EU7 and Theorem EU8 can be extended to allow for stochastic initial wealth, or even stochastic wealth increments.

It should be pointed out that the wealth increment may confound the agent's portfolio choice problem. To illustrate, let r and $\tilde{z} = r+\tilde{\eta}$ with $E(\tilde{\eta}) \geqslant 0$ be the two assets in our simple portfolio set-up. Suppose an agent's demand for $\tilde{z}$ is x_o when his risky initial wealth is $\tilde{y}_o$. His final wealth will be $\tilde{y}_o r+x_o\tilde{\eta}$. If his demand for $\tilde{z}$ changes to $x_1 = x_o+\Delta x$ (Δx may be positive or negative) when his initial wealth increases to $\tilde{y}_1 = \tilde{y}_o+\Delta y$ ($\Delta y > 0$), his final wealth will be $(\tilde{y}_o+\Delta y)r+(x_o+\Delta x)\tilde{\eta}$ if Δy is investible, and $\tilde{y}_o r+(x_1+\Delta x)\tilde{\eta}+\Delta y$ if Δy is not investible.

When Δy is investible, Δx is a consequence of two effects. One may be called a 'resource effect' -- an effect caused by an increase in his investible resources. The other may be called a 'risk attitude effect' -- an effect caused by the change in his attitudes toward risk which in turn is caused by an increase in his 'consumable' income. We shall only be concerned with the risk attitude effect in this section. In other words, we assume that the anticipated wealth increment will be available only after the investment decision is made, therefore noninvestible. For simplicity, we say the wealth increment is <u>ex post</u>. In contrast, the unrestricted wealth increment in Theorems EU8, WU8.1, and WU8.2 is <u>ex ante</u>. Note that, when wealth increment is ex ante but can only be invested in the safe asset, the result in this section still holds.

This section contains two parts. In Subsection 6.1, Theorem EU9 and Theorem EU10, due to Ross (1981), extend Theorem EU7 and Theorem EU8, respectively, to allow for stochastic initial wealth. In Theorem EU10, which is in terms of DRA, the wealth increment is assumed deterministic.

In Subsection 6.2, we first illustrate that Ross' strong concavity index does not have a WU or LGU counterpart. Theorems LGU9 and LGU10 are then presented as special cases of Machina (1982b)'s Theorem 1 extended to LGU by imposing additional structure on the linear Gâteaux derivative ζ (Chew, 1985). Again, we assume stochastic initial wealth and deterministic wealth increments. Both initial wealth and wealth increment are allowed to be stochastic in Theorem LGU11. This case apparently involves too many risks for EU to handle. Hence, there is no Theorem EU11.

Since the results gathered in this section are either from Ross (1981) or based on Machina (1982b) and Chew (1985), their proofs will be discussed but not reproduced. The presence of this section is mainly for the completeness of comparisons among EU, WU and LGU under different assumptions about wealth levels, namely from $(y_0, \Delta y)$, to $(\tilde{y}_0, \Delta y)$, and then to $(\tilde{y}_0, \Delta\tilde{y})$.

6.1 CRA and DRA with Stochastic Wealth under Expected Utility

Suppose two EU agents u_A and u_B have the same risky initial wealth $\tilde{y}_o$. Let π_A and π_B be their respective insurance premia for a risk $\tilde{\varepsilon}$ uncorrelated with $\tilde{y}_o$. Let x_A and x_B be their respective risky-asset demands in a simple portfolio set-up with assets r and $\tilde{z} = r+\tilde{\eta}$, where $E(\tilde{\eta}|y_o) = E(\tilde{\eta}) \geq 0$ for all y_o. If u_A is more risk averse than u_B, we expect u_A to be willing to pay a higher premium than u_B for the insurance against a given risk $\tilde{\varepsilon}$. Similarly, we anticipate u_B to invest more in the risky asset than u_A will. What is the proper condition for being 'more risk averse' in this sense? The answer is given in Theorem EU9 below:

Theorem EU9 (CRA with $\tilde{y}_o$, Ross): The following properties of a pair of continuous, strictly increasing, twice-differentiable von Neumann-Morgenstern utility functions u_A and u_B are equivalent:

(a) (Strong Arrow-Pratt Index) $-\dfrac{u_A''(y+k)}{u_A'(y)} \geq -\dfrac{u_B''(y+k)}{u_B'(y)}$ (6.1)

for all k.

(a') (Strong Concavity) There exist a positive constant λ and a decreasing concave function h such that

$$u_A(y) = \lambda u_B(y) + h(y). \qquad (6.2)$$

(b) (Insurance Premium) Suppose π_A and π_B are the respective insurance premia for risk $\tilde{\varepsilon}$ of u_A and u_B who have identical initial wealth $\tilde{y}_o$. Then, $\pi_A \geq \pi_B$ for all $\tilde{\varepsilon}$ satisfying $E(\tilde{\varepsilon}|y_o) = E(\tilde{\varepsilon})$ for all y_o.

In addition, each of the above implies the following condition:

(c) (Risky-Asset Demand) Suppose u_A and u_B have identical wealth $\tilde{y}_o$ and

x_A, x_B are their respective risky-asset demands in a simple portfolio set-up with assets r and $\tilde{z} = r+\tilde{\eta}$, where $E(\tilde{\eta}|y_o) = E(\tilde{\eta}) \geq 0$ for all y_o. Then, $x_A \leq x_B$.

Note that the risky-asset demand condition is implied but not equivalent to the other conditions. This is another evidence that the nature of insurance premium and risky-asset demand is not quite the same. In this case, $x_A \leq x_B$ implies

$$\begin{aligned}
0 &\geq E[u_A'(\tilde{y}_o r+x_B\tilde{\eta})\tilde{\eta}] \\
&= E[\lambda u_B'(\tilde{y}_o r+x_B\tilde{\eta})\tilde{\eta}] + E[h'(\tilde{y}_o r+x_B\tilde{\eta})\tilde{\eta}] \\
&= E[h'(\tilde{y}_o r+x_B\tilde{\eta})\tilde{\eta}] \\
&= \mathrm{Cov}[h'(\tilde{y}_o r+x_B\tilde{\eta}),\tilde{\eta}] + E[h'(\tilde{y}_o r+x_B\tilde{\eta})]E[\tilde{\eta}]
\end{aligned}$$

which may be satisfied by a function h not simultaneously decreasing and concave.

The strong Arrow-Pratt index condition in Theorem EU9 implies, but is not implied by, the Arrow-Pratt index condition in Theorem EU8. Due to its stronger form here, the initial wealth is allowed to be random. The randomness is however not arbitrary. It must be uncorrelated to the risk $\tilde{\varepsilon}$ to be insured or the risk $\tilde{\eta}$ from the risky asset.

To gain some insight into Ross' strong Arrow-Pratt index, consider an infinitesimal risk $\tilde{\varepsilon}$ which is contingent on the realization of the initial wealth $\tilde{y}_o$. Suppose $\tilde{\varepsilon}$ is likely to occur only if $y_o \in Y$ and will definitely not occur if $y_o \in \bar{Y}$; $E(\tilde{\varepsilon}|y_o) = 0$ and $\mathrm{Var}(\tilde{\varepsilon}|y_o) = \sigma^2$ for every $y_o \in Y$. Moreover, let F and G be the distributions of $\tilde{y}_o$ and $\tilde{y}_o+\tilde{\varepsilon}$, respectively. The insurance premium π for this contingent risk is defined by equation (6.3) below:

$$\int u(y_o-\pi)dF = \int_Y u(y_o+\varepsilon)dG + \int_{\bar{Y}} u(y_o)dF. \tag{6.3}$$

Expand both sides via Taylor's series as follows:

$$\begin{aligned}\int u(y_o-\pi)dF &= \int[u(y_o)-\pi u'(y_o)+O(\pi^2)]dF \\ &= \int u(y_o)dF - \pi\int u'(y_o)dF + O(\pi^2).\end{aligned}$$

$$\begin{aligned}&\int_Y u(y_o+\varepsilon)dG + \int_{\bar{Y}} u(y_o)dF \\ &\quad = \int_Y [u(y_o)+\varepsilon u'(y_o)+\frac{\varepsilon^2}{2}u''(y_o)+o(\varepsilon^2)]dG + \int_{\bar{Y}} u(y_o)dF \\ &\quad = \int_Y u(y_o)dF + \frac{\sigma^2}{2}\int_Y u''(y_o)dF + \int_{\bar{Y}} u(y_o)dF + o(\sigma^2) \\ &\quad = \int u(y_o)dF + \frac{\sigma^2}{2}\int_Y u''(y_o)dF + o(\sigma^2).\end{aligned}$$

Therefore,

$$\pi(\tilde{y}_o, \tilde{\varepsilon}) \approx \frac{\sigma^2}{2}\frac{\int_Y u''(y_o)dF}{-\int u'(y_o)dF}. \tag{6.4}$$

The term $\int_Y u''(y_o)dF$ is an expected diminishing rate of the marginal utility which measures the disutility from \$1 uninsured loss. Since the risk $\tilde{\varepsilon}$ will likely be present only at $y_o \in Y$, the integral is taken over the set Y only. The term $-\int u'(y_o)dF$ on the other hand gives the expected disutility from paying one extra dollar in premium for the insurance. The expectation is taken over the union of Y and $\bar{Y}$ because the premium has to be paid no matter $y_o \in Y$ or $y_o \in \bar{Y}$. The ratio of these two terms is the modified Arrow-Pratt index for the case where the agent's wealth is stochastic, and has the interpretation of "twice the insurance premium per unit of conditional variance" of the infinitesimal contingent risk to be insured.

For two EU maximizers u_A and u_B with identical $\tilde{y}_o$ distributed as F, we have $\pi_A \geq \pi_B$ for all such risks only if

$$-\frac{\int_Y u_A''(y_o)dF}{\int u_A'(y_o)dF} \geq -\frac{\int_Y u_B''(y_o)dF}{\int u_B'(y_o)dF} \tag{6.5}$$

for all Y. Furthermore, under EU the risk $\tilde{\varepsilon}$ can be generalized to noninfinitesimal ones. It is straightforward to check that (6.5) holds for all $F_{\tilde{y}_o}$ and Y if and only if

$$-\frac{u_A''(y+k)}{u_A'(y)} \geq -\frac{u_B''(y+k)}{u_B'(y)} \tag{6.1}$$

for all k.

Condition (a') of Theorem EU9 says u_A can be obtained by transforming u_B via (6.2). Recall that, if u_A is more concave than u_B, then u_A can be obtained by "concavifying" u_B via an increasing, concave function. Does (6.2) require u_A be even more concave? Given an increasing, concave function u_B, suppose

$$u_A(y) = \lambda u_B(y) + h(y),$$

and

$$\bar{u}_A(y) = \bar{h}[u_B(y)],$$

where $\lambda > 0$, $h' < -\lambda u_B' < 0$, $h'' \leq 0$, $\bar{h}' > 0$ and $\bar{h}'' \leq 0$. Then u_A', $\bar{u}_A' > 0$, u_A'', $\bar{u}_A'' \leq 0$ and $-u_A''/u_A' \geq -\bar{u}_A''/\bar{u}_A'$. In other words, in order to have the CRA characterization in Theorem EU7 carry through to the case with stochastic initial wealth, the utility function of the more risk averse individual A must be more concave than in the case where wealth is deterministic. Note that, because u_A and u_B are not lottery-specific, neither will be λ and $h(y)$. This is crucial for the proof of Theorem EU9.

In the context of DRA, suppose an agent's insurance premium for risk $\tilde{\varepsilon}$ at initial wealth $\tilde{y}_o$ is π_o. When $\tilde{y}_o$ is increased by a constant Δy in every state, how will his insurance premium change accordingly? If he is decreasingly risk averse, we expect him to become more reluctant to purchase insurance. Thus, the premium he will be prepared to pay for the same policy should decrease. Similarly, we expect him to increase his holding of the risky asset when he gets richer. The condition for this sense of DRA is given in Theorem EU10 which is simply a rephrasing of Theorem EU9.

Theorem EU10 (DRA with $\tilde{y}_o$ and Δy, Ross): The following properties of a continuous, strictly increasing, twice-differentiable von Neumann-Morgenstern utility function $u(y)$ are equivalent:

(a) (Strong Arrow-Pratt Index) $-\frac{u''(y+k)}{u'(y)}$ is decreasing in y for all k.

(b) (Insurance Premium) Suppose $\pi_o = \pi(\tilde{y}_o,\tilde{\varepsilon})$ and $\pi_1 = \pi(\tilde{y}_1,\tilde{\varepsilon})$ are u's insurance premia for risk $\tilde{\varepsilon}$ at stochastic initial wealth levels $\tilde{y}_o$ and $\tilde{y}_1 = \tilde{y}_o+\Delta y$, respectively. Then, $\Delta y > 0$ implies $\pi_o > \pi_1$ for all $\tilde{\varepsilon}$ satisfying $E(\tilde{\varepsilon}|y_o) = E(\tilde{\varepsilon})$ for all y_o.

In addition, each of the above implies the following property:

(c) (Risky-Asset Demand) Suppose x_o and x_1 are u's risky-asset demands at initial wealth levels $\tilde{y}_o$ and $\tilde{y}_1 = \tilde{y}_o+\Delta y$, respectively, in a simple portfolio set-up with safe asset r and risky asset $\tilde{z} = r+\tilde{\eta}$, where $E(\tilde{\eta}|y_o) = E(\tilde{\eta}) > 0$ for all y_o. Then, $x_o < x_1$ if $\Delta y > 0$.

6.2 CRA and DRA with Stochastic Wealth Beyond Expected Utility

To derive the condition for comparative and decreasing GIPRA under WU, we apply the same approach as in the preceding subsection. Consider an infinitesimal risk $\tilde{\varepsilon}$ contingent on $y_o \in Y$ with $E(\tilde{\varepsilon}|y_o) = 0$ and $Var(\tilde{\varepsilon}|y_o) = \sigma^2$ for every $y_o \in Y$. Let F and G be the distributions of $\tilde{y}_o$ and $\tilde{y}_o+\tilde{\varepsilon}$, respectively. A WU agent's insurance premium is defined by (6.6) below:

$$\frac{\int v(y_o-\pi)w(y_o-\pi)dF}{\int w(y_o-\pi)dF} = \frac{\int_Y v(y_o+\varepsilon)w(y_o+\varepsilon)dG+\int_{\bar{Y}} v(y_o)w(y_o)dF}{\int_Y w(y_o+\varepsilon)dG+\int_{\bar{Y}} w(y_o)dF}. \tag{6.6}$$

Expanding both sides via Tayor's series after cross multiplication yields

$$\pi(\tilde{y}_o, \tilde{\varepsilon}) \approx \frac{\sigma^2}{2}\frac{\int_Y \zeta''(y_o;F)dF}{-\int \zeta'(y_o;F)dF}. \tag{6.7}$$

In view of the similarity between (6.7) and (6.4), one is tempted to speculate that $-\zeta''(y+k;F)/\zeta'(y;F)$ decreasing in y for all k and F is the condition we are seeking for. This conjecture turns out to be incorrect. It is true that

$$-\frac{\zeta_A''(y+k;F)}{\zeta_A'(y;F)} \geq -\frac{\zeta_B''(y+k;F)}{\zeta_B'(y;F)}$$

for all k and F if and only if, for every $F \in D_J$, there exist a constant $\lambda_F > 0$ and a decreasing, concave function $h_F(y)$ such that

$$\zeta_A(y;F) = \lambda_F \zeta_B(y;F) + h_F(y).$$

Since ζ_A and ζ_B are F-specific, λ and h must also depend on F. As a result, the proof of Theorem EU9 will not go through for WU.

From (6.7) we know that, if π_A is to be greater than π_B for all Y, it must be true that

$$-\frac{\zeta_A''(y;F)}{\int\zeta_A'(F)dF} \geq -\frac{\zeta_B''(y;F)}{\int\zeta_B'(F)dF} \tag{6.8}$$

for all y, where F is the distribution of $\tilde{y}_o$. Although (6.8) is derived for small risks, it turns out similar to Machina's condition for general risks.

To extend Machina's results from FDU to LGU, we impose additional structure on LOSUF ζ as below (Chew, 1985):

Assumption 6.1: The linear Gâteaux derivative $\zeta(\cdot;\cdot)$: $J \times D_J \to R$ of the preference functional $V(\cdot)$: $D_J \to R$ is continuously differentiable and ex ante bounded, i.e., there exists $M > 0$ such that $|\zeta(x;F)| \leq M$ for all $x \in J$ and $F \in D_J$.

In this section, we suppose Assumption 6.1 is satisfied. In what follows, the theorems are stated in terms of LGU only.

Theorem LGU9 (CRA with $\tilde{y}_o$): The following properties of two LGU functionals V_A, V_B with increasing, concave LOSUFs ζ_A and ζ_B satisfying Assumption 6.1 are equivalent:

(a) $$-\frac{\zeta_A''(y;F)}{\int\zeta_A'(F)dF} \geq -\frac{\zeta_B''(y;F)}{\int\zeta_B'(F)dF} \tag{6.8}$$

for all y and F.

(b) (Insurance Premium) Suppose π_A and π_B are the respective insurance premia for risk $\tilde{\varepsilon}$ of ζ_A and ζ_B who have identical initial wealth $\tilde{y}_o$. Then, $\pi_A \geq \pi_B$ for all $\tilde{\varepsilon}$ satisfying $E(\tilde{\varepsilon}|y_o) = E(\tilde{\varepsilon})$ at all y_o.

In addition, if both V_A and V_B are diversifers, then each of the above is equivalent to:

(c) (Risky-Asset Demand) Suppose ζ_A and ζ_B have identical wealth $\tilde{y}_o$ and

x_A, x_B are their respective risky-asset demands in a simple portfolio set-up with safe asset r and risky asset $\tilde{z} = r+\tilde{\eta}$, where $E(\tilde{\eta}|y_o) = E(\tilde{\eta}) \geq 0$ for all y_o. Then, $x_A \leq x_B$.

<u>Theorem LGU10</u> (DRA with $\tilde{y}_o$ and Δy): The following properties of an LGU functional V with increasing, concave LOSUF ζ satisfying Assumption 6.1 are equivalent:

(a) $$-\frac{\zeta''(y;F)}{\int \zeta'(F)dF} \geq -\frac{\zeta''(y^*;F^*)}{\int \zeta'(F^*)dF^*} \qquad (6.9)$$

for all $y^* \geq y$ and $F^*(s) \equiv F(s-\Delta)$, $\Delta \geq 0$.

(b) (Insurance Premium) Suppose π_o and π_1 are ζ's insurance premia for risk $\tilde{\varepsilon}$ at initial wealth levels $\tilde{y}_o$ and $\tilde{y}_1 = \tilde{y}_o+\Delta y$, respectively. Then, $\pi_o \geq \pi_1$ for all $\tilde{\varepsilon}$ satisfying $E(\tilde{\varepsilon}|y_o) = E(\tilde{\varepsilon})$ for all y_o if $\Delta y \geq 0$.

In addition, if V is a diversifier, then each of the above is equivalent to the following:

(c) (Risky-Asset Demand) Let x_o and x_1 be ζ's respective risky-asset demands at initial wealth levels $\tilde{y}_o$ and $\tilde{y}_1 = \tilde{y}_o+\Delta y$ in a simple portfolio set-up with safe asset r and risky asset $\tilde{z} = r+\tilde{\eta}$, where $E(\tilde{\eta}|y_o) = E(\tilde{\eta}) \geq 0$ for all y_o. Then, $x_o \leq x_1$ if $\Delta y \geq 0$.

<u>Theorem LGU11</u> (DRA with $\tilde{y}_o$ and $\Delta\tilde{y}$): The following properties of an LGU functional V with increasing, concave LOSUF ζ satisfying Assumption 6.1 are equivalent:

(a) $$-\frac{\zeta''(y;F)}{\int \zeta'(F)dF} \geq -\frac{\zeta''(y^*;F^*)}{\int \zeta'(F^*)dF^*} \qquad (6.10)$$

for all $y^* \geq y$ and $F^* \geq^i F$.

(b) (Insurance Premium) Suppose π_o and π_1 are ζ's insurance premia for risk $\tilde{\varepsilon}$ at initial wealth levels $\tilde{y}_o$ and $\tilde{y}_1 = \tilde{y}_o+\Delta\tilde{y}$, respectively, and $\Delta\tilde{y} \geq 0$. Then, $\pi_o \geq \pi_1$ for all $\tilde{\varepsilon}$ satisfying $E(\tilde{\varepsilon}|y_o) = E(\tilde{\varepsilon}|y_o+\Delta y) = E(\tilde{\varepsilon})$ for all y_o and $y_o+\Delta y$.

In addition, if V is a diversifer, then each of the above conditions is equivalent to:

(c) (Risky-Asset Demand) Suppose x_o and x_1 are ζ's respective risky-asset demands at initial wealth levels $\tilde{y}_o$ and $\tilde{y}_1 = \tilde{y}_o+\Delta\tilde{y} \geq \tilde{y}_o$ in a simple portfolio set-up with safe asset r and risky asset $\tilde{z} = r+\tilde{\eta}$, where $E(\tilde{\eta}|y_o) = E(\tilde{\eta}|y_o+\Delta y) = E(\tilde{\eta}) \geq 0$ for all y_o and $y_o+\Delta y$. Then $x_o \leq x_1$.

Several points are worth noting in the above theorems. First, the risk $\tilde{\varepsilon}$ in insurance premium condition and the risk $\tilde{\eta}$ in the risky-asset demand condition are required to be uncorrelated with $\tilde{y}_o$ and $\Delta\tilde{y}$ in such a manner that $E(\tilde{\varepsilon}|y_o) = E(\tilde{\varepsilon}|y_o+\Delta y) = E(\tilde{\varepsilon})$ and $E(\tilde{\eta}|y_o) = E(\tilde{\eta}|y_o+\Delta y) = E(\tilde{\eta}) \geq 0$ for all y_o and $y_o+\Delta y$. When his wealth is risky, an agent's insurance premium and risky-asset demand will naturally depend on how different sources of risks interact. The purpose of the uncorrelatedness restriction is to eliminate any possible offsetting or aggravating effect among risks.

Secondly, the result in Theorem LGU11 was originally proved by Machina for Fréchet differentiable utility. Essentially, the proof is comparative statics utilizing path derivative. For example, if $F_{\tilde{y}_o+\tilde{\varepsilon}}$ is indifferent to $F_{\tilde{y}_o-\pi_o}$, then there exists a path from $F_{\tilde{y}_o-\pi_o}$ to $F_{\tilde{y}_o+\tilde{\varepsilon}}$ along which the same utility level is maintained. In other words, this path is an indifference curve. Consider a scenario similar to Kahneman and Tversky (1979)'s probabilistic insurance. Suppose an agent can purchase an insurance for $\tilde{\varepsilon}$ at a premium $\pi(\alpha)$ lower than the complete insurance premium π_o. If the risk $\tilde{\varepsilon}$ occurs, lots will be drawn to determine whether the insurer or the insured is to absorb the risk in the event it occurs. With α chance, the insurer will be responsible for $\tilde{\varepsilon}$; with $1-\alpha$ chance, the insurer will walk away with the premium, leaving the agent to absorb the risk. Naturally, $\pi(\alpha)$ increases in α.

Along the indifference path $\{F^{\alpha} \equiv \alpha F_{\tilde{y}_o - \pi(\alpha)} + (1-\alpha) F_{\tilde{y}_o - \pi(\alpha) + \tilde{\varepsilon}}\}$, $\frac{d}{d\alpha} V(F^{\alpha}) = 0$ at all $\alpha \in [0,1]$. Similarly, given $\Delta\tilde{y} \geq 0$, there is another indifference path $\{F^{*\alpha} \equiv \alpha F_{\tilde{y}_o + \Delta\tilde{y} - \pi^*(\alpha)} + (1-\alpha) F_{\tilde{y}_o + \Delta\tilde{y} - \pi^*(\alpha) + \tilde{\varepsilon}}\}$. If the agent is decreasingly risk averse, then $\pi(\alpha) \geq \pi^*(\alpha)$ at each $\alpha \in [0,1]$. Define a path $\{\bar{F}^{\alpha} \equiv \alpha F_{\tilde{y}_o + \Delta\tilde{y} - \pi(\alpha)} + (1-\alpha) F_{\tilde{y}_o + \Delta\tilde{y} - \pi(\alpha) + \tilde{\varepsilon}}\}$. Since $\pi(\alpha)$ is too high to be optimal for wealth $\tilde{y}_o + \Delta\tilde{y}$, a lower α will be preferred. Hence $\frac{d}{d\alpha} V(\bar{F}^{\alpha}) \leq 0$.

When ζ satisfies Assumption 6.1, V is path differentiable on all generalized smooth paths (see Chew (1985)). As such, Machina's results will continue to hold.

Recall that the CRA condition we derived for infinitesimal risks is:

$$- \frac{\zeta_A''(y;F)}{\int \zeta_A'(F)dF} \geq - \frac{\zeta_B''(y;F)}{\int \zeta_B'(F)dF} \tag{6.8}$$

for all y and F. In the context of DRA, we must take into consideration the effect on the ratio caused by the shift of distribution. After a deterministic increase in ex post wealth Δy, the distribution of the final wealth will be $F_{\tilde{y}+\Delta y}$. The condition for DRA therefore becomes

$$- \frac{\zeta''(y;F_{\tilde{y}+\Delta y})}{\int \zeta'(F_{\tilde{y}+\Delta y})dF_{\tilde{y}+\Delta y}} \tag{6.11}$$

decreasing in both y and Δy. This is the case of Theorem LGU10.

When the wealth increment is stochastic, it is required that the ratio (6.11) decrease in y as well as in distribution in the sense of first-degree stochastic dominance. Hence, condition (a) of Theorem LGU11.

Obviously (6.10) implies (6.9). To see the distinction between them, consider a positive stochastic wealth increase $\Delta\tilde{y} = \Delta+\tilde{\theta}$ with $\Delta > 0$, $E(\tilde{\theta}|y_o) = 0$ for all y_o and $\tilde{\theta}$ is bounded from below by $-\Delta$. Consider an agent who is decreasingly risk averse in the sense of Arrow. As his wealth increases by a positive, deterministic amount Δ, he will become less risk averse -- the implication of condition (6.9). When an uncorrelated, zero-mean risk is added to his wealth, he will feel worse-off, therefore become more risk averse. The net effect of an uncorrelated zero-mean risk and a simultaneous deterministic increase in wealth however is ambiguous in general. Condition (6.10) is stronger than (6.9) in the sense that it further requires that the effect on an agent's risk attitude caused by Δ not be offset by the opposite effect of any zero-mean risk $\tilde{\theta}$ bounded from below by $-\Delta$.

This stronger measure can be rephrased to characterize CRA for the case where agents have indentical stochastic initial wealth $\tilde{y}_o$. In such a case, agent V_A is said to be more risk averse than agent V_B <u>up to Δ</u> if

$$- \frac{\zeta_A''(y;F)}{\int \zeta_A'(F)dF} \geqslant - \frac{\zeta_B''(y;G)}{\int \zeta_B'(G)dG}$$

for all y, $\tilde{\theta}$ and F, where $\tilde{\theta}$ satisfies $E(\tilde{\theta}) = 0$ and $\min\{\theta\} \geq -\Delta$ and G is the distribution of $\tilde{s}+\tilde{\theta}$ if F is that of $\tilde{s}$.

Also note that there is no Theorem EU11. This is because the von Neumann-Morgenstern utility function u(y) does not depend on distribution, rendering EU incapable of handling the situation where both initial wealth and wealth increment are stochastic. On the other hand, Theorems WU9, WU10 and WU11 are omitted because they will be identical to their LGU counterparts without the diversifier requirement.

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CONCLUSION

After an extended period of the predominance of expected utility in economics and finance, there is a sense of excitement in terms of new directions being contemplated. Descriptive validity has provided the primary impetus behind many attempts to construct theories beyond expected utility. They include the theories of Allais (1953; 1979), Edwards (1954), Handa (1977), Meginniss (1977) and Karmarkar (1978), Kahneman and Tversky's prospect theory (1979), the regret theory of Bell (1982) and Looms and Sugden (1983), Machina's Fréchet differentiable preference functional analysis (1982a; 1982b) and weighted utility (Chew and MacCrimmon, 1979a; 1979b; Chew, 1980; 1981; 1982; 1983; Fishburn, 1983; Nakamura, 1984). In order to discriminate among these alternative preference theories, further experimental studies will be needed to delineate their respective domains of empirical validity.

Another way of discriminating among them is via their applicability to the economics of uncertainty and information. In comparison with expected utility, few such applications of alternative theories have been reported to date. Of the 'misperception-of-probability' theories, Thaler (1980) applied prospect theory to account for several puzzles in consumer behavior. Shefrin and Statman (1984) partially applied prospect theory to model investors' preference for cash dividends over stock dividends.

Among the theories of general preference functionals, implications of both weighted utility and Fréchet differentiable preference functional approach for income inequality were presented in the respective papers of Chew (1983) and Machina (1982b). Weber (1982) derived for homogeneous weighted utility agents a Nash equilibrium bidding strategy that is compatible with the 'discrepancies' in the observed bids under the Dutch auction and the first-price sealed-bid auction reported in the experiments conducted by Cox, Roberson and Smith (1982). Machina (1982a; 1982b) applied Fréchet differentiable utility theory to obtain conditions for comparative and decreasing risk aversion, as well as for the normality of risky-asset demands. Epstein (1984) applied Fréchet differentiable utility to mean-variance analysis and provided a refreshing and most powerful defense of its theoretical soundness since the scathing attack by Borch (1969) and Feldstein (1969).

With the exception of Machina's and Epstein's works, the above investigations are rather fragmentary in nature. Among them, the studies of Shefrin and Statman, Machina, and Epstein have direct relevance to finance. In order to increase our understanding of the applicability of the numerous alternative preference theories to financial economics, two lines of research appear worthwhile. The first is to directly apply a given theory to model specific situations in finance and obtain implications that can be compared to those based on expected utility in the financial markets. The other is directed towards the derivation of conditions for preference properties relevant to finance such as risk aversion and the normality of risky-asset demands. This essay is intended towards the latter.

We focus our attention on weighted utility and contrast it with expected utility and linear Gâteaux utility which is Fréchet differentiable when restricted to a bounded domain. Weighted utility and linear Gâteaux utility are selected because, unlike other proposed alternatives, both are analytically tractable.

Under expected utility, if two lotteries are indifferent, then when separately mixed with a third lottery at the same proportion, the two new compound lotteries must also be indifferent. This implies that the indifference curves in any simplex involving 3-outcome lotteries are parallel straight lines. To accommodate Allais-type choice behavior, these two compound lotteries must be allowed to lie on two distinct indifference curves. Intuitively, the most liberal compromise is to permit a set of indifference curves that do not intersect (or transitivity will be violated), and will behave in conformance with the law of the-more-the-better (i.e. consistent with first-degree stochastic dominance). For technical convenience, we may also require the indifference curves to be continuous and smooth.

It is not surprising that with so little structure imposed on the preference ordering, risk aversion in different problem contexts might not be equivalent. It is therefore necessary to specify the sense of risk aversion being referred to. We defined, among others, risk aversion in terms of conditional certainty equivalent, unconditional certainty equivalent, mean preserving spread, and risky-asset demand. Without specifying any preference theory, we proved that risk aversion in the sense of conditional certainty equivalent and risk aversion in the sense of mean preserving spread are equivalent as long as the underlying preferences are complete, transitive, continuous in distribution, and consistent with first-degree stochastic dominance. This was first showed for finite lotteries involving rational probabilities, then extended to general monetary lotteries. This also holds in the comparative context, i.e., agent A is more risk averse than agent B in the sense of conditional certainty equivalent if and only if A is more risk averse than B in the sense of simple compensated spread. We also showed that, regardless of the utility theory, A being more risk averse than B in terms of simple compensated spread implies that A will demand less of the risky asset in a world with one safe asset and one risky asset.

In expected utility, properties of a preference ordering are largely captured in the agent's von Neumann-Morgenstern utility function $u(x)$. If we can identify its non-expected utility counterpart, the analytical tractability of a general utility functional will be greatly enhanced. For this purpose, we imposed linear Gâteaux differentials on utility functionals and called such a functional a linear Gâteaux utility. Its Gâteaux derivative $\zeta(x;F)$ is termed a lottery-specific utility function (abbrev. LOSUF) which in many ways plays the role of the von Neumann-Morgenstern utility. For instance, consistency with the first-degree stochastic dominance requires an increasing ζ; global risk aversion in terms of conditional certainty equivalents and mean preserving spreads is characterized by a concave ζ. Unlike expected utility, however, the concavity of ζ is not equivalent to pointwise local risk aversion. This gap is welcome because it can be used to explain why people purchase insurance and gamble at the same time.

If linear Gâteaux utility can resolve major controversies under expected utility, why should we be interested in weighted utility which is a special case of linear Gâteaux utility? At least three arguments can be made in response. First of all, linear Gâteaux utility is not axiomatic. It is unclear what preference properties are embedded in linear Gâteaux utility. In contrast, weighted utility is a consequence of specific assumptions about preferences, namely completeness, transitivity, continuity, monotonicity, and weak substitution. As long as a decision maker's preferences conform to these axioms, the analysis via weighted utility will be valid. Note that the only axiom that departs from expected utility is the weak substitution. This appears to render weighted utility a natural replacement for expected utility when a nonlinear preference functional is called for.

Secondly, to generate all the indifference curves in a simplex of 3-outcome lotteries (cf. Figure 1.1), the amount of information needed under linear Gâteaux utility might prove insurmountable as any smooth, nonintersecting indifference curves are permissible. In comparison, weighted utility is far more efficient. It only requires the knowledge of one arbitrary indifference curve and the point at which all indifference curves intersect. Of course, this also means that there will exist paradoxes that can be explained by linear Gâteaux utility but not by weighted utility. Nevertheless, when the problem context does not require generality at the level of linear Gâteaux utility, the much greater efficiency of weighted utility may appear attractive.

Most importantly, the specific functional form of weighted utility allows us to solve explicitly optimizing problems such as portfolio selection, intertemporal consumption decision, etc. For instance, some implications in this essay are obtainable under weighted utility but not under linear Gâteaux utility. One is the observation that, no matter how risk averse he may be, a weighted utility agent, like his expected utility counterpart, will invest a stictly positive amount in the risky asset as long as the expected rate of return on the risky asset is strictly greater than the safe rate of return. Also, we are assured that a risk averse (in the sense of mean preserving spread) weighted utility agent's not-worse-than sets are always convex so that his weighted utility is quasiconcave in his risky-asset demands. Under linear Gâteaux utility, this need be assumed.

Another result unique to weighted utility is the necessary and sufficient condition for the risky asset to be a normal good (cf. Theorem WU8). With weighted utility, this condition is obtained by first optimizing the agent's weighted utility to yield the first and second order conditions, then performing conventional comparative statics. This approach is not applicable to linear Gâteaux utility without assuming a specific functional form. Even though some comparative statics can be carried out under linear Gâteaux utility via path differentiation, explicit solutions are not obtainable without imposing more structures.

The above discussions point out a natural direction for further research. It should be interesting to see how market behavioral implications obtained under expected utility in some specific financial economic problems such as intertemporal consumption choice, information value and competitive bidding strategy will be altered under weighted utility. It is possible that the results obtained under the expected utility hypothesis are sensitive to agents' preferences. For example, it is well known that under expected utility the Dutch auction and the first-price sealed-bid auction are isomorphic, so are the English auction and the second-price sealed-bid auction. Nonetheless, Weber (1982) was able to demonstrate that they might not be perceived as isomorphic by weighted utility maximizers with decreasing, concave weight functions. Will the second-price sealed-bid auction remain isomorphic to the English auction under weighted utility? Will the demand-revealing property of the second-price sealed-bid auction continue to hold under weighted utility?

On the other hand, some results obtained under expected utility might prove robust to preference hypotheses. The equivalence between risk aversion in the sense of mean preserving spread and risk aversion in the sense of conditional certainty equivalent, proved in this essay, is such an example. Most likely, introducing weighted utility will call for some extent of modification in the results obtained under expected utility. The necessary and sufficient condition for the normality of risky-asset demand is such an example. This condition reveals that, when an agent's utility function depends on the underlying distribution, the attributes of the risk he is facing might affect his market choice behavior. We learn from it that the equivalence between decreasing concavity of the utility function and the normality of risky-asset demand under expected utility has much to do with the linearity of the preference functional -- an analogy of state-independence.

No matter in which of the above categories the findings turn out to be, they should help us understand the nature of the related market behavioral implications.

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ESSAY II

COMPETITIVE BIDDING AND INTEREST RATE FORMATION IN AN INFORMAL FINANCIAL MARKET

0

INTRODUCTION

One problem with the economy of a developing country is its capital scarcity, which makes bank loans relatively inaccessible to most small busi-nessmen and ordinary consumers. As a result, a broad class of spontaneously arising arrangements for the mutual provision of credit and savings is wide-spread in the developing world. The forms of the bulk of these informal financial institutions are of the rotating credit type, which, in the anthropological literature, are most frequently referred to as the 'rotating credit association'. In a rotating credit association, members make regular deposits into a pool which is available to satisfy the borrowing needs of individual members in a rotating manner. The actual organization, including recruitment policy, size of deposit, and the method of determining the order by which members receive funds, exhibits a remarkable degree of variations in adaptation to local socio-economic and cultural conditions. Examples range from an association in Keta, Ghana (Little, 1957), in which the order of rotation was determined by seniority and the size of deposit was not fixed, to those popular among the Chinese and Japanese where the organizer gets the first loan interest-free and the subsequent loans are auctioned off to the highest bidder among the participants who have not yet received loans (Geertz, 1962).

While rotating credit associations with an explicit finance focus are prevalent among the developing countries (given their relatively under-developed capital markets), there are other commonly observed non-market, expectation-based exchange activities that carry a significant rotating-credit component. Neighborliness, gift exchange, poker clubs, etc. are obvious examples.

Table 0.1 summarizes the characteristics of several variations of the rotating credit association as well as neighborliness and gift exchange. The various types of the rotating credit association differ mainly in their methods of determining the order of rotation among members. In the Chinese and Japanese versions, members compete for funds by submitting sealed-bids. The African version is not very interesting since its rotational order is determined by seniority or other 'sociological' criteria rather than some form of interest rates (Little, 1957). The rotating credit association found in the Middle East is mainly for the purpose of purchasing durable goods such as automobiles (among Israelis) and refrigerators (among Lebanese). The last example of the rotating credit association refers to the bilateral private arrangement popular among some Indian laborers.[1] Under this arrangement, a fixed amount is alternated between two individuals at fixed intervals, often on pay days.

In Asian agriculture-based communities, neighbors gather efforts to accomplish their seasonal harvest in rotation. In America, during the pioneering days, similar arrangements were common for building houses, fighting fires, etc.

The rotational nature inherent in gift exchange is particularly clear in the context of wedding gifts. Most people begin by saving (giving wed-

[1] These examples were provided by the author's colleagues and friends.

Table 0.1: Examples of Rotational Exchange

Examples / Characteristics	Rotating Credit Associations: Chinese	Japanese	African	Middle East	Indian Laborer	Neighborliness	Gift Exchange
Organization							
nature	an informal financial market				a bilateral private arrangement	tradition	tradition
explicit organizer	yes				no	no	
purpose	satisfying borrowing or lending needs					mutual aids	friendship, etc.
Membership							
qualification	mutual selection					self-selection	mutual selection
relationship	multiple bilateral				bilateral	multilateral	
Information (credit-worthiness based on)	kinship, job immobility, etc.					locational immobility	status immobility
Deposit							
form	regular contribution ($ or kind)					aid to neighbors	gifts given out
size	fixed or variable		fixed			variable	
frequency	fixed					variable	
Withdrawal							
form	loan (partially refund, $ or kind)					aid received	gifts received
size	fixed or variable		fixed			variable	
frequency	fixed and once per cycle						
Assignment							
mechanism	by competitive bidding		by seniority	by lot	alternating	by needs	by tradition e.g. weddings
precision of rotation	precise					imprecise	

ding gifts) towards their wedding days when they 'withdraw' their 'savings' and take on a loan of gifts that are paid back over the subsequent marriages of other eligible members of the 'wedding club' consisting mostly of friends and relatives. Interestingly, membership in a wedding club is at least partially exogenous. Of similar nature is a poker club which serves certain social functions without an explicit organizer.

Some arguments can be made to see insurance as a special form of the rotating credit association where the insurance company acts as the organizer by selling policies and processing claims. The purpose of the arrangement is of course risk sharing for the members and profit making for the organizer. From the insurance company an individual is associated with, we may infer some private information. For instance, an automobile owner insured by Preferred Risk Mutual Insurance Inc. in the United States must be a non-drinker and non-smoker (assuming that people tell the truth when applying for insurance).

On the deposit and withdrawal side, insurance policy holders make fixed deposits by paying insurance premium periodically. The withdrawal assignment mechanism is the key difference from the recognized financial rotating credit association. First of all, the withdrawal is prompted by the occurrence of presumably exogenous events which is random in cases without moral hazards. Consequently, the rotation is imprecise in the sense that a member may never get withdrawals. Secondly, even if insured hazards do occur and withdrawals are granted, the size of withdrawals will in general depend on the actual losses which again are randomly determined.

Even though the anthropological literature concerning the rotating credit association is about a century's old (Geertz, 1962), the associated repeated intertemporal competitive bidding process has never been the subject of a rigorous microeconomic study. The focus of this essay is on the intertemporal bidding process in the Chinese version of the rotating credit association. Because of the interdependency between bids across periods, the intertemporal bidding process for the rotating credit association is distinct from having repeated auctions independently across time. Nevertheless, results in the competitive (single-period) bidding literature will be of help in the development of our results.

Since the poineering work of Vickrey (1961), there have been numerous studies of the resource allocation role of various forms of auction markets (Oren and Williams, 1975; Oren and Rothkoph, 1975; Green and Laffont, 1977; Milgrom, 1979; Wilson, 1979; Coppinger, Smith and Titus, 1980; Forsythe and Isaac, 1980; Myerson, 1981; Harris and Raviv, 1981; Riley and Samuelson, 1981; Cox, Roberson and Smith, 1982; Cox, Smith and Walker, 1982; Milgrom and Weber, 1982). (The reader is also referred to Stark and Rothkopf (1979) and Engelbrecht-Wiggans (1980) for extensive surveys. Cassady (1967) is a good source of anecdotal historical examples.) Many studies were concerned with four types of auction market forms - the English auction, the Dutch auction, the first-price sealed bid auction and the second-price sealed bid auction.

In the English auction, prices move upwards in progressively smaller intervals. The purchaser pays the price that nobody is willing to bid over. In contrast, prices in the Dutch auction move downward. The bidder who stops the downward price movement purchases the object at that price.

Vickrey argued that the English auction is isomorphic to the second-price sealed bid auction where the highest bidder pays the price of the highest rejected bid. This is supported in the experimental work of Cox, Roberson and Smith (1982). Vickrey's other conjecture - the Dutch auction is isomorphic to the first-price sealed bid auction (where the highest bidder pays the price of his own bid) - however, is falsified in the same experimental study.

The organization of the rest of this essay is outlined below. A more detailed description of the structure of the Chinese version rotating credit association, called 'Hui',[2] is given in Section 1 where we introduce useful terms and notations, and describe eight actual cases of Hui.

Section 2 contains some preliminary analyses of several small, hypothetical Hui (with 2 or 3 members only). The main objective is to investigate in a preliminary way the rationale for the existence of an informal financial institution amid the more sophisticated, western-derived banking system and at the same time familiarize the reader with the workings of Hui.

Section 3 presents the model's assumptions on an individual's incomes, preferences and expectations. We also state a definition of an agent's intertemporal reservation discount vector, which is compatible with, but does not depend on, agents' having access to some interest rate

[2] The label 'Hui' is used in both the singular and the plural form.

in a formal financial market, and prove its existence and uniqueness. This allows us to derive, in Section 4, the individual optimal bidding strategy under the observed first-price auction (and the hypothetical second-price auction) with the additional restriction of concavity and time-additivity on his von Neumann-Morgenstern utility function, and a decreasing marginal outbidden rate (increasing marginal outbidding rate) on his subjective probability distribution of winning at each period. We also discuss some comparative statics and efficiency implications of the individual optimal bidding strategy.

In order to obtain a tractable form for a Nash equilibrium bidding strategy, we, in Section 5, impose further restrictions, including risk neutrality. This yields, for each agent, his ex post Hui borrowing and lending interest rates. These rates depend on the history of the realized winning bid, including the one for the period in which he wins the auction. Weighted by the Nash-equilibrium-induced probability of winning in each period, corresponding ex ante (nondeterministic) interest rates result. Section 6 describes an application of the model built in Section 5 to a tacit collusion among a small group of suppliers selling an indivisible commodity to a single buyer (e.g. the federal government). Section 7 concludes this essay by suggesting some potential directions for further research.

1

THE GENERAL STRUCTURE AND ACTUAL CASES OF HUI

1.1 The General Structure of Hui

A Hui consists of an organizer and N voluntary members of his choice whom he brings together to form an informal market to satisfy their borrowing and lending needs. For operating the market[3] and bearing the default risk of each member, the organizer receives an interest-free loan of NA, which is repaid in N equal installments of A at each of the N subsequent periods. The organizer, on the other hand, poses a common risk shared by the N members collectively. Consequently, an otherwise multilateral exchange relation is replaced effectively by bilateral ones.

Let 0 denote the organizer and n (= 1, 2,..., N) denote the participant who succeeds in bidding for the pool at the nth period. Let b_{in} be the bid submitted by participant i at period n, and $b_n = \max_i\{b_{in}\}$ be the highest bid submitted at period n. We denote by A the 'size' of the per-period, before-discount (or before-premium) deposit into the pool. The

member's actual payment at each period is related to A. In a 'discount-bid' Hui, each member who has already received funds pays A at every

[3] The services provided by the organizer include competitive recruitment and selection of members, the execution of auctions, and the collection and delivery of deposits.

subsequent period. At period n, those who have not yet obtained loans pay $A-b_n$ apiece. In a 'premium-bid' Hui, each member pays A at every period before he succeeds in bidding for the pool. Once he wins, say at period n by bidding b_n, he has to pay $A+b_n$ at every subsequent period. Although both discount-bid and premium-bid Hui are observed, the former is more popular among the Chinese while the latter seems more to the Japanese's liking. The cash flow patterns for the organizer and the N members in both a discount-bid Hui and a premium-bid Hui are summarized in Table 1.1.

From Table 1.1, it is clear that the number of participants, N, in a Hui is also the number of periods this Hui is to last. Note that, among the N members in a discount-bid Hui, member 1, who wins the pool $NA-(N-1)b_1$ in the first bidding, is a pure borrower, whereas member N, who receives NA at period N, is a pure lender. The other members lie somewhere in between. In general, a member remains a lender until he receives loans, at which point his status changes to a borrower. At each period, only lenders are eligible to bid. Since the fund available at each period must be granted to one member,[4] the number of bidders decreases by one every period, leaving the last member to collect NA at the end without bidding. Obviously, an attractive Hui consists of a 'good' mix of borrowers and lenders. A Hui formed by a homogeneous group of borrowers will

[4] When no bids are submitted (or, equivalently, all bids are zero), the winner is determined by lot. In the case of tie-bids, either the fund is shared equally (consequently, future repayments are also shared equally), or a second-stage bidding is conducted to select the winner.

Table 1.1: Cash Flow Patterns of Hui Participants

a. Discount-bid Hui

participant \ period	0	1	...	n	...	N-1	N
0	NA	-A	...	-A	...	-A	-A
1	-A	$A+(N-1)(A-b_1)$	...	-A	...	-A	-A
.	.	.		.		.	.
n	-A	$-(A-b_1)$	...	$nA+(N-n)(A-b_n)$	...	-A	-A
.	.	.		.		.	.
N-1	-A	$-(A-b_1)$	...	$-(A-b_n)$	...	$(N-1)A+(A-b_{N-1})$	-A
N	-A	$-(A-b_1)$	...	$-(A-b_n)$	...	$-(A-b_{N-1})$	NA

b. Premium-bid Hui

participant \ period	0	1	...	n	...	N-1	N
0	NA	-A	...	-A	...	-A	-A
1	-A	NA	...	$-(A+b_1)$	...	$-(A+b_1)$	$-(A+b_1)$
.	.	.		.		.	.
n	-A	-A	...	$NA+\Sigma_{i=1}^{n-1}b_i$	...	$-(A+b_n)$	$-(A+b_n)$
.	.	.		.		.	.
N-1	-A	-A	...	-A	...	$NA+\Sigma_{i=1}^{N-2}b_i$	$-(A+b_{N-1})$
N	-A	-A	...	-A	...	-A	$NA+\Sigma_{i=1}^{N-1}b_i$

provide little room for trades. Similarly, a group of lenders will find a Hui to be quite a boredom since no one will be bidding actively. This suggests a dual problem to the one treated here -- the organizer's problem. An effective (i.e. competitive) organizer presumably maximizes as his objective function the 'surplus' generated from members' participation. His choice variable consists of the mix of membership in terms of their degrees of borrowing or lending needs. We shall not study the organizer's problem in this essay except to point out from time to time his salient features such as the role of default risk.

1.2 Actual Cases of Hui

In November 1983, the financial sector in Taiwan was startled by the largest-ever-scale Hui default in her history. This occurred in a small town of 100,000 people named Chia-Li. Allegedly, over one thousand people were involved for a total amount of four billion New Taiwan Dollars (NT$) (approximately US$100 million based on the current official fixed exchange rate US$1 = NT$40).

This incidence has led several legislators to urge for governmental regulation on Hui operation in Taiwan and has prompted at least one survey on Hui statistics. Qualifying his figures as conservative due to subjects' reluctance to reveal their actual involvement, Wen Li Chung estimated that the total Hui membership approximates 85% of the island's population; the credit provided by Hui is roughly US$237.5 million per month, or US$2.85 billion annually, which is about 21.92% of the island's national income (Chao-Ming, 1983).

In order to familarize the reader with the workings of Hui, we collect in Table 1.2 eight actual examples found in Taiwan. Hui 1 was formed among the employees of a CPA firm; its organizer was the personnel administor who co-signed all other employees' pay checks. Hui 2 - 8 were formed among the employees of the state-owned Taiwan Power Company (known as Tai Power) which experiences very low turn-over. Their organizers were uniformly senior, tenured employees. Each Hui is characterized by its starting and ending time, the predetermined fixed payment A, the size of its membership N (excluding the organizer), its type (discount-bid or premium-bid), and the actual winning bids.

Organizing

According to the current practice in Taiwan, a prospective organizer will draw up and then circulate among potential participants a Hui formation proposal with proposed size of payment (A) and membership (N), date and frequency of meetings (e.g. every two weeks or every month) and other features such as the minimum amount of bids, rounding-off policy (e.g., \$901 and \$904 bids will be considered as \$900 and \$905 respectively), etc.

All interested parties are invited to sign up and suggest alterations of terms. Based on the response and suggestions, initially proposed terms may be revised. When terms and memberships are finalized, a form containing the agreed-upon terms and the names of members will be distributed to all members. Usually the form is designed with space to fill in the winning bid and the amount of the resulting pool at each period. The data in Table 1.2 are taken from such forms. Because the individuals who provided these forms stopped recording the winning bids after they obtained funds,

Table 1.2: Actual Cases of Hui

Hui	1	2	3	4	5	6	7	8
start	Oct. 75	July 77	July 77	Mar. 78	Oct. 79	July 80	Jan. 81	July 81
end	June 77	Mar. 79	Jan. 79	Feb. 80	Apr. 82	June 84	May 83	Mar. 83
A	1,000*	1,000	2,000	2,000	2,000	5,000	5,000	5,000
N	20	20	18	23	30	47	28	25
type	disc.	disc.	disc.	prem.	prem.	disc.	disc.	prem.
min b	300						600	900
time‡	winning bids							
0								
1	450	120	200	400	460	1,530	900	
2	470	140	210	350	500	1,680	960	
3	401	150	200	360	600	1,680	920	
4	500	155	150	360	680	1,720	920	
5	300**	150	160	360	600	1,610	950	
6		135	170	320	550	1,680	960	
7		125	170	310	530	1,590	980	
8		110	150	340	500	1,750	1,000	
9		110	160	330	560	1,610		
10		100	150	310	500	1,620		
11		115	160	280	500	1,640		
12		125	150	280	490	1,520		
13		140	160	230	510	1,660		
14		130	170	250	600	1,670		
15		100	160	250	500			
16		50	120	220	430			
17		110		210	450			
18				210	400			
19					350			
20					400			
21					400			
22					420			
23					450			

‡ All Hui are on monthly basis.

* All amounts are in New Taiwan Dollars (NT$); US$1 ≈ NT$40.

** The winner was determined by lot due to absence of bids.

the data are incomplete.

Although this form is informal (e.g., it is not notarized, not signed by the organizer and the members, etc.) and is provided mainly for the recording convenience of the participants, it is, according to a recent court ruling, acceptable as evidence for the existence of financial claims and liabilities among Hui participants (Chao-Ming, 1983).

Organizer

In return for his services including recruiting members, conducting auctions, collecting payments from each member, delivering the pooled proceeds to the winner, and most importantly assuming the default risk posed by all members, the organizer obtains an interest-free loan at the start of the Hui. Should any member default, the organizer must take over the defaulting member's share and the Hui will continue without interruption.

To be an acceptable organizer, one must be believed to be trustworthy and financially capable of assuming the defaulting shares. 'Informal creditworthiness' (in the sense that his gain from defaulting from his obligations once will be more than outweighed by the loss from all future Hui and other informal transactions) is a necessary attribute of any Hui organizer at the time a Hui is formed. We have yet to know a case in which an individual is able to organize a Hui (or to participate as a member) in a circle where he is known to have defaulted before either as a Hui organizer or a Hui member. Such enforcement by the discipline of continuous dealings depends of course critically on some generalized definition of immobility of which formal collateralization is an example.

As such, organizers need not be wealthy. The organizers in our 8 cases were either in a position to deter defaults, or, if default did occur, had the ability to assume the loss. For example, the organizer of Hui 1 had access to all members' paychecks. As to Hui 2 - 8, the likelihood of the organizers' running away is rather slim as a job with Tai Power is considered more valuable than the pool in most cases.

Besides the organizer's character and creditworthiness, the risk of a Hui depends to a large extent on its orientation and the mix of its membership. For instance, because all Hui in Table 1.2 were formed mainly for the saving purpose and involved no businessmen, they were virtually of no default risk. By now, all the 8 Hui have ended without dispute.

According to our observation, the risk level of a Hui increases if the organizer and/or any members are small businessmen. This is because in many instances the businessman members, with higher opportunity costs of capital, would bid higher and draw funds from the pool first while their ability of paying up their shares depends on the subsequent success of their business or their ability to borrow from other Hui.

It is worth noting that the recent boom of Hui in Taiwan has introduced the so-called 'professional organizers' who have profited from their entrepreneurship in arbitraging across Hui or channelling funds to lucrative ventures. This new breed of professionals usually have good and broad connections with friends, relatives, colleagues, ex-colleagues, neighbors, etc., and consequently have special access to valuable information about either profitable investment opportunities or people's creditworthiness. Being a professional organizer however does not necessitate full-time involvement. It is an occupation often taken up by housewives.

To minimize his risk, a prudent organizer must be selective and is often reluctant to accept people whom he is not familar with as members.

In immigrant Chinese communities in North America, a new immigrant seeking membership is often required to either produce a guarantor acceptable to the organizer, or be a saver in the initial periods to establish his ability to make periodic payments (Light, 1972). In Africa and Middle East where lottery is the allocational mechanism for rotating credit associations, it is a common practice that new faces are not allowed to draw lots and have to be the last ones to withdraw funds until they are better known and have established their creditability (Ardener, 1964).

Members

Hui has reportedly existed among the Chinese for at least 800 years (Geertz, 1962).[5] In a sense, the presence of this informal financial institution reflects a more formal aspect of the existing and generally immobile relationships among the members. Today, most saving-oriented Hui (which are viewed as more conservative but safer) are still formed among individuals who know each other fairly well either directly or indirectly. Naturally, some people are willing to take risk for higher returns by joining Hui that have greater involvement with businessmen. With some 'superior' information and by careful search and other measures,[6] it is not uncommon for one to realize a 30% - 40% annualized rate of return without much risk.

[5] In earlier Hui, especially those found in agriculture-based communities, the exchange commodity was often in kind, being rice in many cases. Today, money is the only known currency traded in Hui prevalent in Taiwan.

[6] For instance, pick a Hui whose organizer is your next door neighbor who runs a TV shop and has enough TV sets in stock which you can lay your hand on in time if situation calls for such auctions.

Frequency of Meeting (Bidding)

The frequency of Hui meeting and bidding depends to a large extent on the size of the payment (A), the size of the membership (N) and the financial background of the members. Most saving-oriented, middle-class-based Hui such as the eight cases we presented are on monthly basis. Bi-weekly meeting is also quite common.

It is noticed that speculative Hui tend to have shorter intervals. Many Hui involved in the Chai-Li scandal were allegedly on a daily or bi-daily basis.

Size of Membership

The size of membership in Table 1.2 ranges from 18 to 47. A membership of 47 is uncommon for a saving-oriented, monthly Hui, especially given the size of payment NT$5,000.[7] In general, the size of membership, the size of payment and the frequency of meeting are interdependent.

It is believed that the longer a Hui lasts, the greater is its risk of default. Most people tend to prefer a duration of one and half years to two years so that the pool NA is not too small and yet the default risk is not unaffordable.

It is quite common that more than one individual share one membership or one individual assumes more than one share. The latter case is of particular interest as it is tantamount to permitting coalition among bidders. Such practices allow Hui to display a somewhat greater degree of

[7] A college graduate's starting monthly salary was about NT$12,000 in 1980.

flexibility in accommodating more desirable size of saving and borrowing. This seems to parallel the insurance market where an individual chooses among a fixed menu of policies rather than specify the size of his own needs together with the price he is willing to pay (Rothschild & Stiglitz, 1976).

A member is also allowed to sell his share to other members or some outsiders before the Hui ends as long as he is able to obtain the approval from the organizer. This can also be done if the original member guarantees the creditworthiness of the member(s) he introduces.

Discount-Bid vs. Premium-Bid

The bulk of Hui in Taiwan are of the discount-bid type. Most Hui found among Chinese in North America, on the contrary, are premium-bid ones. Although these two types are similar in substance, there are institutional differences:

a. The payment made by a member who has not withdrawn funds is A in a premium-bid Hui and A minus the current winning bid in a discount-bid Hui. This provides an incentive for a member to take a more active part in bidding. As a result, a discount-bid is likely to encourage greater participation. This is consistent with the general impression that discount-bid Hui are more 'exciting'.

b. The interest-free loan an organizer obtains is the same (NA) in both types of Hui. In the event a member defaults, the per-period amount the organizer would be responsible for however is different, being A in a discount-bid Hui and A plus the defaulting member's winning bid in a premium-bid Hui. Moreover, bids in a discount-bid Hui are bounded from below by structure (it cannot go beyond A), but may in principle be very large in a premium-bid Hui. The organizer therefore has reasons to prefer discount-bid Hui over premium-bid ones. This can at least partly explain the fact that most 'speculative' Hui are of the discount-bid type.

c. The pool available at period n is $NA+\Sigma_{i=1}^{n-1}b_i$ in a premium-bid Hui and $NA-(N-n)b_n$ in a discount-bid Hui. The former appeals to people who wish to obtain at least NA when they win. The latter is preferred by those who wish to pay less. This argument is at least to some extent superficial since, before joining a Hui, a participant can select a Hui of the characteristics that suit his preference.

d. In a society with wide-spread illiteracy, discount-bid arrangements have the additional advantage of requiring less record keeping as each participant's future payments are independent of past winning bids.

<u>**Bidding**</u>

On the bidding day (often during lunch time of the pay day), members wishing to bid would submit their bids to the organizer. The loan allocational mechanism for Hui has tended to be via first-price sealed-bid auctions.

Until the recent boom of Hui in Taiwan, the bidding had been relatively 'calm'. For example, many saving-minded members simply did not bother to bid. A member who could not show up for the bidding often authorized another member to bid on his behalf or informed the organizer of his bid in advance. After the bidding, only the winning bid was revealed.

Recently, Hui bidding has become more competitive. The following phenomena attest to it:

a. The information about needs for loans is guarded as top personal secret to prevent strategic competition.

b. Eligible bidders not in need of funds would strategically give false signals (usually by talking casually about the amount he intends to bid) to induce higher bids from rival bidders.

c. Members would withhold their bids until the bidding meeting to ensure that their friends or the organizer cannot leak the information.

d. Increasingly, organizers announce all submitted bids without revealing the identity of the bidders (except the winner). This practice may have implications for the role of information and learning in the Hui setting.

It should however be noted that, in less-commercialized Hui such as those reported in Table 1.2, some non-market economic factors, e.g. friendship and social norms, still play a role. For instance, if a member needs funds to hospitalize his ailing parent, it is very likely that all eligible bidders will agree upon a low, nominal bid and effectively grant a subsidized loan to him (provided of course that he is not too unpopular). This corresponds to the insurance function of a Hui and other informal market mechanisms.

Bids

What factors affect the level of bids? Intuitively, we expect bids to increase in A and N as suggested by the data in Table 1.2. Also, premium-bids are expected to be higher than discount-bids since the size of loan in the former is larger.

The economic determinants of a Hui member's bid include at least his investment opportunity cost of capital, his stochastic or nonstochastic non-investment needs for funds, and a strategic component inherent in most game behavior.

First, consider the investment cost of capital in the formal financial markets. In this essay, a financial institution is considered 'formal' if it relies on externally recognized and collateralized evidence to enforce non-default from without (e.g. the legal system). In contrast, Hui is 'informal' in making use of creditworthiness information generated within the (informal) institution to enforce non-default (i.e. enforcement from within). The most familiar formal financial market is the conventional banking system. Almost everybody can save with banks. Therefore the relevant opportunity cost of capital for Hui participants who do not have other investment opportunities would be the bank interest rate for saving which is the same for most individuals.

For those people who have other investment opportunities, the cost of capital in the formal financial market is the bank lending interest rate applicable to him. The reality is however more complicated due to the imperfection of the formal financial market. First of all, capital rationing does exist. Banks, which cannot demand more than the regulated interest rate, prefer to deal with large corporations due to risk consideration and economy of scale. Loans for small businesses and consumption are available, but the process could be forbiddingly costly and the requirements difficult to fulfill. For example, a standard requirement for small business loan is two or more noncorporate guarantors. The interest rate for an unobtainable loan is effectively infinity. It is then not surprising that many small businessmen offer a rate as high as 50% annually for loans from Hui.[8]

For people who participate in more than one Hui (which is a common practice), the interest rate of other Hui might be the relevant opportunity cost of capital. This is especially true for the professional Hui arbitragers.

It should however be noted that for an informal financial mechanism such as Hui to sustain over a long period of time, the pooled capital must be eventually channelled to economically productive activities which yield real returns. In other words, the 'rate of return'[9] on Hui must be supported by growth in real economic activities outside of the Hui system. If the funds keep circulating within a Hui system giving high returns and never flow to the real economic production sector, then this Hui system would eventually lead to a pyramid. This is evident in light of the 1983 Chia-Li fiasco, in which the most striking and devastating feature of the Hui involved is a widespread practice dubbed 'feed-Hui-with-Hui', i.e., a participant draws funds from a Hui to make payment in another Hui. Feed-Hui-with-Hui operation was also blamed for the failure of Chit Fund Corporations in Singapore during 1972 and 1973 (Chua, 1981).

[8] Ironically, the popularity of Hui has rendered an individual's ability to finance through Hui a signal of his creditability. How can a person who cannot be accepted into a Hui expect someone else to guarantee his loan? But if one can borrow from Hui, why should he need to borrow from the bank?

[9] The 'rate of return' from Hui suggests a possible general equilibrium model which is beyond the scope of this essay.

As to the stochastic or non-stochastic non-investment needs for funds, the single most important use of loans from Hui has been house purchase and renovation.[10] Purchases of durable goods, children's education expenses, wedding expenses and foreign travel are other uses of funds from Hui.

It seems reasonable to say that Hui participants have a 'reservation price' for loans available at each period largely determined by his opportunity cost of capital which depends on (a) his investment opportunity elsewhere, (b) his cost of capital in the formal financial market or other financial sources, and (c) his personal needs for funds. This is consistent with the way Hui participants calculate the upper bound of their bids. Usually, an individual whose only alternative is saving with banks will use the bank saving interest rate to calculate the highest discount he can afford to give up. If he has other use of funds, a premium will be added to the basic saving rate and the maximum affordable bid is calculated accordingly. Later in Section 3, we will formally define this reservation price for loans as the 'reservation discount'.

Would a Hui member bid his reservation discount? Not in general. How his bid deviates from his reservation discount will be considered in Section 4 where we study the optimal bidding strategy for Hui members.

Due to the saving-orientation, the bids in Table 1.2 are lower than those in average Hui. For instance, the internal rate of return of the

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[10] The required minimum downpayment is often more than 60% of the price. Usually, it will take more than one Hui to obtain enough funds for this purpose.

first winner of Hui 6 is approximately 1.33% per month, whereas it is not uncommon to have 30% - 50% annualized ex post Hui borrowing interest rate (to be defined in Section 5, Definition 5.1) in many Hui where small businessmen are involved. The latter of course have higher default risk. Whether the perceived default risk leads to higher bids or the other way around has yet to be investigated.

Note that not any two Hui on Table 1.2 are strictly comparable. For example, Hui 1 and 2 had the same A and N and were both of the discount-bid type, but took place two years apart. As discussed above, bids are to a large extent determined by interest rates in the formal financial market, which vary over time. Even if they had existed contemporarily, market segmentation might still prevent bids from attaining parity.

Even though Hui 3 was smaller than Hui 2 by two members, we would expect the winning bids in Hui 3 to approximately double those of Hui 2. The fact that the bids in Hui 3 were much lower than expected could be because of the existence of the organizer. The organizer is expensive to keep and the fewer members a Hui has, the more costly the organizer is to each member, everything else being constant. We will show in Section 2 that a Hui with only two members can not afford to have an organizer.

If all members' expectations and opportunity interest rates remain constant over time, one would expect their bids on the average to be increasing over time. This is mainly due to the decreasing number of eligible bidders who (with payment $A-b_n$) will enjoy the discount. Although none of the winning bid streams in Table 1.2 are monotone over time, it could be the result of bidders' changing expectations or opportunity rates.

Learning

As long as a member knows or is able to estimate his opportunity cost of capital, deriving the reservation discount is straightforward. The spread between his reservation discount and his bid is a more strategic issue. The discussion under Bidding tells us that Hui participants do behave strategically, including actively seeking information on rivals' reservation discounts. Obviously, other bidders' opportunity cost of capital, risk attitudes, bidding strategies, etc. are valuable information.

A question of interest here is: Do participants abstract useful information from the distribution of past winning bids? In repeated single-period auctions where the same group of bidders compete for, say government defense projects, it is evident that bidders learn about their rivals' reservation prices or bidding strategies from past biddings (Green and Laffont, 1977; Milgrom, 1979; Myerson, 1981; Milgrom and Weber, 1982). Therefore, a realistic bidding model must allow for bidders' Bayesian learning behavior.

In the context of Hui, we feel, based on the following reasons, that the learning issue is not as critical as in repeated single-period auctions. First of all, once a Hui member wins a bidding, he drops out the competition for the rest of the Hui duration. Secondly, when a Hui bidder loses in bidding, he still gains in his dual role as a 'seller' of the loan. This suggests a weaker incentive for costly information search.

To keep our analysis simple in this essay, we will assume that Hui bidders' current bids do not depend on past winning bids or bids in general.

Default

Roughly speaking, there are two types of Hui default, one incurred by the organizer, the other by the member. Most organizer-caused defaults are well-planned. The plot usually goes as follows. The organizer will recruit as many members as possible and add to the list of participants a few nonexistent names. At the start of Hui, he collects NA as the organizer. At the next few periods, he submits high bids in the name of those nonexistent members and obtains loans. After his list of nonexistent names is exhausted, he simply disappears. A Hui member can avoid this type of default by insisting on knowing all other members.

The other type of default results from one or more members' not being able to make their shares of contributions after they have drawn funds from the pool, possibly due to unfavorable outcomes of their investments elsewhere. Although the organizer is in principle the only one who assumes all default risks posed by members, it is a fact that a participant does not completely ignore this type of default risk when he makes the decision to join a Hui.

While we recognize the presence of the default risk in the Hui setting, we will not in this essay attempt a model formally incorporating it. Instead, it will be assumed that default risk is negligible.

2

THE ECONOMICS OF HUI WITH TWO OR THREE MEMBERS

The purpose of this section is to provide some basic understanding of Hui by performing some preliminary economic analyses on several simplified Hui examples. Specifically, we assume that all Hui considered here are default riskfree and of the discount-bid type, and that all agents have an explicit opportunity interest rate which remain constant throughout the Hui duration. Moreover, the Hui are small in size, with only two or three members, with or without an organizer.

Following the preceding section, we use the following notations:

A : the size of the per-period, before-discount deposit into the pool;

n (= 1, 2, 3): the participant who succeeds in bidding for the pool at the nth period;

0 : the organizer who receives an interest-free loan of NA repaid in N equal installments of A at each of the N subsequent periods;

b_{in}: the bid submitted by participant i at period n;

$b_n = \max\limits_i \{b_{in}\}$: the highest bid submitted at period n;

r_i : member i's opportunity interest rate, $i = 1, 2, 3$.

Two-Member Hui Without An Organizer

Consider first a Hui with 2 members and no organizer. The ex post cash flows for its members are

time	1	2
member		
1	$A-b_1$	$-A$
2	$-(A-b_1)$	A

Suppose member i's only alternative is to save with banks at a rate r_i. How much should he bid at time 1?

If he is to receive funds at time 1, the proceeds will go to his bank account. The (gross) return after one period must not be less than A, the amount he has to pay at time 2. If he is to lend at time 1 and get the money back at time 2, the Hui lending rate must be at least equal to the bank saving rate r_i. Let v_i denote a bid that satisfies the above conditions. Then,

$$(A-v_i)(1+r_i) = A,$$

which implies

$$v_i = A - \frac{A}{1+r_i} = \frac{Ar_i}{1+r_i}. \tag{2.1}$$

The v_i in (2.1) can be interpreted as the 'price' for getting a loan of A at time 1. Expression (2.1) says that the price (in time 1's dollar) of a loan A for one period is the present value of the one-period interest earned on A. Let's call v_i member i's <u>reservation discount</u> for the loan available at time 1. Note that

$$\frac{dv_i}{dr_i} = \frac{A}{(1+r_i)^2} > 0.$$

In fact, regardless of the size of membership, a Hui member's reservation discount for funds available at each period is an increasing function of his opportunity interest rate r_1. We therefore have the following proposition:

Proposition 2.1: A Hui member's reservation discount increases in his opportunity interest rate.

Obviously, if the two members have the same opportunity interest rate and each knows this fact, then there is no point to form a Hui. It will be better off for both members to save with the bank to avoid possibility of default and retain the flexibility of making withdrawals any time they please. This argument can easily be extended to an N-member Hui:

Proposition 2.2: Suppose a group of individuals have an identical opportunity interest rate which is a common knowledge shared by every individual. Then, it is to everyone's advantage not to form a Hui.

What if $r_1 > r_2$ and both r_1 and r_2 are known to each member? In this case, both members know that

$$v_1 = \frac{Ar_1}{1+r_1} > \frac{Ar_2}{1+r_2} = v_2 .$$

It is easy to see that member 1 has incentive to lower his cost of Hui borrowing by bidding under v_1. On the other hand, member 2 have no incentive to bid under v_2 and would bid over v_2 only in an attempt to push up member 1's bid. If member 2 wins with a bid between v_1 and v_2, both members are worse-off. To illustrate this, assume A = \$100, r_1 = 20% and r_2 = 10%. Accordingly, v_1 = \$16.67 and v_2 = \$9.09. We can then calculate the ex post interest rates and profits from Hui for both members depending on the actual winning bid.

Table 2.1: Ex Post Interest Rates and Profits for A 2-Member Hui -- An Example

bids	member	cash flows		ex post interest rate	ex post profit*
$b_1 = 16.67 > b_2$	1	+83.33	-100	20% (borrow)	0
	2	-83.33	+100	20% (lend)	8.34
$b_1 = 12.00 > b_2$	1	+88	-100	13.6% (borrow)	5.6
	2	-88	+100	13.6% (lend)	3.2
$b_1 = 9.09 > b_2$	1	+90.91	-100	10% (borrow)	9.09
	2	-90.91	+100	10% (lend)	0
$b_1 < b_2 = 9.09$	1	-90.91	+100	10% (lend)	-9.09
	2	+90.91	-100	10% (borrow)	0
$b_1 < b_2 = 12$	1	-88	+100	13.6% (lend)	-5.6
	2	+88	-100	13.6% (borrow)	-3.2
$b_1 < b_2 = 16.67$	1	-83.33	+100	20% (lend)	0
	2	+83.33	-100	20% (borrow)	-8.34

* The profit is $(A-b)(1+r_i)-A$ for the borrower and $A-(A-b)(1+r_i)$ for the lender, where $b = \max\{b_1, b_2\}$.

It follows from the analysis in Table 2.1 that:

Proposition 2.3: For a 2-member Hui with $r_1 > r_2$, it is not Pareto-optimal for member 2 to win the only bidding at time 1.

The actual split of the potential profit depends on these two members' relative bargaining positions.

If each member knows only the distribution of his rival's reservation discount, the bidding strategy becomes much more complicated and will be examined in Sections 3 - 5.

Two-Member Hui With An Organizer

How would the introduction of an organizer change the picture? With an organizer, the cash flows for the participants become the following:

participant \ period	0	1	2
0	2A	-A	-A
1	-A	$2A-b_1$	-A
2	-A	$-(A-b_1)$	2A

Given the opportunity interest rate r_i, member i (i = 1, 2) will demand at least a rate of return equal to r_i, which means his reservation discount v_i would be such that

$$A(1+r_i) + \frac{A}{(1+r_i)} = 2A - v_i,$$

which implies $v_i = - Ar_i^2/(1+r_i) < 0$. Hence,

Proposition 2.4: It is not possible for a 2-member Hui to support an organizer (who obtains an interest-free loan) and yet yield a positive rate of return to both members.

Three-Member Hui Without An Organizer

In this case, the cash flows are as follows:

member \ time	1	2	3
1	$2(A-b_1)$	$-A$	$-A$
2	$-(A-b_1)$	$2A-b_2$	$-A$
3	$-(A-b_1)$	$-(A-b_2)$	$2A$

Now, because there will be two biddings, one at time 1, the other at time 2, each member i will have a reservation discount for the funds available at time 1 (i.e. v_{i1}) and another for funds available at time 2 (i.e. v_{i2}). If his opportunity interest rate is r_i, what will be his reservation discounts v_{i1} and v_{i2}? Given his fall back position (i.e. obtaining 2A at time 3), v_{i1} and v_{i2} should satisfy the following conditions:

$$2(A-v_{i1})(1+r_i)^2 = A(1+r_i) + A, \qquad (2.2)$$

$$(A-v_{i1})(1+r_i)^2 + A = (2A-v_{i2})(1+r_i), \qquad (2.3)$$

$$(A-v_{i1})(1+r_i)^2 + (A-v_{i2})(1+r_i) = 2A. \qquad (2.4)$$

Relations (2.2) - (2.4) together guarantee a rate of return r_i and imply

$$v_{i2} = \frac{3}{2} A \left[1 - \frac{1}{1+r_i}\right], \qquad (2.5)$$

$$v_{i1} = \frac{1}{2} A \left[2 - \frac{1}{1+r_i} - \frac{1}{(1+r_i)^2}\right]. \qquad (2.6)$$

Note that v_{i2} does not depend on v_{i1} and can be obtained by solving (2.3) and (2.4). The interpretation of v_{i2} becomes clearer if we take the incremental cash flows of $(-A+v_{i1}, 2A-v_{i2}, -A)$ over $(-A+v_{i1}, -A+v_{i2}, 2A)$, i.e., $(0, 3A-2v_{i2}, -3A)$. In a sense, $2v_{i2}$ can be interpreted as the price (in the form of pre-paid interest) of getting an implicit loan of 3A at time 2, payable at time 3. To verify this, rewrite (2.5) as (2.7):

$$2v_{i2} = 3Ar_i/(1+r_i). \tag{2.7}$$

The RHS of (2.7) gives the present value of the one-period interest on 3A. Similarly, (2.6) can be rewritten as (2.8):

$$3v_{i1} = [3Ar_i+v_{i2}]/(1+r_i), \tag{2.8}$$

which offers a similar interpretation for v_{i1}.

Note that, as stated in Propostion 2.1, $dv_{i1}/dr_i > 0$ and $dv_{i2}/dr_i > 0$. Moreover, $v_{i2} > v_{i1}$ due to the decreasing number of members who would benefit from the discount.

We know from Proposition 2.2 that, if $r_1 = r_2 = r_3$ is a common knowledge to all members, then there is no reason for them to form a Hui. Suppose members have heterogeneous opportunity interest rates, and assume without loss of generality that $r_1 > r_2 > r_3$. Proposition 2.3 can be generalized to Proposition 2.5 below:

Proposition 2.5: Suppose the N members of a Hui have hetergeneous opportunity interest rates r_i, $i = 1, \ldots, N$, and assume without loss of generality that $r_1 > r_2 > \ldots > r_N$. Then, it is to the group's advantage that, at each period, the bidder with the highest opportunity interest rate obtains loan first.

The term 'advantage' in Proposition 2.5 needs clarification. For a 2-member Hui, both 'profit' and 'ex post interest rate' given in Table 2.1 is well defined because a member is either a pure lender or a pure borrower. It is not so for a Hui with 3 or more members due to the 'ill-behaved' cash flows which twice change signs, consequently might have non-unique internal rates of return.

For a Hui interest rate to be meaningful, it is important that borrowing rates be distinguished from lending rates. For each member (except the pure lender who obtains funds at the last period), there are one Hui borrowing rate and one Hui lending rate, both derived from the same cash flow stream. For instance, given his opportunity interest rate r_2 and cash flow $(-A+b_1, 2A-b_2, -A)$, member 2's Hui borrowing and lending rates, denoted by γ_b and γ_ℓ respectively, are such that

$$(A-b_1)(1+\gamma_\ell) = (2A-b_2) - \frac{A}{1+r_2}$$

and

$$(2A-b_2) - (A-b_1)(1+r_2) = \frac{A}{1+\gamma_b} .$$

In other words, to calculate his Hui borrowing rate, we assume that he borrows only after he has drawn funds from the pool; before that, he lends at his opportunity interest rate r_2. Symmetrically, he lends in Hui up to the time he obtains funds; thereafter, he borrows at his opportunity rate.

As an example, assume r_1 = 20%, r_2 = 16%, r_3 = 10% and A = \$100. The reservation discounts for each member will be as follows:

i	r_i	v_{i1}	v_{i2}
1	20%	\$23.6	\$25.0
2	16	19.7	20.7
3	10	13.2	13.6

Suppose bidders shade their bids under their reservation discount (This is an optimal bidding strategy under a set of assumptions. See Sections 3 - 5.) so that the actual winning bids are $b_1 = b_2$ = \$20. Then, the actual cash flows and the Hui borrowing and lending rates for each member are given below:

		time				
i	r_i	1	2	3	γ_b	γ_ℓ
1	20%	\$160	-\$100	-\$100	16.3%	-
2	15	- 80	+ 180	- 100	13.6	16.3%
3	10	- 80	- 80	+ 200	-	15.8

This table tells a happy story. Member 1 borrowed at 16.3%, lower than his opportunity rate 20%. Member 3 lends at 15.8%, higher than his opportunity rate 10%. Member 2's outcome depends on his position. If he put the \$180 proceeds in his bank savings account, he is more likely to see himself as having saved with Hui for one period. The interest rate he earned from lending \$80 for one period is 16.3%, higher than 16%. If he needed a loan at time 2 for some purpose and considered himself as borrowing a one-period loan with a maturity value of \$100, his Hui borrowing rate is 13.6%, less than 16%.

We shall not consider 3-member Hui with an organizer here because it is a special case of the model to be studied in Sections 3 - 5.

3

THE MODEL FOR AN N-MEMBER HUI WITH AN ORGANIZER

Notations

As in Sections 1 and 2, we will use the following notations:

A : the size of the per-period, before-discount deposit into the pool;

n (= 1, ..., N): the participant who succeeds in bidding for the pool at the nth period;

0 : the organizer who receives an interest-free loan of NA repaid in N equal installments of A at each of the N subsequent periods;

b_{in}: the bid submitted by participant i at period n;

b_n^f : the highest bid submitted at period n;

b_n^s : the second highest bid submitted at period n.

Other notations will be defined and explained as they arise.

Assumptions

Unless otherwise stated, the following assumptions are made throughout the paper.

Assumption 0: There is no possibility of default.

Assumption 1: There is no Bayesian learning from past winning bids by individual members in deciding their current bids.

Assumption 2: Each individual i has a deterministic and known income stream I_i over the duration of Hui prior to the participation, where

$$I_i = (I_{i0}, I_{i1}, \ldots, I_{iN}),\ i = 1, 2, \ldots, N.$$

Assumption 3: Each individual i has a continuous and strictly increasing von Neumann-Morgentern utility function U_i defined over his income stream.

Assumption 4: To each bidder i, the bids from all other bidders at period n are drawn independently from probability distribution F_{in} with support contained in some interval $[\underline{b}_n, \bar{b}_n]$.

To investigate an expected utility maximizer's optimal bidding strategy, we need to define both the 'reservation price' in the context of Hui and 'positive time preference':

Definition 3.1: The reservation discount vector v_i of participant i is any vector $v_i = (v_{i1}, v_{i2}, \ldots, v_{i,N-1})$ such that participant i is indifferent to the N alternative cash flow patterns listed in Table 3.1. In other words,

$$U_i(Y_{in}) = \bar{U}_i \text{ for all } n = 1,\ 2,\ \ldots,\ N, \tag{3.1}$$

where

$$Y_{i1} = (I_{i0}-A, I_{i1}+NA-(N-1)v_{i1}, I_{i2}-A, \ldots, I_{iN}-A),$$

$$\ldots$$

$$Y_{in} = (I_{i0}-A, I_{i1}-(A-v_{i1}), \ldots, I_{in}+NA-(N-n)v_{in}, \ldots, I_{iN}-A),$$

$$\ldots$$

$$Y_{iN} = (I_{i0}-A, I_{i1}-(A-v_{i1}), \ldots, I_{i,N-1}-(A-v_{i,N-1}), I_{iN}+NA).$$

We shall refer to $\bar{U}_i$ as participant i's reservation utility at the beginning of the Hui.

Definition 3.2: An agent i is said to exhibit positive time preference if his preference for income streams of the form $(x_0, \ldots, x_N)$ is such that

Table 3.1: Participant i's Indifferent Cash Flow Patterns in a Discount-bid Hui

period / alternative	0	1	...	n	...	N-1	N
0	NA	-A	...	-A	...	-A	-A
1	-A	$A+(N-1)(A-v_{i1})$	...	-A	...	-A	-A
.	.	.		.		.	.
n	-A	$-(A-v_{i1})$	...	$nA+(N-n)(A-v_{in})$	...	-A	-A
.	.	.		.		.	.
N-1	-A	$-(A-v_{i1})$	...	$-(A-v_{in})$	...	$(N-1)A+(A-v_{i,N-1})$	-A
N	-A	$-(A-v_{i1})$	...	$-(A-v_{in})$	...	$-(A-v_{i,N-1})$	NA

$(x_0,\ldots,x_h,\ldots,x_k,\ldots,x_N)$ is strictly preferred to $(x_0,\ldots,x_k,\ldots,x_h,\ldots,x_N)$ if and only if $x_h > x_k$ for $h = 0, 1, \ldots, N$ and $k > h$.

A set of sufficient conditions for the existence and uniqueness of v_i is given in Lemma 3.1.

<u>Lemma 3.1</u>: The reservation discount vector v_i exists and is unique if the continuous, strictly increasing von Neumann-Morgenstern utility U_i exhibits positive time preference.

Proof: See the Appendix.

The proof of Lemma 3.1 involves the construction of a reservation discount vector, which is then shown to be unique. Note that, in Table 3.1, all entries in every two adjacent rows, say rows n and n+1, are identical except the n<u>th</u> and the (n+1)<u>th</u> ones. This feature allows the convenient backward construction of reservation discounts, starting with periods N and N-1 to obtain $v_{i,N-1}$. From the way v_i is constructed, v_{in} depends only on $v_{i,n+1}$, which in turn depends on $v_{i,n+2}$, ... In other words, as auctions are conducted and bids revealed at periods 1, 2, ..., n, v_{ik} for all $k > n$ does not change as long as bidder i's preference and income stream remain unchanged. Consequently, the property of reservation utility previously mentioned also applies to the Hui cash flow at any intermediate period k. We may call it member i's <u>conditional reservation utility</u> at period k, denoted by $\bar{U}_i^k$, given some realized cash flow $(c_0,c_1,\ldots,c_{k-1})$. In other words,

$$U_i(Y_{i,k+t}^k) = \bar{U}_i^k, \; t = 0, 1, \ldots, N-k, \tag{3.2}$$

where

$$Y^k_{i,k+t} = (c_0,c_1,\ldots,c_{k-1},I_{ik}-(A-v_{ik}),\ldots,$$
$$I_{i,k+t}+NA-(N-k-t)v_{i,k+t},I_{i,k+t+1}-A,\ldots). \qquad (3.3)$$

The <u>conditional reservation discount vector</u> v^k_i then refers to $(v_{ik}, \ldots,v_{i,N-1})$, irrespective of the realized cash flow $(c_0,c_1,\ldots,c_{k-1})$.

The reservation discount has a familiar interpretation when the individual is risk neutral with time preference being completely described by an interest rate available in some formal financial sector. In this particular case, individual i's reservation discounts represent his market-based opportunity cost of Hui dealing, and can be obtained by equating the present values of the N alternative cash flow patterns listed in Table 3.1. By tendering bids equal to his reservation discounts at each period, a Hui participant can at least attain a utility level equal to his reservation utility, which is what he can obtain from the formal financial market. In this sense, reservation discounts describe an individual's 'fall back' position with respect to joining the Hui. It is clear that, if a member bids his reservation discounts throughout the duration, his ex post utility from participating a Hui will never be less than his ex ante reservation utility.

When an individual has no access to the formal financial market, or if the formal financial market does not exist, his reservation discounts can be interpreted as related to some sort of 'as if' interest rate which reflects his 'internal' opportunity cost such as time preference for consumptions.

4

OPTIMAL INDIVIDUAL BIDDING STRATEGIES

It is shown in Section 3 that a Hui participant can ensure a utility level, called 'reservation utility', by submitting a bid equal to his reservation discount. In this section, we consider an expected utility maximizer's optimal bidding strategies. To make the problem tractable, we impose, via Assumption 3', the requirements of concavity, time-additivity, and positive time preference on individuals' preferences.

Assumption 3': The von Neumann-Morgenstern utility of each individual i, U_i, is continuous, strictly increasing, concave, and time-additive with (strictly) positive time preference.

We can write a time-additive von Neumann-Morgenstern utility function in the following way:

$$U_i(x_0,\ldots,x_N) = \Sigma_{n=0}^{N}\lambda_{in}u_i(x_n). \tag{4.1}$$

It is easy to see that strictly positive time preference is, in this case, equivalent to requiring the time preference coefficients to be strictly decreasing, i.e.,

$$1 = \lambda_{i0} > \lambda_{i1} > \ldots > \lambda_{iN} > 0. \tag{4.2}$$

With time-additive utility, our problem can be solved working backward through dynamic programming. From Table 1.1a, we note that, at the last period N, the only member who has not yet received funds obtains NA. There is no need for bidding and therefore no uncertainty involved. As we proceed backward to periods N-1, N-2, ..., the number of bidders increases by one each time. At each period, the prospect of losing entails the expected utility gain from participation in subsequent periods. In general, there are N-n+1 bidders at period n, each of whom submits a bid that maximizes his current expected utility which incorporates, in a nested way, the potential subsequent expected utility gain from future periods. The actual form of the expected utility function depends on the auctioning method (i.e. first-price or second-price). We shall first consider the case of first-price competitive bidding as the allocation mechanism.

4.1 First-Price Competitive Bidding

In a first-price sealed bid Hui where Assumptions 0 (no default), 1 (no learning), 2 (deterministic and known pre-Hui income stream), 3' (continuous, strictly increasing, concave, and time-additive von Neumann-Morgenstern utility with strictly positive time preference) and 4 (independent and identical bid distribution for each agent) hold, bidder i's expected utility at period n of a bid b_{in} is given by expression (4.3):

$$\begin{aligned} EU^f_{in}(b_{in}) = {} & \{\lambda_{in}u_i(I_{in}+NA-(N-n)b_{in}) \\ & + \textstyle\sum_{k=n+1}^{N}\lambda_{ik}u_i(I_{ik}-A)\}[F_{in}(b_{in})]^{N-n} \\ & + \int_{b_{in}}^{\bar{b}_n}\{\lambda_{in}u_i(I_{in}-A+x)+EU^{*f}_{i,n+1}\}d[F_{in}(x)]^{N-n}, \end{aligned} \tag{4.3}$$

where

$$EU^{*f}_{i,n+1} = EU^f_{i,n+1}(b^*_{i,n+1}) = \text{Max } EU^f_{i,n+1}(b_{i,n+1}). \tag{4.4}$$

The first term of the RHS of expression (4.3) is the utility for bidder i if he wins at period n, whereas the second term gives his utility in the case of losing. The component $\lambda_{in}u_i(I_{in}-A+x)$, where $x \in (b_{in},\bar{b}_n]$, is his utility from post-bidding income at the current period. $EU^{*f}_{i,n+1}$ is independent of the winning bid at period n and represents the expected utility at the next period providing that bidder i submits his expected utility maximizing bid at each of the subsequent periods.

Sufficient conditions for an individual's bid to maximize the expected utility globally are formally stated in Theorem 4.1 below after we define the terms 'marginal outbidden rate' and 'increasing (decreasing) marginal-outbidden-rate distribution', which ensures that the expected utility function is pseudo-concave.

Definition 4.1: The marginal outbidden rate for a probability distribution F is given by F'/F.

An interpretation of the marginal outbidden rate will be provided shortly after Theorem 4.1.

Definition 4.2: A probability distribution that yields an increasing (decreasing) marginal outbidden rate is called an increasing (decreasing) marginal-outbidden-rate distribution.

In Theorem 4.1, we suppress the individual and time subscripts (i and n) to F, b, I and v without fear of ambiguity.

Theorem 4.1: Suppose Assumptions 0, 1, 2, 3' and 4 hold. Then, for the class of decreasing marginal-outbidden-rate distributions, the individual's expected utility maximizing bid at the nth period, b, exists uniquely; and is given by the solution to the following equation:

$$\frac{F'(b)}{F(b)} = \frac{u'(I+NA-(N-n)b)}{[u(I+NA-(N-n)b)-u(I+NA-(N-n)v)]-[u(I-A+b)-u(I-A+v)+E^*]} \quad (4.5)$$

with

$$E^* = (1/\lambda_{in})(EU^{*f}_{i,n+1} - \bar{U}_i^{n+1}), \tag{4.6}$$

where $EU^{*f}_{i,n+1}$ is given in expression (4.4) and $\bar{U}_i^{n+1}$ is bidder i's conditional reservation utility at period n+1.

Proof: See the Appendix.

Equation (4.5) is the first order condition for b to maximize the difference between the expected utility and the conditional reservation utility at period n, which, by construction, is a constant. The uniqueness of the expected utility maximizing bid is established by showing that the first order condition is satisfied uniquely (since the LHS of equation (4.5) is decreasing by assumption, and the RHS increases in b by the concavity of u), and that the second order condition is satisfied locally.

The decreasing marginal outbidden rate F'(b)/F(b) has the interpretation as the conditional probability that an individual bidding b is 'marginally' outbid by his second rival given that he has outbid his first rival. A decreasing marginal outbidden rate means that as an individual raises his bid, the increase in the chance of outbidding his rival must not be slower than the increase in the chance of being marginally outbid. A common class of decreasing marginal-outbidden-rate distributions is given by the power probability distribution, of which the uniform distribution used in Section 5 is a special case.

When bids are uniformly distributed and bidders are risk neutral, the individual expected utility maximizing bid will have a closed-form solusolution (given in expression (5.3)). We will have more detailed discussions on this special case in Section 5.

4.2 Second-Price Competitive Bidding

The formulation of a competitive bidding model for the hypothetical second-price sealed bid Hui is similar to that of the first-price model, with minor modifications to incorporate the feature that the effective (i.e. implemented) discount is the second highest bid. Under Assumptions 0, 1, 2, 3' and 4, the problem of the expected utility maximizer i at period n is stated below:

$$\text{Max } EU^s_{in}(b_{in}) =$$

$$\int_{\underline{b}_n}^{b_{in}} \{\lambda_{in} u_i(I_{in}+NA-(N-n)x)+\sum_{k=n+1}^{N} \lambda_{ik} u_i(I_{ik}-A)\} d[F_{in}(x)]^{N-n}$$

$$+ (N-n)\{\lambda_{in} u_i(I_{in}-A+b_{in})+EU^{*s}_{i,n+1}\}[F_{in}(b_{in})]^{N-n-1}[1-F_{in}(b_{in})]$$

$$+ (N-n)\int_{b_{in}}^{\bar{b}_n} \int_{b_{in}}^{x} \{\lambda_{in} u_i(I_{in}-A+y)+EU^{*s}_{i,n+1}\} d[F_{in}(y)]^{N-n-1} dF_{in}(x), \quad (4.7)$$

where

$$EU^{*s}_{i,n+1} = EU^s_{i,n+1}(b^*_{i,n+1}) = \text{Max } EU^s_{i,n+1}(b_{i,n+1}). \quad (4.8)$$

The first term in expression (4.7) is the utility for bidder i if he succeeds in bidding. The second term is his utility if he fails and turns out to be the second highest bidder. The last term accommodates the case where his bid is the third or below.

Suppose an individual who bids b knows that he has been outbid by one of his rivals. It is then to his advantage to be outbid by another rival, so that he, as well as other losers, can benefit from a higher second bid. Parallel to the 'marginal outbidden rate' in the first-price case, we need a 'marginal outbidding rate' for our second-price model.

Definition 4.3: The marginal outbidding rate for a probability distribution F is given by $F'/(1-F)$.

$F'(t)/[1-F(t)]$ is commonly known in the reliability literature as the 'failure rate' (or 'hazard rate'), which is the conditional probability that a product will fail at time t^+ given that it has survived up to time t. In the bidding context, the marginal outbidding rate $F'(b)/[1-F(b)]$ has the interpretation of the conditional probability that an individual bidding b will marginally outbid his second rival given that he has been outbid by his first rival. To make sure that the expected utility function is single-peaked, we require the bid distribution F to yield an increasing marginal outbidding rate. It is easy to check that the power probability distribution with index greater than or equal to one belongs in this category.

Definition 4.4: A probability distribution that yields an increasing (decreasing) marginal outbidding rate is called an increasing (decreasing) marginal-outbidding-rate distribution.

After suppressing the subscripts i and n, we state in Theorem 4.2 a set of sufficient conditions for an individual's bid to maximize his expected utility globally.

Theorem 4.2: Suppose Assumptions 0, 1, 2, 3' and 4 hold. Then, for the class of increasing marginal-outbidding-rate distributions, the individual's expected utility maximizing bid at the nth period, b, exists uniquely; and is given by the solution to the following equation:

$$\frac{F'(b)}{1-F(b)} = \frac{-u'(I-A+b)}{[u(I+NA-(N-n)b)-u(I+NA-(N-n)v)]-[u(I-A+b)-u(I-A+v)+E^*]}, \quad (4.9)$$

with

$$E^* = (1/\lambda_{in})(EU^{*s}_{i,n+1}-\bar{U}^{n+1}_{i}), \quad (4.10)$$

where $EU^{*s}_{i,n+1}$ is given in expression (4.8) and $\bar{U}^{n+1}_i$ is his conditional reservation utility at period n+1.

Proof: Omitted.

The proof of Theorem 4.2 would be similar to that of Theorem 4.1. We can first obtain the first order condition for a problem equivalent to (4.7), then show that, with an increasing marginal-outbidding-rate distribution, the first order condition is uniquely satisfied by a bid which also satisfies the second order condition.

4.3 Implications

In this subsection, we will discuss several implications of Theorems 4.1 and 4.2, making relevant comparisons when appropriate. First of all, the following corollary tells us that the standard thinking in the bidding literature that bidders will shade their bids in first-price auctions remains true of agents' behavior in a Hui.

Corollary 4.1: Under the hypotheses of Theorem 4.1, an agent's expected utility maximizing bid is less than his reservation discount.

Proof: Suppose this is not true. Then $u_i' > 0$ implies that the RHS of equation (4.5) is negative. But the LHS of the equation is positive. This is a contradiction.

Q.E.D.

The well-known result that second-price auctions are demand revealing, however, does not in general apply to Hui according to the corollary below.

Corollary 4.2: Under the hypotheses of Theorem 4.2, an expected utility maximizer will shade his bid under his reservation discount only if the expected utility gain from the next and subsequent periods will more than compensate his utility loss at the current period from shading his bid.

Proof: Omitted.

In a first-price sealed bid Hui, an individual has no incentives to bid in excess of his reservation discount since, when he wins, the amount he obtains is a decreasing function of his bid. If he loses, his bid does not affect what he has to pay. In contrast, due to the disparity between the winning bid and the effective discount, an individual in a second-price sealed bid Hui has some incentive to bid over his reservation discount. When a bidder wins, the amount he obtains depends on the highest rejected bid rather than his own bid. When he loses, the higher the second bid, the less he will have to pay. Because there is a chance that his bid might turn out to be the second highest, he has some incentive to bid over his reservation discount. He, however, will inflate his bid only up to a point beyond which his expected utility gain from future biddings will outweigh the current benefit. If he perceives sufficiently high expected utility gain from subsequent periods, he might even bid under his reservation discount. Obviously, he will overbid at the last second period due to the absence of uncertainty at the final period. Hence, Corollaries 4.3 and 4.4.

Corollary 4.3: Under the hypotheses of Theorem 4.1 (Theorem 4.2), an expected utility maximizer in a first- (second-) price sealed bid Hui will lower his bid at the current period if his perceived expected utility gain in the future increases.

Corollary 4.4: Under the hypotheses of Theorem 4.2, an expected utility maximizer will submit a bid higher than his reservation discount at the last second period.

Regardless of the auctioning methods, possible gains from future biddings tend to drive down the present bid. The possible gain in the current bidding will push the bid even lower in the first-price Hui, but has the opposite effect in the second-price Hui. Therefore, a member's bid in the first-price Hui is unambiguously lower than his reservation discount, but this is not necessarily the case for the second-price Hui.

Suppose each bidder's reservation discounts correctly measure the true value he places on loans being auctioned off each period. Then, if all bidders follow the demand-revealing bidding strategy, i.e., bid exactly their reservation discounts, the allocation of loans will be efficient in the sense that the loan always goes to the most needing agent (indicated by the highest reservation discount), leaving no room for Pareto improvement. On the contrary, if someone else wins the loan, the most needing person can presumably make a side-payment to 'purchase' the loan. An allocation upon which no Pareto improvement can be made is called a Pareto-efficient allocation of the loan provided within the Hui.

It is well known in the competitive bidding literature that a first-price sealed bid auction is not a Pareto-efficient allocation mechanism while the demand-revealing second-price sealed bid auction is. Do we have parallel results in Hui auctions? The answer is negative (except for the last second period in a second-price Hui). We construct an example to illustrate this.

Example 4.1: Suppose (i) individuals 1 and 2 are competing for a \$100 before-discount loan in a first-price Hui, (ii) individual 1 is risk averse whereas individual 2 is risk neutral with $u(x_1,x_2) = x_1+\lambda x_2$, where $\lambda = 1/(1+r)$, r is individual 2's non-Hui borrowing as well as lending interest rate, (iii) both individuals have the same pre-Hui income stream (100,0), (iv) their respective reservation discounts, actual bids and ex post incomes are given below:

individual	reservation discount	bid	ex post income
1	25	12	(+16,+100)
(winner) 2	20	16	(+184,-100)

By the definition of reservation discounts and the dominance argument, we have

$$(+175,-100) \sim (+25,+100) \succ (+16,+100) \quad \text{for individual 1,}$$

$$(+184,-100) \succ (+180,-100) \sim (+20,+100) \quad \text{for individual 2.}$$

Suppose individual 1 makes a side-payment of \$9 to individual 2 to 'buy' the loan, the resulting income streams for individuals 1 and 2 will be (+175,-100) and (+25,+100), respectively. Since

$$u(+25,+100) = 105 > 104 = u(+184,-100)$$

both individuals 1 and 2 are better off.

Similar examples can be constructed for a second-price Hui with $N \geq 3$. ∎

Hence Proposition 4.1 below:

Proposition 4.1: The expected utility maximizing bids given in Theorems 4.1 and 4.2 will not in general yield a Pareto-efficient allocation of credits among Hui participants.

This non-Pareto-efficiency comes from the observation that neither the first- nor the second-price sealed bid Hui exhibits the demand-revealing property. The amount by which an individual will shade or inflate his bid depends on his preferences and expectations. When individuals' preferences and expectations are allowed to differ, we can not in general expect the highest bidder to have the highest reservation discount. However, by Corollary 4.5 below, we can expect an individual to submit a higher bid when his reservation discount increases.

Corollary 4.5: An individual's optimal first- (second-) price sealed bid, given in Theorem 4.1 (Theorem 4.2) increases in his current-period reservation discount.

Proof: See the Appendix.

Cox, Roberson and Smith (1982) showed that, when all bidders have identical preferences and expectations, the ordinary first-price sealed bid auction will be Pareto-efficient. Corollary 4.5 tells us that this is also true for Hui auctions. Hence, Corollary 4.6:

Corollary 4.6: Under the hypotheses of Theorem 4.1 (Theorem 4.2), the first- (second-) price sealed bid Hui will be a Pareto-efficient allocation mechanism for credit if all individuals possess the same tastes (i.e. utility functions and time preference coefficients) and expectations (i.e. bid distributions at the current and each of the subsequent periods).

It was mentioned previously that the actual outcome of a Hui depends on the composition of its members. One question of interest is: How does a member's time preference affect his optimal bid? Other things being equal, a higher preference for current consumption (characterized by a higher time preference coefficient for the current period) should lead to a higher bid for loans currently available. The result that supports this intuition is formally stated in Corollary 4.7.

Corollary 4.7: Under the hypotheses of Theorem 4.1 (Theorem 4.2), an individual optimal bid in a first- (second-) price Hui increases as his relative time preference for current consumption increases.

Proof: See the Appendix.

5

A NASH PROCESS OF INTEREST RATE FORMATION

Having derived the individual optimal bidding strategy for an agent with a time-additive von Neumann-Morgenstern utility and studied some of its implications in Section 4, we are ready to explore the question of how the winning bids of utility maximizing agents will determine for Hui members their respective interest rates ex ante and ex post. The definitions of interest rates which are appropriate in the Hui context will be given after the derivation of the Nash equilibrium individual bidding programs which will yield the relevant interest rate in an endogenous manner. In addition to tractability, the Nash model of strategic interaction has found empirical support in experimental studies of the competitive bidding behavior for single as well as multiple object auctions (Cox, Roberson and Smith, 1982; Cox, Smith and Walker, 1982). Furthermore, the Nash assumption may be more plausible given the auctioning mechanism (sealed bid) used by Hui, which helps members to remain anonymous.

In this section, we will demonstrate that the optimal individual bidding strategies derived in Section 4 will lead to a Nash equilibrium under risk neutrality and linear bid distributions. The risk neutrality assumption, stated below as Assumption 3", replaces Assumption 3'.

Assumption 3": Each individual i is risk neutral with the time-additive utility U_i defined over his income stream $x_i = (x_{i0},\ldots,x_{iN})$ and given by expression (5.1) below:

$$U_i(x_i) = \Sigma_{n=0}^{N}\lambda_{in}x_{in}, \tag{5.1}$$

where $1 = \lambda_{i0} > \lambda_{i1} > \ldots > \lambda_{iN} > 0$.

We have two justifications for the risk neutrality assumption. First, it helps us avoid confounding the risk attitude in individuals' bidding behavior which is the main focus of this essay. Second, observations of the actual dispersions of discount bids tend to be relatively small compared with either the loan available or the pre-Hui incomes so that we are virtually looking at gambles whose outcomes cluster within a small interval for which a linear function is a good approximation. This seems to accord with observations that Hui participants tend to use monetary value, rather than some subjective utility worth, in their bid calculation (Huang, 1981).

Under risk neutrality and uniform bid distributions, the individual optimal first- (second-) price bid given by expression (4.5) (expression (4.9)) will be linear in the reservation discount. Since a uniform distribution is preserved via a linear transformation, we can replace Assumption 4 by Assumption 4'.

Assumption 4': Each individual i's reservation discount for the fund available at period n, v_{in}, is independently drawn from a common uniform distribution H_n with support $[\underline{v}_n, \bar{v}_n]$, where $n = 1, \ldots, N-1$; i.e.,

$$H_n(v_{in}) = (v_{in} - \underline{v}_n)/(\bar{v}_n - \underline{v}_n) \tag{5.2}$$

for all i, where $n = 1, \ldots, N-1$. Furthermore, each individual i observes his own reservation discount v_{in} before he submits his bid b_{in}.

Assumption 4' implies that each individual i's bid for the fund available at the nth period, b_{in}, will be uniformly distributed over the interval $[\underline{b}_{in}, \bar{b}_{in}]$, where $\underline{b}_{in}$ and $\bar{b}_{in}$ correspond, through a linear relationship, to $\underline{v}_n$ and $\bar{v}_n$, respectively, and are individual-specific because different bidders might perceive different gains from future biddings. In order to produce consistent expectation, we will assume that each agent believes that every other bidder has the same discounted expected gain from future biddings as his own, contingent on losing at this period. Since an agent's expected gain is discounted using his appropriate time preference coefficient, we shall refer to it as <u>subjectively discounted expected gain</u> from future bidding participation. This restriction is formally stated as Assumption 5:

<u>Assumption 5</u>: Each individual i believes that all other bidders' subjectively discounted expected gains from future bidding participation, contingent on losing in the current bidding, are equal to his own.

We shall denote member i's subjectively discounted expected gain at period n by G^*_{in}. Note that G^*_{in} is his expected gains from biddings taking place in periods n+1, n+2, ..., N-1, subjectively discounted to period n, and that G^*_{i0} can be interpreted as member i's expected 'surplus' for joining the Hui.

Given the modified assumptions, the Nash equilibrium bidding programs for the first-price and second-price sealed bid Hui are given in Theorem 5.1 and Theorem 5.2, respectively. To simplify notations, we let m = N-n+1 denote the number of eligible bidders (those who have not yet received funds) at period n.

<u>Theorem 5.1</u>: Suppose Assumptions 0, 1, 2, 3", 4' and 5 hold. Then, the Nash equilibrium bidding program for individual i in a first-price sealed bid Hui is given by

$$b^{*f}_{in} = v_{in} - \frac{1}{m+1}(v_{in} - \underline{v}_n) - \frac{1}{m} G^{*f}_{in}, \tag{5.3}$$

where G^{*f}_{in} is the subjectively discounted expected gain from future bidding participation, given iteratively by

$$G^{*f}_{in} = (\lambda_{i,n+1}/\lambda_{in})[(m-2)/(\bar{b}_{n+1} - \underline{b}_{n+1})^{m-2}] \cdot$$
$$\{(v_{i,n+1} - b^*_{i,n+1})(b^*_{i,n+1} - \underline{b}_{n+1})^{m-2}$$
$$+ \int_{b^*_{i,n+1}}^{\bar{b}_{n+1}} (x - v_{i,n+1} + G^{*f}_{i,n+1})(x - \underline{b}_{n+1})^{m-3} dx\}, \tag{5.4}$$

for $n = 1, 2, \ldots, N-1$.

Proof: See the Appendix.

The proof involves identifying the bid function (5.3) as a candidate for a Nash equilibrium and verifying that it is indeed the case.

Theorem 5.2: Suppose Assumptions 0, 1, 2, 3", 4' and 5 hold. Then, the Nash equilibrium bidding program for individual i in a second-price sealed bid Hui is given by:

$$b^{*s}_{in} = v_{in} + \frac{1}{m+1}(\bar{v}_n - v_{in}) - \frac{1}{m} G^{*s}_{in}, \tag{5.5}$$

where G^{*s}_{in} is the subjectively discounted expected gain from future bidding participation, given iteratively by:

$$G^{*s}_{in} = (\lambda_{i,n+1}/\lambda_{in})[(m-2)/(\bar{b}_{n+1} - \underline{b}_{n+1})^{m-2}] \cdot$$
$$\{(m-2)\int_{\underline{b}_{n+1}}^{b^*_{i,n+1}} (x - \underline{b}_{n+1})^{m-3}(v_{i,n+1} - x) dx$$
$$+ (b^*_{i,n+1} - v_{i,n+1} + G^{*s}_{i,n+1})(b^*_{i,n+1} - \underline{b}_{n+1})^{m-3}(\bar{b}_{n+1} - b^*_{i,n+1})$$
$$+ (m-3)\int_{b^*_{i,n+1}}^{\bar{b}_{n+1}} \int_{b^*_{i,n+1}}^{x} (y - v_{i,n+1} + G^{*s}_{i,n+1})(y - \underline{b}_{n+1})^{m-4} dy dx\}, \tag{5.6}$$

for $n = 1, 2, \ldots, N-1$.

Proof: Omitted since it is similar to the proof of Theorem 5.1.

We summarize in Table 5.1 the Nash bidding strategy for period n, as well as the corresponding expected values and variances of the effective discounts (i.e. winning bid under the first-price method and second highest bid under the second-price method).

With risk neutrality and uniform distributions, Vickrey (1961) showed that, in a single-object auction, the expected prices under the first- and the second-price auctioning methods are the same, but the corresponding variance is higher under the latter. In our Hui bidding model, similar results do not appear obvious. It is straightforward to check that, for all $m > 2$, the variance of the effective discount (i.e. the bid that is implemented) is higher under the second-price than under the first-price auctioning method. If we further assume that $G^{*f}_{in} = G^{*f}_{n} = G^{*s}_{n} = G^{*s}_{in}$ for all i, then the expected effective discount will be higher under the second-price than under the first-price method for all m. Therefore, from a borrower's standpoint, it seems reasonable to conjecture that the first-price method will be preferable to the second-price method. When $m = 2$, the second-price method, from the lender's point of view, dominates the first-price method since the former yields a higher expected effective discount with the same variance. However, given the multi-period nature of Hui, the double role played by its members, and the fact that the same auctioning method must be followed consistently throughout a cycle, it is not presently clear how to carry this line of argument to favor one method over another.

Since the auctioned object in Hui is a homogeneous monetary loan, it will be desirable to have some measure by which one can compare the per-

Table 5.1: Nash Bidding Strategies and Their Derivatives

Restrictions:
- no default, no learning
- risk neutrality with positive time preference
- linear distribution of reservation discount v over $[\underline{v},\bar{v}]$
- identical subjectively discounted expected gains G*

	First-Price	Second-Price
Nash bidding strategy, b	$v - \frac{1}{m+1}(v-\underline{v}) - \frac{1}{m}G^*$	$v + \frac{1}{m+1}(\bar{v}-v) - \frac{1}{m}G^*$
$\underline{b}$	$\underline{v} - \frac{1}{m}G^*$	$\underline{v} + \frac{1}{m+1}(\bar{v}-\underline{v}) - \frac{1}{m}G^*$
$\bar{b}$	$\bar{v} + \frac{1}{m+1}(\bar{v}-\underline{v}) - \frac{1}{m}G^*$	$\bar{v} - \frac{1}{m}G^*$
Bid distribution, F(b)	$(v-\underline{v})/(\bar{v}-\underline{v})$	$(v-\underline{v})/(\bar{v}-\underline{v})$
$E(b^f)$, $E(b^s)$	$\underline{v} + \frac{m^2}{(m+1)^2}(\bar{v}-\underline{v}) - \frac{1}{m}G^*$	$\bar{v} - \frac{2m}{(m+1)^2}(\bar{v}-\underline{v}) - \frac{1}{m}G^*$
		$= \underline{v} + \frac{m^2+1}{(m+1)^2}(\bar{v}-\underline{v}) - \frac{1}{m}G^*$
$Var(b^f)$, $Var(b^s)$	$\frac{m^3}{(m+1)^4(m+2)}(\bar{v}-\underline{v})^2$	$\frac{2(m-1)m^2}{(m+1)^4(m+2)}(\bar{v}-\underline{v})^2$

m = N-n+1: number of bidders at period n formance of different Hui from one's point of interest (e.g., borrowing or lending). This measure would preferably be in some form of interest rates. One is attempted to consider the rate r_n that solves equation (5.7) below:

$$\Sigma_{k=0}^{n-1}(A-b_k)(1+r_n)^{n-k} + \Sigma_{k=n+1}^{N}A(1+r_n)^{n-k} = NA+(N-n)(A-b_n), \qquad (5.7)$$

where $b_0 = 0$ and $b_n = b_n^f$ (b_n^s) under the first- (second-) price auctioning method.

This measure implicitly assumes an indentical interest rate for borrowing and lending -- r_n which misleadingly looks like some kind of ex post interest rate for participant n who won the bidding at period n. This internal rate of return however turns out a poor choice given the typical Hui cash flows which, except for the organizer and participant N, change signs twice, yielding non-unique solutions for r_n.

In Section 2, we motivated, using a numerical example, the distinction between Hui borrowing and lending rates. We now formally define them.

Definition 5.1: The ex post Hui borrowing interest rate to member n (who won the loan at period n), n = 1, ..., N, with opportunity cost of capital r_n is the rate γ_n^b that solves equation (5.8) below:

$$NA+(N-n)(A-b_n) - \Sigma_{k=0}^{n-1}(A-b_k)(1+r_n)^{n-k} = \Sigma_{k=n+1}^{N}A(1+\gamma_n^b)^{n-k}, \qquad (5.8)$$

where $b_0 = b_N = 0$ and $b_k = b_k^f$ (b_k^s), k = 1, ..., n, under the first- (second-) price auctioning method.

Definition 5.2: The ex post Hui lending interest rate to member n (who won the loan at period n), n = 1, ..., N, with opportunity cost of capital

r_n is the rate γ_n^ℓ that solves equation (5.9) below:

$$\Sigma_{k=0}^{n-1}(A-b_k)(1+\gamma_n^\ell)^{n-k} = NA+(N-n)(A-b_n) - \Sigma_{k=n+1}^{N}A(1+r_n)^{n-k}, \qquad (5.9)$$

where $b_0 = b_N = 0$ and $b_k = b_k^f$ (b_k^s), $k = 1, \ldots, n$, under the first- (second-) price auctioning method.

Note that the ex post Hui interest rates to the <u>nth</u> member defined above are functions of the winning bids (or the second highest bids) at periods 1, 2, ..., n, and are independent of those at periods n+1, ..., N-1. Therefore, one member's ex post Hui interest rates will in general differ from another's.

For an individual deciding on whether to join a Hui, an ex ante (expected) interest rate is perhaps more relevant.

<u>Definition 5.3</u>: Given member i's opportunity cost of capital r_i and planned bidding program $(b_{i1}, b_{i2}, \ldots, b_{i,N-1})$, his <u>ex ante Hui borrowing interest rate</u> R_i^b is given by expression (5.10) below:

$$R_i^b = \Sigma_{n=1}^{N-1}\{\Pi_{k=1}^{n-1}[1-H_k(b_{ik})]\}H_n(b_{in})E(\gamma_n^b), \qquad (5.10)$$

where $H_k(b_{ik})$, $k = 1, \ldots, N-1$, is the probability that member i will win the loan at period k by submitting a bid equal to b_{ik} and $E(\gamma_n^b)$ solves equation (5.11):

$$\Sigma_{k=0}^{n-1}[A-E(b_k)][1+r_i]^{n-k}+\Sigma_{k=n+1}^{N}A[1+E(\gamma_n^b)]^{n-k} = NA+(N-n)[A-E(b_n)] \qquad (5.11)$$

for $n = 1, 2, \ldots, N-1$; $E(b_n)$ is the expected effective discount given in Table 5.1, and $b_0 = 0$.

<u>Definition 5.4</u>: Given member i's opportunity cost of capital r_i and planned bidding program $(b_{i1}, b_{i2}, \ldots, b_{i,N-1})$, his <u>ex ante Hui lending</u>

interest rate R_i^ℓ is given by expression (5.12) below:

$$R_i^\ell = \Sigma_{n=1}^{N-1}\{\Pi_{k=1}^{n-1}[1-H_k(b_{ik})]\}H_n(b_{in})E(\gamma_n^\ell), \qquad (5.12)$$

where $E(\gamma_n^\ell)$ solves equation (5.13):

$$\Sigma_{k=0}^{n-1}[A-E(b_k)][1+E(\gamma_n^\ell)]^{n-k}+\Sigma_{k=n+1}^{N}A[1+r_i]^{n-k} = NA+(N-n)[A-E(b_n)] \qquad (5.13)$$

for $n = 1, 2, \ldots, N-1$; $E(b_n)$ and $H_n(b_{in})$ are the same as in Definition 5.3.

Note that the bids that are used to calculate the ex ante interest rates in Definitions 5.3 and 5.4 are expected effective discounts while those used in Definitions 5.1 and 5.2 are realized effective discounts. Essentially, the ex ante Hui borrowing (lending) interest rate to a member is his winning-probability-weighted average of the expected borrowing (lending) interest rates induced by the expected effective discounts throughout the duration. In general, we would expect different members to have different ex ante Hui interest rates since their expectations, income streams, as well as time preference characteristics, may differ, giving rise to different reservation discounts.

6

AN APPLICATION TO COLLUSION AMONG SEVERAL SELLERS UNDER REPEATED AUCTIONS[11]

The rotating credit association studied in previous sections can be applied to a form of tacit collusion among a small group of sellers in a sequential bidding setting with a single agent buying at regular intervals, via sealed bid auctions, an indivisible commodity from a fixed group of eligible sellers. The form of conspiracy is non-cooperative in the sense that once the rules are agreed upon and followed, the individual behavior is profit-maximizing in the usual Nash sense.

6.1 The Structure of Rotating Credit Collusion

Consider a buyer buying at regular intervals via sealed-bid auctions a single indivisible commodity supplied by a seller from a fixed group of N sellers.[12] Actual bids submitted are not to exceed the buyer's reservation price denoted by b^r. The sellers enter into bidding at no costs. In the absence of any collusion, sellers maximize their discounted expected profits given the actions of others. In a collusion with side payments, the seller with the lowest cost will be pre-selected to win the auction at any period and the spoils will be split according to some pre-agreed sharing rule.

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[11] This section is essentially taken from Chew, Mao and Reynolds (1984).

[12] A similar structure can be defined for a single seller and several buyers repeated bidding setting.

In rotating credit collusion, the N sellers agree, prior to the start of an N-period bidding cycle, on a withdrawal bid level b^w which is less than b^r. At period n during an N-period bidding cycle, only the m (= N-n+1) 'living' bidders (those who have yet to win an auction in any given cycle of N periods) are allowed to bid at or lower than b^w. The n-1 'dead' sellers (those who have won once prior to the given period) <u>withdraw</u> by submitting bids above b^w, forfeiting their chance to win but giving an appearnace to the buyer of still being active bidders. The incentive to do this is provided by the knowledge that the game will restart after each of the remaining living sellers has won once. The final 'winner' will receive b^w at period N. This is accomplished by submitting a bid of b^w under first-price auctions or submitting a bid at less than b^w but receiving b^w from the next highest bid submitted by a predetermined (e.g., the winner at period 1) dead seller under second-price auctions.

6.2 <u>Assumptions</u>

The following notation will be used in this section.

N : the number of sellers;

b_{in}: <u>ith</u> seller's bid at period n;

b_n^f : the lowest (i.e. winning) bid at period n;

b_n^s : the second lowest bid at period n;

b_n = b_n^f (b_n^s) in a first-price (second-price) setting;

b^r : the single buyer's reservation price;

b^w : the withdrawal bid level, $b^w < b^r$.

Other notations will be explained when they appear.

Assumption 0: There is no default.

Assumption 1: There is no Bayisian learning from past winning bids.

Assumption C2: Each seller i has a deterministic and known costs stream over time: $c_i = [c_{i1}, c_{i2}, \ldots, c_{it}, \ldots]$.

Assumption C3: Each seller maximizes his expected net present value of his profits stream given a market discount factor ρ.

We define below a reservation price v_{in} for the ith seller at period n based on the certain knowledge (Assumption 0) that the worst he can do is to receive b^w at the Nth auction:

$$\rho^{-(N-n)}(v_{in} - c_{in}) = b^w - c_{iN}. \tag{6.1}$$

We will refer to $v_i^n = \{v_{in}, \ldots, v_{iN}\}$, n = 1, 2, ..., N, as the ith seller's reservation price vector at period n. Obviously, $v_{iN} = b^w$ for all i.

Assumption C4: Every seller at period n believes that the reservation price of the other sellers are drawn independently from a common uniform distribution over $[\underline{v}_n, \bar{v}_n]$.

The discounted expected gain from future participation in bidding for seller i at period n-1 (n = 1, ..., N-1), denoted by $G_{i,n-1}$, is defined by:

$$G_{i,n-1} = \rho\{P_{in}(b_{in})(b_n - v_{in}) + [1 - P_{in}(b_{in})]G_{in}\}, \tag{6.2}$$

and

$$G_{i,N-1} = 0, \tag{6.2'}$$

where $P_{in}(b_{in})$ denotes the ith seller's subjective probability of winning the auction with a bid of b_{in} at period n. The probabilities of winning can be derived after we obtain the Nash equilibrium bidding strategies in the next subsection. In particular, G_{i0} represents the overall discounted expected gain from participation in the conspiracy at the start of each bidding cycle.

Assumption C5: Every seller believes that the discounted expected gain from participating in future biddings for all other sellers, contingent on losing in the current period, are equal to his own, i.e., $G_{in} = G_n$ for all i, where n = 1, ..., N.

Assumptions C4 and C5 are symmetry assumptions making the m (= N-n+1) sellers more alike. They are imposed in order to avoid using the traditional homogeneous sellers assumption.

6.3 Nash Equilibrium Bidding Strategies

Consider first the case of second-price auctions. Since in this rotation credit collusion model, a bidder is purely a seller (unlike the Hui bidder who is both a seller and a buyer), it is clear that the demand-revealing property of second-price auctions applies here. (See Vickrey (1961) and Cox, Roberson and Smith (1982).) Hence,

Theorem 6.1 (Second-Price): Under Assumptions 0 - 1 and C2 - C5, the second-price Nash equilibrium bidding program b_{in} for seller i is given by:

$$b_{in} = v_{in} + G_{in} \quad \text{for } n = 1, \ldots, N-1, \tag{6.3}$$

and

$$b_{iN} = v_{iN} - \varepsilon < b^{W}, \tag{6.3'}$$

where

$$G_{in} = \Sigma_{k=n+1}^{N-1}[(\bar{v}_k - v_{ik})/(\bar{v}_k - \underline{v}_k)]^{N-k}\rho^{k-n}[(\bar{v}_k - v_{ik})/(N-k+1)] \tag{6.4}$$

for n = 1, ..., N-2, and

$$G_{i,N-1} = 0. \tag{6.4'}$$

Proof: Omitted.

For the case of first-price auctions, we posit the following recursively defined bidding program:

$$b_{in} = v_{in} + G_{in} + (1/m)(\bar{v}_n - v_{in}), \quad \text{for } n = 1, \ldots, N-1, \tag{6.5}$$

and

$$b_{iN} = v_{iN} = b^W. \tag{6.5'}$$

The distribution of b_{in}, F_{in}, is given by:

$$1-F_{in}(b_{in}) = (\bar{v}_n - v_{in})/(\bar{v}_n - \underline{v}_n) = \eta(\bar{v}_n - b_{in} + G_{in}),$$

where $\eta = m/[(m-1)(\bar{v}_n - \underline{v}_n)]$. Clearly, the probability of winning P_{in} is given by

$$P_{in}(b_{in}) = [1-F_{in}(b_{in})]^{m-1}. \tag{6.6}$$

That (6.5) is a Nash equilibrium bidding strategy can be demonstrated by showing that it maximizes the discounted expected gain from participation in the current and future biddings, $G_{i,n-1}$, given by expression (6.2). Hence,

<u>Theorem 6.2</u> (First-Price): Under Assumptions 0 - 1 and C2 - C5, the first-price Nash equilibrium bidding program b_{in} for the seller i is given recursively by expression (6.5) with G_{in} given by expression (6.4).

Proof: Omitted.

Note that the expected gains from participation in the bidding conspiracy G_{in} are equal under first-price or second-price auctions. Consequently, the per period expected costs from the buyer's point of view under either auction institution are the same. This appears to correspond with Vickrey's results for single-object auctions. We do not expect this to remain the case if we extend the model to incorporate risk aversion in light of the results of Cox, Roberson and Smith (1982)'s extension of Vickrey's model.

7

CONCLUSION

This is a first attempt from the rational choice perspective at a rigorous understanding of the 'rotational' competitive bidding process as a mechanism for interest rate formation, as well as loans and savings allocations. To familarize the reader with the Hui structure, we began by giving several actual examples and using small Hui to illustrate the interworking of its structural parameters. We then introduced a definition of an agent's reservation discount vector, and demonstrated its existence and uniqueness under fairly general circumstances. It turned out to be particularly useful for the derivation of an optimal individual bidding strategy. In order to acquire tractable result, we gave up a lot of generality to obtain the Nash equilibrium bidding program and the associated ex post borrowing and lending interest rates as well as the ex ante winning-probability-weighted interest rates.

A side-reward from the Nash exercise is an almost explicit expression of members' surpluses for joining the bidding market and, hence, of the total members' surplus, which provides a natural candidate for an efficiency measure of different rotating credit markets. For example, loan allocations may be determined by lot, by seniority, or by other 'sociological' criteria (Little, 1957), not forgetting the neighborhood loan shark or pawn shop, the credit union, and the insurance companies.[13] The

[13] We may also examine a discriminative version of the rotational bidding process where the winner of an auction at period n collects $A-b_{jn}$ rather than $A-b_n$ from bidder j.

total members' surplus also provides the natural objective function for the dual problem -- the organizer's problem -- of the optimal combination of the membership. These are promising topics for immediate follow-up research.

Another direction for future work concerns the risk sharing aspect of the rotational bidding process. We can introduce an insurance dimension to the problem by adding an exogenous probability of a large loss in income at any one period, and study how the optimal bidding strategy may be modified to reflect the need to cover unexpected losses over time.

Another refinement of the model is to introduce risk aversion into the Nash model (e.g., agents may have time-additive, constant relative risk aversion von Neumann-Morgenstern utility function with their risk aversion indexes drawn from some known probability distribution).

What about introducing the possibility of default? This, if properly done, would considerably enrich the current model and should provide a theoretical explanation behind the typically observed risk sharing structure of a rotating credit association where the organizer bears the default risk of each member, but poses a common risk to the membership collectively. There appears to be a curious parallel between agency theory and the rotational bidding problem. The principal's problem is to explicitly manipulate the agent's payoff structure relative to the information-monitoring technology available, while the Hui organizer implicitly manipulates the member's payoff and his share of default risk by optimizing over the time preference, risk attitude, and riskiness of the other members over available membership pool. The organizer can also attempt to change the rules such as bearing only half of the default risk instead.

In the opposite direction, a rotating credit association in Vancouver, with a monthly loan pool of approximately $10,000, was formed jointly by its 20 members who are mutual friends, completely doing away with the organizer (Chang, 1981).

Beyond the immediate horizon is the question of the role of the informal financial sector whose capital is financed mainly by numerous rotating credit associations with organizers acting as information and risk arbitragers between the formal and the informal sectors. This seems to be a reasonable approximation to the capital market of Taiwan, Hong Kong, Singapore, and other east Asian countries and many African nations. One estimate puts the size of Ethopia's informal financial sector at 8% of its GNP (Miracle, Miracle and Cohen, 1980). Such a description may even apply, in the developed countries, to certain pocket of the pupolation within the overall economy. In Hawaii today, the rotating credit associations, called 'ko' among the Japanese, are sufficiently prevalent as to be declared illegal.[14]

Since Akerlof's "the market for lemons" paper (1970), we have understood a lot more about the cause of market failures due to informational asymmetry which makes the formal financial sector inherently imperfect. A not unreasonable conjecture may be that the informal sector complements the formal sector in financing smaller, shorter-term capital, by its relatively greater efficiency in information, monitoring, and even enforcement (which may, at times, be rather unorthodox). This seems to find at least

[14] This information came from conversations with a Japanese American student from Hawaii.

superficial support in the coexistence of the sophisticated financial ins- financial institutions of the formal sector in the West along with the rotating credit association type financial markets (among black West Indian immigrants in Brooklyn, New York; 'the very poor' in San Diego; and in centers of Asian immigration in the developed countries (Miracle, Miracle and Cohen, 1980)) and other forms of informal financial institutions, such as the above-mentioned intra-family loans, loan sharks and pawn shops.

In Section 6, we described a simple way in which collusion among a small group of sellers (since the gain from the rotating credit collusion is inversely related to the number of participants) in a repeated auctions setting can take place with minimal monitoring. This model may be modified or extended in a few ways. Instead of the private but known and deterministic cost streams, we may assume that every seller draws from a distribution his cost prior to each bidding period and investigate the effects of various distributional and informational assumptions about the costs on the resulting bidding strategies. We may also assume that sellers have additive intertemporal utilities which are not necessarily linear to study the effects of intertemporal risk aversion.

A different question related to the collusion model is whether there is an incentive to default on the agreement once a seller has won in a particular bidding period. In Chew and Reynolds (1984), it is shown that rotating credit collusion is consistent with noncooperative Nash behavior in an infinite horizon framework under a symmetric uncertain variable costs assumption. The length-of-collusion analysis of Radner (1980) and the collusion-with-bonding analysis of Eswaran and Lewis (1983) may also be adapted to shed light on this question.

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APPENDIX

Proof of Lemma 3.1:

By the definition of v_i,

$$U_i(Y_{i,N-1}) = U_i(Y_{iN}) = \bar{U}_i,$$

where

$$Y_{i,N-1} = (I_{i0}-A, I_{i1}-(A-v_{i1}), \ldots, I_{i,N-1}+NA-v_{i,N-1}, I_{iN}-A),$$

and

$$Y_{iN} = (I_{i0}-A, I_{i1}-(A-v_{i1}), \ldots, I_{i,N-1}-(A-v_{i,N-1}), I_{iN}+NA).$$

By monotonicity of U_i, $U_i(Y_{i,N-1})$ decreases in $v_{i,N-1}$ and $U_i(Y_{iN})$ increases in $v_{i,N-1}$. Given monotonicity and positive time preference, we have

$$\underline{v}_{i,N-1} \leqslant v_{i,N-1} \leqslant \bar{v}_{i,N-1},$$

where $\underline{v}_{i,N-1} = -\left|I_{iN}-I_{i,N-1}\right|$ and $\bar{v}_{i,N-1} = (N+1)A$. Let

$$\bar{a} = U_i(Y_{i,N-1} \mid v_{i,N-1}=\underline{v}_{i,N-1}), \qquad \underline{a} = U_i(Y_{i,N-1} \mid v_{i,N-1}=\bar{v}_{i,N-1}),$$

$$\underline{d} = U_i(Y_{iN} \mid v_{i,N-1}=\underline{v}_{i,N-1}), \qquad \bar{d} = U_i(Y_{iN} \mid v_{i,N-1}=\bar{v}_{i,N-1}).$$

Again, positive time preference and monotonicity imply that

$$\bar{a} > \underline{d}, \tag{A.1}$$

$$\bar{a} > \underline{a}, \tag{A.2}$$

$$\underline{d} < \bar{d}, \tag{A.3}$$

and

$$\underline{a} < \bar{d}. \tag{A.4}$$

Since U_i is assumed continuous, inequalities (A.1), (A.2), (A.3) and (A.4) imply that $v_{i,N-1}$ exists and is unique.

By similar arguments, it can be established that, in general, there exists a unique $v_i = (v_{i1}, v_{i2}, \ldots, v_{i,N-1})$ with $v_{in} \in [\underline{v}_{in}, \bar{v}_{in}]$, where

$$\underline{v}_{in} = \min\{I_{i,n+1} - I_{in}, \frac{1}{N-n}[I_{in} - I_{i,n+1} + (N-n-1)v_{i,n+1}]\},$$

$$\bar{v}_{in} = \frac{N+1}{N-n+1} A,$$

for all $n \in \{1, \ldots, N-1\}$, such that equation (3.1) holds.

Q.E.D.

Proof of Theorem 4.1:

At period n, b_k, $k = 1, \ldots, n-1$, are known. Given time-additive utility, the conditional reservation utility of member i at period n is

$$\bar{U}_i^n = \Sigma_{k=n}^{j-1} \lambda_{ik} u_i(I_{ik} - (A - v_{ik})) + \lambda_{ij} u_i(I_{ij} + NA - (N-j)v_{ij}) + \Sigma_{k=j+1}^{N} \lambda_{ik} u_i(I_{ik} - A),$$

$$j = n, n+1, \ldots, N. \tag{A.5}$$

Under assumptions 0, 1, 2, 3' and 4, the problem of member i at period n is choosing b_{in} to maximize $EU_{in}^f(b_{in})$ given in expression (4.3). Note that maximizing $EU_{in}^f(b_{in})$ is equivalent to maximizing

$$\begin{aligned}
E_{in}(b_{in}) &= \frac{1}{\lambda_{in}} [EU_{in}^f(b_{in}) - \bar{U}_i^n] \\
&= \frac{1}{\lambda_{in}} EU_{in}^f(b_{in}) \\
&\quad - \frac{1}{\lambda_{in}} \{\lambda_{in} u_i(I_{in} + NA - (N-n)v_{in}) + \Sigma_{k=n+1}^{N} \lambda_{ik} u_i(I_{ik} - A)\}[F_{in}(b_{in})]^{N-n} \\
&\quad - \frac{1}{\lambda_{in}} \{\lambda_{in} u_i(I_{in} - (A - v_{in})) + \bar{U}_i^{n+1}\}\{1 - [F_{in}(b_{in})]^{N-n}\}, \\
&= \{u_i(I_{in} + NA - (N-n)b_{in}) - u_i(I_{in} + NA - (N-n)v_{in})\}[F_{in}(b_{in})]^{N-n} \\
&\quad + \int_{b_{in}}^{\bar{b}_n} [u_i(I_{in} - A + x) - u_i(I_{in} - A + v_{in}) + \frac{\lambda_{i,n+1}}{\lambda_{in}} E_{i,n+1}^*] d[F_{in}(x)]^{N-n},
\end{aligned} \tag{A.6}$$

where

$$E^*_{i,n+1} = E_{i,n+1}(b^*_{i,n+1}) = \text{Max } E_{i,n+1}(b_{i,n+1}).$$

The first order condition for b_{in} to maximize expression (A.6), after suppressing the individual and time subscripts, is

$$0 = E'(b) = -(N-n)F(b)^{N-n-1}\{u'(I+NA-(N-n)b)F(b)-\phi F'(b)\}, \tag{A.7}$$

where

$$\phi = u(I+NA-(N-n)b) - u(I+NA-(N-n)v) - [u(I-A+b)-u(I-A+v)+E^*], \tag{A.8}$$

and

$$E^* = (\lambda_{i,n+1}/\ \lambda_{in})E^*_{i,n+1}.$$

Note that (A.7) implies equation (4.5).

With increasing utility, the RHS of equation (4.5) is increasing in b. With decreasing marginal-outbidden-rate distribution F(b), the LHS of equation (4.5) is decreasing in b. Therefore, the bid that satisfies (4.5) is unique.

The second order condition for b to maximize (A.6) requires that

$$(N-n)u''(I+NA-(N-n)b)F(b) - [(N-n+1)u'(I+NA-(N-n)b)+u'(I-A+b)]F'(b)$$
$$+ \phi F''(b) < 0,$$

which is satisfied locally by the b that solves equation (4.5) if F yields decreasing marginal outbidden rate. Hence, the bid that maximizes (A.6) is unique.

Q.E.D.

Proof of Corollary 4.5:

We want to show that the sign of $\frac{\partial b^*}{\partial v}$ is positive. Take total derivative of both sides of equation (A.7) to obtain

$$0 = E''(b^*)(\partial b^*) + \frac{\partial E'(b^*)}{\partial v}(\partial v)$$
$$= E''(b^*)(\partial b^*) + (N-n)[F(b^*)]^{N-n-1}F'(b)(\frac{\partial \phi}{\partial v})(\partial v).$$

Therefore

$$\frac{\partial b^*}{\partial v} = -\{(N-n)[F(b^*)]^{N-n-1}F'(b^*)(\frac{\partial \phi}{\tilde{v}})\}/E''_{in}(b^*),$$

where

$$\frac{\partial \phi}{\partial v} = (N-n)u'(I+NA-(N-n)v) + u'(I-A+v) > 0.$$

Since $E''(b^*) < 0$ and $F'(b) > 0$, we have $\frac{\partial b^*}{\partial v} > 0$ as desired.

The proof for the second-price case is similar.

Q.E.D.

Proof of Corollary 4.7:

Take the total derivative of (A.7) to obtain

$$0 = E''_{in}(b^*_{in})(\partial b^*_{in}) + [\frac{\partial E'_{in}(b^*_{in})}{\partial \lambda_{in}}](\partial \lambda_{in})$$

which implies

$$\frac{\partial b^*_{in}}{\partial \lambda_{in}} = -\frac{\partial E'_{in}(b^*_{in})}{\partial \lambda_{in}} \frac{1}{E''_{in}(b^*_{in})}.$$

Since

$$\frac{\partial E'_{in}(b^*_{in})}{\partial \lambda_{in}} = (N-n)[F_{in}(b^*_{in})]^{N-n-1}F'_{in}(b^*_{in})\cdot E^*_{i,n+1} \cdot \frac{\lambda_{i,n+1}}{(\lambda_{in})^2} > 0,$$

and $E''_{in}(b^*_{in}) < 0$, we have shown that $\partial b^*_{in}/\partial \lambda_{in} > 0$.

The proof of the second-price case is similar.

Q.E.D.

Proof of Theorem 5.1:

In this proof, without fear of confusion, we will omit the time index n to simplify the notation.

Suppose, at period n, bidder j believes that all his rivals adopt the bidding strategy function (5.3). By assumption 4', v_i, for all $i \neq j$, is uniformly distributed over $[\underline{v},\bar{v}]$. Since b_i is linear in v_i, by Assumption 4', b_i will be uniformly distributed over $[\underline{b},\bar{b}]$, where

$$\underline{b} = \underline{v} - \frac{1}{m} G_i^*,$$

$$\bar{b} = \bar{v} - \frac{1}{m+1} (\bar{v}-\underline{v}) - \frac{1}{m} G_i^*.$$

Recall that $m = N-n+1$ is the number of bidders at period n and that G_i^* is the identical subjectively discounted expected gain from future bidding participation.

The inverse of (5.3) is

$$v_i = f(b_i) = \frac{m+1}{m} [b_i - \frac{1}{m+1} \underline{v} + \frac{1}{m} G_i^*].$$

Therefore,

$$F_b(b_i) = F_v(v_i) = \int_{\underline{v}}^{f(b_i)} d[\frac{v-\underline{v}}{\bar{v}-\underline{v}}] = \eta\, [b_i - \underline{v} + \frac{1}{m} G_i^*],$$

and

$$F_b'(b_i) = \eta = (\frac{m+1}{m})/(\bar{v}-\underline{v}).$$

Under Assumptions 0, 1, 2, 3", 4' and 5, the expected utility difference for bidder j of a bid b_j is

$E_j(b_j)$

$$= (m-1)(v_j-b_j)[\eta(b_j-\underline{v}+ \frac{1}{m} G_j^*)]^{m-1} + (m-1)\int_{b_j}^{\bar{b}} (x-v_j+G_j^*)[\eta(x-\underline{v}+ \frac{1}{m} G_j^*]^{m-2}\eta dx,$$

$$= (m-1)\eta^{m-1}\{(v_j-b_j)(b_j-\underline{v}+\frac{1}{m}G_j^*)^{m-1} + \int_{b_j}^{\bar{b}}(x-v_j+G_j^*)(x-\underline{v}+\frac{1}{m}G_j^*)^{m-2}dx\}. \quad (A.9)$$

The first order condition for b_j^* to maximize E_j is

$$0 = E_j'(b_j^*)$$

$$= (m-1)\eta^{m-1}(b_j^*-\underline{v}+\frac{1}{m}G_j^*)^{m-2}\cdot[-(b_j-\underline{v}+\frac{1}{m}G_j^*)+(m-1)(v_j-b_j)-(b_j-v_j+G_j^*)]$$

$$= (m-1)\eta^{m-1}(b_j^*-\underline{v}+\frac{1}{m}G_j^*)^{m-2}[-(m+1)b_j^*+mv_j+\underline{v}-\frac{m+1}{m}G_j^*], \quad (A.10)$$

which implies

$$b_j^* = \frac{1}{m+1}(mv_j + \underline{v} - \frac{m+1}{m}G_j^*) = v_j - \frac{1}{m+1}(v_j-\underline{v}) - \frac{1}{m}G_j^*. \quad (A.11)$$

It is straightforward to check that

$$E_j''(b_j^*) = -(m-1)(m+1)\eta^{m-1}[b_j^* - \underline{v} + \frac{1}{m}G_j^*]^{m-2} < 0$$

since $m > 1$. Thus b_j^* also satisfies the second order condition. Since bidder j's optimal strategy (A.11) is the same as all his rivals', we have shown that the strategy function (5.3) is indeed a Nash equalibrium bidding strategy at period n.

It then follows that the vector $b_i = (b_{i1}, b_{i2}, \ldots, b_{i,N-1})$, with b_{in} given by (5.3), where $n = 1, 2, \ldots, N-1$, is a Nash equilibrium bidding program.

Q.E.D.